Chapter 2
Singular Integral Operators on Closed Lipschitz Curves

In Chap. 1, we state a theory of convolution singular integral operators and Fourier multipliers on infinite Lipschitz curves. A natural question is whether there exists an analogy on closed Lipschitz curves. In this chapter, we establish such a theory for starlike Lipschitz curves. A curve is called a starlike Lipschitz curve if the curve has the following parameterization: $\widetilde{\gamma} = \{\exp(iz) : z \in \gamma\}$, where

$$\gamma = \Big\{x + ig(x) : g' \in L^{\infty}([-\pi, \pi]), g(-\pi) = g(\pi)\Big\}.$$

It can be proved that the starlike Lipschitz curves defined using such parameterization are the same as those defined as star-shaped and Lipschitz in the ordinary sense.

In the same pattern as in the infinite Lipschitz graph case, we can define Fourier series of L^2 functions on γ. The question can now be specified into the following two:

The first, what kind of holomorphic kernels give rise to L^2-bounded operators on starlike Lipschitz curves γ?

The second, is there a corresponding Fourier multiplier theory? In other words, what complex number sequences act as L^p-bounded Fourier multipliers on the curves?

It should be pointed out that these questions are not trivial even for the case $p = 2$, as the Plancherel theorem does not hold in this case. However, on the other hand, the case $p = 2$ is essential, as the boundedness for $1 < p < \infty$ can be deduced from the L^2 theory using the standard Calderón-Zygmund techniques.

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2.1 Preliminaries

Let γ be a Lipschitz curve defined on the interval $[-\pi, \pi]$ with the parameterization

$$\gamma(x) = x + ig(x),\ \ g : [-\pi, \pi] \to \mathbb{R},$$

where $\mathbb{R}$ denotes the real number field, $g(-\pi) = g(\pi)$, $g' \in L^{\infty}([-\pi, \pi])$ with $\|g'\|_{\infty} = N$. Denote by $p\gamma$ the 2π-periodic extension of γ to $-\infty < x < \infty$, and by $\widetilde{\gamma}$ the closed curve

$$\widetilde{\gamma} = \Big\{\exp(iz) : z \in \gamma\Big\} = \Big\{\exp(i(x + ig(x))) : -\pi \leqslant x \leqslant \pi\Big\}.$$

We call $\widetilde{\gamma}$ the starlike Lipschitz curve associated with γ.

We use f, F and $\widetilde{F}$ to denote the functions defined on $p\gamma$, γ and $\widetilde{\gamma}$, respectively. For $\widetilde{F} \in L^2(\widetilde{\gamma})$, the nth coefficient of $\widetilde{F}$ on $\widetilde{\gamma}$ is defined as

$$\widehat{\widetilde{F}}_{\widetilde{\gamma}}(n) = \frac{1}{2\pi i}\int_{\widetilde{\gamma}} z^{-n}\widetilde{F}(z)\frac{dz}{z}.$$

In the case of no confusion, we will sometimes suppress the subscript and write $\widehat{\widetilde{F}}(n)$.

Set

$$\sigma = \exp(-\max g(x)),\ \tau = \exp(-\min g(x)).$$

We consider the following dense subclass of $L^2(\widetilde{\gamma})$:

$$\mathcal{A}(\widetilde{\gamma}) = \Big\{\widetilde{F}(z) : \widetilde{F}(z) \text{ is holomorphic in } \sigma - \eta < |z| < \tau + \eta \text{ for some } \eta > 0\Big\}.$$

Without loss of generality, we assume that $\min g(x) < 0$ and $\max g(x) > 0$. In the case, the domains of the functions in $\mathcal{A}(\widetilde{\gamma})$ contain the unit circle $\mathbb{T}$, and by Cauchy's theorem, we know $\widehat{\widetilde{F}}_{\widetilde{\gamma}}(n) = \widehat{\widetilde{F}}_{\mathbb{T}}(n)$. If $\widetilde{F}$ and $\widetilde{G}$ belong to $\mathcal{A}(\widetilde{\gamma})$, by the Laurent series, we can obtain the inverse Fourier transform formula

$$\widetilde{F}(z) = \sum_{n=-\infty}^{\infty} \widehat{\widetilde{F}}_{\widetilde{\gamma}}(n)z^n, \tag{2.1}$$

where z is in the annulus where $\widetilde{F}$ is defined. We apply Cauchy's theorem to get the Parseval identity

$$\frac{1}{2\pi i}\int_{\widetilde{\gamma}} \widetilde{F}(z)\widetilde{G}(z)\frac{dz}{z} = \sum_{n=-\infty}^{\infty} \widehat{\widetilde{F}}_{\widetilde{\gamma}}(n)\widehat{\widetilde{G}}_{\widetilde{\gamma}}(-n). \tag{2.2}$$

Similar to Chap. 1, we will use the following half and double sectors on the complex plane $\mathbb{C}$. For $\omega \in (0, \pi/2]$, define the sets

$$S^0_{\omega,+} = \Big\{z \in \mathbb{C} : \ |\arg(z)| < \omega, z \neq 0\Big\},$$

$$S^0_{\omega,-} = -S^0_{\omega,+}, \ \ S^0_{\omega} = S^0_{\omega,+} \cup S^0_{\omega,-},$$

and

$$C^0_{\omega,+} = S^0_{\omega} \cup \Big\{z \in \mathbb{C} : \ \operatorname{Im}(z) > 0\Big\},$$

$$C^0_{\omega,-} = S^0_{\omega} \cup \Big\{z \in \mathbb{C} : \ \operatorname{Im}(z) < 0\Big\},$$

where $S^0_{\omega,\pm}$, S^0_{ω}, $C^0_{\omega,\pm}$ and C^0_{ω} are shown in Figs. 1.2, 1.3 and 1.4. Let X be one of the sets defined above. Denote by

$$X(\pi) = X \cap \Big\{z \in \mathbb{C} : \ |\operatorname{Re}(z)| \leqslant \pi\Big\}$$

the truncated set and by

$$pX(\pi) = \bigcup_{k=-\infty}^{\infty} \Big\{X(\pi) + 2k\pi\Big\}$$

the periodic set associated with the truncated set. The graphs of $S^0_{\omega,\pm}(\pi)$, $S^0_{\omega}(\pi)$ and $C^0_{\omega,\pm}(\pi)$ are shown in Figs. 2.1, 2.2 and 2.3.

(1) The figures of the sets $S^0_{\omega,+}$ and $S^0_{\omega,-}$ are as follows:

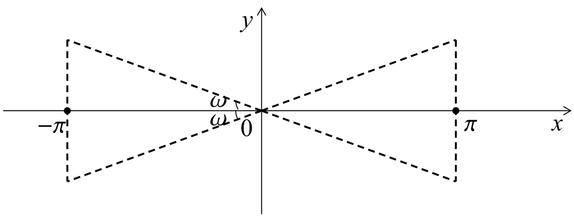

Fig. 2.1 $S^0_{\omega,-}(\pi) \cup S^0_{\omega,+}(\pi)$

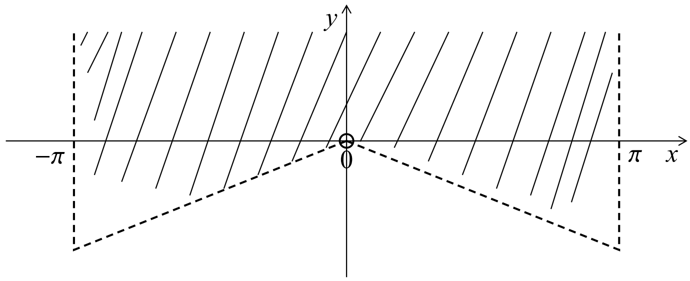

Fig. 2.2 $C^0_{\omega,+}(\pi)$

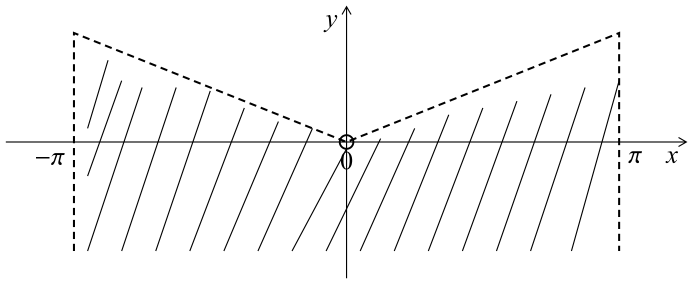

Fig. 2.3 $C^0_{\omega,-}(\pi)$

(2) The sets $C^0_{\omega,+}(\pi)$ and $C^0_{\omega,-}(\pi)$ are shown in the following figures:
We also use the sets of the form $\exp(iO) = \{\exp(iz) : \ z \in O\}$, where O is the truncated set defined above. Let Q be a double or half sector defined above. $H^\infty(Q)$ denotes the function space

$$\Big\{ f : Q \to \mathbb{C} : f \text{ is bounded and holomorphic in } Q \Big\}.$$

If no confusion occurs, we write $\|\cdot\|_\infty$ as $\|\cdot\|_{H^\infty(Q)}$.

Let $b \in H^\infty(S^0_\omega)$, $\omega \in (0, \pi/2]$. Then b can be divided into two parts: $b = b^+ + b^-$, where

$$\begin{cases} b^+ = b\chi_{\{z:\ \mathrm{Re}(z)>0\}}, \\ b^- = b\chi_{\{z:\ \mathrm{Re}(z)<0\}}. \end{cases} \tag{2.3}$$

Hence, $b^\pm \in H^\infty(S^0_{\omega,\pm})$.

In each of the following statements, the symbol "±" should be read as either all "+", or all "−". The following transform has been used in Sect. 1.3:

$$G^\pm(b^\pm)(z) = \phi^\pm(z) = \frac{1}{2\pi} \int_{\rho^\pm_\theta} \exp(iz\zeta) b(\zeta) d\zeta, \ z \in C^0_{\omega,\pm},$$

where $\rho_\theta^\pm$ denotes the ray $s\exp(i\theta), 0 < s < \infty, \theta$ is a constant which depends on $z \in C^0_{\omega,\pm}$ and satisfies $\rho_\theta^\pm \subset S^0_{\omega,\pm}$. Also

$$G_1^\pm(b^\pm)(z) = \phi_1^\pm(z) = \int_{\delta^\pm(z)} \phi^\pm(\zeta)d\zeta,\ z \in S^0_{\omega,\pm},$$

where the integral is along any path from $-z$ to z in $C^0_{\omega,\pm}$.

In what follows, we denote by c_0, c_1, C the fixed constants, and by $C_{\omega,\mu}$ the constants which depend on ω, μ and so on. These constants may vary from one occurrence to another. For $b \in H^\infty(S^0_\omega)$, using the decomposition $b = b^+ + b^-$ and Theorem 1.3.2, and letting

$$\phi = \phi^+ + \phi^-, \quad \phi_1 = \phi_1^+ + \phi_1^-,$$

we can see that the following two theorems are the main results obtained in Sect. 1.3. We reformulate them for the sake of convenience.

Theorem 2.1.1 *Let $\omega \in (0, \pi/2]$ and $b \in H^\infty(S^0_\omega)$. Then there exists a pair of holomorphic functions (ϕ, ϕ_1) defined in S^0_ω and $S^0_{\omega,+}$ such that for any $\mu \in (0, \omega)$,*

(i) $|\phi(z)| \leqslant C_{\omega,\mu}\|b\|_\infty/|z|,\ z \in S^0_\mu$;
(ii) *$\phi_1 \in H^\infty(S^0_{\mu,+})$, $\|\phi_1\|_{H^\infty(S^0_{\mu,+})} \leqslant C_{\omega,\mu}\|b\|_\infty$, and $\phi_1'(z) = \phi(z) + \phi(-z),\ z \in S^0_{\omega,+}$;*
(iii) *for all $f \in \mathcal{S}(\mathbb{R})$*

$$(2\pi)^{-1}\int_{-\infty}^{\infty} b(\zeta)\hat{f}(-\zeta)d\zeta = \lim_{\epsilon\to 0}\left\{\int_{|x|\geqslant\epsilon}\phi(x)f(x)dx + \phi_1(\epsilon)f(0)\right\}.$$

Theorem 2.1.2 *Let $\omega \in (0, \pi/2]$ and $b \in H^\infty(S^0_\omega)$. There exists a pair of holomorphic functions (ϕ, ϕ_1) defined in S^0_ω and $S^0_{\omega,+}$ satisfying*

(i) *there exists a constant c_0 such that*

$$|\phi(z)| \leqslant \frac{c_0}{|z|},\ z \in S^0_\omega;$$

(ii) *there exists a constant c_1 such that $\|\phi_1\|_{H^\infty(S^0_{\omega,+})} < c_1$, and*

$$\phi_1'(z) = \phi(z) + \phi(-z),\ z \in S^0_{\omega,+}.$$

Then for any $\mu \in (0, \omega)$, there exists a unique function $b \in H^\infty(S^0_\mu)$ such that

$$\|b\|_{H^\infty(S^0_\mu)} \leqslant C_{\omega,\mu}(c_0 + c_1),$$

and the function pair determined by b according to Theorem 2.1.1 is identical to (ϕ, ϕ_1). Moreover, for all complex numbers $\xi \in S^0_\omega$, the function b is given by

$$b(\xi) = \lim_{\epsilon\to 0}\lim_{N\to\infty}\left(\int_{\epsilon<|x|<N}\exp(-i\xi x)\phi(x)dx + \phi_1(\epsilon)\right).$$

2.2 Fourier Transforms Between S^0_ω and $pS^0_\omega(\pi)$

Theorem 2.2.1 *Let $\omega \in (0, \pi/2]$ and $b \in H^\infty(S^0_\omega)$, and let (ϕ, ϕ_1) be the function pair associated with b in the pattern of Theorem 1.3.2. Then there exists a pair of holomorphic functions (Φ, Φ_1) defined in $S^0_\omega(\pi)$ and $S^0_{\omega,+}(\pi)$, respectively, satisfying, for every $\mu \in (0, \omega)$,*

(i) *Φ can be holomorphically and periodically extended to $pS^0_\omega(\pi)$ and*

$$|\Phi(z)| \leqslant \frac{C_{\omega,\mu}\|b\|_\infty}{|z|},\ z \in S^0_\mu(\pi).$$

Moreover, $\Phi(z) = \phi(z) + \phi_0(z),\ z \in S^0_\mu(\pi)$, where ϕ_0 is a bounded holomorphic function in $S^0_\mu(\pi)$;

(ii) *$\Phi_1 \in H^\infty(S^0_{\mu,+}(\pi))$, $\|\Phi_1\|_{H^\infty(S^0_{\mu,+})} \leqslant C_{\omega,\mu}\|b\|_\infty$, and*

$$\Phi_1'(z) = \Phi(z) + \Phi(-z),\ z \in S^0_\omega(\pi);$$

(iii) *Φ and Φ_1 are uniquely determined (modulo constants) by the Parseval formula. Precisely, for any continuous 2π-periodic function F defined on $\mathbb{R}$,*

$$2\pi\sum_{n=-\infty}^{\infty} b(n)\widehat{F}(-n) = \lim_{\epsilon\to 0}\left(\int_{\epsilon\leqslant|x|\leqslant\pi}\Phi(x)F(x)dx + \Phi_1(\epsilon)F(0)\right),$$

where $\widehat{F}(n)$ denotes the nth Fourier coefficient of F, and $b(0) = \frac{1}{2\pi}\Phi_1(\pi)$.

Proof By the Poisson summation formula, we define Φ as

$$\Phi(z) = 2\pi\sum_{k=-\infty}^{\infty}\phi(z+2k\pi), z \in pS^0_\omega(\pi), \tag{2.4}$$

where the summation takes the following sense: there is a subsequence $\{n_l\}$ of $\{n\}$ such that for all $z \in S^0_\omega(\pi)$, when $l \to \infty$, the partial sum locally uniformly converges to a 2π-periodic and holomorphic function satisfying the assertion (i). In the sequel, we call such sequences applicable sequences. Moreover, we shall show that the limit functions defined through different applicable sequences differ from one another by constants which are bounded by $c\|b\|_\infty$.

We use the following decomposition

$$\sum_{k=-n}^{n} \phi(z+2k\pi) = \phi(z) + \sum_{k\neq 0}^{\pm n} (\phi(z+2k\pi) - \phi(2k\pi)) + \sum_{k=1}^{n} \phi_1'(2k\pi)$$
$$= \phi(z) + \sum_1 + \sum_2 .$$

We will prove that the series $\sum_1$ locally uniformly converges to a bounded holomorphic function in $S^0_\mu(\pi)$, and some subsequence of the partial sums of $\sum_2$ converges to a constant dominated by $C_\mu \|b\|_\infty$.

By (i) of Theorem 2.1.1, Cauchy's theorem and the fact that ϕ is a holomorphic function, we can deduce the estimate:

$$|\phi'(z)| \leqslant \frac{C_\mu}{|z|^2},\ z \in S^0_\mu,$$

so the convergence of $\sum_1$ is proved. For $\sum_2$, we use the mean value theorem for the integrals to get

$$\sum_{k=1}^{n} \phi_1'(2k\pi) = \int_{2\pi}^{2(n+1)\pi} \phi_1'(r)dr + \sum_{k=1}^{n} \Big[\phi_1'(2k\pi) - \mathrm{Re}(\phi_1'(\xi_k)) - i\mathrm{Im}(\phi_1'(\eta_k))\Big]$$
$$= \phi_1(2(n+1)\pi) - \phi_1(2\pi)$$
$$+ \sum_{k=1}^{n} \Big[\phi_1'(2k\pi) - \mathrm{Re}(\phi_1'(\xi_k)) - i\mathrm{Im}(\phi_1'(\eta_k))\Big],$$

where $\xi_k, \eta_k \in (2k\pi, 2(k+1)\pi)$. By the estimate of ϕ', the above series converges absolutely. It can be deduced from the boundedness of ϕ_1 that there exists an applicable subsequence $\{n_l\}$ such that $\phi_1(2(n_l+1)\pi)$ converges to a constant c_0. Therefore, we have

$$\frac{1}{2\pi}\Phi(z) = \phi(z) + \sum_{k\neq 0} \Big[\phi(z+2k\pi) - \phi(2k\pi)\Big] + \lim_{l\to\infty} \sum_{n=1}^{n_l} \phi_1'(2n\pi)$$
$$= \phi(z) + \phi_0(z) + c_0,$$

where ϕ_0 is a bounded holomorphic function in $S^0_\mu(\pi)$, c_0 is a constant depending on the subsequence $\{n_l\}$ chosen. At the same time, Φ can be extended holomorphically to $pS^0_\omega(\pi)$, and the different Φ's associated with different applicable sequences may differ from one another by constants dominated by $c\|b\|_\infty$.

Now we prove (ii) and (iii). We use the decomposition $b = b^+ + b^-$ given in (2.3). Define

$$b^{\pm,\alpha}(z) = \exp(\mp\alpha z)b^\pm(z),\ \alpha > 0.$$

Let $\phi^{\pm}$ and $\phi^{\pm,\alpha}$ be the functions associated with $b^{\pm}$ and $b^{\pm,\alpha}$, respectively. By Remark 1.3.1, $\phi^{\pm,\alpha}(\cdot) = \phi^{\pm}(\cdot \pm i\alpha)$, and the latter are inverse Fourier transforms of $b^{\pm,\alpha}$. Now we define the corresponding periodic functions $\Phi^{\pm,\alpha}$ and holomorphic functions $\Phi^{\pm}$ in $pC^0_{\omega,\pm}(\pi)$, respectively, which satisfy the size condition in the assertion (i). It is to be noted that for all $\Phi^{\pm,\alpha}$, we choose the same applicable sequence $\{n_l\}$ as we have chosen for $\Phi^{\pm}$. By the estimate in (i) of Theorem 2.1.1 and the fact that ϕ is holomorphic, we can prove that when $\alpha \to 0$, $\sum_1$ is locally uniformly and absolutely convergent. Let

$$\frac{1}{2\pi}\Phi^{\pm,\alpha}(z) = \phi^{\pm,\alpha}(z) + \phi_0^{\pm,\alpha}(z) + c_0^{\pm,\alpha}$$

and

$$\frac{1}{2\pi}\Phi^{\pm}(z) = \phi^{\pm}(z) + \phi_0^{\pm}(z) + c_0^{\pm},$$

where $\phi_0^{\pm,\alpha}$ and $\phi_0^{\pm}$ are holomorphic and uniformly bounded in $C^0_{\mu,\pm}(\pi)$. Since the convergence as $n_l \to \infty$ is uniform for $\alpha \to 0$, we can change the order of taking limits $n_l \to \infty$ and $\alpha \to 0$, and conclude that $\phi^{\pm,\alpha}$, $\phi_0^{\pm,\alpha}$ and $c_0^{\pm}$ are convergent locally uniformly in $C^0_{\omega,\pm}(\pi)$. Hence,

$$\lim_{\alpha\to 0}\Phi^{\pm,\alpha}(z) = \Phi^{\pm}(z).$$

Notice that for fixed α, $\Phi^{\pm,\alpha} \in L^{\infty}([-\pi,\pi])$, and when $n_l \to \infty$, the series which define $\Phi^{\pm,\alpha}$ converges uniformly in $x \in [-\pi,\pi]$. For all non-zero real ξ in the sense of (3) in Theorem 2.1.2, we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(-i\xi x)\Phi^{\pm,\alpha}(x)dx &= \int_{-\pi}^{\pi}\exp(-i\xi x)\lim_{l\to\infty}\sum_{k=-n_l}^{n_l}\phi^{\pm,\alpha}(x+2k\pi)dx \\ &= \int_{-\infty}^{\infty}\exp(-i\xi x)\phi^{\pm,\alpha}(x)dx = b^{\pm,\alpha}(\xi).\end{aligned}$$

In particular, $\{b^{\pm,\alpha}(n)\}$, $n \neq 0$, are the standard Fourier coefficients of $\Phi^{\pm,\alpha}$. If F is any smooth periodic function on $[-\pi,\pi]$, then Parseval's identity holds:

$$2\pi\sum_{n=-\infty}^{\infty} b^{\pm,\alpha}(n)\widehat{F}(-n) = \int_{-\pi}^{\pi}\Phi^{\pm,\alpha}(x)F(x)dx,$$

where

$$b^{\pm,\alpha}(0) = (2\pi)^{-1}\int_{-\pi}^{\pi}\Phi^{\pm}(x \pm i\alpha)dx.$$

Let $\epsilon > 0$. Since $\widehat{F}(n)$ decays rapidly as $n \to \pm\infty$, on letting $\alpha \to 0+$, we have

$$2\pi \sum_{n=-\infty}^{\infty} b^{\pm}(n)\widehat{F}(-n) = \lim_{\alpha\to 0+} \left\{ \int_{[-\pi,\pi]\setminus(-\epsilon,\epsilon)} \Phi^{\pm}(x \pm i\alpha)F(x)dx \right.$$
$$+ \int_{|x|\leqslant\epsilon} \Phi^{\pm}(x \pm i\alpha)(F(x) - F(0))dx$$
$$\left. + \int_{|x|\leqslant\epsilon} \Phi^{\pm}(x \pm i\alpha)F(0)dx \right\}.$$

Then we can get

$$\lim_{\alpha\to 0+} \int_{[-\pi,\pi]\setminus(-\epsilon,\epsilon)} \Phi^{\pm}(x + i\alpha)F(x)dx = \int_{[-\pi,\pi]\setminus(-\epsilon,\epsilon)} \Phi^{\pm}(x)F(x)dx$$

and

$$\limsup_{\alpha\to 0+} \int_{|x|\leqslant\epsilon} |\Phi^{\pm}(x + i\alpha)| \cdot |F(x) - F(0)|dx \leqslant \limsup_{\alpha\to 0} \int_{|x|\leqslant\epsilon} \frac{1}{|x|} \cdot |x|dx \leqslant C\epsilon.$$

Define

$$\Phi_1^{\pm}(z) = \int_{\delta^{\pm}(z)} \Phi^{\pm}(\eta)d\eta,$$

where $\delta^{\pm}(z)$ is a path from $-z$ to z in $C^0_{\omega,\pm}(\pi)$. Hence for $\Phi_1^{\pm}$, (ii) holds and

$$\lim_{\alpha\to 0+} \int_{|x|\leqslant\epsilon} \Phi^{\pm}(x \pm i\alpha)F(0)dx = \Phi_1^{\pm}(\epsilon)F(0).$$

This gives Parseval's identity associated with $b^{\pm}$:

$$2\pi \sum_{n=-\infty}^{\infty} b^{\pm}(n)\widehat{F}(-n) = \lim_{\epsilon\to 0} \left(\int_{[-\pi,\pi]\setminus(-\epsilon,\epsilon)} \Phi^{\pm}(x)F(x)dx + \Phi_1^{\pm}(\epsilon)F(0) \right),$$

where $b^{\pm}(0) = \frac{1}{2\pi}\Phi_1^{\pm}(\pi)$. Note that if we replace $\Phi^{\pm}$ by $\Phi^{\pm} + c^{\pm}$ in the above formulas, then, correspondingly, we need to replace $b^{\pm}(0)$ by $b^{\pm}(0) + c^{\pm}$ in order to make the formulas still hold. Since $\Phi = \Phi^+ + \Phi^-$, on letting $\Phi_1 = \Phi_1^+ + \Phi_1^-$, we see that (ii) and (iii) hold. This completes the proof. □

Remark 2.2.1 When we prove Parseval's identity related to $b \in H^\infty(S^0_\omega)$, the value of b at the origin is naturally involved. For the sake of convenience, we take $b(0) = \frac{1}{2\pi}\Phi_1(\pi)$ in consistency with the formula as shown in the theorem. The proof of the theorem indicates that adding a constant to Φ does not change the Fourier coefficients $\widehat{\Phi}(n) = b(n), n \neq 0$, but we should add the same constant to $b(0)$.

Theorem 2.2.2 *Let* $\omega \in (0, \pi/2]$ *and* (Φ, Φ_1) *be a pair of holomorphic functions defined on* $pS^0_\omega(\pi)$ *and* $S^0_{\omega,+}(\pi)$*, respectively, satisfying*

(i) *Φ is 2π-periodic, and there exists a constant c_0 such that*

$$|\Phi(z)| \leqslant \frac{c_0}{|z|},\ z \in S^0_\omega(\pi);$$

(ii) *there exists a constant c_1 such that $\|\Phi_1\|_{H^\infty(S^0_{\omega,+}(\pi))} < c_1$, and*

$$\Phi_1'(z) = \Phi(z) + \Phi(-z),\ z \in S^0_{\omega,+}(\pi).$$

Then for any $\mu \in (0,\omega)$, there exists a function b^μ such that $b^\mu \in H^\infty(S^0_\mu)$ and

$$\|b^\mu\|_{H^\infty(S^0_\mu)} \leqslant C_\mu(c_0 + c_1).$$

By Theorem 2.2.1, the function pair determined by b^μ is identical with (Φ, Φ_1) (modulo constants). Moreover, $b^\mu = b^{\mu,+} + b^{\mu,-}$,

$$b^{\mu,\pm}(\eta) = \frac{1}{2\pi}\lim_{\epsilon\to 0}\left(\int_{A^\pm(\epsilon,\theta,|\eta|^{-1})} \exp(-i\eta z)\Phi(z)dz + \Phi_1(\epsilon)\right),\ \eta \in S^0_{\mu,\pm}, \quad (2.5)$$

where $\theta = (\mu+\omega)/2$, $A^\pm(\epsilon,\theta,\varrho) = l(\epsilon,\varrho) \cup c^\pm(\theta,\varrho) \cup \Lambda^\pm(\theta,\varrho)$. Here when $\varrho \leqslant \pi$,

$$\begin{aligned} l^\pm(\epsilon,\varrho) &= \Big\{z = x + iy:\ y = 0, \epsilon \leqslant \pm x \leqslant \varrho\Big\},\\ c^\pm(\theta,\varrho) &= \Big\{z = \varrho\exp(i\alpha):\ \alpha \textit{ from } \pi \pm \theta \textit{ to } \pi,\ \textit{and then from } 0 \textit{ to } \mp\theta\Big\},\\ \Lambda^\pm(\theta,\varrho) &= \Big\{z \in C^0_{\omega,\pm}(\pi): z = r\exp(i(\pi \pm \theta)), r \textit{ from } \pi\sec\theta \textit{ to } \varrho,\\ &\qquad \textit{and } z = r\exp(\mp i\theta), r \textit{ from } \varrho \textit{ to } \pi\sec\theta\Big\}, \end{aligned}$$

when $\varrho > \pi$,

$$l(\epsilon,\varrho) = l^\pm(\epsilon,\pi),\ c^\pm(\theta,\varrho) = c^\pm(\theta,\pi),\ \Lambda^\pm(\theta,\varrho) = \Lambda^\pm(\theta,\pi).$$

Proof The integral is along the path $A^\pm(\epsilon,\theta,|\eta|^{-1})$, see Figs. 2.4 and 2.5.

Fix $\mu \in (0,\omega)$ and write b^μ as b in the rest of the proof. For all $\epsilon \in (0,\pi)$ and $\eta \in S^0_\omega \cup \{0\}$, define $b_\epsilon(\eta) = b^+_\epsilon(\eta) + b^-_\epsilon(\eta)$, where $b^\pm_\epsilon$ are the functions in the definition of $b^\pm$ in the theorem before taking the limit as $\epsilon \to 0$. We see that for all ϵ, $b_\epsilon(0) = \frac{1}{2\pi}\Phi_1(\pi)$.

For $|\eta|^{-1} \leqslant \pi$, applying the estimate in Theorem 1.3.3, we can prove that $b_\epsilon(\eta)$ is uniformly bounded, and $\lim\limits_{\epsilon\to 0+} b_\epsilon(\eta) = b(\eta)$ exists.

If $|\eta|^{-1} > \pi$, for the integral over the contour $l(\epsilon,\pi)$, we use the same argument as to the integral over $l(\epsilon,|\eta|^{-1})$ for the case $|\eta|^{-1} \leqslant \pi$. To estimate the integrals

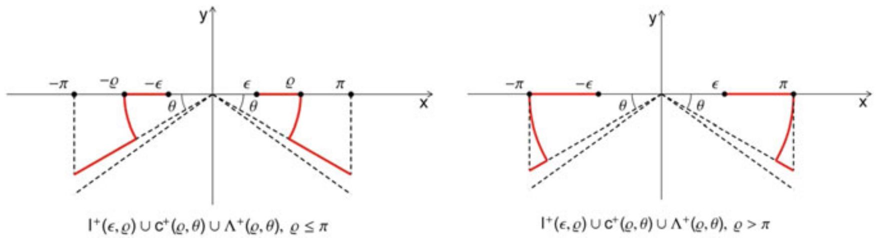

Fig. 2.4 $l^+(\epsilon,\varrho)\cup c^+(\varrho,\theta)\cup\Lambda^+(\varrho,\theta)$

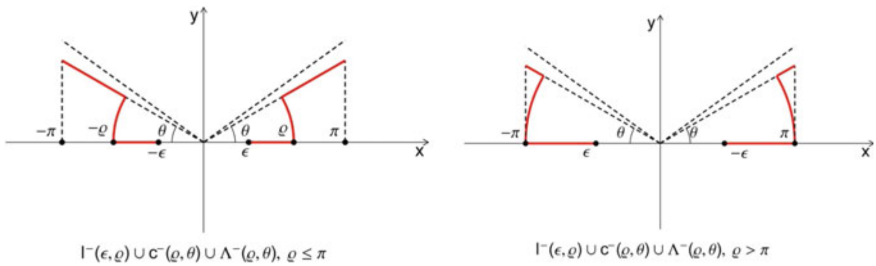

Fig. 2.5 $l^-(\epsilon,\varrho)\cup c^-(\varrho,\theta)\cup\Lambda^-(\varrho,\theta)$

over $c^{\pm}(\theta,\pi)$ and $\Lambda^{\pm}(\theta,\pi)$, we use Cauchy's theorem to change the contour of integration and so to integrate over the set

$$\Big\{z=x+iy: x=-\pi,\ y \text{ from } -(\pm\pi)\tan\theta \text{ to } 0,\ \text{ and } x=\pi,\ y \text{ from } 0 \text{ to } -(\pm\pi\tan\theta)\Big\}.$$

However, by the condition $\pm\mathrm{Re}(z)>0$, it is easy to prove that the integral over the above contour is bounded. Then b is well-defined and bounded.

Let F be any 2π-periodic continuous function on $[-\pi,\pi]$. Expanding F in a Fourier series and using the definition of b_ϵ, we have

$$2\pi\sum_{n=-\infty}^{\infty} b_\epsilon(n)\widehat{F}_{[-\pi,\pi](-n)} = \int_{\epsilon<|x|\leqslant\pi}\Phi(x)F(x)dx+\Phi_1(\epsilon)F(0).$$

On letting $\epsilon\to 0$, we get

$$2\pi\sum_{n=-\infty}^{\infty} b(n)\widehat{F}_{[-\pi,\pi]}(-n) = \lim_{\epsilon\to 0}\left(\int_{\epsilon<|x|\leqslant\pi}\Phi(x)F(x)dx+\Phi_1(\epsilon)F(0)\right).$$

Denote by $(G(b),G_1(b))$ a pair of holomorphic functions associated to b in the pattern of Theorem 2.2.1. It can be deduced from Parseval's identity that

$$\lim_{\epsilon\to 0}\left(\int_{\epsilon<|x|<\pi}(G(b)(x)-\Phi(x))F(x)dx+[G_1(\epsilon)-\Phi_1(\epsilon)]F(0)\right)$$
$$=2\pi\Big(b_1(0)-b(0)\Big)\widehat{F}_{[-\pi,\pi]}(0),$$

where $b_1(0)$ is the function associated with $(G(b), G_1(b))$ in Parseval's identity of Theorem 2.2.1. By Theorem 2.2.1, we can add any constant to $G(b)$ and accordingly adjust the value of $b_1(0)$ such that (iii) of Theorem 2.2.1 holds. In particular, we can take a constant such that $b_1(0)-b(0)=0$. The right hand side of the last displayed equality then becomes 0. Using an approximation to identity $\{F_n\}$ with the property $F_n(0)=0$, we conclude that $G(b)(x)=\Phi(x)$ for $x\neq 0$. Because of analyticity, we know for all $z\in S^0_\omega(\pi)$, $G(b)(z)=\Phi(z)$. For $G_1(b)$, using (iii) of Theorem 2.2.1 together with the assumption (ii) of Φ_1, we can get $\Phi_1'=G_1'(b)$ and Φ_1-G_1 is a constant. Then by the use of

$$\lim_{\epsilon\to 0}\Big[G_1(b)(\epsilon)-\Phi_1(\epsilon)\Big]=0,$$

we have $\Phi_1=G_1(b)$. The uniqueness of b can be proved similarly. □

2.3 Singular Integrals on Starlike Lipschitz Curves

The results obtained in Sect. 2.2 can be applied to study the relation between the singular integral operators defined on periodic Lipschitz curves in Sect. 2.1 and the Fourier multipliers. Taking the change of variable $z\to\exp(iz)$ and substituting $\widetilde{\Phi}=\Phi\circ(\frac{1}{i}\ln)$ and $\widetilde{\Phi_1}=\Phi_1\circ(\frac{1}{i}\ln)$ in Theorems 2.2.1 and 2.2.2, we obtain the following theorem.

Theorem 2.3.1 *Let $\omega\in(0,\pi/2]$ and $b\in H^\infty(S^0_\omega)$. There exists a pair of functions $(\widetilde{\Phi},\widetilde{\Phi_1})$ such that $\widetilde{\Phi}$ and $\widetilde{\Phi_1}$ are holomorphic in $\exp(iS^0_\omega(\pi))$ and $\exp(iS^0_{\omega,+}(\pi))$, respectively. Moreover, for any $\mu\in(0,\omega)$,*

(i) $|\widetilde{\Phi}(z)|\leqslant C_{\omega,\mu}\|b\|_\infty/|1-z|,\ z\in\exp(iS^0_\mu(\pi))$;

(ii) $\widetilde{\Phi}_1\in H^\infty(\exp(iS^0_\mu(\pi))),\ \|\widetilde{\Phi}_1\|_{H^\infty(\exp(iS^0_\mu(\pi)))}<C_{\omega,\mu}\|b\|_\infty$ *and*
$$\widetilde{\Phi}'_1(z)=\frac{1}{iz}\Big(\widetilde{\Phi}(z)+\widetilde{\Phi}(z^{-1})\Big),\ z\in\exp(iS^0_{\omega,+}(\pi));$$

(iii) *For all continuous functions $\widetilde{F}$ defined on $\mathbb{T}$,*

$$2\pi\sum_{n=-\infty}^{\infty}b(n)\widehat{\widetilde{F}}_{\mathbb{T}}(-n)=\lim_{\epsilon\to 0}\left(\int_{|\ln z|>\epsilon, z\in\mathbb{T}}\widetilde{\Phi}(z)\widetilde{F}(z)\frac{dz}{z}+\widetilde{\Phi}_1(\exp(i\epsilon))\widetilde{F}(1)\right).$$

where $\widehat{\widetilde{F}}_{\mathbb{T}}(n)$ is nth Fourier coefficient of $\widetilde{F}$, and $b(0)=\frac{1}{2\pi}\widetilde{\Phi}_1(\exp(i\pi))$.

Theorem 2.3.2 *Let $\omega \in (0, \pi/2]$ and $(\widetilde{\Phi}, \widetilde{\Phi}_1)$ be a pair of functions defined on* $\exp(iS^0_\omega(\pi))$ *and* $\exp(iS^0_{\omega,+}(\pi))$*, respectively, satisfying*

(i) *there exists a constant c_0 such that*

$$|\widetilde{\Phi}(z)| \leqslant \frac{c_0}{|1-z|},\ z \in \exp(iS^0_\omega(\pi));$$

(ii) *there exists a constant c_1 such that $\|\widetilde{\Phi}_1\|_{H^\infty(\exp(iS^0_{\omega,+}(\pi)))} < c_1$, and*

$$\widetilde{\Phi}'_1(z) = \frac{1}{iz}\Big(\widetilde{\Phi}(z) + \widetilde{\Phi}(z^{-1})\Big),\ z \in \exp(iS^0_{\omega,+}(\pi)).$$

Then for any $\mu \in (0, \omega)$, there exists a function b^μ in $H^\infty(S^0_\mu)$ such that

$$\|b^\mu\|_{H^\infty(S^0_\mu)} \leqslant C_\mu(c_0 + c_1).$$

The function pair determined by b^μ according to Theorem 2.3.1 equals to $(\widetilde{\Phi}, \widetilde{\Phi}_1)$ (modulo constants). Moreover, $b^\mu = b^{\mu,+} + b^{\mu,-}$,

$$b^{\pm}(\eta) = \frac{1}{2\pi}\lim_{\epsilon\to 0}\left(\int_{-i\ln z \in A^{\pm}(\epsilon,\theta,\varrho)} z^{-\eta}\frac{dz}{z} + \widetilde{\Phi}_1(\exp(i\epsilon))\right), \eta \in S^0_{\mu,\pm},$$

where $A^{\pm}(\epsilon, \theta, \varrho)$ is the path defined in Theorem 2.2.2, and

$$\widetilde{\Phi}_1(\exp(i\epsilon)) = \int_{l(\epsilon)} \widetilde{\Phi}(\exp(iz))dz,$$

where $l(\epsilon)$ is any path from $-\epsilon$ to ϵ lying in $C^0_{\omega,\pm}$.

The following corollaries are in terms of holomorphic extension of series with positive and negative powers and can be deduced from Theorems 2.3.1 and 2.3.2 immediately.

Corollary 2.3.1 *Let $\{b_n\}_{n=\pm1}^{\pm\infty} \in l^\infty$, $\widetilde{\Phi}(z) = \sum_{n=\pm1}^{\pm\infty} b_n z^n$, $|z^{\pm1}| < 1$, and $\omega \in (0, \pi/2)$. If there exists $\delta > 0$ such that $\omega + \delta \leqslant \pi/2$, and there exists a function $b \in H^\infty(S^0_{\omega+\delta,\pm})$ such that for all $\pm n = \pm1, \pm2, \ldots$, $b(n) = b_n$, then the function $\widetilde{\Phi}$ can be extended holomorphically to the domain* $\exp(iC^0_{\omega+\delta,\pm}(\pi))$*. Moreover, we get*

$$|\widetilde{\Phi}(z)| \leqslant \frac{C_{\omega,\delta}}{|1-z|},\ z \in \exp(iC^0_{\omega,\pm}(\pi)).$$

Corollary 2.3.2 *Let $\omega \in (0, \pi/2)$ and let $\widetilde{\Phi}$ be a holomorphic function satisfying*

$$|\widetilde{\Phi}(z)| \leqslant \frac{C}{|1-z|},\ z \in \exp(iC^0_{\omega,\pm}(\pi)).$$

Then for any $\mu \in (0, \omega)$*, there exists a function* b^{μ} *such that* $b^{\mu} \in H^{\infty}(S^0_{\mu,\pm})$ *and* $\widetilde{\Phi}(z) = \sum_{n=\pm 1}^{\pm\infty} b_n z^n$*. Moreover,* $b^{\mu} = b^{\mu,+} + b^{\mu,-}$*, and for* $\eta \in S^0_{\mu,\pm}$*,*

$$b^{\mu,\pm}(\eta) = \frac{1}{2\pi} \lim_{\epsilon \to 0+} \left(\int_{-i \ln z \in A^{\pm}(\epsilon,\theta,\varrho)} \exp(-i\eta z)\widetilde{\Phi}(\exp(iz))dz + \widetilde{\Phi}_1(\exp(i\epsilon)) \right),$$

where $A^{\pm}(\epsilon, \theta, \varrho)$ *is defined by Theorem 2.2.2, and*

$$\widetilde{\Phi}_1(\exp(i\epsilon)) = \int_{l(\epsilon)} \widetilde{\Phi}(\exp(iz))dz,$$

where $l(\epsilon)$ *is any path from* $-\epsilon$ *to* ϵ *in* $C^0_{\omega,\pm}$*.*

Remark 2.3.1 As indicated in Corollary 2.3.2, the mapping $\widetilde{\Phi} \to b$ satisfying $\widetilde{\Phi}(z) = \sum b(n)z^n$ is not single-valued. In fact, if $\mu_1 \neq \mu_2$, then both b^{μ_1} and b^{μ_2} satisfy the requirement. In general, $b^{\mu_1} \neq b^{\mu_2}$. This can be verified by using $\widetilde{\Phi}(z) = z^n, n \in \mathbb{Z}^+$.

Corollary 2.3.3 *For any* $\omega \in (0, \pi/2)$*, there does not exist any function b such that* $b \in H^{\infty}(S^0_{\omega,+})$ *and satisfies* $b(n) = 1$ *for* $n = 2^k, k = 1, 2, \ldots,$ *and* $b(n) = 0$ *for the other positive integers.*

Proof Consider the function

$$\widetilde{\Phi}(z) = z + z^2 + z^{2^2} + \cdots + z^{2^k} + \cdots$$

It is well known that $\widetilde{\Phi}$ does not have any holomorphic extension across any interval on the unit circle, and according to Corollary 2.3.1, it is not induced by a function b in $H^{\infty}(S^0_{\omega,+})$. □

For the functions b and $\widetilde{F}$ defined in Theorem 2.3.1, by the Laurent series theory, the series

$$\sum_{n=-\infty}^{\infty} b(n)\widehat{\widetilde{F}}_{\mathbb{T}}(n)z^n$$

locally uniformly converges to a holomorphic function in the annulus on which $\widetilde{F}$ is defined. Noticing that $\widehat{\widetilde{F}}_{\mathbb{T}}(n) = \widehat{\widetilde{F}}_{\widetilde{\gamma}}(n)$, we can define an operator $\widetilde{M}_b : \mathcal{A}(\widetilde{\Gamma}) \longrightarrow \mathcal{A}(\widetilde{\gamma})$ as

$$\widetilde{M}_b(\widetilde{F})(z) = 2\pi \sum_{n=-\infty}^{\infty} b(n)\widehat{\widetilde{F}}_{\widetilde{\gamma}}(n)z^n.$$

On the other hand, for the function pair $(\widetilde{\Phi}, \widetilde{\Phi}_1)$ occurring in Theorem 2.3.2, there holds

$$T_{(\widetilde{\Phi},\widetilde{\Phi}_1)}\widetilde{F}(z) = \lim_{\epsilon\to 0}\left(\int_{\pi\geqslant|Re(i^{-1}\ln(\eta z^{-1}))|>\epsilon,\eta\in\widetilde{\gamma}}\widetilde{\Phi}(z\eta^{-1})\widetilde{F}(\eta)\frac{d\eta}{\eta}+\widetilde{\Phi}_1(\exp(i\epsilon t(z)))\widetilde{F}(z)\right),$$

where $t(z)$ is the unit tangent vector of γ at z in $S^0_{\omega,+}(\pi)$. We have the following theorem:

Theorem 2.3.3 *Let* $\omega\in(\arctan N,\ \pi/2]$, $b\in H^\infty(S^0_\omega)$ *and let* $(\widetilde{\Phi},\ \widetilde{\Phi}_1)$ *be the pair of functions corresponding to* b *in the pattern of Theorem 2.3.1. Then the following conclusions hold.*

(i) $T_{(\widetilde{\Phi},\widetilde{\Phi}_1)}$ *is a well-defined operator from* $\mathcal{A}(\widetilde{\gamma})$ *to* $\mathcal{A}(\widetilde{\gamma})$, *and in the sense of modulo constants,*

$$T_{(\widetilde{\Phi},\widetilde{\Phi}_1)} = \widetilde{M}_b.$$

(ii) $\widetilde{M}_b$ *can be extended to a bounded operator on* $L^2(\widetilde{\gamma})$, *and the norm is dominated by* $c\|b\|_\infty$.

Proof (i) For any $\alpha>0$, define $b_z^{\pm,\alpha}(\xi)=-z^{-\xi}b^{\pm,\alpha}(-\xi)$, where $b^{\pm,\alpha}$ is the function defined in Theorem 2.2.1. Let $(\widetilde{\Phi}_z^{\pm,\alpha},\ (\widetilde{\Phi}_z^{\pm,\alpha})_1)$ be the pair of functions corresponding to b in the pattern of Theorem 2.3.1. By (iii) of Theorem 2.3.1 and Cauchy's theorem, we have

$$\begin{aligned}\widetilde{M}_{b_z^{\pm,\alpha}}\widetilde{F}(z) &= 2\pi\sum_{n=-\infty}^{\infty}b_z^{\pm,\alpha}(n)\widehat{\widetilde{F}}_{\widetilde{\gamma}}(n)\\ &= 2\pi\sum_{n=-\infty}^{\infty}b_z^{\pm,\alpha}(n)\widehat{\widetilde{F}}_{\mathbb{T}}(n)\\ &= \int_{\mathbb{T}}\widetilde{\Phi}_z^{\pm,\alpha}(\eta^{-1})\widetilde{F}(\eta)\frac{d\eta}{\eta}\\ &= \int_{\widetilde{\gamma}}\widetilde{\Phi}_z^{\pm,\alpha}(\eta^{-1})\widetilde{F}(\eta)\frac{d\eta}{\eta}.\end{aligned}$$

Similar to the proof of Theorem 2.2.1, taking the limit $\alpha\to 0$ and noticing that

$$\widetilde{\Phi}_z^{\pm}(\eta^{-1}) = \widetilde{\Phi}^{\pm}(z\eta^{-1}),$$

we can get the desired equality for $b^\pm$ and b.

(ii) Now we prove the boundedness of the following operator:

$$T_{(\Phi,\Phi_1)}F(z)=\lim_{\epsilon\to 0}\left\{\int_{\epsilon<|\mathrm{Re}(z-\eta)|\leqslant\pi}\Phi(z-\eta)F(\eta)d\eta+\Phi_1(\epsilon t(z))F(z)\right\},\ F\in\mathcal{A}(\gamma),$$

where $t(z)$ is the unit tangent vector of γ at z in $S^0_{\omega,+}(\pi)$. Here $\mathcal{A}(\gamma)$ denotes the class of all 2π-periodic holomorphic functions satisfying: $F\in\mathcal{A}(\gamma)$ if and only if $\widetilde{F}=F\circ(i^{-1}\ln)\in\mathcal{A}(\widetilde{\gamma})$. By the decomposition of (i) of Theorem 2.2.1, we have

$$
\begin{aligned}
T_{(\Phi,\Phi_1)}F(z) = \lim_{\epsilon_n\to 0}\Big\{ &\int_{\pi\geqslant|\mathrm{Re}(z-\eta)|>\epsilon_n} \phi(z-\eta)F(\eta)d\eta \\
&+ \int_{\pi\geqslant|Re(z-\eta)|>\epsilon_n} \phi_0(z-\eta)F(\eta)d\eta\Big\} \\
&+ c_1\int_{-\pi}^{\pi} F(\eta)d\eta + c_2F(z),
\end{aligned}
$$

where $\epsilon_n \to 0$ is a subsequence of the sequence $\epsilon \to 0$, c_1 and c_2 are constants.

The second and the third integrals are dominated by the L^2-norm of F, while the first integral is dominated by

$$\sup_{\epsilon>0}\left|\int_{|\mathrm{Re}(z-\eta)|>\epsilon} \phi(z-\eta)F_1(\eta)d\eta\right| + c\mathcal{M}F_1(z),\ \mathrm{Re}(z)\in[-\pi,\pi],$$

where for $|\mathrm{Re}(\eta)| \leqslant 2\pi$, $F_1(\eta) = F(\eta)$; otherwise $F_1(\eta) = 0$. $\mathcal{M}F_1$ is the Hardy-Littlewood maximal function of F_1 on the curve . By the boundedness of the operators introduced by (ϕ, ϕ_1) and that of $\mathcal{M}$, we obtain the desired boundedness. □

Theorem 2.3.4 *Let ϕ be a holomorphic function satisfying $|\phi(z)| \le C/|z|$ on S^0_ω. Assume that $\gamma = x + iA(x)$ is a Lipschitz curve, $\|A'\|_\infty < \tan\omega$. If there exists a $L^2(\gamma)$-bounded operator T such that*

$$T(f)(z) = \int_\gamma \phi(z-\zeta)f(\zeta)d\zeta,\ \forall f \in C_c(\gamma), z \notin \ supp\ f,$$

where $C_c(\gamma)$ denotes the class of continuous functions with compact support on γ, then there exists a function $\phi_1 \in H^\infty(S^0_\omega)$ such that $\phi_1' = \phi(z) + \phi(-z)$, $z \in S^0_\omega$.

Proof Because T is bounded on $L^2(\gamma)$, the formula for T can be extended to

$$T(f)(z) = \int_\gamma \phi(z-\zeta)f(\zeta)d\zeta,$$

where $f = \chi_Q$, Q is any finite interval on γ and $z \notin \overline{Q}$. Define a new family of functions $\phi_\epsilon = \phi\chi_{\{z\in\mathbb{C}:\ |z|>\epsilon\}}$ and the corresponding operators:

$$T_\epsilon(f)(z) = \int_\gamma \phi_\epsilon(z-\zeta)f(\zeta)d\zeta.$$

By a standard argument, we can get the operator norm $\|T_\epsilon\|_{L^2(\gamma)\to L^2(\gamma)}$ is uniformly bounded. This implies that for any interval Q on γ and any ϵ, we have uniformly:

$$\int_Q |T_\epsilon\chi_Q||d\zeta| < c|Q|. \tag{2.6}$$

For $z \in \gamma$, denote by γ_z the curve $\gamma - z$. Because γ is a Lipschitz curve, then γ_z is also a Lipschitz curve passing the origin. We also write $Q_{z,\eta} = \{\zeta \in \gamma_z : |\zeta| < \eta\}$. Fix $z_0 \in \gamma$. For $z_1 \in Q_{z_0,\eta/2}$, we will prove the following estimate:

$$\left|T_\epsilon(\chi_{Q_{z_0,\eta}})(z_1) - \int_{\zeta\in\gamma_0,\epsilon<|\zeta|<\eta} \phi(\zeta)d\zeta\right| \leq C < \infty, \tag{2.7}$$

where C is a constant independent of ϵ, η and $z_1 \in Q_{z_0,\eta/2}$.

In fact, denote by $\gamma^{\pm}(z_0, \eta)$ the right and the left endpoints of $Q_{z_0,\eta}$. Define

$$S_1 =: \{\zeta \in \gamma_{z_1}, \ \text{ from } z_1 + \gamma^{+}(z_0, \eta) \text{ to } z_1 + \gamma^{-}(z_0, \eta), \ |\zeta| > \epsilon\}$$

and

$$S_2 =: \{\zeta \in \gamma_0, \ \text{ from } \gamma^{-}(z_0, \eta) \text{ to } \gamma^{+}(z_0, \eta), |\zeta| > \epsilon\}.$$

We have

$$\begin{aligned} &T_\epsilon \chi_{Q_{z_0,\eta}}(z_1) - \int_{\zeta\in\gamma_0,\epsilon<|\zeta|<\eta} \phi(\zeta)d\zeta \\ &= \int_{S_1} \phi(\zeta)d\zeta + \int_{S_2} \phi(\zeta)d\zeta. \end{aligned}$$

Using the Cauchy theorem, we can reduce the above integrals to the integrals along circles of radius η and ϵ and along the directions of radius within $\{z \in \mathbb{C} : \ \eta \leqslant |z| \leqslant 3\eta/2\}$. Then from the condition $|\phi(z)| \leq C/|z|$, we can conclude (2.7).

From (2.7), we have

$$\left|\int_{\zeta\in\gamma_{z_0},\epsilon<|\zeta|<\eta} \phi(\zeta)d\zeta\right| \leqslant C + |T\chi_{Q_{z_0,\eta}}|.$$

Taking average to both sides of this inequality w.r.t. $z_1 \in Q_{z_0,\eta/2}$ and using (2.6), we obtain for any $0 < \epsilon < \eta < \infty$,

$$\left|\int_{\zeta\in\gamma_{z_0},\epsilon<|\zeta|<\eta} \phi(\zeta)d\zeta\right| \leqslant C. \tag{2.8}$$

From the Cauchy theorem, the condition $|\phi(z)| \leqslant C/|z|$ and the inequality (2.8), we have

$$\left|\left(\int_{l^-(z_1^-,z_2^-)} + \int_{l^+(z_1^+,z_2^+)}\right)\phi(\zeta)d\zeta\right| \leqslant C,$$

where $z_1^{\pm}, z_2^{\pm} \in S^0_{\omega,\pm}$, and $l^-(z_1^-, z_2^-)$ is a contour lying in $S^0_{\omega,-}$ from z_1^- to z_2^-, $l^+(z_1^+, z_2^+)$ is a contour lying in $S^0_{\omega,+}$ from z_2^+ to z_1^+, and $|z_1^-| = |z_1^+|$, $|z_2^-| = |z_2^+|$. For $z \in S^0_{\omega,\pm}$, we let

$$\phi_1(z) = \frac{1}{2}\Big(\int_{l^-(-1,\mp z)} \phi(\zeta)d\zeta + \int_{l^+(1,\pm z)} \phi(\zeta)d\zeta\Big).$$

Now it is easy to check that $\phi_1 \in H^\infty(S^0_\omega)$ and for $z \in S^0_\omega$,

$$\phi'(z) = \frac{1}{2}\Big(\phi(z) + \phi(-z)\Big).$$

□

We state the following theorem.

Theorem 2.3.5 *Assume that* $\omega \in (\arctan N,\ \pi/2]$. *Let* $\widetilde{\Phi}$ *be holomorphic in* $\exp(iS^0_\omega(\pi))$ *and satisfy (i) of Theorem 2.3.2 with respect to* ω. *If* T *is a bounded operator on* $L^2(\widetilde{\gamma})$ *and for all* $\widetilde{F}$ *belonging to the class of continuous functions* $C_0(\widetilde{\gamma})$,

$$T(\widetilde{F})(z) = \int_{\widetilde{\gamma}} \widetilde{\Phi}(z\xi^{-1})\widetilde{F}(\xi)\frac{d\xi}{\xi},\ z \notin \ supp\ (\widetilde{F}),$$

then there exists a unique function $\widetilde{\Phi}_1 \in H^\infty(\exp(iS^0_{\mu,+}))$, $\mu \in (0,\omega)$ *such that for* $\widetilde{F} \in C_0(\widetilde{\gamma})$,

$$\widetilde{\Phi}_1'(z) = \frac{1}{iz}\Big(\widetilde{\Phi}(z) + \widetilde{\Phi}(z^{-1})\Big),\ z \in \exp(iS^0_{\omega,+}(\pi))$$

and

$$T(\widetilde{F}) = T_{(\widetilde{\Phi},\widetilde{\Phi}_1)}(\widetilde{F}).$$

Proof On $S^0_{\omega,+}(\pi)$, we define the function ϕ as

$$\phi(\eta) =: \widetilde{\Phi}(e^{i\eta}).$$

Then, on the one hand, we can get for $\eta \in S^0_{\omega,+}(\pi)$,

$$|\phi(\eta)| \leqslant |\widetilde{\Phi}(e^{i\eta})| \leqslant \frac{C}{|1-e^{i\eta}|} \leqslant \frac{C}{|\eta|}.$$

On the other hand, let $f(z) = \widetilde{F}(e^{iz})$. For $z \notin$ supp $(\widetilde{F})$ and $\xi \in \widetilde{\gamma}$, without loss of generality, we can write $z = e^{i\eta}$ and $\xi = e^{iw}$, where $z \notin$ supp f and $w \in \gamma$. If the operator

$$T(\widetilde{F})(z) = \int_{\widetilde{\gamma}} \widetilde{\Phi}(z\xi^{-1})\widetilde{F}(\xi)\frac{d\xi}{\xi},\ z \notin \text{ supp } (\widetilde{F}),$$

is bounded on $L^2(\widetilde{\gamma})$, then by change of variables, we can see that

$$T(\widetilde{F})(z) = \int_{\gamma} \widetilde{\Phi}(e^{i(\eta-w)})\widetilde{F}(e^{iw})\frac{de^{iw}}{e^{iw}}$$

$$= i \int_{\gamma} \phi(\eta - w) f(w) dw$$
$$=: T_{\phi}(f),$$

which implies that the operator T_ϕ is also bounded on $L^2(\gamma)$. By Theorem 2.3.4, there exists a function $\phi_1 \in H^\infty(S^0_\omega)$ such that $\phi_1'(\eta) = \phi(\eta) + \phi(-\eta)$, $\eta \in S^0_\omega$. For $z = e^{i\eta} \in \exp(iS^0_{\omega,+}(\pi))$ with $\eta \in S^0_{\omega,+}(\pi)$, define

$$\widetilde{\Phi}_1(z) = \widetilde{\Phi}_1(e^{i\eta}) =: \phi_1(\eta).$$

Then

$$\begin{aligned}
\widetilde{\Phi}_1'(z) &= \frac{d\eta}{dz}\frac{d}{d\eta}(\widetilde{\Phi}_1(e^{i\eta})) \\
&= \frac{1}{ie^{i\eta}}\frac{d}{d\eta}(\phi_1(\eta)) \\
&= \frac{1}{ie^{i\eta}}[\phi(\eta) + \phi(-\eta)] \\
&= \frac{1}{ie^{i\eta}}\left[\widetilde{\Phi}(e^{i\eta}) + \widetilde{\Phi}(e^{-i\eta})\right] \\
&= \frac{1}{iz}\left[\widetilde{\Phi}(z) + \widetilde{\Phi}(z^{-1})\right].
\end{aligned}$$

This completes the proof of Theorem 2.3.5. □

2.4 Holomorphic H^∞-Functional Calculus on Starlike Lipschitz Curves

The purpose of this section is to clarify that the theory of holomorphic H^∞-functional calculus on infinite Lipschitz curves established by A. McIntosh in [1] can also be established in the case of closed curves. Precisely, we study the relations between the operator classes $\widetilde{M}_b$, $T_{(\widetilde{\Phi},\widetilde{\Phi}_1)}$ and the holomorphic H^∞-functional calculus, see also [2, 3] for further information.

For the functions $\widetilde{F} \in \mathcal{A}(\widetilde{\gamma})$, we define the differential operator $\frac{d}{dz}|_{\widetilde{\gamma}}$ as

$$\frac{d}{dz}|_{\widetilde{\gamma}}\, \widetilde{F}(z) = \lim_{h\to 0,\ z+h\in\widetilde{\gamma}} \frac{\widetilde{F}(z+h) - \widetilde{F}(z)}{h}, \; z \in \widetilde{\gamma}.$$

For $1 < p < \infty$, $\langle L^p(\widetilde{\gamma}),\ L^{p'}(\widetilde{\gamma})\rangle$ is the dual of Banach spaces defined as follows:

$$\langle \widetilde{F},\ \widetilde{G}\rangle = \int_{\widetilde{\gamma}} \widetilde{F}(z)\widetilde{G}(z)dz,$$

where $p' = (1 - p^{-1})^{-1}$. Now by duality, we define $D_{\widetilde{\gamma},p}$ as the closed operator with the largest domain in $L^p(\widetilde{\gamma})$ which satisfies

$$\left\langle D_{\widetilde{\gamma},p}\widetilde{F},\ \widetilde{G}\right\rangle = \left\langle \widetilde{F},\ -z\frac{d}{dz}\ |_{\widetilde{\gamma}}\ \widetilde{G}\right\rangle$$

for all $\widetilde{F}$ and $\widetilde{G}$ in $\mathcal{A}(\widetilde{\gamma})$.

Let $\omega \in (\arctan N,\ \pi/2]$ and $\lambda \notin S^0_\omega$. It is easy to prove $D_{\widetilde{\gamma},p}$ is the surface Dirac operator on $\widetilde{\gamma}$ and $\frac{1}{2\pi}\widetilde{\Phi}_\lambda$ is the functions defined below.

Let $\lambda \notin S^0_\omega$. Then on any starlike Lipschitz curve , $b(z) = \frac{1}{z-\lambda}$ corresponds to the resolvent of the surface Dirac operator. If $\mathrm{Im}(\lambda) > 0$, by (1.1) and (1.2), we have

$$\phi_\lambda(z) = \begin{cases} i\exp(i\lambda z), & \mathrm{Re}(z) > 0, \\ 0, & \mathrm{Re}(z) < 0. \end{cases}$$

If $\mathrm{Im}(\lambda) < 0$, then we have

$$\phi_\lambda(z) = \begin{cases} 0, & \mathrm{Re}(z) > 0, \\ i\exp(i\lambda z), & \mathrm{Re}(z) < 0. \end{cases}$$

It is easy to prove that for every case, ϕ_λ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Hence for the two cases, we can use the remark made after Theorem 2.1.2.

For $\mathrm{Im}(\lambda) > 0$, we can deduce from the definition that

$$\Phi_\lambda(z) = \begin{cases} \frac{i\exp(i\lambda(z+2\pi))}{1-\exp(i\lambda 2\pi)}, & \text{if } -\pi < \mathrm{Re}(z) < 0, \\ \frac{i\exp(i\lambda z)}{1-\exp(i\lambda 2\pi)}, & \text{if } 0 < \mathrm{Re}(z) < \pi. \end{cases}$$

For $\mathrm{Im}(\lambda) < 0$,

$$\Phi_\lambda(z) = \begin{cases} \frac{-i\exp(i\lambda(z-2\pi))}{1-\exp(-i\lambda 2\pi)}, & \text{if } 0 < \mathrm{Re}(z) < \pi, \\ \frac{-i\exp(i\lambda z)}{1-\exp(-i\lambda 2\pi)}, & \text{if } -\pi < \mathrm{Re}(z) < 0. \end{cases}$$

For $\mathrm{Im}(\lambda) > 0$,

$$\widetilde{\Phi}_\lambda(z) = \begin{cases} \frac{i\exp(i\lambda 2\pi)z^\lambda}{1-\exp(i\lambda 2\pi)}, & \text{if } -\pi < \mathrm{Re}(\frac{\ln z}{i}) < 0, \\ \frac{iz^\lambda}{1-\exp(i\lambda 2\pi)}, & \text{if } 0 < \mathrm{Re}(\frac{\ln z}{i}) < \pi. \end{cases}$$

For $\mathrm{Im}(\lambda) < 0$,

$$\widetilde{\Phi}_\lambda(z) = \begin{cases} \frac{-i\exp(-i\lambda 2\pi)z^\lambda}{1-\exp(-i\lambda 2\pi)}, & \text{if } 0 < \mathrm{Re}(\frac{\ln z}{i}) < \pi, \\ \frac{-iz^\lambda}{1-\exp(-i\lambda 2\pi)}, & \text{if } -\pi < \mathrm{Re}(\frac{\ln z}{i}) < 0. \end{cases}$$

We can verify that $D_{\widetilde{\gamma},p}$ is the surface Dirac operator on $\widetilde{\gamma}$, and in the sense of Theorem 2.3.3, the function $\frac{1}{2\pi}\widetilde{\Phi}_\lambda$ is the convolution kernel of the resolvent operator $(D_{\widetilde{\gamma},p}-\lambda)^{-1}$. Moreover,

$$\begin{aligned}\|(D_{\widetilde{\gamma},p}-\lambda)^{-1}\| \leqslant \|\frac{1}{2\pi}\widetilde{\Phi}_\lambda\| &\leqslant \sum_{n=-\infty}^{\infty} \|\phi_\lambda(\cdot+2\pi n)\|_{L^1(\gamma)} \\ &= \|\phi_\lambda\|_{L^1(p\gamma)} \leqslant \sqrt{1+N^2}\{\mathrm{dist}(\lambda, S^0_\omega)\}^{-1}.\end{aligned}$$

The above estimate implies that $D_{\widetilde{\gamma},p}$ is a type ω operator. For the H^∞ functions b with good decay properties at both 0 and ∞, we can define $b(D_{\widetilde{\gamma},p})$ via spectral integrals as follows:

$$b(D_{\widetilde{\gamma},p}) = \frac{1}{2\pi}\int_\delta b(\eta)(D_{\widetilde{\gamma},p}-\eta I)^{-1}d\eta.$$

Here δ is a path consisting of four rays:

$$\begin{cases} \{s\exp(-i\theta):\ s \text{ from } \infty \text{ to } 0\}; \\ \{s\exp(i\theta):\ s \text{ from } 0 \text{ to } \infty\}; \\ \{s\exp(-i(\pi-\theta)):\ s \text{ from } \infty \text{ to } 0\}; \\ \{s\exp(i(\pi+\theta)):\ s \text{ from } 0 \text{ to } \infty\}, \end{cases}$$

where $\arctan N < \delta < \omega$.

By the above estimates, it is easy to prove that any $b(D_{\widetilde{\gamma},p})$ is a bounded operator, and

$$b(D_{\widetilde{\gamma},p}) = \widetilde{M}_b = \widetilde{T}(\widetilde{\Phi}, 0).$$

Taking limits of the sequences of Calderón-Zygmund operators, we can extend the definition of $b(D_{\widetilde{\gamma},p})$ to all functions in $H^\infty(S^0_\omega)$, and prove that

$$b(D_{\widetilde{\gamma},p}) = \widetilde{M}_b = \widetilde{T}(\widetilde{\Phi}, \widetilde{\Phi}_1).$$

Alternative proofs of the boundedness of the operators can be found in [2] by G. Gaudry, T. Qian and S. Wang. In addition, when $b_1, b_2 \in H^\infty(S^0_\omega)$ and $\alpha_1,\ \alpha_2$ are complex numbers,

$$\|b(D_{\widetilde{\gamma},p})\| \leqslant C_\omega \|b\|_\infty,$$

$$(b_1b_2)(D_{\widetilde{\gamma},p}) = b_1(D_{\widetilde{\gamma},p})b_2(D_{\widetilde{\gamma},p})$$

and

$$(\alpha_1 b_1 + \alpha_2 b_2)(D_{\widetilde{\gamma},p}) = \alpha_1 b_1(D_{\widetilde{\gamma},p}) + \alpha_2 b_2(D_{\widetilde{\gamma},p}).$$

Below we shall not restrict ourselves to H^∞-multipliers. It should be pointed out that all the results and methods of the Fourier multiplier theory for infinite Lipschitz curves can be adapted to the present case. The main difference is that the function class $\mathcal{A}(\widetilde{\gamma})$ has even better properties. When we deal with the kernels on γ, we refer to its corresponding kernel on $p\gamma$ via the Poisson summation formula. The following theorem can be proved via the corresponding Schur lemma, and we omit the proofs.

For $b = \{b_n\}_{n=-\infty}^{\infty} \in l^\infty$, define

$$\|b\|_{M_p(\widetilde{\gamma})} = \sup\left\{\left\|\sum b_n \widehat{\widetilde{F}}(n)z^n\right\|_{L^p(\widetilde{\gamma})} : \|\widetilde{F}\|_{L^p(\widetilde{\gamma})} \leqslant 1\right\},$$

and

$$M_p(\widetilde{\gamma}) = \left\{b : \|b\|_{M_p(\widetilde{\gamma})} < \infty\right\}.$$

We call the functions b in $M_p(\widetilde{\gamma})$ the $L^p(\widetilde{\gamma})$−Fourier multipliers.

Theorem 2.4.1 *Let $\widetilde{\Phi}$ be a holomorphic function defined on a simple connected open neighborhood of the set*

$$\widetilde{\gamma} - \widetilde{\gamma} = \left\{z - \xi : z, \xi \in \widetilde{\gamma}\right\}$$

satisfying $|\widetilde{\Phi}(r\exp(i\theta))| \leqslant \psi(\exp(i\theta))$, where $\int_{-\pi}^{\pi} \psi(\exp(i\theta))d\theta < \infty$. Then

$$b = (\widehat{\widetilde{\Phi}}(n))_{n=-\infty}^{\infty} \in M_p(\widetilde{\gamma}),\ 1 < p < \infty,$$

and the corresponding convolution operator $T_{\widetilde{\Phi}}$ can be represented as

$$T_{\widetilde{\Phi}}\widetilde{F}(z) = \int_{\widetilde{\gamma}} \widetilde{\Phi}(z\eta^{-1})\widetilde{F}(\eta)\frac{d\eta}{\eta},\ \widetilde{F} \in \mathcal{A}(\widetilde{\gamma}).$$

Let $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ be two curves of the type under consideration. Define

$$M_p(\widetilde{\gamma}_1, \widetilde{\gamma}_2) = \left\{b \in l^\infty : \|b\|_{M_p(\widetilde{\gamma}_1,\widetilde{\gamma}_2)} < \infty\right\},$$

where

$$\|b\|_{M_p(\widetilde{\gamma}_1,\widetilde{\gamma}_2)} = \sup\left\{\frac{\|\sum b_n \widehat{\widetilde{F}}(n)z^n\|_{L^p(\widetilde{\gamma}_2)}}{\|\widetilde{F}\|_{L^p(\widetilde{\gamma}_1)}} : \widetilde{F} \in \mathcal{A}(\widetilde{\gamma}_1) \cap \mathcal{A}(\widetilde{\gamma}_2)\right\}.$$

If $\widetilde{\gamma}_3$ is the third such curve, and $b_1 \in M_p(\widetilde{\gamma}_1, \widetilde{\gamma}_2)$, $b_2 \in M_p(\widetilde{\gamma}_2, \widetilde{\gamma}_3)$, then $b_2 b_1 \in M_p(\widetilde{\gamma}_1, \widetilde{\gamma}_3)$, and

$$\|b_2 b_1\|_{M_p(\widetilde{\gamma}_1,\widetilde{\gamma}_3)} \leqslant \|b_2\|_{M_p(\widetilde{\gamma}_2,\widetilde{\gamma}_3)} \|b_1\|_{M_p(\widetilde{\gamma}_1,\widetilde{\gamma}_2)}.$$

Theorem 2.4.2 *Let* $b \in l^\infty$ *and* $f_\beta(n) = b(n)\exp(2\beta|n|)$. *If for some* $\beta > M = \max A(x)$, $f_\beta \in M_p(\mathbb{T})$, *where* $\mathbb{T}$ *is the unit circle and* $1 < p < \infty$, *then* $b \in M_p(\widetilde{\gamma})$ *and*

$$\|b\|_{M_p(\widetilde{\gamma})} \leqslant (2\pi\beta)^2(\beta^2 - M^2)^{-1}(1 + N^2)^{1/2}\|f_\beta\|_{M_p(\mathbb{T})}.$$

For flat curves γ, it is obvious that $\|b\|_{M_2(\widetilde{\gamma})} \leqslant C_{\widetilde{\gamma}}\|b\|_\infty$. But the following example indicates that in general case, this fact may not hold.

Take $\gamma(x) = x + iA(x)$ to be a piece of the Lipschitz curve defined on $[-\pi, \pi]$ with $g(0) > 0$ and $m = \min g(x) < 0$. For any integer S, let b_S be a l^∞-sequence satisfying $b_S(n) = 1$ for $n \leqslant S$ and $b_S(n) = 0$ otherwise. Using $F(z) = \frac{1}{1-\exp(iz)}$ as the test function, we can prove that for any $\epsilon > 0$,

$$\|b_S\|_{M_2(\widetilde{\gamma})} \geqslant C_\epsilon \exp(-S(m + \epsilon)).$$

2.5 Remarks

Remark 2.5.1 We can obtain the following generalizations of Theorems 2.3.1 and 2.3.2. Let γ be a closed starlike Lipschitz curve. Suppose that the multiplier b satisfies $|b(z)| \leqslant C|z \pm 1|^s$ in any $S_{\mu,\pm}$, $0 < \mu < w$. Then it can be proved that $\phi(z) = \sum_{n=\pm 1}^{\pm\infty} b(n)z^n$ satisfies

$$|\phi(z)| \leqslant \frac{C_\mu}{|1 - z|^{1+s}},\ z \in C_{\mu,\pm},\ 0 < \mu < w. \tag{2.9}$$

Conversely, if the holomorphic function ϕ satisfies the estimate (2.9), then there exists a function b such that $|b(z)| \leqslant C|z \pm 1|^s$ and $\phi(z) = \sum_{n=\pm 1}^{\pm\infty} b(n)z^n$, see Sect. 7.1 for the details.

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