Chapter 19
Teaching Simultaneous Linear Equations: A Case of Realistic Ambitious Pedagogy

Yew Hoong Leong, Eng Guan Tay, Khiok Seng Quek and Sook Fwe Yap

Abstract In this chapter, we present a conceptualisation of mathematics teaching and learning which we term realistic ambitious pedagogy. We locate this pedagogy within the domains of teaching goals and teaching enactment, and the interactions between them. We argue that it is a suitable pedagogy for use in teacher development enterprises because it takes into deliberate consideration the realistic constraints within which teachers work while pursuing ambitious goals of mathematics teaching. To illustrate, we provide an example taken from our work of redesigning a curriculum unit on simultaneous linear equations in two variables with some Year 8 mathematics teachers in Singapore.

Keywords Realistic ambitious pedagogy · Teaching goals · Teacher professional development

19.1 Introduction

The research that is reported here is part of an ongoing school-based teacher development work that we began with Singapore secondary schools about a decade ago. Through this rather long process of learning, we formalised and refined certain innovations that we argue would advance the goals of the teacher development enterprise. One such innovation is the Replacement Unit Strategy. We have reported on this

Y. H. Leong (✉) · E. G. Tay · K. S. Quek · S. F. Yap
National Institute of Education, Singapore, Singapore
e-mail: yewhoong.leong@nie.edu.sg

E. G. Tay
e-mail: engguan.tay@nie.edu.sg

K. S. Quek
e-mail: khiokseng.quek@nie.edu.sg

S. F. Yap
e-mail: sookfwe.yap@nie.edu.sg

T. L. Toh et al. (eds.), *Mathematics Education in Singapore*,
Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-13-3573-0_19

strategy in other publications (e.g. Leong et al. 2016a, b). In this paper, we highlight another innovation: realistic ambitious pedagogy (RAP).

When we think of "pedagogy", we do not have in mind the rather narrow conceptions of the term as used in popular discourses—such as captured in condensed phrases like "direct teaching", "cooperative learning", or "student-centred pedagogy". We define pedagogy more broadly as a theoretical conceptualisation of teaching that takes into account the main complexities which may hinder or advance the goals of instruction. We think this framing of pedagogy is more suited to our purpose (and the purpose of other researchers who share our stance), which is to provide a theoretical framework of discussion with mathematics teachers across a wide range of schools.

We begin by explicating RAP before reporting empirical findings based on a specific instantiation of the pedagogy in the case of teaching simultaneous linear equations.

19.2 Realistic Ambitious Pedagogy (RAP)

The term "ambitious" in RAP is inspired by current literature on ambitious teaching (e.g. Kazemi et al. 2009; Lampert et al. 2010). These authors conceived of "ambitious" in the sense of teachers striving for ambitious goals of teaching. They can include goals that target the "big ideas" of mathematics (e.g. Charles 2005; Schielack and Chancellor 2010), the use of problem-solving in learning mathematics (e.g. Hiebert et al. 1996; Lester 2003; Schoen 2003), and the attainment of these goals consistently for all students (Kazemi et al. 2009; Lampert et al. 2010).

We think of "ambitious" as that which targets the teaching of disciplinarity in mathematics. This is influenced by a more discipline-based view of mathematics teaching. Following a long tradition (e.g. Lakatos 1976; Lampert 1990) of seeing pedagogy in the mathematics classroom as rooted in how mathematics is done by mathematicians, we share the commitment of helping students learn the disciplinarity of mathematics in actual mathematics classrooms in Singapore. This means the teaching of big ideas of mathematics and of problem-solving, as mentioned in the previous paragraph. But a reflection on "disciplinarity" would reveal that other mathematical dispositions and skills should be considered too. As examples, it includes the key strands of inductive and deductive reasoning (Lakatos 1976) as a process of reaching conclusions in mathematics; it also includes the need for sense-making when connecting mathematical ideas.

In summary, ambitious mathematics teaching to us is a commitment to go beyond presenting mathematics as a set of arbitrary rules to follow—which, sadly to us, remains the popular image of mathematics almost everywhere; it must target the essential elements of disciplinarity relevant to the mathematical topics under study.

But we think that for ambitious teaching to be a sustainable reality in schools, efforts to engage mathematics teachers in PD settings for this purpose must also take into serious considerations the "realistic" constraints of teaching. In fact, we think that much of the failures in actualising the vision of ambitious teaching at scale are due to an inappropriate treatment of the realistic constraints of practice faced by teachers on a day-to-day basis.

A common contextual constraint experienced by teachers is time pressure (e.g. Assude 2005; Jones 2012; Keiser and Lambdin 1996; Leong and Chick 2011; Meek 2003; National Education Commission on Time and Learning 1994/2005). These studies explicated the role time pressure played in teachers' decision-making: the effect was not merely over isolated situations of little consequence to instructional pathways; rather, constraints in time can significantly hinder the fulfilment of worthy (and ambitious) goals of teaching.

Closely related to time pressure is the commitment by many teachers to prepare their students to do well in high-stakes examinations (Amador and Lamberg 2013; Barksdale-Ladd and Thomas 2000; Diamond and Spillane 2004; Plank and Condliffe 2013; Valli and Buese 2007; Wu and Zhang 2006). Many teachers see it as their implicit social responsibility to help their students achieve high examination scores in school mathematics as a way to help them realise their career choices in life.

The contextual constraints of teaching are not usually thought of as significant enough to be brought into play in the theoretical considerations of "pedagogy". This is the point of departure for RAP: For efficacious teacher development that is committed to teachers' buy into (ambitious) pedagogical changes, we argue that realistic constraints of teaching should not be thought of as a theoretical afterthought that we need to put up with; rather, they ought to be incorporated into the theoretical conceptualisation and development of pedagogy. RAP is thus a teacher-sensitive pedagogy that stays true to its mathematics discipline-driven tradition within the contextual givens of teaching in schools. By pedagogy, we are not restricted to a set of specific teaching methods. Instead, it is a useful theoretical framework within the context of teacher development to discuss teaching goals in the classroom.

19.3 Teaching Goals and Teaching Enactment in RAP

The theoretical premise of RAP is that much of teaching actions and decision-making in the classroom is driven by the teacher's instructional goals (Lampert 2001; Leong and Chick 2007/2008; Leong et al. 2007). As such, while pedagogical studies must include the examination of actual classroom teaching, we think that the starting point of the pedagogical inquiry is not teaching enactment; rather it is the set of teaching goals. This is also substantiated from the standpoint of teacher development: In PD work, the aim is to help teachers reflect upon their pedagogy in ways that would challenge existing conceptions and methods of teaching. To facilitate this inquiry, we think that the discourse with teachers should begin with identifying and examining the goals of teaching mathematics.

It is in the examination of instructional goals that both the realistic givens and the disciplinary ideals of RAP can find their places in the theorisation of pedagogy.

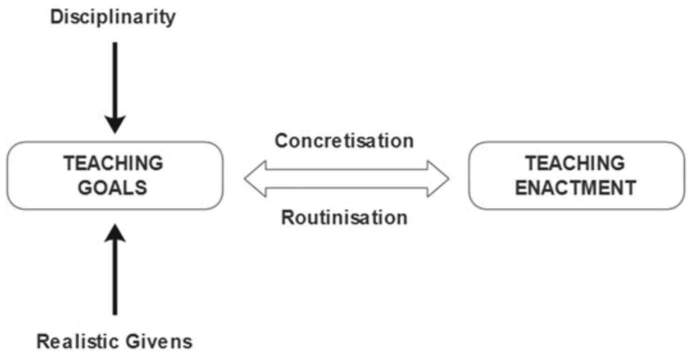

Fig. 19.1 Components and interactions involved in realistic ambitious pedagogy

As discussed in the previous section, some realistic givens discussed in the previous section can be translated into these goal-statements:

R1. Goal-exam: Help students gain fluency with exam-type questions
R2. Goal-time: Balance the use of classroom time judiciously to satisfactorily fulfil all the other instructional goals

Likewise, the ambitious part of the equation can be represented in the form of written instructional goals:

D1. Goal-sense: Help students experience the learning of mathematics as sense-making
D2. Goal-problems: Help students learn mathematics through problem-solving

Thus, in RAP, there is recognition that realistic givens (expressed in the R-goals, R1, R2, …) and aspirations for teaching that mirrors practices in the discipline (expressed in the D-goals, D1, D2, …) are to be included when considering worthy goals of teaching.

In seeking to enact the formalised set of R- and D-goals in the classroom, it is critical that the pedagogical considerations be translated into a form that teachers can use or carry out directly in classroom instruction. For this purpose, we propose the following constructs: concretisation and routinisation. With these components included, the proposed model of RAP is shown in Fig. 19.1.

Concretisation refers to the process of making ideas concrete to teachers. These tangible implements are not merely theoretical conceptualisations of good practices; they are imbued with concreteness so that they can be seen and/or acted on by students and teachers for the purpose of mathematical learning. Examples of these concretised objects are: Activities, including the actual tools for use in the activities and the activity sequences; student tasks, including the task sheets that student work on; and whiteboard working, including the organisation, diagrams, and chronological sequence. Routinisation refers to the process whereby the intended goals are captured in classroom routines in a way that, through repeated rehearsals, they support the fulfilment of these goals. Students' orientations of what school mathematics is and how it is learnt are not easily changed by a one-off innovative task. There is a need for habituation towards this intended instructional goal, and hence the need for routine formation.

The model, as summarised in Fig. 19.1, does not prescribe details as is usually found in recognisable pedagogy. However, for the purpose of teacher development, we think the model contains elements that are essential in the sense that if any of these components are not carefully attended to, ambitious teaching in typical classroom settings is unlikely to be sustainable. RAP is broad enough to encompass a range of different forms of actual classroom enactment yet rooted in worthwhile educational values (as expressed in the teaching goals) and in the commitment to realise these values in class (as expressed in the goals-enactment interaction).

In RAP, we ask these organising questions: (a) How do we bring the R- and D-goals together? (b) How can they be concretised and routinised for classroom use? (c) To what extent is the teaching enactment supportive of the goals? We may further ask the extent in which the goals are realised in the students. But since RAP is developed primarily with teacher PD in view, the focus of inquiry is on teacher enactment. In the next section of the paper, we address these questions in the context of a specific case of unit design that was framed by RAP.

19.4 Context and Method

We are involved in a project with the mathematics department of a secondary school to redesign curricula units for their mathematics classes. This project has been ongoing for more than eight years and we have worked together on a number of secondary level units. The mathematics teachers who participate as co-designers in these units consider the work of planning, developing, and refining the curricula materials a form of professional development. To us, one other goal of the enterprise is to trial and study RAP. We use the intellectual journey of co-designing the unit on simultaneous linear equations in two variables for the two Year 8 classes of the school as a case of RAP.

Chronologically, Questions (a) and (b) relate mainly to activities before the start of the lessons; and Question (c) is aimed at the teaching activities in class during the lessons.

For Questions (a) and (b), the data we drew upon include all the video records of discussion meetings we held with the teachers and the various refinements of teaching materials leading up to the final version for use in class. As the design and PD process is significant in RAP, the focus is on documenting the chronological and intellectual process involved in overcoming the key challenges in the enterprise. Seen in this way, these two questions are not "Research Questions" in the usual sense of requiring a rigorous analytical process to address them. We have nevertheless included them to provide a more complete portrait of RAP.

Question (c) targets the degree to which the teaching goals are realised in teaching enactment. The analysis comes from video records of teaching actions in both classes for each lesson throughout the entire unit. We also draw upon the video records of discussion meetings we held with the teachers and the various refinements of teaching materials during the period of the class enactment. The inquiry is on whether the instructional work of the teachers was supportive of both the R- and D-goals. Since the goals were by design captured—through concretisation and routinisation—into concrete tangibles and routines, the inquiry began with how the teachers utilised these implements to fulfil the intended goals. Further evidence was obtained at other regions of classroom practice where we noticed that teachers appeared to exhibit consciousness in advancing a careful integration of the R- and D-goals. We were able to confirm (or refute) these hunches through the discussion data.

19.5 RAP in the Case of Teaching Simultaneous Linear Equations

We organise this section according to the questions (a)–(c) that we seek to address.

(a) **How do we bring the R- and D-goals together**?

We started by discussing the instructional goals of the topic. As we had worked with the teachers over a number of previous similar curriculum development units, they were comfortable enough to share freely about the realistic demands in the coverage of the topic, summarised as:

r1. To help students develop proficiency with both the technique of "Elimination" and "Substitution"[1]
r2. To inculcate in students the habit of checking by substituting the obtained solution into a suitable equation
r3. To complete the coverage of the topic within five 45-min lessons

In our conception, the topic provides the opportunity to experience something of what learning mathematics within the discipline is like. First, since solving simultaneous linear equations builds on an earlier topic on solving linear equations in one variable, there is an opportunity, in the structuring of the unit, to help students make connections among the mathematical content strands. Second, the concept of "simultaneous" is a recurring idea in Algebra at the secondary school levels. We think it is important that students are not only able to perform the steps in solving the equations but also to develop the disposition of seeking to make sense of critical concepts. In this case, the critical concepts are the idea of "simultaneous" as same

[1] Here, the lower case letters r and d are used to label the goals. This is to make a distinction between this set of goals and the goals listed in the previous section of this paper where they were labelled with capital letters R and D. The relationship between the r- and d-goals and the R- and D-goals respectively is roughly one of subordination. For example, fulfilling r1 within the teaching of this unit supports the fulfilment of some more broad-grained R-goals (such as R1).

variables that are subjected to constraints by both equations, and "solving" as obtaining the solution that satisfies both equations. Third, as the solution steps for this topic tend to be considered long and tedious for many students, it is an opportunity for students to learn to keep track of their thinking and decision-making at critical junctures throughout the solution process. This is an important problem-solving disposition. Thus, through discussions with the teachers and with their agreement, the ambitious set of goals for the topic may be summarised as:

d1. To help students make connections between solving linear equation in one variable and solving simultaneous linear equations in two variables
d2. To make explicit the meaning of "simultaneous" and "solving" as used in this topic
d3. To develop in students the disposition of keeping track of their solution steps and decision-making

It is perhaps important at this juncture to clarify that the r- and d-goals listed do not exhaust the instructional goals for the teaching of this topic. A goal such as helping students correct their errors in algebraic manipulation is certainly necessary for this topic and is indeed included in the teachers' consideration in the planning process. We list only the goals that are directly relevant to this topic under consideration and are overarching across the whole instructional unit. Similarly, it is easy to point out other ambitious goals that are relevant to this topic that are not listed. For example, should we not try to be more ambitious and generalise to the solution of a system of linear equations with more variables? This would certainly be in line with the learning of mathematics within the discipline; but we think that doing so would be more ambitious than is workable. In particular, it would conflict directly with Goal r3 which specifies a fixed number of lessons for the completion of the unit. Thus, the ambitiousness of our goals is also tempered by the realistic situation we work in.

These sets of r- and d-goals were then carefully weighed when we planned the structure of the unit. In the planning process, we discussed possibilities and projected them into the imagined vision of classroom enactment. In particular, we placed heavy emphasis on how the students would respond to certain instructional moves, the difficulties they may face, and how they can be alleviated. The broad-grained trajectory of the unit is summarised in Table 19.1.

By "prominent goals", we mean the goals the teachers should consciously foreground in their classroom instructional moves. This does not mean, however, that no other goals were at play. As an example, though only r1, r2, and d3 are listed as prominent in Lesson 2 as shown in Table 19.1, it does not mean that relating the solution method to earlier methods learnt or the concepts of simultaneity (embodied in Goals d1 and d2, respectively) became unimportant. Rather, it means that these other goals were implicitly embedded in the lesson sequence and were not emphasised to the same degree as the prominent goals. Another example is Goal r3. Although it is not listed in Table 19.1 as prominent in any of the lessons, it tacitly guided the planning of the whole unit in terms of the content to include/exclude and the balance of goals to pursue within the constraint of limited time.

Table 19.1 Overall plan of the unit on solving simultaneous linear equations

Lesson	Key moves	Prominent goals	Main considerations
1	• Recall solution of linear equation in one variable • Consider a single linear equation in two variables—to introduce many solutions • Consider a pair of linear equations in two variables—to introduce the meaning of simultaneous in this context • Motivate a method to solve a pair of simultaneous linear equations in two variables • Introduce method of substitution as a way to reduce the problem to the familiar setting of solving one equation in one variable	d1, d2	This can be considered an introductory lesson that connects prior knowledge with the contents of this topic. But due to r3, we cannot devote the entire lesson to motivational elements. We include in this lesson an overall structure of the method of substitution as a way to deal with the problem presented by the need to solve simultaneous equations
2	• Familiarise students with the method of substitution for simpler equations • Emphasise the overall strategy of reducing the problem into one equation with one variable • Demonstrate the usefulness of substituting the found values into a suitable equation as a way to check that they satisfy both equations—to reinforce simultaneity	r1, r2, d3	This lesson is on the method of substitution. The focus is on helping students take a more zoomed-out view of the process: While solving the equations, they learn to keep track of when and why they apply the required strategies
3	• Use students' work to address key conceptual and methodical errors in their use of the method of substitution • Same focus as Lesson 2, with harder equations that require more careful manipulation	r1, r2, d3	This lesson is a follow-up on the method of substitution. The focus shifts more to procedural fluency at the more fine-grained level as they practise the details of the method over a range of suitably gradated examples
4	• Motivate the learning of method of elimination by contrasting it against the method of substitution—to show relative ease for certain types of simultaneous linear equations • Reinforce the same overall strategy of reducing the problem into one equation with one variable • Familiarise students with the method of elimination for simpler equations	d1, d2, r1, d3	The introduction of another method provides the motivation to revisit and re-emphasise the disciplinary goals The main considerations for Lessons 4 and 5 mirror that of Lessons 2 and 3 respectively, but for the method of elimination instead

(continued)

Table 19.1 (continued)

Lesson	Key moves	Prominent goals	Main considerations
5	• Use students' work to address key conceptual and methodical errors in their use of the method of elimination • Same focus as Lesson 4, with harder equations that require more careful manipulation	r1, r2, d3	

We have also learnt the need to keep the number of listed goals in each lesson small. Beyond a certain number of ostensible goals, it becomes unproductive both for the planning and discussion of teaching. The more goals we load into a lesson, the greater the tendency during enactment for the emphasis on each of the goals to thin out, lessening the success of their fulfilment in the lesson. For the purpose of teacher professional development, it is also helpful to limit the number of goals being emphasised so that teachers can focus their attention on instructional innovations in response to a few goals per lesson instead of having their attention diffused over many instructional goals.

(b) **How can the goals be concretised and routinised**?

The next stage is the concretisation of the goals and plan into a form that would enhance the potential of fulfilment of these goals during classroom enactment. We carried this out together with the teachers by designing student task sheets that embodied the intended learning trajectory of the students. Each lesson in the module was supported by a task sheet. In some lessons, homework task sheets were also developed. Since space is limited, we only discuss selected sections of some task sheets to illustrate how the goals were concretised.

Figure 19.2 shows a section extracted from the task sheet in Lesson 1. The blanks in the extract were meant to draw students' attention to how the solution of simultaneous equations differs from their prior experience of solving linear equations in one variable. The ellipsis "…" signalled an opportunity for students to conjecture and discuss what they think "simultaneous" in this context could possibly mean. This section exemplifies a concretisation of Goal d1.

A follow-up section within the same task sheet illustrates the link to an explicit consideration of what "simultaneous" means (Goal d2), as shown in Fig. 19.3.

In subsequent lessons, the task sheets take on a more familiar form—consisting of exercise questions in which students were required to develop fluency (Goal r1). A typical item in these task sheets features a pair of simultaneous equations, which students are to solve. The item shown in Fig. 19.4 is taken from a task sheet used in Lesson 2.

The space given below the equations in Fig. 19.4 were for students to write the usual steps involved in solving simultaneous equations; the column under "What's

Solving "***simultaneous***" linear equations

What does it mean to solve this pair of simultaneous linear equations?

$$x + y = 7$$
$$x - y = 3$$

I notice: (a) each equation has _____ unknowns; (b) it is now a ***pair*** of equations instead of a single equation; and (c) "simultaneous" here means ____________

To "solve" this pair of simultaneous equations means …

Fig. 19.2 An extract from the task sheet prepared for Lesson 1

An illustration:

Values of x and y in each table satisfy the respective equations

$x + y = 7$	
x	y
7	0
6	1

$x - y = 3$	
x	y
3	0
4	1

…

The same pair of x and y values that I circle in both tables is the solution to the simultaneous equations: ______________

Fig. 19.3 Another extract from the task sheet prepared for Lesson 1

Example 1 Solve the following simultaneous equations $5x - 9y = 17$ $3x - 8y = 5$	What's going on?

Fig. 19.4 An extract from the task sheet prepared for Lesson 2

going on?" was a provision for students to make visible the rationale or prompts corresponding to suitable junctures of the working on the left. This is in line with Goal d3. Under this column, students were expected to write phrases such as "Label equations", "Form one equation, one unknown", "Obtain both x and y values", and "Check" at suitable points of the solution process. Clearly, the task sheet design was not merely to facilitate opportunities to practise (Goal r1), to examine one's solutions steps (Goal d3), and to check the provisional answer by substitution (Goal r2) as ostensible goals. There was also an implicit intent to reiterate the meaning of

Example 1 Solve the following simultaneous equations $x + y = 13$ $x - y = 5$	What's going on?

Fig. 19.5 An extract from the task sheet prepared for Lesson 4

"solving" "simultaneous" equations as finding the same solution for both equations (Goal d2) and to reiterate the connection between this process of solving to the process of finding the solution of one linear equation with one unknown (Goal d1). Thus, within one exercise item in the student's task sheet was a concretisation and an integration of the goals intended for this module.

The item as presented in this layout was not a one-off rarity. In fact, an item of this nature was a standard feature in most of the in-class task sheets as well as homework task sheets. Figure 19.5 shows an extract[2] of the task sheet used in Lesson 4. In other words, routinisation was also built into design of the task sheets.

(c) **To what extent is the teaching enactment supportive of the goals**?

We base our analysis on 14 video recordings of two classes (seven each), transcript of a meeting where all teachers and researchers reviewed the simultaneous equations unit, and a student perception survey of the 80 students in the two classes that was conducted after the unit was taught. We shall focus only on the enactment of the "What's going on" column as a concretisation of r1, r2, and d3.

The two classes A and B were taught by Teacher X and Teacher Z, respectively. The videos were full recordings of seven 40-min lessons for each class. Teacher X started the first lesson with the "What's going on" column already prominently written on the whiteboard (see Fig. 19.6). He consistently used the column from the beginning to the end for all of his lessons (see Fig. 19.7), except for Lesson 5, when students did board work for the first half of the class. Teacher Z started using the column about 18 min into the first lesson. In Lesson 2, he waited until the student's board work was completed before writing the column on the board. Lessons 3, 5, and 6 were similar to Lesson 1—he started using the column a quarter way into the lesson. He did not use the column at all in Lesson 4. In the final lesson, he had the column written on the board from the beginning of class.

We can see that the two teachers enacted the concretisation of the "What's going on" column faithfully. We now turn to the summary meeting to see what all the teachers, including those who observed the lessons in a lesson study context, have to say about the use of this concretisation towards achieving the desired R- and D-goals.

[2] The task may appear at first look to be a repetition of contents taught in earlier lessons. In Lesson 4, the teacher has moved to teaching the method of elimination. Here, students were asked to use this newly learnt method to solve equations they would have solved in earlier lessons using the substitution method.

Fig. 19.6 Beginning of Lesson 1 in Class A

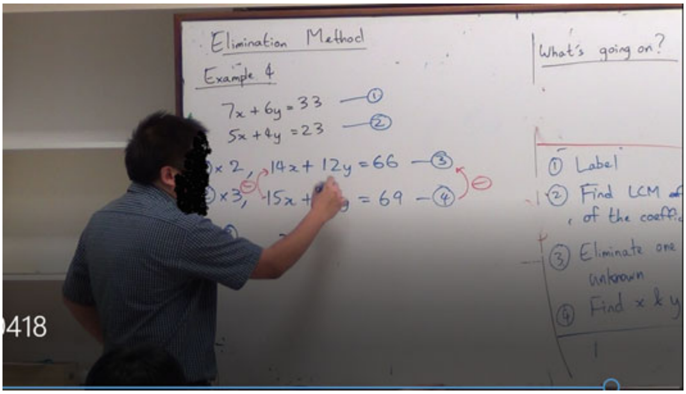

Fig. 19.7 End of Lesson 7 in Class A

They observed that the students treated the column initially as space in which to copy verbatim what the teacher wrote in the same column on the board; then some appreciated its use; and towards the end, some because they had internalised the procedure and others because they had given up, stopped filling in the column:

Teacher Z: *Basically, I think … it is a trend whereby starting there are more people who will write down and follow what you write as you go on, I think once they grasp the concept that column is as good as blank especially for the homework.*
Teacher A: *I observed two lessons. One is Teacher X's first lesson but I think because it's starting so it was getting them to know what that column is about and so when he asked them to write, they did ask them to write down a few things but it's mostly what you wrote on the white board. So they just copied it down. After that I went on to Teacher Z's third lesson. So his third lesson, I think as he said, it's nearing the back, so the two that I observed they didn't write anything down even when he wrote*

it, even when Teacher Z wrote it on the column on the white board, they didn't copy it down.

Teacher B: *I only observed the first two of Teacher Z's lesson. That was the substitution method. Generally the students they just copy and even when … okay, first lesson they basically copied everything that Teacher Z mentioned. Second lesson when Teacher Z wrote things, then for the first example they also copy, but once you moved on to the second example, I saw them flipping because the questions are similar so I saw them flipping to the previous example but their focus is on the steps. How to solve and not so much on "What's going on". So what I gather was the note that they take actually at the column was not very helpful to them.*

Teacher C: *I went for Teacher X's lesson. The first lesson and the second last lesson. The first lesson was Teacher X introducing it. Then the two students that I observed, they were copying down whatever Teacher X is writing down. Then I just asked, "Why are you copying down?" Then the girl said, "It's is very useful for me." And she even do it in different colours. But subsequently, the next lesson that I went for Teacher X's class right, the faster … I noticed that those boys at the back, those two boys, they didn't take down any notes but they were able to do it very fast. They didn't fill up the "What's going on" column at all.*

Teacher D: *I went in for Teacher X's second class. And what basically the two of them were doing was, they were just listing the steps they need to do. So step number 1, "label", step number 2, "write 'x' in terms of 'y'", and step number 3, "solve for 'x' and 'y'" and step number 4, "check". So before they even attempt the question, they will quickly fill up what's going on first. So every single page they will fill up that four steps first and then they will start doing. So I don't know whether maybe by doing this, the four steps went into their heads. So they will remember to do these four steps every time they do. I don't know because I didn't observe them after that lesson so I don't know how useful … But when I went in for Teacher Z's last lesson, almost none around me were writing what's going on anymore. They were already done with this "What's going on" thing.*

Teacher E: *Okay, I observed quite a number … three or four … I actually move down the row, so I managed to observe random students, many random students. Mostly when Teacher X asked them to write down, they will write down, whatever, most of them will write down. Well I'm actually very impressed with one of the students but I don't know his name. The Indonesian one. So that student… he suddenly understand … then when I go to the second example, "What's going on", he write, "Step number one", then he do, then "Step number two", then he could do, "Step number three", then he could do. After he finished ah, he checked you know. After that, you can feel and say that, he is happy. Very happy. But only one student is like that. Then … I also see the two boys behind. These two boys were very fast. They already finished already but nothing there lah. I observed one boy like that. Then I … I mean … It's the front boy … You can see really happiness. He followed everything you know. After that he re-checked. When his answer is correct, so he was so happy. But I do feel that they do benefit some of the students. Maybe not all, maybe very few number but I feel that it is good and I see some writing summary.*

Teacher Z felt that the "What's going on" column helped make his thinking visible to the students and allowed for some permanence unlike if he relied solely on "talking" out the steps: *Okay, I mean, like err, it is the same consensus as everyone that I don't really know whether my students have benefitted from it or not but I do buy Researcher L's idea that it has benefitted the teacher meaning myself in the sense that, I think I'd spoken about this during one of the post meetings as well—that when my thought is being portrayed on the board, it makes my explanation clearer to myself even in that sense too so that I have that consciousness that I need to explain certain things. And to make it visible for the students whether … and also to help students whereby they have those one, two seconds just switched off and suddenly they missed out that bit that I was trying to say but once they switched back on and see the thing that is on the board, they are able to somewhat connect back to that lesson. I thought that was something that is useful. So whether or not they have written it down, and it helps them or not I'm not too sure but at least by seeing something that is on the board, I think that will help them. If it's one word that I am trying to go by audio with my voice, there's another mode whereby you can see the visual of whatever is going on in my mind is actually on the board. That's one thing. The other thing is I think as I look through let's say for example, in a particular lesson after you do example one, you can refer to the steps to do example two, maybe tomorrow if I ask them to do again, they will forget some steps here and there.*

Teacher Z also realised that he had used the column to write down procedural steps rather than exploit its full potential for giving reason to the steps: *Ultimately I think, for my class, what I could focus on more is the whole essence behind solving simultaneous equation. Like what most of you correctly pointed out is that I focused quite a bit on procedural steps. So even my "What's going on" column right is ninety per cent of the procedures—"do this", "do this", "do this" … So procedures without exactly the reason behind it in that sense, I may or may not have said it in class but it wasn't on the "What's going on" column. I can say lah, about ninety per cent about the "What's going on". So in that sense, it ends up, even if students right now, it will be a case of they will have to memorise the step one, step two, step four in order to get the answer instead of let's say if I were to throw in the idea that, "Hey, we are doing elimination because we are trying to eliminate one of the variable whether it is 'x' or 'y'." If they go back with the idea they internalised the idea then when they see that question again, then they will know, "Oh, I need to somehow eliminate one of the variables. How am I going to do it? Err, okay, let's remember the steps." Rather than straight away come in and "What is the first step? 'Label'." As I reflect I thought. The first step shouldn't be 'label'. The first step should be, "Hey I've got two equations, I want to eliminate one of the variables in order to solve the other one." That sort of thing instead of just very procedural, first one, "label", second one …*

Teacher Z felt that teaching this way was better than what he did in the previous year: *My gut feel is that they did better in that sense. I mean, they may not be good. But with my limited teaching capability, what I taught last year and what I taught this year, with the addition of the "What's going on" column, I felt that my students of this batch, they learn better. In the sense that, at least, because I focused a lot on*

Table 19.2 Responses to two items from student perception survey

Type	Description	Class 1	Class 2
1	Writing down notes and my thoughts in the "What's going on" column has helped in my understanding	34/38	35/42
		89.5%	83.3%
2	Writing down notes and my thoughts in the "What's going on" column has helped in my homework	34/38	35/41
		89.5%	85.4%

procedure, so at least, they know what are the procedures to do as compared to my batch of students last year who were not very sure even when to begin with. So they mix up all the steps in that sense. So to put it in that way, I don't know it helps or not to make the procedures clearer in that sense. Step one, step two, step three, step four, rather than just one whole chunk of working, and I don't know what the hell is going on.

Teacher X was the chair of the meeting and gave his own views at the end. He felt that the column was important to show the students the reasoning behind the procedures: *Most of the things he mentioned already but the thing I added was the, I think the reasoning part to tell them the reason behind what we are doing is I guess, very important. Not just teach them the steps.*

Teacher E agreed: *I find your lesson quite good. I find the first few lessons, you were very procedural, then after that you talked more on their understanding.*

The discussion above suggests strongly the two teachers were conscientiously trying to teach better with the affordance of the "What's going on" column. The encouragement to students to write down their thoughts in the column both as the teacher was teaching and in their homework was intended to develop in students the disposition of keeping track of their solution steps and decision-making (d3). What seems interesting also is that the teachers also realised the affordance of the column to make their thinking clear, both procedurally and more importantly, the reasons for the procedures.

With the addition of other parts of the transcript that we have not deemed necessary to add, we think it is clear that both the teachers were focused on achieving the R-goals of helping students develop proficiency with both the technique of "Elimination" and "Substitution" (r1) and inculcating in students the habit of checking by substituting the obtained solution into a suitable equation (r2).

We finally refer to the results of the student perception survey. Table 19.2 shows the number of positive responses (Strongly Agree and Agree) to two related items out of the total class number for the two classes. In spite of the generally pessimistic view of the teachers regarding the use of the column by the students, the results suggest that at least from the students' point of view, the "What's going on" column has helped their understanding and in their homework.

19.6 Conclusion

Ambitious instructional goals provide a vision for teacher educators and teachers to strive towards desirable educational outcomes for students. However, goals that are too far removed from the real constraints within which teachers work on a regular basis can result in a disconnect between teacher educators and teachers—the former being driven by theoretical ideals and the latter by immediate solutions to practical problems of teaching.

In this chapter, we present an alternative approach—a frame of thinking and acting upon teaching goals and enactment—that would draw both teacher educators and teachers into a common ground for teacher development enterprises. This requires careful consideration of the realistic demands of teaching while aiming for goals that would still be considered ambitious for teachers. RAP is a way of conceiving instructional work that would both engage and challenge teachers' existing teaching practices. As such, it is a suitable pedagogy to ground sustainable teacher development programmes, the success of which is dependent on strong teacher motivations and buy-in.

It may be argued that the pedagogy that we advocate here lacks a matching classroom image—unlike the case, for example, of the "pedagogy" of cooperative learning, where one can easily imagine a class of students engaging in collaborative discussions as the main mode of learning. As such, it can be further contended that it is hard for teachers to conceive of how RAP looks like in their classrooms. Indeed, RAP does not map narrowly into certain fixed instructional forms in the classroom. In other words, teachers who purportedly adhere to this pedagogy can enact their classroom work in a variety of forms that nevertheless fulfil both the ambitious and the realistic goals of teaching. In fact, this feature of broad interpretation is intentional so that teachers participate in genuine pedagogical inquiry as co-designers of their lessons. It allows for teachers with different pedagogical starting points to benefit from a reconsideration of their existing ways of teaching in terms of ambitious yet realistic instructional goals.

Realistic ambitious pedagogy is not constructed as an immediately transformative pedagogy. It does not require teachers to make a big leap into the unknown. It allows teachers to take a small but decisive step into familiar and reconstructed territory. If that small step brings satisfaction in terms of better fulfilment of instructional goals, then it is a beginning of a sustained teacher development journey into the elusive but worthwhile target of ambitious teaching.

References

Amador, J., & Lamberg, T. (2013). Learning trajectories, lesson planning, affordances, and constraints in the design and enactment of mathematics teaching. *Mathematical Thinking and Learning, 15*(2), 146–170. https://doi.org/10.1080/10986065.2013.770719.

Assude, T. (2005). Time management in the work economy of a class, a case study: Integration of Cabri in primary school mathematics teaching. *Educational Studies in Mathematics, 59*(1–3), 183–203. https://doi.org/10.1007/s10649-005-5888-0.

Barksdale-Ladd, M. A., & Thomas, K. F. (2000). What's at stake in high-stakes testing: Teachers and parents speak out. *Journal of Teacher Education, 51*(5), 384–397. https://doi.org/10.1177/0022487100051005006.

Charles, R. I. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. *NCSM Journal of Mathematics Education Leadership, 7*(3), 9–24.

Diamond, J. B., & Spillane, J. P. (2004). High-stakes accountability in urban elementary schools: Challenging or reproducing inequality. *Teachers College Record, 106*(6), 1145–1176.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65–97). New York, NY: Macmiillan Publishing Company.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., et al. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Education Researcher, 25*(4), 12–21. https://doi.org/10.3102/0013189X025004012.

Jones, A. M. (2012). *Mathematics teacher time allocation* (Master's thesis, Brigham Young University, Provo, UT). Retrieved from http://scholarsarchive.byu.edu/cgi/viewcontent.cgi?article=4492&context=etd

Kazemi, E., Lampert, M., & Franke, M. (2009). Developing pedagogies in teacher education to support novice teacher's ability to enact ambitious instruction. In R. Hunter, B. Bicknell, & T. Burgess (Eds.), *Crossing Divides: Proceedings of the 32nd Annual Conference of the Mathematics Education Research Group of Australasia.* Wellington, New Zealand (Vol. 1, pp. 11–29). Palmerston North, New Zealand: Mathematics Education Research Group of Australasia.

Keiser, J. M., & Lambdin, D. V. (1996). The clock is ticking: Time constraint issues in mathematics teaching reform. *The Journal of Educational Research, 90*(1), 23.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge, United Kingdom: Cambridge University Press.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27*(1), 29–63. https://doi.org/10.3102/00028312027001029.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.

Lampert, M., Beasley, H., Ghousseini, H., Kazemi, E., & Franke, M. (2010). Using designed instructional activities to enable novices to manage ambitious mathematics teaching. In M. K. Stein & L. Kucan (Eds.), *Instructional Explanations in the Disciplines* (pp. 129–141). New York, NY: Springer Science + Business Media.

Leong, Y. H., & Chick, H. L. (2007/2008). An insight into the 'Balancing Act' of teaching. *Mathematics Teacher Education and Development, 9*, 51–65.

Leong, Y. H., & Chick, H. L. (2011). Time pressure and instructional choices when teaching mathematics. *Mathematics Education Research Journal, 23*(3), 347–362.

Leong, Y. H., Chick, H. L., & Moss, J. (2007). Classroom research as teacher-researcher. *The Mathematics Educator, 10*(2), 1–26.

Leong, Y. H., Tay, E. G., Toh, T. L., Quek, K. S., Toh, P. C., & Dindyal, J. (2016a). Infusing mathematical problem snolving in the mathematics curriculum: Replacement Units. In P. Felmer, E. Perhkonen, & J. Kilpatrick (Eds.), *Posing and solving mathematical problems: Advances and new perspectives* (pp. 309–326). Geneva: Springer.

Leong, Y. H., Tay, E. G., Toh, T. L., Yap, R. A. S, Toh, P. C., Quek, K. S., & Dindyal, J. (2016b). Boundary objects within a replacement unit strategy for mathematics teacher development. In B. Kaur, O. N. Kwon, & Y. H. Leong (Eds.), *Professional development of mathematics teachers: An Asian perspective* (pp. 189–208). Singapore: Springer.

Lester, F. K. (Ed.). (2003). *Teaching mathematics through problem solving: Prekindergarten - Grade 6*. Reston, VA: National Council of Teachers of Mathematics.

Meek, C. (2003). Classroom crisis: It's about time. *Phi Delta Kappan, 84*(8), 592.

National Education Commission on Time and Learning. (1994/2005). *Prisoners of time: Report of the National Education Commission on Time and Learning.* Washington, DC: Education Commission of the States.

Plank, S. B., & Condliffe, B. F. (2013). Pressures of the season: An examination of classroom quality and high-stakes accountability. *American Educational Research Journal, 50*(5), 1152–1182. https://doi.org/10.3102/0002831213500691.

Schielack, J. F., & Chancellor, D. (2010). *Mathematics in focus, K-6: How to help students understand big ideas and make critical connections*. Portsmouth, NH: Heinemann.

Schoen, H. L. (Ed.). (2003). *Teaching mathematics through problem solving: Grades 6-12*. Reston, VA: National Council of Teachers of Mathematics.

Valli, L., & Buese, D. (2007). The changing role of teachers in an era of high-stakes accountability. *American Educational Research Journal, 44*(3), 519–558. https://doi.org/10.3102/0002831207306859.

Wu, M., & Zhang, D. (2006). An overview of the mathematics curricula in the West and East. In F. K. S. Leung, K.-D. Graf, & F. Lopez-Real (Eds.), *Mathematics education in different cultural traditions: A comparative study of East Asia and the West* (pp. 181–193). New York, NY: Springer.

Yew Hoong Leong is an Associate Professor at the National Institute of Education, Nanyang Technological University. He began his academic career in mathematics education with the motivation of improving teaching by grappling with the complexity of classroom instruction. Along the journey, his research has broadened to include mathematics problem solving and teacher professional development. Together with his project teammates, they developed "Realistic Ambitious Pedagogy" and its accompanying plan of action—the "Replacement Unit Strategy".

Eng Guan Tay is an Associate Professor and Head in the Mathematics and Mathematics Education Academic Group of the National Institute of Education at Nanyang Technological University, Singapore. Dr. Tay obtained his Ph.D. in the area of Graph Theory from the National University of Singapore. He has continued his research in Graph Theory and Mathematics Education and has had papers published in international scientific journals in both areas. He is co-author of the books *Counting*, *Graph Theory*: *Undergraduate Mathematics*, and *Making Mathematics Practical*. Dr. Tay has taught in Singapore junior colleges and also served a stint in the Ministry of Education.

Khiok Seng Quek is a Senior Lecturer in the Psychological Studies Academic Group, National Institute of Education, Nanyang Technological University.

Sook Fwe Yap is a Senior Lecturer in the Mathematics and Mathematics Education Academic Group, National Institute of Education, Nanyang Technological University. Her research interest includes Statistics and Statistics Education.