Chapter 8
Innovative Pedagogical Practices

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Abstract This chapter describes some innovative pedagogical practices in Singapore. It is divided into two main sections: pedagogies that engage the minds, and those that engage the hearts, of mathematics learners. Examples of such classroom practices include the Singapore Model Method to solve word problems in primary schools, the Singapore AlgeDisc™ to teach algebra in secondary schools, and guided-discovery learning. The main principle that underlies all these pedagogies that engage the minds of mathematics students is the Concrete-Pictorial-Abstract (C-P-A) approach. We also describe a theoretical framework on engaging the hearts of mathematics learners and the use of various strategies to make lessons interesting. Examples of such strategies include the use of mathematics songs and videos, television shows and movies, mathematics storybooks, drama, magic tricks, and mathematics puzzles and games. Some of these practices are not unique to Singapore but many local teachers are using them in their classrooms. Finally, this chapter also reviews limited local research on these pedagogical practices, and where there is no local research, we suggest some directions for future research.

Keywords Innovative pedagogical practices · Engaging minds and hearts · Model Method · AlgeDisc™ · Guided-discovery learning · Investigation

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8.1 Introduction

Innovative and powerful pedagogical practices in mathematics education include innovative and powerful mathematical learning environments, innovative practices that promote mathematics teaching and learning as inquiry, and mathematical tools that promote deep learning (Hunter et al. 2016). Innovative and powerful mathematical learning environments are formed when teachers establish classroom cultures (Leach et al. 2014), promote productive discourse (ibid.), and promote and maintain student engagement (Marshman and Brown 2014) to support productive mathematical activity. In order to promote mathematics teaching and learning as inquiry, the teachers may have to change their beliefs about social interactions within the classroom, their role and purpose, and classroom dynamics (Murphy 2015) so that they can notice and respond to student reasoning productively (Choy 2013). Mathematical tools that promote deep learning include challenging and ill-structured tasks with multiple entry and exit points that can sustain thinking and argumentation (Sullivan and Davidson 2014), and digital tools that can support and enhance learning (Lowrie and Jorgensen 2015).

In this chapter, we will describe some innovative pedagogical practices in the Singapore mathematics classrooms. Other than the well-known Singapore Model Method, AlgeDisc™ is a recent invention by the Ministry of Education of Singapore (MOE). Another teaching strategy that has been in use for many years is Bruner's (1961) guided-discovery learning. But the main principle that underlies all these innovative practices to engage the minds of mathematics learners in Singapore is still the Concrete-Pictorial-Abstract (C-P-A) approach, which was adapted from Bruner's (1964, 1966) enactive–iconic–symbolic model. We will also describe some classroom practices in Singapore that engage the hearts of mathematics learners, including the use of mathematics songs and videos, television shows and movies, mathematics storybooks, drama and art, magic tricks, and mathematics puzzles and games. Some of these pedagogies are unique or distinctive features of the Singapore education system, e.g. the Model Method and the AlgeDisc™, but the rest may be common practices in other countries as well. Lastly, we draw on some limited local research to examine the effectiveness of such pedagogical practices, and where there is no local research in this area, we suggest some research questions for future studies.

However, this chapter does not describe innovative pedagogies such as a mathematical problem-solving approach to teaching and learning, problems in real-world contexts and mathematical modelling, comics and the use of technology, because they are dealt with in Chaps. 7, 9, 13 and 14 of this book, respectively.

8.2 Engaging the Minds of Mathematics Learners: The Concrete-Pictorial-Abstract (C-P-A) Approach

As mentioned in Chap. 2, the Teach Less, Learn More (TLLM) initiative was launched in the education system in 2005 (Shanmugaratnam 2005). It aims to touch the hearts and engage the minds of our learners, to prepare them for life. To engage the minds of mathematics learners, MOE has used the Concrete-Pictorial-Abstract (C-P-A) approach, mentioned in Chap. 3, extensively in the development of primary school mathematics concepts (MOE 2007, 2012a). This approach is evident in both the MOE syllabus documents and the MOE-approved school textbooks. In Singapore, schools only use textbooks that are approved by MOE. Therefore, schools that use MOE-approved textbooks will also use the resources provided in the textbooks. Chang et al. (2017) gave the example of a teaching sequence in one of the textbooks where pictorial representation (P) in the forms of rectangular and circular models is used to introduce the abstract concept of equivalent fractions, i.e., $\frac{a}{b} = \frac{c}{d}$ (A). In addition, MOE has provided fraction strips and fraction discs to all primary schools so that students can use these concrete manipulatives (C) to learn the concept of equivalent fractions.

As mentioned in Chap. 3, the C-P-A approach has its roots in Bruner's notion of enactive, iconic, and symbolic representations of cognitive growth (Leong et al. 2015; Wong 2015). According to Bruner (1964, 1966), conceptual learning begins when a person undertakes and experiences some actions (enactive), which are then translated into images of the experience (iconic). Subsequently, links are formed to connect the iconic representations into a collective structure governed by a rule derived from organising common attributes found embedded in the representations. Eventually, this rule stands exclusively by itself and is denoted by a symbol.

The MOE syllabus documents specify the use of 'manipulatives or other resources' (MOE 2012a, p. 23, b, p. 23) in activity-based learning to construct meanings and understandings, and from 'concrete manipulatives and experiences' (ibid.), students are guided to uncover abstract mathematical concepts or results. So far, the examples given in the syllabus documents for 'manipulatives or other resources' include the use of paper cut-out of rectangles for primary school mathematics and the virtual balance to learn the concept of equations for secondary school mathematics. In other words, 'manipulatives or other resources' include both concrete and virtual manipulatives. However, 'concrete' in the C-P-A approach does not only mean manipulatives, but concrete 'experiences' (ibid.) derived from playing with the manipulatives.

For instance, using algebra discs or AlgeDisc™ to learn algebraic manipulation such as expansion and factorisation (which we will elaborate in Sect. 8.4) is an example of utilising the C-P-A approach from concrete to pictorial to abstract. But the Singapore Model Method (which we will elaborate in Sect. 8.3) is a pictorial representation and there is no concrete manipulative. Although AlgeBar™ was developed at a later stage as a virtual tool for students to draw the models, the computer application does not function as a manipulative in the sense that students cannot manipulate the models but they just use the application to draw the models. But there is really no need to always rely on concrete or virtual manipulatives because sometimes pictorial representations are good enough to provide students with the necessary concrete experiences to abstract mathematical concepts or to solve problems.

Similarly, when it comes to higher level mathematics, it is not always possible to find suitable manipulatives, whether concrete or virtual, for students to manipulate in order to abstract the concepts. Therefore, the first author has extended the C-P-A approach to include the use of concrete examples, which often involve numerical values for secondary school mathematics. Numerical examples are another way to offer students concrete experiences from which they can abstract the underlying concepts. In the Extended C-P-A approach, there may not be any pictorial representation. The main idea behind the Extended C-P-A approach is that what is abstract at one level may become more concrete at a higher level. For example, concrete objects are concrete to lower primary school students but numbers are more abstract. But when the students reach lower secondary level, numbers have become concrete and what is abstract is algebra. At university level, algebra has become concrete to many undergraduates and what is abstract for them is abstract algebra. We will illustrate the Extended C-P-A approach at the secondary school level with some examples in Sect. 8.5.

In the next three sections, we will describe three main innovative pedagogical practices used in Singapore to engage the minds of mathematics learners: the Singapore Model Method, the Singapore AlgeDisc™, and guided-discovery learning. Only the first two practices are unique features of the Singapore education system, while the last one has been in use in some other countries as well.

8.3 The Singapore Model Method

The Singapore Model Method, or simply the Model Method, was developed in the 1980s to address students' difficulties in understanding and solving mathematics word problems (Kho 1987; Kho et al. 2014). First introduced in 1983, the Model Method has since become a signature problem-solving heuristic in the Singapore primary school mathematics curriculum (Kho et al. 2014). The Model Method uses bar models to represent quantities and the relationships among the quantities given in a word problem. The fundamental idea underpinning the method is the assumption that if pupils were provided with a means to represent the relationships between quantities, then the structure of the problem would be made clear to the pupils, making visible the solution pathways (Kho 1987; Ng and Lee 2009). The method revolves around pupils drawing rectangles of appropriate lengths to represent quantitative relationships (Kho 1987). Despite its simplicity, the method offers pupils a visual way to solve complex word problems without the use of symbolic algebra because the bar models function as visual representations of algebraic equations (Kho et al. 2009). Hence, the method has also been used as a bridge to support primary and secondary school students in the learning of algebra. In this section, we will present

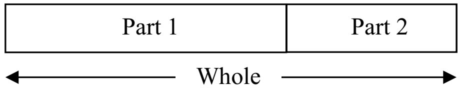


Fig. 8.1 Basic part-whole model

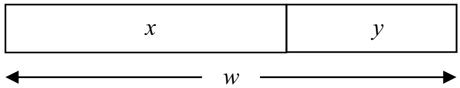


Fig. 8.2 Model representation of $w = x + y$

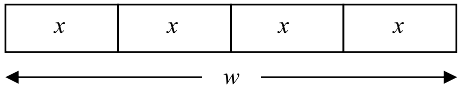


Fig. 8.3 Model representation of $w = 4x$

the two basic models of the Model Method, highlight its connection with algebra, review some of the challenges pupils faced when using the method, and suggest some possible directions for future research in this area.

8.3.1 Two Basic Models

There are two basic models in the Singapore mathematics curriculum, namely the part-whole models, and the comparison models. The Model Method builds on the pictorial representation of part-whole and comparison schemas—the building blocks of mental and cognitive processes for addition and subtraction (Kintsch and Greeno 1985; Nesher et al. 1982)—and extends the part-whole and comparison models to include multiplication, division, fractions, ratios, and percentages. In this section, we will introduce the two basic models. Interested readers may refer to the monograph by Kho et al. (2009).

The part-whole model shows the relationship between a whole and its part, or simply, a whole as comprising of two parts (see Fig. 8.1). Mathematically, this is represented as the whole w is divided into two parts x and y, i.e., $w = x + y$, as shown in Fig. 8.2. A part-whole model may also involve more than two parts.

In some problems, we may have to divide the whole into equal parts. For instance, the pictorial representation in Fig. 8.3 shows that the whole w is divided into four equal parts, i.e., $w = 4x$.

There are two other quantitative relationships that students can make use of depending on the information given. First, given two parts, students can find the whole by adding the two parts given, i.e., Part 1 + Part 2 = Whole (see Fig. 8.1). Next, when students are given the whole and one of the parts, students can perform subtraction to find the other part: Whole − Part 1 = Part 2.

The comparison model shows the relationship between two quantities when they are compared. The model involves three variables: the larger quantity, the smaller quantity and the difference (see Fig. 8.4). Mathematically, this model represents the

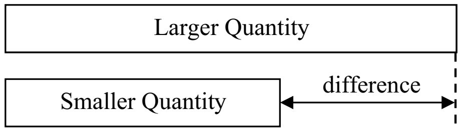


Fig. 8.4 The basic comparison model

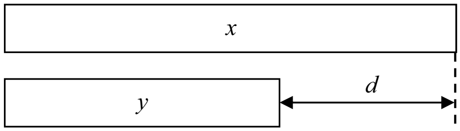


Fig. 8.5 Model representation of $x - y = d$

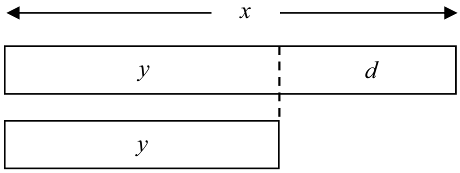


Fig. 8.6 Model representation of $x = y + d$

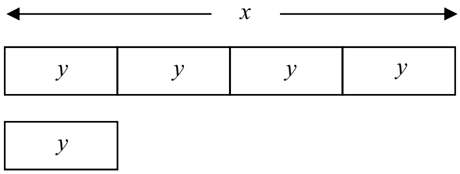


Fig. 8.7 Model representation of $x = 4y$

equation $x - y = d$, as shown in Fig. 8.5. As with the part-whole model, we can use this model to compare three or more quantities.

Depending on the information given, there are a few other quantitative relationships between the three variables. First, students can find the larger quantity by adding the difference to the smaller quantity, i.e., $x = y + d$ (see Fig. 8.6). Second, students can find the smaller quantity if they were given the larger quantity and the difference, i.e., $y = x - d$. In some cases, the sum of two or more quantities may be given.

Last but not least, the comparison model can be used to make this relationship visible to students when one quantity is a multiple of the other, e.g. $x = 4y$ in Fig. 8.7.

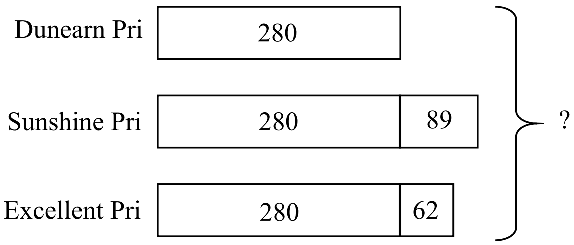
Referring to Figs. 8.3 and 8.7, we see that the part-whole model and the comparison model can be used to represent multiplication and division problems. This provides a means to represent fractions, ratios, and percentages. For example, Fig. 8.7 can be used to represent the relationship $y = \frac{x}{4}$, $x : y = 4 : 1$, or y is 25% of x. By using the models, students can represent complex quantitative relationships given in a word problem and use the visual models to find the unknowns. In the next section, we present some of the typical word problems in the Singapore Mathematics Curriculum and highlight how the Model Method can be used to solve these questions.

8.3.2 *Using the Model Method to Solve Word Problems*

The main strength of the Model Method lies in its affordance to represent quantitative relationships in word problems visually so that students can process the given information and use them to solve for the unknowns. The method can be used for arithmetical word problems, in which the relationship between the unknown and the known is clear, as well as algebraic word problems, in which an unknown needs to be introduced in the solution (Ng and Lee 2005, 2009). In this section, we will illustrate the use of the Model Method to solve word problems.

Example 1
Dunearn Primary School has 280 pupils. Sunshine Primary School has 89 pupils more than Dunearn Primary. Excellent Primary has 62 pupils more than Dunearn Primary. How many pupils are there altogether? (Ng and Lee 2005, p. 63)

Solution

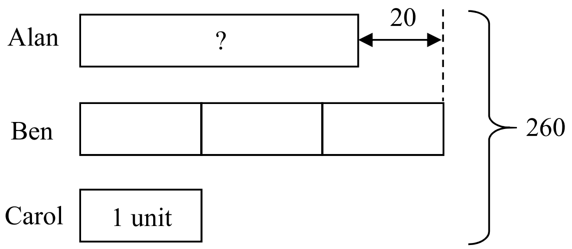


$$
\begin{aligned}
\text{Total number of pupils} &= 280 + 280 + 89 + 280 + 62 \\
&= 280 \times 3 + 89 + 62 \\
&= 991
\end{aligned}
$$

Example 2
$260 was shared among Alan, Ben and Carol. If Alan received $20 less than Ben, and Ben received 3 times as much money as Carol, how much money did Carol receive?

Solution

Let the amount of money Carol received be 1 unit.

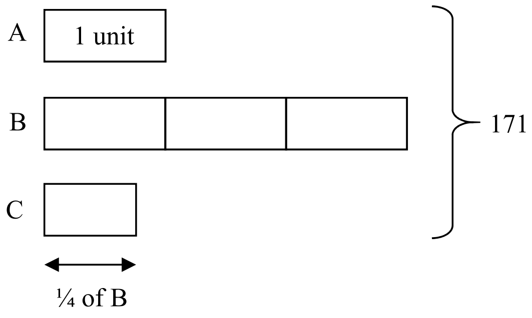


$$
\begin{aligned}
7\text{ units} &= \$260 + \$20\left(\text{by assuming Alan had \$20 more}\right)\\
&= \$280\\
1\text{ unit} &= \$280 \div 7\\
&= \$40
\end{aligned}
$$

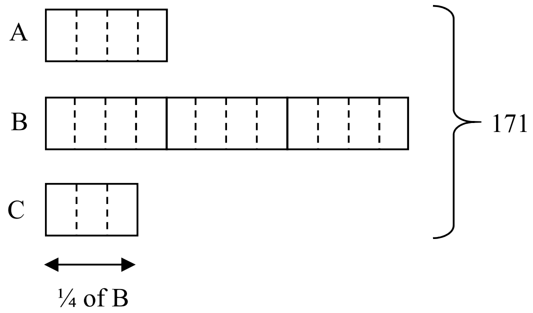
Therefore, Carol received $40.

Example 3
A tank of water with 171 l of water is divided into three containers, A, B and C. Container B has three times as much water as container A. Container C has $\frac{1}{4}$ as much water as container B. How much water is there in container B? (Ng and Lee 2005, p. 63)

Solution



Representing the information given in the problem, we see that the bar representing container C is less than the amount in container A, which is 1 unit. To make all the units the same, we divide the bar representing A into four smaller units (left as an exercise for the reader to figure out) and arrive at the following model:



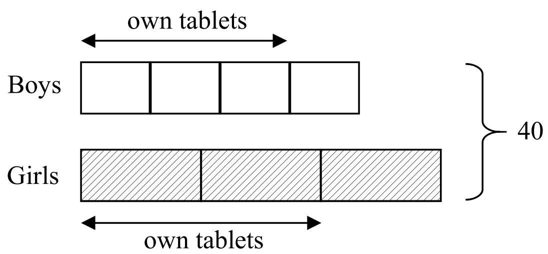
As we can see, the units are now of the same size. Therefore,

$$
\begin{aligned}
19 \text{ units} &= 171 \text{ litres of water} \\
1 \text{ unit} &= 171 \div 19 \\
&= 9 \text{ litres} \\
12 \text{ units} &= 9 \times 12 \\
&= 108 \text{ litres}
\end{aligned}
$$

There are 108 litres of water in Container B.

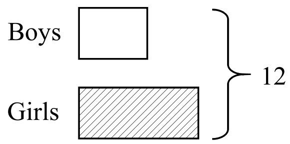
Example 4
In a class of 40 pupils, 75% of the boys and 2/3 of the girls owned a tablet computer. If 30% of the pupils do not own tablet computers, find the number of girls who own tablet computers.

Solution



This is a challenging problem because the units representing the boys and girls are different in sizes. However, the Model Method can be extended to solve simultaneous linear equations in two variables without using symbolic algebra.

Since one unit of boys and one unit of girls do not own a tablet computer, we know that one unit of boys plus one unit of girls equal 30% $\times$ 40 = 12 pupils. This can be illustrated with another model below if need to, although it is not necessary to do so.



Referring to the first model above, we can see that there are 3 groups of (1 unit of boys + 1 unit of girls) = 3 × 12 pupils.

$$\begin{aligned}\text{Thus one unit of boys} &= 40 - 3 \times 12 \\ &= 4\end{aligned}$$

So there are 4 × 4 = 16 boys, and 40 − 16 = 24 girls.
Therefore, 16 girls own tablet computers.

8.3.3 Linking the Model Method to Algebra

The Model Method solution to Example 4 resembles a typical algebraic approach. As Kho et al. (2014) highlight, many Secondary One pupils continue to use the Model Method to solve algebraic problems because they have difficulties formulating the equations. Students may then see the algebraic approach as redundant partly because they do not see the need to learn another method when they could solve the question using the Model Method. How do teachers support students to learn the algebraic method? One way is to guide students see the versatility of the algebraic approach by giving them word problems in which the Model Method may not be the best way to solve the problems. One common type of word problem that could not be solved easily using the Model Method is shown below:

Example 5
I have some sweets. If I give each student in my class six sweets each, I have five sweets left. If I give each of them seven sweets each, I am short of four sweets. How many students do I have?

The elementary solution for Example 5 usually involves students in presupposing certain conditions (we will leave the elementary solution to Example 5 as an exercise for the reader). This is often difficult for many students. However, with algebra, this question is trivial.

Solution
Let x be the number of students in my class.
Then $6x + 5 = 7x - 4$.
Therefore, $x = 9$.

Another way to convince pupils of the need for the algebraic method is to give them a word problem that involves a quadratic equation and let them try to solve

Table 8.1 Parallel presentation of Model Method and Algebraic Method

Model Method	*Algebraic Method*
Let the amount of money Carol received be 1 unit.	Let \$$x$ be the amount of money received by Carol.
[Draw model here. In addition, write letter 'x' in each box representing 1 unit.]	Then Ben had \$$3x$ and Alan had \$$(3x - 20)$.
7 units = \$260 + \$20 (assuming Alan had \$20 more) = \$280 1 unit = \$280 ÷ 7 = \$40 Therefore, Carol received \$40.	Thus $x + 3x + (3x - 20) = 260$ $7x = 260 + 20$ $= 280$ $x = 280 \div 7$ $= 40$ Therefore, Carol received \$40.

using the Model Method. Then they will realise the limitation of the Model Method: it can only be used to solve word problems that involve linear equations or even simultaneous linear equations in two variables, but not those that involve quadratic equations. Therefore, they will have to learn the algebraic method so that they can solve other types of word problems later on.

To tackle the issue of Secondary One pupils having difficulties formulating the equations, the Model Method can serve as a way to smoothen the transition from arithmetic to algebra (Kho et al. 2009, 2014; Ng 2003). Ng (2003) suggests that teachers should support students in making explicit links between the Model Method and its algebraic representation. One way to do this is through a parallel presentation of the two methods (Ng 2003). For example, referring to the question on sharing of money (Example 2), we could present the solution as shown in Table 8.1. Notice that the teacher should write the letter 'x' in each box representing one unit in the model on the left after letting \$$x$ be the amount of money received by Carol. Then, from both the given information in the word problem and from the model, students are led to see that Ben had \$$3x$ and Alan had \$$(3x - 20)$, so that they can see the explicit links between the Model Method on the left and the algebraic method on the right. The critical difference comes in the next step of forming the linear equation in the algebraic method, which has no equivalence in the Model Method. But the teacher can link the next step in the simplification of the linear equation to the Model Method, thus demonstrating to the students that the algebraic method is not very different from the Model Method.

For more examples on parallel presentations, the interested reader should refer to Kho et al. (2009, pp. 115–136). However, it is important to note that there are times in which the unit may not correspond to the unknown in the algebraic representation (Ng 2003). For example, referring to Table 8.2 which shows the parallel presentation for the Tablet Computer Problem (Example 4), we see that the unknown is one unit of boys (=4) in the Model Method, but the unknown is the number of boys b (=16)

Table 8.2 Different unknowns in Model Method and Algebraic Method

Model Method	*Algebraic Method 1:*
[Draw model here] 1 unit of boys + 1 unit of girls = 12 From the model, 1 unit of boys = 40 – 3 × 12 = 4 So there are 4 × 4 = 16 boys, and 40 – 16 = 24 girls. Therefore, 16 girls own tablet computers.	Let b be the number of boys and g be number of girls. Then $\frac{3}{4}b + \frac{2}{3}g = 28$ ------- (1) $b + g = 40$ ------- (2) (1) × 12: $9b + 8g = 336$ ------- (3) (2) × 8: $8b + 8g = 320$ ------- (4) (3) – (4): $b = 16$ Subst. in (2): $16 + g = 40$ $g = 24$ Therefore, 16 girls own tablet computers. *Algebraic Method 2:* Let b be the number of boys. Then there are 40 – b girls. So $\frac{3}{4}b + \frac{2}{3}(40 - b) = 28$ $9b + 8(40 - b) = 28 \times 12$ $9b + 320 - 8b = 336$ $b = 16$ ∴ 40 – 16 = 24 girls own tablet computers.

in the algebraic method. This may cause confusion to students, and teachers should be aware of the different unknowns when making links between the Model Method and the algebra method.

There are several variants of the canonical Model Method introduced by MOE. Many of these methods used a pseudo-Model Method approach, which is actually the algebraic method in disguise. For example, many experienced teachers, who did not learn the Model Method when they were primary school students, often write the following after they have drawn the model in Table 8.1: 1 unit + 3 units + (3 units − \$20) = \$260.

This is *not* the Model Method but the formation of a linear equation in disguise: one that involves ‘1 unit’ as the unknown instead of ‘x’. In fact, we have observed that some trainee teachers, who have learnt the algebraic method after learning the Model Method in primary school, also use this approach of forming a linear equation involving ‘1 unit’ when using the Model Method: somehow, the new knowledge of the algebraic method has interfered with the old knowledge of the Model Method. How such pseudo-Model Methods support or hinder the learning of algebra remains an open question.

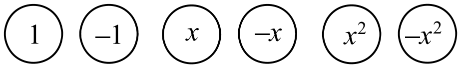
8.3.4 Local Research on Model Method

The Model Method works on the basis that students could translate the textual information given in word problems into a pictorial form, which provides a visual representation of the quantitative relationships involved. However, as Goh (2009) has found, middle-achieving and lower-achieving students may find it difficult to do the transformation, especially for multi-step word problems involving multiplicative relationships, or before–after contexts. Similarly, Poh (2007) also found her lower-achieving students struggling with the interpretation of word problems, transformation of textual information into pictorial forms, understanding the quantitative relationships encapsulated in the models, and seeing the connections between models and solution methods. In both cases, it appeared that students relied heavily on previously taught methods of drawing the models and their familiarity with problem types to work out the solutions. They may not have fully understood the relationships between the quantities and were unable to comprehend and solve the problem when the problem structure is unfamiliar, or when the problem involves an algebraic approach.

These issues suggest that it is necessary for teachers to provide opportunities for students to make sense of the quantitative relationships given in word problems, see the connections between the Model Method and operations involved in the solution, and think about the reasons behind the procedure (Goh 2009; Poh 2007). There may be a need to consider when, and how, the Model Method is introduced, especially for the lower-achieving students (Poh 2007). In particular, teachers should not assume that the translation process is intuitive, and may need to model the thinking process rather than merely presenting the solutions. In addition, teachers may want to open up the solution space of word problems and highlight other solution methods, when appropriate. This is important as the Model Method is not the only heuristic available to the students, even though many students may try to apply the method without understanding the problem.

Despite the various issues and difficulties, students may face when using the Model Method, the prevalence of the Model Method in our classrooms is definitely one of the unique features of our mathematics curriculum. Although the efficacy of the method has never been systematically studied on a large scale, anecdotal evidence suggests that the Model Method provides a way for students to think about quantitative relationships (Ng, S. F., personal communication, 22 November, 2017). It remains to be seen whether such a study would be carried in the future. The findings of such studies may be of interest because algebraic methods of solving simple linear equations involving one variable will be taught under the current syllabus for Primary Six students with effect from 2018. How the Model Method supports or hinders students' learning of algebraic methods may be a fertile area for research.

Fig. 8.8 Algebra discs or AlgeDisc™



8.4 The Singapore AlgeDisc™

While the Model Method has been used extensively in primary schools in Singapore since 1983, the use of AlgeDisc™ in secondary schools only began with Secondary One students three decades later in 2013. The current secondary school mathematics syllabus document (MOE 2012b) stipulates as learning experiences that Secondary One students should have opportunities to 'use algebra discs or the AlgeDisc™ application in AlgeTools™ to make sense of addition, subtraction and multiplication involving negative integers and develop proficiency in the 4 operations of integers' (ibid., p. 34), and Secondary Two students should have opportunities to 'use algebra manipulatives, e.g. algebra discs, to explain the process of expanding the product of two linear expressions of the form $px + q$, where p and q are integers, to obtain a quadratic expression of the form $ax^2 + bx + c$' (ibid., p. 40) and 'use the AlgeDisc™ application in AlgeTools™ to factorise a quadratic expression of the form $ax^2 + bx + c$ into two linear factors where a, b and c are integers' (ibid., p. 40).

Algebra discs or AlgeDisc™ consist of discs as shown in Fig. 8.8. When the '1' disc is flipped over, it will show '-1'. Similarly, when the 'x' and 'x^2' discs are flipped over, they will show '$-x$' and '$-x^2$', respectively. Flipping over only occurs when we take the negative of the number or the term shown on the disc.

AlgeTools™ is a dynamic software produced by the Ministry of Education of Singapore (Yeo et al. 2008). It contains the AlgeBar™ application and the AlgeDisc™ application: the former is used to draw models for the Model Method while the latter is used to draw algebra discs. In this section, we will not use the AlgeDisc™ application but we will just describe how algebra discs can be used to expand and factorise quadratic expressions in the manner specified in the current secondary school mathematics syllabus document (MOE 2012b), followed by some suggestions for research in this area.

8.4.1 Using AlgeDisc™ to Expand and Factorise Quadratic Expressions

We now turn our attention to the use of concrete manipulatives to teach expansion and factorisation of quadratic expressions. Many countries have been, and are still, using algebra tiles to represent quadratic expressions in pictorial form. For example, Fig. 8.9 shows an arrangement of algebra tiles used to represent the quadratic expression $x^2 + 5x + 6$. The large square tile has length x units, so its area is x^2 square units, while each small square tile has length 1 unit, so its area is 1 square unit. Each rectangular tile is of dimensions x units by 1 unit, so its area is x square units. Therefore, the

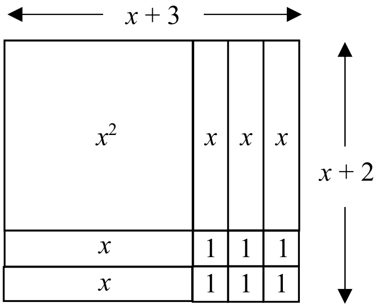


Fig. 8.9 Factorisation of $x^2 + 5x + 6$ using algebra tiles

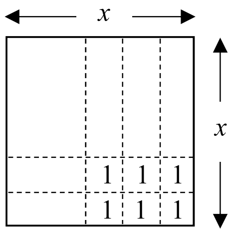


Fig. 8.10 Factorisation of $x^2 - 5x + 6$ using algebra tiles

total area of all the tiles in Fig. 8.9 is $x^2 + 5x + 6$ square units. It is important to take note that the algebra tiles do not have x^2, x or 1 written on them, but what is shown in Fig. 8.9 is for illustration purpose only. In order to factorise $x^2 + 5x + 6$, we have to arrange the tiles to form a rectangle. In this case, the rectangle in Fig. 8.9 has a length of $x + 3$ units and a breadth of $x + 2$ units, so its area is also given by $(x + 3)(x + 2)$, i.e., $x^2 + 5x + 6 = (x + 3)(x + 2)$.

But what happens when it comes to negative terms? For example, how do we use algebra tiles to represent $x^2 - 5x + 6$? Yes, there is a way to do this by covering one side of the x^2 tile with two x tiles, and an adjacent side of the x^2 tile with another three x tiles, and then add six '1' tiles because we subtract 6 twice, as shown in Fig. 8.10. However, many students fail to see why they have to add six '1' tiles.

Algebra discs or AlgeDisc™ do not have this problem because they do not use the concept of area. Figure 8.11 shows the arrangement of algebra discs for factorising $x^2 + 5x + 6$ and $x^2 - 5x + 6$. The arrangement of algebra discs for factorising $x^2 + 5x + 6$ in Fig. 8.11a is very similar to the arrangement of algebra tiles for factorising $x^2 + 5x + 6$ in Fig. 8.9, except that one uses discs while the other uses tiles. On the other hand, the arrangement of algebra discs for factorising $x^2 - 5x + 6$ in Fig. 8.11b is also very similar to the arrangement of algebra discs for factorising $x^2 + 5x + 6$ in Fig. 8.11a. In this way, algebra discs can help students deal with negative terms more easily than the method shown in Fig. 8.10 for algebra tiles.

But the use of the algebra discs is not an end in itself. There is still a need to abstract the algebraic manipulations of expansion and factorisation. To this end, the Ministry of Education of Singapore has introduced what they call the 'multiplication frame' as shown in Fig. 8.12. The multiplication frames in Fig. 8.12 look very similar

(a)

$\times$	x	1	1	1
x	x^2	x	x	x
1	x	1	1	1
1	x	1	1	1

(b)

$\times$	x	-1	-1	-1
x	x^2	$-x$	$-x$	$-x$
-1	$-x$	1	1	1
-1	$-x$	1	1	1

Fig. 8.11 Factorisation of $x^2 + 5x + 6$ and $x^2 - 5x + 6$ using AlgeDisc™

Fig. 8.12 Factorisation of $x^2 + 5x + 6$ and $x^2 - 5x + 6$ using multiplication frame

(a)

$\times$	x	$+3$
x	x^2	$+3x$
$+2$	$+2x$	$+6$

(b)

$\times$	x	-3
x	x^2	$-3x$
-2	$-2x$	$+6$

Fig. 8.13 Factorisation of $x^2 + x - 6$ using multiplication frame

(a)

$\times$	x	
x	x^2	$?x$
	$?x$	-6

$\rightarrow$

(b)

$\times$	x	$+3$
x	x^2	$+3x$
-2	$-2x$	-6

to the arrangement of the algebra discs in Fig. 8.11, which is why it is more natural to progress from algebra discs to the multiplication frame than to the traditional cross multiplication method used to factorise quadratic expressions.

We will now illustrate how students are taught to factorise $x^2 + x - 6$ using the multiplication frame method without the use of algebra discs anymore. Just like the traditional cross multiplication method, students will need to find the corresponding factors of -6, i.e., $-6 = \pm 1 \times \mp 6 = \pm 2 \times \mp 3$. They will start with the multiplication frame shown in Fig. 8.13a, where the coefficients of x, represented by ?, are corresponding factors of -6 such that the sum of the coefficients is $+1$. The students can use guess and check, or some deduction to reason that the corresponding factors of -6 must be $+3$ and -2 since $3 - 2 = 1$. Then, they will obtain the multiplication frame shown in Fig. 8.13b. Therefore, $x^2 + x - 6 = (x + 3)(x - 2)$.

8.4.2 Local Research on AlgeDisc™

Prior to the implementation of AlgeDisc™ with Secondary One students in 2013, the Ministry of Education of Singapore have piloted the use of AlgeDisc™ with some classes and have found that the students benefited from the intervention programme. But they have not published any of their findings. As the use of AlgeDisc™ only began in recent years, there is currently no other local research in this area. For example, both Leong (2015) and Huang (2016) made use of algebra tiles, instead of AlgeDisc™, in their doctoral and Master's study, respectively. The only local paper on the multiplication frame method is a book chapter by Chua (2017), where he described how to use the multiplication frame effectively, without even mentioning algebra discs or AlgeDisc™ at all.

Therefore, we will highlight some issues related to the use of AlgeDisc™ in the teaching and learning of algebra and suggest possible directions for future research. Firstly, algebra discs may help students deal with negative terms better than algebra tiles but students may not understand the idea behind factorisation if there is no concept of area. How will this affect their learning? Secondly, one way to have the best of both worlds is to start with algebra tiles using the concept of area for positive terms and then change to algebra discs for negative terms. But will this be too confusing for students? Thirdly, Leong et al. (2010) have fused algebra tiles and algebra discs to become AlgeCards, which are similar to algebra discs except that they are in the shape of squares and rectangles. Unlike algebra tiles, algebra cards have two sides: on one side is written 1, x or x^2, but on the other side is -1, $-x$ or $-x^2$, respectively. Thus, AlgeCards retain the concept of area for positive terms, but for negative terms, the students can just use the other sides of the cards. However, a problem may arise if we have, e.g. x^2 and y^2 terms: the square cards will have to be of different sizes, but will this be confusing for students? For AlgeDisc™, this is not an issue as all the discs have the same size. Nevertheless, Leong et al. had tried out AlgeCards with some schools and found them to be effective, but there was no comparison with the other two types of manipulatives.

As we can see, the main issue of using manipulatives for learning algebra is the evaluation of the effectiveness of such methods. This is an unexplored area of research, at least in Singapore. One or more of the following research questions can frame research in the use of such manipulatives when learning and teaching algebra:

1. Do the combined use of algebra discs with algebra tiles develop both students' procedural skills and conceptual understanding?
2. Is the use of more than one type of manipulatives confusing for students? If so, why?
3. What can we say about the effectiveness of the three types of manipulatives—algebra tiles, algebra discs and algebra cards—in the learning and teaching of algebra?

8.5 Guided-Discovery Learning and Investigation

The use of AlgeDisc™ in the above manner described by the learning experiences in the current secondary school mathematics syllabus document (MOE 2012b) is to guide students to discover certain mathematical concepts or skills. In fact, many of these learning experiences make use of Bruner's (1961) guided-discovery learning, which is another distinctive feature of local classroom practices although it is not unique to Singapore. Even before the stipulation of learning experiences in the current primary and secondary school mathematics syllabus documents (MOE 2012a, b), MOE-approved textbooks have already been using activities or investigation to guide students to discover mathematical concepts or skills.

Ernest (1991) contrasted the differences among three inquiry methods for teaching mathematics, namely, problem solving, guided discovery and investigation. Guided discovery is different from mathematical investigation (Jaworski 1994) in that guided discovery is like trail-blazing to a desired location while investigation is like exploring an unknown land where 'the journey, not the destination is the goal' (Pirie 1987, p. 2). But Yeo and Yeap (2010) distinguished between the process of investigation and the activity of using an open investigative task to investigate. As a process, investigation involves examining specific examples or special cases (i.e., specialising) in order to generalise; it is an inductive process, in contrast to the use of deductive reasoning. So Yeo and Yeap argued that problem solving and guided discovery-learning utilise the process of investigation if students specialise by using specific examples, instead of trying to solve the problem by using a deductive approach. On the other hand, an investigative approach to teaching and learning mathematics involves the use of open investigative tasks where students are free to explore and pose any problems to investigate or solve (Ernest 1991). However, the latter approach is seldom used in Singapore schools (Yeo 2013).

Guided-discovery learning can be traced to Bruner (1961). Because guided-discovery learning usually starts with specific examples for students to investigate, these examples become concrete experiences for students to abstract the mathematical concept. In other words, the underlying principle behind guided-discovery learning is still the C-P-A approach or the Extended C-P-A approach, which again is based on Bruner's (1964, 1966) idea of enactive–iconic–symbolic representations of cognitive growth discussed in Sect. 8.2. Because the term 'guided-discovery learning' is from the perspective of the teacher, some textbooks use the term 'investigation' while others use the term 'activity' to describe this kind of investigation or activity that the students will do. In what follows, we will describe two exemplars of guided-discovery learning using concrete manipulatives or concrete examples (guided-discovery learning using virtual manipulatives will be discussed in Chap. 14 while the problem-solving approach has already been dealt with in Chap. 7 of this book).

In secondary schools, paper folding can be used to guide students to discover that the perpendicular bisector of a chord will always pass through the centre of a circle. Figure 8.14 shows part of an investigation in a school textbook by Yeo et al.

6. Using the same circle as in Question 5, fold along a chord *AB* that is not a diameter of the circle and then fold it into two equal halves as shown in Fig. 11.3(**c**).

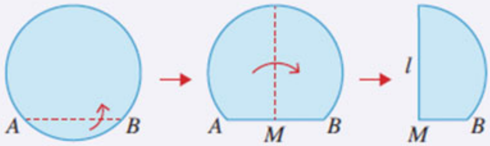


Fig. 11.3(c)

Open up the paper as shown in Fig. 11.3(**d**), where the dotted lines indicate the lines obtained from the above paper folding.

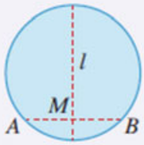


Fig. 11.3(d)

As the paper is folded into two equal halves, the line l bisects the chord *AB* and $\angle AMB$. Since $\angle AMB = 180°$, l is perpendicular to the chord *AB*.

(a) Which two of the three conditions on page 361 are satisfied?

(b) Does the line l pass through the centre *O* of the circle that you have marked in Question 5?

Fig. 8.14 Textbook investigation on symmetric property of circle (reproduced with permission from Shing Lee Publishers Pte Ltd.)

(2015). Prior to what is shown in Fig. 8.14, the students have already folded the paper circle to mark out the centre of the circle. Then, the students will follow the steps in Fig. 8.14 to discover the said symmetric property of a circle.

At higher level mathematics where there may not be any suitable concrete or virtual manipulatives to use, we can just use concrete examples. For secondary school mathematics, these concrete examples usually make use of numerical values. For example, to guide students to discover the product law of logarithms, the teacher can set up a table of values of x and y and get the class to use a calculator to evaluate $\lg x + \lg y$ and $\lg xy$ for different values of x and y. In this manner, the students will discover that $\lg x + \lg y = \lg xy$. This particular guided-discovery learning can be found in the school textbook by Yeo et al. (2013a). Only after gaining some concrete experiences of the product law, the students will then be guided at a later stage to prove it because, according to C-P-A, what is abstract (i.e. the product law) should be developed later.

Although there is some research from other countries on inquiry learning such as guided-discovery learning (Franke et al. 2007; Hunter et al. 2016), there is a dearth of local research in this area. From the online repository of the National Institute of Education (NIE), which is the sole teacher education institution in Singapore, there are no Ph.D. dissertations or Master's theses that are directly related to guided-discovery learning. But there are some local Master's theses that compared the use of ICT and traditional teacher-directed teaching, and the pedagogy behind the use of ICT is actually guided-discovery learning. Even though most of these studies, which will be described in Chap. 14 of this book, suggest the effectiveness of ICT and guided-discovery learning over traditional teacher-directed teaching, it is not known which of these two variables (i.e. ICT and guided-discovery learning) may be the cause of the effectiveness of the intervention programme in these studies. Therefore, we suggest the following questions for future research.

1. How does a guided-discovery approach compare with a teacher-directed approach for teaching mathematics?
2. What are some of the pedagogical principles for teachers to think about when they use a guided-discovery learning approach?
3. What is the role of ICT in guided-discovery learning, especially in light of the country's move towards a more ICT-integrated learning environment?

8.6 Engaging the Hearts of Mathematics Learners: LOVE Mathematics Framework

While it is essential to engage the minds of mathematics learners through various pedagogies described in the previous sections, it is also important to engage their hearts because several studies have demonstrated that students' emotions have a profound influence on learning. For instance, students' epistemic emotions triggered by cognitive problems are critical when learning with new non-routine tasks (Pekrun 2014). However, the study of the affective domain is complicated partly because there is no common agreement on the definitions of terms, and partly because affective constructs are more difficult to describe and measure than cognition (McLeod 1992). Aiken (1972) used the term attitude to mean 'approximately the same thing as *enjoyment*, *interest*, and to some extent, *level of anxiety*' (p. 229) while Hart (1989) used the word attitude towards an object as a general term to refer to emotional (or affective) reactions to the object, behaviour towards the object and beliefs about the object. But Simon suggested the use of affect as a more general term in 1982 (cited in McLeod 1992), and McLeod (1989) further divided the affective variables into three categories: beliefs, attitudes, and emotions.

To measure affective constructs, the traditional paradigm for research on affect often relied on questionnaires and quantitative methods (McLeod 1992). But most of these studies focused on students' existing attitude and their effect on other variables such as test performance. There are very few intervention studies, such as on how to change students' attitude (Yeo 2018a). Moreover, the literature on affective education is mainly confined to affective variables in general, such as improving students' personal development and self-esteem, interpersonal relationships and social skills, and their feelings about themselves as learners and about their academic subjects (Lang et al. 1998), but when it comes to making mathematics lessons interesting and helping students to appreciate mathematics, there is not much literature on this (Yeo 2018a).

Some common issues that teachers face when trying to make mathematics lessons interesting are (a) not every student will find the same thing fascinating; (b) it is not possible to make every part of a lesson engaging; and (c) the enjoyment does not necessarily translate to learning. To address these issues, Yeo (2018a) proposed the LOVE Mathematics framework (Linking Opportunities in a Variety of Experiences to the learning of Mathematics) to engage the hearts of mathematics learners. The framework consists of three principles: variety, opportunity and linkage. Firstly, students have different tastes, so there is a need to use a variety of resources to interest different students in the hope that all the students will find something intriguing, although that 'something' may be different for different students. Secondly, there is actually no need to make every part of every lesson engaging. Yeo suggested that teachers should just provide ample opportunities to engage their students, e.g. in at least one part of most mathematics lessons. Thirdly, Yeo believed that the main purpose of engaging the hearts of students is not just to make them laugh and have fun, but to link the resources to the learning of mathematics.

In this section, we will use the lens of the LOVE Mathematics framework to illustrate how mathematics teachers in Singapore use a variety of resources to provide opportunities for students to engage in mathematical sense making, although these resources may not be unique to Singapore.

8.6.1 Variety Principle

In Singapore, mathematics teachers use a variety of different resources to heighten students' interest in the subject. Types of resources used include songs such as the 'Polygon Song' (see Yeo et al. 2013b; Yeo 2018b), television shows such as *NUMB3RS*, story books such as 'The Doorbell Rang' by Pat Hutchins and 'How Big is a Foot' by Rolf Myller, drama, magic tricks, puzzles, and games. Although the choice of these resources may seem eclectic at times, the idea of using different types of resources beyond mathematical tasks is supported by the Theory of Multiple Intelligences (Gardner 2006). For example, there has also been research to suggest that different parts of the brain are stimulated when children engage in dramatic plays (Hough and Hough 2012). Studies similar to Hough and Hough (2012) form the basis of programmes such as Teaching Through the Arts Programme (TTAP), which was initiated by the National Arts Council (NAC) (National Arts Council 2017; Yuen 2016). Several schools have benefitted from this programme, in which they integrated drama into the teaching of perimeter and area with inputs from drama educators. These schools have found it a fun and engaging approach to teaching.

Similarly, Yeo (2018a) described a lesson where the teacher showed the class a 10-min excerpt of *Splash Splash Love*, a Korean drama, with English subtitles. The teacher chose the excerpt because there was an incident when the king and his subjects were unable to solve a mathematical problem but a student called Dan Bi solved it for them using Pythagoras' theorem. However, there was really not much linkage in the drama to the learning or application of the theorem. So the teacher designed three problems for the class to do. These three problems contain contexts from the drama, e.g. the first problem described how the king shot a deer across the river and wanted his hunting trip to be recorded in history, so he needed Dan Bi to work out the distance the arrow travelled, which the students had to calculate using Pythagoras' theorem. Then, the students were assigned homework from the textbook, which consisted of typical questions on Pythagoras' theorem. The teacher was surprised when one of her students, who had not been handing in homework on time, unexpectedly handed in her homework punctually the following day. It seems that the student was motivated enough by the Korean drama that she even did the routine homework promptly. Although the evidence base is largely anecdotal, teachers' implementation in schools suggests that using a variety of resources is more likely to engage and motivate students. Whether, and if so, how these varieties of resources help support students in learning mathematical concepts will be an interesting area of research for mathematics educators.

8.6.2 Opportunity Principle

As highlighted in the *Splash Splash Love* example, using a variety of resources alone is not sufficient for engaging the hearts of students. Instead, teachers need to incorporate these resources meaningfully into their lessons to provide more opportunities for students to make sense of mathematics through these resources. The idea is to embed different types of resources into different parts of a lesson. For example, one could use a storybook or a mathematics trick to motivate the study of a topic, such as fraction; use a movie clip to illustrate the use of fractions, or explain the operations involving fractions, and use games or plan questions around a video clip to encourage students to practise the skills taught. It may not be realistic to play an entire show or movie during a lesson but clips of 5–10 min, showing excerpts from the shows or movies focusing on a particular concept would allow the teacher to capitalise on its affordance without too much intrusion on the curriculum time. Realising the affordance of such resources can potentially open up new possibilities to engage the hearts of students and possibly enhance students' learning.

In a small-scale local study conducted by Lim et al. (2014), Mr. Fu Siqiang, a teacher of Fairfield Methodist School (Primary), used a rope trick to illustrate that average is a representative value of the set of items. He started with three ropes of different lengths, manipulated them into three ropes of the same length to teach the concept of average, and then changed them back into three ropes of different lengths to emphasise that the lengths of the ropes do not actually change when we take the average of the lengths of the ropes. The rope trick was carried out in two Primary Five classes. The quiz results of students who saw this rope illustration and students who did not were compared. Students who saw the rope trick performed significantly better on the quiz requiring them to find the average of given sets of values and they were also more engaged during the lesson. Their findings concur with other studies which highlighted that the use of magic tricks may enthuse students and provide more opportunities for students to dig deeper into the concepts presented (Koirala and Goodwin 2000; Lesser and Glickman 2009). However, it remains to be seen whether there is an optimum structure in which these resources are sequenced within a lesson. More importantly, the design decisions surrounding the choice and implementation of such resources deserve more attention in research.

8.6.3 Linking to Mathematics

Another important insight gained from the use of resources, such as the *Splash Splash Love* example, revolves around the importance of designing tasks around these resources to connect students to the mathematics concepts. Without this critical connection to mathematics, it would be difficult for teachers to go beyond making lessons fun to making lessons effective. To illustrate this principle, we refer to Yeo (2018b), in which he described the use of an amusing video found on YouTube called '25 divided by 5 equals 14'. The video clip shows three erroneous proofs that 25 divided by 5 is equal to 14: the first proof is by division, the second one by multiplication and the last one by addition. Although most students will laugh at the slapstick humour in the video, the third principle of the LOVE Mathematics framework suggests the need for the teacher to get the students to explain why the proofs are incorrect, namely, the issue of the wrong place value of some of the digits. In this manner, the teacher can link the video to the learning of mathematics: the importance of the place value system. Another illustration of this principle is the use of a movie snippet from *NUMB3RS*, in which Charlie Eppe explained the classic The Monty Hall problem. The teacher could first ask students to answer the question based on their intuition or current understanding of probability before working on the problem to prove (or disprove) their decision. The students could then watch the show where Charlie explained the problem.

Here, we argue that the critical aspect of using different resources to engage the hearts of our learners lies in how teachers design tasks around these resources to provide opportunities for students to engage in mathematical processes. For instance, appropriate games can provide meaningful situations for students to (1) practice their mathematical skills, (2) develop mathematical thinking, (3) test out their intuitive ideas and problem-solving strategies, (4) communicate and reason mathematically through the actions and decisions they make during a game, and (5) develop positive attitudes towards mathematics (Burns 2003; Davies 1995). But how teachers can design these tasks around different resources has not been well studied in Singapore. This is certainly an important area for research for mathematics educators.

8.6.4 Local Research on Engaging the Hearts of Mathematics Learners

Although there are some research done in other countries on affective education in the mathematics classroom, there is a dearth of local research in this area. To gain a more comprehensive understanding of how affect may influence the learning of mathematics, we suggest the following issues for investigation.

Firstly, there is a need for more rigorous studies to evaluate the effectiveness of the above-mentioned resources in motivating local students to learn mathematics and in enhancing their learning. The main problem lies in the measurement of students' motivation and interest: Do we get the students to fill in a questionnaire? Or do we base our judgements on the teacher's observation of their enthusiasm in class? Or do we use a combination of methods? Another important issue regarding measurement is the measurement of students' learning of the subject matter: Do we use the usual class test based on procedural knowledge and skills? Or one that tests on conceptual understanding as well?

Secondly, there is a need to study why these interventions work. Knowing what works, and why it works, is critical for teachers to implement these strategies effectively. Several questions arise with regard to the implementation of these strategies or interventions:

1. Whether the duration of the intervention matters, and if so, how long does it take for the intervention to take effect?
2. Do these strategies have any lasting effect on students' motivation, and more importantly, on students' learning?
3. What are the factors that affect the effectiveness of such strategies? For example, does the types of resources used, the topics to be taught, the belief and knowledge of individual students, or the teacher make a difference?

What are the pedagogical principles that can be derived from these interventions? Knowing these principles may help us develop more targeted strategies for the different students.

8.7 Conclusion

This chapter reviews the key pedagogical innovations that have been implemented, and researched on, in Singapore classrooms. There seems to be limited research evidence supporting the use of such innovations to improve mathematics learning in the local context. However, it is not simply about knowing whether a particular intervention works. But rather, it is crucial for mathematics educators to know the conditions for such interventions to work. As discussed in this chapter, there are three main research problems on the design and use of pedagogical innovations in Singapore:

1. The effectiveness, and the underlying theoretical perspectives, of these innovations;
2. The measurement of effectiveness of these innovations; and
3. The design and development of such innovations for the variations encountered in the different classroom contexts.

As argued persuasively by Lewis (2015), the idea of improvement science (Langley et al. 2009) may be useful for us to consider. The main difficulty of using experimental approaches to investigate the effectiveness of pedagogical innovations lies in the need to control for the different variables in classroom practices. Minimising variations in an experimental setup is unlikely to ensure the transferability of innovations because classrooms are complex ecological systems. As highlighted by Bryk (2015),

> Such studies, however, are not primarily designed to tell us what it will take to make the intervention work for different subgroups of students and teachers or across varied contexts. At base here is *the difference between knowledge that something can work and knowledge of how to actually make it work reliably over diverse contexts and populations*. Yet the latter is what practitioners typically want to know—what will it take to make it work for me, for my students, and in my circumstances? Unfortunately, policy actors who see evidence-based practice as today's answer typically miss this critical distinction. (p. 469)

The way forward is to accept the challenge of making 'this critical distinction' as we implement pedagogical innovations, and as we design and develop new ones. An improvement science approach distinguishes two types of knowledge essential for improving teaching practices: knowledge about the discipline of mathematics education (Lewis 2015), and 'a system of profound knowledge' needed to enact basic disciplinary knowledge within organisations (Deming, cited in Langley et al. 2009, p. 75). This system of profound knowledge is structured around 'knowledge of systems, psychology, variations, and how knowledge grows' (Lemire et al. 2017, p. 24). According to Langley et al. (2009), the improvement science approach, which consists of rapid cycles of *plan-do-study-act* (PDSA), is framed by three key questions:

1. What are we trying to accomplish?
2. How will we know that a change is an improvement?
3. What change can we make that will result in an improvement?

There is evidence to suggest that what teachers and mathematics educators have been doing in Singapore seems to improve students' learning and motivation to some extent. However, questions regarding its effectiveness and transferability to other contexts remain. Research into improving mathematics learning and teaching through an improvement science paradigm may be one way to address these issues. Building on the good work done in Singapore, we suggest that it is time for us, as a community of inquiry, to look deeply into the design and implementation of pedagogical innovations so that we can learn how these innovations can be applied through a variety of contexts.

References

Aiken, L. R. (1972). Research on attitudes toward mathematics. *Arithmetic Teacher, 19,* 229–234.

Bruner, J. S. (1961). The act of discovery. *Harvard Educational Review, 31,* 21–32.

Bruner, J. S. (1964). The course of cognitive growth. *American Psychologist, 19,* 1–15.

Bruner, J. S. (1966). *Toward a theory of instruction*. MA: Harvard University Press.

Bryk, A. S. (2015). Accelerating how we learn to improve. *Educational Researcher, 44*(9), 467–477.

Burns, M. (2003). Using games in your math teaching. *Connect Magazine, 17*(2), 1–4.

Chang, S. H., Lee, N. H., & Koay, P. L. (2017). Teaching and learning with concrete-pictorial-abstract sequence—A proposed model. *The Mathematics Educator, 17*(1), 1–28.

Choy, B. H. (2013). Productive mathematical noticing: What it is and why it matters. In V. Steinle, L. Ball, & C. Bardini (Eds.), *Proceedings of the 36th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 186–193). Melbourne: MERGA.

Chua, B. L. (2017). Empowering learning in an algebra class: The case of expansion and factorisation. In B. Kaur & N. H. Lee (Eds.), *Empowering mathematics learners* (Association of Mathematics Educators 2017 Yearbook, pp. 9–29). Singapore: World Scientific.

Davies, B. (1995). The role of games in mathematics. *Square One, 5*(2), 7–17.

Ernest, P. (1991). *The philosophy of mathematics education*. London: Falmer Press.

Franke, M. L., Kazemi, E., & Battey, D. (2007). Understanding teaching and classroom practice in mathematics. In J. Frank & K. Lester (Eds.), *Second handbook of research on mathematics teaching and learning* (pp. 225–256). Charlotte, NC: Information Age Publishing.

Gardner, H. (2006). *Multiple intelligences: New horizons*. New York: Basic Books.

Goh, S. P. (2009). *Primary 5 pupils' difficulties in using the Model Method for solving complex relational word problems* (Unpublished master's thesis). National Institute of Education, Nanyang Technological University, Singapore.

Hart, L. E. (1989). Describing the affective domain: Saying what we mean. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving: A new perspective* (pp. 37–45). New York: Springer.

Hough, B. H., & Hough, S. (2012). The play was always the thing: Drama's effect on brain function. *Psychology, 3*(6), 454–456.

Huang, Y. (2016). *Effects of crafted video lessons incorporating multi-modal representations on learning of factorization of quadratic expressions* (Unpublished master's thesis). National Institute of Education, Nanyang Technological University, Singapore.

Hunter, R., Hunter, J., Jorgensen, R., & Choy, B. H. (2016). Innovative and powerful pedagogical practices in mathematics education. In K. Makar, S. Dole, J. Visnovska, M. Goos, A. Bennison, & K. Fry (Eds.), *Research in mathematics education in Australasia 2012–2015* (pp. 213–234). Singapore: Springer.

Jaworski, B. (1994). *Investigating mathematics teaching: A constructivist enquiry*. London: Falmer Press.

Kho, T. H. (1987). *Mathematical models for solving arithmetic problems.* Paper presented at the 4th Southeast Asian Conference on Mathematics Education, Singapore.

Kho, T. H., Yeo, S. M., & Fan, L. (2014). Model method in Singapore primary mathematics textbooks. In K. Jones, C. Bokhove, G. Howson, & L. Fan (Eds.), *Proceedings of the International Conference on Mathematics Textbook Research and Development (ICMT-2014)* (pp. 275–282). Southampton: University of Southampton.

Kho, T. H., Yeo, S. M., & Lim, J. (2009). *The Singapore Model Method for learning mathematics*. Singapore: EPB Pan Pacific.

Kintsch, W., & Greeno, J. G. (1985). Understanding and solving arithmetic word problems. *Psychological Review, 92*(1), 109–129.

Koirala, H. P., & Goodwin, P. M. (2000). Teaching algebra in the middle grades using math magic. *Mathematics Teaching in the Middle School, 5*(9), 562–566.

Lang, P., Katz, Y. J., & Menezes, I. (Eds.). (1998). *Affective education: A comparative view*. London: Cassell.

Langley, G. J., Moen, R. D., Nolan, K. M., Nolan, T. W., Norman, C. L., & Provost, L. P. (2009). *The improvement guide: A practical approach to enhancing organizational performance*. San Francisco: Jossey-Bass.

Leach, G., Hunter, R., & Hunter, J. (2014). Teachers repositioning culturally diverse students as doers and thinkers of mathematics. In J. Anderson, M. Cavanagh, & A. Prescott (Eds.), *Proceedings of the 37th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 381–388). Sydney: MERGA.

Lemire, S., Christie, C. A., & Inkelas, M. (2017). The methods and tools of improvement science. In C. A. Christie, M. Inkelas, & S. Lemire (Eds.), *Improvement science in evaluation: Methods and uses. new directions for evaluation* (pp. 23–33).

Leong, S. L. (2015). *Effects of a mathematics instructional sequence on the conceptual and procedural understanding of algebraic expressions for secondary students with mathematics difficulties* (Unpublished doctoral dissertation). National Institute of Education, Nanyang Technological University, Singapore.

Leong, Y. H., Ho, W. K., & Cheng, L. P. (2015). Concrete-Pictorial-Abstract: Surveying its origins and charting its future. *The Mathematics Educator, 16*(1), 1–18.

Leong, Y. H., Yap, S. F., Teo, M. L., Thilagam, S., Karen, I., Quek, E. C., & Tan, K. L. K. (2010). Concretising factorisation of quadratic expressions. *The Australian Mathematics Teacher, 66*(3), 19–24.

Lesser, L. M., & Glickman, M. E. (2009). Using magic in the teaching of probability and statistics. *Model Assisted Statistics and Applications, 4,* 265–274.

Lewis, C. (2015). What is improvement science? Do we need it in education? *Educational Researcher, 44*(1), 54–61.

Lim, P., Fu, S., & Foong, P. H. S. (2014). Magic tricks in the teaching of the arithmetic mean. In J. Vincent, G. FitzSimons, & J. Steinle (Eds.), *Proceedings of the 51st Mathematical Association of Victoria Annual Conference: Maths Rocks* (pp. 94–101). Victoria, Australia: MAV.

Lowrie, T., & Jorgensen (Zevenbergen), R. (Eds.). (2015). *Digital games and mathematics learning: Potential, promises and pitfalls.* Dordrecht, The Netherlands: Springer.

Marshman, M., & Brown, R. (2014). Coming to know and do mathematics with disengaged students. *Mathematics Teacher Education and Development, 16*(2), 71–88.

McLeod, D. B. (1989). Beliefs, attitudes, and emotions: New views of affect in mathematics education. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving: A new perspective* (pp. 245–258). New York: Springer.

McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575–596). New York: MacMillan.

Ministry of Education of Singapore. (2007). *Mathematics syllabus: Primary.* Singapore: Curriculum Planning and Development Division.

Ministry of Education of Singapore. (2012a). *Mathematics syllabus: Primary.* Singapore: Curriculum Planning and Development Division.

Ministry of Education of Singapore. (2012b). *Mathematics syllabus: Secondary.* Singapore: Curriculum Planning and Development Division.

Murphy, C. (2015). Changing teachers' practices through exploratory talk in mathematics: A discursive pedagogical perspective. *The Australian Journal of Teacher Education, 40*(5), 61–84.

National Arts Council. (2017). *Teaching through the arts programme.* Retrieved from https://aep.nac.gov.sg.

Nesher, P., Greeno, J. G., & Riley, M. S. (1982). The development of semantic categories for addition and subtraction. *Educational Studies in Mathematics, 13*(4), 373–394.

Ng, S. F. (2003). How Secondary Two Express stream students used algebra and the Model Method to solve problems. *The Mathematics Educator, 7*(1), 1–17.

Ng, S. F., & Lee, K. (2005). How Primary Five pupils use the Model Method to solve word problems. *The Mathematics Educator, 9*(1), 60–83.

Ng, S. F., & Lee, K. (2009). The Model Method: Singapore children's tool for representing and solving algebraic word problems. *Journal for Research in Mathematics Education, 40*(3), 282–313.
Pekrun, R. (Ed.) (2014). *Emotions and learning* (Vol. 24). Switzerland: UNESCO.
Pirie, S. (1987). *Mathematical investigations in your classroom: A guide for teachers.* Basingstoke, England: Macmillan.
Poh, B. K. (2007). *Model method: Primary three pupils' ability to use models for representing and solving word problems* (Unpublished master's thesis). National Institute of Education, Nanyang Technological University, Singapore.
Shanmugaratnam, T. (2005). *Teach less learn more (TLLM).* Speech by Mr Tharman Shanmugaratnam, Minister of Education, at the MOE Work Plan seminar 2005. Singapore: National Archives of Singapore.
Sullivan, P., & Davidson, A. (2014). The role of challenging mathematical tasks in creating opportunities for student reasoning. In J. Anderson, M. Cavanagh, & A. Prescott (Eds.), *Proceedings of the 37th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 605–612). Sydney: MERGA.
Wong, K. Y. (2015). *Effective mathematics lessons through an eclectic Singapore approach* (Association of Mathematics Educators 2017 Yearbook, pp. 219–248). Singapore: World Scientific.
Yeo, J. B. W. (2013). *The nature and development of processes in mathematical investigation* (Unpublished doctoral thesis). National Institute of Education, Nanyang Technological University, Singapore.
Yeo, J. B. W. (2018a). Engaging the hearts of mathematics learners. In P. C. Toh & B. L. Chua (Eds.), *Mathematics instruction: Goals, tasks and activities* (Association of Mathematics Educators 2018 Yearbook, pp. 115–132). Singapore: World Scientific.
Yeo, J. B. W. (2018b). *Maths songs, videos and games for secondary school maths.* Retrieved from http://math.nie.edu.sg/bwjyeo/videos.
Yeo, J. B. W., Teh, K. S., Loh, C. Y., & Chow, I. (2013a). *New syllabus additional mathematics* (9th ed.). Singapore: Shinglee.
Yeo, J. B. W., Teh, K. S., Loh, C. Y., Chow, I., Neo, C. M., & Liew, J. (2013b). *New syllabus mathematics 1* (7th ed.). Singapore: Shinglee.
Yeo, J. B. W., Teh, K. S., Loh, C. Y., Chow, I., Ong, C. H., & Liew, J. (2015). *New syllabus mathematics 3* (7th ed.). Singapore: Shinglee.
Yeo, J. B. W., & Yeap, B. H. (2010). Characterising the cognitive processes in mathematical investigation. *International Journal for Mathematics Teaching and Learning*. Retrieved from http://www.cimt.org.uk/journal.
Yeo, S. M., Thong, C. H., & Kho, T. H. (2008, July). *Algebra discs: Digital manipulatives for learning algebra*. Paper presented at the 11th International Congress on Mathematics Education, Monterrey, Mexico.
Yuen, S. (2016, March 14). Drama-based teaching on the rise. *The Straits Times.* Retrieved from www.straitstimes.com.

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