Chapter 11
Metacognition in the Teaching and Learning of Mathematics

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Abstract This chapter first presents the evolving conceptualisation of metacognition since it was first coined by Flavell in 1976. In particular, the issue of awareness, monitoring, and regulation of both cognitive and affective resources was examined. The role that metacognition plays in mathematical problem-solving was also examined, leading to a discussion of the role of metacognition in the Singapore School Mathematics Curriculum which has mathematical problem-solving as its central aim. In view of this, the conceptualisation of metacognition as well as the how's of addressing metacognition in the Singapore mathematics classrooms were discussed from the intended curriculum point of view. Some of the local postgraduate works on metacognition and teaching, and learning of mathematics was also presented to provide an overview of the landscape of the work in this area that has been undertaken thus far. In addition, examples of ongoing works on metacognitive approaches, which have made some inroads in some local schools, were shared to give the reader a glimpse of how research in this area has impacted school practices locally. The chapter concludes with implications for addressing metacognition in the Singapore Mathematics classrooms from the perspective of teachers' professional development.

Keywords Singapore School Mathematics Curriculum · Cognition · Metacognition · Offline metacognition · Online metacognition · Metacognitive instructional strategies · Mathematical problem-solving · Teaching and learning of mathematics · Reflection · Reflective practice model · Meta-metacognition · Theory of mind · Social metacognition

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11.1 Introduction: Metacognition

11.1.1 Conceptions of Metacognition

Metacognition is a term that was first coined by Flavell in 1976 as:

> Metacognition refers to one's knowledge concerning one's own cognitive processes and products or anything related to them … metacognition refers, among many other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects on which they bear, usually in the service of some concrete goal or objective. (Flavell 1976, p. 232)

Flavell's idea of metacognition involved knowledge about and of cognition as well as processes of monitoring and regulating of cognitive actions. He then developed other related metacognitive constructs following that, such as metacognitive knowledge and metacognitive experience (Flavell 1979, 1987), contributing to a body of work that provided the foundational knowledge for the theory of metacognition. However, Schoenfeld (1992) observed that there is no agreement among researchers on a single definition of the term metacognition. Flavell et al. (2002, p. 164) refined metacognition further by referring it to 'metacognitive knowledge and, metacognitive monitoring and self-regulation'. Though a rather extensive survey of the related literature, Loh (2015, p. 29) found that researchers 'tend to agree that the main elements of metacognition are knowledge or awareness of cognition, monitoring and regulation'.

In fact, the Singapore Ministry of Education (MOE) (2012a, p. 19) describes metacognition as 'thinking about thinking' and that it involves 'the awareness of, and the ability to control one's thinking processes… It includes monitoring of one's own thinking and self-regulation of learning'. In fact, Swartz and Perkins (1990, p. 109) in viewing metacognition as 'thinking about thinking', referred to it as 'a crosscutting superordinate kind of thinking relevant to all the others'.

In addition, the work on brain research in the 1990s, which has been labelled as the Decade of the Brain by then US president George H. W. Bush as part of a larger effort involved to enhance public awareness of the benefits to be derived from brain research, has also contributed to conception of metacognition. Freeman (1995, p. 89) pointed out that while the frontal lobes of the brain allow us to elaborate on the details of our goals and plans, it's emotions that generate them and drive their execution. So, Chang and Ang (1999) proposed at a presentation at the 8th International Conference on Thinking at Edmonton, Canada, to add the knowledge of 'personal affective resources' to the knowledge of 'personal cognitive resources' as proposed by Schmitt and Newby (1986). The term 'affective resources' rather than 'emotions' has been used so as to be more encompassing in the conceptualisation. Leder (1993, p. 46) defined 'affect' as a term 'used to denote a wide range of concepts and phenomena including feelings, emotions, moods, motivation and certain drives and instincts'. McLeod (1992, p. 579) states that:

> Beliefs are largely cognitive in nature, and are developed over a relatively long period of time. Emotions, on the other hand, may involve little cognitive appraisal and may appear and disappear rather quickly. Therefore we can think of beliefs, attitudes and emotions as representing increasing levels of affective involvement, decreasing levels of cognitive involvement.

In fact, Ng (2009), in addressing the role of metacognition in the Singapore School Mathematics Curriculum, refers to it as the 'children's awareness of and the ability to monitor and control their thinking'. And, she added that it involves children learning about 'the dynamic use of mathematics and about themselves as problem solvers, their attitudes towards mathematics, mathematics teaching and learning, as well as their own monitoring capabilities' (Ng 2009, p. 20). This is similar to Dweck's (2012) idea of a growth mindset. Dweck (2006) refers mindset as a predisposition or fixed mental attitude that determines how an individual will respond to and interpret situations and make decisions. In a growth mindset, students understand that their talents and abilities can be developed through effort, good teaching, and persistence.

11.1.2 Metacognition and Problem-Solving

Making use of the distinction between novice-expert problem solver by Sweller and Low (1992, p. 84), Barkatsas and Hunting (1996) found that the level of employment of metacognitive strategies appeared to be different between the two groups. Successful problem solvers tend to reflect on their problem-solving activities, have available powerful strategies for dealing with complex and unknown problems, and regulate (even subconsciously) powerful strategies efficiently. On the other hand, though novices have acquired problem-solving strategies, they were observed to be less aware of the utility of them and do not use them effectively in the acquisition of new learning.

In fact, Loh's (2015) detailed analysis of Polya's four-phase problem-solving model (Pólya 1957) pointed towards notions of metacognitive activities to be evident in all the four phases of the model. As she pointed out, the absence of the explicit use of the word 'metacognition' in Polya's work could easily be attributed to the fact that the word was only coined by Flavell in 1976. It is no wonder that Silver (1987) noted that 'no process model of problem solving in any domain can be complete without an adequate account of the role of metacognition'.

Davidson et al. (1994) observed that all problems contain three important components: givens, a goal, and obstacles. They referred 'givens' to the elements, their relations, and the conditions that compose the initial state of the problems situation, while the 'goal' is the solution or desired outcome of the problem. The obstacles, from their perspective, are the characteristics of both the problem solver and the problem situation that make it difficult for the solver to transform the initial state of the problem into the desired state. They perceive problem-solving as the active process of trying to transform the initial state of a problem into the desired one, and metacognition helps the problem solver to:

- recognise that there is a problem to be solved
- figure out what exactly the problem is, and
- understand how to reach a solution

They also identified four metacognitive processes that are important contributors to problem-solving performance across a wide range of domains:

- identifying and defining the problem
- mentally representing the problem
- planning how to proceed
- evaluating what you know about your performance.

11.2 Metacognition and the Singapore School Mathematics Curriculum

Given the importance of metacognition in problem-solving and that problem-solving is central to the Singapore School Mathematics Curriculum (see Chap. 2), it is not surprising that metacognition is one of the five interrelated aspects of the School Mathematics Curriculum Framework (SMCF) (Fig. 1.5 in Chap. 1). Lee, Ng, and Lim observed in Chap. 2 that despite metacognition being coined by Flavell only in 1976, metacognition has been featured as one of the five aspects of the original version of the SMCF which was developed in the 1980s (MOE 1990a, b), reflecting a curriculum that is forward looking and informed by theory and research.

11.2.1 Metacognition and the Evolving Singapore School Mathematics Curriculum

It was highlighted in Chap. 2 that, though there were no major changes made to the metacognition aspect of the SMCF, there was conscious effort to refine and operationalise the construct. In the original version of the SMCF (MOE 1990a, p. 3, 1990b, p. 3), metacognition is referred to as 'monitoring of one's own thinking'. There is also a short accompanying paragraph which describes further metacognition as 'the ability to control one's own thinking processes in problem solving' and includes:

- 'constant monitoring of strategies used in carrying out a task
- seeking alternative ways of performing a task
- checking the appropriateness and reasonableness of answers' (MOE 1990a, p. 4, 1990b, p. 4).

The short paragraph captured the essence and the key actions of control and monitoring of thinking processes involved in metacognition as proposed by Flavell in 1976. The other two points about 'seeking alternative ways of performing a task' and 'checking the appropriateness and reasonableness of answers' are metacognitive strategies that teachers are encouraged to develop as students' productive Habits of Mind.

This depiction of metacognition remained unchanged during the first revision of the Curriculum that was implementation in 2001 (MOE 2000a, p. 11, 2000b, p. 12). However, in the subsequent revision of the Curriculum that was implemented in 2007 (MOE 2006a, b), there were two distinctions made to the way metacognition is featured in the Curriculum.

Firstly, the construct was further elaborated as 'monitoring of one's own thinking and self-regulation of learning' (MOE 2006a, p. 12), capturing not only the monitoring aspect but also the regulatory aspect of metacognition. There was also an accompanying paragraph to elaborate on the various sub-processes involved in metacognition (MOE 2006a, p. 15):

> Metacognition, or "thinking about thinking", refers to the awareness of, and the ability to control one's thinking processes, in particular the selection and use of problem-solving strategies. It includes monitoring of one's own thinking, and self-regulation of learning.

This is reflective of the impact of the continued work mentioned in Sect. 11.1.1 to refine and clarify the construct in the Curriculum.

Secondly, there was an explicit attempt to separate an understanding of the construct from instructional experiences that teachers are encouraged to provide students with to develop metacognitive Habits of Mind. Teachers were encouraged to provide metacognitive experiences to help students develop their problem-solving skills. The following five types of activities were advocated to develop the metacognitive awareness of students and to enrich their metacognitive experience (MOE 2006a, p. 15):

(a) Expose students to general problem-solving skills, thinking skills, and heuristics, and how these skills can be applied to solve problems
(b) Encourage students to think about the strategies and methods they use to solve particular problems
(c) Provide students with problems that require planning (before solving) and evaluation (after solving)
(d) Encourage students to seek alternative ways of solving the same problem and to check the appropriateness and reasonableness of the answer
(e) Allow students to discuss how to solve a particular problem and to explain the different methods that they use for solving the problem.

These five types of activities involved students not only in being aware of (a, b, and e), but also encourages students to monitor (b, c, d, and e) and regulate (a, c, and e) their thinking processes. These activities, though more specific, are well aligned with the four clusters of metacognitive instructional strategies that Lee (2008, pp. 70–71) found in the literature to be effective in improving students' problem-solving ability:

i. Mathematics log writing—students use writing activities to cultivate a more metacognitive approach towards mathematical problem-solving.

ii. Effective questioning techniques—teachers establish an environment in which both teachers and students continuously ask questions with regard to the problem-solving process so as to better understand, monitor, and direct students' cognitive processes.
iii. Identification of structural properties of problems—teachers consistently ask students to identify similarities and differences among methods of solution and structural properties of problems that involve different contexts.
iv. Pair and group problem-solving—students worked in pairs and/or groups, reasoning aloud and interviewing each other so as to be more aware of and thus more conscious of regulating the thought processes of the problem solver.

In the latest revised Curriculum, implemented since 2013, there was no further change in the way the construct of metacognition is being presented (MOE 2012a, p. 19, 2012b, p. 17, 2012c, p. 17), reflecting a more stabilised conceptualisation of the construct. However, in line with the emphasis of this revised Curriculum, activities that develop the metacognitive awareness of students and to enrich their metacognitive experience are now articulated in the like of learning experiences (see Chap. 2):

> To develop metacognitive awareness and strategies, and know when and how to use the strategies, students should have opportunities to solve non-routine and open-ended problems, to discuss their solutions, to think aloud and reflect on what they are doing, and to keep track of how things are going and make changes when necessary. (MOE 2012a, p. 19)

While the five types of activities have been more generically summarised in a paragraph, the essence for the need to encourage students to be more aware of, and continuously monitor and regulate their thinking process is still inherent. The importance in selecting appropriate tasks—non-routine and open-ended ones that encourage the development of metacognitive awareness and strategies is highlighted. There is also an encouragement for teachers to help students to make their thinking audible (think aloud) and visible (reflection writing) to heighten students' awareness, monitoring, and regulating of their thinking.

11.2.2 Operationalisation of Metacognition for Teaching and Learning of Mathematics

Brown (1980) and Markman (1977) found that metacognition is a developmental skill that does not automatically increase with age, while Schmitt and Newby (1986) also noted that supplementing instruction with metacognitive aspects would prove beneficial to most learners. In the Singapore context, Wong (1992) found that students need guided instruction in the use of metacognitive strategies for (mathematical) problems-solving. Lee (2015, 2016b) argued that metacognition need to be supported by explicit instruction whereby related skills/processes are explicitly labelled and discussed, and students are guided throughout their repeated distributed practice of these developmentally within the context teaching and learning of mathematics.

This is in line with previous findings that critical thinking skills can be learned and transferred to novel situations when pupils receive explicit instruction designed to foster transfer (Bangert-Drowns and Bankert 1990; Cotton 1991; Dweck 2002, Halpern 1998, 2003; Marin and Halpern 2011). Lee further proposed that, like for the case in teaching content, there is a need to carry out a 'task analysis' for one to explicitly address metacognition in the mathematics classroom. As was presented earlier in this chapter, metacognition involves an awareness of, monitoring of, and regulation of one's both cognitive and affective resources in the context of carrying out goal-oriented processes. Lee observed that regulating of one's cognitive and affective resources assumed the ability of one to monitor such resources, while the ability to monitor such resources assumed the ability of one to be first aware of one's own cognitive and affective resources. He therefore suggested that following taxonomy in addressing metacognition in the mathematics classroom given the hierarchical nature of these three aspects of metacognition:

- Awareness
- Monitoring
- Regulating

While it is true that one may be exercising all these three aspects of metacognition during the course of carrying out goal-oriented processes, he felt that teachers should first create an awareness of students' own cognitive and affective resources first. This would include getting students to be able label and describe these resources in relation to both themselves and the related task(s) at hand. Such an awareness would then allow students to monitor short episodes of their cognitive and affective processes before they are able to regulate such process. In particular, Lee emphasised that regulating such processes need not always result in a change of course of action. The regulation may further affirm one's current course of action if an evaluation of the processes deems fit.

At the same time, Lee also alerted to the fact that there were effort put into encourage students to employ metacognitive practices online—while performing a task, as well as offline—after the completion of a task. Online metacognitive practices involve an awareness, active monitoring, and constant regulating of one's thinking processes while performing a task with the goal of more efficiently and effectively attaining the goal of completing the task at hand. On the other hand, offline metacognitive practices, though also involve an awareness, monitoring, and regulating of one's thinking processes, these are carried out in a post-mortem manner. The purpose of such practices is to improve on future performance of similar task(s), or a transfer of the learning to other similar task(s) with different context.

Taking into consideration both the taxonomy and types of metacognitive practices involved in the classroom, Lee (2015, 2016b) proposed a two-dimensional conceptualisation of metacognition for teaching and learning. While both types of metacognitive practices—offline and online involve awareness, monitoring, and regulating of one's own cognitive and affective resources, the offline and online practice turn these into reflective and interactive practices, respectively (as shown in Fig. 11.1). Lee also observed that a greater level cognitive load is involved in interactive metacognitive

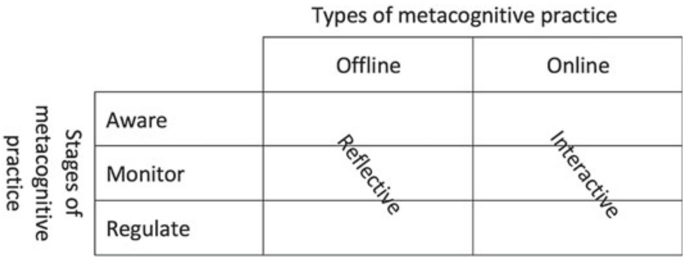

Fig. 11.1 Operationalisation of metacognition for teaching and learning

practices—given that the students need to juggle between metacognitive and cognitive practices at the same time. He suggested that teachers may want to consider addressing offline metacognitive practices first before involving students in both offline and online metacognitive practices.

In arguing for students to be guided throughout their repeated distributed practice of metacognitive practices developmentally within the context teaching and learning of mathematics, Lee also referred to Costa and Kallick's (2000, p. 26) reference of metacognition as one of the Habits of Mind. Costa (2001) refers the Habits of Mind as thinking dispositions skilfully and mindfully displayed by characteristically intelligent people when confronted with problems the solutions to which are not immediately apparent. The Habits of Mind are not thinking tools, rather they are dispositions that one inclines to adopt. Just as Costa and Kallick (2009, p. xi) proposed, Lee also felt that such productive Habits of Mind, as in the case of physical habits, are formed only through continuous practice with teachers providing 'generative, rich, and provocative opportunities for using' such Habits of Mind.

11.3 Local Research Studies on Metacognition and the Teaching and Learning of Mathematics

As was observed by Loh (2015, p. 48), there are few research studies on metacognition in teaching and learning of mathematics that involved local subjects. Wong's (1989) pioneering work in this area on investigating metacognition in mathematical problem-solving is based on the data drawn from a questionnaire used in a larger study (Chang 1989). The main finding of the study revealed that students practiced metacognitive activities at least half of the time when they were solving mathematics problems. However, it was found that the lower-achieving students were less frequent in the usage of metacognitive strategies than those exhibited by the higher-achieving students. Another pioneering research work in this area was undertaken by Foong (1990, 1993). She developed a taxonomy of mathematical problem-solving behaviour that included metacognitive behaviour. These pioneering works have also become foundational work for others to pursue their postgraduate studies on. Yeap's

Fig. 11.2 Loh (2015, p. 98) PSM framework

(1997) master study tapped on Foong's work to develop a catalogue of metacognitive behaviour that teachers could use to detect metacognitive activities in the mathematics classrooms. Lo's (1995) master study on the other hand followed on the footsteps of Wong to further improved on the Study Skills Questionnaire by Chang (1989) by taking into consideration the Learning Process Questionnaire by Biggs (1987). Lo aimed to establish the relationship between the metacognitive strategies employed in mathematics problem-solving and mathematics achievement of students at a Singapore Junior college, as well as the relationship between their learning approaches and mathematics achievement. Loh's Ph.D. study (2015) provided further refinement of Wong's idea to establish her problem-solving metacognitive (PSM) framework (as shown in Fig. 11.2) to analyse students' use of metacognitive strategies by phase and levels, and to gain insights to the nature and frequency of lower secondary school students' use of metacognitive strategies during problem-solving.

There are also studies which target at specific topics in the teaching and learning of mathematics. Teo (2006), for example, carried out a small-scale study on the effects of metacognition and beliefs on students in the study of A-level sequences and series. Yap (2016), on the other hand, carried out an intervention study on metacognitive-heuristic approach, also based on Foong's work on metacognition (1995), to help low attainers in ratio word problems. Lee (2008) also carried out an intervention study for his Ph.D. study to investigate the impact of metacognitive instructional strategies—instructional strategies that help to activate students' metacognitive practices (Mevarech et al. 2006), on the mathematical learning and achievement of secondary one students. Lee's pioneering work on the use of a mixed methods research design in such a study helps to address the many issues that surrounded the validity of the various data collection method involving the study of metacognition (Loh 2015, pp. 64–80). The research drew upon quantitative data from survey questionnaire and problem-solving test, and qualitative data from self-report in the problem-solving test and student interview. The complementary nature of quantitative and qualitative approaches explored the different dimensions of metacognition, and therefore, produced more insights on metacognition and problem-solving through triangulation of data. The approach forms the basis for the research design adopted by Loh's Ph.D. work (Loh 2015), which was mentioned earlier.

The next two sub-sections detailed two areas of work that have made their way into some local school practices to address metacognition in teaching and learning. Of these two areas, one will address offline metacognitive practice while the other will deal with online metacognitive practice (see Fig. 11.1).

11.3.1 The A-Cube Change 2-Dimensional Reflective Practice Model

Lee (2003), based on his interaction with both future and practicing teachers, developed the EmC^2 or Change Model for mathematics teachers to reflect on their teaching so as to develop teachers' offline metacognitive habit of mind in learning from their teaching episodes. The Model consisted of three types of reflection, namely the emotive reflection (EMR), the critical reflection (CIR), and the creative reflection (CER). EMR refers to the awareness of the teacher with regard to his/her 'gut feeling' of how successful a lesson as a whole has been upon completing a lesson episode. CIR engages the teacher in a detailed analysis of his/her lesson episode; it encourages the teacher to examine the various parts of the lesson. CER helps the teacher to bring the reflective process to fruition by inviting the teacher to create a new lesson episode based on the reflection during EMR and CIR stages. Figure 11.3 provided a diagrammatic representation of the EmC^2 or Change Reflection Model, reflecting the hierarchical nature of the three types of reflection and includes the key questions that teachers could use to guide them through each type of reflection in the Model.

In an attempt to also include the learning aspect of the mathematics classroom, Lee (2010) further refined the question prompts in the Model to adapt to both teaching and learning of mathematics, as shown in Fig. 11.4.

However, Lee (2015) shared that based on his further work with teachers on reflective practices, he felt also a need to examine the depth of reflection, and not just the type of reflection. He postulated that while teachers/learners may undergo creative reflection and profess of new/reinforced understanding, such may exist only at an articulated level. There is a need to encourage the teachers/learners to examine such articulated new/reinforced understanding against their personal belief so that such new/reinforced understanding may be truly assimilated by the teachers/learners. He also believes that for the teachers/learners to truly own the assimilated new schema of understanding—be it reinforced or renewed, the teachers/learners need to know when and why, not just the how, of wielding the new knowledge/skill. In other words, there is a need for the teachers/learners to appraise the new schema for effective regulated use of such new/reinforced understanding. He presented the depth of reflection using the A^3 Reflection Model as shown in Fig. 11.5, together with the accompanying sample question prompts that teachers may use. As shown in Fig. 11.5, the three

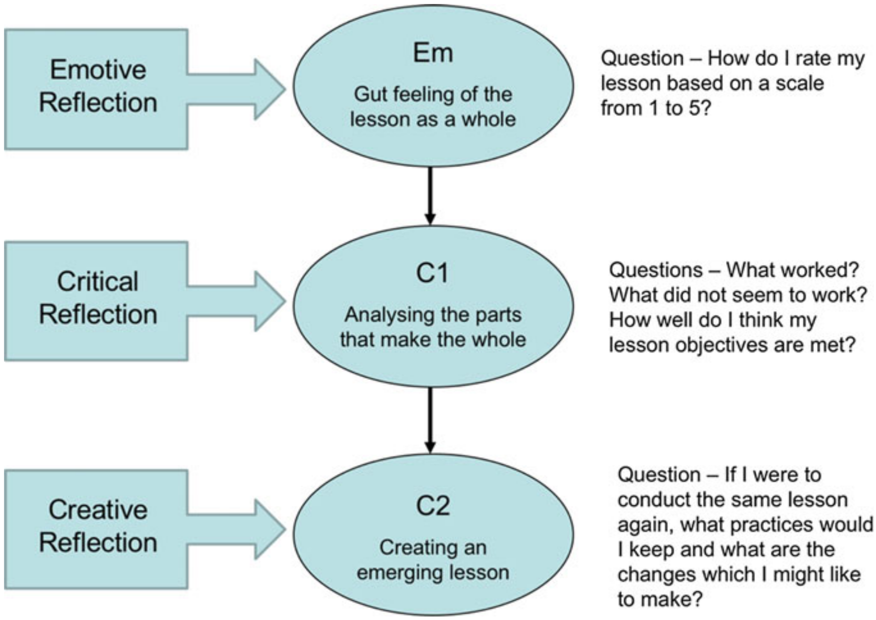

Fig. 11.3 The EmC^2 or Change Reflection Model for teaching Mathematics

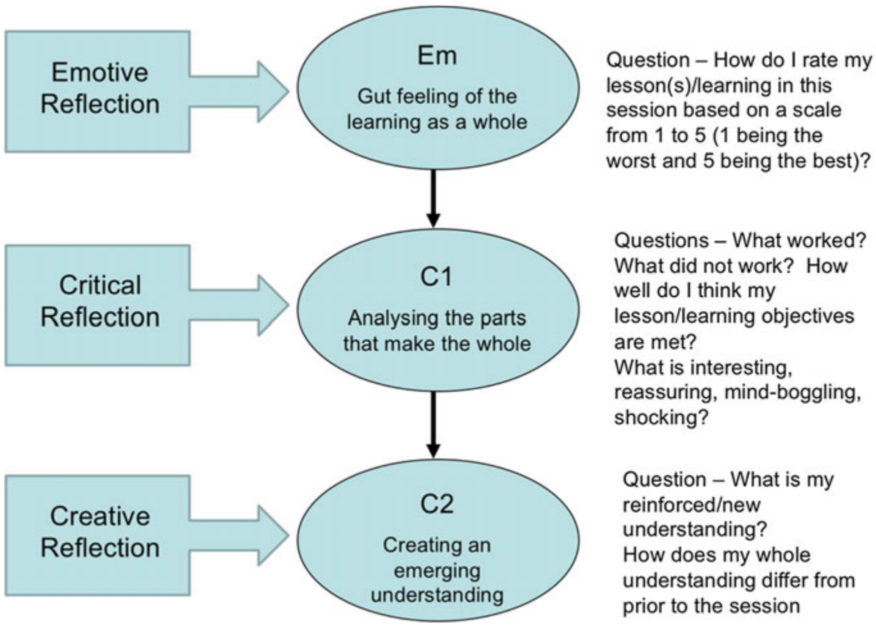

Fig. 11.4 The EmC^2 or Change Reflection Model for teaching and learning of Mathematics

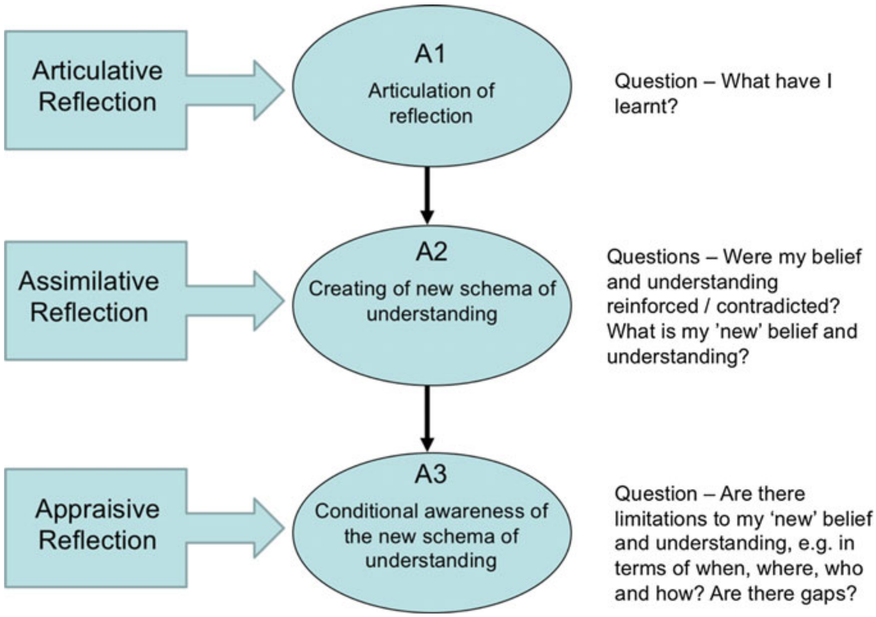

Fig. 11.5 A^3 Reflection Model for teaching and learning of Mathematics

depths of reflection, also in hierarchical manner, are articulative, assimilative, and appraisive.

Based on the types and depth of reflection, Lee (2015) presented the A-Cube Change 2-Dimensional Reflection Model as shown in Fig. 11.6, to illustrate how the two dimensions of reflection interact for effective offline metacognitive practice or reflective practice to occur. As can be observed from Fig. 11.6, the interaction of both the dimensions also demonstrated how the three aspects of metacognition, namely awareness, monitoring, and regulation, play out in the process. He also developed sample question prompts to accompany this 2-Dimensional Reflection Model (Fig. 11.7). Lee shared that this 2-Dimensional Reflection Model promotes deep reflection. Such deep reflective practice allows for effective transfer of learning to new/novel situations as it involves one critically and creatively examining one's practice against one's knowledge, skills, and beliefs. Furthermore, the practice also promotes the establishment of connections to make sense of one's practice as it encourages linking the examination of one's practice to not only principles of teaching and learning but also one's belief system. Such deep reflective practice, he pointed out, promotes the evolvement of a more connected and robust schema of practice that better aligns practice, knowledge, and beliefs.

The 2-Dimensional Reflection Model has been shared and adopted in pre-service, in-service, and postgraduate courses in mathematics education as well as schools (Lee 2017a). One such example is reported in the newsletter SingTeach, whereby a postgraduate student, after learning about the Model in her study, attempted to

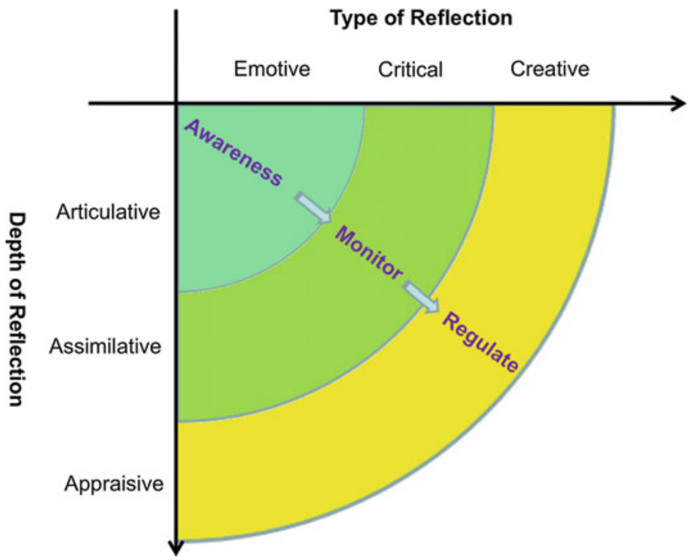

Fig. 11.6 The A-Cube change 2-Dimensional Reflection Model for teaching and learning

Depth/Stages of Reflection	Examples of reflective question prompt
Articulative – The Emotive Reflection	How do I rate my lesson(s)/learning in this session based on a scale from 1 to 5 (1 being the worst and 5 being the best)?
Articulative – The Critical Reflection	What worked? What did not work? How well do I think my lesson/learning objectives are met? What is interesting, reassuring, mind-boggling, shocking?
Articulative – The Creative Reflection	What is my reinforced / new understanding? How does my whole understanding differ from prior to the session?
Assimilative	Were my belief and understanding reinforced/contradicted? What is my 'new' belief and understanding?
Appraisive	Are there limitations to my 'new' belief and understanding, e.g. in terms of when, where, who and how? Are there gaps?

Fig. 11.7 Sample question prompts for the A-Cube change 2-Dimensional Reflection Model

adopt the Model for the primary school that she is teaching in developing reflective culture among the teachers using a whole-school approach towards (The Discoveries of Reflective Practice 2015). In fact, the teacher shared that although 'the Model was originally designed with Math in mind', she 'saw the potential for widespread adoption in her school, regardless of subject'. As pointed out by the teacher, the reflection that was commonly carried out in her school used to be focusing on 'reporting and accounting instead of critical analysis'. The Model, she noted, 'lends structure' to the school's pre-existing reflection routine, thus helping the teachers to develop metacognitive habit of mind, as pointed out earlier. A Chinese language teacher of the school, who translated the Model into Chinese, described her previous form of reflection as 'akin to recalling the lesson than detailed reflection'. Her new challenge, she shared, is 'coming to terms with' her 'own belief system though reflective practice, about what works and what does not'. As pointed out by another teacher, 'you will have to address your belief, adjust your belief, maybe to also let go of your belief at certain times', achieving what Lee (2015) had wanted the Model to achieve—a better alignment of practice, knowledge, and belief.

Lee (2017a) observed that such deep reflective practice, though offline in nature, aids the transition from offline metacognitive practice to online metacognitive practice as it deals with:

- Regulation of cognitive/affective resources that is data-based
- An heightened awareness of one's belief system which often plays up during online metacognition.

11.3.2 The Problem Wheel

As was reflected in Sect. 11.2.1, based on a survey of the relevant literature, Lee (2008, pp. 70–71) found that there are four clusters of metacognitive instructional strategies that have been shown to be effective in improving students' problem-solving ability. One of the clusters, the use of effective questioning techniques, refers to the case whereby teachers establish an environment in which both teachers and students continuously ask questions with regard to the problem-solving process so as to better understand, monitor, and direct students' cognitive resources. To realise such an approach towards the development of online metacognition, he tapped upon the Problem Wheel (Fig. 11.8), developed as a graphic organiser by Chang et al. (2001). The Problem Wheel, which is an adaptation of the Reasoning Wheel (Paul 1993) and based on the work by Lee et al. (1998), involves getting students to make sense of the basic structure of a mathematical problem through the use of systematic question prompts. The set of question prompts corresponding to the respective components of the Problem Wheel is shown in Fig. 11.9. The question prompts serve as a means for students to be more aware of their understanding of the problem context, and through such an awareness better select, match, and/or discriminate, i.e. regulate, their own cognitive resources to initiate the problem-solving process.

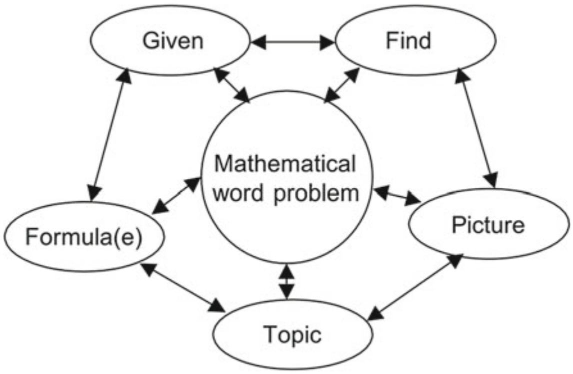

Fig. 11.8 The problem wheel

Component of the Problem Wheel	Examples of Question Prompts
Given	• What is / are given to us in this problem? • What do we know about the problem? • What value(s) is / are we given to us in this problem?
Find	• What are we supposed to find in this problem? • Which value(s) is /are we supposed to find as the answer to this problem?
Picture	• Can we draw a picture to represent this problem? • What would we draw to represent this problem?
Topic	• Which is / are the topic(s) we have learnt that might help us solve this problem?
Formula(e)	• Which is / are the formula(e) we have learnt that would help us to solve this problem?

Fig. 11.9 Question prompts for the problem wheel

The components are depicted as a wheel with double-headed arrows linking the various components to convey the idea that these are not to be perceived linearly though sometimes they may occur as such. Students may go through the various components of the wheel non-sequentially. They may go back to earlier components of the wheel to revise the information gathered and translated as they move round the wheel to gain a better understanding of the problem and try to translate the information into the mathematical concepts. The interactivity of the various components of the wheel reflects the dynamism exhibited during both the monitoring and regulatory aspects of the online metacognition.

Based on Lee's (2008) study, the wheel seemed to have a positive impact in kick-starting students problem-solving process and improved problem-solving performance. Lee (2008, p. 353) noted that, with the use of the wheel, the students were observed ‘to be more focused in their problem solving attempts by starting from the familiar ground of the ‘givens’ in a problem and working more productively and purposefully towards what they are to ‘find’ through visual representations of the problem structure’. In other words, the wheel provided a means for students to be more aware of, be actively monitoring, and constant regulating their thinking process during problem-solving, i.e. the wheel serves to guide students to actively carry out online metacognition, instead of being too concerned about just getting the right answer. This, as pointed out by Lee (2008, p. 369), results in students becoming ‘less judgmental, and appeared to be more creative in their approach towards learning and doing mathematics’. It is thus not surprising that Lee also observed a positive change in the intellectual self-concept and mathematics self-efficacy of the students in the study, as the students become more aware of their cognitive/affective resources and learn to kick-start their new ‘encounters’ with mathematics through extending and connecting, i.e. regulating, from their available resources.

Again, as in the case of the 2-D Reflection Model, the wheel has been shared with teachers in schools through conferences and seminars (e.g. Lee 2015, 2016a, b; Hong et al. 2012). There was even a workshop that was specifically conducted for acquainting the secondary school mathematics teacher to the wheel and organised by MOE (Lee 2016a). However, it should also be pointed out that as schools adopt the wheel for their mathematics lessons, some have further modified the wheel to better fit into their school programmes. Lee et al. (2014) provide a detailed description of such an undertaking by a school. The school embarked on an intervention programme to encourage their students to be more aware of their thought processes by thinking aloud to initiate students’ problem-solving process, i.e. the understanding and planning phase of problem-solving. The teachers concerned made use of a metacognitive questioning scheme that is called the STARt Understand and Planning (STARtUP) (Fig. 11.10), which is an adaptation of the Problem Wheel.

A comparison of the Problem Wheel and the STARtUp scheme will reveal that ‘Topic’ and ‘Formula(e)’ components of the wheel have been replaced with ‘Heuristic(s)’ and ‘Start’ in the STARtUP scheme. As the school has put in place a programme in the explicit teaching of Heuristics and the teachers concerned felt that solving of non-routine problems may not be topic specific, replacing the ‘Topic’ component of the wheel by ‘Heuristics’ was initiated. And since, ‘Topic’ component of the wheel has been removed, ‘Start’ was introduced as the new component to replace ‘Formula(e)’ in the wheel, to further emphasise the objective of the intervention in kick-starting students’ problem-solving process. Though the intervention was a mere six contact hours, and so might not have been sufficient for students to internalise the STARtUP scheme as a habit of mind, the findings did show that students improved in the way they initiated the problem-solving process. In fact, to some extent, the students have developed a more metacognitive approach towards mathematical problem-solving, and the school has integrated the scheme into the School’s mathematics programme (Lee et al. 2014).

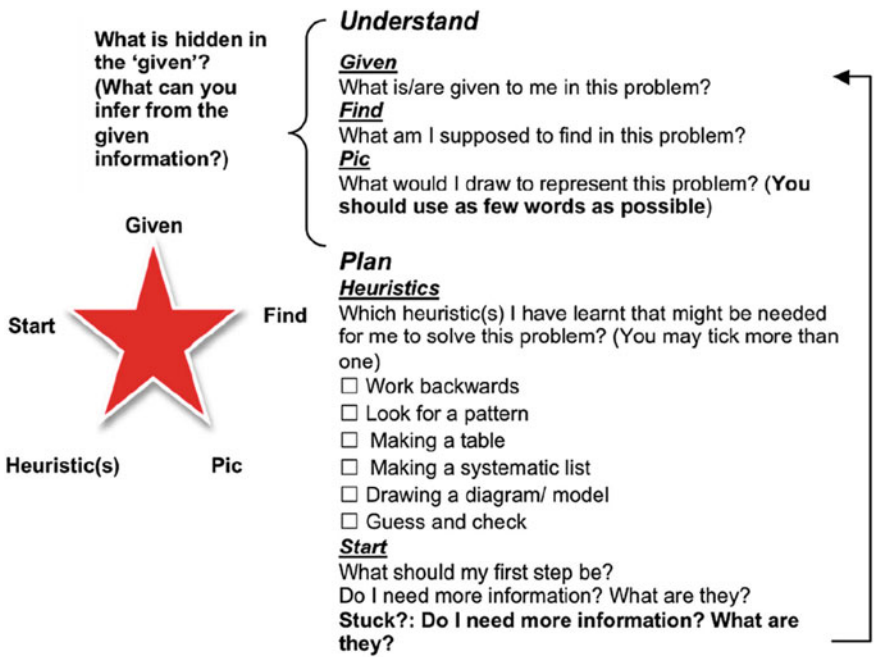

Fig. 11.10 The STARtUP scheme

11.4 Enactment of Metacognitive Instructional Practices

As was pointed out Sect. 11.2.1, metacognition has been an aspect of the SMCF since 1992. However, despite the fact that metacognition has been a key feature of the SMCF for more than twenty years, there has been limited effort to examine its impact in the mathematics classrooms from both the teaching and learning perspective. Through Singapore's participation in PISA 2009 (OECD 2010), there were efforts to collect data through student survey data to obtain a profile of our 15-year-old students' control strategies as a form of metacognitive practice in learning. However, there is also a need for identification and classification of the conceptions of metacognition and metacognitive instructional practices among mathematics teachers—the frontline practitioners who enact the curriculum and principally responsible for the curriculum experienced by the students. The Core 2 Research Programme undertaken by the National Institute of Education (NIE) (Jonid et al. 2014) made a first attempt to determine the level of metacognitive instructional practice employed by teachers in Primary 5 and Secondary 3 mathematics classrooms by analysing the mathematics teachers' instruction. However, it is extremely difficult to differentiate between cognitive and metacognitive processes (Garofalo and Lester 1985; Perkins et al. 1990; Loh and Lee 2017). The key issue lies with the difficulty to distinguish clearly between what is meta and what is cognition (Brown et al. 1983; Baker 1991;

Cheng 1999) and that the interactions between the various mental processes are complex (Yeo 2013; Loh and Lee 2017). In a review of the literature on cognition and metacognition, Tarricone (2011, p. 1) observed that the main distinction between the two is that 'cognition is a constant flow of information and metacognition is knowledge and awareness of processes and the monitoring and control of such knowledge and processes … metacognition is considered to be second-order cognitions'. Thus the task of accurately coding a teaching act as metacognitive instructional practice without other supporting input to triangulate the data, as in the case by Jonid et al. (2014), may result in an over-generalisation and thus may not accurately portray the current situation in the primary mathematics classroom.

In 2015, MOE awarded a group of mathematics education researchers in NIE a research project (AFR 04/14 LNH) to systematically use a multi-site, multi-case-based approach to collect and triangulate data through survey, classroom observations and interviews to develop preliminary teacher conceptions of metacognition and metacognitive practices grounded based on the phenomenon under observation. Ng et al. (2016) shared the following findings from the project:

- Teachers' conception of metacognition is superficial and/or they are confused between cognition and metacognition. Teachers' description of metacognition may contain elements of metacognition, e.g. awareness, monitoring, regulation, reflection, but there is a lack of precise description; some confused metacognition with other cognitive skills (e.g. critical thinking and creative thinking skills), while others confused it with teaching approaches (e.g. engaged learning).
- In terms of metacognitive instructional practices, practices are vague; metacognition is still not explicitly addressed in the classroom. Though there were some understanding and linking of metacognition to reflection and monitoring, there is a lack of evidence that the instructional practices foster nor activate students' metacognition.
- The data suggest the following cases are possible:
  - Teachers could have unconsciously and subtly address metacognition in the classroom.
  - Teacher thought metacognition has been addressed (explicitly or implicitly), but there is a lack of actual instructional evidence of this.

Of the three schools that participated in the study, the six participating teachers in one of the schools, despite being unclear of the conception of metacognition, appeared to all possess—observed to both profess and practice, some elements of metacognitive instructional practice (Lee et al. 2016). When probed further, these participating teachers attribute the practice to the 2-year-old school programme on promoting talk moves (Chapin et al. 2013). It appeared that there existed a differential impact of a more than 20-year-old curriculum versus a 2-year-old school programme on teachers' metacognitive instructional practices. After an insightful interview with the Head of Mathematics Department of the said school, the researchers attributed the differential impact to the following factors that favoured the enactment of metacognitive instructional practices in the school:

- Existence of localised expert—there is buy-in by the Head of Department
- Strong theory-practice link—the Head of Department carried out actual classroom demonstration to exemplify the enactment of the theory
- Actual hand-holding—under the guidance of the Head of Department, teachers worked in groups to collaboratively plan the lessons
- Enculturation versus performance—the Head of Department encouraged peer observation and discussion of the lessons rather than evaluative observation by school leaders for performance.

11.5 Conclusion: Implications for Teaching and Learning of Mathematics

While this chapter has presented the extensive work done both internationally and locally that have been carried out on the address of metacognition in the teaching and learning of mathematics, there still exists challenges faced in the actual enactment of metacognitive instructional practices. The following are the three key challenges:

i. developing a clear and functional conception of metacognition for teaching and learning
ii. availing a set of practical metacognitive instructional strategies for teaching and learning
iii. conducting related and appropriate professional development for teachers.

11.5.1 Developing a Clear and Functional Conception of Metacognition for Teaching and Learning

As pointed out in Sect. 11.1.1, Schoenfeld (1992) observed that there is no agreement among researchers on a single definition of the term metacognition. Given the fuzziness of the construct of metacognition, it may be challenging for teachers to address metacognition in the classroom explicitly. Lee's (2015, 2016b) two-dimensional conceptualisation of metacognition for teaching and learning (Sect. 11.2.2) and Loh's (2015) problem-solving metacognitive (PSM) framework (Sect. 11.3) might help to lend some clarity to the conception of metacognition for teaching and learning in the mathematics classroom.

However, both the literature and the research, including those locally, have shown that teachers are generally confused between cognition and metacognition (Sect. 11.4). Swartz and Perkins' (1990, p. 109) (Sect. 11.1.1) Map of the Thinking Domain provided a schematic representation of the relationships between thinking skills, goal-oriented process, and metacognition, making clear distinction between cognition and metacognition. In addition, in Yeo's (2013) doctoral study, he researched on the nature and development of cognitive and metacognitive processes in mathematical investigation. His refined investigation model for metacognitive processes based on local data provided a scheme to show how metacognitive processes and cognitive processes interact during mathematical investigation. Loh and Lee (2017) also provided examples to demonstrate how to identify notions of cognition or metacognition in students' work in mathematics classrooms. These works provided teachers not only with a clearer distinction between cognition and metacognition, but also helped to make clearer how the two constructs interact and play out in the actual classroom context.

11.5.2 Availing a Set of Practical Metacognitive Instructional Strategies for Teaching and Learning

As mentioned in Sect. 11.4, despite the fact that metacognition is featured in the SMCF for more than twenty years, Singapore mathematics teachers' metacognitive instructional practices still appeared to be vague. Not only does it seem that metacognition is still not explicitly addressed in the classroom, there was also a lack of evidence that the claimed metacognitive instructional practices foster or activate students' metacognition. Furthermore, as was noted in Sect. 11.2.2, as a productive habit of mind, as in the case of physical habits, metacognition is formed only through continuous practice with teachers providing 'generative, rich, and provocative opportunities for using' such Habits of Mind. Thus, there is a need to avail a set of practice-oriented metacognitive instructional schemes/models/approaches that could bridge the theory and practice nexus on the development of metacognition and which may also be easily adapted for the cultivation of such metacognitive habit of mind in daily mathematics lessons.

The two works that were presented in Sects. 11.3.1 and 11.3.2, namely the A-Cube Change Two-Dimensional Reflection Model (Lee 2015) and the Problem Wheel (Chang et al. 2001), have shown the tractability of such schemes in the Singapore mathematics classrooms. The offline metacognitive scheme—the A-Cube Change 2-Dimensional Reflective Practice Model has not only been adopted at mathematics education courses but also teaching and learning in general—supporting the teaching and learning beyond that for mathematics. The online metacognitive scheme, on the other hand, not only have been adopted but also adapted for the Singapore Mathematics classrooms. However, these schemes as they have been presented, addressed the regulatory phase of both offline and online metacognition. Given the taxonomy of metacognition (Lee 2015, 2016b) that is presented in Sect. 11.2.2, schemes for the other aspects of the taxonomy need also to be investigated and developed for teachers to fully address metacognition in completeness in the classroom.

11.5.3 Conducting Related and Appropriate Professional Development for Teachers

It is a common knowledge that a teacher should possess not only a good grasp of the content to be taught as well as a set of instructional strategies that enable him/her to help the learner to develop a good grasp of the content. Some find it reasonable to suggest, however, that, at a bare minimum, teachers should possess knowledge and deep understanding of the subject matter recommended for students at the level of their teaching and, preferably, one grade level category above their particular teaching level (National Research Council 2010). Applying this similar argument to the teaching of metacognition, Lee (2016b) argued for the address of the following for professional development of teaching in addressing metacognition in teaching and learning, on top and above of a clear conception of metacognition and an accompanying repertoire of metacognitive instructional strategies for teaching of the various aspects of metacognition:

- Meta-metacognition
- Theory of mind
- Social metacognition

Lee et al. (2013) examined the design and implementation of the series of mathematical modelling lessons to determine how the development of metacognition was addressed during planning as well as during implementation of the lessons. They observed that the rich opportunities for the metacognitive development of the students afforded by mathematical modelling tasks require teachers' explicit offline and online interventions through task design, lesson planning, and strategic scaffolding during lesson implementation, or meta-metacognition (Stillman 2007). They argued that such meta-metacognition knowledge—a meta-knowledge of one's metacognition may constitute as a key pedagogical content knowledge for effective address of metacognition in the mathematics classrooms as they observed that the lacking of such knowledge in the mathematics classrooms may be detrimental to the mathematical development of the students.

Misailidi (2010) observed that metacognition and theory of mind have 'evolved over the past 20 years as two distinct and unconnected research fields' though Flavell (2002) maintains that the two fields share the same overall objective—'to investigate the development of children's knowledge and cognition about mental phenomena' (p. 106). Unlike metacognition, which is concerned with thinking about one's thinking, theory of mind deals with the ability to think about or make inferences about the thoughts and feelings of another person (Kuhn 2000a, b; Lockl and Schneider 2006). Kuhn (2000a, p. 302) describes metacognition or 'meta-knowing' as 'any cognition that has cognition … as its object'. According to Kuhn (1999, 2000a, b), theory of mind corresponds to the metacognitive knowing that includes children's knowledge about the mind, i.e. knowledge of mental state exist. Such knowledge can be both personal and impersonal. Personal metacognitive knowledge is knowledge about one's own mental states, whereas impersonal metacognitive knowing is knowledge about others' mental states. One of Kuhn's key ideas is that theory of mind serves as the foundation for the development of other dimensions of meta-knowing, i.e. children need to acquire a theory of mind first, before they begin to develop the other dimensions of meta-knowing. In other words, teachers may tap on theory of mind to further develop students' metacognition. In fact, Lee (2016b) further argued that teachers themselves also need to tap on theory of mind to better make sense of students' thinking so as to enhance their ability in addressing the metacognitive development of the students in their classrooms.

Chiu and Kuo (2009) compare and contrasted individual metacognition and social metacognition as follows:

> Individual metacognition is monitoring and controlling one's own knowledge, emotions, and actions, while social metacognition consists of group members' monitoring and control of one another's knowledge, emotions, and actions.

In other words, social metacognition as compared to metacognition, as it is presented thus far in this chapter, shifted the awareness, monitoring, and regulation of thoughts from the individual to that of a social context (Brinol and DeMarree 2012). Chiu and Kuo (2009) observed that social metacognition 'distributes metacognitive responsibilities across group members' and it aids group members' 'identification of errors, construction of shared knowledge, and maintenance of group members' motivation. As pointed out in Chap. 1, one of the outcomes of the education in Singapore is to develop each student into 'an active contributor who is able to work effectively in teams', so an inclusion of social metacognition in our address of metacognition in the context of teaching and learning in Singapore is certainly well aligned with the national curriculum. In fact, Chiu and Kuo (2009) have also noted that several programmes have showed that 'improving students' social metacognition skills aids their learning and academic performance', further reinforcing the need to address both individual and social metacognition simultaneously. Lee (2016b) further proposed that for teachers to effectively teach metacognition in the classroom context, which is in fact a group or social context, it is essential for teachers themselves to be better equipped with social metacognitive skills.

Lee conducted two 12-h in-service courses (Lee 2017b, c), one for primary and another for secondary mathematics teachers, to address the issue of metacognition in the teaching and learning of mathematics. In response to the findings of the research project (AFR 04/14 LNH) mentioned in Sect. 11.4, the emphases of these two courses are:

- Providing teachers with an operationalised conception of metacognition for teaching and learning mathematics
- Equipping teachers with some metacognitive instructional strategies for teaching and learning of mathematics.

Lee's (2015, 2016b) proposed a two-dimensional conceptualisation of metacognition for teaching, and learning was presented as an operationalised conception of metacognition to the participating teachers. Furthermore, the participants were also introduced to an offline metacognitive approach—the A-Cube Change

2-Dimensional Reflection Model (Lee 2015) and an online metacognitive approach—Problem Wheel (Chang et al. 2001).

The feedback from the participating teachers indicated that most of them has developed a better understanding of the 'difference between cognition and metacognition' and an awareness that 'there are 3 components of metacognition—awareness, monitoring and regulation'. While the participants also appreciated and valued both the offline and online metacognitive approaches shared, a number of the participants have reflected that they would like these courses to be longer so that more metacognitive instructional strategies and classroom cases could be examined and discussed. It reflects a need to better equip teachers with a more comprehensive set of metacognitive instructional strategies as well as the necessary and related knowledge and skills, as discussed in this section. While a longer in-service course may not be a practical response, there may be a need to design in-service courses to address the various aspects in preparing these teachers to be teachers of metacognition. In addition, based on the factors that favoured the enactment of metacognitive instructional practices in schools, as discussed in Sect. 11.4, the planned professional development may need to include some form of hand-holding for actual implementation in the classroom as well as the establishing of a school-based expert. In other words, from the work thus far in the Singapore context, these are a need to adopt a more holistic approach towards the professional development of teachers of metacognition.

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