Chapter 10
Patterns Across the Years—Singapore Learners' Epistemology

Swee Fong Ng and Boon Liang Chua

Abstract Pattern has a prominent position in the Singapore mathematics curriculum. This chapter reports how learners across the grades thought about patterns, how they recognised patterns, and how they constructed rules to describe the structure underpinning specific patterns. The corpus of data came from four studies. Primary children participated in the first three studies: Age and Individual Differences, Forward and Backward Rule, Colour Contrast whilst Secondary 2 students participated in the fourth, Strategies and Justifications in Mathematical Generalization. All these studies used the mathematics curriculum to design grade-specific mathematical tasks. In general, two types of pattern tasks were used, number patterns presented in tandem with figures and figural patterns. Data with primary children were collected using paper-and-pencil task and clinical interviews were used to collaborate their responses. The fourth study analysed the written responses of the secondary students to paper-and-pencil task. These studies found that learners focused on the surface features to arrive at a rule to describe these number patterns. In the colour-contrast study, compared with monochromatic presentation, those using two colours encouraged learners to present possible general rules. The more able academic stream secondary students were able to arrive at general rules for linear figural patterns. However, all students across the academic spectrum were challenged by quadratic patterns. Findings from the four suggest that it important for teachers to know how to move learners to look for the structure underpinning patterns, numerical and figural, and to construct the all-important general rule.

Keywords Colour contrast · Linear figural · Recursive rule · Predictive rule · Structure · Number patterns · Quadratic patterns

---

S. F. Ng (✉) · B. L. Chua
National Institute of Education, Singapore, Singapore
e-mail: sweefong.ng@nie.edu.sg

B. L. Chua
e-mail: boonliang.chua@nie.edu.sg

T. L. Toh et al. (eds.), *Mathematics Education in Singapore*,
Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-13-3573-0_10

10.1 Introduction

A large part of the mathematics taught to pupils is really about seeing patterns, interpreting what is seen, and expressing those patterns in words and symbols (Mason 1990). The important patterns are the ones that are not just particular to one situation, but apply to many different but similar situations and are therefore generalisations. Expressing these generalities is the root of algebra. Pupils can be encouraged to express generalities themselves and can have them pointed out by others.

To engage in mathematical thinking, to appreciate the strengths and limitations of mathematics, it is essential to express perceived patterns and generalities and offer these for others to consider, challenge, and where appropriate, modify. These conjectures need to be tested on our peers and our adversaries, with the specific objective of trying to convince them that generalities perceived by us are acceptable to them, too. This is how mathematical thinking develops.

Expressing generality is an important part of mathematics, and of almost all aspects of living. It appears in many guises, but the most basic and clear instance is in pattern spotting, particularly in seeing links, and connecting between things. Algebraic thinking gets going when you try to express patterns in words and pictures, so that others can see what you see.

10.2 Algebraic Thinking, Patterns, and Functions

Algebraic thinking "defies simple definition" (Driscoll 1999, p. 1). Historical origins of algebraic thinking emerged from "proportional thinking as a short, direct and alternative way of solving 'non-practical' problems" Radford (2001, p. 13). *Algebraic Thinking: Grades K-12* sets out the theoretical discussion on what is algebraic thinking and how it differs from algebra—that there is "an algebraic *way of thinking*" (Moses 1999, p. 3, original emphasis). Such thinking incorporates forming "generalizations from experiences with number and computation, formalizing these ideas with the use of a meaningful symbol system, and exploring the concepts pattern and functions" (Van de Walle and Bay-Williams 2014, p. 276). The three strands algebraic thinking infuses the key ideas of generalisation and symbolisation (Kaput 2008). The first strand involves the study of structures in the number system, including those used in arithmetic, described by Usiskin (1988) as generalised arithmetic. Strand two explores the study of patterns, relations, and functions. The third strand seeks to study how best to capture the information or to model the situation symbolically. The human mind seeks to organise the huge amount of information present in the environment by constructing meaningful relations with the various inputs and outputs and capturing the information symbolically (Fosnot and Jacob 2010). Therefore, to be able to organise information meaningfully, the human mind detects what remains the same and what is changing and to construct an appropriate rule. This reduces the demands on human attention so that the mind can function economically (Mason 1996).

10.3 Some Examples of Pattern Tasks

The Singapore primary mathematics curriculum places a heavy emphasis on (i) understanding of patterns, relations, and functions and (ii) representing and analysing mathematical situations and structures using algebraic symbols (Cai et al. 2005). The primary mathematics curriculum introduces and develops numerical and geometrical patterns of varied nature. It is customary to engage young children, e.g. those in Primary 1, with tasks such as those in Fig. 10.1. Here, the stimuli comprise four animals and the children are required to decide which animal comes after the last rabbit. The children have to see that this string of animals is constructed using four animals: penguin, frog, rabbit and rabbit. After the fourth rabbit, the pattern repeats itself. Thus, the structure underpinning this task is a string of four objects, which repeats itself in the same order. Numbers 1, 4, 7, 7, 1, 4, 7, 7, 1, 4, 7, _, could replace the animal stimuli. Completing such number sequences could be challenging to young children. Competing knowledge could prevent children from completing this number sequence. When asked what comes after the number 7, some children would reply 8 and some others, 11.

The complexity of number patterns could be increased to include skip counting such as completing number sequences 2, 4, 6, 8, _, 12, _. Such tasks encourage children to recognise this number sequence as part of the two times tables. Pattern task presented in Fig. 10.2 challenges children to analyse how both repeating and growing patterns are generated.

Here, the geometric shapes above any given number are a function of the nature of the number: triangles are above odd numbers and circles are above even numbers. This task requires children to analyse that the shapes are alternating between triangles and circles, i.e. a repeating pattern of string two. The objective of the task in Fig. 10.2 is to relate the geometrical objects in the pattern to their positions in a pattern and generalise those relationships. Thus, triangles are above odd numbers and circles are above even numbers. To predict what comes next, i.e. what shape follows a circle is relatively simple. A child who understands the demands of the task would be able to

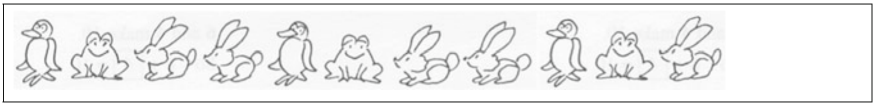


Fig. 10.1 An example of a simple pattern task introduced at Primary 1

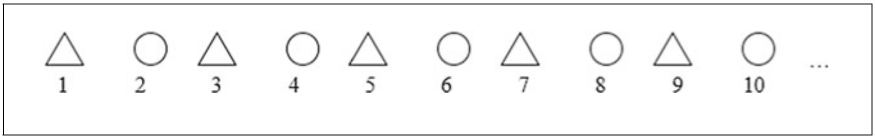


Fig. 10.2 Number patterns constructed using repeating shapes of length 2 and growing number sequence

Shapes	△	○	△	○	△	○	△	○	△	○
Numbers	1	2	3	4	5	6	7	8	9	10
Remainder when number is divided by 2	1	0	1	0	1	0	1	0	1	0

Fig. 10.3 Shapes task reflecting the deeper structures underpinning such beguilingly simple pattern tasks

state a triangle follows a circle. Similarly, children are likely to say that the number eleven is after the number ten. However, the task becomes more challenging when children are asked to predict what shape is above the number 23 or the number 46. Children need to understand that the 23 is an odd number and triangles are above odd numbers. Because 46 is an even number and circles are above even numbers, the shape above 46 must be a circle. Such tasks may seem beguilingly simple are not trivial. Children could answer such questions by writing out the entire number sequences until they arrive at the required number. However, this is not desired. Children may state correctly that circles are above even numbers, but what if they were asked what shape is above 146 or 1023? Attending to superficial features such as the digits in the ones place may get them the correct answer but it is more important for children to notice that triangles are above odd numbers and circles are those above even numbers. Patterns come with clockwork regularity. A more convincing justification that could be generalised to other numbers which are not possible to list is to offer the response that triangles are found above numbers which leave a remainder one when divided by 2 and circles are above numbers with remainder zero and the corresponding numbers are written in one row as in Fig. 10.3. Such function tasks are precursors to periodic functions expressed formally as "$f(x) = f(x + a), x \geq 0$".

Figure 10.4 presents a configuration of strings of shapes, in this case a string of stick houses, starting with Diagram 1 as the smallest configuration. It is necessary to identify that there is a pattern underpinning the construction of these string of diagrams. This is achieved by identifying the recursive rule used to generate each successive diagram. In this case, the recursive rule is to add 4 sticks to the number of sticks in the previous diagram. With the correct identification of the recursive rule, it is possible to state the rule for Diagram 10. The number of sticks for this diagram can be found using the recursive rule of adding 4 sticks to the number of sticks in Diagram 9. Therefore, a possible recursive rule for the number of sticks in Diagram 10 is 5 + 9 times 4. Although some may give the erroneous solution as twice the total number of sticks in Diagram 5. Thus, for any diagram, the correct rule can then be generalised to 5 + 4 times one less than the current diagram number. However, the challenge is to provide a more informative rule, the predictive rule that can be used to state the general rule for any diagram number. The predictive rule provides a direct relationship between the number of sticks and the diagram number.

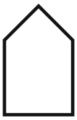
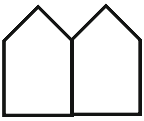
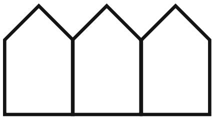
					
	Diagram 1	Diagram 2	Diagram 3	Diagram 5	Diagram 10
Number of sticks					

Fig. 10.4 Ordered configurations of strings of houses, starting from the smallest configuration

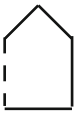
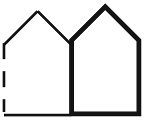
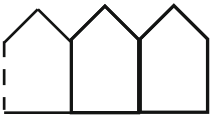
					
	Diagram 1	Diagram 2	Diagram 3	Diagram 5	Diagram 10
Number of sticks					

Fig. 10.5 A dashed line is used to represent the freestanding wall

The total number of sticks for Diagram 10 = 1 free-standing wall + 10 times 4 sticks
The total number of sticks for any diagram number = 1 free-standing wall + diagram number times 4
The total number of sticks for Diagram $n = 1 + n \times 4$

It is hypothesised that construction of the predictive rule may not be so easy if nothing is used to differentiate the freestanding wall from the rest of the sticks. If a dashed line is used to represent the freestanding wall then the predictive rule for any diagram, which provides information on the panels and the diagram number, could be the sum of the freestanding wall and the number of prefabricated panels, which is a function of the figure number. Hence, the predictive rule would be (in this case)

$$\text{The total number of sticks for Diagramn} = 1 + n \times 4$$

Compared to the recursive rule, the predictive rule provides direct information. In this case, the 4 represents the four sticks which remain the same irrespective of the diagram number (Fig. 10.5).

10.4 The Four Studies on Patterns

Using data from four studies, this chapter documents how learners recognised the structure underpinning specific patterns, their construction of the rules underpinning the pattern tasks and the difficulties they have with different pattern-type tasks. The participants of Study One—the Age and Individual Differences in Mathematical Abilities, Study Two—Forward and Backward Rule Study, Study Three—Colour-Contrast Study, were primary pupils, and Study Four—Strategies and Justification in Pattern Generalization (JuStraGen), secondary students. This chapter discusses each study in turn and the Conclusion discusses the overall implications these findings have on the teaching of patterns, in particular, how the teachers can help sensitise learners to patterns.

All children participated with consent. For this chapter, pupils are used for primary school participants, students for those from secondary schools.

10.4.1 Study One: The Age and Individual Differences in Mathematical Abilities: From Kindergarten to Secondary Schools Study

In 2005, the longitudinal study (henceforth, Age and Individual Differences Study) examined the relationships amongst cognitive abilities, socio-motivational beliefs, and mathematical performance of Singapore children from kindergarten to secondary schools. A range of mathematical tasks including arithmetic word problems, algebraic word problems, number and geometric pattern-type tasks, function–machine type tasks, and function tasks were constructed to track children's mathematical performance across the years. All the mathematical tasks were constructed to reflect the expectations of the mathematics curriculum and the spiral structure of the mathematics curriculum. A number of publications emerged from the Age and Individual Differences Study (e.g. Lee et al. 2017). Primary children's performance with function–machine tasks was reported in Ng (2018).

This chapter reports the findings from four different studies, the findings from two grade levels, Primary 3 and Primary 4 from the Age and Individual Differences Study, are reported here. Two of these function tasks are similar in presentation, with one based on Primary 3 pupils' knowledge of the five times tables (henceforth, the five function task, Fig. 10.6), and the other based on Primary 4 pupils' application knowledge of long division and the three times tables (the three function task, Fig. 10.7).

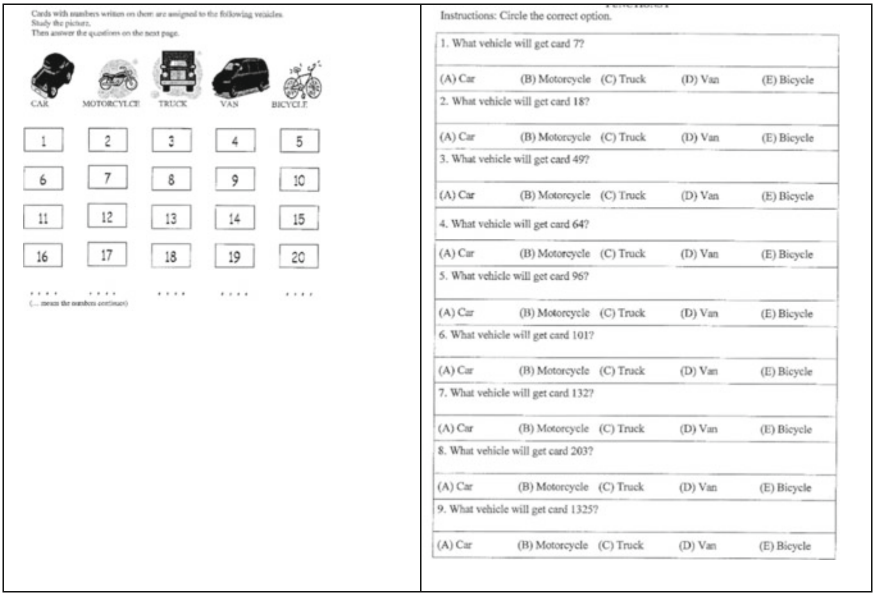


Fig. 10.6 The five function task for Primary 3 pupils

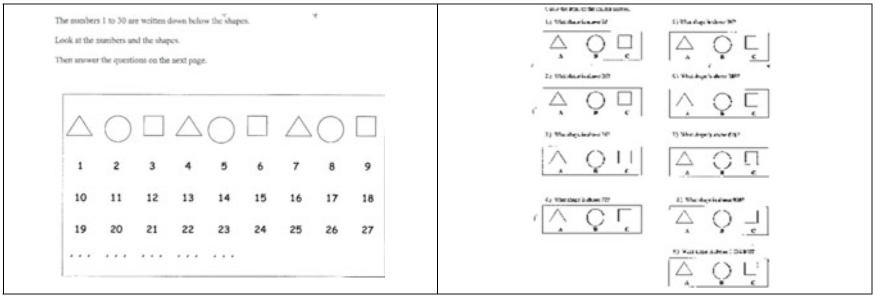


Fig. 10.7 The three function task used with Primary 4 pupils

10.4.1.1 The Participants

Two groups of pupils, ten each from Primary 3 (age 9+) and Primary 4 (age 10+), participated in the supplementary study for the function task. The Primary 3 function task involved core repeating patterns of five while the Primary 4 pupils worked with function tasks involving core-repeating of 3. Although these two tasks required pupils to use their knowledge of five times and three times table, the Primary 3 pupils were more fluent with the five times tables than the Primary 4 with the three times tables.

10.4.1.2 The Instrument

In the Age and Individual Differences Study, in any one year, the function instrument comprised nine questions, divided into four subparts. Part (a) consisted of two base questions (Q1 and Q2), the solutions of which could be read directly from the diagram itself. Part (b) were three near-prediction questions (Q3, Q4, and Q5) the solutions of which could not be found directly from the diagram but could be found either by continuing with the number sequence or from some beginnings of pattern recognition in the structure of the numbers. The solutions to the three Part (c), far-prediction questions, could not be found by continuing writing out the numbers. Rather the solutions to these questions (Q6, Q7, and Q8) could be found by identifying the structure underpinning the numbers. The single question in Part (d) used a seven-digit number which was beyond counting but required abstraction of the structure underpinning the pattern and then generalising that structure any number beyond counting. This four-part structure meant that it was possible to identify pupils' current knowledge of number structure. Those pupils who could solve all four parts indicated that they had good knowledge of structure underpinning numbers. However, those who could complete only Parts (a) meant that they could understand the demands of the function tasks and knew where to look for the solutions. Those who proceeded to Part (c) had better command of the structure of numbers and could identify that although the numbers were getting bigger, certain parts of the numbers remained the same. By identifying what remained the same and what changed, they could provide the appropriate solutions to the questions without having to write out the entire number sequence.

10.4.1.3 The Interview Protocol

This section is technical as it sets out the process how the interviewer engaged the participants with the function tasks. All function tasks have a similar structure. To gain insights into how the participant understood a given task, the interviewer used the following questions to engage with the tasks. It is necessary to provide this level of detail so that others may choose to follow the same protocol to see if they could arrive at the similar findings. Otherwise, others could amend the interview protocol to ascertain if they would achieve different findings.

To ensure that the participant understood the demand of the task, each participant was asked to read aloud

(i) the instructions provided at the beginning of each question;
(ii) the numbers listed below each of the five vehicles; and
(iii) the numbers below each vehicle. For example, "What number cards are below the car? The motorcycle? The truck? The van? The bicycle?"

The participants were asked to provide answers to

(i) Q1 and Q2 (base items). Participants' correct responses suggested that they could see that the numbers were the function of the shapes.

(ii) Q3, Q4, and Q5 (near prediction). If participants were able to answer without continuing the number list provided, then this suggested that participants may have developed a sense of the patterns underpinning the task.
(iii) Q6, Q7, and Q8 (far-prediction questions). Participants' correct responses suggested that they were confident of the rule they had constructed that helped them answer the near-prediction Q3, Q4, and Q5.
(iv) Q9. Participants' correct response meant that the size does not matter. They were likely to have abstracted and were willing to generalise the structure the number pattern to any number.

However, those participants who continued the number list to help them answer Q3, Q4, and Q5 were asked to explain their strategy and to offer alternative strategies, if possible.

(i) Why do you write down all these numbers?
(ii) Can you try to answer these questions without writing down these numbers?

The interviewer terminated the interview when the pupils were seen struggling with a task. When the participants were unable to answer Q3, Q4, and Q5, they did not proceed with Q6, Q7, Q8, and Q9. However, if they were able to secure two out of three correct responses, they proceeded with Q6, Q7, and Q8. As well, the interview was discontinued at any time a child expressed a wish to stop, but no child chose to do so.

For each category of numbers, the interviewer followed up with the following epistemological questions.

- How do you know these are the number cards received by each vehicle?
- How do you know you are correct?
- Did you learn to do such questions? In school or at home?

10.4.1.4 Findings

The three function task was offered to the Primary 4 pupils and the five function task to the Primary 3 because based on task analysis, the former was deemed more challenging than the five function task. In the five function task, the numbers belonging to the bicycle had the digit 5 in the ones place or were multiples of ten. However, for the three function task, the pattern was less obvious. Shapes with numbers that were multiples of ten moved from column to column. The number 10 was in the column for triangle, 20 for the column for circle and 30, the square. The interviews with the pupils showed that the Primary 3 and Primary 4 pupils used two different surface strategies to answer the questions. These surface strategies are (i) the nature of the digit in the ones place, chunking, and counting on. Five Primary 4 pupils used the more sophisticated strategy whereby they looked at the "deep" structure of looking for the remainder when the number was divided by 3 for the three function task. No Primary 3 pupil used the remainder strategy because long division with remainder was not taught until Primary 4.

The nature of the digit in the ones place, chunking, and counting on. All Primary 3 and five Primary 4 pupils explained that they looked at the digits in the ones place to identify which object (vehicle or shape) would receive a specific number card. For example, to answer Q7: What vehicle will get card 132, the common strategy was to first look at the first row and identify that motorcycle has the number 2, and 12 in the ones place. Then they could either add in chunks of 10 until they reached 130 and then they added 2 to arrive at the total of 132. The other strategy was to count in chunks of 10 from the last number in the row, i.e. 5. They then applied skip counting of 10. Ten, twenty, thirty, forty, fifty, sixty, … till they reached 130. Then they moved across to the first column where the digit in the ones place is one, they counted on two more from there, 130, 131, and 132. The motorcycle has the number 132 (Fig. 10.8).

Deep structure of looking for the remainder: Primary 4 pupils are taught the long division algorithm. However, the strategy of looking for deep structure did not come naturally to these children. It was necessary to draw pupils' attention to what remained the same and what changes. Pupils found the following scaffolding questions helped them in seeing the deep structure, i.e. there is a relationship between the number of shapes which kept repeating (three shapes, triangles, circles, squares) and the remainder of the numbers in the columns below each shape. How many shapes are there? Name these shapes. What shape comes after the triangle? The circle, the square? What do you notice when you divide number 3 below the square by 3? The number 4 below the triangle? The number 5 below the circle?

Pupils used the scaffolding questions to look for patterns. PCA used the scaffolding questions to help her come up with a conjecture. Her annotations showed that she listed the remainders of numbers that were within writing. For Q2, What shape is above 26, the annotations showed that PCA noticed that the shape triangle was associated with the remainder 1, circle with the remainder 2, and the shape square had no remainder. Solution to the right in Fig. 10.9 showed how this pupil used the scaffolding questions to support her in constructing the rule that the shape associated with any number is a function of its remainder when it was divided by 3. The solution offered by PR was the most intriguing and his solution showed how some children might see patterns where others did not. For numbers below 100 m, PR used the remainder of 3 to predict which shape was above the number. However, with numbers above 100, PR began testing this alternative rule. Divide the number the number by 3 and also by 9. Were the two remainders the same? He divided the number 388 by 3 and also by 9 and found that the remainders were the same. This strategy was tested with the numbers 621 and 920 and the remainders were the same whether the number was divided by 3 or by 9. In PR's case, he could have noticed that the remainder of the number cards in the second and beyond were functions of the numbers in the first row regardless whether these numbers were divided by 3 or by 9. PR was the only participant who saw this deeper structure, i.e. the relationship between the remainders when the numbers were divided by 3 and then by 9 (Fig. 10.10).

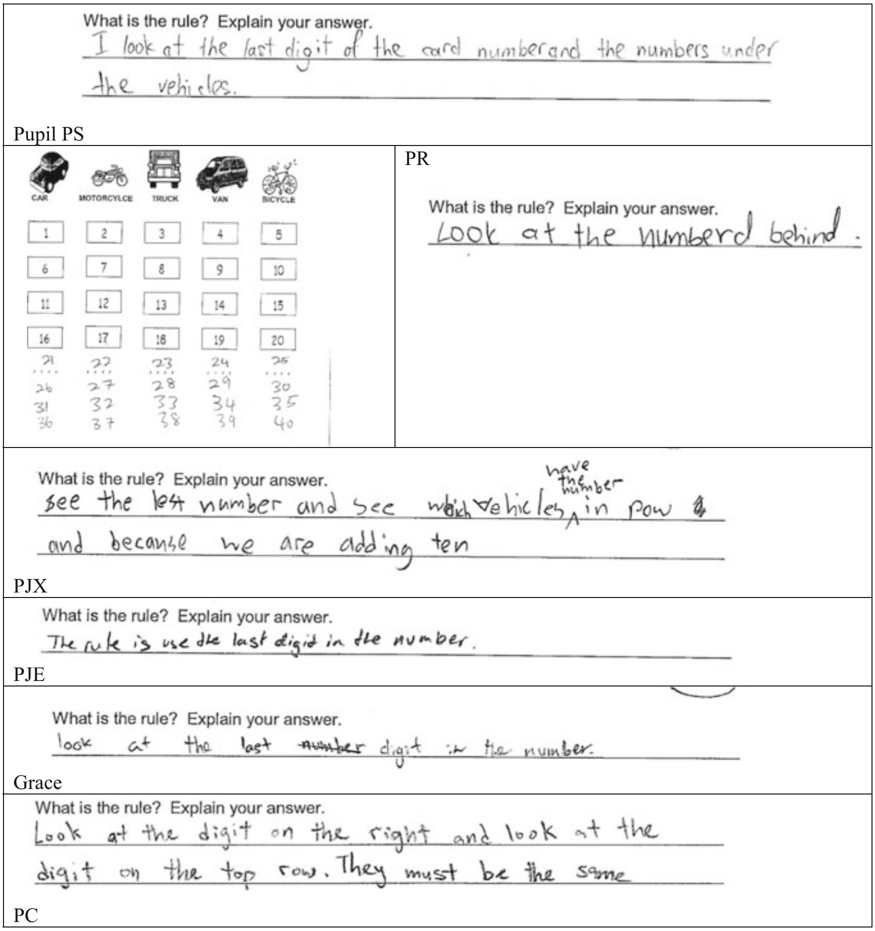


Fig. 10.8 Some of the pupils' written responses explaining how they saw the structure underpinning the pattern task

10.4.2 Study Two: Forward and Backward Rule Study

This study, conducted in 2014, investigated how young pupils navigated the elusive pattern generalisation process. In particular, it examined how Primary 4 pupils (age 10+) determined specific terms that were both near and far from the last given term in a pattern generalising task, how they worked out the position of a term when given the term itself, and how successful they were in establishing the predictive rule that described the pattern depicted in the task. Three pattern tasks were used, and in each task, the pattern was presented figurally as a sequence of four consecutive configurations: Diagram 1 to Diagram 4. So Diagram 5 to Diagram 10 were then taken to be a *near* term whereas any term beyond Diagram 10 was considered as a

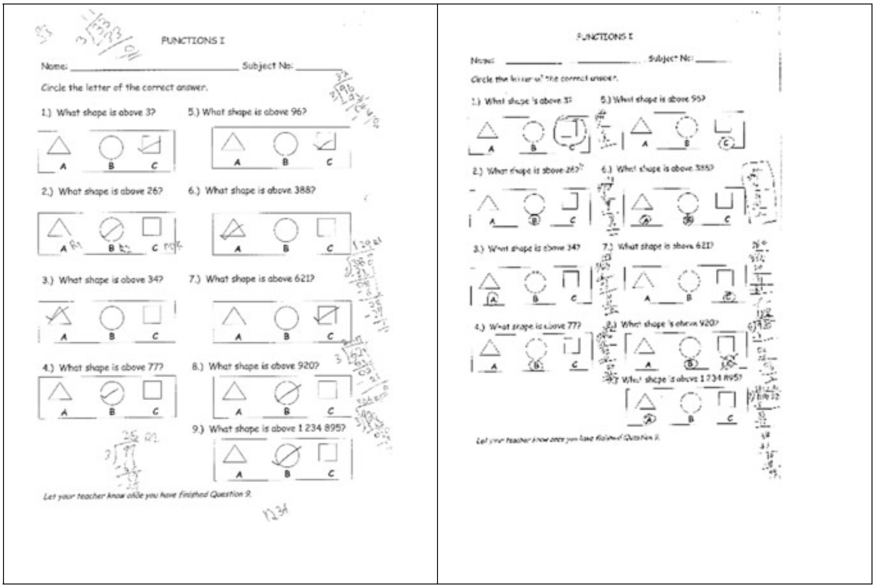


Fig. 10.9 Solution by PCA on the left and PR on the right

	Shapes	△	○	□	△	○	□	△	○	□
	Card numbers	1	2	3	4	5	6	7	8	9
Deep structure	Remainder when divisor is 3	1	2	0	1	2	0	1	2	0
Deeper structure	Remainder when divisor is 9	1	2	3	4	5	6	7	8	0

Fig. 10.10 PR was the only participant who saw the relationship between the two remainders

far term. The predictive rules in all the three pattern tasks had the linear structure $an+b$, with a and b as constants and n the figure number. Hence, all the three patterns depicted a linear relationship between the term and its position.

10.4.2.1 The Participants

Fifty-seven Primary 4 pupils, (34 boys, 23 girls) from one primary school participated in this study. They were from different classes and, based on their performance in their year-end Primary 3 mathematics examination, were identified as *high progressing*, *middle progressing,* and *low progressing.* In Singapore schools, high progressing Primary 3 and Primary 4 pupils were those who scored 85 marks and above, middle progressing were those who scored between 84 marks and 50 marks, and low progressing were those who scored 50 marks and below. In this study, there were 25 high progressing pupils, 14 middle progressing pupils, and 18 low progressing pupils. As part of the mathematics curriculum, under the generic problem-solving heuristic of looking for patterns, these pupils were taught how to (i) continue the pattern and identify the shapes in a figural repeating pattern sequence, (ii) continue the pattern in a growing pattern presented as a sequence of numbers or figural configurations, and (iii) find the position of a given term in a pattern sequence.

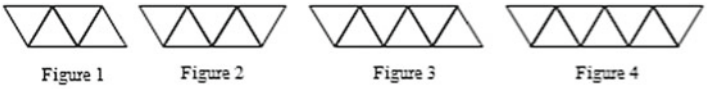
A pilot study was conducted with 30 Primary 4 pupils from another primary school with the objectives of checking the clarity and comprehensibility of the three pattern tasks and gauging the time needed to complete all the three tasks. These pupils' performance in the pilot study showed no further modification of the tasks was needed. Calculators were not necessary as the pupils were able to work out all numerical computations manually. Because more answer space invited more elaborations from the pupils, hence, more space was provided for each subpart in the actual study. Based on the evidence from the pilot study, these pupils were given 75 min to complete the task in the actual study.

10.4.2.2 The Instrument

FUN-PATS Test: The paper-and-pencil test instrument used in this study, henceforth *FUN-PATS* for short, comprised three linear figural pattern tasks, each part divided into four subparts. Pupils were asked to predict a near term in Part (a) and make a far prediction in Part (b). Part (c) required them to determine the position of a given term, and Part (d) asked for the predictive rule. The difficulty level of the three tasks was graduated with the first two tasks, *Making Triangles* and *Triangular Chains*, depicting a 1-step linear rule: $3n$ and$n + 3$, respectively, where n is the figure number. The last task, *Square Tiles Extension*, involved a more challenging 2-step linear rule: $4n + 1$. Figure 10.11 shows the *Triangular Chains* task.

In the *Triangular Chains* task, figure number 7 in Part (a) was chosen given its close proximity to figure number (i.e. 4) of the last configuration in the figural sequence. To find the number of triangles in Figure 7, pupils could easily extend the pattern by adding 1 successively or draw out Figure 7 configuration and then count the number of triangles. In Part (b), figure number 21, which is a multiple of the figure number in Part (a), was deliberately chosen for two reasons: (i) to see how pupils determined the number of triangles when the figure number was farther away and using the "add 1" rule might be inconvenient and (ii) to see whether pupils would erroneously think that Figure 21 had three times as many triangles as Figure 7 given that 21 is thrice of 7. The term in Part (c), which in this case referred to the number of triangles, was made manageable for pupils to manipulate without the aid of calculators. The aim of Part (d) was to examine the pupils' innate ability to articulate the predictive rule when they had not even been taught formally how to do it. The two other pattern tasks were set in a similar context and Table 10.1 shows the respective patterns alongside the specific terms and the figure number to be determined.

Benny used triangles to make the following chains to form a pattern.



For example:

Benny used 4 triangles in Figure 1.

He used 5 triangles in Figure 2.

He used 6 triangles in Figure 3.

As the pattern is continued, more triangles are used.

Study the figures carefully then answer the questions that follow.

No.	Question and Working	Explain clearly how you obtained the answers. You may use words and diagrams.
2a	Find the number of triangles Benny would need to form the chain in Figure 7. Benny would need _____ triangles to form the chain in Figure 7.	
2b	Find the number of triangles Benny would need to form the chain in Figure 21. Benny would need _____ triangles to form the chain in Figure 21.	
2c	Benny used 34 triangles to form a chain. What is the figure number of this chain? The figure number of this chain is _____.	

Fig. 10.11 Triangular chains task

Table 10.1 Making triangles and square tiles extension pattern tasks

Parts	Making triangles	Square tiles extension
(a) Find the near term	8	6
(b) Find the far term	18	12
(c) Find the figure number	63	81

10.4.2.3 Interview Protocol

To understand better the thinking and reasoning processes of these pupils, nine pupils (four boys and five girls) were selected for individual interviews after they had completed the *FUN-PATS* task. The aim of the interview was to gain further insights into the pupils' choice and their epistemology of strategies for finding the near and far terms, and the relationship between the figure number and the predictive rule. The following semi-structured questions guided the interviews:

(i) Can you think of another way to get the number of (objects) for Figure N? How did you figure out this? Does this method work for other figure numbers?
(ii) How did you decide on what the rule is?

Before asking the pupils about each task, they were given sufficient time to look at their written responses in the tasks. Every pupil, except for one, was interviewed on two of the three pattern tasks. Only one pupil, who was articulate and swift in responding to the interview questions within the stipulated time, was interviewed for all three tasks.

10.4.2.4 Findings

The performance of the Primary 4 pupils across the three pattern tasks fell with the increasing complexity of the predictive rules in this order: $3n$ in *Making Triangles*, followed by $n + 3$ in *Triangular Chains*, and then $4n + 1$ in *Square Tiles Extension*. For instance, the percentages of pupils who made the far prediction correctly were 74% in *Making Triangles*, 44% in *Triangular Chains,* and 30% in *Square Tiles Extension*. Within each pattern task, the pupil performance across the four subparts also followed a similar trend, with the highest success rate in making near prediction and the lowest in constructing the predictive rule. Although the pupils were found to employ various generalising strategies to find both the near and far terms, the majority seemed to favour the recursive approach of adding the common difference between two consecutive terms to the previous term in both *Triangular Chains* and *Square Tiles Extension*. But in *Making Triangles*, the pupils employed predominantly a functional approach. A possible reason for their choice of a functional approach is that the predictive rule corresponded to the three times tables, which they were all familiar with.

Finding the figure number seems tough for many Primary 4 pupils. Only about a third of the pupils answered correctly in *Triangular Chains* and *Square Tiles Extension* although there were twice as many pupils in *Making Triangles*. Amongst the successful pupils, a sizeable number had recognised the inherent pattern structures when predicting the far term and performed reversal thinking of operations by applying the *Undoing* strategy to work out the figure number: for instance, $(81 - 1) \div 4 = 20$ in *Square Tiles Extension* in which the predictive rule was $4n + 1$. However, a small number of pupils obtained the correct figure number by listing out the terms until the one being considered was found. The figure number was then established by counting its position.

The construction of the predictive rule proved tricky for the Primary 4 pupils, with success rates of below 15% for all three pattern tasks. Expressing generality is not a straightforward task for primary school pupils and remains elusive for even secondary school pupils and adults. Thus, it was not surprising to find the majority of the successful Primary 4 pupils coming from the high progressing group and none of the low progressing pupils made it. Another noteworthy finding emerging from the data analysis was that not every Primary 4 pupils who recognised the inherent pattern structure and predicted correctly the far term succeeded in rule construction. This finding resonates with the remarks made by Blanton and Kaput (2004) and Mason (2008) that the ability to make a far prediction does not always lead to successful rule construction. The generalising skill is not acquired in a day or two; it takes time as well as guidance from teachers and repeated exposure to develop.

10.4.3 Study Three: Colour-Contrast Study

This study was conducted in 2015 to explore the use of colour contrast in a linear figural pattern in assisting pre-algebra Primary 6 pupils (12+) in establishing the predictive rule. It involved collecting both quantitative and qualitative data through a paper-and-pencil task and clinical task-based interviews with selected pupils.

10.4.3.1 Participants

Thirty Primary 6 pupils (15 boys and 15 girls) from an intact class of a typical Singapore primary school participated in the main study. Two were high progressing pupils and the rest were either low progressing or middle progressing pupils, who had weak number sense and basic operation manipulation skills, in particular, multiplication and division. This class was the sample of choice because the intent was to ascertain whether low and middle progressing pupils could generate the predictive rule when colours were used to construct the structures in a figural pattern task. These pupils had no formal knowledge of algebra. Thus, they were unfamiliar with the use of letters to represent unknowns in algebraic expressions. Further, they had very little experience in working with pattern tasks that involved rule construction.

Based on their performance in their Primary 5 summative mathematics examination, the Primary 6 pupils were divided evenly into two groups: single-colour group and colour-contrasted group. The 15 pupils in the single-colour group were largely middle progressing with one high progressing and two low progressing because they were considered to have better generalising skills. The colour-contrasted group was made up of nine low progressing pupils with one high progressing pupil and five middle progressing pupils. After they have completed the task, four pupils from each group were selected to participate in the clinical task-based interviews. The eight

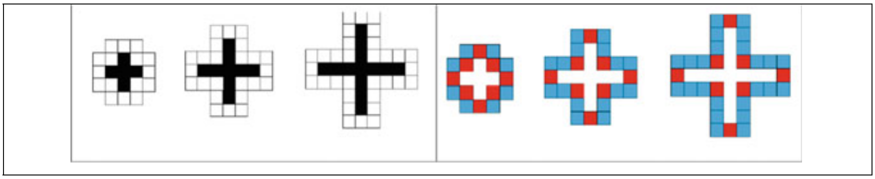


Fig. 10.12 The white tiles task, single colour to the left, colour contrast, to the right

pupils (4 boys, 4 girls) comprised two high progressing, three middle progressing, and another three low progressing.

10.4.3.2 The Instruments

The paper-and-pencil *White Tiles* task was used in a number of studies (Amit and Neria 2008; Rivera and Becker 2005). The *White Tiles* task came in two versions: single colour versus colour contrasted (Fig. 10.12).

The pattern to the left was given to the single-colour group and the one to the right, to the colour-contrasted group. The pupils in both groups answered the same questions: (i) write down in words a rule to find the number of tiles in any figure, (ii) write the rule in mathematical symbols, and (iii) find the figure that is made up of 272 tiles. The pupils had access to the calculator and were given 45-min to complete the task that was conducted after curriculum.

10.4.3.3 Clinical Task-Based Interview Protocol

The interviews were conducted the day after the written task. Each pupil completed the interviews in one sitting and the interviews for the entire group took two consecutive days to complete. The interviews meant that it was possible to (i) understand these pupils' written responses, (ii) provide insights into pupils' thinking as they explained and worked out the tasks, and (iii) test whether it was possible to support those pupils who failed to construct the predictive rule in the paper-and-pencil task via a chance to do so through a series of questions which directed their attention to the relationship between input and output variables. The semi-structured interviews meant that it was possible to modify the interview questions depending on the pupil responses. Some of the questions asked were:

- As the pattern grows, what stays the same and what changes in each figure?
- How many objects stay the same from one figure to the next?
- Is there a relationship between the figure number and the number of tiles in the figure?
- Explain what your pattern rule means from the figures.

During the interview, pupils had access to coloured counters that they could use to build the patterns and describe their rules.

10.4.3.4 Findings

The use of colour contrast in linear figural pattern task had helped pupils of all learning abilities, in particular the low progressing pupils, to

(i) count the quantitative parts of the figures accurately,
(ii) visualise the growth of the figural pattern correctly, and
(iii) identify parts of the figures that remained the same and parts that changed in the linear pattern task.

The colour-contrasted pattern task shows potential in making the multiplicative relationship between the figure number and the parts of the figures that change in step with the figure number more salient. Additionally, such a task seems to ground pupils with a strong structural understanding of the predictive rule. Pupils in the colour-contrasted group were more likely to be able to interpret their rules geometrically than those in the single-colour group.

The Primary 6 pupils were more successful than the Primary 4 pupils in the *Forward and Backward Rule* study in finding the figure number. About half of the Primary 6 pupils found correctly the figure with 272 tiles. This finding was noteworthy considering that these pupils were mostly low to average performers in their school mathematics examination. There is also strong evidence to suggest that the use of colour contrast had enabled the low progressing pupils in the colour-contrast group to determine the correct figure number.

10.4.4 Study Four: Strategies and Justifications in Mathematical Generalization (JuStraGen) Study

This large-scale study sought to (i) investigate how Secondary 2 students (age 14+) made and justified generalisations of figural patterns when the format of pattern display and the type of functions underpinning the pattern were varied and (ii) probe systematically the effect of the format of pattern display and the type of functions on the students' generalisations. The format of pattern display is concerned with whether the figural pattern is presented as a sequence of successive or non-successive configurations. The type of functions considers whether the term-to-position relationship describes a linear or non-linear rule. In this study, the non-linear relationship is of the quadratic type. Data were collected in 2011 through administering a paper-and-pencil task that comprised four linear pattern tasks and four quadratic pattern tasks. Several publications emerged from this *JuStraGen* study (e.g. Chua and Hoyles 2009, 2011, 2012, 2014). This chapter will only focus on the students' competence in rule construction and the effect of the two task features on their generalisation of the predictive rules.

10.4.4.1 Participants

Based on their performance in a national examination taken at the end of their primary education, secondary school students in Singapore are placed in one of the three tracks Express, Normal (Academic), and Normal (Technical), in order of decreasing academic performance. The curricula for each track places are designed to suit the students' learning abilities and interests. Of the entire cohort in the first grade at the secondary level (Secondary 1), the top 60% makes it to the Express track, the next 30%, and the remaining 10% to the Normal (Academic) and Normal (Technical) tracks, respectively.

In the *JuStraGen* study, 515 students from the Express and Normal (Academic) courses from three schools participated with consent. They were 14-year-old Secondary 2 students (242 boys, 273 girls), with 337 from the Express track and 178 from the Normal (Academic) track. The Express and Normal (Academic) students were divided into two groups, *Successive* or *Non-successive*, based on three criteria: their mathematics grades at the national examination, their scores in a baseline test taken before the study commenced, and their gender. There were 266 students (170 Express, 96 Normal (Academic)) in the *Successive* group and 249 students (167 Express, 82 Normal (Academic)) in the *Non-successive* group. Students in the *Successive* group were given pattern tasks that showed a sequence of three successive configurations whereas those in the *Non-successive* group received the same tasks, but with configurations in a non-successive order. The Secondary 1 mathematics curriculum had introduced these students to pattern-type tasks where they had to recognise and represent number patterns and to derive the predictive rule for finding any term in the pattern. However, they had very little experience with work which required the construction of quadratic equations because the concept of quadratic functions was introduced only in Secondary 2.

10.4.4.2 The Instrument

The *JuStraGen* task was developed specifically to inquire into the effect of the format of pattern display and of the type of functions on the students' pattern recognition and their ability to generalise. The paper-and-pencil task consisted of four pattern tasks involving one linear pattern structure and four involving a quadratic structure. Each pattern task assumed two different formats, with its pattern depicted as (i) a sequence of three consecutive diagrams and (ii) a single configuration or a sequence of two or three non-successive configurations. A full discussion of the design of the *JuStraGen* test is provided in Chua and Hoyles (2013). To make the task manageable to the students, the eight pattern tasks were divided into two sets of four tasks and these were administered on two separate days. Each set consisted of two linear and two quadratic pattern tasks. All the pattern tasks were unstructured to allow students scope for exploration so that they could produce their own interpretations of the patterns. Students had to derive the predictive rules and justify how the rules were obtained for all the pattern tasks. The students were given 45 min to complete the

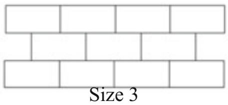
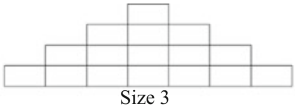
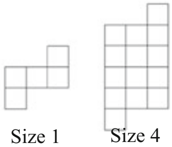
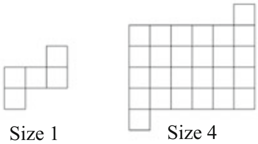
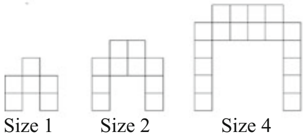
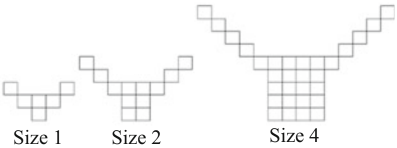
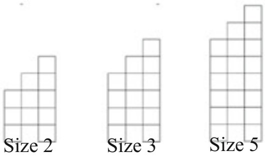
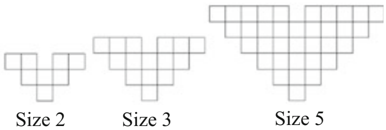
<table>
<tr><th>Linear</th><th>Quadratic</th></tr>
<tr><td>Bricks
For successive format: Sizes 1, 2, 3 were given

</td><td>Wall Design
For successive format: Sizes 1, 2, 3 were given

</td></tr>
<tr><td>Birthday Party Decorations
For successive format: Sizes 1, 2, 3 were given

</td><td>Christmas Party Decorations
For successive format: Sizes 1, 2, 3 were given

</td></tr>
<tr><td>Towers
For successive format: Sizes 2, 3, 4 were given

</td><td>Oh Deer!
For successive format: Sizes 2, 3, 4 were given

</td></tr>
<tr><td>High Chairs
For successive format: Sizes 2, 3, 4 were given

</td><td>Tulips
For successive format: Sizes 2, 3, 4 were given

</td></tr>
</table>

Fig. 10.13 Linear and quadratic pattern tasks in JuStraGen test

task and they had access to calculators. Figure 10.13 shows the eight figural patterns given in the non-successive version of the *JuStraGen* test. The corresponding figure numbers of the three successive configurations for each pattern task are also indicated.

10.4.4.3 Findings

The academically more able Express pupils far outperformed the Normal (Academic) pupils in the *JuStraGen* test. The success rates for all the eight tasks ranged from 50 to 70% for Express pupils in both the successive and non-successive groups, whereas those for Normal (Academic) pupils in the two groups ranged from a low of 2 to 18%. This stark contrast in success rates shows that pattern generalisation remains a stern challenge to a vast majority of the academically less able students.

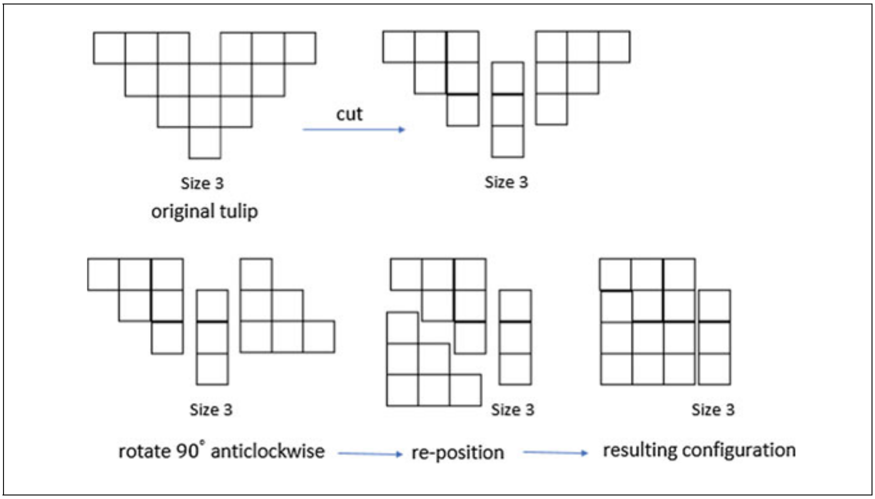


Fig. 10.14 The Tulips task that proved to be most challenging for the students

The study had also shown that the Express students were more likely than the Normal (Academic) students to produce a predictive rule for the pattern tasks. The predictive rules that were constructed by the Express students for each pattern task had a far more diverse range of structurally distinct-looking but equivalent rules, predominantly expressed in algebraic notations. The prevalence of a wide diversity of equivalent rules reflects the express students' flexible thinking and discernment of the pattern structure in multiple ways.

The Express students were not affected by the format of pattern display. Be the display one of the typical and familiar formats of three successive configurations or one involving non-successive configurations, the Express students were still able to construct the predictive rules. Their ability in rule construction seems to be supported very much by their keen awareness of the pattern structure. This finding lends credence to the view that it is extremely crucial to teach students to identify structure (Küchemann 2010; Mason et al. 2009) and that their attention will no longer be drawn to focus on the usual counting of tiles but on using the figure number as a *generator* (Chua and Hoyles 2014) to abstract a relationship between the figure number and the parts of the configuration that change in step with the figure number, then followed by articulating a rule that captures this relationship.

However, the type of functions seems to matter to both the Express and Normal (Academic) students. Compared to linear pattern tasks, quadratic pattern tasks were found to be more challenging for these students. A plausible reason is that spotting the relationship between the figure number and the parts of the configuration that change in step with the figure number in a quadratic pattern is not a straightforward process. Consider the *Tulips* task in Fig. 10.14.

One way to envision the tulip pattern geometrically is to cut each configuration into three parts: the petal on the left, the stalk in the middle, and the petal on the right. The left and right petals are identical "staircases" that are mirror images of each other. Next, establish a link between the figure number and each part of the tulip configuration. Students may easily notice that the number of tiles in the stalk corresponds to the size number. However, the link between the size number and each petal is *not* conspicuous and hard to establish. The number of tiles in a "3-step staircase" is $1 + 2 + 3$, which is easy to determine. However, for bigger figure number, the calculation of the number of tiles then becomes tedious. Furthermore, students also struggle to work out the general expression for the number of tiles in any figure. This problem is resolved if the right petal is rotated 90° anticlockwise and repositioned below the left petal to form a rectangle. The resulting configuration reveals the pattern structure of the tulip that can then be interpreted in two different ways: a n by $(n + 1)$ rectangle with an additional column of n tiles or a $(n + 1)$ by $(n + 1)$ square with one missing tile at the top right corner. Such a strategy of rearranging one or more parts of the original configuration to form a shape more familiar may not be apparent and clear-cut to many pupils. This is why quadratic pattern tasks such as Tulips is not easy for secondary school students.

10.5 Conclusion

The suite of studies reported in this chapter looked at the performances of learners from different age groups and academic abilities with pattern tasks. Constructing rule that captures the information inherent in any given pattern task is not an arbitrary process. Learners need to have good number sense and proficient command of the four binary operations. Only then can these learners notice the surface structures underpinning a pattern. Skilful questioning can help learners look for deeper structures. However, there are some learners who are able to notice more structural relationships than others. Pupil PR (Fig. 10.10) is a case in point. Researchers and teachers should be aware of such possible structures so that they do not dismiss the responses of such "gem" learners as nonsense. Rather unusual noticing raise questions of the epistemology of such "gem" learners: How did these learners notice such structures? Why did they notice such structures? Such unusual responses offer teachers opportunities to encourage learners not to be satisfied with the first rule they constructed but rather to encourage learners to wonder whether there are other possibilities.

Processes such as pattern spotting, identifying the underlying structures and generating the predictive have to be taught, not told. Selecting appropriate examples and asking good questions would help. The evidence from the colour-contrast activity showed that low performing children benefited from the way the tasks were presented to them. Perhaps normal academic and normal technical students, too, may benefit from such colour-contrast tasks and teachers can use the questions in their teaching.

This chapter reported how the different studies utilised the three strands algebraic thinking to infuse the key ideas of generalisation and symbolisation (Kaput 2008). It is important to bear in mind that passing examinations should not be the mainstay of cultivation of algebraic thinking but rather the emphasis is to help the human mind organise the huge amount of information present in the environment enabling individuals to function meaningfully (Fosnot and Jacob 2010).

References

Amit, M., & Neria, D. (2008). Rising to the challenge: Using generalization in pattern problems to unearth the algebraic skills of talented pre-algebra students. *Zentralblatt für Didaktik der Mathematik, 40,* 111–129.

Blanton, M. L., & Kaput, J. J. (2004). Elementary grades students' capacity for functional thinking. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 135–142). Bergen, Norway.

Cai, J., Lew, H. C., Morris, A., Moyer, J. C., Ng, F. F., & Schmittau, J. (2005). The development of students' algebraic thinking in earlier grades: A cross-cultural comparative perspective. *Zentralblatt für Didaktik der Mathematik, 37*(1), 5–15.

Chua, B. L., & Hoyles, C. (2009). Generalisation and perceptual agility: How teachers fared in a generalising problem. In *Proceedings of the British Society for Research in Learning Mathematics (BSRLM) Conference* (pp. 13–18). Bristol, The United Kingdom: BSRLM.

Chua, B. L., & Hoyles, C (2011). The interplay between format of pattern display and expressing generality. In B. Ubuz (Ed.) *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (p. 281). Ankara, Turkey: PME.

Chua, B. L., & Hoyles, C. (2012). The effect of different pattern formats on secondary two students' ability to generalise. In T. Y. Tso (Ed.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol 2, pp. 155–162). Taipei, Taiwan: PME.

Chua, B. L., & Hoyles, C. (2013). Rethinking and researching task design in pattern generalisation. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol 2, pp. 193–200). Kiel, Germany: PME.

Chua, B. L., & Hoyles, C. (2014). Generalisation of linear figural patterns in Secondary School. *The Mathematics Educator, 15*(2), 1–30.

Driscoll, M. (1999). *Fostering algebraic thinking: A guide for teachers grades 6–10.* Portsmouth: NH: Heinemann.

Fosnot, C. T., & Jacob, B. (2010). *Young mathematicians at work: Constructing algebra*. Reston, Virginia: NCTM.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (p. 5–17). Reston, VA: NCTM.

Küchemann, D. (2010). Using patterns generically to see structure. *Pedagogies: An International Journal, 5*(3), 233–250.

Lee, K., Ng, S. F., & Bull, R. (2017). Learning and solving more complex problems: The roles of working memory, updating, and prior skills for general mathematical achievement and algebra. In D. C. Geary, D. B. Berch, R. Ochsendorf, & K. M. Koepke (Eds.), *Acquisition of complex arithmetic skills and higher-order Mathematics concepts* (pp. 197–220). http://dx.doi.org/10.1016/B978-0-12-805086-6.0009-6.

Mason, J. (1990). *Supporting primary Mathematics: Algebra.* Milton Keynes, UK Open University: The Open University.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee, (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65–86). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Mason, J. (2008). Making use of children's powers to produce algebraic thinking. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 57–94). New York: Lawrence Erlbaum Associates.

Mason, J., Stephens, M., & Watson, A. (2009). Appreciating Mathematical structures for all. *Mathematics Education Research Journal, 21*(2), 10–32.

Moses, B. (Ed.). (1999). *Algebraic thinking. Grades K-12.* Reston, VA: NCTM.

Ng, S. F. (2018). Function tasks, input, output and the predictive rule is: How some Singapore primary children construct the rule. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds the global evolution of an emerging field of research and practice.* Springer International Publishing AG.

Radford, L. G. (2001). The historical origins of algebraic thinking. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives in school algebra* (pp. 13–36). Dordrecht: The Netherlands: Kluwer Academic Publishers.

Rivera, F. D., & Becker, J. R. (2005). Figural and numerical modes of generalising in algebra. *Mathematical Teaching in the Middle School, 11*(4), 198–203.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. Coxford (Ed.), *Ideas of algebra: K-12* (pp. 8–19). Reston, VA: NCTM.

Van De Walle, J., & Bay-Williams, J. M. (Eds.). (2014). *Elementary and middle school Mathematics: Teaching developmentally* (8th ed.). Essex, UK: Pearson.

Swee Fong Ng is an Associate Professor at the National Institute of Education, Nanyang Technological University, Singapore. She holds a master and a Ph.D. in mathematics education, both from the University of Birmingham, United Kingdom. Her general interests include how teacher's pedagogical content knowledge influence the nature of questions and the choice of examples used to support the teaching and learning of mathematics across the curriculum. The teaching and learning of algebra are her special interest.

Boon Liang Chua is an Assistant Professor in Mathematics Education at the National Institute of Education, Nanyang Technological University in Singapore. He holds a Ph.D. in Mathematics Education from the Institute of Education, University College London, UK. His research interests cover mathematical reasoning and justification, task design, and pattern generalisation. Given his experience as a classroom teacher, head of the department, and teacher educator, he seeks to help mathematics teachers create a supportive learning environment that promotes understanding and inspire their students to appreciate the beauty and power of mathematics. With his belief that students' attitudes towards mathematics are shaped by their learning experiences, he hopes to share his passion of teaching mathematics with the teachers so that they make not only their teaching more interesting but also learning mathematics an exciting and enjoyable process for their students. He feels honoured to have been awarded *Excellence in Teaching* by the National Institute of Education in 2009 and 2013.