Chapter 15
The National Institute of Education and Mathematics Teacher Education: Evolution of Pre-service and Graduate Mathematics Teacher Education

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Abstract The National Institute of Education (NIE) is the sole teacher education institution in Singapore. This chapter considers the various kinds of knowledge that make up the knowledge base of a mathematics teacher of substance, and illustrates in detail the various programme designs which NIE implements to ensure the development of such a knowledge base in pre-service and in-service mathematics teachers. The special features of NIE, such as the symbiotic relationship of its mathematicians and mathematics educators and its adaptation of good practices from around the world, are described in some detail to cast light on how it has been generally successful in carrying out this purpose. Challenges ahead and possible future directions for improvement are also discussed.

Keywords Teacher education · Teacher education programmes · Singapore · Mathematics education · Teacher knowledge · Pedagogical content knowledge · Subject knowledge · Postgraduate mathematics teacher programmes

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15.1 The National Institute of Education and Its Teacher Education Programmes

15.1.1 Teacher Education Programmes

What is the best teacher education programme structure?

A review of American teacher education programme structures (Arends and Winitzky 1996) lists six types of teacher education programmes. The predominant structure is that of the 4-year undergraduate model, commonly known as the Bachelor of Education programme. The other models are: (a) an extended 5-year bachelor's programme, (b) an extended 5-year bachelor *and* master's programme, (c) a fifth-year programme that leads to a master's, (d) a 6-year master's programme and (e) alternative certification programmes. Zeichner and Conklin (2005) state that the most common teacher education programme issues in the teacher education literature then were about whether 4 years were enough for the preparation of a teacher, making teacher preparation a completely postgraduate programme, and the establishment of alternative routes into teaching. Their paper continues with a quite comprehensive review of the literature and summarizes the results for 4-year versus 5-year programmes, and alternative versus traditional programmes. They conclude wistfully that "few definitive statements can be made about the effects of different structural models of preservice teacher education based on this body of research" (p. 698). They, however, cite the Teacher Education and Learning to Teach (TELT) study in suggesting that "it is programme substance and not structure that is key in influencing prospective teachers" (p. 701). TELT researchers differentiated programmes by their substance into two categories, viz. traditional programmes that emphasized the organization of students and classroom activities, and reform programmes that encouraged more learner-centred practices with much emphasis on subject-specific teaching. It was the latter that had more influence on the prospective teacher.

We turn to Finland, a country highly regarded for its education system, for another perspective of teacher education. The Finnish system is a two-tier 5-year teacher education programme. There are three main components in the structure: (a) academic disciplines, i.e. subject majors such as mathematics or history for secondary school teachers, and educational sciences for primary school teachers, (b) research studies that consist of methodological studies, a BA thesis and an MA thesis and (c) pedagogical studies that include teaching practice. According to decrees issued in 1979, 1995 and 2005, all teachers require a master's degree (Niemi 2012). The master's requirement is the result of the Finnish teacher education policy that necessitates education on research-based foundations. A Web-based survey conducted by Niemi (2012) among students in teacher education programmes at two Finnish universities found that students rated themselves highly in designing of instruction, critical reflection of own work, awareness of ethical basis of teaching profession, lifelong professional growth, self-evaluation of own teaching, using teaching methods and development of own educational philosophy. Students rated themselves lowest for administrative tasks, management of tasks outside the classroom, cooperation with parents and acting in conflict situations. The split in competencies fairly mirrors the "subject-specific against the administration" dichotomy of the American situation.

Put together, the American and Finnish situations set the context for viewing Singapore's teacher preparation programmes, mainly through highlighting two considerations, i.e. the "subject-specific against the administration" dichotomy, and the fact that "few definitive statements can be made about the effects of different structural models of preservice teacher education based on … research" (Arends and Winitzky 1996, p. 698). These help us to understand how Singapore has positioned itself in relation to good international practices and local constraints.

Singapore is a young nation, independent for barely half a century. It inherited its teacher training programmes from its British colonial masters and evolved it according to the needs of the nation. Singapore has always been known for its pragmatism. Mahbubani (2015) wrote the following in the Huffington Post:

> So why did Singapore succeed so comprehensively? The simple answer is exceptional leadership … Lee Kuan Yew, the founding prime minister …, Goh Keng Swee, the architect of Singapore's economic miracle, and S. Rajaratnam, Singapore's philosopher par excellence. Together, they made a great team. This exceptional team also implemented three exceptional policies: Meritocracy, Pragmatism and Honesty. Indeed, I share this "secret" MPH formula with every foreign student at the Lee Kuan Yew School, and I assure them that if they implement it, their country will succeed as well as Singapore. Meritocracy means a country picks its best citizens, not the relatives of the ruling class, to run a country. *Pragmatism means that a country does not try to reinvent the wheel. As Dr. Goh Keng Swee would say to me, "Kishore, no matter what problem Singapore encounters, somebody, somewhere, has solved it. Let us copy the solution and adapt it to Singapore."* [italics added]

In education, and in teacher training, Singapore has pragmatically looked at the systems in the world and adapted the right "wheel" for its use. In teacher education, "the strength of [Singapore's] initial teacher preparation lies in the strong integration between content and pedagogical preparation, the design and development of which is backed by evidence-based educational research" (Gopinathan 2010, p. 140). Here, we see that all the positive aspects of subject-specific teaching and research have been assimilated into a coherent teacher training framework. Indeed, Chen and Koay (2010) in their preface describe teacher education in Singapore as being "guided by pragmatic principles, a blend of philosophy of the East and West unique to Singapore, which evolved as the newly established nation responded to the challenges of the times" (p. xii). Singapore education officials, school leaders and researchers regularly travel to other countries to learn best practices for adaptation at home. In addition, education officials and significant academics visit Singapore to see what is happening here and to share their views and recommendations.

15.1.2 Evolution of Teacher Education in NIE: Programmes, Students and Teacher Educators

We have today in Singapore the National Institute of Education (NIE) as the sole teacher education institution in the country. The following is a short history of the evolution of the teacher education programmes summarized from *Transform-*

ing Teaching, Inspiring Learning (Chen and Koay 2010). The Teachers Training College (TTC) was established in 1950. It conducted certificate courses in education for non-graduates. In the same year, a School of Education was established in the local university to train graduates for teaching on a full-time basis. Students were conferred a Diploma in Education (Dip. Ed.). In 1971, the School of Education was closed and TTC became the only institution responsible for teacher training. It entered into a new relationship with the university, whereby besides certificate courses, it also prepared graduate students for the Dip. Ed. In 1973, the Institute of Education (IE) was established from the TTC. It offered a 2-year full-time or a 3-year part-time Certificate in Education (Cert. Ed.) programme for non-graduates, and a 1-year full-time or an 18-month part-time Dip. Ed. programme for graduates. On 1 July 1991, the NIE was established as an institute of the Nanyang Technological University (NTU). As part of the university, new 4-year degree programmes were offered to matriculated students. These programmes, Bachelor of Arts with Education and Bachelor of Science with Education (collectively called B.A./B.Sc. (Ed.)), imparted both subject matter knowledge and pedagogical knowledge to student teachers. Non-graduates and graduates training to be teachers took the two-year Dip. Ed. and the one-year Postgraduate Diploma in Education (PGDE) programmes, respectively.

We turn our attention now to focus mainly on the student teacher intake for the years since the establishment of IE in 1973. The first decade of IE continued with the providing of quality training of teachers against the backdrop of the rising demand for a larger number of qualified teachers in the schools. Compared to colonial times when only a fraction of the population could afford school, the new nation of Singapore was intent on educating all its children. Schools could be and were built quickly to house all the children, but getting enough quality teachers was not as easy. Most student teachers in TTC in the past were trained for the primary school. Thus, to cope with the demand particularly in the secondary schools, graduates of the two local universities at that time were trained in part-time programmes in IE under the teaching cadetship scheme and were awarded the Dip. Ed. on completion. These teacher cadets studied and taught at the same time, assuming two-thirds of a regular teacher's workload during their 18-month cadetship. This makeshift approach lasted until 1980 when all pre-service programmes in IE became full time. In the early years of the TTC, candidates for teacher training did not all possess high academic qualifications. Some had not even completed twelve years of school themselves. In IE, however, only candidates with GCE "A"-Level qualifications were considered for the 2-year Cert. Ed. programme. When IE became NIE in 1991, teacher education in Singapore was re-established within a university framework. The recruitment of teachers was ramped up greatly, and NIE took in students through the PGDE, Dip. Ed. (for non-graduates) and a totally new 4-year degree programme (B.A./B.Sc. (Ed.)) which combined subject matter knowledge and pedagogical knowledge within the same university setting. The intakes for each year up to 2012 were large and comprised about 2000 new student teachers each year from the three programmes with about 500 from the degree programme (Gopinathan 2010; MOE 2016, p. 34). Although the prerequisites for enrolment in the programmes were not brought down, the variation in the quality and experience of the students was great. For example, the "A"-Level results of candidates for the degree programme differed widely. Also, student teachers in the PGDE could come from degree backgrounds as diverse as engineering, law and accountancy. Many of the engineering graduates were to be trained as mathematics teachers. This would have an effect on the mathematics teacher education programmes, as we will read later. With a target of 33,000 teachers for Singapore (MOE 2012), the intake tapered off after 2012 until the present intake of about 1000 a year.

Finally, in this section, we shall describe the evolution of the teaching staff in teacher education. IE staff in 1975 consisted of 5 (4.8%) with doctorates, 34 (32.7%) with masters, 23 (22.1%) with a first degree and 42 (40.4%) non-graduates. As a result of the intentional upgrading of staff and the recruitment of better-qualified staff, in 1982, the staff composition was now 19 (12.0%) with doctorates, 88 (55.3%) with masters, 27 (17.0%) with a first degree and 25 (15.7%) non-graduates (Chin 2010). Believing that "the most important single factor for the quality of education is the quality of the teachers' training" (Barber and Mourshed 2007) and that the quality of teacher training depends heavily on the quality of the teacher educator, NIE continued to recruit strong academics until currently, for example, in the Mathematics and Mathematics Education (MME) Academic Group, all 27 full-time teaching staff hold doctorates. The B.A./B.Sc. (Ed.) programme made a very significant change in NIE staff recruitment. Academics with doctorates in content specializations were now recruited to teach the subject matter knowledge. Since NIE is part of a world-class university, NTU, staff on the professorial tenure track would be required to publish in high-quality journals in their area of specializations. For example again, there are 14 academics in MME who hold doctorates in mathematics across a number of fields such as Graph Theory, Number Theory, Integration Theory, Domain Theory and Statistics. The symbiosis of mathematicians and mathematics educators would be a major factor in an exceptional mathematics teacher training programme.

15.2 Knowledge Base for the Mathematics Teacher

15.2.1 Three Forms of Knowledge for the Mathematics Teacher

What should a mathematics teacher know to be able to teach effectively?

Shulman (1986) in his seminal paper on teacher knowledge contrasted the examinations for Californian elementary school teachers in 1875 and in 1986. While the former emphasized assessment of subject matter knowledge covering topics such as Written Arithmetic, Mental Arithmetic, Written Grammar, Geography, History of the United States, Theory and Practice of Teaching, Algebra, Physics, Constitution of the USA and California, School Law of California, Biology, Reading and Vocal Music, the latter emphasized the assessment of capacity to teach, covering categories such as organization in preparing and presenting instructional plans, evaluation, recognition of individual differences, cultural awareness, understanding youth, management and educational policies and procedures. He pointed out the "absence of focus on subject matter among the various research paradigms for the study of teaching" (p. 6) at that time in the 1980s. He noted the cleavage between content and pedagogy apparent from the emphasis on one and then the other in the 1870s and the 1980s. He then traced the apparent dichotomy to the time in the medieval universities when they were not considered separately but were both considered essential to be a university doctor or "dottore" which means teacher. "The tradition of treating teaching as the highest demonstration of scholarship" (p. 7) was derived from Aristotle who "distinguish[ed] the man who knows from the ignorant man [by] an ability to teach" (p. 7).

Thus, we have at least two forms of teacher knowledge, viz. "content knowledge" (i.e. subject matter knowledge) and "general pedagogical knowledge" (Shulman 1986, p. 9). General pedagogical knowledge, by its emphasis in many teacher education programmes today, is well known to include aspects of educational psychology, classroom management, general teaching craft such as questioning techniques and organizing learning in collaborative groups, and assessment. These are generally independent of the subject matter, for example, assessing the validity and reliability of biology tests and those of mathematics tests are basically the same.

Shulman (1986) then proposed a perspective on subject matter knowledge in teaching that encompassed three kinds: (a) content knowledge, which refers to "the amount and organization of knowledge per se in the mind of the teacher" (p. 9), (b) pedagogical content knowledge, which "goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge *for teaching*" (p. 9) and (c) curricular knowledge, so as to draw from the curriculum and its associated materials almost as a "pharmacopeia" (p. 10) of tools for presentation, exemplifying, remediation and evaluation. For example, with (a), a teacher knows that "-1" is the additive inverse of "1" in the field of real numbers; with (b), the teacher can represent "-1" as a "concrete" object, such as a symbol on a card that when put together with another card labelled "1", *eliminates* both, or as a point on the number line equidistant from "0" as "1"; and with (c), the teacher will avail herself of concrete materials such as AlgeCards, or computer software that teaches operations with negative numbers, if such are available. Figure 15.1 shows how $3 + (-4)$ can be represented in the two different ways mentioned in (b), with the first representation availing itself of materials as in (c). A teacher with good pedagogical content knowledge would know different representations and can decide wisely which to use in different class settings.

The further differentiation of subject matter knowledge by others such as Ball et al. (2008) has shown that proficiency in this area is much more than getting a degree in the subject. For our purpose of explaining the structure and objectives of mathematics teacher education in Singapore, it suffices for us to focus on the main differentiation of subject matter knowledge into content knowledge and pedagogical content knowledge as first perceived by Shulman. Thus, using Shulman's lens, NIE teacher education is organized along three main components: educational studies for general pedagogical knowledge, academic subject for content knowledge and curriculum studies for pedagogical content knowledge.

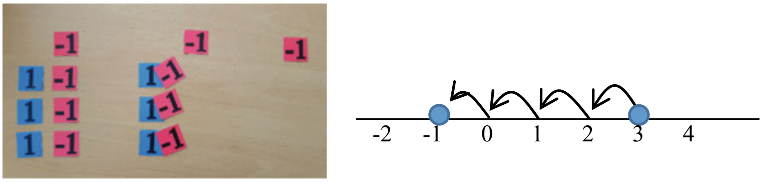

Fig. 15.1 Different representations of 3 + (−4)

15.2.2 Preparing Mathematics Teachers of Substance: Issues for Consideration

We agree with Shulman that "mere content knowledge is likely to be as useless pedagogically as content-free skill" (Shulman 1986, p. 8). Thus, we agree that general pedagogy knowledge is essential for a teacher and that it should be a significant part of a teacher education programme. In this chapter, however, we shall leave further exposition of this component and instead focus on subject matter knowledge and its two manifestations, content knowledge and pedagogical content knowledge. From here on, we shall only discuss the preparation of mathematics teachers. Shulman continued his work with like-minded colleagues, and together they asserted that an examination of subject matter knowledge of teachers is important to teacher educators (Grossman et al. 1989). To prepare "teachers of substance", they argued that "teacher educators must share the responsibility for the transmission of subject matter knowledge to prospective teachers" (p. 24).

The first reason they gave was that it had become "increasingly clear that … [teacher educators] can no longer assume that the subject matter component of teacher preparation is fulfilled by undergraduate coursework in other departments" (Grossman et al. 1989, p. 24). The realities that they state to support this assertion resonate even in Singapore today. They state that within an academic major, in different universities and even in the same university, requirements vary for different specializations. In Chap. 5 of this book, Ho et al. (2018) interviewed a number of professors of mathematics in three Singapore university mathematics departments and they concurred that "content reduction is one of the most significant changes that took place at the university level for undergraduate mathematics degree programmes" (p. 94). It would seem that the training in mathematics rigour would be adversely affected and mathematics graduates may have problems with analysis concepts such as convergence and limits, which would manifest themselves in secondary school calculus.

Grossman et al. (1989) also observed that the overlap between the content of courses at the university level and at the school level is "tenuous at best" (p. 24). For example, Geometry is hardly offered in undergraduate mathematics courses nowadays, though it is an important topic in the school. In the Singapore universities, it is also possible for a mathematics major to avoid courses in Statistics. Although one can possibly learn on the job when required to teach Statistics in the schools, this ability for all teachers cannot be taken for granted. It would be better that the deficiencies are addressed in the teacher education programme.

Finally, Grossman et al. (1989) point to the difference between content knowledge and pedagogical content knowledge. It is highly unlikely that future teachers would learn pedagogical content knowledge from their professors in their undergraduate courses if the content department were divorced from the teacher education department, which is the case in the majority of universities. As it is quite impossible to convince the mathematics professor in the mathematics department to focus a little on pedagogical content knowledge, it would seem more feasible that teacher educators take up the mantle to teach pedagogical content knowledge and to fill up deficiencies in content knowledge.

The establishment of NIE as an autonomous institute of the NTU in the 1990s gave Singapore teacher education an exceptional opportunity to plan programmes that take into account the new understandings of teacher knowledge from Shulman and his colleagues. NIE's placement within a university setting allowed it to design a bachelor's programme that combined a subject matter specialization and a teaching certification. Within the preparation of mathematics teachers, faculty with doctorates in mathematics and with doctorates in mathematics education were recruited in equal numbers to enable a symbiosis of expertise crucial to actualizing a holistic learning environment for the three forms of knowledge. It helped also that many faculty members with doctorates in mathematics were also qualified school teachers before.

The majority of pre-service teachers would still go through the PGDE programme. They would come into NIE with degrees from other universities. Some would have gone through undergraduate mathematics programmes that did not overlap well with the mathematics content in schools. Of greater concern would be the fact that some of the student teachers were not mathematics majors. Some had engineering degrees and had been accepted by the Ministry of Education to teach mathematics when there was a need to quickly ramp up the number of secondary school teachers. These student teachers lacked content knowledge, but there was no time in the one-year PGDE programme to conduct content upgrading. A stopgap measure was implemented to at least raise the awareness of the student teachers about their own level in content knowledge by making them take a School Mathematics Mastery Test. The inadequacy of the PGDE programme with regard to content knowledge would be partly ameliorated with the provision of in-service content courses as well as a postgraduate Masters in Science (Mathematics for Educators).

The final consideration was for primary school teachers who had to teach mathematics but who were not mathematics majors. Keenly aware that some of these teachers actually disliked mathematics, could only follow procedures in mathematics and thus would only teach procedures subsequently, a component called Subject Knowledge was included in the programme for primary school teachers. This component resonated with Ma's (2010) assertion that elementary mathematics is a "field of depth, breadth, and thoroughness" (p. 122).

Each of the programmes mentioned in the paragraphs above—their motivations, evolutions, structures and implementations—will be fleshed out in the following sections of this chapter. The sections will follow the distinction between pedagogical content knowledge and content knowledge. All these developments are a result of NIE's aspiration to have programmes that would prepare mathematics teachers of substance.

15.3 Pedagogical Content Knowledge (PCK)

15.3.1 Pre-service PCK

We begin with curriculum studies (CS) in mathematics, which is designed to give student teachers the pedagogical skills in teaching mathematics in Singapore schools from the perspective of PCK. This component is offered in the pre-service PGDE, B.A./B.Sc. (Ed.) and the Dip. Ed. programmes. In the CS Mathematics courses for the Dip. Ed. programme, student teachers specialize in the methodology for teaching mathematics at the primary level, while the CS Mathematics courses for the B.A./B.Sc. (Ed.) programme prepare student teachers to teach mathematics at either the primary or secondary/junior college level. The CS Mathematics courses for the PGDE programme also offer courses for teaching at either primary or secondary/junior college level. In this section, we describe the key characteristics of our CS primary and secondary mathematics courses, namely (i) a common foundation, (ii) mathematical problem-solving and school mathematics topics, (iii) relevance to Singapore schools and (iv) responsiveness to the changing educational landscape.

15.3.1.1 Common Foundation

Key common course content. One of the changes in the primary CS Mathematics courses as a result of the curriculum review of 2004 was keeping the contents of the CS Mathematics courses in the different primary programmes similar (Lim-Teo 2009). Samples of course outlines for the various CS Mathematics courses offered at NIE, Singapore, over the last decade (from 2007 to 2017) were analysed, and the analysis shows key common course content and assessment modes across CS Mathematics primary and secondary courses in the Dip. Ed. programme, B.A./B.Sc. (Ed.) (Primary), B.A./B.Sc. (Ed.) (Secondary), PGDE (Primary) and PGDE (Secondary) programmes. The common contents of the CS Mathematics courses include the Singapore mathematics curriculum, general psychological theories for learning mathematics, lesson planning, mathematical problem-solving, test construction, student misconceptions or errors and the teaching of the various school mathematics topics spelt out in the Singapore mathematics curriculum.

Most of the CS Mathematics courses are conducted through lectures and tutorials, supported by a technology-enhanced environment through the blackboard course management system (Wong et al. 2012). E-lectures on various mathematics topics (e.g. matrices, probability, real-life applications of mathematics) and topics, e.g. van

Hiele Theory in the CS Secondary Mathematics courses, encourage student teachers to “learn things for themselves”. E-learning is also implemented in some of the CS Mathematics courses to encourage student teachers to be self-directed learners.

Key common assessment modes. Student teachers taking the CS Mathematics courses as described above are assessed through common key assessment components such as design of mathematics lesson plans, PCK written test, design of mathematical problem-solving task, test construction and use of technology in the teaching and learning of mathematics. In addition, one of the assessment components requires student teachers to teach an assigned concept, algorithm, word problem, etc. (e.g. microteaching). This is usually followed by reflections on their teaching. Having such a range of assessment modes in the CS Mathematics courses exposes the student teachers to the realities of teaching.

Anchoring the above CS Mathematics courses in a common foundation is important because it provides a common language for the mathematics education community in Singapore. Furthermore, the common foundation was carefully constructed to provide our pre-service teachers with the critical capabilities supported by sound theoretical groundings to make prudent pedagogical judgements and decisions in their mathematics classrooms. Common modes of assessment are used to ensure that key skills (such as pedagogical skills, reflective skills and thinking dispositions) and knowledge (such as knowledge of student, pedagogy, curriculum and assessment) are evenly developed through the CS Mathematics courses across the different programmes.

15.3.1.2 Mathematical Problem-Solving and School Mathematics Topics

Central to the Singapore mathematics curriculum is mathematical problem-solving. The Singapore mathematics syllabus also encourages exposure to problem-solving approaches such as Pólya’s model (Pòlya 1971). The emphasis on mathematical problem-solving and reasoning is reflected in the CS Mathematics course outline, but it may be interwoven into the strands, as well as taught separately in the CS Mathematics courses. The problem-solving approaches and heuristics used are based on Pólya’s model and approaches (Lim-Teo 2009).

Another characteristic of the CS Mathematics courses is that the main bulk of the course is spent on the teaching of various mathematics topics in the school curriculum (school mathematics topic structure). For example, for the primary CS Mathematics courses, topics such as whole numbers, fractions and decimals (classified as Number and Algebra Strand in the Singapore primary mathematics curriculum) are included. This characteristic of the CS Mathematics courses enables student teachers to re-examine their understanding of the mathematics topics, in particular, from the perspectives of students in Singapore mathematics classrooms. Indeed, the school mathematics topic structure provides opportunities for CS Mathematics course instructors to delve more deeply into students’ learning difficulties and diagnosis of students’ errors and explore strategies and teaching approaches to remedy misconceptions specific to each of the school mathematics topics. These learning experiences are crucial for student teachers as they need these experiences to sharpen their abilities to identify and address those learning difficulties and errors. The experiences provided will also develop their capacities to anticipate possible learning difficulties and errors that their students may have and get student teachers "ready" to address those errors when they encounter those situations in their teaching. Wong et al.'s (2012) analysis of primary student teachers' performance in mathematics PCK assessed by the TEDS-M study shows that student teachers may "need more opportunities to learn about ways to deal with pupils' misconceptions in mathematics" (p. 304). In addition to dealing with the common students' errors in the teaching of various mathematics topics, the first batch of student teachers in the 16-month PGDE programme (inaugurated in December 2016) also had the opportunity to complete an error analysis task during their Teaching Assistantship (TA) school stint.

The school mathematics topic structure also provides instructors and the student teachers more opportunities to share, discuss, unpack and reflect on examples of classroom practice specific to the various mathematics topics, and this brings us to the next characteristic of the CS Mathematics courses, that is, direct relevance to the realities of local classroom teaching.

15.3.1.3 Relevance to Singapore Schools

The Singapore model method (MOE 2009), a pedagogical strategy developed by a team of curriculum specialists in the Singapore Ministry of Education, is a unique feature in the Singapore mathematics curriculum, and it is used widely in the Singapore primary classrooms. By unpacking the model method and how the model method can be integrated with the algebraic method to help students formulate algebraic equations to solve problems, student teachers are empowered to translate this pedagogical approach directly into the Singapore mathematics classrooms, thus enhancing the links of the CS Mathematics courses to the local classroom teaching. Pedagogy is partially culturally situated in this sense. Pedagogy is also partially universal (Cai et al. 2009). As such, resources that integrate local experiences and research with international "best practices", for example, Lee and Lee (2009) and Lee (2009), were also published for the CS Mathematics courses (Wong et al. 2012).

To establish the theory-practice links, student teachers are assisted to reflect critically on several aspects of their learning during their CS Mathematics courses. For example, during their microteaching, opportunities are created for student teachers to reflect and relate to the theories and any classroom experiences that they already had (such as in their previous practicums or in their pre-enrolment contract teaching). The critical reflection is vital for student teachers to develop multiple perspectives of teaching and learning mathematics, identify potential challenges and suggest alternative solutions to overcome those challenges. With the TA school stint (in addition to the original final teaching practicum or field-based experience) and developmental nature (spread throughout the entire programme) of practicum for the B.A./B.Sc.

(Ed.) programme, student teachers can tap upon their multiple field experiences to make greater personal sense of the theory-practice links as they engage in critical reflection during the CS Mathematics courses.

A suite of videos of authentic teaching—authentic in that the lessons were conducted in a naturalistic classroom context—is also used in some of the CS Secondary Mathematics courses to facilitate the link between the realities of actual classroom practice and theories gleaned from the courses. "A critical aspect of bridging theory and practice involves strengthening the link between research and practice" (NIE 2012, p. 13). As such, CS Mathematics course instructors have "integrated mathematics pedagogical principles from international research and practices with local contexts and lessons learned from local implementations" (Wong et al. 2012, p. 297).

15.3.1.4 Responsiveness to the Changing Education Landscape

CS Mathematics courses have been revised over the years, with the core foundations still intact, to keep abreast of the rapid changes both locally and internationally. Responsiveness to changes due to recruitment and education initiatives launched by the Ministry of Education, Singapore, will be elaborated below.

As mentioned earlier, student teachers in the PGDE could come from various degree backgrounds. The PGDE secondary student teachers specialize in teaching two subjects at the secondary school level, namely CS1 as the first teaching subject and CS2 as the second teaching subject. Applicants for the PGDE secondary programme taking up CS Secondary Mathematics will be designated either one of the three tracks: CS1 Mathematics, CS2 Mathematics and CS2 Lower Secondary Mathematics. Those in the CS2 Lower Secondary Mathematics "are not required to have studied tertiary mathematics, but they must have good grades in their O-level or A-level mathematics" (Wong et al. 2013, p. 206). They "meet lower criteria than for CS2 [Secondary] Mathematics and they will only be prepared to teach mathematics at lower secondary level because of their lack of mathematical background" (Lim-Teo 2009, p. 53). There was some concern over the mastery of mathematics content for teaching at secondary levels among the teachers when the majority of PGDE secondary student teachers doing CS1 Mathematics were not mathematics majors and when the number of CS2 Lower Secondary Mathematics student teachers grew tremendously. To address this concern, a School Mathematics Mastery Test (SMMT) was introduced in 2003 to provide student teachers the opportunity to be aware of their current state of mathematical knowledge so as to start them on self-improvement for mastery of secondary school mathematics content. There was also a concern about primary teachers' subject understanding (Lim-Teo 2009, p. 64). A Subject Knowledge (SK) component was introduced in some programmes to address this problem (Lim-Teo 2009).

CS Mathematics courses have adapted over the years to remain relevant to the needs of teachers. The adaptation was necessary for our student teachers to acquire the skills, knowledge and disposition to implement the initiatives by the Singapore Ministry of Education (MOE). For example, one of the key changes in the 2012

Singapore secondary mathematics curriculum is the explication of Problems in Real-World Context (PRWC). As such, PRWC was included as one of the topics in the CS Secondary Mathematics courses to prepare student teachers to implement PRWC tasks in their mathematics classrooms. Guest lectures by MOE, Curriculum Planning and Development Division, were also arranged for our student teachers on topics such as AlgeDiscs and Algebars, and Learning Experiences—new features in the 2012 Singapore mathematics curriculum.

15.3.2 Postgraduate PCK

It is well known that pre-service training cannot be sufficient for the needs of the teacher. In the first place, there is not enough time to cover all aspects of teaching. In addition, learning without the information of actual practice is deficiently one-dimensional. We close this section on PCK with a description of NIE's postgraduate programme to further develop teachers' pedagogical content knowledge through reflective practice and research.

The Master of Education (Mathematics) programme, or M.Ed. (Maths) for short, is the mathematics education specialization within the NIE wide Master of Education programme, designed for mathematics educators in Singapore schools and other mathematics education professionals. The main aim of this specialization is to develop within the participants the capacity to reflect deeply upon their own mathematics instructional practices, so as to prepare them for career development in leadership positions in schools. The duration of the programme is between 2 and 4 years for those participants studying part time, and between 1½ and 2 years for those studying on a full-time basis. To graduate from the programme, participants need to earn 30 academic units, or AU's for short, where 1 AU corresponds to 1 h per week of instruction over a 13 week semester, and most of these academic units are gained from taught courses. For the more academically inclined participants, the programme also serves as an induction into contemporary mathematics education research, by exposing them to the latest scholarly and professional work in this area. For this purpose, candidates can opt to replace two taught courses with a 6 AU dissertation, running over multiple semesters, which is an independent piece of research work conducted under the guidance of a supervisor appointed from the faculty.

Since its inception, the M.Ed. (Maths) programme has achieved very high levels of success, and the reader is directed to Lim-Teo (2009) and Tay et al. (2017) for more detailed descriptions of its history and structure. Nevertheless, it is worth highlighting here some of the more significant factors which have contributed to its success. First, with the exception of a small number of foundational courses which introduce the participant to educational research in general, the great majority of the course studies are in mathematics education. Second, all the courses are taught by members of the faculty who are themselves active researchers in the fields of mathematics and mathematics education, and who are often also qualified school teachers, which means that they are able to pass on to participants the latest theories and perspectives.

Third, the great majority of participants are themselves active teachers of mathematics in schools, who often come to their lessons directly from their own classrooms and who therefore possess very high levels of motivation to apply what they are learning in the M.Ed. (Maths) programme to improve their own day-to-day instructional practices. Thus, the participants and faculty together form a community of focus similar to the "teaching research groups" engaged in "intensive study" which Ma (2010) identified as being a principal characteristic contributing to the strength of mathematics education in China.

Despite these successes, a growing imbalance began to emerge between the content and pedagogic strengths of the participants, just as Shulman (1986) warned, albeit seen from the following more positive perspective. Specifically, as NIE became better and better at imparting strong pedagogic skills to its pre-service teachers, exigencies of deployment during the "expansion" era meant that it became feasible to deploy into classrooms teachers whose content background did not necessarily match the subjects they were expected to teach. For example, it became very common in secondary schools to have mathematics taught by teachers who were not mathematics majors, or even majors in closely related subjects such as Physics or Chemistry, but rather in other quantitative disciplines such as engineering or economics, and such teachers naturally sought to upgrade their skills in programmes such as the M.Ed. (Maths). A second issue was that roughly half the participants were primary school teachers, and given the generalist nature of primary school teaching, it was common to find such teachers who despite being graduates and having a strong interest in mathematics were non-science majors. To partially address these issues, some entry restrictions into the M.Ed. (Maths) programme were imposed, namely that participants should have taken at least two mathematics courses during their undergraduate studies and that they should be active teachers of mathematics. Since the whole point of the M.Ed. (Maths) programme is to encourage teachers to upgrade themselves, these restrictions were necessarily quite mild, and other more positive measures to address the content weakness of participants were taken as follows.

The first positive measure was to include alongside the elective courses in mathematics education an equal number of elective courses which were intended to be a synergy of mathematics content and pedagogy. Generically entitled "X and the Teaching of X", these courses covered most of the subjects taught in schools up to secondary level, including Arithmetic, Algebra, Geometry, Statistics and Discrete Mathematics. Ideally taught by a pair of faculty, one staff member specializing in content and one specializing in pedagogy, the aim of these courses was to simultaneously reinforce content mastery, while at the same time drawing out the pedagogic implications for classroom practice. While the intentions behind the crafting of these courses were sound, in practice they were challenging to teach, since to succeed well required very high levels of cooperation between the content and pedagogy staff involved. Unfortunately, even to arrange for a pair of staff to co-teach was often quite difficult, and the exigencies of staff deployment often constrained the assignment of a single member of staff to teach the course in its entirety. Despite the best of intentions of the staff assigned, this inevitably resulted in a weakening of the synergy in the direction of the area of specialization of the staff member involved, be it content or pedagogy.

The second positive measure was to include alongside the core course in mathematics education a stand-alone content course entitled “Fundamental Concepts in Mathematics”. Compulsory for all participants in the M.Ed. (Maths) programme, the principal aim of this course was to level up the participants’ mastery of fundamental concepts, so as to provide a foundation for the later “X and the Teaching of X” courses. From the outset, this course took a broad historical perspective and introduced participants to the evolution of the key mathematical ideas and concepts underpinning primary and secondary school mathematics. This broad historical perspective, which contributed significantly to the popularity of the course, also made it rather challenging for one staff member to teach alone, so for many years it was co-taught very successfully by a pair of staff, both in content. Nevertheless, despite its popularity, this breadth made it very difficult to cater to the needs of both primary and secondary school teachers, since historically these relate to quite different eras in the history of mathematics. Therefore, as part of the recent major restructuring described below, a reluctant decision was taken to break up this course and instead to reincorporate its components into the newly recrafted “X and the Teaching of X” courses.

Starting in 2016 the M.Ed. (Maths) programme was subject to major reviews, one internally by the academic department for the purposes of quality improvement and the other externally as part of an NIE wide programme review. The main aim of the external review was to concentrate the 30 AUs required to graduate into a smaller number of larger courses (essentially 4 AU as opposed to 3 AU) so as to make it feasible for full-time participants to graduate within one calendar year. One of the major aims of the internal review was to better serve the particular needs of primary school teachers by creating a “primary track” within the M.Ed. (Maths) programme. This is not a stand-alone programme, but rather consists of a sequence of courses which, despite being open to all participants, are marked out as being of particular relevance and interest to those teaching at primary level. Another aim of the internal review was to further strengthen the content components of the programme, by making a renewed effort to allow the “X and the Teaching of X” courses to fulfil their potential. This consisted of reversing the previous practice, that is, first crafting the courses and only then searching for a pair of staff willing to co-teach it; instead, pairs of staff were hand-picked for their proven ability to cooperate well, and then together they recrafted the course description with that cooperation built in from the very start.

15.4 Content Knowledge (CK)

15.4.1 Academic Subject (AS)

The Academic Subject courses, abbreviated as AS courses, are a set of courses offered to degree programme students who either major (AS1 Mathematics) or minor in mathematics (AS2 Mathematics).

The AS1 Mathematics students comprise two groups in the B.Sc. (Ed.) programme whose first teaching subject is mathematics: specialists in teaching primary school mathematics and specialists in teaching secondary school mathematics. Since their first teaching subject of the AS1 Mathematics students is mathematics, these students are expected to have a larger base of subject matter knowledge. Thus, AS1 Mathematics students are required to complete a total of 17 courses for AS1 during their four years of tertiary mathematics education. AS1 student must complete a core set of compulsory courses: 4 in the first year, 6 in the second year and an academic exercise at the end of the fourth year. The remaining courses are all electives. In contrast, AS2 Mathematics students will be deployed to teach secondary mathematics as their second teaching subject. (We note that these students may have their AS1 subject in either the Arts or the Sciences; their programmes will then be respectively, B.A. (Ed.) and B.Sc. (Ed.).) Consequently, AS2 Mathematics student teachers are only required to read tertiary mathematics in their first year of study, which comprises four compulsory courses. Table 15.1 displays the courses to be completed for the degree requirement of the AS students under the respective tracks.

A look at the range of the mathematics courses that AS Mathematics students are required to take in the B.A./B.Sc. (Ed.) programme, as shown in Table 15.1, shows that the degree programme at NIE has courses with similar, if not identical, titles offered in traditional mathematics programmes at other universities worldwide. These courses give a comprehensive coverage of the "subject matter component of teacher preparation" (Grossman et al. 1989, p. 24) which may not be fulfilled by a student teacher taking the PGDE route. At this point, we highlight a question often raised by student teachers in the primary track—"Why do we need to study such difficult mathematics which will not be used in primary school?" To this question, we justify as follows. Firstly, as a subject major one is expected to possess the disciplinarity of that subject. For example, an English Literature major must know Shakespeare even though Shakespeare is never taught in primary school; a mathematics major ought to know the notion of infinite countability even though primary school children rarely count beyond a million. Secondly, there are aspects of mathematics study that undergird the content per se, such as problem-solving disposition, rigour and the ability to read new mathematics. These disciplinarity aspects need to be transferred to the primary school students; indeed, a non-mathematics major will most likely emphasize on the procedural aspects of mathematics, while a mathematics major is likely to engage students in problem-solving, problem posing, understanding symbols (reading mathematics) and some rigour.

Table 15.1 Course structure for AS1 and AS2 Mathematics in the B.A./B.Sc. (Ed.)

Year	Courses	Remarks
Year 1	Linear Algebra I	Compulsory Year 1 core subjects common to both tracks
	Calculus I	
	Finite Mathematics	
	Number Theory	
Year 2	Linear Algebra II	Core for all AS1 Mathematics students
	Calculus II	
	Statistics I	
	Computational Mathematics	
	Differential Equations	
	Complex Analysis	
Year 3	Special Topics in Mathematics I	AS1: Three electives
	Statistics II	
	Real Analysis	
	Modern Algebra	
	Modelling with Differential Equations	
	Statistics III	
	Combinatorial Analysis	
Year 4	Academic Exercise: Mathematics	AS1: Core course
	Special Topics in Mathematics II	AS1: Three electives
	Statistical Theory	
	Applied Statistics	
	Techniques in Operations Research	
	Mathematical Programming and Stochastic Processes	
	Metric Spaces	
	Galois Theory	
	Geometry	
	Advanced Mathematical Modelling	

Crucially, the B.A./B.Sc. (Ed.) programme at NIE offered to mathematics student teachers is fundamentally *distinctive* in its design and implementation that targets at certain aims, which we now elaborate.

In his efforts to modernize mathematics education in Germany during the early 1900s, Felix Klein, one of the leading mathematicians of his time, deplored what he termed as the "double discontinuity"—a problematic experience faced by mathematics students as they move from high school to university, and then back again to the profession of school mathematics teachers:

> The young university student found himself, at the outset, confronted with problems, which did not suggest, in any particular, the things which he had been concerned at school. Naturally, he forgot these things quickly and thoroughly. When, after finishing his course of study, he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way; and since he was scarcely able, unaided, to discern any connection between this task and his university mathematics, he soon fell in with the time honoured way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching. (Klein 1908/1932, p. 1)

The first "discontinuity" highlights the well-known problems of transition which mathematics students struggle with as they learn mathematics at the tertiary level for the first time (Gueudet 2008; Thomas 2008). The major problems encountered by undergraduate mathematics students at this stage concern reading and writing mathematical texts, understanding and constructing rigorous mathematical proofs. This situation is not helped as the student struggles to simultaneously grapple with the new content knowledge and the aforementioned deficiency in skills. The second "discontinuity" concerns the difficulties experienced by mathematics teachers in transferring "academic knowledge gained at university to relevant knowledge for a teacher" (Winsløw and Grøbæk 2014). Cognizant that the "double discontinuity" continues to plague student teachers of our present age, MME designs and implements the AS courses to achieve two overarching objectives. Firstly, the degree programme should facilitate a smooth and effective transition into tertiary mathematics for the AS Mathematics students. Secondly, learning opportunities should be available to AS Mathematics students allowing them to look at the school mathematics from a higher standpoint, and crucially, together with training received in the CS courses, these teachers-to-be will be equipped with the ability to transfer the mathematical content knowledge gained from the AS courses to relevant pedagogical content knowledge for a teacher.

As mentioned earlier, MME has a natural advantage over many other mathematics departments in terms of the composition of its academic staff; i.e. the staff members comprise both mathematicians and mathematics educators (indeed, there are those who are both!). The synergy between the mathematicians and the mathematics educators, simple as it sounds, manifests as follows. Mathematicians identify those "higher standpoints", that is, those mathematical concepts and results at the tertiary level that can impact strongly on the understanding of mathematical concepts taught and learnt in schools, while mathematics educators inform mathematicians of the salient pedagogical theories that underpin the learning of mathematics at the tertiary level. Zeroing in on the vexing problem of the "double discontinuity", MME looked to the pedagogical findings gleaned from rigorous educational research concerning the problems encountered in teaching tertiary mathematics. A work of Alcock and Simpson (2009) was brought to the attention of the Academic Group, and therein it was reported that mathematics students at high school take a number of years for "development from an action through a process to an object conception before they begin to use the concept at university" (p. 22). Furthermore, at the university level "a similar development is necessary, but a much shorter time period is available" (ibid., p. 22). This research finding alerted that there was simply not enough time to teach reading and writing, understanding and construction of proofs, when the knowledge content needs to be covered *concurrently*.

For this reason, MME saw that a possible solution to the identified problem would seem to be a total curriculum review that rightly involved all the academic staff who were teaching the curriculum. Tyler (1949) proposed a basic model that provides the needful framework for such a major curriculum review. In short, Tyler's model demands, first, that the objectives of the curriculum are positioned in a matrix with the existing modules of the programme so that the design can ascertain which cell in the table will be activated, i.e. which module can be used to attain the objective (Tay and Ho 2016).

In this curriculum redesign that began in July 2015, MME considered carefully the trinity of learning objective, learning experience and assessment. Accordingly, learning objectives were classified under six domains: (1) content, (2) cognition, (3) problem-solving, (4) computation, (5) communication and (6) disposition. Pertaining to "cognition", one of the objectives read as "At the end of the degree mathematics programme, the learner should be able to read mathematical text or language with understanding". This learning objective was further unpacked into four sub-objectives: the ability to read (i) a definition, (ii) a theorem, (iii) a proof and (iv) a mathematical text, with corresponding learning experiences and assessments. In the actual implementation, for instance, the learning experiences for reading a definition was realized by the requirement that "given a definition, come up with examples/non-examples", "compare related definitions and identify the differences" and "come up with special cases" and "visualize definitions". Correspondingly, assessments were 'given a new definition (which may not be covered in the course), determine whether a given object satisfies the definition; come up with an example/non-example', and 'given several definitions, determine whether some given objects satisfy each of the definitions'.

For each learning sub-objective, courses across the four years in the programme (see Table 15.1) were then designated to meet it. For example, "be able to read a definition" was designated to Linear Algebra I (Year 1, Semester 1), Number Theory (Year 1, Semester 2), Calculus II (Year 2, Semester 2) and Complex Analysis and Linear Algebra II (Year 2, Semester 2). The philosophy behind such a curriculum design and its implementation is that things are delivered in bite-size which are reinforced over a period of time. Each lecturer took ownership of enacting the curriculum with the aim of achieving the designated learning sub-objectives while ensuring a comprehensive coverage of the required content—each sub-objective would be covered in five courses over four semesters.

Recently, a systematic review of the curriculum implementation for the Year 1, Semester 1 (July 2016 Semester), was carried out with promising findings. For instance, Calculus I students performed extremely well in answering traditionally difficult ε-δ/M definitions for continuity and limits. Importantly, the same report underscores that "success over four years has to be a team effort (of the entire Academic Group of MME) with each doing his or her part in line with a well-thought-out curricular plan" (Ho et al. 2017).

We now turn to elaborate how the curriculum redesign addresses the second "discontinuity", i.e. the difficulties of transferring the mathematical content knowledge to the relevant pedagogical content knowledge of a mathematics teacher. Under the domain "content" that was identified in the classification of learning objectives in the degree programme, it was stressed that the NIE Degree Programme will equip the student teacher with a solid foundation of school mathematics and the canons of undergraduate mathematics. Specifically, this learning objective was unpacked into three sub-objectives: (i) possess deep understanding of all the different topics in school mathematics (up to A-Level Mathematics) from a higher standpoint (by the end of Year III), (ii) possess fluency in carrying out standard mathematical procedures in school mathematics and (iii) possess the mathematical background practice and the mathematical rigour needed for them to be able to proceed to postgraduate studies in mathematics. To help the reader better understand how the "content" sub-objective (i) was realized in the degree programme, we extract an episode of a lesson in Calculus I as an illustration. This episode was contributed by the lecturer who taught Calculus I in the July 2016 Semester in an interview.

In Week 8 (after the definition of the differentiability of a function and the derivative of a differentiable function have been taught), the lecturer introduced the concept of turning point on the curve $y = f(x)$ of a continuous function f defined over some open interval I, whose definition is given below:

> **Definition (Turning points).** Let f be a continuous real-valued function defined on an open interval I, and $a \in I$. A point $(a, f(a))$ on the curve C: $y = f(x)$ about which there is no change in sign for $f(x) - f(a)$ for some open deleted neighbourhood of a is called a *turning point* of C.

At this juncture, the students were tasked to give two examples of turning points on the graph of continuous functions. When asked if it is necessarily true that the gradient of the graph at a turning point is zero, *all* the students responded positively and affirmed their claims using the examples they came up with. It is worth noting that the students' examples were all polynomial functions which are not only continuous but also differentiable on the interval of definition. Without giving further comments on the students' examples, the lecturer showed two examples. The first one was the maximum turning point (0, 0) on the parabola $y = -x^2$, and the second one was the minimum turning point (0, 0) on the graph of $y = |x|$. The students validated the two given examples against the above definition of a turning point. The students were then asked if there were any confusion or conflict with their pre-existing understanding of the concept of turning points. To this question, several remarked that they were taught in A-Level Mathematics (during their times as junior college students) to determine the turning points of a curve by setting $\frac{dy}{dx} = 0$ and to solve the equation for the x-coordinate of the turning point(s). The students realized that "the function may not be even differentiable at the turning point, let alone requiring its derivative to be 0 [pointing at the second example given by the lecturer]". The students then quickly responded by saying that "it is not *necessarily* the case that a turning point be stationary (a term they have used since high school)". In response to their remark, the lecturer then displayed the following theorem and its corollary:

> **Theorem.** Let f be a continuous function defined on an interval I and $a \in I$. If $(a, f(a))$ is a turning point on the curve C: $y = f(x)$, then $f'(a) = 0$ if the derivative at a exists.
>
> **Corollary.** If f is differentiable on an interval I, then every turning point of the curve C: $y = f(x)$ in I is a stationary point (i.e. a point at which $f'(a) = 0$).

At this point of the lesson, one of the students responded as follows:

> Now I realise why the A-Level method for finding turning points work … these [functions] are restricted to only the differentiable ones …

This episode illustrates how student teachers in their AS courses were guided to acquire for themselves a deeper understanding of the school mathematics at a higher standpoint. In this case, the *implicit* requirement that the functions involved in the determination of turning points via differentiation (an A-level mathematics technique) are restricted to only the differentiable ones is the salient CK required of the teacher.

What we have elaborated concerning the AS curriculum review, as guided by Tyler's framework, was just one of the several instances of synergy between the mathematicians and the mathematics educator colleagues in MME. One notable teaching innovation, among many others, that MME implements in the AS curriculum is that of Mathematics Problem-Solving (MPS). For more information on the role of MPS in NIE Degree Programme, we refer the reader to Chap. 7 of this book.

15.4.2 Subject Knowledge (SK)

The Subject Knowledge courses, or SK courses for short, are a set of three courses covering all of the content taught at primary level, namely Number Topics (arithmetic and number operations), Geometry Topics (properties of figures and mensuration) and Further Topics (Algebra, discrete mathematics and elementary statistics). These courses are offered to primary student teachers across all programmes (diploma, degree and PGDE) except that degree students majoring in mathematics take an abbreviated version, and the reader is directed to Lim-Teo (2009) and Tay et al. (2017) for comprehensive accounts of the structure and evolution of these courses at a programme level. The main aim of the SK courses is to raise the level of understanding of the mathematical foundations of these topics at primary level among pre-service teachers so that they can themselves go on to teach these topics both confidently and correctly. A secondary aim is to ensure the smooth and efficient delivery of the curriculum studies, or CS, courses, uninterrupted by any prior content weakness among the pre-service teachers, and for this purpose the SK courses are normally scheduled to take place immediately before the corresponding CS course. Finally, as pointed out by Lim-Teo (2009, p. 65), although the CS staff at NIE are exemplary teachers, what they are teaching is pedagogy, not content, making it hard for trainees to model their own classroom practice directly on what they experience in the CS courses. So, a final unintended but quite significant benefit of the SK courses is that they are an opportunity for the pre-service teachers to experience for themselves what it is like to learn mathematics in an exemplary pedagogic environment.

Due to the generalist nature of primary school teaching in the USA, where all primary teachers are expected to be able to teach all of the core subjects, the market in the USA for textbooks generically titled “Mathematics for Elementary School Teachers” is very large. For the sake of efficiency, therefore, when the SK courses were first being offered at NIE, it was decided to adopt one of these texts, and of the many titles available the one authored by Billstein et al. (2001) was chosen as the most suitable. As a commercially produced text, this brought many immediate advantages, such as its comprehensive coverage (one text covered all three SK courses), reliability (it was then in its seventh edition) and quality (especially the figures and use of colour printing). It also brought several minor inconveniences, principally the weight of the text (many students resorted to tearing it into three parts), the cost (trainees had to purchase their own copy) and also the peculiar US habit of retaining imperial units (Singapore like the UK having converted to the metric system long ago). A major issue, however, was that the level of rigour, while appropriate for the US market, fell below what it was felt student teachers at NIE were capable of, since they are able to specialize to a greater degree than primary teachers in the USA, so after a few years of careful use this textbook was set aside. Although more advanced texts existed, none was deemed a suitable replacement, so the decision was taken to produce a set of notes in house, which as well as addressing the issue of rigour would also neatly resolve the minor issues of weight, cost and the use of non-metric units.

These new notes were written by the content staff, all of whom possess doctorates in mathematics content, divided into groups according to areas of specialization (in fact one staff member was a researcher expert in both content and pedagogy) so the levels of rigour were unimpeachable. Once these new notes were put into use, however, the danger highlighted by Tay et al. (2017) soon became apparent, which in fact afflicted a whole generation of authors in the 1960s and the 1970s (Keedy 1969; Griffiths and Hilton 1970; Campbell 1970; Hunter et al. 1971; Mendelson 1973) who attempted to write mathematically rigorous accounts of primary topics. This danger is the very large gap which exists between the modern language of mathematical rigour, which is generally abstract, axiomatic and deductive, and the language of the primary mathematics classroom, which is concrete, constructivist and inductive. The practical outcome, therefore, was that although the student teachers were generally able to master the content of the notes without great difficulty, they were unable to make the connection between the SK courses and the primary mathematics curriculum. Accordingly, in 2010 it was attempted to draft an entirely new set of notes, to be used as an alternative alongside the existing set, which would attempt to bridge this gap by adopting a completely different approach. This new approach sought to draw upon "other ways of understanding" the material, more suited to the needs of school teachers, so in a sense constituted a "postmodern" approach to mathematics teaching, based on the following three points of focus.

The first point of focus was historical and observed that the modern language of mathematical rigour did not exist at the time when the originators of most of the topics in the primary mathematics curriculum made their contribution. They must have had an entirely different way of thinking about the material, one which, given the very early date in history when these topics first appeared, was probably much closer to the constructivist mathematics classroom environment. For example, the modern "definition" of Hindu–Arabic numeration, expressed in terms of sums of single-digit multiples of powers of ten, uses the kind of Algebra and exponent notation unknown to the Hindu mathematicians who first evolved this system almost 2000 years ago. So, in the revised set of notes, historical numeration systems were treated in much greater detail than is usually the case in the generic "Mathematics for Elementary School Teachers" texts, especially the reasons motivating the choice of symbology and size of base in the different simple grouping and place value systems studied. Surprisingly, it became clear that ciphered place value systems are a very natural evolutionary step on from simple grouping systems, in the sense that several "obvious" measures taken to make simple grouping systems symbolically and computationally more efficient naturally lead to ciphered place value. In particular, it became evident that the key feature of place value systems is the regrouping property, a point highlighted by Ma (2010), and not the absolute size of the quantities represented; that is, it is essentially a recursive system, in which the distinction between whole numbers and fractions is secondary.

The second point of focus, which follows from the first, is that the historical development of mathematics is far from linear and that in many cases essentially the same mathematical concepts arose independently in several different eras and locations, and expressed in different ways. This suggests an alternative route to Ma's (2010) "profound understanding of fundamental mathematics"; namely that by exposing student teachers to these alternative expressions of the same mathematical concept, they are able to distinguish better between what is essential and what is incidental, than if they only studied a single instance, namely what is taught in the contemporary primary classroom. For example, in the revised version of the notes, heavy emphasis was placed on getting the trainees to become proficient at carrying out the main arithmetic algorithms ($+ - \div \times$) not only in a variety of different number bases other than base ten, but also using a variety of different synthetic symbologies, created by analogy with Hindu–Arabic digits, in some cases created by the student teachers themselves. Initially, it was quite a shock for the student teachers, as they realized quite how much of what they had previously thought of as their "understanding" of arithmetic algorithms was merely rote learning, but by deconstructing their prior knowledge, and then reconstructing it in this more general context, their understanding was deepened. Another example would be the contrast between the classical Greek approach to Geometry, based on the concept of parallel lines, and the modern concept of Geometry based on the notion of a global sense of direction, which although apparently quite different can easily be shown to be logically equivalent. This modern approach is in fact closer to the constructivist classroom environment, such as making use of children's natural belief that the sum of the exterior angles of a polygon is 360°, and the concept of logical equivalence enables trainees to see that the distinction between what counts as a "definition" and what is classified merely as a "property" of a geometric figure is often quite arbitrary or a matter of convenience; e.g. parallelograms can just as easily be "defined" to be figures with opposite equal sides as figures with opposite parallel sides.

The third point of focus, following on from the other two, is to understand that the evolution of mathematics is far from over and that many potential improvements remain to be made, in both the near and the far future. From this perspective, student teachers come to appreciate that often what makes certain parts of mathematics difficult to teach is not a lack of ability on their part, but rather deficiencies in the mathematics itself. For example, it is widely appreciated in the research literature, but hardly at all among teachers that the size of the base ten in Hindu–Arabic numeration is too large and that use of a lower base around base five would be both computationally more efficient and allow for more meaningful symbology. Although a global shift to base five is not a realistic possibility in the near future, reasoning along these lines uncovers ways in which even base ten arithmetic might be improved, such as inventing a symbol for "ten" for use with regrouping. Similarly, the classical Greek approach to Geometry could certainly be improved upon by a shift to a more modern approach based on the concept of symmetry, which is more appealing to children and also aligns better with how Geometry is taught at tertiary level. Symmetry transformations also naturally lend themselves to a dynamic hands-on approach, should share the same advantages over traditional logic-based approaches to Geometry that research has shown to be the case for dynamic geometry software (DGS), and are clearly much closer to the constructivist primary mathematics classroom.

15.4.3 Postgraduate Content Knowledge

Three common traits among countries with top school systems identified in the executive summary of the 2007 McKinsey report are (i) getting the right people to become teachers, (ii) developing them into effective instructors and (iii) ensuring that the system is able to deliver the best possible instruction for every child (McKinsey & Company 2007, p. 1). Regarding (ii), the 2007 McKinsey report highlighted two case studies which exercised deliberate emphasis on professional development for teachers: (1) policymakers in Finland raised the status of the teaching profession by requiring that all teachers possess a master's degree. (2) Singapore policymakers have achieved a similar result by ensuring the *academic rigour* of their teacher education courses, as well as providing all teachers with a substantial number of hours of fully paid professional development training each year.

The Ministry of Education (MOE) in Singapore takes a serious view when it comes to developing its teachers professionally and has since gone beyond the 100 hours of professional development scheme mentioned in McKinsey (2007). Started in 2005, the Professional Development Continuum Model (PDCM) scheme provides graduate teachers with alternative pathways to higher certification. Cognizant of the fact that beginning teacher preparation courses are merely a first step to ensure "getting the right persons to become teachers", MOE intends that the PDCM scheme motivates teachers to keep relevant in content proficiency and pedagogy, thereby "developing them into effective instructors" (McKinsey & Company 2007, p. 1). Here, we go into the specifics of what this statement means concerning the mathematics teachers recruited by MOE and the professional development made available to them. For Singapore, although all mathematics teachers in government schools, by requirement of MOE, are PGDE graduates and thus have received preparation in pedagogical matters concerning their teaching subjects, not all of these are mathematics graduates. The truth is that there is a large number of engineering graduates and/or graduates from other mathematics-related disciplines (e.g. Computer Science) joining MOE as mathematics teachers. Many of such teachers, in their reflections, sounded out their need to deepen their content knowledge in mathematics to be confident classroom teachers. To illustrate this lack of solid mastery of content knowledge, we give some concrete examples of questions, contributed by school students, teachers and NIE mathematics faculty staff, which pose difficulties with regard to the mathematics content of school mathematics taught in Singapore.

- Why is partial fraction decomposition of a rational function always unique? ['O'-Level Additional Mathematics]
- Does definite integration give an approximation to the area under a graph or its exact value? ['O'-Level Additional Mathematics]
- Does $\frac{\overline{X}-\mu}{S/\sqrt{n}}$ always have a t-distribution with $n-1$ (and why not n?) degrees of freedom? ['A'-Level H2 Mathematics]
- How can we produce different integral triplets (a, b, c) so that they always represent the length of the sides of a right-angled triangle? Standard textbooks present $(3, 4, 5)$, $(5, 12, 13)$, $(8, 15, 17)$ and their integral multiples, are these all? ['O'-Level Mathematics]
- How can the value of π be computed? Can we write $\pi = \frac{22}{7}$ and then conclude that it is a rational number? But we know from the textbook that π is irrational. Do we have a contradiction? How can one be certain that π is indeed irrational? ['O'-Level Mathematics]
- Angles taught in primary school are measured in degrees, why do we need to introduce the radian measure as an alternative unit of measurement for angles? ['O'-Level Mathematics]
- The Binomial Theorem seems an isolated topic classified as further Algebra in "O"-Level Additional Mathematics. How often is it relevant to a mathematics learner to be able to find the term independent of x in the expansion of some expression such as $\left(x-\frac{1}{x}\right)^6$? Where else is the Binomial Theorem used? ['O'-Level Additional Mathematics]

- How come differentiation and integration are inverses of each other? When I integrate the derivative of x^2, I get $x^2 + C$, but the constant C need not be zero? [‘O’-Level Additional Mathematics]

Additionally, there are other aspects of work such as curriculum planning and matters related to gifted education that demands understanding of mathematics from a higher and more holistic perspective.

In view of the aforementioned subject matter knowledge demands on mathematics educators, NIE offers a Master of Science (Mathematics for Educators), M.Sc. (MfE) for short, by coursework to in-service teachers with the aim of making mathematics teachers content proficient and, as a result, developing them to be effective instructors of mathematics. For this reason, the programme bears a meaningful name “Mathematics for Educators” and is designed specifically to provide rigorous preparation in advanced mathematics for mathematics teachers. The distinctive feature of this programme is its emphasis of the acquisition of rigorous mathematics at postgraduate level, together with the deep connection with the mathematical topics taught in schools. The programme is backed by the belief that an effective mathematics teacher must be, first and foremost, a teacher who himself/herself must have a *sound* knowledge of mathematics and can teach *correct* mathematics to his/her students.

At this juncture, it is natural to ask: “How is this deep connection with the school mathematics realized in this graduate programme?” The answer to this question lies in the course structure. By factoring in some of the content-related questions raised earlier, some of the courses are intentionally designed to impart subject matter knowledge that can be used to address these questions. Such courses are categorized meaningfully by practising mathematicians (most of whom have prior school teaching experience) as Level 1 courses since they specifically highlight the deeper mathematical structure underlying the topics of Mathematics and Additional Mathematics listed in the Singapore school mathematics syllabi. Table 15.2 shows how Differentiation and Integration in a Level 1 course, Elements of Mathematical Analysis with Applications in the Teaching of Calculus, is aligned to equip the mathematics teacher with higher mathematics to teach the calculus in “O”-Level Additional Mathematics and “A”-Level H2 Mathematics.

After a student has attained a firm foundation in that part of the tertiary mathematics which is related to the mathematics he/she is teaching in schools, he/she would proceed to deepen his/her understanding of the topic further. This is made possible by the placement of Level 2 courses. Often requiring certain prerequisite Level 1 course(s), the mathematics content covered in the Level 2 courses is more abstract and sophisticated in nature. For example, the Level 1 course, Elements of Mathematical Analysis with Applications in the Teaching of Calculus, can lead to its corresponding Level 2 courses: Real Analysis, Functional Analysis and Topology. In order to maintain a high level of academic rigour in the programme, a student is required to complete at least four Level 2 courses.

In order to graduate from this programme, a student must complete a total of ten courses, including the capstone course, Mathematical Inquiry. This capstone course is a mandatory course where a student works with a practicing mathematician in a

Table 15.2 Alignment with school mathematics

Topic	Contents	Connection with school mathematics
Differentiation and Integration	Limits and continuity of functions	Continuity of certain standard functions is assumed, e.g. in the determination of range of functions
	Differentiation	Rules of differentiation (O-Level Additional Mathematics and A-Level H2 Mathematics)
	Rolle's theorem, mean value theorem	Increasing and decreasing functions determined by sign of derivative (O-Level Additional Mathematics and A-Level H2 Mathematics)
	Definite integral as a limit of a sum, fundamental theorem of calculus, indefinite integrals	Riemann sum, definite integral as the limit of Riemann sum, Differentiation and Integration are "inverses" of each other (O-Level Additional Mathematics), finding indefinite integrals of standard functions (O-Level Additional Mathematics), using standard forms, substitution, integration by parts (A-Level H2 Mathematics)

specific research field of Pure Mathematics or Applied Mathematics. In this course, the student is expected to carry out independent study and give an oral presentation on the research work performed. The main intention of this course is for the student to finally be able to put together all his/her learning in an academic exercise. The M.Sc. (MfE) programme distinguishes itself from many other masters by coursework programmes by insisting that the students go through the experience of reading and writing mathematics—which should be the hallmark of a literate mathematics graduate.

Apart from the course structure, the way in which the substance of the programme is delivered is also of paramount importance in realizing the aims and objectives of the M.Sc. (MfE) programme. Again, the pedagogical theories espoused by the MME mathematics educators informed their mathematician colleagues how best to teach the content of mathematics crafted out in the various courses. Notably, useful research findings in problem-solving derived from the Mathematics Problem-Solving for Everyone (MProSE)—a funded research project in MME—guided the lecturers of Number Theory and the Teaching of Arithmetic, Real Analysis and Theory and Applications of Differential Equations in their classroom didactics by taking advantage of the practical paradigm of Mathematics Problem-Solving (Ho et al. 2014). Another example was the innovative use of flipped classroom pedagogy in teaching Topology as an attempt to resolve problems caused by the heavy cognitive load of set theory in learning Topology and to raise learner's motivation in the course (Ho and Chan 2016). The point here to make is that the distinctive symbiosis of mathematicians and mathematics educators in NIE realizes a high-quality professional development programme that benefits in-service mathematics teachers in Singapore schools. Readers may wish to peruse the positive feedback given by students who graduated from the M.Sc. (MfE) programme in Tay et al. (2017, pp. 126–128).

15.5 Conclusion

In this chapter, we have painted in broad brushstrokes the role NIE has been playing in mathematics teacher education in Singapore. By identifying the various knowledge domains relevant to mathematics teacher education, i.e. subject matter knowledge and pedagogical knowledge, we see how NIE has designed and implemented a holistic spectrum of teacher training and professional development programmes that equip both pre-service and in-service mathematics teachers in Singapore with the twenty-first-century competencies specific to teaching and learning of mathematics in schools.

15.5.1 Summary

From our preceding development, the alert reader might have already been aware of two important earlier works completed by NIE mathematics educators that touched on the issues of preparing mathematics teachers in Singapore. The first work by Lim-Teo (2009) elaborates on the evolution and development of mathematics teacher education via the pre-service and in-service programmes offered in NIE in the period from the late 1990s to 2009. The second work by Tay et al. (2017) examines the issue of Mathematics Content Knowledge in connection with mathematics teacher education in Singapore. Our current work not only extends Lim-Teo (2009) in the sense that we give account of the various NIE Mathematics and Mathematics Education courses since 2009 but also follows up on areas of discussion left open by Tay et al. (2017). We close this chapter by summarizing what was accomplished recently and what could be done in the near future to improve the quality of mathematics teacher education provided by NIE; along the way, we also address some of the remarks made in Tay et al. (2017).

Academic Subject. A major curriculum revamp based on Tyler's 1949 framework is currently being implemented for the AS courses, as an AG-wide effort, with focus on six domains: content, cognition, problem-solving, computation, communication and disposition. Each AS course has been assigned to address one or two of these domains along with delivering the subject matter knowledge under its coverage. As the implementation of the new AS curriculum is still ongoing, it is important to stay vigilant to see that the student teachers receiving the training in these six domains are indeed acquiring the targeted domains through the various AS courses.

This can be achieved by constant monitoring of the student teachers' performance in the assessment tasks as well as through the feedback via interviews. Interviews with the AS course lecturers have also been carried out to obtain feedback on the implementation of the revised curriculum.

Subject Knowledge. Tay et al. (2017) remarked that as to "what type of CK (Content Knowledge) is actually needed or appreciated by the teachers remains not completely understood" (p. 128). This statement was made pertaining to the design of SK as NIE's attempt of achieving "better understanding of Mathematics related to elementary Mathematics as conceptualised by Ma (2010)" (p. 128). The proposal put forward in Tay et al. (2017) was then to motivate the SK course with "actual elementary school problems and concepts and building the course materials directly by which prospective and practising teachers can be made to see the relationship of what they are learning to their teaching" (p. 128). In response to this proposal, the SK lecturers collaborated to collate a new set of notes that aims at bridging the gap between the topics covered in SK and the Primary School Mathematics Syllabus. They adopted a "postmodern approach" in the style of writing these notes with three specific points of focus. These points of focus include (1) the *historical nature* of mathematical development: the modern language of mathematical rigour was actually brought into existence when the originators of most of the topics in the primary mathematics curriculum made their contribution; (2) the *nonlinearity* of the historical development of mathematics: simultaneous and independent emergence of new mathematical concepts; and (3) the *ongoing nature* of mathematical development. For the next step, it is of paramount importance to examine to what extent the use of the new in-house notes has improved student teachers' connection of SK topics with the Primary School Mathematics Syllabus. One of the ways is to formally study the effects of SK courses on student teachers who had graduated from the programme and are currently teaching in primary schools. The question to answer is "Does the revised SK course produce better primary school teachers?" In addition, just as the AS curriculum is being revamped based on Tyler's 1949 framework, the SK courses have also been reviewed in Tyler's framework, with special emphasis on getting the learning objectives and their attendant learning experiences and assessments right.

Master of Science (Mathematics for Educators). Tay et al. (2017) suggested that the M.Sc. (MfE) programme "will benefit from a review of its courses from a perspective of Usiskin's teachers' Mathematics, perceived as a generalisation of Ma's profound understanding of fundamental Mathematics" (p. 128). Indeed, at around the time of writing of this present work, the mathematicians who have been directly involved in teaching the M.Sc. (MfE) courses are proposing a restructured M.Sc. (MfE) programme that will be launched in August 2018. This restructuring will result in the number of Academic Units (AU) for each course (i.e. academic credit points earned by students upon completing a course) increasing from 3 AU to 4 AU. An increase in the AU per course not only allows for a deeper treatment of the subject matter in the course but also creates the opportunity for the course lecturer to consider the use of modern computer and video technologies for non-face-to-face instructions. In order to avoid a compromise of breadth for depth of coverage, the collection of courses is streamlined into a coherent body of mathematical knowledge organized along four different strands: Analysis–Geometry, Algebra–Number Theory, Discrete Applied Mathematics and Statistics. Maintaining the crucial feature of connecting advanced mathematics to school mathematics, each strand offers one to two courses at "Foundation Level" with the intention of equipping non-mathematics majors with the essential subject matter knowledge needed to move up higher along the strand, i.e. to read courses pegged at the "Advanced Level".

Recognizing that a majority of the twenty-first-century competencies are skill-based, the restructured programme proposes a new 2 AU course that focuses on the honing of mathematics research skills for mathematics educators. It is hoped that by imparting a wide scope of skills, which includes literature review and citation, problem-solving, reading pertaining to mathematics, typesetting mathematical texts, posing research questions and communication, mathematics teachers become more confident in teaching research-related skills to their students along the disciplinarity of a mathematician.

15.5.2 The Way Forward

This chapter has considered the various kinds of knowledge that make up the knowledge base of a mathematics teacher of substance, and illustrated in detail the various programme designs which NIE implemented to ensure the development of such a knowledge base in pre-service and in-service teachers. While NIE teacher educators can focus on designing top-quality teacher preparation courses and professional developmental programmes, the ultimate responsibility of translating the salient knowledge acquired in these courses into effective enactment of the mathematics curriculum still rests upon the shoulders of the *competent* mathematics teacher—whose characteristics form the subject of discussion in the next chapter.

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