Chapter 3
Clifford Analysis, Dirac Operator and the Fourier Transform

In this chapter, we state basic knowledge, notations and terminologies in Clifford analysis and some related results. These preliminaries will be used to establish the theory of convolution singular integrals and Fourier multipliers on Lipschitz surfaces. In Sect. 3.1, we give a brief survey on basics of Clifford analysis. In Sect. 3.2, we state the monogenic functions on sectors introduced by Li, McIntosh, Qian [1]. Section 3.3 is devoted to the Fourier transform theory on sectors established by [1]. Section 3.4 is based on the Möbius covarian of iterated Dirac operators by Peeter and Qian in [2]. In Sect. 3.5, we give a generalization of the Fueter theorem in the setting of Clifford algebras [3]. In Chaps. 6 and 7, this generalization will be used to estimate the kernels of holomorphic Fourier multiplier operators on closed Lipschitz surfaces.

3.1 Preliminaries on Clifford Analysis

In this section, n and M denote the positive integers, L equals to 0 or $n+1$, and $M \geqslant \max\{n, L\}$. The real 2^M-dimensional Clifford algebra $\mathbb{R}_{(M)}$ or the complex 2^M-dimensional Clifford algebra $\mathbb{C}_{(M)}$ has basis vectors e_S, where S is any subset in $\{1, 2, \ldots, M\}$. Under the identifications

$$\begin{cases} e_0 = e_{\emptyset}, \\ e_j = e_{\{j\}}, \ 1 \leqslant j \leqslant M, \end{cases}$$

the multiplication of basis vectors satisfies

$$\begin{cases} e_0 = 1, \ e_j^2 = -e_0 = -1, \ 1 \leqslant j \leqslant M, \ e_0 = 1; \\ e_j e_k = -e_k e_j = e_{\{j,k\}}, \ 1 \leqslant j < k \leqslant M; \\ e_{j_1} e_{j_2} \ldots e_{j_s} = e_S, \ 1 \leqslant j_1 < j_2 < \cdots < j_s \leqslant M \text{ and } S = \{j_1, j_2, \ldots, j_s\}. \end{cases}$$

T. Qian and P. Li, *Singular Integrals and Fourier Theory on Lipschitz Boundaries*,
https://doi.org/10.1007/978-981-13-6500-3_3

Let $u = \sum_S u_S e_S$ and $v = \sum_T v_T e_T$ be two elements in $\mathbb{R}_{(M)}$ (or $\mathbb{C}_{(M)}$). Then the product of u and v can be expressed as

$$uv = \sum_{S,T} u_S v_T e_S e_T,$$

where $u_S, v_T \in \mathbb{R}$ (or $\mathbb{C}$), $u_\emptyset e_\emptyset$ is usually written as $u_0 e_0$ or u_0, and is called the scalar part of u.

By identifying the standard basis vectors $e_1,\ e_2, \ldots, e_n$ of $\mathbb{R}^n$ with their counterparts in $\mathbb{R}_{(M)}$ or $\mathbb{C}_{(M)}$, we embed the vector space $\mathbb{R}^{n+1}$ in the Clifford algebras $\mathbb{R}_{(M)}$ and $\mathbb{C}_{(M)}$. There are two usual methods to embed $\mathbb{R}^{n+1}$ in Clifford algebras. We treat them together by denoting standard basis vectors of $\mathbb{R}^{n+1}$ by $e_1, e_2, \ldots, e_n, e_L$, and identifying e_L with either e_0 or e_{n+1}.

On $\mathbb{R}_{(M)}$ and $\mathbb{C}_{(M)}$, we use the Euclidean norm $|u| = (\sum_S |u_S|^2)^{1/2}$. For a constant C depending only on M, $|uv| \leqslant C|u||v|$. If $u \in \mathbb{R}^{n+1}$, then we can take the constant C as 1. If $u \in \mathbb{C}^{n+1}$, the constant C is taken as $\sqrt{2}$.

We write $x \in \mathbb{R}^{n+1}$ as $x = \underline{x} + x_L e_L$, where $\underline{x} \in \mathbb{R}^n$ and $x_L \in \mathbb{R}$. We also write the Clifford conjugate of x as $\bar{x} = -\underline{x} + x_L \bar{e}_L$, where $\bar{e}_L e_L = 1$. Then

$$\bar{x}x = x\bar{x} = \sum_{j=1}^{n} x_j^2 + x_L^2 = |x|^2.$$

The Clifford algebras $\mathbb{R}_{(0)}$, $\mathbb{R}_{(1)}$ and $\mathbb{R}_{(2)}$ are the real numbers, the complex numbers and the quaternions, respectively. An important property of the three algebras is that every non-zero element has an inverse. Although this is not true in general, but every non-zero element $x = \underline{x} + x_L e_L$ in $\mathbb{R}^{n+1}$ has an inverse x^{-1} in $\mathbb{R}_{(M)}$. In fact, $x^{-1} = |x|^{-2}\bar{x} \in \mathbb{R}^{n+1} \subset \mathbb{R}_{(M)}$.

For the sake of convenience, we recall some basic knowledge in Clifford analysis. The differential operator

$$D = \underline{D} + \frac{\partial}{\partial x_L} e_L, \text{ where } \underline{D} = \sum_{k=1}^{n} \frac{\partial}{\partial x_k} e_k,$$

acts on C^1 functions $f = \sum_S f_S e_S$ of $n+1$ variables to give

$$Df = \sum_{k=1}^{n} \frac{\partial f_S}{\partial x_k} e_k e_S + \frac{\partial f_S}{\partial x_L} e_L e_S$$

and

$$fD = \sum_{k=1}^{n} \frac{\partial f_S}{\partial x_k} e_S e_k + \frac{\partial f_S}{\partial x_L} e_S e_L.$$

Let f be a C^1 function defined on an open subset of $\mathbb{R}^{n+1}$ with values in $\mathbb{R}_{(M)}$ or $\mathbb{C}_{(M)}$. If $Df = 0$, then f is called a left-monogenic function. If $fD = 0$, then f is called a right-monogenic function. If f is both left-monogenic and right-monogenic, we call f a monogenic function. For the left-monogenic function and the right-monogenic function, each component is harmonic. It is easy to prove that for fixed ζ, the function $e(x, \zeta)$ is a left-monogenic and right-monogenic function of variable x because

$$\begin{aligned}\frac{\partial}{\partial x_L} e_L e(x,\zeta) &= -e_L i\zeta e_L e(x,\zeta)\\ &= -e_L i\overline{e_L}\zeta e(x,\zeta)\\ &= -i\zeta e(x,\zeta) = -\underline{D}e(x,\zeta).\end{aligned}$$

Define a function E on $\mathbb{R}^{n+1} \setminus \{0\}$ as

$$k(x) = \frac{1}{\sigma_n}\frac{\overline{x}}{|x|^{n+1}},\ x \neq 0,$$

where σ_n is the volume of the unit n-sphere in $\mathbb{R}^{n+1}$. In Clifford analysis, for the above function E, the corresponding Cauchy integral formula holds.

Theorem 3.1.1 *Let Ω be a bounded open subset of $\mathbb{R}^{n+1}$ with the Lipschitz boundary $\partial\Omega$ and the exterior unit normal $\mathfrak{n}(y)$ defined for almost all $y \in \partial\Omega$. Assume that f is a left-monogenic function on the neighborhood of $\Omega \cup \partial\Omega$ and g is a right-monogenic function on the neighborhood of $\Omega \cup \partial\Omega$. Then*

$$\text{(i)}\ \int_\Sigma g(y)\mathfrak{n}(y)f(y)dS_y = 0;$$

$$\text{(ii)}\ \int_{\partial\Omega} g(y)\mathfrak{n}(y)E(x-y)dS_y = \begin{cases} g(x), & x \in \Omega,\\ 0, & x \notin \Omega \cup \partial\Omega;\end{cases}$$

$$\text{(iii)}\ \int_{\partial\Omega} E(x-y)\mathfrak{n}(y)f(y)dS_y = \begin{cases} f(x), & x \in \Omega,\\ 0, & x \notin \Omega \cup \partial\Omega.\end{cases}$$

Proof (i) is a direct corollary of Gauss's divergence theorem, while (ii) and (iii) can be deduced from (i) and the following identity which is easily verified:

$$\int_{|y-x|=r} \mathfrak{n}(y)E(y-x)dS_y = \int_{|y-x|=r} E(y-x)\mathfrak{n}(y)dS_y = 1,\ r > 0.$$

□

We also need the following result.

Theorem 3.1.2 *Let f be a right-monogenic function on $\mathbb{R}^{n+1} \setminus \{0\}$ and satisfy $|f(x)| \leqslant C/|x|^n$ for $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Then for some constant $c \in \mathbb{C}_{(n)}$, $f(x) = c\overline{x}/|x|^{n+1}$.*

For $\xi \in \mathbb{R}^n$, $\xi \neq 0$, define

$$\chi_{\pm}(\xi) = (1 \pm i\xi e_L|\xi|^{-1})/2$$

such that $\chi_+(\xi) + \chi_-(\xi) = 1$. By $(i\xi e_L)^2 = |\xi|^2$, we get

$$\begin{cases} \chi_+(\xi)^2 = \chi_+(\xi), \\ \chi_-(\xi)^2 = \chi_-(\xi), \\ \chi_+(\xi)\chi_-(\xi) = 0 = \chi_-(\xi)\chi_+(\xi). \end{cases}$$

Moreover, $i\xi e_L = |\xi|\chi_+(\xi) - |\xi|\chi_-(\xi)$, and in fact, for any polynomial $P(\lambda) = \sum a_k\lambda_k$ in one variable with scalar coefficients, we have

$$P(i\xi e_L) = \sum_k a_k(i\xi e_L)^k = P(|\xi|)\chi_+(\xi) + P(-|\xi|)\chi_-(\xi).$$

Hence, the polynomial p in m variables defined by $p(\xi) = P(i\xi e_L)$ satisfies $p(0) = P(0)$ and

$$p(\xi) = P(i\xi e_L) = P(|\xi|)\chi_+(\xi) + P(-|\xi|)\chi_-(\xi), \ \xi \neq 0.$$

It is natural to associate every function B of one real variable with a function b of n real variables. Precisely, if $|\xi|$ and $-|\xi|$ are in the domain of B,

$$b(\xi) = B(i\xi e_L) = B(|\xi|)\chi_+(\xi) + B(-|\xi|)\chi_-(\xi).$$

When 0 is in the domain of B, $b(\underline{0}) = B(0)$, where $\underline{0}$ denotes the 0-vector in $\mathbb{R}^n$.

We repeat this procedure for holomorphic functions of complex variables. At first for $\zeta = \xi + i\eta \in \mathbb{C}^n$, define

$$|\zeta|^2_{\mathbb{C}} = \sum_{j=1} \zeta_j^2 = |\xi|^2 - |\eta|^2 + 2i\langle\xi, \eta\rangle,$$

where $\xi, \eta \in \mathbb{R}^n$, and note that $(i\zeta e_L)^2 = |\zeta|^2_{\mathbb{C}}$. Hence, we extend $|\xi|^2$ holomorphically to $\mathbb{C}^n$. When $|\zeta|^2_{\mathbb{C}} \neq 0$, take $\pm|\zeta|_{\mathbb{C}}$ as its two square roots and define

$$\chi_{\pm}(\zeta) = \frac{1}{2}\left(1 \pm \frac{i\zeta e_L}{|\zeta|_{\mathbb{C}}}\right)$$

such that

$$\begin{cases} \chi_+(\zeta) + \chi_-(\zeta) = 1, \\ \chi_+(\zeta)^2 = \chi_+(\zeta), \\ \chi_-(\zeta)^2 = \chi_-(\zeta), \\ \chi_+(\zeta)\chi_-(\zeta) = 0 = \chi_-(\zeta)\chi_+(\zeta) \\ i\zeta e_L = |\zeta|_{\mathbb{C}}\chi_+(\xi) - |\zeta|_{\mathbb{C}}\chi_-(\xi). \end{cases}$$

Let $P(\lambda) = \sum a_k \lambda_k$ be a polynomial in one variable with complex coefficients, the corresponding polynomial in n variables is defined by

$$p(\zeta) = P(i\zeta e_L) = \sum_k a_k (i\zeta e_L)^k$$

and satisfies if $|\zeta|^2_{\mathbb{C}} \neq 0$, then

$$\begin{aligned} p(\zeta) &= P(i\zeta e_L) = P(|\zeta|_{\mathbb{C}})\chi_+(\zeta) + P(-|\zeta|_{\mathbb{C}})\chi_-(\zeta) \\ &= \frac{1}{2}\Big(P(|\zeta|_{\mathbb{C}}) + P(-|\zeta|_{\mathbb{C}})\Big) + \frac{1}{2}\frac{\Big(P(|\zeta|_{\mathbb{C}}) - P(-|\zeta|_{\mathbb{C}})\Big)i\zeta e_L}{|\zeta|_{\mathbb{C}}}; \end{aligned}$$

if $|\zeta|^2_{\mathbb{C}} = 0$, then

$$p(\zeta) = P(0) + P'(0)i\zeta e_L.$$

Let B be any holomorphic function in one variable defined on the open subset S in $\mathbb{C}$ and let b be the holomorphic Clifford-valued function in n variables. For all $\zeta \in \mathbb{C}^n$, $\{\pm|\zeta|_{\mathbb{C}}\} \subset S$. The correspondence between B and b can be defined as follows naturally. If $|\zeta|^2_{\mathbb{C}} \neq 0$, then

$$\begin{aligned} b(\zeta) &= B(i\zeta e_L) = B(|\zeta|_{\mathbb{C}})\chi_+(\zeta) + B(-|\zeta|_{\mathbb{C}})\chi_-(\zeta) \\ &= \frac{1}{2}\Big(B(|\zeta|_{\mathbb{C}}) + B(-|\zeta|_{\mathbb{C}})\Big) + \frac{1}{2}\frac{\Big(B(|\zeta|_{\mathbb{C}}) - B(-|\zeta|_{\mathbb{C}})\Big)i\zeta e_L}{|\zeta|_{\mathbb{C}}}. \end{aligned}$$

If $|\zeta|^2_{\mathbb{C}} = 0$, then

$$b(\zeta) = B(0) + B'(0)i\zeta e_L.$$

The reason that the above correspondence is natural because not only b is the desired polynomial when B is a polynomial, but also the mapping from B to b is an algebra homomorphism. In other words, if F is another holomorphic function defined on S and $c_1, c_2 \in \mathbb{C}$, then

$$(c_1F + c_2B)(i\zeta e_L) = c_1F(i\zeta e_L) + c_2B(i\zeta e_L)$$

and

$$(FB)(i\zeta e_L) = F(i\zeta e_L)B(i\zeta e_L).$$

We give an example. For any real number t, define the holomorphic function $E_t(\lambda) = e^{-t\lambda}$ with variable $\lambda \in \mathbb{C}$. The corresponding function in n variables is defined as follows. If $|\zeta|^2_{\mathbb{C}} \neq 0$,

$$\begin{aligned} e(te_L,\ \zeta) &= E_t(i\zeta e_L) = e^{-t|\zeta|_{\mathbb{C}}}\chi_+(\zeta) + e^{t|\zeta|_{\mathbb{C}}}\chi_-(\zeta) \\ &= \cosh(t|\zeta|_{\mathbb{C}}) - \sinh(t|\zeta|_{\mathbb{C}})|\zeta|^{-1}_{\mathbb{C}} i\zeta e_L. \end{aligned}$$

If $|\zeta|^2_{\mathbb{C}} = 0$,

$$e(te_L, \zeta) = 1 - ti\zeta e_L.$$

Then

$$e(te_L, \zeta)e(se_L, \zeta) = e((t+s)e_L, \zeta)$$

and $e(te_L, -\zeta) = e(-te_L, \zeta)$. In addition,

$$\frac{d}{dt}e(te_L, \zeta) = -i\zeta e_L e(te_L, \zeta) = -e(te_L, \zeta)i\zeta e_L.$$

For any complex number α, another example is the function defined by $R_\alpha(\lambda) = (\lambda - \alpha)^{-1}$, $\lambda \neq \alpha$. Then

$$R_\alpha(i\zeta e_L) = (i\zeta e_L - \alpha)^{-1} = (i\zeta e_L + \alpha)(|\zeta|^2_{\mathbb{C}} - \alpha^2)^{-1},\ \ |\zeta|^2_{\mathbb{C}} \neq \alpha^2.$$

From now on, although we assume that $|\zeta|^2_{\mathbb{C}} \notin (-\infty, 0]$ and $\mathrm{Re}|\zeta|_{\mathbb{C}} > 0$, it has been unimportant which sign we assign to each square root of $|\zeta|^2_{\mathbb{C}}$. Now we prove some estimates.

Theorem 3.1.3 *Let $\zeta = \xi + i\eta \in \mathbb{C}^n$, where $\xi, \eta \in \mathbb{R}^n$, and assume that $|\zeta|^2_{\mathbb{C}} \notin (-\infty, 0]$. Let*

$$\theta = \tan^{-1}\left(\frac{|\eta|}{\mathrm{Re}|\zeta|_{\mathbb{C}}}\right) \in [0,\ \pi/2).$$

Then

(a) $0 < \mathrm{Re}|\zeta|_{\mathbb{C}} \leqslant |\xi| \leqslant \sec\theta \mathrm{Re}|\zeta|_{\mathbb{C}}$,
(b) $\mathrm{Re}|\zeta|_{\mathbb{C}} \leqslant ||\zeta|_{\mathbb{C}}| \leqslant \sec\theta \mathrm{Re}|\zeta|_{\mathbb{C}} \leqslant |\zeta| \leqslant (1 + 2\tan^2\theta)^{1/2}\mathrm{Re}|\zeta|_{\mathbb{C}}$,
(c) $-\theta \leqslant \arg|\zeta|_{\mathbb{C}} \leqslant \theta$,
(d) $|\chi_\pm(\zeta)| \leqslant \sec\theta/\sqrt{2}$.

Proof It is easy to prove

$$||\zeta|_{\mathbb{C}}|^2 = ||\zeta|^2_{\mathbb{C}}| = \left((|\xi|^2 - |\eta|^2)^2 + 4\langle\xi,\ \eta\rangle^2\right)^{1/2} \leqslant |\xi|^2 + |\eta|^2 = |\zeta|^2.$$

Hence

$$\mathrm{Re}|\zeta|_{\mathbb{C}} \leqslant ||\zeta|_{\mathbb{C}}| \leqslant |\zeta|. \tag{3.1}$$

Taking the real part of the identity

$$-(\xi + i\eta)^2 = -\zeta^2 = |\zeta|_{\mathbb{C}}^2 = \left(\mathrm{Re}|\zeta|_{\mathbb{C}} + i\mathrm{Im}|\zeta|_{\mathbb{C}}\right)^2,$$

we obtain

$$|\xi|^2 - |\eta|^2 = (\mathrm{Re}|\zeta|_{\mathbb{C}})^2 - (\mathrm{Im}|\zeta|_{\mathbb{C}})^2 \tag{3.2}$$

or

$$2|\xi|^2 - |\zeta|^2 = 2(\mathrm{Re}|\zeta|_{\mathbb{C}})^2 - ||\zeta|_{\mathbb{C}}|^2.$$

Therefore, by (3.1), we get $\mathrm{Re}|\zeta|_{\mathbb{C}} \leqslant |\xi|$. In addition, by (3.2), we have

$$|\xi|^2 \leqslant |\eta|^2 + (\mathrm{Re}|\zeta|_{\mathbb{C}})^2 = (\tan^2\theta + 1)(\mathrm{Re}|\zeta|_{\mathbb{C}}^2).$$

This means $|\xi| \leqslant \sec\theta \mathrm{Re}|\zeta|_{\mathbb{C}}$ and (a) is proved.

Another corollary of (3.2) is

$$|\zeta|^2 = 2|\eta|^2 + (\mathrm{Re}|\zeta|_{\mathbb{C}})^2 - (\mathrm{Im}|\zeta|_{\mathbb{C}})^2 \leqslant (1 + 2\tan^2\theta)(\mathrm{Re}|\zeta|_{\mathbb{C}}^2),$$

which implies (b).

(c) is a direct corollary of the inequality $||\zeta|_{\mathbb{C}}| \leqslant \sec\theta \mathrm{Re}|\zeta|_{\mathbb{C}}$, and (d) can be deduced from $|\zeta| \leqslant (1 + 2\tan^2\theta)^{1/2}||\zeta|_{\mathbb{C}}|$. □

Define

$$S_\mu^0 = \left\{\zeta = \xi + i\eta \in \mathbb{C}^n : \ |\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0] \text{ and } |\eta| < \mathrm{Re}(|\zeta|_{\mathbb{C}}) \tan\mu\right\}.$$

By (c) of Theorem 3.1.3, we know that $|\zeta|_{\mathbb{C}} \in S_{\mu,+}^0(\mathbb{C})$ and $-|\zeta|_{\mathbb{C}} \in S_{\mu,-}^0$ when $\zeta \in S_\mu^0(\mathbb{C}^m)$. So for any holomorphic function B defined on $S_\mu^0(\mathbb{C}) = S_{\mu,+}^0(\mathbb{C}) \cup S_{\mu,-}^0$, the corresponding holomorphic function b in n variables:

$$b(\zeta) = B(i\zeta e_L) = B(|\zeta|_{\mathbb{C}})\chi_+(\zeta) + B(-|\zeta|_{\mathbb{C}})\chi_-(\zeta)$$

is defined on $S_\mu^0(\mathbb{C}^n)$. In addition, by (d) of Theorem 3.1.3, if B is bounded, then

$$\|b\|_\infty \leqslant \sqrt{2}\sec\mu\|B\|_\infty.$$

Let

$$H^\infty(S_\mu^0(\mathbb{C}^n)) = H^\infty(S_\mu^0(\mathbb{C}^n), \mathbb{C}_{(M)})$$

be the Banach space of bounded Clifford-valued holomorphic functions on $S_\mu^0(\mathbb{C}^n)$. We have the following result.

Theorem 3.1.4 *The mapping $B \to b$ defined above is a one-one bounded algebra homomorphism from $H^\infty(S_\mu^0(\mathbb{C}))$ to $H^\infty(S_\mu^0(\mathbb{C}^n))$.*

Proof We only need to prove the mapping is one-one. In fact, this point can be deduced from the following formula, and the reverse result from b to B still holds:

$$B(\lambda) = \frac{2}{\sigma_{n-1}} \int_{|\xi|=1} b(\lambda\xi)\chi_{\pm}(\xi)d\xi, \ \lambda \in S^0_{\mu,\pm}(\mathbb{C}),$$

where σ_{n-1} is the volume of the unit $(n-1)$-sphere in $\mathbb{R}^n$. □

So far we have considered Clifford-valued holomorphic functions of n complex variables. What is called Clifford analysis is the study of monogenic functions of $n+1$ real variables. In the next section, we will relate these two concepts via the Fourier transform. We need to introduce the following generalized exponential function:

$$\begin{aligned} e(x,\zeta) &= e(\underline{x} + x_L e_L, \zeta) \\ &= e^{i\langle \underline{x},\ \zeta\rangle} e(x_L e_L, \zeta) \\ &= e^{i\langle \underline{x},\ \zeta\rangle}(e^{-x_L|\zeta|_{\mathbb{C}}}\chi_+(\zeta) + e^{x_L|\zeta|_{\mathbb{C}}}\chi_-(\zeta)). \end{aligned}$$

For any $x = \underline{x} + x_L e_L \in \mathbb{R}^{n+1}$, this function is holomorphic on $\zeta \in \mathbb{C}^n$ and is a left-monogenic function of $x \in \mathbb{R}^{n+1}$ for any $\zeta \in \mathbb{C}^n$. This function satisfies

$$\begin{cases} e(x,\zeta)e(y,\zeta) = e(x+y,\zeta), \\ e(x,-\zeta) = e(-x,\zeta). \end{cases}$$

Specially, when $\underline{x} \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$, $e(\underline{x},\ \xi) = e^{i\langle \underline{x},\ \xi\rangle}$, i.e., the usual exponential function in the Fourier theory. Moreover, for any $\zeta \in \mathbb{C}^n$, $e(x,\zeta)\overline{e_L}$ is also a right-monogenic function of $x \in \mathbb{R}^{n+1}$. We point out that

$$e(x,\zeta) = \exp i(\langle \underline{x},\zeta\rangle - x_L\zeta e_L) = \sum_{k=0} \frac{1}{k!}(i(\langle \underline{x},\zeta\rangle - x_L\zeta e_L))^k.$$

3.2 Monogenic Functions on Sectors

On the Lipschitz surface, to establish the relation between holomorphic multipliers and the functional calculus of Dirac operator, Li, McIntosh, Qian [1] introduced the monogenic functions on sectors. In this section, we will give a detailed statement for the function classes $K(S_{N_\mu})$, $K(C^+_{N_\mu})$ and $K(C^-_{N_\mu})$ which will be used in Sect. 3.3 and Chap. 5 below.

We start by specifying some sets of unit vectors in

$$\mathbb{R}^{n+1}_+ = \left\{x = \underline{x} + x_L e_L \in \mathbb{R}^{n+1} : \ x_L > 0\right\}.$$

For these unit vectors, we use the metric $\angle(\mathfrak{n}, y) = \cos^{-1}\langle \mathfrak{n}, y\rangle$.

Let N be a compact set of unit vectors in $\mathbb{R}^{n+1}_+$ which contains e_L and let

$$\mu_N = \sup_{\mathfrak{n}\in N} \angle(\mathfrak{n}, e_L).$$

Then $0 \leqslant \mu_N < \pi/2$. For $0 < \mu < \pi/2 - \mu_N$, define the open neighborhood N_μ of N in the unit sphere by

$$N_\mu = \left\{ y \in \mathbb{R}_+^{n+1} : \ |y| = 1, \ \angle(y, \mathfrak{n}) < \mu \text{ for some } \mathfrak{n} \in N \right\}.$$

For every unit vector n, let C_n^+ be the open half space

$$C_\mathfrak{n}^+ = \left\{ x \in \mathbb{R}^{n+1} : \ \langle x, \mathfrak{n} \rangle > 0 \right\},$$

and define the open cones in $\mathbb{R}^{n+1}$ as follows. Let

$$\begin{cases} C_{N_\mu}^+ = \bigcup \{ C_n^+ : \ \mathfrak{n} \in N_\mu \}, \\ C_{N_\mu}^- = -C_{N_\mu}^+, \\ S_{N_\mu} = C_{N_\mu}^+ \cap C_{N_\mu}^-. \end{cases}$$

Definition 3.2.1 $K(C_{N_\mu}^+)$ is defined as the Banach space of all right monogenic functions Φ from $C_{N_\mu}^+$ to $\mathbb{C}_{(M)}$ satisfying

$$\|\Phi\|_{K(C_{N_\mu}^+)} = \frac{1}{2}\sigma_n \sup_{x\in C_{N_\mu}^+} |x|^n |\Phi(x)| < \infty.$$

Similarly, we can define $K(C_{N_\mu}^-)$.

Definition 3.2.2 Define $K(S_{N_\mu})$ as the Banach space of all function pairs $(\Phi, \underline{\Phi})$, where Φ is a right-monogenic function from S_{N_μ} to $\mathbb{C}_{(M)}$, and $\underline{\Phi}$ is continuous on $(0, +\infty)e_L$ such that $(\Phi, \underline{\Phi})$ satisfies

$$\underline{\Phi}(Re_L) - \underline{\Phi}(re_L) = \int_{r\leqslant|\underline{x}|\leqslant R} \Phi(\underline{x}) d\underline{x} e_L,$$

and

$$\|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})} = \frac{1}{2}\sigma \sup_{x\in S_{N_\mu}} |x|^n |\Phi(x)| + \sup_{r>0} |\underline{\Phi}(re_L)| < +\infty.$$

Notice that $\underline{\Phi}$ is determined by Φ up to an additive constant, and

$$\underline{\Phi}'(re_L) = \int_{|\underline{x}|=r} \Phi(\underline{x}) d\underline{x} e_L.$$

In addition, $\underline{\Phi}$ has a unique and continuous extension to the cone

$$T_{N_\mu} = \left\{ y = \underline{y} + y_L e_L \in \mathbb{R}^{n+1}_+ : \ y^\perp \subset S_{N_\mu} \right\}.$$

This extension satisfies

$$\underline{\Phi}(y) - \underline{\Phi}(z) = \int_{A(y,z)} f(x)\mathfrak{n}(x) dS_x,$$

where $A(y, z)$ is a smoothly oriented n-manifold in S_{N_μ} jointing the $(m-1)$-sphere

$$S_y = \left\{ x \in \mathbb{R}^{n+1} : \ \langle x, y \rangle = 0 \text{ and } |x| = |y| \right\}$$

to the $(n-1)$-sphere S_x, in which case, for all $y \in T_{N_\mu}$,

$$|\underline{\Phi}(y)| \leqslant \|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})}.$$

If N is rotationally symmetric, i.e.,

$$N = \left\{ \mathfrak{n} = \underline{\mathfrak{n}} + \mathfrak{n}_L e_L \in \mathbb{R}^{n+1}_+ : \ |\mathfrak{n}| = 1, \mathfrak{n}_L \geqslant |\underline{\mathfrak{n}}| \cot \omega \right\},$$

we use the symbol

$$T^0_\mu = T_{N_{\mu-\omega}} = \left\{ y = \underline{y} + y_L e_L \in \mathbb{R}^{n+1} : \ y_L > |x| \cot \mu \right\}.$$

Now we state the relationship between these spaces. Here $H_{y,\pm}$ denote the hemispheres

$$H_{y,\pm} = \left\{ x \in \mathbb{R}^{n+1} : \ \pm\langle x, y \rangle \geqslant 0 \text{ and } |x| = |y| \right\}.$$

with the boundaries S_y.

Theorem 3.2.1 *(i) Let $\Phi_\pm \in K(C^\pm_{N_\mu})$. Define the function $\underline{\Phi_\pm}$ on T_{N_μ} as*

$$\underline{\Phi_\pm}(y) = \pm \int_{H_{y,\pm}} \Phi_\pm(x)\mathfrak{n}(x) dS_x, \ \ y \in T_{N_\mu},$$

where $\mathfrak{n}(x) = x/|x|$ is the normal to the hemisphere $H_{y,\pm}$. Then

$$(\Phi, \underline{\Phi}) = (\Phi_+ + \Phi_-, \ \underline{\Phi_+} + \underline{\Phi_-}) \in K(S_{N_\mu})$$

and

$$\|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})} \leqslant \|\Phi_+\|_{K(C^+_{N_\mu})} + \|\Phi_-\|_{K(C^-_{N_\mu})}.$$

(ii) Conversely, assume that $(\Phi, \underline{\Phi}) \in K(S_{N_\mu})$. There exists unique functions $\Phi_\pm \in K(C^\pm_{N_\mu})$ satisfying $\Phi = \Phi_+ + \Phi_-$ and $\underline{\Phi} = \underline{\Phi_+} + \underline{\Phi_-}$. For all $n \in N_\mu$ and $x \in C^\pm_n \subset C^\pm_{N_\mu}$,

$$\Phi_{\pm}(x) = \pm \lim_{\epsilon \to 0} \left(\int\limits_{\langle y, \mathfrak{n} \rangle = 0,\ |y| \geqslant \epsilon} \Phi(y)\mathfrak{n}(x)E(x-y)dS_y + \underline{\Phi}(\epsilon e_L)k(x) \right),$$

where $E(x) = \overline{x}/\sigma|x|^{n+1}$, *and* $(\Phi, \underline{\Phi})$ *satisfies*

$$\|\Phi_{\pm}\|_{K(C^{\pm}_{N_\mu})} \leqslant c\|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})},$$

where c *only depends on* μ_N, μ *and the dimension* n.

Proof (i) In order to prove

$$\underline{\Phi_{\pm}}(y) - \underline{\Phi_{\pm}}(z) = \int\limits_{A(y,z)} \Phi_{\pm}(x)\mathfrak{n}(x)dS_x,$$

we apply Cauchy's theorem to the right monogenic functions $\Phi_{\pm}$. The bound is straightforward.

(ii) This is a direct corollary of the results of [4]. In other words, there exists a natural isomorphism:

$$K(S_{N_\mu}) \simeq K(C^{+}_{N_\mu}) \oplus K(C^{-}_{N_\mu}).$$

We also need the closed linear subspaces $M(C^{\pm}_{N_\mu})$ of $K(C^{\pm}_{N_\mu})$. The functions in $M(C^{\pm}_{N_\mu})$ are both left monogenic and right monogenic. The subspaces $M(S_{N_\mu})$ of $K(S_{N_\mu})$ satisfying

$$M(S_{N_\mu}) \simeq M(C^{+}_{N_\mu}) \oplus M(C^{-}_{N_\mu})$$

are

$$M(S_{N_\mu}) = \left\{(\Phi, \underline{\Phi}) \in K(S_{N_\mu}) :\ \Phi \text{ is left monogenic and satisfies (3.3)}\right\},$$

where for $r > 0$,

$$\int\limits_{|\underline{y}|=r} \langle \underline{y}, \underline{x}\rangle r^{-1}(e_L\Phi(\underline{y})\underline{y} - \underline{y}\Phi(\underline{y}e_L))dS_y + \underline{x}\underline{\Phi}(re_L) - e_L\underline{\Phi}(re_L)\underline{x}e_L = 0. \quad (3.3)$$

It is easy to see that

(i) the value of the integral is independent of r,
(ii) if $\Phi \in M(C^{\pm}_{N_\mu})$, the integral equals to 0.

We only need to prove that when $(\Phi, \underline{\Phi}) \in M(S_{N_\mu})$, the function $\Phi_{\pm}$ defined in (ii) of Theorem 3.2.1 is left monogenic. We omit the details and refer to [4]. □

Now we consider convolutions. Assume that $\Phi \in K(C^{+}_{N_\mu})$, $\Psi \in M(C^{+}_{N_\mu})$ and $x \in C^{+}_{\mathfrak{n}} \subset C^{+}_{N_\mu}$. Define $(\Phi * \Psi)$ as

$$(\Phi * \Psi)(x) = \int\limits_{\langle y, \mathfrak{n}\rangle=\delta} \Phi(x-y)\mathfrak{n}(y)\Psi(y)dS_y$$
$$= \int\limits_{\langle y,\ \mathfrak{n}\rangle=0,\ |y|\geqslant\epsilon} \Phi(x-y)\mathfrak{n}(y)\Psi(y)dS_y + \underline{\Phi}(\epsilon e_L)\Psi(x),$$

where $0 < \delta < \langle x,\ \mathfrak{n}\rangle$. By Cauchy's theorem and the assumptions that Φ is right monogenic and Ψ is left monogenic, we can deduce that the integral is independent of the choice of the surfaces. On the other hand, because Ψ is right monogenic, $\Phi * \Psi$ is right monogenic. In fact, we can see from the following Theorem 3.3.1 that for all $\nu < \mu$,

$$\|\Phi * \Psi\|_{K(C^+_{N_\nu})} \leqslant c_{\nu,\mu}\|\Phi\|_{K(C^+_{N_\mu})}\|\Psi\|_{K(C^+_{N_\mu})}.$$

Moreover, if $\Psi_1 \in M(C^+_{N_\mu})$, then $\Psi * \Psi_1$ is both left monogenic and right monogenic, and

$$\Phi * (\Psi * \Psi_1) = (\Phi * \Psi) * \Psi_1.$$

For the functions defined on $C^-_{N_\mu}$, we have a similar result.

If $(\Phi, \underline{\Phi}) \in K(S_{N_\mu})$ and $(\Psi, \underline{\Psi}) \in M(S_{N_\mu})$, define

$$(\Phi, \underline{\Phi}) * (\Psi, \underline{\Psi}) \in M(S_{N_\mu}) = (\Phi_+ * \Psi_+ + \Phi_- * \Psi_-,\ \underline{\Phi_+ * \Psi_+} + \underline{\Phi_- * \Psi_-}).$$

Hence we can get for all $\nu < \mu$,

$$\|(\Phi, \underline{\Phi}) * (\Psi, \underline{\Psi})\|_{K(S^0_{\nu,+})} \leqslant C_{\nu,\mu}\|(\Phi, \underline{\Phi})\|_{K(S^0_{\mu,+})}\|(\Psi, \underline{\Psi})\|_{K(S^0_{\mu,+})}.$$

Let K^+_N be the linear space of all functions Φ on $\mathbb{R}^n \setminus \{0\}$ which can be extended monogenically to $\Phi \in K(C^+_{N_\mu})$ for some $\mu > 0$. Similarly, we define K^-_N, K_N, M^+_N, M^-_N and M_N, such that

$$K_N \simeq K^+_N \oplus K^-_N$$

and

$$M_N \simeq M^+_N \oplus M^-_N,$$

while M_N, M^+_N and M^-_N are all convolution algebras. The functions which belong to both K^+_N and K^-_N are of the form $\Phi(\underline{x}) = ck(\underline{x})$ for some $c \in \mathbb{C}_{(M)}$, where

$$E(\underline{x}) = \frac{1}{\sigma_n}\frac{-\underline{x}}{|\underline{x}|^{n+1}},\ \underline{x} \in \mathbb{R}^n \setminus \{0\},$$

with the monogenic extension

$$k(x) = \frac{1}{\sigma_n}\frac{\overline{x}}{|x|^{n+1}}.$$

The embedding of K_N^+ into K_N is defined as $ck \in K_N^+ \to (ck,\ c/2) \in K_N$, while the embedding K_N^- in K_N is defined as $ck \in K_N^- \to (ck,\ -c/2) \in K_N$.

3.3 Fourier Transforms on the Sectors

The section is devoted to the Fourier transform $\mathcal{F}_{\pm}(\Phi)$ of the function $\Phi \in K_N^{\pm}$ and the Fourier transform $\mathcal{F}(\Phi, \underline{\Phi})$ of $(\Phi, \underline{\Phi})$ introduced by Li, McIntosh, Qian [1]. We will prove these transforms are bounded holomorphic functions defined on the cones in $\mathbb{C}^m$. We also prove that $\mathcal{F}_+, \mathcal{F}_-$ and $\mathcal{F}$ are algebra homomorphism from the convolution algebras M_N^+, M_N^- and M_N to holomorphic functions.

We first associate with every unit vector $\mathfrak{n} = \underline{\mathfrak{n}} + \mathfrak{n}_L e_L \in \mathbb{R}^{n+1}$ satisfying $\mathfrak{n}_L > 0$, a real n-dimensional surface $\mathfrak{n}(\mathbf{C}^m)$ in $\mathbb{C}^n$, defined as follows.

$$\begin{aligned}\mathfrak{n}(\mathbb{C}^n) &= \left\{\zeta = \xi + i\eta \in \mathbb{C}^n \,:\, \xi \neq 0 \text{ and } \mathfrak{n}_L \eta = (\mathfrak{n}_L^2|\xi|^2 + \langle x, \underline{\mathfrak{n}}\rangle^2)^{1/2}\underline{\mathfrak{n}}\right\}\\ &= \left\{\zeta = \xi + i\eta \in \mathbb{C}^n \,:\, |\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0] \text{ and } \mathfrak{n}_L \eta = \mathrm{Re}(|\zeta|_{\mathbb{C}})\underline{\mathfrak{n}}\right\}\\ &= \left\{\zeta = \xi + i\eta \in \mathbb{C}^n \,:\, |\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0] \text{ for some } \kappa > 0, \eta + \mathrm{Re}(|\zeta|_{\mathbb{C}})e_L = \kappa\mathfrak{n}\right\},\end{aligned}$$

where

$$|\zeta|_{\mathbb{C}}^2 = \sum_{j=1}^{n} \zeta_j^2 = |\xi|^2 - |\eta|^2 + 2i\langle x, \eta\rangle.$$

The surfaces associated with distinct unit vectors are disjoint. In particular, $e_L(\mathbb{C}^n) = \mathbb{R}^n \setminus \{0\}$. On these surfaces, $|\xi|$, $|\zeta|$, $\mathrm{Re}(|\zeta|_{\mathbb{C}})$ and $||\zeta|_{\mathbb{C}}|$ are all equivalent. In fact, by Theorem 3.1.3,

$$\mathrm{Re}|\zeta|_{\mathbb{C}} \leqslant |\xi| \leqslant (\mathfrak{n}_L)^{-1}\mathrm{Re}|\zeta|_{\mathbb{C}},$$

and for all $\zeta \in \mathfrak{n}(\mathbb{C}^n)$,

$$\mathrm{Re}|\zeta|_{\mathbb{C}} \leqslant ||\zeta|_{\mathbb{C}}| \leqslant (\mathfrak{n}_L)^{-1}\mathrm{Re}|\zeta|_{\mathbb{C}} \leqslant |\zeta| \leqslant (n_L)^{-1}(1 + |\underline{\mathfrak{n}}|^2)^{1/2}\mathrm{Re}|\zeta|_{\mathbb{C}}.$$

Further, the parametrization used in the first definition of $\mathfrak{n}(\mathbb{C}^n)$ is smooth, with

$$\left|\det\left(\frac{\partial\zeta_j}{\partial\xi_k}\right)\right| \leqslant \frac{1}{\mathfrak{n}_L},\ \xi \neq 0.$$

To prove this, without loss of generality, we can assume that $\mathfrak{n} = \mathfrak{n}_1 e_1 + \mathfrak{n}_L e_L$. So

$$\zeta = \xi + i\frac{\mathfrak{n}_1}{\mathfrak{n}_L}(|\xi|^2\mathfrak{n}_L^2 + \xi_1^2\mathfrak{n}_1^2)^{1/2}e_1.$$

Then if $j \geqslant 2$, $\partial\zeta_j/\partial\xi_k = \delta_{jk}$ and

$$\frac{\partial\zeta_1}{\partial\xi_k} = \delta_{1k} + \frac{i\mathfrak{n}_1\xi_k(\mathfrak{n}_L^2 + \delta_{1k}\mathfrak{n}_1^2)}{\mathfrak{n}_L(|\xi|^2\mathfrak{n}_L^2 + \xi_1^2\mathfrak{n}_1^2)^{1/2}}.$$

Hence when $k \geqslant 2$,

$$\left|\frac{\partial\zeta_1}{\partial\xi_1}\right| \leqslant \frac{1}{\mathfrak{n}_L} \text{ and } \left|\frac{\partial\zeta_1}{\partial\xi_k}\right| \leqslant \mathfrak{n}_1.$$

The estimate for the Jacobian follows.

For the open set N_μ of the unit vectors defined above, we define the open cones $N_\mu(\mathbb{C}^n)$ in $\mathbb{C}^n$ as follows:

$$\begin{aligned} N_\mu(\mathbb{C}^n) &= \bigcup_{\mathfrak{n}\in N_\mu} \mathfrak{n}(\mathbb{C}^n) \\ &= \Big\{\zeta = \xi + i\eta \in \mathbb{C}^n :\ |\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0] \text{ for some } \kappa > 0 \text{ and } \mathfrak{n} \in N_\mu, \\ &\qquad \eta + \mathrm{Re}(|\zeta|_{\mathbb{C}})e_L = \kappa\mathfrak{n}\Big\}. \end{aligned}$$

Because $N_\mu(\mathbb{C}^n) \subset S^0_{\mu_N+\mu}(\mathbb{C}^n)$, the estimates of Theorem 3.1.3 all hold, where $\theta = \mu_N + \mu$.

When N is rotationally symmetric, namely

$$N = \Big\{\mathfrak{n} = \underline{\mathfrak{n}} + n_L e_L \in \mathbb{R}_+^{n+1} :\ |\mathfrak{n}| = 1, \mathfrak{n}_L \geqslant |\underline{\mathfrak{n}}| \cot w\Big\},$$

we have $S^0_\mu(\mathbb{C}^n) = N_{\mu-w}(\mathbb{C}^n)$. We let the functions take their values in the complex Clifford algebra $\mathbb{C}_{(M)}$, so for example $H_\infty(N_\mu(\mathbb{C}^n))$ denotes the Banach space of all bounded holomorphic functions from $N_\mu(\mathbb{C}^n)$ to $\mathbb{C}_{(M)}$ with the norm defined as

$$\|b\|_\infty = \sup\Big\{|b(\zeta)| :\ \zeta \in N_\mu(\mathbb{C}^n)\Big\}.$$

The exponential functions are defined as

$$e(x,\zeta) = e_+(x,\zeta) + e_-(x,\zeta),$$

where

$$e_+(x,\zeta) = e^{i\langle \underline{x},\ \zeta\rangle} e^{-x_L|\zeta|_{\mathbb{C}}} \chi_+(\zeta)$$

and

$$e_-(x,\zeta) = e^{i\langle \underline{x},\ \zeta\rangle} e^{x_L|\zeta|_{\mathbb{C}}} \chi_-(\zeta).$$

For fixed ζ, these functions are entire left monogenic functions of $x \in \mathbb{R}^{n+1}$. For fixed x, these functions are holomorphic functions of $\zeta \in N_\mu(\mathbb{C}^n)$ which satisfy

$$|e_+(x,\zeta)| = e^{-\langle \underline{x},\eta\rangle - x_L \mathrm{Re}|\zeta|_{\mathbb{C}}}|\chi_+(\zeta)| \leqslant \frac{\sec(\mu_N+\mu)}{\sqrt{2}} e^{-\langle x,\ \mathfrak{n}\rangle \mathrm{Re}|\zeta|_{\mathbb{C}}/\mathfrak{n}_L},\ \zeta \in \mathfrak{n}(\mathbb{C}^n)$$

and

$$|e_-(x,\zeta)| = e^{-\langle \underline{x},\ \eta\rangle + x_L \mathrm{Re}|\zeta|_{\mathbb{C}}}|\chi_-(\zeta)| \leqslant \frac{\sec(\mu_N+\mu)}{\sqrt{2}} e^{-\langle x,\ \mathfrak{n}\rangle \mathrm{Re}|\zeta|_{\mathbb{C}}/\mathfrak{n}_L},\ \zeta \in \overline{\mathfrak{n}}(\mathbb{C}^n).$$

Let

$$H_\infty^{\pm}(N_\mu(\mathbb{C}^n)) = \Big\{ b \in H_\infty(N_\mu(\mathbb{C}^n)) :\ b\chi_\pm = b \Big\}.$$

Then any function $b \in H_\infty(N_\mu(\mathbb{C}^n))$ can be uniquely decomposed as

$$b = b_+ + b_-,\ \text{where } b_\pm = b\chi_\pm \in H_\infty^{\pm}(N_\mu(\mathbb{C}^n)).$$

$H_\infty^{\pm}(N_\mu(\mathbb{C}^n))$ is the closed linear subspace of $H_\infty(N_\mu(\mathbb{C}^n))$. Actually, because for all $b \in H_\infty(N_\mu(\mathbb{C}^n))$,

$$\|b\chi_\pm\|_\infty \leqslant \sqrt{2}\|b\|_\infty \|\chi_\pm\|_\infty \leqslant \sec(\mu_N+\mu)\|b\|_\infty,$$

then

$$H_\infty(N_\mu(\mathbb{C}^n)) = H_\infty^{+}(N_\mu(\mathbb{C}^n)) \oplus H_\infty^{-}(N_\mu(\mathbb{C}^n)).$$

We also introduce the subalgebra

$$\mathcal{A}(N_\mu(\mathbb{C}^n)) = \Big\{ b \in H_\infty(N_\mu(\mathbb{C}^m)) :\ \zeta e_L b(\zeta) = b(\zeta)\zeta e_L \text{ for all } \zeta \Big\}.$$

Similarly, we can define $\mathcal{A}^{\pm}(N_\mu(\mathbb{C}^n))$. Notice that if $b \in \mathcal{A}(N_\mu(\mathbb{C}^n))$, then $b_\pm = b\chi_\pm \in \mathcal{A}^{\pm}(N_\mu(\mathbb{C}^n))$ such that

$$\mathcal{A}(N_\mu(\mathbb{C}^n)) = \mathcal{A}^{+}(N_\mu(\mathbb{C}^n)) \oplus \mathcal{A}^{-}(N_\mu(\mathbb{C}^n)).$$

Particular functions b belonging to $\mathcal{A}(N_\mu(\mathbb{C}^n))$ are those of the form

$$b(\zeta) = B(i\zeta e_L) = B(|\zeta|_{\mathbb{C}})\chi_+(\zeta) + B(-|\zeta|_{\mathbb{C}})\chi_-(\zeta),$$

where $B \in H_\infty(S^0_{\mu_N+\mu}(\mathbb{C}))$. All scalar-valued holomorphic functions in $H_\infty(N_\mu(\mathbb{C}^n))$ belong to $\mathcal{A}(N_\mu(\mathbb{C}^n))$. One of the simplest examples is $r_k(\zeta) = i\zeta_k/|\zeta|_{\mathbb{C}},\ k = 1, 2, \ldots, n$.

Let H_N^+ be the algebra of all functions b on $\mathbb{R}^n \setminus \{0\}$ which can be holomorphically extended to $b \in H_\infty^+(N_\mu(\mathbb{C}^n))$ for some $\mu > 0$. Let H_N^- denote the algebra of all functions b on $\mathbb{R}^n \setminus \{0\}$ which can be holomorphically extended to $b \in H^-_\infty(\overline{N}_\mu(\mathbb{C}^n))$, where $\overline{N} = \{\overline{\mathfrak{n}} \in \mathbb{R}^{n+1} : \mathfrak{n} \in N\}$. Then $H^+_N \cap H^-_N = \{0\}$.

Define H_N as $H_N = H^+_N + H^-_N$. Then $H_N = H^+_N \oplus H^-_N$. Let $\mathcal{A}^+_N$, $\mathcal{A}^-_N$ and $\mathcal{A}_N$ be the spaces of the functions in H^+_N, H^-_N and H_N satisfying $\xi e_L b(\xi) = b(\xi)\xi e_L$, $\xi \neq 0$. Then

$$\mathcal{A} = \mathcal{A}^+_N \oplus \mathcal{A}^-_N.$$

If we assume that N is connected, these holomorphic extensions are unique. In fact we assume that the compact set N of unit vectors in $\mathbb{R}^{n+1}_+$ satisfies a stronger condition:N are starlike about e_L, that is, if $\mathfrak{n} \in N$ and $0 \leqslant \tau \leqslant 1$, then

$$\frac{(\tau \mathfrak{n} + (1 - \tau e_L))}{|\tau \mathfrak{n} + (1 - \tau e_L)|} \in N.$$

Under this case, the open set N_μ is also starlike about e_L and $N_\mu(\mathbb{C}^n)$ is the connected open subset in $\mathbb{C}^n$.

Theorem 3.3.1 *Let N be a compact set of unit vectors in $\mathbb{R}^{n+1}_+$ and starlike about e_L. For any $(\Phi, \underline{\Phi}) \in K_N$, there exists a unique function $b \in H_N$ such that for all u in the Schwarz space $\mathcal{S}(\mathbb{R}^n)$, we have the Parseval identity*

$$\begin{aligned}(2\pi)^{-n} \int_{\mathbb{R}^n} b(\xi)\hat{u}(-\xi)d\xi &= \lim_{\alpha \longrightarrow 0+} \int_{\mathbb{R}^n} \Phi(\underline{x} + \alpha e_L)e_L u(\underline{x})d\underline{x} \\ &= \lim_{\varepsilon \to 0} \Big(\int_{|\underline{x}| \geqslant \varepsilon} \Phi(\underline{x})e_L u(\underline{x})d\underline{x} + \underline{\Phi}(\varepsilon e_L)u(\underline{0}) \Big).\end{aligned} \tag{3.4}$$

Hence $b\overline{e_L}$ is the Fourier transform of the distributions of $(\Phi, \underline{\Phi})$. We write $b = \mathcal{F}(\Phi, \underline{\Phi})e_L$.

The Fourier transform $\mathcal{F}$ is linear and satisfies the following properties.

(i) *$\mathcal{F}$ is one-one from K_N to H_N. In other words, for any $b \in H_N$, there exists unique functions $(\Phi, \underline{\Phi}) \in K_N$ such that $b = \mathcal{F}(\Phi, \underline{\Phi})e_L$. Actually, if $b = b_+ + b_-$ and $b_\pm = b\chi_\pm \in H^\pm_N$, then*

$$(\Phi, \underline{\Phi}) = (\Phi_+, \underline{\Phi_+}) + (\Phi_-, \underline{\Phi_-}),$$

where $\Phi_\pm = \mathcal{G}_\pm(b_{\pm\overline{e_L}}) \in K^\pm_N$. We write $(\Phi, \underline{\Phi}) = \mathcal{G}(b\overline{e_L})$ and call $\mathcal{G}$ the inverse Fourier transform.

(ii) *If $0 < \nu < \mu \leqslant \pi/2 - \mu_N$ and $(\Phi, \underline{\Phi}) \in K(S_{N_\mu})$, then $b_+ \in H^+_\infty(N_\nu(\mathbb{C}^n))$, $b_- \in H^-_\infty(\overline{N}_\nu(\mathbb{C}^m))$ and*

$$\|b_\pm\|_\infty \leqslant c_\nu \|(\Phi, \underline{\Phi})\|_{K(S_{N_\mu})},$$

where the constant c_ν only depends on ν.

(iii) *If $0 < \nu < \mu \leqslant \pi/2 - \mu_N$, $b_+ \in H^+_\infty(N_\mu(\mathbb{C}^n))$ and $b_- \in H^-_\infty(\overline{N}_\mu(\mathbb{C}^n))$, then $(\Phi, \underline{\Phi}) \in K(S_{N_\nu})$ and*

$$\|(\Phi, \underline{\Phi})\|_{K(S_{N_\nu})} \leqslant c_\nu(\|b_+\|_\infty + \|b_-\|_\infty),$$

where the constant c_ν depends on ν.

(iv) $(\Phi, \underline{\Phi}) \in M_N$ *if and only if* $b \in \mathcal{A}_N$.

(v) *If* $(\Phi, \underline{\Phi}) \in K_N$, $(\Psi, \underline{\Psi}) \in M_N$, $b = \mathcal{F}(\Phi, \underline{\Phi})e_L$ *and* $f = \mathcal{F}(\Psi, \underline{\Psi})e_L$, *then*

$$bf = \mathcal{F}((\Phi, \underline{\Phi}) * (\Psi, \underline{\Psi}))e_L.$$

(vi) *The mapping* $(\Phi, \underline{\Phi}) \longmapsto b$ *is an algebra homomorphism from the convolution algebra* M_N *to the function algebra* $\mathcal{A}_N$.

(vii) *If* $(\Phi, \underline{\Phi}), (\Psi, \underline{\Psi}) \in K_N$, $b = \mathcal{F}(\Phi, \underline{\Phi})e_L$, $f = \mathcal{F}(\Psi, \underline{\Psi})e_L$ *and if* $f = pb$, *where* p *is a polynomial in* n *variables with values in* $\mathbb{C}_{(M)}$, *then*

$$\Psi = p\Big(-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_n}\Big)\Phi.$$

(viii) *Let* $0 < \nu < \mu \leqslant \pi/2 - \mu_N$, $s > -n$. *If* b_+ *(or* b_-*) can be holomorphically extended to a bounded function for some* c_s *and all* $\zeta \in N_\mu(\mathbb{C}^n)$ *(correspondingly,* $\zeta \in \overline{N}_\mu(\mathbb{C}^n)$*), which satisfies* $|b_\pm(\zeta)| \leqslant c_s|\zeta|^s$, *then there exists* $c_{s,\nu}$ *such that for all* $x \in C^+_{N_\nu}$,

$$|\Phi(x)| \leqslant c_{s,\nu}|x|^{-n-s};$$

For all $y \in T_{N_\mu}$,

$$|\underline{\Phi}(y)| \leqslant c_{s,\nu}|y|^{-s}.$$

In particular, when $-n < s < 0$, *we have* $\lim_{y\to 0}\underline{\Phi}(y) = 0$.

Proof Without loss of generality, we only verify (i)-(viii) for the case $C^+_{N_\mu}$, $N_\mu(\mathbb{C}^n)$, K^+_N, M^+_N, $H^+_\infty(N_\mu(\mathbb{C}^n))$, H^+_N, $\mathcal{A}^+_N$ and $\mathcal{F}_+$. The case $C^-_{N_\mu}$, $\overline{N}_\mu(\mathbb{C}^n)$, K^-_N, M^-_N, $H^-_\infty(\overline{N}_\mu(\mathbb{C}^n))$, H^-_N, $\mathcal{A}^-_N$ and $\mathcal{F}_-$ can be dealt with similarly. In the proof, the constant c may depend on μ_N, μ and the dimension n, and may vary from line to line. We denote by c_ν a constant if the constant only depends on ν. Let $\Phi \in K(C^+_{N_\mu})$. Either form of the Parseval identity uniquely determines b on $\mathbb{R}^n$. Because $N_\mu(\mathbb{C}^n)$ is a connected open set, Parseval's identity also determines b on $N_\mu(\mathbb{C}^n)$.

We construct b as follows. For $\alpha > 0$, define $\Phi_\alpha(x) = \Phi(x + \alpha e_L)$, $x + \alpha e_L \in C^+_{N_\mu}$. We have

$$\begin{aligned}\|\Phi_\alpha\|_{K(C^+_{N_\mu})} &= \frac{1}{2}\sigma_n \sup\Big\{|x|^n|\Phi(x + \alpha e_L)| : x \in C^+_{N_\mu}\Big\}\\ &\leqslant \sup\Big\{|y|^n|\Phi(y)| : y \in C^+_{N_\nu} + \alpha e_L\Big\} \leqslant \|\Phi\|_{K(C^+_{N_\mu})}.\end{aligned}$$

For

$$\zeta \in \mathfrak{n}(\mathbb{C}^n) \subset N_\nu(\mathbb{C}^n) \subset N_\mu(\mathbb{C}^n),\ \nu < \mu,$$

define

$$b_\alpha(\zeta) = \int_\sigma \Phi_\alpha(x)\mathfrak{n}(x)e_+(-x,\zeta)dS_x,$$

where the surface σ is defined as

$$\sigma = \Big\{x \in \mathbb{R}^{n+1} :\ \langle x, \mathfrak{n}\rangle = -|x|\sin(\mu-\nu)\Big\}.$$

Note that the function in the integral is continuous and exponentially decreasing at infinity. As usual, $\mathfrak{n}(x)$ denotes the normal of σ and $\mathfrak{n}_L(x) > 0$. In fact, for $x \in \sigma$,

$$\begin{aligned}|e_+(-x,\zeta)| &\leqslant \frac{\sec(\mu_N+\mu)}{\sqrt{2}}e^{\langle x,\ \mathfrak{n}\rangle \mathrm{Re}|\zeta|_{\mathbb{C}}/\mathfrak{n}_L}\\ &\leqslant \frac{\sec(\mu_N+\mu)}{\sqrt{2}}e^{-|x||\xi|\sin\theta},\end{aligned}$$

where $\theta = \mu - \nu$.

By this fact and Cauchy's theorem for monogenic functions, noticing that Φ_α is right monogenic and $e_+(-x,\zeta)$ is left monogenic in x, we can see that the definition of $b_\alpha(\zeta)$ is independent of the choice of the surfaces σ. So $b_\alpha(\zeta)$ depends on $\zeta \in N_\mu(\mathbb{C}^n)$ holomorphically. Hence for all $\alpha, \beta > 0$,

$$\begin{aligned}b_\alpha(\zeta)e^{\alpha|\zeta|_{\mathbb{C}}} &= \int_\sigma \Phi(x+\alpha e_L)\mathfrak{n}(x)e_+(-(x+\alpha e_L),\ \zeta)dS_x\\ &= \int_\sigma \Phi(x+\beta e_L)\mathfrak{n}(x)e_+(-(x+\beta e_L),\ \zeta)dS_x\\ &= b_\beta(\zeta)e^{\beta|\zeta|_{\mathbb{C}}}.\end{aligned}$$

Then we define b as the holomorphic function on $N_\mu(\mathbb{C}^n)$ which satisfies the following condition:

$$b(z) = b_\alpha(\zeta)e^{\alpha|\zeta|_{\mathbb{C}}}\ \forall\, \alpha > 0.$$

We shall prove that for all $z \in N_\mu(\mathbb{C}^n)$,

$$|b_\alpha(\zeta)| \leqslant c_\nu\|\Phi\|_{K(C^+_{N_\mu})}, \tag{3.5}$$

where c_ν is independent of α and

$$(2\pi)^{-n}\int_{\mathbb{R}^n} b_\alpha(\xi)\hat{u}(-\xi)d\xi = \int_{\mathbb{R}^n}\Phi(\underline{x}+\alpha e_L)u(\underline{x})d\underline{x}. \tag{3.6}$$

As the estimate in (ii), the first form of Parseval's identity (3.4) can be deduced as a corollary.

We prove (3.5). Let

$$\zeta \in n(\mathbb{C}^n) \subset N_\nu(\mathbb{C}^n) \subset N_\mu(\mathbb{C}^n)$$

and $\theta = \mu - \nu$. Changing the surface in the integral by Cauchy's theorem, we can get

$$b_\alpha(\zeta) = \left(\int\limits_{\sigma(0,0,|\zeta|^{-1})} + \int\limits_{\tau(\theta,|\zeta|^{-1})} + \int\limits_{\sigma(\theta,|\zeta|^{-1},\infty)} \right) \Phi_\alpha(x)\mathfrak{n}(x)e_+(-x,\zeta)dS_x,$$

where

$$\sigma(\theta, r, R) = \left\{ x \in \mathbb{R}^{n+1} : \langle x, \mathfrak{n}\rangle = |x|\sin\theta,\ r \leqslant |x| \leqslant R \right\},$$
$$\tau(\theta, R) = \left\{ x \in \mathbb{R}^{n+1} : |x| = R, 0 \geqslant \langle x, \mathfrak{n}\rangle \geqslant -R\sin\theta \right\}.$$

We need the following estimates.

(i) For $R \leqslant |\zeta|^{-1}$,

$$\begin{aligned} &\Big| \int\limits_{\sigma(0,0,R)} \Phi_\alpha(x)\mathfrak{n}(x)e_+(-x,\zeta)dS_x \Big| \\ &\leqslant c\Big| \int\limits_{\sigma(0,0,R)} \Phi_\alpha(x)\mathfrak{n}(x)e(-x,\zeta)dS_x \Big| \\ &\leqslant c\Big| \int\limits_{\sigma(0,0,R)} \Phi_\alpha(x)\mathfrak{n}(x)[e(-x,\zeta)-1]dS_x \Big| + c\Big| \int\limits_{\langle x,\mathfrak{n}\rangle \geqslant 0, |x|=R} \Phi_\alpha(x)\mathfrak{n}(x)dS_x \Big| \\ &\leqslant c\|\Phi_\alpha\|_{K(C^+_{N_\mu})}\Big(\sup\Big\{ |\nabla_y e(-y,\zeta)| : y \in \sigma(0,0,R) \Big\} \int_{\sigma(0,0,R)} |x|^{1-n} dS_x + 1 \Big) \\ &\leqslant c\|\Phi_\alpha\|_{K(C^+_{N_\mu})}(R|\zeta|+1) \leqslant c\|\Phi_\alpha\|_{K(C^+_{N_\mu})}. \end{aligned} \tag{3.7}$$

(ii) $R \geqslant |\zeta|^{-1}$,

$$\begin{aligned} \Big| \int_{\tau(\theta,R)} \Phi_\alpha(x)\mathfrak{n}(x)e_+(-x,\zeta)dS_x \Big| &\leqslant c\|\Phi_\alpha\|_{K(C^+_{N_\mu})} R^{-n} \int_{\tau(\theta,R)} e^{\langle x,\mathfrak{n}\rangle \mathrm{Re}|\zeta|_{\mathbb{C}}/\mathfrak{n}_L} dS_x \\ &= c\|\Phi_\alpha\|_{K(C^+_{N_\mu})} \int_{\tau(\theta,1)} e^{\langle x,\mathfrak{n}\rangle R\ \mathrm{Re}|\zeta|_{\mathbb{C}}/\mathfrak{n}_L} dS_x \quad (3.8) \\ &= c\|\Phi_\alpha\|_{K(C^+_{N_\mu})} \int_{-\theta}^{0} e^{R\ \mathrm{Re}|\zeta|_{\mathbb{C}} \sin\Phi/\mathfrak{n}_L} d\Phi \end{aligned}$$

$$\leqslant \frac{c}{R|\zeta|}\|\Phi_\alpha\|_{K(C^+_{N_\mu})}$$
$$\leqslant c\|\Phi_\alpha\|_{K(C^+_{N_\mu})}.$$

(iii) $R \geqslant |\zeta|^{-1}$,

$$\begin{aligned}
&\Big|\int_{\sigma(\theta,R,\infty)} \Phi_\alpha(x)\mathfrak{n}(x)e_+(-x,\zeta)dS_x\Big| \\
&\leqslant c\|\Phi_\alpha\|_{K(C^+_{N_\mu})}\int_{\sigma(\theta,R,\infty)} |x|^{-n}e^{\langle x,\ \mathfrak{n}\rangle \mathrm{Re}|\zeta|_{\mathbb{C}}/\mathfrak{n}_L}dS_x \qquad (3.9)\\
&= c\|\Phi_\alpha\|_{K(C^+_{N_\mu})}\int_R^\infty s^{-1}e^{-s\sin\theta \mathrm{Re}|\zeta|_{\mathbb{C}}/\mathfrak{n}_L}dS_x \\
&\leqslant \frac{c_\nu}{R|\zeta|}\|\Phi_\alpha\|_{K(C^+_{N_\mu})}.
\end{aligned}$$

In the above estimates (i)–(iii), taking $R = |\zeta|^{-1}$, we can use the representation of b to obtain (3.5).

Now we prove (3.6). For $\zeta \in \mathbb{R}^n$, we define $b_{\alpha,N}$ as

$$b_{\alpha,N}(\xi) = \int_{|\underline{x}|\leqslant N} \Phi_\alpha(\underline{x})e_L e^{i\langle \underline{x},\ \xi\rangle}d\underline{x}.$$

Then it can be deduced from Parseval's identity that for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$(2\pi)^{-n}\int_{\mathbb{R}^n} b_{\alpha,N}(\xi)\hat{u}(-\xi)d\xi = \int_{|\underline{x}|\leqslant N} \Phi(\underline{x}+\alpha e_L)u(\underline{x})d\underline{x}.$$

We will prove

$$\text{for all } \xi \in \mathbb{R}^n \text{ and } N > 0,\ |b_{\alpha,N}(\xi)| \leqslant c\|\Phi\|_{K(C^+_{N_\mu})}, \qquad (3.10)$$

$$\text{for any } \xi \in \mathbb{R}^n,\quad \text{when } N \to \infty,\ b_{\alpha,N}(\xi)\chi_+(\xi) \to b_\alpha(\xi) \qquad (3.11)$$

and

$$\text{for any } \xi \in \mathbb{R}^n,\quad \text{when } N \to \infty,\ b_{\alpha,N}(\xi)\chi_-(\xi) \to 0_\alpha(\xi). \qquad (3.12)$$

Then (3.6) can be deduced from the above estimates and the Lebesgue dominate convergence theorem.

To prove (3.10) and (3.11), letting $\mathfrak{n} = e_L$ in the definitions of σ, $\sigma(\theta, \tau, R)$ and $\tau(\theta, R)$, we use the estimates (3.7)–(3.9).

At first, when $|\xi|^{-1} \leqslant N$, we prove (3.10). Taking $0 < \theta < \mu$ and using Cauchy's theorem, we have

$$b_{\alpha,N}(\xi)\chi_+(\xi) = \left(\int_{\sigma(0,0,|\xi|^{-1})} + \int_{\tau(\theta,|\xi|^{-1})} + \int_{\sigma(\theta,|\xi|^{-1},N)} - \int_{\tau(\theta,N)}\right)\Phi_\alpha(x)\mathfrak{n}(x)e_+(-x,\xi)dS_x.$$

So we can apply (3.7)–(3.9) to prove that $b_{\alpha,N}(\xi)\chi_+(\xi)$ is uniformly bounded for ξ and N. On the other hand, similar to the proof of (3.8), we get

$$|b_{\alpha,N}(\xi)\chi_-(\xi)| \leqslant \frac{c}{N|\xi|}\|\Phi\|_{K(C^+_{N\mu})} \leqslant c\|\Phi_\alpha\|_{K(C^+_{N\mu})}.$$

When $|\xi|^{-1} \geqslant N$, to prove (3.10), only (3.7) is needed. To prove (3.11), fix $\xi \in \mathbb{R}^m$, $\xi \neq 0$, and apply Cauchy's theorem to write

$$b_\alpha(\xi) - b_{\alpha,N}(\xi)\chi_+(\xi) = \left(\int_{\tau(\theta,N)} + \int_{\sigma(\theta,N,\infty)}\right)\Phi_\alpha(x)\mathfrak{n}(x)e_+(-x,\xi)dS_x.$$

So, by (3.8) and (3.9), as $N \to 0$,

$$\left|b_\alpha(\xi) - b_{\alpha,N}(\xi)\chi_+(\xi)\right| \leqslant \frac{c}{N|\xi|}\|\Phi_\alpha\|_{K(C^+_{N\mu})} \to 0.$$

Moreover, (3.12) follows from the estimate given above.

As noted previously, the first version of Parseval's identity (3.4) holds. The next aim is to prove the second version of (3.4). Let $\varepsilon > 0$. Then

$$\begin{aligned}(2\pi)^{-n}\int_{\mathbb{R}^n} b(\xi)\hat{u}(-\xi)d\xi &= \lim_{\alpha\to 0+}\left(\int_{|\underline{x}|\geqslant\varepsilon}\Phi_\alpha(\underline{x}+\alpha e_L)e_L u(\underline{x})d\underline{x} + \int_{|\underline{x}|\leqslant\varepsilon}\Phi_\alpha(\underline{x}+\alpha e_L)e_L u(\underline{0})d\underline{x}\right.\\ &\quad\left.+\int_{|\underline{x}|\leqslant\varepsilon}\Phi_\alpha(\underline{x}+\alpha e_L)e_L(u(\underline{x})-u(\underline{0}))d\underline{x}\right)\\ &= \int_{|\underline{x}|\geqslant\varepsilon}\Phi(\underline{x})e_L u(\underline{x})d\underline{x} + \underline{\Phi}(\varepsilon)u(\underline{0})\\ &\quad+\lim_{\alpha\to 0+}\left(\int_{|\underline{x}|\leqslant\varepsilon}\Phi_\alpha(\underline{x}+\alpha e_L)e_L(u(\underline{x})-u(\underline{0}))d\underline{x}\right),\end{aligned}$$

where in the second integral we have used Cauchy's theorem.

Now when $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\overline{\lim_{\varepsilon\to 0}}\lim_{\alpha\to 0+}\left(\int_{|\underline{x}|\leqslant\varepsilon}\left|\Phi(\underline{x}+\alpha e_L)e_L(u(\underline{x})-u(\underline{0}))\right|d\underline{x}\right)$$

$$\leqslant\overline{\lim_{\varepsilon\to 0}}\lim_{\alpha\to 0+}\left(C\int_{|\underline{x}|\leqslant\varepsilon}|\underline{x}+\alpha e_L|^{-n}|u(\underline{x})-u(\underline{0})|d\underline{x}\right)$$

$$\leqslant\overline{\lim_{\varepsilon\to 0}}\left(C\int_{|\underline{x}|\leqslant\varepsilon}|\underline{x}|^{-n}|u(\underline{x})-u(\underline{0})|d\underline{x}\right)=0,$$

so

$$(2\pi)^{-n}\int_{\mathbb{R}^n}b(\xi)\hat{u}(-\xi)d\xi=\lim_{\varepsilon 0}\left(\int_{|\underline{x}|\geqslant\varepsilon}\Phi(\underline{x})e_L u(\underline{x})d\underline{x}+\underline{\Phi}(\varepsilon)u(\underline{0})\right).$$

This gives (ii).

We prove (i) and (iii). It is easy to verify that $\mathcal{F}_+$ is one-one. By constructing the inverse Fourier transform $\mathcal{G}_+$, we prove the mapping is onto H_N^+.

Consider the function $b\in H_\infty^+(N_\mu(\mathbb{C}^n))$. For $\mathfrak{n}\in N_\mu$ and

$$x=\underline{x}+x_L e_L\in C_{\mathfrak{n}}^+\subset C_{N_\mu}^+,$$

define

$$\Phi_n(x)=(2\pi)^{-n}\int_{\mathfrak{n}(\mathbb{C}^n)}b(\zeta)e(x,\zeta)d\zeta_1\wedge d\zeta_2\wedge\cdots\wedge d\zeta_n\overline{e_L}$$

$$=(2\pi)^{-n}\int_{\mathfrak{n}(\mathbb{C}^n)}b(\zeta)e_+(x,\zeta)d\zeta_1\wedge d\zeta_2\wedge\cdots\wedge d\zeta_n\overline{e_L},$$

where in the last equality we have used the facts that $e(x,\zeta)=e_+(x,\zeta)+e_-(x,\zeta)$ and $\langle b(\zeta),e_-(x,\zeta)\rangle=0$ for $b\in H_\infty^+(N_\mu(\mathbb{C}^n))$. On the surface $\mathfrak{n}(\mathbb{C}^n)$, the function in the integral is exponentially decreasing at infinity. In fact, when $\zeta\in\mathfrak{n}(\mathbb{C}^n)$, then

$$|e^{i\langle\mathbf{x},\zeta\rangle}e^{-x_L|\zeta|_{\mathbb{C}}}|\leqslant ce^{-\langle x,\mathfrak{n}\rangle\mathrm{Re}|\zeta|_{\mathbb{C}}/\mathfrak{n}_L}$$

and $\langle x,\mathfrak{n}\rangle>0$. Moreover, $e(x,\zeta)\overline{e_L}$ is right monogenic and Φ_n is a right monogenic function on $C_{\mathfrak{n}}^+$ satisfying

$$|\Phi_{\mathfrak{n}}(x)|\leqslant\frac{c\|b\|_\infty}{\langle x,\mathfrak{n}\rangle^n},$$

where c only depends on μ_N and μ.

Moreover the integrand depends holomorphically on the single complex variable $z = \langle \zeta, \underline{\mathfrak{n}} \rangle$. So by the starlike nature of N_μ and Cauchy's theorem in the z-plane, we find that for all $x \in C^+_{\mathfrak{n}}$ satisfying $x_L > 0$, $\Phi_{\mathfrak{n}}(x) = \Phi_{e_L}(x)$. Hence there exists unique right monogenic function Φ on $C^+_{N_\mu}$ which coincides with $\Phi_{\mathfrak{n}}(x)$ on $C^+_{\mathfrak{n}}$. We call Φ the Fourier transform of $b\overline{e_L}$ and denote $\Phi = \mathcal{G}_+(b\overline{e_L})$. The above estimates for $\Phi_{\mathfrak{n}}$ indicate that for all $\nu < \mu$, $\Phi \in K(C^+_{N_\nu})$ and

$$\|\Phi\|_{K(C^+_{N_\nu})} \leqslant c_\nu \|b\|_\infty.$$

For the special case $x_L = 0$ and all $\zeta \in N_\mu(\mathbb{C}^n)$,

$$|b(\zeta)| \leqslant \frac{c}{1 + |\zeta|^{n+1}}.$$

Then by Cauchy's theorem, we can change the surface of integration to obtain

$$\mathcal{G}_+(b\overline{e_L})(\underline{x}) = \Phi(\underline{x}) = (2\pi)^{-n} \int_{\mathbb{R}^n} b(\xi) e^{i\langle \underline{x}, \xi \rangle} d\xi \overline{e_L} = \check{b}(\underline{x})\overline{e_L},$$

which is the usual inverse Fourier transform of $b\overline{e_L}$.

We prove that b and $\Phi = \mathcal{G}_+(b\overline{e_L})$ satisfy Parseval's identity (3.4). Hence we can deduce that $\mathcal{G}_+$ is the inverse of the Fourier transform $\mathcal{F}_+$, and complete the proofs of (i) and (iii).

For $\alpha > 0$, let $b_\alpha(\zeta) = b(\zeta)e^{-\alpha|\zeta|_{\mathbb{C}}}$. Then for $\underline{x} \in \mathbb{R}^n$,

$$\Phi(\underline{x} + \alpha e_L) = \mathcal{G}_+(b\overline{e_L})(\underline{x} + \alpha e_L) = \mathcal{G}_+(b_\alpha \overline{e_L})(\underline{x}) = (b_\alpha)\check{}(\underline{x})\overline{e_L}.$$

By the usual Parseval's identity, we can obtain

$$(2\pi)^{-n} \int_{\mathbb{R}^n} b_\alpha(\xi)\hat{u}(-\xi) d\xi = \int_{\mathbb{R}^n} \Phi(\underline{x} + \alpha e_L) e_L u(\underline{x}) d\underline{x},$$

and for all $u \in \mathcal{S}(\mathbb{R}^n)$,

$$(2\pi)^{-n} \int_{\mathbb{R}^n} b(\xi)\hat{u}(-\xi) d\xi = \lim_{\alpha \to 0+} \int_{\mathbb{R}^n} \Phi(\underline{x} + \alpha e_L) e_L u(\underline{x}) d\underline{x}.$$

Now we prove (iv). Take $\Phi \in K(C^+_{N_\mu})$. Then Φ is left monogenic (and is also right monogenic) if and only if for all $x \in C^+_{N_\mu}$,

$$\underline{D} e_L \Phi(x) = (\Phi e_L)\underline{D}(x),$$

where the both sides all equal to $-\partial \Phi / \partial x_L(x)$.

Let $b\overline{e_L} = \mathcal{F}_+(\Phi)$ and define b_α as above. Using twice Parseval's identity for b_α, we can see that for all $u \in \mathcal{S}(\mathbb{R}^n)$,

$$(2\pi)^{-n}\int_{\mathbb{R}^n} \xi e_L b_\alpha(\xi)\hat{u}(-\xi)d\xi = -i\int_{\mathbb{R}^n} (\underline{D}e_L\Phi)(\underline{x}+\alpha e_L)e_L u(\underline{x})d\underline{x}$$

and

$$(2\pi)^{-n}\int_{\mathbb{R}^n} b_\alpha(\xi)\xi e_L\hat{u}(-\xi)d\xi = -i\int_{\mathbb{R}^n} (\Phi e_L\underline{D})(\underline{x}+\alpha e_L)e_L u(\underline{x})d\underline{x}.$$

Hence $\Phi \in M(C^+_{N_\mu})$ if and only if for all $u \in \mathcal{S}(\mathbb{R}^n)$, $\underline{D}e_L\Phi(x) = (\Phi e_L)\underline{D}(x)$. So the above equality holds if and only if

$$\underline{D}e_L\Phi(\underline{x}+\alpha e_L) = (\Phi e_L)\underline{D}(\underline{x}+\alpha e_L) \text{ for all } x \in \mathbb{R}^n \setminus \{0\}.$$

The above equality is equivalent to $\xi e_L b_\alpha(\xi) = b_\alpha(\xi)\xi e_L$. This equation is equivalent to $\zeta e_L b(\zeta) = b(\zeta)\zeta e_L$ for all $z \in N_\mu(\mathbb{C}^n)$. This proves (iv).

The remaining part can be proved in a similar way with the estimates in (viii) requiring a modification of the proof of (iii). □

Denote by $\mathcal{G}_- : H^-_N \to K^-_N$ the inverse of $\mathcal{F}_-$. We call $\mathcal{F}_-$ the Fourier transform and $\mathcal{G}_-$ the inverse Fourier transform.

Remark 3.3.1 When $N = \overline{N}$, $b_+ \in H^+_\infty(N_\mu(\mathbb{C}^n))$ and $b_- \in H^-_\infty(\overline{N}_\mu(\mathbb{C}^n))$ if and only if

$$b \in H_\infty(N_\mu(\mathbb{C}^n)).$$

Let $B \in H^\infty(S^0_\mu(\mathbb{C}))$, where $0 < \mu < \pi/2$. We have seen that B is associated with the function $b \in H^\infty(S^0_\mu(\mathbb{C}^n))$ defined as

$$b(\zeta) = B(i\zeta e_L) = B(|\zeta|_{\mathbb{C}})\chi_+(\zeta) + B(-|\zeta|_{\mathbb{C}})\chi_-(\zeta).$$

In fact,

$$b \in \mathcal{A}(S^0_\mu(\mathbb{C}^n)) = \Big\{b \in H^\infty(S^0_\mu(\mathbb{C}^n)) : \zeta e_L b(\zeta) = b(\zeta)\zeta e_L \text{ for all } \zeta\Big\},$$

and the mapping $B \longmapsto b$ is a one-one algebra homomorphism from $H^\infty(S^0_\mu(\mathbb{C}))$ to $\mathcal{A}(S^0_\mu(\mathbb{C}^n))$. Recall that

$$C^0_{\mu,+}(\mathbb{C}) = \Big\{Z = X + iY \in \mathbb{C} : Z \neq 0, Y > -|X|\tan\mu\Big\},$$

$$C^0_{\mu,-}(\mathbb{C}) = -C^0_{\mu,+}(\mathbb{C}),$$

$$S^0_{\mu,+}(\mathbb{C}) = \Big\{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg(\lambda)| < \mu\Big\},$$

$$S^0_{\mu,-}(\mathbb{C}) = -S^0_{\mu,+}(\mathbb{C}),$$

$$C^0_{\mu,+} = \left\{x = \underline{x} + x_L e_L \in \mathbb{R}^{n+1} : x_L > -|\underline{x}| \tan \mu\right\},$$

$$C^0_{\mu,-} = -C^0_{\mu,+}, \quad S^0_\mu = C^0_{\mu,+} \cap C^0_{\mu,-},$$

$$T^0_\mu = \left\{y = \underline{y} + y_L e_L \in \mathbb{R}^{n+1} : y_L > |\underline{y}| \cot \mu\right\},$$

$$S^0_\mu(\mathbb{C}^n) = \left\{\zeta = \xi + i\eta \in \mathbb{C}^n : |\zeta|^2_{\mathbb{C}} \notin (-\infty, 0] \text{ and } |\eta| < \mathrm{Re}(|\zeta|_{\mathbb{C}}) \tan \mu\right\}.$$

We find the inverse Fourier transform of b in terms of the inverse Fourier transform of B. We first assume that $B \in H^\infty(S^0_{\mu,+}(\mathbb{C}))$. In this case, the inverse Fourier transform of B, $\Phi = \mathcal{G}(B)$, is a complex-valued holomorphic function defined on $C^0_{\mu,+}(\mathbb{C})$. Specially, when $\mathrm{Im}(Z) > 0$,

$$\Phi(Z) = \frac{1}{2\pi} \int_0^\infty B(r) e^{irZ} dr.$$

When $x_L > 0$,

$$\begin{aligned}
\Phi(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} B(|\xi|) e_+(x, \xi) d\xi \overline{e_L} \\
&= \frac{1}{2(2\pi)^n} \int_{\mathbb{R}^n} (\overline{e_L} + \frac{i\xi}{|\xi|}) B(|\xi|) e^{-x_L|\xi|} e^{i\langle \underline{x},\ \xi\rangle} d\xi \\
&= \frac{1}{2(2\pi)^n} \int_{S^{m-1}} (\overline{e_L} + i\tau) \int_0^\infty B(r) e^{-x_L r} e^{i\langle \underline{x},\ \tau\rangle r} r^{n-1} dr dS_\tau \qquad (3.13) \\
&= \frac{1}{2(2\pi i)^{n-1}} \int_{\mathbb{S}^{n-1}} (\overline{e_L} + i\tau) \Phi^{(n-1)}(\langle \underline{x},\ \tau\rangle + i x_L) dS_\tau \\
&= \frac{1}{2(2\pi i)^{n-1}} \int_{\mathbb{S}^{n-1}} (\overline{e_L} + i\langle \underline{x},\ \tau\rangle \underline{x}|\underline{x}|^{-2}) \Phi^{(n-1)}(\langle \underline{x},\ \tau\rangle + i x_L) dS_\tau \\
&= \frac{\sigma_{n-2}}{2(2\pi i)^{n-1}} \int_{-1}^1 (1-t^2)^{(n-3)/2} \left(\overline{e_L} + \frac{it\underline{x}}{|\underline{x}|}\right) \Phi^{(n-1)}(|\underline{x}|t + i x_L) dt,
\end{aligned}$$

where $\Phi^{(n-1)}$ is the $(n-2)$th derivative of Φ. On $C^0_{\mu,+}$, Φ extends to a left and right monogenic function. For all $\nu < \mu$, this function belongs to $M(C^0_{\nu,+})$.

For $B \in H^\infty(S^0_{\mu,-}(\mathbb{C}))$, $\Phi = \mathcal{G}(B)$ and

$$b(\zeta) = B(i\zeta e_L) = B(-|\zeta|_{\mathbb{C}}) \chi_-(\zeta).$$

Then $b \in H^-_\infty(S^0_\mu(\mathbb{C}^n))$. Hence we can construct $\Phi = \mathcal{G}_-(b\overline{e_L})$. We see that if $x_L < 0$,

$$
\begin{aligned}
\Phi(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^m} B(-|\xi|)e_-(x, \xi) d\xi \overline{e_L} \\
&= \frac{1}{2(2\pi)^n} \int_{\mathbb{R}^n} (\overline{e_L} - \frac{i\xi}{|\xi|}) B(-|\xi|) e^{-x_L|\xi|} e^{i\langle \underline{x},\ \xi\rangle} d\xi \\
&= \frac{1}{2(2\pi)^n} \int_{S^{n-1}} (\overline{e_L} - i\tau) \int_0^{+\infty} B(-r) e^{x_L r} e^{i\langle \underline{x},\ \tau\rangle r} r^{n-1} dr dS_\tau \\
&= \frac{(-1)^{n-1}}{2(2\pi)^n} \int_{S^{n-1}} (\overline{e_L} + i\tau) \int_{-\infty}^{0} B(-r) e^{-x_L r} e^{i\langle \underline{x},\ \tau\rangle r} r^{n-1} dr dS_\tau \\
&= \frac{1}{2(-2\pi i)^{n-1}} \int_{S^{n-1}} (\overline{e_L} + i\tau) \Phi^{(n-1)}(\langle \underline{x},\ \tau\rangle + i x_L) dS_\tau \\
&= \frac{\sigma_{n-2}}{2(-2\pi i)^{n-1}} \int_{-1}^{1} (1-t^2)^{(n-3)/2} \left[\overline{e_L} + \frac{it\underline{x}}{|\underline{x}|}\right] \Phi^{(n-1)}(|\underline{x}|t + i x_L) dt.
\end{aligned}
$$

When $B \in H_\infty(S^0_\mu(\mathbb{C}))$, write $B = B_+ + B_-$, where $B_+ = B\chi_{\text{Re}>0} \in H_\infty(S^0_{\mu,+}(\mathbb{C}))$ and $B_- = B\chi_{\text{Re}<0} \in H_\infty(S^0_{\mu,-}(\mathbb{C}))$. Then $b = b_+ + b_-$, where $b_\pm$ is the function with respect to $B_\pm$. We can use this decomposition to relate the inverse Fourier transform $\mathcal{G}(b\overline{e_L}) = (\Phi,\ \underline{\Phi})$ of $b\overline{e_L}$ to the inverse Fourier transform $\mathcal{G}(B) = (\Phi,\ \Phi_1)$ of B.

In the end of this section, we give two examples to make the reader to understand the relation between $(\Phi(z),\ \Phi_1(z))$ and (b, B), and between $(\Phi(z),\ \Phi_1(z))$ and $(\Phi(x), \underline{\Phi}(y))$, see [1] for more examples.

Example 3.3.1 As usual,

$$E(x) = \frac{1}{\sigma_n} \frac{\overline{x}}{|x|^{n+1}}.$$

(i) $(\Phi(z),\ \Phi_1(z)) = (0, 1)$, $B(\lambda) = 1$, $b(\zeta) = 1$;
(ii) $(\Phi(z),\ \Phi_1(z)) = (\frac{i}{2\pi z}, \frac{1}{2})$, $B(\lambda) = \chi_{\text{Re}>0}$, $b(\zeta) = \chi_+(\zeta)$;
(iii) $(\Phi(z),\ \Phi_1(z)) = (\frac{i}{2\pi z}, \frac{1}{2})$, $B(\lambda) = \chi_{\text{Re}<0}$, $b(\zeta) = \chi_-(\zeta)$;
(iv) $(\Phi(z),\ \Phi_1(z)) = (\frac{i}{\pi z}, 0)$, $B(\lambda) = \text{sgn}(\lambda)$, $b(\zeta) = \frac{i\zeta e_L}{|\zeta|_{\mathbb{C}}}$;

The above example describes the relation between the function pair $(\Phi(z),\ \Phi_1(z))$ and the function pair $(\Phi(x), \underline{\Phi}(y))$.

Example 3.3.2 (i) Let $(\Phi(z),\ \Phi_1(z)) = \left(\frac{1}{(z+it)}, -i\pi + \log(\frac{z+it}{z-it})\right)(t > 0)$. Then

$$(\Phi(x), \underline{\Phi}(y)) = (k(x + te_L),\ \underline{\Phi}(y)),\ \lim_{y\to 0} \underline{\Phi}(y) = 0.$$

(ii) Let $(\Phi(z),\ \Phi_1(z)) = \frac{t}{2\pi}\left(\frac{-1}{(z+it)^2},\ \frac{2z}{z^2+t^2}\right)(t > 0)$. Then

$$(\Phi(x), \underline{\Phi}(y)) = \Big(-t\frac{\partial k}{\partial t}(x + te_L),\ \underline{\phi}(y)\Big), \lim_{y\to 0} \underline{\Phi}(y) = 0.$$

(iii) Let $(\Phi(z),\ \Phi_1(z)) = \Gamma(1+is)\Big(\frac{i}{2\pi}e^{-\pi s/2}z^{-1-is},\, (\pi s)^{-1}\sinh(\pi s/2)z^{-is}\Big)$. Then

$$(\Phi(x), \underline{\Phi}(y)) = \left(\frac{-1}{\Gamma(1-is)}\int_0^\infty t^{-is}\frac{\partial k}{\partial t}(x+te_L)dt,\ \underline{\Phi}_s(y)\right),$$

where the function $\underline{\Phi}_s$ is represented as

$$\underline{\Phi}_s(r\mathfrak{n}) = \frac{r^{-is}}{\Gamma(1-is)}\int_0^\infty t^{is-1}F(n,\mathfrak{n}_L,\tau)d\tau\overline{e_L}\mathfrak{n},$$

where $r>0$, $|\mathfrak{n}|=1$, and F is real-valued and satisfies

$$|F(n,\mathfrak{n}_L,t)| \leqslant c(n,\mathfrak{n}_L)\frac{t^n}{(1+t)^{n+1}}.$$

In particular, if $\mathfrak{n}=e_L$, then

$$\underline{\Phi}_s(re_L) = \frac{\sigma_{n-1}r^{-is}}{\Gamma(1-is)}\int_0^\infty \frac{t^{n+is-1}}{(1+t^2)^{(n+1)/2}}dt,\ r>0.$$

(To prove this, first show that the function $\underline{\Phi}$ in the preceding row has the form $\underline{\Phi}(r\mathfrak{n}) = F(n,\mathfrak{n}_L, r/t)\overline{e_L}\mathfrak{n}$.)

The functions Φ_1 and $\underline{\Phi}$ are really only of interest near zero, and when they tend to 0, these functions do not enter into Parseval's identity or the convolution formulae. It has been proved in Chap. 1 that if $|B(\lambda)| \leqslant c_s|\lambda|^s$ holds for all $\lambda \in S^0_{\mu,+}(\mathbb{C})$ and some $s<0$, then when $z\to 0$ ($z\in S_{\nu,+}(\mathbb{C}),\ \nu<\mu$), $\Phi_1(z)\to 0$, and for all $\zeta \in S^0_\mu(\mathbb{C})$, $|b(\zeta)| \leqslant c_s|\zeta|^s$. Hence by (viii) of Theorem 3.3.1, we conclude that $y\to 0$ ($y\in T^0_\nu,\ \nu<\mu$), $\underline{\Phi}(y)\to 0$. Therefore for $|B(\lambda)|\leqslant c_s|\lambda|^s,\ s<0$, there is no need to find Φ_1 and $\underline{\Phi}$.

Let us turn our attention to the function $B = B_+ = B\chi_{\mathrm{Re}>0}$, and substitute the corresponding values of Φ and Φ in (3.13). Using the fact that

$$(\overline{e_L}+i\tau)(a+ib)^k = (\overline{e_L}+i\tau)(a-be_L\tau)^k$$

for $\tau\in\mathbb{S}^{n-1}$ and $a,b\in\mathbb{R}$, we obtain

$$\begin{aligned}\frac{\overline{x}}{\sigma_n|x|^{n+1}} &= \frac{1}{2(2\pi i)^{n-1}}\int_{\mathbb{S}^{n-1}}(\overline{e_L}+i\tau)\frac{i}{2\pi}\frac{(-1)^{n-1}(n-1)!}{(\langle \underline{x},\ \tau\rangle + ix_L)^n}dS_\tau\\ &= \frac{(n-1)!}{2}\left(\frac{i}{2\pi}\right)^n\int_{\mathbb{S}^{n-1}}(\overline{e_L}+i\tau)(\langle\underline{x},\ \tau\rangle - x_Le_L\tau)^{-n}dS_\tau,\end{aligned}$$

where $x_L > 0$. If we take the real part of the right hand side, the above result is the plane wave decomposition of the Cauchy kernel obtained by Sommen in [5]. For the function $B = \chi_{\mathrm{Re}<0}$, we obtain

$$\frac{\overline{x}}{\sigma_n |x|^{n+1}} = \frac{-(n-1)!}{2}\left(\frac{-i}{2\pi}\right)^n \int_{\mathbb{S}^{n-1}} (\overline{e_L} + i\tau)(\langle \underline{x},\ \tau\rangle - x_L e_L \tau)^{-n} dS_\tau,$$

where $x_L > 0$. This coincides with Sommen's formula, see Ryan [6] for the details.

3.4 Möbius Covariance of Iterated Dirac Operators

In this section, we deduce the fundamental solutions of Dirac operators

$$\underline{D}^l = \sum_{i=1}^n e_i \frac{\partial}{\partial x_i}$$

and

$$D^l = \left(\frac{\partial}{\partial x_0} + \underline{D}\right)^l$$

in the setting of Clifford algebras. In [2], Peeter and Qian obtained the Möbius covariance of iterated Dirac operators.

For $\alpha > 0$, define the operator $\underline{D}^{-\alpha}$ as

$$\underline{D}^{-\alpha} f(\underline{x}) = c_n \int_{\mathbb{R}^n} e^{-i\langle \underline{x},\ \underline{\xi}\rangle} (i\underline{\xi})^{-\alpha} \hat{f}(\underline{\xi}) d\underline{\xi},$$

where $(i\underline{\xi})^{-\alpha}$ is defined by

$$(i\underline{\xi})^{-\alpha} = |\underline{\xi}|^{-\alpha} \chi_+(\underline{\xi}) + (-|\underline{\xi}|)^{-\alpha} \chi_-(\underline{\xi})$$

and

$$\chi_\pm(\underline{\xi}) = \frac{1}{2}\left(1 \pm i\underline{\xi}/|\underline{\xi}|\right).$$

Hence if $\alpha = l$ is a positive integer, we have

$$(i\underline{\xi})^{-l} = \begin{cases} 1/|\underline{\xi}|^l, & \text{if } l \text{ is even,} \\ i\underline{\xi}/|\underline{\xi}|^{l+1}, & \text{if } l \text{ odd .} \end{cases}$$

Therefore

$$\underline{D}^{-\alpha} f(\underline{x}) = \frac{c_n}{2}\Big[\int_{\mathbb{R}^n} e^{-i\langle \underline{x},\, \underline{\xi}\rangle}|\underline{\xi}|^{-\alpha}\hat{f}(\underline{\xi})d\underline{\xi} + \underline{D}\int_{\mathbb{R}^n} e^{-i\langle \underline{x},\, \underline{\xi}\rangle}|\underline{\xi}|^{-\alpha-1}\hat{f}(\underline{\xi})d\underline{\xi}$$
$$+\int_{\mathbb{R}^n} e^{-i\langle \underline{x},\, \underline{\xi}\rangle}(-|\underline{\xi}|)^{-\alpha}\hat{f}(\underline{\xi})d\underline{\xi} + \underline{D}\int_{\mathbb{R}^n} e^{-i\langle \underline{x},\, \underline{\xi}\rangle}(-|\underline{\xi}|)^{-\alpha-1}\hat{f}(\underline{\xi})d\underline{\xi}\Big].$$

If $0 < \alpha, \alpha + 1 < n$, by the formula

$$\left(\frac{1}{|\underline{\xi}|^{\beta}}\right)^{\vee} = c_{n,\beta}\frac{1}{|\underline{x}|^{n-\beta}},$$

we can deduce

$$\underline{D}^{-\alpha} f(\underline{x}) = K_{n,\alpha} * f(\underline{x}),$$

where

$$K_{n,\alpha}(\underline{x}) = c_{n,\alpha}(1+e^{-i\alpha\pi})\frac{1}{|\underline{x}|^{n-\alpha}} + d_{n,\alpha}(1-e^{-i\alpha\pi})\underline{D}\left(\frac{1}{|\underline{x}|^{n-\alpha-1}}\right).$$

For general $\alpha > 0$, by the same method, we can get

$$K_{n,\alpha}(\underline{x}) = c_{n,\alpha}(1+e^{-i\alpha\pi})G_{n,\alpha}(\underline{x}) + d_{n,\alpha}(1-e^{-i\alpha\pi})\underline{D}G_{n,\alpha+1}(\underline{x}),$$

where $G_{n,\beta}$ is the fundamental solution of the operator $|\underline{D}|^{\beta}$ with the symbol $|\underline{\xi}|^{\beta}$. Then for odd n,

$$K_{n,l}(\underline{x}) = \begin{cases} c_{n,l}\frac{\underline{x}}{|\underline{x}|^{n-l+1}}, & \text{if } l \text{ is odd;} \\ c_{n,l}\frac{1}{|\underline{x}|^{n-l}}, & \text{if } l \text{ is even.} \end{cases} \tag{3.14}$$

For the even n,

$$K_{n,l}(\underline{x}) = \begin{cases} c_{n,l}\frac{\underline{x}}{|\underline{x}|^{n-l+1}}, & \text{if } l \text{ is odd and } l < n; \\ c_{n,l}\frac{1}{|\underline{x}|^{n-l}}, & \text{if } l \text{ is odd and } l < n; \\ (c_{n,l}\log|\underline{x}| + d_{n,l})\frac{\underline{x}}{|\underline{x}|^{n-l+1}}, & \text{if } l \text{ odd and } l > n; \\ (c_{n,l}\log|\underline{x}| + d_{n,l})\frac{1}{|\underline{x}|^{n-l}}, & \text{if } l \text{ is even and } l > n; \end{cases} \tag{3.15}$$

Now we consider the fundamental solutions of the operators $D^l, l \in \mathbb{Z}_+$. Write $D_0 = \frac{\partial}{\partial x_0}$. Then

$$D^{-l} = (D_0 + \underline{D})^{-l} = (D_0 - \underline{D})^l (D_0^2 - \underline{D}^2)^{-l}.$$

By the Fourier transform, the symbol of $(D_0^2 - \underline{D}^2)^{-l}$ is $|\underline{\xi}|^{-2l}$. For $0 < 2l < n+1$, the inverse Fourier transform of $|\underline{\xi}|^{-2l}$ is $c_{n,l}|x|^{-(n+1-2l)}$. This indicates that the kernel of the operator D^{-l} is

$$L_{n,l}(x) = c_{n,l}(D_0 - \underline{D})^l \left(\frac{1}{|x|^{n+1-2l}} \right), \ 0 < 2l < n+1.$$

A direct computation gives

$$L_{n,l}(x) = c_{n,l} \frac{x_0^{l-1} \overline{x}}{|x|^{n+1}}, \ l \in \mathbb{Z}_+. \tag{3.16}$$

For any $x = x_0 + x_1 e_1 + \cdots + x_n e_n$, we write $x = x_0 + \underline{x}$ and $\underline{x} = x_1 e_1 + \cdots + x_n e_n \in \mathbb{R}^n$. We define two elementary operations

$$(e_{i_1} \cdots e_{i_l})^* =: e_{i_l} \cdots e_{i_1},$$

$$(e_{i_1} \cdots e_{i_l})' = (-1)^l (e_{i_1} \cdots e_{i_l}).$$

Let Γ_n be the multiplicative group of all elements in the Clifford algebra which can be written as products of non-zero vectors in $\mathbb{R}^n$. For any $a, b \in \Gamma_n \cup \{0\}$, $\overline{a}a = |a|^2$ and $|ab| = |a| \cdot |b|$. If $a \in \Gamma_n$, then $a = \prod_{j=1}^{M(a)} a_j$, where $a_j \in \mathbb{R}^n$. Generally speaking, such a representation and $M(a)$ are not unique. Denote by $m(a)$ the minimum of $M(a)$ over all such representations. If $a \in \mathbb{R} \setminus \{0\}$, we set $m(a) = 0$. Hence, $m(\underline{x}) = 1$, and for $a \in \Gamma_n$, $aa^* = a^*a = (-1)^{m(a)}|a|^2$. We call a group to be a Möbius group if this group consists of orientation preserving transforms acting in the Euclidean spaces. All Möbius transforms from $\mathbb{R}^n \cup \{\infty\}$ to $\mathbb{R}^n \cup \{\infty\}$ can be represented as

$$\phi(\underline{x}) = (a\underline{x} + b)(c\underline{x} + d)^{-1},$$

where $a, b, c, d \in \Gamma \cup \{0\}$ and

$$ad^* - bc^* \in \mathbb{R} \setminus \{0\}, \ a^*c, cd^*, d^*b, ba^* \in \mathbb{R}^n.$$

In addition, under 2×2 block matrix multiplication, the identification between the ϕ's and Clifford matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gives a homomorphism. For simplicity, we take $ad^* - bc^* = 1$ to normalize the Möbius transform. We consider the following multipliers:

$$T_l(\phi) f(\underline{x}) = J_{l,\phi} \cdot f(\phi(\underline{x})),$$

where for $l \in \mathbb{Z}$,

$$J_{l,\phi}(\underline{x}) = \begin{cases} \frac{(c\underline{x}+d)^*}{|c\underline{x}+d|^{n-l+1}}, \ l \text{ is odd }, \\ \frac{1}{|c\underline{x}+d|^{n-l}}, \ l \text{ is even.} \end{cases} \tag{3.17}$$

We will use the closed relation between $K_{n,l}$, $\underline{D}^l$ and the conformal weights $J_{l,\phi}$ to prove the following result.

Theorem 3.4.1 *For $l \in \mathbb{Z}_+$, the iterated Dirac operator $\underline{D}^l$ intertwines the representations T_l, T_{-l} of the Möbius transform group, that is, for $c \neq 0$,*

$$\underline{D}^l(T_l f) = \begin{cases} (-1)^{m(c)+1} T_{-l}(\underline{D}^l f), l \text{ is odd}; \\ T_{-l}(\underline{D}^l f), l \text{ is even}. \end{cases} \tag{3.18}$$

If $c = 0$, then $d \neq 0$ and the factor $(-1)^{m(c)+1}$ in the last formula should be replaced by $(-1)^{m(d)}$.

Proof We only prove the case $c \neq 0$. The case $c = 0$ can be dealt with similarly and is easier, so we omit the proof. We only need to prove

$$(T_l f) = \begin{cases} (-1)^{m(c)+1} \underline{D}^{-l} T_{-l}(\underline{D}^l f), l \text{ is odd}; \\ \underline{D}^{-l} T_{-l}(\underline{D}^l f), l \text{ is even}. \end{cases} \tag{3.19}$$

At first, we assume that n is odd or n is even and $l < n$. Denote by ψ the inverse of ϕ. If $\underline{y} = \phi(\underline{x}) = (a\underline{x} + b)(c\underline{x} + d)^{-1} \in \mathbb{R}^n$, then $\underline{y}(c\underline{x} + d) = a\underline{x} + b$. Hence we have $\underline{x} = \psi(\underline{y}) = (\underline{y}c - a)^{-1}(-\underline{y}d + b)$. Let $z = z(\underline{y}) = \underline{y} - a$ and $A = b - ac^{-1}d$. We can get

$$\underline{x} = z^{-1}A - c^{-1}d. \tag{3.20}$$

On the other hand, because $\underline{x} = \underline{x}^*$, $\underline{y} = \underline{y}^*$, (3.20) is equivalent to

$$\underline{x} = A^*(z^*)^{-1} - d^*(c^*)^{-1}. \tag{3.21}$$

By the Möbius transform and the formula (3.20), we deduce from $c \neq 0$ that $A \neq 0$ and

$$\begin{aligned} \underline{D}^{-l}(T_{-l}(\underline{D}^l f))(\psi(\underline{x})) &= c_{n,l} \int K_{n,l}(\psi(\underline{x}) - \underline{y}) \cdot J_{-l,\phi}(\underline{y})(\underline{D}^l f)(\phi(\underline{y}))d\underline{y} \\ &= c_{n,l} \int K_{n,l}(\psi(\underline{x}) - \psi(\underline{y})) \cdot J_{-l,\phi}(\psi(\underline{y}))(\underline{D}^l f)(\underline{y}) \left| \frac{d\psi(\underline{y})}{d\underline{y}} \right| d\underline{y}, \end{aligned} \tag{3.22}$$

where $|d\psi(\underline{y})/d\underline{y}|$ is the Jacobian matrix. Noticing that $\underline{x} = \psi(\underline{y})$ is also a Möbius transform, by the formula (2.4) in [7] and the condition $a\overline{d^*} - b\overline{c^*} = 1$, we can obtain the Jacobian matrix equals to $|z(y)|^{-2n}$. By equalities (3.15), (3.17) and

$$\begin{aligned} \psi(\underline{x}) - \psi(\underline{y}) &= (z^{-1}(\underline{x}) - z^{-1}(\underline{y}))A, \\ z^{-1}(\underline{x}) - z^{-1}(\underline{y}) &= -z^{-1}(\underline{x})(z(\underline{x}) - z(\underline{y}))z^{-1}(\underline{y}) \\ z(\underline{x}) - z(\underline{y}) &= (\underline{x} - \underline{y})c, \end{aligned}$$

we can deduce that (3.22) is equivalent to

$$
\begin{aligned}
&-c_{n,l}\frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-l+1}}\int\frac{(\underline{x}-\underline{y})}{|\underline{x}-\underline{y}|^{n-l+1}}\frac{c}{|c|^{n-l+1}}\frac{z^{-1}(\underline{y})}{|z^{-1}(\underline{y})|^{n-l+1}}\frac{A}{|A|^{n-l+1}}\\
&\cdot\frac{A^*}{|A|^{n-l+1}}\frac{(z^{-1}(\underline{y}))^*}{|z^{-1}(\underline{y})|^{n-l+1}}\frac{c^*}{|c|^{n-l+1}}(\underline{D}^l f)(\underline{y})\frac{1}{|z(\underline{y})|^{2n}}d\underline{y}\\
&=c_{n,l}\frac{1}{|A|^{2n}}\frac{(-1)^{m(c)}}{|c|^{2n}}\frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-l+1}}\int\frac{(\underline{x}-\underline{y})}{|\underline{x}-\underline{y}|^{n-l+1}}(\underline{D}^l f)(\underline{y})(\underline{y})\\
&=\frac{1}{|A|^{2n}}\frac{(-1)^{m(c)}}{|c|^{2n}}\frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-l+1}}\int K_{n,l}(\underline{x}-\underline{y})(\underline{D}^l f)(\underline{y})(\underline{y})\\
&=\frac{1}{|A|^{2n}}\frac{(-1)^{m(c)}}{|c|^{2n}}\frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-l+1}}f(\underline{x}),
\end{aligned}
$$

where in the above estimate we have used $m(z^{-1}A)=1$. Replacing $\underline{x}$ by $\phi(\underline{x})$ and noticing that $(\underline{x}+d^*(c^*)^{-1})=z^{-1}(\phi(\underline{x}))A$, we get

$$
\underline{D}^{-l}(T_{-l}(\underline{D}^l f))(\underline{x})=\frac{(-1)^{m(c)}cA^*}{|cA|^{n+l+1}}\frac{(cx+d)^*}{|cx+d|^{n-l+1}}f(\phi(\underline{x})).
$$

By $bc^*=ad^*-1$ and $c^{-1}d\in\mathbb{R}^n$, we can deduce that $b=-(c^*)^{-1}+ac^{-1}d$ and $A=-(c^*)^{-1}$, which gives (3.19).

The case for even l can be proved similarly. The only difference is that we should use the formulas (3.14), (3.15) and (3.17) for the case l even. Now we consider the case $l\geqslant n$, where n is even. Similar to the case l being odd, it can be deduced from (3.15) that

$$
\begin{aligned}
&\underline{D}^{-l}(T_{-l}(\underline{D}^l f))(\psi(\underline{x}))\\
&=\frac{1}{|A|^{2n}}\frac{(-1)^{m(c)}}{|c|^{2n}}\frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-l+1}}\int\Big[(-c_{n,l})\log|z(\underline{x})|+(c_{n,l}\log|\underline{x}-\underline{y}|+d_{n,l})\\
&+c_{n,l}\log|c|+(-c_{n,l}\log|z(\underline{y})|)\Big]\frac{\underline{x}-\underline{y}}{|\underline{x}-\underline{y}|^{n-l+1}}(\underline{D}^l f)(\underline{y})d\underline{y}\\
&=\sum_{i=1}^{4}I_i.
\end{aligned}
$$

When n is even and l is odd satisfying $l\geqslant n$, $(\underline{x}-\underline{y})/|\underline{x}-\underline{y}|^{n-l+1}=\pm(\underline{x}-\underline{y})^{l-n}$, $I_1=I_3=0$. For I_2, by the property of fundamental solution, we can deduce

$$
I_2=\frac{1}{|A|^{2n}}\frac{(-1)^{m(c)}}{|c|^{2n}}\frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-l+1}}f(\underline{x}).
$$

We shall prove $I_4=0$. In fact, because

$$\frac{\underline{x}-\underline{y}}{|\underline{x}-\underline{y}|^{n-l+1}} = \pm[(\underline{x}-ac^{-1})-(\underline{y}-ac^{-1})]^{l-n} = \sum_{k+j=l-n} h_{kj}(\underline{x}-ac^{-1})^k(\underline{y}-ac^{-1})^j,$$

by integration by parts, we have

$$\begin{aligned}
I_4 &= -c_{n,l}\sum_{k+j=n-l} h_{kj}(\underline{x}-ac^{-1})^k\int(\log|\underline{y}-ac^{-1}|+\log|c|)(\underline{y}-ac^{-1})^j(\underline{D}^l f)((\underline{y}))d\underline{y}\\
&= -c_{n,l}\sum_{k+j=n-l} h_{kj}(\underline{x}-ac^{-1})^k\int\log|\underline{y}-ac^{-1}|(\underline{y}-ac^{-1})^j(\underline{D}^l f)((\underline{y}))d\underline{y}\\
&= -c_{n,l}\sum_{k+j=n-l,\,j<l-n} h_{kj}(\underline{x}-ac^{-1})^k\int(\log|\underline{y}-ac^{-1}|+\log|c|)(\underline{y}-ac^{-1})^j(\underline{D}^l f)((\underline{y}))d\underline{y}\\
&\quad -c_{n,l}h_{0,l-n}\int\log|\underline{y}-ac^{-1}|(\underline{y}-ac^{-1})^{l-n}(\underline{D}^l f)((\underline{y}))d\underline{y}\\
&= \pm h_{0,l-n}\int c_{n,l}(\log|\underline{y}-ac^{-1}|+d_{n,l})\frac{(\underline{y}-ac^{-1})}{|\underline{y}-ac^{-1}|^{n-l+1}}(\underline{D}^l f)((\underline{y}))d\underline{y}\\
&= \pm h_{0,l-n}f(ac^{-1})\\
&= 0,
\end{aligned}$$

where in the last step we have used the following fact: the function $f\circ\phi$ is compactly supported and hence

$$f(ac^{-1}) = f\circ\phi\circ\psi(ac^{-1}) = f\circ\phi(\infty) = 0.$$

As above, we still obtain

$$\underline{D}^{-l}(T_{-l}(\underline{D}^l f))(\psi(\underline{x})) = \frac{1}{|A|^{2n}}\frac{(-1)^{m(c)}}{|c|^{2n}}\frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-l+1}}f(\underline{x}).$$

Replacing $\underline{x}$ by $\phi(\underline{x})$, we get (3.19) for the case l odd and $l \geqslant n$ with n even. The case l even can be obtained similarly. □

Now we consider the following question: for the operator D^l, if we have the similar conformal covariance. In fact, if we replace $\mathbb{R}^n$ in the Möbius transform, the identification relation and the certain Clifford matrices by $\mathbb{R}^n_1$, respectively, all conclusions still hold. Now let ϕ denote a Möbius transform from $\mathbb{R}^n_1\cup\{\infty\}$ to $\mathbb{R}^n_1\cup\{\infty\}$ and let g be a fixed function from $\mathbb{R}^n_1\cup\{\infty\}$ to $\mathbb{R}^n_1\cup\{\infty\}$. Define

$$g(x) = \frac{x^*}{|x|^{n+1}},\ x = x_0+\underline{x}.$$

Define the representations

$$S_1(\phi)f(x) := L_{n,1}((cx+d)^*)f(\phi(x))$$

and

$$S_{-1}(\phi)f(x) := g(cx+d)f(\phi(x)).$$

We have the following result.

Theorem 3.4.2

$$D(S_1 f) = S_{-1}(Df). \tag{3.23}$$

Proof By the fundamental solution of the operator D and (3.16), replacing $\underline{x}$ and $\underline{y}$ in Theorem 3.4.1 by x and y, respectively, we have

$$\begin{aligned}
D^{-1}(S_{-1}(Df))(\psi(x)) &= \int L_{n,1}(\psi(x)-\psi(y))g(cz^{-1}(y)A)(Df)(y)\frac{1}{|z(y)|^{2(n+1)}}dy \\
&= c_{n,1}\frac{-\overline{z^{-1}(x)}}{|z^{-1}(x)|}\int \frac{\overline{(x-y)cz^{-1}(y)A}}{|x-y|^{n+1}|c|^{n+1}|z^{-1}(y)|^{n+1}|A|^{n+1}} \\
&\quad \times \frac{A^*(z^{-1}(y))^*c^*}{|A|^{n+3}|z^{-1}(y)|^{n+3}|c|^{n+3}}\frac{1}{|z(y)|^{2(n+1)}}dy \\
&= \frac{-1}{|Ac^*|^{2n+2}}\frac{-\overline{z^{-1}(x)}}{|z^{-1}(x)|}\int L_{n,1}(x-y)(Df)(y)dy \\
&= \frac{-1}{|Ac^*|^{2n+2}}\frac{-\overline{z^{-1}(x)}}{|z^{-1}(x)|}f(x).
\end{aligned}$$

Replacing x by $\phi(x)$ and using $Ac^* = -1$, we get (3.23). □

3.5 The Fueter Theorem

In this section, we elaborate Qian's work on the generalization of Fueter's mapping theorem, see [3]. We shall work in $\mathbb{R}^{n+1}$, the real-linear span of $\{e_0, e_1, \ldots, e_n\}$, where e_0 is identical with 1 and $e_ie_j + e_je_i = -2\delta_{ij}$. $\mathbb{R}^{n+1}$ is embedded into Clifford algebra $\mathbb{R}_1^n$ generated by $e_1, \ldots e_n$. The elements in $\mathbb{R}^{n+1}$ are represented as $x = x_0 + \underline{x}$, where $x_0 \in \mathbb{R}$ and $\underline{x} = x_1e_1 + \cdots + x_ne_n$ with $x_j \in \mathbb{R}$. If $x \neq 0$, there exists an inverse x^{-1}: $x^{-1} = \frac{\overline{x}}{|x|^2}$, where $\overline{x} = x_0 - \underline{x}$. We will study the $\mathbb{R}^{n+1}$-valued and Clifford-valued functions, and the left and right monogeneity introduced by the Dirac operator

$$D = \frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} + \cdots + e_n\frac{\partial}{\partial_n}.$$

The Kelvin inversion of a function f is $I(f)(x) = E(x)f(x^{-1})$. The symbols $\mathbb{Z}$ and $\mathbb{Z}^+$ denote the sets of all integers and positive integers, respectively.

For a function f on $\mathbb{R}^{n+1}$, the Fourier transform of f is defined by

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^{n+1}} e^{2\pi i\langle x,\ \xi\rangle} f(x)dx.$$

A useful result associated with Fourier transform is

$$\mathcal{F}\Big(\frac{P_k(\cdot)}{|\cdot|^{k+n+1-\alpha}}\Big)(\xi) = \gamma_{k,\alpha}\frac{P_k(\xi)}{|\xi|^{k+\alpha}}, \tag{3.24}$$

where $0 < \alpha < n+1$, $k \in \mathbb{Z}^+$, P_k is the homogeneous harmonic polynomial of degree k, and

$$\gamma_{k,\alpha} = i^k \pi^{(n+1)/2-\alpha}\frac{\Gamma(k/2+\alpha/2)}{\Gamma(k/2+(n+1)/2-\alpha/2)},$$

(Γ denotes the usual Gamma function).

For a function g, the inverse Fourier transform $\mathcal{R}(g)$ is defined as

$$\mathcal{R}(g)(x) = \int_{\mathbb{R}^{n+1}} e^{-2\pi i\langle x,\ \xi\rangle} g(\xi)d\xi.$$

The Fourier transform of a function in the Schwartz class still belongs to the Schwartz class. In this case, the Fourier inversion formula holds: $\mathcal{RF}(f) = f$. In the sequel, the Fourier transform and the inverse Fourier transform will be used in the distributional sense.

For the function g defined on $\mathbb{R}^{n+1}$, we can introduce the Fourier multiplier M_g as $M_g f = \mathcal{R}(g\mathcal{F}f)$. It is easy to prove that the Fourier multiplier induced by $-4\pi^2|\xi|^2$ is identical to the Laplace operator.

Let f^0 be a complex-valued function defined on an open set O in the upper-half complex plane. Write $f^0 = u + iv$, where u and v are real-valued. For $x \in \overrightarrow{O}$, set

$$\overrightarrow{f}^0(x) = u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}v(x_0, |\underline{x}|),$$

where

$$\overrightarrow{O} = \Big\{x \in \mathbb{R}^{n+1} : (x_0, |\underline{x}|) \in O\Big\}.$$

$\overrightarrow{f}^0$ is called the function induced from f^0, and $\overrightarrow{O}$ is call the set induced from O.

We shall work with the functions of the form

$$g(x) = p(x_0, |\underline{x}|) + i\frac{\underline{x}}{|\underline{x}|}q(x_0, |\underline{x}|),$$

where p and q are real-valued. We call p and q the real part and the imaginary part of g, respectively.

The concepts of intrinsic functions and intrinsic sets naturally fit to our theory. On the complex plane $\mathbb{C}$, if an open set is symmetric with respect to the real axis, then the set is called an intrinsic set. If a function is defined on an intrinsic set and satisfies $\overline{f^0(z)} = f^0(\overline{z})$ within its domain, then the function is called an intrinsic function. For $f^0 = u + iv$, the above condition is equivalent to requiring that u is even respect to the second variable, and v is odd respect to the second variable. In particular, $v(x_0, 0) = 0$, i.e., if the domain f^0 is restricted on the real axis, then f^0 is real-valued.

Denote by τ the mapping

$$\tau(f^0) = \Delta^{(n-1)/2}\overrightarrow{f}^0,$$

where f^0 is any holomorphic intrinsic function and the differential operation is in the distributional sense. For the sake of convenience, outside the intrinsic set $\overrightarrow{O}$, we take $\overrightarrow{f}^0 = 0$.

Note that for odd $n \in \mathbb{Z}^+$, the operator $\Delta^{(n-1)/2}$ is a pointwise differential operator, while for even $n \in \mathbb{Z}^+$, $\Delta^{(n-1)/2}$ is the Fourier multiplier induced by $(2\pi i|\xi|)^{n-1}$ mapping some functions to the distributions. If b is a complex-valued function defined on an intrinsic set, then

$$\begin{cases} g^0(z) = \frac{1}{2}\Big[b(z) + \overline{b}(\overline{z})\Big], \\ b^0(z) = \frac{1}{2i}\Big[b(z) - \overline{b}(\overline{z})\Big] \end{cases}$$

both are intrinsic functions defined on the same set, and $b = g^0 + ib^0$.

The above observation enables us to extend the domain of τ to the sets of the complex-valued functions b on the intrinsic set. These functions b may not be intrinsic functions. For such a function b, we define

$$\tau(b) = \tau(g^0) + i\tau(b^0).$$

The mapping τ extended in such a way is linear under addition and real-scalar multiplication. In the sequel, for the mapping τ, we only need to consider the holomorphic intrinsic functions. For intrinsic functions, the coefficients of their Laurent series expansions in annuli centered at real points in their domains are all real. Hence we only need to consider the functions $\tau((\cdot)^{-k}),\ k \in \mathbb{Z}$. For $k \in \mathbb{Z}^+$, define

$$P^{(-k)} = \tau((\cdot)^{-k}), \quad P^{(k-1)} = I(P^{(-k)}).$$

We have the following result.

Theorem 3.5.1 *Let $k \in \mathbb{Z}^+$. Then*

(i) *$P^{(-k)}$ and $P^{(k-1)}$ are monogenic functions;*
(ii) *$P^{(-k)}$ is homogeneous of degree $(n+1-k)$ and $P^{(k-1)}$ is homogeneous of degree $(k-1)$;*
(iii) *If n is odd, then $P^{(k-1)} = \tau((\cdot)^{n+k-2})$.*

Proof (i) By the Fourier transform and the following relation:

$$\overrightarrow{(\cdot)^{-k}}(x) = \left(\frac{\overline{x}}{|x|^2}\right)^k = \frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_0}\right)^{k-1}\left(\frac{\overline{x}}{|x|^2}\right),$$

we get

$$\begin{aligned}P^{(-k)}(x) &= \tau((\cdot)^{-k})(x) \\ &= \frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_0}\right)^{k-1} \mathcal{RF}\left(\Delta^{(n-1)/2}\frac{\overline{x}}{|x|^2}\right) \\ &= \frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_0}\right)^{k-1} \mathcal{R}\left(\gamma_{1,n}(2\pi i|\xi|)^{n-1}\frac{\overline{\xi}}{|\xi|^{n+1}}\right) \\ &= \frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_0}\right)^{k-1} \gamma_{1,n}^2(2\pi i)^{n-1}\frac{\overline{x}}{|x|^{n+1}} \\ &= \frac{(-1)^{k-1}}{(k-1)!}\kappa_n\left(\frac{\partial}{\partial x_0}\right)^{k-1} E(x),\end{aligned} \tag{3.25}$$

where

$$\kappa_n = (2\pi i)^{n-1}\gamma_{1,n}^2 = (2i)^{n-1}\Gamma^2((n+1)/2).$$

This means that for $k \in \mathbb{Z}^+$, $P^{(-k)}$ is monogenic. The monogeneity of $P^{(k)}$ can be deduced by the property of the Kelvin inversion, or the result of Bojarski, see [2].

The conclusion (ii) can be obtained by the expression of $P^{(-k)}$ and the property of the Kelvin inversion.

(iii) Let $n = 2m + 1$. We have

$$\kappa_n = (-1)^m 2^{2m}(m!)^2 = (-1)^m((2m)!!)^2.$$

We use the mathematical induction. The case $k = 1$ reduces to verifying $\Delta^m(x^{2m}) = (-1)^m(2m)!!$. We need the following lemma.

Lemma 3.5.1 *Let $f^0(z) = u(x_0, y) + iv(x_0, y)$ be a holomorphic function defined on an open set U in the upper-half complex plane. Write $u_0 = u$, $v_0 = v$, and for $s \in \mathbb{Z}^+$, write*

$$u_s = 2s\frac{\partial u_{s-1}}{\partial y}\frac{1}{y}$$

and

$$v_s = 2s\left(\frac{\partial v_{s-1}}{\partial y}\frac{1}{y} - \frac{v_{s-1}}{y^2}\right) = 2s\frac{\partial}{\partial y}\frac{v_{s-1}}{y}.$$

Then

$$\Delta^s\overrightarrow{f}^0(x) = u_s(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}v_s(x_0, |\underline{x}|),\ x_0 + i|\underline{x}| \in U.$$

This lemma can be proved using mathematical induction via a computation of $\Delta(u_{s-1} + iv_{s-1})$ invoking the following relation proved in [8]:

$$\frac{\partial u_{s-1}}{\partial x_0} = \frac{\partial v_{s-1}}{\partial y} + 2(s-1)\frac{v_{s-1}}{y}, \quad \frac{\partial u_{s-1}}{\partial y} = -\frac{\partial v_{s-1}}{\partial x_0}.$$

We will frequently use the formula given in [8]: for any function $f^0 = u + iv$ and $r \in \mathbb{Z}^+$,

$$(\overrightarrow{f}^0)^r(x) = \sum_{l=0}^{[r/2]}(-1)^l C_r^{2l} u^{r-2l} v^{2l} + \frac{\underline{x}}{|\underline{x}|}\sum_{l=0}^{[r/2]}(-1)^l C_r^{2l+1} u^{r-2l-1} v^{2l+1}, \quad (3.26)$$

where C_r^l are binomial coefficients with the convention that $C_r^l = 0$ for $l > r$, and $[s]$ denotes the largest integer that does not exceed s.

For $f^0(z) = z$, using the formula (3.26), by $r = 2m$ and Lemma 3.5.1, we can obtain $\Delta^m(x^{2m}) = (-1)^m((2m)!!)^2$, which proves the case $k = 1$. Now assume that $P^{(k)} = \tau((\cdot)^{n+k-1})$. We need to prove $P^{(k+1)} = \tau((\cdot)^{n+k})$. This is equivalent to proving

$$\frac{-1}{k+1}\frac{\partial}{\partial x_0}(I(\Delta^m((\cdot)^{2m+k}))) = I(\Delta^m((\cdot)^{2m+k+1})), \quad (3.27)$$

where $k \in \mathbb{Z}^+$ or $k = 0$.

By (3.26) and Lemma 3.5.1, we have

$$\begin{aligned}
&\Delta((\cdot)^{2m+k})(x)\\
&= (2m)!!\Big[\sum_{l=0}^{m+[k/2]}(-1)^l C_{2m+k}^{2l}(2l)(2l-2)\cdots(2l-2m+2)x_0^{2m+k-2l}y^{2l-2m}\\
&+\frac{\underline{x}}{y}\sum_{l=0}^{m+[k/2]}(-1)^l C_{2m+k}^{2l+1}(2l)(2l-2)\cdots(2l-2m+2)x_0^{2m+k-2l-1}y^{2l+1-2m}\Big],
\end{aligned}$$

where we take $y = |\underline{x}|$.

By the Kelvin inversion, we replace x_0, y and $\underline{x}/y$ by $x_0|x|^{-2}$, $y|x|^{-2}$ and $-\underline{x}/y$, respectively. It follows that the above becomes

$$\begin{aligned}
&(2m)!!\frac{\overline{x}}{|x|^{n+2k+1}}\Big[\sum_{l=0}^{m+[k/2]}(-1)^l C_{2m+k}^{2l} \quad (3.28)\\
&(2l)(2l-2)\cdots(2l-2m+2)x_0^{2m+k-2l}y^{2l-2m} + \frac{\underline{x}}{y}\sum_{l=0}^{m+[k/2]}(-1)^{l+1}C_{2m+k}^{2l+1}\\
&\times(2l)(2l-2)\cdots(2l-2m+2)x_0^{2m+k-2l-1}y^{2l+1-2m}\Big].
\end{aligned}$$

Applying the differential operator $[-1/(k+1)]\partial/\partial x_0$ to (3.28), we have

$$\frac{-(2m)!!}{k+1}E(x)\frac{1}{|x|^{2k+2}}\Big\{\Big(-(n+2k)x_0+\frac{\underline{x}}{y}y\Big)[\cdots]+(x_0^2+y^2)\frac{\partial}{\partial x_0}[\cdots]\Big\}, \tag{3.29}$$

where $[\cdots]$ is as $[\cdots]$ in (3.28).

Now we have

$$\begin{aligned}
&\Big(-(n+2k)x_0+\frac{\underline{x}}{y}y\Big)[\cdots]\\
&=\Bigg\{\sum_{l=0}^{m+[k/2]}(-1)^{l+1}C_{2m+k}^{2l}(n+2k)(2l)(2l-2)\cdots(2l-2m+2)x_0^{2m+k-2l}y^{2l-2m}\\
&+\sum_{l=0}^{m+[k/2]}(-1)^{l}C_{2m+k}^{2l+1}(2l)(2l-2)\cdots(2l-2m+2)x_0^{2m+k-2l-1}y^{2l+1-2m}\Bigg\}\\
&+\frac{\underline{x}}{y}\Bigg\{\sum_{l=0}^{m+[k/2]}(-1)^{l+1}C_{2m+k}^{2l+1}(n+2k)(2l)(2l-2)\cdots(2l-2m+2)x_0^{2m+k-2l}y^{2l+1-2m}\\
&+\sum_{l=0}^{m+[k/2]}(-1)^{l}C_{2m+k}^{2l}(2l)(2l-2)\cdots(2l-2m+2)x_0^{2m+k-2l}y^{2l+1-2m}\Bigg\}
\end{aligned}$$

and

$$\begin{aligned}
&(x_0^2+y^2)\frac{\partial}{\partial x_0}[\cdots]\\
&=\Bigg\{\sum_{l=0}^{m+[k/2]}(-1)^{l}C_{2m+k}^{2l}(2l)(2l-2)\cdots(2l-2m+2)(2m+k-2l)\\
&\quad\times\ (x_0^{2m+k-2l+1}y^{2l-2m}+x_0^{2m+k-2l-1}y^{2l-2m+2})\Bigg\}\\
&+\frac{\underline{x}}{y}\Bigg\{\sum_{l=0}^{m+[k/2]}(-1)^{l+1}C_{2m+k}^{2l+1}(2l)(2l-2)\cdots(2l-2m+2)(2m+k-2l-1)\\
&\quad\times(x_0^{2m+k-2l}y^{2l+1-2m}+x_0^{2m+k-2l-2}y^{2l+1-2m+2})\Bigg\}.
\end{aligned}$$

By comparing the coefficients of a general nomial $x_0^{2m+k+1-2l}y^{2l-2m}$ in the real part of (3.29) with those in the real part of

$$I(\Delta^m((\cdot)^{2m+k+1}))(x)=E(x)(\Delta^m((\cdot)^{2m+k+1}))(x^{-1}),$$

the latter being of the expression (3.28) but with $k+1$ in place of k, we are reduced to verifying

$$
\begin{aligned}
&-2l(n+2k)C_{2m+k}^{2l} + (2m-2l)C_{2m+k}^{2l-1} \\
&+2l(2m+k-2l)C_{2m+k}^{2l} + (2m-2l)(2m+k-2l+2)C_{2m+k}^{2l-2} \\
&= -(k+1)2lC_{2m+k+1}^{2l}.
\end{aligned} \tag{3.30}
$$

By $(s-l)C_s^l = (l+1)C_s^{l+1}$, the second and fourth entries on the left hand side of (3.30) add up to

$$2l(2m-2l)C_{2m+k}^{2l-1}, \tag{3.31}$$

while the first and third to

$$
\begin{aligned}
-2l(2l+k+1)C_{2m+k}^{2l} &= [-4l^2 - 2l(k+1)]C_{2m+k}^{2l} \\
&= -2l(2m+k-2l+1)C_{2m+k}^{2l-1} - 2l(k+1)C_{2m+k}^{2l}.
\end{aligned} \tag{3.32}
$$

Combining (3.31) with the right hand side of (3.32) and using $C_s^l + C_s^{l-1} = C_{s+1}^l$, we get (3.30). Similarly, we can prove that the imaginary part of (3.29) is equivalent to the imaginary part of $I(\Delta^m((\cdot)^{2m+k+1}))$. This proves (iii). □

In [9], Kou, Qian and Sommen obtained the following generalization of Theorem 3.5.1. For any $x = x_0 + \underline{x} \in \mathbb{R}^n$, let P_k be a homogeneous polynomial of $\underline{x}$ of degree k and satisfy

$$\underline{\partial} P_k(\underline{x}) = 0.$$

We consider the following question: if

$$D\Delta^{k+(n-1)/2}\Big(\Big(u(x_0,\underline{x}) + \frac{\underline{x}}{|\underline{x}|}v(x_0,\underline{x})P_k(\underline{x})\Big)\Big) = 0.$$

We first prove that if $l \in \mathbb{Z}$, the function

$$\Delta^{k+(n-1)/2}\Big((x_0+\underline{x})^l P_k(\underline{x})\Big) \tag{3.33}$$

is still a left monogenic function.

At first, we assume that l is negative. By a simple computation, we can see that

$$(x_0+\underline{x})^{-l} = \left(\frac{\overline{x}}{|x|^2}\right)^l = \frac{(-1)^{l-1}}{(l-1)!}\left(\frac{\partial}{\partial x_0}\right)^l\left(\frac{\overline{x}}{|x|^2}\right),\ l = 1, 2, \ldots.$$

Hence we only need to prove

$$\Delta^{k+(n-1)/2}\left(\frac{\overline{x}}{|x|^2}P_k(\underline{x})\right)$$

is left monogenic.

Lemma 3.5.2 *$Q_{k+1}(x) = \overline{x}P_k(\underline{x})$ is harmonic and homogeneous of degree $k+1$.*

Proof By definition, it can be verified directly that

$$\left(\frac{\partial}{\partial x_0}\right)^2 Q_{k+1}(x) = 0.$$

Using Leibniz's formula for second derivative, we can get

$$\left(\frac{\partial}{\partial x_i}\right)^2 Q_{k+1}(x) = 2\left(\frac{\partial}{\partial x_i}\right)(\overline{x})\left(\frac{\partial}{\partial x_i}\right) P_k(\underline{x}) + \overline{x}\left(\frac{\partial}{\partial x_i}\right) P_k(\underline{x})^2.$$

This implies that

$$\Delta Q_{k+1}(x) = -2\underline{\partial} P_k(\underline{x}) + \overline{x}\underline{\Delta} P_k(\underline{x}) = 0.$$

□

In the proof of Theorem 3.5.1, we use the following Bochner type formula: in the sense of tempered distributional sense,

$$\left(\frac{Q_j(\cdot)}{|\cdot|^{j+(n+1)-\alpha}}\right)^{\wedge}(\xi) = \gamma_{j,\alpha}\frac{Q_j(\cdot)(\xi)}{|\xi|^{j+\alpha}},\ j \in \mathbb{Z}_+, 0 < \alpha < n+1, \tag{3.34}$$

where Q_j is a harmonic homogeneous polynomial of degree j, and

$$\gamma_{j,\alpha} = i^j \pi^{(n+1)/2-\alpha}\frac{\Gamma(j/2+\alpha/2)}{\Gamma(j/2+(n+1)/2-\alpha/2)}.$$

By the Fourier transform, (3.34) is equivalent to the following equality: for any Schwartz function ϕ on $\mathbb{R}_1^n$ and $j \in \mathbb{Z}_+, 0 < \alpha < n+1$,

$$\int_{\mathbb{R}_1^n}\frac{Q_j(x)}{|x|^{j+(n+1)-\alpha}}\hat{\phi}(x)dx = i^j \pi^{(n+1)/2-\alpha}\frac{\Gamma(j/2+\alpha/2)}{\Gamma(j/2+(n+1)/2-\alpha/2)}\int_{\mathbb{R}_1^n}\frac{Q_j(x)}{|x|^{j+\alpha}}\phi(x)dx.$$

Now we will generalize the above formula to the case $\mathrm{Re}(\alpha) > -j$ and $j \in \mathbb{Z}_+$.

Lemma 3.5.3 *Let $-j < \beta, \alpha < (n+1)+j, \alpha+\beta = n+1$ and $j \in \mathbb{Z}_+$. For any Schwartz function ϕ on $\mathbb{R}_1^n$, we have*

$$\pi^{\beta/2}\Gamma\left(\frac{j+\beta}{2}\right)\int_{\mathbb{R}_1^n}\frac{Q_j(x)}{|x|^{j+\beta}}\hat{\phi}(x)dx = i^j\pi^{\alpha/2}\Gamma\left(\frac{j+\alpha}{2}\right)\int_{\mathbb{R}_1^n}\frac{Q_j(x)}{|x|^{j+\alpha}}\phi(x)dx. \tag{3.35}$$

Proof For $0 < \alpha < n+1$, both sides of (3.34) are holomorphic. For $j \geqslant 1$, by the orthogonality of the spherical harmonic polynomials, there follows

$$\int_{\mathbb{R}_1^n} \frac{Q_j(x)}{|x|^{j+\beta}} \hat{\phi}(x) dx = \lim_{\epsilon \to 0+} \frac{Q_j(x)}{|x|^{j+(n+1)-\alpha}} \left(\hat{\phi}(x) - \hat{\phi}(0) - \frac{1}{(j-1)!} \left(\sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i} \right)^{j-1} \hat{\phi}(0) \right) dx$$
$$+ \int_{|x|>1} \frac{Q_j(x)}{|x|^{j+(n+1)-\alpha}} \hat{\phi}(x) dx,$$

that can be extended to all complex numbers α with $\mathrm{Re}(\alpha) > -j$ holomorphically. Similarly, the right hand side of (3.34) can also be extended holomorphically to all complex numbers α such that $\mathrm{Re}(\alpha) > -j$. □

Proposition 3.5.1 *Let $l \in \mathbb{Z}_+$, where $n+1$ is odd and k is non-negative. Then the functions*

$$\Delta^{k+(n-1)/2} \left(\left(\frac{\overline{x}}{|x|^2} \right)^l P_k(\underline{x}) \right), \ l \in \mathbb{Z}_+$$

are all left monogenic.

Proof In Lemma 3.5.3, letting $\alpha = 2 - j$, we have

$$\lim_{\epsilon \to 0+} \int_{|x|>\epsilon} \frac{Q_j(x)}{|x|^{j+(n+1)+j-2}} \hat{\phi}(x) dx = \frac{i^j \pi^{(n+1)/2+(j-2)}}{\Gamma((n+1)/2+j-1)} \int_{\mathbb{R}_1^n} \frac{Q_j(x)}{|x|^2} \phi(x) dx.$$

Replacing ϕ by $\Delta^{k+(n+1)/2}\phi$ and j by $k+1$, we get

$$\lim_{\epsilon \to 0+} \int_{|x|>\epsilon} \frac{Q_{k+1}(x)}{|x|^{(n+1)+k}} |x|^{2k+(n-1)} \hat{\phi}(x) dx = \beta_k \int_{\mathbb{R}_1^n} \Delta^{k+(n-1)/2} \left(\frac{Q_{k+1}(x)}{|x|^2} \right) \phi(x) dx,$$

where

$$\beta_k = 2^{1-n-2k} i^{2-n-k} \pi^{-k-(n-1)/2} \frac{1}{\Gamma((n+1)/2+k)}.$$

Hence we obtain

$$\int_{\mathbb{R}_1^n} \frac{Q_{k+1}(x)}{|x|^2} \hat{\phi}(x) dx = \beta_k \int_{\mathbb{R}_1^n} \Delta^{k+(n-1)/2} \left(\frac{Q_{k+1}(x)}{|x|^2} \right) \phi(x) dx.$$

Replacing Q_{k+1} by $\overline{x} P_k(\underline{x})$, we have

$$\int_{\mathbb{R}_1^n} \frac{Q_{k+1}(x)}{|x|^2} \hat{\phi}(x) dx = \int_{\mathbb{R}_1^n} \left(\frac{\bar{\cdot}}{|x|^2} P_k(\underline{\cdot}) \right)^{\wedge} (x) \phi(x) dx$$
$$= \gamma_{1,n}^{-1} \int_{\mathbb{R}_1^n} E * (P_k(\underline{\partial})\delta)(x) \phi(x) dx,$$

where $E(x) = \frac{\overline{x}}{|x|^{n+1}} = \gamma_{1,n} (\frac{\bar{\cdot}}{|\cdot|^2})^{\wedge}(x)$ is the Cauchy kernel on $\mathbb{R}_1^n$, δ is the Dirac function. Therefore,

$$\Delta^{k+(n-1)/2}\left(\frac{\overline{x}}{|x|^2}P_k(\underline{x})\right)=\gamma_{1,n}^{-1}\beta_k^{-1}E*(P_k(\underline{\partial})\delta)(x)=\gamma_{1,1}\beta_k^{-1}EP_k(\underline{\partial})(x).$$

This implies

$$\Delta^{k+(n-1)/2}\left(\frac{\overline{x}}{|x|^2}P_k(\underline{x})\right)$$

is left monogenic. In addition,

$$\begin{aligned}\Delta^{k+(n-1)/2}\left(\left(x_0+\underline{x}\right)^{-l}P_k(\underline{x})\right)&=\Delta^{k+(n-1)/2}\left(\left(\frac{\overline{x}}{|x|^2}\right)^{l}P_k(\underline{x})\right)\\&=\Delta^{k+(n-1)/2}\left(\frac{(-1)^{l-1}}{(l-1)!}\left(\frac{\partial}{\partial x_0}\right)^{l-1}\left(\frac{\overline{x}}{|x|^2}\right)P_k(\underline{x})\right)\\&=\gamma_{1,1}\beta_k^{-1}\left(\frac{\partial}{\partial x_0}\right)^{l-1}EP_k(\underline{\partial})(x).\end{aligned}\tag{3.36}$$

So

$$\Delta^{k+(n-1)/2}\left(\left(x_0+\underline{x}\right)^{-l}P_k(\underline{x})\right),\ l\in\mathbb{Z}_+,$$

are all left monogenic. □

Now for the case $l\geqslant 0$, we prove the function

$$\Delta^{k+(n-1)/2}\Big[(x_0+\underline{x})^lP_k(\underline{x})\Big]\tag{3.37}$$

is left monogenic function. We first discuss the fundamental solution of the operator $D\Delta^{k+(n-1)/2}$. Below we assume that $2s=2k+(n-1)$. Hence $2s$ may be even or odd. It is even if and only if $n+1$ is even.

Lemma 3.5.4 *The operator $D|D|^{2s}$ in $\mathbb{R}_1^n$ has a fundamental solution of the same form as those in the above list for $\underline{\partial}^{2s+1}$ in $\mathbb{R}^{n+1}$, except that the term $\underline{x}$ in the latter is replaced by $\overline{x}$.*

Proof We divide the proof into two cases based on the parity of $2s$.

(i) Case 1: $2s$ is even. The Fourier multiplier corresponding to the fundamental solution of $D|D|^{2s}$ is

$$c_{n,k}\frac{1}{\xi}\frac{1}{|\xi|^{2s}}=c_{n,k}\frac{\overline{\xi}}{|s|^{2s+2}},$$

where $c_{n,k}$ is a constant depending on n and k. A fundamental solution of $|D|^{2s+2}$ is a radial function and is the same as the one in the above list for $\underline{\partial}^{2s+2}$ in $\mathbb{R}^{n+1}$. We denote the fundamental solution by $K(x)$. By (3.14) and (3.15), when $n+1$ is even and $2s+2<n+1$,

$$K(x) = \frac{1}{|x|^{n-2s-1}}.$$

When $n+1$ is even and $2s+2 \geqslant n+1$,

$$K(x) = (c \log |x| + d) \frac{1}{|x|^{n-2s-1}}.$$

Then $\overline{D}K$ is a fundamental solution of $D|D|^{2s}$. Hence the function $\overline{D}K$ can be represented as follows. When $n+1$ is even and $2s+2 < n+1$,

$$K(x) = \frac{\overline{x}}{|x|^{n-2s+1}}.$$

When $n+1$ is even and $2s+2 \geqslant n+1$,

$$K(x) = (c \log |x| + d) \frac{\overline{x}}{|x|^{n-2s+1}}.$$

(ii) Case 2: $2s$ is odd. At first, because $\xi\overline{\xi} = |\xi|^2$, $\frac{1}{\xi}\frac{1}{|\xi|^{2s}} = \frac{1}{|\xi|}\frac{\overline{\xi}}{|\xi|^{2s+1}}$. Also, the Fourier multiplier corresponding to a fundamental solution of $D|D|^{2s-1}$ is $\frac{\overline{\xi}}{|\xi|^{2s+1}}$. By (3.14) and (3.15), when $n+1$ is odd, the fundamental solution is $\overline{x}/|x|^{n-2s+2}$. Because the Fourier transform of $1/|\xi|$ is the Riesz potential $1/|x|^n$, then in the tempered distributional sense, the fundamental solution of $D|D|^{2s}$ can be represented via convolution:

$$\frac{1}{|\cdot|^n} * \frac{\overline{(\cdot)}}{|\cdot|^{n-2s+2}}.$$

It is easy to see that the convolution is a locally integrable function away from the origin. In fact, after being applied a certain times Laplace operator, the above distribution becomes a locally integrable function away from the origin. Secondly, as a distribution, the convolution is homogeneous of degree $2s-n$. To show this, letting M and N denote the distributions induced by $\frac{1}{|x|^n}$ and $\frac{\overline{x}}{|x|^{n-2s+2}}$, respectively, then for any Schwartz function ϕ, we have

$$\langle M * N(x),\ \phi(x/\delta)\rangle = \delta^{(n+1)+(2s-n)}\langle M * N(x),\ \phi(x)\rangle.$$

Write $\tau_\delta f(x) = f(\delta x)$. By the homogeneous properties of M and N, we know

$$\begin{aligned}\Big\langle M * N(x),\ \phi(x/\delta)\Big\rangle &= \Big\langle M * N,\ \tau_{\delta^{-1}}\phi(x)\Big\rangle \\ &= \Big\langle N(x),\ M * (\tau_{\delta^{-1}}\phi)(x)\Big\rangle \\ &= \delta\Big\langle N(x),\ \tau_{\delta^{-1}} M * \phi(x)\Big\rangle\end{aligned}$$

$$= \delta^{1+2s}\Big\langle N,\ M * \phi\Big\rangle$$
$$= \delta^{1+2s}\Big\langle M * N,\ \phi\Big\rangle.$$

Let ρ denote the rotation about the origin in $\mathbb{R}_1^n$. The representation matrix of ρ is (ρ_{ij}) and the operation of ρ on x is denoted by $\rho^{-1}x$. The operation of ρ on the functions is denoted by $\rho(f)(x) = f(\rho^{-1}x)$. Because M is scalar-valued and N is vector-valued, the function $M * N$ is vector-valued and homogeneous with degree $2s - n$. Write this vector-valued function as $K(x) = M * N(x)$. Then we can get

$$\begin{aligned}
\Big\langle \rho\overline{K(x)},\ \phi(x)\Big\rangle &= \Big\langle \overline{K(x)},\ \rho^{-1}\phi(x)\Big\rangle \\
&= \Big\langle \overline{N(x)},\ M * \rho^{-1}\phi(x)\Big\rangle \\
&= \Big\langle \overline{N(\rho^{-1}x)},\ M * \phi(x)\Big\rangle \\
&= \Big\langle (\rho_{ij})\overline{(N(x))},\ M * \phi(x)\Big\rangle \\
&= (\rho_{ij})\Big\langle \overline{K(x)},\ \phi(x)\Big\rangle \\
&= \Big\langle (\rho_{ij})\overline{(K(x))},\ \phi(x)\Big\rangle,
\end{aligned}$$

that is, $\overline{K}(\rho^{-1}x) = \rho(\overline{K(x)})$. Applying the lemma obtained in [10, Chap. 3, Sect. 1.2] to $\overline{K(x)}/|x|^{2s-n}$, we get $\overline{K(x)}/|x|^{2s-n} = Cx/|x|$, Hence

$$M * N(x) = \frac{C\overline{x}}{|x|^{n-2s+1}}.$$

□

We prove when $l \in \mathbb{Z}_+$, the function

$$\Delta^{k+(n-1)/2}\Big(\Big(x_0 + \underline{x}\Big)^{-l} P_k(\underline{x})\Big)$$

is left monogenic. We need the intertwining relation for the operator.

Lemma 3.5.5 *Let n be any positive integer. Then for $s = k + (n-1)/2$ and any infinitely differentiable function g in $\mathbb{R}_1^n \setminus \{0\}$, we have*

$$(D\Delta^s)\Big(\frac{\overline{x}}{|x|^{(n+1)-2s}} g(x^{-1})\Big) = \alpha_{n,s} \frac{x}{|x|^{(n+1)+2s+2}} (D\Delta^s)(g)(x^{-1}), \tag{3.38}$$

where $\alpha_{n,s}$ is a constant depending on n and s.

Proof Write $L = D\Delta^s = D|D|^{2s}$. Because $n+1$ is odd, by Case 2 of Lemma 3.5.4, the fundamental solution of L is $G(x) = \frac{C\overline{x}}{|x|^{n-2s+1}}$. We have

$$
\begin{aligned}
& L^{-1}\left(\frac{(\cdot)}{|(\cdot)|^{(n+1)+2s+2}}(Lg)((\cdot)^{-1})\right)(x^{-1}) \\
&= \int_{\mathbb{R}_1^n} G(x^{-1}-y^{-1})\frac{y^{-1}}{|y^{-1}|^{(n+1)+2s+2}}\frac{1}{|y|^{2n+2}}(Lg)(y)dy \\
&= \frac{C\overline{x^{-1}}}{|x^{-1}|^{n-2s+1}}\int_{\mathbb{R}_1^n}\frac{-\overline{(x-y)}}{|x-y|^{n-2s+1}}\frac{\overline{y^{-1}}}{|y^{-1}|^{n-2s+1}} \\
&\quad \times \frac{y^{-1}}{|y^{-1}|^{(n+1)+2s+2}}\frac{1}{|y|^{2n+2}}(Lg)(y)dy \\
&= \frac{C\overline{x^{-1}}}{|x^{-1}|^{n-2s+1}}\int_{\mathbb{R}_1^n}\frac{\overline{x-y}}{|x-y|^{n-2s+1}}(Lg)(y)dy \\
&= \frac{C\overline{x^{-1}}}{|x^{-1}|^{n-2s+1}}g(x).
\end{aligned}
$$

Then we can deduce that

$$
L\left(\frac{\overline{(\cdot)}}{|\cdot|^{(n+1)-2s}}g((\cdot)^{-1})\right)(x) = C\frac{x}{|x|^{(n+1)+2s+2}}(Lg)(x^{-1}).
$$

□

In Lemma 3.5.5, take $g(x)=\left(\frac{\bar{x}}{|x|^2}\right)^l P_k(\underline{x}),\ l\in\mathbb{Z}_+$. Because $g(x^{-1}) = (-1)^k x^l |x|^{-2k}P_k(\underline{x})$, we have

$$
\left(D\Delta^{k+(n-1)/2}\right)\left((-1)^k x^{l-1}P_k(\underline{x})\right) = \alpha_{n,s}\frac{x}{|x|^{2n+2k+2}}\left(D\Delta^{k+(n-1)/2}\right)\left(\left(\frac{\bar{\cdot}}{|\cdot|^2}\right)^l P_k(\underline{\cdot})\right)(x^{-1}). \tag{3.39}
$$

By Proposition 3.5.1, we can see that the right hand side of (3.39) is zero and conclude that

$$
\left(D\Delta^{k+(n-1)/2}\right)\left((x_0+\underline{x})^{l-1}P_k(\underline{x})\right)=0,\ l\in\mathbb{Z}_+.
$$

Based on the following preliminary lemma, we give a generalization of Theorem 3.5.1.

Theorem 3.5.2 *Let f be a holomorphic function defined on an open set B in the upper half complex plane. Define the set*

$$
\overrightarrow{B} = \left\{x = x_0+\underline{x}\in\mathbb{R}_1^n,\ (x_0,|\underline{x}|)\in B\right\}.
$$

(i) Let $P_k(\underline{x})$ be left-monogenic and homogeneous of degree k. If $k+(n-1)/2$ is a non-negative integer, then in the set $\overrightarrow{B}$, the function

$$\Delta^{k+(n-1)/2}[f(x_0+\underline{x})P_k(\underline{x})]$$

is left monogenic.

(ii) If $(n-1)/2$ is odd and k is a non-negative integer and $P_k(\underline{x})$ is monogenic and homogeneous of degree k, then in the set $\overrightarrow{B}$, the function

$$\Delta^{k+(n-1)/2}[f(x_0+\underline{x})P_k(\underline{x})]$$

is left monogenic.

Proof We only need to prove that if the function

$$\Delta^{k+(n-1)/2}((x_0+\underline{x})^l P_k(\underline{x})),\ l\in\mathbb{Z},$$

is monogenic, then the function

$$\Delta^{k+(n-1)/2}(f(x_0+\underline{x})P_k(\underline{x}))$$

is also monogenic. Through a translation, we may assume that the function f is holomorphic in a disc centered at the origin of the complex plane. Further, we define the holomorphic function

$$\begin{cases} g(z)=\frac{1}{2}[f(z)+\overline{f}(\overline{z})],\\ h(z)=\frac{1}{2i}[f(z)-\overline{f}(\overline{z})]. \end{cases}$$

It is easy to see that $f(z)=g(z)+ih(z)$. Then we can further assume that the Taylor series expansion of f is of real coefficients. We will prove:

(i) the series $\sum_{l=-\infty}^{-1} c_l z^l$ and

$$\sum_{l=-\infty}^{-1} c_l\Delta^{k+(n-1)/2}[(x_0+\underline{x})^l P_k(\underline{x})]$$

have the same convergence radius;

(ii) the series $\sum_{l=0}^{\infty} c_l z^l$ and

$$\sum_{l=0}^{\infty} c_l\Delta^{k+(n-1)/2}[(x_0+\underline{x})^l P_k(\underline{x})]$$

have the same convergence radius.

For (i), it can be deduced from (3.36) and (3.25) that

$$|\Delta^{k+(n-1)/2}[(x_0+\underline{x})^l P_k(\underline{x})]|\leqslant C(1+|l|)^{n+2k}\frac{1}{|x|^{n+k+|l|-1}},$$

which implies that the two series in (i) have the same convergence radius.

At last, we prove (ii). For this case, n is even. Because $\Delta^s = |D|^{-1}\Delta^{k+n/2}$, the fundamental solution of Δ^s can be represented as the convolution of Riesz potential $1/|x|^n$ and the fundamental solution of $\Delta^{k+n/2}$. Under the case that the spatial dimension is odd, the fundamental solution of $\Delta^{k+n/2}$ is $C/|x|^{(n+1)-2s-1}$, where C is a constant depending on n and k. By Lemma 3.5.4, the fundamental solution of Δ^s can be represented as $C/|x|^{n+1-2s}$. Then applying Lemma 3.5.5, we can get

$$(\Delta^s)\left(\frac{1}{|x|^{(n+1)-2s}}g(x^{-1})\right) = \frac{C}{|x|^{(n+1)+2s+2}}(\Delta^s)(g)(x^{-1}).$$

Let $g(x) = \left(\frac{\overline{x}}{|x|^2}\right)^l P_k(\underline{x})$. Then $g(x^{-1}) = (-1)^k x^l P_k(\underline{x})$. Replacing s by $s+1$, we have

$$\Delta^{k+1+(n-1)/2}\Big((-1)^k x^l P_k(\underline{x})\Big) = \frac{C}{|x|^{2n+2k+2}}\Delta^{(k+1)+(n-1)/2}\left(\left(\frac{\overline{x}}{|x|^2}\right)^l P_k(\underline{x})\right)(x^{-1}).$$

By the Newton potential and (3.36), in the sense of distributions,

$$\Delta^{k+1+(n-1)/2}\Big(x^l P_k(\underline{x})\Big) = \frac{C}{(l-1)!}\int_{\mathbb{R}_1^n}\frac{1}{|x-y|^{n-1}}\frac{1}{|y|^{2n+2k-2}}\partial_0^{l-1}\Delta E P_k(\underline{x})(y^{-1})dy.$$

By Lemma 3.5.4,

$$|\Delta^{k+(n-1)/2}[(x_0+\underline{x})^l P_k(\underline{x})]| \leqslant C(1+|l|)^{n+2k}|x|^{l-k-n+1}.$$

□

3.6 Remarks

Remark 3.6.1 The idea of Theorem 3.5.1 is to investigate the similarity between the Clifford analysis and the complex analysis of single variable. Via the correspondence $z^k \to P^{(k)}$, some similarity has been obtained in [11].

The quaternionic space does not coincide with our result for $n = 3$. The quaternion forms a complete algebra, and the latter is not a complete algebra. Fueter's theorem implies that τ maps a holomorphic function of one variable to a regular function of variables in the quaternionic space. M. Sce generalized Fueter's result and proved that if n is odd, then τ maps the holomorphic functions defined on the subset in the upper-half complex plane to the monogenic functions. Theorem 3.5.1 (iii) indicates that if n is odd, the result obtained by the Kelvin inversion coincides with the result for $f^0(z) = z^k,\ k \in \mathbb{Z}$ obtained by Sce.

However, for even n, the method of using the the differential operator $\Delta^{(n-1)/2}$ introduced by Fueter and Sce is not valid. By the Fourier multiplier transform, the results of Fueter and Sce can be extended to the case of the power function with negative index, that is, $f^0(z) = z^k$, $-k \in \mathbb{Z}^+$; while for the power function with non-negative index, this method is not directly valid.

Remark 3.6.2 There is the following generalization of the result in Sect. 3.5. In [12], F. Sommen proved that if $n + 1$ is a positive even integer, P_k is any homogeneous polynomial in $\underline{x}$ of degree k, and is left monogenic for the Dirac $\underline{D}$: $\underline{D}P_k(\underline{x}) = 0$, then

$$D\Delta^{k+(n-1)/2}\left(\left(u(x_0, \underline{x}) + \frac{\underline{x}}{|\underline{x}|}v(x_0, \underline{x})P_k(\underline{x})\right)\right) = 0.$$

It is readily seen that the above result is a special case of Theorem 3.5.2.

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