Chapter 7
Problem Solving in the Singapore School Mathematics Curriculum

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Abstract Problem solving has been the heart of the Singapore school mathematics curriculum since the early 1990s after being adopted as the goal of school mathematics education. Since its adoption, it has captured the interest of many Singapore educators and researchers. It appears that problem solving will continue to be a very active research area since there is great interest in the very high level of performance of Singapore students in international comparative studies such as TIMSS and PISA. This chapter begins with a re-categorization of the research work done to date on problem solving in Singapore using the Singapore mathematics curriculum framework by integrating two classifications done by Foong in 2009 and Chan

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in 2014, respectively, and including work done since 2011 that was not reported in either survey. The earlier research focused on addressing the readiness of students for mathematical problem solving (MPS) from the perspective of the Singapore mathematics curriculum framework; the later research tended to emphasize the enactment of MPS in the Singapore mathematics classroom and teacher education. This chapter gives more detail to this later research with an emphasis on the enactment of Pólya's stages in solving structured problems.

Keyword Mathematical problem solving · Pólya's model · Pre-service teacher education · Real-world context

7.1 Mathematical Problem Solving and the Singapore School Mathematics Curriculum

The central goal of the school mathematics curriculum in Singapore is mathematical problem solving (MPS), as reflected in the School Mathematics Curriculum Framework shown in Fig. 3.1 in Chap. 3. MPS has remained as the central goal of the curriculum since its inception in the early 1990s in spite of the changing educational landscape over the decades.

The curriculum documents across several revisions (e.g. Ministry of Education (MOE) 1990, 2006) describe problem solving in terms of what it encompasses rather than as a definition of what problem solving is.

> Mathematical problem solving includes using and applying mathematics in practical tasks, in real life problems and within mathematics itself. In this context, a problem covers a wide range of situations from routine mathematical problems to problems in unfamiliar contexts and open-ended investigations that make use of the relevant mathematics and thinking processes.
>
> (MOE 1990, p. 6)
>
> Mathematical problem solving is central to mathematics learning. It involves the acquisition and application of mathematics concepts and skills in a wide range of situations, including non-routine, open-ended and real-world problems.
>
> (MOE 2006, p. 3)

7.2 Why Research on MPS?

At the beginning of this millenium, Dong et al. (2002) did a short survey on more than 100 students from five junior colleges across Singapore on their readiness to handle non-routine problems in mathematics. The preliminary study showed that there was a considerable difference in the achievement among the students on routine and non-routine problems. These students lacked skills and techniques which are considered crucial for MPS. Moreover, MPS at the primary school level in Singapore is usually equated with solving word problems which constitute at least 60% of the high-stakes national mathematics examination taken at Primary Six (sixth grade) (Lee 2014). The close link between assessment and curriculum led to the belief that word problems are the focus in the Singapore primary mathematics curriculum and has led many primary schools to teach problem solving by teaching word problems. In addition, although the international comparative studies PISA and TIMSS have shown that Singapore students generally have achieved a very high level of competence in school mathematics, an in-depth analysis of the results of these studies has also noted a relatively weaker performance of Singapore students on solving unfamiliar problems (Kaur 2009).

As evidence from research suggests that Singapore students might not be very well-prepared in handling non-routine problems, it is not surprising that research in Singapore on MPS in school mathematics in order to support classroom practice or inform curricular policy with research-based evidence continues to receive much attention in the Singapore context. This was also evident in Teong et al. (2009). The study included more than 150 mathematics lessons from several primary and secondary schools in Singapore, with this as one of the findings:

> [teachers] generally read the problems, executed the solution and checked the answers. There was very little dwelling on the exploration or the planning aspect of the solutions. … The emphasis appeared to be more to address the skills and procedures needed to solve problems than to tackle fresh problems anew where students have more chance of grappling with understanding and thinking about how to solve the problems. (p. 84)

Two reviews of MPS research have been carried out thus far. Foong (2009) categorized the research from the 1980s up to 2007 into three broad strands, namely (a) approaches and tasks, (b) teachers' beliefs and practices, and (c) students' problem solving behaviours. Chan (2015) looked at MPS involving students from 2001 to early 2011. He categorized the research into six broad categories as follows: (a) model-drawing method, (b) choice of heuristics, (c) open-ended and real-world problems, (d) metacognition, (e) sense-making, and (f) affective domain. This chapter summarises the various MPS research from the two reviews and adds details on more comprehensive research on MPS carried out recently since the later part of the last decade.

Numerous local studies (e.g. Foong et al. 1996; Foong 2009) indicate that up to 2009, MPS was still mostly theoretical talk and not common in classroom enactments. Foong et al. (1996) reported that some mathematics teachers have expressed their inadequacy in implementing the intended curriculum for MPS. Alan Schoenfeld wrote in the 2007 special issue on problem solving of the journal ZDM that the prevailing focus should lie in translating decades of theory building about problem solving into workable practices in the classroom:

> That body of research—for details and summary, see Lester (1994) and Schoenfeld (1985, 1992)—was robust and has stood the test of time. It represented significant progress on issues of problem solving, but it also left some very important issues unresolved. … The theory had been worked out; all that needed to be done was the (hard and unglamorous) work of following through in practical terms. (Schoenfeld 2007, p. 539)

This has spurred the interest of researchers in the Singapore National Institute of Education (NIE) to research on the feasibility of doing the “hard and unglamorous” work of realising the ideals of mathematical problems solving—as envisioned to be at the heart of the Singapore mathematics curriculum—into the daily practices of mathematics classrooms. This has opened a new dimension of research, i.e. in enacting the problem solving curriculum in mathematics classrooms.

7.3 Studies on MPS from 2001 to 2011

Figure 7.1 summarizes the research related to MPS from Foong (2009) and Chan (2015). The categorization provides a clearer depiction of the current state of research compared to the previous reviews in that it takes into account some of the overlapping strands for which the previous reviews did not capture and consolidated only those from 2001 onwards. Moreover, several studies recently located were appended to the list and the list of research is sequenced chronologically from 2001 to get a sense of the different strands of problem solving research carried out in the twenty-first century.

Figure 7.1 shows that studies involving problem solving heuristics are popular among researchers with 13 studies located. This is followed by 11 studies related to open-ended problems including solving real-world and modelling problems. Other strands like cognition and sense making, metacognition, affect and ICT make up about five studies each. Studies on the affective domain began from 2005 while studies related to ICT appeared to have dwindled after 2004 with only one study in 2009.

From 2009 onwards, there is a surge of research publications on MPS under the category of problem solving curriculum. This is an indication of a new focus from 2009 onwards on enacting MPS in the mathematics classroom and is a response to the disturbing observation that MPS is seldom realized in the classroom.

At that time, the authors, who were either teaching mathematics content or mathematics pedagogy at the tertiary level and who had vast school teaching experience prior to joining the university, felt strongly that mathematical problem solving had not been enacted according to its true spirit in school mathematics classrooms. The processes of problem solving were not stressed sufficiently because such processes were not eventually assessed in the high-stakes national assessment. Extensive anecdotal evidence suggested that teachers were mainly focused on preparing their students for examinations by equipping them with the ability to solve a fixed repertoire of exam-type problems. In fact, even challenging problems and problems that involve application in the real world were “routinized”—taught in a way that they eventually became routine—for the students.

From the perspective of mathematicians, this is the incorrect sequence of teaching mathematics and, more unfortunately, defeats the purpose of mathematics education, which is to prepare students to be able to handle unseen problems instead of memorizing various algorithms to tackle different types of mathematics problems. Figure 7.2

Strand	**References**	**Total number**
Heuristics and Model Drawing	Wong (2002); Wong & Lim-Teo (2002); Ho & Hedberg (2005); Hedberg, Wong, Ho, Lioe, & Tiong (2005); Ng & Lee (2005); Ng (2006); Yeo (2006); Ho (2007); Poh (2007); Wong (2008); Goh (2009); Looi & Lim (2009); Yeo (2011).	13
Open-ended / real world problems	Seoh (2002); Lee (2002); Ng (2003); Chow (2004); Chang (2005); Chan (2005); Chua & Fan (2007); Fan & Zhu (2008); Ng (2010); Chan (2010); Kaur & Toh (2011).	11
Cognition & Sense Making	Ho, Lee & Yeap (2001); Chang (2004); Foong (2005); Teo (2005); Heng (2007)	5
Metacognition	Teong (2003); Teo (2006); Lioe, Ho & Hedberg (2006); Wong (2007); Lee (2008)	5
Affect	Tan (2002); Teo (2005); Yeo (2005); Toh (2009); Chan (2011)	5
ICT	Hung (2001); Lee (2002); Teong (2003); Ibrahim (2004); Looi & Lim (2009)	5
Problem solving curriculum	Fan & Zhu (2007); Quek, Toh, Leong, Dindyal, & Tay (2009); Dindyal, Toh, Quek, Leong, & Tay (2009); Leong, Quek, Toh, Dindyal, & Tay (2009); Chan (2010); Quek et al. (2010); Dindyal et al. (2010); Leong, Toh, Quek, Dindyal, & Tay (2010); Toh, Quek, Leong, Dindyal, & Tay (2009); Toh (2010)	10
Others (problem posing, language proficiency; CL)	Hung (2001); Ho, Lee & Yeap (2001); Foong (2002); Yeap & Lee (2002); Quek (2002); Fan & Zhu (2007); Ho (2007); Chua & Fan (2007)	8

Fig. 7.1 Classification of studies on MPS from 2001 to 2011

shows the relationship between mathematics problems and exercises, and learning mathematics.

The original intent of various types of mathematical questions (both exercises and non-routine problems, or “hard problems”) serves to facilitate the students to learn mathematics. With the acquisition of the new mathematical knowledge, students are then prepared to solve questions on applications. However, teachers who anticipate that their students would not be able to do the hard problems would end up teaching them how to solve these instead—and often through “routinizing” them. The solution to this conundrum is for students to first learn *about* problem solving so that they will be able to attend to the hard problems independently and so learn much more about the mathematics content through personal ownership and reflection (Pólya’s fourth stage).

To emphasise again, the reality in mathematics classrooms was that teachers were teaching students all the mathematical content knowledge so as to solve the exercises, the hard problems and problems on applications. This was driven by the high-stakes school and national assessments, as teachers would not want to disadvantage their

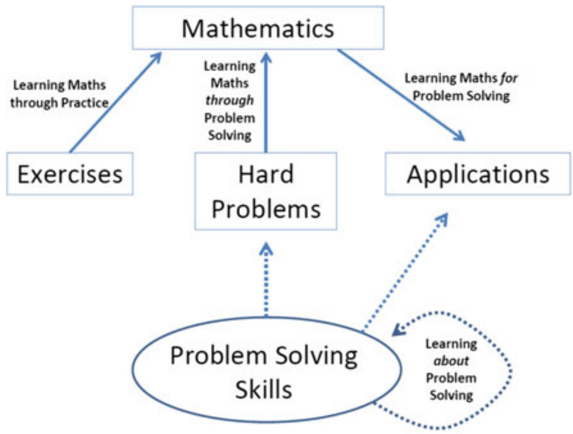

Fig. 7.2 Learning mathematics and problem solving

students with less preparation for these examinations. This, unfortunately, ran in direct contradiction to the spirit of MPS.

Driven by a desire to restore the original spirit of MPS, NIE researchers began to work on how to successfully enact MPS in the mathematics classroom.

7.3.1 Enactment of MPS in Mathematics Classroom

The surge in research publications under the "Problem Solving Curriculum" strand in Fig. 7.2 was mainly the result of the work of a team of researchers comprising Toh, Quek, Leong, Tay, and Dindyal who from 2008 to 2011, embarked on studies to actualize the intent of the problem solving curriculum. This research project was named MProSE (an acronym for Mathematical Problem Solving for Everyone)—a reflection of the researchers' belief that MPS should be meant for the general student population, rather than reserved for the elite few.

MProSE was a design experiment that focused on the secondary school mathematics curriculum with the intention of infusing problem solving into the regular mathematics curriculum and pedagogical practices across all levels in the school as a long-term plan. At the phase from 2008 to 2011, MProSE was implemented in a school specializing in mathematics and science as the testbed for its design. This was based on the "best-case scenario" method to start the investigation with high-ability mathematics students in a school that clearly emphasized the development of the mathematical ability of its students to the fullest. It was argued that the testbed for the initialization phase of an innovation should be at the school that is most conducive for success. With the "success case" achieved using the best-case scenario, researchers would then be able to understand the critical factors that led to its success, thereby how problem solving could be tweaked to meet the demand of the mainstream schools.

7.3.1.1 Theoretical Framework of MProSE

MProSE was developed based on the classical four-phase Pólya's problem solving model (1954) with an overlay of Schoenfeld's four components (1985). MProSE introduces a new paradigm of perceiving mathematical problem solving in the school mathematics curriculum as similar to the science practical lessons in the school science curriculum. It envisages the need for this new "practical paradigm" to convince school leaders and teachers of the need for curriculum time for problem solving.

The MProSE curriculum consists of specialized lessons introducing mathematical problem solving in which students learn the various aspects of problem solving through these specialized lessons. The distinct characteristics of these specialized lessons are that (1) each lesson focuses on solving one particular problem with an emphasis on the problem solving strategies in each lesson and (2) the mathematical content forms the background of each lesson with the objective of providing the context to introduce the various aspects of mathematical problem solving.

Schroeder and Lester (1989) provide three ways to understand the role and purposes of problem solving with respect to the overall mathematic curriculum:

1. Teaching *for* problem solving
2. Teaching *about* problem solving
3. Teaching *through* problem solving.

MProSE stresses the importance of teachers modelling and explicitly teaching the language and strategies of problem solving to students (i.e. teaching *about* problem solving), with its long-term plan to make mathematical problem solving as a new pedagogical approach to teach new mathematical content (i.e. teaching *through* problem solving).

Another aspect of MProSE that should be noted is the accompanying teacher scaffolding via Pólya's model and the holistic assessment strategy included in the MProSE teaching package. A full description of the teacher scaffolding provided via the mathematics "practical worksheet" is described in Toh et al. (2011a). The guiding principle for the teacher scaffolding in the MProSE specialized lesson is that teachers should start helping a student stuck in a particular mathematics problem with a general guide based on Pólya stages, rather than beginning by prescribing how students should solve the problem. The next level of scaffolding is for the teacher to provide heuristic-related prompts. Only when even this level of help fails, the teacher provides problem-specific hints. Throughout the entire scaffolding process, teachers will reinforce the use of problem solving language à la Pólya, in order to develop in students the thinking and habits of problem solving.

The assessment strategy in MProSE evaluated students' processes in solving a problem in addition to the accuracy of their solution of the mathematics problem since it was strongly felt that in problem solving the processes are as valuable as the product. A full description of the assessment strategy is found in Toh et al. (2011b).

7.3.1.2 Research Methodology

MProSE uses design experiments (Brown 1992; Collins 1999; Wood and Berry 2003) as the methodological backbone. Design experiments arose from attempts by the education research community to address the demands of conducting research in real-life school settings in all their complexity. It works on designing an educational product that is adapted for use in the school via a series of implementation-research feedback loops. Design experiments use multiple methods, such as participant observation, interview, video recording, and paper-and-pencil testing to provide corroborative evidence for findings. The envisaged outcome of MProSE was to produce a workable "design" for learning MPS in all Singapore schools. Starting in one school in Phase I, MProSE scaled up to another four schools in Phase II, following Gorard (2004): "[t]he emphasis [in design experiments], therefore, is on a general solution that can be *transported* to any working environment where others might determine the final product within their particular context [italics added]" (p. 101).

7.3.1.3 Findings of MProSE

Several papers were published on the findings related to implementing problem solving in the mainstream secondary school mathematics curriculum, beginning from 2008. The first paper that describes MProSE appeared in Toh et al. (2008), and a detailed description of the entire MProSE curriculum was published in Toh et al. (2011a, b). An outline of the MProSE problem solving module, together with the sample mathematical problems, is available in the website http://math.nie.edu.sg/mprose. In the papers, the researchers showed the importance of explicitly teaching a problem solving model to students, complemented with a common "problem solving" language for discussion.

The MProSE approach of enacting problem solving in the mathematics curriculum encompasses a wide range of issues: (1) students' belief and response to MPS; (2) teachers' belief and pedagogical practices related to MPS; (3) professional development of teachers related to MPS. We shall report on the various findings from the various papers related to the MProSE approach to MPS.

Students' Belief and Response to MPS

The students in the MProSE research school went through the entire module of problem solving. It was reported that most of the students could complete the first three stages and apply the heuristics in solving the problem. Many of them had also demonstrated Pólya's fourth stage to some extent: checked the reasonableness of their solution, provided alternative solutions, or generalised the given problem by offering at least one related problem (Toh et al. 2011b).

As reported in Dindyal et al. (2012), students in the MProSE research school found the MProSE lessons useful. Prior to the publication of Dindyal et al. (2012), the MProSE team conducted an interview on three students from the initial MProSE research school (Toh et al. 2011b). These three students represent each of three bands (higher, middle, and lower abilities) in that school. To the highest ability students in mathematics, MProSE was seen to be complementary to their higher level of mathematical training in that it helped them to regulate their "cognitive resource" that they had been equipped through other mathematics content training. The middle-ability students appreciated the problem solving process skills, in particular, Pólya's stage four of Check and Expand (Pólya used "look back" to describe stage although "check and expand" reflects more clearly the original spirit of Pólya), which was initially perceived by them as only belonging to the domain of mathematicians. The lower-ability students felt "coerced" to learn the entire problem solving process, which was seen by them initially as redundant. However, as these processes were being assessed, they had to go through the processes and they eventually realized the value of being equipped with these problem solving skills.

7.4 Studies on MPS from 2012 Onwards

MPS has continued to receive emphasis in the Singapore schools. A similar emphasis on MPS was also introduced in teacher education, so as to prepare the teachers to be ready for Singapore schools. MPS was also introduced into the teaching of undergraduate mathematics education for student teachers. This strong cohesion in the development of MPS in schools, teacher education and undergraduate mathematics education is the key feature of research on MPS that has been taking place from 2012 onwards.

This section discusses the research done on MPS that took place from 2012 onwards. Three main directions on MPS took place at this juncture, some following from the research that had been conducted in the earlier years and some in response to the revision of the school mathematics curriculum then:

(1) MProSE, an effort to introduce the holistic approach to teaching MPS, was seen to be successful and sustainable in the first research school. Further funding was obtained to scale up the MProSE research model to other mainstream schools.
(2) With the emphasis on Modelling and Application of Mathematics in the real world in the curriculum revision of 2009, there was an increased interest in introducing students to MPS in real-world contexts.
(3) Although pre-service mathematics teachers had always been exposed to MPS in their teacher education programme, there was a heightened interest in infusing MPS in the student teachers' undergraduate content course. This was done with the intention of enabling them to have first-hand experience of struggle and success in MPS.

This chapter focuses on (1) and (3), and item (2) on Modelling and Application of Mathematics in the real world will be discussed in Chap. 8.

7.4.1 *Scaling up the Enactment of an MPS Curriculum in Singapore Schools*

The first MProSE project in 2008 was carried out in one school specializing in mathematics and science. The success in that school managed to attract the attention of a range of Singapore schools who were eager to participate in this study on MPS. This led to the eventual scaling up of MProSE to four new schools, which covered the whole range of Singapore schools (independent school, autonomous school, and mainstream school), with the initial MProSE school continuing in the second phase of this research which focused on the sustainability of MPS in its regular mathematics curriculum.

As the MProSE research design was transported to a wider range of Singapore schools, the package was tweaked to meet the needs of the schools. Although the core MPS design of MProSE was content appropriate to the demands of the school and aligned to the Singapore mathematics curriculum, it still had to be adapted to meet the particular needs, student ability, and teacher readiness of the new schools.

Firstly, the researchers worked in collaboration with the teachers of the participating schools in crafting appropriate mathematical problems to be used in the adapted MProSE lessons. The criteria for crafting appropriate problems for MProSE MPS lessons were that: (1) the mathematical content of the problems must be aligned to the school mathematics curriculum, so that the students would have the "cognitive resource" to tackle these problems; and (2) the problems must not be the typical examination-type questions, as this would defeat the purpose of introducing the importance of MPS to students.

Secondly, the original lesson plans were modified to meet the constraints of the individual schools. The original proposal by Toh et al. (2011a) of using 10 lessons for MProSE was subsequently modified to meet the constraints of the schools: unlike the first MProSE School, the other more mainstream schools were less ready to allocate a total of 10 additional hours for introducing MPS. Eventually, the 10-hour lesson MProSE package was condensed to 6–8 h, without compromising on the coverage on the various aspects of MPS.

Despite the customization and adaptation of MProSE to the other schools, the following parameters of the design (see Gorard 2004) could not be and were not compromised:

1. MPS is meant for *every* student, rather than for the elite few. As such, if the MProSE package is to be implemented, it should be meant for every student in the particular level.
2. MPS must be assessed. The students' performance in the MProSE lessons must count towards a significant part of their continual assessment.

7.4.1.1 Findings in the Second Phase of MProSE

In the study in the five participating schools, it was observed that generally the students were able to meet the MPS demands of MProSE. The students who were interviewed after participating in MProSE generally asserted that MPS has enabled them to solve mathematical problems which they were unable to solve initially (Ho et al. in-print).

The teacher interviewees also agreed that it was important that MPS be introduced in their school curriculum. In particular, the interviews revealed that visible success of the MProSE as an educational intervention as well as the facilitation in the MPRoSE lessons brought about positive changes to both the students' and teachers' competencies with regard to MPS. Such visible successes also showed up at the school level, i.e. these produced deliverables that were aligned with the vision, mission, and goals of the schools, as well as professionally developed teachers and students' growth.

In turn, visible successes produced by MProSE helped to promote a state of "perpetual" flow of positive factors such as (1) earning support from school leaders, (2) gaining higher degree of autonomy and flexibility in planning for MProSE and its implementations, (3) nurturing positive attribution of teachers and students towards the second phase of MProSE, (4) making suitable modifications, adaptations and inventions made by teachers of the problem solving lessons, and (5) putting in place a continual professional development model for problem solving teachers.

7.4.1.2 Further Development of MPS

The problem solving approach developed in MProSE was adapted for another project Mathematical Progress and Value for Everyone (MProVE). This project focused on helping students who were making slower progress in the learning of mathematics compared to the majority of their peers. The typical academic profile of these students with respect to MPS was that of avoidance, low levels of persistence, and over-reliance on teachers' step-by-step instruction. In response to this, the MProVE team designed a suite of lessons to help students develop a problem solving disposition, which is marked by a willingness to try problems, improving on the strategy, and moving beyond the solution by extending the method. The students were mostly able to proceed positively with the problem solving attempts—including making relevant modifications—all the way to pushing beyond the original problems with little direct intervention by the teachers. A fuller description of the problem solving lessons and the students' responses is found in Leong et al. (2013).

With the emphasis on Mathematical Modelling and Application of Mathematics in the prescribed school curriculum (MOE 2006, p 16), studies began to be conducted on MPS using problems in real-world context. This is a relatively new domain in problem solving in the local context and Chap. 8 is devoted to discussing teacher education and solving problems in real-world contexts.

7.4.2 Teacher Education Programme

Mathematics teachers must be familiar with MPS, which is the core of the Singapore mathematics curriculum. Thus, it is not surprising that researchers from the NIE, being the sole pre-service teacher education provider in Singapore, also conducted several studies on MPS on its student teachers. To introduce MPS to the Singapore schools, it is crucial to initiate student teachers into this entire paradigm on MPS in their pre-service teacher education. This section describes several studies that have been carried out on student teachers in both the curriculum studies (CS) and academic subjects (AS) component of their pre-service teacher education.

Student teachers in NIE are introduced to MPS firstly through lectures explicitly disseminating related knowledge and facts about MPS. In addition, the student teachers clarify the concepts and skills of MPS by being *engaged* in the processes of MPS during actual solving of non-routine mathematics problems in class. A typical process of engaging student teachers in MPS during pre-service teacher education is described in Kaur and Toh (2011). They are given an *authentic* mathematical problem (which is non-routine to the student teachers, and which has multiple plausible solutions) and expected to solve the problem without first being introduced to any theory of problem solving. They are to reflect on the processes of solving the problem: (1) number of attempts up to the first successful attempts and (2) the strategies and heuristics that they have used in attempting to solve the problems. The student teachers are required to share their processes and, with the instructor as the guide, derive at the definition of a "mathematics problem". The student teachers are also given several additional problems to solve and are asked to generalize the processes they have used to solve these additional problems. This way of engaging the student teachers in MPS not only provides them with the theory and knowledge of MPS, but also their first-hand experience in MPS.

Dovetailing with the work in the secondary schools, the MProSE model was extended to the teaching of undergraduate mathematics in two undergraduate mathematics content courses at the NIE: (1) number theory course at Year 1 and (2) differential equations course at Year 3. In the two courses, the instructors used the MProSE design to different extents in teaching the courses. The intent was to equip the student teachers with MPS skills for their own acquisition of new undergraduate mathematical content knowledge (learning *through* problem solving).

Following the successful experimentation of infusing MPS into the content courses in pre-service teacher education programme and an undergraduate mathematics programme curriculum review, a new MPS module was introduced to all Year 1 student teachers as a general elective. The three subsections that follow describe these three infusions of MPS into the pre-service teacher education programme in the NIE. We give substantial detail in these sections because we think that these are somewhat unique in pre-service teacher education.

7.4.2.1 Undergraduate Number Theory Course

The origin of using problem solving approach of the MProSE design arose from the instructor's (Toh et al. 2014) prior experience in teaching undergraduate mathematics courses and frustration when student teachers waited passively for the instructor for answers and solutions when they encountered difficult questions which they could not make much progress. Undergraduate students in Year 1 have not read courses on mathematics curriculum studies (CS); hence, they have not been exposed to Pólya's model of problem solving at this point.

Along the line of thought of weaving in problem solving approach into the undergraduate number theory course, the instructor did not compromise the rigour that is expected of any typical first undergraduate mathematics course. First, the instructor identified the types of questions in the number theory courses that are suitable or otherwise for problem solving—questions that provide a clear approach of tackle do not belong to this category of questions for problem solving.

The instructor was careful not to introduce many structural changes to the course because he wanted to ensure that this course covered the usual content of the first undergraduate number theory course. Instead, he used the theorems in number theory as a *context* for introducing Pólya's problem solving model.

He distinguished the problems in his course into two categories: (1) those that were "straightforward" problems and (2) those that were amenable to problem solving. Category (1) consisted of those problems in which the method is prescribed in a direct manner while (2) consisted of those problems in which the methods of solution were not immediately obvious (see Fig. 7.3 for an example). As the method of solution of problems from Category (2) was usually not immediately obvious, it was an opportunity for the instructor to introduce the entirety of Pólya's stages, beginning with Stage I: the importance of *understanding the problem*.

For Category (1) problems, the instructor would teach using the usual exposition since the method of tackling the problem had been clearly prescribed. For Category (2) problems, he would seize the opportunity to introduce Pólya's problem solving model and demonstrate the Pólya's stages and model the use of problem solving heuristics to solve these problems. This was done through the instructor thinking aloud, consistently using the language of Pólya in solving the problem. MPS was then weaved into the number theory course gradually throughout the semester.

The study anchored on the analysis of the student teachers' performance in one Number Theory problem, which, according to the instructor, was an unseen problem

An example of Category 1 problem: Prove, using mathematical induction, that $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$.
An example of Category 2 problem: Prove that if the product of two integers is odd, then both integers must be odd.

Fig. 7.3 Examples of problems belonging to categories 1 and 2

Let a, b, c be natural numbers satisfying $a + b + c = 2012$.
If we have $a!b!c! = m\ 10^n$ for some integers m and n,
where 10 does not divide m, find the smallest possible value of n.

Fig. 7.4 Problem to assess problem solving in the number theory course

whose genre was not taught explicitly in the lectures, but one in which a student with good problem solving disposition should be able to handle (Fig. 7.4).

In analysing the students' performance in the above item, it was reported that most of the students (46 out of 55) demonstrated their attempt to understand the problem and also the use of heuristics to understand the problem. The student teachers also demonstrated the use of more than one heuristics in attempting to solve the problem. It was also found that 48 out of 55 student teachers also attempted Pólya's Stage Four to "Check and Expand" the given problem. Fewer students (33 out of 55) attempted some form of generalizing and extending the given problem. It was also found that the student teachers' attempt to generalize and extend a given problem was only mainly restricted to changing the value of 2012 or increasing the number of variables. However, none of them attempted to discuss how their solutions could be adapted to solve the proposed extended problem.

The instructor attributed the student teachers' attempt in demonstrating the use of Pólya's stages in solving the problem to the assessment criteria of the course. The student teachers had been informed that they were being assessed on their problem solving assignment, and that they would also be assessed based on their exhibition of problem solving behaviour in addition to the correctness of their solution. The instructor also admitted that due to the attempt to balance the delivery of the mathematical content with incorporating elements of MPS, the full range of processes related to Pólya's Stage Four might not have been sufficiently emphasized to the student teachers, hence the limited variety demonstrated in Pólya's Stage Four.

7.4.2.2 Undergraduate Course on Differential Equations

At the same time that problem solving was infused in the teaching of first-year undergraduate number theory course as described in the preceding paragraph, a similar study was also carried out to infuse problem solving in the teaching of the third-year undergraduate differential equations course (Toh et al. 2013). The authors recognized this as an opportunity to model the teaching of problem solving through a mathematics content course.

The infusion of MPS into this course differs from the number theory course described in the previous subsection in that the course structure was re-designed to accommodate eight "mathematics practical lessons" in the sense of Toh et al. (2011a), which proposed the use of these specialized "practical lessons" to focus on teaching *about* problem solving. Consequently, the total number of lectures of this course was reduced by eight to sixteen lectures (instead of the originally allocated 24 lectures of one hour each).

In each practical lesson, the instructor first introduced one aspect of problem solving (see Toh et al. 2011a for the detailed lesson plan of the eight practical lessons) and engaged the students to solve a relatively challenging problem on differential equations, based on the content of the corresponding lecture in that or the preceding week. The student teachers were allocated 40 min to solve a given problem. The instructor then went over the solution of the problem while the student teachers performed peer marking. The instructor consciously highlighted the use of problem solving heuristics at various junctures while discussing the mathematical content.

Before the first practical lesson, the instructor revised the stages of Pólya's model and demonstrated how the stages could be applied to solve a problem in differential equations. The assessment rubric was introduced at the beginning of the first practical lesson, so that the student teachers were aware of how they would be assessed in these practical lessons. Each practical lesson was centred on one particular problem on differential equations, called the *Problem of the Day*. The student teachers were to assess their peers' solutions of the *Problem of the Day*. The instructor rode on this opportunity to introduce peer assessment as numerous researches has shown that any opportunity for student teachers to assess their own understanding of mathematical knowledge and that of their peers could be beneficial in their early professional development (e.g. McTighe and Wiggins 2004).

The instructor carried out the six problem solving practical lessons (the last two practical lessons were used to consolidate the students' learning about the mathematical content of this course). Their overall performance in the six *Problems of the Day* was summarised in Table 7.1.

It was encouraging to the researchers to notice that *all* the student teachers who submitted the practical worksheet after the practical lessons exhibit behaviour of problem solving in minimally demonstrating appropriate use of heuristics (except one student in Problem Three).

Despite problems three and six (two relatively difficult problems on mathematical proofs) being the relatively more challenging problems for the mathematics practical lessons, it is clear from Table 7.1 that practically all the student teachers exhibited the MPS behaviour using Pólya's stages and most of them arrived at least a partially correct solution. Most students were also able to exhibit some behaviour of Pólya's Stage Four (check and expand) to a certain extent for most of the problems. Generally, the students found it more difficult to check and expand problems that involved mathematical proof, as mathematical proofs involve mainly intensive deductive reasoning, and thus requires a thorough understanding of the mathematical concepts.

Despite the reduction of the number of lectures in this course, it was reported that the content covered in the course using this approach was not reduced, and the rigour of the course was not compromised. In fact, some of the intricate details of parts of the course were transported to the *Problems of the Day* of the mathematics practical lessons, during which the student teachers had more opportunity to explore in greater depth using their MPS tools (learning *through* problem solving). By engaging in MPS

Table 7.1 Student teachers' performance in the six problems introduced in the practical lessons

Correctness of solution	No. of students for problem					
	One	Two	Three	Four	Five	Six
Completely correct solution	35	41	24	45	45	22
Partially correct solution with appropriate use of heuristics	15	9	13	5	6	25
Incorrect solution with appropriate use of heuristics	1	0	13	1	0	4
Incorrect solution without use of appropriate heuristics	0	0	1	0	0	0
Stage IV: checking and expanding	No. of students for problem					
	One	Two	Three	Four	Five	Six
No attempt in Stage IV	27	0	31	3	1	11
Attempt to check reasonableness of answer	21	8	4	14	7	16
Attempt to check answer + either alternative solution or generalize the problem	3	11	13	13	7	15
Attempt to check answer + alternative solution + generalize the problem	0	11	3	21	36	9

during the lesson of the "new" problem, the student teachers in fact had first-hand experience and exploring with the mathematical content, which was traditionally covered by the typical lecture delivery mode.

7.4.2.3 Undergraduate MPS Course Introduced in Year 1

In a curriculum review ongoing since 2015, it was agreed that student teachers would benefit from the direct experience of MPS in their undergraduate mathematics education. A problem solving general elective module for student teachers parked under the academic subject was conceptualized and developed using the secondary school MProSE design as a template.

The course consisted of twelve 3-hour face-to-face lectures. In the first four lessons, the student teachers were introduced to the general principle of MPS. Pólya's model and the MProSE scaffolding worksheet were explicitly introduced to the students. The choice of the problems in these four lessons was made in consideration for the various aspects of MPS disposition and heuristics that were desired of students to do MPS. In the remaining eight lessons, the problem solving course focused more on tackling the challenging problems from the other mathematics content courses (finite mathematics and number theory) that the student teachers were pursuing at that time. This was an opportunity, not easily available in the secondary school context, for the students to appreciate how MPS can facilitate them to better learn their own mathematical content from specific fields.

A typical lesson consisted of the following structure: (1) discussion of the homework problems given in the previous lesson; (2) solving two problems in the class; and (3) assignment of two problems for homework (to be discussed in the next lesson). Throughout the entire lesson, various problem solving dispositions, habits, and heuristics were reinforced.

The instructor insisted on the use of Pólya's model throughout the course. Rather than perceiving this as an imposition on the part of the lecturer, the students actually appreciated this as they saw how the model enabled them to understand and solve the problem more efficiently. The interviewed student teachers highlighted that the lecturers assisted them with the direction on how to proceed to solve a problem, using the various levels of scaffold proposed in Toh et al. (2009). The levels of scaffold range from the most generic suggestions using Pólya's language (e.g. "Have you understood the problem?" "What is your plan?") to problem-specific hints.

Three student teachers were interviewed (Tay et al. 2016) and asked about how the problem solving course had helped them in the learning of the other undergraduate mathematics content course. The general response was that they were able to go through the entire MPS process when faced with a difficult mathematical problem. According to the student teachers, the heuristics learnt in the problem solving course were particularly useful.

However, the student teachers also pointed out that unlike the problem solving course, they were not required to write out explicitly the problem solving processes in solving the problems in other mathematics courses. However, the interview with several student teachers seemed to suggest that the problem solving processes had already been assimilated by them, as "it happens in the mind". When faced with a problem that cannot be solved immediately, they would record down the applicable Pólya stages almost immediately.

7.4.3 Infusion into the School Mathematics Curriculum

The research carried out on enacting MPS in the school mathematics classroom has resulted in the establishment of certain permanent features in the mathematics curriculum at various school levels. For example, in the MProSE research schools, the MProSE problem solving module has become permanent features in those schools. For the initial research school, the MPS module is a compulsory module for all Year 2 students. The school has tweaked the module to include several e-lessons for their students. The rationale is that selected theory portions of the lesson are to be viewed by the students before attending the face-to-face MProSE lessons. This would allow students to have ample time for hands-on experience in authentic problem solving.

In the other MProSE mainstream research schools, the problem solving module has remained a compulsory module for all their Sec One express stream students. In fact, one of the schools has built on the MProSE module to establish a common set of mathematical language grounded in MPS for all their students in their subsequent years. Another mainstream school has worked with several MProSE researchers to extend the MPS experience for upper-level students by infusing MPS through the use of replacement units. Readers can obtain more details about replacement units in Leong et al. (2016a, b). Not only that, the MProSE approach has influenced the development of the new H3 mathematics curriculum, in which one significant component of this subject emphasizes MPS. Team members of MProSE are commissioned by MOE to conduct four 2-hour workshops to teach *about* problem solving to a large segment of H3 mathematics students.

From professional development workshops on MPS conducted by NIE, several other secondary schools have developed their own problem solving modules and MPS approach to teaching mathematics, although their concepts are not entirely congruent to MProSE approach.

7.5 Conclusion

This chapter traces the development of MPS research carried out in Singapore since it became the heart of the Singapore mathematics curriculum. The earlier research focused on addressing the readiness of students for MPS from the perspective of the Singapore mathematics curriculum framework; the later research tended to emphasize the enactment of MPS in the Singapore mathematics classroom. Research on MPS has also moved beyond the schools to the teacher education programme in NIE. To a certain extent, these design research projects have made an impact on the implementation of MPS in the school curriculum.

The research done on enacting MPS in mathematics classrooms described above tends to be carried out in secondary level and above. Some educators and school leaders have reflected to the researchers that MPS disposition is best developed in children at the upper primary level, which is a crucial stage for habit formation. Perhaps a future direction for research on MPS should be research on the enactment of MPS in the primary school mathematics classroom.

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