Mathematics Education – An Asian Perspective

Tin Lam Toh
Berinderjeet Kaur
Eng Guan Tay *Editors*

Mathematics Education in Singapore

Springer

Mathematics Education – An Asian Perspective

Series editors

Aims and Scope

Mathematics Education – An Asian Perspective facilitates high quality publications on rigorous aspects of mathematics education in Asia. This will be achieved by producing thematic books that capture knowledge and practices on mathematics education in Asia from both the insider and outsider perspectives. The series helps to establish a much needed Asian perspective to mathematics education research in the international landscape.

Over the last decade or so, several international comparative studies have shed light on systems of schooling that were otherwise not very much sought after. Several educational systems in Asia, in particular East Asia have consistently produced stellar outcomes for mathematics in both TIMSS and PISA despite the fact that both studies measure achievement in mathematics in distinct ways that are very much orthogonal to each other, while other Asian systems have not been able to replicate the same level of success. Though one may occasionally chance upon a publication on some aspect of mathematics education in Asia, there appears to be in general a dearth of publications on mathematics education in Asian countries from the perspectives of scholars from Asia. Hence it is apparent that there is a gap in the availability of knowledge on mathematics education from the region in the international space.

This series has a wide scope with emphasis on relevancy and timeliness. It encompasses the general trends in educational research such as theory, practice and policy. Books in the series are thematic and focus both on macro and micro topics. An example of a themed book on a macro topic could be one on "School mathematic curricula – An Asian perspective" while a themed book on a micro topic could be one on "The pedagogy of 'simultaneous equations' in Asian classrooms".

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Foreword

Since Confucius and Socrates, educators have recognised the double purpose of education: to impart the meaning and significance of the past, and to prepare young people for the challenges of the future.

When we could still assume that what we learn in school will last for a lifetime, teaching routine cognitive skills was rightly at the centre of education. These days, the dilemma for educators is that the skills that are easiest to teach and easiest to test have become the skills that are easiest to digitise and automate. Today, when we can access content via search engines, and when routine cognitive tasks are being digitised and outsourced, the focus must shift to enabling people to become lifelong learners. Lifelong learning is about constantly learning, unlearning and relearning when the contexts change. It entails continuous processes of reflection, anticipation and action. Reflective practice is needed to take a critical stance when deciding, choosing and acting, by stepping back from what is known or assumed and by taking different perspectives. Anticipation mobilises cognitive skills, such as analytical or critical thinking, to foresee what may be needed in future or how actions taken today might have consequences for the future. Both reflective practice and anticipation contribute to the willingness to take responsible actions, in the belief that it is within the power of all of us to shape and change the course of events. This is how agency is built.

Modern schools need to help students constantly evolve and grow, and to find and adjust their right place in a changing world. They need to prepare students for rapid change, to learn for jobs that have not been created, to tackle societal challenges that we cannot yet imagine and to use technologies that have not yet been invented. And they need to prepare students for an interconnected world in which students understand and appreciate different perspectives and world views, interact successfully and respectfully with others, and take responsible action towards sustainability and collective well-being.

No country I know of is constantly reimaging the education of tomorrow's students as systematically and rigorously as Singapore does. This book is about the why, the what and the how of mathematics education in Singapore. But beyond that, it provides a unique window into the ways in which Singapore designs its instructional system and builds the capacity to deliver it consistently in every classroom. Having gone through this book, readers will understand that the stellar performance of Singapore in the global PISA mathematics and science tests is not an artefact of context and culture, but the result of carefully designed policy and practice.

This book begins with providing Singapore's answer to what students should learn in mathematics and why, and how this answer evolved as the world kept changing. It presents a mathematics curriculum that is characterised by rigour (building what is being taught on a high level of cognitive demand), by focus (aiming at conceptual understanding by prioritising depth over breadth of content) and by coherence (sequencing instruction based on a scientific understanding of learning progressions and human development). While great attention is paid to remain true to the mathematics discipline, much thought is also given to interdisciplinary learning and building students' capacity to see problems through multiple lenses. Singapore's curriculum also carefully balances knowledge of mathematics content with knowledge about the underlying nature and principles of the discipline; to help students address unknown future problems, it gives priority to knowledge, skills and attitudes that can be learned in one context and applied to others. A great strength of this part is that it goes beyond laying out curricular intentions and also provides a critical analysis of the challenges that lie in translating those intentions into classroom practice.

Building on this, the second part shows how a demanding curriculum can be consistently realised in diverse classrooms. Bringing teachers along with the ideas of a twenty-first-century curriculum is the heart of Singapore's success. This is not simply about teachers knowing mathematics, teachers knowing how students learn mathematics and teachers knowing their students, but it is about enabling teachers to design innovative pedagogical practice, with due attention to the needs of all learners. It is about framing learning in relevant and realistic contexts, and using approaches that are problem-based, project-based and centred around co-creation with their colleagues and their students. Professionalising teaching in this way has ensured that teachers have a deep understanding not only of the curriculum as a product, but of the process of designing pedagogies that will best communicate the ideas behind the curriculum.

As readers will see, Singapore does whatever it takes to develop ownership of professional practice by the teaching profession. In turn, professional discretion accorded to teachers allows them greater latitude in developing student creativity and critical thinking skills that are central to success in the twenty-first century and that are much harder to develop in highly prescriptive learning environments. And when teachers feel a sense of ownership over their classrooms, when students feel a sense of ownership over their learning, that is when productive teaching takes place. Singapore's answer to this has been to strengthen trust, transparency, professional autonomy and the collaborative culture of the profession all at the same time. This is how Singapore has created an open-source community of teachers and unlocked teachers' creativity simply by tapping into the desire of people to contribute, collaborate and be recognised for their contributions.

What makes this book a fascinating read is to see how all the pieces fit together, the why, the what and the how. It underlines how Singapore's success in education is a story about leadership and alignment between policy and practice; about setting ambitious standards; about focusing on building teacher and leadership capacity to develop vision and strategy at the school level; and about a culture of continuous improvement that benchmarks practice against the best in the world.

Paris, France

Andreas Schleicher
Director for the Directorate of Education and Skills
The Organisation for Economic Co-operation
and Development (OECD)

Series Editors' Introduction

The third volume of the book series Mathematics Education: An Asian Perspective, entitled "Mathematics Education in Singapore" and edited by Tin Lam Toh, Berinderjeet Kaur and Eng Guan Tay offers a one-stop resource on the why, the what and the how of the current state of mathematics education in Singapore. As noted by Andreas Schleicher in the foreword, it provides a unique window into the ways in which Singapore designs its instructional system and builds the capacity to deliver it consistently in every classroom.

The chapters in this book provide a rich source of information and analyses from a scholarly insider's view. Myths about the exotic East continue to circulate. Do Singapore teachers drill to kill? Do Singapore schools resemble a factory production line? Are Singapore students and people unimaginative and dour? This book, like the country, will surprise the reader with its energy, openness, humility and pragmatism. This volume is thus a much needed and worthy contribution to the mathematics education literature.

There is no doubt that this book contributes towards reducing the dearth in the availability of knowledge about mathematics education in Asia for the international audience. We hope researchers will find it a valuable resource and, for all, an enjoyable read.

Singapore — Berinderjeet Kaur
Manila, Philippines — Catherine Vistro-Yu

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About the Editors

Tin Lam Toh is an Associate Professor and currently the Deputy Head of the Mathematics and Mathematics Education Academic Group in the National Institute of Education, Nanyang Technological University of Singapore. He obtained his Ph. D. from the National University of Singapore in 2001. He continues to do research in mathematics as well as mathematics education. He has published papers in international scientific journals in both areas.

Berinderjeet Kaur is a Professor of Mathematics Education at the National Institute of Education in Singapore. She holds a Ph.D. in Mathematics Education from Monash University in Australia. She has been with the institute for the last 30 years and is one of the leading figures of mathematics education in Singapore. In 2010, she became the first full professor of mathematics education in Singapore. She has been involved in numerous international studies of mathematics education and was the Mathematics Consultant to TIMSS 2011. She was also a core member of Mathematics Expert Group (MEG) for PISA 2015. She is passionate about the development of mathematics teachers and in turn the learning of mathematics by children in schools. Her accolades at the national level include the public administration medal in 2006 by the President of Singapore, the long public service with distinction medal in 2016 by the President of Singapore and in 2015, in celebration of 50 years of Singapore's nation building, recognition as an outstanding educator by the Sikh community in Singapore for contributions towards nation building.

Eng Guan Tay is an Associate Professor and the Head in the Mathematics and Mathematics Education Academic Group of the National Institute of Education at Nanyang Technological University, Singapore. He obtained his Ph.D. in the area of Graph Theory from the National University of Singapore. He has continued his research in graph theory and mathematics education and has had papers published in international scientific journals in both areas. He is the co-author of the books *Counting*, *Graph Theory: Undergraduate Mathematics* and *Making Mathematics Practical*. He has taught in Singapore junior colleges and also served a stint in the Ministry of Education.

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Acronyms

AME	Association of Mathematics Educators
AS	Academic subject
AST	Academy of Singapore Teachers
AU	Academic unit
BA (Ed)	Bachelor of Arts with Education
BSc (Ed)	Bachelor of Science with Education
CAI	Computer-assisted instruction
CAS	Computer algebra system
21CC	21st Century Competencies
CCA	Co-curricular activity
CCT	Critical and creative thinking
CDIS	Curriculum Development Institute of Singapore
CER	Creative Reflection
Cert. Ed	Certificate in Education
CIR	Critical Reflection
CK	Content knowledge
CLT	Cognitive load theory
CPA	Computer application
C-P-A	Concrete–pictorial–abstract
CS	Curriculum studies
C-V	Concrete-Virtual
DI	Differentiated instruction
Dip Ed	Diploma in Education
DOE	Desired Outcomes of Education
EmC^2	Change reflection model
EMR	Emotive Reflection
FM	Further Mathematics
GC	Graphing calculator
GCE—N(A) Level	General Certificate of Education (Normal [Academic]) Level

GCE—N(T) Level	General Certificate of Education (Normal [Technical]) Level
GCE—O Level	General Certificate of Education (Ordinary) Level
GDP	Gross domestic product
gRAT	Group readiness assurance test
GSP	Geometer's Sketchpad
ICAN	Improving Confidence And Numeracy
ICT	Information and communication technology
ICTMA	International Community of Teachers of Mathematical Modelling and Applications
IE	Institute of Education
IEA	International Association for the Evaluation of Educational Achievement
IHL	Institutes of higher learning
IP	Integrated Programme
IPW	Interdisciplinary Project Work
iRAT	Individual readiness assurance test
ITE	Institute of Technical Education
JC	Junior college
LAPM	Low Attainers in Primary Mathematics
LC	Laboratory class
LOVE Mathematics framework	Linking Opportunities in a Variety of Experiences to the learning of Mathematics
LPS	Learner's Perspective Study
LS	Lesson study
LT	Lead teacher
MAM	Mathematical Applications and Modelling
MAS	Mathematical Sciences
MATHLET	Mathematical Learning Experience Toolkit
MC	Module credit
MCK	Mathematics Content Knowledge
MEd (Math)	Master of Education (Mathematics)
MEd	Master of Education
MME	Mathematics and Mathematics Education Academic Group
MMO	Mathematical Modelling Outreach
MOE	Ministry of Education, Singapore
mp 1	ICT Masterplan 1
mp 2	ICT Masterplan 2
mp 3	ICT Masterplan 3
mp 4	ICT Masterplan 4
MPCK	Mathematics pedagogical content knowledge
MProSE	Mathematical Problem Solving for Everyone
MPS	Mathematical Problem Solving

MSc (MfE)	Master of Science (Mathematics for Educators)
MTC	Mathematics Teachers Conference
MTT	Master Teacher
N(A)	Normal (Academic)
N(T)	Normal (Technical)
NE	National Education
NES	New education system
NHSMS	NUS High School of Mathematics and Science
NIE	National Institute of Education, Singapore
NLCs	Networked Learning Communities
NTU	Nanyang Technological University
NUS	National University of Singapore
OECD	Organisation for Economic Co-operation and Development
OPAL	One Portal All Learners
PAP	People's Action Party
PCK	Pedagogical content knowledge
PD	Professional development
PDCM	Professional Development Continuum Model
PGDE	Postgraduate Diploma in Education
PIRLS	Progress in International Reading Literacy Study
PISA	Programme for International Student Assessment
PLCs	Professional learning communities
PMT	Pre-class Milestone Tasks
PRWC	Problems in real-world context
PSLE	Primary School Leaving Examination
R&D	Research and development
RAP	Realistic ambitious pedagogy
RAT	Readiness assurance test
SBDP	School-based development programme
SK	Subject knowledge
SDT	Self-determination theory
SMAPP	Singapore Mathematics Assessment and Pedagogy Project
SMCF	School mathematics curriculum framework
SMMT	School Mathematics Mastery Test
SPMS	School of Physical and Mathematical Sciences
ST	Senior teacher
STEM	Science, technology, engineering and mathematics
SUSS	Singapore University of Social Science
TA	Teaching assistantship
TBL	Team-based learning

TE 21	Teacher Education in the 21st Century
TEDS-M	Teacher Education and Development Study in Mathematics
TELT	Teacher Education and Learning to Teach
TfU	Teaching for understanding
TGM	Teacher Growth Model
TIMSS	Trends in International Mathematics and Science Study
TLLM	Teach Less, Learn More
TRAISI	Training Administration System on Internet
TRU	Teaching for Robust Understanding
TSLN	Thinking Schools, Learning Nation
TSP	NTU-NIE Teaching Scholars Programme
TTC	Teachers' Training College
UCLES	University of Cambridge Local Examinations Syndicate
V-P-A	Virtual–pictorial–abstract

Chapter 1
Surprising Singapore

Eng Guan Tay, Tin Lam Toh and Berinderjeet Kaur

Abstract Singapore is widely acknowledged to have been successful in its duty to educate its citizens, especially with regard to science and mathematics. The focus of the world on this small nation state came about because of Singapore's consistently high performance across many years of Trends in International Mathematics and Science Study (TIMSS) and Programme for International Student Assessment (PISA) assessments. Although reluctantly cast into the spotlight, Singapore has taken this opportunity of greater interaction with international education and policy experts to review and improve its education system as well as to share what seems to work in Singapore with those who would want to consider if it would also be appropriate for their own systems. This book is intended in this direction of sharing Singapore's mathematics education with the world fraternity. This chapter gives an overview of the book.

Keywords Mathematics education · Schools · Teachers · Students · Policies · Research · Singapore

1.1 Singapore—Modestly in the Spotlight

Science and Mathematics education in Singapore was put in the spotlight when this small nation state consistently placed within the top places of Trends in International Mathematics and Science Study (TIMSS) and Programme for International Student Assessment (PISA) assessments. Although Singapore also did well in Progress in International Reading Literacy Study (PIRLS)—for example, placing 4th in

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Reading for PIRLS 2011—the world's attention seemed to focus more on its science and mathematics achievements, perhaps because of its more direct association with Science, Technology, Engineering and Mathematics (STEM) education and its economic implications.

Some books have been published recently to give a closer view of the education system in Singapore. In particular, Tan, Low and Hung (2017) give a perspective of the system as a legacy of Singapore's revered founding prime minister Lee Kuan Yew, and Tan, Liu and Low (2017) examine teacher education on the premise that the quality of an education system is dependent on the quality of its teachers and by extension the quality of its teacher education. More general issues such as streaming are discussed in generic books such as these. For example, in Tan (2017, p. 21), a succinct evaluation of streaming is offered:

> Aimed to reduce educational wastage, Dr. Goh Keng Swee (the then deputy Prime Minister) and his team overhauled the education system with the introduction of streaming in 1979, where students were separated into groups based on their academic achievement (Goh Keng Swee and the Education Study Team 1979). The rationale was to have a system that best addressed the needs of each student according to his or her academic ability. Although the virtues of streaming are much debated, it was successful in reducing attrition rate (Goh and Gopinathan 2008).

Streaming continues in Singapore today under the premise that it allows for pacing and suitably pitched tasks that match perceived capability. These tasks are not restricted to content alone but include process skills such as problem-solving and mathematical modelling. In addition, the practice of streaming itself has been tweaked over the years. It was refined in 2008 into subject-based banding in primary schools (Ministry of Education 2015). Thus, a primary school student is not coarsely branded as low-achieving but is acknowledged to have both strengths and weaknesses. In secondary school, a student can now move across to a "higher" stream when his performance in his stream shows that he has benefited from the different pacing and that he is ready to "move up".

While the influence, positive or negative, of non-school factors, such as tuition outside of school, or a culture of achievement, cannot be denied, the extent to which they affect mathematics learning in general cannot be more than the impact of school mathematics learning enabled by government policy and funding. Tan (2017, p. 22–23) argues this point passionately by comparing Singapore across the years.

> Singapore's approach and achievement in education have given our children freedom—freedom from poverty, freedom from injustice, and freedom from illiteracy. In the 1950s, youths and children roamed the streets with little prospect of their future in a society of poverty, mudflats, gang fights and racial unrest. However, with a strong conviction that people are our most precious resources, Lee, and Singapore, went about with a pragmatic approach in trying to provide equal opportunity for all citizens, and maximizing their potential. Singapore's success can be seen in key indicators … unemployment rate 9.2% in 1966 to 1.8% in 2015 … literacy rate 82.9% in 1980 to 96% in 2015 … 94% pursue post-secondary education today in contrast to 50% barely 40 years ago.

In this book then, we do not intend to scrutinize all the factors that have contributed to the state of mathematics education in Singapore today. Instead, being reluctantly cast into the spotlight, we are of the view that we want to share what seems to work in Singapore with those who would want to consider if it would also be appropriate for their own systems. This book is thus intended in this direction by describing Singapore's mathematics education with the world fraternity from the perspectives of its educators, researchers, mathematicians and policy makers. It is also intended as an update to Wong et al.'s (2009) volume on mathematics education in Singapore.

Again, though Singapore has had stellar results in international benchmark assessments, it has always kept its feet firmly on the ground and has continually reflected on its practices to see if they can be further improved. To this end, one can understand why many of the following chapters in this book highlight deficiencies in policy and pedagogy (with clear suggestions for improvement) while acknowledging the gains of working within the system set up by the top, i.e. the Ministry of Education (MOE). This tension engendered by studied self-criticism and external affirmation of what seems to be working is maintained healthily by ministry officials, mathematics educators and researchers, and teachers in the schools.

1.2 An Overview of This Book

This book comprises three sections. Drawing on official documents, research works and knowledge and experiences of mathematics educators, it presents to readers a one stop resource on the why, what and how of mathematics education in Singapore. The scope of the book is framed by three significant and closely interrelated components (shown in Fig. 1.1), namely the Singapore mathematics curriculum, mathematics teacher education and professional development, and learners in Singapore mathematics classrooms.

Excluding this introductory chapter and the final summing up chapter, there are altogether 19 chapters in the book. In Sect. 1—The Singapore School Mathematics Curriculum—there are five chapters. Chapter 2 provides the reader with an overview of Singapore's education system, milestones in the development of the system and school mathematics curriculum. Chapter 3 details the school mathematics curriculum, covering its philosophy, framework, learning objectives, assessment modes

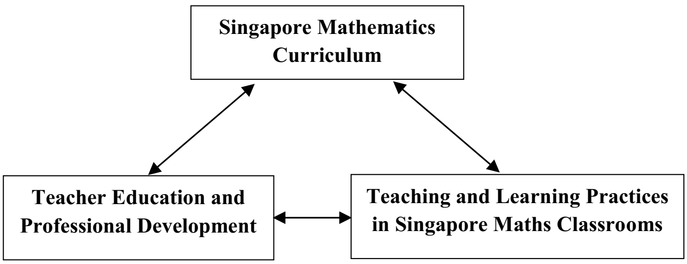

Fig. 1.1 The three interlinked sections

and evolution to its present state. While Chap. 3 is about the intended curriculum, Chap. 4 takes a more critical look at what is really enacted of the curriculum in the schools.

Chapter 5 is entitled: Beyond school mathematics. It provides insights on the scope of the pre-university mathematics curriculum, preparation for university mathematics and university mathematics instruction. This chapter is somewhat special within the scope of this book because it captures what happens after K-12 mathematics thus seeing what happens on the other side of Klein's "double discontinuity". The last chapter in this section reviews Singapore's achievements in international benchmark studies, namely Trends in International Mathematics and Science Study (TIMSS), Programme for International Student Assessment (PISA) and Teacher Education and Development Study in Mathematics (TEDS-M). It presents snapshots of significant data and findings of Singapore's participation in the benchmark studies.

Section 2 of the book progresses from the macro perspective of Sect. 1 by providing the readers with more detailed insights into the teaching and learning practices in the Singapore mathematics classroom. Here, we get to see from research done in the schools what happens in the Singapore classroom with regard to specific pedagogies and processes. This section contains eight chapters and forms a large bulk of this book. Chapter 7 updates the state of mathematical problem-solving in the Singapore mathematics classroom by reporting on the various studies on problem-solving that have been carried out since the first review by Foong (2009).

Chapter 8 presents to readers some key innovative practices and signature pedagogies in the Singapore mathematics classroom. The well-known Model Method is described here. Chapter 9 reports some new trends in the Singapore mathematics classrooms: that of mathematical modelling and problem-solving in the real-world context. This chapter discusses teacher preparation for these efforts, analyses the research outcomes in this domain and proposes the way towards a more holistic understanding of Modelling and Application of Mathematics in the Mathematics Curriculum. Despite pattern recognition being an important aspect of the Singapore mathematics curriculum across the various levels, little is known about how students of various grades think about patterns, how they recognize patterns or how they construct rules to describe the structures underpinning specific patterns. Chapter 10 addresses this gap by presenting the recent research done.

Chapter 11 describes the research projects that have been carried out in Singapore in addressing metacognition in relation to mathematical problem-solving, which is considered the heart of the Singapore mathematics curriculum. Several frameworks to study metacognition in relation to problem-solving are described in this chapter. This chapter concludes with presenting some implications to address metacognition in the Singapore Mathematics classrooms from the perspective of teachers' professional development as well as classroom practices. Chapter 12 switches perspective by recording students' perspectives of good mathematics lessons and how they learn mathematics. The data for this chapter is drawn from two specific studies: the Learner's Perspective Study and the more recent study of school mathematics curriculum enacted by competent teachers in Singapore secondary schools.

Chapter 13 presents how Singapore teachers use various approaches to help low attaining students learn mathematics. It begins with a baseline study which identifies the mismatch between how students think they learn mathematics well and how teachers teach mathematics. The chapter continues with several studies in the Singapore context on how various research groups target different approaches to help low attaining students from the Normal (Academic) and Normal (Technical) secondary mathematics students learn the subject. This includes co-operative learning strategies, use of the Concrete-Pictorial-Abstract (C-P-A) approach and the use of storytelling and comics to motivate students to learn mathematics. Chapter 14 discusses the use of Information and Communications Technology (ICT) in the mathematics classrooms. This includes the use of handheld technology, computing and programming tools. The chapter also reports on how the National Institute of Education (NIE) in Singapore prepares her teachers for the use of ICT in mathematics classrooms, in response to the ICT Masterplans of the MOE. The chapter also presents a theoretical framework on analysing the processes of "flipped classroom", a recent use of ICT for learning "on the go" and to optimize preparation for face-to-face teaching.

Section 3 of the book completes the triad by elaborating on the aspect of teacher education and professional development. We concur with Barber and Mourshed (2007) that the most important single factor for the quality of education is the quality of the teachers' training. There are six chapters in this section. The first chapter in this section describes in some detail the development of the NIE, which is the sole teacher education institution in Singapore, and how Singapore's pragmatic approach to issues has leveraged well on international mathematics teacher education research and practice to develop the teacher education programmes of NIE today. While this chapter expounds a little on the forms of knowledge necessary for a mathematics teacher, the next chapter further elaborates on exemplary characteristics of mathematics teachers as seen in Singapore classrooms.

The last four chapters are about teachers' continuing professional development (PD), a key feature of a teacher education model that sees learning for teachers as a lifelong endeavour. Chapter 17 describes the framework of in-service courses for teachers that are organized by the MOE and conducted mostly by academics from the NIE and Master Teachers from the Academy of Singapore Teachers (AST). Chapter 18 reports on three models of teacher development researched in Singapore. The first is an amalgamation of the "training model of PD" with sustained support for teachers to integrate knowledge gained from the PD into their classroom practice. The second model is the laboratory class cycle which adapts the Lesson Study process while working with primary school mathematics teachers in school-based settings. The third model is networked learning communities. All three exemplify a gradual shift in the centre of gravity away from the university-based, "supply-side", "off-line" forms of knowledge production conducted by university researchers for teachers towards an emergent school-based, demand-side, on-line, in situ forms of knowledge production by teachers with support from university scholars and knowledgeable others.

The final two chapters are of two approaches to teacher professional development that are being pioneered by Singapore mathematics educators, that is, through school-based Replacement Units and facilitating teachers for productive noticing. Chapter 19 describes a particular school-based professional development approach called Replacement Units. Guided by the principles of a Realistic Ambitious Pedagogy, such Replacement Units for topics, which teachers deem hard to teach, are crafted by university academics and teachers within a school setting to achieve a strong semblance of mathematical discipline within the time afforded for the topic. The last chapter in this section argues that participating in teacher inquiry activities—the process of investigating and thinking about learning and teaching, encapsulated in many professional learning activities such as lesson study, action research, video club study and microteaching—does not necessarily lead to better practice. Instead, teachers need to hone their teacher *noticing* expertise, which is a kind of professional vision, to examine artefacts of teaching and learning so as to make sense of their own practice and gain a better awareness of their teaching. The chapter describes some of the studies related to understanding and developing teachers' noticing expertise.

1.3 Surprising Singapore

From the brief overview of the chapters in the preceding section, we can see that there is a comprehensive coverage of policy, teacher preparation, teaching enactment and research in mathematics education in Singapore in an effort to develop an effective system. This comprehensive coverage is enabled by the pragmatic approach described in Chap. 15: *Pragmatism means that a country does not try to reinvent the wheel. As Dr. Goh Keng Swee* [the founding Deputy Prime Minister] *would say to me,* "*Kishore* [former ambassador to the United Nations]*, no matter what problem Singapore encounters, somebody, somewhere, has solved it. Let us copy the solution and adapt it to Singapore*". Much can be done in Singapore because things often do not start from scratch.

As one reads chapter after chapter of this book, one will be struck by how every endeavour is framed by a solid internationally reputable concept, idea or system and then adapted to work within the Singapore context. For example, Chap. 2 describes the Singapore Mathematics Curriculum Framework with problem-solving at its heart, the result of adopting Schoenfeld's (1985) expansion of Polya's (1954) work in problem-solving. As an additional example, the Realistic Ambitious Pedagogy of Chap. 19 is the result of reconciling the ambitious disciplinarity ideas of Lampert (1990) and the classroom constraints apparent to many others (see, e.g., Assude 2005; Plank and Condliffe 2013). Indeed, while it may be the tendency for some to regard anything "old as cold", mathematics education in Singapore, as in a general Singaporean outlook, regards traditional practices as "gold" to be burnished or refined.

Did you know that a Singapore company was the first in the year 2000 to produce commercially a USB drive? Although it has failed to consistently beat the big boys in the courts (where big money rules), the fact that Trek Technology, a miniscule technology company could get its name among the claimants in Wikipedia (n. d.), probably means that it was de facto the first. Without controversy however is the fact that Sim Wong Hoo, a Singapore polytechnic graduate, revolutionized sound coming out from a computer with his range of Sound Blaster cards. Very recently, Sim was in the news again when his Super X-Fi, a technology that can be used in headphones to recreate the listening experience of a multi-speaker surround sound system, won a Best of CES 2018 award at the electronics trade show in Las Vegas (Kwang 2018). While it seems fair game to some to summarily dismiss Asian education as capable only of producing rote learners and copycats, the fact that high-tech Asian companies, such as Samsung, Sony, Alibaba and Sim Wong Hoo's Creative Technology, are innovative, creative and world leading show that such dismissals comes more from prejudice than profundity. Singapore education, like that of other Asian nations, throws up more surprises when one probes deeper to understand how it works. Chapter 4 looks deeper into the enacted curriculum and debunks a standard categorization of what happens in Singapore classrooms as "direct instruction" or "teacher-dominated instruction" within an East-Asian culture of excessive drill and practise for success in examinations. Instead, from the research findings a "Singapore pedagogy" with its unique features is revealed. Specialised mathematics and science schools (Chap. 5), a focus on making mathematical problem-solving work in schools (Chap. 7), innovative practices (Chap. 8), problems in real-world contexts and mathematical modelling (Chap. 9) all testify to a diversity and a deepening of mathematics learning in Singapore schools.

Finally, in Singapore, it is no surprise to see that teachers are well respected in society for their generally good work. In a public perception survey commissioned by the Ministry of Education (MOE) in 2010 on the most respected professions in terms of its contribution to society, respondents ranked teachers second, just below doctors (MOE 2012). The whole of Sect. 3 gives insight into the preparation and continual professional development of a teacher in Singapore. The mathematics curriculum is reviewed every six years by the MOE to keep learning relevant and to give innovative pedagogies space within the curriculum. Teachers in Singapore respond to the changing landscape by constantly being in touch with mathematics and mathematics pedagogy through such learning opportunities. Thus, in Singapore, wholesome mathematics learning comes much from insightful and inspiring mathematics teaching by dedicated mathematics teachers.

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Part I
The Singapore School Mathematics Curriculum

Chapter 2
Overview of Singapore's Education System and Milestones in the Development of the System and School Mathematics Curriculum

Berinderjeet Kaur

Abstract The first part of the chapter acquaints the reader with Singapore's education system, mainly the goals of education, primary school, secondary school and post-secondary schools/institutions. The courses of study at appropriate years of schooling and the relationships between the corresponding mathematics syllabuses in the courses of study are presented. It also illuminates the possible lateral transfers between courses of study at the secondary school and various pathways to post-secondary education and institutes of higher learning. The second part of the chapter traces chronologically the milestones of the education system for the last six decades, which fall into four well-marked phases in the development of the system. The phases are (i) survival-driven phase (1959–1978), (ii) efficiency-driven phase (1979–1996), (iii) ability-based, aspiration-driven phase (1997–2011), and (iv) values-based, student-centric phase (2012–present). Alongside these developmental phases, the milestones in the development of the school mathematics curriculum are also elaborated.

Keywords School mathematics curriculum · Singapore's education system · Survival-driven phase · Efficiency-driven phase · Ability-based aspiration-driven phase · Values-based student-centric phase · Development milestones

2.1 Singapore's Education System

Singapore is an island, with an area of 719.1 km^2. The population is approximately 5.5 million of which more than one million are foreigners working in the country. The GDP per capita as of August 2017 was Singapore Dollars $71, 300. The two largest budget items of the government expenditure are defence and education, respectively. It is apparent that people are the only natural resource of Singapore and the nation spares no effort to actualize its simple objective of education that is:

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> … to educate a child to bring out his greatest potential so that he will grow into a good man and a useful citizen (Lee 1979, p. iii)

In Singapore, education is also a key enabler of social mobility and the system provides equal opportunity for every child by:

- Ensuring that no child is deprived of educational opportunities because of their financial situations;
- Leveraging on the school system to provide more support for families from poorer backgrounds;
- Investing in preschool education targeted at children with families with poorer backgrounds; and
- Investing in levelling up programmes in primary schools that attempt to level up academically weaker students in both English and Mathematics, so as to improve their foundations for future learning (Heng 2012).

In the fast-changing world of today, the system also imbues school leavers with the knowledge and motivation to be lifelong and adaptable learners through a "values-driven, student-centric" focus (Heng 2012). The framework for twenty-first-century competencies and student outcomes introduced by the Ministry of Education (MOE) in 2010 encapsulates this focus. The framework, shown in Fig. 2.1, is founded on the belief that knowledge and skills must be underpinned by values. The core values, namely respect, responsibility, integrity, care, resilience and harmony define a person's character and shape the beliefs, attitudes and actions of the person. Thus, these values form the core of the framework. The middle ring represents the social and emotional competencies. These competencies concern firstly with how a student understands and manages him or herself and subsequently how a student relates to others. The outer ring of the framework represents emerging twenty-first-century competencies that are necessary for success in the globalized world. The framework aims to develop a young person into:

- a confident person who has a strong sense of right and wrong, is adaptable and resilient, knows himself, is discerning in judgment, thinks independently and critically, and communicates effectively.
- a self-directed learner who questions, reflects, perseveres and takes responsibility for his own learning.
- an active contributor who is able to work effectively in teams is innovative, exercises initiative, takes calculated risks and strives for excellence.
- a concerned citizen who is rooted to Singapore has a strong sense of civic responsibility, is informed about Singapore and the world, and takes an active part in bettering the lives of others around him (MOE n.d.).

Education for primary, secondary and tertiary levels is mostly supported by the state. All institutions, private and public, must be registered with MOE. English is the language of instruction in all public schools and all subjects are taught and examined in English except for the "Mother Tongue" language paper. While "Mother Tongue"

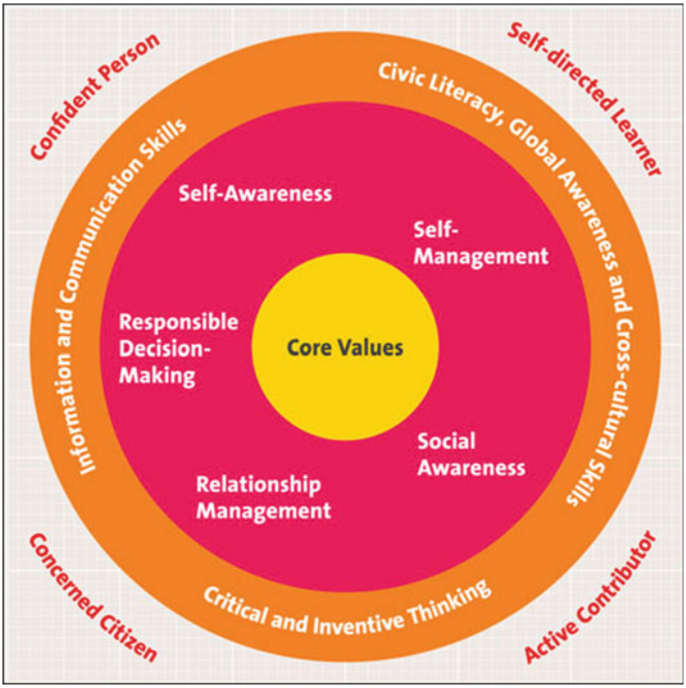

Fig. 2.1 Framework for twenty-first-century competencies and student outcomes (MOE n.d.)

generally refers to the first language internationally, in Singapore's education system, it is used to refer to the second language as English is the first language. Education takes place in three stages: "Primary education", "Secondary education" and "Post-secondary education". Detailed and most current information on Singapore's Education System is available at https://www.moe.edu.sg/. The following sections provide briefs about the system.

2.1.1 Primary School

In Singapore, every child receives a six-year compulsory primary school education made up of a four-year foundation stage and a two-year orientation stage. The primary school curriculum provides children with a strong foundation in subject disciplines such as languages, humanities and the arts, and mathematics and science; knowledge skills focussing on thinking and communication skills; and character development. Subject-based banding begins in Primary 5 and continues till Primary 6. It provides greater flexibility for children as they can take a combination of standard and/or foundation subjects depending on their strengths. This helps the child focus on and stretch

Primary 5 - Decimals	
Mathematics	***Foundation Mathematics***
1.1 multiplying and dividing decimals (up to 3 decimal places) by 10, 100, 1000 and their multiples without calculator 1.2 converting a measurement from a smaller unit to a larger unit in decimal form, and vice-versa • kilometres and metres • metres and centimetres • kilograms and grams • litres and mililitres 1.3 solving word problems involving the four operations	1.1 notation, representations and place values (tenths, hundredths, thousandths) 1.2 comparing and ordering decimals 1.3 converting decimals to fractions 1.4 converting fractions to decimals when the denominator is a factor of 10 or 100 1.5 rounding decimals to • the nearest whole number • 1 decimal place • 2 decimal places 2.1 adding and subtracting decimals (up to 2 decimal places) without calculator 2.2 multiplying and dividing decimals (up to 3 decimal places) by 10, 100, 1000 2.3 converting a measurement from a smaller unit to a larger unit in decimal form, and vice-versa • kilometres and metres • metres and centimetres • kilograms and grams • litres and mililitres 2.4 solving word problems involving addition and subtraction

Fig. 2.2 An extract of content for decimals for Primary 5 Mathematics and Foundation Mathematics (MOE 2012a)

his potential in the subjects (standard) he is strong in while building up the fundamentals in the subjects (foundation) in which he needs more support. For foundation, subjects support is available in the form of smaller class size, where teachers focus on helping students close gaps in their deficits and progress at a pace that is suited to their needs. The decision to take a foundation subject is often based on a child's achievement in the subject at the end of Primary 4 with close consultations by a school and the parents/guardians of the child. At the end of six years of primary school, students take the Primary School Leaving Examination (PSLE). The subjects tested in the PSLE are English Language/Foundation English Language, Mother Tongue Language/Foundation Mother Tongue, Mathematics/Foundation Mathematics and Science/Foundation Science. They may also take an optional subject that is Higher Mother Tongue Language.

The mathematics syllabus (curriculum) for all students from Primary 1 to Primary 4 (P1–4) is the same (MOE 2012a). However, in Primary 5 and Primary 6 (P5–6), there is differentiation in the content for Mathematics and Foundation Mathematics. The P5–6 mathematics syllabus continues the development of the same in P1–4, while the P5–6 Foundation Mathematics syllabus revisits some of the important concepts and skills in the P1–4 syllabus. The new concepts and skills introduced in the Foundation Mathematics is a subset of the Mathematics syllabus. Figure 2.2 shows content for Mathematics and Foundation Mathematics at the Primary 5 level for the topic decimals.

It is apparent from Fig. 2.2 that for P5 Foundation Mathematics syllabus items 1.1–2.1 are part of the P4 mathematics syllabuses that are revisited and items 2.2–2.4 are a subset of the P5 Mathematics syllabus. Note that in item 2.2, the Foundation Mathematics students are allowed to use calculators and they only multiply and divide decimals (up to 3 decimal places) by 10, 100, 1000 unlike those doing Mathematics who multiply and divide decimals (up to 3 decimal places) by 10, 100, 1000 and their multiples without calculator. Similarly as shown in item 2.4 for Foundation Mathematics, students only solve word problems involving addition and subtraction while those doing Mathematics solve word problems involving the four operations.

2.1.2 Secondary School

Following six years of primary schooling, learning at secondary schools is tailored to different abilities. The PSLE is a placement examination. The score obtained by the student in the PSLE and other indicators such as special talent and or interest helps teachers and parents guide students in taking an appropriate course of study at a secondary school. There are three courses of study at the secondary school. They are the Express Course (including the Integrated Programme (IP)), Normal (Academic) [N(A)] Course and the Normal (Technical) [N(T)] Course. Table 2.1 shows the enrolments of Secondary 1 students by the course of study for the past five years (2012–2016).

It is apparent from Table 2.1, for the period 2012–2016, that the percentage of students in the Express Course of study ranged from 60.0 to 64.0. The percentage of girls in the course of study ranged from 50.6 to 51.8. For the same period, the percentage of students in the N(A) and N(T) courses ranged from 23.0 to 26.5 and

Table 2.1 Secondary one enrolment by course (2012–2016)

Year	Sex	Express	Normal (Academic)	Normal (Technical)	Total
2012	All	27,293 (60.4%)	11,848 (26.2%)	6,057 (13.4%)	45,198 (100%)
	F	13,803 (50.6%)	5,636 (47.6%)	2,289 (37.8%)	21,728 (48.1%)
2013	All	28,870 (60.0%)	12,747 (26.5%)	6,477 (13.5%)	48,094 (100%)
	F	14,802 (51.3%)	5,955 (46.7%)	2,346 (36.2%)	23,103 (48.0%)
2014	All	27,490 (64.0%)	9,873 (23.0%)	5,606 (13.0%)	42,969 (100%)
	F	13,963 (50.8%)	4,713 (47.7%)	2,080 (37.1%)	20,756 (48.3%)
2015	All	26,736 (63.3%)	9,972 (23.6%)	5,509 (13.1%)	42,217 (100%)
	F	13,841 (51.8%)	4,556 (45.7%)	2,191 (39.8%)	20,588 (48.8%)
2016	All	24,613 (62.2%)	10,033 (25.4%)	4,904 (12.4%)	39,550 (100%)
	F	12,568 (51.1%)	4,795 (47.8%)	1,899 (38.7%)	19,262 (48.7%)

Source of data Education Statistics Digest 2017 (MOE 2017, p. 28)

12.4 to 13.5, respectively. The percentage of girls in the N(A) and N(T) courses ranged from 45.7 to 47.8 and 36.2 to 39.8, respectively.

While a student may be initially placed in a particular course based on his ability to cope with the learning pace and style, there are opportunities at every stage for him or her to make a lateral transfer to another course if it is more suited to his or her interests and abilities. He or she can also take specific subjects at an academically higher level in upper secondary. For example, if he or she is in the N(T) course, he or she may be able to take some subjects at N(A) level. Figure 2.3 shows an overview of the pathways and possible lateral transfers amongst the courses of study. Lateral transfers are possible as the curriculum for the three courses is related to each other.

From Fig. 2.3, it is also apparent that students in the Express Course of study take the General Certificate of Education (Ordinary Level) (GCE-O Level) examination after four years of secondary schooling. However, students who are in the IP, which provides an integrated six-year Secondary and Junior College education, do not take the GCE-O Level examination. Their six years of schooling culminates in General Certificate of Education (Advanced Level), International Baccalaureate or other diploma qualifications. The IP is for academically strong students who can benefit from programmes that provide broader learning experiences. The IP aims to stretch their potential in non-academic aspects that are beyond the formal academic curriculum. Schools that offer the IP admit students in secondary 1. Students in the Express Course can also join in the IP at Secondary 3.

Students in the N(A) course of study take the General Certificate of Education (Normal[Academic] Level) (GCE-N(A) Level) examination after four years of secondary schooling. Based on their results in the GCE-N(A) Level examinations, they may continue with another year of secondary school and take the GCE-O Level examination at the end of their fifth year in a secondary school or continue with their post-secondary education at a polytechnic or Institute of Technical Education (ITE). Students in the N(T) course of study take the General Certificate of Education (Normal[Technical] Level) (GCE-N(T) Level) examination after four years of secondary schooling. Based on their results in the GCE-N(T) Level examinations, they may continue with another year of secondary school and take the GCE-N(A) Level examination at the end of their fifth year in a secondary school or continue with their post-secondary education at ITE. As shown in Fig. 2.3, there are diverse pathways for students of all abilities to realize their potential and attain desired qualifications.

As mentioned earlier, lateral transfers are possible as the curriculum for the three courses is related to each other. From Fig. 2.4, it is apparent that the O-Level Mathematics syllabus builds on the Standard Mathematics syllabus. The N(A)-Level Mathematics syllabus is a subset of O-Level Mathematics, except that it revisits some of the topics in Standard Mathematics syllabus. The N(T)-Level Mathematics syllabus builds on the Foundation Mathematics syllabus. It is also obvious from Fig. 2.4 gaps exist between the curriculums of the courses. Therefore when lateral transfers do take place, bridging of knowledge is undertaken by teachers in schools during additional curriculum time.

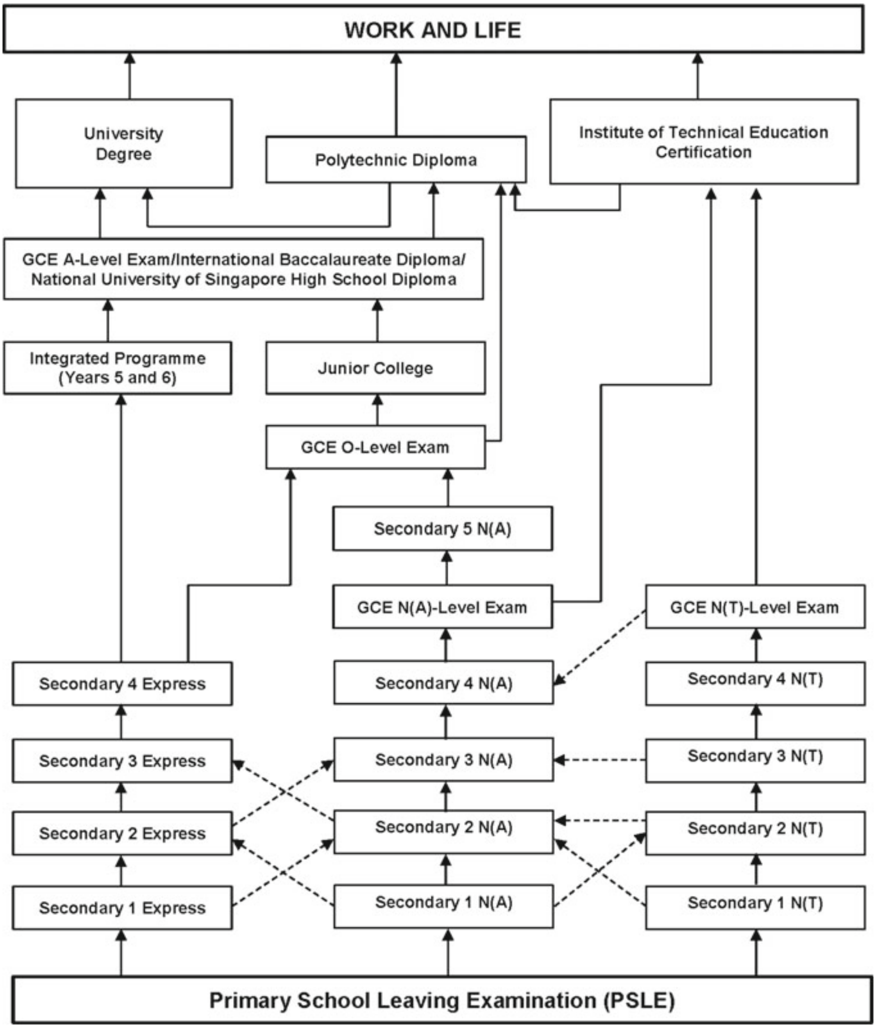

Fig. 2.3 An overview of the pathways and possible lateral transfers amongst the courses of study

Figure 2.5 shows content for the topic Algebra for the three courses at the secondary one level. As part of the Standard Mathematics syllabus students in Primary 6 do Algebra which involves the following:

1.1 using a letter to represent an unknown number;
1.2 notation, representations and interpretation of simple algebraic expressions such as $a \pm 3$, $a \times 3$ or 3a, $a \div 3$ or a/3;
1.3 simplifying simple linear expressions excluding brackets;
1.4 evaluating simple linear expressions by substitution;

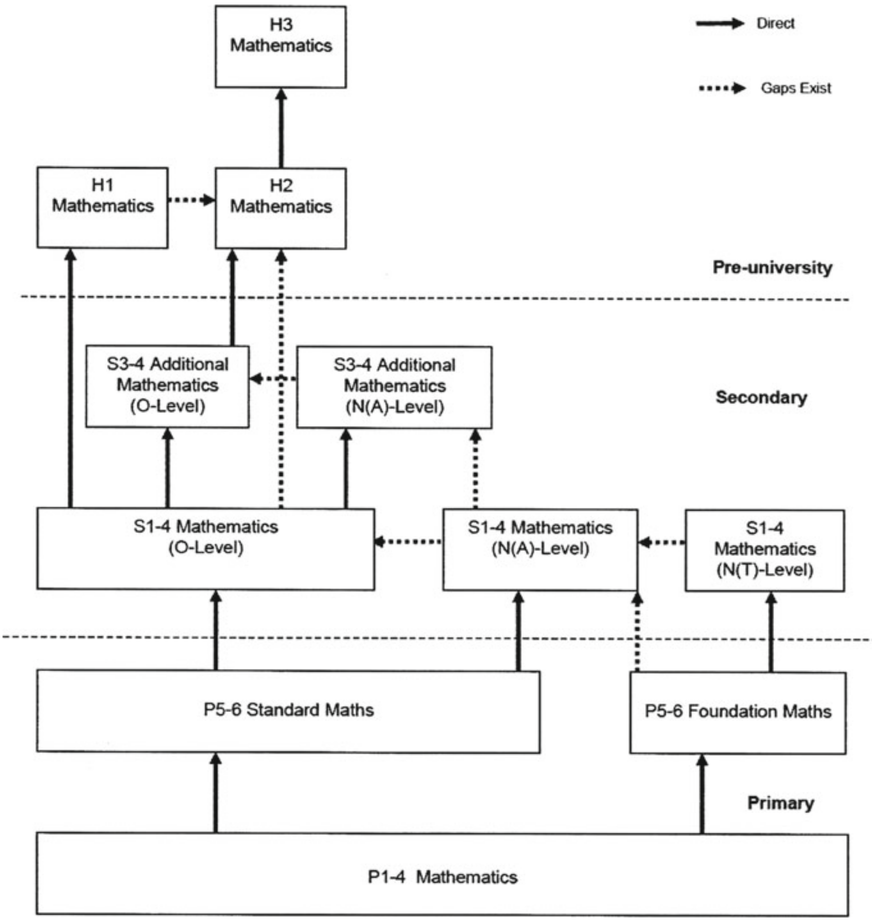

Fig. 2.4 An overview of the connected Mathematics syllabuses

1.5 solving simple linear equations involving whole number coefficient only in simple context.

From Fig. 2.5, it is apparent that the Express Course syllabus builds on the content of Algebra in the Primary school for students who took the Standard Mathematics. It spans from using letters to represent numbers to solution of simple inequalities, fractional linear equations and formulation of linear equations in one variable to solve problems. The N(A) Course syllabus is a subset of the Express Course syllabus. It essentially focuses on strengthening the foundation in Algebra and does not involve work on algebra fractions, simple inequalities and solution of simple fractional linear equations at the secondary one level.

Secondary One - Algebra		
<u>Express Course</u>	<u>Normal (Academic) Course</u>	<u>Normal (Technical) Course</u>
Algebraic expressions and formulae o Using letters to represent numbers o Interpreting notations - ab as a×b, $\frac{a}{b}$ as a÷b, or a× $\frac{1}{b}$ - a^2 as a×a, a^3 as a×a×a, a^2b as a×a×b, … - 3y as y+y+y or 3×y - 3(x + y) = 3 × (x + y) - $\frac{3+y}{5}$ as (3+y) ÷ 5 or $\frac{1}{5}$ × (3+y) o Evaluation of algebraic expressions and formulae o Translation of simple real-world situations into algebraic expressions o Recognizing and representing patterns / relationships by finding an algebraic expression for the nth term o Addition and subtraction of linear expressions o Simplification of linear expressions with integral coefficients such as - $-2(3x-5)+4x$; $\frac{2x}{3}-\frac{3(x-5)}{2}$ o Use brackets and extract common factors Equations and inequalities o Concepts of equation and inequality o Solving linear equations with integral coefficients in one variable o Solving simple inequalities in the form ax ≤ b, ax ≥ b, ax < b and ax > b where a and b are integers. o Solving simple fractional equations that can be reduced to linear equations such as - $\frac{x}{3}+\frac{x-2}{4}=3$; $\frac{3}{x-2}=6$ o Formulating a linear equation in one variable to solve problems	Algebraic expressions and formulae o Using letters to represent numbers o Interpreting notations - ab as a×b, $\frac{a}{b}$ as a÷b, or a× $\frac{1}{b}$ - a^2 as a×a, a^3 as a×a×a, a^2b as a×a×b, … - 3y as y+y+y or 3×y - 3(x + y) = 3 × (x + y) - $\frac{3+y}{5}$ as (3+y) ÷ 5 or $\frac{1}{5}$ × (3+y) o Evaluation of algebraic expressions and formulae o Translation of simple real-world situations into algebraic expressions o Recognizing and representing patterns/relationships by finding an algebraic expression for the nth term o Addition and subtraction of linear expressions o Simplification of linear expressions with integral coefficients such as - 2(x – y); 4x – 2(3x – 5); 3(x – y) (2y + x) – y Equations and inequalities o Concept of equation o Solving linear equations with integral coefficients in one variable o Formulating a linear equation in one variable to solve problems	Algebraic expressions and formulae o Using letters to represent numbers o Interpreting notations - ab as a×b, $\frac{a}{b}$ as a÷b, or a× $\frac{1}{b}$ - a^2 as a×a, a^3 as a×a×a, a^2b as a×a×b, … - 3y as y+y+y or 3×y - 3(x + y) = 3 × (x + y) - $\frac{3+y}{5}$ as (3+y) ÷ 5 or $\frac{1}{5}$ × (3+y) o Evaluation of algebraic expressions and formulae o Translation of simple real-world situations into algebraic expressions o Recognizing and representing number sequences (including finding an algebraic expression for the nth term for simple cases such as n + 3, 2n + 1.

Fig. 2.5 An extract of content for algebra for the three courses Express, N(A) and N(T) (MOE 2012b, c)

As part of the Foundation Mathematics syllabus, students do not do any Algebra in the primary school. Therefore, the syllabus for Algebra for the N(T) course in secondary 1 is very basic but nevertheless also somewhat what students in the two other courses would be re-visiting.

2.1.3 Post-secondary Schools/Institutions

Post-secondary education takes place from two to three years at junior colleges, Polytechnics and Institutes of Technical Education (see Fig. 2.3). For junior colleges, the curriculum comprises of specialized subjects and a contrasting subject for a broad-based education. Chapter 5 describes the mathematics curriculum for junior colleges (see also Fig. 2.4). At the junior college level, mathematics is not a compulsory subject. Furthermore, most of the students who go on to junior colleges are university bound. Therefore, it is expected that the mathematics curriculum at the junior college level prepares students for further study of mathematics or related subjects at the university. For Polytechnics and Institutes of Technical Education, the curriculum is specialized and specific to the course of study the students are undergoing, for example, business studies, mass communication, and engineering.

2.2 Milestones in the Development of the Education System and School Mathematics Curriculum in Singapore

Though schools were in existence well before 1959, due to a lack of documentation of the educational system for the period prior to 1959, this chapter limits the development of the system from 1959 onwards. The developments from 1959 to present that have shaped the present School Mathematics Curricula in Singapore are direct consequences of developments in the Education System of Singapore during the same period. Major changes in the education system during the last six decades fall into four well-marked phases in the development of the system. The phases are the survival-driven phase, the efficiency-driven phase, the ability-based aspiration-driven phase and the values-based student-centric phase. In the following subsections, the milestones in the development of mathematics curriculum are presented alongside key developments in the education system of Singapore.

2.2.1 Survival-Driven Phase (1959–1978)

According to Yip et al. (1990), this period was a conflict resolution and quantitative expansion one. Two major thrusts and priorities of this period stood out in bold relief. The first was the use of education, in the period after 1959 to resolve some of the pressing conflicts and dilemmas Singapore faced in the 1950s. The second concerns the pressure to rapidly expand educational opportunities in Singapore with a view not only to democratizing education, but also to using education as a device for achieving national cohesion and the economic restructuring of the society. In 1959 when the People's Action Party (PAP) came to power, it acted upon the White Paper of 1956 and put in place a Five-Year Plan in education. The main features of this plan were:

- Equal treatment of the four language streams of education: Malay, Chinese, Tamil and English;
- The establishment of Malay as a national language of the new state;
- Emphasis on the study of Mathematics, Science and Technical Subjects.

The government embarked on an accelerated school building programme with the objective of providing a place in school for every child of school-going age in Singapore. 1965 witnessed the end of Singapore's merger with Malaysia and marked the beginning of a transformation from statehood to nationhood. Under the leadership of PAP, education remained a key to its survival and there was a shift in emphasis from academic to technical education to provide the manpower base for industrialization. In this phase, all the ethnically diverse educational steams were also gradually merged into a single national system together with the adoption of a bilingual education policy. Under this policy, all students are educated in English as their first language and also learn their Mother Tongue as a second language.

Up to the late 1950s, schools in Singapore were mainly vernacular in nature, i.e. there were Chinese, Malay, Tamil and English schools. The language of instruction in Chinese schools was Chinese and their curricula were adopted from China. Likewise the language of instruction in English schools was English and their curricula were adopted from Britain. Therefore, several mathematics syllabuses were in use across Singapore, with each school adopting its own.

The first local set of syllabuses for mathematics was drafted in 1957 and published in 1959 (Lee 2008). This set of syllabuses, contained in a single booklet available in English, was for all primary and secondary schools irrespective of their language streams. The syllabuses adopted a spiral approach and factored no consideration for differences in the mathematical abilities of students. The secondary school mathematics syllabuses referred to as Syllabus B prepared students for the mathematics examinations of the Cambridge Certificate of Education conducted by the University of Cambridge Local Examinations Syndicate (UCLES). This set of syllabuses marked the first step towards the localization of mathematics education in Singapore (Lee 2008). The first MOE designated mathematics syllabuses for primary and secondary schools were available for use in Singapore schools is 1960. It was only ten years later, in 1970 that mathematics was made a compulsory subject for all primary and secondary schools.

A revision of the first local set of syllabuses for both the primary and secondary schools took place in the late 1960s in response to the global "Math Reform of the 1960s". The primary school mathematics syllabus was revised in 1971 with emphasis on outcomes-based approach to the teaching of mathematics in the primary schools (Wong and Lee 2010). It was again revised in 1979 and algebra became part of the curriculum for grades 5 and 6 (Lee 2008). For secondary school mathematics, the revised syllabus known as Syllabus C was implemented in the early 1970s (Lee 2008). Towards the end of the 1970s, the syllabus underwent yet another revision resulting in Syllabus D. At the secondary level, all students take the mathematics (elementary) course. At the upper secondary level, the more able students take the additional mathematics course. Both courses are based on the "Ordinary" level syllabuses of UCLES. Since the 1980s, Singapore secondary students have been following the Syllabus D. The Ministry of Education issues the syllabus for the Lower Secondary levels. This syllabus covers topics in Arithmetic, Mensuration, Algebra, Graphs, Geometry, Statistics and Trigonometry. For each topic, the syllabus describes the instructional objectives and lists the main concepts and learning outcomes. These topics are a subset of the syllabus for the "Ordinary" level UCLES mathematics examination.

2.2.2 Efficiency-Driven Phase (1979–1996)

By the late 1970s, certain "cracks" and weaknesses in the system had begun to manifest themselves. Amongst the weaknesses identified by the MOE's Study, Team led by Dr. Goh Keng Swee (Goh and the Education Study Team 1979) was the high education wastage resulting in low literacy levels in the country. In line with the "simple objective" of education in Singapore,

> … to educate a child to bring out his greatest potential so that he will grow into a good man and a useful citizen. (Lee 1979, p. iii)

as spelt out by the then Prime Minister of Singapore in 1979 and the findings of Goh's Report (Goh and the Education Study Team 1979), the New Education System (NES) was introduced in February 1979. The NES introduced ability-based streaming both at the primary and secondary levels of education on the grounds that in the past a common curriculum in the primary and secondary schools had failed to take into consideration variations in the learning capacities of children. Streaming, according to Goh's report, would provide an opportunity for less capable students to develop at a slower pace and would also enable a child to go as far as he can. Students who are not academically inclined could still acquire basic literacy and numeracy required for skills training. The NES was implemented in 1981. Students were streamed in Primary 4 and Secondary 1.

In June 1980, the Curriculum Development Institute of Singapore (CDIS) was established. It replaced the Education Development Division of the Ministry of Education, which spearheaded the pioneering efforts in curriculum development for Singapore schools. The main function of the CDIS was development of curriculum and teaching materials. It was directly involved in the implementation of syllabuses and systematic collection of feedback at each stage of implementation for the next cycle of syllabus revision (Ang and Yeoh 1990).

1985 marked a watershed in the economic development of Singapore. Based on two key reports, one in Singapore (Economic Committee 1986) and another in the

USA (Tan 1986), the Minister for Education in 1986 enunciated that future education policies in Singapore would be guided by three principles. These were:

- Education policy must keep in pace with the economy and society;
- Basics—Languages, Science, Mathematics and Humanities will be stressed to encourage logical thinking and life-long learning;
- Creativity in schools must be boosted through a "bottom-up" approach whereby the initiative must come from principals and teachers instead of from the Ministry (Tan 1986).

As part of an ongoing process of self-improvement, in 1987 based on the report, Towards Excellence in Schools (MOE 1987), schools became the centre of attention. This was a result of the premise that the goal of excellence in education could only be achieved through better schools (Tan 1987). Several refinements to the NES have been made since its implementation in 1981. In 1991, the level at which streaming in the primary school was carried out was changed to Primary 5. In 1994, the Secondary Normal Course was further refined into the N(A) and N(T) courses and students were streamed accordingly following the outcome of their PSLE.

In the NES, the primary mathematics curriculum (detailed syllabuses, textbooks, workbooks and teacher guides) was developed by experienced mathematics teachers from schools and the Ministry of Education under the guidance of international experts and curriculum writers at CDIS. The revised primary mathematics curriculum was produced in 1981. The curriculum adopted the Concrete-Pictorial-Abstract (C-P-A) approach (see Chaps. 3 and 8 for details) to the teaching and learning of mathematics. This approach provides students with the necessary learning experiences and meaningful contexts, using concrete hands-on materials and pictorial representations to construct abstract mathematical knowledge.

In 1983, the mathematics team writing the primary curriculum materials, led by Dr. Kho, at CDIS made a breakthrough by addressing the difficulties students were having with word problems. They introduced the "Model Method" (Kho 1987) (see Chaps. 3 and 8 for details) in the curriculum for Primary 5 and 6 students in the late 1980s. In the current curriculum, students are introduced to the model method in Primary 1. This method is now synonymous with Singapore maths worldwide. In line with the goal of the NES, i.e. to provide for every student in the system, from 1981 onwards, there was differentiation in the mathematics content to match the ability of students in their respective course of study, both at the primary and secondary schools. How the contents of the various courses of study are connected to one another are exemplified in the earlier sections of the chapter.

The Curriculum Development Division of the MOE in 1988 set up a mathematics syllabus review committee to review and revise the mathematics syllabuses in use since 1981. The goal of the committee was to study the adequacy of the syllabuses in meeting the needs of the students and to revise the syllabuses to reflect appropriate recent trends in mathematics education (Wong 1991). It was during this review that the committee felt that besides elaborating the aims and objectives, a framework was necessary to describe the philosophy of the revised curriculum. Hence, the framework, shown in Fig. 2.6, that spells out ***mathematical problem-solving*** as the

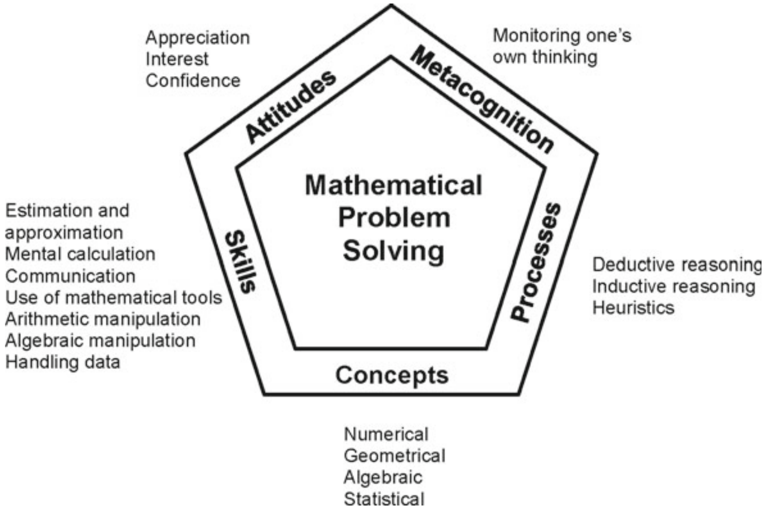

Fig. 2.6 School Mathematics curriculum framework (MOE 1990)

primary focus of the school mathematics curriculum was born. This coherent framework connects the "product" conception of mathematics and its "process" aspect and links both of them to the five factors that facilitate the development of mathematical problem-solving (Wong and Lee 2010). The framework "presents a balanced, integrated vision that connects and describes the skills, concepts, processes, attitudes and metacognition" (Leinwand and Ginsburg 2007, p. 32). The five components of the framework—concepts, skills, attitudes, metacognition and processes—have remained steadfast although some refinements have been made to their attributes at periodic subsequent revisions of the school mathematics curriculum. These refinements are elaborated in Chap. 3 of the book. In 1991, the revised Mathematics syllabuses were implemented. The revised syllabuses for both the primary and secondary schools placed emphasis on problem-solving.

In 1992, the mathematics syllabus for the N(T) course students was produced by MOE (1992). The Normal Course mathematics syllabus was also renamed as N(A) course mathematics syllabus A (4010). The N(T) course mathematics syllabus is a subset of the N(A) course syllabus. The N(T) course mathematics syllabus T (4012) was implemented in 1994 when the N(T) course came into being at the secondary one level for the first time.

2.2.3 Ability-Based, Aspiration-Driven Phase (1997–2011)

In 1997, the Prime Minister, Mr. Goh Chok Tong in his speech (Goh 1997) at the opening of the Seventh International Conference on Thinking held in Singapore signalled that changes had to be made to the existing education system. These were necessary to prepare young Singaporeans for the new circumstances and new problems that they will face in the new millennium. He emphasized that we must ensure that our young can think for themselves, so that the next and future generations can find their own solutions to whatever new problems they may encounter. He also announced at the opening of the conference that Singapore's vision for meeting this challenge is encapsulated in four words: Thinking Schools, Learning Nation (TSLN). The desired outcomes of education document were also published in 1998. The Desired Outcomes of Education (DOE) available at https://www.moe.gov.sg/ are attributes that educators aspire for every Singaporean to have by the completion of his formal education. These outcomes establish a common purpose for educators, drive our policies and programmes and allow us to determine how well our education system is doing. Thus, begun the transition from an efficiency-driven education system to an ability-driven one that witnessed school leaders creating environments conducive to learning and innovation and teachers imbuing the vision—TSLN as thinking and caring professionals (MOE 2000).

Three initiatives were launched in Singapore's education system in 1997. They were National Education (NE), Information and Communications Technology (ICT) and Critical and Creative Thinking (CCT) (MOE 1998). These initiatives are elaborated in Chap. 3. To forge the vision TSLN and to push forward the initiatives of ICT and CCT, changes were recommended in four main areas, namely curriculum, teaching, teachers and assessment (MOE 1997). To accommodate the recommendations, the MOE initiated a content reduction of all curricular subjects. Every subject underwent a content reduction ranging from 10 to 30% and the reduced content syllabuses became effective in 1999. The amount of curriculum time for each subject remained the same. The instruction time freed up by the reduced content supported the implementation of the three initiatives.

The Teach Less, Learn More (TLLM) initiative was launched in the education system in 2005 (Shanmugaratnam 2005). TLLM builds on the groundwork laid in place by the systemic and structural improvements under TSLN, and the changes in mindset encouraged in schools. It continues the TSLN journey to improve the quality of interaction between teachers and learners so that our learners can be more engaged in real learning and achieve better the desired outcomes of education. TLLM aims to touch the hearts and engage the minds of our learners, to prepare them for life. It reaches into the core of education—why we teach, what we teach and how we teach. It is about shifting the focus from "quantity" to "quality" in education. It emphasizes "more quality" in terms of classroom interaction, opportunities for expression, the learning of life-long skills and the building of character through innovative and effective teaching approaches and strategies. It also emphasizes "less quantity" in terms of rote-learning, repetitive tests, and following prescribed answers and set formulae. To provide for ability-based education, the integrated programme was introduced in some secondary schools. Also to provide for aspiration-driven education, the National University of Singapore High School of Mathematics and Science was set up in 2005 and the School of the Arts in 2008.

Following the launch of the three initiatives: NE, ICT and CCT in 1997, the mathematics syllabus underwent a content reduction exercise. The following rationale guided the content reduction.

- The learning of mathematics is sequential and hierarchical in nature. Therefore, essential topics and skills removed from one level were transferred to another level in order to ensure continuity in the learning of the subject.
- Topics that were core content, i.e. essential as the foundation for further mathematics learning, developed the desired outcomes of the syllabuses and provided continuity and completeness were retained.
- Topics that were less fundamental and not connected to other topics in the syllabus, which placed heavy emphasis on mechanical computation, which overlapped with those taught at other levels, that were too abstract for the intended level, and concepts/skills that were taught in other subjects were removed from the syllabus.

In 1998, following the content reduction exercise, a revision of the syllabuses was undertaken to:

- Update the content to keep abreast with the latest developments and trends in mathematics education.
- Explicate the thinking processes inherent in the subject and to encourage the use of ICT tools in the teaching and learning of mathematics.
- Ensure the content meets the needs of the country in the next millennium (twenty-first century).

As a result of the revision, changes were made to the reduced content syllabus. It must be noted that the revised syllabus and reduced content syllabus were almost the same. A reorganization of the content was mainly carried out. There was minimal increase in the content to emphasize the development of thinking skills and help in the attainment of the objectives. A critical appraisal of the framework was also undertaken. Two changes were made to the framework of the 1990 syllabus. Under the arm of processes "Deductive reasoning and Inductive reasoning" were replaced by "Thinking skills", which covered a much wider range of skills that students were encouraged to use when solving problems. As an additional attribute, perseverance was added to the arm of Attitudes.

The revised curriculum was implemented in 2001. In 2001, textbooks for the primary school mathematics were privatized. This was done so that schools would have more choices of curriculum materials though the scope of the content remained the same. All the books that are available for use in schools must have been approved by the Ministry of Education for a specified period of time. CDIS never produced curriculum materials for secondary school mathematics. The first local textbook series for secondary schools was published in 1969 by Teh (1969).

Since 2001, the school mathematics curriculum undergoes revision every six years. This ensures that the curriculum remains relevant in this rapidly changing and highly competitive and technologically driven world. In 2006, the syllabuses were revised. The revised syllabuses were implemented in 2007. The revised primary syllabus introduced the use of calculators in the Primary 5 and Primary 6 teaching and learning of mathematics. In 2009, the use of calculators was part of the mathematics examination during the PSLE. The revised secondary mathematics syllabuses placed greater emphasis on algebraic manipulation skills.

2.2.4 Values-Based, Student-Centric Phase (2012–Present)

In 2008, the Curriculum 2015 committee was set up to study twenty-first century-skills and mindsets needed to prepare future generations in Singapore for a globalized world (MOE 2009). The committee unveiled the twenty-first-century competencies framework, shown in Fig. 2.1, in 2010. In 2012 the then Minister of Education in his keynote speech, at the Singapore Conference in the USA, noted that the key foci of the education system moving forward were

(i) to help every child access the new economic future,
(ii) to make the system centred on students' aspirations and interests, and
(iii) to build fundamental values and skills (Heng 2012).

At the same meeting, the Minister also made apparent that the education system had embarked on a "values-driven, student-centric" phase. In 2012, the Academy of Singapore Teachers was set up to develop professional excellence with a focus on student-centric, values-driven education.

Following the introduction of the twenty-first-century competencies framework in 2010, the review of all mathematics syllabuses from primary to secondary to pre-university level was undertaken. The review resulted in the 2012 revised syllabuses which were implemented in 2013. The revised syllabuses of 2012 (MOE 2012a, b, c) make explicit that learning mathematics is a twenty-first century-necessity and it is a key fundamental in every education system that aims to prepare its citizens for a productive life in the twenty-first century. It also notes that for Singapore as a nation the development of a highly skilled and well-educated manpower is critical to support an innovation- and technology-driven economy. Therefore, it is the goal of the national mathematics curriculum to ensure that all students will achieve a level of mastery of mathematics that will serve them well in their lives, and for those who have the interest and ability, to pursue mathematics at the highest possible level.

The syllabuses place heightened emphasis on the role of learning experiences for mathematics learning. They state that:

> Learning mathematics is more than just learning concepts and skills. Equally important are the cognitive and metacognitive process skills. These processes are learned through carefully constructed learning experiences. For example, to encourage students to be inquisitive, the learning experiences must include opportunities where students discover mathematical results on their own. To support the development of collaborative and communication skills, students must be given opportunities to work together on a problem and present their ideas using appropriate mathematical language and methods. To develop habits of self-directed

Primary Two (Sub-Strand: Whole Numbers)	
Content	Learning Experience
1. Numbers up to 1000	Students should have opportunities to:
1.1 counting in tens/hundreds 1.2 number notation, representations and place values (hundreds, tens, ones) 1.3 reading and writing numbers in numerals and in words 1.4 comparing and ordering numbers 1.5 patterns in number sequences 1.6 odd and even numbers	a) give examples of numbers in everyday situations, and talk about how and why the numbers are used. b) work in groups using concrete objects/the base-ten set/play money to - count in tens/hundreds to establish 10 tens make 1 hundred and 10 hundreds make 1 thousand. - represent and compare numbers. c) make sense of the size of 100 and use it to estimate the number of objects in the size of hundreds. d) use the base-ten set/play money to represent a number that is 1, 10 or 100 more than/less than a 3-digit number. e) use place-value cards to illustrate and explain place values, e.g. the digit 3 stands for 300, 30 or 3 depending on where it appears in a number. f) use place-value cards to compare numbers digit by digit from left to right, and use language such as 'greater than', 'greatest', 'smaller than', 'smaller than', 'smallest' and 'the same as' to describe the comparison. g) describe a given number pattern before continuing the pattern or finding the missing number(s).

Fig. 2.7 An excerpt from the primary Mathematics syllabus (MOE 2012a, p. 37)

> learning, students must be given opportunities to set learning goals and work towards them purposefully. A classroom rich with these opportunities, will provide the platform for students to develop twenty first century competencies (MOE 2012a, p. 22; 2012b, p. 20; 2012c, p. 18).

Learning experiences are explicitly stated in the mathematics syllabuses to influence the ways teachers teach and students learn so that the curriculum objectives can be achieved. Figure 2.7 shows an excerpt from the primary mathematics syllabus and Fig. 2.8 shows an excerpt from the secondary mathematics syllabus. From both figures, it is apparent that statements expressed in the form "students should have the opportunities to …" focus teachers on the student-centric aspect of learning mathematics. The statements describe actions that would allow students to (i) engage in co-creation of knowledge (ii) make sense of the knowledge they acquired and (iii) work collaboratively and communicate their reasoning using mathematical vocabulary.

In 2011, nationwide professional development of mathematics teachers was carried out to prepare them for the implementation of the 2012 revised mathematics curriculum. The implementation of these syllabuses began in 2013. Following the 6-year cycle of review and revision of mathematics curriculum, in 2018, the revised syllabuses will be announced and implemented in 2019.

Secondary Three/Four (O-Level Mathematics) (Strand – Geometry and Measurement)	
Content	Learning Experience
G3 Properties of Circles	Students should have opportunities to:
3.1 symmetry properties of circles - equal chords are equidistant from the centre - the perpendicular bisector of a chord passes through the centre - tangents from an external point are equal in length - the line joining an external point to the centre of the circle bisects the angle between the tangents 3.2 angle properties of circles - angle in a semicircle is a right angle - angle between tangent and radius of a circle is a right angle - angle at the centre is twice the angle at the circumference - angles in the same segment are equal - angles in opposite segments are supplementary	a) use paper folding to visualise symmetric properties of circles, e.g. "the perpendicular bisector of a chord passes through the centre". b) Use GSP or other dynamic geometry software to explore the properties of circles, and use geometrical terms correctly for effective communication.

Fig. 2.8 An excerpt from the secondary Mathematics syllabus (MOE 2012b, pp. 57–58)

2.3 Conclusion

It is apparent from the goals of the education system that they are shaped by the needs of Singapore for its economic survival. As part of the school curriculum, the study of mathematics has been critical since the late 1950s. It is a compulsory school subject, which takes into consideration the differing abilities and needs of students. It provides differentiated pathways and choices to support every learner in order to maximize their potential.

The achievement of Singapore students in benchmark studies such as Trends in International Mathematics and Science Study (TIMSS) and Programme for International Student Assessment (PISA), which is presented in Chap. 6, affirms that the school mathematics curriculum is robust and in tandem with global trends. The consistent and commendable achievement of the students also show that the enactment of the curriculum places emphasis on mastery learning and problem-solving. Chapter 4 provides some insights into the enactment of the school mathematics curriculum in Singapore schools.

There is no doubt that the education system and in tandem the school mathematics curriculum will continue to evolve as every new day brings new challenges and opportunities in the nation and also the global arena. This evolution is critical in order to prepare the young to access the new economic future.

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Chapter 3
The Intended School Mathematics Curriculum

Ngan Hoe Lee, Wee Leng Ng and Li Gek Pearlyn Lim

Abstract This chapter examines the changes to the intended Singapore School Mathematics Curriculum since 1990 to the present that resulted from reviews carried out periodically. Special features and key approaches are identified to gain better insights of the curriculum. The curriculum is also examined from the perspective of the three educational initiatives that were implemented in 1997: The Critical and Creative Thinking (CCT) Initiative, the National Education (NE) Initiative, and the Information and Communications Technology (ICT) Initiative. A short discussion on textbooks is also included as they contain and communicate the intended School Mathematics Curriculum. The chapter concludes with an examination of the intended School Mathematics Curriculum from two levels: national versus school. This discussion is taken from the perspective of the process of curriculum development.

Keywords Singapore School Mathematics Curriculum · School Mathematics Curriculum Framework · Intended School Mathematics Curriculum · Nation-building initiatives and School Mathematics Curriculum · 21CC and mathematics education · ICT in mathematics education · Textbook in mathematics education · Mathematics curriculum development

3.1 The Problem-Solving Mathematics Curriculum

As noted by Kaur in Chap. 2, the Singapore Ministry of Education's (MOE) goal of setting up of the mathematics syllabus review committee to review and revise the mathematical syllabuses in use since 1981 was to study the adequacy of the

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syllabuses in meeting the needs of the students as well as to reflect relevant newer trends in mathematics education. One major outcome of this effort was the positioning of developing students' ability in mathematical problem-solving as the primary aim of the Singapore School Mathematics Curriculum (MOE 1990a, p. 3), reflecting the impact of the then considerable amount of research on mathematical problem-solving (Lester 1994) on the Singapore Mathematics Curriculum. This Problem-Solving Mathematics Curriculum of Singapore was first implemented in 1992, and though it has undergone several rounds of review of revision (MOE 1990a, b, 2000a, b, 2006a, b, 2012a, b, c), problem-solving remains central to the Singapore Mathematics Curriculum.

Lee (2016) in an analysis of the Singapore School Mathematics Curriculum (MOE 2012a, b, c), identified the key approaches and key features of it that exemplify a connected curriculum (MOE 2012a, p. 11). In the sections, that follow these are elaborated, and henceforth in this chapter, the Singapore School Mathematics Curriculum would also be referred to as the School Mathematics Curriculum.

3.1.1 Key Approaches

Lee (2016) identified two key approaches in the School Mathematics Curriculum, namely the curriculum development approach and the pedagogical approach.

3.1.1.1 Curriculum Development Approach—Spiral Curriculum Development Approach

The School Mathematics Curriculum recognizes the 'hierarchical' nature of mathematics and adopts a 'spiral approach' to the design of the curriculum (MOE 2012a, p. 11). Each topic is revisited and introduced in increasing depth from one level to the next to enable students to consolidate the concepts and skills learned and to develop these concepts and skills further. This is basically aligned with Bruner's (1960) idea of readiness for learning wherein he believed that a spiral curriculum can foster or scaffold that readiness by 'deepening the child's powers where you find him here and now'. An example of how the spiral curriculum is exemplified in the teaching of addition and subtraction of fractions at the primary levels is shown in Table 3.1. The table illustrates how clearly and refined the spiralling of the content is specified in the curriculum document (MOE 2012a).

Garland (2013) reported that based on a 2012 study by William Schmidt and Richard Houang, it is found that the (USA) Common Core Math Standards were highly correlated with those of high-performing countries, including Singapore. In fact, she noted that '[A]n analysis by Achieve, a nonprofit organization that has supported the Common Core, found that Singapore's math curriculum was similar to Common Core, but that in Singapore, students more quickly reach a higher level of

Table 3.1 Spiralling of the teaching of addition and subtraction of fractions at the primary level (MOE 2012a)

Level	Topic
Primary 2	Addition and subtraction of fractions: Adding and subtracting like fractions within one whole with denominators of given fractions not exceeding 12
Primary 3	Addition and subtraction of fractions: Adding and subtracting two related fractions within one whole with denominators of given fractions not exceeding 12
Primary 4	Addition and subtraction of fractions: • Adding and subtracting fractions with denominators of given fractions not exceeding 12 and not more than two different denominators • Solving up to two-step word problems involving addition and subtraction
Primary 5	Addition and subtraction of mixed numbers

math proficiency', reflecting on the efficiency of Singapore's spiral approach towards curriculum development.

3.1.1.2 Pedagogical Approach—The Concrete–Pictorial–Abstract (C-P-A) Approach

The School Mathematics Curriculum also recognizes the need for 'age-appropriate strategies' such as through 'the use of concrete manipulatives and pictorial representations to scaffold the learning and for sense making' (MOE 2012a, p. 33). Consequently, the key pedagogical approach advocated by the curriculum document is the 'concrete–pictorial–abstract' (C-P-A) approach, particularly for the teaching of the number and algebra strand.

As was observed by Leong et al. (2015), this approach is an adaptation of Bruner's conception of the 'enactive-iconic-symbolic' modes of representation (Bruner 1966). They also argued that Bruner is interested in the external representations of knowledge when putting forth these three modes. Ng (2009), in advocating the use of the C-P-A development of concepts, advised teachers to 'structure' the external representations in the learning environment, wherever possible, to enable students to progress from 'concrete and pictorial levels to abstract representation'. Lee and Tan (2014) observed that in fact it is a common practice for teachers adopting the C-P-A approach not only to present mathematical ideas in concrete, pictorial and abstract representations, but also encourage students to establish linkages among these external representations to aid students in their development of their internal representation system of an abstract mathematical idea.

While the key curriculum approach—the spiral curriculum approach—promotes connecting to extend existing knowledge and skills, i.e. inter-conceptual connection, the key pedagogical approach, the C-P-A approach encourages connecting to make sense of learning through multiple representations, i.e. intra-conceptual connections.

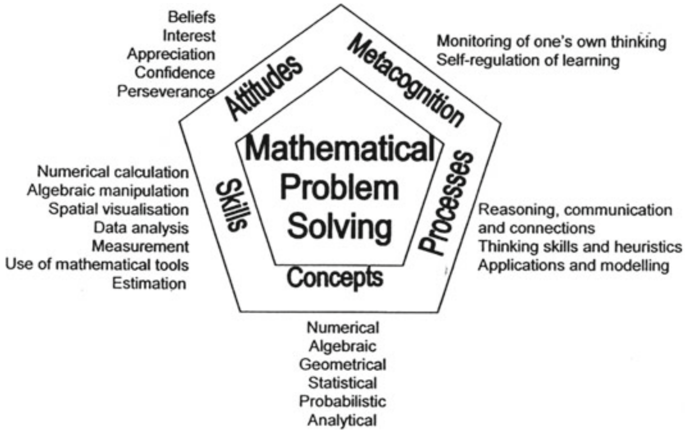

Fig. 3.1 School Mathematics Curriculum Framework (MOE 2012a, p. 16)

3.1.2 Key Features

Lee (2016) also identified two key features of the School Mathematics Curriculum, namely the School Mathematics Curriculum Framework (SMCF) and the pedagogical tool—the Model Method.

3.1.2.1 The School Mathematics Curriculum Framework (SMCF)

The SMCF, shown in Fig. 2.6 in Chap. 2, has problem-solving as its 'central focus'. The framework 'stresses conceptual understanding, skills proficiency and mathematical processes and gives due emphasis to attitudes and metacognition', with these five components being viewed as 'inter-related'. As observed by Wong (1991), the framework describes 'the philosophy' of the curriculum and integrates the 'aspects about mathematics learning and teaching'.

As pointed out by Kaur in Chap. 2, these five components of the framework, concepts, skills, attitudes, metacognition and processes have remained 'steadfast although some refinements have been made of their attributes at periodic subsequent revisions' of the curriculum. In the following, we detail each of these components and trace the refinements made during the periodic subsequent revisions of the curriculum. Figure 3.1 shows the present version of the SMCF (MOE 2012a, p. 16).

Concepts: To encourage the development of deep understanding of mathematical concepts, which forms the foundation of the SMCF, the syllabus document advocate teaching that 'make sense of various mathematical ideas as well as their connections and applications' so as to help students to 'relate abstract mathematical concepts with concrete experiences' (MOE 2012a, p. 17). The two key approaches mentioned reflects the emphasis on promoting conceptual understanding through conceptual inter- and conceptual intra-connectedness. The original SMCF (MOE 1990a, b) and that follows from the immediate revision of the curriculum for implementation in 2001 (MOE 2000a, b) included only numerical, geometrical, algebraic and statistical concepts. However, in subsequent revisions (MOE 2006a, b, 2012a, b, c), probabilistic and analytical concepts were also included. The inclusion of these two groups of concepts also marks the move away from relying purely on examination syllabuses for upper secondary and pre-university levels, which are tied closely to the emphasis placed on the national examinations at these levels then. For the first time, the SMCF articulated the philosophy of the curriculum from primary to pre-university levels when the revised syllabus was implemented in 2007 (MOE 2006a, b). In fact, the latest teaching and learning curriculum documents articulate clearly not only the generic philosophy of the curriculum but also interpretations of the curriculum approaches and features in relation to the respective year levels and courses of study (MOE 2012a, b, c, 2015).

Skills: As reflected in the curriculum document (MOE 2012a, p. 17), these are skills 'specific to mathematics and are important in the learning and application of mathematics. The set of skills have not changed much over the various rounds of revision of the curriculum, except for the way these skills are grouped. Numerical calculation, for example, now encompasses mental calculation and arithmetic manipulations (elaborated in the earlier versions of the SMCF—MOE 1990a, b, 2001a, b). Other skill sets included algebraic manipulation, spatial visualization, data analysis, measurement, use of mathematics tools and estimation. Though many of these skills are procedural in nature, the curriculum places an emphasis for these to be taught with 'an understanding of the underlying mathematical principles'. In other words, the curriculum advocates the promotion of relations understanding, as purported by Skemp (1976), thus encouraging the address of conceptual-procedural connections.

Processes: From the perspective of the curriculum document, processes refer to the 'process skills involved in the process of acquiring and applying mathematical knowledge' (MOE 2012a, p. 17). This aspect of the SMCF has undergone the greatest refinement over the years. In the first version (MOE 1990a, b), this aspect only included heuristics and deductive and inductive reasoning—the two most common types of reasoning involved in the learning and doing of mathematics. In the subsequent revision of the curriculum (MOE 2000a, b), heuristics remained but deductive and inductive reasoning were then subsumed under the generic group of thinking skills. The twelve heuristics listed in the original curriculum were reduced to eleven with 'use of tabulation' and 'make a systematic list' combined as 'make a systematic list'. At the same time, a list of eight core thinking skills was listed under the more generic term 'thinking skills', and the eight included both induction and deduction, reflecting then the impact of the vision for 'Thinking School Learning Nation' (see Chap. 2). In the next revision of the curriculum (MOE 2006a, b), the process aspect continued to undergo further refinement. Firstly, communication, both written and verbal, was reclassified from being a mathematical skill to a mathematical process skill, signifying an increased emphasis of the role classroom discourse have in the teaching and learning of mathematics. Furthermore, the distinction was made between the microthinking skills versus reasoning as a process. Emphasis was also placed to explicitly address the issue of encouraging students to see and make connections 'among mathematical ideas' as well as 'between mathematics and other subjects, and between mathematics and everyday life' (MOE 2006a, p. 17). In the latest revision of the curriculum (MOE 2012a, b, c), this emphasis on getting students to make sense of what they learn in mathematics and relate to real life is given a further boost with the explicit inclusion of applications and modelling to the process aspect of the SMCF. The continued refinement of the process aspect of the SMCF with emphasis both on the necessary cognitive skills—thinking skills, heuristics, communication, reasoning, connections, as well as the actual process of addressing real-life problems using mathematics—solving problems in real-world contexts and mathematical modelling (see Chap. 8) reflects the importance of real-life connections in the SMCF. In fact, the importance of the process aspect of the curriculum has been further elevated by the recognition of this aspect as one of the strands, others being the content strands—number and algebra, measurement and geometry, and statistics—that cut across the three content strands (MOE 2012a, p. 32).

Metacognition: Despite the fact that metacognition was a term that is coined by Flavell only in 1976, metacognition was featured as one of the five aspects of the original version of the SMCF (MOE 1990a, b), which was developed in the 1980s. This reflects that the Singapore Mathematics Curriculum is not only a forward-looking curriculum but also one that is informed by theory and research, of which the impact of Bruner's theories (Bruner 1960, 1966) has already been discussed earlier. Though there were no major changes made to this aspect of the SMCF, there was a conscious effort to refine and operationalize the construct, reflecting the continuous work on addressing this aspect of learning and doing mathematics (see Chap. 11). In the first version and subsequent first revision of the SMCF, metacognition was explained to be 'the ability to control one's own thinking processes', and it includes the 'constant (and conscious) monitoring of the strategies (and thinking processes)' (MOE 1990a, p. 4, 2000a, p. 11). In the subsequent revisions, metacognition was further refined and elaborated as 'thinking about thinking' that involves 'awareness of', 'monitoring of' and 'regulation of' one's thinking and learning. Such an emphasis on the executive processes over cognition (Tarricone 2011, p. 147) points towards the SMCF's promotion of the executive control connections.

Attitude: As with the other aspects of the SMCF, the attitudes aspect, which refers to 'the affective aspects of mathematics learning' (MOE 2012a, p. 19), underwent constant refinement over the various rounds of curriculum reviews. In the first version of the SMCF (MOE 1990a, p. 4), attitudes encompassed three affective aspects, namely (interest in and) enjoying doing mathematics, appreciating the beauty and power of mathematics, and showing confidence in using mathematics. In the following round of revision, persevering in solving a problem was added (MOE 2001a, p. 11), and with the next revision that follows, beliefs about mathematics and its usefulness were added (MOE 2006a, p. 15). The most significant address to this aspect of the SMCF occurs in the latest revised School Mathematics Curriculum—the introduction of learning experiences in the curriculum documents (MOE 2012a, b, c, 2015). Learning experiences are stated explicitly in the curriculum documents 'to influence the ways teachers teach and students learn so that curriculum objectives can be achieved' (MOE 2012a, p. 22). Though the learning experiences stipulated in the curriculum documents are not meant to be exhaustive, a conscious effort was made to include them as the key learning experiences in each of the topics addressed in the curriculum. Textbooks endorsed by MOE for use in schools, for example, are required to reflect these so that these learning experiences are addressed in all mathematics classes in Singapore. The idea is to 'remind teachers of the student-centric nature' (MOE 2012a, p. 22) of learning mathematics. The five attitudinal components of the attitudes aspect of the SMCF do provide guidance in the choice and design of learning experiences to help students develop a more positive attitude towards and in the process of learning mathematics. However, with the explicit and deliberate address of these learning experiences in MOE sanction instructional materials, there appears to be a conscious effort to level up the learning experiences resulting from possible differences in teachers' level of expertise. This is aligned with the philosophy held by MOE—that 'Every School a Good School', a slogan popularized by then Minister of Education, Mr. Heng Swee Kiat, when he spoke during the 2014 Committee of Supply debates on 7 March 2014. Minister Heng elaborated that 'Every School a Good School does not mean Every School the Same School, but it does mean Every School Good in its Own Way, seeking to bring out the Best in Every Child'. The various pathways and possible lateral transfers among the course of study shown in Fig. 2.3 in Chap. 2 are another reflection of MOE's student-centric approach towards learning of mathematics. MOE regularly updates and conducts workshops for teachers who may be involved in the teaching of students of different courses of study so as to ensure that students continue to be well supported and provided with positive learning experiences even when they switch course of study. A secondary one mathematics teacher teaching secondary 1 normal (academic) mathematics, for example, may have students who studied mathematics as well as those who studied foundation mathematics. Such teachers are not only informed of the differential entry knowledge of these students, but also invited to attend workshops to help them level up the two different groups of students in their classes. All in all, the attitude aspect of the SMCF seeks to address the affective aspects of learning so as to achieve a more holistic learning experience in the mathematics classroom. In other words, the attitudes aspect of the SMCF promotes holistic learning connections.

3.1.2.2 The Model Method

The Model Method refers to the use of rectangular drawings to represent a problem situation and to visualize and explore relationships among the quantities related to the problem situation. The introduction of the Model Method is 'an essential element of the concrete–pictorial–abstract approach', students progress from 'use of concrete objects', to 'drawing of rectangular bars as pictorial representations of the models', to using the models 'solve abstract mathematics word problems' (MOE 2009, p. 15). A more detailed treatment of the Model Method can be found in Chap. 8. Here, we present the role and benefits of the Model Method in the intended Singapore Mathematics Curriculum.

As explicated in MOE (2009), the Model Method serves to:

- Exemplify and make visible the part-whole thinking that is key to the learning of, particularly primary, mathematics (see Chaps. 3 and 4 in MOE 2009).
- Provides pupils with an efficient and effective problem-solving heuristic (see Chap. 5 in MOE 2009).
- Expose pupils at the primary levels to informal algebra by promoting algebraic thinking years before they are ready for formal algebra (see Chap. 6 in MOE 2009).

It is thus not surprising that teachers generally find the Model Method to be beneficial in the following ways:

- The model is a simplifying tool; many constraints can be handled simultaneously. Fraction problems, for example, can be solved without cumbersome computations involving fractions.
- Students are able to solve challenging problems without the use of form algebra.
- Students are able to engage in algebraic thinking years before they are ready for formal algebra. It can subsequently help students to make sense of formal algebra.

The Model Method that is highly emphasized in the primary school mathematics classrooms appears to lend well as a strong connection between primary and secondary mathematics learning. This key feature of the School Mathematics Curriculum points towards an address of transitional connections.

3.1.3 The Connected School Mathematics Curriculum

In the above analysis of the School Mathematics Curriculum through an examination of the key approaches and key features of the curriculum, it is also apparent that the intended School Mathematics Curriculum is a multidimensional connected curriculum that promotes:

- Intra-conceptual connections
- Inter-conceptual connections
- Conceptual-procedural connections
- Real-life connections
- Executive control connections
- Holistic learning connections
- Transitional learning connections.

This multidimensional approach towards a connected curriculum is similar to the one proposed by Kaur and Toh (2012):

> Teachers must provide students with opportunities to experience connection in the mathematics they learn. This is possible through links between the conceptual and procedural knowledge, connections among mathematics topics and equivalent representations of the same concept. Similarly, teachers must also provide students with opportunities to experience connections between mathematics and other disciplines of the school curriculum and daily life needs. (pp. 6–7)

In fact, Perkins has as early as 1993 argued for the need of a connected curriculum:

> A good deal of the typical curriculum does not connect – not to practical applications, nor to personal insights, nor to much of anything else. It's not the kind of knowledge that would connect. Or its not taught in a way that would help learners to make connections … What is needed is a connected rather than a disconnected curriculum – one full of knowledge of the right kind, one taught in a way to connect richly to future insights and applications. (p. 91)

3.2 The Impact of Nation-Building Initiatives on the Intended Mathematics Curriculum

To better appreciate the modifications and refinement to the School Mathematics Curriculum over the years that was first implemented in 1992 (MOE 1990a, b), there is a need to examine these from the perspective of the three education initiatives (see Chap. 2) that were introduced in 1997:

1. Critical and Creative Thinking (CCT)
2. National Education (NE)
3. Information and Communications Technology (ICT).

These three initiatives have a major and significant impact on the school curriculum as they were nation-building initiatives based on the concerns that plagued the nation then (Lee 2008). All school subjects, including mathematics, were required to respond to the initiatives accordingly.

3.2.1 Impact of the CCT Initiative

As was noted by Lee (2008), one of the approaches taken to respond to the CCT Initiative was the infusion of teaching of thinking skills into the core subjects, English, science, mathematics, geography and history. About 30% of curriculum time for these subjects consisted of such infusion lessons. Thinking skills and teaching strategies that promoted thinking were integrated into content instruction. To accommodate for the extra time needed to cope with such an approach, these subjects, including mathematics, underwent a content reduction ranging from 10 to 30% and reduced content syllabuses became effective in 1999 (see Chap. 2).

In the meantime, a more systematic review of the mathematics curriculum was carried out in 1998 to take into consideration both the content reduction that occurred in the interim curriculum implemented in 1999 and the teaching of thinking. This resulted in the refinement of the process aspect of the SMCF for the version of the curriculum to be implemented in 2001 (MOE 2000a, b) as mentioned previously. Instead of deductive reasoning and inductive reasoning for the process aspect of the SMCF, a list of eight core thinking skills were listed under the more generic term 'thinking skills'. The eight core thinking skills, which are not meant to be exhaustive, are:

1. Classifying
2. Comparing
3. Sequencing
4. Analysing parts and whole
5. Identifying patterns and relationships
6. Induction
7. Deduction
8. Spatial Visualization.

This is aligned with the intention of the CCT Initiative to get teachers more deliberate in the address of the teaching of thinking in the mathematics classrooms. To establish a more common understanding of what these thinking skills are, an operationalization of these thinking skills was provided in an appendix of the curriculum documents (MOE 2000a, p. 131; 2000b, p. 87). Thus, the School Mathematics Curriculum was refined, not displaced, with minor refinements in response to the CCT Initiative. The list of thinking skills continued to be refined, and the eight thinking skills that are reflected in the latest curriculum documents (2012a, b, c) are:

1. Classifying
2. Comparing
3. Sequencing
4. Generalizing
5. Induction
6. Deduction
7. Analysing (from whole to parts)
8. Synthesizing (from parts to whole).

The refined list is essentially the same as the original list except for 'spatial visualization' missing. In fact, 'spatial visualization' is not missing from the curriculum document, it was removed from the list of generic thinking skills but explicitly mentioned and addressed in the teaching of measurement and geometry, in view of its relevance in the teaching of this content strand. In other words, the CCT Initiative has a lasting impact on the intended mathematics curriculum till today.

3.2.2 Impact of the NE Initiative

Lee (2008) observed that the NE Initiative, in its original form, only requires its infusion across a core group of subjects, namely social studies, history and geography, where the NE values have been identified as being especially suited for infusion into. Though mathematics does not belong to this group of subjects, mathematics teachers were still encouraged to incorporate NE in their teaching (MOE 2000a):

> National Education is part of Total Education; therefore every teacher has a role to play. In the context of mathematics, National Education can be integrated into instruction by drawing examples from the prevailing national and current issues during mathematics lessons. These examples can be expressed in the problem context during problem solving or incorporated into practical work. (p. 18)

The call for application of mathematics to problems in real-world contexts seems to have its roots in the NE Initiative.

As was noted in Chap. 2, in facing a more globalized world in the twenty-first century, MOE introduced the 21CC framework in 2010 (MOE n.d.a), consisting of a circle at the centre surrounded by two concentric rings (see Fig. 2.1 in Chap. 2). At the centre of this framework is a circle that captures the core of 21CC. The first ring that encircles this core represents the social and emotional competencies, namely self-awareness, self-management, social awareness, relationship management and responsible decision-making. The outer ring that goes round the first ring encompasses the three main clusters of emerging 21CC:

1. Civic literacy, global awareness and cross-cultural skills
2. Critical and inventive thinking
3. Communication, collaboration and information Skills.

Clearly, both the CCT Initiative and NE Initiative, in fact even the ICT Initiative (to be elaborated in this chapter), are encapsulated in the 21CC framework. The explicit inclusion of learning experiences in the latest curriculum documents is not just, as mentioned above, to address the affective aspect of learning mathematics; these are also carefully chosen and designed to ensure that mathematics classrooms are rich with opportunities 'to provide the platform for students to develop these twenty first century competencies' (MOE 2012a, p. 22), including values related to the NE Initiative and skills related to the CCT Initiative (and ICT Initiative). The explicit inclusion of mathematical modelling and applications to problems in real-world contexts within the process aspect of the SMCF (MOE 2012a, p. 16) allow the address the contexts not only related to the NE Initiative, but now expanded to that face by globalization.

3.2.3 Impact of the ICT Initiative

The ICT Initiative in the education system of Singapore has evolved over the years, from the development of masterplan 1 for ICT in education (officially abbreviated as mp1) to mp2 and then mp3 (which are, respectively, the abbreviations for the second and third ICT masterplans in education) and in 2015, the fourth masterplan for ICT in education—mp4. The goal of mp4 is to put 'quality learning in the hands of every learner empowered with technology' (MOE n.d.b).

While mp1, implemented from 1997 to 2002, laid a strong foundation for schools to harness ICT, mp2 implemented from 2003 to 2008 built on mp1 to strive for effective and pervasive use of ICT in schools. mp3, implemented from 2009 to 2014, harnessed the developments of mp1 and mp2 and enriched and transformed the learning environments of students equipping them with critical competencies and dispositions to succeed in a knowledge economy (MOE n.d.b).

When MOE rolled out mp1 in 1997, the ICT masterplan was guided by the principle of 'appropriate and judicious use of technology in teaching and learning' (MOE n.d.b). The initiative laid strong foundations for schools to embrace ICT in their respective curriculums, particularly in the provision of basic ICT infrastructure and equipping teachers with a basic level of ICT integration competency. In other words, the foundations were laid to harness ICT through building the infrastructure and developing resources including ICT competency for teachers.

As outlined by Koh and Koh (2006), at the end of mp1, effective use of ICT tools in the mathematics curriculum in Singapore could be classified as follows:

(1) Productivity tool to help teachers and students to manage and speed up administrative tasks associated with teaching and learning mathematics;
(2) Informational tool to facilitate students' access to information on mathematics;
(3) Instructional or assessment tool to assist teachers to automate aspects of teaching mathematics and assessing learning;
(4) Visualization or simulation tool to facilitate learners in recognizing patterns, trends or relationships and in visualizing or simulating abstract mathematical phenomena;
5) Connection tool to allow teachers and students to engage one another on mathematical learning anytime and anywhere; and
6) Reconstruction tool to provide students with an integrated learning environment that is equipped with a suite of ICT-based tools for the reconstruction and experience of some subdomain of mathematics.

The second ICT masterplan, mp2, launched in 2003, built on the foundation laid by mp1 to establish baseline ICT standards for students and seeding innovative use of ICT among schools. Indeed, as part of MOE's continual effort to level up the ICT competency, mp2 focused on the pervasiveness of ICT in the classroom through the amalgamation with the educational curriculums. The charting of directions of the first two masterplans was primarily influenced by Singapore's economic development, from a survival-driven industrialization phase to the current knowledge and ability-based phase (see also Chap. 2), working towards an innovation and values-driven future (MOE n.d.b). According to Ng and Leong (2009) during the progression from mp1 to mp2, the use of ICT in the mathematics classroom could be classified as follows:

(1) ICT-use as a better way for teaching mathematics;
(2) ICT-use as a better way for learning mathematics; and
(3) ICT-use in relation to other factors in the instructional environment.

The third ICT masterplan, mp3, launched in 2009, was a continuum of the vision of mp1 and mp2, which is to enrich and transform the learning environments of the students and equip them with the critical competencies and dispositions to succeed in a knowledge economy. It focused on promoting self-directed learning and collaborative learning for learners through strengthening and scaling the potential of individuals to leverage on technology effectively, with the intention of such ICT-enabled learning being delivered anytime and anywhere. The initiative also empowered and supported teachers to have the capacity to plan and deliver ICT-enriched lessons. Students were able to use ICT extensively for school work, and teachers were able to adapt a wide variety of ICT tools.

As part of the goals and objectives of the mathematics curriculum, students are expected to 'use technology to present and communicate mathematical ideas' (MOE 2012a, b, c) and undergo specific learning experiences through the use of ICT tools so as to enhance conceptual understanding. The presence of such instructions on students' learning experiences across the syllabi for all levels of mathematics is the culmination of the development of mp3.

At the primary level, teachers are expected to use digital manipulatives, in addition to other learning tools, to illustrate the various algorithms for the four operations on whole numbers and fractions so that students can better make connections between the operations for whole numbers and those for fractions. In addition, teachers could include activities in which pupils construct bar charts, pie charts and line graphs using a spreadsheet software and make connections among the different graphical representations (MOE 2012a).

Virtual manipulatives could be used as a visual image like a static picture, manipulated like a concrete manipulative, or linked with verbal and symbolic notations (Goldin and Shteingold 2001). Virtual manipulatives, being capable of embodying several representations, thus lend itself to supporting the learner in connecting different mathematical concepts and ideas. In addition, virtual manipulative, if used appropriately, could be a powerful cognitive tool for learners (Moyer-Packenham et al. 2008) because learners would need to remain focussed within a virtual mathematical environment and constantly interact with the visual, verbal and/or symbolic feedback in relation to their actions on the virtual manipulative.

As discussed above, the C-P-A approach has helped learners to relate their concrete experiences with the abstract mathematical ideas, thus closing the cognitive gap between the two representations. According to Lee (2014), virtual manipulatives could help to further narrow the cognitive gap between the concrete and pictorial representations. However, as noted by Lee and Tan (2014), to incorporate the use of virtual manipulatives in the C-P-A approach, it would be unwise to simply replace the 'C' with 'V' (where V refers to external representation arising from the use of virtual manipulatives) or add 'V' into the original approach. As such, they proposed a revision of the C-P-A approach into a two-part approach: C-V and V-P-A approach. The authors elaborated that the advantages of using the aforesaid two-part revised model of the C-P-A approach include helping teachers to better understand the role of virtual manipulatives as a technological tool within the context of the commonly used C-P-A approach. Furthermore, using an integrative rather than additive approach to revising the C-P-A approach not only would increase receptivity of the revised model among teachers, but also improve the effectiveness and efficiency of lesson delivery in applying the revised model which in turn aid learners in developing conceptual understanding.

For mathematics at the secondary level, examples of ICT opportunities students are expected to receive include use of spreadsheets (e.g. Microsoft Excel) to explore the concept of variables and evaluate algebraic expressions, compare and examine the differences between pairs of expressions such as $2n$ and $2 + n$, n^2 and $2n$, $2n^2$ and $(2n)^2$ and study how the graph of $y = ax + b$ changes when either a or b varies or how the graph of $y = ax^2 + bx + c$ changes when either a, b or c varies. In addition, teachers are expected to use the AlgeDisc™ application in AlgeTools™ to help students make sense of addition, subtraction and multiplication involving negative integers and develop proficiency in the four operations of integers, make sense of and interpret linear expressions with integral coefficients such as $4x - 3y$ and $-3(x - 2)$, construct and simplify linear expressions with integral coefficients and factorize a quadratic expression of the form $ax^2 + bx + c$ into two linear factors where a, b and c are integers. Teachers could also explore the use of other ICT tools in helping students develop understanding mathematical concepts. For instance, the AlgeBar™ application in AlgeTools™ could be used to formulate linear equations to solve problems; Graphmatica, applets or other graphing software could be used to explore the characteristics of various graph functions, draw the graph of $ax + by = c$, check that the coordinates of a point on the straight line satisfy the equation and explain why the solution of a pair of simultaneous linear equations is the point of intersection of two straight lines. Furthermore, computer simulations could be used to compare and discuss the experimental and theoretical values of probability (MOE 2012b).

At the pre-university level, the use of graphing calculators, which has been integrated into the advanced level mathematics curriculum since 2006, has impacted the teaching and learning mathematics in various ways. In particular, in examinations, students are expected to use graphing calculators to graph a given function, solve an equation exactly or approximately, solve a system of linear equations, find the approximate value of a definite integral, locate maximum and minimum points and find the approximate value of a derivative at a given point (MOE 2007). More details regarding the effects of the graphing calculator are discussed in Chap. 14.

The above examples illustrated how the ICT Initiative has widened the choices of tools and platforms that mathematics teachers may employ to better achieve conceptual understanding and procedural skills fluency. Thus, the initiative enriched the pathways to better realize both the concepts and skills aspects of the SMCF.

Unveiled in 2015, the fourth ICT masterplan mp4 aims to nurture 'future-ready and responsible digital learners' with the productive and efficient use of ICT in support of the total curriculum in order to deepen subject mastery and develop the twenty-first-century competencies (MOE n.d.b). Its focus is on deepening learning through quality ICT-enabled learning and design, addressing cyber-wellness issues, developing new media literacies and sharpening the use of ICT in teaching practices. It serves a greater mission to prepare our nation's only natural resource—people, to be ICT-savvy besides having subject-specific knowledge. This helps to further realize the development of the 21CC within the mathematics classrooms, providing further impetus to the realization of the attitudes, process and metacognition aspects of the SMCF.

The four ICT masterplans have collectively set the direction for schools to plan, design, implement and evaluate ICT-integrated mathematics curriculum.

3.2.4 Overall Impact of the Three Education Initiatives

In the above examination of the impact of the three education initiatives on the School Mathematics Curriculum, it is clear that the curriculum was modified and refined but not displaced. In fact, the CCT Initiative and the NE Initiative appear to provide the necessary contexts for the refinement and clarifications, while the ICT Initiatives expanded and enriched the pathways towards realizing the curriculum. In fact, Lee (2008, 2015) observed that the SMCF, developed in the 1980s, has remained steadfast, undergoing only minor changes resulting from the numerous curriculum reviews undertaken to date. This is in part due to the rigour and robustness of the philosophy and principles underlying the decisions made about what mathematics education should equip students within the SCMF.

3.3 Textbooks and the Intended Mathematics Curriculum

A chapter on the intended mathematics curriculum would be incomplete without a discussion on the role of textbooks as they contain and communicate the intended School Mathematics Curriculum (Schmidt et al. 1997). In fact, Ang (2008) used the word 'textbook' and 'curriculum' interchangeably as reflected by the high occurrence of 'textbook (curriculum)' in the article. It seems that she saw textbook as equivalent to curriculum. She even further elaborated with an example in primary mathematics:

> When the new 'part-whole' model method was introduced in the syllabus, textbooks were specifically designed to incorporate this and its associated teaching approaches and strategies. (p. 81)

This is not surprising as textbooks in Singapore must be formally approved by MOE before they could be adopted by schools.

When the Problem-Solving Mathematics Curriculum was first implemented in 1992, primary mathematics textbooks continued to be produced by MOE based on the materials developed by CDIS in the 1980s (see Chap. 2). However, mathematics textbooks for secondary schools were published by commercial publishers. As pointed out in Chap. 2, for the mathematics curriculum that was implemented in 2001, both the primary and secondary mathematics textbooks were all produced by commercial publishers. Despite the involvement of the commercial publishers in the production of the mathematics textbooks, MOE continues its rigorous process of vetting these books for alignment with the intended curriculum, involving both mathematics teachers and curriculum specialists in the process.

As Singapore has consistently performed well in TIMSS since 1995, Singapore mathematics textbooks have also been of interest to researchers around the world. Oates' (2014) policy paper reported that Singapore mathematics textbooks clearly conveyed key concepts, provided systematic learning progression, included a variety of examples and applications and encouraged learner reflection. The paper also opined that while textbooks in Singapore had to be approved MOE, the textbooks did not dictate teachers' teaching styles. Instead, teachers used textbooks in different ways: teachers might ask their pupils to read the text in class or at home and then discuss the main concepts as a whole class. Some teachers used the textbooks as a guide when structuring their lessons and others selected assessment items from the textbooks for the pupils to attempt. The policy paper also reported that 70% of students in Singapore had mathematics teachers who used textbooks as a basis for instruction, as evidenced from TIMSS 2011.

From the perspective of the SMCF, Low (2011), as part of his master's study, investigated the extent that the framework is represented in secondary school textbooks. In the study, Low and two other coders analysed chapters categorized under the topic algebra in a Secondary Three Mathematics textbook used in Singapore. They coded sections of the chapters according to the five aspects of the SMCF. Of the contents coded, 31.5% were classified as concepts, 44.4% as skills, 11.7% as processes, 3.1% as attitudes, 7.4% as metacognition. Although there are limitations to the study, especially when it only examined the teaching of algebra, the study did show that all the five aspects of the framework were represented to a certain extent in the textbook selected. However, the glean of the distribution of the coded content across the five aspects of the SMCF raise another pertinent issue on the intended curriculum—Is it important, reasonable or even sensible to discuss about what is the ideal distribution of such codes across the five aspects of the SMCF from the perspective of the intended curriculum?

3.4 Conclusion: National Versus School Intended Mathematics Curriculum

In this chapter, we have presented the intended School Mathematics Curriculum from the perspective of the national curriculum. Singapore has a national School Mathematics Curriculum, and the philosophy, principles, goals and objectives are articulated through the curriculum documents which were produced by and disseminated by MOE and that this chapter has made reference to (MOE 1990a, b, 2000a, b, 2006a, b, 2012a, b, c, 2015).

Olivia (2013) observed that models of curriculum development are generally deductive or inductive. Deductive models of curriculum development proceed 'from the general (e.g. examining the needs of society) to the specific (e.g. specifying instructional objectives)', as pointed out by Lunenburg (2011a). Tyler's (1949) classic work is an excellent example of a deductive model of curriculum development. The way nation building in Singapore has impacted the national School Mathematics Curriculum, as presented in this chapter, also shows that it follows the deductive model of development. In fact, Lunenburg (2011a) also noted that most curricular makers adhere to the deductive approach of curriculum development as it allows the broader needs of society to be addressed. With Singapore being a young nation, a deductive approach would help ensure necessary changes to the education system are effected to meet nation-building needs (Lee 2008).

However, as Singapore enters into an ability-based, aspiration-driven phase (1997–2011) (see Chap. 2), there is a greater focus on the development of the individual student. In fact, in response to the 'Teach Less Learn More' (TLLM) Initiative mentioned in Chap. 2, Lee (2014) reported that many school teachers have embarked on a number of interesting school-based curriculum innovations, with generous support from MOE, to cater to the specific needs of the students in their respective schools. Furthermore, under the earlier mentioned Minister Heng's vision of 'Every School A Good School' for the values-based, student-centric phase of the Singapore education system (see Chap. 2), 'schools have been resourced to offer customized programmes … Different schools also offer a variety of programmes to develop the varied interests and abilities of their students' (MOE n.d.c). All these school-based curriculum innovations and programmes appear to be more aligned with an inductive approach, where the process starts with the actual 'development of curriculum materials and leading to generalization' (Lunenburg 2011b). As Lunenburg (2011b) noted, such an approach has incorporated 'a postmodern view of curriculum, because they are temporal and naturalistic'.

Lee (2014) observed that the centrally controlled national mathematics curriculum coupled with school-based mathematics curriculum innovations and programmes have created a new mathematics curriculum that is evolving in Singapore schools. This new intended School Mathematics Curriculum 'starts with the actual development of curriculum materials to target the specific needs of the pupils from the respective schools, but that is also aligned with the national mathematics curriculum' (Lee 2014). The approach taken to the development of such a mathematics curriculum appears, as Lee (2014) proposed, to be a mixed model one—one containing the elements of both the deductive and inductive models.

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Chapter 4
The Enacted School Mathematics Curriculum

Yew Hoong Leong and Berinderjeet Kaur

Abstract This chapter comprises three sections. In the first section, we make reference to the previous chapter on "The intended school mathematics curriculum". We broaden the discussion to the age-old question of bridging the intended–enacted curriculum gap. Here we draw on the international literature corpus to highlight how this gap is faced everywhere before coming back to the Singapore setting—with her unique challenges and affordances. In the second section, we draw on multi-sites research projects that are of scale on how mathematics is taught in Singapore classrooms and map the way mathematics is taught across a number of Singapore schools with a view of representing broadly the enacted curriculum. While these larger scale research can be seen as giving us a broad overview—the "airplane" view of Singapore mathematics teaching—the next section can be regarded as zoomed-in views of specific sites where the research focuses on how various contextual elements come into play to render the carrying out of curricular goals of teaching in actual classrooms challenging. We do this by drawing on classroom research studies of relatively small scale that reveal interesting complexities that are too fine-grained in the bigger studies. We end this section with a description of a current research project that draws upon both these lenses of looking at enactment. We conclude the chapter by reflecting on the question: Is there a distinctly "Singapore pedagogy"?

Keywords Enacted school mathematics curriculum · Mathematics classrooms · Mathematics teachers · Singapore

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4.1 Introduction

In Chap. 3, we are presented with a discussion of the Singapore intended school mathematics curriculum. In this chapter, our attention shifts to the enacted school mathematics curriculum. They are distinguished by both the framers of the curriculum and the place where the curriculum is accessed. For the intended curriculum, the principal framers are policy-makers—normally in consultation with other stakeholders—and they are found in curricula documents and textbooks; as to the enacted curriculum, the framers are mainly the teachers and the arena of enactment is the classrooms that these teachers conduct their instructional work. These two ways of viewing curriculum are now commonly adopted in the literature (e.g., Seitz 2017).

That there are studies on alignment of curricula (Porter 2002) presupposes that curricula, such as the intended curriculum and the enacted curriculum, are not fully aligned. The closing of this "gap" between the curricula is seen as a continual dialogic work-in-progress: the intended curriculum provides an explication of the vision for schools and teachers to work towards; research on the enacted curriculum informs the ongoing work of curriculum revisions towards nearness to practice.

We acknowledge that there are other perspectives of "curricula", such as the experienced curriculum (curriculum as experienced from the point of view of students, distinguished from the curriculum perceived to be enacted by the teacher), assessed curriculum (such as expressed in high-stakes examination, which may be distinguished from both the intended curriculum and the enacted curriculum), and even "unofficial curriculum" (which includes teaching and learning that takes place outside the spatial and temporal bounds associated with the formal curriculum). However, the focus of this chapter is on the "enacted school curriculum", and we define it as the curriculum that has its place of enactment located primarily in the mathematics classroom and where claims of enactment are based on evidence arising from research.

The PISA 2009 data showed that Singapore was first among 18 countries when it came to the proportion of students who had private tuition. Forty-three per cent of students who were tested said they had one-on-one tuition while in primary school. But they did not do any better than those with no tuition in the PISA test (OECD 2010). The usefulness of tuition was called into question, in the Singapore parliament by members in September 2013. Responding to questions, Senior Minister of State (Law and Education) Indranee Rajah said that Singapore's education system is "run on the basis that tuition is not necessary". For students who need additional support, comprehensive levelling-up programmes are in place to ensure students develop a good foundation in English and mathematics. At the same time, teachers provide remedial and supplementary classes on top of community tuition schemes such as those run by self-help groups (Straits Times 2013). Her view is supported by a 2005 exploratory study on secondary students in Singapore, which found that tuition might be counterproductive (Cheo and Quah 2005). Cheo and Quah noted that the high prevalence of tuition does not indicate a lack of confidence in the education system as it could simply be a consequence of an increased climate of competition.

A survey of 500 parents conducted by Nexus Link for the Straits Times (Davis 2015) found that only a third agreed that tuition actually pulled up academic performance of their children by a noticeable extent.

Due to a lack of substantial research on how out-of-school private tutoring has contributed towards the enactment of the school mathematics curriculum or continues to do so, for the rest of this chapter, we focus on studies conducted by Singapore researchers on the enacted mathematics curriculum in schools by teachers.

4.2 The Enacted School Mathematics Curriculum: Seen from Broad Categories

We start the review with the work done by Kaur and Yap (1997) undertaken as part of the KASSEL project. This study is the first of its kind in Singapore in terms of an attempt to systematically inquire into "what actually goes on in mathematics lessons?" across a rather large sample of lessons. Prior to this, the answer to this question is largely based on anecdotal evidences or self-reflection of one's instructional practices.

The researchers developed a lesson observation sheet which was used to record the happenings of 43 secondary mathematics lessons. Predetermined categories such as "teacher exposition", "teacher demonstration", "direct question", "seatwork" were used to code the lesson happenings. Based on these categories, narratives were written for each lesson. The conglomerated image that arose from these narratives was that the teachers' predominant mode of instruction was presentation of knowledge through explaining and demonstration. They placed emphasis on procedures, answers, accuracy more than concepts and processes. Seen through the school mathematics curriculum framework (see Chap. 2, Fig. 2.6), it appeared that the teachers were heavy on "skills" but light on "concepts" and "processes". The study did not directly address "attitudes" and "metacognition" although the researchers recorded observations that both teachers and students seemed enthusiastic about participating in the activities in the classroom. Overall, the apparently skill-biased instruction in the classes was interpreted as consistent with an examination-oriented culture (Kaur and Yap 1998), especially when common examinations had heavier assessment weighting on the skills component (Leong 2008).

A study by Chang et al. (2001) may be considered to be the first that utilized substantially video-recorded mathematics lessons and investigated the pedagogical practices of four Grade 5 mathematics teachers. For each teacher, a lesson was recorded. Teachers were also interviewed about their lessons. It was found that lessons were mainly teacher directed with two-thirds of the lesson time devoted to teacher talk and a third to student work (individually or group work). Student talk consisted of answering teacher-initiated questions or seeking clarifications. The tasks enacted during the lessons mainly encouraged comprehension and application of knowledge.

Furthermore, classwork and homework focused mainly on the development of skills and use of knowledge to complete routine tasks and prepare for examinations.

Ho and Hedberg (2005) studied the pedagogical practices of three Grade 5 mathematics teachers using a similar approach as Chang et al. (2001). They found that the practice of the teachers centred around explaining concepts, demonstrating skills and engaging students in practice of mathematical tasks to hone skills and application of concepts to solve word problems. It was apparent that teachers were teaching for problem-solving and emphasis was placed on preparation for periodic examinations of the schools.

Another study on actual classroom practices was by Kaur and her colleagues as part of the Learner's Perspective Study (Kaur and Loh 2009). The method used in this study differed from the previous research in these significant ways: (1) instead of observing a single lesson from each teacher, a whole suite of 10 lessons per teacher was recorded as representative of the instructional mode of the teacher. Due to this heavy investment of resources on one teacher, the study limited the data collection on three secondary teachers; (2) instead of basing the analysis on classroom observation, video records of all lessons were used; (3) apart from parsing the instructional sequences of the teachers into broad categories, there were zoomed-in analyses into the nature of discourse during whole-class instructional segments; (4) the selected teachers were deemed by the community as experienced and competent.

The results of this study did not at first look different substantially from the results of the earlier studies in that these teachers were found to remain highly committed to the goal of skill acquisition in the students; however, there were further insights uncovered: Kaur (2009) reported that the three teachers proceeded systematically mainly in cycles of "whole-class demonstration", "seatwork", and "whole-class review of students' work". The contents within cycle and across adjacent cycles were linked and usually formed a gradual build-up of mathematical complexity. In other words, the image that emerged from these competent teachers is one of deliberate tight-knitting of content development packed together using the familiar classroom activities of teacher explanation, students' practices, and feedback; moreover, although the mode of instruction can be seen as predominantly teacher-centred, there were intentional drawing upon students' actual work (including errors) as resources to build upon whole-class discussions.

The stereotype of the Singapore mathematics classroom as one where the teacher focuses on helping students attain proficiency of prescribed procedures appears further supported by the project undertaken by Hogan and his colleagues (Hogan et al. 2013a, b). The methods adopted by this project differed yet again from the earlier studies: in setting out to map the instructional work of mathematics teachers by using a nationally representative sample, the scope of data collection included 120 classes across 32 secondary schools. A student survey, focused on students' perceptions of instructional practices in mathematics, was used and 1166 students in these schools completed it.

The scales that recorded high means corresponded to the types of instructional practices characterized as "Traditional Teaching" (3.69 out of 5) and "Direct Instruction" (3.67). The descriptors of these modes include familiar categories such as use of exercises from textbooks and worksheets, repeated practice for fluency, focus on examination preparation, and clear organization of content.

However, the researchers also reported a relatively high mean (3.38) for "Teaching for Understanding (TfU)", as well as high correlations among these three categories. The descriptors for TfU include focus on understanding, monitoring of students' understanding and use of quality questions by the teacher. The data set collected did not include "the full range of exchanges that takes place in the classroom … and the relationship between the modes …" (Hogan et al. 2012, pp. 171–172) which would potentially contribute to explaining the seeming incoherence between TfU and the other instructional modes.

Nevertheless, these results pressed the researchers to re-examine conventions such as the sharp separation between teaching for skills and teaching for conceptual understanding, and the necessity of dialogic talk (of the kind that engages students in prolonged genuine inquiry, and which appears rare from the students' data) for TfU.

To be sure, these revisionary ideas that challenge (1) the traditional modes of teaching as necessarily ineffective in achieving more ambitious goals of teaching and (2) the very image of "traditional teaching" as being monolithically focusing on "drill-n-practise" are shared by other writers (e.g., Leung 2001; Putnam et al. 1990) and have arisen in part against the backdrop of high performance of East Asian (including Singaporean) students in international comparison studies such as TIMSS and PISA.

Going further, we think that simplistic castings of instructional approaches as being "traditional teaching", "direct instruction", "teaching for understanding" (and others) do not capture the complexity of teaching (Lampert 2001). Quality teaching can be a complex mix-and-match of different instructional forms whose choice is dependent on various factors and competing priorities. This point is eloquently expressed by Kilpatrick et al. (2001):

> Much debate centers on forms and approaches to teaching: 'direct instruction' versus 'inquiry,' 'teacher-centred' versus 'student-centred,' 'traditional' versus 'reform'. These labels make rhetorical distinctions that often miss the point regarding the quality of instruction. Our review of the research makes plain that the effectiveness of mathematics teaching and learning does not rest in simple labels …. Moreover, effective teaching—teaching that fosters the development of mathematical proficiency over time—can take a variety of forms. (p. 315)

In summary, the earlier research on the enactment of the Singapore mathematics curriculum presented an overall image of classroom work that is strategically directed towards preparing students for examination. It was done in a way that did not lower demands for students' understanding and conceptual development. Perhaps, due to the methods and analytical tools used in these studies, it remains unclear how Singapore mathematics teachers manage these seemingly opposing goals in a coherent manner. To complement this broad-grained view of the enactment of the mathematics curriculum obtained through these reviewed studies, we now turn to studies that trade scale (in terms of the number of participating schools) with more in-depth analyses into the work of teaching mathematics.

4.3 The Enacted Mathematics Curriculum: Zoomed-in Views

Developing from a common metaphor of teaching as a "balancing act" (e.g., Wood et al. 1995), Leong and Chick (2007) studied the tensions of juggling multiple goals of teaching in the classroom. These goals include reasoning towards mathematical formula, helping students to be less reliant on the teacher for answers, teach *every* student, cover syllabus and keep to time. Undertaking the task as teacher–researcher, the first author presented zoomed-in analyses of various junctures—in what might occur to an outside observer as non-problematic moments—where he experienced irreconcilable conflicts among worthy goals of teaching. He summarized the work of balancing goals:

> Despite careful planning beforehand to carry out those goals in practice, the actual occurrences during classroom instruction produced situations that caused some goals to appear in competition with each other. Those conflicting priorities posed serious challenges to the work of teaching. Nevertheless, at points when I needed to suppress some goals, it was not to abandon the whole act; rather, the giving up of a goal was to allow other goals to be met so that I could still proceed with the lesson (p. 62).

Following up on this study, Leong and Chick (2011) examined further the particular challenge of Singapore (and presumably, settings in other jurisdictions that are similarly committed) where there is pronounced pressure to prepare students to master vast amount of mathematical content given limited curriculum time. The analysis undertaken revealed that seemingly insignificant insertions into an already goal-dense instructional plan could actually trigger a chain of tensions in lessons that resulted in unintended departures from the original hypothetical learning trajectories.

These studies provided useful insights into answering the "why" of the enactment of the mathematics curriculum in Singapore. From the perspective of analysing teaching as an attempt to fulfil the goals of instruction, we explore the view that Singapore mathematics teachers are pragmatic when considering classroom teaching: they are less tied down to theoretical models of teaching; rather, their primary focus is to structure lessons in such a way as to support the achievement of their intended goals. In other words, their classroom work is goal-based rather than tightly adhering to a particular theoretical pedagogical model. This may partially explain the difficulty of previous model-based approaches (using categories such as "Traditional Teaching" and "TfU") at determining a dominant single mode of instruction. Perhaps, Singapore mathematics teachers do not see themselves nor structure their instructional work according to any particular standard classification; they are led by "what works" for them. By this is meant the motivation to harness whatever tools that are necessary—even when others may view them as derived from theoretically divergent sources—to get "the job" (that is, of fulfilling the goals of teaching) done.

Moreover, the practical work of teaching is indeed tension-filled due to competing and sometimes conflicting goals. This coheres with the "paradoxical" image mentioned earlier about how the enactment of the curriculum can appear merely skill-oriented and yet can "result" in high student performances, including tasks that are considered requiring high-order thinking. From the zoomed-in studies, we see that while the teacher could not "solve" the conflicting-goals problem at those moments when they occurred (such as between teaching a skill and encouraging reasoning, and so might have to temporarily suppress the latter), yet he persevered with teaching, keeping some of the abandoned goals in mind at the background—so that he could find opportunities at a later juncture within a module to carry them out. This perspective of "enactment" is missed out in most broad-grained studies partly due to the choice of "unit of analysis". In most earlier studies that seek to partition lesson activities into natural episodes of classroom happenings, (1) the "lens" is too broad-grained to capture the fine-grained acts of the teacher—perhaps within temporal duration of seconds—which he uses to fulfil other goals of teaching apart from the ones coded at the broad level; (2) the generic labels used to categorize these episodes (such as "Teacher demonstration") does not provide the trajectory for inter-unit tracing of goal development and hence teachers' attempt at backgrounding goals in one unit and then foregrounding these goals in another unit becomes invisible in the process of analysis.

We think that both the views of enactment—the broad categories and the in-depth examination—are complementary: the former gives the zoomed-out picture and the latter the zoomed-in nuances. Metaphorically speaking, we see different things when the same object is viewed from an aerial shot as compared to a macro-capture. But both are needed in providing a more rigorous study of the enactment of the mathematics curriculum in Singapore. We now turn to an ongoing project that draws upon these two "lenses".

4.4 The Enactment Project

A more detailed description of the project is found in Kaur et al. (2018). We provide a brief summary here. The main goal of this research project is to examine the enacted curriculum of experienced secondary school teachers. The project aims to focus on "the instructional core" (City et al. 2009) of Singapore mathematics teaching. By "core" is meant the nature of instructional practice that is determined by "the relationship between the teacher, the student, and the content—not the qualities of any one of them by themselves" (City et al. 2009, pp. 22–23). Aligned to the study of this core, the project is about the interactions between secondary school mathematics teachers and their students. It also examines the content through the instructional materials used—their preparation, use in classroom, and as homework. In other words, the project can be seen as comprising of two-related studies: Study 1—Pedagogies adopted by experienced mathematics teachers when enacting the curriculum; and Study 2—Experienced secondary school mathematics teachers' use of instructional materials for the enactment of the curriculum. This project is also the first to be classified as "programmatic" by the Office of Educational Research at the National Institute of Education, Singapore. It is programmatic in that it goes beyond the scope of a narrow research agenda—in the case of this project, it aims to draw from the data, analyses and findings of both Study 1 and Study 2 to address the bigger challenge of mapping the enacted mathematics curriculum.

There are six main research questions within the scope of Study 1:

- How do experienced mathematics teachers introduce *concepts* to students or engage students in constructing concepts?
- How do experienced mathematics teachers engage students in developing *fluency* with *skills* in computing or manipulating mathematical tasks?
- What are the *mathematical processes* commonly emphasized by experienced mathematics teachers?
- How do experienced mathematics teachers facilitate the development of *metacognitive* strategies amongst their students?
- How do experienced mathematics teachers imbue desired *attitudes* for the learning of mathematics amongst students?
- What are the perceptions of students about good mathematics lessons?

As highlighted by the italicized words, the first five questions correspond to the five sides of the school mathematics curriculum framework in the intended curriculum. It directly addresses how—and what factors influence—teachers respond in class to enact these five emphases of the curriculum. The last question covers the perception of the enactment from the students' point of view.

With respect to Study 2, there are three main research questions:

- How do experienced teachers select instructional materials for use in their lessons preparation and/or classroom work?
- What are the experienced teachers' guiding principles as they modify the selected instructional materials?
- What are the characteristics of instructional materials that will fulfil the twin objectives of (i) helping experienced teachers enact worthy instructional goals of teaching mathematics and (ii) helping students improve desirable outcomes?

These questions presuppose that mathematics teachers draw heavily from base materials (such as textbooks) in preparation for lessons but they do not merely offload from these sources, they intentionally modify them (Brown 2009) into a form that is classroom ready (that is, into "instructional materials") that fulfil their instructional goals of the respective lessons. Also, as in Study 1, the third question in Study 2 reveals the commitment to examine research objects not just from the teachers' perspective, but also in relation to how they impact the students in the classes.

Methodologically, this project draws upon the techniques used in previously reviewed studies mentioned in the earlier sections in order to develop both the zoomed-out view and the zoomed-in perspectives. It used both a video segment and a survey segment, where the survey segment is dependent on the findings of the video segment. The methods guiding the video segment of the study is influenced by the complementary accounts methodology developed by Clarke (2001) and widely used in the study of classrooms such as in the previously discussed Learner's Perspective Study. This methodology recognizes that only by seeing classroom situations from the perspectives of all participants (teachers and students) can we come to an insight into the motivations and meanings that underlie their participation.

The project is currently at the data collection phase. The video segment is expected to involve the recording of lesson units of some thirty competent teachers taken across the whole spectrum of school types in Singapore. In the context of the project, a competent teacher is defined as one who has taught a same course of study for a minimum of 5 years, and one who is recognized by the professional community as being effective in the teaching of mathematics. Selected students from the classes taught by these teachers will also be interviewed. For the survey segment of the project, some 600 secondary school mathematics teachers, purposefully sampled and representative of the profile of mathematics teachers in Singapore secondary schools, will be expected to take part. The findings from the video segment will shape the contents of the survey.

4.5 A Singapore Pedagogy for Mathematics Lessons?

Closely related to the study of the enacted mathematics curriculum is the question of whether there is a discernible distinctive Singapore way of teaching mathematics; or, is there a Singapore pedagogy? This question can also be seen as being inspired by the findings of the TIMSS Video Study as reported in Stigler and Hiebert (1999). They presented three portraits: the US classroom, the German classroom, and the Japan classroom, arguing that it is meaningful to speak of a xxx classroom (where xxx stands for a well-defined geographical region) as if there is a single portrait because, while there are definitely variations in how mathematics is taught in each country, this variation is small compared to the variations across countries. Following this argument, it seems natural to ask, “Is there a Singapore mathematics classroom?”

In the context of the strong performance of East Asian countries in TIMSS and PISA, a related question would be whether it is more appropriate to consider the commonalities among high-performing East Asian classrooms instead of merely looking for distinctives within each country. Leung (2001), for example, suggested the notion of an East Asian identity.

The answer is in part—again—to do with the degree of zoom in which we view the enactment situation. If one is content with putting classroom instructional work into boxes like “performative orientation”, “teacher domination”, or “traditional teaching”, then these labels would perhaps describe in common language the East Asian “pedagogy”. However, as we have argued repeatedly in this chapter, these broad-grained strokes do not tell the whole story—including critical pieces of the story—of what goes on in the minds and actions of the teachers and the students. And, it is for the reason of doing justice to the enacted curriculum—in all its complexity—that recent research efforts, including the ongoing enactment project as described in the previous section, are directed. [More details of the enactment project and examples of distinctive features of the Singapore mathematics classroom are provided in Chap. 16 and will not be repeated here].

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Chapter 5
Beyond School Mathematics

Weng Kin Ho, Pee Choon Toh, Kok Ming Teo, Dongsheng Zhao and Kim Hoo Hang

Abstract In this chapter, we unfold a two-sided painting of the tertiary mathematics education landscape in Singapore. One side displays how the education system in Singapore prepares her students for further learning of mathematics at university, while the other portraits how mathematics is taught at the tertiary level in the mathematics department of some Singapore-based universities. Regarding the pre-university mathematics education at 'A'-level, we examine some of the major syllabus changes for Mathematics, making sense of these changes through the analytical lens of curriculum orientation. In passing, we also looked at the H3 Mathematics curriculum and its implementation, and the niche school NUS High School of Mathematics and Science. For the tertiary mathematics education, we rely on the collective wisdom of seven mathematics professors who have rich experience in teaching undergraduate mathematics from the top four local universities. The story of what goes beyond school mathematics in Singapore brings forth an important message, that is, tertiary mathematics education is responsive to shifts in educational policies occurring at schools—one which is unique of Singapore.

Keywords Tertiary mathematics · A-level mathematics syllabi · Curriculum orientation · H3 Mathematics · Niche-area schools · Undergraduate mathematics courses

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5.1 Introduction

The turn of the twenty-first century saw a global movement to building SMART nations, nations where people are empowered by technology to lead meaningful and fulfilled lives, and technological advancements in Engineering. Responding to this worldwide trend, universities began re-looking at ways of equipping their graduates to meet the expanding demands in the areas of Science, Technology, Engineering and Mathematics (STEM), and hence a re-emphasis on STEM education. As the vehicular language for Science, Technology and Engineering, Mathematics as an academic subject is of central importance, starting from primary, through secondary and pre-university, and culminating at tertiary education. With regards to this recent movement towards STEM education, the Ministry of Education in Singapore (MOE) has stipulated in the official document of the mathematics syllabus at the Advanced Level that 'H2 Mathematics is designed to prepare students for a range of university courses, including *mathematics*, sciences, engineering and related courses, where a good foundation in mathematics is required. It develops mathematical thinking and reasoning skills that are essential for *further learning of mathematics*.' The focus of this present chapter was on this further learning of mathematics—tertiary mathematics education. More precisely, we describe how Singapore is moving *beyond school mathematics*. We expound on this aspect in the ensuing three sections.

A discussion of the changes in the way mathematics have been taught at the tertiary level in Singapore can never begin without tracing the relevant parts of the major changes that occurred in the Singapore mathematics syllabi at the pre-university level, i.e. A-level mathematics taught in junior colleges and centralised institutes. MOE constantly tapped on expert advice from junior college teachers and university professors regarding the reshaping and re-crafting of the mathematics syllabi at the A-level. We begin Sect. 5.2, with a brief chronological recount of the significant education policies introduced at different junctures of time that directly brought about the changes in the A-level Mathematics syllabi. These changes are then analysed through the lens of curriculum orientation—in this case, it represents a shift from scholar academic to social efficiency. This critical analysis provides insight into how Singapore, as a young nation, endeavours to equip the next generation of her citizens with the needful twenty-first century life-skills, amongst which mathematical competencies take central status.

With the chronicle of the changes in the mathematics syllabus landscape at the A-level as the backdrop, in Sect. 5.3, we zoom into the microscopic aspects of the pre-university mathematics education in Singapore that have direct bearings on the way mathematics will be taught and learnt at the tertiary level. Here, we focus on two specific domains. The first domain deals with the H2 Mathematics syllabus implemented by MOE. Further equipping students to study mathematics at a more rigorous level is one key area that MOE has identified at A-level. Students who are mathematically inclined and capable are encouraged to take up Mathematics at Higher 3 level, which aims to develop their advanced mathematical problem-solving skills, mathematical reasoning and communication skills through mastery of precise mathematical language. In Sect. 5.3.1, we examine the implementation of the H3 Mathematics syllabus, following the revamp of the GCE A-level curriculum by MOE in 2006. More specifically, we exhibit the various options available to students taking H3 Mathematics, such as modules offered by the National University of Singapore (NUS), the Nanyang Technological University (NTU) and MOE, and describe the syllabi and content coverage of these modules, as well as major changes in the syllabi by MOE.

The second domain concerns MOE's designation of certain schools that provide specific learning to students in certain niche-areas—with particular emphasis on mathematics and science. In Sect. 5.3.2, we give an example of such a niche-area school—the NUS High School of Mathematics & Science (NHSMS). In particular, we look at how education strategies deployed in NHSMS achieves her vision and mission statement.

Section 5.4, the climax of this chapter, is where we compare and contrast the different ways tertiary mathematics education has evolved in selected local universities. Traditionally, professors delivered mathematics lessons at the tertiary level in lecture-cum-tutorial style. However, in recent years, many mathematics departments worldwide moved from teacher-centric didactics to student-centric pedagogies. We interviewed seven professors from three different mathematics departments of the aforementioned local universities and provide some insight into the new ways of teaching university mathematics with special focus on Singapore context. Finally, in Sect. 5.5, we conclude with what lessons can be gleaned from the narrative we have presented so far with regard to the tertiary education landscape.

5.2 Changes in A-Level Mathematics Syllabi in Singapore

An in-depth exploration of the changes that have taken place in the Singapore tertiary mathematics education so far makes sense only if we understand what changes had taken place in the education experience of students *prior to* their enrolment into university mathematics courses. This is clear since one aim of the A-level H2 Mathematics is to enable students 'to acquire mathematical concepts and skills to prepare for their tertiary studies in *mathematics*...' (MOE 2017a, p. 2).

Through the years, MOE has been conscientious in revising the A-level Mathematics syllabi to keep up with national and global issues, needs and trends. It is important for us to trace this journey to elucidate the major revisions in order to ascertain the landscape of issues and trends affecting the curriculum orientation, i.e. what is valued by the Singaporean society, which underpins the curriculum decisions. Here, curriculum decisions include content selection, high-stake examinations, scope and sequence of scheme of work, and classroom teaching and learning experiences.

5.2.1 A Brief Chronicle of Major Educational Policies that Shaped A-Level Mathematics Landscape

Since the 1990s, several significant education-related initiatives have impacted the A-level mathematics curriculum. These initiatives are detailed in Chaps. 2 and 3 of this book. Table 5.1 gives an overview of the initiatives and their impact on the A-level Mathematics syllabi.

5.2.1.1 Thinking Schools Learning Nation (TSLN) and Teach Less Learn More (TLLM)

Responding to the Thinking Schools Learning Nation (TSLN) initiative, the MOE undertook a fundamental review of the school curriculum and assessment system to allow for the development of creative thinking and learning skills required for the future. To achieve this, an important step was taken to reduce the amount of content knowledge that the students needed to learn so that both teachers and students can free up more time to engage in activities that develop the aforementioned skills (e.g. Project Work). An outcome of this content reduction in the A-level Mathematics curriculum was the Mathematics Syllabus 9233 which took its final form in 2001 after its 1999 interim version was phased out.

In 2002, the Junior College (JC)/Upper Secondary Education Review committee was set up to look into a major reshaping of the pre-university education landscape in the junior colleges and centralised institutes. While the aforementioned review was in progress, another independent and major education initiative the 'Teach Less Learn More' (TLLM) was introduced by MOE in 2005.

Independently, but happening in parallel to the JC/Upper Secondary Education Review, the development of the new A-level syllabi took place and was largely completed by the end of 2004. These new A-level syllabi took effect in 2006. In order to broaden the learning experience of the student, the notion of a contrasting subject came into being—it refers to a content-based subject taken outside a student's main area of specialisation. Under the new system, subjects are offered at either H1, H2 or H3 level. H2-level subjects are equivalent to the previous A-level subjects in terms of demand and intellectual challenge but would have their content reduced to free up the curriculum time for contrasting subjects and non-academic pursuits. H1-level subjects can be seen as half of their H2 counterparts in terms of curriculum time. For a subject taken at the H3 level, specific H3 programmes are offered to allow academically exceptional students to pursue that subject or area in which they have passion and aptitude. Unlike the previous Special Papers (commonly known as 'S' Papers), H3 programmes run on a separate syllabi which go beyond the H2 syllabi. We shall elaborate on the various H3 Mathematics programmes in Sect. 5.3.1. Notably, a majority of the A-level students sit for examinations under the H2 Mathematics Syllabus 9740.

Table 5.1 Initiatives since 1990s and their impact on the A-level Mathematics syllabi

Initiative	General description	Syllabus change
TSLN (Thinking Schools, Learning Nation), 1997	Lifelong learning, collective tolerance for change, schools as learning organisations, students develop both lower and higher thinking skills and processes	Mathematics 9205 changed to Mathematics 9233 (Interim), 1999
ICT Masterplan I (mp 1), 1997	Equip schools with ICT hardware, LCD projector in every classroom, whole school networking, ICT use in 30% curriculum time	• Graphing Calculators (GCs) were introduced in A-level mathematics and used in the 2001 Further Mathematics (FM) 9234 (Revised) 'GC-neutral' examination • Mathematics 9233 (Interim) changed to 9233 (Revised), 2001
Review of Junior College (JC)/Upper Secondary Education, 2002	Reviewing at, both the macro and the micro levels, the curricula of all subjects for Junior College	The development of the 2006 A-level curriculum took place parallel to the Junior College (JC)/Upper Secondary Education Review and was largely completed by 2004
ICT Masterplan II (mp 2), 2002	Integrate ICT with curriculum design, student-centred learning environment, evaluation of the use of ICT in education	GCs are used for teaching and learning as well as assessment, i.e. in the H2 Mathematics (9740) examinations
TLLM (Teach Less, Learn More), 2005		New A-level curriculum, 2006 • Broader and more flexible JC curriculum • Subjects offered at H1, H2 or H3 • Contrasting subjects for H1 and H2 • Mathematics 9233 (Revised) changed to H2 Mathematics 9740 • Removal of the subject Further Mathematics 9234
ICT Masterplan III (mp 3), 2009	Transforming the learning environment, a continuum of mp1 and mp 2	
SMART Nation, 2014	Making full use of new technologies to develop sustainable and innovative solutions, not just to run the place better but to make a difference to people's live	A re-emphasis on STEM education

(continued)

Table 5.1 (continued)

Initiative	General description	Syllabus change
ICT Masterplan IV (mp 4), every school a good school, 2012–2015	4 every's in MOE education initiatives: • Every School a Good School • Every Student an Engaged Learner • Every Teacher a Caring Educator • Every Parent a Supportive Partner	• H2 Mathematics 9740 changed to H2 Mathematics 9758 – Twenty-first century competencies (e.g. creative and inventive thinking) – Re-emphasis on STEM education – Focus on 'disciplinarity'; constructivist pedagogies • Reintroduction of the subject Further Mathematics 9649

Aligning with the call for a more broad-based education, Further Mathematics 9234 was removed in 2005. Expert advice from the syllabi review and steering committees was sought to decide which essential topics in Further Mathematics were to be included in the H2 Mathematics Syllabus 9740, and which topics in Mathematics Syllabus 9233 were to be removed (and in some instances, moved to O-level Additional Mathematics) to create sufficient space for the additional topics. Strictly speaking, the change from 9233 to 9740 involved more intricate restructuring of topics so that coherence of the new syllabus was ensured.

5.2.1.2 Information and Communications Technology Masterplans and SMART Nation

The Information and Communications Technology (ICT) Masterplans are described in Chap. 3. There are four altogether. A significant impact ICT Masterplan I (mp 1) had on the A-level mathematics curriculum was the introduction of the use of Graphing Calculators (GCs), i.e. hand-held scientific calculators that have facilities for plotting graphs, computing terms of sequences, solving equations and performing other tasks with variables or simple data structures, e.g. lists. GCs were first allowed in examinations for the Further Mathematics 9234 (Revised) syllabus in 2001, where it was maintained that the question items were set to be 'GC-neutral', meaning that students who used GCs would have no absolute advantage over those who did not. ICT Masterplan II (mp 2) advanced the role of GCs in the teaching and learning of A-level mathematics. Since 2006 GCs became a mode of assessment under the H2 Mathematics Syllabus 9740 in that 'the examination papers will be set with the assumption that candidates will have access to GCs. As a general rule, unsupported answers obtained from GCs are allowed unless the question states otherwise. … For questions where graphs are used to find a solution, candidates should sketch these graphs as part of their answers. … if there is written evidence of using GCs correctly, method marks may be awarded' (MOE 2017c, p. 3).

The SMART Nation initiative in Singapore was launched by her Prime Minister Lee Hsien Loong during the 2014 National Day Rally (Lee 2014). It called for Singaporeans to be empowered by technology to lead meaningful and fulfilled lives through technological advancements in Engineering. This brought about a renewed focus on STEM initiatives. The timely development of ICT Masterplan 4 (mp 4), in 2015, supported the impetus to develop life-skills and competencies relevant to the twenty-first century. In particular, the interconnection among the disciplines of Science, Technology, Engineering and Mathematics (STEM) now takes central stage, and the nation's education system must gear itself towards equipping the next generation with STEM-related knowledge.

5.2.1.3 Every School a Good School

Beginning in 2012, the MOE set a new goal in education to provide every child with the opportunity to develop holistically and maximise his or her potential; and to do that, MOE must ensure that 'every school is a good school'. What then makes a good school? Over four Ministry of Education Work Plan Seminars from 2012 to 2015, MOE rolled out in stages the four 'every's:

(i) *Every school a good school.* A good school cares for its students, studying and knowing the needs, interests and strengths of her students and motivates them to learn and grow. The call for 'every school to be a good school' was first formalised by Minister of Education, Mr. Heng Swee Keat, in 2012 and subsequently expanded into the following three aspects (ii), (iii) and (iv) (Heng 2015).
(ii) *Every student an engaged learner.* A good school ensures all students acquire strong fundamentals of literacy and numeracy and develops them holistically, in character, knowledge and critical competencies. A good school creates a positive school experience for each student, making him a confident and lifelong learner.
(iii) *Every teacher a caring educator.* A good school has caring and competent teachers who are steadfast in their mission to impact lives.
(iv) *Every parent a supportive partner.* A good school has the support of parents and the community, working together to bring out the best in our children.

Bearing in mind the importance of a STEM-based education as well as catering for a wider variety of students' needs and interests, MOE created in 2016 an expanded suite of A-level syllabi including, in particular, Further Mathematics. The H2 Further Mathematics Syllabus 9649 is specifically designed for 'students who are mathematically-inclined and who intend to specialise in mathematics, sciences or engineering or disciplines with higher demand on mathematical skills. It extends and expands on the range of mathematics and statistics topics in H2 Mathematics and provides these students with a head start in learning a wider range of mathematical methods and tools that are useful for solving more complex problems in mathematics and statistics' (MOE 2017a). To better engage learners and to immerse them in authentic and relevant learning, H2 Mathematics and Further Mathematics syllabi included two new components, one is assessed in the 'A'-level examinations and the other to augment the 'examinable'. The part which will be assessed in the 'A'-level examinations appears in the form of *Problem in Real-World Contexts* (PRWC), where real-world situations are mathematised via suitable mathematical models and question items are designed to allow students solve problems pertaining the given real-world situation. The augmenting component targeted to enhance teaching and learning is *Learning Experience*, where students are immersed into meaningful discussions, giving them appropriate platforms in which they actively reason and communicate their understanding of concepts. Learning experiences typically manifest as lessons designed to make connections between ideas in different topics, between abstract mathematics and real-life applications, and between mathematics and other disciplines. An exemplar of learning experience created by and for teachers' use is given in the Appendix; it illustrates the relevant mathematical content knowledge that connects two different topics in the Mathematics 9758 Syllabus. It is hoped that, through learning experiences, students may learn to form their own understanding of concepts independently before these are formally taught to them. The detailed record of all content adjustments that took place in A-level mathematics syllabi from 1997 to 2017 can be found in Ho and Ratnam-Lim (2018).

5.2.1.4 Some Concerns Raised by Junior College Teachers

Having discussed the major education initiatives put forth by the MOE and how these initiatives have shaped the A-level Mathematics curriculum, we now turn to look briefly at what some experienced JC teachers have to say about the curriculum changes that occurred in the period 1997–2017. For the purpose of this chapter, we only focus on the comments that have implications on tertiary mathematics education. Though the sample of JCs teachers is small, they are holding or have held middle-leadership roles (e.g. level heads, head of departments) in JCs during the stated period, and hence, their views on the syllabi changes are definitely representative of the views of most JC teachers. Each participating teacher answered a questionnaire pertaining to the A-level mathematics syllabi changes and their perceived outcomes. We present, in summarised form, the data we obtained:

- *Is there a real need for changes in the A-level mathematics syllabi? If so why?*

All the participants of the interview expressed that there is a genuine need to have timely change in the syllabi. Moreover, they listed some criteria to consider as far as syllabi changes are concerned:

> I think the syllabus should change according to the changing needs. Beauty, suitability, relevance and applicability are some parameters for consideration when deciding on the content of the syllabus.
>
> Yes. To respond to changes occurring in the world such as technological advances.

- *Describe the changes that you see in the learning outcomes of the students over these years of changes in the syllabus.*

Some teachers perceived that the changes in the learning outcomes of their students were generally positive, i.e. they are beneficial to the students' academic development:

> The changes made to the syllabus over the last few years made the learning of mathematics more relevant to the student by relating it to the real-world contexts. The implementation of 'Learning Experience' and the introduction of 'Application Questions' in the examination aim to achieve this goal.

Some teachers held different opinions:

> I do agree that mathematics should be seen as an effective tool in solving real-world problems but I also do not discount the fact that studying mathematics should be an end itself. That is, mathematics should be pursued regardless of whether it has potential for applications. But it almost always turns out that some obscure piece of mathematics holds the key to the answer of some deeper questions in science.
>
> No mathematical rigour. Can be seen from their [students'] work. Poor algebraic skills. Students are mostly performance-driven.

- *Sum up your experience/opinions concerning the changes in A-level mathematics syllabi over the stated period.*

JC teachers are also concerned whether the content reduction in the A-level Mathematics syllabi may result in weakening students' mathematical content knowledge, and thus, whether students will be ready for learning mathematics at the tertiary level. Here are some of their voices:

> I think the H2 Math syllabus should provide opportunities for students to think logically and articulate mathematically. One area where students can develop these good qualities is 'Proofs'. I am actually saddened that Mathematical Induction has been removed from the H2 Math syllabus. Mathematical Induction is an important tool in proving mathematical statements.
>
> Students can also appreciate the beauty of this technique and its logical foundation. It's a beautiful piece of mathematics. I feel that some of the 'proofs' in the current H2 Math syllabus are not rigorous enough. For example, to 'proof' that a function is 1-1, one uses the horizontal line test which is incorrect.
>
> Mechanics is another subject which in my opinion lends itself perfectly to mathematical modelling and applications in real-world contexts. It also has cross-disciplinary interaction with Physics.
>
> I strongly maintain that traditional pure mathematics topics like 'Group Theory' should be brought back to the 'A' level FM syllabus to let students have a taste of handling mathematical proofs and understanding what a mathematical structure is. This idea of structure in mathematics is an important one and permeates almost all branches of mathematics – Group structure, Measure Spaces, Normed Spaces, and Topological Spaces etc. are all mathematical structures.
>
> I remember the Math B and Further Math B syllabus I did as a student have given me a strong foundation to study mathematics in the university. I can't really say the same about our current syllabus.

Experienced JC teachers also showed concern towards the deteriorating standards of mathematical content knowledge of JC teachers in the current state as well as in the future, and this observed phenomenon may have implications on JC teachers' competencies and professional development.

> Removal of FM results in loss of expertise, experience and resources in higher math.
>
> Many of the ex-FM teachers have retired or just teaching H2 Math due to small candidature.
>
> The quality of teachers' knowledge is lacking – content, assessment, etc. There are gaps which the teachers cannot see themselves, and they wonder what is wrong. It seems to be getting worse.
>
> Re-introducing FM – seems to place more emphasis on engineering, and hence engineer-trained teachers are teaching FM.

While the JC teachers' feedback and concerns from their implementation of the various revised A-level Mathematics syllabus are genuine and truthful, we need to remain objective in understanding the current situation of the A-level mathematics syllabi and the readiness of the next generation of Singapore mathematics learners to progress beyond the school mathematics. To achieve this objectivity in our analysis, we must acquaint ourselves with relevant and established curricular theories. The framework we choose is the spectrum of curriculum ideologies as presented by Schiro (2013), and in Sect. 5.2.2, we follow closely his interpretation of the central curricular ideologies. In Sect. 5.2.2, we analyse the shifts of curricular orientation that Singapore A-level Mathematics experienced from 1997 to 2017 so as to better understand how well Singapore students are equipped for learning mathematics at tertiary level.

5.2.2 Curriculum Ideologies

We give a brief introduction of the four main curriculum ideologies below: Scholar Academic, Social Efficiency, Learning Centred and Social Reconstruction. For each ideology, wherever possible, we tease out those syllabi aims of the different 'A'-level Mathematics Syllabi over the past two decades, which will provide evidence for the presence of influence of that ideology.

5.2.2.1 The Scholar Academic Ideology

Scholar academics advocate that the human culture has amassed a body of important knowledge which has been organised into academic disciplines institutionalised in universities. Education, hence, is aimed at inducting young children into the system of acquisition of such knowledge, i.e. the different academic disciplines. An academic discipline is perceived to be a hierarchy of people in search of truth and knowledge. The *top* comprises scholars who discover new truths, i.e. university professors and researchers, the *middle* teachers of the discipline who disseminate the truths discovered by the scholars and the *bottom* school students whose responsibility is to learn the truth so as to become more proficient members of the academic community. Mathematics as a long-established discipline requires of its learners a specific *disciplinarity*—the way a mathematician think and the way a mathematician works—the very characterisation of the practices of a working mathematician. The aim of education in mathematics, according to the Scholar Academic ideology, is to equip young members of the mathematical discipline with the disciplinarity of mathematics, moving them from the bottom (primary mathematics) towards the top (tertiary mathematics) of the aforementioned hierarchy.

Regarding thinking and working, both the 9205 and 9233 Syllabi are explicit about the need for clarity in thinking, and accuracy and carefulness in working in the wording of the syllabus aims:

> [item (c), 9205] encourages *clear thinking* and *accurate working;*
>
> [item (6), 9233] develop their [students'] ability to *think clearly*, *work carefully* and …;

Additionally, both these syllabi highlight logic and coherence as important aspects of the mathematical disciplinarity:

> [item (g), 9205] develops a *logical and coherent view of mathematics;*
>
> [item (4), 9233] appreciate mathematics as a *logical* and *coherent* subject with rich interconnections;

All the Mathematics Syllabi (9205, 9233, 9758) emphasised on the upward movement of youth members of the mathematics discipline from the school level to the university level, and this is evidenced in the explicit statements found in their respective syllabus aims:

> [item (d) & (e), 9205] provides as much as possible of the mathematics necessary for the student's concurrent study at A-level; provides a suitable foundation for *beginning a degree level course in mathematics* or a related discipline;
>
> [item (9), 9233] acquire a suitable foundation for *further study of mathematics* and related disciplines;
>
> [item (a), 9758] acquire mathematical concepts and skills to prepare for their *tertiary studies in mathematics*, sciences, engineering and other related disciplines;

5.2.2.2 The Social Efficiency Ideology

The Social Efficiency Ideology advocates that the purpose of schooling is to efficiently meet the needs of the society by training its youth to function as future mature contributing members of society. Skills and procedures needed at workplace and at home are deemed as of paramount importance to ensure productive lives and to perpetuate the functioning of society.

Through the lens of Social Efficiency Ideology, mathematics learnt in schools must be functional and useful at the workplace. Thus, certain fields of mathematics at the tertiary level that are inclined towards this ideology include financial mathematics, engineering mathematics, econometrics, mathematical biology and operations research methods; all of these have a natural tendency to be interdisciplinary and focus more on real-life applications. As the emphasis shifts towards STEM education, one observes that the changes in the 'A'-level Mathematics Syllabi lean towards the applicability of mathematics at the workplace.

> [item (h), 9205] presents at least one major area of application of mathematics—either particle mechanics or probability and statistics—so that students can see examples of the *usefulness* of mathematics in the *real world*;
>
> [item (8), 9233] appreciate how mathematical ideas can be *applied in everyday world;*

In order that mathematics learnt at school is truly functional at the workplace, the entirety of skills related to applicability to real-life situations must be realised by effective social interactions and output, i.e. collaboration, communication and invention. Traces of this social efficiency aspect can be found in the 'A'-level Mathematics Syllabi:

> [9740] produce imaginative and *creative* work arising from mathematical ideas; develop abilities to reason logically, to *communicate* mathematically, and to learn *cooperatively* and independently; and make effective use of variety of mathematical tools (including information and *communication* technology tools) in the learning and *application* of mathematics;
>
> [9758] develop thinking, reasoning, *communication* and modelling skills through a mathematical approach to problem solving;

5.2.2.3 The Learner Centred Ideology

The Learner Centred Ideology anchors itself on the needs and concerns of the individual learners. Thus, the goal of education is the growth of individuals, each in harmony with his or her own unique intellectual, social, emotional and physical attributes. In the lens of Learner Centered Ideology, the emphasis *shifted* from personal enjoyment to a holistic experiential appreciation of the subject over the recent years.

> [9205] develops further the mathematical knowledge of students in a way that encourages *confidence* and provides understanding and *enjoyment*;
>
> [items (1)-(2), 9233] develop further their understanding of mathematics and mathematical processes in a way that encourages *confidence* and *enjoyment*; develop a *positive attitude* to learning and applying mathematics;
>
> [9740] develop *positive attitudes* towards mathematics;
>
> [9758] *experience* and *appreciate* the nature and beauty of mathematics and its value in life and other disciplines;

Already mentioned in Sect. 5.2.1.4, 'Learning experiences' are explicitly incorporated into the 9758 Mathematics and 9649 Further Mathematics Syllabi to promote such an experiential appreciation of Mathematics in the classroom.

5.2.2.4 The Social Reconstruction Ideology

The Social Reconstruction Ideology comes from a social perspective in that it assumes that the current state of the society is unjust and plagued with certain societal problems such as racial, ethnic, gender or economic inequalities, or some form of threat to the society. To resolve these societal problems, Social Reconstruction Ideologists advocate education is the way to facilitate the construction of a new and more just society that offers maximum satisfaction of its members.

Although Singapore has enjoyed many years of social peace and harmony under the leadership of a corrupt-free and efficiency government, the perpetuation of social justice and stability must never be taken for granted, especially given the current volatile world trends and rampant terrorist threats. Staying vigilant and ready to respond towards potential crisis and threats to Singapore's survival, the education system must be robust and quick enough to respond to changing demands, constantly positioning Singapore at the competitive front. Top-quality education is the only way to safeguard Singapore's regional competitiveness and social stability, and this must be available not just to a few elite schools but every school in Singapore—whence, 'every school a good school' (see Sect. 5.2.1.4). Although no part of the 'A'-level Mathematics Syllabi articulates the young nation's uncompromising stand for social justice or stability, it is the subtle insistence on the virtues of diligence, persistence and resilience to be developed in students during their course of the mathematical training that nurtures these students to be ready in times of crisis in the future.

5.2.3 Analysis of the Changes in the A-Level Mathematics Syllabi Based on Curriculum Orientations

Scanning through the evolutionary history of the 'A'-level Mathematics Syllabi, one observes a gradual but clear change in curriculum orientation, namely from Scholar Academic, through Social Efficiency to Learner Centred. Other than the *statements of the syllabi aims*, the actual changes in the selection and alignment of topics found in each syllabus revision witness this change in curriculum orientation. Here, we focus on three major 'A'-level Mathematics syllabus revision which resulted in three different syllabi: (R1) Revised 2001 Mathematics Syllabus (9233), (R2) 2006 'A'-level Mathematics Syllabus (H2 9740) and (R3) 2016 'A'-level Mathematics Syllabus (H2 9758). Since Further Mathematics was withdrawn as an 'A'-level subject from 2006 to 2016, we choose to focus on the syllabi change occurring for Mathematics (and not Further Mathematics) as our main purpose is to highlight the shift in the syllabi orientation throughout a continuous time interval.

5.2.3.1 Revised 2001 'A'-Level Mathematics Syllabus (9233)

As mentioned earlier, the MOE TSLN initiative pushed forward the reduction in content to provide schools with more time to incorporate more thinking activities and infuse ICT into lessons. This content reduction 'movement' supports the view that MOE began to value the higher-order cognitive processes in teaching and learning across all subjects, departing from the old Scholar Academic stance. The most notable change was a substantial removal of trigonometry from the syllabus (e.g. general solution of trigonometrical equations, etc.). Graphing Calculators (GCs) was introduced to Further Mathematics students, though it was maintained that examination questions in the 'A'-level Further Mathematics paper were 'GC-neutral', i.e. students who used GCs would have no unfair advantage over those who did not. The use of GCs was in line with the ICT Masterplans mentioned in Sect. 5.2.1.2 as well as to expose students to the use of a powerful computational tool. In order to create a greater flexibility of choice of topics for the students, the assessment format in the 'A'-level examination changes: an 'Either-Or' option was available for the last question of Paper 1. This first syllabus revision spells the first step of departure from Scholar Academic, emphasising usability and learner-centredness.

5.2.3.2 2006 'A'-Level Mathematics Syllabus (H2 Mathematics 9740)

Emphasis was placed on solving real-world problems, and activities including communication about the mathematics involved in solving a problem and interpretation of the solution in the context of the problem were encouraged. GCs took up a more significant role in teaching and learning of 'A'-level Mathematics in the classrooms, and crucially GCs had since been officially required in the examinations. In the second wave of content reduction, significantly more topics were removed from the syllabus (see Table 5.2); with regards to assessment, this reduction resulted in no question choice in the examination format. At this stage, the curriculum orientation for Mathematics at 'A'-level was then steering towards the Social Efficiency Ideology, where applicability of mathematics in the real world, e.g. at the workplace, in the community, etc., is emphasised.

Table 5.2 Da Vinci programme structure

Year	Course item	Requirement
1 and 2	Creative thinking	Participate in activities that stimulate creative thinking
3 and 4	Independent Research Studies	Complete a Research Methodology module; encourage to work on a research project under the guidance of a teacher-mentor; have the option to participate in external research programmes at universities
5 and 6	Advanced Research Project in either Mathematics or Science	Completed within 9–18 months, dependent on the nature of the project, including at least two weeks of full-time research to be mentored by professors at leading research institutions, universities or polytechnics; research project must be showcased at the annual Research Congress held in March; possible grades received: distinction, merit, pass or fail

5.2.3.3 2016 'A'-Level Mathematics Syllabus (H2 Mathematics 9758)

An expanded suite of syllabi, with H2 Further Mathematics, was introduced to give students more options to choose from, thus catering better to their diversified needs. An emphasis was placed on mathematical processes such as mathematical reasoning, mathematical modelling and communication. Learning experiences, which are stated in the syllabus, are instituted to positively influence the ways teachers teach and how students learn so that curriculum objectives can be achieved. It is stated explicitly in the revised syllabus that teachers are also encouraged to use pedagogies that are constructive in nature. At this stage, we witness a shift of the curriculum orientation from Social Efficiency to Learner Centred Ideology.

5.3 Preparing for Tertiary Mathematics Education at Schools

The analysis given in the previous section provides insight into how education system is moving towards, the general 'big' direction. Now we are ready to see whether our findings and deductions account for the ways Singapore prepare her students for tertiary education, in our interest area—tertiary mathematics education. In this section, we look at these preparatory processes in two ways: H3 Mathematics syllabus and its implementation, and the set-up of niche-area schools—a case study of NHSMS.

5.3.1 H3 Mathematics Syllabus and Its Implementation

With the revamp of the A-level curriculum in 2006, H3 Mathematics was introduced to provide opportunity for students who have an exceptional aptitude and passion for mathematics to pursue it at a higher level than that of H2 Mathematics. Students offering H3 Mathematics have several options available to them. They can choose to read the mathematics module Linear Algebra I or Numbers and Matrices offered by NUS or NTU, respectively; undertake research projects supervised by academic staff from NUS or NTU; or read H3 Mathematics offered by MOE.

The module Linear Algebra I offered by NUS to pre-university students taking it as a H3 module is also a regular module that is offered to NUS undergraduates, and is taught by NUS lecturers. It is a typical first course in linear algebra which covers systems of linear equations, matrices and matrix algebra, vector spaces and linear transformations. In contrast, the module Numbers and Matrices offered by NTU was specially designed as a H3 module for pre-university students. It covered basic number theory, and basic linear algebra topics such as matrix algebra and vector spaces, and was taught by NTU lecturers.

We shall now focus on the H3 Mathematics offered by MOE. Since its implementation in 2007, the MOE H3 Mathematics syllabi have undergone minor and major changes. Even though the syllabi have changed over the years, the focus of MOE H3 Mathematics has been to develop students' abilities to solve non-routine problems and write mathematical proofs. The students are expected to develop their fluency with mathematical language and notation, and the concepts of proposition and its converse, contrapositive and inverse. They also need to have knowledge of the different methods of proof. In terms of content coverage, there has been a shift from new additional content on top of H2 Mathematics to content that builds on the knowledge acquired in H2 Mathematics. We elaborate below the H3 Mathematics syllabi and the changes since 2007.

5.3.1.1 2007 'A'-Level Mathematics Syllabus (H3 Mathematics 9810)

There were four topics, namely, Differential Equations, Plane Geometry, Graph Theory and Combinatorics. While Differential Equations, Combinatorics built on the topics of differential equations and permutations and combinations in H2 Mathematics, and Plane Geometry expanded on plane geometry topic in 'O'-level Additional Mathematics, Graph Theory was a completely new content that was not related to any topic in H2 Mathematics. From past examination papers, it was apparent that Graph Theory and Plane Geometry were used as medium to develop students' mathematical reasoning and proof-writing skills, while Combinatorics focus on honing students problem-solving skills through applying basic principles of counting to solve a variety of counting problems. On the other hand, the emphasis of Differential Equations was on analytical and numerical methods of solving first-order differential equations, and its applications in modelling population dynamics. The examination paper consisted of two sections: Section A, which contained four questions on Differential Equations totalling 40 marks, and candidates had to answer all the questions; and Section B, with two questions on each of Plane Geometry, Graph Theory and Combinatorics, and candidates were required to answer any four questions, with each question worth 14 marks. There were 4 marks allocated for the clarity of presentation.

5.3.1.2 2010 Revised 'A'-Level Mathematics Syllabus (H3 Mathematics 9810)

A notable change in this revised syllabus was the removal of Plane Geometry. Other than this, there were not much changes in the syllabi of Differential Equations, Graph Theory and Combinatorics, except that digraphs and tournament were removed from Graph Theory. There were no significant changes to the examination format, except that candidates had to answer all questions in Section B, with two questions on each of Graph Theory and Combinatorics, and each question still worth 14 points.

5.3.1.3 2013 'A'-Level Mathematics Syllabus (H3 Mathematics 9824)

This syllabus represented a significant change in content coverage from Syllabus 9810, as Graph Theory was no longer included. In its place were topics from H2 Mathematics: Functions and Graphs, Sequence and Series, and Calculus. The other two topics from Syllabus 9810, namely, Combinatorics and Differential Equations, were still in this syllabus, though the latter had been renamed as Differential Equations as Mathematical Models. The topics in Combinatorics and Differential Equations as Mathematical Models remained largely unchanged from Syllabus 9810, although second-order homogeneous linear differential equations and mathematical models of vibrating springs had been added in the latter. As noted earlier, the content built on knowledge acquired in H2 Mathematics, but in greater depth and breadth. There were also notable changes to the examination format. The examination paper consisted of eight questions of varying lengths and marks, with three questions on each of Functions and Graphs and Differential Equations, and two questions from Combinatorics, and candidates were required to answer all questions. No marks were allocated for clarity of presentation.

5.3.1.4 2017 'A'-Level Mathematics Syllabus (H3 Mathematics 9820) (MOE 2017b)

With the introduction of Further Mathematics, most topics in Differential Equations in the previous H3 Mathematics syllabi are now covered in Further Mathematics. Therefore, Differential Equations no longer features prominently in this syllabus and is subsumed under the broad topic of Functions, which also includes graphs, symmetries, derivatives and integrals. The other broad topics in this syllabus are as follows: Numbers, Sequences and Series, Inequalities and Counting. Comparing with Syllabus 9824, the topic on Numbers is new. Although it is new additional content, the amount of materials it covers is substantially less than that in Graph Theory in Syllabus 9810. Further, Numbers builds on the topics of prime and composite numbers and greatest common divisor in secondary mathematics. Nevertheless, students need to learn the formal definitions and properties of divisibility, prime numbers and greatest common divisors, as well as new concepts on congruence and modular arithmetic. That said, the content of Numbers is elementary and well-suited to serve as a means for students to learn mathematical reasoning and proofs. There are also changes to the examination format. The examination paper will consist of eight to ten questions of different lengths, with each question worth 8–16 marks, and candidates will be expected to answer all questions. The scheme of examination paper does not spell out how many questions there are for each broad topic.

5.3.2 A Case Study of a Niche-Area School: NUS High School of Mathematics and Science

The NUS (National University of Singapore) High School of Math and Science (NHSMS, for short) is a specialised independent high school in Singapore offering a six-year Integrated Programme (IP) leading to the NUS High School Diploma. The school offers a highly accelerated mathematics and science curriculum integrated with language, arts, humanities, sports, in a modular system. It is estimated that about 90% of its graduates have pursued Science, Technology, Engineering and Medicine-related courses in University.

5.3.2.1 Academic Curriculum

Although the NUS High School is an Integrated Programme school, which means students need not take the O-levels, it does not offer A-level or International Baccalaureate programmes, unlike other Integrated Programme schools in Singapore. In place of these, an NUS High School Diploma is conferred onto her graduates, and this diploma is recognised by all universities both locally and worldwide by virtue of its high level of academic rigour that is comparable to the above-mentioned qualifications. What makes NHSMS a niche school is its accelerated curriculum for mathematics and science curriculum. Honours courses in the Specialization Stage for mathematical and scientific disciplines are offered to further stretch the academic abilities of able students beyond the already-accelerated curriculum.

The graduation requirement for the NUS High School Diploma mandates that the students take Mathematics and at least two science subjects (including computing studies) at the major (basic) level in the Advancement Stage. Students are given the option to read a fourth subject from any subject group (sciences, humanities and the arts), and take any math/science subject at the honours level. In addition, students must complete an Advanced Research Project under the school's Da Vinci Research Programme. This is a mandatory research curriculum programme that every NHSMS student must go through. The Office of Research, Innovation and Enterprise is the primary body responsible for developing and implementing this research curriculum, with the programme structure given below in Table 5.2. In their senior years, students are encouraged to sit for Advanced Placement and Scholastic Assessment Test (SAT) examinations for credits for admission into foreign universities. Note that these additional sittings of examinations are not part of the graduation requirement.

Talent programmes are a central hallmark of a student's school experience at NUS High School. Apart from the Da Vinci Programme, four other specially featured talent programmes include (IP) Internationalisation Programme (exchange programmes with other math and science schools, Summer Academic Programmes), (E+P) Einstein+Programme (academic mentorship by NUS Professors, Olympiad training programme), (SP) Socrates Programme (for talented students in the humanities) and (AAP) Aesthetic Appreciation Programme.

Table 5.3 Different undergraduate mathematics programmes offered by universities in Singapore

Conferring university	College/department	Programme title	Duration of candidature
Nanyang Technological University (NTU)	National Institute of Education (Mathematics and Mathematics Education Academic Group)	BA/BSc (Ed)	4 years
	School of Physical and Mathematical Sciences (division of mathematical sciences)	B.Sc./B.Sc. (Hons) [Mathematical Sciences]	4 years
National University of Singapore (NUS)	Department of Mathematics	B.Sc. (Hons) with Major in Mathematics (MA)	4 years
Singapore University of Social Sciences (SUSS)	School of Science and Technology	B.Sc. Mathematics	3 years

5.4 Tertiary Mathematics Education in Singapore

At the time of writing this chapter, there are three universities in Singapore offering mathematics at undergraduate level under different programme titles (see Table 5.3). Note that Nanyang Technological University offers two distinct Bachelor of Science degree programmes under the School of Physical and Mathematical Sciences and the National Institute of Education, respectively (Table 5.5).

5.4.1 Programme Structures

5.4.1.1 Nanyang Technological University/National Institute of Education (NIE)

We point the reader to Chap. 15 (Sect. 15.4.1) for the detailed programme structure for B.A/B.Sc. (Ed) offered by NIE. In that section, the reader will also find more information about the *distinctive* quality of the NIE undergraduate programme, i.e. teaching and learning of tertiary mathematics, unlike the other programmes mentioned herein, is guided and shaped by the pedagogical principles as advised by the mathematics educator colleagues of the Mathematics and Mathematics Education Academic Group.

5.4.1.2 Nanyang Technological University/School of Physical and Mathematical Sciences (SPMS)

Based on a social efficiency orientation, the curriculum for the undergraduate mathematics programme in NTU/SPMS is designed with the objective of equipping the graduate with rigorous training needed for the new economy. The approach is also backed up with the belief for continual lifelong learning so that the graduates can be adaptive individuals that can contribute towards the society. Both breadth and depth in knowledge and competencies are emphasised: breadth in knowledge and competency in useful skills such as communication as well as depth in knowledge domains rooted to the discipline of mathematics that is required of a mathematics major. For the Major in Mathematical Sciences (MAS), students will be trained in analytical and reasoning skills, together with problem-solving skills, through the acquisition of rigorous mathematical concepts. Additionally, undergraduate mathematics students are trained in computing, technical communication, and exposed to the interdisciplinary nature of mathematics, especially with other disciplines such as biology, computer science, economics and finance. Deeper investigations in the subject can be taken up by students via special courses, supervised independent study and research projects.

Given the broadness of mathematical sciences, four distinct tracks: (1) Pure Mathematics, (2) Applied Mathematics, (3) Statistics and (4) Business Analytics, which are offered within the Major in Mathematical Science, cater to the varying interests of students. The summarised programme structure of MAS is given in Table 5.4.

For the purpose of comparison, we look more closely at the courses offered for Track (1) Pure Mathematics only (see Table 5.5).

Table 5.4 Summarised Programme Structure for MAS (NTU/SPMS)

Courses	AU	Remarks
MAS Core Courses for all track	48	
MAS Core Courses for a specific track (Pure Math/Applied Math/Statistics)	11	
MAS Prescribed Electives for a specific track, including project (Pure Math/Applied Math/Statistics)	25	A grade of A− or better in the Final Year Project (MH4900, 8 AU) is compulsory for the award of Honours (Highest Distinction)
GER: General Elective Requirement GER Core Courses GER Elective Courses	12 15	
Unrestricted Electives	21	
Total	132	

AU—Academic Unit

Table 5.5 Courses overview for MAS Track (1) Pure Mathematics

Year/courses	AU
Year 1	
Calculus I & II, Linear Algebra I & II, Foundations of Mathematics, Discrete Mathematics, Algorithms and Computing I & II	27
Year 2	
Calculus III, Groups and Symmetries, Algorithms & Computing III, Probability and Introduction to Statistics, Real Analysis I, Ordinary Differential Equations	21
One core course of Track (1)	3–4
Year 3	
Two core courses of Track (1)	7–8
Prescribed electives of Track (1)	See below
Year 4	
Prescribed electives of Track (1)	See below
Note A grade of A- or better in the Final Year Project (MH4900, 8 AU) is compulsory for the award of Honours (Highest Distinction)	
Track in Pure Mathematics (1)	
Courses offered	
Core courses: Complex Analysis, Knots And Surfaces: Introduction To Topology, Abstract Algebra	4/4/3AU
Prescribed electives: List 1: Real Analysis II, Algebraic Topology, Differential Geometry, Continuous Methods List 2: Number Theory, Abstract Algebra II, Set Theory and Logic, Algebraic Methods List 3: Coding Theory, Cryptography, Combinatorics, Discrete Methods, Algorithms and Theory of Computing, Algorithms for the Real World List 4: Final Year Project, Professional Internship	4 AU each 8AU/11AU

AU—Academic Unit

5.4.1.3 National University of Singapore/Department of Mathematics

The B.Sc. (Hons) with Major in Mathematics (MA) is advertised as the flagship major that any leading university of the world is obliged to offer. The objective of the programme is to expose students to all the important areas of mathematical knowledge including algebra, logic, number theory and combinatorics, real and complex analysis, differential equations, geometry and topology with focus on mathematical foundations and fundamental techniques. The prerequisite to the programme is a pass in the 'A'-level H2 Mathematics; a lack of basic background may be made up for by reading a certain 'bridging' module pegged at Module Level 1000.

To graduate with a B.Sc. (respectively, B.Sc. (Hons)) with primary major in Mathematics, a student must complete a total of 120 (respectively, 160) Modular Credits

Table 5.6 NUS B.Sc./B.Sc. (Hons) programme degree requirement

Module level	Major requirements	Cumulative MCs
1000	Fundamental concepts of Mathematics or Discrete Structures Linear Algebra I, Calculus, Programming Methodology	16
2000	Linear Algebra II. Multivariate Calculus, Mathematical Analysis I Algebra I, Probability One additional module from List II, III, IV	40–44
3000	Mathematical Analysis II, Complex Analysis I, Two modules from List MA3 Pass one additional module from List III, IV	60–66
4000	Honours Project in Mathematics Four modules from List MA4 Pass one additional module from List IV	92–98
Lists II, III and IV are not available on the public domain		
List MA3: Algebra II, Set Theory, Mathematical Analysis III, Ordinary Differential Equations, Introduction to Number Theory, Introduction to Fourier Analysis		
List MA4: Galois Theory, Mathematical Logic, Functional Analysis, Partial Differential Equations, Complex Analysis II, Measure and Integration, Topology, Differential Geometry of Curves and Surfaces		

MC—Modular Credit

(MC) of courses, inclusive of 20 MC of university requirements, 4–8 (respectively, 4–12) MC of faculty requirements, 60–66 (92–98) MC of major requirements and 26–36 (respectively, 30–44) MC of free electives. Furthermore, the major requirements in Table 5.6 must be satisfied.

5.4.1.4 Singapore University of Social Science (SUSS)/The School of Science and Technology

The SUSS mathematics programme offers graduates a rigorous and broad foundation in the three main pillars of pure mathematics, applied mathematics and statistics. The programme aims to have her graduates explore in greater depth any of a combination of these three important pillars via a range of elective courses that includes abstract algebra, financial mathematics, mathematical modelling, mathematics in computing, mathematical logic, number theory, probability and statistics.

An interesting and seemingly attractive feature of the programme in SUSS is that all foundational mathematics is re-examined and reviewed within the compulsory core Level 1 mathematics courses. This facilitates anyone who meets the general university entry requirement with an interest in learning mathematics to be eligible for the programme; in particular, an ‘A’-level pass in H2 Mathematics is *not* a prerequisite of the BSc Mathematics programme.

To graduate with a basic degree, students are required to complete a total of 130 Credit Units (CU) of courses, inclusive of 10 CU of university core courses. The breakdown of the CU's to be completed for the BSc Mathematics programme is as follows: (1) 70 CU of Compulsory Courses; (2) 50 CU of Elective Courses; and (3) 10 CU of University Core Courses. The curriculum has a three-tier structure. Level 1 courses comprise a basic suite of four courses covering all aspects of foundational mathematics and statistics. Level 2 courses consist of a set of core courses in pure mathematics, applied mathematics and statistics that will prepare students for higher level mathematics courses, together with a number of elective courses in financial mathematics, mathematics in computing and computer programming in C++. Level 3 courses consist of a collection of advanced elective courses such as graph theory, complex analysis, optimisation, logic, number theory and applied probability, where students can choose courses to suit their own interests and abilities.

5.4.2 University Professors' Viewpoints on the Changes in Tertiary Mathematics

It is not our purpose here to compare the different programme structures of the above lists of undergraduate degree programmes since we trust that it is the *substance* of the programme rather than its *structure* which makes the difference in the quality of programme. Here, the 'substance of the programme' is characterised by the manner and quality of the teaching and learning that take place in the undergraduate mathematics courses offered under each programme. Based on this view, we believe it is more meaningful to interview professors, seven in total, who have taught or are currently teaching mathematics courses offered under these programmes in the aforementioned three universities. It is hoped that their responses will offer insights into the way tertiary mathematics is imparted to the mathematics majors by these interviewees. Our ensuing analysis of the interview data uses a qualitative approach. Admittedly, the small number of professors yields data that are far from being representative of the general approach taken at the respective universities. Nevertheless, what we compromise for numbers we make up by the rich teaching experience of these professors—the minimum being 12 years; the maximum 57 years. Table 5.7 summarises the profiles of the seven interviewees, whom we label as Professors A-G.

In the remaining of this subsection, we shall summarise the information gathered from the interview data based on the inputs of the professors we interviewed. The information is categorised accordingly to (a) the major changes in tertiary mathematics education system and their objectives and (b) the future of tertiary mathematics education, with special focus on whether schools students are ready to read mathematics at university level and/or to become mathematicians.

Table 5.7 Profiles of the seven interviewees (Professors A-G)

Professor	Universities Taught/Teaching	Teaching experience (Years)	Undergraduate math courses taught	Teaching Philosophy (P)/Teaching Approach (A)
A	NTU/NIE	19	Graph Theory Number Theory Computational Math	P: Guide students to understand and be infused by the disciplinarity of mathematics—rigour, proof, problem-solving, beauty A: Awareness of the different capabilities of students and adjust accordingly
B	NUS NTU/NIE	57	All undergraduate courses	P: Teach students to have sharp observation and critical analysis A: Mindful of pitching teaching at different levels for different students
C	NTU/SPMS	12	Discrete math Real analysis Abstract algebra	P: Motivate students in thinking and practicing mathematics A: Use teaching methods that target on students' motivation and interest levels
D	NTU/NIE NTU/SPMS	19	Calculus Multivariate Calculus Real Analysis Complex Analysis	P: Teaching and learning is a social activity and involves intellectual exchange, where the teacher and the students have to commit their attention so that meaningful learning takes place A: Work out details in classes; not keen about using power-point
E	NUS NTU/NIE	34	Calculus (Engineer) Analysis Measure Theory Functional Analysis Mathematical Methods	P: Awareness of students' background knowledge; progress from fundamental concepts; go deep rather than broad A: Start with examples and end with examples/counterexamples; use whiteboard to teach small class; power-point for mass lectures; use computer animations

(continued)

Table 5.7 (continued)

Professor	Universities Taught/Teaching	Teaching experience (Years)	Undergraduate math courses taught	Teaching Philosophy (P)/Teaching Approach (A)
F	NUS	21	Fundamental Concepts of Mathematics Calculus (Engineer) Linear Algebra Multivariate Calculus Quantitative Reasoning	P: Help students transit smoothly from A-level to tertiary mathematics; changing students' mindset/attitude towards learning math A: Engaging students intellectually in class; teacher–students and student–student interactions
G	NUS SSU	45	Calculus Advanced Calculus Linear Algebra Modern Algebra Combinatorics Graph Theory Discrete Math	P: Teaching is an interaction between the teacher and his/her audience; this meaningful interaction contributes to the growth of both parties A: Present abstract ideas in concrete or geometrical ways; use historical remarks/picture of mathematicians to motivate topics; talk to students: obtain feedback on teacher's teaching and students' difficulties

5.4.2.1 From Past to Present

Finding out from the various professors what the major changes took place in the respective university mathematics department in terms of undergraduate programme in mathematics is a key step in our current undertaking to understand what goes beyond school mathematics, particularly what is the tertiary mathematics education landscape like. The interview question below was intended to tease out exactly this required information:

> What are the major changes in the tertiary education system over the last 10-20 years in the university stated you are teaching in, in terms of an undergraduate programme in mathematics? Be more specific in terms of the description, e.g., change in course structures, assessment modes, modular system, honors-year thesis, etc., and the estimated year of occurrence.

The interview responses all revealed both changes to the programme structures and to the ways degree programme courses in mathematics were taught, as well as the manner in which students were assessed.

Let us note the changes in programme structure that took place in the National Institute of Education over the last two decades. According to Professor A, three major programmatic changes occurred at NIE in around 1998, 2003, 2008 and most recently in 2017. While these changes might be brought about due to direct impact from new education policies made and perhaps be justified or understood at a programmatic level, additional insight can be obtained by matching the years in which these changes took place (1998, 2003, 2008 and 2014) with the years in which a revision of the mathematics syllabus at 'A'-level took place (1997, 2001, 2006, 2016). With the exception of the 2014 programme revamp, a major structural change in the degree programme at NIE occurred one or two years *after* a major revision in the 'A'-level Mathematics syllabus. This approximate correspondence is an indication that the decision making at the policy level in NIE was *responding just in time* to important educational policy changes that took place at the school level in much the same way as schools responding to the major changes in the educational directives (e.g. TSLN, TLLM and ICT Masterplans) initiated by the government through the MOE. In 1997, a new 'interim' Mathematics Syllabus C (9205) was introduced which was applicable to Singapore 'A'-level candidates only. In this revised syllabus, more emphasis was put on *higher-order thinking (H.O.T.) questions*. Responding to this was the major change in NIE that took place around 1998 which saw more hours pumped into the AS component of the programme to equip students with a more complete coverage of content mathematics at the tertiary level.

In 2001, the new Mathematics Syllabus 9233 was introduced with the main aim of reducing the content, freeing up for more space in thinking and infusing Information Technology into teaching and learning of mathematics. Graphing Calculators made their first appearance in the Further Mathematics Syllabus. Interestingly, this cut in content at the 'A'-level brought about a similar change in the mathematics degree programme at NIE as witnessed by Professor A:

> The most major changes are the *loss of the* 5^{th} *Honours year* (and the academic exercise) for Maths majors (2008), and … – Prof. A.

We now saw that the 'emphasis more on other aspects of teacher education' was in actuality a systems response to the changing demands in the school education landscape, namely, higher-order thinking and using Information Technology.

The year 2006 saw the removal of the Further Mathematics as an 'A'-level subject, and the H2 Mathematics Syllabus introduced put emphasis on solving real-world problems, i.e. solving a problem and interpreting its solutions. To ensure that the student teachers acquire more content knowledge to meet with the aforementioned change in educational emphasis in schools, the NIE degree programme for mathematics experienced a third major change in 2008 with an increase in the number of Academic Units for AS courses but the Academic Exercise did not return until later:

The fourth change in the NIE degree programme for mathematics was noted by Prof. A as follows:

> … and the 'revival' of the proportionate emphasis on AS1 (51 AUs compared to 39 AUs) with the academic exercise (2014). – Prof. A.

This change unlike other previous changes did not take place after a change in the Mathematics syllabus at ‘A’-level. Crucially, this 2014 change brought about a renewed emphasis on content mathematics and the Academic Exercise was reinstated in the fourth year of the degree programme.

In the NUS, degree programme in mathematics has undergone several changes. Notably, in around 2005, a group-work approach to the Honours Year Project started (as opposed to individual work in the past). Concerning the reduction of content, Professor F has the following to say:

> Reduced syllabus - Over the years, there have been a rebalance between the breadth and depth in university education, with a *reduction of the major program requirement*, and an *increase in the general education requirement*. – Prof. F.

In both NTU/SPMS and SUSS, an emphasis was placed on content mathematics through Honours Year Thesis; for instance:

> For honor’s projects, we have one-year FYP projects, to make sure that students can have deep understanding of the research area.- Prof. C
>
> Introducing more applied modules & honors-yr thesis – Prof. G.

Apart from the changes in the programme structures of mathematics undergraduate programmes, we also see changes that cater to the need of using technology in teaching and learning. We may interpret such changes as the impact of the ICT Masterplans rolled in the period 1997–2009. Three professors made special mention about the use of technology in their tertiary mathematics teaching and related concerns:

> Use of ICT in teaching - Both top down and ground up; this is a natural trend with the advancement of technology and new generation of learners. – Prof. F.
>
> “We provide chances for students to use Mathematica and Matlab in our teaching of calculus and linear algebra, since year 2005.” – Prof C.
>
> I think the main change is the push of using TEL. It aims to use technology to enhance learning. But I believe that some struggles are still necessary. Over reliance on technology to relief the growing pains may end up not growing at all. Google and other search engine also brought forth an important change in how students obtain information. It used to be hard to get information (e.g., proof of a theorem) but now it is readily available. – Prof. D.

Looking for new pedagogies/methods of delivering mathematics at the tertiary level has now received more attention than in the past. Teacher belief in this aspect has also started to change:

> I relied on the use of mathematics software to illustrate concepts especially exploiting computing animations. – Prof. E.
>
> Blended learning/flipped classroom - More traditional lectures are being replaced by recorded video. Students come to classes for hands on, practical, group discussion. – Prof. F.

Changes also took place in the form of assessment; with regard to assessment, many of the interviewees put forth their views (and sometimes quite different within the same institution):

> In Singapore, the major changes took place in 1971 and in the last 20 years. In 1971, it was the introduction of new courses. Recently, it was *assessment*. Roughly, the *change in assessment* is from the British system to the American. – Prof. B.
>
> For my course on Abstract Algebra II, students have presentations on some topics they are interested, and presentation is part of the assessment, since 2014. – Prof. C.
>
> Grade-free modules - Probably something unique in NUS; students in their first year have the option of not counting the grade of the module, but opt for satisfactory/unsatisfactory.
>
> Open book exam – Though this has become more common in university exam to discourage students from memorizing, not many examiners are adopting it. It is more common for students to bring help sheets to math exams. – Prof. F.
>
> less closed-book exams – Prof. G.

Let us summarise what we have heard from the seven mathematics professors. Common to the responses of all the interviewees is the phenomenon of constant change in the tertiary mathematics education landscape. These changes usually took place at the university level as a response to significant initiative changes that occurred at the school level—in particular, about two years after a major revision in the 'A'-level Mathematics syllabus. Such changes in the tertiary mathematics education landscape ranged from structural changes in undergraduate degree programmes to the manner mathematics was taught and assessed at the university. We also see the impact the ICT Masterplans had on university teaching as mathematics professors looked for innovative ways to convey mathematical ideas to students by relying on computer and video technology. For an elaboration on the use of ICT in teaching and learning of mathematics at all levels in Singapore, the reader is pointed to Chap. 12. These changes have their repercussions whether for better or for worse as pointed out by Professor A below:

> Both changes significantly affected the ability of maths majors to get deeper into the disciplinarity, the first adversely [referring to the "C" series] and the latter, positively [referring to the new "A" series]. Time and content for reading and writing mathematics were affected. – Prof. A.

5.4.2.2 From Present to Future

In the interviews, all the professors indicated that content reduction is one of the most significant changes that took place at the university level for undergraduate mathematics degree programmes; generally, the coverage of pure mathematics at the tertiary level changed as a response to the content reduction in the 'A'-level Mathematics syllabus and has been reduced over the years. This then begs the following questions:

(a) What will the future of tertiary mathematics education in Singapore look like?
(b) Are the younger generation (school students) better prepared to read mathematics at the university level?
(c) Are they better prepared to become mathematicians in the future?

Here, we have three camps: the optimistic, the realistic and the less-than-optimistic.

The *optimistic* camp holds the view that the mathematics education students receive at school equips them sufficiently so that they may, if the situation allows, take up mathematics as a career.

> Tertiary mathematics education can provide a platform for students to learn math-related subjects, like finance, business analytics, modelling, etc. Quite a lot of students from our division find their jobs in financial industry. Yes. Students from JC are well-prepared to read math at university level.—Prof. C.

Optimism sometimes comes with bold creativity in that mathematics need to be redefined in order to enlarge the scope of its meaning. By so doing, this allows ones to see that many other skills and knowledge domains need to be imparted to students at school so that they can become a new generation of mathematicians:

> We need to redefine mathematics and mathematician. We no longer need to produce the same kind of mathematicians. I believe the same breeds are equally good. – Prof. B.

Most interviewees recognised that it is only *realistic* that not all people become mathematicians, and so tertiary mathematics education should not be solely aimed at producing mathematicians. Indeed, many careers call for analytical skills, problem-solving skills and logical reasoning which are expected attributes of mathematics graduates. A point to add here is that STEM students, in some of these universities, have the option to read Mathematics as a second major, and so tertiary mathematics education ought to be more inclusive to cater for the needs of this new group of students.

> I think students now are no longer the same as what we had. *Not many reading mathematics intend to be research mathematicians.* Nonetheless, there are some good mathematicians who came from other backgrounds. So I am not too dogmatic about that. – Prof. D.
>
> The trend seems to be students are becoming more "pragmatic" and choose applied math over pure math. There will still be a small group of students who will go for pure math but majority will choose to do applied math. I am not worried about this. What I hope to see is for more STEM students to do math as their second major (if their first major is science, computing, engineering etc.) to build a stronger foundation for their analytical skill. – Prof. F.

Awareness of the wide difference between making mathematics available for the majority and training the mathematical elite to be researchers in mathematics, the challenge here is how tertiary mathematics education can position itself in middle ground. Professor A proposed a realistic opinion about this:

> The future looks like it will be severely bifurcated – mathematics for applications for the majority and 'hard-core' research publishable mathematics for the elite. I think this is happening in NUS and NTU/SPMS. I hope that it will not happen in NTU/NIE – attempts are being made to review the curriculum to achieve the objective of a mathematics major who can read and write maths, can tackle unfamiliar problems, is exposed to the 'canon' of mathematics, can code, and have a positive attitude towards mathematics. – Prof. A.

Some interviewees held a *less-than-optimistic* view about the future of tertiary mathematics education, as far as the undergraduate degree programme is concerned.

I don't think the A-level mathematics syllabus prepares student sufficiently for mathematics education at the tertiary level. Not enough rigour. – Prof. E.

I hope with the return of 'A' level further math, students entering university will be better prepared. In general no [answering (b)]. Our education caters to the mass, which aims at equipping students enough math skills for the job market. To be mathematicians, this requires more in depth training, and only selected few with the passion and aptitude will make it. – Prof. F.

No for the second [answering (a)] and third [answering (b)] questions. – Prof. G.

5.4.3 Interpretation of Findings About Tertiary Mathematics Education in Singapore

From the classification of the interview responses by the professors, we see how education initiatives and policies have their impact, through the different school levels ('O'-level and 'A'-level education), on tertiary education—focusing on mathematics as a subject at school level and Mathematics as a discipline at the tertiary level. Shaping the tertiary mathematics took the form of policy-driven changes in undergraduate programme structures as well as the self-directed changes in teachers' beliefs which later translate into various classroom implementation, e.g. alternative pedagogies and the use of technology in teaching and learning tertiary mathematics. We have learnt from the responses of the eight professors that teaching style has slowly moved from chalk-and-talk to more student-centred learning. Putting on our curriculum ideological lens, it is not difficult to see that a trend shift has taken place at the tertiary mathematics education landscape: there is a significant shift from Scholar Academic through Social Efficiency to Learner Centred; following more or less a similar movement as that in schools.

Perhaps this finding is not surprising if one considers the output of schools to be the students who graduated from the school system, having attained the intended level of content knowledge and skills in mathematics. The mathematical competency of these students who have graduated from schools and ready to enter university is a part of the Learned Curriculum (as opposed to the Intended Curriculum articulated in the detailed syllabus). We see that the changes at the tertiary level are ways in which the university, as a system, handle the effects of the changes that took place upstream, i.e. at the school level.

However, this is only one direction of the flow. Now this is where the NIE stands in contrast to the rest of the other local universities as far as tertiary mathematics is concerned. As the sole teacher training institute, NIE is responsible of ensuring a high-quality teaching force is ready to be feedback into the school system. As such, mathematics student teachers must be master of *both* mathematical content knowledge and pedagogical knowledge. In the next subsection, we shall look at a snapshot of how MME makes use of teaching innovations which are backed by sound pedagogical theories to enhance teaching and learning in an undergraduate core course in mathematics.

5.5 Conclusion

In this chapter, we have attempted to paint the two sides of the tertiary mathematics education landscape in Singapore: the preparatory side at the pre-university level and teaching and learning of mathematics at the university level. Through the lens of curriculum ideologies, we have begun to understand the observed trend, i.e. there is an evident shift of the curriculum orientation from Academic Scholar through Social Efficiency to Learner Centred. Changes at the school level and at the university level are a manifestation of systemic response to the changing demands of the society and the world through the seemingly more direct ‘top-down’ impact of new educational initiatives.

For mathematicians-educators (mathematicians who are passionate about mathematics education) in NIE, it is perhaps time for us to reflect on what we, as teachers at the tertiary level, want and what the society needs insofar as mathematics learning is concerned. Not everyone needs to be a mathematician or even needs to love mathematics. Not every student needs to excel in mathematics, not every student who excels in mathematics needs to major in mathematics, and not every mathematics major needs to end up as a research mathematician. However, it will be in our interest to see that as long as mathematics is taught in the schools, for whatever purpose, it is taught correctly and in the way and spirit that it should be taught.

At a round-table discussion at the ICIAM 2003 (The International Congress on Industrial and Applied Mathematics) in Sydney, Australia, in response to a heated argument on why Australian students coming to the universities were not well prepared mathematically, the late Professor Renfrey Potts (1925–2005) stood up to say, ‘It is the duty of *every mathematician* at the universities of this country to help and make sure that mathematics is taught right in the schools.’s We believe tertiary mathematics education has this to take care of, especially at NIE.

Appendix

An exemplar for creating learning experience in H2 Mathematics Syllabus 9758

Complex Numbers

Lesson Objective

Based on an ‘old’ idea of $C + iS$, this learning experience involves the students to create the imaginary counterpart of a sinusoidal voltage function across a resistor arising from an alternating current source. By so doing, the students reinvent the phasor of the voltage function, which takes advantage of the vector nature of complex numbers, and exploit it to calculate the resultant voltage function that results from adding in series two alternating current sources that are not necessarily in phase. Engineers use this method, called *phasor analysis*, to think and reason about alternating current voltages, and related quantities.

Problem

An alternating current source has the following voltage function:

$$V_1 = 3\cos\left(2t + \frac{\pi}{4}\right),$$

where V_1 is the voltage (V) across a given resistor, and t is the time lapsed (s) since the source was turned on.

Another alternating current source whose voltage function is given by

$$V_2 = 4\cos\left(2t + \frac{\pi}{6}\right)$$

is now placed in series with the above-mentioned source so that the resultant voltage is calculated by their sum:

$$V_1 + V_2.$$

What is the amplitude and the period of the resultant voltage?

Mathematical Content Knowledge

For the first voltage V_1, we create the imaginary sine counterpart of the function $3\sin\left(2t + \frac{\pi}{4}\right)$ and construct the complex voltage function:

$$3\left(\cos\left(2t + \frac{\pi}{4}\right) + i\sin\left(2t + \frac{\pi}{4}\right)\right) = 3e^{i\left(2t+\frac{\pi}{4}\right)}.$$

Similarly, for the second voltage V_2, we have the complex voltage function:

$$4\left(\cos\left(2t + \frac{\pi}{6}\right) + i\sin\left(2t + \frac{\pi}{6}\right)\right) = 4e^{i\left(2t+\frac{\pi}{6}\right)}.$$

Now, we sum these two complex voltages together:

$$3e^{i\left(2t+\frac{\pi}{4}\right)} + 4e^{i\left(2t+\frac{\pi}{6}\right)}.$$

A preliminary investigation using a GCs reveals that the above sum can be reduced to a single trigonometric function.

From the vector geometry of complex numbers, one can show rigorously that $3e^{i\left(2t+\frac{\pi}{4}\right)} + 4e^{i\left(2t+\frac{\pi}{6}\right)} = \mathrm{R}e^{i(2t+\alpha)}$, where $R = \sqrt{\left(4\sin\frac{\pi}{6} + 3\sin\frac{\pi}{4}\right)^2 + \left(4\cos\frac{\pi}{6} + 3\cos\frac{\pi}{4}\right)^2}$ and $\tan\alpha = \frac{4\sin\frac{\pi}{6}+3\sin\frac{\pi}{4}}{4\cos\frac{\pi}{6}+3\cos\frac{\pi}{4}}$, which in particular are independent of t.

Further exploration

The phasor addition works because the two voltages are of the same angular frequency. A natural question to ask is how one can tackle the case when the angular frequencies are different. Use a GCs to investigate this situation.

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Chapter 6
Singapore's Participation in International Benchmark Studies—TIMSS, PISA and TEDS-M

Berinderjeet Kaur, Ying Zhu and Wai Kwong Cheang

Abstract Large-scale international assessments of schooling effects attempt to provide comparative data for participating countries. Two such assessments are the Trends in International Mathematics and Science Study (TIMSS) conducted by the International Association for the Evaluation of Educational Achievement (IEA) and the Programme for International Student Assessment (PISA) conducted by the Organization for Economic Co-operation and Development (OECD). Singapore has participated in TIMSS since 1995 and PISA since 2009. These studies use student outcomes as measures of school effectiveness and educational achievement. They focus on student achievement mainly in three school subjects: mathematics, science and language. Other international studies like the Teacher Education and Development Study in Mathematics (TEDS-M) also provide comparative data on teachers of mathematics and related matters. Singapore participated in TEDS-M. The results of TEDS-M were available in 2012. This chapter presents snapshots of significant data and findings of Singapore's participation in TIMSS 2015, PISA 2009 and 2015 and TEDS-M. For TIMSS 2015, it focuses on the performance of Singapore students and their engagement and attitudes for mathematics. For PISA 2009 and 2015, it focuses on the performance of Singapore students and their exposure to mathematics content and their drive and motivation to learn mathematics. For TEDS-M, it focuses on the national contexts and policies for teacher education and nature of mathematics teacher education programmes in Singapore. It also examines the performance of future teachers from Singapore in mathematics content knowledge (MCK) and mathematics pedagogical content knowledge (MPCK) and their beliefs and perceptions of opportunities to learn. The chapter concludes with possible reasons about the commendable performance of Singapore students in TIMSS and PISA.

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Keywords International benchmark studies · Mathematics · PISA · Singapore students · Singapore mathematics teachers · TEDS-M · TIMSS

6.1 Introduction

Large-scale international assessments of schooling effects attempt to provide comparative data for participating countries. Two such assessments are the Trends in International Mathematics and Science Study (TIMSS) conducted by the International Association for the Evaluation of Educational Achievement (IEA) and the Programme for International Student Assessment (PISA) conducted by the Organization for Economic Co-operation and Development (OECD). Singapore has participated in both of them. These studies use student outcomes as measures of school effectiveness and educational achievement. They focus on student achievement mainly in three school subjects: mathematics, science and language. Singapore participates in TIMSS and PISA for four main purposes, which according to Kaur (2013b) are as follows:

- to benchmark the outcomes of schooling, vis-à-vis the education system against international standards;
- to learn from educational systems that are excelling;
- to update school curriculum and keep abreast of global advances; and
- to contribute towards the development of excellence in education internationally.

Other international studies like the Teacher Education and Development Study in Mathematics (TEDS-M) also provide comparative data on teachers of mathematics and related matters. Singapore participated in TEDS-M.

This chapter presents snapshots of significant data and findings of Singapore's participation in TIMSS 2015 (Mullis et al. 2016), PISA 2009 (OECD 2010a), PISA 2012 (OECD 2013a), PISA 2015 (OECD 2015) and TEDS-M (Tatto et al. 2012). For TIMSS 2015, it focuses on the performance of Singapore students and their engagement and attitudes for mathematics (Mullis et al. 2016). For PISA it focuses on the performance of overall Singapore students in PISA 2009, 2012 and 2015 and specifically for PISA 2012 the performance of students from Singapore on some released sample items and students' motivation to learn mathematics (OECD 2013a). For TEDS-M, it focuses on the national contexts and policies for teacher education and nature of mathematics teacher education programmes in Singapore. It also examines the performance of future teachers from Singapore in mathematics content knowledge (MCK) and mathematics pedagogical content knowledge (MPCK) and their beliefs and perceptions of opportunities to learn (Tatto et al. 2012). The chapter concludes with possible reasons about the commendable performance of Singapore students in TIMSS and PISA.

6.2 Trends in International Mathematics and Science Study (TIMSS)

Trends in International Mathematics and Science Study (TIMSS) is a series of international mathematics and science assessments conducted every four years by the International Association for the Evaluation of Educational Achievement (IEA). TIMSS is designed to provide trends in fourth- and eighth-grade mathematics and science achievement in an international context. TIMSS 2015 was the sixth and most recent cycle of assessment. Forty-six countries participated at the eighth-grade level, and 56 countries participated at the fourth-grade level. Data were collected from representative samples of students, in participating countries, at the respective grade levels. However, the teacher participants may not constitute representative samples as they were the teachers of the students. The TIMSS 2015 International Results in Mathematics (Mullis et al. 2016) contain analysis of data that spans from achievement of participants to home environment that supports mathematics and science achievement, school resources for teaching mathematics and science, school climate, teacher preparation and classroom instruction. Singapore has participated in all the six cycles of TIMSS so far. Several publications have focused on the performance of Singapore's students in TIMSS 1995, 1999, 2003, 2007 and 2011 (Kaur 2005, 2009a, b, 2013a; Boey 2009; Kaur et al. 2012, 2013).

In this section, we focus on the performance of Singapore students and their engagement and attitudes for mathematics in TIMSS 2015. The data and findings reported in this chapter are drawn from the respective international mathematics reports of TIMSS 2015 (Mullis et al. 2016), TIMSS 2011 (Mullis et al. 2012) and TIMSS 2007 (Mullis et al. 2008). All of these reports are available at the IEA TIMSS and PIRLS International Study Centre website (http://timssandpirls.bc.edu).

6.2.1 Performance of Singapore Students in TIMSS 2015

The performance of Singapore students in TIMSS in the six cycles held so far has been consistently outstanding and captured the attention of many educators and politicians worldwide. Table 6.1 shows the rank of Singapore in the last six cycles of TIMSS for both grades 4 and 8.

The international benchmarks presented as part of the TIMSS data help to provide participating countries with a distribution of the performance of their students in an international setting. For a country, the proportions of students reaching these benchmarks are perhaps telling of certain strengths and weaknesses of mathematics education programmes of the country. The benchmarks delineate performance at four points of the performance scale. Characteristics of students at each of the benchmarks are shown in Fig. 6.1.

Students who participated in TIMSS 2015 at the grade 8 level are from the same cohort of grade 4 students who participated in TIMSS 2011. Similarly, the 8th graders

Advanced International Benchmark - 625	
Grade 4 *Students can apply their understanding and knowledge in a variety of relatively complex situations and explain their reasoning*. They solve a variety of multi-step word problems involving whole numbers. Students at this level show an increasing understanding of fractions and decimals. They can apply knowledge of a range of two- and three-dimensional shapes in a variety of situations. They can interpret and represent data to solve multi-step problems.	Grade 8 *Students can apply and reason in a variety of problem situations, solve linear equations, and make generalizations*. They can solve a variety of fraction, proportion, and percent problems and justify their conclusions. Students can use their knowledge of geometric figures to solve a wide range of problems about area. They demonstrate understanding of the meaning of averages and can solve problems involving expected values.
High International Benchmark - 550	
Grade 4 *Students can apply their understanding and knowledge to solve problems.* They can solve word problems involving operations with whole numbers, simple fractions, and two-place decimals. Students demonstrate understanding of geometric properties of shapes and of angles that are less than or greater than a right angle. Students can interpret and use data in tables and a variety of graphs to solve problems.	Grade 8 *Students can apply their understanding and knowledge in a variety of relatively complex situations*. They can use information to solve problems involving different types of numbers and operations. They can relate fractions, decimals, and percentages to each other. Students at this level show basic procedural knowledge related to algebraic expressions. They can solve a variety of problems with angles including those involving triangles, parallel lines, rectangles, and similar figures. Students can interpret data in a variety of graphs and solve simple problems involving outcomes and probabilities.
Intermediate International Benchmark	
Grade 4 *Students can apply basic mathematical knowledge in simple situations*. They demonstrate an understanding of whole numbers and some understanding of fractions and decimals. Students can relate two- and three-dimensional shapes and identify and raw shapes with simple properties. They can read and interpret bar graphs and tables.	Grade 8 *Students can apply basic mathematical knowledge in a variety of situations.* They can solve problems involving negative numbers, decimals, percentages, and proportions. Students have some knowledge of linear expressions and two- and three-dimensional shapes. They can read and interpret data in graphs and tables. They have some basic knowledge of chance.
Low International Benchmark	
Grade 4 *Students have some basic mathematical knowledge*. They can add and subtract whole numbers, have some understanding of multiplication by one-digit numbers, and can solve simple word problems. They have some knowledge of simple fractions, geometric shapes, and measurement. Students can read and complete simple bar graphs and tables.	Grade 8 *Students have some knowledge of whole numbers and basic graphs*.

Fig. 6.1 Descriptions of the TIMSS 2015 International Benchmarks. *Source* Mullis et al. (2016) Exhibits 2.1 and 2.8

Table 6.1 Ranking of Singapore's students for Mathematics in TIMSS

TIMSS	Rank	
	Grade 4	Grade 8
1995	1	1
1999	–	1
2003	1	1
2007	2	3
2011	1	2
2015	1	1

Source http://timssandpirls.bc.edu

Table 6.2 Percentage of Singapore students in last three cycles of TIMSS at the respective benchmarks for mathematics achievement

TIMSS	Grade	International benchmarks			
		Advanced (625)	High (550)	Intermediate (475)	Low (400)
2015	4	50 (2.1)	80 (1.7)	93 (0.9)	99 (0.3)
2011	4	43 (2.0)	78 (1.4)	94 (0.7)	99 (0.2)
2007	4	41 (2.1)	74 (1.7)	92 (0.9)	98 (0.3)
2015	8	54 (1.8)	81 (1.5)	94 (0.9)	99 (0.2)
2011	8	48 (2.0)	78 (1.8)	92 (1.1)	99 (0.3)
2007	8	40 (1.9)	70 (2.0)	88 (1.4)	97 (0.6)

( )—standard errors
Source Mullis et al. (2016) Exhibits 2.3 and 2.10

in TIMSS 2011 were from the same cohort of 4th graders in TIMSS 2007. Table 6.2 shows the percentage of grade 4 and 8 students from Singapore at the benchmarks for the past three cycles of TIMSS, namely TIMSS 2007, TIMSS 2011 and TIMSS 2015.

It is apparent from Table 6.2 that as the cohorts of students progressed from 4th to 8th grade, higher proportions of the students reached the advanced international benchmark. 41% of grade 4 students at the advanced international benchmark in TIMSS 2007 compared to 48% grade 8 at the same benchmark in TIMSS 2011 and 43% grade 4 at the advanced international benchmark in TIMSS 2011 compared to 54% grade 8 at the same benchmark in TIMSS 2015. Table 6.2 also shows that percentages of grade 4 and 8 students reaching the high and advanced benchmarks have steadily increased over the last three cycles of TIMSS. The periodic revisions of the school mathematics curriculum from the year 2000 onwards placing heightened emphasis on problem-solving and mathematical processes such as thinking skills and reasoning appear to have contributed towards improved student learning of mathematics (Ministry of Education 2016a).

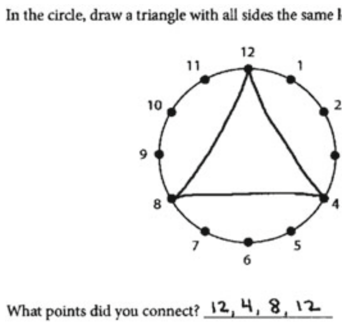
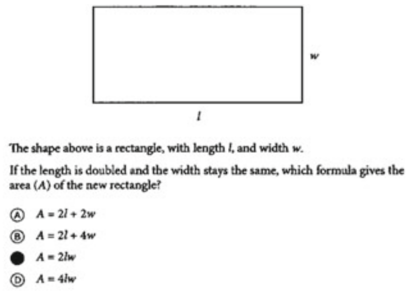
Grade 4	Grade 8
Content domain: Geometric Shapes and Measures Cognitive domain: Reasoning Description: Draws a specified geometric shape by connecting dots on a circle	Content domain: Algebra Cognitive domain: Applying Description: Identifies the formula that represents a situation involving area
	
Percent full credit: 64 (1.8) International average: 58 (0.3)	Percent full credit: 82 (1.6) International average: 51 (0.3)

Fig. 6.2 Examples of items of the High International Benchmark Level. *Source* Mullis et al. (2016) Exhibits 2.6.3 and 2.13.2

However, for the low international benchmark level, the proportion of students reaching it improved by 1% from 2007 to 2011 but remained the same at 99% from 2011 to 2015. These findings have been of concern to policy makers and educators in Singapore. It may be said that the revisions of the curriculum have had limited impact on these students. Since 2013, teachers of low attainers in mathematics have received additional support in the form of resources and self-development (see Chap. 13 for details).

Figures 6.2 and 6.3 show items of the High International Benchmark and Advanced International Benchmark Levels, respectively, for TIMSS 2015. For each item in the figures, the per cent correct for Singapore and the international average are stated. The grade 4 item, shown in Fig. 6.2, is a non-routine and challenging one for 4th graders in Singapore. Study of circles and equilateral triangles is beyond the scope of the mathematics curriculum in grade 4. As such, Singapore's 4th graders performed reasonably well on the item. Their counterparts from Republic of Korea (76%) and Japan (73%) did better than them. The grade 8 item, shown in Fig. 6.2, may be said to be a routine one for 8th graders in Singapore schools. Students from Singapore were ranked the best for the item.

Figure 6.3 shows items of the Advanced International Benchmark Level. In the figure, the grade 4 item is a non-routine one for Singapore's 4th graders. The multi-step word problem is a higher-order thinking task. Nevertheless, the students performed reasonably well on it and were ranked fourth. Their counterparts from Republic of Korea (77%), Hong Kong SAR (71%) and Japan (66%) did better than them.

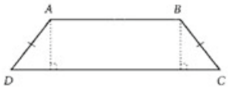
Grade 4	Grade 8
Content domain: Number Cognitive domain: Reasoning Description: Solves a multi-step reason problem involving division Sally has 12 lengths of wire, 40 round beads, and 48 flat beads. She uses 1 length of wire, 10 round beads, and 8 flat beads to make 1 bracelet. If Sally makes all her bracelets the same, how many bracelets can she make? Ⓐ 40 Ⓑ 12 Ⓒ 5 ● 4 Percent full credit: 65 (2.1) International average: 37 (0.3)	Content domain: Geometry Cognitive domain: Reasoning Description: Uses the Pythagorean theorem in finding the perimeter of a trapezoid  *ABCD* is a trapezoid with $AB = 10$ cm and $CD = 16$ cm. $AD = BC$. The distance between the parallel lines, *AB* and *CD*, is 4 cm. What is its perimeter? ● 36 cm Ⓑ 34 cm Ⓒ 32 cm Ⓓ 30 cm Percent full credit: 68 (1.8) International average: 32 (0.3)

Fig. 6.3 Examples of items of the Advanced International Benchmark Level. *Source* Mullis et al. (2016) Exhibits 2.7.1 and 2.14.4

The grade 8 item may be said to be a routine one for 8th graders in Singapore schools. They were ranked second to Chinese Taipei (72%) for the item.

6.2.2 Engagement and Attitudes of Singapore Students in TIMSS 2015

As part of TIMSS 2015, students completed tests on mathematics and science and also a student questionnaire that collected data on students' views about their mathematics instruction and attitudes towards mathematics. Grade 4 students were asked to indicate their degrees of agreement to statements on the *Students' Views on Engaging Teaching in Mathematics Lessons Scale, Students Like Learning Mathematics Scale* and *Students Confident in Mathematics Scale*. The grade 8 students were asked to indicate their degrees of agreement to statements on four scales, the same three scales as the grade 4 and the Students Value Mathematics Scale. In this section, we present data for both grades 4 and 8 for the three common scales that were part of their student questionnaires.

6.2.2.1 Students' Views on Engaging Teaching in Mathematics Lessons

The student questionnaire asked students about how engaging their mathematics lessons were. Students were scored according to their degree of agreement with ten

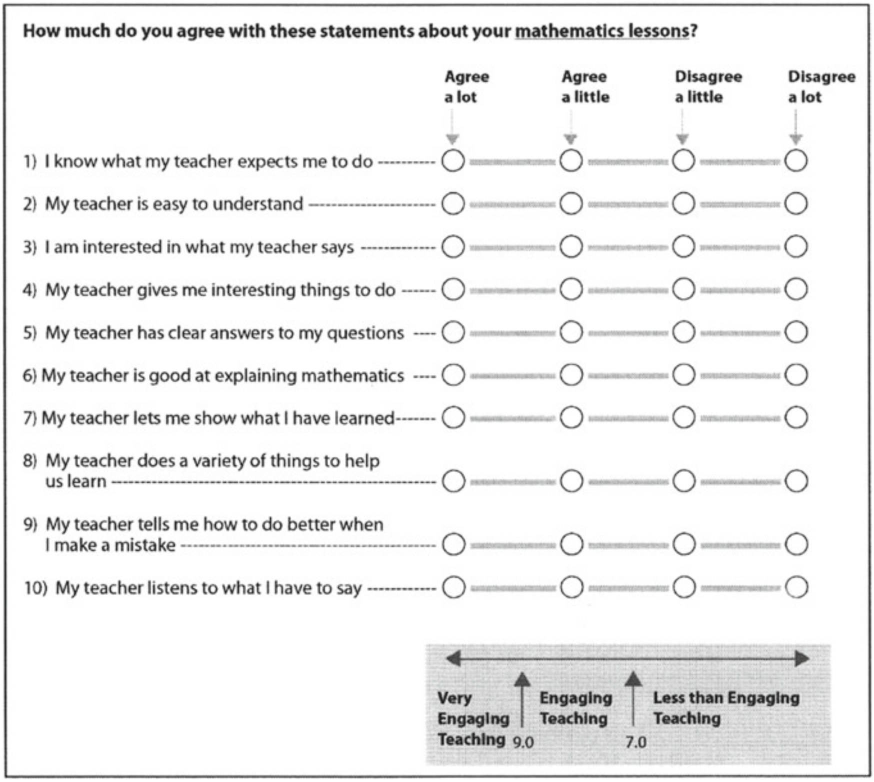


Fig. 6.4 Engaging Teaching in Mathematics Lessons Scale. *Source* Mullis et al. (2016) Exhibits 10.1 and 10.2

statements on the *Students' Views on Engaging Teaching in Mathematics Lessons* Scale shown in Fig. 6.4.

Students who experienced Very Engaging Teaching in mathematics lessons had a score on the scale of at least 9.0, which corresponds to their "agreeing a lot" with five of the ten statements and "agreeing a little" with the other five, on average. Students who experienced teaching that was Less than Engaging had a score no higher than 7.0, which corresponds to their "disagreeing a little" with five of the ten statements and "agreeing a little" with the other five, on average. All other students experienced Engaging Teaching in mathematics lessons. Table 6.3 shows students' views on Engaging Teaching in Mathematics Lessons for students from grades 4 and 8 from Singapore and the international averages in TIMSS 2015.

The 4th graders' average scale score for views on Engaging Teaching in Mathematics Lessons ranged from 11.2 for Bulgaria to 8.2 for Japan, while that for 8th graders ranged from 11.2 for Jordon to 8.4 for Republic of Korea. It is apparent from Table 6.3 that 55% of 4th graders from Singapore found their mathematics lessons very engaging. Just like their peers from the top-performing countries, this result is

Table 6.3 Students' views on Engaging Teaching in Mathematics Lessons

Grade/Country	Very Engaging Teaching		Engaging Teaching		Less than Engaging Teaching		Average scale score
	Per cent of students	Average achieve-ment	Per cent of students	Average achieve-ment	Per cent of students	Average achieve-ment	
Grade 4							
Singapore	55 (1.0)	625 (4.0)	37 (0.7)	613 (4.3)	7 (0.5)	592 (6.7)	9.3 (0.04)
International average	68 (0.2)	510 (0.4)	26 (0.1)	498 (0.6)	5 (0.1)	481 (1.2)	
Grade 8							
Singapore	33 (1.0)	633 (3.6)	52 (0.8)	620 (3.4)	16 (0.8)	596 (6.3)	9.7 (0.04)
International average	43 (0.2)	494 (0.7)	41 (0.2)	478 (0.6)	17 (0.2)	464 (0.9)	

( )—standard errors
Source Mullis et al. (2016) Exhibits 10.1 and 10.2

lower than the international average of 68%, which contrasts with their achievement on the test items. Also for the 8th graders, 33% found their mathematics lessons very engaging compared to the international average of 43%. As the data collected represent students' perceptions, it appears that more 4th graders compared with 8th graders in Singapore mathematics lessons perceived that their mathematics lessons were very engaging. A perception of an engaging lesson may be one where students use manipulatives or carry out activities such as measuring lengths and volumes. Such lessons are more prevalent in the primary school than secondary school mathematics lessons in Singapore. It is noteworthy that the percentages of students at both grade levels are close to the international averages for "Less than Engaging Teaching" though the average achievements of the students from Singapore are much higher than the international averages. Such a finding prompts one to speculate if achievement on the test items is solely an outcome of "teaching" during mathematics lessons.

6.2.2.2 Students Like Learning Mathematics

The student questionnaire also asked students about their liking of learning mathematics. Students were scored according to their degree of agreement with nine statements on the *Students Like Learning Mathematics* Scale shown in Fig. 6.5.

Students who very much Like Learning Mathematics had a score of at least 10.1, which corresponds to their "agreeing a lot" with five of the nine statements and "agreeing a little" with the other four, on average. Students who do not Like Learning Mathematics had a score no higher than 8.3, which corresponds to their "disagreeing

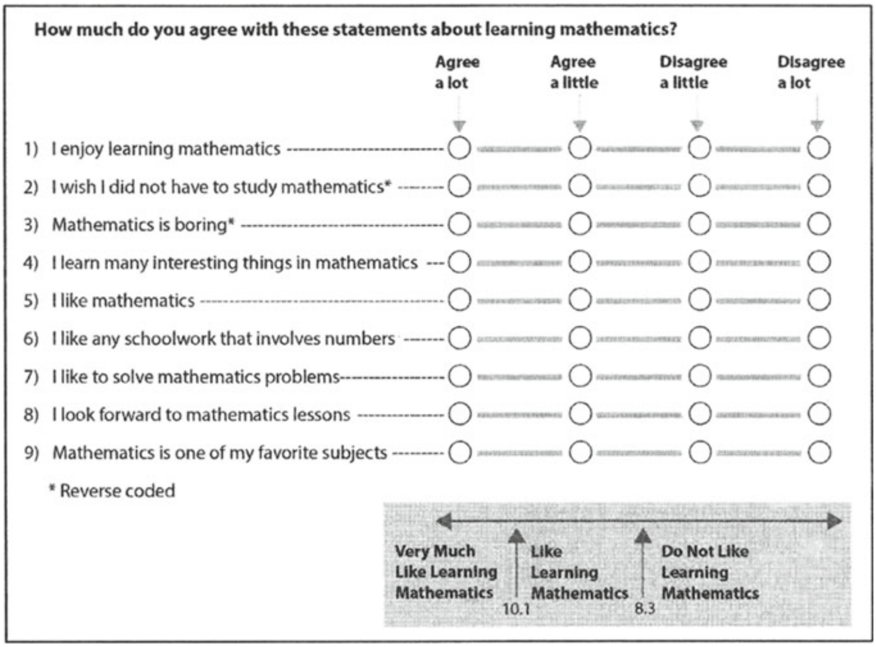


Fig. 6.5 Like Learning Mathematics Scale. *Source* Mullis et al. (2016) Exhibits 10.3 and 10.4

a little" with five of the nine statements and "agreeing a little" with the other four, on average. All other students Like Learning Mathematics. Table 6.4 shows the data for students' Like Learning Mathematics for students from grades 4 and 8 from Singapore and the international averages in TIMSS 2015.

The 4th graders' average scale score for Like Learning Mathematics ranged from 11.3 for Turkey to 8.9 for Republic of Korea, while that for 8th graders ranged from 11.4 for Botswana to 8.7 for Slovenia. It is apparent from Table 6.4 that only 39% of 4th graders from Singapore very much like learning mathematics. Just like their peers from the top-performing countries, this result is lower than the international average of 46%, which again contrasts with their achievement on the test items. However, for the 8th graders 24% very much like learning of mathematics and this result was marginally higher than the international average of 22% unlike that for the other top-performing countries. The push for mastery in the learning of mathematics in Singapore schools may have produced good achievement scores but certainly have not provided all students with enjoyment that translated into feelings of "like".

Table 6.4 Students Like Learning Mathematics

Grade/Country	Very much Like Learning Mathematics		Like Learning Mathematics		Do not Like Learning Mathematics		Average scale score
	Per cent of students	Average achievement	Per cent of students	Average achievement	Per cent of students	Average achievement	
Grade 4							
Singapore	39 (0.8)	640 (4.1)	38 (0.7)	611 (4.1)	23 (0.8)	591 (4.5)	9.6 (0.03)
International average	46 (0.2)	521 (0.5)	35 (0.1)	495 (0.5)	19 (0.1)	483 (0.8)	
Grade 8							
Singapore	24 (0.7)	654 (3.2)	42 (0.8)	625 (3.5)	33 (0.8)	592 (4.3)	10.1 (0.03)
International average	22 (0.1)	518 (0.8)	39 (0.1)	485 (0.6)	38 (0.2)	462 (0.6)	

( )—standard errors
Source Mullis et al. (2016) Exhibits 10.3 and 10.4

6.2.2.3 Students Confident in Mathematics

The student questionnaire also asked students about their confidence in mathematics. Students were scored according to their degree of agreement with nine statements on the *Students Confident in Mathematics* Scale shown in Fig. 6.6.

Students Very Confident in Mathematics had a score of at least 10.6, which corresponds to their "agreeing a lot" with five of the nine statements and "agreeing a little" with the other four, on average. Students who were Not Confident in Mathematics had a score no higher than 8.5, which corresponds to their "disagreeing a little" with five of the nine statements and "agreeing a little" with the other four, on average. All other students were Confident in Mathematics. Table 6.5 shows the data for students' Confidence in Mathematics for students from Singapore and the international averages in TIMSS 2015.

The 4th graders' average scale score for Confident in Mathematics ranged from 10.6 for Kazakhstan to 8.9 for Chinese Taipei, while that for 8th graders ranged from 10.7 for Israel to 9.1 for both Thailand and Chinese Taipei. It is apparent from Table 6.5 that 19% of 4th graders from Singapore reported that they were very confident in mathematics. Just like their peers from the top-performing countries, this result is lower than the international average of 32% despite their commendable achievement on the test items. For 8th graders, the international average was 14% for students claiming that they were very confident in mathematics and the per cent for the same was marginally lower, i.e. 13%, for Singapore students. Asian students, including those from Singapore, are always modest in making claims of achievement.

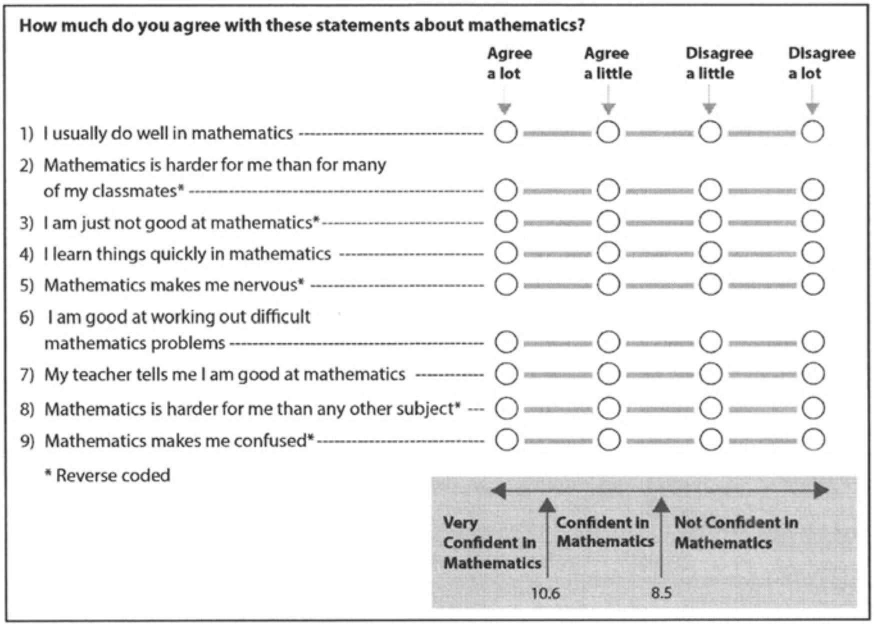


Fig. 6.6 Students Confident in Mathematics Scale. *Source* Mullis et al. (2016) Exhibits 10.5 and 10.6

Table 6.5 Students Confident in Mathematics

Grade/Country	Very Confident in Mathematics		Confident in Mathematics		Not Confident in Mathematics		Average scale score
	Per cent of students	Average achievement	Per cent of students	Average achievement	Per cent of students	Average achievement	
Grade 4							
Singapore	19 (0.8)	681 (3.6)	42 (0.6)	633 (3.6)	39 (1.1)	572 (4.0)	9.2 (0.05)
International average	32 (0.1)	546 (0.5)	45 (0.1)	502 (0.5)	23 (0.1)	460 (0.6)	
Grade 8							
Singapore	13 (0.5)	675 (3.0)	41 (0.7)	642 (2.8)	46 (0.8)	588 (4.0)	9.7 (0.04)
International average	14 (0.1)	554 (0.8)	43 (0.1)	494 (0.6)	43 (0.2)	449 (0.6)	

( )—standard errors
Source Mullis et al. (2016) Exhibits 10.5 and 10.6

Therefore, it is not alarming that students from the top 5 education systems in Asia both at grades 4 and 8 do not agree a lot or agree a little with the nine statements in Fig. 6.6.

6.3 Programme for International Student Assessment (PISA)

Programme for International Student Assessment (PISA) was launched by the OECD in 1997. It aims to evaluate education systems worldwide every three years by assessing 15-year-olds' competencies in the key subjects: reading, mathematics and science. Most importantly, the PISA assessments focus on literacy and the use of knowledge by participants. Although in every cycle, all the three subjects are assessed, only one of the subjects is the focus. For example in PISA 2009, reading was the focus; in PISA 2012, mathematics was the focus; and in PISA 2015, science was the focus. Initially, participants of PISA were OECD countries, but at present, non-OECD countries like Singapore and economies like Shanghai are also participating. More than 70 economies participated in PISA 2009. Singapore participated in PISA for the first time in 2009. PISA collects data from students and their school leaders. After every cycle of PISA, the myriad analysis of the data is publically available for everyone through the OECD web pages (http://www.oecd.org/pisa/) and also in the form of reports such as *PISA 2009 Results: What Students Know and Can Do* (OECD 2010a); *PISA 2009 Results: Overcoming Social Background* (OECD 2010b); and *PISA 2009 Results: Learning to Learn* (OECD 2010c).

6.3.1 Performance of Singapore Students in PISA

As one of the world's best-performing school education systems in a 2007 Mckinsey study of teachers (Barber and Mourshed 2007), Singapore has been among the top-performing countries in PISA for the last three cycles. Table 6.6 shows that Singapore has moved up rapidly in PISA overall rankings from fifth in 2009 to first in 2015. It was noted that the results of the 2015 and past PISA cycles reflected the deliberate curricular shifts made over the years towards a greater emphasis on higher-order, critical thinking skills, and pedagogical shifts in moving learning beyond content to mastery and application of skills to solve authentic problems in various contexts (Ministry of Education 2016b).

The PISA 2012 focused on mathematics. Singapore ranked second with a mean score of 573 points that was significantly lower than Shanghai, China, and significantly higher than Hong Kong that ranked third. For PISA 2012, Table 6.7 shows that on average across OECD countries, 13% of students were top performers in mathematics with proficiency Level 5 or 6. These students have capacity of develop-

Table 6.6 Global features of Singapore performance in PISA 2009, 2012 and 2015

Year	Focus	Rank (overall)	Mathematics		Reading	Science
			Average score	Rank	Average score	Average score
2009	Reading	5	562	2	526	542
2012	Mathematics	2	573	2	542	551
2015	Science	1	564	1	535	556

Source OECD (2009, 2012, 2015)

Table 6.7 Percentage of students from Singapore and the OECD average in PISA 2012 at each level of mathematics proficiency

Country	Rank	Average	International Benchmarks				
			Above level 2 (420)	Above level 3 (482)	Above level 4 (545)	Above level 5 (607)	Above level 6 (669)
Singapore	2	573 (1.3)	91.7	79.5	62.0	40.0	19.0
OECD average		490 (0.4)	77.0	54.5	30.8	12.6	3.3

( )—standard errors
Source OECD (2012)

ing and working with model for complex situations, and they can work strategically using broad, well-developed thinking and reasoning skills (OECD 2013a). Two-fifths (40%) of students from Singapore were at these levels. On the other side, 23% of students in OECD countries did not achieve Level 2 in PISA mathematics. Level 2 is the baseline level on the mathematics proficiency scale that is required for full participation in modern society (OECD 2013a). The percentage of low achievers who were below Level 2 was 8.3% for Singapore.

6.3.2 Students Performance on Mathematics Released Sample Items of PISA 2012

For mathematics, PISA assesses mathematical literacy that is defined as an individual's capacity to formulate, employ and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognize the role that mathematics plays in the world and to make the well-founded judgements and decisions needed by constructive, engaged and reflective citizens (OECD 2013d, p. 17).

The PISA mathematics assessment framework has three dimensions, which are:

1. Processes (three categories and seven fundamental capabilities)

Categories (i) formulating situations mathematically, (ii) employing mathematical concepts, facts, procedures and reasoning and (iii) interpreting, applying and evaluating mathematical outcomes.

Fundamental mathematical capabilities (i) communicating, (ii) mathematizing, (iii) representation, (iv) reasoning and argument, (v) devising strategies for problem-solving, (vi) using symbolic, formal and technical language and operations and (vii) using mathematical tools.

2. Content (four overarching ideas)

(i) quantity, (ii) space and shape, (iii) change and relationships and (iv) uncertainty and data.

3. Contexts (four categories)

(i) personal, (ii) occupational, (iii) societal and (iv) scientific (OECD 2013d, p. 18).

This section presents two examples, Drip Rate and Revolving Door, with accompanying released sample items from the PISA 2012. These items illustrate the dimensions of the PISA assessment framework and also highlight the performance of students from Singapore.

Figure 6.7 shows Example 1 (Drip Rate), comprising items (Questions 1 and 3) categorized as change and relationships. The key to Question 1 lies in students being able to relate the change in drip rate to the change in time, given the variables drop factor and volume are held constant. This question intends to model the change and relationships with appropriate algebra functions, as well as interpreting symbolic representations of relationships. A form of proportional reasoning is needed. This is a question at mathematics proficiency Level 5, and the challenge is that it requires students to give a brief explanation of the effect of specified change to one variable on a second variable if other variables remain constant. In particular, students' explanation needs to describe both the direction of the effect (i.e. getting smaller) and its size (i.e. 50%).

On average across OECD, less than one-quarter of the students answered this question correctly. Only 33.42% of the students from Singapore could state both the direction and size of the effect correctly and obtained full credit. Another 26.97% of the Singapore students could state either the direction or the size of the effect, but not both, and obtained partial credit.

Question 3 requires students to transpose an equation to find expression for volume v so as to obtain the required result by substituting values of two variables into the expression. This is a question at Level 5 proficiency. The question also makes certain demand on interpreting formula linking three variables in a medical context and translating from natural language to symbolic language. Students from Singapore did well with 63.86% obtaining full credit.

Figure 6.8 shows Example 2 (Revolving Door) with three accompanying released items. The first two questions are space and shape items. Question 1 is a proficiency Level 3 question. It requires some basic factual knowledge about circle geometry and

Example 1: Drip Rate

Infusions (or intravenous drips) are used to deliver fluids and drugs to patients. Nurses need to calculate the drip rate, D, in drops per minute for infusions. They use the formula $D = \frac{dv}{60n}$ where

d is the drop factor measured in drops per millilitre (mL)
v is the volume in mL of the infusion
n is the number of hours the infusion is required to run.

Question 1:
Description: Explain the effect that doubling one variable in a formula has on the resulting value if other variables are held constant.
Content area: change and relationships; Context: Occupational; Process: Employ

A nurse wants to double the time an infusion runs for.
Describe precisely how *D* changes if n is **doubled** but d and *v* do not change.

	Full credit	Partial credit
Singapore	33.42% (1.1)	26.35% (0.98)
OECD average	16.32% (0.18)	11.82% (0.14)

( ) – standard errors

Question 3:
Description: Transpose an equation and substitute two given values
Content area: Change and relationships; Context: Occupational; Process: Employ

Nurses also need to calculate the volume of infusion, v, from the drip rate, D. An infusion with a drip rate of 50 drops per minute has to be given to a patient for 3 hours. For this infusion the drop factor is 25 drops per millilitre.

What is the volume in mL of the infusion?
Volume of the infusion: ……………mL

	Full credit
Singapore	63.86% (1.2)
OECD average	25.70% (0.21)

( ) – standard errors

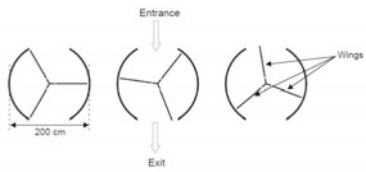
Fig. 6.7 Drip Rate example with accompanying released items and Singapore students' achievement in PISA 2012. *Source* OECD (2013c, pp. 6–8); OECD (2012)

spatial understanding of the diagrams. Students need to recognize the relevance of the information about equal sectors in order to find the central angle of a sector of a circle. The performance of students from Singapore on this question was commendable (76%), which was far above the OECD average (58%).

Question 2 requires students to interpret a geometrical model in a real-life situation and then calculate the length of an arc. It requires substantial geometry reasoning about the design features of revolving door that enable it to perform its function as a doorway while maintaining a sealed space that prevents air flowing between the entrance and exit. This is a novel question and it requires some creative thought, not just the application of any textbook knowledge they would have learnt. Classified as formulate for the dimension process, this item draws very heavily on the fundamental mathematical capability of reasoning and argument, because the problem in the real situation has to be carefully analysed and transformed into a mathematical problem

Example 2: Revolving Door

A revolving door includes three wings which rotate within a circular-shaped space. The inside diameter of this space is 2 metres (200 centimetres). The three door wings divide the space into three equal sectors. The plan below shows the door wings in three different positions viewed from the top.



Question 1:
Description: Compute the central angle of a sector of a circle.
Content area: Space and shape; Context: Scientific; Process: Employ.

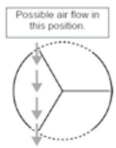
What is the size in degrees of the angle formed by two door wings?
Size of the angle: ……………… °

	Full credit
Singapore	75.72% (1.1)
OECD average	57.67% (0.25)

( ) – standard errors

Question 2:
Description: Interpret a geometrical model of real life situation to calculate the length of an arc.
Content area: Space and shape; Context: Scientific; Process: Formulate

The two door openings (the dotted arcs in the diagram) are the same size. If these openings are too wide the revolving wings cannot provide a sealed space and air could then flow freely between the entrance and the exit, causing unwanted heat loss or gain. This is shown in the diagram opposite.
What is the maximum arc length in centimetres (cm) that each door opening can have, so that air never flows freely between the entrance and the exit?



Maximum arc length: ……………… cm

	Full credit
Singapore	13.17% (0.81)
OECD average	3.47% (0.08)

( ) – standard errors

Question 3:
Description: Identify information and construct an (implicit) quantitative model to solve the problem.
Content area: Quantity; Context: Scientific; Process: Formulate

The door makes 4 complete rotations in a minute. There is room for a maximum of two people in each of the three door sectors.
What is the maximum number of people that can enter the building through the door in 30 minutes? (A) 60 (B) 180 (C) 240 (D) 720

	Full credit
Singapore	59.30% (0.99)
OECD average	46.42% (0.24)

( ) – standard errors

Fig. 6.8 Revolving Door example with accompanying released items and Singapore students' achievement in PISA 2012. *Source* OECD (2013c, pp. 33–35); OECD (2012) in geometric terms and then back again to the contextual situation of the problem. This question was one of the most challenging questions in the PISA 2012 test and it belongs to the upper end of Level 6 on the mathematics proficiency scale. Less than 15% of the students from Singapore were able to complete this question correctly. On average across OECD countries, only 3.5% of the students answered this question correctly.

Question 3 addresses a different type of challenge, involving rates and proportional reasoning, and it lies at mathematics proficiency Level 4. Students are required to identify relevant information and construct an implicit quantitative model to solve the problem. The content category of the question is quantity category because of the way in which the multiple relevant quantities have to be combined by number operations to produce the required number of persons to enter in 30 min. The question also makes considerable demand on the formulating process. A student needs to understand the real-world problem so as to assemble the data provided in the right way. Students from Singapore did reasonably well on this item with 59.3% obtaining full credit. On average across OECD countries, almost half of the students answered this question correctly.

6.3.3 Singapore Students' Exposure to Mathematics Content and Their Drive and Motivation to Learn Mathematics in PISA 2012

As part of PISA 2012, each student took a two-hour handwritten test on reading, mathematics and science (with a focus on mathematics). The tests were a mixture of open-ended and multiple-choice questions that were organized in groups based on a passage setting out a real-life situation. Following the cognitive test, students spend nearly one more hour answering a questionnaire about themselves, their family and home, general aspects of learning mathematics, problem-solving experiences, and specific aspects of learning mathematics as in 2012 the focus of PISA was mathematics.

6.3.3.1 Students' Exposure to Mathematics Content

Research shows that students' exposure to subject content in school, known as "opportunity to learn", is associated with student performance (Schmidt et al. 2001; Sykes et al. 2009). The PISA 2012 questionnaire asked students how often they encountered various types of mathematics problems or tasks during their time at school and also how familiar they were with mathematical concepts such as exponential function, divisor, quadratic function, proper number, linear equation, vectors, complex number, rational number, radicals, subjunctive scaling, polygon, declarative fraction, congruent figure, cosine, arithmetic mean and probability. Responses

Table 6.8 Index of Singapore students' Exposure to Mathematics Content in PISA 2012

Country	Exposure to word problems	Exposure to formal mathematics	Exposure to applied mathematics
Singapore	1.56 (0.016)	2.23 (0.010)	2.00 (0.010)
OECD average	1.87 (0.003)	1.70 (0.003)	1.92 (0.002)

( )—standard errors
Source OECD (2014) Table 1.3.1

to the questionnaire were used to create three categories: exposure to word problems, exposure to formal mathematics and exposure to applied mathematics and respective indices created (OECD 2014). The values of these indices range from 0 to 3, with 0 corresponding to no exposure and 3 to frequent exposure.

Table 6.8 shows Singapore students' indices for Exposure to Mathematics Content and the corresponding OECD averages. Singapore stood out among all PISA participating countries as having the strongest relationship between the index of exposure to formal mathematics and students' mathematics performance (OECD 2014, p. 153). This result suggests that opportunities to learn formal mathematics are associated with PISA performance. Furthermore, exposure to more advanced mathematics content, such as algebra and geometry, seems to be related to high performance on the PISA mathematics performance. Exposure to word problems, which are usually represented in textbooks as applications of mathematics, is also related to performance, but was found to be less strong when compared to the OECD average. From the index of exposure to applied mathematics, it is apparent that students in Singapore are exposed to a wide range of problems (including with real-world contexts) to solve during their study of mathematics. In this way, students learn to apply mathematics in varying contexts and develop necessary skills for future use.

6.3.3.2 Students' Drive and Motivation to Learn Mathematics

In PISA 2012, students' perseverance, openness to problem-solving, and students' intrinsic and instrumental motivation to learn mathematics were measured to assess Students' Drive and Motivation to Learn Mathematics (OECD 2013b). Perseverance and Openness to Problem-Solving are two new scaled indices in 2012 PISA. They were developed in recognition of the increasing importance of problem-solving in the cognitive part of the assessment. Based on students' self-reports, PISA results show that drive and motivation are essential for students' to realize their potential. Students' Perseverance was gauged by their responses to the five statements shown in Table 6.9. Students responded with one of the following: "very much like me", "mostly like me", "somewhat like me", "not much like me" or "not at all like me". Across OECD countries, 56% of students indicated that they do not give up easily when confronted with a problem, 49% indicated that they remain interested in the tasks that they start, and 44% indicated that they continue working on tasks until

Table 6.9 Items measuring students' Perseverance and students' Openness to Problem-Solving

How well does each of the following statements below describe you?		Percentage of students	
Students' perseverance		OECD average	Singapore
a	When confronted with a problem, I give up easily[b]	56.0 (0.2)	61.8 (0.7)
b	I put off difficult problems[b]	36.9 (0.1)	43.8 (0.7)
c	I remain interested in tasks that I start[a]	48.9 (0.2)	57.9 (0.8)
d	I continue working on tasks until everything is perfect[a]	43.8 (0.2)	61.1 (0.9)
e	When confronted with a problem I do more than what is expected of me[a]	34.5 (0.1)	45.3 (0.9)
Openness to problem-solving			
a	I can handle a lot of information[a]	53.0 (0.2)	44.2 (0.9)
b	I am quick to understand things[a]	56.6 (0.1)	50.4 (0.9)
c	I seek explanations for things[a]	60.7 (0.2)	68.5 (0.7)
d	I can easily link facts together[a]	56.7 (0.2)	52.4 (1.0)
e	I like to solve complex problems[a]	33.1 (0.1)	39.1 (0.9)

( )—standard errors

[a]Percentage of students who reported that the statements describe someone "very much like me" or "mostly like me"

[b]Percentage of students who reported that the statements describe someone "not much like me" or "not at all like me"

Source OECD (2013b) Tables III.3.1a and III.3.2a

everything is perfect. The percentage of Singapore students showing perseverance for each individual statement is higher than the international average, with 62% of students indicating that they do not give up easily when confronted with a problem, 58% indicating that they remain interested in the tasks that they start, and 61% indicating that they continue working on tasks until everything is perfect.

PISA 2012 also measured students' Openness to Problem-Solving through their responses to the five statements shown in Table 6.9. The questions asked students about the extent to which they feel they resemble someone who can handle a lot of information, is quick to understand things, seeks explanations for things, can easily link facts together and likes to solve complex problems.

Student's responses to each question could range from: the statement describing someone "very much like me", "mostly like me", "somewhat like me", "not much

Table 6.10 Index of Singapore students' Perseverance and Openness to Problem-Solving in PISA 2012

	Index of students' Perseverance				Index of students' Openness to Problem-Solving			
	All students	By proficiency level			All students	By proficiency level		
		Below level 2	Level 4	Level 5 or 6		Below level 2	Level 4	Level 5 or 6
Singapore	0.29 (0.02)	0.09 (0.04)	0.35 (0.03)	0.36 (0.03)	0.01 (0.02)	−0.30 (0.06)	0.02 (0.03)	0.20 (0.03)
OECD average	0.00 (0.00)	−0.28 (0.01)	0.19 (0.01)	0.43 (0.01)	0.00 (0.00)	−0.40 (0.01)	0.29 (0.01)	0.70 (0.01)

( )—standard errors

Source OECD (2013b) Tables III.3.1d, III.3.2d and III.3.8

like me" or "not at all like me". Across OECD countries, 53% of students indicated that they can handle a lot of information, 57% reported that they are quick to understand things, and 61% reported that they seek explanation for things, 57% reported that they can easily link facts together, and only 33% indicated that they like to solve complex problems. Singapore students showed higher intention in seeking explanation for things and solving complex problems than the international average with 69% reporting that they seek explanation for things and 39% indicating that they like to solve complex problems, but showed lower self-belief of being able to handling lots of information, quickly understanding things and easily linking facts together.

The responses to the items in Table 6.9 were used to create the index of students' Perseverance and index of students' Openness to Problem-Solving. The indices were standardized to have a mean of 0 and a standard deviation of 1 across the OECD countries and other economies and countries that participated in PISA 2012. Table 6.10 shows the indices for perseverance and openness to problem-solving for Singapore students in PISA 2012. The mean index of perseverance ranged from 0.77 for Kazakhstan to −0.59 for Japan. Singapore had an index of 0.29, which was the best among the top-performing East Asian countries/economies that participated in PISA 2012.

It is apparent from Table 6.10 that students with proficiency Level 5 or 6 reported higher levels of perseverance than those with lower proficiency levels, which indicates a strong association between perseverance and mathematics performance in terms of proficiency level achieved in PISA 2012. However, the perseverance index of Singapore students with proficiency Level 5 or 6 was lower than the international average of 0.43.

The index of students' openness to problem-solving ranged from 0.62 for Jordan and Montenegro to −0.73 for Japan. Table 6.10 shows that index for Singapore students was 0.01 just above the OECD average. There appears to be generally an inverse relationship between openness to problem-solving and mathematics performance among students who participated in PISA 2012. Singapore students with proficiency Level 5 or 6 reported relatively higher levels of openness to problem-solving than those with lower proficiency levels. For Levels 4, 5 and 6 of proficiency,

Table 6.11 Items measuring Intrinsic and Instrumental Motivation to Learn Mathematics

Thinking about your views on mathematics: to what extent do you agree with the following statements?		Percentage of students*	
Intrinsic motivation to learn mathematics (mathematics interest)		OECD average	Singapore
a	I enjoy reading about mathematics	30.6 (0.2)	68.1 (0.9)
b	I look forward to my mathematics lessons.	36.2 (0.2)	76.8 (0.8)
c	I do mathematics because I enjoy it	38.1 (0.2)	72.2 (0.8)
d	I am interested in the things I learn in mathematics	53.1 (0.2)	77.1 (0.8)
Instrumental motivation for mathematics			
a	Making an effort in mathematics is worth it because it will help me in the work that I want to do later on	75.0 (0.1)	90.4 (0.6)
b	Learning mathematics is worthwhile for me because it will improve my career prospects and chances	78.2 (0.1)	88.2 (0.6)
c	Mathematics is an important subject for me because I need it for what I want to study later on	66.3 (0.2)	87.4 (0.6)
d	I will learn many things in mathematics that will help me get a job	70.5 (0.2)	85.5 (0.7)

( )—standard errors
*Percentage of students who "strongly agree" or "agree"
Source OECD (2013b) Tables III.3.4a and III.3.5a

their indices of openness to problem-solving were lower than the OECD average. It is interesting to note that for PISA 2012, creative problem-solving students from Singapore were ranked first and yet their perceptions of openness to problem-solving suggest that they do not have attributes of good problem solvers. This mismatch could be attributed to their inability to self-assess their abilities or sheer over modesty, as often portrayed by Asian students.

PISA measures students' Intrinsic Motivation to Learn Mathematics and Instrumental Motivation to Learn Mathematics through their responses "strongly agree", "agree", "disagree" or "strongly disagree" with the statements shown in Table 6.11.

As shown in Table 6.11, on average across OECD countries, students who participated in PISA 2012 have shown relatively low levels of intrinsic motivation to learn mathematics. Only 31% of students indicated that they agree or strongly agree that they enjoy reading about mathematics, 36% reported that they look forward to their mathematics lessons, 38% reported that they do mathematics because they enjoy it, and 53% reported that they are interested in the things they learn in mathematics. However, Singapore students seem to have high levels of intrinsic motivation to learn mathematics, with 68% of students indicating that they enjoy reading about mathematics, 77% indicating that they look forward to their mathematics lessons, 72%

Table 6.12 Index of Singapore students' Intrinsic Motivation to Learn Mathematics and Instrumental Motivation to Learn Mathematics in PISA 2012

	Index of students' Intrinsic Motivation to Learn Mathematics				Index of students' Instrumental Motivation to Learn Mathematics			
	All students	By proficiency level			All students	By proficiency level		
		Below level 2	Level 4	Level 5 or 6		Below level 2	Level 4	Level 5 or 6
Singapore	0.84 (0.02)	0.60 (0.06)	0.88 (0.03)	0.88 (0.02)	0.40 (0.02)	0.38 (0.06)	0.44 (0.03)	0.34 (0.03)
OECD average	0.00 (0.00)	−0.21 (0.01)	0.17 (0.01)	0.51 (0.01)	0.00 (0.00)	−0.22 (0.01)	0.15 (0.01)	0.40 (0.01)

( )—standard errors
Source OECD (2013b) Tables III.3.4d, III.3.5d and III.3.8

reporting that they do mathematics because they enjoy it, and 77% reporting that they are interested in the things they learn in mathematics.

From Table 6.11, it is also apparent that students who participated in PISA 2012 appreciate the instrumental value of mathematics. On average across OECD countries, 75% of students responded that they agree or strongly agree that making an effort in mathematics is worthwhile because it will help them in the work that they want to do later on in life. 78% of students responded that learning mathematics will improve their career prospects, and 71% of students believed that learning many things in mathematics will help them get a job. Likewise, Singapore students have also shown very high levels of instrumental motivation to learn mathematics, with 90% of students responding that they agree or strongly agree that making an effort in mathematics is worth it because it will help them in the work that they want to do later on, 88% of students responding that learning mathematics is worthwhile because it will improve their career, 87% reporting that mathematics is an important subject because they need it for what they want to study later on, and 86% believing that many things they learnt in mathematics will help them get a job.

The responses were used to create standardized indices, with mean of 0 and standard deviation of 1, for students' Intrinsic Motivation and Instrumental Motivation to Learn Mathematics. The index for students' intrinsic motivation to learn mathematics ranged from 0.96 for Albania to −0.35 for Austria, while that for instrumental motivation ranged from 0.56 for Peru to −0.57 for Romania. Table 6.12 shows the indices for Singapore students' intrinsic motivation and instrumental motivation to learn mathematics and the corresponding OECD averages in PISA 2012. For intrinsic motivation to learn mathematics, Singapore had an index of 0.84. For instrumental motivation, the index was 0.40. Both indices were the highest compared with the other top-performing East Asian countries/economies in PISA 2012.

From Table 6.12, it is also apparent that students with proficiency Level 5 or 6 showed significantly higher index of intrinsic motivation than those with proficiency level below 2. The results suggest an association between students' intrinsic motivation and mathematics performance in terms of proficiency level achieved in PISA 2012. However, for instrumental motivation to learn, it appears that Singapore is an exception as students at all proficiency levels show high indices of instrumental motivation to learn mathematics.

6.4 Teacher Education and Development Study in Mathematics (TEDS-M)

TEDS-M is the first international comparative study on the training of future mathematics teachers carried out by IEA. Seventeen countries including Singapore participated in the study. Singapore participated in the study to compare teacher education at the National Institute of Education (NIE), the sole teacher education institute in Singapore, and performance of NIE student teachers in mathematics content knowledge (MCK) and mathematics pedagogical content knowledge (MPCK) against international benchmarks.

TEDS-M was a survey study that used specific questionnaires to collect data from educators and future mathematics teachers. The theoretical framework of the study is detailed in Tatto et al. (2008). The study comprises three components. Component 1 is about the national contexts and policies for teacher education. The national research coordinators of the participating countries provided country reports explaining these contexts and policies. Component 2 is specific to the nature of mathematics teacher education programmes. Coordinators of the institutes that were sampled in each country completed the Institution questionnaire. Educators from the institutes did the same for the Educator questionnaire that sought their beliefs about pedagogy and activities offered by their courses for future mathematics teachers. Information about the school mathematics curricula and mathematics teacher education courses was also collected and analysed.

Component 3 examines the outcomes of teacher education in terms of the performance of future mathematics teachers in MCK and MPCK and their beliefs and perceptions of opportunities to learn (OTL) about mathematics and pedagogy. The mathematics for teaching test comprised MCK and MPCK items. The MCK items covered four content knowledge domains (Number, Geometry, Algebra and Data) and three cognitive domains (Knowing, Applying and Reasoning). These domains are based on the corresponding domains used in the TIMSS 2007 framework (Mullis et al. 2007). The MPCK items measured three types of mathematics knowledge for teaching: mathematical curricular knowledge; knowledge of planning for mathematics teaching and learning (pre-active); and enacting mathematics for teaching and learning (interactive). The test comprised 24 items and 30 items for the primary and lower secondary future mathematics teachers, respectively, and teachers had 60 min to complete it. The beliefs and OTL survey comprised 53 Likert-type items, and teachers had 30 min to complete it. The survey sought their beliefs about the nature of mathematics, learning mathematics and mathematics achievement and perceptions about content and skills relating to seven broad areas hypothesized to influence knowledge for teaching mathematics: tertiary-level mathematics, school-level mathematics, mathematics education pedagogy, general pedagogy, teaching diverse students, learning through school-based experiences, and coherence of their teacher education programme.

As NIE is the sole teacher education institute in Singapore, it provided a census sample to represent Singapore. Altogether 380 primary (263 primary generalist + 117 primary math specialist) and 393 (142 lower secondary + 251 upper secondary) secondary future mathematics teachers completed the TEDS-M tests and surveys. Seventy-seven NIE mathematicians, mathematics educators and teacher educators who taught at least one course to the future teachers participating in TEDS-M also completed the Educator questionnaire in 2007.

The TEDS-M international report detailing the data and findings related to the three components was published in 2012 (Tatto et al. 2012). Several publications by Wong Khoon Yoong, who was the National Research Coordinator for Singapore, and his colleagues provide us with insights about findings that are Singapore centric (Wong et al. 2011, 2012a, b, 2013a, b, c; 2014). In the following sections, we draw on the international report and also publications by Wong and colleagues and present a brief overview of findings that provide us with a glimpse of where mathematics teacher education sits in the international arena and how future mathematics teachers rank in the same.

6.4.1 National Contexts and Policies for Teacher Education

Wong et al. (2012) noted that teacher education policies varied widely across the 17 countries that participated in TEDS-M and it was not possible to draw definitive implications about the effects of these policies on the performance of future teachers. Furthermore in several countries, including Singapore, policies have changed since the country reports were submitted in 2008. Nevertheless, Tatto et al. (2012) reported that in both Chinese Taipei and Singapore where future mathematics teachers scored high on the TEDS-M tests:

- there were strong controls over the number of entrants accepted into teacher education programmes;
- there were specific policies to ensure that teaching is an attractive career; and
- teacher education programmes were able to recruit able high school graduates.

6.4.2 Nature of Mathematics Teacher Education Programmes

The primary mathematics education programmes were classified along a generalist–specialist continuum. In Singapore, data were collected in November 2007 and May/June 2008, from four different types of pre-service programmes for primary teachers at the National Institute of Education (NIE):

- Diploma in Education, Dip Ed (A) or Dip Ed (C);
- Bachelor of Arts with Education, BA (Ed) (C-series);
- Bachelor of Science with Education, B.Sc. (Ed) (C-series);
- Postgraduate Diploma in Education (Primary), PGDE (P) (A) or PGDE (P) (C).

At the time of the study, the Dip Ed and PGDE (P) programmes offered two options: option A covered two teaching subjects (one of which was mathematics) and option C covered three teaching subjects (one of which was mathematics). The C-series Bachelor programmes trained only primary school teachers and covered four teaching subjects, including mathematics. Teachers who were training to teach two subjects were classified as primary mathematics specialists, while those who were training to teach more than two subjects were classified as generalists.

Secondary mathematics teacher education programmes covered either lower secondary up to grade 10 or upper secondary up to grade 12. In Singapore, data were collected in November 2007 and May/June 2008 from two cohorts of the Postgraduate Diploma in Education (PGDE) (Secondary) programme. Future mathematics teachers in the programme prepared to teach either lower secondary mathematics or all secondary mathematics. Those preparing to teach lower secondary mathematics are generally weaker in mathematics compared to those who are preparing to teach all secondary mathematics. In TEDS-M, the lower secondary mathematics teachers from Singapore were classified as those preparing to teach lower secondary to grade 10, while those preparing to teach all secondary mathematics were classified as upper secondary up to grade 12. It is reported by Wong et al. (2012) that Singapore and Chinese Taipei had the highest requirements for the mathematics courses that future teachers must complete in order to enter the professional component of their teacher education programmes. However, secondary future teachers in Chinese Taipei and Russia were prepared to teach only one subject, while those in NIE were prepared to teach one major and one minor subject.

6.4.3 Performance of Future Teachers in MCK and MPCK and Their Beliefs and Perceptions of Opportunity to Learn

Future mathematics teachers from Singapore performed well on the MCK and MPCK tests and Singapore ranked among the top countries. Table 6.13 gives an overview

Table 6.13 Ranking and score of NIE student teachers

	Primary generalist	Primary math specialist	Lower secondary	Upper secondary
N	263	117	142	251
MCK	2 (586)	2 (600)	1 (544)	3 (587)
MPCK	2 (588)	1 (604)	2 (539)	4 (562)

NB. International mean = 500
Source of data Tatto et al. (2012, pp. 139, 143, 147, 150)

of their performance and Table 6.14 gives a detailed breakdown of the same by programmes of study at NIE. From the tables, it is apparent that among the primary student teachers at NIE, those who were trained to teach only two subjects performed better than those who were trained to teach more than two subjects. The secondary student teachers who were trained to teach upper secondary performed better than those who were trained to teach lower secondary. This result was expected. Table 6.14 shows that for future primary teachers when the performance is analysed by NIE programmes, student teachers in the BSc (Ed) programme in fact topped both tests in the Primary Generalist group. One reason could be some student teachers in this programme were doing undergraduate mathematics as their academic subject. Among the six NIE programmes for future primary mathematics teachers in Table 6.14, the performance of student teachers in Dip Ed (C) was the lowest for both tests. This is not unexpected because the Diploma programme admits student teachers not qualified for the Bachelor programmes.

6.4.3.1 Primary MCK and MPCK Test Items

As an illustration of the performance of future primary teachers from Singapore, we consider two released items, one from MCK and the other from MPCK. Figure 6.9 shows an item MFC 204 of the MCK Geometry-Knowing domain. This item requires knowledge of the relationships among quadrilaterals. For example, a square is both a rectangle and a rhombus. Wong et al. (2012b) reported that 66% of NIE student teachers had chosen the correct option C. This is slightly higher than the international level of 64%. As these geometric relationships have been covered in the Subject Knowledge (SK) courses at NIE, they expected the student teachers "to perform better in this task than the result reported here". A better performance entails a deeper understanding of these relationships. An approach worth considering is to reinforce student teachers' ability to differentiate the defining properties of quadrilaterals from the other properties. Knowing the roots of these relationships should help in the understanding of *why* a square is both a rectangle and a rhombus, for example.

Figure 6.10 shows two items of the MPCK Enacting domain: MFC 208A at the intermediate level and MFC 208B at the advanced level. MFC 208A tests the ability to recognize the two common misconceptions that multiplication will always produce

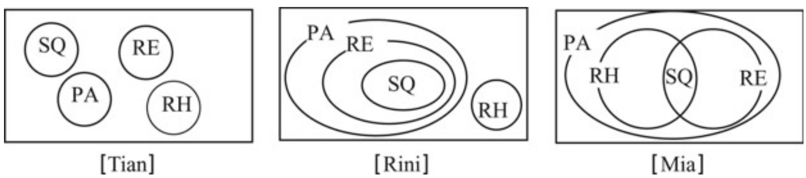
Table 6.14 Performance of Singapore student teachers in MCK and MPCK tests

Programme group/country	MCK		MPCK	
	Rank	Mean	Rank	Mean
Primary generalist (Grade 6 maximum)				
Singapore (All)	2	586	2	588
B.Sc. (Ed)		625		626
PGDE (P) (C)		593		596
BA (Ed)		586		587
Dip. Ed (C)		567		568
Primary mathematics specialist				
Singapore (All)	2	599	1	603
PGDE (P) (A)		600		601
Dip Ed (A)		598		607
Lower secondary				
Singapore	1	544	2	539
Upper secondary				
Singapore	3	587	4	562
International mean = 500*; standard deviation* = 100				

Source of data Wong et al. (2012b, p. 300); Tatto et al. (2012 pp. 139, 143, 147, 150)

MFC 204

Three students have drawn the following Venn diagrams showing the relationships between four quadrilaterals: Rectangles (RE), Parallelograms (PA), Rhombuses (RH), and Squares (SQ).



Which student's diagram is correct? (A) [Tian] (B) [Rini] (C) [Mia]

Fig. 6.9 Released item (MFC 204) of the MCK Geometry-Knowing domain (Wong et al. 2013a, p. 300)

a larger product and division will always make a number smaller. MFC 208B tests the competency to "translate" an abstract operation into a visual model to help pupils correct these misconceptions.

It is reported in Wong et al. (2012) that 67% of NIE student teachers could state at least one misconception in MFC 208A. Although this is much higher than the corresponding international performance of 41%, "that about one-third … could not recognize these misconceptions, giving irrelevant responses" was still "truly

[Jeremy] notices that when he enters 0.2 × 6 into a calculator his answer is smaller than 6, and when he enters 6 ÷ 0.2 he gets a number greater than 6. He is puzzled by this, and asks his teacher for a new calculator!
MFC 208A (a) What is [Jeremy's] most likely misconception?
MFC 208B (b) Draw a visual representation that the teacher could use to model 0.2 × 6 to help [Jeremy] understand WHY the answer is what it is?

Fig. 6.10 Released items (MFC 208A and 208B) of the MCK Enacting domain (Wong et al. 2012b, p. 302)

MFC 610D
Determine whether the following is an irrational number always, sometimes or never?
D. Result of dividing 22 by 7.

MFC 705A
We know that there is only one point on the real number line that satisfies the equation $3x = 6$, namely $x = 2$.

A. Suppose now that we consider this same equation in the plane, with coordinates x and y. What does this set of points that satisfy the equation $3x = 6$ look like in this setting?

□ one point □ one line □ one plane □ other

Fig. 6.11 Released MCK items (MFC 610D and MFC 705A) of the Number-Knowing and Geometry-Knowing domains, respectively (Wong et al. 2012a, p. 3; 2012b, pp. 14, 22)

surprising" to them. This is because these two misconceptions are covered in the NIE Curriculum Studies (CS) courses. Overall, only 23% of NIE student teachers could answer both parts of MFC 208 correctly. For mathematics educators at NIE, a question worth pondering is thus how student teachers' ability to recognize and deal with misconceptions can be strengthened through their CS courses.

6.4.3.2 Secondary MCK and MPCK Test Items

As an illustration of the performance of future secondary teachers from Singapore, we consider some released items, two from MCK and one from MPCK. Figure 6.11 shows two MCK items that NIE student teachers found rather difficult though both belonged to the domain of knowing.

The performance of NIE student teachers on items MFC 610D and MFC 705A is reported in Wong et al. (2012a). For item MFC 610D, 66% of NIE student teachers knew that the result of diving 22 by 7 is never an irrational number, whereas 31% thought it is always an irrational number, probably not realizing that 22/7 is only an approximation for π. The corresponding international averages were 40 and 51%, respectively. The performance of NIE student teachers on item MFC 705A was also weak. About 69% knew that the solution to the equation $3x = 6$ in the plane is a line, but 27% thought it was a point thinking that the solution $x = 2$, which give a single value. The corresponding international averages were 58 and 33%, respectively. The

MFC 712C A mathematics teacher wants to show some students how to prove the quadratic formula. Determine whether the following types of knowledge is needed in order to understand a proof of this result. C. How to complete the square of a trinomial. □ needed □ not needed

Fig. 6.12 Released MPCK item (MFC 712C) of the Algebra-Planning domain (Wong et al. 2012a, p. 3; 2013a, p. 30)

performance of NIE student teachers on both of the above items suggests weak conceptual understanding of some basic mathematics.

Figure 6.12 shows a MPCK item that student teachers at NIE found difficult too. The item belongs to the content domain—Algebra—and type of mathematics knowledge for teaching—planning. Students' performance of the item is reported in Wong et al. (2012a). In proving the quadratic formula, only 37% knew that the proof requires the knowledge to complete the square of a trinomial; this result was much worse than the international average of 55%. As noted by Wong and colleagues, two major reasons may account for this poor result. First, the term trinomial is rarely used in Singapore textbooks, and second, some secondary teachers do not teach this formula using "complete the square" approach. As such, student teachers who had not encountered this proof during their school days may not have the opportunity to learn it their post-secondary mathematics courses.

6.4.3.3 Singapore Student Teachers' Beliefs and Perceptions of Opportunity to Learn

In Wong et al. (2011), the outcomes of a questionnaire on the reasons for student teachers to become a teacher and their beliefs about teaching as a lifetime career were presented. Drawing on their findings, among the nine reasons given in the questionnaire for becoming a teacher, the two most important reasons were "I like working with young people" and "I want to have an influence on the next generation". In line with these reasons, 86% of primary and 82% of secondary student teachers at NIE either expected teaching as a lifetime career or believed in this possibility. These levels of commitment are higher than the corresponding international levels of 72 and 77%. These findings are consistent with the "career-based" model of teacher employment in Singapore. The student teachers at NIE have a distinctive status as employees of the Ministry of Education, and Wong et al. (2013c) noted that "this distinctive system of 'paying' people to be trained as teachers is indicative of a career-based system in its fullest sense".

Wong et al. (2012a) also noted that NIE student teachers and educators generally endorsed the conceptual approaches to learning mathematics compared to the procedural ones. 78% of the educators believed that mathematics should be learned through student activity, compared to 72% of the primary and 66% of the secondary student teachers. Student teachers who held conceptual orientations tended to have higher MCK and MPCK scores compared to those with procedural or fixed ability beliefs. Wong and colleagues reported that on a scale of 0–1, the coverage of mathematics education pedagogy in NIE programmes ranged from 0.68 to 0.72 and this was similar to the international mean. However, the coverage of general pedagogy was in the range of 0.57–0.65 and this was low when compared to Russia, Switzerland and the USA. NIE student teachers also scored below the international mean for opportunities to learn about teaching diverse students, and they rated "rarely having opportunities" to read about research on mathematics and mathematics education; write mathematical proofs; and develop research projects to test teaching strategies for pupils of diverse abilities. A significant difference between the perceptions of student teachers and educators at NIE was that educators felt that they had provided fairly frequent opportunities for the student teachers to engage in interactive learning experiences, such as to ask questions, participate in class discussion, work in groups and make presentations to the class. However, with the exception of group work, the student teachers rated the other three interactive experiences lower than the educators. Nevertheless, 90% of NIE students rated their programmes as effective or very effective in preparing them to teach mathematics.

6.4.4 What Are the Implications of the Main Findings of the TEDS-M for Educators in Singapore?

The findings of comparative studies like TEDS-M (Tatto et al. 2012) provide participating countries with ample scope to make comparisons with other participating countries and glean valuable insights. Drawing on the data and findings of Tatto et al. (2012), Wong et al. (2011), five key implications arising from the findings of TEDS-M were noted by Wong and colleagues for educators in Singapore (Wong et al. 2011). They are as follows.

6.4.4.1 Recruit Future Mathematics Teachers with Strong Mathematics Background

The results of the TEDS-M study in general (Tatto et al. 2012) and Singapore's data (Wong et al. 2011) in particular affirm that teachers with sound mathematical knowledge demonstrate high performance in MCK and MPCK, which are necessary requisites for them to teach mathematics competently in schools. Hence, it is important that the Ministry of Education in Singapore continue to recruit future mathematics teachers with strong entry qualifications.

6.4.4.2 Stress Sound Grounding in Mathematics-Related Knowledge in NIE Programmes

Generally, the good performance of NIE future teachers in MCK and MPCK affirms a strong grounding in mathematics-related knowledge that has been acquired while undergoing study at the NIE to be a mathematics teacher. Although generally the performance of the future teachers was commendable, there were differences in performance across the different programmes (see Table 6.14). Therefore, there is a need to look at the structure of these programmes and make revisions that would help future teachers of mathematics learn more mathematics while preparing to teach mathematics at NIE. Compared to some other countries that also participated in TEDS-M, future mathematics teachers at NIE reported relatively low coverage of validation/structuring/ abstracting topics such as Boolean algebra, mathematical induction, logical connectives and linear space. This could be an area for consideration at least for the Bachelor Degree curriculum as these topics are important for the development of mathematical thinking.

Another gap is the relatively low attention given to teaching mathematics to diverse students. Paying attention to the teaching of students with diverse backgrounds is important as it is in line with differentiated instruction in schools advocated by the Ministry of Education in Singapore. Yet another area that warrants attention is the general agreement among educators and future teachers about the low frequency requiring future teachers to read about research in mathematics and mathematics education. Given the recent trends towards evidence-based practices, it is imperative that educators engage future teachers to read, discuss and experiment with researched practices.

6.4.4.3 Align Opportunities to Learn from the Perceptions of Educators and Future Teachers

Some mismatches were found between perceptions of opportunities to learn some components of the NIE programmes as reported by the educators and future teachers. One significant area to probe further is the frequency of using interactive learning experiences such as future teachers asking questions and discussions during lessons. Periodic surveys like the one used by TEDS-M for opportunities to learn by educators at NIE may help them keep NIE programmes relevant and prepare future teachers who are also ready for the rapidly changing learning spaces of the future.

6.4.4.4 Strengthen Commitment to Teaching as a Lifetime Career

Although NIE student teachers had expressed more favourable commitment to teaching as a lifetime career when compared to their international counterparts, there were 20% of first-career future teachers who were not fully committed. Steps should be taken while these future teachers are at NIE to acquaint them with the various challenges and achievements of being a teacher.

6.4.4.5 Learn from Other High-Performing Countries

Chinese Taipei and Russia performed better than Singapore in MCK and MPCK for some groups of future mathematics teachers. When they did perform better, the differences in scores were much larger than those of NIE student teachers. It is valuable for educators at NIE to learn about teacher education systems of these two countries through study visits and research collaborations.

6.5 Conclusion—Why Singapore Students Do Well in TIMSS and PISA?

This chapter has put forth the performance in mathematics for both students and future mathematics teachers in Singapore, in international benchmark studies TIMSS, PISA and TEDS-M. As noted by Barber and Mourshed (2007) in the McKinsey report "The quality of an Education System cannot exceed the quality of its teachers" (p. 16), it is apparent from the findings of the TEDS-M study that mathematics teachers in Singapore are one of the contributory factors for the commendable performance of their students in Mathematics. An analogy to Barber and Mourshed's claim that the quality of teachers in any education system is significantly dependent on the quality of teacher educators in that system is also supported by the findings of the TEDS-M for Singapore. Therefore, it appears that the quality of both mathematics teacher educators and mathematics teachers partly explains the performance of Singapore students in TIMSS and PISA.

Since the introduction of the New Education System (NES) in 1979 (Goh and The Education Study Team 1979), Singapore has dedicatedly pursed the vision of a high-quality education system that devotes attention and resources not only to high achievers, but also to lower level achievers. In line with the vision, the school mathematics curriculum has undergone periodic revisions since the 1980s, to remain relevant and keep abreast of development in the world around. It has been detailed in Chap. 2 how the education system has developed so far and in tandem how the school mathematics curriculum has also evolved into one that provides for every child in school. The curriculum lays a solid foundation in mathematics for all students in elementary grades, which seems to play a core role in students' later success. From upper primary onwards, students are assigned specialist teachers in mathematics. From upper secondary onwards, a range of specialized mathematics courses at higher levels are available for students who are interested to build up their strengths. It is apparent that the government invests wholeheartedly in education. One may say that the school levels the playing field for all students. Students who are lacking in progress are identified almost immediately and helped to overcome difficulties and allowed to achieve.

Since 1981, the curriculum has adopted the Concrete-Pictorial-Abstract approach to the teaching and learning of mathematics. This approach provides students with the necessary learning experiences and meaningful contexts, using concrete hands-on materials and pictorial representations to construct abstract mathematical knowledge. The system-wide guides of the intended curriculum issued by the Curriculum Planning and Development Division of the Ministry of Education place emphasis on the scope and sequence of topics taught at the respective grade levels. It makes clear the nature of the spiral curriculum and the student-centric learning experiences necessary for the acquisition of deep mathematical knowledge. The principles of teaching and phases of learning detailed in the guides make apparent that deep conceptual knowledge and procedural fluency must be the goals of mathematics instruction. Students must through exploration, clarification, practice and application over time represent mathematical concepts in multiple ways and apply them to solve problems in unfamiliar situations. Therefore, it also appears that the education system and school mathematics curriculum contribute in part towards the success of Singapore's students in TIMSS and PISA.

Singapore students' strong drive and motivation to learn mathematics are key to their performance in the subject. In addition, the high expectations of students by teachers and parents certainly impact their performance. In short we may say that society, in Singapore, as a whole places a premium on education.

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Part II
Teaching and Learning Practices in Singapore Mathematics Classrooms

Chapter 7
Problem Solving in the Singapore School Mathematics Curriculum

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Abstract Problem solving has been the heart of the Singapore school mathematics curriculum since the early 1990s after being adopted as the goal of school mathematics education. Since its adoption, it has captured the interest of many Singapore educators and researchers. It appears that problem solving will continue to be a very active research area since there is great interest in the very high level of performance of Singapore students in international comparative studies such as TIMSS and PISA. This chapter begins with a re-categorization of the research work done to date on problem solving in Singapore using the Singapore mathematics curriculum framework by integrating two classifications done by Foong in 2009 and Chan

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in 2014, respectively, and including work done since 2011 that was not reported in either survey. The earlier research focused on addressing the readiness of students for mathematical problem solving (MPS) from the perspective of the Singapore mathematics curriculum framework; the later research tended to emphasize the enactment of MPS in the Singapore mathematics classroom and teacher education. This chapter gives more detail to this later research with an emphasis on the enactment of Pólya's stages in solving structured problems.

Keyword Mathematical problem solving · Pólya's model · Pre-service teacher education · Real-world context

7.1 Mathematical Problem Solving and the Singapore School Mathematics Curriculum

The central goal of the school mathematics curriculum in Singapore is mathematical problem solving (MPS), as reflected in the School Mathematics Curriculum Framework shown in Fig. 3.1 in Chap. 3. MPS has remained as the central goal of the curriculum since its inception in the early 1990s in spite of the changing educational landscape over the decades.

The curriculum documents across several revisions (e.g. Ministry of Education (MOE) 1990, 2006) describe problem solving in terms of what it encompasses rather than as a definition of what problem solving is.

> Mathematical problem solving includes using and applying mathematics in practical tasks, in real life problems and within mathematics itself. In this context, a problem covers a wide range of situations from routine mathematical problems to problems in unfamiliar contexts and open-ended investigations that make use of the relevant mathematics and thinking processes.
>
> (MOE 1990, p. 6)
>
> Mathematical problem solving is central to mathematics learning. It involves the acquisition and application of mathematics concepts and skills in a wide range of situations, including non-routine, open-ended and real-world problems.
>
> (MOE 2006, p. 3)

7.2 Why Research on MPS?

At the beginning of this millenium, Dong et al. (2002) did a short survey on more than 100 students from five junior colleges across Singapore on their readiness to handle non-routine problems in mathematics. The preliminary study showed that there was a considerable difference in the achievement among the students on routine and non-routine problems. These students lacked skills and techniques which are considered crucial for MPS. Moreover, MPS at the primary school level in Singapore is usually equated with solving word problems which constitute at least 60% of the high-stakes national mathematics examination taken at Primary Six (sixth grade) (Lee 2014). The close link between assessment and curriculum led to the belief that word problems are the focus in the Singapore primary mathematics curriculum and has led many primary schools to teach problem solving by teaching word problems. In addition, although the international comparative studies PISA and TIMSS have shown that Singapore students generally have achieved a very high level of competence in school mathematics, an in-depth analysis of the results of these studies has also noted a relatively weaker performance of Singapore students on solving unfamiliar problems (Kaur 2009).

As evidence from research suggests that Singapore students might not be very well-prepared in handling non-routine problems, it is not surprising that research in Singapore on MPS in school mathematics in order to support classroom practice or inform curricular policy with research-based evidence continues to receive much attention in the Singapore context. This was also evident in Teong et al. (2009). The study included more than 150 mathematics lessons from several primary and secondary schools in Singapore, with this as one of the findings:

> [teachers] generally read the problems, executed the solution and checked the answers. There was very little dwelling on the exploration or the planning aspect of the solutions. … The emphasis appeared to be more to address the skills and procedures needed to solve problems than to tackle fresh problems anew where students have more chance of grappling with understanding and thinking about how to solve the problems. (p. 84)

Two reviews of MPS research have been carried out thus far. Foong (2009) categorized the research from the 1980s up to 2007 into three broad strands, namely (a) approaches and tasks, (b) teachers' beliefs and practices, and (c) students' problem solving behaviours. Chan (2015) looked at MPS involving students from 2001 to early 2011. He categorized the research into six broad categories as follows: (a) model-drawing method, (b) choice of heuristics, (c) open-ended and real-world problems, (d) metacognition, (e) sense-making, and (f) affective domain. This chapter summarises the various MPS research from the two reviews and adds details on more comprehensive research on MPS carried out recently since the later part of the last decade.

Numerous local studies (e.g. Foong et al. 1996; Foong 2009) indicate that up to 2009, MPS was still mostly theoretical talk and not common in classroom enactments. Foong et al. (1996) reported that some mathematics teachers have expressed their inadequacy in implementing the intended curriculum for MPS. Alan Schoenfeld wrote in the 2007 special issue on problem solving of the journal ZDM that the prevailing focus should lie in translating decades of theory building about problem solving into workable practices in the classroom:

> That body of research—for details and summary, see Lester (1994) and Schoenfeld (1985, 1992)—was robust and has stood the test of time. It represented significant progress on issues of problem solving, but it also left some very important issues unresolved. … The theory had been worked out; all that needed to be done was the (hard and unglamorous) work of following through in practical terms. (Schoenfeld 2007, p. 539)

This has spurred the interest of researchers in the Singapore National Institute of Education (NIE) to research on the feasibility of doing the "hard and unglamorous" work of realising the ideals of mathematical problems solving—as envisioned to be at the heart of the Singapore mathematics curriculum—into the daily practices of mathematics classrooms. This has opened a new dimension of research, i.e. in enacting the problem solving curriculum in mathematics classrooms.

7.3 Studies on MPS from 2001 to 2011

Figure 7.1 summarizes the research related to MPS from Foong (2009) and Chan (2015). The categorization provides a clearer depiction of the current state of research compared to the previous reviews in that it takes into account some of the overlapping strands for which the previous reviews did not capture and consolidated only those from 2001 onwards. Moreover, several studies recently located were appended to the list and the list of research is sequenced chronologically from 2001 to get a sense of the different strands of problem solving research carried out in the twenty-first century.

Figure 7.1 shows that studies involving problem solving heuristics are popular among researchers with 13 studies located. This is followed by 11 studies related to open-ended problems including solving real-world and modelling problems. Other strands like cognition and sense making, metacognition, affect and ICT make up about five studies each. Studies on the affective domain began from 2005 while studies related to ICT appeared to have dwindled after 2004 with only one study in 2009.

From 2009 onwards, there is a surge of research publications on MPS under the category of problem solving curriculum. This is an indication of a new focus from 2009 onwards on enacting MPS in the mathematics classroom and is a response to the disturbing observation that MPS is seldom realized in the classroom.

At that time, the authors, who were either teaching mathematics content or mathematics pedagogy at the tertiary level and who had vast school teaching experience prior to joining the university, felt strongly that mathematical problem solving had not been enacted according to its true spirit in school mathematics classrooms. The processes of problem solving were not stressed sufficiently because such processes were not eventually assessed in the high-stakes national assessment. Extensive anecdotal evidence suggested that teachers were mainly focused on preparing their students for examinations by equipping them with the ability to solve a fixed repertoire of exam-type problems. In fact, even challenging problems and problems that involve application in the real world were "routinized"—taught in a way that they eventually became routine—for the students.

From the perspective of mathematicians, this is the incorrect sequence of teaching mathematics and, more unfortunately, defeats the purpose of mathematics education, which is to prepare students to be able to handle unseen problems instead of memorizing various algorithms to tackle different types of mathematics problems. Figure 7.2

Strand	**References**	**Total number**
Heuristics and Model Drawing	Wong (2002); Wong & Lim-Teo (2002); Ho & Hedberg (2005); Hedberg, Wong, Ho, Lioe, & Tiong (2005); Ng & Lee (2005); Ng (2006); Yeo (2006); Ho (2007); Poh (2007); Wong (2008); Goh (2009); Looi & Lim (2009); Yeo (2011).	13
Open-ended / real world problems	Seoh (2002); Lee (2002); Ng (2003); Chow (2004); Chang (2005); Chan (2005); Chua & Fan (2007); Fan & Zhu (2008); Ng (2010); Chan (2010); Kaur & Toh (2011).	11
Cognition & Sense Making	Ho, Lee & Yeap (2001); Chang (2004); Foong (2005); Teo (2005); Heng (2007)	5
Metacognition	Teong (2003); Teo (2006); Lioe, Ho & Hedberg (2006); Wong (2007); Lee (2008)	5
Affect	Tan (2002); Teo (2005); Yeo (2005); Toh (2009); Chan (2011)	5
ICT	Hung (2001); Lee (2002); Teong (2003); Ibrahim (2004); Looi & Lim (2009)	5
Problem solving curriculum	Fan & Zhu (2007); Quek, Toh, Leong, Dindyal, & Tay (2009); Dindyal, Toh, Quek, Leong, & Tay (2009); Leong, Quek, Toh, Dindyal, & Tay (2009); Chan (2010); Quek et al. (2010); Dindyal et al. (2010); Leong, Toh, Quek, Dindyal, & Tay (2010); Toh, Quek, Leong, Dindyal, & Tay (2009); Toh (2010)	10
Others (problem posing, language proficiency; CL)	Hung (2001); Ho, Lee & Yeap (2001); Foong (2002); Yeap & Lee (2002); Quek (2002); Fan & Zhu (2007); Ho (2007); Chua & Fan (2007)	8

Fig. 7.1 Classification of studies on MPS from 2001 to 2011

shows the relationship between mathematics problems and exercises, and learning mathematics.

The original intent of various types of mathematical questions (both exercises and non-routine problems, or "hard problems") serves to facilitate the students to learn mathematics. With the acquisition of the new mathematical knowledge, students are then prepared to solve questions on applications. However, teachers who anticipate that their students would not be able to do the hard problems would end up teaching them how to solve these instead—and often through "routinizing" them. The solution to this conundrum is for students to first learn *about* problem solving so that they will be able to attend to the hard problems independently and so learn much more about the mathematics content through personal ownership and reflection (Pólya's fourth stage).

To emphasise again, the reality in mathematics classrooms was that teachers were teaching students all the mathematical content knowledge so as to solve the exercises, the hard problems and problems on applications. This was driven by the high-stakes school and national assessments, as teachers would not want to disadvantage their

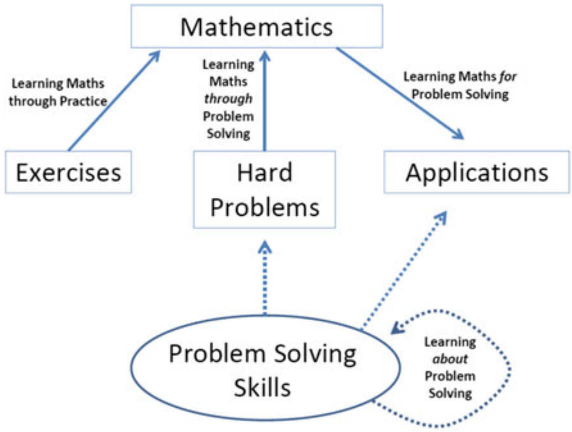

Fig. 7.2 Learning mathematics and problem solving

students with less preparation for these examinations. This, unfortunately, ran in direct contradiction to the spirit of MPS.

Driven by a desire to restore the original spirit of MPS, NIE researchers began to work on how to successfully enact MPS in the mathematics classroom.

7.3.1 Enactment of MPS in Mathematics Classroom

The surge in research publications under the "Problem Solving Curriculum" strand in Fig. 7.2 was mainly the result of the work of a team of researchers comprising Toh, Quek, Leong, Tay, and Dindyal who from 2008 to 2011, embarked on studies to actualize the intent of the problem solving curriculum. This research project was named MProSE (an acronym for Mathematical Problem Solving for Everyone)—a reflection of the researchers' belief that MPS should be meant for the general student population, rather than reserved for the elite few.

MProSE was a design experiment that focused on the secondary school mathematics curriculum with the intention of infusing problem solving into the regular mathematics curriculum and pedagogical practices across all levels in the school as a long-term plan. At the phase from 2008 to 2011, MProSE was implemented in a school specializing in mathematics and science as the testbed for its design. This was based on the "best-case scenario" method to start the investigation with high-ability mathematics students in a school that clearly emphasized the development of the mathematical ability of its students to the fullest. It was argued that the testbed for the initialization phase of an innovation should be at the school that is most conducive for success. With the "success case" achieved using the best-case scenario, researchers would then be able to understand the critical factors that led to its success, thereby how problem solving could be tweaked to meet the demand of the mainstream schools.

7.3.1.1 Theoretical Framework of MProSE

MProSE was developed based on the classical four-phase Pólya's problem solving model (1954) with an overlay of Schoenfeld's four components (1985). MProSE introduces a new paradigm of perceiving mathematical problem solving in the school mathematics curriculum as similar to the science practical lessons in the school science curriculum. It envisages the need for this new "practical paradigm" to convince school leaders and teachers of the need for curriculum time for problem solving.

The MProSE curriculum consists of specialized lessons introducing mathematical problem solving in which students learn the various aspects of problem solving through these specialized lessons. The distinct characteristics of these specialized lessons are that (1) each lesson focuses on solving one particular problem with an emphasis on the problem solving strategies in each lesson and (2) the mathematical content forms the background of each lesson with the objective of providing the context to introduce the various aspects of mathematical problem solving.

Schroeder and Lester (1989) provide three ways to understand the role and purposes of problem solving with respect to the overall mathematic curriculum:

1. Teaching *for* problem solving
2. Teaching *about* problem solving
3. Teaching *through* problem solving.

MProSE stresses the importance of teachers modelling and explicitly teaching the language and strategies of problem solving to students (i.e. teaching *about* problem solving), with its long-term plan to make mathematical problem solving as a new pedagogical approach to teach new mathematical content (i.e. teaching *through* problem solving).

Another aspect of MProSE that should be noted is the accompanying teacher scaffolding via Pólya's model and the holistic assessment strategy included in the MProSE teaching package. A full description of the teacher scaffolding provided via the mathematics "practical worksheet" is described in Toh et al. (2011a). The guiding principle for the teacher scaffolding in the MProSE specialized lesson is that teachers should start helping a student stuck in a particular mathematics problem with a general guide based on Pólya stages, rather than beginning by prescribing how students should solve the problem. The next level of scaffolding is for the teacher to provide heuristic-related prompts. Only when even this level of help fails, the teacher provides problem-specific hints. Throughout the entire scaffolding process, teachers will reinforce the use of problem solving language à la Pólya, in order to develop in students the thinking and habits of problem solving.

The assessment strategy in MProSE evaluated students' processes in solving a problem in addition to the accuracy of their solution of the mathematics problem since it was strongly felt that in problem solving the processes are as valuable as the product. A full description of the assessment strategy is found in Toh et al. (2011b).

7.3.1.2 Research Methodology

MProSE uses design experiments (Brown 1992; Collins 1999; Wood and Berry 2003) as the methodological backbone. Design experiments arose from attempts by the education research community to address the demands of conducting research in real-life school settings in all their complexity. It works on designing an educational product that is adapted for use in the school via a series of implementation-research feedback loops. Design experiments use multiple methods, such as participant observation, interview, video recording, and paper-and-pencil testing to provide corroborative evidence for findings. The envisaged outcome of MProSE was to produce a workable "design" for learning MPS in all Singapore schools. Starting in one school in Phase I, MProSE scaled up to another four schools in Phase II, following Gorard (2004): "[t]he emphasis [in design experiments], therefore, is on a general solution that can be *transported* to any working environment where others might determine the final product within their particular context [italics added]" (p. 101).

7.3.1.3 Findings of MProSE

Several papers were published on the findings related to implementing problem solving in the mainstream secondary school mathematics curriculum, beginning from 2008. The first paper that describes MProSE appeared in Toh et al. (2008), and a detailed description of the entire MProSE curriculum was published in Toh et al. (2011a, b). An outline of the MProSE problem solving module, together with the sample mathematical problems, is available in the website http://math.nie.edu.sg/mprose. In the papers, the researchers showed the importance of explicitly teaching a problem solving model to students, complemented with a common "problem solving" language for discussion.

The MProSE approach of enacting problem solving in the mathematics curriculum encompasses a wide range of issues: (1) students' belief and response to MPS; (2) teachers' belief and pedagogical practices related to MPS; (3) professional development of teachers related to MPS. We shall report on the various findings from the various papers related to the MProSE approach to MPS.

Students' Belief and Response to MPS

The students in the MProSE research school went through the entire module of problem solving. It was reported that most of the students could complete the first three stages and apply the heuristics in solving the problem. Many of them had also demonstrated Pólya's fourth stage to some extent: checked the reasonableness of their solution, provided alternative solutions, or generalised the given problem by offering at least one related problem (Toh et al. 2011b).

As reported in Dindyal et al. (2012), students in the MProSE research school found the MProSE lessons useful. Prior to the publication of Dindyal et al. (2012), the MProSE team conducted an interview on three students from the initial MProSE research school (Toh et al. 2011b). These three students represent each of three bands (higher, middle, and lower abilities) in that school. To the highest ability students in mathematics, MProSE was seen to be complementary to their higher level of mathematical training in that it helped them to regulate their "cognitive resource" that they had been equipped through other mathematics content training. The middle-ability students appreciated the problem solving process skills, in particular, Pólya's stage four of Check and Expand (Pólya used "look back" to describe stage although "check and expand" reflects more clearly the original spirit of Pólya), which was initially perceived by them as only belonging to the domain of mathematicians. The lower-ability students felt "coerced" to learn the entire problem solving process, which was seen by them initially as redundant. However, as these processes were being assessed, they had to go through the processes and they eventually realized the value of being equipped with these problem solving skills.

7.4 Studies on MPS from 2012 Onwards

MPS has continued to receive emphasis in the Singapore schools. A similar emphasis on MPS was also introduced in teacher education, so as to prepare the teachers to be ready for Singapore schools. MPS was also introduced into the teaching of undergraduate mathematics education for student teachers. This strong cohesion in the development of MPS in schools, teacher education and undergraduate mathematics education is the key feature of research on MPS that has been taking place from 2012 onwards.

This section discusses the research done on MPS that took place from 2012 onwards. Three main directions on MPS took place at this juncture, some following from the research that had been conducted in the earlier years and some in response to the revision of the school mathematics curriculum then:

(1) MProSE, an effort to introduce the holistic approach to teaching MPS, was seen to be successful and sustainable in the first research school. Further funding was obtained to scale up the MProSE research model to other mainstream schools.
(2) With the emphasis on Modelling and Application of Mathematics in the real world in the curriculum revision of 2009, there was an increased interest in introducing students to MPS in real-world contexts.
(3) Although pre-service mathematics teachers had always been exposed to MPS in their teacher education programme, there was a heightened interest in infusing MPS in the student teachers' undergraduate content course. This was done with the intention of enabling them to have first-hand experience of struggle and success in MPS.

This chapter focuses on (1) and (3), and item (2) on Modelling and Application of Mathematics in the real world will be discussed in Chap. 8.

7.4.1 *Scaling up the Enactment of an MPS Curriculum in Singapore Schools*

The first MProSE project in 2008 was carried out in one school specializing in mathematics and science. The success in that school managed to attract the attention of a range of Singapore schools who were eager to participate in this study on MPS. This led to the eventual scaling up of MProSE to four new schools, which covered the whole range of Singapore schools (independent school, autonomous school, and mainstream school), with the initial MProSE school continuing in the second phase of this research which focused on the sustainability of MPS in its regular mathematics curriculum.

As the MProSE research design was transported to a wider range of Singapore schools, the package was tweaked to meet the needs of the schools. Although the core MPS design of MProSE was content appropriate to the demands of the school and aligned to the Singapore mathematics curriculum, it still had to be adapted to meet the particular needs, student ability, and teacher readiness of the new schools.

Firstly, the researchers worked in collaboration with the teachers of the participating schools in crafting appropriate mathematical problems to be used in the adapted MProSE lessons. The criteria for crafting appropriate problems for MProSE MPS lessons were that: (1) the mathematical content of the problems must be aligned to the school mathematics curriculum, so that the students would have the "cognitive resource" to tackle these problems; and (2) the problems must not be the typical examination-type questions, as this would defeat the purpose of introducing the importance of MPS to students.

Secondly, the original lesson plans were modified to meet the constraints of the individual schools. The original proposal by Toh et al. (2011a) of using 10 lessons for MProSE was subsequently modified to meet the constraints of the schools: unlike the first MProSE School, the other more mainstream schools were less ready to allocate a total of 10 additional hours for introducing MPS. Eventually, the 10-hour lesson MProSE package was condensed to 6–8 h, without compromising on the coverage on the various aspects of MPS.

Despite the customization and adaptation of MProSE to the other schools, the following parameters of the design (see Gorard 2004) could not be and were not compromised:

1. MPS is meant for *every* student, rather than for the elite few. As such, if the MProSE package is to be implemented, it should be meant for every student in the particular level.
2. MPS must be assessed. The students' performance in the MProSE lessons must count towards a significant part of their continual assessment.

7.4.1.1 Findings in the Second Phase of MProSE

In the study in the five participating schools, it was observed that generally the students were able to meet the MPS demands of MProSE. The students who were interviewed after participating in MProSE generally asserted that MPS has enabled them to solve mathematical problems which they were unable to solve initially (Ho et al. in-print).

The teacher interviewees also agreed that it was important that MPS be introduced in their school curriculum. In particular, the interviews revealed that visible success of the MProSE as an educational intervention as well as the facilitation in the MPRoSE lessons brought about positive changes to both the students' and teachers' competencies with regard to MPS. Such visible successes also showed up at the school level, i.e. these produced deliverables that were aligned with the vision, mission, and goals of the schools, as well as professionally developed teachers and students' growth.

In turn, visible successes produced by MProSE helped to promote a state of "perpetual" flow of positive factors such as (1) earning support from school leaders, (2) gaining higher degree of autonomy and flexibility in planning for MProSE and its implementations, (3) nurturing positive attribution of teachers and students towards the second phase of MProSE, (4) making suitable modifications, adaptations and inventions made by teachers of the problem solving lessons, and (5) putting in place a continual professional development model for problem solving teachers.

7.4.1.2 Further Development of MPS

The problem solving approach developed in MProSE was adapted for another project Mathematical Progress and Value for Everyone (MProVE). This project focused on helping students who were making slower progress in the learning of mathematics compared to the majority of their peers. The typical academic profile of these students with respect to MPS was that of avoidance, low levels of persistence, and over-reliance on teachers' step-by-step instruction. In response to this, the MProVE team designed a suite of lessons to help students develop a problem solving disposition, which is marked by a willingness to try problems, improving on the strategy, and moving beyond the solution by extending the method. The students were mostly able to proceed positively with the problem solving attempts—including making relevant modifications—all the way to pushing beyond the original problems with little direct intervention by the teachers. A fuller description of the problem solving lessons and the students' responses is found in Leong et al. (2013).

With the emphasis on Mathematical Modelling and Application of Mathematics in the prescribed school curriculum (MOE 2006, p 16), studies began to be conducted on MPS using problems in real-world context. This is a relatively new domain in problem solving in the local context and Chap. 8 is devoted to discussing teacher education and solving problems in real-world contexts.

7.4.2 Teacher Education Programme

Mathematics teachers must be familiar with MPS, which is the core of the Singapore mathematics curriculum. Thus, it is not surprising that researchers from the NIE, being the sole pre-service teacher education provider in Singapore, also conducted several studies on MPS on its student teachers. To introduce MPS to the Singapore schools, it is crucial to initiate student teachers into this entire paradigm on MPS in their pre-service teacher education. This section describes several studies that have been carried out on student teachers in both the curriculum studies (CS) and academic subjects (AS) component of their pre-service teacher education.

Student teachers in NIE are introduced to MPS firstly through lectures explicitly disseminating related knowledge and facts about MPS. In addition, the student teachers clarify the concepts and skills of MPS by being *engaged* in the processes of MPS during actual solving of non-routine mathematics problems in class. A typical process of engaging student teachers in MPS during pre-service teacher education is described in Kaur and Toh (2011). They are given an *authentic* mathematical problem (which is non-routine to the student teachers, and which has multiple plausible solutions) and expected to solve the problem without first being introduced to any theory of problem solving. They are to reflect on the processes of solving the problem: (1) number of attempts up to the first successful attempts and (2) the strategies and heuristics that they have used in attempting to solve the problems. The student teachers are required to share their processes and, with the instructor as the guide, derive at the definition of a "mathematics problem". The student teachers are also given several additional problems to solve and are asked to generalize the processes they have used to solve these additional problems. This way of engaging the student teachers in MPS not only provides them with the theory and knowledge of MPS, but also their first-hand experience in MPS.

Dovetailing with the work in the secondary schools, the MProSE model was extended to the teaching of undergraduate mathematics in two undergraduate mathematics content courses at the NIE: (1) number theory course at Year 1 and (2) differential equations course at Year 3. In the two courses, the instructors used the MProSE design to different extents in teaching the courses. The intent was to equip the student teachers with MPS skills for their own acquisition of new undergraduate mathematical content knowledge (learning *through* problem solving).

Following the successful experimentation of infusing MPS into the content courses in pre-service teacher education programme and an undergraduate mathematics programme curriculum review, a new MPS module was introduced to all Year 1 student teachers as a general elective. The three subsections that follow describe these three infusions of MPS into the pre-service teacher education programme in the NIE. We give substantial detail in these sections because we think that these are somewhat unique in pre-service teacher education.

7.4.2.1 Undergraduate Number Theory Course

The origin of using problem solving approach of the MProSE design arose from the instructor's (Toh et al. 2014) prior experience in teaching undergraduate mathematics courses and frustration when student teachers waited passively for the instructor for answers and solutions when they encountered difficult questions which they could not make much progress. Undergraduate students in Year 1 have not read courses on mathematics curriculum studies (CS); hence, they have not been exposed to Pólya's model of problem solving at this point.

Along the line of thought of weaving in problem solving approach into the undergraduate number theory course, the instructor did not compromise the rigour that is expected of any typical first undergraduate mathematics course. First, the instructor identified the types of questions in the number theory courses that are suitable or otherwise for problem solving—questions that provide a clear approach of tackle do not belong to this category of questions for problem solving.

The instructor was careful not to introduce many structural changes to the course because he wanted to ensure that this course covered the usual content of the first undergraduate number theory course. Instead, he used the theorems in number theory as a *context* for introducing Pólya's problem solving model.

He distinguished the problems in his course into two categories: (1) those that were "straightforward" problems and (2) those that were amenable to problem solving. Category (1) consisted of those problems in which the method is prescribed in a direct manner while (2) consisted of those problems in which the methods of solution were not immediately obvious (see Fig. 7.3 for an example). As the method of solution of problems from Category (2) was usually not immediately obvious, it was an opportunity for the instructor to introduce the entirety of Pólya's stages, beginning with Stage I: the importance of *understanding the problem*.

For Category (1) problems, the instructor would teach using the usual exposition since the method of tackling the problem had been clearly prescribed. For Category (2) problems, he would seize the opportunity to introduce Pólya's problem solving model and demonstrate the Pólya's stages and model the use of problem solving heuristics to solve these problems. This was done through the instructor thinking aloud, consistently using the language of Pólya in solving the problem. MPS was then weaved into the number theory course gradually throughout the semester.

The study anchored on the analysis of the student teachers' performance in one Number Theory problem, which, according to the instructor, was an unseen problem

An example of Category 1 problem: Prove, using mathematical induction, that $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$.
An example of Category 2 problem: Prove that if the product of two integers is odd, then both integers must be odd.

Fig. 7.3 Examples of problems belonging to categories 1 and 2

Let a, b, c be natural numbers satisfying $a + b + c = 2012$.
If we have $a!b!c! = m\ 10^n$ for some integers m and n,
where 10 does not divide m, find the smallest possible value of n.

Fig. 7.4 Problem to assess problem solving in the number theory course

whose genre was not taught explicitly in the lectures, but one in which a student with good problem solving disposition should be able to handle (Fig. 7.4).

In analysing the students' performance in the above item, it was reported that most of the students (46 out of 55) demonstrated their attempt to understand the problem and also the use of heuristics to understand the problem. The student teachers also demonstrated the use of more than one heuristics in attempting to solve the problem. It was also found that 48 out of 55 student teachers also attempted Pólya's Stage Four to "Check and Expand" the given problem. Fewer students (33 out of 55) attempted some form of generalizing and extending the given problem. It was also found that the student teachers' attempt to generalize and extend a given problem was only mainly restricted to changing the value of 2012 or increasing the number of variables. However, none of them attempted to discuss how their solutions could be adapted to solve the proposed extended problem.

The instructor attributed the student teachers' attempt in demonstrating the use of Pólya's stages in solving the problem to the assessment criteria of the course. The student teachers had been informed that they were being assessed on their problem solving assignment, and that they would also be assessed based on their exhibition of problem solving behaviour in addition to the correctness of their solution. The instructor also admitted that due to the attempt to balance the delivery of the mathematical content with incorporating elements of MPS, the full range of processes related to Pólya's Stage Four might not have been sufficiently emphasized to the student teachers, hence the limited variety demonstrated in Pólya's Stage Four.

7.4.2.2 Undergraduate Course on Differential Equations

At the same time that problem solving was infused in the teaching of first-year undergraduate number theory course as described in the preceding paragraph, a similar study was also carried out to infuse problem solving in the teaching of the third-year undergraduate differential equations course (Toh et al. 2013). The authors recognized this as an opportunity to model the teaching of problem solving through a mathematics content course.

The infusion of MPS into this course differs from the number theory course described in the previous subsection in that the course structure was re-designed to accommodate eight "mathematics practical lessons" in the sense of Toh et al. (2011a), which proposed the use of these specialized "practical lessons" to focus on teaching *about* problem solving. Consequently, the total number of lectures of this course was reduced by eight to sixteen lectures (instead of the originally allocated 24 lectures of one hour each).

In each practical lesson, the instructor first introduced one aspect of problem solving (see Toh et al. 2011a for the detailed lesson plan of the eight practical lessons) and engaged the students to solve a relatively challenging problem on differential equations, based on the content of the corresponding lecture in that or the preceding week. The student teachers were allocated 40 min to solve a given problem. The instructor then went over the solution of the problem while the student teachers performed peer marking. The instructor consciously highlighted the use of problem solving heuristics at various junctures while discussing the mathematical content.

Before the first practical lesson, the instructor revised the stages of Pólya's model and demonstrated how the stages could be applied to solve a problem in differential equations. The assessment rubric was introduced at the beginning of the first practical lesson, so that the student teachers were aware of how they would be assessed in these practical lessons. Each practical lesson was centred on one particular problem on differential equations, called the *Problem of the Day*. The student teachers were to assess their peers' solutions of the *Problem of the Day*. The instructor rode on this opportunity to introduce peer assessment as numerous researches has shown that any opportunity for student teachers to assess their own understanding of mathematical knowledge and that of their peers could be beneficial in their early professional development (e.g. McTighe and Wiggins 2004).

The instructor carried out the six problem solving practical lessons (the last two practical lessons were used to consolidate the students' learning about the mathematical content of this course). Their overall performance in the six *Problems of the Day* was summarised in Table 7.1.

It was encouraging to the researchers to notice that *all* the student teachers who submitted the practical worksheet after the practical lessons exhibit behaviour of problem solving in minimally demonstrating appropriate use of heuristics (except one student in Problem Three).

Despite problems three and six (two relatively difficult problems on mathematical proofs) being the relatively more challenging problems for the mathematics practical lessons, it is clear from Table 7.1 that practically all the student teachers exhibited the MPS behaviour using Pólya's stages and most of them arrived at least a partially correct solution. Most students were also able to exhibit some behaviour of Pólya's Stage Four (check and expand) to a certain extent for most of the problems. Generally, the students found it more difficult to check and expand problems that involved mathematical proof, as mathematical proofs involve mainly intensive deductive reasoning, and thus requires a thorough understanding of the mathematical concepts.

Despite the reduction of the number of lectures in this course, it was reported that the content covered in the course using this approach was not reduced, and the rigour of the course was not compromised. In fact, some of the intricate details of parts of the course were transported to the *Problems of the Day* of the mathematics practical lessons, during which the student teachers had more opportunity to explore in greater depth using their MPS tools (learning *through* problem solving). By engaging in MPS

Table 7.1 Student teachers' performance in the six problems introduced in the practical lessons

Correctness of solution	No. of students for problem					
	One	Two	Three	Four	Five	Six
Completely correct solution	35	41	24	45	45	22
Partially correct solution with appropriate use of heuristics	15	9	13	5	6	25
Incorrect solution with appropriate use of heuristics	1	0	13	1	0	4
Incorrect solution without use of appropriate heuristics	0	0	1	0	0	0
Stage IV: checking and expanding	No. of students for problem					
	One	Two	Three	Four	Five	Six
No attempt in Stage IV	27	0	31	3	1	11
Attempt to check reasonableness of answer	21	8	4	14	7	16
Attempt to check answer + either alternative solution or generalize the problem	3	11	13	13	7	15
Attempt to check answer + alternative solution + generalize the problem	0	11	3	21	36	9

during the lesson of the "new" problem, the student teachers in fact had first-hand experience and exploring with the mathematical content, which was traditionally covered by the typical lecture delivery mode.

7.4.2.3 Undergraduate MPS Course Introduced in Year 1

In a curriculum review ongoing since 2015, it was agreed that student teachers would benefit from the direct experience of MPS in their undergraduate mathematics education. A problem solving general elective module for student teachers parked under the academic subject was conceptualized and developed using the secondary school MProSE design as a template.

The course consisted of twelve 3-hour face-to-face lectures. In the first four lessons, the student teachers were introduced to the general principle of MPS. Pólya's model and the MProSE scaffolding worksheet were explicitly introduced to the students. The choice of the problems in these four lessons was made in consideration for the various aspects of MPS disposition and heuristics that were desired of students to do MPS. In the remaining eight lessons, the problem solving course focused more on tackling the challenging problems from the other mathematics content courses (finite mathematics and number theory) that the student teachers were pursuing at that time. This was an opportunity, not easily available in the secondary school context, for the students to appreciate how MPS can facilitate them to better learn their own mathematical content from specific fields.

A typical lesson consisted of the following structure: (1) discussion of the homework problems given in the previous lesson; (2) solving two problems in the class; and (3) assignment of two problems for homework (to be discussed in the next lesson). Throughout the entire lesson, various problem solving dispositions, habits, and heuristics were reinforced.

The instructor insisted on the use of Pólya's model throughout the course. Rather than perceiving this as an imposition on the part of the lecturer, the students actually appreciated this as they saw how the model enabled them to understand and solve the problem more efficiently. The interviewed student teachers highlighted that the lecturers assisted them with the direction on how to proceed to solve a problem, using the various levels of scaffold proposed in Toh et al. (2009). The levels of scaffold range from the most generic suggestions using Pólya's language (e.g. "Have you understood the problem?" "What is your plan?") to problem-specific hints.

Three student teachers were interviewed (Tay et al. 2016) and asked about how the problem solving course had helped them in the learning of the other undergraduate mathematics content course. The general response was that they were able to go through the entire MPS process when faced with a difficult mathematical problem. According to the student teachers, the heuristics learnt in the problem solving course were particularly useful.

However, the student teachers also pointed out that unlike the problem solving course, they were not required to write out explicitly the problem solving processes in solving the problems in other mathematics courses. However, the interview with several student teachers seemed to suggest that the problem solving processes had already been assimilated by them, as "it happens in the mind". When faced with a problem that cannot be solved immediately, they would record down the applicable Pólya stages almost immediately.

7.4.3 Infusion into the School Mathematics Curriculum

The research carried out on enacting MPS in the school mathematics classroom has resulted in the establishment of certain permanent features in the mathematics curriculum at various school levels. For example, in the MProSE research schools, the MProSE problem solving module has become permanent features in those schools. For the initial research school, the MPS module is a compulsory module for all Year 2 students. The school has tweaked the module to include several e-lessons for their students. The rationale is that selected theory portions of the lesson are to be viewed by the students before attending the face-to-face MProSE lessons. This would allow students to have ample time for hands-on experience in authentic problem solving.

In the other MProSE mainstream research schools, the problem solving module has remained a compulsory module for all their Sec One express stream students. In fact, one of the schools has built on the MProSE module to establish a common set of mathematical language grounded in MPS for all their students in their subsequent years. Another mainstream school has worked with several MProSE researchers to extend the MPS experience for upper-level students by infusing MPS through the use of replacement units. Readers can obtain more details about replacement units in Leong et al. (2016a, b). Not only that, the MProSE approach has influenced the development of the new H3 mathematics curriculum, in which one significant component of this subject emphasizes MPS. Team members of MProSE are commissioned by MOE to conduct four 2-hour workshops to teach *about* problem solving to a large segment of H3 mathematics students.

From professional development workshops on MPS conducted by NIE, several other secondary schools have developed their own problem solving modules and MPS approach to teaching mathematics, although their concepts are not entirely congruent to MProSE approach.

7.5 Conclusion

This chapter traces the development of MPS research carried out in Singapore since it became the heart of the Singapore mathematics curriculum. The earlier research focused on addressing the readiness of students for MPS from the perspective of the Singapore mathematics curriculum framework; the later research tended to emphasize the enactment of MPS in the Singapore mathematics classroom. Research on MPS has also moved beyond the schools to the teacher education programme in NIE. To a certain extent, these design research projects have made an impact on the implementation of MPS in the school curriculum.

The research done on enacting MPS in mathematics classrooms described above tends to be carried out in secondary level and above. Some educators and school leaders have reflected to the researchers that MPS disposition is best developed in children at the upper primary level, which is a crucial stage for habit formation. Perhaps a future direction for research on MPS should be research on the enactment of MPS in the primary school mathematics classroom.

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Chapter 8
Innovative Pedagogical Practices

Joseph B. W. Yeo, Ban Heng Choy, Li Gek Pearlyn Lim and Lai Fong Wong

Abstract This chapter describes some innovative pedagogical practices in Singapore. It is divided into two main sections: pedagogies that engage the minds, and those that engage the hearts, of mathematics learners. Examples of such classroom practices include the Singapore Model Method to solve word problems in primary schools, the Singapore AlgeDisc™ to teach algebra in secondary schools, and guided-discovery learning. The main principle that underlies all these pedagogies that engage the minds of mathematics students is the Concrete-Pictorial-Abstract (C-P-A) approach. We also describe a theoretical framework on engaging the hearts of mathematics learners and the use of various strategies to make lessons interesting. Examples of such strategies include the use of mathematics songs and videos, television shows and movies, mathematics storybooks, drama, magic tricks, and mathematics puzzles and games. Some of these practices are not unique to Singapore but many local teachers are using them in their classrooms. Finally, this chapter also reviews limited local research on these pedagogical practices, and where there is no local research, we suggest some directions for future research.

Keywords Innovative pedagogical practices · Engaging minds and hearts · Model Method · AlgeDisc™ · Guided-discovery learning · Investigation

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8.1 Introduction

Innovative and powerful pedagogical practices in mathematics education include innovative and powerful mathematical learning environments, innovative practices that promote mathematics teaching and learning as inquiry, and mathematical tools that promote deep learning (Hunter et al. 2016). Innovative and powerful mathematical learning environments are formed when teachers establish classroom cultures (Leach et al. 2014), promote productive discourse (ibid.), and promote and maintain student engagement (Marshman and Brown 2014) to support productive mathematical activity. In order to promote mathematics teaching and learning as inquiry, the teachers may have to change their beliefs about social interactions within the classroom, their role and purpose, and classroom dynamics (Murphy 2015) so that they can notice and respond to student reasoning productively (Choy 2013). Mathematical tools that promote deep learning include challenging and ill-structured tasks with multiple entry and exit points that can sustain thinking and argumentation (Sullivan and Davidson 2014), and digital tools that can support and enhance learning (Lowrie and Jorgensen 2015).

In this chapter, we will describe some innovative pedagogical practices in the Singapore mathematics classrooms. Other than the well-known Singapore Model Method, AlgeDisc™ is a recent invention by the Ministry of Education of Singapore (MOE). Another teaching strategy that has been in use for many years is Bruner's (1961) guided-discovery learning. But the main principle that underlies all these innovative practices to engage the minds of mathematics learners in Singapore is still the Concrete-Pictorial-Abstract (C-P-A) approach, which was adapted from Bruner's (1964, 1966) enactive–iconic–symbolic model. We will also describe some classroom practices in Singapore that engage the hearts of mathematics learners, including the use of mathematics songs and videos, television shows and movies, mathematics storybooks, drama and art, magic tricks, and mathematics puzzles and games. Some of these pedagogies are unique or distinctive features of the Singapore education system, e.g. the Model Method and the AlgeDisc™, but the rest may be common practices in other countries as well. Lastly, we draw on some limited local research to examine the effectiveness of such pedagogical practices, and where there is no local research in this area, we suggest some research questions for future studies.

However, this chapter does not describe innovative pedagogies such as a mathematical problem-solving approach to teaching and learning, problems in real-world contexts and mathematical modelling, comics and the use of technology, because they are dealt with in Chaps. 7, 9, 13 and 14 of this book, respectively.

8.2 Engaging the Minds of Mathematics Learners: The Concrete-Pictorial-Abstract (C-P-A) Approach

As mentioned in Chap. 2, the Teach Less, Learn More (TLLM) initiative was launched in the education system in 2005 (Shanmugaratnam 2005). It aims to touch the hearts and engage the minds of our learners, to prepare them for life. To engage the minds of mathematics learners, MOE has used the Concrete-Pictorial-Abstract (C-P-A) approach, mentioned in Chap. 3, extensively in the development of primary school mathematics concepts (MOE 2007, 2012a). This approach is evident in both the MOE syllabus documents and the MOE-approved school textbooks. In Singapore, schools only use textbooks that are approved by MOE. Therefore, schools that use MOE-approved textbooks will also use the resources provided in the textbooks. Chang et al. (2017) gave the example of a teaching sequence in one of the textbooks where pictorial representation (P) in the forms of rectangular and circular models is used to introduce the abstract concept of equivalent fractions, i.e., $\frac{a}{b} = \frac{c}{d}$ (A). In addition, MOE has provided fraction strips and fraction discs to all primary schools so that students can use these concrete manipulatives (C) to learn the concept of equivalent fractions.

As mentioned in Chap. 3, the C-P-A approach has its roots in Bruner's notion of enactive, iconic, and symbolic representations of cognitive growth (Leong et al. 2015; Wong 2015). According to Bruner (1964, 1966), conceptual learning begins when a person undertakes and experiences some actions (enactive), which are then translated into images of the experience (iconic). Subsequently, links are formed to connect the iconic representations into a collective structure governed by a rule derived from organising common attributes found embedded in the representations. Eventually, this rule stands exclusively by itself and is denoted by a symbol.

The MOE syllabus documents specify the use of 'manipulatives or other resources' (MOE 2012a, p. 23, b, p. 23) in activity-based learning to construct meanings and understandings, and from 'concrete manipulatives and experiences' (ibid.), students are guided to uncover abstract mathematical concepts or results. So far, the examples given in the syllabus documents for 'manipulatives or other resources' include the use of paper cut-out of rectangles for primary school mathematics and the virtual balance to learn the concept of equations for secondary school mathematics. In other words, 'manipulatives or other resources' include both concrete and virtual manipulatives. However, 'concrete' in the C-P-A approach does not only mean manipulatives, but concrete 'experiences' (ibid.) derived from playing with the manipulatives.

For instance, using algebra discs or AlgeDisc™ to learn algebraic manipulation such as expansion and factorisation (which we will elaborate in Sect. 8.4) is an example of utilising the C-P-A approach from concrete to pictorial to abstract. But the Singapore Model Method (which we will elaborate in Sect. 8.3) is a pictorial representation and there is no concrete manipulative. Although AlgeBar™ was developed at a later stage as a virtual tool for students to draw the models, the computer application does not function as a manipulative in the sense that students cannot manipulate the models but they just use the application to draw the models. But there is really no need to always rely on concrete or virtual manipulatives because sometimes pictorial representations are good enough to provide students with the necessary concrete experiences to abstract mathematical concepts or to solve problems.

Similarly, when it comes to higher level mathematics, it is not always possible to find suitable manipulatives, whether concrete or virtual, for students to manipulate in order to abstract the concepts. Therefore, the first author has extended the C-P-A approach to include the use of concrete examples, which often involve numerical values for secondary school mathematics. Numerical examples are another way to offer students concrete experiences from which they can abstract the underlying concepts. In the Extended C-P-A approach, there may not be any pictorial representation. The main idea behind the Extended C-P-A approach is that what is abstract at one level may become more concrete at a higher level. For example, concrete objects are concrete to lower primary school students but numbers are more abstract. But when the students reach lower secondary level, numbers have become concrete and what is abstract is algebra. At university level, algebra has become concrete to many undergraduates and what is abstract for them is abstract algebra. We will illustrate the Extended C-P-A approach at the secondary school level with some examples in Sect. 8.5.

In the next three sections, we will describe three main innovative pedagogical practices used in Singapore to engage the minds of mathematics learners: the Singapore Model Method, the Singapore AlgeDisc™, and guided-discovery learning. Only the first two practices are unique features of the Singapore education system, while the last one has been in use in some other countries as well.

8.3 The Singapore Model Method

The Singapore Model Method, or simply the Model Method, was developed in the 1980s to address students' difficulties in understanding and solving mathematics word problems (Kho 1987; Kho et al. 2014). First introduced in 1983, the Model Method has since become a signature problem-solving heuristic in the Singapore primary school mathematics curriculum (Kho et al. 2014). The Model Method uses bar models to represent quantities and the relationships among the quantities given in a word problem. The fundamental idea underpinning the method is the assumption that if pupils were provided with a means to represent the relationships between quantities, then the structure of the problem would be made clear to the pupils, making visible the solution pathways (Kho 1987; Ng and Lee 2009). The method revolves around pupils drawing rectangles of appropriate lengths to represent quantitative relationships (Kho 1987). Despite its simplicity, the method offers pupils a visual way to solve complex word problems without the use of symbolic algebra because the bar models function as visual representations of algebraic equations (Kho et al. 2009). Hence, the method has also been used as a bridge to support primary and secondary school students in the learning of algebra. In this section, we will present

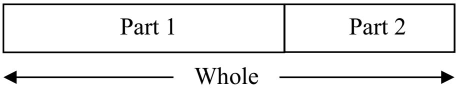


Fig. 8.1 Basic part-whole model

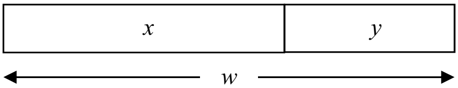


Fig. 8.2 Model representation of $w = x + y$

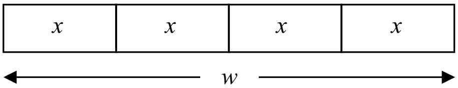


Fig. 8.3 Model representation of $w = 4x$

the two basic models of the Model Method, highlight its connection with algebra, review some of the challenges pupils faced when using the method, and suggest some possible directions for future research in this area.

8.3.1 Two Basic Models

There are two basic models in the Singapore mathematics curriculum, namely the part-whole models, and the comparison models. The Model Method builds on the pictorial representation of part-whole and comparison schemas—the building blocks of mental and cognitive processes for addition and subtraction (Kintsch and Greeno 1985; Nesher et al. 1982)—and extends the part-whole and comparison models to include multiplication, division, fractions, ratios, and percentages. In this section, we will introduce the two basic models. Interested readers may refer to the monograph by Kho et al. (2009).

The part-whole model shows the relationship between a whole and its part, or simply, a whole as comprising of two parts (see Fig. 8.1). Mathematically, this is represented as the whole w is divided into two parts x and y, i.e., $w = x + y$, as shown in Fig. 8.2. A part-whole model may also involve more than two parts.

In some problems, we may have to divide the whole into equal parts. For instance, the pictorial representation in Fig. 8.3 shows that the whole w is divided into four equal parts, i.e., $w = 4x$.

There are two other quantitative relationships that students can make use of depending on the information given. First, given two parts, students can find the whole by adding the two parts given, i.e., Part 1 + Part 2 = Whole (see Fig. 8.1). Next, when students are given the whole and one of the parts, students can perform subtraction to find the other part: Whole − Part 1 = Part 2.

The comparison model shows the relationship between two quantities when they are compared. The model involves three variables: the larger quantity, the smaller quantity and the difference (see Fig. 8.4). Mathematically, this model represents the

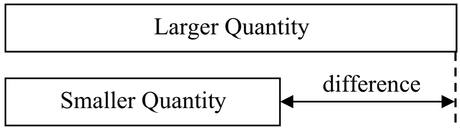


Fig. 8.4 The basic comparison model

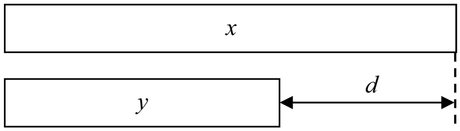


Fig. 8.5 Model representation of $x - y = d$

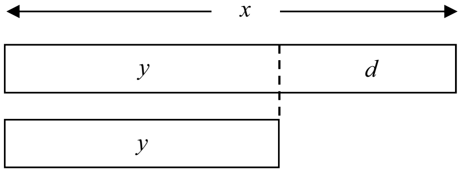


Fig. 8.6 Model representation of $x = y + d$

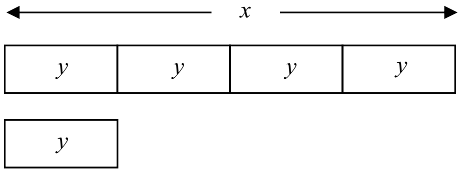


Fig. 8.7 Model representation of $x = 4y$

equation $x - y = d$, as shown in Fig. 8.5. As with the part-whole model, we can use this model to compare three or more quantities.

Depending on the information given, there are a few other quantitative relationships between the three variables. First, students can find the larger quantity by adding the difference to the smaller quantity, i.e., $x = y + d$ (see Fig. 8.6). Second, students can find the smaller quantity if they were given the larger quantity and the difference, i.e., $y = x - d$. In some cases, the sum of two or more quantities may be given.

Last but not least, the comparison model can be used to make this relationship visible to students when one quantity is a multiple of the other, e.g. $x = 4y$ in Fig. 8.7.

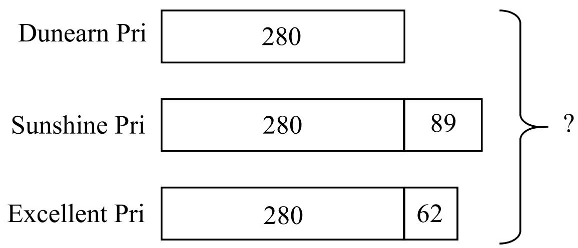
Referring to Figs. 8.3 and 8.7, we see that the part-whole model and the comparison model can be used to represent multiplication and division problems. This provides a means to represent fractions, ratios, and percentages. For example, Fig. 8.7 can be used to represent the relationship $y = \frac{x}{4}$, $x : y = 4 : 1$, or y is 25% of x. By using the models, students can represent complex quantitative relationships given in a word problem and use the visual models to find the unknowns. In the next section, we present some of the typical word problems in the Singapore Mathematics Curriculum and highlight how the Model Method can be used to solve these questions.

8.3.2 *Using the Model Method to Solve Word Problems*

The main strength of the Model Method lies in its affordance to represent quantitative relationships in word problems visually so that students can process the given information and use them to solve for the unknowns. The method can be used for arithmetical word problems, in which the relationship between the unknown and the known is clear, as well as algebraic word problems, in which an unknown needs to be introduced in the solution (Ng and Lee 2005, 2009). In this section, we will illustrate the use of the Model Method to solve word problems.

Example 1
Dunearn Primary School has 280 pupils. Sunshine Primary School has 89 pupils more than Dunearn Primary. Excellent Primary has 62 pupils more than Dunearn Primary. How many pupils are there altogether? (Ng and Lee 2005, p. 63)

Solution

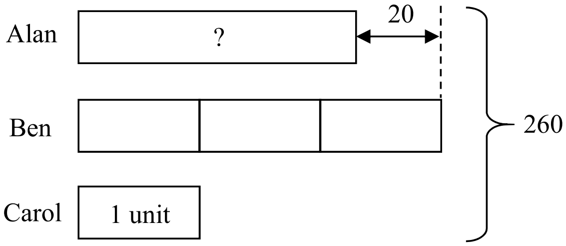


$$\begin{aligned}\text{Total number of pupils} &= 280 + 280 + 89 + 280 + 62\\ &= 280 \times 3 + 89 + 62\\ &= 991\end{aligned}$$

Example 2
$260 was shared among Alan, Ben and Carol. If Alan received $20 less than Ben, and Ben received 3 times as much money as Carol, how much money did Carol receive?

Solution

Let the amount of money Carol received be 1 unit.

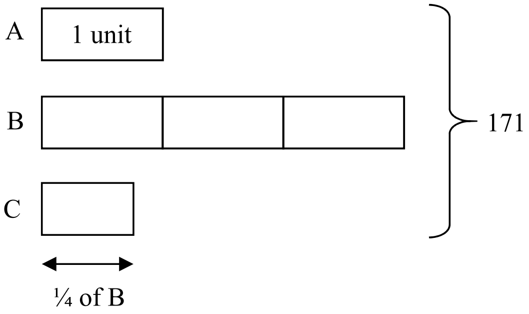


$$7\text{ units} = \$260 + \$20(\text{by assuming Alan had }\$20\text{ more})$$
$$= \$280$$
$$1\text{ unit} = \$280 \div 7$$
$$= \$40$$

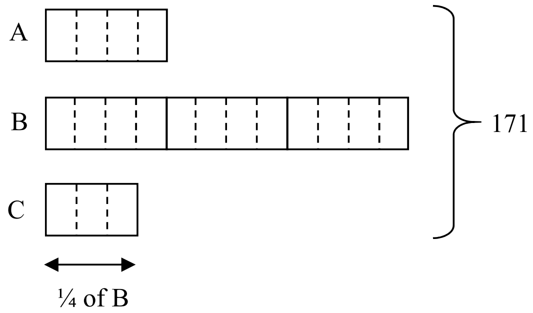
Therefore, Carol received $40.

Example 3
A tank of water with 171 l of water is divided into three containers, A, B and C. Container B has three times as much water as container A. Container C has $\frac{1}{4}$ as much water as container B. How much water is there in container B? (Ng and Lee 2005, p. 63)

Solution



Representing the information given in the problem, we see that the bar representing container C is less than the amount in container A, which is 1 unit. To make all the units the same, we divide the bar representing A into four smaller units (left as an exercise for the reader to figure out) and arrive at the following model:



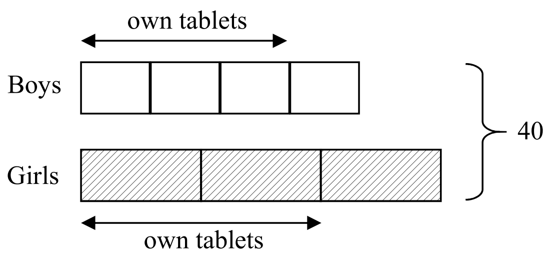
As we can see, the units are now of the same size. Therefore,

$$
\begin{aligned}
19\,\text{units} &= 171\,\text{litres of water}\\
1\,\text{unit} &= 171 \div 19\\
&= 9\,\text{litres}\\
12\,\text{units} &= 9 \times 12\\
&= 108\,\text{litres}
\end{aligned}
$$

There are 108 litres of water in Container B.

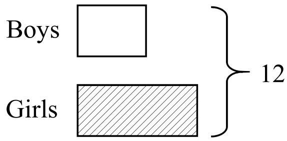
Example 4
In a class of 40 pupils, 75% of the boys and 2/3 of the girls owned a tablet computer. If 30% of the pupils do not own tablet computers, find the number of girls who own tablet computers.

Solution



This is a challenging problem because the units representing the boys and girls are different in sizes. However, the Model Method can be extended to solve simultaneous linear equations in two variables without using symbolic algebra.

Since one unit of boys and one unit of girls do not own a tablet computer, we know that one unit of boys plus one unit of girls equal $30\% \times 40 = 12$ pupils. This can be illustrated with another model below if need to, although it is not necessary to do so.



Referring to the first model above, we can see that there are 3 groups of (1 unit of boys + 1 unit of girls) = 3 × 12 pupils.

$$\begin{aligned}\text{Thus one unit of boys} &= 40 - 3 \times 12\\ &= 4\end{aligned}$$

So there are 4 × 4 = 16 boys, and 40 − 16 = 24 girls.
Therefore, 16 girls own tablet computers.

8.3.3 Linking the Model Method to Algebra

The Model Method solution to Example 4 resembles a typical algebraic approach. As Kho et al. (2014) highlight, many Secondary One pupils continue to use the Model Method to solve algebraic problems because they have difficulties formulating the equations. Students may then see the algebraic approach as redundant partly because they do not see the need to learn another method when they could solve the question using the Model Method. How do teachers support students to learn the algebraic method? One way is to guide students see the versatility of the algebraic approach by giving them word problems in which the Model Method may not be the best way to solve the problems. One common type of word problem that could not be solved easily using the Model Method is shown below:

Example 5
I have some sweets. If I give each student in my class six sweets each, I have five sweets left. If I give each of them seven sweets each, I am short of four sweets. How many students do I have?

The elementary solution for Example 5 usually involves students in presupposing certain conditions (we will leave the elementary solution to Example 5 as an exercise for the reader). This is often difficult for many students. However, with algebra, this question is trivial.

Solution
Let x be the number of students in my class.
Then $6x + 5 = 7x - 4$.
Therefore, $x = 9$.

Another way to convince pupils of the need for the algebraic method is to give them a word problem that involves a quadratic equation and let them try to solve

Table 8.1 Parallel presentation of Model Method and Algebraic Method

Model Method	*Algebraic Method*
Let the amount of money Carol received be 1 unit.	Let \$$x$ be the amount of money received by Carol.
[Draw model here. In addition, write letter 'x' in each box representing 1 unit.]	Then Ben had \$$3x$ and Alan had \$$(3x - 20)$.
7 units = \$260 + \$20 (assuming Alan had \$20 more) = \$280 1 unit = \$280 ÷ 7 = \$40 Therefore, Carol received \$40.	Thus $x + 3x + (3x - 20) = 260$ $7x = 260 + 20$ $= 280$ $x = 280 \div 7$ $= 40$ Therefore, Carol received \$40.

using the Model Method. Then they will realise the limitation of the Model Method: it can only be used to solve word problems that involve linear equations or even simultaneous linear equations in two variables, but not those that involve quadratic equations. Therefore, they will have to learn the algebraic method so that they can solve other types of word problems later on.

To tackle the issue of Secondary One pupils having difficulties formulating the equations, the Model Method can serve as a way to smoothen the transition from arithmetic to algebra (Kho et al. 2009, 2014; Ng 2003). Ng (2003) suggests that teachers should support students in making explicit links between the Model Method and its algebraic representation. One way to do this is through a parallel presentation of the two methods (Ng 2003). For example, referring to the question on sharing of money (Example 2), we could present the solution as shown in Table 8.1. Notice that the teacher should write the letter 'x' in each box representing one unit in the model on the left after letting \$$x$ be the amount of money received by Carol. Then, from both the given information in the word problem and from the model, students are led to see that Ben had \$$3x$ and Alan had \$$(3x - 20)$, so that they can see the explicit links between the Model Method on the left and the algebraic method on the right. The critical difference comes in the next step of forming the linear equation in the algebraic method, which has no equivalence in the Model Method. But the teacher can link the next step in the simplification of the linear equation to the Model Method, thus demonstrating to the students that the algebraic method is not very different from the Model Method.

For more examples on parallel presentations, the interested reader should refer to Kho et al. (2009, pp. 115–136). However, it is important to note that there are times in which the unit may not correspond to the unknown in the algebraic representation (Ng 2003). For example, referring to Table 8.2 which shows the parallel presentation for the Tablet Computer Problem (Example 4), we see that the unknown is one unit of boys (=4) in the Model Method, but the unknown is the number of boys b (=16)

Table 8.2 Different unknowns in Model Method and Algebraic Method

Model Method	*Algebraic Method 1:*
[Draw model here] 1 unit of boys + 1 unit of girls = 12 From the model, 1 unit of boys = 40 – 3 × 12 = 4 So there are 4 × 4 = 16 boys, and 40 – 16 = 24 girls. Therefore, 16 girls own tablet computers.	Let b be the number of boys and g be number of girls. Then $\frac{3}{4}b + \frac{2}{3}g = 28$ ------- (1) $b + g = 40$ ------- (2) (1) × 12: $9b + 8g = 336$ ------- (3) (2) × 8: $8b + 8g = 320$ ------- (4) (3) – (4): $b = 16$ Subst. in (2): $16 + g = 40$ $g = 24$ Therefore, 16 girls own tablet computers. *Algebraic Method 2:* Let b be the number of boys. Then there are $40 - b$ girls. So $\frac{3}{4}b + \frac{2}{3}(40 - b) = 28$ $9b + 8(40 - b) = 28 \times 12$ $9b + 320 - 8b = 336$ $b = 16$ ∴ 40 – 16 = 24 girls own tablet computers.

in the algebraic method. This may cause confusion to students, and teachers should be aware of the different unknowns when making links between the Model Method and the algebra method.

There are several variants of the canonical Model Method introduced by MOE. Many of these methods used a pseudo-Model Method approach, which is actually the algebraic method in disguise. For example, many experienced teachers, who did not learn the Model Method when they were primary school students, often write the following after they have drawn the model in Table 8.1: 1 unit + 3 units + (3 units − \$20) = \$260.

This is *not* the Model Method but the formation of a linear equation in disguise: one that involves '1 unit' as the unknown instead of 'x'. In fact, we have observed that some trainee teachers, who have learnt the algebraic method after learning the Model Method in primary school, also use this approach of forming a linear equation involving '1 unit' when using the Model Method: somehow, the new knowledge of the algebraic method has interfered with the old knowledge of the Model Method. How such pseudo-Model Methods support or hinder the learning of algebra remains an open question.

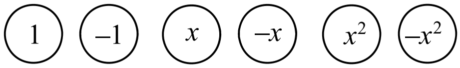
8.3.4 Local Research on Model Method

The Model Method works on the basis that students could translate the textual information given in word problems into a pictorial form, which provides a visual representation of the quantitative relationships involved. However, as Goh (2009) has found, middle-achieving and lower-achieving students may find it difficult to do the transformation, especially for multi-step word problems involving multiplicative relationships, or before–after contexts. Similarly, Poh (2007) also found her lower-achieving students struggling with the interpretation of word problems, transformation of textual information into pictorial forms, understanding the quantitative relationships encapsulated in the models, and seeing the connections between models and solution methods. In both cases, it appeared that students relied heavily on previously taught methods of drawing the models and their familiarity with problem types to work out the solutions. They may not have fully understood the relationships between the quantities and were unable to comprehend and solve the problem when the problem structure is unfamiliar, or when the problem involves an algebraic approach.

These issues suggest that it is necessary for teachers to provide opportunities for students to make sense of the quantitative relationships given in word problems, see the connections between the Model Method and operations involved in the solution, and think about the reasons behind the procedure (Goh 2009; Poh 2007). There may be a need to consider when, and how, the Model Method is introduced, especially for the lower-achieving students (Poh 2007). In particular, teachers should not assume that the translation process is intuitive, and may need to model the thinking process rather than merely presenting the solutions. In addition, teachers may want to open up the solution space of word problems and highlight other solution methods, when appropriate. This is important as the Model Method is not the only heuristic available to the students, even though many students may try to apply the method without understanding the problem.

Despite the various issues and difficulties, students may face when using the Model Method, the prevalence of the Model Method in our classrooms is definitely one of the unique features of our mathematics curriculum. Although the efficacy of the method has never been systematically studied on a large scale, anecdotal evidence suggests that the Model Method provides a way for students to think about quantitative relationships (Ng, S. F., personal communication, 22 November, 2017). It remains to be seen whether such a study would be carried in the future. The findings of such studies may be of interest because algebraic methods of solving simple linear equations involving one variable will be taught under the current syllabus for Primary Six students with effect from 2018. How the Model Method supports or hinders students' learning of algebraic methods may be a fertile area for research.

Fig. 8.8 Algebra discs or AlgeDisc™



8.4 The Singapore AlgeDisc™

While the Model Method has been used extensively in primary schools in Singapore since 1983, the use of AlgeDisc™ in secondary schools only began with Secondary One students three decades later in 2013. The current secondary school mathematics syllabus document (MOE 2012b) stipulates as learning experiences that Secondary One students should have opportunities to 'use algebra discs or the AlgeDisc™ application in AlgeTools™ to make sense of addition, subtraction and multiplication involving negative integers and develop proficiency in the 4 operations of integers' (ibid., p. 34), and Secondary Two students should have opportunities to 'use algebra manipulatives, e.g. algebra discs, to explain the process of expanding the product of two linear expressions of the form $px + q$, where p and q are integers, to obtain a quadratic expression of the form $ax^2 + bx + c$' (ibid., p. 40) and 'use the AlgeDisc™ application in AlgeTools™ to factorise a quadratic expression of the form $ax^2 + bx + c$ into two linear factors where a, b and c are integers' (ibid., p. 40).

Algebra discs or AlgeDisc™ consist of discs as shown in Fig. 8.8. When the '1' disc is flipped over, it will show '−1'. Similarly, when the 'x' and 'x^2' discs are flipped over, they will show '$-x$' and '$-x^2$', respectively. Flipping over only occurs when we take the negative of the number or the term shown on the disc.

AlgeTools™ is a dynamic software produced by the Ministry of Education of Singapore (Yeo et al. 2008). It contains the AlgeBar™ application and the AlgeDisc™ application: the former is used to draw models for the Model Method while the latter is used to draw algebra discs. In this section, we will not use the AlgeDisc™ application but we will just describe how algebra discs can be used to expand and factorise quadratic expressions in the manner specified in the current secondary school mathematics syllabus document (MOE 2012b), followed by some suggestions for research in this area.

8.4.1 Using AlgeDisc™ to Expand and Factorise Quadratic Expressions

We now turn our attention to the use of concrete manipulatives to teach expansion and factorisation of quadratic expressions. Many countries have been, and are still, using algebra tiles to represent quadratic expressions in pictorial form. For example, Fig. 8.9 shows an arrangement of algebra tiles used to represent the quadratic expression $x^2 + 5x + 6$. The large square tile has length x units, so its area is x^2 square units, while each small square tile has length 1 unit, so its area is 1 square unit. Each rectangular tile is of dimensions x units by 1 unit, so its area is x square units. Therefore, the

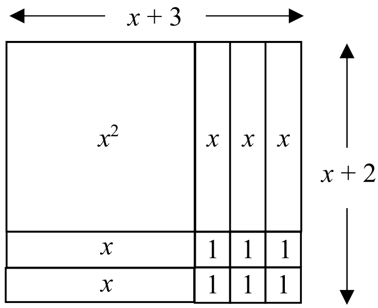


Fig. 8.9 Factorisation of $x^2 + 5x + 6$ using algebra tiles

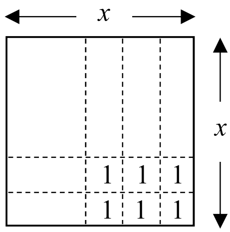


Fig. 8.10 Factorisation of $x^2 - 5x + 6$ using algebra tiles

total area of all the tiles in Fig. 8.9 is $x^2 + 5x + 6$ square units. It is important to take note that the algebra tiles do not have x^2, x or 1 written on them, but what is shown in Fig. 8.9 is for illustration purpose only. In order to factorise $x^2 + 5x + 6$, we have to arrange the tiles to form a rectangle. In this case, the rectangle in Fig. 8.9 has a length of $x + 3$ units and a breadth of $x + 2$ units, so its area is also given by $(x + 3)(x + 2)$, i.e., $x^2 + 5x + 6 = (x + 3)(x + 2)$.

But what happens when it comes to negative terms? For example, how do we use algebra tiles to represent $x^2 - 5x + 6$? Yes, there is a way to do this by covering one side of the x^2 tile with two x tiles, and an adjacent side of the x^2 tile with another three x tiles, and then add six '1' tiles because we subtract 6 twice, as shown in Fig. 8.10. However, many students fail to see why they have to add six '1' tiles.

Algebra discs or AlgeDisc™ do not have this problem because they do not use the concept of area. Figure 8.11 shows the arrangement of algebra discs for factorising $x^2 + 5x + 6$ and $x^2 - 5x + 6$. The arrangement of algebra discs for factorising $x^2 + 5x + 6$ in Fig. 8.11a is very similar to the arrangement of algebra tiles for factorising $x^2 + 5x + 6$ in Fig. 8.9, except that one uses discs while the other uses tiles. On the other hand, the arrangement of algebra discs for factorising $x^2 - 5x + 6$ in Fig. 8.11b is also very similar to the arrangement of algebra discs for factorising $x^2 + 5x + 6$ in Fig. 8.11a. In this way, algebra discs can help students deal with negative terms more easily than the method shown in Fig. 8.10 for algebra tiles.

But the use of the algebra discs is not an end in itself. There is still a need to abstract the algebraic manipulations of expansion and factorisation. To this end, the Ministry of Education of Singapore has introduced what they call the 'multiplication frame' as shown in Fig. 8.12. The multiplication frames in Fig. 8.12 look very similar

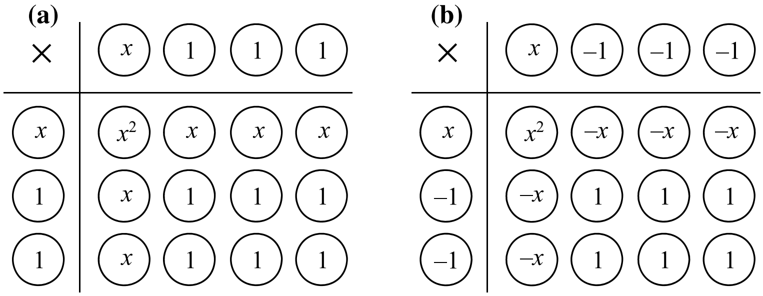


Fig. 8.11 Factorisation of $x^2 + 5x + 6$ and $x^2 - 5x + 6$ using AlgeDisc™

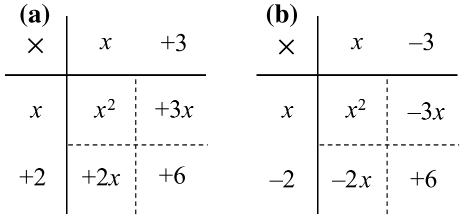


Fig. 8.12 Factorisation of $x^2 + 5x + 6$ and $x^2 - 5x + 6$ using multiplication frame

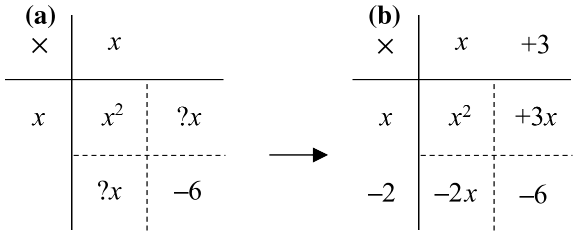


Fig. 8.13 Factorisation of $x^2 + x - 6$ using multiplication frame

to the arrangement of the algebra discs in Fig. 8.11, which is why it is more natural to progress from algebra discs to the multiplication frame than to the traditional cross multiplication method used to factorise quadratic expressions.

We will now illustrate how students are taught to factorise $x^2 + x - 6$ using the multiplication frame method without the use of algebra discs anymore. Just like the traditional cross multiplication method, students will need to find the corresponding factors of -6, i.e., $-6 = \pm 1 \times \mp 6 = \pm 2 \times \mp 3$. They will start with the multiplication frame shown in Fig. 8.13a, where the coefficients of x, represented by ?, are corresponding factors of -6 such that the sum of the coefficients is +1. The students can use guess and check, or some deduction to reason that the corresponding factors of -6 must be +3 and -2 since $3 - 2 = 1$. Then, they will obtain the multiplication frame shown in Fig. 8.13b. Therefore, $x^2 + x - 6 = (x + 3)(x - 2)$.

8.4.2 Local Research on AlgeDisc™

Prior to the implementation of AlgeDisc™ with Secondary One students in 2013, the Ministry of Education of Singapore have piloted the use of AlgeDisc™ with some classes and have found that the students benefited from the intervention programme. But they have not published any of their findings. As the use of AlgeDisc™ only began in recent years, there is currently no other local research in this area. For example, both Leong (2015) and Huang (2016) made use of algebra tiles, instead of AlgeDisc™, in their doctoral and Master's study, respectively. The only local paper on the multiplication frame method is a book chapter by Chua (2017), where he described how to use the multiplication frame effectively, without even mentioning algebra discs or AlgeDisc™ at all.

Therefore, we will highlight some issues related to the use of AlgeDisc™ in the teaching and learning of algebra and suggest possible directions for future research. Firstly, algebra discs may help students deal with negative terms better than algebra tiles but students may not understand the idea behind factorisation if there is no concept of area. How will this affect their learning? Secondly, one way to have the best of both worlds is to start with algebra tiles using the concept of area for positive terms and then change to algebra discs for negative terms. But will this be too confusing for students? Thirdly, Leong et al. (2010) have fused algebra tiles and algebra discs to become AlgeCards, which are similar to algebra discs except that they are in the shape of squares and rectangles. Unlike algebra tiles, algebra cards have two sides: on one side is written 1, x or x^2, but on the other side is -1, $-x$ or $-x^2$, respectively. Thus, AlgeCards retain the concept of area for positive terms, but for negative terms, the students can just use the other sides of the cards. However, a problem may arise if we have, e.g. x^2 and y^2 terms: the square cards will have to be of different sizes, but will this be confusing for students? For AlgeDisc™, this is not an issue as all the discs have the same size. Nevertheless, Leong et al. had tried out AlgeCards with some schools and found them to be effective, but there was no comparison with the other two types of manipulatives.

As we can see, the main issue of using manipulatives for learning algebra is the evaluation of the effectiveness of such methods. This is an unexplored area of research, at least in Singapore. One or more of the following research questions can frame research in the use of such manipulatives when learning and teaching algebra:

1. Do the combined use of algebra discs with algebra tiles develop both students' procedural skills and conceptual understanding?
2. Is the use of more than one type of manipulatives confusing for students? If so, why?
3. What can we say about the effectiveness of the three types of manipulatives—algebra tiles, algebra discs and algebra cards—in the learning and teaching of algebra?

8.5 Guided-Discovery Learning and Investigation

The use of AlgeDisc™ in the above manner described by the learning experiences in the current secondary school mathematics syllabus document (MOE 2012b) is to guide students to discover certain mathematical concepts or skills. In fact, many of these learning experiences make use of Bruner's (1961) guided-discovery learning, which is another distinctive feature of local classroom practices although it is not unique to Singapore. Even before the stipulation of learning experiences in the current primary and secondary school mathematics syllabus documents (MOE 2012a, b), MOE-approved textbooks have already been using activities or investigation to guide students to discover mathematical concepts or skills.

Ernest (1991) contrasted the differences among three inquiry methods for teaching mathematics, namely, problem solving, guided discovery and investigation. Guided discovery is different from mathematical investigation (Jaworski 1994) in that guided discovery is like trail-blazing to a desired location while investigation is like exploring an unknown land where 'the journey, not the destination is the goal' (Pirie 1987, p. 2). But Yeo and Yeap (2010) distinguished between the process of investigation and the activity of using an open investigative task to investigate. As a process, investigation involves examining specific examples or special cases (i.e., specialising) in order to generalise; it is an inductive process, in contrast to the use of deductive reasoning. So Yeo and Yeap argued that problem solving and guided discovery-learning utilise the process of investigation if students specialise by using specific examples, instead of trying to solve the problem by using a deductive approach. On the other hand, an investigative approach to teaching and learning mathematics involves the use of open investigative tasks where students are free to explore and pose any problems to investigate or solve (Ernest 1991). However, the latter approach is seldom used in Singapore schools (Yeo 2013).

Guided-discovery learning can be traced to Bruner (1961). Because guided-discovery learning usually starts with specific examples for students to investigate, these examples become concrete experiences for students to abstract the mathematical concept. In other words, the underlying principle behind guided-discovery learning is still the C-P-A approach or the Extended C-P-A approach, which again is based on Bruner's (1964, 1966) idea of enactive–iconic–symbolic representations of cognitive growth discussed in Sect. 8.2. Because the term 'guided-discovery learning' is from the perspective of the teacher, some textbooks use the term 'investigation' while others use the term 'activity' to describe this kind of investigation or activity that the students will do. In what follows, we will describe two exemplars of guided-discovery learning using concrete manipulatives or concrete examples (guided-discovery learning using virtual manipulatives will be discussed in Chap. 14 while the problem-solving approach has already been dealt with in Chap. 7 of this book).

In secondary schools, paper folding can be used to guide students to discover that the perpendicular bisector of a chord will always pass through the centre of a circle. Figure 8.14 shows part of an investigation in a school textbook by Yeo et al.

6. Using the same circle as in Question 5, fold along a chord *AB* that is not a diameter of the circle and then fold it into two equal halves as shown in Fig. 11.3(**c**).

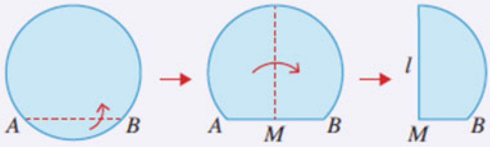


Fig. 11.3(c)

Open up the paper as shown in Fig. 11.3(**d**), where the dotted lines indicate the lines obtained from the above paper folding.

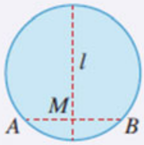


Fig. 11.3(d)

As the paper is folded into two equal halves, the line l bisects the chord *AB* and $\angle AMB$. Since $\angle AMB = 180°$, l is perpendicular to the chord *AB*.

(a) Which two of the three conditions on page 361 are satisfied?

(b) Does the line l pass through the centre *O* of the circle that you have marked in Question 5?

Fig. 8.14 Textbook investigation on symmetric property of circle (reproduced with permission from Shing Lee Publishers Pte Ltd.)

(2015). Prior to what is shown in Fig. 8.14, the students have already folded the paper circle to mark out the centre of the circle. Then, the students will follow the steps in Fig. 8.14 to discover the said symmetric property of a circle.

At higher level mathematics where there may not be any suitable concrete or virtual manipulatives to use, we can just use concrete examples. For secondary school mathematics, these concrete examples usually make use of numerical values. For example, to guide students to discover the product law of logarithms, the teacher can set up a table of values of x and y and get the class to use a calculator to evaluate $\lg x + \lg y$ and $\lg xy$ for different values of x and y. In this manner, the students will discover that $\lg x + \lg y = \lg xy$. This particular guided-discovery learning can be found in the school textbook by Yeo et al. (2013a). Only after gaining some concrete experiences of the product law, the students will then be guided at a later stage to prove it because, according to C-P-A, what is abstract (i.e. the product law) should be developed later.

Although there is some research from other countries on inquiry learning such as guided-discovery learning (Franke et al. 2007; Hunter et al. 2016), there is a dearth of local research in this area. From the online repository of the National Institute of Education (NIE), which is the sole teacher education institution in Singapore, there are no Ph.D. dissertations or Master's theses that are directly related to guided-discovery learning. But there are some local Master's theses that compared the use of ICT and traditional teacher-directed teaching, and the pedagogy behind the use of ICT is actually guided-discovery learning. Even though most of these studies, which will be described in Chap. 14 of this book, suggest the effectiveness of ICT and guided-discovery learning over traditional teacher-directed teaching, it is not known which of these two variables (i.e. ICT and guided-discovery learning) may be the cause of the effectiveness of the intervention programme in these studies. Therefore, we suggest the following questions for future research.

1. How does a guided-discovery approach compare with a teacher-directed approach for teaching mathematics?
2. What are some of the pedagogical principles for teachers to think about when they use a guided-discovery learning approach?
3. What is the role of ICT in guided-discovery learning, especially in light of the country's move towards a more ICT-integrated learning environment?

8.6 Engaging the Hearts of Mathematics Learners: LOVE Mathematics Framework

While it is essential to engage the minds of mathematics learners through various pedagogies described in the previous sections, it is also important to engage their hearts because several studies have demonstrated that students' emotions have a profound influence on learning. For instance, students' epistemic emotions triggered by cognitive problems are critical when learning with new non-routine tasks (Pekrun 2014). However, the study of the affective domain is complicated partly because there is no common agreement on the definitions of terms, and partly because affective constructs are more difficult to describe and measure than cognition (McLeod 1992). Aiken (1972) used the term attitude to mean 'approximately the same thing as *enjoyment*, *interest*, and to some extent, *level of anxiety'* (p. 229) while Hart (1989) used the word attitude towards an object as a general term to refer to emotional (or affective) reactions to the object, behaviour towards the object and beliefs about the object. But Simon suggested the use of affect as a more general term in 1982 (cited in McLeod 1992), and McLeod (1989) further divided the affective variables into three categories: beliefs, attitudes, and emotions.

To measure affective constructs, the traditional paradigm for research on affect often relied on questionnaires and quantitative methods (McLeod 1992). But most of these studies focused on students' existing attitude and their effect on other variables such as test performance. There are very few intervention studies, such as on how to change students' attitude (Yeo 2018a). Moreover, the literature on affective education is mainly confined to affective variables in general, such as improving students' personal development and self-esteem, interpersonal relationships and social skills, and their feelings about themselves as learners and about their academic subjects (Lang et al. 1998), but when it comes to making mathematics lessons interesting and helping students to appreciate mathematics, there is not much literature on this (Yeo 2018a).

Some common issues that teachers face when trying to make mathematics lessons interesting are (a) not every student will find the same thing fascinating; (b) it is not possible to make every part of a lesson engaging; and (c) the enjoyment does not necessarily translate to learning. To address these issues, Yeo (2018a) proposed the LOVE Mathematics framework (Linking Opportunities in a Variety of Experiences to the learning of Mathematics) to engage the hearts of mathematics learners. The framework consists of three principles: variety, opportunity and linkage. Firstly, students have different tastes, so there is a need to use a variety of resources to interest different students in the hope that all the students will find something intriguing, although that 'something' may be different for different students. Secondly, there is actually no need to make every part of every lesson engaging. Yeo suggested that teachers should just provide ample opportunities to engage their students, e.g. in at least one part of most mathematics lessons. Thirdly, Yeo believed that the main purpose of engaging the hearts of students is not just to make them laugh and have fun, but to link the resources to the learning of mathematics.

In this section, we will use the lens of the LOVE Mathematics framework to illustrate how mathematics teachers in Singapore use a variety of resources to provide opportunities for students to engage in mathematical sense making, although these resources may not be unique to Singapore.

8.6.1 Variety Principle

In Singapore, mathematics teachers use a variety of different resources to heighten students' interest in the subject. Types of resources used include songs such as the 'Polygon Song' (see Yeo et al. 2013b; Yeo 2018b), television shows such as *NUMB3RS*, story books such as 'The Doorbell Rang' by Pat Hutchins and 'How Big is a Foot' by Rolf Myller, drama, magic tricks, puzzles, and games. Although the choice of these resources may seem eclectic at times, the idea of using different types of resources beyond mathematical tasks is supported by the Theory of Multiple Intelligences (Gardner 2006). For example, there has also been research to suggest that different parts of the brain are stimulated when children engage in dramatic plays (Hough and Hough 2012). Studies similar to Hough and Hough (2012) form the basis of programmes such as Teaching Through the Arts Programme (TTAP), which was initiated by the National Arts Council (NAC) (National Arts Council 2017; Yuen 2016). Several schools have benefitted from this programme, in which they integrated drama into the teaching of perimeter and area with inputs from drama educators. These schools have found it a fun and engaging approach to teaching.

Similarly, Yeo (2018a) described a lesson where the teacher showed the class a 10-min excerpt of *Splash Splash Love*, a Korean drama, with English subtitles. The teacher chose the excerpt because there was an incident when the king and his subjects were unable to solve a mathematical problem but a student called Dan Bi solved it for them using Pythagoras' theorem. However, there was really not much linkage in the drama to the learning or application of the theorem. So the teacher designed three problems for the class to do. These three problems contain contexts from the drama, e.g. the first problem described how the king shot a deer across the river and wanted his hunting trip to be recorded in history, so he needed Dan Bi to work out the distance the arrow travelled, which the students had to calculate using Pythagoras' theorem. Then, the students were assigned homework from the textbook, which consisted of typical questions on Pythagoras' theorem. The teacher was surprised when one of her students, who had not been handing in homework on time, unexpectedly handed in her homework punctually the following day. It seems that the student was motivated enough by the Korean drama that she even did the routine homework promptly. Although the evidence base is largely anecdotal, teachers' implementation in schools suggests that using a variety of resources is more likely to engage and motivate students. Whether, and if so, how these varieties of resources help support students in learning mathematical concepts will be an interesting area of research for mathematics educators.

8.6.2 Opportunity Principle

As highlighted in the *Splash Splash Love* example, using a variety of resources alone is not sufficient for engaging the hearts of students. Instead, teachers need to incorporate these resources meaningfully into their lessons to provide more opportunities for students to make sense of mathematics through these resources. The idea is to embed different types of resources into different parts of a lesson. For example, one could use a storybook or a mathematics trick to motivate the study of a topic, such as fraction; use a movie clip to illustrate the use of fractions, or explain the operations involving fractions, and use games or plan questions around a video clip to encourage students to practise the skills taught. It may not be realistic to play an entire show or movie during a lesson but clips of 5–10 min, showing excerpts from the shows or movies focusing on a particular concept would allow the teacher to capitalise on its affordance without too much intrusion on the curriculum time. Realising the affordance of such resources can potentially open up new possibilities to engage the hearts of students and possibly enhance students' learning.

In a small-scale local study conducted by Lim et al. (2014), Mr. Fu Siqiang, a teacher of Fairfield Methodist School (Primary), used a rope trick to illustrate that average is a representative value of the set of items. He started with three ropes of different lengths, manipulated them into three ropes of the same length to teach the concept of average, and then changed them back into three ropes of different lengths to emphasise that the lengths of the ropes do not actually change when we take the average of the lengths of the ropes. The rope trick was carried out in two Primary Five classes. The quiz results of students who saw this rope illustration and students who did not were compared. Students who saw the rope trick performed significantly better on the quiz requiring them to find the average of given sets of values and they were also more engaged during the lesson. Their findings concur with other studies which highlighted that the use of magic tricks may enthuse students and provide more opportunities for students to dig deeper into the concepts presented (Koirala and Goodwin 2000; Lesser and Glickman 2009). However, it remains to be seen whether there is an optimum structure in which these resources are sequenced within a lesson. More importantly, the design decisions surrounding the choice and implementation of such resources deserve more attention in research.

8.6.3 Linking to Mathematics

Another important insight gained from the use of resources, such as the *Splash Splash Love* example, revolves around the importance of designing tasks around these resources to connect students to the mathematics concepts. Without this critical connection to mathematics, it would be difficult for teachers to go beyond making lessons fun to making lessons effective. To illustrate this principle, we refer to Yeo (2018b), in which he described the use of an amusing video found on YouTube called '25 divided by 5 equals 14'. The video clip shows three erroneous proofs that 25 divided by 5 is equal to 14: the first proof is by division, the second one by multiplication and the last one by addition. Although most students will laugh at the slapstick humour in the video, the third principle of the LOVE Mathematics framework suggests the need for the teacher to get the students to explain why the proofs are incorrect, namely, the issue of the wrong place value of some of the digits. In this manner, the teacher can link the video to the learning of mathematics: the importance of the place value system. Another illustration of this principle is the use of a movie snippet from *NUMB3RS*, in which Charlie Eppe explained the classic The Monty Hall problem. The teacher could first ask students to answer the question based on their intuition or current understanding of probability before working on the problem to prove (or disprove) their decision. The students could then watch the show where Charlie explained the problem.

Here, we argue that the critical aspect of using different resources to engage the hearts of our learners lies in how teachers design tasks around these resources to provide opportunities for students to engage in mathematical processes. For instance, appropriate games can provide meaningful situations for students to (1) practice their mathematical skills, (2) develop mathematical thinking, (3) test out their intuitive ideas and problem-solving strategies, (4) communicate and reason mathematically through the actions and decisions they make during a game, and (5) develop positive attitudes towards mathematics (Burns 2003; Davies 1995). But how teachers can design these tasks around different resources has not been well studied in Singapore. This is certainly an important area for research for mathematics educators.

8.6.4 Local Research on Engaging the Hearts of Mathematics Learners

Although there are some research done in other countries on affective education in the mathematics classroom, there is a dearth of local research in this area. To gain a more comprehensive understanding of how affect may influence the learning of mathematics, we suggest the following issues for investigation.

Firstly, there is a need for more rigorous studies to evaluate the effectiveness of the above-mentioned resources in motivating local students to learn mathematics and in enhancing their learning. The main problem lies in the measurement of students' motivation and interest: Do we get the students to fill in a questionnaire? Or do we base our judgements on the teacher's observation of their enthusiasm in class? Or do we use a combination of methods? Another important issue regarding measurement is the measurement of students' learning of the subject matter: Do we use the usual class test based on procedural knowledge and skills? Or one that tests on conceptual understanding as well?

Secondly, there is a need to study why these interventions work. Knowing what works, and why it works, is critical for teachers to implement these strategies effectively. Several questions arise with regard to the implementation of these strategies or interventions:

1. Whether the duration of the intervention matters, and if so, how long does it take for the intervention to take effect?
2. Do these strategies have any lasting effect on students' motivation, and more importantly, on students' learning?
3. What are the factors that affect the effectiveness of such strategies? For example, does the types of resources used, the topics to be taught, the belief and knowledge of individual students, or the teacher make a difference?

What are the pedagogical principles that can be derived from these interventions? Knowing these principles may help us develop more targeted strategies for the different students.

8.7 Conclusion

This chapter reviews the key pedagogical innovations that have been implemented, and researched on, in Singapore classrooms. There seems to be limited research evidence supporting the use of such innovations to improve mathematics learning in the local context. However, it is not simply about knowing whether a particular intervention works. But rather, it is crucial for mathematics educators to know the conditions for such interventions to work. As discussed in this chapter, there are three main research problems on the design and use of pedagogical innovations in Singapore:

1. The effectiveness, and the underlying theoretical perspectives, of these innovations;
2. The measurement of effectiveness of these innovations; and
3. The design and development of such innovations for the variations encountered in the different classroom contexts.

As argued persuasively by Lewis (2015), the idea of improvement science (Langley et al. 2009) may be useful for us to consider. The main difficulty of using experimental approaches to investigate the effectiveness of pedagogical innovations lies in the need to control for the different variables in classroom practices. Minimising variations in an experimental setup is unlikely to ensure the transferability of innovations because classrooms are complex ecological systems. As highlighted by Bryk (2015),

> Such studies, however, are not primarily designed to tell us what it will take to make the intervention work for different subgroups of students and teachers or across varied contexts. At base here is *the difference between knowledge that something can work and knowledge of how to actually make it work reliably over diverse contexts and populations*. Yet the latter is what practitioners typically want to know—what will it take to make it work for me, for my students, and in my circumstances? Unfortunately, policy actors who see evidence-based practice as today's answer typically miss this critical distinction. (p. 469)

The way forward is to accept the challenge of making 'this critical distinction' as we implement pedagogical innovations, and as we design and develop new ones. An improvement science approach distinguishes two types of knowledge essential for improving teaching practices: knowledge about the discipline of mathematics education (Lewis 2015), and 'a system of profound knowledge' needed to enact basic disciplinary knowledge within organisations (Deming, cited in Langley et al. 2009, p. 75). This system of profound knowledge is structured around 'knowledge of systems, psychology, variations, and how knowledge grows' (Lemire et al. 2017, p. 24). According to Langley et al. (2009), the improvement science approach, which consists of rapid cycles of *plan-do-study-act* (PDSA), is framed by three key questions:

1. What are we trying to accomplish?
2. How will we know that a change is an improvement?
3. What change can we make that will result in an improvement?

There is evidence to suggest that what teachers and mathematics educators have been doing in Singapore seems to improve students' learning and motivation to some extent. However, questions regarding its effectiveness and transferability to other contexts remain. Research into improving mathematics learning and teaching through an improvement science paradigm may be one way to address these issues. Building on the good work done in Singapore, we suggest that it is time for us, as a community of inquiry, to look deeply into the design and implementation of pedagogical innovations so that we can learn how these innovations can be applied through a variety of contexts.

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Chapter 9
Problems in Real-World Context and Mathematical Modelling

Chun Ming Eric Chan, Kit Ee Dawn Ng, Ngan Hoe Lee and Jaguthsing Dindyal

Abstract This chapter discusses teacher education efforts and analyses the research outcomes in the domain of solving problems in real-world contexts, particularly in the field of mathematical modelling among other tasks situated in real-world contexts in Singapore mathematics classrooms. The first part of this chapter begins with an understanding of "applications and modelling" from the perspective of the Singapore school mathematics curriculum framework. The second part of this chapter reports on the efforts made in supporting applications and modelling in teacher education through professional development opportunities. This chapter continues with a discussion of findings from local research in solving problems in real-world contexts (applications and/or modelling) carried out with students in primary and secondary schools to add to the repertoire of knowledge in this domain. Challenges are surfaced in the light of the preceding sections with implications for teacher education and research with the acknowledgement that there is still some distance to go to know more about applications and modelling and actualizing the curriculum in a more holistic sense through teacher education and the implementing of modelling lessons. This chapter discusses the way forward in supporting the advancement of mathematical modelling in the mathematics curriculum.

Keywords Applications and mathematical modelling · Interdisciplinary project work · Teacher education · Professional development

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9.1 Perspective of Applications and Modelling from the Singapore School Mathematics Curriculum Framework

Applications and modelling is a relatively new domain under the process component of the Singapore school mathematics curriculum framework (MOE 2006a, b, 2012a). The curriculum document articulates that applications and mathematical modelling tasks are crucial in helping students draw connections between school mathematics and the real world, enhance understanding of mathematical knowledge and skills, and develop mathematical competencies (MOE 2012a, p. 15). *Connections* in the document refer to the "ability to see and make linkages among mathematical ideas, between mathematics and other subjects, and between mathematics and the real world" (p. 15).

Stillman et al. (2008) noted the difference between applications tasks and mathematical modelling activities. While both are problems situated within real-world contexts, the nature of use of mathematical knowledge and skills in each is different. An applications task involves the teacher selecting real-world situations for students to apply predetermined mathematics learned in class. Such tasks can be open-ended in terms of solutions and answers as long as students can draw appropriate connections and interpretations of mathematics they know within the real-world situation presented in the problem context. One adaptation of an applications task could be what the Singapore Ministry of Education refers to as "Problems in Real-World Contexts" (PRWC). In PRWC tasks, students solve a multi-part mathematics problem where the stem of the problem presents the context and key variables (MOE 2015). Each question part that follows requires students to apply the mathematics they know to find an answer. Real-world interpretation of the problem context and, often, assumption making are involved during sense-making in the solution process. For the purpose of assessment, PRWC tasks are often open-ended to a limited extent. Since October 2016, PRWC tasks have been incorporated into the GCE "O" levels mathematics examinations. Teachers have started to familiarize themselves with PRWC task design for teaching and learning in secondary mathematics according to the assessment guidelines provided by the Singapore Ministry of Education (see MOE 2015). Other Singapore literature on the use of PRWC for teaching, learning, and assessment is still limited to date (e.g. Yeo et al. 2018).

In contrast, a mathematical modelling activity starts with a real-world problem situation where modellers (or problem solvers) use mathematical lenses to solve the problem. In doing so, different modellers may apply different mathematics to develop a mathematical model to solve the problem or even use mathematics new to them in the process. In this regard, a mathematical modelling activity is truly open-ended, possibly right from the beginning during problem posing where modellers may craft their own mathematical problem from the real-world situation. Literature records different types of modelling activities from various theoretical stances (see Kaiser and Sriraman 2006). The Singapore school mathematics curriculum defines mathematical modelling as "the process of formulating and improving a mathemat-

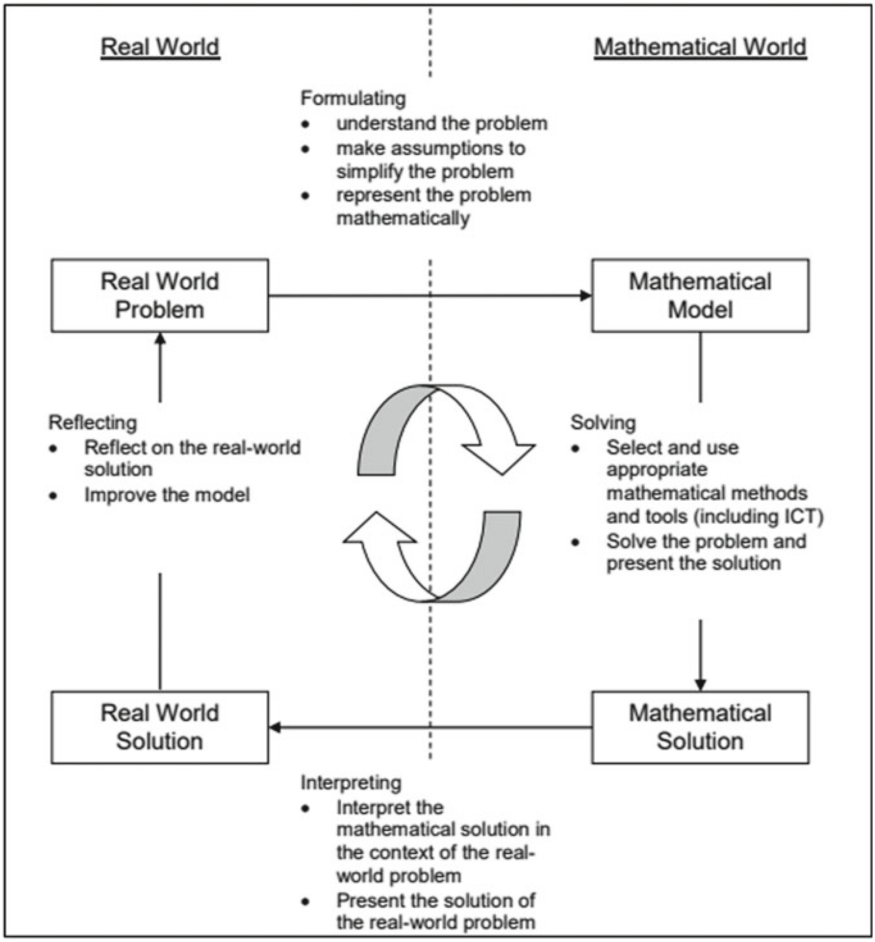

Fig. 9.1 A simplified diagram of the mathematical modelling process (MOE 2012b)

ical model to represent and solve real-world problems". A mathematical model is a "mathematical representation or idealisation of a real-world situation" which can be presented in numerical, algebraic, geometrical, or statistical forms (MOE 2012b, p. 2). As outlined in various key modelling literature in this established field (e.g. Blum et al. 2007; Lesh and Doerr 2003; Niss 2010; Stillman et al. 2016), four stages in the modelling process are also recognized in the Singapore mathematics curriculum documents (MOE 2015): real-world problem, mathematical model, mathematical solution, and real-world solution. The Singapore curriculum adapted the generic mathematical modelling process by incorporating connecting *elements*, namely formulating, solving, interpreting, and reflecting between the various modelling stages as shown in Fig. 9.1 (MOE 2012b).

The expected modelling competencies of students when they attempt modelling activities in Singapore schools are articulated for each element and for each stage of the modelling process. The *formulating* element is activated when the modeller crafts a mathematical model to represent the real-world problem as a start of the solution process. It is expected that the modeller attempts to understand the real-world context and lists the assumptions involved as the initial model evolves. Key mathematical variables and the relationships between these variables are also defined along with the parameters for model development. The *solving* element involves the modeller selecting and using appropriate mathematical methods and tools to solve the mathematical problem. The *interpreting* element requires modellers to interpret the mathematical solution in the context of the real-world problem by checking if they have indeed addressed the problem and whether their solution makes sense within the real-world constraints within the context. Finally, the *reflecting* element calls for modellers to reflect on the reasonableness of their solutions, assumptions made, and appropriateness of mathematics applied. Modellers can investigate more strategies to be used as part of the solution process and even examine the applicability and generalizability of the mathematical models developed on other similar contexts. Limitations of the models can be discussed, leading to the development of more sophisticated mathematical models for the real-world problem.

9.2 Teacher Education on Mathematical Modelling

Applications in the mathematics curriculum have been part of the students' problem-solving endeavours in school and therefore is a familiar content area. On the other hand, although mathematical modelling was incorporated into the Singapore mathematics curriculum framework since 2007, teacher development programmes on mathematical modelling have only begun in earnest in 2009. To date, mathematical modelling activities have been implemented in many Singapore secondary schools and all pre-university institutions (i.e. junior colleges). Such activities range from simple classroom discussions of real-world problems resulting in the use of selected elements from the modelling stages mentioned above to large-scale inter-school modelling challenges held in Singapore schools or overseas (see Kwek and Ko 2011; Lee and Ng 2015; Ng and Lee 2012). The following subsections highlight the inception of teacher education on mathematical modelling through mass outreach efforts and subsequently in the form of in-service professional development courses. Research efforts in teacher development with the help of emerging frameworks are also discussed.

9.2.1 Mathematics Teachers Conference

On 4 June 2009, the Association of Mathematics Educators (AME) in collaboration with the Mathematics and Mathematics Education (MME) Academic Group organized a Mathematics Teachers Conference (MTC). The theme of the conference was Mathematical Applications and Modelling with international experts presenting the keynote addresses and plenaries. In addition, there were workshops on various aspects of applications and modelling intended for the primary schools, secondary schools, and junior colleges. Two booklets written by MME AG authors were specifically prepared for the MTC, one for primary teachers entitled *Applications and Modelling for the Primary Mathematics Classroom* (Dindyal 2009) and the other for secondary teachers entitled *Mathematical Modelling in the Secondary and Junior College Classroom* (Ang 2009). These booklets contained several examples that focused on some of the concerns of the teachers about the approaches to use for teaching applications and modelling in schools and included several modelling examples for the teachers to use in their teaching.

9.2.2 Mathematical Modelling Outreach

The Mathematics and Mathematics Education (MME) Academic Group at the National Institute of Education (NIE) organized a Mathematical Modelling Outreach (MMO) event on June 2010. The event was attended by at least 320 participants, including teachers and students from 31 primary and secondary schools in Singapore, Australia, and Indonesia. The event involved primary and secondary school students working with inter-school discussion groups on mathematical modelling activities facilitated by pre-service teachers. Students were seen to have benefitted from the modelling activities in various ways:

(i) They learned new concepts while working on the modelling activities.
(ii) They learned to be innovative and were not be constrained by the limits of the modelling task.
(iii) They performed at different levels.
(iv) They worked collaboratively on the task and thereby developed better social skills.
(v) They learned to manage meaningful experiences with complex systems.

Various professional development programmes were also conducted for the accompanying teachers by MME teacher educators (see Ng and Lee 2012) and internationally renowned researchers in the field. Such an event was inspired by similar ones held in Australia (Brown et al. 2015) and Germany (Kaiser and Schwarz 2010). The event culminated in a publication of an academic book “Mathematical Modelling—From Theory to Practice” (Lee and Ng 2015)—a collection of thoughts and ideas shared and deliberated by the presenters, both international and local. Learning gleaned from the MMO was presented at the ICTMA 15 Conference in the form of a symposium consisting of three papers which were later published (Chan 2013; Lee 2013; Ng 2013). The papers provided insights into teacher professional development on mathematical modelling from the Singapore perspective, covering areas that included conceptions, task design, and facilitation. In addition, the tasks that were designed and employed at this event have also been collated and consolidated as a task book for teachers' use (Ng and Lee 2012). The book features simple and enriching modelling tasks, and each task is discussed in detail with prompting questions for students as well as teachers' notes for facilitation. It serves as a rich resource to support teachers' address of mathematical modelling in the Singapore mathematics classrooms.

9.2.3 Promoting a Framework of Instruction for Mathematical Modelling

A research project was started in 2015 to design and test strategies that will enable secondary mathematics teachers develop competencies needed in the teaching of mathematics modelling through the use of a new proposed instructional framework (Ang 2015; Lee and Ang 2015). The project led to a seminar that included talks by both international and local experts on mathematical modelling. The 3-hour seminar was attended by 75 participants, including teachers from 17 schools as well as officers from the relevant units of the Ministry of Education. One of the collaborators of the project took the project further by completing a Ph.D. study on "Professional Development for Teachers of Mathematical Modelling in Singapore" (Tan 2015), further contributing to the expertise in teacher education on mathematical modelling. The tried-and-tested mathematical modelling instructional framework (Ang 2015) is now made more accessible through a digital platform (www.mathmodelling.sg) allowing teachers not only access to relevant online resources for the teaching and learning of mathematical modelling but also to be able to contribute to the collection of resources.

9.2.4 In-Service Professional Development Courses

Teacher education on mathematical modelling turned towards the provision of in-service training because of the pedagogical demands on the teachers in facilitating modelling activities. Recently, the Ministry of Education rolled out intensive mathematical modelling professional development courses for in-service mathematics teachers representing the majority of Singapore secondary schools. These courses focus on helping teachers understand what mathematical modelling is, how individual elements of mathematical modelling can be activated within a mathematics classroom, and how to design and incorporate simple modelling activities in mathematics classrooms. To complement these courses, the Ministry of Education commissioned another in-service course on mathematical modelling conducted by MME so that secondary mathematics teachers who are interested in experiencing the full modelling cycle and those who lead mathematical modelling events in their schools can fully immerse in the potentials of modelling activities. The MME in-service course allows teachers to experience being a modeller during real-world problem-solving facilitated by the course instructor before unpacking the modelling stages and elements in more detail. Modelling task design involving full modelling cycles and rubrics for evaluating students' modelling attempts is also discussed.

9.3 Research Involving Teachers in Mathematical Modelling

Several case studies have been carried out in primary schools with respect to building teachers' capacity in the designing and facilitating of modelling activities. Case studies allow the researchers to understand more deeply the teachers' thinking and facilitative actions that led to the successful completion of their own as well as the students' modelling endeavours. Research designs based on an adaptation of the multi-tiered teaching experiment design (Lesh and Kelly 2000) alongside a design experiment method (Dolk et al. 2010) formed the theoretical basis in framing the teacher development process at the primary school level. The multi-tiered teaching experiment (Fig. 9.2) was a three-tiered design that enabled the researcher (Tier 1) to collaborate with the teachers (Tier 2) in the discussion and design of the modelling tasks as well have the teacher to facilitate modelling activities to have an understanding of the features of the modelling task as well as the experience of going through the modelling process. They would also get to see how the students (Tier 3) worked on the modelling task and generated models. The incorporation of design experiment methodology, in particular, during the retrospective analysis phase, enabled the researchers (Tier 3) to guide the analysis and interpretation of data within an interactive cycle to find out the teacher's (Tier 2) rationales during reflection sessions as to why certain instructional approaches were taken, which then formed the basis for feedback and knowledge for the next cycle of facilitation of students' (Tier 3) modelling endeavours. The main principle underlying the multi-tiered teaching experiment framework was to seek corroboration through triangulation where non-prescriptive conditions were provided for the development of new conceptions of participants' experiences, interactions were structured to test and refine constructs, tools were provided to facilitate the construction of models, and the formative feedback and consensus building were integrated into the learning process.

Findings from Chan et al. (2015) on the enactment of the research designs through interviews and retrospective analyses sessions with a novice teacher–modeller revealed that the teacher development process had enabled him to have a better understanding of the different phases and the iterative nature of the modelling pro-

Tier 3 - Researchers	* Development of conceptual framework (model) to develop teachers' knowledge and capacity in facilitating modelling tasks in two cycles. This involved creating learning situations for teachers and students through describing, explaining, predicting teachers' and students' behaviours. * Researchers collaborate with teachers to test and review modelling activity. * Researchers reflect on their own evolving knowledge of the participants' learning experiences for the development of tools to scaffold teachers.
Tier 2 - Teachers	* Teachers collaborate with researchers to test and review modelling activity. * Teachers review feedback for designing own modelling tasks. * Teachers reflect on their own evolving knowledge of the students' learning experiences for the development of tools to scaffold their learning.
Tier 1 - Students	* Students engage in model-eliciting tasks in small groups where they will be involved in constructing and refining models that reveal their interpretation of the problem situation. They will describe, represent, explain, justify and document their mathematical constructions.

Fig. 9.2 A three-tiered teaching experimental framework

cess. The interaction between the tiers (Tiers 1 and 2) played a crucial part in enabling the teacher–modeller to complete the modelling experience successfully. Moreover, the knowledge acquired through the modelling experience would serve to help him become familiar with the students' evolving ways of thinking about important ideas and abilities that he would want the students to develop (Tier 3). As well, the interaction between the teacher–modeller and the researchers (Tiers 2 and 1) was also seen as a model development process for putting the theoretical framework into practice and reviewing how each party was learning through the express-test-revise cycles of the multi-tiered teaching experiment.

In another study involving a novice teacher–modeller, Ng et al. (2013) reported that three teacher competencies surfaced as crucial in the incorporation of mathematical modelling during the retrospective analysis phase of the study, and these were striking an appropriate balance between questioning and listening during facilitation of student discussions, use of metacognitive strategies, and (c) fostering the setting of assumptions in the modelling process. It was inferred that developing these competencies would pave the way for cultivating a pattern of listening–observing–questioning behaviours for a better understanding of students' thinking and interpretation of the real-world problem, overcoming blockages through metacognitive strategies as well as learning to work within parameters by being aware of the need for and setting assumptions when solving real-world problems. There were two follow-up studies on the same teacher–modeller where the teacher's reflections were sought to determine the critical moments of learning and her perceptions of the modelling experience, respectively. For the former, the teacher reported that the deliberate focus in the use of questions to refine students' models, encourage articulation of student ideas in self-evaluation of the models, and clarify and understand student reasoning were core to helping the students in the successful completion of the modelling activity (Ng et al. 2012). It revealed how the retrospective analysis phase, employed in conjunction with the other phases, could serve as a platform for eliciting critical moments of learning for the teacher, building upon the careful selection and discussion of appropriate stimuli. For the latter, based on the teacher's reflections about the modelling experience, the teacher found the modelling activity to be mathematically rich and that it provided a platform for the identification of key variables and their relationships, relating school-based mathematical knowledge and skills to the real-world experience, and justification of mathematical models developed (Seto et al. 2012). In the teacher's view, carrying out modelling activities would make for a more student-centred mathematics lesson that allowed for richer communication of ideas as they would have to justify the models and make their thinking visible compared to a regular mathematics class.

The case studies above also led the researchers to examine the effective use of videos in developing teacher competencies (Ng et al. 2015). It was noted in that using videos as a means for stimulated recall was sufficiently effective in eliciting teacher reflection towards identifying crucial competencies for self-development as one went through the playback instances in the negotiation of meaning and sense-making. Ng and her associates (2015) stressed the importance of timely and appropriate teacher scaffolding when facilitating student group discussions during mathematical modelling activities, in particular the need for teacher to listen in during student talk and mediating between teacher questions and prompts. The findings from Ng's research with her colleagues extended those from Blomhøj and Kjeldsen (2006) as well as Julie and Mudaly (2007) which can be summarized into three interrelated dilemmas faced by teachers when facilitating modelling activities: (a) balancing between a holistic and a reductionist approach, (b) using modelling as a vehicle versus as content, and (c) providing sufficient and appropriate student autonomy during the modelling process.

On developing teachers in the teaching of mathematical modelling at the secondary school level, Tan and Ang (2016) designed a School-Based Development Programme (SBDP) (Fig. 9.3) aimed at influencing teachers' knowledge and resources, goals and orientations in planning, designing, and enacting modelling learning experiences. The SBDP framework sought to help teachers acquire knowledge of modelling and modelling instruction (content) and take them through the transformative learning cycle of modelling (process) within the school context and culture (context). Teachers began with the planning and designing of modelling lessons adopted from Ang's (2015) framework which put forth a set of decision procedures in scaffolding novice teachers towards translating their modelling ideas into a series of modelling learning tasks pitched at different levels of learning experiences. The enactment of the teachers' lessons was video-recorded, and modelling issues encountered were analysed and discussed with the researcher. The cognitive dissonance that surfaced from the teachers' reflections was used to revise and reorganize their learning experiences in anticipation of the enactment of the next modelling activity.

Findings from Tan's case study of the enactment of a teacher participant in the SBDP revealed that the teacher was able to plan and structure developmentally appropriate modelling learning experiences for their students as evidenced by factoring increasing demands on students' modelling skills and competencies in the planning of the students' modelling experiences. Through focused group discussions, the development of the teacher's knowledge of the modelling task solution space

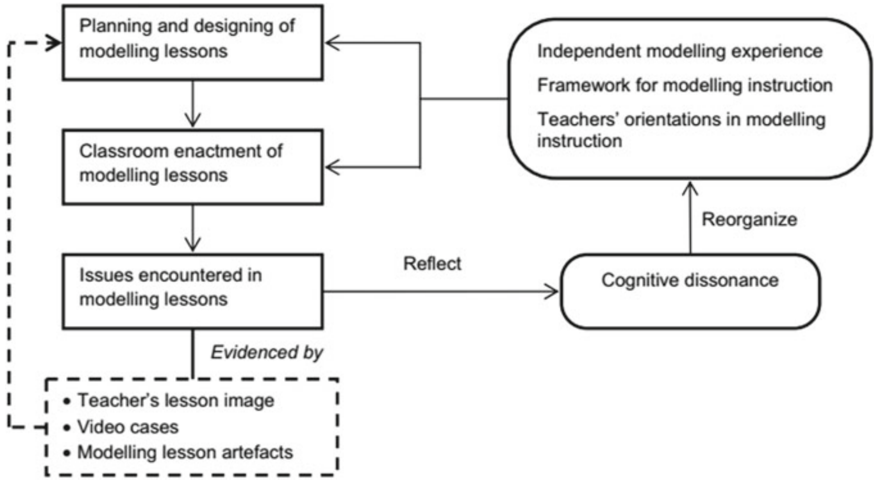

Fig. 9.3 School-based development programme for mathematical modelling (Tan and Ang 2016)

was heightened and it was inferred that the formation of the lesson image in the SBPD programme served as an important resource to support their enactment of modelling instruction practice. The patterns of the teacher's instruction were found to be driven by goals to develop students' modelling competencies as evidenced from the various lesson episodes, and the teaching orientations were mainly viewed as "modelling as content".

9.4 Research Involving Students in Solving Real-World Problems

Research involving school students in solving problems in real-world contexts in the Singapore classrooms took place only in recent years. We take real-world contexts to mean problems that include modelling activities, interdisciplinary project work as well as word problems that include authentic data. Though limited, the research involving primary school students covered different aspects such as the students' mathematical reasoning, competencies, attitudes, problem-solving, and empowerment. Most of the researches were qualitative case studies of selected groups of Primary 5 students to determine what they were capable of during mathematical modelling except for the research on attitudes which included both quantitative and qualitative aspects. As these studies were carried out in conjunction with research on developing teachers' capacity in mathematical modelling as well, the studies therefore embraced the multi-tiered teaching experiment design framework similar to what had been discussed in the previous section. One other study reported in this section is that of engaging secondary school students in interdisciplinary project work which has a history in terms of developing students to make connections with the real world.

9.4.1 Research on Students' Mathematical Modelling in Schools

A recent case study on fostering students' mathematical reasoning through engaging a modelling activity showed that students generated several models (seen as systems) in their decision-making process of selecting school swimmers for the national swimming meet (Chan et al. 2016). It was found that students came up with a point-and-elimination system where they awarded points to winners of various races and eliminated participants who exceeded a certain time in several races to narrow the number of selection choices; students also used averages to find the average time of the participants as a means to validate their findings through the point-and-elimination system. In generating these models, it was found that there appeared to be a cyclical pattern during the modelling endeavour where students exhibited reasoning behaviours such as comparing and analysing data, drawing logical conclusions, justifying decisions and procedures, explaining the mathematical concept, and then back to comparing and analysing data and so on with the testing of hypothesis surfacing whenever the teacher attempted to scaffold and extend their thinking. These reasoning aspects are highly valued as part of reformed pedagogy in mathematics education. In this respect, mathematical modelling is found to empower students not just in terms of eliciting their mathematical reasoning but has also provided opportunities for students to engage in exploration, metacognitive thinking, making decision, and interpretations (Chan et al. 2017).

It can be said that all primary school students in the research were novice modellers. It was not unexpected that they faced some difficulties in making assumptions as a modelling competence during their modelling endeavours. This was revealed in a study on assessing students' mathematical modelling competencies (Chan et al. 2012). The modelling task required students to plan the most efficient route for a teacher to travel from her home to her new school, and various bus service routes were given in an authentic map. In this regard, making a fair comparison of the various bus services plying different routes was an essential aspect of assumption making. Some students articulated that "all buses start and end at the same time" and "we assume that there are no junctions" which were seen as examples of flawed assumptions. Flawed assumptions could impede successful completion of the modelling activity. Learning to make valid assumptions is a crucial aspect of mathematical modelling as it acts as a bridge that connects the real world and the mathematical world. As such, there was a need for teacher–facilitators to scaffold the students' thinking which in turn encourage revisions of thinking and generating better models. On the other hand, students were found to be able to mathematize the problem situations, make interpretations, and justify why their model worked.

From the perspective of viewing modelling as problem-solving, it was found that students go through stages of describing the modelling task by breaking down task information, making geometrical considerations through manipulation, making predictions through revising their solutions when they were tasked to build a box with the greatest volume (Chan 2009). The modelling process forced students to manifest problem-solving behaviours, formulate important relationships between variables, and assess the appropriateness and limitations of the models and communicate their results.

One case study reported by Ang (2013) involved Secondary 3 students who worked on designing a possible layout for a car park with at least 100 parking spaces while leaving as much space as possible to the remainder of a field for other activities. From the written feedback, there were students who found the modelling activity refreshing and the experience enabled them to apply ideas of trigonometry in real-world situations. Others found the experience stressful, while one student preferred classroom time to be teaching time instead of doing projects. Video observations showed that the modelling activity was still very much guided by the teacher through promptings who later revealed that he was concerned that the students would not be able to arrive at the expected solution. Ang stressed the need for a strong framework to guide teachers who otherwise might have the tendency to over-facilitate.

One other research carried out with Primary 6 students was to find out their perception of problem-based learning after the completion of a series of six modelling activities (Chan 2011). The questionnaire showed positive responses in the attributes of interest, perseverance, and confidence. The mixed-ability students were found to have slightly more positive attitudes as they registered higher but statistically not significant mean scores in the three areas mentioned than high-ability students. It suggests that the use of modelling activities could impact how learning environments might be designed towards shaping desirable learning behaviours.

9.4.2 Research on Interdisciplinary Project Work

Research into real-world problem-solving is an established field. Singapore's journey into this could be said to have started sometime in the late 1990s where Interdisciplinary Project Work (IPW) was implemented nation-wide in all primary and secondary schools as well as pre-university colleges (MOE 1999). IPW are real-world problems which are designed to incorporate at least two content subjects as knowledge anchors in the solution process. It could be perceived that an IPW involving mathematics is somewhat similar to an applications task and could be seen as the prelude that set the stage for targeted incorporation of applications and mathematical modelling in the Singapore mathematics curriculum framework in later years of curriculum revisions (evidenced in MOE 2006a, b, 2012a). Deliberate timetabling changes were made then in schools so that student groups could spend up to 10 weeks working on one IPW, meeting with two or more facilitating teachers from different subject specializations during a common curriculum time slot. The final outcome expected from the student groups would be that of a solution to the problem and their oral presentation of their solution process, examining the content knowledge and skills applied, group collaboration efforts, communication of ideas, and independent learning and research. A key goal of using IPW in schools is the recognition of interdisciplinarity in real-world problem-solving where different content knowledge and skills come together in appropriate ways for real-world decision-making in connection to the context.

Research into the use of IPW involving mathematics has been and still is limited. Ng (2009) investigated secondary students' mathematical knowledge application, their use of metacognitive strategies, and their perception of interconnectedness in an IPW involving mathematics, science, and design and technology as anchor subjects. As reported in Ng (2011), students' application of mathematical knowledge in real-world problems was at times purely mathematical and devoid of real-world interpretations. Their solution process could be technically correct in terms of computations but lacks sense-making in real-life application purposes. Ng (2010) also noted incidences of partial metacognitive blindness in group problem-solving attempts during IPW where a dominant group member brushed aside metacognitive monitoring behaviours of others, resulting in incorrect mathematical outcomes. Ng et al. (2007) presented the development of two new scales through factor analysis for measuring students' perceptions of interconnectedness during real-world problem-solving. It was revealed that Express students tend to perceive the interconnectedness of mathematics (within mathematics, between mathematics and other subjects, and between mathematics and the real-world) more than Normal Academic students and are likely to make efforts at making connections such as engaging in using mathematics for inter-subject learning.

9.4.3 Research on Students' Solving Problems Using Authentic Information

Cheng's (2013a) study focused on solving problems in real-world context. She asserted that such problems for young children at the primary level can be designed to embed the three stages of pre-task, actual task, and extension task. The problem posing activities in Cheng (2013b) required primary students to use a variety of real-life situations to construct their own word problems. The students were also required to solve the problems that they constructed. The problems posed by the students suggested gaps in their understanding of fraction addition and multiplication concepts. For example, the inappropriate context was found in the problems posed by students suggesting that students' understanding of fraction concepts can be deepened through classroom discussion of the choice and appropriateness of the real-life context used in fraction word problems.

In another study, Cheng and Toh (2015) reported on the advantages of designing and using mathematical problems in real-world context for both teachers and young children. The teachers would deepen their own mathematical knowledge for teaching in areas such as knowledge of the curriculum, task design, and children's thinking. The students would be able to develop their mathematical processes and computational skills and become more flexible in their thinking. In addition, real-world problems provide the opportunity for students to acquire twenty-first-century competencies through critical and inventive thinking. Cheng and Toh also reported the challenges that teachers faced in using real-world problems with young children and cautioned against dressing the context too thickly such that the opportunities to unpack the mathematics are compromised.

9.5 Challenges and the Way Forward

Mathematical applications and modelling (MAM) are relatively new additions in the school mathematics curriculum in Singapore (MOE 2006a, b). Changes in the curriculum always bring to the fore some implementation issues. How well a curriculum meets its goals ultimately depends on the level of preparation of the teachers using this curriculum and the extent to which these teachers are confident to implement that curriculum in the classroom. Applications and modelling are not located within specific content domains, but rather it is expected that they cut across different areas in the curriculum. Conversely, mathematics teachers are more confident with problem-solving (see Kaur and Dindyal 2010). This is to be expected, as problem-solving is the central organising idea of the local mathematics curriculum since the 1990s, and over time, teachers have developed greater expertise in working with problem-solving as compared to MAM. While the syllabus documents from MOE have provided some implementation guidelines, it is not sufficient without having the teachers to actually go through some form of professional development courses in order for them to be ready to take on such activities with the students. The last few years have seen the inception of teacher education efforts towards advancing MAM. Nonetheless, challenges with the MAM persist.

First, let us look at *applications* in the mathematics curriculum. This is not new, as applications have been emphasized in the local curriculum in various ways and more so through its emphasis on problem-solving. Applications are not meant only for the more able students. There are aspects of applications that can be tailored to the needs of even low-performing students. However, a "teacher who is himself or herself confident in the applications of mathematics and has a strong content background with a sound pedagogical content knowledge has better chances of implementing the applications of mathematics in his or her classroom" (Dindyal and Kaur 2010, p. 328). Generally, most mathematics teachers are familiar with the idea of applications in mathematics and they do understand that this involves higher-order thinking. It is also easier for the mathematics teachers to search and adapt applications tasks for use in classrooms with students of various abilities.

Second, regarding modelling, the issues are more pronounced. Blum et al. (2007) have stated that:

> Yet while applications and modelling play more important roles in many countries' classrooms than in the past, there still exists a substantial gap between the ideas expressed in educational debate and innovative curricula on the one hand, and everyday teaching practice on the other. In particular, genuine modelling activities are still rare in mathematics classrooms. (p. xi)

Mathematical modelling details in the 2006 mathematics syllabus were scant, and understandably, the lack of awareness and understanding in this domain was a logical reason why mathematical modelling did not quite take off. In 2012, the production of the mathematical modelling resource kit by MOE (MOE 2012b) detailed essential aspects of mathematical modelling such as its definition, process, benefits, and facilitation guides alongside a series of modelling examples and students' samples solutions. The booklet was meant as a starter kit to support secondary mathematics teachers in the implementation of mathematical modelling. While the booklet may enhance the awareness of mathematical modelling, the challenge still remains with respect to how to design and facilitate modelling activities, assess performance and how to address issues that surface. Teachers need practical information about orchestrating their lessons. Ang (2010), primarily from a secondary perspective, identified three main challenges for implementing mathematical modelling in Singapore schools. First, teachers in Singapore have not had much exposure to mathematical modelling, although they are otherwise very well trained. Very few primary mathematics teachers in Singapore are specialists in the teaching of mathematics, and most of them do not have a mathematics degree, although they teach mathematics. It is quite challenging of the typical mathematics teacher to be well versed in sourcing for modelling tasks, modifying and implementing these tasks for use in the classroom. Even at the secondary level, where the mathematics teachers have followed post-secondary-level mathematics courses, the situation is not much better. Second, students are driven by assessments, and if mathematical modelling does not form an assessable component, then students are not motivated to engage in modelling activities. And third, there is a perceived lack of resources for teachers to use in the classroom. These issues were echoed by Chan et al. (2015) for teachers at the primary level. Chan et al. also highlighted the paucity of research in mathematical modelling at the local level. Although in recent years outreach platforms like the MTC and the MMO have been carried out, it would be too simplistic to assume that issues about the implementation of mathematical modelling by teachers in schools would be resolved. While teacher education and skilling of teachers in this domain have begun, the professional development work is still very much in its infancy, and it is reasonable to say that there is still some way to go before teachers get to have a deeper sense of awareness, appreciation, and confidence in implementing mathematical modelling activities in the classroom. There needs to be a sustained effort in preparing pre-service teachers and providing professional development opportunities to in-service teachers.

The following suggestions could be worth considering as the way forward in advancing teacher education and professional development efforts in the field of mathematical modelling:

(a) Problem-solving seen from a modelling perspective. While teachers are considered to be conversant in teaching mathematical problem-solving from the perspective of imparting heuristics skills for solving problems using definitive procedures, the notion of problem-solving needs to evolve towards one that is closer to solving unstructured problems where the problem solver has to develop a more productive way of thinking through cycles of expressing, testing, and revising of solutions (Lesh and Zawojewski 2007). Such cycles convey a more realistic process of problem-solving that depict what scientists and engineers do in generating models and conceptual tools towards problem resolution instead of one seen as the direct mappings between the structure of the problem situation and the structure of a symbolic expression that leads to only one way of interpreting the problem (Lesh and Doerr 2003). This perspective, which is synonymous to a modelling perspective, suggests the modelling process as is a non-trivial and thought-revealing problem-solving process. In this regard, teachers will also need to learn to get into a different pedagogical mode with less of frontal prescriptive teaching and more of being facilitators and mediators of learning.
(b) Adopting a developmental framework for instruction. It has been highlighted that having an awareness of mathematical modelling is not enough. Ang (2015) called for a practical instructional framework grounded in design principles that will help and guide teachers in preparing modelling lessons, activities, and learning experiences in the classroom. There is a variety of frameworks in the literature, and some of these arise out of different modelling interpretations and research agendas. For the local curriculum, and especially since mathematical modelling is relatively young, the adoption of a sound and user-friendly framework within any teacher development course on mathematical modelling is a welcome move.
(c) Researcher–teacher collaboration. Having the necessary resources (e.g. syllabus documents, literature, frameworks) and attending mathematical modelling courses are helpful means to acquire the knowledge in the related field. However, teacher competence in mathematical modelling can only be strengthened over time through facilitating more modelling activities (Chan et al. 2012). A worthwhile endeavour to deepen one's competence is for teachers to work with experts through the adoption of established research frameworks like those of the multi-tiered teaching experiment framework (Lesh and Kelly 2000) or the design research methodology (Dolk et al. 2010). Adopting such design frameworks provides affordances for close collaboration for exploring, adapting, and refining lesson plans in bringing about intended learning. As Gravemeijer and van Eerde (2015, p. 523) put it, they help "researchers and teachers experience the project as a collective effort in which they together analyse video footage, student work, and other data to decide on the next steps" leading towards ownership and understanding.
(d) Tried-and-tested materials. The recent research findings in local classrooms highlighted earlier pointed to positive learning outcomes, particularly with respect to what students are capable of when confronted with modelling activ-

ities. While some may argue that the research carried out are case studies and are not generalizable to be indicative of positive learning of the masses, they nonetheless have provided useful information and may be seen as encouraging signs with respect to students' displaying problem-solving and reasoning behaviours valued in reformed pedagogy. One way forward is to translate research materials and findings into instructional materials as part of professional development efforts (e.g. Ng and Lee 2012). The research–practice nexus may be an influential way in helping teachers to appreciate mathematical modelling more and at the same time give them the confidence to carry out similar modelling activities with the knowledge of what to expect using tried-and-tested materials.

(e) Greater balance in assessing process and product of learning outcomes. While there have been some discussions to downplay the emphasis on grade chasing in assessment, it remains a challenge since performance in high-stake examinations is still an important criterion in student selection purposes. In order not to relegate the solving of real-world or modelling problems as a side dish to be carried out as time fillers after examinations, the values and benefits of engaging students in such activities need to be seen as upholding student learning and even performances through integrating them within lessons. Findings from qualitative research that show the value of what students go through as mathematical processes can play a greater role in heightening awareness with respect to eliciting those processes that are valued in mathematics education.

(f) Professional learning communities (PLCs). Many education institutions worldwide are leveraging on innovative ways for teachers to network and develop themselves professionally. The Ministry of Education has acknowledged that PLCs form an integral part of in-service teacher professional development (MOE 2014). In this light, an interest group that focuses on mathematical modelling may be set up at the school cluster or zonal level for teachers to work and learn collaboratively by sharing resources and experiences in this domain. Ang from NIE in 2017 launched an online mathematical modelling resource centre (www.mathmodelling.sg) through a funded research project in support of advancing the field of mathematical modelling for teachers. There is a prospective advantage that incorporating PLCs with digital technology using such online resources may prove to be a flexible way for professional growth.

9.6 Conclusion

The central aim of the Singapore mathematics curriculum is mathematical problem-solving. Professional development of teachers and research have been focused on numerous aspects of problem-solving in the last 30 years. In the last decade, there has been increased attention paid to solving real-world problems, in particular, with a focus on the mathematical modelling aspect of the process component of the mathematics curriculum framework. With the recognition that problem-solving abilities in today's world are more than just about applying a heuristic or getting the right answer, there is a need to expose and equip students to complex situations where they would have to describe, explain, analyse, construct, manipulate, and interpret those situations and to develop competencies that are valued in twenty-first-century economies. Many countries have embraced Science, Technology, Engineering and Mathematics (STEM) instructional approaches as a means to engage students learning in real-world situations. Through expanding the idea of problem-solving to incorporate applications and modelling, the Singapore mathematics curriculum has poised itself in being relevant by implementing reformed pedagogies and developing twenty-first-century skills with applications and modelling during a time of change.

This chapter has outlined the teacher development as well as the research efforts carried out in advancing the cause of applications and modelling in Singapore schools. Locally, this domain is still in its infancy and what has been implemented has left a trial of documentation for reflections and refinements. There is still much to learn in this domain as there are challenges to overcome. Nonetheless, it has been a meaningful start in the last 10 years to reinforce the importance of solving problems in real-world situations as a fundamental process of mathematical literacy.

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Chapter 10
Patterns Across the Years—Singapore Learners' Epistemology

Swee Fong Ng and Boon Liang Chua

Abstract Pattern has a prominent position in the Singapore mathematics curriculum. This chapter reports how learners across the grades thought about patterns, how they recognised patterns, and how they constructed rules to describe the structure underpinning specific patterns. The corpus of data came from four studies. Primary children participated in the first three studies: Age and Individual Differences, Forward and Backward Rule, Colour Contrast whilst Secondary 2 students participated in the fourth, Strategies and Justifications in Mathematical Generalization. All these studies used the mathematics curriculum to design grade-specific mathematical tasks. In general, two types of pattern tasks were used, number patterns presented in tandem with figures and figural patterns. Data with primary children were collected using paper-and-pencil task and clinical interviews were used to collaborate their responses. The fourth study analysed the written responses of the secondary students to paper-and-pencil task. These studies found that learners focused on the surface features to arrive at a rule to describe these number patterns. In the colour-contrast study, compared with monochromatic presentation, those using two colours encouraged learners to present possible general rules. The more able academic stream secondary students were able to arrive at general rules for linear figural patterns. However, all students across the academic spectrum were challenged by quadratic patterns. Findings from the four suggest that it important for teachers to know how to move learners to look for the structure underpinning patterns, numerical and figural, and to construct the all-important general rule.

Keywords Colour contrast · Linear figural · Recursive rule · Predictive rule · Structure · Number patterns · Quadratic patterns

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10.1 Introduction

A large part of the mathematics taught to pupils is really about seeing patterns, interpreting what is seen, and expressing those patterns in words and symbols (Mason 1990). The important patterns are the ones that are not just particular to one situation, but apply to many different but similar situations and are therefore generalisations. Expressing these generalities is the root of algebra. Pupils can be encouraged to express generalities themselves and can have them pointed out by others.

To engage in mathematical thinking, to appreciate the strengths and limitations of mathematics, it is essential to express perceived patterns and generalities and offer these for others to consider, challenge, and where appropriate, modify. These conjectures need to be tested on our peers and our adversaries, with the specific objective of trying to convince them that generalities perceived by us are acceptable to them, too. This is how mathematical thinking develops.

Expressing generality is an important part of mathematics, and of almost all aspects of living. It appears in many guises, but the most basic and clear instance is in pattern spotting, particularly in seeing links, and connecting between things. Algebraic thinking gets going when you try to express patterns in words and pictures, so that others can see what you see.

10.2 Algebraic Thinking, Patterns, and Functions

Algebraic thinking "defies simple definition" (Driscoll 1999, p. 1). Historical origins of algebraic thinking emerged from "proportional thinking as a short, direct and alternative way of solving 'non-practical' problems" Radford (2001, p. 13). *Algebraic Thinking: Grades K-12* sets out the theoretical discussion on what is algebraic thinking and how it differs from algebra—that there is "an algebraic *way of thinking*" (Moses 1999, p. 3, original emphasis). Such thinking incorporates forming "generalizations from experiences with number and computation, formalizing these ideas with the use of a meaningful symbol system, and exploring the concepts pattern and functions" (Van de Walle and Bay-Williams 2014, p. 276). The three strands algebraic thinking infuses the key ideas of generalisation and symbolisation (Kaput 2008). The first strand involves the study of structures in the number system, including those used in arithmetic, described by Usiskin (1988) as generalised arithmetic. Strand two explores the study of patterns, relations, and functions. The third strand seeks to study how best to capture the information or to model the situation symbolically. The human mind seeks to organise the huge amount of information present in the environment by constructing meaningful relations with the various inputs and outputs and capturing the information symbolically (Fosnot and Jacob 2010). Therefore, to be able to organise information meaningfully, the human mind detects what remains the same and what is changing and to construct an appropriate rule. This reduces the demands on human attention so that the mind can function economically (Mason 1996).

10.3 Some Examples of Pattern Tasks

The Singapore primary mathematics curriculum places a heavy emphasis on (i) understanding of patterns, relations, and functions and (ii) representing and analysing mathematical situations and structures using algebraic symbols (Cai et al. 2005). The primary mathematics curriculum introduces and develops numerical and geometrical patterns of varied nature. It is customary to engage young children, e.g. those in Primary 1, with tasks such as those in Fig. 10.1. Here, the stimuli comprise four animals and the children are required to decide which animal comes after the last rabbit. The children have to see that this string of animals is constructed using four animals: penguin, frog, rabbit and rabbit. After the fourth rabbit, the pattern repeats itself. Thus, the structure underpinning this task is a string of four objects, which repeats itself in the same order. Numbers 1, 4, 7, 7, 1, 4, 7, 7, 1, 4, 7, _, could replace the animal stimuli. Completing such number sequences could be challenging to young children. Competing knowledge could prevent children from completing this number sequence. When asked what comes after the number 7, some children would reply 8 and some others, 11.

The complexity of number patterns could be increased to include skip counting such as completing number sequences 2, 4, 6, 8, _, 12, _. Such tasks encourage children to recognise this number sequence as part of the two times tables. Pattern task presented in Fig. 10.2 challenges children to analyse how both repeating and growing patterns are generated.

Here, the geometric shapes above any given number are a function of the nature of the number: triangles are above odd numbers and circles are above even numbers. This task requires children to analyse that the shapes are alternating between triangles and circles, i.e. a repeating pattern of string two. The objective of the task in Fig. 10.2 is to relate the geometrical objects in the pattern to their positions in a pattern and generalise those relationships. Thus, triangles are above odd numbers and circles are above even numbers. To predict what comes next, i.e. what shape follows a circle is relatively simple. A child who understands the demands of the task would be able to

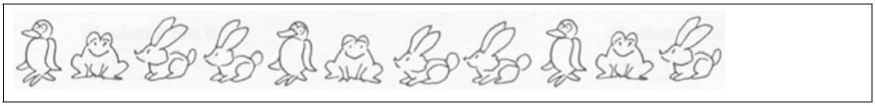


Fig. 10.1 An example of a simple pattern task introduced at Primary 1

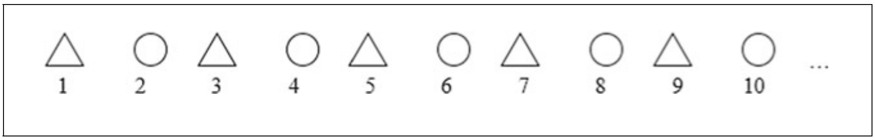


Fig. 10.2 Number patterns constructed using repeating shapes of length 2 and growing number sequence

Shapes	△	○	△	○	△	○	△	○	△	○
Numbers	1	2	3	4	5	6	7	8	9	10
Remainder when number is divided by 2	1	0	1	0	1	0	1	0	1	0

Fig. 10.3 Shapes task reflecting the deeper structures underpinning such beguilingly simple pattern tasks

state a triangle follows a circle. Similarly, children are likely to say that the number eleven is after the number ten. However, the task becomes more challenging when children are asked to predict what shape is above the number 23 or the number 46. Children need to understand that the 23 is an odd number and triangles are above odd numbers. Because 46 is an even number and circles are above even numbers, the shape above 46 must be a circle. Such tasks may seem beguilingly simple are not trivial. Children could answer such questions by writing out the entire number sequences until they arrive at the required number. However, this is not desired. Children may state correctly that circles are above even numbers, but what if they were asked what shape is above 146 or 1023? Attending to superficial features such as the digits in the ones place may get them the correct answer but it is more important for children to notice that triangles are above odd numbers and circles are those above even numbers. Patterns come with clockwork regularity. A more convincing justification that could be generalised to other numbers which are not possible to list is to offer the response that triangles are found above numbers which leave a remainder one when divided by 2 and circles are above numbers with remainder zero and the corresponding numbers are written in one row as in Fig. 10.3. Such function tasks are precursors to periodic functions expressed formally as "$f(x) = f(x + a), x \geq 0$".

Figure 10.4 presents a configuration of strings of shapes, in this case a string of stick houses, starting with Diagram 1 as the smallest configuration. It is necessary to identify that there is a pattern underpinning the construction of these string of diagrams. This is achieved by identifying the recursive rule used to generate each successive diagram. In this case, the recursive rule is to add 4 sticks to the number of sticks in the previous diagram. With the correct identification of the recursive rule, it is possible to state the rule for Diagram 10. The number of sticks for this diagram can be found using the recursive rule of adding 4 sticks to the number of sticks in Diagram 9. Therefore, a possible recursive rule for the number of sticks in Diagram 10 is 5 + 9 times 4. Although some may give the erroneous solution as twice the total number of sticks in Diagram 5. Thus, for any diagram, the correct rule can then be generalised to 5 + 4 times one less than the current diagram number. However, the challenge is to provide a more informative rule, the predictive rule that can be used to state the general rule for any diagram number. The predictive rule provides a direct relationship between the number of sticks and the diagram number.

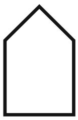
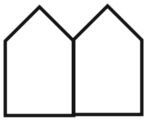
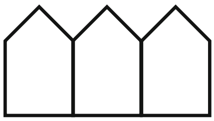
					
	Diagram 1	Diagram 2	Diagram 3	Diagram 5	Diagram 10
Number of sticks					

Fig. 10.4 Ordered configurations of strings of houses, starting from the smallest configuration

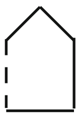
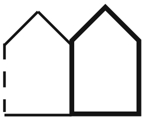
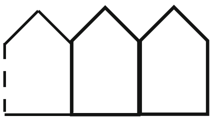
					
	Diagram 1	Diagram 2	Diagram 3	Diagram 5	Diagram 10
Number of sticks					

Fig. 10.5 A dashed line is used to represent the freestanding wall

The total number of sticks for Diagram 10 = 1 free-standing wall + 10 times 4 sticks
The total number of sticks for any diagram number = 1 free-standing wall + diagram number times 4
The total number of sticks for Diagram $n = 1 + n \times 4$

It is hypothesised that construction of the predictive rule may not be so easy if nothing is used to differentiate the freestanding wall from the rest of the sticks. If a dashed line is used to represent the freestanding wall then the predictive rule for any diagram, which provides information on the panels and the diagram number, could be the sum of the freestanding wall and the number of prefabricated panels, which is a function of the figure number. Hence, the predictive rule would be (in this case)

$$\text{The total number of sticks for Diagramn} = 1 + n \times 4$$

Compared to the recursive rule, the predictive rule provides direct information. In this case, the 4 represents the four sticks which remain the same irrespective of the diagram number (Fig. 10.5).

10.4 The Four Studies on Patterns

Using data from four studies, this chapter documents how learners recognised the structure underpinning specific patterns, their construction of the rules underpinning the pattern tasks and the difficulties they have with different pattern-type tasks. The participants of Study One—the Age and Individual Differences in Mathematical Abilities, Study Two—Forward and Backward Rule Study, Study Three—Colour-Contrast Study, were primary pupils, and Study Four—Strategies and Justification in Pattern Generalization (JuStraGen), secondary students. This chapter discusses each study in turn and the Conclusion discusses the overall implications these findings have on the teaching of patterns, in particular, how the teachers can help sensitise learners to patterns.

All children participated with consent. For this chapter, pupils are used for primary school participants, students for those from secondary schools.

10.4.1 Study One: The Age and Individual Differences in Mathematical Abilities: From Kindergarten to Secondary Schools Study

In 2005, the longitudinal study (henceforth, Age and Individual Differences Study) examined the relationships amongst cognitive abilities, socio-motivational beliefs, and mathematical performance of Singapore children from kindergarten to secondary schools. A range of mathematical tasks including arithmetic word problems, algebraic word problems, number and geometric pattern-type tasks, function–machine type tasks, and function tasks were constructed to track children's mathematical performance across the years. All the mathematical tasks were constructed to reflect the expectations of the mathematics curriculum and the spiral structure of the mathematics curriculum. A number of publications emerged from the Age and Individual Differences Study (e.g. Lee et al. 2017). Primary children's performance with function–machine tasks was reported in Ng (2018).

This chapter reports the findings from four different studies, the findings from two grade levels, Primary 3 and Primary 4 from the Age and Individual Differences Study, are reported here. Two of these function tasks are similar in presentation, with one based on Primary 3 pupils' knowledge of the five times tables (henceforth, the five function task, Fig. 10.6), and the other based on Primary 4 pupils' application knowledge of long division and the three times tables (the three function task, Fig. 10.7).

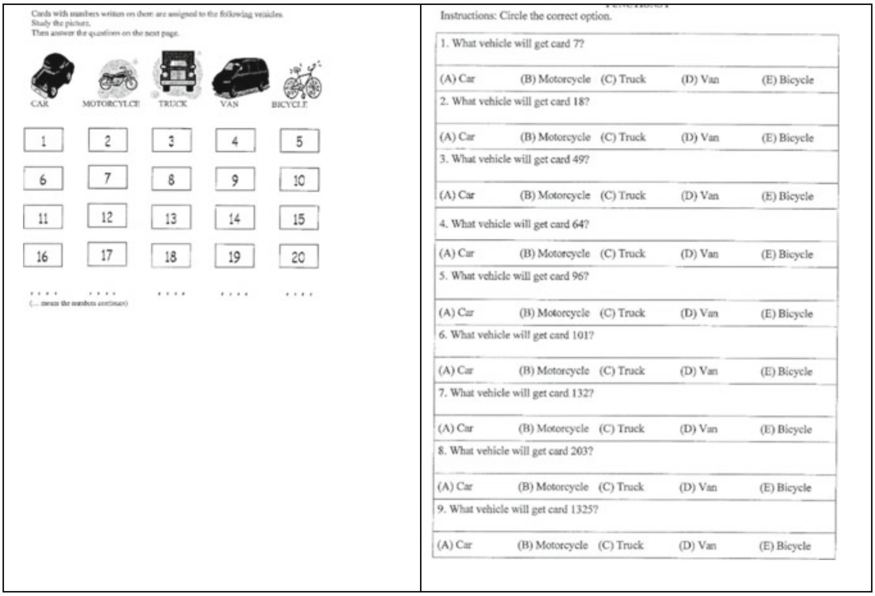


Fig. 10.6 The five function task for Primary 3 pupils

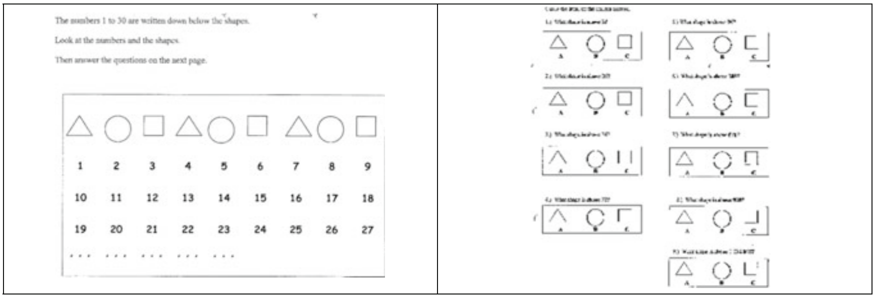


Fig. 10.7 The three function task used with Primary 4 pupils

10.4.1.1 The Participants

Two groups of pupils, ten each from Primary 3 (age 9+) and Primary 4 (age 10+), participated in the supplementary study for the function task. The Primary 3 function task involved core repeating patterns of five while the Primary 4 pupils worked with function tasks involving core-repeating of 3. Although these two tasks required pupils to use their knowledge of five times and three times table, the Primary 3 pupils were more fluent with the five times tables than the Primary 4 with the three times tables.

10.4.1.2 The Instrument

In the Age and Individual Differences Study, in any one year, the function instrument comprised nine questions, divided into four subparts. Part (a) consisted of two base questions (Q1 and Q2), the solutions of which could be read directly from the diagram itself. Part (b) were three near-prediction questions (Q3, Q4, and Q5) the solutions of which could not be found directly from the diagram but could be found either by continuing with the number sequence or from some beginnings of pattern recognition in the structure of the numbers. The solutions to the three Part (c), far-prediction questions, could not be found by continuing writing out the numbers. Rather the solutions to these questions (Q6, Q7, and Q8) could be found by identifying the structure underpinning the numbers. The single question in Part (d) used a seven-digit number which was beyond counting but required abstraction of the structure underpinning the pattern and then generalising that structure any number beyond counting. This four-part structure meant that it was possible to identify pupils' current knowledge of number structure. Those pupils who could solve all four parts indicated that they had good knowledge of structure underpinning numbers. However, those who could complete only Parts (a) meant that they could understand the demands of the function tasks and knew where to look for the solutions. Those who proceeded to Part (c) had better command of the structure of numbers and could identify that although the numbers were getting bigger, certain parts of the numbers remained the same. By identifying what remained the same and what changed, they could provide the appropriate solutions to the questions without having to write out the entire number sequence.

10.4.1.3 The Interview Protocol

This section is technical as it sets out the process how the interviewer engaged the participants with the function tasks. All function tasks have a similar structure. To gain insights into how the participant understood a given task, the interviewer used the following questions to engage with the tasks. It is necessary to provide this level of detail so that others may choose to follow the same protocol to see if they could arrive at the similar findings. Otherwise, others could amend the interview protocol to ascertain if they would achieve different findings.

To ensure that the participant understood the demand of the task, each participant was asked to read aloud

(i) the instructions provided at the beginning of each question;
(ii) the numbers listed below each of the five vehicles; and
(iii) the numbers below each vehicle. For example, "What number cards are below the car? The motorcycle? The truck? The van? The bicycle?"

The participants were asked to provide answers to

(i) Q1 and Q2 (base items). Participants' correct responses suggested that they could see that the numbers were the function of the shapes.

(ii) Q3, Q4, and Q5 (near prediction). If participants were able to answer without continuing the number list provided, then this suggested that participants may have developed a sense of the patterns underpinning the task.
(iii) Q6, Q7, and Q8 (far-prediction questions). Participants' correct responses suggested that they were confident of the rule they had constructed that helped them answer the near-prediction Q3, Q4, and Q5.
(iv) Q9. Participants' correct response meant that the size does not matter. They were likely to have abstracted and were willing to generalise the structure the number pattern to any number.

However, those participants who continued the number list to help them answer Q3, Q4, and Q5 were asked to explain their strategy and to offer alternative strategies, if possible.

(i) Why do you write down all these numbers?
(ii) Can you try to answer these questions without writing down these numbers?

The interviewer terminated the interview when the pupils were seen struggling with a task. When the participants were unable to answer Q3, Q4, and Q5, they did not proceed with Q6, Q7, Q8, and Q9. However, if they were able to secure two out of three correct responses, they proceeded with Q6, Q7, and Q8. As well, the interview was discontinued at any time a child expressed a wish to stop, but no child chose to do so.

For each category of numbers, the interviewer followed up with the following epistemological questions.

- How do you know these are the number cards received by each vehicle?
- How do you know you are correct?
- Did you learn to do such questions? In school or at home?

10.4.1.4 Findings

The three function task was offered to the Primary 4 pupils and the five function task to the Primary 3 because based on task analysis, the former was deemed more challenging than the five function task. In the five function task, the numbers belonging to the bicycle had the digit 5 in the ones place or were multiples of ten. However, for the three function task, the pattern was less obvious. Shapes with numbers that were multiples of ten moved from column to column. The number 10 was in the column for triangle, 20 for the column for circle and 30, the square. The interviews with the pupils showed that the Primary 3 and Primary 4 pupils used two different surface strategies to answer the questions. These surface strategies are (i) the nature of the digit in the ones place, chunking, and counting on. Five Primary 4 pupils used the more sophisticated strategy whereby they looked at the "deep" structure of looking for the remainder when the number was divided by 3 for the three function task. No Primary 3 pupil used the remainder strategy because long division with remainder was not taught until Primary 4.

The nature of the digit in the ones place, chunking, and counting on. All Primary 3 and five Primary 4 pupils explained that they looked at the digits in the ones place to identify which object (vehicle or shape) would receive a specific number card. For example, to answer Q7: What vehicle will get card 132, the common strategy was to first look at the first row and identify that motorcycle has the number 2, and 12 in the ones place. Then they could either add in chunks of 10 until they reached 130 and then they added 2 to arrive at the total of 132. The other strategy was to count in chunks of 10 from the last number in the row, i.e. 5. They then applied skip counting of 10. Ten, twenty, thirty, forty, fifty, sixty, … till they reached 130. Then they moved across to the first column where the digit in the ones place is one, they counted on two more from there, 130, 131, and 132. The motorcycle has the number 132 (Fig. 10.8).

Deep structure of looking for the remainder: Primary 4 pupils are taught the long division algorithm. However, the strategy of looking for deep structure did not come naturally to these children. It was necessary to draw pupils' attention to what remained the same and what changes. Pupils found the following scaffolding questions helped them in seeing the deep structure, i.e. there is a relationship between the number of shapes which kept repeating (three shapes, triangles, circles, squares) and the remainder of the numbers in the columns below each shape. How many shapes are there? Name these shapes. What shape comes after the triangle? The circle, the square? What do you notice when you divide number 3 below the square by 3? The number 4 below the triangle? The number 5 below the circle?

Pupils used the scaffolding questions to look for patterns. PCA used the scaffolding questions to help her come up with a conjecture. Her annotations showed that she listed the remainders of numbers that were within writing. For Q2, What shape is above 26, the annotations showed that PCA noticed that the shape triangle was associated with the remainder 1, circle with the remainder 2, and the shape square had no remainder. Solution to the right in Fig. 10.9 showed how this pupil used the scaffolding questions to support her in constructing the rule that the shape associated with any number is a function of its remainder when it was divided by 3. The solution offered by PR was the most intriguing and his solution showed how some children might see patterns where others did not. For numbers below 100 m, PR used the remainder of 3 to predict which shape was above the number. However, with numbers above 100, PR began testing this alternative rule. Divide the number the number by 3 and also by 9. Were the two remainders the same? He divided the number 388 by 3 and also by 9 and found that the remainders were the same. This strategy was tested with the numbers 621 and 920 and the remainders were the same whether the number was divided by 3 or by 9. In PR's case, he could have noticed that the remainder of the number cards in the second and beyond were functions of the numbers in the first row regardless whether these numbers were divided by 3 or by 9. PR was the only participant who saw this deeper structure, i.e. the relationship between the remainders when the numbers were divided by 3 and then by 9 (Fig. 10.10).

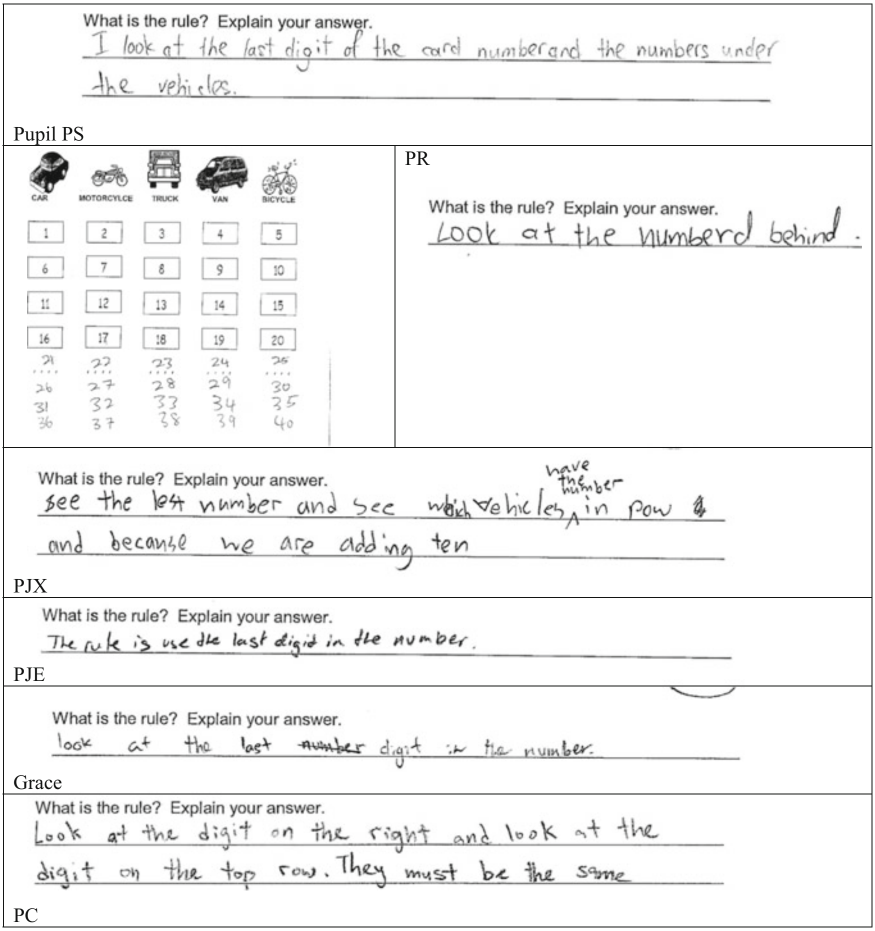


Fig. 10.8 Some of the pupils' written responses explaining how they saw the structure underpinning the pattern task

10.4.2 Study Two: Forward and Backward Rule Study

This study, conducted in 2014, investigated how young pupils navigated the elusive pattern generalisation process. In particular, it examined how Primary 4 pupils (age 10+) determined specific terms that were both near and far from the last given term in a pattern generalising task, how they worked out the position of a term when given the term itself, and how successful they were in establishing the predictive rule that described the pattern depicted in the task. Three pattern tasks were used, and in each task, the pattern was presented figurally as a sequence of four consecutive configurations: Diagram 1 to Diagram 4. So Diagram 5 to Diagram 10 were then taken to be a *near* term whereas any term beyond Diagram 10 was considered as a

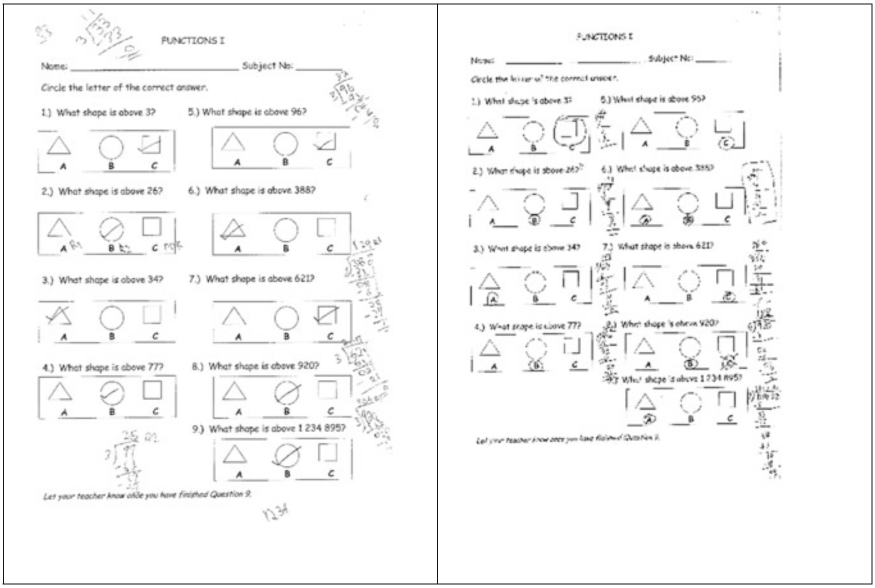


Fig. 10.9 Solution by PCA on the left and PR on the right

	Shapes	△	○	□	△	○	□	△	○	□
	Card numbers	1	2	3	4	5	6	7	8	9
Deep structure	Remainder when divisor is 3	1	2	0	1	2	0	1	2	0
Deeper structure	Remainder when divisor is 9	1	2	3	4	5	6	7	8	0

Fig. 10.10 PR was the only participant who saw the relationship between the two remainders

far term. The predictive rules in all the three pattern tasks had the linear structure $an+b$, with a and b as constants and n the figure number. Hence, all the three patterns depicted a linear relationship between the term and its position.

10.4.2.1 The Participants

Fifty-seven Primary 4 pupils, (34 boys, 23 girls) from one primary school participated in this study. They were from different classes and, based on their performance in their year-end Primary 3 mathematics examination, were identified as *high progressing*, *middle progressing,* and *low progressing.* In Singapore schools, high progressing Primary 3 and Primary 4 pupils were those who scored 85 marks and above, middle progressing were those who scored between 84 marks and 50 marks, and low progressing were those who scored 50 marks and below. In this study, there were 25 high progressing pupils, 14 middle progressing pupils, and 18 low progressing pupils. As part of the mathematics curriculum, under the generic problem-solving heuristic of looking for patterns, these pupils were taught how to (i) continue the pattern and identify the shapes in a figural repeating pattern sequence, (ii) continue the pattern in a growing pattern presented as a sequence of numbers or figural configurations, and (iii) find the position of a given term in a pattern sequence.

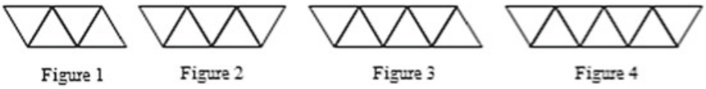
A pilot study was conducted with 30 Primary 4 pupils from another primary school with the objectives of checking the clarity and comprehensibility of the three pattern tasks and gauging the time needed to complete all the three tasks. These pupils' performance in the pilot study showed no further modification of the tasks was needed. Calculators were not necessary as the pupils were able to work out all numerical computations manually. Because more answer space invited more elaborations from the pupils, hence, more space was provided for each subpart in the actual study. Based on the evidence from the pilot study, these pupils were given 75 min to complete the task in the actual study.

10.4.2.2 The Instrument

FUN-PATS Test: The paper-and-pencil test instrument used in this study, henceforth *FUN-PATS* for short, comprised three linear figural pattern tasks, each part divided into four subparts. Pupils were asked to predict a near term in Part (a) and make a far prediction in Part (b). Part (c) required them to determine the position of a given term, and Part (d) asked for the predictive rule. The difficulty level of the three tasks was graduated with the first two tasks, *Making Triangles* and *Triangular Chains*, depicting a 1-step linear rule: $3n$ and$n + 3$, respectively, where n is the figure number. The last task, *Square Tiles Extension*, involved a more challenging 2-step linear rule: $4n + 1$. Figure 10.11 shows the *Triangular Chains* task.

In the *Triangular Chains* task, figure number 7 in Part (a) was chosen given its close proximity to figure number (i.e. 4) of the last configuration in the figural sequence. To find the number of triangles in Figure 7, pupils could easily extend the pattern by adding 1 successively or draw out Figure 7 configuration and then count the number of triangles. In Part (b), figure number 21, which is a multiple of the figure number in Part (a), was deliberately chosen for two reasons: (i) to see how pupils determined the number of triangles when the figure number was farther away and using the "add 1" rule might be inconvenient and (ii) to see whether pupils would erroneously think that Figure 21 had three times as many triangles as Figure 7 given that 21 is thrice of 7. The term in Part (c), which in this case referred to the number of triangles, was made manageable for pupils to manipulate without the aid of calculators. The aim of Part (d) was to examine the pupils' innate ability to articulate the predictive rule when they had not even been taught formally how to do it. The two other pattern tasks were set in a similar context and Table 10.1 shows the respective patterns alongside the specific terms and the figure number to be determined.

Benny used triangles to make the following chains to form a pattern.



For example:

Benny used 4 triangles in Figure 1.

He used 5 triangles in Figure 2.

He used 6 triangles in Figure 3.

As the pattern is continued, more triangles are used.

Study the figures carefully then answer the questions that follow.

No.	Question and Working	Explain clearly how you obtained the answers. You may use words and diagrams.
2a	Find the number of triangles Benny would need to form the chain in Figure 7. Benny would need _____ triangles to form the chain in Figure 7.	
2b	Find the number of triangles Benny would need to form the chain in Figure 21. Benny would need _____ triangles to form the chain in Figure 21.	
2c	Benny used 34 triangles to form a chain. What is the figure number of this chain? The figure number of this chain is _____.	

Fig. 10.11 Triangular chains task

Table 10.1 Making triangles and square tiles extension pattern tasks

Parts	Making triangles	Square tiles extension
(a) Find the near term	8	6
(b) Find the far term	18	12
(c) Find the figure number	63	81

10.4.2.3 Interview Protocol

To understand better the thinking and reasoning processes of these pupils, nine pupils (four boys and five girls) were selected for individual interviews after they had completed the *FUN-PATS* task. The aim of the interview was to gain further insights into the pupils' choice and their epistemology of strategies for finding the near and far terms, and the relationship between the figure number and the predictive rule. The following semi-structured questions guided the interviews:

(i) Can you think of another way to get the number of (objects) for Figure N? How did you figure out this? Does this method work for other figure numbers?
(ii) How did you decide on what the rule is?

Before asking the pupils about each task, they were given sufficient time to look at their written responses in the tasks. Every pupil, except for one, was interviewed on two of the three pattern tasks. Only one pupil, who was articulate and swift in responding to the interview questions within the stipulated time, was interviewed for all three tasks.

10.4.2.4 Findings

The performance of the Primary 4 pupils across the three pattern tasks fell with the increasing complexity of the predictive rules in this order: $3n$ in *Making Triangles*, followed by $n + 3$ in *Triangular Chains*, and then $4n + 1$ in *Square Tiles Extension*. For instance, the percentages of pupils who made the far prediction correctly were 74% in *Making Triangles*, 44% in *Triangular Chains,* and 30% in *Square Tiles Extension*. Within each pattern task, the pupil performance across the four subparts also followed a similar trend, with the highest success rate in making near prediction and the lowest in constructing the predictive rule. Although the pupils were found to employ various generalising strategies to find both the near and far terms, the majority seemed to favour the recursive approach of adding the common difference between two consecutive terms to the previous term in both *Triangular Chains* and *Square Tiles Extension*. But in *Making Triangles*, the pupils employed predominantly a functional approach. A possible reason for their choice of a functional approach is that the predictive rule corresponded to the three times tables, which they were all familiar with.

Finding the figure number seems tough for many Primary 4 pupils. Only about a third of the pupils answered correctly in *Triangular Chains* and *Square Tiles Extension* although there were twice as many pupils in *Making Triangles*. Amongst the successful pupils, a sizeable number had recognised the inherent pattern structures when predicting the far term and performed reversal thinking of operations by applying the *Undoing* strategy to work out the figure number: for instance, $(81 - 1) \div 4 = 20$ in *Square Tiles Extension* in which the predictive rule was $4n + 1$. However, a small number of pupils obtained the correct figure number by listing out the terms until the one being considered was found. The figure number was then established by counting its position.

The construction of the predictive rule proved tricky for the Primary 4 pupils, with success rates of below 15% for all three pattern tasks. Expressing generality is not a straightforward task for primary school pupils and remains elusive for even secondary school pupils and adults. Thus, it was not surprising to find the majority of the successful Primary 4 pupils coming from the high progressing group and none of the low progressing pupils made it. Another noteworthy finding emerging from the data analysis was that not every Primary 4 pupils who recognised the inherent pattern structure and predicted correctly the far term succeeded in rule construction. This finding resonates with the remarks made by Blanton and Kaput (2004) and Mason (2008) that the ability to make a far prediction does not always lead to successful rule construction. The generalising skill is not acquired in a day or two; it takes time as well as guidance from teachers and repeated exposure to develop.

10.4.3 Study Three: Colour-Contrast Study

This study was conducted in 2015 to explore the use of colour contrast in a linear figural pattern in assisting pre-algebra Primary 6 pupils (12+) in establishing the predictive rule. It involved collecting both quantitative and qualitative data through a paper-and-pencil task and clinical task-based interviews with selected pupils.

10.4.3.1 Participants

Thirty Primary 6 pupils (15 boys and 15 girls) from an intact class of a typical Singapore primary school participated in the main study. Two were high progressing pupils and the rest were either low progressing or middle progressing pupils, who had weak number sense and basic operation manipulation skills, in particular, multiplication and division. This class was the sample of choice because the intent was to ascertain whether low and middle progressing pupils could generate the predictive rule when colours were used to construct the structures in a figural pattern task. These pupils had no formal knowledge of algebra. Thus, they were unfamiliar with the use of letters to represent unknowns in algebraic expressions. Further, they had very little experience in working with pattern tasks that involved rule construction.

Based on their performance in their Primary 5 summative mathematics examination, the Primary 6 pupils were divided evenly into two groups: single-colour group and colour-contrasted group. The 15 pupils in the single-colour group were largely middle progressing with one high progressing and two low progressing because they were considered to have better generalising skills. The colour-contrasted group was made up of nine low progressing pupils with one high progressing pupil and five middle progressing pupils. After they have completed the task, four pupils from each group were selected to participate in the clinical task-based interviews. The eight

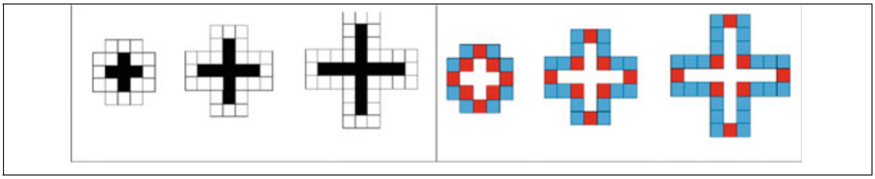


Fig. 10.12 The white tiles task, single colour to the left, colour contrast, to the right

pupils (4 boys, 4 girls) comprised two high progressing, three middle progressing, and another three low progressing.

10.4.3.2 The Instruments

The paper-and-pencil *White Tiles* task was used in a number of studies (Amit and Neria 2008; Rivera and Becker 2005). The *White Tiles* task came in two versions: single colour versus colour contrasted (Fig. 10.12).

The pattern to the left was given to the single-colour group and the one to the right, to the colour-contrasted group. The pupils in both groups answered the same questions: (i) write down in words a rule to find the number of tiles in any figure, (ii) write the rule in mathematical symbols, and (iii) find the figure that is made up of 272 tiles. The pupils had access to the calculator and were given 45-min to complete the task that was conducted after curriculum.

10.4.3.3 Clinical Task-Based Interview Protocol

The interviews were conducted the day after the written task. Each pupil completed the interviews in one sitting and the interviews for the entire group took two consecutive days to complete. The interviews meant that it was possible to (i) understand these pupils' written responses, (ii) provide insights into pupils' thinking as they explained and worked out the tasks, and (iii) test whether it was possible to support those pupils who failed to construct the predictive rule in the paper-and-pencil task via a chance to do so through a series of questions which directed their attention to the relationship between input and output variables. The semi-structured interviews meant that it was possible to modify the interview questions depending on the pupil responses. Some of the questions asked were:

- As the pattern grows, what stays the same and what changes in each figure?
- How many objects stay the same from one figure to the next?
- Is there a relationship between the figure number and the number of tiles in the figure?
- Explain what your pattern rule means from the figures.

During the interview, pupils had access to coloured counters that they could use to build the patterns and describe their rules.

10.4.3.4 Findings

The use of colour contrast in linear figural pattern task had helped pupils of all learning abilities, in particular the low progressing pupils, to

(i) count the quantitative parts of the figures accurately,
(ii) visualise the growth of the figural pattern correctly, and
(iii) identify parts of the figures that remained the same and parts that changed in the linear pattern task.

The colour-contrasted pattern task shows potential in making the multiplicative relationship between the figure number and the parts of the figures that change in step with the figure number more salient. Additionally, such a task seems to ground pupils with a strong structural understanding of the predictive rule. Pupils in the colour-contrasted group were more likely to be able to interpret their rules geometrically than those in the single-colour group.

The Primary 6 pupils were more successful than the Primary 4 pupils in the *Forward and Backward Rule* study in finding the figure number. About half of the Primary 6 pupils found correctly the figure with 272 tiles. This finding was noteworthy considering that these pupils were mostly low to average performers in their school mathematics examination. There is also strong evidence to suggest that the use of colour contrast had enabled the low progressing pupils in the colour-contrast group to determine the correct figure number.

10.4.4 Study Four: Strategies and Justifications in Mathematical Generalization (JuStraGen) Study

This large-scale study sought to (i) investigate how Secondary 2 students (age 14+) made and justified generalisations of figural patterns when the format of pattern display and the type of functions underpinning the pattern were varied and (ii) probe systematically the effect of the format of pattern display and the type of functions on the students' generalisations. The format of pattern display is concerned with whether the figural pattern is presented as a sequence of successive or non-successive configurations. The type of functions considers whether the term-to-position relationship describes a linear or non-linear rule. In this study, the non-linear relationship is of the quadratic type. Data were collected in 2011 through administering a paper-and-pencil task that comprised four linear pattern tasks and four quadratic pattern tasks. Several publications emerged from this *JuStraGen* study (e.g. Chua and Hoyles 2009, 2011, 2012, 2014). This chapter will only focus on the students' competence in rule construction and the effect of the two task features on their generalisation of the predictive rules.

10.4.4.1 Participants

Based on their performance in a national examination taken at the end of their primary education, secondary school students in Singapore are placed in one of the three tracks Express, Normal (Academic), and Normal (Technical), in order of decreasing academic performance. The curricula for each track places are designed to suit the students' learning abilities and interests. Of the entire cohort in the first grade at the secondary level (Secondary 1), the top 60% makes it to the Express track, the next 30%, and the remaining 10% to the Normal (Academic) and Normal (Technical) tracks, respectively.

In the *JuStraGen* study, 515 students from the Express and Normal (Academic) courses from three schools participated with consent. They were 14-year-old Secondary 2 students (242 boys, 273 girls), with 337 from the Express track and 178 from the Normal (Academic) track. The Express and Normal (Academic) students were divided into two groups, *Successive* or *Non-successive*, based on three criteria: their mathematics grades at the national examination, their scores in a baseline test taken before the study commenced, and their gender. There were 266 students (170 Express, 96 Normal (Academic)) in the *Successive* group and 249 students (167 Express, 82 Normal (Academic)) in the *Non-successive* group. Students in the *Successive* group were given pattern tasks that showed a sequence of three successive configurations whereas those in the *Non-successive* group received the same tasks, but with configurations in a non-successive order. The Secondary 1 mathematics curriculum had introduced these students to pattern-type tasks where they had to recognise and represent number patterns and to derive the predictive rule for finding any term in the pattern. However, they had very little experience with work which required the construction of quadratic equations because the concept of quadratic functions was introduced only in Secondary 2.

10.4.4.2 The Instrument

The *JuStraGen* task was developed specifically to inquire into the effect of the format of pattern display and of the type of functions on the students' pattern recognition and their ability to generalise. The paper-and-pencil task consisted of four pattern tasks involving one linear pattern structure and four involving a quadratic structure. Each pattern task assumed two different formats, with its pattern depicted as (i) a sequence of three consecutive diagrams and (ii) a single configuration or a sequence of two or three non-successive configurations. A full discussion of the design of the *JuStraGen* test is provided in Chua and Hoyles (2013). To make the task manageable to the students, the eight pattern tasks were divided into two sets of four tasks and these were administered on two separate days. Each set consisted of two linear and two quadratic pattern tasks. All the pattern tasks were unstructured to allow students scope for exploration so that they could produce their own interpretations of the patterns. Students had to derive the predictive rules and justify how the rules were obtained for all the pattern tasks. The students were given 45 min to complete the

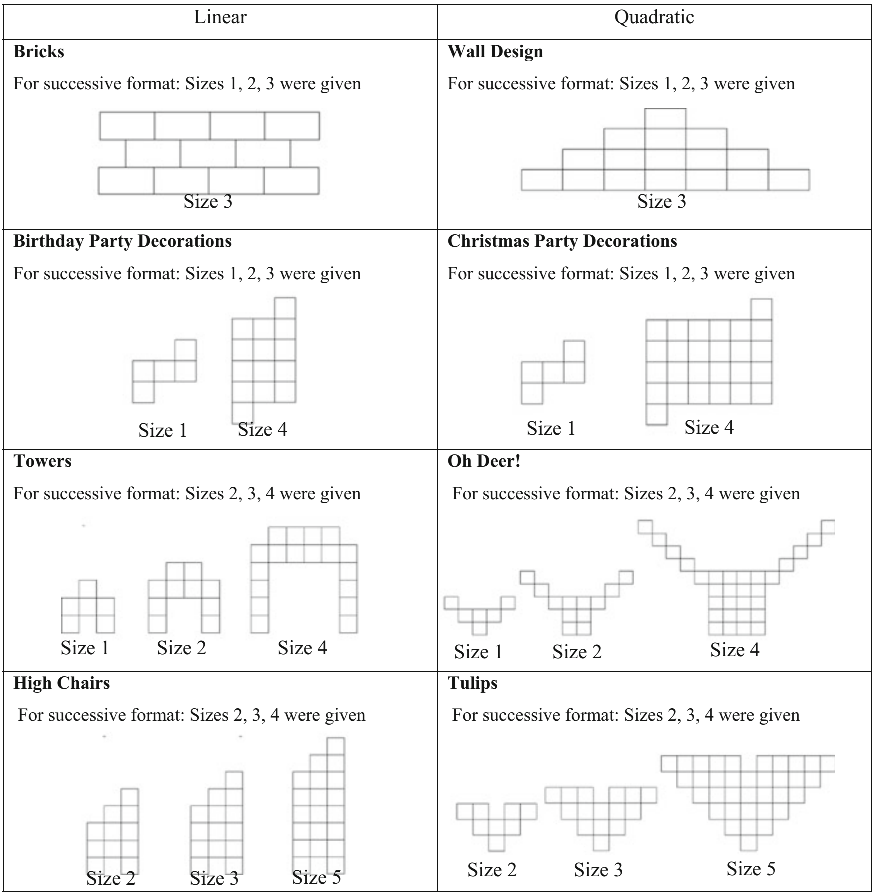


Fig. 10.13 Linear and quadratic pattern tasks in JuStraGen test

task and they had access to calculators. Figure 10.13 shows the eight figural patterns given in the non-successive version of the *JuStraGen* test. The corresponding figure numbers of the three successive configurations for each pattern task are also indicated.

10.4.4.3 Findings

The academically more able Express pupils far outperformed the Normal (Academic) pupils in the *JuStraGen* test. The success rates for all the eight tasks ranged from 50 to 70% for Express pupils in both the successive and non-successive groups, whereas those for Normal (Academic) pupils in the two groups ranged from a low of 2 to 18%. This stark contrast in success rates shows that pattern generalisation remains a stern challenge to a vast majority of the academically less able students.

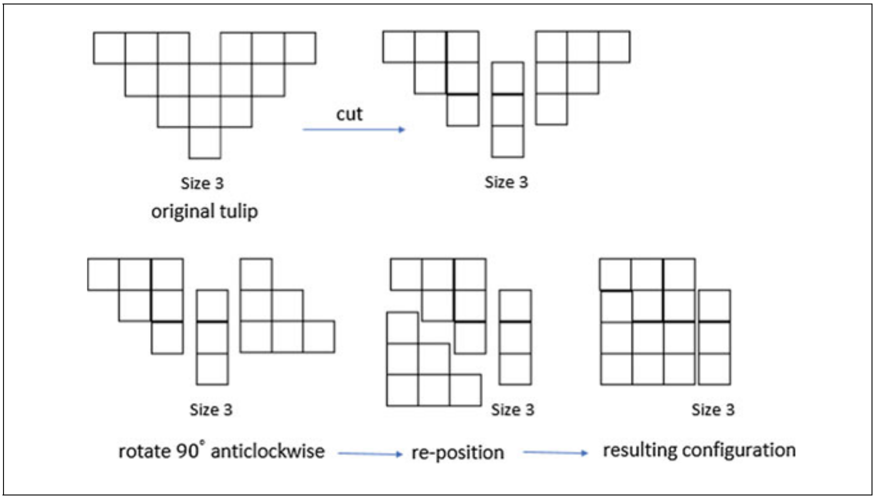


Fig. 10.14 The Tulips task that proved to be most challenging for the students

The study had also shown that the Express students were more likely than the Normal (Academic) students to produce a predictive rule for the pattern tasks. The predictive rules that were constructed by the Express students for each pattern task had a far more diverse range of structurally distinct-looking but equivalent rules, predominantly expressed in algebraic notations. The prevalence of a wide diversity of equivalent rules reflects the express students' flexible thinking and discernment of the pattern structure in multiple ways.

The Express students were not affected by the format of pattern display. Be the display one of the typical and familiar formats of three successive configurations or one involving non-successive configurations, the Express students were still able to construct the predictive rules. Their ability in rule construction seems to be supported very much by their keen awareness of the pattern structure. This finding lends credence to the view that it is extremely crucial to teach students to identify structure (Küchemann 2010; Mason et al. 2009) and that their attention will no longer be drawn to focus on the usual counting of tiles but on using the figure number as a *generator* (Chua and Hoyles 2014) to abstract a relationship between the figure number and the parts of the configuration that change in step with the figure number, then followed by articulating a rule that captures this relationship.

However, the type of functions seems to matter to both the Express and Normal (Academic) students. Compared to linear pattern tasks, quadratic pattern tasks were found to be more challenging for these students. A plausible reason is that spotting the relationship between the figure number and the parts of the configuration that change in step with the figure number in a quadratic pattern is not a straightforward process. Consider the *Tulips* task in Fig. 10.14.

One way to envision the tulip pattern geometrically is to cut each configuration into three parts: the petal on the left, the stalk in the middle, and the petal on the right. The left and right petals are identical "staircases" that are mirror images of each other. Next, establish a link between the figure number and each part of the tulip configuration. Students may easily notice that the number of tiles in the stalk corresponds to the size number. However, the link between the size number and each petal is *not* conspicuous and hard to establish. The number of tiles in a "3-step staircase" is 1 + 2 + 3, which is easy to determine. However, for bigger figure number, the calculation of the number of tiles then becomes tedious. Furthermore, students also struggle to work out the general expression for the number of tiles in any figure. This problem is resolved if the right petal is rotated 90° anticlockwise and repositioned below the left petal to form a rectangle. The resulting configuration reveals the pattern structure of the tulip that can then be interpreted in two different ways: a n by $(n + 1)$ rectangle with an additional column of n tiles or a $(n + 1)$ by $(n + 1)$ square with one missing tile at the top right corner. Such a strategy of rearranging one or more parts of the original configuration to form a shape more familiar may not be apparent and clear-cut to many pupils. This is why quadratic pattern tasks such as Tulips is not easy for secondary school students.

10.5 Conclusion

The suite of studies reported in this chapter looked at the performances of learners from different age groups and academic abilities with pattern tasks. Constructing rule that captures the information inherent in any given pattern task is not an arbitrary process. Learners need to have good number sense and proficient command of the four binary operations. Only then can these learners notice the surface structures underpinning a pattern. Skilful questioning can help learners look for deeper structures. However, there are some learners who are able to notice more structural relationships than others. Pupil PR (Fig. 10.10) is a case in point. Researchers and teachers should be aware of such possible structures so that they do not dismiss the responses of such "gem" learners as nonsense. Rather unusual noticing raise questions of the epistemology of such "gem" learners: How did these learners notice such structures? Why did they notice such structures? Such unusual responses offer teachers opportunities to encourage learners not to be satisfied with the first rule they constructed but rather to encourage learners to wonder whether there are other possibilities.

Processes such as pattern spotting, identifying the underlying structures and generating the predictive have to be taught, not told. Selecting appropriate examples and asking good questions would help. The evidence from the colour-contrast activity showed that low performing children benefited from the way the tasks were presented to them. Perhaps normal academic and normal technical students, too, may benefit from such colour-contrast tasks and teachers can use the questions in their teaching.

This chapter reported how the different studies utilised the three strands algebraic thinking to infuse the key ideas of generalisation and symbolisation (Kaput 2008). It is important to bear in mind that passing examinations should not be the mainstay of cultivation of algebraic thinking but rather the emphasis is to help the human mind organise the huge amount of information present in the environment enabling individuals to function meaningfully (Fosnot and Jacob 2010).

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Chapter 11
Metacognition in the Teaching and Learning of Mathematics

Ngan Hoe Lee, Kit Ee Dawn Ng and Joseph B. W. Yeo

Abstract This chapter first presents the evolving conceptualisation of metacognition since it was first coined by Flavell in 1976. In particular, the issue of awareness, monitoring, and regulation of both cognitive and affective resources was examined. The role that metacognition plays in mathematical problem-solving was also examined, leading to a discussion of the role of metacognition in the Singapore School Mathematics Curriculum which has mathematical problem-solving as its central aim. In view of this, the conceptualisation of metacognition as well as the how's of addressing metacognition in the Singapore mathematics classrooms were discussed from the intended curriculum point of view. Some of the local postgraduate works on metacognition and teaching, and learning of mathematics was also presented to provide an overview of the landscape of the work in this area that has been undertaken thus far. In addition, examples of ongoing works on metacognitive approaches, which have made some inroads in some local schools, were shared to give the reader a glimpse of how research in this area has impacted school practices locally. The chapter concludes with implications for addressing metacognition in the Singapore Mathematics classrooms from the perspective of teachers' professional development.

Keywords Singapore School Mathematics Curriculum · Cognition · Metacognition · Offline metacognition · Online metacognition · Metacognitive instructional strategies · Mathematical problem-solving · Teaching and learning of mathematics · Reflection · Reflective practice model · Meta-metacognition · Theory of mind · Social metacognition

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11.1 Introduction: Metacognition

11.1.1 Conceptions of Metacognition

Metacognition is a term that was first coined by Flavell in 1976 as:

> Metacognition refers to one's knowledge concerning one's own cognitive processes and products or anything related to them … metacognition refers, among many other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects on which they bear, usually in the service of some concrete goal or objective. (Flavell 1976, p. 232)

Flavell's idea of metacognition involved knowledge about and of cognition as well as processes of monitoring and regulating of cognitive actions. He then developed other related metacognitive constructs following that, such as metacognitive knowledge and metacognitive experience (Flavell 1979, 1987), contributing to a body of work that provided the foundational knowledge for the theory of metacognition. However, Schoenfeld (1992) observed that there is no agreement among researchers on a single definition of the term metacognition. Flavell et al. (2002, p. 164) refined metacognition further by referring it to 'metacognitive knowledge and, metacognitive monitoring and self-regulation'. Though a rather extensive survey of the related literature, Loh (2015, p. 29) found that researchers 'tend to agree that the main elements of metacognition are knowledge or awareness of cognition, monitoring and regulation'.

In fact, the Singapore Ministry of Education (MOE) (2012a, p. 19) describes metacognition as 'thinking about thinking' and that it involves 'the awareness of, and the ability to control one's thinking processes… It includes monitoring of one's own thinking and self-regulation of learning'. In fact, Swartz and Perkins (1990, p. 109) in viewing metacognition as 'thinking about thinking', referred to it as 'a crosscutting superordinate kind of thinking relevant to all the others'.

In addition, the work on brain research in the 1990s, which has been labelled as the Decade of the Brain by then US president George H. W. Bush as part of a larger effort involved to enhance public awareness of the benefits to be derived from brain research, has also contributed to conception of metacognition. Freeman (1995, p. 89) pointed out that while the frontal lobes of the brain allow us to elaborate on the details of our goals and plans, it's emotions that generate them and drive their execution. So, Chang and Ang (1999) proposed at a presentation at the 8th International Conference on Thinking at Edmonton, Canada, to add the knowledge of 'personal affective resources' to the knowledge of 'personal cognitive resources' as proposed by Schmitt and Newby (1986). The term 'affective resources' rather than 'emotions' has been used so as to be more encompassing in the conceptualisation. Leder (1993, p. 46) defined 'affect' as a term 'used to denote a wide range of concepts and phenomena including feelings, emotions, moods, motivation and certain drives and instincts'. McLeod (1992, p. 579) states that:

> Beliefs are largely cognitive in nature, and are developed over a relatively long period of time. Emotions, on the other hand, may involve little cognitive appraisal and may appear and disappear rather quickly. Therefore we can think of beliefs, attitudes and emotions as representing increasing levels of affective involvement, decreasing levels of cognitive involvement.

In fact, Ng (2009), in addressing the role of metacognition in the Singapore School Mathematics Curriculum, refers to it as the 'children's awareness of and the ability to monitor and control their thinking'. And, she added that it involves children learning about 'the dynamic use of mathematics and about themselves as problem solvers, their attitudes towards mathematics, mathematics teaching and learning, as well as their own monitoring capabilities' (Ng 2009, p. 20). This is similar to Dweck's (2012) idea of a growth mindset. Dweck (2006) refers mindset as a predisposition or fixed mental attitude that determines how an individual will respond to and interpret situations and make decisions. In a growth mindset, students understand that their talents and abilities can be developed through effort, good teaching, and persistence.

11.1.2 Metacognition and Problem-Solving

Making use of the distinction between novice-expert problem solver by Sweller and Low (1992, p. 84), Barkatsas and Hunting (1996) found that the level of employment of metacognitive strategies appeared to be different between the two groups. Successful problem solvers tend to reflect on their problem-solving activities, have available powerful strategies for dealing with complex and unknown problems, and regulate (even subconsciously) powerful strategies efficiently. On the other hand, though novices have acquired problem-solving strategies, they were observed to be less aware of the utility of them and do not use them effectively in the acquisition of new learning.

In fact, Loh's (2015) detailed analysis of Polya's four-phase problem-solving model (Pólya 1957) pointed towards notions of metacognitive activities to be evident in all the four phases of the model. As she pointed out, the absence of the explicit use of the word 'metacognition' in Polya's work could easily be attributed to the fact that the word was only coined by Flavell in 1976. It is no wonder that Silver (1987) noted that 'no process model of problem solving in any domain can be complete without an adequate account of the role of metacognition'.

Davidson et al. (1994) observed that all problems contain three important components: givens, a goal, and obstacles. They referred 'givens' to the elements, their relations, and the conditions that compose the initial state of the problems situation, while the 'goal' is the solution or desired outcome of the problem. The obstacles, from their perspective, are the characteristics of both the problem solver and the problem situation that make it difficult for the solver to transform the initial state of the problem into the desired state. They perceive problem-solving as the active process of trying to transform the initial state of a problem into the desired one, and metacognition helps the problem solver to:

- recognise that there is a problem to be solved
- figure out what exactly the problem is, and
- understand how to reach a solution

They also identified four metacognitive processes that are important contributors to problem-solving performance across a wide range of domains:

- identifying and defining the problem
- mentally representing the problem
- planning how to proceed
- evaluating what you know about your performance.

11.2 Metacognition and the Singapore School Mathematics Curriculum

Given the importance of metacognition in problem-solving and that problem-solving is central to the Singapore School Mathematics Curriculum (see Chap. 2), it is not surprising that metacognition is one of the five interrelated aspects of the School Mathematics Curriculum Framework (SMCF) (Fig. 1.5 in Chap. 1). Lee, Ng, and Lim observed in Chap. 2 that despite metacognition being coined by Flavell only in 1976, metacognition has been featured as one of the five aspects of the original version of the SMCF which was developed in the 1980s (MOE 1990a, b), reflecting a curriculum that is forward looking and informed by theory and research.

11.2.1 Metacognition and the Evolving Singapore School Mathematics Curriculum

It was highlighted in Chap. 2 that, though there were no major changes made to the metacognition aspect of the SMCF, there was conscious effort to refine and operationalise the construct. In the original version of the SMCF (MOE 1990a, p. 3, 1990b, p. 3), metacognition is referred to as 'monitoring of one's own thinking'. There is also a short accompanying paragraph which describes further metacognition as 'the ability to control one's own thinking processes in problem solving' and includes:

- 'constant monitoring of strategies used in carrying out a task
- seeking alternative ways of performing a task
- checking the appropriateness and reasonableness of answers' (MOE 1990a, p. 4, 1990b, p. 4).

The short paragraph captured the essence and the key actions of control and monitoring of thinking processes involved in metacognition as proposed by Flavell in 1976. The other two points about 'seeking alternative ways of performing a task' and 'checking the appropriateness and reasonableness of answers' are metacognitive strategies that teachers are encouraged to develop as students' productive Habits of Mind.

This depiction of metacognition remained unchanged during the first revision of the Curriculum that was implementation in 2001 (MOE 2000a, p. 11, 2000b, p. 12). However, in the subsequent revision of the Curriculum that was implemented in 2007 (MOE 2006a, b), there were two distinctions made to the way metacognition is featured in the Curriculum.

Firstly, the construct was further elaborated as 'monitoring of one's own thinking and self-regulation of learning' (MOE 2006a, p. 12), capturing not only the monitoring aspect but also the regulatory aspect of metacognition. There was also an accompanying paragraph to elaborate on the various sub-processes involved in metacognition (MOE 2006a, p. 15):

> Metacognition, or "thinking about thinking", refers to the awareness of, and the ability to control one's thinking processes, in particular the selection and use of problem-solving strategies. It includes monitoring of one's own thinking, and self-regulation of learning.

This is reflective of the impact of the continued work mentioned in Sect. 11.1.1 to refine and clarify the construct in the Curriculum.

Secondly, there was an explicit attempt to separate an understanding of the construct from instructional experiences that teachers are encouraged to provide students with to develop metacognitive Habits of Mind. Teachers were encouraged to provide metacognitive experiences to help students develop their problem-solving skills. The following five types of activities were advocated to develop the metacognitive awareness of students and to enrich their metacognitive experience (MOE 2006a, p. 15):

(a) Expose students to general problem-solving skills, thinking skills, and heuristics, and how these skills can be applied to solve problems
(b) Encourage students to think about the strategies and methods they use to solve particular problems
(c) Provide students with problems that require planning (before solving) and evaluation (after solving)
(d) Encourage students to seek alternative ways of solving the same problem and to check the appropriateness and reasonableness of the answer
(e) Allow students to discuss how to solve a particular problem and to explain the different methods that they use for solving the problem.

These five types of activities involved students not only in being aware of (a, b, and e), but also encourages students to monitor (b, c, d, and e) and regulate (a, c, and e) their thinking processes. These activities, though more specific, are well aligned with the four clusters of metacognitive instructional strategies that Lee (2008, pp. 70–71) found in the literature to be effective in improving students' problem-solving ability:

i. Mathematics log writing—students use writing activities to cultivate a more metacognitive approach towards mathematical problem-solving.

ii. Effective questioning techniques—teachers establish an environment in which both teachers and students continuously ask questions with regard to the problem-solving process so as to better understand, monitor, and direct students' cognitive processes.
iii. Identification of structural properties of problems—teachers consistently ask students to identify similarities and differences among methods of solution and structural properties of problems that involve different contexts.
iv. Pair and group problem-solving—students worked in pairs and/or groups, reasoning aloud and interviewing each other so as to be more aware of and thus more conscious of regulating the thought processes of the problem solver.

In the latest revised Curriculum, implemented since 2013, there was no further change in the way the construct of metacognition is being presented (MOE 2012a, p. 19, 2012b, p. 17, 2012c, p. 17), reflecting a more stabilised conceptualisation of the construct. However, in line with the emphasis of this revised Curriculum, activities that develop the metacognitive awareness of students and to enrich their metacognitive experience are now articulated in the like of learning experiences (see Chap. 2):

> To develop metacognitive awareness and strategies, and know when and how to use the strategies, students should have opportunities to solve non-routine and open-ended problems, to discuss their solutions, to think aloud and reflect on what they are doing, and to keep track of how things are going and make changes when necessary. (MOE 2012a, p. 19)

While the five types of activities have been more generically summarised in a paragraph, the essence for the need to encourage students to be more aware of, and continuously monitor and regulate their thinking process is still inherent. The importance in selecting appropriate tasks—non-routine and open-ended ones that encourage the development of metacognitive awareness and strategies is highlighted. There is also an encouragement for teachers to help students to make their thinking audible (think aloud) and visible (reflection writing) to heighten students' awareness, monitoring, and regulating of their thinking.

11.2.2 Operationalisation of Metacognition for Teaching and Learning of Mathematics

Brown (1980) and Markman (1977) found that metacognition is a developmental skill that does not automatically increase with age, while Schmitt and Newby (1986) also noted that supplementing instruction with metacognitive aspects would prove beneficial to most learners. In the Singapore context, Wong (1992) found that students need guided instruction in the use of metacognitive strategies for (mathematical) problems-solving. Lee (2015, 2016b) argued that metacognition need to be supported by explicit instruction whereby related skills/processes are explicitly labelled and discussed, and students are guided throughout their repeated distributed practice of these developmentally within the context teaching and learning of mathematics.

This is in line with previous findings that critical thinking skills can be learned and transferred to novel situations when pupils receive explicit instruction designed to foster transfer (Bangert-Drowns and Bankert 1990; Cotton 1991; Dweck 2002, Halpern 1998, 2003; Marin and Halpern 2011). Lee further proposed that, like for the case in teaching content, there is a need to carry out a 'task analysis' for one to explicitly address metacognition in the mathematics classroom. As was presented earlier in this chapter, metacognition involves an awareness of, monitoring of, and regulation of one's both cognitive and affective resources in the context of carrying out goal-oriented processes. Lee observed that regulating of one's cognitive and affective resources assumed the ability of one to monitor such resources, while the ability to monitor such resources assumed the ability of one to be first aware of one's own cognitive and affective resources. He therefore suggested that following taxonomy in addressing metacognition in the mathematics classroom given the hierarchical nature of these three aspects of metacognition:

- Awareness
- Monitoring
- Regulating

While it is true that one may be exercising all these three aspects of metacognition during the course of carrying out goal-oriented processes, he felt that teachers should first create an awareness of students' own cognitive and affective resources first. This would include getting students to be able label and describe these resources in relation to both themselves and the related task(s) at hand. Such an awareness would then allow students to monitor short episodes of their cognitive and affective processes before they are able to regulate such process. In particular, Lee emphasised that regulating such processes need not always result in a change of course of action. The regulation may further affirm one's current course of action if an evaluation of the processes deems fit.

At the same time, Lee also alerted to the fact that there were effort put into encourage students to employ metacognitive practices online—while performing a task, as well as offline—after the completion of a task. Online metacognitive practices involve an awareness, active monitoring, and constant regulating of one's thinking processes while performing a task with the goal of more efficiently and effectively attaining the goal of completing the task at hand. On the other hand, offline metacognitive practices, though also involve an awareness, monitoring, and regulating of one's thinking processes, these are carried out in a post-mortem manner. The purpose of such practices is to improve on future performance of similar task(s), or a transfer of the learning to other similar task(s) with different context.

Taking into consideration both the taxonomy and types of metacognitive practices involved in the classroom, Lee (2015, 2016b) proposed a two-dimensional conceptualisation of metacognition for teaching and learning. While both types of metacognitive practices—offline and online involve awareness, monitoring, and regulating of one's own cognitive and affective resources, the offline and online practice turn these into reflective and interactive practices, respectively (as shown in Fig. 11.1). Lee also observed that a greater level cognitive load is involved in interactive metacognitive

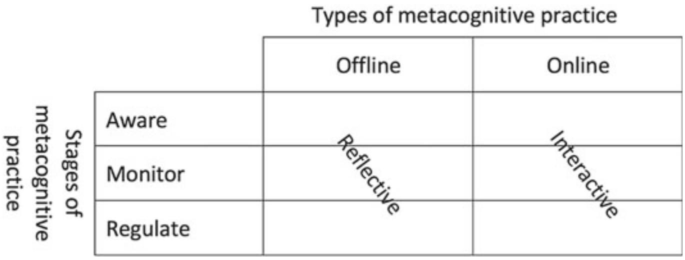

Fig. 11.1 Operationalisation of metacognition for teaching and learning

practices—given that the students need to juggle between metacognitive and cognitive practices at the same time. He suggested that teachers may want to consider addressing offline metacognitive practices first before involving students in both offline and online metacognitive practices.

In arguing for students to be guided throughout their repeated distributed practice of metacognitive practices developmentally within the context teaching and learning of mathematics, Lee also referred to Costa and Kallick's (2000, p. 26) reference of metacognition as one of the Habits of Mind. Costa (2001) refers the Habits of Mind as thinking dispositions skilfully and mindfully displayed by characteristically intelligent people when confronted with problems the solutions to which are not immediately apparent. The Habits of Mind are not thinking tools, rather they are dispositions that one inclines to adopt. Just as Costa and Kallick (2009, p. xi) proposed, Lee also felt that such productive Habits of Mind, as in the case of physical habits, are formed only through continuous practice with teachers providing 'generative, rich, and provocative opportunities for using' such Habits of Mind.

11.3 Local Research Studies on Metacognition and the Teaching and Learning of Mathematics

As was observed by Loh (2015, p. 48), there are few research studies on metacognition in teaching and learning of mathematics that involved local subjects. Wong's (1989) pioneering work in this area on investigating metacognition in mathematical problem-solving is based on the data drawn from a questionnaire used in a larger study (Chang 1989). The main finding of the study revealed that students practiced metacognitive activities at least half of the time when they were solving mathematics problems. However, it was found that the lower-achieving students were less frequent in the usage of metacognitive strategies than those exhibited by the higher-achieving students. Another pioneering research work in this area was undertaken by Foong (1990, 1993). She developed a taxonomy of mathematical problem-solving behaviour that included metacognitive behaviour. These pioneering works have also become foundational work for others to pursue their postgraduate studies on. Yeap's

Fig. 11.2 Loh (2015, p. 98) PSM framework

(1997) master study tapped on Foong's work to develop a catalogue of metacognitive behaviour that teachers could use to detect metacognitive activities in the mathematics classrooms. Lo's (1995) master study on the other hand followed on the footsteps of Wong to further improved on the Study Skills Questionnaire by Chang (1989) by taking into consideration the Learning Process Questionnaire by Biggs (1987). Lo aimed to establish the relationship between the metacognitive strategies employed in mathematics problem-solving and mathematics achievement of students at a Singapore Junior college, as well as the relationship between their learning approaches and mathematics achievement. Loh's Ph.D. study (2015) provided further refinement of Wong's idea to establish her problem-solving metacognitive (PSM) framework (as shown in Fig. 11.2) to analyse students' use of metacognitive strategies by phase and levels, and to gain insights to the nature and frequency of lower secondary school students' use of metacognitive strategies during problem-solving.

There are also studies which target at specific topics in the teaching and learning of mathematics. Teo (2006), for example, carried out a small-scale study on the effects of metacognition and beliefs on students in the study of A-level sequences and series. Yap (2016), on the other hand, carried out an intervention study on metacognitive-heuristic approach, also based on Foong's work on metacognition (1995), to help low attainers in ratio word problems. Lee (2008) also carried out an intervention study for his Ph.D. study to investigate the impact of metacognitive instructional strategies—instructional strategies that help to activate students' metacognitive practices (Mevarech et al. 2006), on the mathematical learning and achievement of secondary one students. Lee's pioneering work on the use of a mixed methods research design in such a study helps to address the many issues that surrounded the validity of the various data collection method involving the study of metacognition (Loh 2015, pp. 64–80). The research drew upon quantitative data from survey questionnaire and problem-solving test, and qualitative data from self-report in the problem-solving test and student interview. The complementary nature of quantitative and qualitative approaches explored the different dimensions of metacognition, and therefore, produced more insights on metacognition and problem-solving through triangulation of data. The approach forms the basis for the research design adopted by Loh's Ph.D. work (Loh 2015), which was mentioned earlier.

The next two sub-sections detailed two areas of work that have made their way into some local school practices to address metacognition in teaching and learning. Of these two areas, one will address offline metacognitive practice while the other will deal with online metacognitive practice (see Fig. 11.1).

11.3.1 The A-Cube Change 2-Dimensional Reflective Practice Model

Lee (2003), based on his interaction with both future and practicing teachers, developed the EmC^2 or Change Model for mathematics teachers to reflect on their teaching so as to develop teachers' offline metacognitive habit of mind in learning from their teaching episodes. The Model consisted of three types of reflection, namely the emotive reflection (EMR), the critical reflection (CIR), and the creative reflection (CER). EMR refers to the awareness of the teacher with regard to his/her 'gut feeling' of how successful a lesson as a whole has been upon completing a lesson episode. CIR engages the teacher in a detailed analysis of his/her lesson episode; it encourages the teacher to examine the various parts of the lesson. CER helps the teacher to bring the reflective process to fruition by inviting the teacher to create a new lesson episode based on the reflection during EMR and CIR stages. Figure 11.3 provided a diagrammatic representation of the EmC^2 or Change Reflection Model, reflecting the hierarchical nature of the three types of reflection and includes the key questions that teachers could use to guide them through each type of reflection in the Model.

In an attempt to also include the learning aspect of the mathematics classroom, Lee (2010) further refined the question prompts in the Model to adapt to both teaching and learning of mathematics, as shown in Fig. 11.4.

However, Lee (2015) shared that based on his further work with teachers on reflective practices, he felt also a need to examine the depth of reflection, and not just the type of reflection. He postulated that while teachers/learners may undergo creative reflection and profess of new/reinforced understanding, such may exist only at an articulated level. There is a need to encourage the teachers/learners to examine such articulated new/reinforced understanding against their personal belief so that such new/reinforced understanding may be truly assimilated by the teachers/learners. He also believes that for the teachers/learners to truly own the assimilated new schema of understanding—be it reinforced or renewed, the teachers/learners need to know when and why, not just the how, of wielding the new knowledge/skill. In other words, there is a need for the teachers/learners to appraise the new schema for effective regulated use of such new/reinforced understanding. He presented the depth of reflection using the A^3 Reflection Model as shown in Fig. 11.5, together with the accompanying sample question prompts that teachers may use. As shown in Fig. 11.5, the three

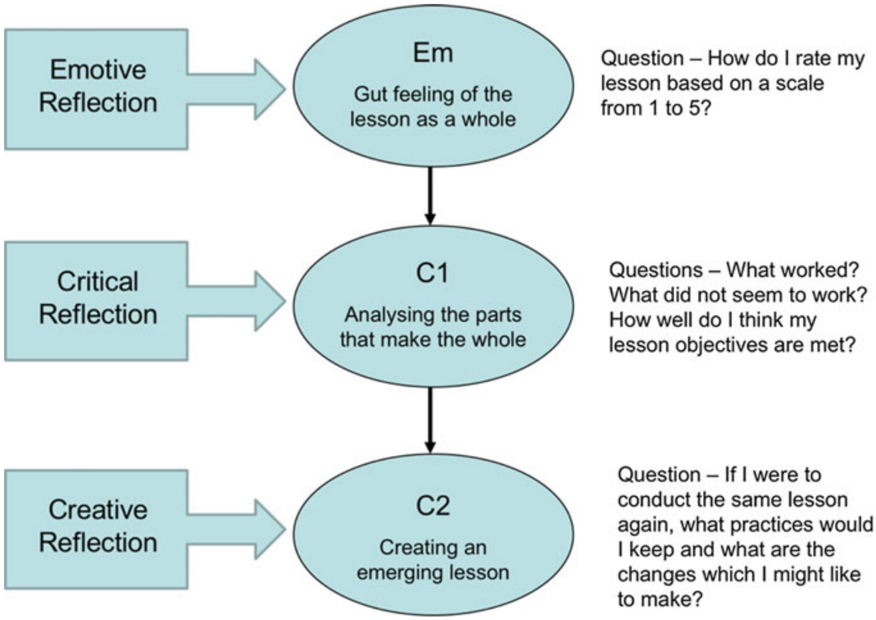

Fig. 11.3 The EmC^2 or Change Reflection Model for teaching Mathematics

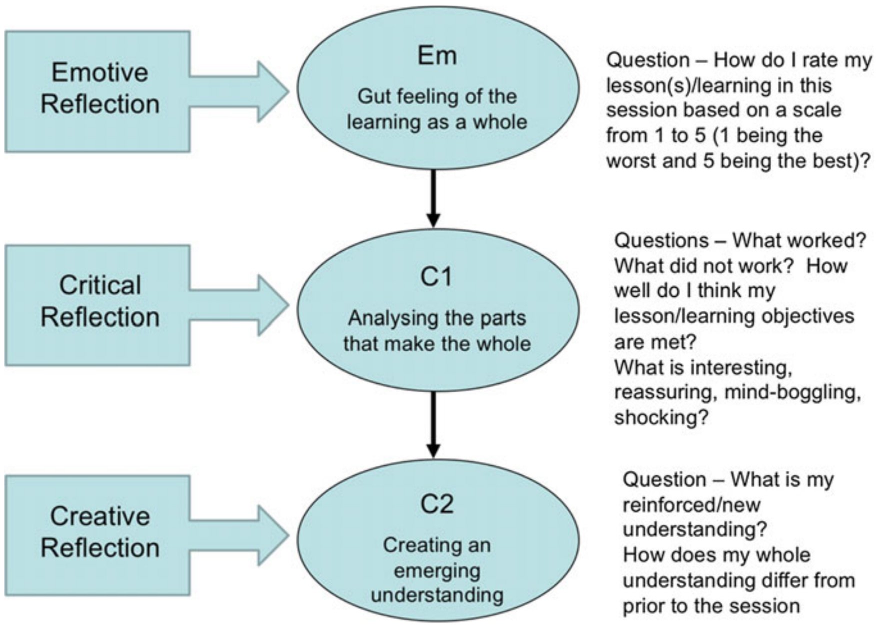

Fig. 11.4 The EmC^2 or Change Reflection Model for teaching and learning of Mathematics

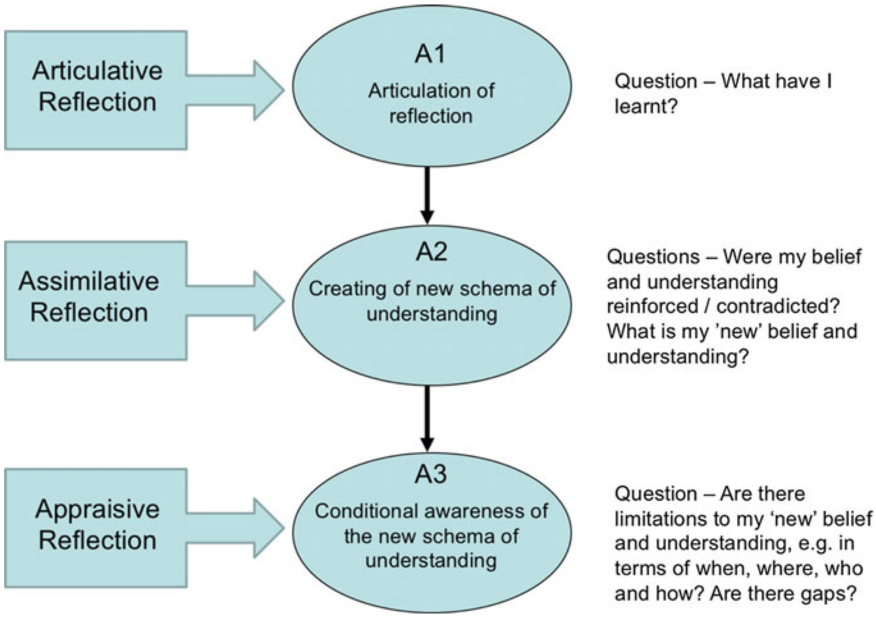

Fig. 11.5 A^3 Reflection Model for teaching and learning of Mathematics

depths of reflection, also in hierarchical manner, are articulative, assimilative, and appraisive.

Based on the types and depth of reflection, Lee (2015) presented the A-Cube Change 2-Dimensional Reflection Model as shown in Fig. 11.6, to illustrate how the two dimensions of reflection interact for effective offline metacognitive practice or reflective practice to occur. As can be observed from Fig. 11.6, the interaction of both the dimensions also demonstrated how the three aspects of metacognition, namely awareness, monitoring, and regulation, play out in the process. He also developed sample question prompts to accompany this 2-Dimensional Reflection Model (Fig. 11.7). Lee shared that this 2-Dimensional Reflection Model promotes deep reflection. Such deep reflective practice allows for effective transfer of learning to new/novel situations as it involves one critically and creatively examining one's practice against one's knowledge, skills, and beliefs. Furthermore, the practice also promotes the establishment of connections to make sense of one's practice as it encourages linking the examination of one's practice to not only principles of teaching and learning but also one's belief system. Such deep reflective practice, he pointed out, promotes the evolvement of a more connected and robust schema of practice that better aligns practice, knowledge, and beliefs.

The 2-Dimensional Reflection Model has been shared and adopted in pre-service, in-service, and postgraduate courses in mathematics education as well as schools (Lee 2017a). One such example is reported in the newsletter SingTeach, whereby a postgraduate student, after learning about the Model in her study, attempted to

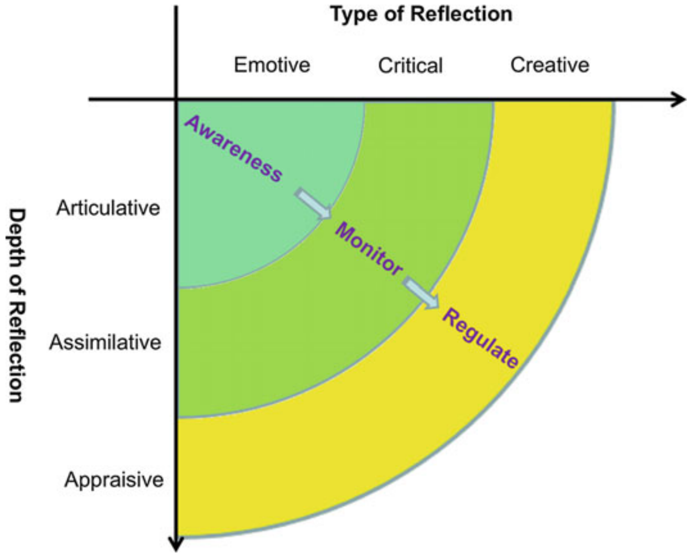

Fig. 11.6 The A-Cube change 2-Dimensional Reflection Model for teaching and learning

Depth/Stages of Reflection	Examples of reflective question prompt
Articulative – The Emotive Reflection	How do I rate my lesson(s)/learning in this session based on a scale from 1 to 5 (1 being the worst and 5 being the best)?
Articulative – The Critical Reflection	What worked? What did not work? How well do I think my lesson/learning objectives are met? What is interesting, reassuring, mind-boggling, shocking?
Articulative – The Creative Reflection	What is my reinforced / new understanding? How does my whole understanding differ from prior to the session?
Assimilative	Were my belief and understanding reinforced/contradicted? What is my 'new' belief and understanding?
Appraisive	Are there limitations to my 'new' belief and understanding, e.g. in terms of when, where, who and how? Are there gaps?

Fig. 11.7 Sample question prompts for the A-Cube change 2-Dimensional Reflection Model

adopt the Model for the primary school that she is teaching in developing reflective culture among the teachers using a whole-school approach towards (The Discoveries of Reflective Practice 2015). In fact, the teacher shared that although 'the Model was originally designed with Math in mind', she 'saw the potential for widespread adoption in her school, regardless of subject'. As pointed out by the teacher, the reflection that was commonly carried out in her school used to be focusing on 'reporting and accounting instead of critical analysis'. The Model, she noted, 'lends structure' to the school's pre-existing reflection routine, thus helping the teachers to develop metacognitive habit of mind, as pointed out earlier. A Chinese language teacher of the school, who translated the Model into Chinese, described her previous form of reflection as 'akin to recalling the lesson than detailed reflection'. Her new challenge, she shared, is 'coming to terms with' her 'own belief system though reflective practice, about what works and what does not'. As pointed out by another teacher, 'you will have to address your belief, adjust your belief, maybe to also let go of your belief at certain times', achieving what Lee (2015) had wanted the Model to achieve—a better alignment of practice, knowledge, and belief.

Lee (2017a) observed that such deep reflective practice, though offline in nature, aids the transition from offline metacognitive practice to online metacognitive practice as it deals with:

- Regulation of cognitive/affective resources that is data-based
- An heightened awareness of one's belief system which often plays up during online metacognition.

11.3.2 The Problem Wheel

As was reflected in Sect. 11.2.1, based on a survey of the relevant literature, Lee (2008, pp. 70–71) found that there are four clusters of metacognitive instructional strategies that have been shown to be effective in improving students' problem-solving ability. One of the clusters, the use of effective questioning techniques, refers to the case whereby teachers establish an environment in which both teachers and students continuously ask questions with regard to the problem-solving process so as to better understand, monitor, and direct students' cognitive resources. To realise such an approach towards the development of online metacognition, he tapped upon the Problem Wheel (Fig. 11.8), developed as a graphic organiser by Chang et al. (2001). The Problem Wheel, which is an adaptation of the Reasoning Wheel (Paul 1993) and based on the work by Lee et al. (1998), involves getting students to make sense of the basic structure of a mathematical problem through the use of systematic question prompts. The set of question prompts corresponding to the respective components of the Problem Wheel is shown in Fig. 11.9. The question prompts serve as a means for students to be more aware of their understanding of the problem context, and through such an awareness better select, match, and/or discriminate, i.e. regulate, their own cognitive resources to initiate the problem-solving process.

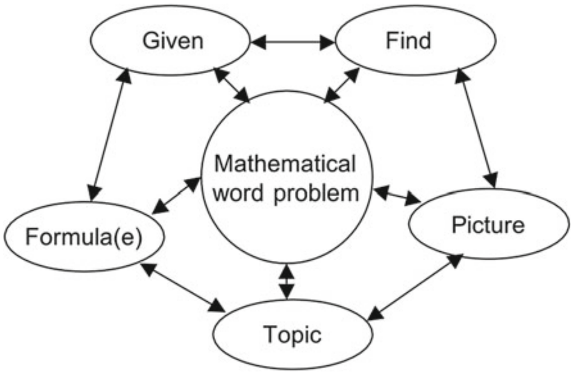

Fig. 11.8 The problem wheel

Component of the Problem Wheel	Examples of Question Prompts
Given	• What is / are given to us in this problem? • What do we know about the problem? • What value(s) is / are we given to us in this problem?
Find	• What are we supposed to find in this problem? • Which value(s) is /are we supposed to find as the answer to this problem?
Picture	• Can we draw a picture to represent this problem? • What would we draw to represent this problem?
Topic	• Which is / are the topic(s) we have learnt that might help us solve this problem?
Formula(e)	• Which is / are the formula(e) we have learnt that would help us to solve this problem?

Fig. 11.9 Question prompts for the problem wheel

The components are depicted as a wheel with double-headed arrows linking the various components to convey the idea that these are not to be perceived linearly though sometimes they may occur as such. Students may go through the various components of the wheel non-sequentially. They may go back to earlier components of the wheel to revise the information gathered and translated as they move round the wheel to gain a better understanding of the problem and try to translate the information into the mathematical concepts. The interactivity of the various components of the wheel reflects the dynamism exhibited during both the monitoring and regulatory aspects of the online metacognition.

Based on Lee's (2008) study, the wheel seemed to have a positive impact in kick-starting students problem-solving process and improved problem-solving performance. Lee (2008, p. 353) noted that, with the use of the wheel, the students were observed ‘to be more focused in their problem solving attempts by starting from the familiar ground of the ‘givens’ in a problem and working more productively and purposefully towards what they are to ‘find’ through visual representations of the problem structure’. In other words, the wheel provided a means for students to be more aware of, be actively monitoring, and constant regulating their thinking process during problem-solving, i.e. the wheel serves to guide students to actively carry out online metacognition, instead of being too concerned about just getting the right answer. This, as pointed out by Lee (2008, p. 369), results in students becoming ‘less judgmental, and appeared to be more creative in their approach towards learning and doing mathematics’. It is thus not surprising that Lee also observed a positive change in the intellectual self-concept and mathematics self-efficacy of the students in the study, as the students become more aware of their cognitive/affective resources and learn to kick-start their new ‘encounters’ with mathematics through extending and connecting, i.e. regulating, from their available resources.

Again, as in the case of the 2-D Reflection Model, the wheel has been shared with teachers in schools through conferences and seminars (e.g. Lee 2015, 2016a, b; Hong et al. 2012). There was even a workshop that was specifically conducted for acquainting the secondary school mathematics teacher to the wheel and organised by MOE (Lee 2016a). However, it should also be pointed out that as schools adopt the wheel for their mathematics lessons, some have further modified the wheel to better fit into their school programmes. Lee et al. (2014) provide a detailed description of such an undertaking by a school. The school embarked on an intervention programme to encourage their students to be more aware of their thought processes by thinking aloud to initiate students’ problem-solving process, i.e. the understanding and planning phase of problem-solving. The teachers concerned made use of a metacognitive questioning scheme that is called the STARt Understand and Planning (STARtUP) (Fig. 11.10), which is an adaptation of the Problem Wheel.

A comparison of the Problem Wheel and the STARtUp scheme will reveal that ‘Topic’ and ‘Formula(e)’ components of the wheel have been replaced with ‘Heuristic(s)’ and ‘Start’ in the STARtUP scheme. As the school has put in place a programme in the explicit teaching of Heuristics and the teachers concerned felt that solving of non-routine problems may not be topic specific, replacing the ‘Topic’ component of the wheel by ‘Heuristics’ was initiated. And since, ‘Topic’ component of the wheel has been removed, ‘Start’ was introduced as the new component to replace ‘Formula(e)’ in the wheel, to further emphasise the objective of the intervention in kick-starting students’ problem-solving process. Though the intervention was a mere six contact hours, and so might not have been sufficient for students to internalise the STARtUP scheme as a habit of mind, the findings did show that students improved in the way they initiated the problem-solving process. In fact, to some extent, the students have developed a more metacognitive approach towards mathematical problem-solving, and the school has integrated the scheme into the School’s mathematics programme (Lee et al. 2014).

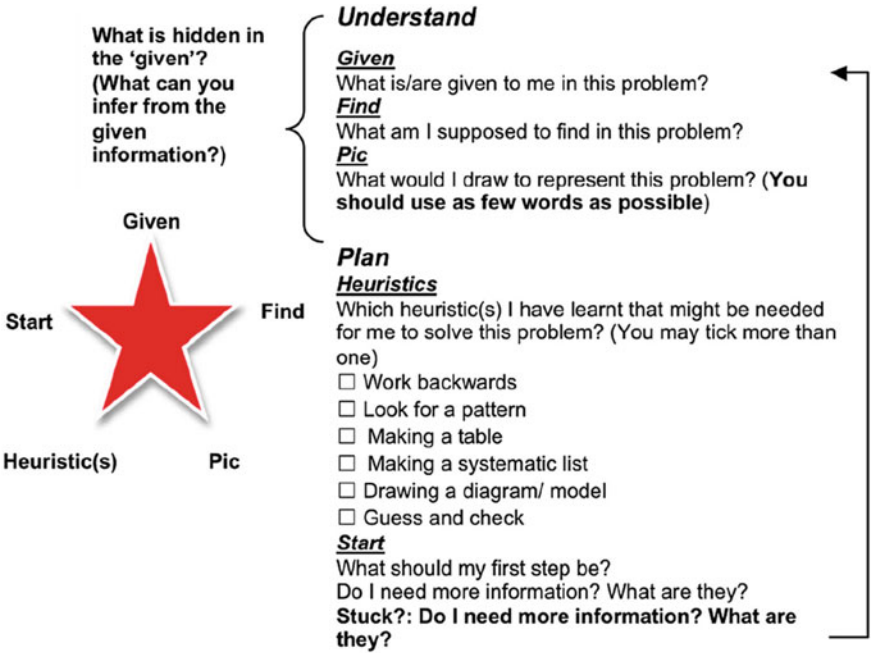

Fig. 11.10 The STARtUP scheme

11.4 Enactment of Metacognitive Instructional Practices

As was pointed out Sect. 11.2.1, metacognition has been an aspect of the SMCF since 1992. However, despite the fact that metacognition has been a key feature of the SMCF for more than twenty years, there has been limited effort to examine its impact in the mathematics classrooms from both the teaching and learning perspective. Through Singapore's participation in PISA 2009 (OECD 2010), there were efforts to collect data through student survey data to obtain a profile of our 15-year-old students' control strategies as a form of metacognitive practice in learning. However, there is also a need for identification and classification of the conceptions of metacognition and metacognitive instructional practices among mathematics teachers—the frontline practitioners who enact the curriculum and principally responsible for the curriculum experienced by the students. The Core 2 Research Programme undertaken by the National Institute of Education (NIE) (Jonid et al. 2014) made a first attempt to determine the level of metacognitive instructional practice employed by teachers in Primary 5 and Secondary 3 mathematics classrooms by analysing the mathematics teachers' instruction. However, it is extremely difficult to differentiate between cognitive and metacognitive processes (Garofalo and Lester 1985; Perkins et al. 1990; Loh and Lee 2017). The key issue lies with the difficulty to distinguish clearly between what is meta and what is cognition (Brown et al. 1983; Baker 1991;

Cheng 1999) and that the interactions between the various mental processes are complex (Yeo 2013; Loh and Lee 2017). In a review of the literature on cognition and metacognition, Tarricone (2011, p. 1) observed that the main distinction between the two is that ‘cognition is a constant flow of information and metacognition is knowledge and awareness of processes and the monitoring and control of such knowledge and processes … metacognition is considered to be second-order cognitions’. Thus the task of accurately coding a teaching act as metacognitive instructional practice without other supporting input to triangulate the data, as in the case by Jonid et al. (2014), may result in an over-generalisation and thus may not accurately portray the current situation in the primary mathematics classroom.

In 2015, MOE awarded a group of mathematics education researchers in NIE a research project (AFR 04/14 LNH) to systematically use a multi-site, multi-case-based approach to collect and triangulate data through survey, classroom observations and interviews to develop preliminary teacher conceptions of metacognition and metacognitive practices grounded based on the phenomenon under observation. Ng et al. (2016) shared the following findings from the project:

- Teachers’ conception of metacognition is superficial and/or they are confused between cognition and metacognition. Teachers’ description of metacognition may contain elements of metacognition, e.g. awareness, monitoring, regulation, reflection, but there is a lack of precise description; some confused metacognition with other cognitive skills (e.g. critical thinking and creative thinking skills), while others confused it with teaching approaches (e.g. engaged learning).
- In terms of metacognitive instructional practices, practices are vague; metacognition is still not explicitly addressed in the classroom. Though there were some understanding and linking of metacognition to reflection and monitoring, there is a lack of evidence that the instructional practices foster nor activate students’ metacognition.
- The data suggest the following cases are possible:
  - Teachers could have unconsciously and subtly address metacognition in the classroom.
  - Teacher thought metacognition has been addressed (explicitly or implicitly), but there is a lack of actual instructional evidence of this.

Of the three schools that participated in the study, the six participating teachers in one of the schools, despite being unclear of the conception of metacognition, appeared to all possess—observed to both profess and practice, some elements of metacognitive instructional practice (Lee et al. 2016). When probed further, these participating teachers attribute the practice to the 2-year-old school programme on promoting talk moves (Chapin et al. 2013). It appeared that there existed a differential impact of a more than 20-year-old curriculum versus a 2-year-old school programme on teachers’ metacognitive instructional practices. After an insightful interview with the Head of Mathematics Department of the said school, the researchers attributed the differential impact to the following factors that favoured the enactment of metacognitive instructional practices in the school:

- Existence of localised expert—there is buy-in by the Head of Department
- Strong theory-practice link—the Head of Department carried out actual classroom demonstration to exemplify the enactment of the theory
- Actual hand-holding—under the guidance of the Head of Department, teachers worked in groups to collaboratively plan the lessons
- Enculturation versus performance—the Head of Department encouraged peer observation and discussion of the lessons rather than evaluative observation by school leaders for performance.

11.5 Conclusion: Implications for Teaching and Learning of Mathematics

While this chapter has presented the extensive work done both internationally and locally that have been carried out on the address of metacognition in the teaching and learning of mathematics, there still exists challenges faced in the actual enactment of metacognitive instructional practices. The following are the three key challenges:

i. developing a clear and functional conception of metacognition for teaching and learning
ii. availing a set of practical metacognitive instructional strategies for teaching and learning
iii. conducting related and appropriate professional development for teachers.

11.5.1 Developing a Clear and Functional Conception of Metacognition for Teaching and Learning

As pointed out in Sect. 11.1.1, Schoenfeld (1992) observed that there is no agreement among researchers on a single definition of the term metacognition. Given the fuzziness of the construct of metacognition, it may be challenging for teachers to address metacognition in the classroom explicitly. Lee's (2015, 2016b) two-dimensional conceptualisation of metacognition for teaching and learning (Sect. 11.2.2) and Loh's (2015) problem-solving metacognitive (PSM) framework (Sect. 11.3) might help to lend some clarity to the conception of metacognition for teaching and learning in the mathematics classroom.

However, both the literature and the research, including those locally, have shown that teachers are generally confused between cognition and metacognition (Sect. 11.4). Swartz and Perkins' (1990, p. 109) (Sect. 11.1.1) Map of the Thinking Domain provided a schematic representation of the relationships between thinking skills, goal-oriented process, and metacognition, making clear distinction between cognition and metacognition. In addition, in Yeo's (2013) doctoral study, he researched on the nature and development of cognitive and metacognitive processes in mathematical investigation. His refined investigation model for metacognitive processes based on local data provided a scheme to show how metacognitive processes and cognitive processes interact during mathematical investigation. Loh and Lee (2017) also provided examples to demonstrate how to identify notions of cognition or metacognition in students' work in mathematics classrooms. These works provided teachers not only with a clearer distinction between cognition and metacognition, but also helped to make clearer how the two constructs interact and play out in the actual classroom context.

11.5.2 Availing a Set of Practical Metacognitive Instructional Strategies for Teaching and Learning

As mentioned in Sect. 11.4, despite the fact that metacognition is featured in the SMCF for more than twenty years, Singapore mathematics teachers' metacognitive instructional practices still appeared to be vague. Not only does it seem that metacognition is still not explicitly addressed in the classroom, there was also a lack of evidence that the claimed metacognitive instructional practices foster or activate students' metacognition. Furthermore, as was noted in Sect. 11.2.2, as a productive habit of mind, as in the case of physical habits, metacognition is formed only through continuous practice with teachers providing 'generative, rich, and provocative opportunities for using' such Habits of Mind. Thus, there is a need to avail a set of practice-oriented metacognitive instructional schemes/models/approaches that could bridge the theory and practice nexus on the development of metacognition and which may also be easily adapted for the cultivation of such metacognitive habit of mind in daily mathematics lessons.

The two works that were presented in Sects. 11.3.1 and 11.3.2, namely the A-Cube Change Two-Dimensional Reflection Model (Lee 2015) and the Problem Wheel (Chang et al. 2001), have shown the tractability of such schemes in the Singapore mathematics classrooms. The offline metacognitive scheme—the A-Cube Change 2-Dimensional Reflective Practice Model has not only been adopted at mathematics education courses but also teaching and learning in general—supporting the teaching and learning beyond that for mathematics. The online metacognitive scheme, on the other hand, not only have been adopted but also adapted for the Singapore Mathematics classrooms. However, these schemes as they have been presented, addressed the regulatory phase of both offline and online metacognition. Given the taxonomy of metacognition (Lee 2015, 2016b) that is presented in Sect. 11.2.2, schemes for the other aspects of the taxonomy need also to be investigated and developed for teachers to fully address metacognition in completeness in the classroom.

11.5.3 Conducting Related and Appropriate Professional Development for Teachers

It is a common knowledge that a teacher should possess not only a good grasp of the content to be taught as well as a set of instructional strategies that enable him/her to help the learner to develop a good grasp of the content. Some find it reasonable to suggest, however, that, at a bare minimum, teachers should possess knowledge and deep understanding of the subject matter recommended for students at the level of their teaching and, preferably, one grade level category above their particular teaching level (National Research Council 2010). Applying this similar argument to the teaching of metacognition, Lee (2016b) argued for the address of the following for professional development of teaching in addressing metacognition in teaching and learning, on top and above of a clear conception of metacognition and an accompanying repertoire of metacognitive instructional strategies for teaching of the various aspects of metacognition:

- Meta-metacognition
- Theory of mind
- Social metacognition

Lee et al. (2013) examined the design and implementation of the series of mathematical modelling lessons to determine how the development of metacognition was addressed during planning as well as during implementation of the lessons. They observed that the rich opportunities for the metacognitive development of the students afforded by mathematical modelling tasks require teachers' explicit offline and online interventions through task design, lesson planning, and strategic scaffolding during lesson implementation, or meta-metacognition (Stillman 2007). They argued that such meta-metacognition knowledge—a meta-knowledge of one's metacognition may constitute as a key pedagogical content knowledge for effective address of metacognition in the mathematics classrooms as they observed that the lacking of such knowledge in the mathematics classrooms may be detrimental to the mathematical development of the students.

Misailidi (2010) observed that metacognition and theory of mind have 'evolved over the past 20 years as two distinct and unconnected research fields' though Flavell (2002) maintains that the two fields share the same overall objective—'to investigate the development of children's knowledge and cognition about mental phenomena' (p. 106). Unlike metacognition, which is concerned with thinking about one's thinking, theory of mind deals with the ability to think about or make inferences about the thoughts and feelings of another person (Kuhn 2000a, b; Lockl and Schneider 2006). Kuhn (2000a, p. 302) describes metacognition or 'meta-knowing' as 'any cognition that has cognition … as its object'. According to Kuhn (1999, 2000a, b), theory of mind corresponds to the metacognitive knowing that includes children's knowledge about the mind, i.e. knowledge of mental state exist. Such knowledge can be both personal and impersonal. Personal metacognitive knowledge is knowledge about one's own mental states, whereas impersonal metacognitive knowing is knowledge about others' mental states. One of Kuhn's key ideas is that theory of mind serves as the foundation for the development of other dimensions of meta-knowing, i.e. children need to acquire a theory of mind first, before they begin to develop the other dimensions of meta-knowing. In other words, teachers may tap on theory of mind to further develop students' metacognition. In fact, Lee (2016b) further argued that teachers themselves also need to tap on theory of mind to better make sense of students' thinking so as to enhance their ability in addressing the metacognitive development of the students in their classrooms.

Chiu and Kuo (2009) compare and contrasted individual metacognition and social metacognition as follows:

> Individual metacognition is monitoring and controlling one's own knowledge, emotions, and actions, while social metacognition consists of group members' monitoring and control of one another's knowledge, emotions, and actions.

In other words, social metacognition as compared to metacognition, as it is presented thus far in this chapter, shifted the awareness, monitoring, and regulation of thoughts from the individual to that of a social context (Brinol and DeMarree 2012). Chiu and Kuo (2009) observed that social metacognition 'distributes metacognitive responsibilities across group members' and it aids group members' 'identification of errors, construction of shared knowledge, and maintenance of group members' motivation. As pointed out in Chap. 1, one of the outcomes of the education in Singapore is to develop each student into 'an active contributor who is able to work effectively in teams', so an inclusion of social metacognition in our address of metacognition in the context of teaching and learning in Singapore is certainly well aligned with the national curriculum. In fact, Chiu and Kuo (2009) have also noted that several programmes have showed that 'improving students' social metacognition skills aids their learning and academic performance', further reinforcing the need to address both individual and social metacognition simultaneously. Lee (2016b) further proposed that for teachers to effectively teach metacognition in the classroom context, which is in fact a group or social context, it is essential for teachers themselves to be better equipped with social metacognitive skills.

Lee conducted two 12-h in-service courses (Lee 2017b, c), one for primary and another for secondary mathematics teachers, to address the issue of metacognition in the teaching and learning of mathematics. In response to the findings of the research project (AFR 04/14 LNH) mentioned in Sect. 11.4, the emphases of these two courses are:

- Providing teachers with an operationalised conception of metacognition for teaching and learning mathematics
- Equipping teachers with some metacognitive instructional strategies for teaching and learning of mathematics.

Lee's (2015, 2016b) proposed a two-dimensional conceptualisation of metacognition for teaching, and learning was presented as an operationalised conception of metacognition to the participating teachers. Furthermore, the participants were also introduced to an offline metacognitive approach—the A-Cube Change

2-Dimensional Reflection Model (Lee 2015) and an online metacognitive approach—Problem Wheel (Chang et al. 2001).

The feedback from the participating teachers indicated that most of them has developed a better understanding of the 'difference between cognition and metacognition' and an awareness that 'there are 3 components of metacognition—awareness, monitoring and regulation'. While the participants also appreciated and valued both the offline and online metacognitive approaches shared, a number of the participants have reflected that they would like these courses to be longer so that more metacognitive instructional strategies and classroom cases could be examined and discussed. It reflects a need to better equip teachers with a more comprehensive set of metacognitive instructional strategies as well as the necessary and related knowledge and skills, as discussed in this section. While a longer in-service course may not be a practical response, there may be a need to design in-service courses to address the various aspects in preparing these teachers to be teachers of metacognition. In addition, based on the factors that favoured the enactment of metacognitive instructional practices in schools, as discussed in Sect. 11.4, the planned professional development may need to include some form of hand-holding for actual implementation in the classroom as well as the establishing of a school-based expert. In other words, from the work thus far in the Singapore context, these are a need to adopt a more holistic approach towards the professional development of teachers of metacognition.

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Chapter 12
Students' Perspectives of Good Mathematics Lessons, Homework and How Their Teachers Facilitate Learning of Mathematics

Berinderjeet Kaur and Wei Yeng Karen Toh

Abstract This chapter presents data and findings from the Learner's Perspective Study (LPS) carried out in Singapore, about students' perspectives of good mathematics lessons and the role of homework in their learning of mathematics. It also presents data and findings from the Study of school mathematics curriculum enacted by competent teachers in Singapore secondary schools that examines students' perspectives of how two competent teachers facilitate the learning of mathematics in their classrooms. Both the studies are motivated by a strong belief that the characterization of the practices of mathematics classrooms must attend to learners' practice with at least the same priority as that accorded to teacher practice. Post-lesson student interview and survey data from LPS and post-lesson interview data from the Study of school mathematics curriculum enacted by competent teachers are used to examine student practice. In the LPS, students deemed mathematics lessons as good when teachers performed specific actions as part of the teachers instruction pattern which is the **D** (Whole class demonstration)—**S** (Seatwork/Out of class assignments)—**R** (Review and feedback) cycle. Students' perspective of homework illuminated six roles it performed which included improving or enhancing understanding of mathematics concepts, preparing for test or examination and extending mathematical knowledge. In the Study of the enacted school mathematics curriculum, students in both the classes of the teachers affirmed that their teachers' carefully prepared instructional materials engaged them in learning mathematical concepts and developing the necessary procedural fluency. Though both teachers, A and B, adopted classroom discourse approaches skewed more towards student-centredness, they facilitated their students' learning differently. Teacher A had a more structured seating for her students, while Teacher B let her students to form their own clusters (friendship oriented) and sit together during the lessons. The activities for the group-based work were also dissimilar for the students in the two classes.

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Keywords Students' perspectives · Good mathematics lessons · Homework · How teachers facilitate learning of mathematics · Learner's perspective study (LPS) · Enacted school mathematics curriculum · Singapore

12.1 Introduction

Attending to students' perspectives serves to enhance our understanding of the ongoing relationship between the teacher and student as co-constructors of knowledge and practice within the classroom (Anthony et al. 2013). Competent teachers like the ones in the Learner's Perspective Study (LPS) (Clarke et al. 2006) and the Study of enacted school mathematics curriculum in Singapore secondary schools (Kaur et al. 2018) are the ones that focus on enhancing student outcomes and achieve their purpose. Achievement outcomes related to mathematical proficiency encompass conceptual understanding, procedural fluency, strategic competence and adaptive reasoning (National Research Council 2001). Anthony and Walshaw (2007) add that there are also the social and cultural outcomes relating to affect, behaviour, communication and participation that underwrite the quality of a mathematical experience. This is evident from the research reported in *Student Voice in Mathematics Classrooms around the World* (Kaur et al. 2013). Therefore, it is important that students' voice is accorded due consideration when delineating culturally and socially situated effective pedagogy of mathematics.

In this chapter, we present data and findings from the Learner's Perspective Study (LPS) carried out in Singapore (Kaur 2008, 2009), about students' perspectives of good mathematics lessons and the role of homework in their learning of mathematics. It also presents data and findings from the Study of school mathematics curriculum enacted by competent teachers in Singapore secondary schools that examines students' perspectives of how two competent teachers facilitate the learning of mathematics in their classrooms. Both the studies are motivated by a strong belief that the characterization of the practices of mathematics classrooms must attend to learners' practice with at least the same priority as that accorded to teacher practice (Clarke et al. 2006). Post-lesson student interview and survey data from LPS and post-lesson interview data from the Study of school mathematics curriculum enacted by competent teachers are used to examine student practice.

12.2 The Learner's Perspective Study in Singapore

The Learner's Perspective Study (LPS) is an international study helmed by Professor David Clarke at the University of Melbourne. It started in 1999 with Australia, Germany, Japan and the USA examining the practices of eighth-grade mathematics classrooms in a more integrated and comprehensive manner than had been attempted in past international studies, in particular the TIMSS Video Studies of 1995 and 1999.

Singapore joined the study in 2004. The LPS is motivated by a strong belief that the characterization of the practices of mathematics classrooms must attend to learners' practice with at least the same priority as that accorded to teacher practice (Clarke et al. 2006). Singapore joined the LPS in 2004. Three mathematics teachers (T1, T2, and T3) recognized for their locally defined 'teaching competence' and their respective grade 8 classes, one per teacher, participated in the study. Details about Singapore's participation in the LPS are given in Chap. 16.

12.2.1 Students' Perspectives of Good Mathematics Lessons

A distinguishing feature of the LPS is the exploration of learner practices using post-lesson video-stimulated interviews. Video records of 13 consecutive lessons (three during the familiarization stage and ten as part of the study) for each teacher were collected using three cameras. The teacher camera captured the teacher's actions and talk during the lesson. The student camera focused on a group of two students, known as the "focus group" and captured their actions and talk during the lesson. Each group of students was only videotaped once. The whole class camera captured the whole class in action. A split-screen video record mixed on-site from the teacher and student camera images was used as a stimulus for students to reconstruct accounts of classroom events during the interviews. Two students from the focus group were interviewed separately after each lesson. The interviews of the "focus students" consisted of two parts. The first part was based on the video record of the lesson for which they were the focus students. The second part was stimulated by several prompts. Student artefacts (e.g. worksheet and homework) from the focus group were also collected after each lesson.

Fifty-nine students were interviewed: 19 from T1's class, 20 from T2's class and 20 from T3's class. Although 59 students were interviewed the transcripts of 57 students were available for study. From the 57 interview transcripts responses to two prompts in the second part of the interview were the source of the data analysed. The two prompts were:

- Would you describe that lesson as a good one for you?
- What has to happen for you to feel that a lesson was a "good" lesson?

For all three teachers, T1, T2 and T3, 94%, 85% and 84% of their students, respectively, felt that the lesson for which they were the "focus students" was a good one. A close-up lens was used and the grounded theory approach adopted to analyse the responses to the second prompt. Three categories and 12 subcategories were derived for coding the responses (see Kaur 2008 for details). Table 12.1 shows the categories and subcategories.

Analysis of the 50 interview responses that related to their lesson being a good one revealed that students deemed a mathematics lesson to be a good one when some of the following characteristics were present. The teacher

- explained clearly the concepts and steps of procedures;

Table 12.1 Categories and subcategories for coding teachers' teaching

Instructional practice	Subcategory
Exposition (Whole Class Instruction)	**EC**: teacher explained **D**: teacher demonstrated a procedure, "taught the method" or showed using manipulative concepts/relationships **NK**: teacher introduced new knowledge **GI**: teacher gave instructions (assigned homework/showed how work should be done/when work should be handed in for grading, etc.) **RE**: teacher used real-life examples during instruction
Seatwork	**IW**: students worked individually on tasks assigned by teacher or made/copied notes **GW**: students worked in groups **M**: material used as part of instruction (worksheet or any other print resource)
Review and feedback	**PK**: teacher reviewed prior knowledge **SP**: teacher used student's presentation or work to give feedback for in-class work or homework **IF**: teacher gave feedback to individuals during lesson **GA**: teacher gave feedback to students through grading of their written assignments

- made complex knowledge easily assimilated through demonstrations, use of manipulative and real-life examples;
- reviewed past knowledge;
- introduced new knowledge;
- used student work/group presentations to give feedback to individuals or the whole class;
- gave clear instructions related to mathematical activities for in-class and after-class work;
- provided interesting activities for students to work on individually or in small groups and
- provided sufficient practice tasks (in class and for homework) for preparation towards examinations.

The main instructional approach of the three teachers may be said to comprise the **D** (Whole class demonstration)—**S** (Seatwork/Out of class assignments)—**R** (Review and feedback) cycle (Kaur 2009). When we juxtapose the findings of students' perspectives of good mathematics lessons with that of the instructional approaches of the teachers, we may infer some of the actions that characterized good teaching in each part of the DSR cycle as follows:

- *Whole class demonstration (exposition)*

  Teacher

– explained clearly the concepts and steps of procedures;

– made complex knowledge easily assimilated through demonstrations, use of manipulative, real-life examples;
– introduced new knowledge.

- *Seatwork/Out of class assignments*

  Teacher

– gave clear instructions, related to mathematical activities for in-class and after-class work;
– provided interesting activities for students to work on individually or in small groups;
– provided sufficient practice tasks for preparation towards examinations.

- *Review and feedback*

  Teacher

– reviewed past knowledge;
– used student work/group presentations to give feedback to individuals or the whole class.

It appears that the above characteristics of good mathematics teaching that have resulted by juxtaposing the teachers instructional approaches and perspectives of students compare well with the component "Building understanding" of Sullivan and Mousley's (2007) model of the components of quality mathematics teaching.

12.2.2 Students' Perspectives of Homework

Homework provides an opportunity for students to extend and consolidate what they have learnt in school and for teachers to extend the time for learning beyond the hours of formal schooling. In Singapore, many schools do not have a policy on homework assignments. However, in most mathematics lessons, teachers assign their students homework on a regular basis, i.e. after every lesson or after every two or more lessons. Homework is usually meant to be done alone but at times teachers do assign tasks for a group of students to do as a homework assignment. Homework is most frequently done at home, but it may be done in school during study periods or after school hours in the library, or anywhere the student so wishes to do.

117 grade 8 students completed a questionnaire as part of the LPS, 37 were from the class of T1, 40 each from the classes of T2 and T3, respectively. The responses to an item of the student questionnaire, "*Do homework assignments given by [name of teacher] help you in the learning of mathematics*?" were analysed to ascertain the role of homework from the students' perspective. All 117 of the students from the three classes completed the questionnaire and 115 (98.3%) of them (36 from the class of T1, 40 students from the class of T2 and 39 from the class of T3) indicated in their responses that homework assignments given by their teachers assisted them

in their learning of mathematics. The qualitative responses of the students were analysed using content analysis. The responses were first scanned through for themes, following which codes were generated and the data coded. Inevitably "a progressive process of sorting and defining and defining and sorting" (Glesne 1999, p. 135) led to the establishment of the final list of *functions of homework* (see Kaur 2011 for details).

Students' perspectives of the six functions of homework, inferred from the data, were as follows:

- Improving/enhancing understanding of mathematics concepts;
- Revising/practising the topic taught;
- Improving problem-solving skills;
- Preparing for test/examination;
- Assessing understanding/learning from mistakes;
- Extending mathematical knowledge.

It was apparent that all of the six inferred functions of homework were direct consequences of the instructional purpose of homework assigned by the teachers. The function "*extending mathematical knowledge*" was solely inferred from the responses of students of T2. Unlike students of T1 and T3, students of T2 were exposed to "challenging" tasks taken from non-textbook sources. This may have provided them with opportunities to extend their mathematics knowledge. These findings mirror those of students in the UK, on the main purposes of doing homework (MacBeath and Turner 1990). All of the six functions of homework inferred from the responses of the students, appear to belong to only one function of homework delineated by Tam (2009), i.e. serving immediate learning goals, which means to review learning, to prepare for quizzes and examinations, to comprehend things learned and to apply learning. This finding is not surprising as the students were asked a very specific question "*Do homework assignments given by [name of teacher] help you in the learning of mathematics*?" and there was no intention to seek data from the students about other aspects of homework involvement.

12.3 A Study of School Mathematics Curriculum Enacted by Competent Teachers in Singapore Secondary Schools

A Study of school mathematics curriculum enacted by competent teachers in Singapore secondary schools examines how competent experienced secondary school teachers implement the designated curriculum prescribed by the Ministry of Education (MOE) in the 2013 revision of curriculum. It does this firstly by examining the video recordings of the classroom instruction and interactions between secondary school mathematics teachers and their students, as it is these interactions that fundamentally determine the *nature* of the actual mathematics learning and teaching that takes place in the classroom (City et al. 2009). The study comprises a video segment and a survey segment. For the purpose of this chapter, we focus on the video segment of the study which adopts the complementary accounts methodology developed by Clarke (1998, 2001), a methodology which is widely used in the study of classrooms across many countries in the world as part of the Learner's Perspective Study (Clarke et al. 2006). This methodology recognizes that only by seeing classroom situations from the perspectives of all participants (teachers and students) can we come to an understanding of the motivations and meanings that underlie their participation. It also facilitates practice-oriented analysis of learning.

Thirty competent experienced teachers and approximately 600 (in each class about 20 students, who volunteer to be the focus students and be interviewed) students in their classrooms are participating in the video segment of the study. In the context of the study, a competent experienced teacher is one who has taught the same course of study for a minimum of five years, is recognized by the school/cluster as a competent teacher who has developed an effective approach of teaching mathematics. At the time of writing this chapter, 20 teachers and 303 students had participated in the study. In this chapter, we examine the interview data of students from the classrooms of two teachers, A and B, who organize student learning in their classrooms in different ways though albeit skewed towards a more student-centred approach. Specifically responses of the students to the interview prompt "How does your teacher help you learn mathematics?" are explored.

12.3.1 Case Study 1 (Teacher A)

Grade 9: Topic—Quadratic Equations

Students seated in orderly clusters of fours

Number of students interviewed: 21

The collaborative learning orientation in Teacher A's Classroom

Research had shown that collaborative work will promote academic achievement and positive social interaction of students in all educational levels and in a big variety of subjects (Slavin and Cooper 1999; Johnson and Johnson 2000). It has also been documented that when small groups worked together, there was greater impact on the transfer of learning compared to individualist learning (Gomez 2016). According to Swan (2006), traditional, direct "transmission" of explanations, examples and exercises "do not promote robust, transferable learning that endures over time". In contrast, when students collaborate and work in small groups based on the model of teaching he designed, teachers could emphasize the interconnected nature of the subject and confront common conceptual difficulties. Such a method would provide students with opportunities to tackle problems before offering them guidance and support. Moreover, it would encourage students to apply pre-existing knowledge and allow teachers to assess and then help them build on that knowledge. Figure 12.1 illustrates the differences between the two models of teaching.

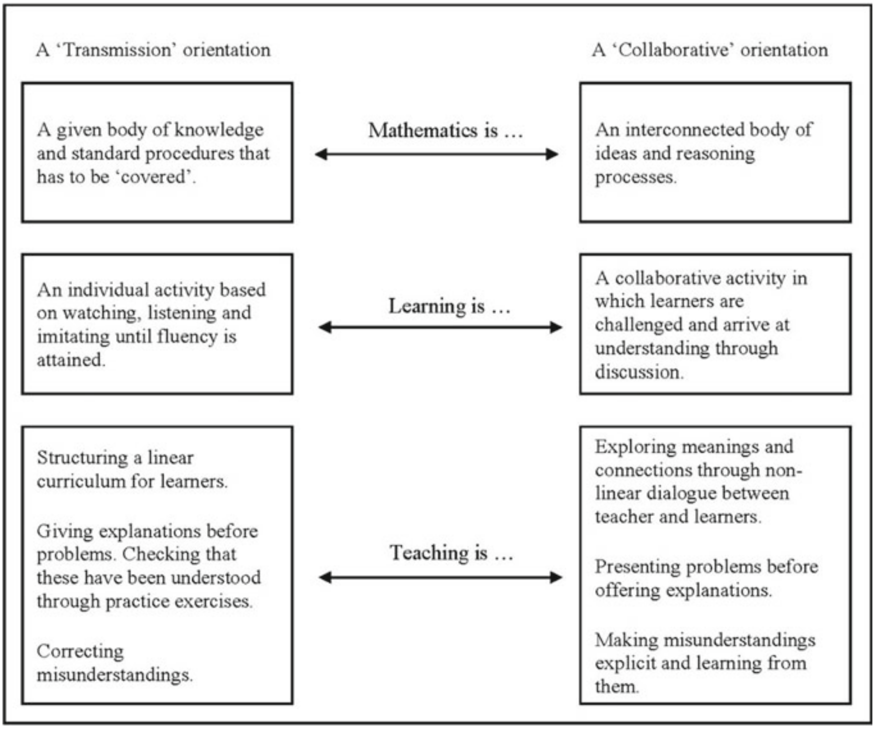

Fig. 12.1 Two models of teaching (Swan 2006)

Teacher A harnessed the collaborative teaching model extensively in her lessons. Students were assigned to sit in groups of four. Every group was provided with a mini whiteboard and a marker pen which they were tasked to use to write down their agreed responses during group discussions. Students had to discuss among themselves, come to a consensus, write down their responses on the mini whiteboard and then present to Teacher A. Next, Teacher A would select a few suitable responses to further explain so as to achieve the instructional objectives of the lesson. Sometimes, Teacher A invited students to explain the reasoning behind their group's answers to help the class understand their responses better.

The questions that Teacher A posed to generate discussion among students were purposefully designed to check students' prerequisite knowledge; stimulate students' thinking and/or to illuminate concepts. For instance, when Teacher A first introduced the term "quadratic equations", and a student classified equations with a x^2 term as "quadratic", she posed the question: Is $\frac{3}{x^2} + 2x - 1 = 0$ a quadratic equation? Students discussed the question in their groups, arrived at a consensus and wrote their responses on the mini whiteboards provided. From the students' responses on the mini whiteboards, Teacher A gathered the misconceptions and went on to discuss the concept of "quadratic equations" further. One of the misconceptions students had was that "$x^{-2} = x^2$", therefore $\frac{3}{x^2}+2x-1=0$ is a quadratic equation. Teacher A took the opportunity to elicit students' responses and probe why they thought x^{-2} and x^2 were equivalent, which was erroneous. When students collectively established that $x^{-2} \neq x^2$, students were asked to discuss whether $\frac{3}{x^2} + 2x - 1 = 0$ was a quadratic equation again. Eventually, they concluded that the given equation was not of a quadratic type. Another example of a question Teacher A posed was: Is $2x^2 = 3$ a quadratic equation? Time was given to students to think and then discuss. It was meant to stimulate a discussion to address students' common misconception—all quadratic equations take the form of $ax^2 + bx + c = 0$. Teacher A ensured that students had a good grasp of mathematical concepts. Throughout the lessons, Teacher A posed questions to elicit students' responses and encouraged them to clarify their thinking/understanding both in their groups and in class. She said repeatedly "it's okay to make mistakes".

The main instructional materials Teacher A provided for students was a set of self-designed "notes cum worksheets", hereafter known as "notes". In the set of notes, Teacher A placed emphasis on key mathematical terms such as "equation", "factorization", "completing the square" by bolding them to draw students' attention. Blank spaces were also provided in the text so that students could fill them in with key points they gathered during class. The notes had sections, such as "Self-Check", "Practice Questions", "Examples", and "Try it!", each addressing specific instructional objectives. Tasks named as "Examples" were used for exemplifying; tasks named as "Practice Questions" and "Try it!" were for students to apply new concepts and practice new skills; and tasks labelled as "Self-Check" were for students to assess themselves. Some parts of the notes were completed during the lessons, while others were set as homework. Prerequisite knowledge and concepts from other topics in the syllabus were also included in the notes. During lessons, Teacher A created opportunities for students recall their previous knowledge and connect new concepts with it so as to understand the new topic better. Besides notes, Teacher A also assigned students to complete selected items from the textbook as homework. In addition, Teacher A designed a few worksheets which had mainly practice items for students to work individually as homework. Numerical answers to these practice items were provided so students could check their work.

How Teacher A helped students learn mathematics

After every lesson, 3 of the focus students were interviewed. As part of the interview, students were asked how their teacher helped them learn mathematics during the lesson. A total of 21 students were interviewed. Table 12.2 shows the content analysis of two students' interview data.

From the interview data, it was apparent that Teacher A facilitated her students learning of mathematics by providing them with opportunities to discuss in groups and present their responses on a mini whiteboard. Half of the students said they enjoyed the lessons because of such opportunities. Some mentioned that they learned from their peers during the discussions. During the Post-Lesson Interview, Student AP02 expressed that "by discussing with friends", she was able "[to] see … different ways they think then I'm able to like learn more"; and Student AP06 articulated that

Table 12.2 Content analysis of data (Students AP02 and AP06)

Student ID	Interview response	Inferences
AP02	[00:10:52] S: My teacher **posed some questions** that I will not think of usually. Because usually, I see the paper, I see the word "factorise", I see the words "quadratic equations", I will just start doing without knowing what it means. So by **teaching us what the words** [like "factorise", "quadratic", "roots"] **actually mean** right, I get, I grasp the concepts faster, then I understand the question. So no matter how my teacher turns the question, I will know what to do, yah [00:12:17] I: You mentioned the mini whiteboard a few times. How does that help you?	Notes and mini whiteboard are useful Merits i. Emphasis was placed on the meaning of mathematical terms
	[00:12:22] I: It actually—because we sit in a cluster of 4, so because I am not that good at Math usually, so when—**by discussing with my friends around me, I see like the different ways they think then I'm able to like learn more**	ii. Discussions among students increased learning
	[00:12:56] I: How do you find it? [00:12:58] S: I actually find it's much better … It's also **fun** 'cause you **can actually interact** with your classmates	iii. Interaction among peers
	[00:18:18] I: Is there any part of the lesson you like best? [00:18:23] S: The whiteboard part… All the whiteboard parts… I just find it fun. Like **more engaging**; if not, we'll just sit there on our own. (And) my teacher will ask us to flip through and do all the questions, it's quite boring	iv. Engaged students through the use of mini whiteboards
AP06	[00:07:20] S: My teacher did not straight away tell us the answer yet, my teacher **lets us answer ourselves, after that discuss among ourselves**, then after that if our answers are wrong, **we can realise our own mistakes. Then like that we can remember better** [00:07:33] I: Was there any material used to help you learn? Was it inside the notes? [00:07:39] S: **The whiteboard** and the marker	Group discussion with the use of mini whiteboard is helpful Merits i. Discussion promotes collation of thoughts among students
	[00:07:47] I: How does that help? [00:07:57] S: It **lets us collaborate with our group members**, then from there, we can have more knowledge–**gain more knowledge from them**. Like **if you don't know something, they will also explain to you** [00:09:56] I: … So this is the notes that helped you to learn. So how do you feel about this whole thing? Today's learning? The use of all these materials?	ii. Discussion promotes collaborative learning iii. Increased students' engagement during lessons
	[00:10:11] S: It's quite **effective** [00:10:12] I: Effective? In what way [00:10:18] S: Ahh… **it'll help you remember**, instead of just my teacher talking then you just sit down there and listen. Actually, if you do that, you can easily forget the things my teacher talked (about)	iv. Increased retention of knowledge and skills
	[00:10:32] I: Then, on your part, how do you make sure that you are learning also? [00:10:51] S: Pay attention. **When my teacher gives us homework, I check the answers at the back to make sure it's correct**. **If it's wrong, I try to find my own mistake**. If really cannot find, then I'll ask my teacher	v. Numerical answers provided for practice items in textbook and worksheets. Students were encourage to regulate their own learning

the discussions allowed her to "collaborate with group members", and thereby she could "have more knowledge–gain more knowledge from them". Some students also found it more engaging to use the mini whiteboards to present their responses as it was "fun" and "more engaging" instead of merely "flip[ing] through [the notes] and do all the questions". One student who expressed a lack of self-confidence felt that it was less embarrassing to present a consolidated group response than an individual's response so the use of the mini whiteboard was "better".

Students also appreciated the notes the teacher prepared as they were the primary source of information and reference for most of them. The emphasis on important mathematical terms in the topic helped students focus on the concepts pertinent for their learning. To comment on Teacher A's deliberate emphasis on the mathematical terms—"factorize", "quadratic", "root"—in the introductory lesson, Student AP02 said that "by teaching us what the words actually mean right, … I grasp the concepts faster".

12.3.2 Case Study 2 (Teacher B)

Grade 8: Topic—Mean, Median and Mode; Probability
Students seated in random clusters of three to five
Number of students interviewed: 18

The learning of mathematics with peers in small groups in Teacher B's Classroom

Research on small group learning in schools started in the early 1970s with prominent reviews by Johnson and Johnson (1974) and Slavin (1977). The findings showed that there were positive effects on student achievement, especially compared to other forms of instruction that involve less interaction between students (e.g. O'Donnell 2006; Slavin 1995). It was observed that students can learn by working with and helping each other. When students learn in small groups, they share knowledge, build on one another's ideas and justify their own. Students also get to recognize and resolve contradictions between their own and other students' perspectives (Bossert 1988–1989; Webb and Palincsar 1996).

Cohen (1994) recommended teachers who implement small group learning in their classrooms to carefully listen to group discussions, to make hypotheses about the groups' difficulties before deciding on what questions to ask or suggestions to make and to keep interventions to a minimum. She argued that students will be more likely to initiate ideas and to take responsibility for their discussions if teachers provide little direct supervision (such as guiding students through tasks, or answering individual student's questions before the group has attempted to work collectively to solve a problem). This will help to foster student reliance on each other rather than on teachers.

In Teacher B's classroom, students sat with classmates whom they feel could help them learn mathematics better. Students sat in clusters of three to five. While working on mathematical tasks assigned by the teacher, students discussed their work and helped each other.

The instructional materials Teacher B provided for the topic was a set of coherent mathematical tasks from the textbook; a teacher made worksheet; an online quiz; and a post-topic test. During lessons, Teacher B would begin explaining new concepts, by introducing definitions and formulae, and then demonstrate a few examples. Following which, Teacher B would assign students to work on selected items from the textbook. Teacher B believed that students should make full use of the mathematical tasks in their textbook as the book is of high quality and the tasks in the book span a good range from simple to complex. After completing the assigned tasks from the textbook, students worked on their worksheets. When students are working at their desks, Teacher B walks around the classroom monitoring students' progress and assisting clusters of students with their difficulties. One of the ground rules Teacher B had put in place was that students must first discuss among themselves their difficulties before approaching the teacher for assistance. According to Teacher B, the mathematical tasks in the online quiz and post-topic test tested mainly the common difficulties that surfaced during the lessons. Teacher B hoped that students would learn from their mistakes during lessons and then apply the knowledge onto summative assessments.

How Teacher B helped students learn mathematics

After every lesson, three students were interviewed. As part of the interview, students were asked how their teacher helped them learn mathematics during the lesson. Altogether 18 students were interviewed. Table 12.3 shows the content analysis of two students' interview data.

From the interview data, it was apparent that teacher B facilitated his students learning of mathematics in two distinct ways. The first was the range of mathematical tasks (items) that helped them consolidate concepts and develop procedural fluency. This was important for the students as they were mindful of how they would perform in their summative assessments like tests and exams. Student BP06 mentioned specifically that being familiar with the items helped to build her confidence—she "will not … panic".

The second was the seating arrangement of the class. 61.1% of the students voiced their preference for it. They felt that they benefitted by sitting with friends who could help them with their seatwork and homework. Student BP06 specifically mentioned, "If I don't understand the concept, I will ask my friends beside me". Students not only could learn from their classmates, teaching another person helped them reinforced their concepts too. BP14 commented that, "When you teach people right, then you can be reminded of the questions. Then you can see if you really understood the question".

Table 12.3 Content analysis of data (Students BP06 and BP14)

Student ID	Interview response	Inferences
BP06	[00.04:00]I: Can you tell me, in what way is the worksheet useful?	Worksheet is the most useful Seating arrangement is helpful Merits
	[00.04:04] S: Like—because they say what, "practice makes perfect", so if you do more this kind of questions, of course you will **familiarise more with this kind of questions**. Then **when exam come out, you will not be so panicked** and all this kind of things. Then you don't know how to do. So five questions is all about this kind of questions. So when we do five questions right, maybe when I do first or second question right, I**'m not really sure of the answer, then I will ask my friend**, but then **when I do the third, fourth, fifth, it is similar to first and second question**, then I will try and do it. So if I find the answer correct, then that means it's useful to me	i. Similar practice questions help student acquire procedural fluency ii. Worksheet helps student prepare for assessments
	[00.04:41]I: So how do you- besides the Math worksheet right, there's something else he gave–which is (questions from) the Math textbook, right? How do you find those questions in the textbook compared to the Math worksheets? [00.04:51] S: Hmmm… textbook? Textbook… my teacher gives examples, and used textbook examples. I think it's OK, textbook examples. But then, I prefer my teacher to print out those questions to let us try because textbook still need to flip those kind of pages then this one just **one worksheet then you can refer (to it for) everything**. Then the (difficulty of the) questions are still about the same	iii. Worksheet is more concise than textbook
	[00.05:26] I: So on your part right, what did you do today to help yourself learn? [00.05: 31] S: Ah…pay attention to my teacher in class, and then if I don't understand the concept, **I will ask my friends beside me**. And if I still don't understand, then I will ask (my teacher) for help	iv. Student benefits from learning with friends
BP14	[00.11:35] I: So on your part, what did you do to help yourself learn? [00.11:41] S: I take down notes. … [00.12:24]I: OK, so is there anything else you did in class, other than taking notes? [00.12:30]S: **Teaching my friends**. … Like when we sit in pairs, **we can help each other out, like we don't understand, we can ask each other**. But if you sit alone, then if you want to ask someone, like it's very troublesome [00.12:58]I: In (your) class, I understand that (your teacher) lets you all sit with the person you are comfortable with. How do you find it? [00.13:07] S: (My teacher) lets us sit with people who can help us… [00.13:20]I: How do you find it? Do you find it useful for yourself? For your learning?	Seating arrangement is helpful. Worksheet is most useful Merits i. Student benefits from learning with friend
	[00.13:27] S: **Yes. When you teach people right, then you can be reminded of the questions. Then you can see if you really understood the question**	ii. Student gets to reinforce own learning while guiding friend

(continued)

Table 12.3 (continued)

Student ID	Interview response	Inferences
	[00.11:35] I: So out of the materials that you have, which one is the most useful when you are learning Math? … [00.18:38] S: Worksheet. … **'Cause (our teacher) will give like all types of questions and then teach us how to do different types of questions which will help us in our exam**	iii. Various practice questions help students prepare for summative assessments

12.3.3 How Competent Teachers A and B Helped Their Students Learn Mathematics

Both teachers A and B used carefully prepared instructional materials to engage their students in learning mathematical concepts and developing the necessary procedural fluency. Teacher A placed emphasis on mathematical vocabulary that helped students develop sound understanding of mathematical concepts such as quadratic expressions, equations and factorization. Teacher B likewise through his carefully chosen learning tasks that he used to demonstrate the concepts of averages (mean, mode and median) to the whole class facilitated his students' development of conceptual knowledge. Both teachers provided adequate practice tasks, from simple to complex, thereby facilitating development of procedural fluency. Practice tasks also had past examination questions so that students were familiar with the expectation of their class quizzes and school examinations. It was apparent in the classes of both the teachers that conceptual knowledge and procedural fluency contributed towards successful problem-solving.

Though both teachers A and B adopted classroom discourse approaches skewed more towards student-centredness, they facilitated their students' learning differently. Teacher A had a more structured seating for her students. They were seated in groups of four and the members were the same for every lesson. Teacher B left to the students to form their own clusters (friendship oriented) and sit together during the lessons. There was no involvement of the teacher with regard to with whom the students sat together. The activities for the group-based work were also dissimilar for the students in the two classes. In the class of Teacher A, the group-based activities were driven by the teacher posing questions to address the possible misconceptions students may have or develop while acquiring conceptual knowledge worked on during the lesson. In the class of Teacher B, the work students did at their desks were mainly practice tasks and when students faced difficulties doing the work they sought help from their peers, who were sitting around them or they helped their peers who had difficulties.

12.4 Conclusion

The data and findings reported in this chapter affirm that students' perspective of good mathematics lessons are in tandem with the practices of their teachers. This is so as students deemed mathematics lessons as good when teachers performed specific actions as part of the teachers instruction pattern which is the **D** (Whole class demonstration)—**S** (Seatwork/Out of class assignments)—**R** (Review and feedback) cycle. It is also illuminating to uncover the six roles of homework perceived by the students. Often when teachers assign their students homework it is received with remarks like "oh no!", "more work to do", etc.

Students perspectives of how the two teachers, A and B, in the Study of the enacted school mathematics curriculum facilitated their learning show us that competent teachers harness the potential of their students by organising learning in ways that are both non-negotiable such as curriculum materials and negotiable such as diversity in organising student–student interactions and student–teacher interactions for student learning.

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Chapter 13
Low Attainers and Learning of Mathematics

Tin Lam Toh and Berinderjeet Kaur

Abstract This chapter describes the main studies which have been carried out in the Singapore mathematics classrooms to identify and address the learning needs of low attainers in mathematics at the primary and secondary levels. This chapter begins with describing two research projects on low attainers: the first is an exploratory study on low attainers at the primary level and the second a survey on teachers' perception of low attainers from the Normal (Technical) stream at the secondary level. These two studies identified the characteristics of low attainers and their content knowledge, teachers' perception about their motivation and competency in mathematics, and provide a preliminary knowledge of how teachers have attempted to facilitate them to learn mathematics better. The chapter further presents three intervention research projects that were conducted by researchers from the Singapore National Institute of Education (NIE) in collaboration with school teachers to facilitate the low attainers in learning mathematics. The first project was an action research proposed by a school to facilitate the mathematics learning of students from the Normal (Academic) stream through the use of cooperative learning strategies. The researchers proposed a framework of cooperative learning that was trialled in their school setting. The second project was another research project initiated by researchers from NIE on using comics and storytelling in teaching mathematics in the Normal (Technical) stream. The study shows that there was an overall positive impact of this approach on students' motivation in learning mathematics and their performance in mathematics achievement test. In the third project, another team of researchers from NIE attempting to use the Concrete–Pictorial–Abstract heuristic to help Normal (Academic) students learn mathematics by assisting them to make abstract algebra meaningful and manageable. This chapter concludes with describing the projected initiated by the Ministry of Education at the national level on building teacher capacity to facilitate learning of mathematics among the low attainers in mathematics.

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13.1 Introduction

Despite the fact that Singapore mathematics education has received much attention from educators throughout the world, there was a concern among the policy makers in the Singapore Ministry of Education (MOE) and the school leaders about the significantly large proportion of Singapore students who are not performing well in mathematics. This concern is not surprising as mathematics is a core subject in the Singapore primary and secondary mathematics curriculum. As Kaur et al. (2012) aptly put it, "failure in mathematics due to factors that may be controlled would be unjust for the [students]" (p. 2).

This chapter reports the various efforts taken by the MOE, mathematics educators and researchers from the Singapore National Institute of Education (NIE) in addressing the learning needs of the low-performing students (or "low attainers") in mathematics. These efforts are crucial and in fact should begin early in students' career as data (such as the TIMSS 2003 data for the USA) have shown that students who fail at a lower grade will very likely fail at subsequent grades if no intervention has been taken.

13.2 Who Are the Low Attainers?

We first begin with defining and explicating the choice of the term "low attainer" that is used throughout this paper. Early researchers used the term "low attainers" to describe those students who fall into the bottom 20% of mathematics attainment in their age group in national examinations (e.g. Denvir et al. 1982). Haylock (1991) used the term "low attainers" to define students who attain very much less in mathematics when compared to their contemporaries. In this chapter, we choose the words "low attainer" over several other terms like "slow learners", "at-risk students", "special needs students" and so on, to describe this group of students at the various levels. As in Kaur et al. (2012), we adopt the use of the term "low attainer" is a purely descriptive term and does not make any judgement about the reasons for the students' low attainment in mathematics.

Teaching low attainers has always been a challenging task faced by school teachers. Low attainers generally show little interest in the various academic subjects. They are usually not mentally focused in the classrooms, tend to be restless in classes and have relatively short attention span (Lui et al. 2009).

What are the reasons that low attainers behave in this manner in the classrooms? Many educators have argued that low attainers are what they are because there is a misfit between their needs and the existing educational programmes; they generally need educational programmes which are more skill-based than theory-based. Most teachers, based on their classroom experience and anecdotes, generally associate low attainers with kinaesthetic learners. In Dunne et al. (2007), a senior member of a high school commented that:

> …A lot of the low-ability pupils are very kinaesthetic and they just can't sit still and that's why they are low-ability pupils. They just don't work like this (p. 62)

Educators and researchers also recognized that many of them are kinaesthetic learners rather than visual or audio learners (Amir and Subramaniam 2007; Rayneri et al. 2003). Empirical studies such as that by Shahrill et al. (2013) concluded that the learning style of students of lower mathematical ability tends to be kinaesthetic. However, educational programmes worldwide have evolved into one that is more theory-based than skill-based, and most of the instructional programmes tend to focus on students who are visual and audio learners (Glass 2003). This has put the lower-attaining students at a disadvantage. It is thus not surprising that low attainers generally do not show interest in the academic subjects or are perceived as lack of competence in these subjects.

In the situation of Singapore, a study conducted by a group of researchers from the Singapore NIE on low attainers in primary four mathematics (Chang et al. 2010; Kaur and Ghani 2012) reported that there was a mismatch between how pupils think they best learn mathematics and how teachers teach them mathematics. Most students in the study preferred to be engaged in group activity during mathematics lessons and believed that the use of computers would help them learn mathematics better. However, the instructional method adopted by the teachers was mainly teacher demonstration, seatwork and review of pupil work. The study recommended that teachers provide pupils with opportunities to talk and clarify their thinking as well as motivate pupils to ask questions and freely share with others their thoughts. This could offer a further clue on the lack of good behaviour of low attainers in the classrooms.

13.2.1 Characteristics of Low Attainers of Mathematics

Education literature abounds with elaborate descriptions of the characteristics of low attainers. Much research has been conducted on identifying the characteristics of low attainers in mathematics. Generally, these characteristics can be broadly classified under three broad categories:

1. *Cognitive and metacognitive factors*: The low attainers generally lack metacognitive strategies (Cardlle-Elawar 1995; Kruteskii, 1976; Mercer and Mercer 2005; Verschaffel and De Corte 1995). Most of them suffer "cognitive overload" and tend to have short-lived memory for mathematical procedures (Keijzer and Terwei 2004; Mercer and Mercer 2005). Also, they lack the ability to apply the appropriate heuristics for different situations (Nelissen and Tomic 1998;

Verschaffel and De Corte 1995) or to apply domain-specific knowledge flexibly (Kraemer 2000). Not only that, they lack appropriate background/prerequisite knowledge (Mercer and Mercer 2005). They also have difficulties in using more sophisticated representations such that schemata and models or in considering numbers as formal objects (Karsenty et al. 2007; Kraemer and Janssen 2000).

2. *Affective factors*: Low attainers in mathematics usually show negative attitude towards learning mathematics. Some signs of negative attitude include feelings of fear, stress, anxiety and resentment towards mathematics (Haylock 1991; Karsenty 2010; Mercer and Mercer 2005; Lehr and Harris 1988). In addition, they have low motivation and academic self-concept (Fong et al. 2012; Karsenty 2010; Mercer and Mercer 2005).
3. *Social factor*: Many of the low attainers have social problems and lack social skills. Many of these problems of the low attainers can be attributed to their family background or immediate environment (Haylock 1991; Lehr and Harris 1988). However, this crosses into the boundary of social work and psychology and is beyond the scope of this chapter. We shall not discuss this here.

This chapter next discusses how low attainers are recognized early in the students' career and how they are integrated into the school system. The discussion will be based on the perspective from the objective of the Singapore education system. We shall next discuss the various efforts that were undertaken by the various mathematics educators and researchers in identifying and addressing the learning needs of the low attainers in the Singapore school system, and a nationwide effort to facilitate the learning of the low attainers by the Singapore Ministry of Education.

13.3 The New Education System—An Ability-Driven System

Singapore, with its unique historical and geopolitical factors, has seen several phases of development since its independence in 1965. Without any natural resources of her own since her independence, the nation, under the visionary leadership of the first Prime Minister Mr Lee Kuan Yew and his likeminded cabinet at that time, placed great emphasis on developing her human resource. Thus, the education of the population was the main priority at that point of consideration. Mr. Lee Kuan Yew stressed that the "simple objective" of education is to "educate a child to bring out his greatest potential so that he will grow into a good man and a useful citizen" (Lee 1979).

In the phase of education development in 1979, the Singapore Ministry of Education's Study Team led by the then Minister of Education Dr. Goh Keng Swee (Goh and The Study Team 1979) identified that one of the key weaknesses of the education system at that time was the high education wastage. A high proportion of the population did not receive the minimum number of years of secondary education. This led to the low literacy level in the country (Goh and The Study Team 1979). With the New Education System (NES) introduced in Singapore in 1979, an education that was ability-driven was implemented. Instead of having a common core curriculum for all students, in which students' various capabilities was not taken into consideration, streaming was introduced. This allowed students of various capabilities to progress at a pace that was suitable for them. All students had the opportunity to complete the minimum number of years of schooling in order to acquire basic literacy and numeracy.

At this phase of the NES, ability-based streaming of students was carried out at primary three into Foundation and Normal programme and at the secondary one level into Express/Special and Normal. Subsequently, several fine-tuning of streaming was done. In 1991, the streaming at primary three level was adjusted to primary four. In 1994, the Normal Stream at the secondary level was further divided into Normal (Academic) and Normal (Technical), paying particular attention to students in the Normal Stream who are more inclined to technical education.

As a consequence of this process of streaming, students from the Foundation programme at the primary level and the Normal (Technical) stream at the secondary level tended to be labelled as "low attainers". Readers are cautioned that labelling students from these streams as "low attainers" is incorrect (although it may be true that a significant proportion of students from the Foundation and the Normal (Technical) streams are less academically inclined). For example, some students in the Normal (Technical) stream could be proficient in mathematics but less inclined to most other academic subjects. In this chapter, we recognize that interventions that were carried out to assist low attainers in mathematics were usually divided according to the streams they were assigned.

Readers should note that streaming in the Singapore school system is not static. Recognizing that the stage of student development varies with individual, channel is available for students to be transferred from one stream (especially those who are classified as "low attainers") to another if they prove to be capable to manage the challenges required in the latter stream.

13.3.1 Identification and Support for Low-Attaining Group of Students

Even before the level of streaming at primary four, efforts are made by MOE to identify low-attaining groups of students, so that additional support can be provided to facilitate their learning. In primary one and primary two schools, identify low-attaining students with the help of school-based School Readiness Test (SRT), school-based entry/diagnostic tests and performance of students in semestral assessment. Schools adopt varying strategies to support these students in their learning. Two strategies adopted by schools are banding based on either the students' overall performance or their performance in the core subjects (English and Mathematics) and placement in the Learning Support Programme where teachers work either with

Table 13.1 TIMSS 2007 Grade 4 mathematics mean scores

Country	Rank	TIMSS 2007 Grade 4 mathematics mean scores of pupils up to the *n*th percentiles				
	n	5th	10th	15th	20th	25th
Hong Kong	1	462	488	503	514	524
Singapore	2	408	440	461	477	490
Chinese Taipei	3	427	453	469	481	490
Japan	4	404	432	449	463	474

individual students or a small group of them for the subject they are identified as low attaining.

In the next two sections, we shall report on two studies targeted to obtain a better understanding of low attainers in mathematics: the first is an exploratory study at the primary level, and the second is a survey on the teachers teaching Normal (Technical) mathematics at the secondary level.

13.4 Exploratory Study One: Low Attainers in Primary Mathematics (LAPM)

Although Singapore students in general topped the various international comparative studies in mathematics (TIMSS and PISA), the findings from 2007 TIMSS on the performance of low attainers, as shown in Table 13.1, indicate that the lowest performing students were not performing as well as their counterparts in several other Asian countries.

Table 13.1 shows the mean score in the mathematics section of Grade 4 students from four Asian countries (Hong Kong, Singapore, Chinese Taipei and Japan) in 2007 TIMSS according to the various *n*th percentile levels ($n = 5, 10, 15, 20, 25$). The table shows that the lowest achieving students at the *n*th percentile (5, 10, 15, 20, 25) are not performing as well as several other countries (Hong Kong, Chinese Taipei and Japan), which are on par with Singapore in the overall student performance in 2007 TIMSS.

This concern among the policy makers in the Ministry of Education and the school leaders about the high proportion of low attainers in mathematics at the primary schools led to the conceptualization of a research project involving an exploratory study named "Low Attainers in Primary Mathematics" (LAPM). The project, which was initiated by researchers in the Singapore NIE and supported by the MOE, was conducted on low attainers in mathematics at the primary level.

In particular, LAPM aims to identify (1) how schools and teachers motivate and inspire low attainers in primary mathematics to learn the subject; and (2) how schools and teachers address the diverse learning needs of low attainers in primary mathematics. The project also serves to determine the characteristics of low attainers in mathematics in their relation to (3) their school (e.g. behaviour during class, absenteeism, interactions with peers); (4) their home (e.g. home support, resources, environment) and (5) their experiences of learning mathematics. We shall next present a brief summary of the findings of the LAPM under five main subcategories:

13.4.1 How Teachers Motivate Low Attainers in Mathematics at the Primary Level

The result of LAPM shows that primary school mathematics teachers used a variety of strategies to motivate and inspire low-attaining students in mathematics. The teachers modified the existing teaching packages by chunking up the resources in each teaching unit into small chunks, and to teach the students using their modified package in smaller groups. In addition, teachers also used various forms of activity-based learnings, such as the use of pictures, games, songs and manipulatives, in their mathematics lessons. Technology was also harnessed during lessons to enhance the students' learning. Regarding the general educational psychological principle, teachers also used games and quizzes, words of encouragement and extrinsic rewards to motivate and inspire the low attainers.

13.4.2 How Schools and Teachers Address the Learning Needs of Low Attainers

According to the study, the teachers strongly believed that large class sizes, the nature of the mathematics curriculum, time constraint and the mode of assessment are the main hurdles in addressing the learning needs of low attainers. All the schools addressed the above hurdles by engaging additional staff (the provision of either allied educators or adjunct teachers) and provided supplementary and remedial lessons for their students.

The teachers also gave three main recommendations in assisting them to further address the needs of low attainers:

1. They should be provided with teacher professional development programme which specifically address the learning needs of low attainers, in addition to the general student population.
2. Enrichment programme should be provided to the low attainers.
3. More curriculum time should be allocated for core subjects (in particular, mathematics).

Table 13.2 Performance of 390 low-attaining students in mathematics content tests

Test	Max. score possible	Mean	Standard deviation
Whole numbers (concept)	30	17.1	4.49
Whole numbers (operations)	29	22.1	3.98
Whole numbers (word problems)	9	3.52	2.45
Fractions	29	13.71	6.98
Measurement	21	10.14	3.69
Measurement (word problems)	8	3.20	2.05
Geometry	10	6.34	2.16
Data analysis	10	6.37	2.55

13.4.3 Low Attainers' Mastery of Primary Three Mathematics Content

One key research area of LAPM involves examining the performance of 390 primary four low attainers in a test with various strands in the primary school mathematics curriculum. It shows that their poor performance in mathematics was also interfered by their language ability and the psychological hurdle of perpetual failure in mathematics. Table 13.2 shows the mean and standard deviation of the students in these tests.

A full description of the pupils' performance in the individual items is described in greater detail by Koay et al. (2012), which will not be elaborated here.

It is clear from Table 13.2 that the students performed best in items related to concept and operations involving whole numbers and worst in word problems on whole numbers and measurement. Koay et al. (2012) attributed this to their poor reading ability. Sufficiently, competent language ability is necessary in understanding and solving word problems. They suggested the strategy of using visual imagery to solve the word problems by engaging by the bar model method and classroom instruction to address these deficits among the low attainers.

It was encouraging that Koay et al. (2012) asserted that most low attainers had the capability to learn. However, perpetual failure in mathematics created a major obstacle along their journey of mathematics education. It was recommended that classroom teachers refine their instructional approach and set achievable learning goals and suitable assessment tasks for this group of pupils.

13.4.4 Characteristics of Low Attainers in Relation to the Schools

With regard to the general characteristics of low attainers, it was found from LAPM that the majority of the low attainers (70%) were seldom or never attentive in class while the teachers were teaching. Few students (15%) sought help from the teachers when in doubt. As they were not attentive and did not seek help from their teachers, less than half of the students (about 39.6%) were able to submit classwork on time.

In relation to their learning of mathematics, the majority of the pupils appreciated the importance (about 88.9%) and usefulness (65%) of mathematics to their daily life. Although the vast majority of the pupils (about 89.6%) believed that they could perform better in mathematics, approximately half of them (54%) believed that they were not good in mathematics. The learning of mathematics aroused a diverse range of emotions from feeling of happiness, dislike, anger, confusion, stress and anxiety to boredom among this group of low-attaining pupils.

Although approximately half of the pupils had parents with secondary (45.3%) and tertiary (8%) education, about one-third of the pupils (28.8%) reported that their parents did not check their mathematics work at home.

13.5 Exploratory Study Two—Teaching and Learning Mathematics at the Normal (Technical) Stream in the Secondary Level

Students from the Normal (Technical) stream at the secondary level are believed to be relatively less inclined towards mathematics. In an effort to understand the perception of teachers teaching Normal (Technical) mathematics about their students, and the strategies they had adopted to help their students better learn mathematics, an exploratory study was carried out by Toh and Lui in 2013. This took place during a teacher professional development workshop that was concurrent with the book launch of one new secondary mathematics textbook series for the Normal (Technical) students for use from 2014 onwards. The findings of this exploratory study were reported in Toh and Lui (2014).

During this survey, the participating teachers were asked to discuss if it was a challenge to teach Normal (Technical) mathematics and to answer three key questions about their

(1) Perception of the reasons of their students' lack of interest in mathematics;
(2) Perception of the reasons for their students' learning difficulties in mathematics;
(3) Strategies and resources that they had used to help their students learn mathematics.

All the teachers' answers were affirmative to the fact that it was a challenge to teach Normal (Technical) students mathematics. The breakdown of the answers to the above three key questions is presented below.

13.5.1 Teachers' Perception of the Reasons of Their Students' Lack of Interest in Mathematics

The reasons provided by the teachers for this question in the survey can be broadly classified under two main categories: (1) cognitive and (2) affective factors.

(1) Cognitive factors

The main reason that students lacked interest in mathematics was that mathematics was difficult for many of them, and not relevant to daily life.

(2) Affective/psychological factors

Mathematics was boring for these students. Not only that, many of these students had never experienced success in mathematics before, as they rarely passed mathematics when they were in primary school.

13.5.2 Teachers' Perception of the Reasons of Their Students' Difficulty in Mathematics

The reasons provided by the teachers on their perception of the reasons of their students' difficulty in mathematics can be classified under two main categories: (1) cognitive and (2) affective factors.

(1) Cognitive factors

Most of the teachers responded that their students had poor mathematics and language foundation, which led them to being unable to understand the mathematics problems. Not only that, the students were easily confused by mathematics problems which involve multiple steps. In addition, the students' style of learning is through numerous and a wide variety of worked examples in order to acquire various mathematical concepts. Several teachers also reflected that the low attainers experienced much difficulty memorizing the various mathematical formulae. It was also reported in the survey that many of these students lacked the perseverance in solving mathematics problems.

(2) Affective/psychological factors

Two key factors cited were that the students were generally not interested in mathematics, and they generally had short attention span. The students' negative attitude towards the subject, manifested by a lack of interest, and short attention span in the classroom could easily lead to disruptive behaviour during the lesson. This further led them to dislike mathematics even more.

13.5.3 Teachers' Use of Strategies and Resources to Help Their Students Learn Mathematics

The numerous responses provided by the teachers for this item in the survey can be broadly classified under six main categories:

(1) Use of manipulatives (including standard manipulative proposed by MOE and others);
(2) Information and communications technology (online learning platforms, content websites and mathematical tools);
(3) Media (newspaper cuttings, existing video clips from YouTube and other resources);
(4) Modification of standard pedagogical practices (managing the pace of lesson, peer coaching, individual explanation of mathematical concepts, use of appropriate language, relate mathematics to everyday life);
(5) Psychology (building up students' confidence in mathematics by providing them with small opportunities of success in the process of learning mathematics);
(6) Alternative pedagogy (use of storytelling, cartoons with humours, games and quizzes).

This preliminary survey (Toh and Lui 2014) clearly indicates that teachers were already engaging their students using a variety of strategies, both utilizing the teaching material available and adopting and adapting existing material outside the traditional teaching resource. It was encouraging to the researchers that many of these strategies, other than the standard manipulatives and teaching approaches proposed by the MOE, were the creative invention or adaptation made by the teachers in addressing the learning needs of their students.

Teachers were clearly aware of the learning difficulties and the lack of interest in mathematics among the low attainers in mathematics in the Normal (Technical) stream. The teachers had devised their own intervention of addressing their students' learning difficulties and lack of interest in the subject. Their intervention strategies can also be broadly classified broadly under two broad categories: (1) that of addressing students' learning difficulties in the subject and (2) that of addressing students' motivation and self-esteem.

In addition to the exploratory studies in understanding the teaching and learning of mathematics among the low attainers, a variety of intervention studies were also conducted by researchers and mathematics educators to address the teaching needs of teachers and the learning needs of students. These interventions include pedagogical approaches that attempt to build up students' interest in the subject and also approaches that attempt to help students learn mathematics better by unpacking the abstractness of mathematics. The next sections shall introduce several of these intervention programmes.

13.6 Rethinking Cooperative Learning Strategies in the Mathematics Classroom

This section reports on a study that was conducted on the learning of mathematics among low attainers by introducing cooperative learning strategies into the mathematics classroom. Mathematics teachers from one mainstream secondary school, in consultation with a team of researchers from NIE, engaged in an action research project on studying the impact of infusing cooperative learning strategies in the Normal (Academic) and Normal (Technical) classrooms.

Earlier, a team of researchers led by Lui (reported in Lui 2003; Lui et al. 2005) conducted a research in Singapore schools in validating an instrument of measuring students' academic self-concept and motivation in the Singapore schools, using mathematics as a context. This collaboration between NIE and schools sparkled the interest of one mainstream secondary school to embark on the action research on introducing cooperative learning strategies into the mathematics classroom. According to Lui et al. (2009), the teachers were keen in helping their students "who [were] weak in mathematics" by using incorporating cooperative learning strategies in the lower secondary mathematics classrooms. The study was conducted with the objective to empower the low attainers in learning mathematics.

The theoretical background of this action research was Vygotsky's (1982) theory of social constructivism. According to Vygotsky (1982), students learn best in group activities, and greater opportunity for such interactions will widen the zone of proximal development for the students. According to Burns (1990), social interaction is one key factor in learning mathematics. Greater opportunity for interaction of students with their peers, parents and teachers will allow them to have more exposure to a more variety of viewpoints, thereby stimulating them to reflect on their own.

13.6.1 Various Levels of Instructions for Cooperative Learning

The action research project was divided into two phases, and cooperative learning activities were conducted for the various mathematical topics for secondary two in the Normal (Academic) stream.

In the first phase, in consultation with mathematics professors from NIE, the teachers crafted three activities for three mathematical topics, with differing level of instructions on cooperation proposed to the students. The lesson objectives of the three activities were clearly stated. Activity one, which involved work in computer laboratory in solving a mathematics task, did not contain explicit instructions for students to work cooperatively. During the lesson, the teachers encouraged the students to work in pair. Activity two contained explicit instructions for students to work cooperatively. The teachers gave the students explicit instructions on how they could do cooperative learning to complete the task. In addition to clear instruction for cooperative learning in activity three, the proposed group interaction was also structured to ensure the accountability of each group member towards the completion of the task. The content of the three activities crafted for the study was checked by mathematics educators from NIE to ensure that it was mathematically and pedagogically sound. The findings for the first phase were used to better plan and refine cooperative learning activities for the second phase.

Findings from the first phase

In activity one, the students did not exhibit much interaction with their partners even though they were verbally encouraged to do so by their teacher. The students had not learnt from their partners nor taught their partners during the interaction. Not only that, most of the students indicated that they had a low level of confidence in completing the task. Thus, the presence of the mere physical infrastructure (in this case, the classroom arrangement being conducive for group interaction) and teacher encouragement is insufficient to ensure students working collaboratively. In order to encourage cooperative learning among students, teachers must provide a clear set of instructions on how to work cooperatively. In addition, teachers should create a spirit of cooperation among the students and make them accountable for the task to be completed.

In activity two, explicit instructions were given to students to work in groups, and the teachers also briefed the students on how to work cooperatively. In this activity, student cooperation and discussion were observed. Furthermore, the students were able to state at least one thing they had learnt from or taught their partners. There was a higher rating on student confidence in completing the task in this activity. However, it was observed that little learning took place among the pairs who were both mathematically weak. Also, the students lost focus when they continued to work in pairs for a long period of time, as students, in particular, the low attainers, generally had short attention span. This taught the teacher researchers the lesson that in executing cooperative learning, the grouping or pairing of students based on their capacity was crucial in order to ensure that learning took place. The tasks for cooperative learning must be designed in a way that it is engaging, and not an unduly lengthy task that demands too much time in view of students' short attention span.

In activity three, in addition to the explicit instructions and teacher briefing, the students were divided into groups of four. The students were given clear briefing on the importance of cooperation, the tasks to be completed and the allocation of the scores for the activity. This activity was conducted in an open area in school. The students were clearly aware of the task that they had to complete and the demand of the task. However, the teachers faced greater difficulty in managing the students in open area compared to conducting lessons in the usual classroom. Moreover, in groups of four, there was evidence of some students leaving most of the work to their

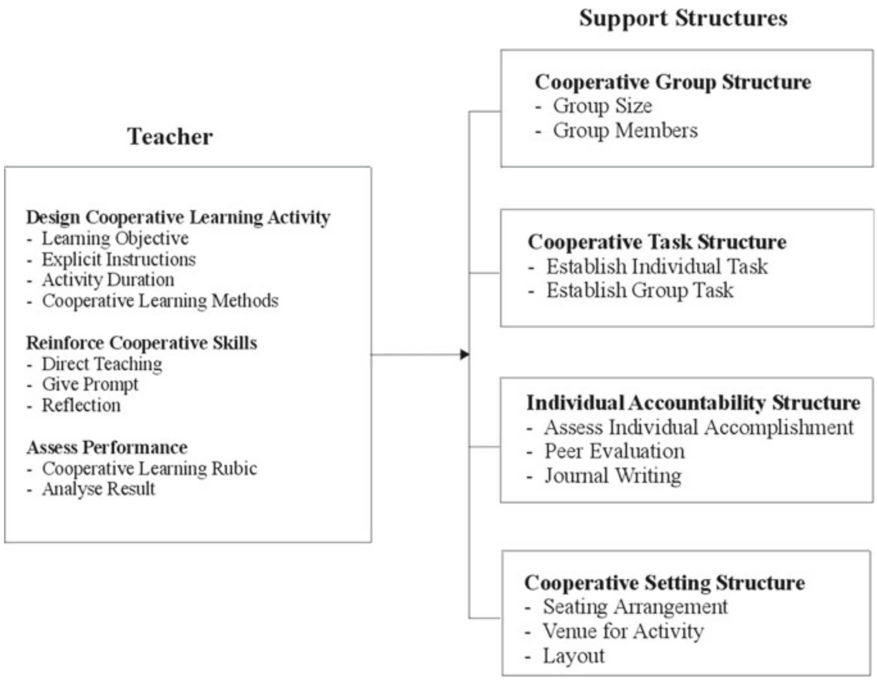

Fig. 13.1 A cooperative learning framework (Lui et al. 2009)

peers and completing minimal work themselves. This appears to point to a higher chance for successful implementation of cooperative learning in pair work rather than in bigger group.

The findings from the first phase provided the researchers with greater understanding of implementing cooperative learning in the second phase.

Implementation of cooperative learning in the second phase

With the learning points in the first phase taken into consideration, the implementation of cooperative learning in second phase incorporated the following arrangement:

- Listing clearly and enforcing class rules;
- Partnering of students of different mathematical abilities;
- Developing a spirit of cooperation in class by constantly encouraging peer tutoring;
- Prompting students to use cooperative skills when offering or asking for help.

The implementation in the second phase involved using cooperative learning strategies for four mathematical activities that were designed by the same team of school teachers, and endorsed by the mathematics professors from the NIE. The activities covered four different subtopics from the secondary two mathematics curriculum.

After the implementation in the second phase, most of the students (about 80%) indicated that they had been able to learn and help one another using cooperative skills. They showed a high degree of positive interdependence, and there was clear evidence of peer tutoring among the students.

Based on the findings of this research project, the researchers proposed a cooperative learning framework that posits the implementation of cooperative learning through the dependence of two main factors: (1) teacher and (2) support structures (Fig. 13.1).

13.7 Contextualizing Mathematics and Comics in Teaching and Learning Mathematics

In the survey conducted by Toh and Lui (2014) described in 12.5, it was seen that there was sporadic effort among the mathematics teachers in using creative teaching strategies to help their students in learning mathematics. For example, they were already using strategies like storytelling and cartoons in teaching mathematics for the low attainers in the Normal (Technical) mathematics classroom. However, it seemed that there was a lack of concerted effort among teachers to explore how these alternative approaches could be used for mathematics instructions. In 2014, a team of researchers from the Singapore NIE initiated a research project with the objective of studying (1) the feasibility of using comics and storytelling for mathematics instruction and (2) the impact on the students' motivation, academic self-concept and performance in mathematics achievement test of using comics and storytelling as a mode of instruction.

Although the idea of using comics in the teaching of mathematics was not entirely new [as the survey in 2013 reported in Toh and Lui (2014)], this marked the beginning of an organized concerted effort headed by researchers in collaboration with teachers from the Singapore school in designing teaching units of secondary one mathematics using the comics and storytelling approach.

Even before the survey in 2013 was carried out by Toh and Lui (2014), Toh (2009) discussed the possibility of using cartoons and comics in teaching mathematics to the less-motivated students. It was argued in the paper that comics not only has the potential to motivate students to yearn to learn mathematics, but it could be useful in teaching abstract concepts in mathematics. Several examples on how comics could be used to expound the various abstract algebra concepts were detailed in the paper. Subsequently, this idea was adopted by a series of textbook for Normal (Technical) stream that was launched in 2014 and adopted in the Singapore schools.

In this research project led by Toh et al. (2017), the researchers designed comics teaching packages for selected secondary one mathematics unit in the Normal (Technical) mathematics curriculum. The package for each unit was complete in both the content coverage and practice questions accompanying the comics. In other words, the package that was developed was used as a "replacement unit" for the existing curriculum resource. The comics package for each unit was offered in both hardcopy versions and online versions for which schools were given the option to select. The characteristics of the two versions and the feedback provided by the teachers after the first cycle of implementation are described below.

Hardcopy version: This version of the comics package was offered in the form of printed worksheets that consisted of both the comic strips and the accompanying practice questions. During each mathematics lesson, the teacher would give up each set of comics relevant to that particular lesson. In each lesson, teacher would go through the comic strips and give students sufficient time to attempt solving the questions. In the feedback provided by the teachers, the hardcopy version had the advantage in that students were provided with some concrete material to rely on and to take notes whenever necessary. However, the comics given to students was in black and white version; the attraction to the comics was reduced compared to the original design which was fully coloured.

Online version: The same set of comic resources was also made available using the online platform (a sample comic strip with the accompanying practice questions can be found in http://math.nie.edu.sg/magical). The differences of the online version from the hardcopy version are that

1. The former is in full-colour form which was designed such that it was compatible with iPad and mobile, and that students were able to swipe the comics across their mobile or iPad in the same way that they read e-comics.
2. The accompanying practice questions in the online version provide immediate feedback to the students' response. Not only that, the package was designed in such a way that the teachers could keep track of their students' responses for any particular practice question.

In addition, the researchers tapped on the affordance of the online comics version and randomize the practice questions each time a different question appeared at different logging in. However, the feedback provided by the teachers was that, as the questions were randomized each time, the teacher was unable to identify a common question to discuss during the classroom instruction, and this led to a messy situation.

The participating teachers covered the entire unit using the comics package. The researchers provided proposed lesson plans that accompanied each set of comics. The teachers executing the lesson were given ideas on the stories to tell at each juncture of the comics and the plausible activities to engage their students during the mathematics lessons. The teachers were also given the liberty to tweak the package according to the needs of the students at various junctures, although they were reminded that the whole research was to study the impact of comics on student motivation and learning of mathematics.

To assist the researchers to better understand what happened in the comics lessons, in particular, how teachers tweaked the package or adapted the proposed lesson plans during the actual lesson implementation, all the comics lessons were video-recorded and analysed by the researchers.

After each cycle of implementation, the researchers discussed with the participating teachers. The latter's feedback of the teaching package provided the researchers ideas on how best to fine-tune the existing package for subsequent implementation within the school. Through this cycle of design, fine-tuning and even redesigning, the comics lesson package that had been designed was customized to the needs of the individual schools.

In the preliminary interview conducted by the Straits Times (Teng 2016, May 30), one teacher from a participating school reported that it was quite challenging to engage his Normal (Technical) students in the mathematics classes. With the use of comics, he could see that there was greater engagement from his students during the lesson. Also, more two-way communication between teacher and students on mathematics started to surface during his mathematics lessons. More importantly, for the unit that he taught using comics, his students performed better in the school assessment compared with the other topics that were taught by the usual textbook resource.

13.7.1 Twenty-First-Century Competencies

Using the twenty-first-century competency framework developed by the MOE, Toh et al. (2017) reported from a preliminary analysis of the video recordings of the comics lessons taught by one team of teachers in one research school. It was interesting to the researchers to notice that, in the process of unpacking the teachers' use of the comics package in teaching mathematics, the teachers (both consciously and subconsciously) attempted to develop in their students other competencies, skills and values in their lesson delivery using the comics package. For example, the teachers rode on the affordance of the comic strips in several instances to facilitate their students to extract information from the visual cues of the comic strips, thereby making sense of the social context provided by the story. This contributed to raising students' social awareness in the lesson delivery of lessons.

In another instance, in a set of comics on the statistics unit for secondary one students, the scene was a typical office setting but the context was not explicitly stated. The teachers made the effort to engage their students to offer a possible interpretation of the context of the comics on the process of carrying out a survey, thereby making the process of data collection and tabulation in statistics sensible to the students. Another point noteworthy to mention is that the teachers went the extra mile to engage their students in communication and collaborative work through the use of comics. There were also instances in which one participating teacher attempted to instil in her students the idea of civic literacy and global awareness (two important attributes of the twenty-first-century competency framework) within the lessons.

13.7.2 Students' Perception of the Comics

Toh et al. (2017) reported on several interviews with the students on their perception of the use of comics in classroom lessons. The students found the comics lessons much more interesting, compared to the regular mathematics lessons. The students described the usual mathematics lessons as "quite boring". Not only that, one student highlighted that the comics lessons had provided them with an opportunity to discuss mathematics in the context of the real world with his friends. In this sense, the comics had also contributed to facilitate the students in communicating their ideas, and this translates to developing them into active contributor who can communicate and work in teams.

Through regularly being engaged in interpreting plausible contexts of the comics, the students became more confident in dealing with mathematics in the real-world context. The students were also engaged in role-play during the comics lessons. They developed a better understanding of different perspectives, and a stronger sense of right and wrong, and being discerning in making judgement.

It was also highlighted in the student interview that the context provided by the comics provided the platforms for them to alter the problems in the original comic strips thereby solving more problems that were created by the students.

Another pleasant surprise to the researchers in the interview with the students was the interview with one student who identified himself as dyslexic. He reflected that he had recognized the importance of mathematics in the real world, although mathematics was a very difficult subject for him. He asserted that he had a positive change in attitude of the student due to the use of comics for mathematics instruction. The student claimed that at first he was neither good nor confident in mathematics. However, the comics got him excited about mathematics. Consequently, he started to read and re-read the comics repeatedly and revised the related mathematics regularly. This constant engagement with comics reading made him develop further interest in mathematics. He claimed that due to being dyslexic, he had difficulty reading mathematics, and each time in attempting to understand a problem, he had to "read a lot of times". As the comics teaching package was presented with colourful pictures and cartoons, it became easier for him to understand the problem and it helped him learn mathematics better.

At the time this chapter was written, the research was still going on in several Singapore secondary schools. The researchers will report more findings on this research sometime in the future.

13.8 Concrete–Pictorial–Abstract Approach in Mathematics Instruction

Algebra presents much difficulty to many students, especially low attainers in mathematics. The difficulty can be attributed to two plausible reasons: (1) the students have poor foundation and might not have met the prerequisite knowledge necessary for the learning of algebra; (2) much of the learning emphasized in schools is based on rote learning rather on facilitating students to make sense of the abstract mathematics (Quek et al. 2016).

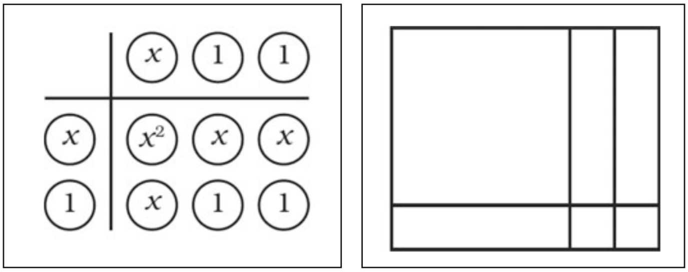

Fig. 13.2 Using AlgeDiscs™ (left) and algebra tiles (right) in factorizing $x^2 + 3x + 2$

The Concrete–Pictorial–Abstract (CPA) approach of teaching mathematics has been highlighted as one of the key instructional strategies in the Singapore mathematics curriculum since its introduction in the 1980s. This is an adaptation of the "enactive–iconic–symbolic" modes of representation first proposed by Bruner in 1966 in his book *Toward a Theory of Instructions*. Leong et al. (2015) provided a survey of the origin of CPA approach used in Singapore in relation to the original proposed idea of Bruner (1966).

According to the Singapore MOE, CPA approach is an activity-based approach of learning mathematics by doing.

> Students engage in activities to explore and learn mathematical concepts and skills …. They could use manipulatives or other resources to construct meanings and understandings. From concrete manipulatives and experiences, students are guided to uncover abstract mathematical concepts or results. … During the activity, students communicate and share their understanding using concrete and pictorial representations. The role of the teacher is that of a facilitator who guides students through the concrete, pictorial and abstract levels of understanding by providing appropriate scaffolding and feedback (MOE 2012, p. 23)

In aligning to CPA as an instructional approach recommended for mathematics instructions, manipulatives have been used in the Singapore mathematics classrooms to make students learn abstract mathematics more sensibly and meaningfully. Two types of manipulatives, the AlgeDisc™ and algebra tiles, were introduced into the Singapore secondary mathematics curriculum as an attempt to help students to "concretise" the otherwise abstract algebra in the teaching and learning processes.

For example, instead of getting students to memorize the procedure of algebraic factorization and completing squares, concrete manipulatives were introduced in the classroom to help students make sense of these and other algebraic processes (Fig. 13.2).

The AlgeDisc™ has the advantage of explicit labelling of the algebraic quantities $(1, -1, x, -x)$ on the discs. The "negative" of this algebraic/numerical quantity can be represented by a "flip" of the disc. On the other hand, the alge-tiles has the added advantage of the geometrical feature of representing each algebraic quantity by the dimensions and the area of a tile.

In adapting the advantages of the algebra discs and algebra tiles, Leong et al. (2010) developed a hybrid form of manipulative that combines the advantages of both the AlgeDisc™ and the algebra tiles which was called the *AlgeCards*.

The AlgeCards involve the use of three types of rectangles (of areas x^2, x and 1), which the length of each rectangle could easily be associated with the algebraic quantity. For example, the rectangle (or square) with area x^2 has dimensions x by x units while the rectangle with area x has dimension x by 1 unit. The flipped side of the rectangle x represents $-x$, and so on. Here, we present two of the studies by Leong et al. (2010), Quek et al. (2016) in their studies on the use of AlgeCards in teaching and learning school algebra.

13.8.1 Study One: Quadratic Factorization

Leong et al. (2010) discusses how the process of factorization of quadratic expressions by the "trial-and-error" method, which has usually been perceived by students as a meaningless arbitrary procedure, can be made meaningful for students. The authors also discuss how the physical use of the manipulative can be meaningfully translated into realistic heuristics that could be applied during the usual paper-and-pencil tests.

Leong et al. (2010) describe in great details in how the AlgeCard, which is used as a scaffolding, can be used to move students from concrete to abstract stage in performing factorization of quadratic expressions during the two lessons on algebraic factorization. In this paper, they described their study on the impact of AlgeCard on the learning of mathematics among Normal (Academic) students from a Singapore mainstream school. They specifically focused on the algebra subtopic on factorization of quadratic expressions.

In the first lesson, the students were introduced to the use of AlgeCard and rectangle diagrams in performing factorization, thereby making sense of the entire otherwise abstract process (factorization of quadratic expressions can be interpreted as forming rectangles with a total fixed area). This parallel use of concrete manipulative and pictorial representation enabled the students to make greater connection of the algebraic process with the pictorial interpretation. Both AlgeCard and rectangle diagrams were used for all the questions in the worksheet in this first lesson.

In the second lesson, the researchers emphasized more on the pictorial representation using rectangle diagrams and downplayed the use of the concrete AlgeCard. In addition, more cumbersome quadratic expressions were introduced in the worksheet to demonstrate the inadequacy of the physical manipulative, although the students were still allowed to use the AlgeCard and the rectangle diagrams if they chose to do so. In the last part of the second lesson on quadratic factorization, quadratic expressions with negative coefficients were introduced in the worksheet. This rendered the use of physical manipulative unnatural.

It was observed that the students were able to respond to this gradual process of scaffolding and most of them eventually moved away from the use of AlgeCard

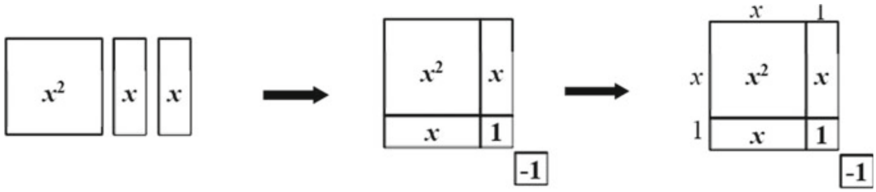

Fig. 13.3 Illustration using AlgeCard on completing squares in $x^2 + 2x$

and resorted to the use of rectangle diagrams and algebraic processes to complete the factorization. What was noteworthy was that this approach increased the level of student engagement among those who were originally uninterested in mathematics.

13.8.2 Study Two: Completing the Squares

Quek et al. (2016) extend the legacy of the AlgeCard experience in Leong et al. (2010) to another Singapore mainstream school to teach the secondary three students from the Normal (Academic) stream the process of completing squares in quadratic expressions. Continuing the line of thought in Leong et al. (2010), the researchers Quek et al. (2016) attempt to assist students to make sense of the process of completing squares and to develop in them the skill to perform typical assessment type items on completing squares. An example provided in Quek et al. (2016) on completing the squares in $x^2 + 2x$ is shown in Fig. 13.3.

They used experiment control approach to test the effect of using the above approach in teaching completing squares compared to the usual procedural emphasis taught by most teachers. The class that was taught using the CPA approach with the use of AlgeCards and another comparable class (that was taught using the usual procedural approach) were required to sit a test, which consisted of typical examination-type items about completing squares in quadratic expressions. The result of the test shows that the experimental group taught using the CPA approach performed significantly better (with higher mean score and smaller standard deviation) than the class that served as control.

13.9 Nationwide Teacher Capacity Building for Teachers Teaching Low Attainers in Mathematics

In recognizing the importance of preparing teachers specifically for low attainers in mathematics, the Singapore Ministry of Education embarked on a project *Improving Confidence And Numeracy* (with the acronym ICAN) to equip her mathematics teachers' proficiency in teaching low attainers in mathematics.

The project ICAN, which started in 2013, targets to assist the low attainers in mathematics (which is identified as the bottom 15% of each cohort) from both primary and secondary schools. The project involves building teachers' capacity to facilitate the learning of mathematics among the low attainers in mathematics. The eight pedagogical principles identified for the teacher building capacity of ICAN are listed below:

1. Establish the classroom norms—getting students ready for the lesson and setting expected behaviour.
2. Check and diagnose students' prerequisite knowledge—to bridge any learning gaps.
3. Create a motivating environment.
4. Focus on fundamentals—during lesson delivery.
5. Giving direct and explicit instruction.
6. Simplify and scaffold.
7. Provide guided practice/communication—oral explanation and reasoning are encouraged.
8. Provide individual practice and review.

It is not difficult to observe that the eight principles of ICAN, which serve to help the low attainers to get the basics right, also serve to address the five dimensions of problem-solving as represented by the five sides of the Singapore school mathematics framework (Fig. 3.1, Chap. 3). Principle 1 relates to setting a conducive environment for learning. Principles 2, 4 and 5 address the importance of skills and concepts dimensions of problem-solving. Principles 7 and 8 address the process of learning mathematics. Principle 3 stresses on the affective dimension of learning—that of attitude. Principle 8, that of providing simplification and scaffolding, addresses the metacognitive dimension of problem-solving.

The support for teachers includes workshops, mentoring, network meetings, pedagogical resources and an annual symposium. To build the capacity at the ground and to sustain ICAN for the longer term, a pool of cluster mathematics mentors from primary schools and secondary schools are supporting the training and mentoring effort of teachers and school mathematics mentors at the cluster level. As the work is still work in progress at the time this chapter is written, we are unable to report the findings of the project at the current stage.

13.10 Conclusion

This chapter presents the various studies that have been conducted in the Singapore education context to provide assistance to low attainers in mathematics at the various levels. In Singapore, the low attainers are identified early in the early years of schooling so that additional assistance can be provided to help them learn better. Not only that, various efforts made by the MOE and the various mathematics education researchers to facilitate the low attainers learn mathematics better are provided at both primary and secondary levels. These additional assistance being offered come in the various types of pedagogical innovation, which include the domain-specific (e.g. the use of mathematics manipulatives in the mathematics lessons), and interpretation and implementation generic pedagogical principles (the use of cooperative learning strategies), and the use of ideas from pop culture (incorporating the use of comics in mathematics instruction).

To ensure that these pedagogical innovations were evidence-based, they were introduced as research projects conducted by the researchers. This chapter provides a brief report of these studies. Readers are encouraged to read up the original research papers if they are keen to have a more detailed understanding of each of the innovative practices to assist the low attainers.

Not only that, these innovative approaches could not have been implemented in classroom if teachers are not well-prepared. Thus, teacher capacity building has always been recognized as an important aspect of better assisting the learning of the low attainers. It is thus not surprising that the MOE embarked on the nationwide effort to build teacher capacity for teachers working with low attainers.

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Chapter 14
Use of Technology in Mathematics Education

Wee Leng Ng, Beng Chong Teo, Joseph B. W. Yeo, Weng Kin Ho and Kok Ming Teo

Abstract This chapter discusses the use of information and communications technology (ICT) in mathematics education from primary to tertiary level. The focus is on how ICT has been, or could be used, in enhancing the teaching and learning of mathematics, particularly in the Singapore context. Four main ICT tools, namely hand-held technology, dynamic geometry software, computing and programming tools and e-learning, are examined. Our examination of each tool entails, the why, what and how of the tool and research on the use of the tool in Singapore schools. Generally, in the context of the sites of research, there is encouraging positive impact of the tools on student learning of mathematics.

Keywords Information and communications technology · Hand-held technology · Graphing calculators · Computer algebra system · Dynamic geometry software · Computing and programming · E-Learning · Flipped classroom

14.1 Introduction

Most research projects conducted in Singapore in recent years tend to be on the exploring and designing of ways of using technology for the classrooms, such as the Singapore Mathematics Assessment and Pedagogy Project (SMAPP) and My Math Homework PAL. The SMAPP aimed to create technology-mediated mathematics

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problems which are non-routine multi-step investigative tasks to assess secondary school students' understanding and applications of mathematical concepts (Fan et al. 2010), while My Math Homework PAL is a project which developed a software application to assist primary school students to solve mathematics word problems by providing scaffold and structured steps to identify data, establish facts, derive new information and write out the solutions (Ministry of Education n.d.).

As technology becomes more pervasive in work and daily living, it is opportune that we look at the way we teach and learn mathematics, especially in balancing mathematical thinking and applications into our curriculum. In this chapter, we will consider ICT tools that are targeting specifically at mathematics as well those that are for broader use.

To facilitate discussions in this chapter, we classify ICT tools into four categories: (1) Hand-held Technology; (2) Dynamic Geometry Software; (3) Computing and Programming Tools; and (4) E-Learning. These are common ICT tools used by mathematics educators (Becta 2003). The following four sections of this chapter will be on the aforementioned four categories.

14.2 Hand-Held Technology

We consider hand-helds to be computing or electronic devices that are compact and portable enough to be held and used in one's palm. Calculators of different levels of sophistication, from scientific calculators to graphing calculators (GCs) and those with computer algebra system (CAS) capabilities, will be given more attention.

In Singapore, scientific calculators have been included into the secondary school education since 1981, and into the final two years in primary schools since 2009. At the pre-university level, GCs without CAS capabilities were first permitted in examinations in 2002 for Further Mathematics. This section provides the details of these developments and presents the main findings of relevant studies including local ones.

14.2.1 Use of Hand-Held Technology in Singapore

In the years 2002 through 2006, students who sat for the Further Mathematics papers in the advanced level (pre-university level) examinations were allowed to use GCs without a built-in CAS. However, in these examinations the questions were constructed in such a way that students who did not use a GC during the examinations would not be disadvantaged. As Further Mathematics was a subject only taken by students with aptitude and interest in mathematics, most pre-university students were not mandated to use GCs.

14.2.1.1 GC-integrated Curriculum

Following a mathematics curriculum revision in 2006, students could read either Higher 1 (H1) Mathematics or Higher 2 (H2) Mathematics, where H2 Mathematics is a subject taken by the majority of pre-university students, while H1 Mathematics is taken by students who are less mathematically inclined. The use of GCs is required for both subjects in all assessments including national examinations. The examination papers are set with the assumption that candidates will have access to GCs and are proficient in solving problems with the aid of GCs under conditions of a timed examination.

In particular, students are expected to use the GC to explore properties of graphs, determine the range of a function from its graph and examine the conditions for the existence of inverse functions and composite functions. Apart from using the GC as a graphing tool, students are also expected to know how to use a GC to solve problems on different topics in the syllabus. For example, students need to learn how to use a GC to solve inequalities by graphical methods; find a numerical solution of an equation; carry out addition, subtraction, multiplication and division of complex numbers; and find the square roots, modulus and argument of a complex number, etc. Students also need to learn to use the GC to compute probabilities pertaining to Binomial and Normal distributions.

14.2.1.2 Emphases of the Revised Curriculum

The revised curriculum places greater emphasis, as compared to the curriculum before 2006, to the development of students' abilities to conjecture, discover, reason and communicate mathematics with the aid of technological tools. To achieve this objective, teachers were required to make suitable adjustment to their classroom practices and pedagogical strategies. For instance, teachers were encouraged to plan lessons such that the GC would play a key role when students engage in stimulating discussions and activities in which they can explore possibilities and make connections. Indeed, the use of GCs was intended to be integral to the learning of mathematics at the advanced level in the revised curriculum.

One other intent of the revised curriculum was that GCs be used in ways which allow students to learn mathematics in practical and meaningful contexts, using analytic methods together with graphical and numerical techniques (Kissane et al. 2015). The computational and graphing capabilities of GCs should be used to enable students to engage in active learning through exploratory work and experiments. Students could work collaboratively with others, share ideas and discuss their findings. GCs should also be used to extend the range of problems accessible to students and enable them to execute routine procedures quickly and accurately, to make connections between algebraic and geometric concepts, and to switch swiftly between numerical, graphical and symbolic representations in mathematical explorations. In a nutshell, it was intended that the routine tasks be relegated to the GC, allowing students more time for thinking, reflection and discovery.

To train the teachers in using the GCs, in-service courses were requested by the Ministry of Education (MOE) to the Singapore National Institute of Education (NIE) (Ng 2005). Two books were subsequently written by Ng (2006, 2009) to provide further support to the teachers. An area of concern at the initial stage of the implementation stems from the limitations of GCs. For instance, the initial graph provided by a GC may not display the complete graph, and thus, window settings may need to be adjusted. To address this concern, sharing sessions and workshops were conducted during which teachers were alerted to the inherent limitations of GCs.

14.2.1.3 Looking Back

Based on anecdotal evidence, benefits brought about by the implementation of the GC-integrated curriculum include allowing teachers access to a wider range of problems (e.g. non-standard functions and matrices of higher orders); opportunities to engage students in active learning through exploratory work and experiments, sharing ideas, discussing findings and working collaboratively; more time spent on discovery, thinking, reflection and making inferences; learning in meaningful contexts, and allowing students quick and accurate execution of routine procedures (e.g. calculations and graphics); connections between algebraic and geometric ideas; and ease of switching between different representations.

Most teachers are now sufficiently skilled in using the GC and more teachers are using them in ways that would encourage students to explore mathematics concepts on their own. However, some teachers still use the GC mainly as a computational tool and a graphing tool rather than as a pedagogical tool. Furthermore, it is observed that there is a shift in teachers' concerns from those which are assessment-related to those related to student misconceptions or learning difficulties as they realise that the use of GCs may bring undesirable outcomes if students misuse the GC, and that conceptual understanding and awareness of the inherent limitations of the GC are required for the appropriate use of the GC.

Most of the learning difficulties faced or common errors made by students are due to their superficial level of mathematical content knowledge and the lack of an analytical habit of the mind as they blindly accept what they see on the GC screen. Teachers could do more to advise the students against relying too much on GC for performing simple mathematical tasks that could be done easily by hand, and promote in-depth thinking with the use of the GC.

In conclusion, to reap more educational benefits from the use of GC, teachers should use the GC as a pedagogical tool to develop conceptual understanding so that students use the GC as a learning tool, as opposed to using it merely as a computational tool or graphing tool. To help teachers develop new pedagogical skills, professional development is important and achievable through a systemic approach to planning pre-service training, in-service training and professional sharing.

14.2.2 Research and Development (R&D) on Hand-held Technology

The use of GCs enables mathematics to be taught in a more coherent way by providing students with the opportunities to connect mathematical concepts within and between topics by offering a learning environment that enables mathematics to be experienced through multiple representations (Roschelle and Singleton 2008). As pointed out by Ng et al. (2009), the pedagogical affordances of the GC have a close relationship with the improved learning of mathematics. Indeed, students learn more when the cognitive load is focused on the most important learning challenge, when linked multiple representations help them to move flexibly from one representation to another, and when they have more time to focus on the strategic and problem-solving aspects of mathematics (Ellington 2003) and on conceptual understanding (Roschelle and Singleton 2008; Ng et al. 2009).

Many studies have demonstrated the benefits of using GCs in mathematics learning (e.g. Sang 2003; Abu-Naja 2008; Chamblee et al. 2008; Lyublinskaya and Zhou 2008). In addition, GCs have a positive effect on students' understanding of graphs and their connection to algebraic representation. Zachariades et al. (2007) proposed that the GC, which has dynamically linked graphical, numerical and symbolic functions, is an appropriate tool for teaching calculus concepts.

The demand on students' cognitive and problem-solving abilities in the mathematics curriculum, coupled with the need to harness the advanced functionality that GCs afford, has led to calls for a change in the way that teachers teach and students learn mathematics (Goos 2004). Other studies on GCs have examined their use as a social tool in the classroom. For instance, Nathan and Knuth (2003) noted that learning mathematics in the classroom can create a community that engages students in social interaction.

In a local study by Yen (2006), a survey questionnaire was administered to 116 first-year and 94 second-year students in a junior college (JC) where the use of the GC was more actively promoted. The survey questionnaire consisted of 15 items with Likert-type responses on a five-point scale. The items were divided into two main categories: items on cognitive effects and affective effects. The items on cognitive effects were further categorised into items on the supportive toolkit approach versus the black box approach; and items on the importance of the GC, its effects on achievement and its role in assessment. For feedback on how the students use the GC, data were also obtained from a school-based test. For the analysis of the survey and test data, the students were grouped into three groups: frequent users, infrequent users and non-users based on their reported frequency of use of the GC. Furthermore, five first-year and five second-year students were also interviewed to obtain some qualitative feedback on their attitudes towards the use of GCs, their usage of the GC and the difficulties they encountered. Survey results indicated that there was consensus among the Further Mathematics students that it was important to learn to use the GC and that it should be allowed in mathematics tests or examinations. There were also significant differences at the 0.05 level among the frequent, infrequent and non-users for the mean scores for the attitudes of the students towards the use of the GC and for the different dimensions: affective, cognitive (supportive toolkit), cognitive (black box) and cognitive (importance and achievement). The use of the GC in a school-based test did not contribute to a significant difference in the mean test scores between the users and non-users but made a difference in helping the users in the sketching of the graphs as reflected in the graph scores. Finally, one of the main difficulties students faced in the use of the GC was unfamiliarity with its features because of underutilisation due to the restriction of its use to Further Mathematics only and also the requirement for supporting working in high-stake examinations.

On the other hand, the use of wireless classroom systems has led to the development of a notion of the mathematics classroom as a community in which learning tasks and tools are used to trigger mathematical thinking and discussions (Nathan and Knuth 2003). A study by Demana et al. (2003) also showed the potential of wireless classroom systems for creating environments that are learner-, knowledge-, assessment- and community-centred.

Roschelle (2003) highlighted that wireless learning networks can connect learners and their devices in a pedagogically sound manner. He distinguished normal social participation in the classroom, such as discussions, from information-based participation among connected devices in a wireless learning classroom as two distinct kinds of participation, both of which are important in teaching and learning. Some wireless classroom systems such as the classroom response system (Roschelle 2003) and participatory simulations (Roschelle 2003; Wilensky and Stroup 2002) allow teachers to conduct formative assessments, monitor student learning and provide role playing and collaborative learning tasks in the mathematics classroom.

Indeed, advanced GCs such as the TI-Nspire has wireless classroom networking capabilities when coupled with the TI-Nspire Navigator, a wireless classroom network system that enables instant and active interaction between students and teachers, and thus has the potential to enhance the teaching and learning process through its flexibility, portability and communication features. In Singapore, Ng (2011) conducted a design experiment to examine the role of the TI-Nspire, an advanced GC, in teaching and learning calculus. The design experiment involves the design and conduct of a TI-Nspire Intervention Programme for an intact class of thirty-five secondary four students (15–16 years) from a secondary school. Use of the TI-Nspire was integrated into teaching and learning Calculus concepts with the aid of the TI-Nspire Navigator. Mathematics attitudes surveys and structured interviews were administered to assess the effects of the use of the TI-Nspire on students' attitudes towards mathematics. It was found that appropriate use of graphical, numerical and algebraic representations of calculus concepts using the TI-Nspire could enable the subjects to better visualise the concepts and make generalisations of relevant mathematical properties. Results of paired samples t-tests and interviews with students suggest that the use of the TI-Nspire has a positive effect on students' confidence in and perceived usefulness of mathematics.

In both cycles of the experiment conducted by Ng (2011), it was found that the students used TI-Nspire as a tool in several different ways. For instance, they not only used it as a visualisation tool to better understand the behaviour of graphs, the new

Table 14.1 Using the TI-Nspire as a tool in teaching and learning calculus

Use of the TI-Nspire	Description
As an exploratory tool	TI-Nspire was used to explore and understand the concepts of differentiation and integration. For example, students explored the differentiation of the products of two functions and derived the product rule
As a confirmatory tool	Students used TI-Nspire to verify their answers to the questions in the exercises. For instance, they first solved problems by hand and then confirmed their answers using the graphing functions of TI-Nspire
As a problem-solving tool	Students used TI-Nspire to try different approaches to solving calculus problems. For example, in solving problems involving turning points, students used algebraic, graphical and numerical approaches
As a visualisation tool	Students used TI-Nspire to better visualise the behaviour of functions, new concepts being taught or problem situations. TI-Nspire was also used by teachers to construct simulations to illustrate problems or new concepts. For example, a simulation was constructed to allow students to explore the concept of rate of change by manipulating variables and observing the dynamic changes in the graphs
As a calculation tool	Students used TI-Nspire to calculate values or evaluate complex expressions
As a graphing tool	Students used TI-Nspire to graph functions and solve problems graphically. For instance, they used the Graph & Geometry application to solve problems related to the area under a curve

concepts being taught, or to solve problems, but also learned how to use TI-Nspire as a confirmatory tool to verify the correctness of their answers. Table 14.1 summarises the ways in which TI-Nspire was used during the intervention programme, both as a pedagogical tool in teaching or as a learning tool by the students.

14.3 Computing and Programming in Singapore

In this section, we present the use of computing and programming tools in the school systems in Singapore. We define computing tools as a set of software to perform numerical computation and graphical representation. Some software applications commonly used are spreadsheets, graphical and statistical packages, and CAS such as MAPLE and MATHEMATICA. For programming tools used, beside popular languages such as C, JAVA, BASIC and many others, new and simple visual and robotic coding tools are also becoming popular among schools in Singapore, such as SCRATCH, TYNKER, LEGO MINDSTORMS and other similar app development tools.

As the primary mathematics curricula in schools focus mainly on fundamental concepts and techniques, few computing tools are introduced and used except the hand-held calculators. Though in recent years, there have been strong interests to bring elementary coding into the primary schools, and these coding programs remain outside the mathematics curriculum and are run primarily as co-curricular activity (CCA) or enrichment programmes. In the secondary schools and JCs, ICT and specialised programming are incorporated into the curricula in subjects such as Computer Application, Design and Technology, Computing and Computer Science, but not within the subject of mathematics. In general, students, who are not taking these subjects, do learn and apply some computing and programming tools where applicable in their project works or inter-school activities. Looking ahead, there is a strong need to re-align and re-connect computing and programming into the mathematics curricula especially in view of the greater interests in STEM education, and 21CC requirements and expectations.

14.3.1 Use of Programming Tools in Mathematics

While the uses of ICT and computers are common in the primary school classrooms as learning aids and resources, programming is not a featured component in the mathematics syllabi. With the recent heightened interests in computational thinking and coding for kids, some schools have provided enrichment programs on top of the standard curriculum on basic programming, visual programming and coding for selected cohort of students to promote awareness and interest in programming with the aim to encourage more youngsters to consider pursuing further studies in game and software design, development and computer science. Beside these, there is also another platform in which programming tools are used which is in the CCA clubs such as Robotics Club, Computer Club, Game Design Club, Innovation Club and Maker Spaces.

In secondary schools, common ICT skills and computing tools are taught to most students in the academic subject called Computer Application (CPA) at the lower secondary levels. However, writing of codes and debugging in standard programming languages are not in the CPA syllabus. Moreover, CPA is usually taught independently from Mathematics. There is no use of programming tools in the mathematics curriculum. General computing tools like spreadsheets, the Geometer's Sketchpad (GSP), GeoGebra (International GeoGebra Institute 2017) and other dynamic mathematics software, statistical and plotting tools are taught and used for illustrations, explorations and computations in various mathematics topics in the syllabus.

There is another specialised academic subject called Computing in the secondary curriculum which is offered at the upper secondary levels. This subject is a beginning course to prepare students who are interested in pursuing study in Computer Science in future. In this subject, programming is taught as a core component in the syllabus. However, this special subject is offered at selected schools only.

In JCs and tertiary institutions, as mathematics subject becomes specialised and disciplinary, the use of programming tools diminishes in most areas except in applied topics such as numerical methods, discrete mathematics, operation research and statistics.

In summary, programming is generally taught outside of mathematics in Singapore schools. As for the use of programming tools, they are introduced and deployed when needed in only specific topics, computational tasks and project work. Whether programming should be taught within the mathematics curricula is debatable, but more appropriately, how should we consider programming, algorithm and computational approach in problem-solving be better promoted, incorporated and align into the mathematics curricula to develop mathematical knowledge, reasoning and problem-solving.

14.3.2 Use of Computing Tools in Mathematics

In comparison with the use of programming tools, the use of computing tools is more common and frequent in mathematics especially those which are designed and built specially for mathematics such as GSP, LiveMath, GeoGebra, MatLab, Maple, SPSS, just to name a few. These tools can be generally classified in terms of geometric representation, numerical and graphical data crunching, algebra symbolic system, modelling and simulation, networking and optimisation, and statistical data analysis.

Although the use of computing tools in primary schools is not specifically built into the curriculum, teachers are encouraged to use ICT and digital resources in their teaching if they were to find the tools appropriate and useful for specific topics. Teachers need to develop their students' understanding in mathematics first before they can apply the tools correctly. While the use of computing tools may not be necessary for primary school students in practising mathematics, teachers should introduce these tools to illustrate, explore and explain ideas. For example, teachers can consider using GSP to teach parallelism, angles, shapes and scales in geometry or using spreadsheets to teach average and percentage, charting of graphs.

For secondary schools, there are more uses of computing tools in mathematics. Most notable tools used are the dynamic geometry software like GSP and GeoGebra. The use of these tools is incorporated into the syllabus and is used explicitly for teachers to illustrate and explain geometrical ideas during some of the lessons in class or for students to explore further by themselves in homework exercises. Other computing tools like Excel spreadsheets and free easy-to-use software like iNZight are used for data exploration in Statistics. The use of computing tools in mathematics can be seen in secondary schools increasingly and is becoming more popular, but its usage is still being limited mainly for illustration purpose and not as working or learning tools. A possible change to this is to bring more use of technology into the teaching and learning of mathematics by introducing mathematical ideas, knowledge, processes and problems solving skills through applications.

At JC level, the use of computing tools likewise is not mandatory. The main objective of the JC curricula is to prepare students for admission into university studies in academic areas of their choice. As the major admission criteria are based on the scores of GCE A-level examination which does not involve the use of technological tools, much of the focus of JC study is on learning and understanding concepts, techniques and solving academic problems. The use of computing tools remains in general for illustrations and demonstrations. Exception happens when selected students undertake special project work or research work in problems which require the use of computing tools of which students learn on the job under the coaching of their supervising teachers or university researchers.

The use of computing tools at the tertiary level is norm. Subjects of study at this level become more specialised, disciplinary and technical, and the use of computing tools is essential and necessary depending on its contents, methods and technicality. Various standard or popular tools are introduced and used in topics like linear algebra, calculus, numerical method, dynamic systems, operation research and statistics.

14.3.3 Research and Development (R&D) on Computing and Programming

14.3.3.1 R&D on Programming and Computing in Mathematics

Not much research in recent time is being undertaken on programming for mathematics. While the number of research projects in this area is relatively small, there are some projects being undertaken involving programming or computing, especially those related to the teaching of coding and the development of computational thinking. For example, one project has a focus on building teachers' capacity in teaching computing using the unplugged approach as introductory activities for teaching computing as pedagogy on helping students to understand the concepts in computational thinking, while another project seeks to use educational robot as tools in the mathematics curriculum. There is also a project which aims to prepare lower secondary students for future enrolment in 'O' Level computing and another which seeks to develop students to be critical and inventive using 'Student as Designer' approach during project work or CPA. It is beneficial for pre-university students to experience basic programming even if they may not eventually choose a related career or university course. One project thus aims to train at least 600 pre-university students in basic programming via a 3-week blended learning course.

14.3.3.2 R&D on the Use of Technology in the Teaching and Learning of Mathematics

Researches in this area are more common in Singapore schools and at the Institutes of Higher Learning (IHL). In this section, we shall report some of the major research work undertaken in the use of technology in the teaching and learning of mathematics in Singapore schools. For latest R&D initiatives in general, readers can refer themselves to online information sites such as Ministry of Education (www.moe.gov.sg), National Research Foundation (www.nrf.gov.sg) and various IHL including the NIE (www.nie.edu.sg).

Two projects were completed in 2016, one of which aimed to make students' thinking visible so that they gain a holistic approach to problem-solving, and in turn, develop into confident problem-solvers. The approach focuses on getting students to talk and communicate in groups and in writing. The use of screen casting applications allows students to communicate clearly the steps they took to solve the problem and thus demonstrate their understanding by making thinking processes for each step of the problem-solving approach visible. This will help students to hone their competencies in holistic problem-solving and skilled communications using mathematical language. The other project targeted Normal (Technical) stream students in addressing weakness in visualising 3D objects in the learning of Geometry and Mensuration in Mathematics. The original goal is to develop a mobile application to diagnose and develop 3D spatial visualisation skills in the areas of mental rotation, perspective taking, and folding and unfolding of objects. Due to delay in the software development outsourcing processes, the project has to switch by using SketchUp for their design.

On the other hand, a project which completed in April 2017 had trained teachers to apply the learning framework in designing and developing resources in various subjects (including mathematics) using open source virtual environment and tools that present challenging tasks for surfacing intuitions and misconceptions within an immersive learning environment.

Finally, a project, which commenced in October 2015, aims to design, develop and pilot test a software for primary school students to solve math word problems. The main objective of the software is to help students, especially slow-progress learners to think and reflect on their own working. The software would guide the students to solve a given problem by simplifying the questions more and more until he can solve it correctly, then going backward to solve the next harder questions until he can solve the original question. By giving students opportunities to succeed, and making senses meta-cognitively in what they are doing, their competence and confidence in solving word problems correctly would improve.

14.4 Exploratory Mathematics Software

The first ICT Masterplan in Education for Singapore was launched in 1997 to provide a strong foundation to harness ICT for teaching and learning (Chap. 2). The foci at that time were to provide ICT infrastructure and educational software and resources for all schools, and core ICT training for all teachers. An example of a mathematics software that the MOE bought for all schools was GSP which is a dynamic geometry software that allows students to explore geometrical concepts. In this section, the term ‘exploratory mathematics software’ is used to describe any software that allows students to explore mathematics.

The theoretical basis for using an exploratory software to explore mathematical concepts is based on Taylor’s (1980) tool mode. Taylor classified the use of the computer in the school in the 1960s and 1970s into three modes: tutor, tool and tutee modes. In the tutor mode, the computer is the tutor and so students learn *from* the computer. In the tool mode, the computer is the tool and so students learn *with* the computer. Students can use the computer as a tool in at least two ways: to solve mathematical problems or to explore mathematical concepts. In the tutee mode, the computer is the tutee and so students learn *through* teaching the computer.

In the present days, there are many other exploratory software which can employ the tool mode to help students learn mathematics. Examples of such software include graphing software such as Desmos, statistical software such as Excel, dynamic geometry software such as GSP and GeoGebra, and interactive CAS such as LiveMath.

14.4.1 Use of Exploratory Mathematics Software in Singapore

14.4.1.1 Graphing Software

A graphing software, such as Desmos (2017), or even a hand-held GC, allows students to explore graphical concepts easily. The use of such a graphing software in this manner follows Taylor’s (1980) idea of a tool mode where students learn with the computer.

An example is shown in Fig. 14.1. In exploring the function $y = 4\sin x + 3\cos x$, teachers would instruct their students to use a graphing software to plot $y = 4\sin x$, $y = 3\cos x$ and $y = 4\sin x + 3\cos x$. Most students would realise that $y = 4\sin x + 3\cos x$ is still a sinusoidal curve and is of the form $y = R\sin(x + \alpha)$. The teachers might also guide their students to observe that $R = 5$ because the amplitude of $y = 4\sin x + 3\cos x$ is 5, but the students would have realised that it is not easy to find α. The teacher would then teach the students how to plot the graph of $y = R\sin(x + a)$, with the sliders for R and a, since Desmos does not allow the users to input α. The students could then drag the sliders until the graph of $y = R\sin(x + a)$ coincides with the graph of $y = 4\sin x + 3\cos x$, which will happen when $R = 5$ and $a = 37°$ (corrected to nearest whole

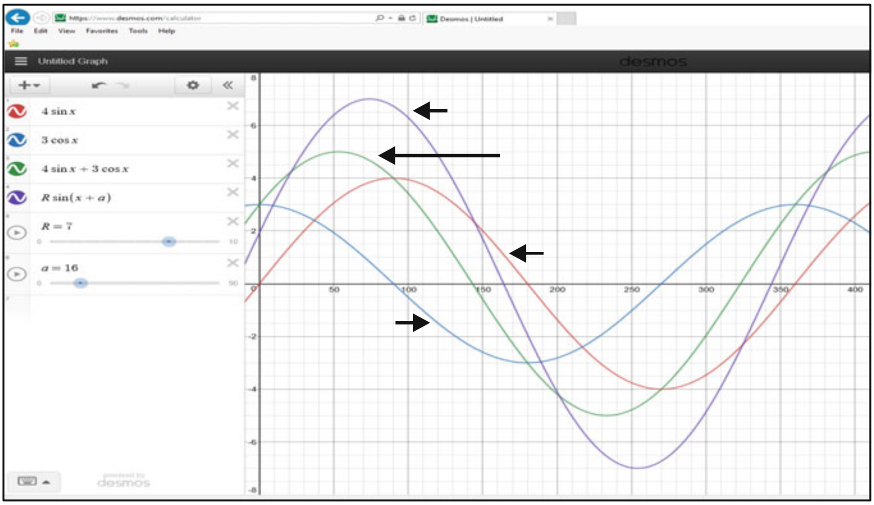

Fig. 14.1 Screenshot of Desmos exploration of R-formula

number). The purpose of this exploration is to get students to realise that 4 sin x + 3 cos x can be expressed as R sin ($x + \alpha$), where $R > 0$ and α is acute. The teacher would then challenge the students to find R and α without the use of the software.

14.4.1.2 Statistical Software

For statistics, teachers in Singapore would have instructed their students to use Excel as a tool to analyse statistical data. For example, they would have asked their students to plot a bar chart or a line graph for some data, such as the number of siblings that they have, or the monthly live births in Singapore for a particular year, and calculate the mean and the mode, or analyse if there is any trend. Real-world data such as monthly live births are available on the website of the Department of Statistics, Singapore (Government of Singapore 2017). Another example is to explore the effect of the vertical scale of a statistical graph. In the Excel template in Fig. 14.2, students in Wu's (2005) research study could change the scale, the minimum value and the maximum value of the vertical axis in order to study its effect on the display of the average monthly temperatures. Again, this is what Taylor (1980) called the tool mode of using the computer in the mathematics classroom.

14.4.1.3 Geometry Software

For geometry, a dynamic geometry software, such as GSP or GeoGebra, allows students to explore geometrical concepts interactively. By changing or dragging a

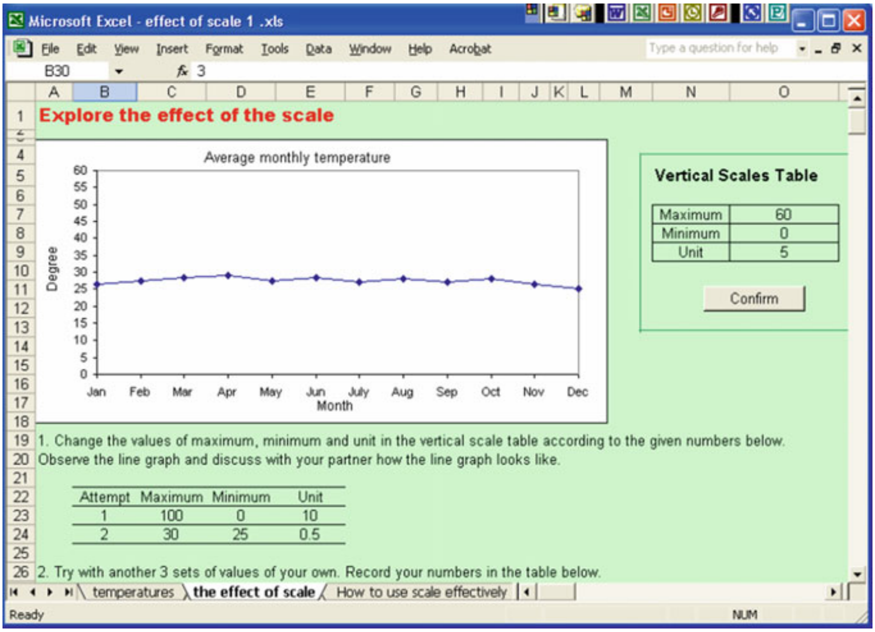

Fig. 14.2 Screenshot of excel template on effect of vertical scale (from Wu 2005)

mathematical object such as a point, a line or a circle in the software, all other objects that are linked to it will also change automatically and instantaneously. For example, many teachers in Singapore would have used a GSP template, such as the one shown in Fig. 14.3, to guide their students to discover the angle properties of a circle. The students are told by the instruction in the template to observe the relationship between the angle at the centre and the angle at the circumference. From the exploration, the students are guided to observe that the angle at the centre is twice the angle at the circumference if both angles are subtended by the same arc of the circle.

Another useful feature of a dynamic geometry software is the ability to trace the path of a point, whether manually by the user or automatically by clicking on the animation button. Figure 14.4 shows a GSP template where the sine curve on the right is being traced out by the unit circle on the left. Without the software, such diagram in the school textbook will just be a static illustration, which some students may not be able to envisage.

14.4.1.4 Computer Algebra System

Some teachers have used an interactive CAS called LiveMath (MathMonkeys 2017) for their students to explore algebraic and calculus concepts. A CAS is able to perform algebraic manipulations such as expansion and factorisation, and calculus operations

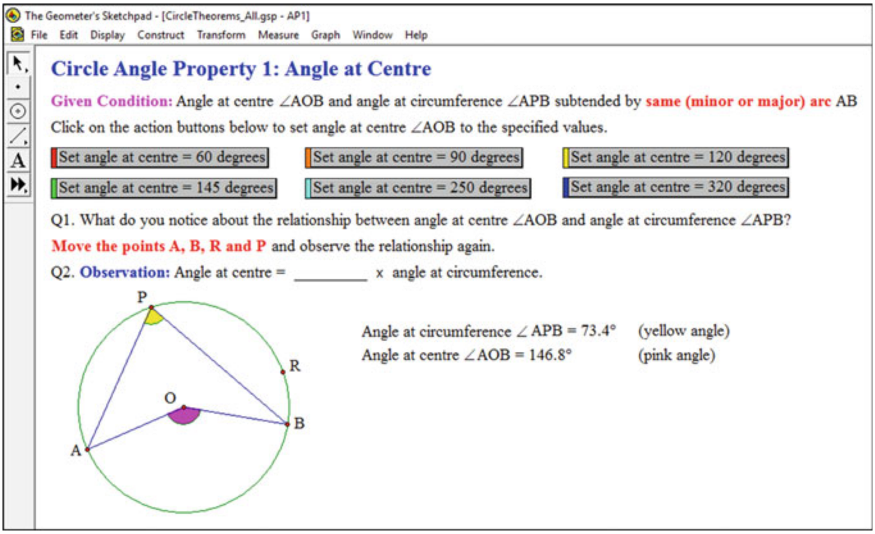

Fig. 14.3 Screenshot of GSP template on angle properties of circle

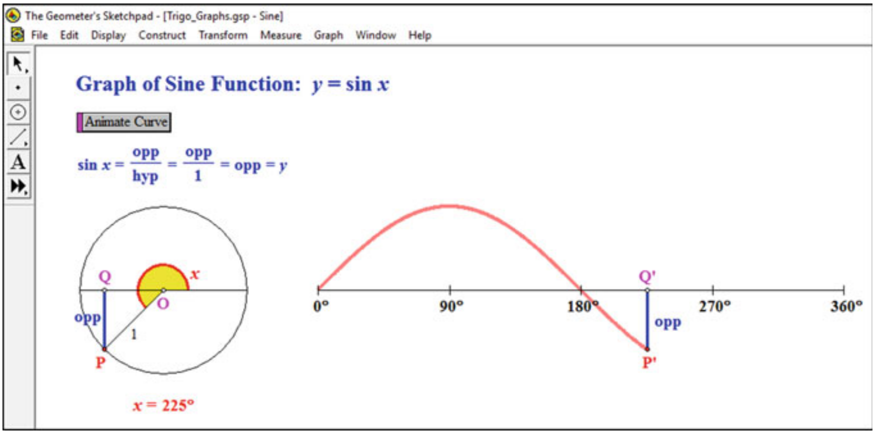

Fig. 14.4 Screenshot of GSP template on graph of sine function

such as differentiation and integration (Yeo 2004, 2015). An interactive CAS like LiveMath is able to effect the change automatically and instantaneously whenever the value of a parameter or a variable is changed. For example, in the LiveMath template shown in Fig. 14.5, students are guided to observe the sign of the gradient of the tangent to a curve around the neighbourhood of a stationary point, leading to the discovery of the first derivative test.

Because LiveMath is expensive, not many teachers in Singapore have access to the software. So Yeo (2001a, b) decided to use the Web-based version of the software

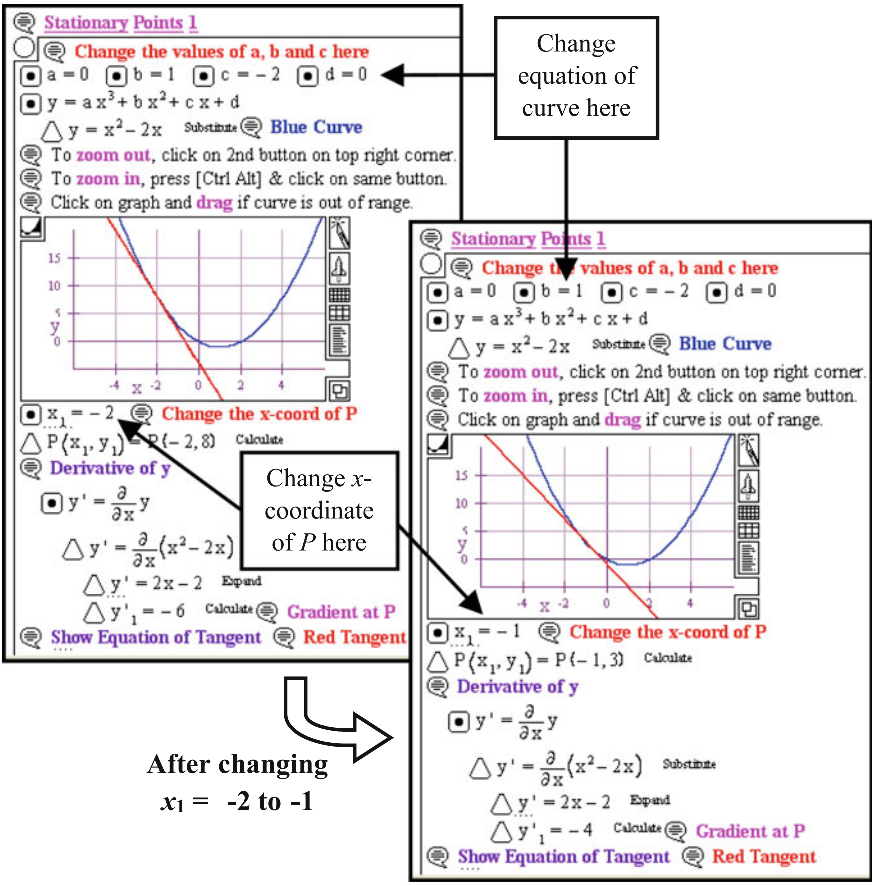

Fig. 14.5 Screenshot of a LiveMath template on stationary points

when he published two ICT workbooks for O-level Elementary and Additional Mathematics students to use. Each workbook came with a CD containing GSP templates and LiveMath Webpages. The end-users just needed to install a free LiveMath plugin in order to view and interact with the LiveMath Webpages. However, in recent years, many browsers do not allow the installation of plug-ins. Therefore, the company behind LiveMath (MathMonkeys 2017) has designed a *free* LiveMath Viewer for end-users to view and interact with LiveMath templates directly, while the software itself is now called LiveMath Maker.

14.4.1.5 Apps

With the advance in smart phone technology, some softwares, such as Desmos and GeoGebra, are now available as apps. This opens another window of learning opportunities for students. Instead of bringing students to the computer laboratory to use the software, teachers can now get their students to explore mathematical concepts on their smart devices in the classroom. This saves curriculum time as students do not have to walk to and from the computer laboratory, and teachers can carry out other parts of the lesson in the classroom that do not involve the use of the computers. However, not all the above-mentioned software are available as apps.

14.4.2 Research and Development on Exploratory Mathematics Software

14.4.2.1 Teaching Approaches

Instead of allowing students to use the computer as a tool to learn mathematics, some teachers have used the above software on their laptop, with projection onto a big screen in front of the classroom, as a teacher-centred demonstration, instead of bringing the students to the computer laboratory for the latter to explore the concepts themselves. Are both teaching approaches effective for Singapore students? In this section, we will look at the findings of local research studies on exploratory mathematics software.

14.4.2.2 Funded Local Research

There were at least 60 local funded research studies on mathematics education between 2001 and 2016 inclusive, some of which are still in progress. Only three of these studies were on ICT, out of which two of them did not involve the use of an exploratory software. The third study by Looi (2007) used SimCalc Math Worlds to support the creation of displacement-time, velocity-time and acceleration-time graphs, which are visually editable by clicking on hot spots as well as algebraically editable. Two schools participated in the preliminary study. The first school used the software package as part of an after-examination enrichment programme for one of their Secondary One classes in the Express stream for 10 lessons spread over two weeks, while the second school integrated the package into the lessons of a group of Secondary Four students in the Normal Academic (NA) stream for nine lessons over three weeks.

Every student had access to a computer to experiment with the software, which helped the student to switch between different representations (i.e. displacement-time, velocity-time and acceleration-time graphs) of movement of objects in the real world. Analysis of test results shows that the students in both schools performed significantly better in the posttest than in the pretest. Although the pretest and posttest scores were not significantly different between the two schools, the students in the second school fared slightly better in questions that require an open ended response. However, the research did not continue beyond the preliminary study.

14.4.2.3 Postgraduate Dissertations

Prior to 1996, postgraduate students had to do their Master of Education (MEd) programmes under the National University of Singapore (NUS). We were able to trace only three MEd dissertations on mathematics education and ICT in the 1980s: they were also mentioned in Chong and Lim-Teo (1992). From 1996 to 2016, there were at least 162 Master's dissertations and Ph.D. theses on mathematics education done by postgraduate students at the NIE in Singapore. Only 23 out of 162 (i.e. about 14.2%) were on ICT, of which one was a Ph.D. thesis, 13 were MEd dissertations, and the remaining 9 were from MA. Therefore, there were a total of 26 dissertations or theses on mathematics education and ICT from 1985 to 2016. The peak occurred in the years 2001–2004. Since the research would have been done earlier than the date of the thesis or dissertation, this would coincide with the late 1990s and early 2000s, just after the conceptualisation of the first ICT Masterplan in 1997. But after 2004, the number of local mathematics education dissertations or theses on ICT has dropped drastically.

On the other hand, there have been a total of 26 local mathematics education dissertations or theses on ICT out of which 14 of them used the computer as an exploratory tool, of which one of them was e-learning (or more specifically, asynchronous online learning). Although some of these 14 dissertations stated that they had used computer-assisted instruction (CAI), they actually used it as an exploratory tool, according to Taylor's (1980) tool mode, rather than CAI, which is classified under Taylor's tutor mode. As for CAI, only 2 of the 26 dissertations used it, and they were both under e-learning: one of them used online lecture notes and tutorials, while the other used online video-recorded lessons.

The focus of research studies in this section is on the 14 local mathematics education dissertations or theses on exploratory mathematics software. The following summarises the breakdown in terms of the types of software used, the contents/topics, the research subjects and the research designs.

- Types of software used: Of the 14 dissertations or theses, 3 used GSP, 2 used Graphmatica, 2 used LOGO, one used LiveMath, one used Excel, 4 used a program written in Basic (mostly in the 1980s), and the last one used web-based Java applets and a graphing simulator.
- Contents/topics: Of the 14 dissertations or theses, 7 explored geometrical concepts, 6 on graphical concepts and the last one on statistical concepts.

- Research Subjects: Of the 14 dissertations or theses, 12 were on secondary school students, one was on primary school students and the last one on pre-service primary school teachers.
- Research designs: Of the 14 dissertations or theses, 8 used a quasi-experimental pretest–posttest control group design, 4 used a pretest–posttest design for an experimental group only, one used a diagnostic posttest and oral defence for an experimental group, and the last one compared three groups undergoing different treatments.

We will now summarise some of the findings. Of the five dissertations that studied only one experimental group and no control group, 4 of them (Ingham 2001; Lee 2002; Puranadharshini 2011; Wu 2005) found that the students had performed significantly better in the posttest than in the pretest, while the last one (Ng 2004), that had no pretest, had found that the students had improved in their ability to explain and to sketch geometrical transformations. One criticism of such a research design is that the subjects will usually perform better after they have learnt a topic as compared to before they learned the topic (Mills and Gay 2016; Soh 2009). One way to resolve this issue is to have a control group.

Of the eight dissertations that used a control group, six of them compared the use of an exploratory mathematics software with traditional teacher-directed teaching. Four of them (Ho 1997; Lee-Leck 1985; Ong 2002; Woo-Tan 1989) found that the experimental group that used ICT had performed significantly better than the control group in the posttest. But one of them (Jee 2003) found that there was no significant difference in the posttest for both the experimental group and the control group. Further analysis reveals that the research subjects were of different academic levels and abilities. In Singapore, students in the Gifted stream are generally of higher ability than students in the Express stream, who in turn are generally more academically inclined than students in the Normal Academic (NA) stream, while students from a top school are generally of higher ability than students from a neighbourhood school.

In general, it seems that the use of an exploratory software had a positive effect on students of lower range and middle range ability in Ho's (1997), Ong's (2002), Woo-Tan's (1989) research, but not on high-ability students in the Gifted stream in Jee's (2003) study. To a certain extent, this appears to agree with the findings from Yeo's (1995) study, where his subjects were from two high, two medium and two low-ability classes, and one of each of the ability group was randomly assigned as the experimental group and the other as the control group. Yeo found that the medium-ability experimental group had performed significantly better than the medium-ability control group in the posttest, but there was no significant difference in the posttest for both the experimental group and the control group for the low-ability and high-ability classes. In other words, the findings for Yeo's high-ability group were similar to the findings for Jee's (2003) high-ability gifted students: one possible reason why there was no significant difference is that the high-ability students were capable of abstracting the graphical concepts themselves, even without the help of the software. The findings for Yeo's (1995) medium-ability group were also similar to the findings for

Woo-Tan's (1989) middle range ability students from the same top school as Yeo's (1995).

However, the findings were mixed for lower ability students because Yeo's (1995) low-ability students from a top school would in general be more academically inclined than Ho's (1997), Ong's (2002) students from neighbourhood schools. This may be due to the different topics or software used: Ho's (1997) and Ong's (2002) topics were on geometrical concepts using LOGO and GSP, respectively, while Yeo's (1995) topic was on graphical concepts using a program written in BASIC. Nevertheless, these were only six small-scale studies and more research needs to be done to investigate the effect of the tool mode on local students of different abilities using different software for various topics.

One criticism of a research design that compares the use of an exploratory software with traditional teacher-directed teaching is that the pedagogy itself is not kept constant. Oppenheimer (1997) opined that the difference in the outcomes from such a research design was most likely due to a difference between student-centred learning and teacher-directed teaching, rather than the effectiveness of ICT. Therefore, Yeo (2003) decided to use student-centred guided-discovery learning for both the experimental and the control group, except that the experimental group used a software to explore the characteristics of the graphs of exponential and logarithmic functions, while the control group used hard copies of pre-printed plots of the graphs to explore their characteristics. The analysis revealed that the experimental group performed significantly better than the control group in their conceptual and procedural knowledge of exponential and logarithmic functions. In fact, Tan's (1987) research also used guided-discovery learning of linear graphs for both the experimental and control group. Although she wrote in her dissertation that she used the traditional expository mode of instruction for the control group, she actually let the students explore the characteristics of linear graphs by getting them to draw on graph papers, which was student-centred guided-discovery learning as well. Tan also found that the experimental group performed significantly better than the control group in the posttest. Again, these were just two small-scale studies and more research needs to be done to investigate whether any significant difference is really due to the effectiveness of an exploratory software, *ceteris paribus* (i.e. all other things being equal).

In the last dissertation, Leong (2001) compared three groups undergoing different treatments. The first class used student-centred guided-discovery learning with a software, the second class used teacher-centred guided-discovery learning with the same software (i.e. the teacher would manipulate the objects on the display on the projected screen what students would for themselves like to do on the computers), and the third class used teacher demonstration with the same software. Leong found that there was no significant difference in the posttest between the first two classes, but both classes performed significantly better than the third class. Therefore, it seems that teacher demonstration was not as effective as guided-inquiry discovery with the exploratory software. Again, this was just a small study. An implication for classroom teaching is that if the teacher does not wish to bring his or her students to a computer laboratory for student-centred guided-discovery learning for whatever reason, he or she can adopt the same pedagogy used for the second class: it is like a teacher demonstration, but instead of the teacher just demonstrating, he or she will ask the class which objects they would like to manipulate and the teacher will then do the manipulation on the computer himself or herself, while the computer display is being projected onto a screen in the classroom.

14.4.2.4 Unfunded Local Research (Others)

For other unfunded local research, we first examined two kinds of publications by the Association of Mathematics Educators (AME) of Singapore: its refereed journal *The Mathematics Educator* (first issue in 1996) and its yearbook (first issue in 2009). There were 16 articles on ICT in the journal from 1996 to 2016: five were from overseas contributors, six were local discussion papers, and five were local research papers. Out of the five local research papers, two of them (Leong and Lim-Teo 2003; Yeo 2006) were reports of local dissertations (Leong 2001; Yeo 2003) described earlier, the third one was a survey on integration of ICT and the fourth one was on video conferencing. Only the last one (Leong 2003) was on an exploratory mathematics software, namely GSP (which we will describe shortly).

From the first AME year book in 2009 to the yearbook in 2017, there were 11 book chapters on ICT, of which only three were reports of research studies. None of these are on the use of exploratory mathematics software.

We next turn our attentions to four major mathematics or ICT conferences that were held in Singapore since 2000. All the local conference papers on ICT were in the first two conferences: EARCOME 2/SEACME 9 in 2002, and the 9th ATCM in 2004. For the 35th MERGA in 2012, there were 11 local research papers and four local short communications, but none on ICT. For PME 41 in 2017, there were four local research papers and five local short communications, but none on ICT. It appears that research interest on ICT in mathematics education has peaked in the early 2000s, and has since dwindled.

Of the four local research papers on exploratory mathematics software in EARCOME 2/SEACME 9, three of them (Lee and Pereira-Mendoza 2002; Leong and Lim-Teo 2002a, b) were reports from local dissertations (Lee 2002; Leong 2001) discussed earlier. We will report on the findings of the other paper (Ho 2002) shortly. For the 9th ATCM, the only local research paper on exploratory mathematics software (Yeo 2004) was from the author's dissertation discussed earlier (Yeo 2003). In short, for local unfunded research, we have managed to trace two reports which were not from local dissertations or theses: a conference paper by Ho (2002) and a journal article by Leong (2003). We will now describe some findings from these two papers.

In the conference paper, Ho (2002) taught 11 Primary Five students from the Gifted stream a two-hour lesson on line and rotational symmetries using GSP. The students were shown an example using GSP by Ho, and then, they were given time to explore and create their own designs, including a snowflake with six evenly spaced branches. Ho observed that the students had no difficulty navigating the software and understanding why they had to 'Mark Centre' in GSP when creating the snowflake design even when they had not learnt about rotational symmetry. Feedback from the students suggests that they were very excited about the software. The majority of them even went on to create snowflakes with more than six branches (although in reality snowflakes only have 6 or 12 branches), thus learning by themselves how much to rotate one branch of the snowflake.

In the other paper, Leong (2003) surveyed 41 students from 10 secondary schools on how they used GSP. Only 33 of the 41 teachers (about 80%) indicated that they had used GSP in teaching mathematics. Of the many geometrical concepts (including mensuration and trigonometrical concepts) in the secondary school syllabus, it was found that the 33 teachers only used GSP in teaching some of the topics. The favourite topics which the teachers had used GSP to teach were ‘angle properties of polygon’, ‘angle properties of points/lines’, ‘angle properties of circles’ and ‘transformations’. At the other end, few teachers had used GSP to teach ‘symmetry’, ‘congruence’, ‘similarity’, ‘vectors’, ‘mensuration’ and ‘trigonometric ratios’. Leong observed that ‘it appeared that certain attributes of the software were piecewise utilised in a fragmented way to fit into bits of geometry, instead of a full integration into the curriculum’ (p. 91). In the survey, Leong also asked the respondents to indicate their modes of using GSP and to rank them. Only 30 of the teachers clearly indicated their most preferred mode of using GSP. The most preferred mode with the highest number of teachers was to ‘draw diagrams for worksheet/test paper’ (seven teachers), followed by ‘teacher click-and-drag pre-designed templates to show some geometrical properties’ (six teachers) and ‘teacher shows animation/movement in front of class to aid students’ visualisation (six students). At the other end, the most preferred mode with the least number of teachers was to ‘provide templates for students to observe and conjecture properties’ (0 teachers), followed by ‘let students explore hands-on freely’ (three teachers). In other words, the local teachers in this study preferred to use an exploratory software as teacher demonstration, rather than letting their students do the exploration themselves, so Leong concluded that ‘the full power of the software and the ‘promise’ of its potential to transform classrooms into lab-like places for students’ inquiries are not realised in most geometry classrooms’ (pp. 92–93).

However, the above report on teachers’ use of GSP was in 2003. The question is whether teachers in Singapore are still using GSP or other exploratory software. Since there were not many local mathematics education studies on ICT since 2004, we will gather the evidence from an ongoing funded mathematics education research that is not directly related to the use of ICT, but a study on the enacted school mathematics curriculum. At the time of writing, 20 competent secondary school mathematics teachers had been videotaped teaching a topic for about two to three weeks. The classes ranged from Secondary One to Secondary Five from the Express, Normal Academic and Normal Technical streams, including students from schools that offer the Integrated Programme (IP). Out of the 20 teachers, 10 of them used ICT in some parts of their lessons. Of these 10 teachers, four of them used an exploratory mathematics software, and all the four of them let their students use the software to explore graphical, geometrical or trigonometrical concepts, either in the computer laboratory or in the classroom. There is some evidence that there are still local teachers who continue to use the computer as a tool for their students to explore mathematical concepts.

14.4.2.5 Summary of Local Research

With the launch of the first ICT Masterplan in Education for Singapore in 1997, there was a growing interest in local research on the use of ICT, but the interest started to dwindle after 2004. Most of the local research on the use of exploratory mathematics software in the classroom were unfunded and most of them were local Master's dissertations. The general findings suggest that Taylor's (1980) tool mode has made learning more effective for local students than traditional teacher-directed teaching. However, more research needs to be done locally to investigate whether a student-centred inquiry approach without the use of ICT can be as effective as one that uses the tool mode. Moreover, as Ng and Leong (2009) have pointed out, the limitations of many of these local studies are in their small sample sizes of one or two classes of students, and their short duration of a few hours of intervention.

Another important finding suggests that local teachers in the early 2000s seem to prefer to use an exploratory software in the teacher-demo mode, rather than letting their students do the exploration themselves. However, in the present day, there are some teachers who let their students use the software in the computer laboratory or in the classroom to explore the concepts themselves. On one hand, '[t]he computer stand betwixt and between the world of formal systems and physical things; it has the ability to make the abstract concrete' (Turkle and Papert 1990, p. 346). On the other hand, '[t]echnology is just a tool. In terms of getting the kids working together and motivating them, the teacher is the most important' (Bill Gates, cited in Wong 2015).

14.5 Flipped Classroom

In this section, we discuss a specific blend of e-learning called 'flipped classroom'. The word 'flipped' or 'inverted' first introduced by Strayer (2007) was originally taken to mean 'events that have traditionally taken inside the classroom now takes place outside the classroom and vice versa' (Lage et al. 2000, p. 32). Recently, more works in flipped pedagogy (Bishop and Verleger 2013; Ho and Chan 2016) have adopted a more focused meaning, i.e. an educational method comprising of two systems: interactive learning system inside the classroom (the face-to-face component) and a direct computer-based instruction outside the classroom (the e-learning component).

We present a theoretical framework, formulated recently by Ho and Chan (2016), which can be used to characterise the processes centred around flipped classroom. In addition, we also discuss issues concerning student motivational and cognitive load that pertain to flipped classroom, following up on works by Abeysekera and Dawson (2015).

Admittedly, flipped classroom requires a heavy investment in terms of time, energy and resources both on the part of the teacher and the student. Despite this, some schools and IHLs in Singapore have taken the first few steps in trying out this new method of teaching and learning. For mathematics, the take-up numbers are low. We report on some research-based innovations that exploits flipped classroom implemented at the pre-university (respectively, university) level to teach A-level mathematics (respectively, tertiary mathematics).

14.5.1 Use of Flipped Classroom in Singapore

14.5.1.1 Definitional Matters

For the purpose of setting up a more focused, and hence meaningful, discussion in this section, we shall use the terminology ‘flipped classroom’ to mean the use of computer technology and the Internet (e.g. video-recorded lecture available online) to enable the ‘movement’ of the information-transmission component of a traditional face-to-face lesson out of class time and replace that with a range of specifically designed activities to engage students and to motivate independent learning. This working definition of ‘flipped classroom’ therefore comprises two subsystems: firstly, the interactive learning system inside the classroom and secondly, a computer-based instruction outside the classroom. Pedagogical theories of grounded image (Ho et al. 2015) highlighted that students who made specific reference to particular junctures of the video-recorded lectures could make more relevant discussions. Because of the above considerations, we exclude those implementations of blended learning that do not make use of videos as the media of out-of-class teaching instruction.

14.5.1.2 A Brief Overview of Flipped Classroom in Singapore Schools

In Chap. 4, we have seen how the Singapore mathematics curriculum has responded to the education trends in the rest of the world—going through rapid shift in the curriculum ideology from the Scholar Academic, through the social efficiency, to the student centred. Locally in the schools, these shifts have a rippled effect on the classroom practices—manifested in four major shifts: (1) from lower-order thinking to higher-order thinking, (2) from analogue to digital, (3) from individualistic to collaborative. These four shifts in teaching practice are simultaneously realised in many Singapore teachers’ attempt to experiment with flipped classroom, or what they understood to be.

More accurately, to most practitioners of flipped classroom, it remained very much at the experimental and exploratory stage. Thus, attempts to implement flipped classroom in Singapore schools were carried out at a small scale, with occasional pockets of news reporting isolated successful episodes of such implementation. There was never a concerted effort among school teachers to understand either the theory of flipped classrooms or the practice of it, and this could be due to a lack of expert knowledge concerning this relatively new pedagogy. However, there are exceptions—in such cases collaboration between school teachers and education researchers are always the essential ingredient. In the ensuing development, we shall briefly recount one such implementation at a local JC, where H2 Mathematics was taught to a selected group of 'A' Level students using the flipped classroom approach. In that pilot study, a theoretical framework for flipped classroom was developed with two major considerations in mind: *self-determination and cognitive load*.

14.5.1.3 Theoretical Framework for Flipped Classroom Pedagogy

Bishop and Verleger (2013) modelled flipped classroom to be a system consisting of two disjoint parts: interactive group learning activities inside the classroom and direct computer-based individual instruction outside the classroom. However, we do not adopt their framework here as recent research in flipped classroom by Abeysekera and Dawson (2015) suggests that flipped classroom approach relies on the *interaction* between in-class human classroom activities and outclass computer-assisted learning, and hence, these two parts are far from disjoint. Backed by self-determination theory (SDT) and cognitive load theory (CLT), Ho and Chan (2016) asserted that 'the teacher plays an important role in *connecting* these two aspects' by intentionally designing pre-class activities and post-class activities, constantly tweaking the lesson design to respond to the learning processes that take place in-class and online. The strong interplay between the interactive classroom activities and the explicit instruction methods assisted by media/computer technology makes up the core of flipped classroom pedagogy, and we depict this characteristic mutual interaction in Fig. 14.6.

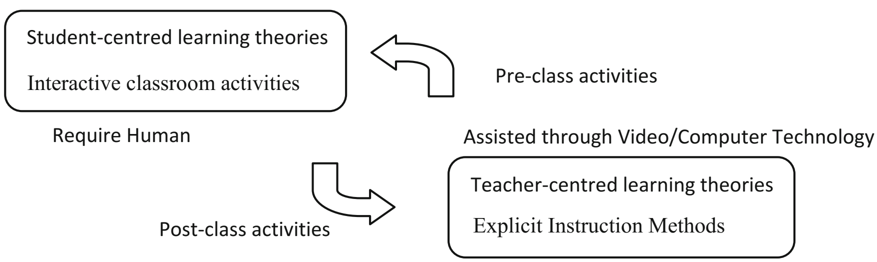

Fig. 14.6 Theoretical framework for flipped classroom

14.5.1.4 Summary of a Case of Flipped Classroom Implementation at a Local School

This implementation took place at a JC. Students (JC1 are 11th graders; JC2 12th graders) who take mathematics as an 'A' Level subject attend lessons given in lecture-tutorial style for two years and sit for a General Certificate Examination (Cambridge-Singapore syndicate) at the end of the final year. For curricular details about 'A' Level Mathematics in Singapore, we refer the reader to Chap. 13 of this book. Being one of the core subjects required in most university courses, academic competency in 'A' Level H2 Mathematics is considered an important performance indicator in most university entry requirement.

The participating students were of mixed abilities and competencies in mathematics. It was reported in Ho and Chan (2016) that the students who were weak in mathematics displayed a lack of interest, motivation and confidence in the subject. During the lectures, the mathematically stronger students found the pace slow and became disengaged from the lectures, while weaker students had problem grasping the basic concepts. Chan felt an urgent need to 'fix' this problem and started to look for an alternative pedagogy—one which offers autonomy to students in learning at their own pace and concurrently optimise students' engagement in-class.

Chan made an adaptation of team-based learning (TBL) inspired by the findings of Haidet et al. (2012). Three specific attributes associated to a learner's feeling about the flipped classroom experience are of top priority, and we briefly define these attributes.

- Autonomy: The student *feels* in control and independent.
- Competence: The student *feels* competent to master the knowledge, skills and behaviours necessary to be successful in a given social context.
- Relatedness: The student gets a *sense* of belonging to a social group in a given context.

Three classes with a total of 63 students of varying abilities were given a separate class to attend; i.e., they did not attend the traditional lecture-cum-tutorial classes. These students were grouped uniformly into teams of six to seven students of mixed mathematical abilities, gender and class. Special arrangement was made in the school timetable to schedule two 100-min mathematics lessons each week, spanning over 18 weeks—an equivalent of two school terms.

A typical flipped lesson of H2 Mathematics planned by the teacher is a three-movement symphony:

(i) *Pre-class reading*—Students of these three classes were expected to self-learn the contents by reading the notes and fill out the blanks via any of the following options.

   (a) Referring to the PowerPoint Slides.
   (b) Referring to video-recorded lectures (these lectures were exactly the same as the ones attended by the non-flipped classroom students).
   (c) Google for additional information to clarify their doubts.

(ii) *In-class activities*

(a) Individual Readiness Assurance Test (iRAT) was conducted at the beginning of the lessons when a new topic (or subtopic) is being introduced via an online software that allows the teacher to obtain immediate feedback on the performance of all students. The area of weakness in understanding the content for each student, as well as the class as a whole, could be surfaced instantly. The students, however, would not get to know the results of the test.

(b) Group Readiness Assurance Test (gRAT) would follow immediately at the end of iRAT, using the same set of multiple-choice questions. Each team would discuss the questions, focusing on the justification of their choice of answer. Online software was used to allow each team to obtain immediate feedback on their selected answer. If the answer given was incorrect, the team had to re-deliberate their choice until a correct choice was made before moving on to the next question.

(c) Class discussion—At the end of the gRAT, the teacher would facilitate the discussion of the MCQs to ensure that all students have gained basic understanding of the concepts learnt.

(d) Applications of Concepts learnt—The class would proceed to solve challenging problem(s) by applying the concepts learnt as a team and present the solutions or answers using mini-white boards or flash cards.

(e) Exit assessment—Conducted towards the end of a lesson to check each student's mastery of concepts and its applications. Feedback on the performance of the assessment would be addressed during the following lesson. Sometimes, due to time constraint, this exit assessment might be given as timed assignment to be submitted on the following day of the lesson.

(iii) *Preparation for the next lesson*—Students would attempt selected tutorial questions given in their lecture notes as preparation for discussion during subsequent lessons.

Since assessment is a key part of teaching and learning, the teacher placed emphasis on the RAT (both the individual and the group) by including a small percentage of the scores into the continual assessment. RATs took the format of multiple-choice questions (only 1 correct answer out of 4 options). Figure 14.7 shows a sample question.

The RAT items were carefully designed, backed by the Bloom's Taxonomy (Bloom 1965), to include questions with varying levels of difficulty, e.g. basic ques-

1. Given that X follows a normal distribution with $\mathrm{E}(X) = \mu$ where $\mu \neq 0$. If $\mathrm{P}(|X| > a) = 0.8$, then $\mathrm{P}(X > a) =?$

(A) 0.4 (B) 0.2 (C) 0.1 (D) Unable to determine.

Fig. 14.7 Sample RAT item for the topic of normal distribution in H2 Mathematics tions that require a mere recalling of facts as well as application questions that demand higher-order thinking skills. The iRAT item takes about a minute per question, and the same item appears in the gRAT where more time has been factored into allow group deliberation on the question.

The teacher had access to the performance of the students' iRAT before conducting the group discussion, and so areas of weakness had been identified a priori. Questions were thus planned just in time to tease out the important aspects, e.g. students' misconceptions. The teacher built in the culture of requiring the students to present their answers in the following format:

- Justify their choice of the correct answer
- Explain why the answer should not be his/her initial choice (The teacher needs to be tactful and not revealing the fact that the student being asked had given a wrong initial response. However, in reality, most students would 'confess' that the wrong answer was his/her initial choice.)
- Share the group's points of discussion.

At the end of the two school terms, a survey was carried out for the students to feedback on the flipped classroom experience.

14.5.2 Research and Development on Flipped Classroom

14.5.2.1 Significant Findings from Chan's Implementation

A number of findings were gleaned from the pilot study conducted at the aforementioned JC. To a large extent, Chan's implementation showed positive effects in most areas of extrinsic motivation due to specific features put in place in the flipped classroom design. We highlight two particular points: (1) teachers who fore-load the lecture content as pre-class activities usually 'compactify' all that need to be covered within one video lecture. But students need time to unpack and digest the materials even as they watched the lesson played online. (2) There is a need for teachers to practise 'wait-and-observe' so that they may respond with appropriate changes that are taking place in class by designing *just-in-time* postclass activities (or pre-class activities prior to the next lesson). Recording lessons and planning too far ahead could be counter-intuitively ineffective as students might have unforeseen difficulties that need to be resolved.

14.5.2.2 Taking It Forward—A Flipped Classroom Implementation at Tertiary Level

The abovementioned implications and lessons learnt in the pilot project conducted in the above JC were taken into consideration by the fourth author of this chapter. In particular as mentioned in Chap. 13 of this book, a graduate mathematics class in

Topology (MSM832) adopted Chan's lesson design principles to implement a flipped topology class. The Topology course ran over 13 weeks, where weekly lessons were each of duration 3 h. Each lesson followed the same sequence of activities used by Chan. However, there are two main differences in this second implementation of flipped classroom. The first one is about (i) pre-class activities outside the classroom, and (ii) Applications of Concepts Learnt during the in-class activities.

For the pre-class activities, each video-recorded lecture for the week is split into bit-size of at most 10 min. Each day, the student watches one or two such video segments. Embedded within the video are Pre-class Milestone Tasks (PMT) to be completed as the student watches the video. There are many kinds of PMTs. For instance, one typical PMT requires the student to pause the video and draw a diagram to represent the topological concept; another PMT requires the student to complete the rest of a proof that has been started by the lecturer in the video; and yet another requires the student to pause the video and give an example/counterexample of a property mentioned in the video segment. CLT informs us that the student's focus on video-lectures begins to waver when the segment exceeds 5 min. This explains why each 10 min-segment is filled with suitably positioned PMTs (each spaced out uniformly about 2 min) so as to engage the students in meaningful learning as they watched the video. This would also prevent them from 'cheating' by streaming the video without actually watching it. In order for the reader to have a better understand of how the PMTs were positioned along the video, we include a portion of the detailed lesson plan/script in Fig. 14.8. Note that sometimes notes (in italics and bold) are included as hints or remarks for students when they attempt the PMTs.

For the Applications of Concepts, a problem-solving approach was used. Here in the last hour of the lesson, an unseen problem that requires the content knowledge acquired earlier on in that particular lesson was given to the groups. Such unseen problems can be of two kinds. (1) A new theorem that students have not encountered in the lecture and/or RATs and each group is to produce the required proof. (2) A new theorem with a given proof (usually tersely written or with some gaps) where each group writes the full proof in their own words and fills in details wherever there are gaps. Figures 14.8 and 14.9 show samples of such problem-solving items.

Although the RAT items are MCQs (there are five options for this implementation—graduate students have a higher competence than JC students in handling MCQs), students in the Topology course reflect that:

> "I am … and challenged by higher-order thinking questions (especially the last three questions)"—Student A.

> "they [the MCQs] are not so straightforward … they can be quite tricky"—Student B.

> "the MCQs in the RAT really tease out my misconceptions. I found these out only when I discuss my answers with my group members during the gRAT and realized other group members had differing views."—Student C.

Student reflected on the group discussion and felt that they had the opportunity to hear people out. They learnt to be more open-minded and receptive to others' views:

Video: Axioms of separation (I) Lecture[Slides 1 to 8]	**PMT**
[Slide 2] Pause at the end of this slide.	**PMT 1** Which one of the following correctly describes the condition for a T_0 separation? (A) Any two distinct points cannot share the same family of open neighbourhoods. (B) Any two distinct points share the same family of open neighbourhoods. (C) Any two distinct points must be the same actually. (D) For any two distinct points, their closures coincide. (E) Two distinct points can be separated by non-intersecting neighbourhoods. Answer: A ***This is the same as saying that for any two distinct points x and y, either there is an open set that contains x but not y, or the other way round.***
[Slide 6] Pause at the end of this slide.	**PMT 2** Let X be a set with more than one point, and endow it with the indiscrete space. Then X cannot be T_0 because (A) the only non-empty open set is the singleton set which cannot be used to separate points. (B) the only non-empty open set is the entire space which will contain all points and hence fail to separate points. (C) the only open set is the empty set which will contain both points and hence fail to separate points. (D) the only non-empty closed set is the entire space which will contain all points and hence fail to separate points. (E) the only closed set is the empty set which cannot be used to separate points. Answer: B

Fig. 14.8 Sample PMTs positioned in a video segment of 10 min

Problem
Let X be a topological space. The apartness map $(\neq)$: $X \times X \to \Sigma$, where Σ is the Sierpinski space is defined by: $(\neq)(x,y) = 0$ if $x = y$; else 1. Prove that the following statements are equivalent: (i) X is Hausdorff. (ii) The apartness map is continuous.

Fig. 14.9 Sample problem-solving item

> "I find my group members very insightful in their thinking and they helped me in answering one another's doubts or misconceptions. I felt that they cared about my learning and the group left no one behind in the discussion."—Student H.

> "Not only did we clarify our doubts, but we reinforced on what we understood. I have time to reflect on things that I thought I did not understand at first, but eventually obtain a clearer understanding after we discussed about it…"—Student E.

This implementation of flipped classroom for a graduate Topology mathematics class also ended with a survey. Similar to Chan's study, it was reported that the students found that they gained greater autonomy, competence and relatedness in the flipped classroom experience.

There were also a number of students who complained that they did not have the time (because of their daytime job as school teachers) to watch all the video segments.

> "the power-point slides were useful and well-paced…but too many slides make the video draggy at times."—Student F.

> "I am challenged by a lack of time to sit down and go through the video lecture and the PMTs. Too long at times."—Student A.

Regarding the element of 'just-in-time' response on the part of the lecturer, in Week 5 the lecturer organised a special session (no flipped classroom for that particular session) to consolidate the students' feelings and opinions about what they felt about the flipped classroom approach. This consolidation session was an essential checkpoint for both the students and the lecturer to make sure that the flipped classroom pedagogy is doing good and not harmful to the participants.

14.5.2.3 Implications Drawn from the Two Implementations

While we are very encouraged in both implementations of flipped classroom by the phenomenal transformations in the learners' motivation, we have to be cautious of many potential risks involved.

Firstly, implementation of flipped classroom requires careful and extensive planning. This inevitably requires an inordinate amount of time and effort put in, e.g. writing the detailed script for the video-recording, the actual recording of the lectures by the crew (i.e. both the voice-over video and the live demonstration of the theorems), the setting of the RATs and the choice of the unseen problems.

Secondly, the effectiveness of the group discussions need to be further examined. Ho and Chan (2016) caution that 'the extent to which an individual finds group discussion beneficial depends on several social factors'. Although the second implementation already factored this consideration in the design of the problem-solving task, it was inevitable that at times there were students who were not as proactive as desired in contributing their views, and others who dominated the discussions most of the time while others kept listening to them. Here the teacher, as facilitator, should intervene and moderate the processes in the group discussion.

Flipped classroom is not the magic bullet to mathematics teaching and learning, and the teacher who considers using this new pedagogical approach must be cognisant of the various challenges and limitations that are reported in the above implementations. Further research certainly needs to be performed to better harness the benefits of flipped classroom in mathematics teaching and learning.

14.6 Conclusion

This chapter tracks how ICT has been used in Singapore mathematics education from all level from the primary to the tertiary level, and the various research projects and classroom anecdotal evidence about the use of these technologies. While encouraging positive impact on student learning of mathematics has been reported in this chapter, readers are also cautioned on the context of their implementation.

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Part III
Teacher Education and Professional Development

Chapter 15
The National Institute of Education and Mathematics Teacher Education: Evolution of Pre-service and Graduate Mathematics Teacher Education

Eng Guan Tay, Weng Kin Ho, Lu Pien Cheng and Paul M. E. Shutler

Abstract The National Institute of Education (NIE) is the sole teacher education institution in Singapore. This chapter considers the various kinds of knowledge that make up the knowledge base of a mathematics teacher of substance, and illustrates in detail the various programme designs which NIE implements to ensure the development of such a knowledge base in pre-service and in-service mathematics teachers. The special features of NIE, such as the symbiotic relationship of its mathematicians and mathematics educators and its adaptation of good practices from around the world, are described in some detail to cast light on how it has been generally successful in carrying out this purpose. Challenges ahead and possible future directions for improvement are also discussed.

Keywords Teacher education · Teacher education programmes · Singapore · Mathematics education · Teacher knowledge · Pedagogical content knowledge · Subject knowledge · Postgraduate mathematics teacher programmes

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15.1 The National Institute of Education and Its Teacher Education Programmes

15.1.1 Teacher Education Programmes

What is the best teacher education programme structure?

A review of American teacher education programme structures (Arends and Winitzky 1996) lists six types of teacher education programmes. The predominant structure is that of the 4-year undergraduate model, commonly known as the Bachelor of Education programme. The other models are: (a) an extended 5-year bachelor's programme, (b) an extended 5-year bachelor *and* master's programme, (c) a fifth-year programme that leads to a master's, (d) a 6-year master's programme and (e) alternative certification programmes. Zeichner and Conklin (2005) state that the most common teacher education programme issues in the teacher education literature then were about whether 4 years were enough for the preparation of a teacher, making teacher preparation a completely postgraduate programme, and the establishment of alternative routes into teaching. Their paper continues with a quite comprehensive review of the literature and summarizes the results for 4-year versus 5-year programmes, and alternative versus traditional programmes. They conclude wistfully that "few definitive statements can be made about the effects of different structural models of preservice teacher education based on this body of research" (p. 698). They, however, cite the Teacher Education and Learning to Teach (TELT) study in suggesting that "it is programme substance and not structure that is key in influencing prospective teachers" (p. 701). TELT researchers differentiated programmes by their substance into two categories, viz. traditional programmes that emphasized the organization of students and classroom activities, and reform programmes that encouraged more learner-centred practices with much emphasis on subject-specific teaching. It was the latter that had more influence on the prospective teacher.

We turn to Finland, a country highly regarded for its education system, for another perspective of teacher education. The Finnish system is a two-tier 5-year teacher education programme. There are three main components in the structure: (a) academic disciplines, i.e. subject majors such as mathematics or history for secondary school teachers, and educational sciences for primary school teachers, (b) research studies that consist of methodological studies, a BA thesis and an MA thesis and (c) pedagogical studies that include teaching practice. According to decrees issued in 1979, 1995 and 2005, all teachers require a master's degree (Niemi 2012). The master's requirement is the result of the Finnish teacher education policy that necessitates education on research-based foundations. A Web-based survey conducted by Niemi (2012) among students in teacher education programmes at two Finnish universities found that students rated themselves highly in designing of instruction, critical reflection of own work, awareness of ethical basis of teaching profession, lifelong professional growth, self-evaluation of own teaching, using teaching methods and development of own educational philosophy. Students rated themselves lowest for administrative tasks, management of tasks outside the classroom, cooperation with parents and acting in conflict situations. The split in competencies fairly mirrors the "subject-specific against the administration" dichotomy of the American situation.

Put together, the American and Finnish situations set the context for viewing Singapore's teacher preparation programmes, mainly through highlighting two considerations, i.e. the "subject-specific against the administration" dichotomy, and the fact that "few definitive statements can be made about the effects of different structural models of preservice teacher education based on … research" (Arends and Winitzky 1996, p. 698). These help us to understand how Singapore has positioned itself in relation to good international practices and local constraints.

Singapore is a young nation, independent for barely half a century. It inherited its teacher training programmes from its British colonial masters and evolved it according to the needs of the nation. Singapore has always been known for its pragmatism. Mahbubani (2015) wrote the following in the Huffington Post:

> So why did Singapore succeed so comprehensively? The simple answer is exceptional leadership … Lee Kuan Yew, the founding prime minister …, Goh Keng Swee, the architect of Singapore's economic miracle, and S. Rajaratnam, Singapore's philosopher par excellence. Together, they made a great team. This exceptional team also implemented three exceptional policies: Meritocracy, Pragmatism and Honesty. Indeed, I share this "secret" MPH formula with every foreign student at the Lee Kuan Yew School, and I assure them that if they implement it, their country will succeed as well as Singapore. Meritocracy means a country picks its best citizens, not the relatives of the ruling class, to run a country. *Pragmatism means that a country does not try to reinvent the wheel. As Dr. Goh Keng Swee would say to me, "Kishore, no matter what problem Singapore encounters, somebody, somewhere, has solved it. Let us copy the solution and adapt it to Singapore."* [italics added]

In education, and in teacher training, Singapore has pragmatically looked at the systems in the world and adapted the right "wheel" for its use. In teacher education, "the strength of [Singapore's] initial teacher preparation lies in the strong integration between content and pedagogical preparation, the design and development of which is backed by evidence-based educational research" (Gopinathan 2010, p. 140). Here, we see that all the positive aspects of subject-specific teaching and research have been assimilated into a coherent teacher training framework. Indeed, Chen and Koay (2010) in their preface describe teacher education in Singapore as being "guided by pragmatic principles, a blend of philosophy of the East and West unique to Singapore, which evolved as the newly established nation responded to the challenges of the times" (p. xii). Singapore education officials, school leaders and researchers regularly travel to other countries to learn best practices for adaptation at home. In addition, education officials and significant academics visit Singapore to see what is happening here and to share their views and recommendations.

15.1.2 Evolution of Teacher Education in NIE: Programmes, Students and Teacher Educators

We have today in Singapore the National Institute of Education (NIE) as the sole teacher education institution in the country. The following is a short history of the evolution of the teacher education programmes summarized from *Transform-*

ing Teaching, Inspiring Learning (Chen and Koay 2010). The Teachers Training College (TTC) was established in 1950. It conducted certificate courses in education for non-graduates. In the same year, a School of Education was established in the local university to train graduates for teaching on a full-time basis. Students were conferred a Diploma in Education (Dip. Ed.). In 1971, the School of Education was closed and TTC became the only institution responsible for teacher training. It entered into a new relationship with the university, whereby besides certificate courses, it also prepared graduate students for the Dip. Ed. In 1973, the Institute of Education (IE) was established from the TTC. It offered a 2-year full-time or a 3-year part-time Certificate in Education (Cert. Ed.) programme for non-graduates, and a 1-year full-time or an 18-month part-time Dip. Ed. programme for graduates. On 1 July 1991, the NIE was established as an institute of the Nanyang Technological University (NTU). As part of the university, new 4-year degree programmes were offered to matriculated students. These programmes, Bachelor of Arts with Education and Bachelor of Science with Education (collectively called B.A./B.Sc. (Ed.)), imparted both subject matter knowledge and pedagogical knowledge to student teachers. Non-graduates and graduates training to be teachers took the two-year Dip. Ed. and the one-year Postgraduate Diploma in Education (PGDE) programmes, respectively.

We turn our attention now to focus mainly on the student teacher intake for the years since the establishment of IE in 1973. The first decade of IE continued with the providing of quality training of teachers against the backdrop of the rising demand for a larger number of qualified teachers in the schools. Compared to colonial times when only a fraction of the population could afford school, the new nation of Singapore was intent on educating all its children. Schools could be and were built quickly to house all the children, but getting enough quality teachers was not as easy. Most student teachers in TTC in the past were trained for the primary school. Thus, to cope with the demand particularly in the secondary schools, graduates of the two local universities at that time were trained in part-time programmes in IE under the teaching cadetship scheme and were awarded the Dip. Ed. on completion. These teacher cadets studied and taught at the same time, assuming two-thirds of a regular teacher's workload during their 18-month cadetship. This makeshift approach lasted until 1980 when all pre-service programmes in IE became full time. In the early years of the TTC, candidates for teacher training did not all possess high academic qualifications. Some had not even completed twelve years of school themselves. In IE, however, only candidates with GCE "A"-Level qualifications were considered for the 2-year Cert. Ed. programme. When IE became NIE in 1991, teacher education in Singapore was re-established within a university framework. The recruitment of teachers was ramped up greatly, and NIE took in students through the PGDE, Dip. Ed. (for non-graduates) and a totally new 4-year degree programme (B.A./B.Sc. (Ed.)) which combined subject matter knowledge and pedagogical knowledge within the same university setting. The intakes for each year up to 2012 were large and comprised about 2000 new student teachers each year from the three programmes with about 500 from the degree programme (Gopinathan 2010; MOE 2016, p. 34). Although the prerequisites for enrolment in the programmes were not brought down, the variation in the quality and experience of the students was great. For example, the "A"-Level results of candidates for the degree programme differed widely. Also, student teachers in the PGDE could come from degree backgrounds as diverse as engineering, law and accountancy. Many of the engineering graduates were to be trained as mathematics teachers. This would have an effect on the mathematics teacher education programmes, as we will read later. With a target of 33,000 teachers for Singapore (MOE 2012), the intake tapered off after 2012 until the present intake of about 1000 a year.

Finally, in this section, we shall describe the evolution of the teaching staff in teacher education. IE staff in 1975 consisted of 5 (4.8%) with doctorates, 34 (32.7%) with masters, 23 (22.1%) with a first degree and 42 (40.4%) non-graduates. As a result of the intentional upgrading of staff and the recruitment of better-qualified staff, in 1982, the staff composition was now 19 (12.0%) with doctorates, 88 (55.3%) with masters, 27 (17.0%) with a first degree and 25 (15.7%) non-graduates (Chin 2010). Believing that "the most important single factor for the quality of education is the quality of the teachers' training" (Barber and Mourshed 2007) and that the quality of teacher training depends heavily on the quality of the teacher educator, NIE continued to recruit strong academics until currently, for example, in the Mathematics and Mathematics Education (MME) Academic Group, all 27 full-time teaching staff hold doctorates. The B.A./B.Sc. (Ed.) programme made a very significant change in NIE staff recruitment. Academics with doctorates in content specializations were now recruited to teach the subject matter knowledge. Since NIE is part of a world-class university, NTU, staff on the professorial tenure track would be required to publish in high-quality journals in their area of specializations. For example again, there are 14 academics in MME who hold doctorates in mathematics across a number of fields such as Graph Theory, Number Theory, Integration Theory, Domain Theory and Statistics. The symbiosis of mathematicians and mathematics educators would be a major factor in an exceptional mathematics teacher training programme.

15.2 Knowledge Base for the Mathematics Teacher

15.2.1 Three Forms of Knowledge for the Mathematics Teacher

What should a mathematics teacher know to be able to teach effectively?

Shulman (1986) in his seminal paper on teacher knowledge contrasted the examinations for Californian elementary school teachers in 1875 and in 1986. While the former emphasized assessment of subject matter knowledge covering topics such as Written Arithmetic, Mental Arithmetic, Written Grammar, Geography, History of the United States, Theory and Practice of Teaching, Algebra, Physics, Constitution of the USA and California, School Law of California, Biology, Reading and Vocal Music, the latter emphasized the assessment of capacity to teach, covering categories such as organization in preparing and presenting instructional plans, evaluation, recognition of individual differences, cultural awareness, understanding youth, management and educational policies and procedures. He pointed out the "absence of focus on subject matter among the various research paradigms for the study of teaching" (p. 6) at that time in the 1980s. He noted the cleavage between content and pedagogy apparent from the emphasis on one and then the other in the 1870s and the 1980s. He then traced the apparent dichotomy to the time in the medieval universities when they were not considered separately but were both considered essential to be a university doctor or "dottore" which means teacher. "The tradition of treating teaching as the highest demonstration of scholarship" (p. 7) was derived from Aristotle who "distinguish[ed] the man who knows from the ignorant man [by] an ability to teach" (p. 7).

Thus, we have at least two forms of teacher knowledge, viz. "content knowledge" (i.e. subject matter knowledge) and "general pedagogical knowledge" (Shulman 1986, p. 9). General pedagogical knowledge, by its emphasis in many teacher education programmes today, is well known to include aspects of educational psychology, classroom management, general teaching craft such as questioning techniques and organizing learning in collaborative groups, and assessment. These are generally independent of the subject matter, for example, assessing the validity and reliability of biology tests and those of mathematics tests are basically the same.

Shulman (1986) then proposed a perspective on subject matter knowledge in teaching that encompassed three kinds: (a) content knowledge, which refers to "the amount and organization of knowledge per se in the mind of the teacher" (p. 9), (b) pedagogical content knowledge, which "goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge *for teaching*" (p. 9) and (c) curricular knowledge, so as to draw from the curriculum and its associated materials almost as a "pharmacopeia" (p. 10) of tools for presentation, exemplifying, remediation and evaluation. For example, with (a), a teacher knows that "-1" is the additive inverse of "1" in the field of real numbers; with (b), the teacher can represent "-1" as a "concrete" object, such as a symbol on a card that when put together with another card labelled "1", *eliminates* both, or as a point on the number line equidistant from "0" as "1"; and with (c), the teacher will avail herself of concrete materials such as AlgeCards, or computer software that teaches operations with negative numbers, if such are available. Figure 15.1 shows how $3 + (-4)$ can be represented in the two different ways mentioned in (b), with the first representation availing itself of materials as in (c). A teacher with good pedagogical content knowledge would know different representations and can decide wisely which to use in different class settings.

The further differentiation of subject matter knowledge by others such as Ball et al. (2008) has shown that proficiency in this area is much more than getting a degree in the subject. For our purpose of explaining the structure and objectives of mathematics teacher education in Singapore, it suffices for us to focus on the main differentiation of subject matter knowledge into content knowledge and pedagogical content knowledge as first perceived by Shulman. Thus, using Shulman's lens, NIE teacher education is organized along three main components: educational studies for general pedagogical knowledge, academic subject for content knowledge and curriculum studies for pedagogical content knowledge.

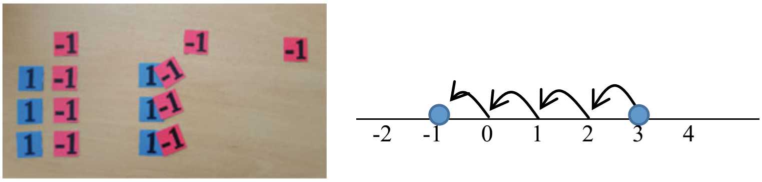

Fig. 15.1 Different representations of 3 + (−4)

15.2.2 Preparing Mathematics Teachers of Substance: Issues for Consideration

We agree with Shulman that "mere content knowledge is likely to be as useless pedagogically as content-free skill" (Shulman 1986, p. 8). Thus, we agree that general pedagogy knowledge is essential for a teacher and that it should be a significant part of a teacher education programme. In this chapter, however, we shall leave further exposition of this component and instead focus on subject matter knowledge and its two manifestations, content knowledge and pedagogical content knowledge. From here on, we shall only discuss the preparation of mathematics teachers. Shulman continued his work with like-minded colleagues, and together they asserted that an examination of subject matter knowledge of teachers is important to teacher educators (Grossman et al. 1989). To prepare "teachers of substance", they argued that "teacher educators must share the responsibility for the transmission of subject matter knowledge to prospective teachers" (p. 24).

The first reason they gave was that it had become "increasingly clear that … [teacher educators] can no longer assume that the subject matter component of teacher preparation is fulfilled by undergraduate coursework in other departments" (Grossman et al. 1989, p. 24). The realities that they state to support this assertion resonate even in Singapore today. They state that within an academic major, in different universities and even in the same university, requirements vary for different specializations. In Chap. 5 of this book, Ho et al. (2018) interviewed a number of professors of mathematics in three Singapore university mathematics departments and they concurred that "content reduction is one of the most significant changes that took place at the university level for undergraduate mathematics degree programmes" (p. 94). It would seem that the training in mathematics rigour would be adversely affected and mathematics graduates may have problems with analysis concepts such as convergence and limits, which would manifest themselves in secondary school calculus.

Grossman et al. (1989) also observed that the overlap between the content of courses at the university level and at the school level is "tenuous at best" (p. 24). For example, Geometry is hardly offered in undergraduate mathematics courses nowadays, though it is an important topic in the school. In the Singapore universities, it is also possible for a mathematics major to avoid courses in Statistics. Although one can possibly learn on the job when required to teach Statistics in the schools, this ability for all teachers cannot be taken for granted. It would be better that the deficiencies are addressed in the teacher education programme.

Finally, Grossman et al. (1989) point to the difference between content knowledge and pedagogical content knowledge. It is highly unlikely that future teachers would learn pedagogical content knowledge from their professors in their undergraduate courses if the content department were divorced from the teacher education department, which is the case in the majority of universities. As it is quite impossible to convince the mathematics professor in the mathematics department to focus a little on pedagogical content knowledge, it would seem more feasible that teacher educators take up the mantle to teach pedagogical content knowledge and to fill up deficiencies in content knowledge.

The establishment of NIE as an autonomous institute of the NTU in the 1990s gave Singapore teacher education an exceptional opportunity to plan programmes that take into account the new understandings of teacher knowledge from Shulman and his colleagues. NIE's placement within a university setting allowed it to design a bachelor's programme that combined a subject matter specialization and a teaching certification. Within the preparation of mathematics teachers, faculty with doctorates in mathematics and with doctorates in mathematics education were recruited in equal numbers to enable a symbiosis of expertise crucial to actualizing a holistic learning environment for the three forms of knowledge. It helped also that many faculty members with doctorates in mathematics were also qualified school teachers before.

The majority of pre-service teachers would still go through the PGDE programme. They would come into NIE with degrees from other universities. Some would have gone through undergraduate mathematics programmes that did not overlap well with the mathematics content in schools. Of greater concern would be the fact that some of the student teachers were not mathematics majors. Some had engineering degrees and had been accepted by the Ministry of Education to teach mathematics when there was a need to quickly ramp up the number of secondary school teachers. These student teachers lacked content knowledge, but there was no time in the one-year PGDE programme to conduct content upgrading. A stopgap measure was implemented to at least raise the awareness of the student teachers about their own level in content knowledge by making them take a School Mathematics Mastery Test. The inadequacy of the PGDE programme with regard to content knowledge would be partly ameliorated with the provision of in-service content courses as well as a postgraduate Masters in Science (Mathematics for Educators).

The final consideration was for primary school teachers who had to teach mathematics but who were not mathematics majors. Keenly aware that some of these teachers actually disliked mathematics, could only follow procedures in mathematics and thus would only teach procedures subsequently, a component called Subject Knowledge was included in the programme for primary school teachers. This component resonated with Ma's (2010) assertion that elementary mathematics is a "field of depth, breadth, and thoroughness" (p. 122).

Each of the programmes mentioned in the paragraphs above—their motivations, evolutions, structures and implementations—will be fleshed out in the following sections of this chapter. The sections will follow the distinction between pedagogical content knowledge and content knowledge. All these developments are a result of NIE's aspiration to have programmes that would prepare mathematics teachers of substance.

15.3 Pedagogical Content Knowledge (PCK)

15.3.1 Pre-service PCK

We begin with curriculum studies (CS) in mathematics, which is designed to give student teachers the pedagogical skills in teaching mathematics in Singapore schools from the perspective of PCK. This component is offered in the pre-service PGDE, B.A./B.Sc. (Ed.) and the Dip. Ed. programmes. In the CS Mathematics courses for the Dip. Ed. programme, student teachers specialize in the methodology for teaching mathematics at the primary level, while the CS Mathematics courses for the B.A./B.Sc. (Ed.) programme prepare student teachers to teach mathematics at either the primary or secondary/junior college level. The CS Mathematics courses for the PGDE programme also offer courses for teaching at either primary or secondary/junior college level. In this section, we describe the key characteristics of our CS primary and secondary mathematics courses, namely (i) a common foundation, (ii) mathematical problem-solving and school mathematics topics, (iii) relevance to Singapore schools and (iv) responsiveness to the changing educational landscape.

15.3.1.1 Common Foundation

Key common course content. One of the changes in the primary CS Mathematics courses as a result of the curriculum review of 2004 was keeping the contents of the CS Mathematics courses in the different primary programmes similar (Lim-Teo 2009). Samples of course outlines for the various CS Mathematics courses offered at NIE, Singapore, over the last decade (from 2007 to 2017) were analysed, and the analysis shows key common course content and assessment modes across CS Mathematics primary and secondary courses in the Dip. Ed. programme, B.A./B.Sc. (Ed.) (Primary), B.A./B.Sc. (Ed.) (Secondary), PGDE (Primary) and PGDE (Secondary) programmes. The common contents of the CS Mathematics courses include the Singapore mathematics curriculum, general psychological theories for learning mathematics, lesson planning, mathematical problem-solving, test construction, student misconceptions or errors and the teaching of the various school mathematics topics spelt out in the Singapore mathematics curriculum.

Most of the CS Mathematics courses are conducted through lectures and tutorials, supported by a technology-enhanced environment through the blackboard course management system (Wong et al. 2012). E-lectures on various mathematics topics (e.g. matrices, probability, real-life applications of mathematics) and topics, e.g. van

Hiele Theory in the CS Secondary Mathematics courses, encourage student teachers to “learn things for themselves”. E-learning is also implemented in some of the CS Mathematics courses to encourage student teachers to be self-directed learners.

Key common assessment modes. Student teachers taking the CS Mathematics courses as described above are assessed through common key assessment components such as design of mathematics lesson plans, PCK written test, design of mathematical problem-solving task, test construction and use of technology in the teaching and learning of mathematics. In addition, one of the assessment components requires student teachers to teach an assigned concept, algorithm, word problem, etc. (e.g. microteaching). This is usually followed by reflections on their teaching. Having such a range of assessment modes in the CS Mathematics courses exposes the student teachers to the realities of teaching.

Anchoring the above CS Mathematics courses in a common foundation is important because it provides a common language for the mathematics education community in Singapore. Furthermore, the common foundation was carefully constructed to provide our pre-service teachers with the critical capabilities supported by sound theoretical groundings to make prudent pedagogical judgements and decisions in their mathematics classrooms. Common modes of assessment are used to ensure that key skills (such as pedagogical skills, reflective skills and thinking dispositions) and knowledge (such as knowledge of student, pedagogy, curriculum and assessment) are evenly developed through the CS Mathematics courses across the different programmes.

15.3.1.2 Mathematical Problem-Solving and School Mathematics Topics

Central to the Singapore mathematics curriculum is mathematical problem-solving. The Singapore mathematics syllabus also encourages exposure to problem-solving approaches such as Pólya’s model (Pòlya 1971). The emphasis on mathematical problem-solving and reasoning is reflected in the CS Mathematics course outline, but it may be interwoven into the strands, as well as taught separately in the CS Mathematics courses. The problem-solving approaches and heuristics used are based on Pólya’s model and approaches (Lim-Teo 2009).

Another characteristic of the CS Mathematics courses is that the main bulk of the course is spent on the teaching of various mathematics topics in the school curriculum (school mathematics topic structure). For example, for the primary CS Mathematics courses, topics such as whole numbers, fractions and decimals (classified as Number and Algebra Strand in the Singapore primary mathematics curriculum) are included. This characteristic of the CS Mathematics courses enables student teachers to re-examine their understanding of the mathematics topics, in particular, from the perspectives of students in Singapore mathematics classrooms. Indeed, the school mathematics topic structure provides opportunities for CS Mathematics course instructors to delve more deeply into students’ learning difficulties and diagnosis of students’ errors and explore strategies and teaching approaches to remedy misconceptions specific to each of the school mathematics topics. These learning experiences are crucial for student teachers as they need these experiences to sharpen their abilities to identify and address those learning difficulties and errors. The experiences provided will also develop their capacities to anticipate possible learning difficulties and errors that their students may have and get student teachers "ready" to address those errors when they encounter those situations in their teaching. Wong et al.'s (2012) analysis of primary student teachers' performance in mathematics PCK assessed by the TEDS-M study shows that student teachers may "need more opportunities to learn about ways to deal with pupils' misconceptions in mathematics" (p. 304). In addition to dealing with the common students' errors in the teaching of various mathematics topics, the first batch of student teachers in the 16-month PGDE programme (inaugurated in December 2016) also had the opportunity to complete an error analysis task during their Teaching Assistantship (TA) school stint.

The school mathematics topic structure also provides instructors and the student teachers more opportunities to share, discuss, unpack and reflect on examples of classroom practice specific to the various mathematics topics, and this brings us to the next characteristic of the CS Mathematics courses, that is, direct relevance to the realities of local classroom teaching.

15.3.1.3 Relevance to Singapore Schools

The Singapore model method (MOE 2009), a pedagogical strategy developed by a team of curriculum specialists in the Singapore Ministry of Education, is a unique feature in the Singapore mathematics curriculum, and it is used widely in the Singapore primary classrooms. By unpacking the model method and how the model method can be integrated with the algebraic method to help students formulate algebraic equations to solve problems, student teachers are empowered to translate this pedagogical approach directly into the Singapore mathematics classrooms, thus enhancing the links of the CS Mathematics courses to the local classroom teaching. Pedagogy is partially culturally situated in this sense. Pedagogy is also partially universal (Cai et al. 2009). As such, resources that integrate local experiences and research with international "best practices", for example, Lee and Lee (2009) and Lee (2009), were also published for the CS Mathematics courses (Wong et al. 2012).

To establish the theory-practice links, student teachers are assisted to reflect critically on several aspects of their learning during their CS Mathematics courses. For example, during their microteaching, opportunities are created for student teachers to reflect and relate to the theories and any classroom experiences that they already had (such as in their previous practicums or in their pre-enrolment contract teaching). The critical reflection is vital for student teachers to develop multiple perspectives of teaching and learning mathematics, identify potential challenges and suggest alternative solutions to overcome those challenges. With the TA school stint (in addition to the original final teaching practicum or field-based experience) and developmental nature (spread throughout the entire programme) of practicum for the B.A./B.Sc.

(Ed.) programme, student teachers can tap upon their multiple field experiences to make greater personal sense of the theory-practice links as they engage in critical reflection during the CS Mathematics courses.

A suite of videos of authentic teaching—authentic in that the lessons were conducted in a naturalistic classroom context—is also used in some of the CS Secondary Mathematics courses to facilitate the link between the realities of actual classroom practice and theories gleaned from the courses. “A critical aspect of bridging theory and practice involves strengthening the link between research and practice” (NIE 2012, p. 13). As such, CS Mathematics course instructors have “integrated mathematics pedagogical principles from international research and practices with local contexts and lessons learned from local implementations” (Wong et al. 2012, p. 297).

15.3.1.4 Responsiveness to the Changing Education Landscape

CS Mathematics courses have been revised over the years, with the core foundations still intact, to keep abreast of the rapid changes both locally and internationally. Responsiveness to changes due to recruitment and education initiatives launched by the Ministry of Education, Singapore, will be elaborated below.

As mentioned earlier, student teachers in the PGDE could come from various degree backgrounds. The PGDE secondary student teachers specialize in teaching two subjects at the secondary school level, namely CS1 as the first teaching subject and CS2 as the second teaching subject. Applicants for the PGDE secondary programme taking up CS Secondary Mathematics will be designated either one of the three tracks: CS1 Mathematics, CS2 Mathematics and CS2 Lower Secondary Mathematics. Those in the CS2 Lower Secondary Mathematics “are not required to have studied tertiary mathematics, but they must have good grades in their O-level or A-level mathematics” (Wong et al. 2013, p. 206). They “meet lower criteria than for CS2 [Secondary] Mathematics and they will only be prepared to teach mathematics at lower secondary level because of their lack of mathematical background” (Lim-Teo 2009, p. 53). There was some concern over the mastery of mathematics content for teaching at secondary levels among the teachers when the majority of PGDE secondary student teachers doing CS1 Mathematics were not mathematics majors and when the number of CS2 Lower Secondary Mathematics student teachers grew tremendously. To address this concern, a School Mathematics Mastery Test (SMMT) was introduced in 2003 to provide student teachers the opportunity to be aware of their current state of mathematical knowledge so as to start them on self-improvement for mastery of secondary school mathematics content. There was also a concern about primary teachers’ subject understanding (Lim-Teo 2009, p. 64). A Subject Knowledge (SK) component was introduced in some programmes to address this problem (Lim-Teo 2009).

CS Mathematics courses have adapted over the years to remain relevant to the needs of teachers. The adaptation was necessary for our student teachers to acquire the skills, knowledge and disposition to implement the initiatives by the Singapore Ministry of Education (MOE). For example, one of the key changes in the 2012

Singapore secondary mathematics curriculum is the explication of Problems in Real-World Context (PRWC). As such, PRWC was included as one of the topics in the CS Secondary Mathematics courses to prepare student teachers to implement PRWC tasks in their mathematics classrooms. Guest lectures by MOE, Curriculum Planning and Development Division, were also arranged for our student teachers on topics such as AlgeDiscs and Algebars, and Learning Experiences—new features in the 2012 Singapore mathematics curriculum.

15.3.2 Postgraduate PCK

It is well known that pre-service training cannot be sufficient for the needs of the teacher. In the first place, there is not enough time to cover all aspects of teaching. In addition, learning without the information of actual practice is deficiently one-dimensional. We close this section on PCK with a description of NIE's postgraduate programme to further develop teachers' pedagogical content knowledge through reflective practice and research.

The Master of Education (Mathematics) programme, or M.Ed. (Maths) for short, is the mathematics education specialization within the NIE wide Master of Education programme, designed for mathematics educators in Singapore schools and other mathematics education professionals. The main aim of this specialization is to develop within the participants the capacity to reflect deeply upon their own mathematics instructional practices, so as to prepare them for career development in leadership positions in schools. The duration of the programme is between 2 and 4 years for those participants studying part time, and between 1½ and 2 years for those studying on a full-time basis. To graduate from the programme, participants need to earn 30 academic units, or AU's for short, where 1 AU corresponds to 1 h per week of instruction over a 13 week semester, and most of these academic units are gained from taught courses. For the more academically inclined participants, the programme also serves as an induction into contemporary mathematics education research, by exposing them to the latest scholarly and professional work in this area. For this purpose, candidates can opt to replace two taught courses with a 6 AU dissertation, running over multiple semesters, which is an independent piece of research work conducted under the guidance of a supervisor appointed from the faculty.

Since its inception, the M.Ed. (Maths) programme has achieved very high levels of success, and the reader is directed to Lim-Teo (2009) and Tay et al. (2017) for more detailed descriptions of its history and structure. Nevertheless, it is worth highlighting here some of the more significant factors which have contributed to its success. First, with the exception of a small number of foundational courses which introduce the participant to educational research in general, the great majority of the course studies are in mathematics education. Second, all the courses are taught by members of the faculty who are themselves active researchers in the fields of mathematics and mathematics education, and who are often also qualified school teachers, which means that they are able to pass on to participants the latest theories and perspectives.

Third, the great majority of participants are themselves active teachers of mathematics in schools, who often come to their lessons directly from their own classrooms and who therefore possess very high levels of motivation to apply what they are learning in the M.Ed. (Maths) programme to improve their own day-to-day instructional practices. Thus, the participants and faculty together form a community of focus similar to the "teaching research groups" engaged in "intensive study" which Ma (2010) identified as being a principal characteristic contributing to the strength of mathematics education in China.

Despite these successes, a growing imbalance began to emerge between the content and pedagogic strengths of the participants, just as Shulman (1986) warned, albeit seen from the following more positive perspective. Specifically, as NIE became better and better at imparting strong pedagogic skills to its pre-service teachers, exigencies of deployment during the "expansion" era meant that it became feasible to deploy into classrooms teachers whose content background did not necessarily match the subjects they were expected to teach. For example, it became very common in secondary schools to have mathematics taught by teachers who were not mathematics majors, or even majors in closely related subjects such as Physics or Chemistry, but rather in other quantitative disciplines such as engineering or economics, and such teachers naturally sought to upgrade their skills in programmes such as the M.Ed. (Maths). A second issue was that roughly half the participants were primary school teachers, and given the generalist nature of primary school teaching, it was common to find such teachers who despite being graduates and having a strong interest in mathematics were non-science majors. To partially address these issues, some entry restrictions into the M.Ed. (Maths) programme were imposed, namely that participants should have taken at least two mathematics courses during their undergraduate studies and that they should be active teachers of mathematics. Since the whole point of the M.Ed. (Maths) programme is to encourage teachers to upgrade themselves, these restrictions were necessarily quite mild, and other more positive measures to address the content weakness of participants were taken as follows.

The first positive measure was to include alongside the elective courses in mathematics education an equal number of elective courses which were intended to be a synergy of mathematics content and pedagogy. Generically entitled "X and the Teaching of X", these courses covered most of the subjects taught in schools up to secondary level, including Arithmetic, Algebra, Geometry, Statistics and Discrete Mathematics. Ideally taught by a pair of faculty, one staff member specializing in content and one specializing in pedagogy, the aim of these courses was to simultaneously reinforce content mastery, while at the same time drawing out the pedagogic implications for classroom practice. While the intentions behind the crafting of these courses were sound, in practice they were challenging to teach, since to succeed well required very high levels of cooperation between the content and pedagogy staff involved. Unfortunately, even to arrange for a pair of staff to co-teach was often quite difficult, and the exigencies of staff deployment often constrained the assignment of a single member of staff to teach the course in its entirety. Despite the best of intentions of the staff assigned, this inevitably resulted in a weakening of the synergy in the direction of the area of specialization of the staff member involved, be it content or pedagogy.

The second positive measure was to include alongside the core course in mathematics education a stand-alone content course entitled "Fundamental Concepts in Mathematics". Compulsory for all participants in the M.Ed. (Maths) programme, the principal aim of this course was to level up the participants' mastery of fundamental concepts, so as to provide a foundation for the later "X and the Teaching of X" courses. From the outset, this course took a broad historical perspective and introduced participants to the evolution of the key mathematical ideas and concepts underpinning primary and secondary school mathematics. This broad historical perspective, which contributed significantly to the popularity of the course, also made it rather challenging for one staff member to teach alone, so for many years it was co-taught very successfully by a pair of staff, both in content. Nevertheless, despite its popularity, this breadth made it very difficult to cater to the needs of both primary and secondary school teachers, since historically these relate to quite different eras in the history of mathematics. Therefore, as part of the recent major restructuring described below, a reluctant decision was taken to break up this course and instead to reincorporate its components into the newly recrafted "X and the Teaching of X" courses.

Starting in 2016 the M.Ed. (Maths) programme was subject to major reviews, one internally by the academic department for the purposes of quality improvement and the other externally as part of an NIE wide programme review. The main aim of the external review was to concentrate the 30 AUs required to graduate into a smaller number of larger courses (essentially 4 AU as opposed to 3 AU) so as to make it feasible for full-time participants to graduate within one calendar year. One of the major aims of the internal review was to better serve the particular needs of primary school teachers by creating a "primary track" within the M.Ed. (Maths) programme. This is not a stand-alone programme, but rather consists of a sequence of courses which, despite being open to all participants, are marked out as being of particular relevance and interest to those teaching at primary level. Another aim of the internal review was to further strengthen the content components of the programme, by making a renewed effort to allow the "X and the Teaching of X" courses to fulfil their potential. This consisted of reversing the previous practice, that is, first crafting the courses and only then searching for a pair of staff willing to co-teach it; instead, pairs of staff were hand-picked for their proven ability to cooperate well, and then together they recrafted the course description with that cooperation built in from the very start.

15.4 Content Knowledge (CK)

15.4.1 Academic Subject (AS)

The Academic Subject courses, abbreviated as AS courses, are a set of courses offered to degree programme students who either major (AS1 Mathematics) or minor in mathematics (AS2 Mathematics).

The AS1 Mathematics students comprise two groups in the B.Sc. (Ed.) programme whose first teaching subject is mathematics: specialists in teaching primary school mathematics and specialists in teaching secondary school mathematics. Since their first teaching subject of the AS1 Mathematics students is mathematics, these students are expected to have a larger base of subject matter knowledge. Thus, AS1 Mathematics students are required to complete a total of 17 courses for AS1 during their four years of tertiary mathematics education. AS1 student must complete a core set of compulsory courses: 4 in the first year, 6 in the second year and an academic exercise at the end of the fourth year. The remaining courses are all electives. In contrast, AS2 Mathematics students will be deployed to teach secondary mathematics as their second teaching subject. (We note that these students may have their AS1 subject in either the Arts or the Sciences; their programmes will then be respectively, B.A. (Ed.) and B.Sc. (Ed.).) Consequently, AS2 Mathematics student teachers are only required to read tertiary mathematics in their first year of study, which comprises four compulsory courses. Table 15.1 displays the courses to be completed for the degree requirement of the AS students under the respective tracks.

A look at the range of the mathematics courses that AS Mathematics students are required to take in the B.A./B.Sc. (Ed.) programme, as shown in Table 15.1, shows that the degree programme at NIE has courses with similar, if not identical, titles offered in traditional mathematics programmes at other universities worldwide. These courses give a comprehensive coverage of the "subject matter component of teacher preparation" (Grossman et al. 1989, p. 24) which may not be fulfilled by a student teacher taking the PGDE route. At this point, we highlight a question often raised by student teachers in the primary track—"Why do we need to study such difficult mathematics which will not be used in primary school?" To this question, we justify as follows. Firstly, as a subject major one is expected to possess the disciplinarity of that subject. For example, an English Literature major must know Shakespeare even though Shakespeare is never taught in primary school; a mathematics major ought to know the notion of infinite countability even though primary school children rarely count beyond a million. Secondly, there are aspects of mathematics study that undergird the content per se, such as problem-solving disposition, rigour and the ability to read new mathematics. These disciplinarity aspects need to be transferred to the primary school students; indeed, a non-mathematics major will most likely emphasize on the procedural aspects of mathematics, while a mathematics major is likely to engage students in problem-solving, problem posing, understanding symbols (reading mathematics) and some rigour.

Table 15.1 Course structure for AS1 and AS2 Mathematics in the B.A./B.Sc. (Ed.)

Year	Courses	Remarks
Year 1	Linear Algebra I	Compulsory Year 1 core subjects common to both tracks
	Calculus I	
	Finite Mathematics	
	Number Theory	
Year 2	Linear Algebra II	Core for all AS1 Mathematics students
	Calculus II	
	Statistics I	
	Computational Mathematics	
	Differential Equations	
	Complex Analysis	
Year 3	Special Topics in Mathematics I	AS1: Three electives
	Statistics II	
	Real Analysis	
	Modern Algebra	
	Modelling with Differential Equations	
	Statistics III	
	Combinatorial Analysis	
Year 4	Academic Exercise: Mathematics	AS1: Core course
	Special Topics in Mathematics II	AS1: Three electives
	Statistical Theory	
	Applied Statistics	
	Techniques in Operations Research	
	Mathematical Programming and Stochastic Processes	
	Metric Spaces	
	Galois Theory	
	Geometry	
	Advanced Mathematical Modelling	

Crucially, the B.A./B.Sc. (Ed.) programme at NIE offered to mathematics student teachers is fundamentally *distinctive* in its design and implementation that targets at certain aims, which we now elaborate.

In his efforts to modernize mathematics education in Germany during the early 1900s, Felix Klein, one of the leading mathematicians of his time, deplored what he termed as the "double discontinuity"—a problematic experience faced by mathematics students as they move from high school to university, and then back again to the profession of school mathematics teachers:

> The young university student found himself, at the outset, confronted with problems, which did not suggest, in any particular, the things which he had been concerned at school. Naturally, he forgot these things quickly and thoroughly. When, after finishing his course of study, he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way; and since he was scarcely able, unaided, to discern any connection between this task and his university mathematics, he soon fell in with the time honoured way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching. (Klein 1908/1932, p. 1)

The first "discontinuity" highlights the well-known problems of transition which mathematics students struggle with as they learn mathematics at the tertiary level for the first time (Gueudet 2008; Thomas 2008). The major problems encountered by undergraduate mathematics students at this stage concern reading and writing mathematical texts, understanding and constructing rigorous mathematical proofs. This situation is not helped as the student struggles to simultaneously grapple with the new content knowledge and the aforementioned deficiency in skills. The second "discontinuity" concerns the difficulties experienced by mathematics teachers in transferring "academic knowledge gained at university to relevant knowledge for a teacher" (Winsløw and Grøbæk 2014). Cognizant that the "double discontinuity" continues to plague student teachers of our present age, MME designs and implements the AS courses to achieve two overarching objectives. Firstly, the degree programme should facilitate a smooth and effective transition into tertiary mathematics for the AS Mathematics students. Secondly, learning opportunities should be available to AS Mathematics students allowing them to look at the school mathematics from a higher standpoint, and crucially, together with training received in the CS courses, these teachers-to-be will be equipped with the ability to transfer the mathematical content knowledge gained from the AS courses to relevant pedagogical content knowledge for a teacher.

As mentioned earlier, MME has a natural advantage over many other mathematics departments in terms of the composition of its academic staff; i.e. the staff members comprise both mathematicians and mathematics educators (indeed, there are those who are both!). The synergy between the mathematicians and the mathematics educators, simple as it sounds, manifests as follows. Mathematicians identify those "higher standpoints", that is, those mathematical concepts and results at the tertiary level that can impact strongly on the understanding of mathematical concepts taught and learnt in schools, while mathematics educators inform mathematicians of the salient pedagogical theories that underpin the learning of mathematics at the tertiary level. Zeroing in on the vexing problem of the "double discontinuity", MME looked to the pedagogical findings gleaned from rigorous educational research concerning the problems encountered in teaching tertiary mathematics. A work of Alcock and Simpson (2009) was brought to the attention of the Academic Group, and therein it was reported that mathematics students at high school take a number of years for "development from an action through a process to an object conception before they begin to use the concept at university" (p. 22). Furthermore, at the university level "a similar development is necessary, but a much shorter time period is available" (ibid., p. 22). This research finding alerted that there was simply not enough time to teach reading and writing, understanding and construction of proofs, when the knowledge content needs to be covered *concurrently*.

For this reason, MME saw that a possible solution to the identified problem would seem to be a total curriculum review that rightly involved all the academic staff who were teaching the curriculum. Tyler (1949) proposed a basic model that provides the needful framework for such a major curriculum review. In short, Tyler's model demands, first, that the objectives of the curriculum are positioned in a matrix with the existing modules of the programme so that the design can ascertain which cell in the table will be activated, i.e. which module can be used to attain the objective (Tay and Ho 2016).

In this curriculum redesign that began in July 2015, MME considered carefully the trinity of learning objective, learning experience and assessment. Accordingly, learning objectives were classified under six domains: (1) content, (2) cognition, (3) problem-solving, (4) computation, (5) communication and (6) disposition. Pertaining to "cognition", one of the objectives read as "At the end of the degree mathematics programme, the learner should be able to read mathematical text or language with understanding". This learning objective was further unpacked into four sub-objectives: the ability to read (i) a definition, (ii) a theorem, (iii) a proof and (iv) a mathematical text, with corresponding learning experiences and assessments. In the actual implementation, for instance, the learning experiences for reading a definition was realized by the requirement that "given a definition, come up with examples/non-examples", "compare related definitions and identify the differences" and "come up with special cases" and "visualize definitions". Correspondingly, assessments were 'given a new definition (which may not be covered in the course), determine whether a given object satisfies the definition; come up with an example/non-example', and 'given several definitions, determine whether some given objects satisfy each of the definitions'.

For each learning sub-objective, courses across the four years in the programme (see Table 15.1) were then designated to meet it. For example, "be able to read a definition" was designated to Linear Algebra I (Year 1, Semester 1), Number Theory (Year 1, Semester 2), Calculus II (Year 2, Semester 2) and Complex Analysis and Linear Algebra II (Year 2, Semester 2). The philosophy behind such a curriculum design and its implementation is that things are delivered in bite-size which are reinforced over a period of time. Each lecturer took ownership of enacting the curriculum with the aim of achieving the designated learning sub-objectives while ensuring a comprehensive coverage of the required content—each sub-objective would be covered in five courses over four semesters.

Recently, a systematic review of the curriculum implementation for the Year 1, Semester 1 (July 2016 Semester), was carried out with promising findings. For instance, Calculus I students performed extremely well in answering traditionally difficult ε-δ/M definitions for continuity and limits. Importantly, the same report underscores that "success over four years has to be a team effort (of the entire Academic Group of MME) with each doing his or her part in line with a well-thought-out curricular plan" (Ho et al. 2017).

We now turn to elaborate how the curriculum redesign addresses the second "discontinuity", i.e. the difficulties of transferring the mathematical content knowledge to the relevant pedagogical content knowledge of a mathematics teacher. Under the domain "content" that was identified in the classification of learning objectives in the degree programme, it was stressed that the NIE Degree Programme will equip the student teacher with a solid foundation of school mathematics and the canons of undergraduate mathematics. Specifically, this learning objective was unpacked into three sub-objectives: (i) possess deep understanding of all the different topics in school mathematics (up to A-Level Mathematics) from a higher standpoint (by the end of Year III), (ii) possess fluency in carrying out standard mathematical procedures in school mathematics and (iii) possess the mathematical background practice and the mathematical rigour needed for them to be able to proceed to postgraduate studies in mathematics. To help the reader better understand how the "content" sub-objective (i) was realized in the degree programme, we extract an episode of a lesson in Calculus I as an illustration. This episode was contributed by the lecturer who taught Calculus I in the July 2016 Semester in an interview.

In Week 8 (after the definition of the differentiability of a function and the derivative of a differentiable function have been taught), the lecturer introduced the concept of turning point on the curve $y = f(x)$ of a continuous function f defined over some open interval I, whose definition is given below:

> **Definition (Turning points).** Let f be a continuous real-valued function defined on an open interval I, and $a \in I$. A point $(a, f(a))$ on the curve C: $y = f(x)$ about which there is no change in sign for $f(x) - f(a)$ for some open deleted neighbourhood of a is called a *turning point* of C.

At this juncture, the students were tasked to give two examples of turning points on the graph of continuous functions. When asked if it is necessarily true that the gradient of the graph at a turning point is zero, *all* the students responded positively and affirmed their claims using the examples they came up with. It is worth noting that the students' examples were all polynomial functions which are not only continuous but also differentiable on the interval of definition. Without giving further comments on the students' examples, the lecturer showed two examples. The first one was the maximum turning point (0, 0) on the parabola $y = -x^2$, and the second one was the minimum turning point (0, 0) on the graph of $y = |x|$. The students validated the two given examples against the above definition of a turning point. The students were then asked if there were any confusion or conflict with their pre-existing understanding of the concept of turning points. To this question, several remarked that they were taught in A-Level Mathematics (during their times as junior college students) to determine the turning points of a curve by setting $\frac{dy}{dx} = 0$ and to solve the equation for the x-coordinate of the turning point(s). The students realized that "the function may not be even differentiable at the turning point, let alone requiring its derivative to be 0 [pointing at the second example given by the lecturer]". The students then quickly responded by saying that "it is not *necessarily* the case that a turning point be stationary (a term they have used since high school)". In response to their remark, the lecturer then displayed the following theorem and its corollary:

> **Theorem.** Let f be a continuous function defined on an interval I and $a \in I$. If $(a, f(a))$ is a turning point on the curve C: $y = f(x)$, then $f'(a) = 0$ if the derivative at a exists.
>
> **Corollary.** If f is differentiable on an interval I, then every turning point of the curve C: $y = f(x)$ in I is a stationary point (i.e. a point at which $f'(a) = 0$).

At this point of the lesson, one of the students responded as follows:

> Now I realise why the A-Level method for finding turning points work … these [functions] are restricted to only the differentiable ones …

This episode illustrates how student teachers in their AS courses were guided to acquire for themselves a deeper understanding of the school mathematics at a higher standpoint. In this case, the *implicit* requirement that the functions involved in the determination of turning points via differentiation (an A-level mathematics technique) are restricted to only the differentiable ones is the salient CK required of the teacher.

What we have elaborated concerning the AS curriculum review, as guided by Tyler's framework, was just one of the several instances of synergy between the mathematicians and the mathematics educator colleagues in MME. One notable teaching innovation, among many others, that MME implements in the AS curriculum is that of Mathematics Problem-Solving (MPS). For more information on the role of MPS in NIE Degree Programme, we refer the reader to Chap. 7 of this book.

15.4.2 Subject Knowledge (SK)

The Subject Knowledge courses, or SK courses for short, are a set of three courses covering all of the content taught at primary level, namely Number Topics (arithmetic and number operations), Geometry Topics (properties of figures and mensuration) and Further Topics (Algebra, discrete mathematics and elementary statistics). These courses are offered to primary student teachers across all programmes (diploma, degree and PGDE) except that degree students majoring in mathematics take an abbreviated version, and the reader is directed to Lim-Teo (2009) and Tay et al. (2017) for comprehensive accounts of the structure and evolution of these courses at a programme level. The main aim of the SK courses is to raise the level of understanding of the mathematical foundations of these topics at primary level among pre-service teachers so that they can themselves go on to teach these topics both confidently and correctly. A secondary aim is to ensure the smooth and efficient delivery of the curriculum studies, or CS, courses, uninterrupted by any prior content weakness among the pre-service teachers, and for this purpose the SK courses are normally scheduled to take place immediately before the corresponding CS course. Finally, as pointed out by Lim-Teo (2009, p. 65), although the CS staff at NIE are exemplary teachers, what they are teaching is pedagogy, not content, making it hard for trainees to model their own classroom practice directly on what they experience in the CS courses. So, a final unintended but quite significant benefit of the SK courses is that they are an opportunity for the pre-service teachers to experience for themselves what it is like to learn mathematics in an exemplary pedagogic environment.

Due to the generalist nature of primary school teaching in the USA, where all primary teachers are expected to be able to teach all of the core subjects, the market in the USA for textbooks generically titled "Mathematics for Elementary School Teachers" is very large. For the sake of efficiency, therefore, when the SK courses were first being offered at NIE, it was decided to adopt one of these texts, and of the many titles available the one authored by Billstein et al. (2001) was chosen as the most suitable. As a commercially produced text, this brought many immediate advantages, such as its comprehensive coverage (one text covered all three SK courses), reliability (it was then in its seventh edition) and quality (especially the figures and use of colour printing). It also brought several minor inconveniences, principally the weight of the text (many students resorted to tearing it into three parts), the cost (trainees had to purchase their own copy) and also the peculiar US habit of retaining imperial units (Singapore like the UK having converted to the metric system long ago). A major issue, however, was that the level of rigour, while appropriate for the US market, fell below what it was felt student teachers at NIE were capable of, since they are able to specialize to a greater degree than primary teachers in the USA, so after a few years of careful use this textbook was set aside. Although more advanced texts existed, none was deemed a suitable replacement, so the decision was taken to produce a set of notes in house, which as well as addressing the issue of rigour would also neatly resolve the minor issues of weight, cost and the use of non-metric units.

These new notes were written by the content staff, all of whom possess doctorates in mathematics content, divided into groups according to areas of specialization (in fact one staff member was a researcher expert in both content and pedagogy) so the levels of rigour were unimpeachable. Once these new notes were put into use, however, the danger highlighted by Tay et al. (2017) soon became apparent, which in fact afflicted a whole generation of authors in the 1960s and the 1970s (Keedy 1969; Griffiths and Hilton 1970; Campbell 1970; Hunter et al. 1971; Mendelson 1973) who attempted to write mathematically rigorous accounts of primary topics. This danger is the very large gap which exists between the modern language of mathematical rigour, which is generally abstract, axiomatic and deductive, and the language of the primary mathematics classroom, which is concrete, constructivist and inductive. The practical outcome, therefore, was that although the student teachers were generally able to master the content of the notes without great difficulty, they were unable to make the connection between the SK courses and the primary mathematics curriculum. Accordingly, in 2010 it was attempted to draft an entirely new set of notes, to be used as an alternative alongside the existing set, which would attempt to bridge this gap by adopting a completely different approach. This new approach sought to draw upon "other ways of understanding" the material, more suited to the needs of school teachers, so in a sense constituted a "postmodern" approach to mathematics teaching, based on the following three points of focus.

The first point of focus was historical and observed that the modern language of mathematical rigour did not exist at the time when the originators of most of the topics in the primary mathematics curriculum made their contribution. They must have had an entirely different way of thinking about the material, one which, given the very early date in history when these topics first appeared, was probably much closer to the constructivist mathematics classroom environment. For example, the modern "definition" of Hindu–Arabic numeration, expressed in terms of sums of single-digit multiples of powers of ten, uses the kind of Algebra and exponent notation unknown to the Hindu mathematicians who first evolved this system almost 2000 years ago. So, in the revised set of notes, historical numeration systems were treated in much greater detail than is usually the case in the generic "Mathematics for Elementary School Teachers" texts, especially the reasons motivating the choice of symbology and size of base in the different simple grouping and place value systems studied. Surprisingly, it became clear that ciphered place value systems are a very natural evolutionary step on from simple grouping systems, in the sense that several "obvious" measures taken to make simple grouping systems symbolically and computationally more efficient naturally lead to ciphered place value. In particular, it became evident that the key feature of place value systems is the regrouping property, a point highlighted by Ma (2010), and not the absolute size of the quantities represented; that is, it is essentially a recursive system, in which the distinction between whole numbers and fractions is secondary.

The second point of focus, which follows from the first, is that the historical development of mathematics is far from linear and that in many cases essentially the same mathematical concepts arose independently in several different eras and locations, and expressed in different ways. This suggests an alternative route to Ma's (2010) "profound understanding of fundamental mathematics"; namely that by exposing student teachers to these alternative expressions of the same mathematical concept, they are able to distinguish better between what is essential and what is incidental, than if they only studied a single instance, namely what is taught in the contemporary primary classroom. For example, in the revised version of the notes, heavy emphasis was placed on getting the trainees to become proficient at carrying out the main arithmetic algorithms ($+ - \div \times$) not only in a variety of different number bases other than base ten, but also using a variety of different synthetic symbologies, created by analogy with Hindu–Arabic digits, in some cases created by the student teachers themselves. Initially, it was quite a shock for the student teachers, as they realized quite how much of what they had previously thought of as their "understanding" of arithmetic algorithms was merely rote learning, but by deconstructing their prior knowledge, and then reconstructing it in this more general context, their understanding was deepened. Another example would be the contrast between the classical Greek approach to Geometry, based on the concept of parallel lines, and the modern concept of Geometry based on the notion of a global sense of direction, which although apparently quite different can easily be shown to be logically equivalent. This modern approach is in fact closer to the constructivist classroom environment, such as making use of children's natural belief that the sum of the exterior angles of a polygon is 360°, and the concept of logical equivalence enables trainees to see that the distinction between what counts as a "definition" and what is classified merely as a "property" of a geometric figure is often quite arbitrary or a matter of convenience; e.g. parallelograms can just as easily be "defined" to be figures with opposite equal sides as figures with opposite parallel sides.

The third point of focus, following on from the other two, is to understand that the evolution of mathematics is far from over and that many potential improvements remain to be made, in both the near and the far future. From this perspective, student teachers come to appreciate that often what makes certain parts of mathematics difficult to teach is not a lack of ability on their part, but rather deficiencies in the mathematics itself. For example, it is widely appreciated in the research literature, but hardly at all among teachers that the size of the base ten in Hindu–Arabic numeration is too large and that use of a lower base around base five would be both computationally more efficient and allow for more meaningful symbology. Although a global shift to base five is not a realistic possibility in the near future, reasoning along these lines uncovers ways in which even base ten arithmetic might be improved, such as inventing a symbol for "ten" for use with regrouping. Similarly, the classical Greek approach to Geometry could certainly be improved upon by a shift to a more modern approach based on the concept of symmetry, which is more appealing to children and also aligns better with how Geometry is taught at tertiary level. Symmetry transformations also naturally lend themselves to a dynamic hands-on approach, should share the same advantages over traditional logic-based approaches to Geometry that research has shown to be the case for dynamic geometry software (DGS), and are clearly much closer to the constructivist primary mathematics classroom.

15.4.3 Postgraduate Content Knowledge

Three common traits among countries with top school systems identified in the executive summary of the 2007 McKinsey report are (i) getting the right people to become teachers, (ii) developing them into effective instructors and (iii) ensuring that the system is able to deliver the best possible instruction for every child (McKinsey & Company 2007, p. 1). Regarding (ii), the 2007 McKinsey report highlighted two case studies which exercised deliberate emphasis on professional development for teachers: (1) policymakers in Finland raised the status of the teaching profession by requiring that all teachers possess a master's degree. (2) Singapore policymakers have achieved a similar result by ensuring the *academic rigour* of their teacher education courses, as well as providing all teachers with a substantial number of hours of fully paid professional development training each year.

The Ministry of Education (MOE) in Singapore takes a serious view when it comes to developing its teachers professionally and has since gone beyond the 100 hours of professional development scheme mentioned in McKinsey (2007). Started in 2005, the Professional Development Continuum Model (PDCM) scheme provides graduate teachers with alternative pathways to higher certification. Cognizant of the fact that beginning teacher preparation courses are merely a first step to ensure "getting the right persons to become teachers", MOE intends that the PDCM scheme motivates teachers to keep relevant in content proficiency and pedagogy, thereby "developing them into effective instructors" (McKinsey & Company 2007, p. 1). Here, we go into the specifics of what this statement means concerning the mathematics teachers recruited by MOE and the professional development made available to them. For Singapore, although all mathematics teachers in government schools, by requirement of MOE, are PGDE graduates and thus have received preparation in pedagogical matters concerning their teaching subjects, not all of these are mathematics graduates. The truth is that there is a large number of engineering graduates and/or graduates from other mathematics-related disciplines (e.g. Computer Science) joining MOE as mathematics teachers. Many of such teachers, in their reflections, sounded out their need to deepen their content knowledge in mathematics to be confident classroom teachers. To illustrate this lack of solid mastery of content knowledge, we give some concrete examples of questions, contributed by school students, teachers and NIE mathematics faculty staff, which pose difficulties with regard to the mathematics content of school mathematics taught in Singapore.

- Why is partial fraction decomposition of a rational function always unique? ['O'-Level Additional Mathematics]
- Does definite integration give an approximation to the area under a graph or its exact value? ['O'-Level Additional Mathematics]
- Does $\frac{\overline{X}-\mu}{S/\sqrt{n}}$ always have a t-distribution with $n-1$ (and why not n?) degrees of freedom? ['A'-Level H2 Mathematics]
- How can we produce different integral triplets (a, b, c) so that they always represent the length of the sides of a right-angled triangle? Standard textbooks present $(3, 4, 5)$, $(5, 12, 13)$, $(8, 15, 17)$ and their integral multiples, are these all? ['O'-Level Mathematics]
- How can the value of π be computed? Can we write $\pi = \frac{22}{7}$ and then conclude that it is a rational number? But we know from the textbook that π is irrational. Do we have a contradiction? How can one be certain that π is indeed irrational? ['O'-Level Mathematics]
- Angles taught in primary school are measured in degrees, why do we need to introduce the radian measure as an alternative unit of measurement for angles? ['O'-Level Mathematics]
- The Binomial Theorem seems an isolated topic classified as further Algebra in "O"-Level Additional Mathematics. How often is it relevant to a mathematics learner to be able to find the term independent of x in the expansion of some expression such as $\left(x - \frac{1}{x}\right)^6$? Where else is the Binomial Theorem used? ['O'-Level Additional Mathematics]

- How come differentiation and integration are inverses of each other? When I integrate the derivative of x^2, I get $x^2 + C$, but the constant C need not be zero? [‘O’-Level Additional Mathematics]

Additionally, there are other aspects of work such as curriculum planning and matters related to gifted education that demands understanding of mathematics from a higher and more holistic perspective.

In view of the aforementioned subject matter knowledge demands on mathematics educators, NIE offers a Master of Science (Mathematics for Educators), M.Sc. (MfE) for short, by coursework to in-service teachers with the aim of making mathematics teachers content proficient and, as a result, developing them to be effective instructors of mathematics. For this reason, the programme bears a meaningful name “Mathematics for Educators” and is designed specifically to provide rigorous preparation in advanced mathematics for mathematics teachers. The distinctive feature of this programme is its emphasis of the acquisition of rigorous mathematics at postgraduate level, together with the deep connection with the mathematical topics taught in schools. The programme is backed by the belief that an effective mathematics teacher must be, first and foremost, a teacher who himself/herself must have a *sound* knowledge of mathematics and can teach *correct* mathematics to his/her students.

At this juncture, it is natural to ask: “How is this deep connection with the school mathematics realized in this graduate programme?” The answer to this question lies in the course structure. By factoring in some of the content-related questions raised earlier, some of the courses are intentionally designed to impart subject matter knowledge that can be used to address these questions. Such courses are categorized meaningfully by practising mathematicians (most of whom have prior school teaching experience) as Level 1 courses since they specifically highlight the deeper mathematical structure underlying the topics of Mathematics and Additional Mathematics listed in the Singapore school mathematics syllabi. Table 15.2 shows how Differentiation and Integration in a Level 1 course, Elements of Mathematical Analysis with Applications in the Teaching of Calculus, is aligned to equip the mathematics teacher with higher mathematics to teach the calculus in “O”-Level Additional Mathematics and “A”-Level H2 Mathematics.

After a student has attained a firm foundation in that part of the tertiary mathematics which is related to the mathematics he/she is teaching in schools, he/she would proceed to deepen his/her understanding of the topic further. This is made possible by the placement of Level 2 courses. Often requiring certain prerequisite Level 1 course(s), the mathematics content covered in the Level 2 courses is more abstract and sophisticated in nature. For example, the Level 1 course, Elements of Mathematical Analysis with Applications in the Teaching of Calculus, can lead to its corresponding Level 2 courses: Real Analysis, Functional Analysis and Topology. In order to maintain a high level of academic rigour in the programme, a student is required to complete at least four Level 2 courses.

In order to graduate from this programme, a student must complete a total of ten courses, including the capstone course, Mathematical Inquiry. This capstone course is a mandatory course where a student works with a practicing mathematician in a

Table 15.2 Alignment with school mathematics

Topic	Contents	Connection with school mathematics
Differentiation and Integration	Limits and continuity of functions	Continuity of certain standard functions is assumed, e.g. in the determination of range of functions
	Differentiation	Rules of differentiation (O-Level Additional Mathematics and A-Level H2 Mathematics)
	Rolle's theorem, mean value theorem	Increasing and decreasing functions determined by sign of derivative (O-Level Additional Mathematics and A-Level H2 Mathematics)
	Definite integral as a limit of a sum, fundamental theorem of calculus, indefinite integrals	Riemann sum, definite integral as the limit of Riemann sum, Differentiation and Integration are "inverses" of each other (O-Level Additional Mathematics), finding indefinite integrals of standard functions (O-Level Additional Mathematics), using standard forms, substitution, integration by parts (A-Level H2 Mathematics)

specific research field of Pure Mathematics or Applied Mathematics. In this course, the student is expected to carry out independent study and give an oral presentation on the research work performed. The main intention of this course is for the student to finally be able to put together all his/her learning in an academic exercise. The M.Sc. (MfE) programme distinguishes itself from many other masters by coursework programmes by insisting that the students go through the experience of reading and writing mathematics—which should be the hallmark of a literate mathematics graduate.

Apart from the course structure, the way in which the substance of the programme is delivered is also of paramount importance in realizing the aims and objectives of the M.Sc. (MfE) programme. Again, the pedagogical theories espoused by the MME mathematics educators informed their mathematician colleagues how best to teach the content of mathematics crafted out in the various courses. Notably, useful research findings in problem-solving derived from the Mathematics Problem-Solving for Everyone (MProSE)—a funded research project in MME—guided the lecturers of Number Theory and the Teaching of Arithmetic, Real Analysis and Theory and Applications of Differential Equations in their classroom didactics by taking advantage of the practical paradigm of Mathematics Problem-Solving (Ho et al. 2014). Another example was the innovative use of flipped classroom pedagogy in teaching Topology as an attempt to resolve problems caused by the heavy cognitive load of set theory in learning Topology and to raise learner's motivation in the course (Ho and Chan 2016). The point here to make is that the distinctive symbiosis of mathematicians and mathematics educators in NIE realizes a high-quality professional development programme that benefits in-service mathematics teachers in Singapore schools. Readers may wish to peruse the positive feedback given by students who graduated from the M.Sc. (MfE) programme in Tay et al. (2017, pp. 126–128).

15.5 Conclusion

In this chapter, we have painted in broad brushstrokes the role NIE has been playing in mathematics teacher education in Singapore. By identifying the various knowledge domains relevant to mathematics teacher education, i.e. subject matter knowledge and pedagogical knowledge, we see how NIE has designed and implemented a holistic spectrum of teacher training and professional development programmes that equip both pre-service and in-service mathematics teachers in Singapore with the twenty-first-century competencies specific to teaching and learning of mathematics in schools.

15.5.1 Summary

From our preceding development, the alert reader might have already been aware of two important earlier works completed by NIE mathematics educators that touched on the issues of preparing mathematics teachers in Singapore. The first work by Lim-Teo (2009) elaborates on the evolution and development of mathematics teacher education via the pre-service and in-service programmes offered in NIE in the period from the late 1990s to 2009. The second work by Tay et al. (2017) examines the issue of Mathematics Content Knowledge in connection with mathematics teacher education in Singapore. Our current work not only extends Lim-Teo (2009) in the sense that we give account of the various NIE Mathematics and Mathematics Education courses since 2009 but also follows up on areas of discussion left open by Tay et al. (2017). We close this chapter by summarizing what was accomplished recently and what could be done in the near future to improve the quality of mathematics teacher education provided by NIE; along the way, we also address some of the remarks made in Tay et al. (2017).

Academic Subject. A major curriculum revamp based on Tyler's 1949 framework is currently being implemented for the AS courses, as an AG-wide effort, with focus on six domains: content, cognition, problem-solving, computation, communication and disposition. Each AS course has been assigned to address one or two of these domains along with delivering the subject matter knowledge under its coverage. As the implementation of the new AS curriculum is still ongoing, it is important to stay vigilant to see that the student teachers receiving the training in these six domains are indeed acquiring the targeted domains through the various AS courses.

This can be achieved by constant monitoring of the student teachers' performance in the assessment tasks as well as through the feedback via interviews. Interviews with the AS course lecturers have also been carried out to obtain feedback on the implementation of the revised curriculum.

Subject Knowledge. Tay et al. (2017) remarked that as to "what type of CK (Content Knowledge) is actually needed or appreciated by the teachers remains not completely understood" (p. 128). This statement was made pertaining to the design of SK as NIE's attempt of achieving "better understanding of Mathematics related to elementary Mathematics as conceptualised by Ma (2010)" (p. 128). The proposal put forward in Tay et al. (2017) was then to motivate the SK course with "actual elementary school problems and concepts and building the course materials directly by which prospective and practising teachers can be made to see the relationship of what they are learning to their teaching" (p. 128). In response to this proposal, the SK lecturers collaborated to collate a new set of notes that aims at bridging the gap between the topics covered in SK and the Primary School Mathematics Syllabus. They adopted a "postmodern approach" in the style of writing these notes with three specific points of focus. These points of focus include (1) the *historical nature* of mathematical development: the modern language of mathematical rigour was actually brought into existence when the originators of most of the topics in the primary mathematics curriculum made their contribution; (2) the *nonlinearity* of the historical development of mathematics: simultaneous and independent emergence of new mathematical concepts; and (3) the *ongoing nature* of mathematical development. For the next step, it is of paramount importance to examine to what extent the use of the new in-house notes has improved student teachers' connection of SK topics with the Primary School Mathematics Syllabus. One of the ways is to formally study the effects of SK courses on student teachers who had graduated from the programme and are currently teaching in primary schools. The question to answer is "Does the revised SK course produce better primary school teachers?" In addition, just as the AS curriculum is being revamped based on Tyler's 1949 framework, the SK courses have also been reviewed in Tyler's framework, with special emphasis on getting the learning objectives and their attendant learning experiences and assessments right.

Master of Science (Mathematics for Educators). Tay et al. (2017) suggested that the M.Sc. (MfE) programme "will benefit from a review of its courses from a perspective of Usiskin's teachers' Mathematics, perceived as a generalisation of Ma's profound understanding of fundamental Mathematics" (p. 128). Indeed, at around the time of writing of this present work, the mathematicians who have been directly involved in teaching the M.Sc. (MfE) courses are proposing a restructured M.Sc. (MfE) programme that will be launched in August 2018. This restructuring will result in the number of Academic Units (AU) for each course (i.e. academic credit points earned by students upon completing a course) increasing from 3 AU to 4 AU. An increase in the AU per course not only allows for a deeper treatment of the subject matter in the course but also creates the opportunity for the course lecturer to consider the use of modern computer and video technologies for non-face-to-face instructions. In order to avoid a compromise of breadth for depth of coverage, the collection of courses is streamlined into a coherent body of mathematical knowledge organized along four different strands: Analysis–Geometry, Algebra–Number Theory, Discrete Applied Mathematics and Statistics. Maintaining the crucial feature of connecting advanced mathematics to school mathematics, each strand offers one to two courses at "Foundation Level" with the intention of equipping non-mathematics majors with the essential subject matter knowledge needed to move up higher along the strand, i.e. to read courses pegged at the "Advanced Level".

Recognizing that a majority of the twenty-first-century competencies are skill-based, the restructured programme proposes a new 2 AU course that focuses on the honing of mathematics research skills for mathematics educators. It is hoped that by imparting a wide scope of skills, which includes literature review and citation, problem-solving, reading pertaining to mathematics, typesetting mathematical texts, posing research questions and communication, mathematics teachers become more confident in teaching research-related skills to their students along the disciplinarity of a mathematician.

15.5.2 The Way Forward

This chapter has considered the various kinds of knowledge that make up the knowledge base of a mathematics teacher of substance, and illustrated in detail the various programme designs which NIE implemented to ensure the development of such a knowledge base in pre-service and in-service teachers. While NIE teacher educators can focus on designing top-quality teacher preparation courses and professional developmental programmes, the ultimate responsibility of translating the salient knowledge acquired in these courses into effective enactment of the mathematics curriculum still rests upon the shoulders of the *competent* mathematics teacher—whose characteristics form the subject of discussion in the next chapter.

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Chapter 16
Exemplary Practices of Mathematics Teachers

Yew Hoong Leong, Berinderjeet Kaur, Ngan Hoe Lee and Tin Lam Toh

Abstract In the first section of this chapter, we review the growing literature on "practices", focusing on the purpose of studying teacher practices in actual classrooms in view of its potential in teacher professional development. Following that, we zoom in to the Singapore situation by reviewing other studies here on mathematics teacher practices. In the second section, we describe an ongoing project and its contribution to research on exemplary practices of Singapore mathematics teachers. In the final section, we discuss the usefulness of this review in relation to the effort of building portraits of Singapore mathematics teacher practices.

Keywords Exemplary teaching · Instructional practices · Mathematics · Singapore

16.1 Introduction

Currently, one of the challenges faced by the Singapore mathematics education community, especially those involved in professional development (PD) work for teachers, is that we do not yet have a coherent portrait of exemplary practices from which to take reference when considering areas of mathematics instruction that can be improved. This can result in teachers having the impression that a myriad of disconnected pedagogical innovations—often introduced simultaneously—are running parallel to each other. As an example, one may advocate improvement in questioning

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techniques, another in alternative assessment modes, among others. This can result in the dilution of the effects of PD and ultimately to PD fatigue. In this chapter, we review research on exemplary instructional practices carried out by Singapore mathematics teachers. We begin by drawing from the international literature to clarify the term “practices”.

16.2 Exemplary Practices

“Practices” within the context of education has gained interest as seen from the recent literature (e.g. Lampert 2010). They can be thought of as a set of easily recognizable units of work that mathematics teachers commonly carry out in the classroom. By “practices”, we have in mind the following characteristics—drawn from the international literature (Ball et al. 2009; Hatch and Grossman 2009) and our experiences with mathematics teacher professional development, especially school-based designs of instructional innovations: (1) they are professional units of work that teachers do on a regular basis in school. Seen in this way, “instructional practices” are analogous to “medical practices” or “legal practices”—the work practitioners do as closely identifiable to the image of their respective professions; (2) they are units of work that are sufficiently isolable so as to allow for analysis, rehearsal and honing for improvement. In this sense, “instructional practices” carry the meaning of practices similar to routines—such as in a sports arena (e.g. specific skill drills in football)—that through repeated trials and fine-tuning become increasingly part of the overall work of high-quality teaching.

This leads naturally to the question of the kind of practices that ought to be upheld as exemplary for analysis and learning by teachers. The community is in need of a clear articulation of the standards of exemplary practices that are worth pursuing.

Calls for reforms in mathematics teaching towards exemplary practices are often expressed using contrastive pairs to present the traditional new distinction. Kirshner (2002) observed that, in the USA, “the Learning Principle propounded in Principles and Standards for School Mathematics (NCTM 2000) rehearses the familiar distinction between facts/procedures and understanding as a central guiding principle of teaching reform” (p. 46). Boaler (2002) presented the distinction as one between “skill-oriented” and “reform-oriented” teaching approaches. Other researchers, who avoided association with prescriptive methods, sought rather to describe methods that teachers use in their classroom practices. Some of them have also used contrasting dualistic descriptions, as in “calculationally oriented mathematics teacher” versus “conceptually oriented mathematics teacher” (Thompson et al. 1994) and teaching by “procedural instruction” versus teaching by “inquiry” (Cobb et al. 1998).

There have, however, been calls to move away from this simplistic traditional new dualistic casting of the teaching enterprise. In this alternative perspective, enactment of exemplary practices is not about merely applying a single teaching approach but a variety of instructional methods suited for different contexts and purposes. Quality teaching can be a complex mix and match of different instructional forms whose choice is dependent on various factors and competing priorities. Apart from this eclectic stance in considering exemplary practice, we advocate that a pragmatic dimension is added into the dialectic. In other words, instead of thinking about exemplary practices along universal categories, we should ask the question about exemplary practices for who? Would the images of exemplary practices differ between a mathematics classroom in urban low-resource USA and a mathematics classroom in "neighbourhood" Singapore schools? To deny the need to make these distinctions is to run the risk of divorcing teaching from its context. Teaching is a cultural activity (Stigler and Hiebert 1999). This provides the basis for studying exemplary practices within the cultural context of teaching in Singapore—and we should not be surprised that while there are features that would resonate globally, there would be characteristics distinctive to the Singapore context.

In the next sections of this chapter, we review two studies—one recent and one ongoing—on exemplary practices of Singapore mathematics teachers.

16.3 The Learner's Perspective Study—A Study of Competent Grade 8 Mathematics Teachers

Singapore's participation in the Learner's Perspective Study (LPS) may be marked as the beginning of research with a focus on exemplary practices of mathematics teachers in Singapore schools. The Learner's Perspective Study (LPS) is an international study helmed by David Clarke at the University of Melbourne. It started in 1999 with Australia, Germany, Japan and the USA examining the practices of Grade 8 mathematics classrooms in a more integrated and comprehensive manner than had been attempted in past international studies, in particular the TIMSS Video Studies of 1995 and 1999. The study has several distinguishing features among which are (a) documentation of a sequence of lessons rather than just single lessons, (b) the exploration of learner practices and (c) use of the complementary accounts methodology developed by Clarke (1998) for data collection of classroom practice—an activity where both teacher and students are key participants (Clarke et al. 2006).

Three Grade 8 mathematics teachers, T1, T2 and T3, recognized for their locally defined "teaching competence" and their respective classes of students participated in the study in 2005. These teachers are from a pool of teachers deemed as "experienced and competent", where experience was a measure of the number of years they have taught mathematics in secondary schools and competency was a composite measure of their students' performance at examinations and their performance in class in the eyes of their students. The teachers were nominated by their respective school leaders and the LPS research team in Singapore followed up on the nominations and interviewed the teachers. A strict requirement for participation in the study was that the teacher had to teach the way he/she did all the time; i.e. no special preparation was allowed. Details about the study are reported elsewhere (Kaur 2008, 2009; Kaur and Loh 2009). Data and findings of the study have also been reported in numerous publications (Kaur 2008, 2009, 2010, 2011, 2013, 2014; Kaur and Loh 2009; Kaur et al. 2006; Seah et al. 2006; Mok and Kaur 2006). In the following subsections, some selected data and findings on exemplary practices of three competent Grade 8 mathematics teachers, specifically their instructional patterns, nature of mathematical tasks used and purpose of homework, are presented.

16.3.1 Instructional Approaches

The video records of the 10-lesson sequence for each of the teachers in the study were the main source of the data analysed. On average, there are about six 45-minute lessons allocated to mathematics in the Singapore classrooms per week. For the first phase of the data analysis, a wide-angle lens was adopted. The researchers viewed the video records and located global features related to the patterns of instruction of the three teachers. For the second phase of the data analysis, a close-up lens was used and the grounded theory approach was adopted. An activity segment, "the major division of the lessons", served as an appropriate unit of analysis for examining the structural patterns of lessons since it allowed us "to describe the classroom activity as a whole" (Stodolsky 1988, p. 11).

For the purpose at hand, the activity segments were distinguished mainly by the instructional format that characterized them, although there were other segment properties, such as materials that differed among the various activity segments identified. Six categories of activity segments emerged through reiterative viewing of the video data. These mutually exclusive segments were found to account for most of the 30 lessons, 10 each from T1, T2 and T3. Table 16.1 shows the categories, and Table 16.2 shows the analysis of lesson structure with mathematical content of T2.

Coding of the video data revealed patterns of instructional cycles that consisted mainly of combinations of the three main categories of classroom activity: whole-

Table 16.1 Categories of activity segments

Category	Description
Whole-class demonstration [D]	Whole-class mathematics instruction that aimed to develop students' understanding of mathematical concepts and skills
Seatwork [S]	Students were assigned questions to work on either individually or in groups at their desks
Whole-class review of student work [R]	Teachers' primary focus was to review the work done by students or the task assigned to them
Miscellaneous [M]	A catch-all category during which the class was involved in managerial and administrative activities
Group quiz [Q]	Found in T2's lessons; students solved tasks in groups in a competitive manner
Test [T]	Found only in the lessons of T1 and T3

Table 16.2 Analysis of lesson structure for T2

Lesson No.	Activity segment code	Mathematical content	Instructional objective	Cycle No.
1	[D]	Worked example: $(3x + 2y)^2 - 6x - 4y$	Factorization by grouping	1
	[S]	Practice task: $2x + 4y - 3(x + 2y)^2$		
	[R]	Student wrote answers for practice task on board		
1	[D]	Worked examples: $x^2 - 9$, $y^2 - 1/16$, $9y^2 - 4z^2$	Factorization of expression in the form of difference of two squares	2
	[S]	Practice tasks: $a^2x^2 - 16y^2$, $50x^2 - 2p^2$		
	[R]	Teacher and students worked out practice tasks on board		
	[S]	Practice tasks: $18m^2 - 8n^4$ $(x - 1)^2 - (2x + 3)^2$		
	[R]	Teacher and students worked out practice tasks on board		
	[Q]	Quiz tasks $4x^2 - 25$ $121 - 36x^2$ $49x^2 - 1$ $\pi R^2 - \pi r^2$		
2	[R]	Reviewed solutions of $6p^4 - 24q^2$ $32xy^4 - 2x^2$ $16n^2 + 8ne + e^2$ $49y^2 + 42yz + 9z^2$ $9f^2 + 24fg + 16g^2$	Factorization of expressions by grouping and difference of two squares	1

class demonstration [D], seatwork [S] and whole-class review of student work [R] for the sequences of ten lessons each for T1, T2 and T3. Figure 16.1 shows the segment sequence for the ten lessons each for T1, T2 and T3. Activity segments that served different instructional objectives were separated by a dotted vertical line. In an instructional cycle, the mathematical tasks shared the same instructional objective.

To understand the instructional approaches further, it is necessary to go beyond structural patterns of the lesson sequence. The key features of the classroom talk through which the teachers realized their roles in not just the teaching of mathematics but also in engaging students to learn it are described elsewhere (see Kaur 2009).

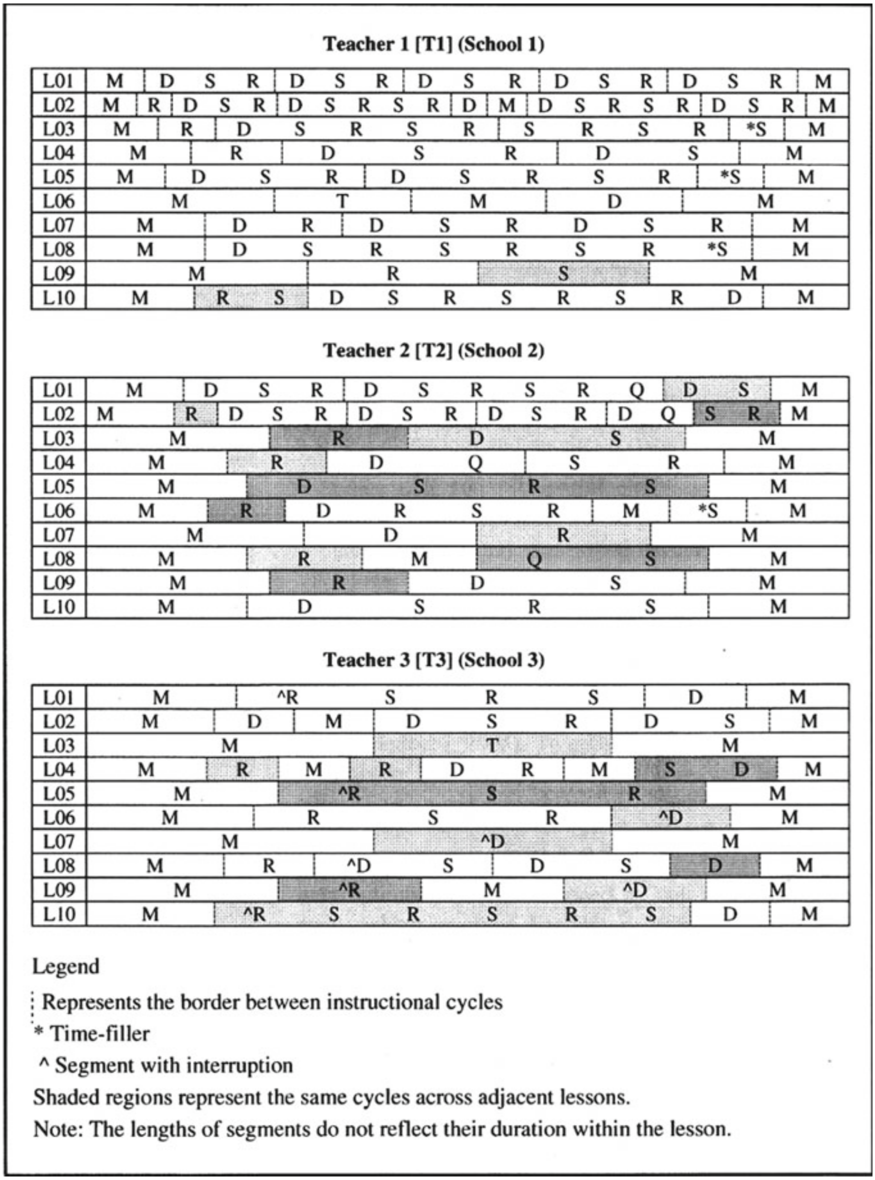

Fig. 16.1 Structural patterns of the lesson sequences of T1, T2 and T3 (Kaur 2009, p. 338)

The wide-angle lens findings show that the pattern of instruction in the Grade 8 classrooms of the three competent teachers was as follows: (1) set the stage for a topic/review past knowledge, (2) present a concept/procedure and show how to work out the solution of a problem, (3) do seatwork, and (4) correct seatwork and assign homework. Lessons were also deemed to be teacher-centred, mainly comprising teacher exposition coupled with student practice. This is often interpreted as "drill and practice" by many who have no other information about the what and the how of the lessons. On the contrary, the close-up lens findings show that lessons consisted of instructional cycles that were highly structured combinations of D, S and R. Specific instructional objectives guided each instructional cycle, with subsequent cycles building on the knowledge. Carefully selected examples that systematically varied in complexity from low to high were used during whole-class demonstrations. There was also active monitoring of student's understanding during seatwork (teachers moved from desk to desk guiding those with difficulties and selecting appropriate student work for subsequent whole-class review and discussion). Most importantly, student understanding of knowledge expounded during whole-class demonstrations was reinforced by detailed review of student work done in class or as homework, and lessons were both teacher- and student-centred.

16.3.2 Nature of Mathematical Tasks

There were three main types of tasks, learning, practice and assessment, used by the teachers. A learning task (Mok 2004) is an example the teacher uses to teach the students a new concept or skill. A review task is a task used by the teacher to review previously learnt concepts and/or skills so as to facilitate the learning of new concepts and skills. Practice tasks are tasks used during the lesson to either illuminate the concept or demonstrate the skill further and tasks the teacher asks students to work through during the lesson either in groups or individually or during out of class time. Assessment tasks are tasks used to assess the performance of the students. Based on these considerations, the tasks used by the teachers, in particular the source of the tasks and aspects of the demands the tasks make on the learners, were studied (see Kaur 2010 for details).

It was found that learning tasks used by the teachers either introduced new concepts and skills, made connections between new and old concepts or skills, or introduced students to knowledge or information that might excite them (Example T3 showing them the history of Pythagoras via the Internet) or explained some of their observations (Example T3 working through the generalized representations of Pythagorean triplets). These tasks were either taken from the textbook or sourced by teachers from their personal resources.

Practice tasks often preceded a learning task, and there was emphasis on "practice makes perfect". They were either taken from the textbook or sourced by teachers from other books. The ones from the textbook were procedural in nature. The textbooks used in the three classrooms adopted the exposition–examples–exercises

10 000	=	10^4	Generate Pythagorean triplets (a, b, c) Such that $a^2 + b^2 = c^2$
1000	=	10^3	
100	=	10^2	
10	=	10^1	
1	=	$10^?$	
$0.1 = \frac{1}{10} = \frac{1}{10^1}$	=	$10^?$	
$0.01 = \frac{1}{100} = \frac{1}{10^2}$	=	$10^?$	
$0.001 = \frac{1}{1000} = \frac{1}{10^3}$	=	$10^?$	
$0.0001 = \frac{1}{10000} = \frac{1}{10^4}$	=	$10^?$	
$0.00001 = \frac{1}{100000} = \frac{1}{10^5}$	=	$10^?$	
Task T1-L01-P3			Task T3-L02-P1

Fig. 16.2 Practice tasks

model (Love and Pimm 1996), and therefore, the exercises of the textbook for the relevant topic formed the bulk of the practice tasks. These tasks were mainly procedural and algorithmic in nature. Tasks from other books were word problems contextualized in some "real-world" context or like those shown in Fig. 16.2 that provided students with opportunities to engage in thinking skills such as comparing, inductive reasoning and systematic listing.

The assessment tasks were taken from past examination papers. These tasks mainly tested the reproduction of facts or procedures, manipulation of algebraic expressions, computations and application of mathematical concepts and procedures to solve simple and routine problems. Bearing in mind the limitations of pencil–paper tests, these items appeared to largely test for concepts and skills.

16.3.3 Homework

All the three teachers assigned their students' homework for instructional purposes. An analysis of the nature and source of mathematics homework was carried out. The details are described elsewhere (see Kaur 2011). It was found that the goal of the homework was to engage students in consolidating what they were taught in class and prepare them for upcoming tests and examinations. The homework only involved paper and pencil, was compulsory and often due for submission within a week from being assigned. It was homogenous for the whole class and meant for individual work.

The homework assignments were of only two types, i.e. Type I and Type II. Type I homework was meant to review, practise and drill same-day content, while Type

II was meant to amplify, elaborate and enrich previously learnt information. For all three teachers, the main source of homework assignments was the textbook that the students used for the study of mathematics at school. Teachers also gave their students homework from past examination papers and non-school textbooks so that they would experience a wide range of questions, of varying levels of difficulty, for a particular mathematical topic. All three teachers monitored their students' homework and graded the assignments, giving them feedback. They also helped their students with their homework. When several students faced a common difficulty in their assignments, the teachers convened a focused discussion of the homework task and demonstrated the solution on the whiteboard.

The perspectives of the teachers regarding the role of homework they assigned their students were also explored. Again, the details were described elsewhere (see Kaur 2011). From the perspectives of the teachers, the role of homework they assigned their students was threefold. Firstly, "practice makes perfect" appears to be an underlying belief of all three teachers when rationalizing why they gave their students homework assignments. For all of them, it was important that their students "*hone their skills and comprehend the concepts*" of mathematical knowledge they were taught. Secondly, T2 also gave her students homework with the view that it was an extension of the lesson during which students were engaged in individual seatwork. Thirdly, T1 and T2 also gave their students homework to cultivate a sense of responsibility towards their learning. Certainly, the main underlying belief that "practice makes perfect" resonates with the finding of Macbeath and Turner (1990) about the most important functions of homework according to secondary school teachers, i.e. reinforcement, review and practice of work so that students perform well in tests and examinations. Inferring from the types of homework the teachers assigned their students, it is apparent that the homework was related to ongoing classroom work. T2, specifically, assigned her students challenging tasks as part of homework, and T3 was mindful of the fact that if he gave his students too much homework they were unable to cope with it. These findings resonate with that of Hallam's (2004) about homework being related to ongoing classroom work, be manageable, be challenging but not too difficult and that there be guidance and support to complete the work.

16.3.4 A Juxtaposition of Teachers' Practice and Students' Perception

Findings about how competent teachers teach Grade 8 mathematics reported here as well as students' perceptions about a good mathematics lesson (presented in Chap. 12) are essential for the creation of an image of exemplary instructional practices. This is exactly what the data and nature of analysis adopted in the Singapore LPS allowed the researchers to do. In so doing, the researchers questioned the stereotype of East Asian mathematics teaching (Leung 2001) and have been motivated to delve deeper into their classrooms and create a model of mathematics teaching in Singapore schools.

The next section reports on research that is presently underway to document the enactment of the school mathematics curriculum in secondary schools. This project involves the study of exemplary practices that are carried out by some 30 secondary mathematics teachers that are viewed as competent by the professional community in Singapore. As the project is ongoing, for the purpose of this chapter, we report preliminary findings based on a subset of the data corpus.

16.4 Enactment Project: Exemplary Practices in Relation to the Intended Curriculum

For this section on exemplary practices in relation to the intended curriculum, we examined 21 lessons from four teachers. The design of the study is such that two of these teachers were from School A while the other two were from school B so that any possible inter- and intra-school issues, if they exist at all, may be investigated.

As each enacted lesson was about an hour long, the enacted lessons were segmented into phases to facilitate comparison of the lessons and to examine in detail how the intended curriculum was enacted by these teachers. Since these Singapore teachers are familiar with the segmentation of lessons into the four phases—introduction, development, consolidation and closure (Lee 2009)—these phases were used and operationalized as follows:

- Introduction: Teacher setting the stage for current learning, such as checking for mastery of prerequisite knowledge (linkages to other subjects) and use of motivating stories/contexts.
- Development: Teaching for the attainment of the objective of the current lesson (alignment with other subjects).
- Consolidation: Teacher providing opportunities for students to practise on tasks related directly to the objective of the current lesson. It entails:
  - Students' independent work
  - Teacher selects and explains questions
  - Teacher asks students to explain their work
  - Teacher draws connections between previous lesson's tasks done in class or at home, and goals of the present lesson.
- Closure: Summary of lesson, setting of homework and/or assigning follow-up activity to set the stage for the next lesson.

These four phases of lessons also correspond closely to the phases of learning reflected in the syllabus document (Ministry of Education 2012), as presented in Table 16.3.

To gain further insights into how the intended curriculum was enacted by these teachers, each segment of these lessons was examined from the perspective of each

Table 16.3 Phases of lesson and phases of learning

Phases of lesson	Phases of learning
Introduction	Readiness (R)—In the readiness phase of learning, teachers prepare students so that they are ready to learn. This requires considerations of prior knowledge, motivating contexts and learning environment
Development	Engagement (E)—This is the main phase of learning where teachers use a repertoire of pedagogical approaches to engage students in learning new concepts and skills
Consolidation	Mastery (M)—This is the final phase of learning where teachers help students consolidate and extend their learning. The mastery approaches include motivated practice, reflective review and extended learning
Closure	

of the five interrelated aspects of the School Mathematics Curriculum Framework (SMCF) (Ministry of Education 2012), namely concepts, skills, processes, metacognition and attitude (see Fig. 3.1 in Chap. 3). However, it is observed that the level of enactment of the various aspects of the SMCF was very much dependent on the nature of the lessons. A skill-based lesson, for example, naturally yielded more codes under the skills aspect of the SMCF, while a concept-based lesson correspondingly yielded more codes under the concepts aspect of the SMCF. Consequently, the lessons were classified into the following five types to better reflect the nature of each lesson for further comparison:

- Type 1: Introducing new concepts
- Type 2: Revisiting learnt concepts
- Type 3: Introducing new skills
- Type 4: Revisiting learnt skills
- Type 5: Problem-solving (Barkatsas and Hunting 1996):
  - Type 5A: The application of learnt concepts and skills to solve either complex/non-routine problems (there must be a blockage to the students in general)
  - Type 5B: The application of learnt concepts and skills to solve either complex/non-routine problems (there must be a blockage to the students in general) demonstrated through implicit or explicit enactment of Polya's four-step approach.

The distribution of the types of lesson that were enacted in the 21 lessons by the 4 teachers is shown in Table 16.4.

From Table 16.4, it can be seen that these experienced mathematics teachers enacted a good spread of the different types of lesson. In particular, there is a good mix of addressing conceptual understanding and teaching of procedural skills; while slightly more than half of the occurrences are introducing and revisiting mathematics skills, a fifth of them were on introducing and revisiting learnt concepts. In particular, all these teachers were observed to weave in many short cycles of development and consolidation phases within each and between lessons, i.e. Engagement Mastery

Table 16.4 Distribution of the types of lesson

Type of lesson	Number of occurrences	Percentage
Introducing new concept	6	13.6
Revisiting learnt concept	4	9.1
Introducing new skills	11	25.0
Revisiting learnt skills	14	31.8
Problem-solving	9	20.5
Total	44	100.0

Note The total count is more than 21 as some of the lessons were coded as more than one type of lessons

cycles, to ensure that students reach a reasonable level of mastery in the relevant mathematical skills with a good grasp of the underlying conceptual understanding.

Furthermore, all these teachers were also very selective in their choice of questions to be used for teacher modelling, guided practice and independent practice. There appeared to be generally a good alignment for these questions to ensure that the students have sufficient practice to acquire the relevant mathematical skills to tackle such questions.

All in all, these teachers seemed to focus much on promoting conceptual understanding and fluency in procedural skills.

It is also observed that all these teachers tapped on the affordance of information and communications technology (ICT) to achieve their lesson objectives, though the role of ICT use might differ. There was use of YouTube videos to facilitate flipped classroom teaching, while some animations and videos were used to create motivating teaching materials. There was also use of graphing tool and commercially produced technological resources to exemplify mathematics ideas for proving or promoting of understanding of mathematical concepts. There was also an attempt to use a digital textbook to facilitate the teaching of and to make visible the problem-solving process. These seemed to reflect an impact on these teachers' enactment of the intended curriculum as a result of the four ICT master plans that have been put in place (see also Chap. 3).

From the perspective of the concepts and skills of the SMCF, these teachers seemed to be pedagogically strong in promoting conceptual understanding through the various use of technological aids and learning experiences, and procedural skills were taught alongside an understanding of the underlying principles/concepts.

In addition, there was also a reasonable good emphasis on problem-solving, as can be seen from Table 16.4 that about a fifth of the occurrences of the lesson types are on problem-solving. In other words, these teachers also provided opportunities for students to apply their learnt concepts and skills to solve either complex and/or non-routine problems.

Furthermore, these teachers also made conscious efforts to teach mathematical language explicitly and both written and verbal communication were encouraged. The teachers were also observed to promote reasoning by getting students to explain/justify their work through such mathematical communication. The thinking skills induction and deduction were employed in many of the lessons. The most common heuristics that were observed to be employed in these lessons were drawing a diagram, guess and check, and working backwards.

Thus, from the perspective of the process aspect of the SMCF, these teachers provided rich opportunities for the enactment of these mathematical processes.

In terms of the metacognitive aspect of the SMCF, all the teachers were observed to encourage students not to think impulsively but instead pause to monitor their thoughts. There were also some attempts to make visible the problem-solving process through digital means, as noted earlier. In addition, all the teachers were also observed to encourage offline metacognition, i.e. reflection (see Chap. 10), among the students.

From the attitude aspect of the SMCF, these teachers have certainly established a rather impressive rapport with their students; students were generally observed to be interested and enjoyed the lessons. The teachers were also seen to be consciously structuring their teaching by breaking the learning and doing into smaller chunks to boost students' confidence.

Thus, these experienced mathematics teachers' classroom practices seemed to be very well informed and guided by the intended curriculum.

16.5 Enactment Project: Exemplary Case Studies on Teachers' Use of Instructional Materials

The first is a case of a teacher's use of instructional material in "making things explicit" to his students. [The more detailed version of this case study can be found in Leong et al. (in press). We provide a summary here.] In searching the international literature, it is found that a popular conception of "explicit" is found in "explicit instruction" and it is seen as closely associated with other methods of instruction such as "teacher-directed instruction" (Doabler et al. 2015) and "direct instruction" (Gersten and Carnine 1984). The former highlights the primary role of the teacher in structuring lesson sequences; the latter focusses on the direct manner in which procedural steps "pass from" teacher to students. But in the case of "making things explicit" that we studied in the project, we began with a different starting point: we were not limiting "explicit" to these forms of instruction; but we started with the teacher's conception of explicitness; in particular, we examined his use of instructional materials as an instrument for making things explicit.

Our findings revealed that the teacher's attempt at using instructional materials for making things explicit can be summarized along these lines: explicit–from base; explicit–within materials and explicit–to instruction. These three conceptions correspond roughly to the three arrows shown in Fig. 16.3.

Explicit–from base. The teacher referred extensively from the school-subscribed textbook as his base curricular material. However, the transference from textbook to the instructional materials he used was not merely one of the direct lifting nor minor

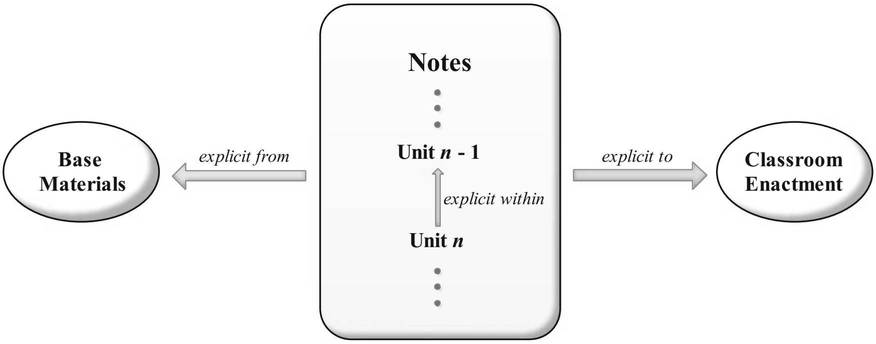

Fig. 16.3 Illustration of how the notes were used for explicit–from, explicit–within and explicit–to

adaptations. He saw the move between these material domains as primarily one of "making explicit". This explicitation can be further categorized into: filling gaps in the textbook content, making links between representations given in the textbook and highlighting critical ideas—without which students may inadvertently develop misconceptions—not emphasized in the textbook.

Explicit–within materials. The teacher used each unit within the notes he prepared to focus on one main concept. As is usually the case in mathematics, the focused concept is tightly linked to other related ideas. Instead of highlighting all the ideas in one-go within a unit, he used the strategy of foregrounding a particular idea while holding the other related ones as "supporting cast" at the background. This inter-unit implicit-to-explicit strategy reveals a level of sophistication in the crafting of instructional materials that we had not previously studied. The common anecdotal portrayal of Singapore mathematics teachers' use of materials is one of the numerous similar routine exercise items for students to repetitively practise the same skill to gain fluency. In the case of this teacher's notes, it was not pure repetitive practice that was in play; rather, students were given the opportunity to revisit similar tasks and representations but with added richness of perspective each time. In other words, each revisit allowed students to reinforce previously introduced ideas and to connect to new ones.

Explicit–to instruction. The teachers recognized the limitations to the extent in which the notes by itself can help make things explicit to the students. The explicitation strategy went beyond the contents contained in the notes. In particular, he used the notes as a springboard to connect to further examples and explanations he would provide during in-class instruction. He drew students' attention to questions spelt out in the notes, created opportunities for students to formulate initial thoughts and used these preparatory moves to link to the explicit content he subsequently covered in class.

From the point of view of students' learning experience, the chronology of first prompting their thoughts followed by the teacher's explicitation inverts the more traditional order of teacher–teach proceeded by student–practice. While the latter

Example 3	Solve the inequality $2x^2 - 7x + 6 < 0$.
Example 4	Solve the following inequality using a graphical approach: (a) $x^2 - 4x + 3 > 0$ (b) $3x^2 - 4x - 7 < 0$ (c) $4 - x^2 < 0$

Fig. 16.4 Examples in the instructional materials

tends to foster a passive adherence to teacher-demonstrated steps, the former allows students to carry out their first-cut thought experiments before the teacher points out the salient ideas or demonstrate some canonical methods. This sequence provides students the opportunity to contrast their more naive preliminary ideas against the explicit treatment provided by the teacher and thus learn to better appreciate the mathematical explicitation.

The second case features the principles a teacher used to sequence examples in his notes in such a way as to support mathematical reasoning. This is a significant study both in terms of the place that "reasoning" holds in the Singapore mathematics curriculum (see Fig. 3.1 in Chap. 3) and also in the ongoing interest in "reasoning" within the international mathematics education community (e.g. Jeanotte and Kieran 2017; Lampert 1990).

It is less surprising to find that the teachers sequenced the examples to "advance a method" (the teacher's own words) that he had demonstrated to the students. Figure 16.4 provides an illustration of a sequence of examples he gave within the topic of solving quadratic inequalities.

The method that was demonstrated to the students—for Example 3—was a series of steps that involved quadratic factorization followed by the use of graphical representation to show that the solution to the quadratic inequality corresponded to the x-values of the portion of the graph that is below the x-axis. This method was "advanced" as the subsequent examples retained the main thrust of the method but with refinements to deal with tweaks—such as the switch to ">" in Example 4(a), to non-strict inequality in Example 4(b) and to an inequality with zero coefficient for the x-term. The advancing of method principle is further reinforced as he proceeded with subsequent examples (see Fig. 16.5) as he modified the method to handle quadratic expressions that are not factorizable over the rationals.

Through the post-interview and classroom videos, it became also clear that "advance the method" was not the teacher's only goal in his use of this sequence of examples. The teacher expanded the examples systematically to a whole suite of what he called non-standard cases in Examples 5 and 6.

Analysis of the teacher's progression from Example 6(a) to 6(b) in his lessons showed that while he demonstrated how the same method applied, he also advocated an alternative method as he advocated that students "think flexibly". In other words, he wanted students not merely to follow strictly to the method he demonstrated but to constantly exercise reasoning behind the method and the procedural steps.

Example 5	Solve the inequality $2x^2 + x - 4 > 0$. Solution: We observe (or check) that the expression $2x^2 + x - 4$ is not easily factorized. In this case, we have to find the x-intercepts using the quadratic formula. We present our working in this way:
Example 6	Solve the following inequalities, giving exact answers: (a) $x^2 + 4x - 7 > 0$ (b) $2x^2 < 5$ (c) $x^2 + 2x + 11 > 0$ (d) $3x^2 - 30x + 75 < 0$

Fig. 16.5 Examples 5 and 6 in the instructional materials

This goal to encourage students' habitual reasoning is more obvious in Example 6(c). In this case, the solution of the associated equations is "no real roots". The students were unable to simply apply the method used in previous examples. They were thus "forced" to reason their way out of the quandary. That reasoning was inbuilt into the design of the examples was attested by the teacher during the interview: "today the focus is on the non-standard examples [Examples 5 and 6] … . So here is to promote reasoning in general, because here the basic idea is … to get the sketch of the graph, [then] use the graph to deduce a solution … . This way we make sure that they know the thinking behind the particular graphical method, and we put in all these parts to make sure that they are actually applying the reasoning behind the graphical method" (emphases added). The teacher was not merely using the sequence of examples to advance a method; he also wanted students to attend to the mathematical reasoning behind the (advancement of the) method. In other words, the advancement of the method "pulled along" the underlying mathematical reasoning.

The two cases described enabled us to uncover complex design considerations behind what may look to a casual observer as "simply drill-and-practice" instructional materials. In the enactment project, we are just beginning to examine these exemplary practices that are helpful in developing portraits of high-quality teaching in Singapore mathematics classrooms.

16.6 Discussion

This chapter reports on two significant mathematics education research projects that have been conducted in the Singapore mathematics classroom in identifying pedagogical approaches and exemplary practices exhibited by mathematics teachers in their enactment of the school mathematics curriculum. The researchers have moved away from the traditionally prescriptive approach in identifying classroom practices or using contrasting dualistic lens in the study of the mathematics classrooms. The researchers recognize that classroom teaching, being culture- and context-bound, is a much more complex process than it has traditionally been perceived by researchers. In the study of the Singapore mathematics classrooms reported in this chapter, the researchers have also questioned the stereotyped "East Asian pedagogy" (Leung 2001) in favour of delving deeper into the authentic mathematics classroom.

The message of the various studies reported in this chapter can be summarized in the following key points. To identify teachers' instructional approaches and exemplary practices, it is essential to

- Transcend the superficial patterns of the lesson sequence, but to take into consideration the totality of teacher instruction and role in engaging students in the entire process of learning during the class;
- Take into account the local factors in the educational landscape. In particular, it is crucial to study the classroom lessons or lesson segments using the lens of the underlying reasons and principles of the intended school curriculum, which is one of the key factors that drives the way lessons are conducted in the classrooms (in the researchers' experience with the Enactment Project described above, the lessons that were examined using the mathematical problem-solving framework in the Singapore mathematics curriculum document); and
- examine the instructional materials that are used by the teachers. As described in the preceding sections, teachers did not use the existing teaching resource wholesale in delivering a lesson. The teachers made many careful considerations in adapting or developing the instructional resource for lesson delivery. This aspect, though not directly visible in classroom observations, contributes to an extremely important component in the study and identification of teachers' exemplary practices.

At the time that this chapter is written, the enactment project is still work in progress. After identifying the exemplary practices of this relatively small sample of experienced mathematics teachers, the next step for the researchers is to identify how widespread these exemplary practices are among the mathematics teachers in the Singapore education system in general. This will allow the researchers to have a fuller picture of the overall mathematics classrooms in Singapore. A study of how these exemplary practices among mathematics teachers impact on students' learning (cognitive, metacognitive and affective dimensions) of the subject is another area which will likely attract international attention on Singapore mathematics.

The researchers of the enactment project used a coding scheme that attempted to explain in great depth the intent of the teacher. Besides the Singapore mathematics curriculum document, Schoenfeld's Teaching for Robust Understanding (TRU) framework is one of the theoretical frameworks that was used at least in the initial phase in designing the coding scheme (Kaur et al. 2018). As teaching has been recognized to be cultural and it is very much context-dependent, perhaps what we need next is to develop a local Singapore teaching framework. Although we would not go so far as to suggest to develop a prescriptive list of "exemplary" practices, such a local teaching framework would be useful for researchers in understanding the specific pedagogical approaches of the teacher which could be unique to Singapore in recognition of its unique social–cultural factors.

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Chapter 17
Continuing from Pre-service: Towards a Professional Development Framework for Mathematics Teachers in the Twenty-First Century

Kit Ee Dawn Ng, Joseph Kai Kow Yeo, Boon Liang Chua and Swee Fong Ng

Abstract Quality teachers have long been recognised as key to preparing the future generation for the nation. Hence, a robust teacher education system with collaborative support from major stakeholders is crucial. Singapore has adopted a complex yet integrated approach in teacher education thus far. This chapter moves on from an earlier report presented in 2009 on the Singapore teacher education system. To pave the way forward, the chapter introduces the visions of key stakeholders in the professional development landscape for teacher education in the twenty-first century. Various factors of influence are analysed before presenting the structure of professional development for mathematics teachers at the National Institute of Education. Current mathematics professional development courses are classified according to aspects of teacher knowledge derived from research so as to gain insights into the content, pedagogical and assessment focuses. Finally, a proposed conceptual framework amidst the multidimensional and multifaceted teacher education landscape is outlined to describe mathematics teacher professional development for the twenty-first century.

Keywords Twenty-first-century competencies · Mathematics pedagogical content knowledge · Mathematics education · Professional development framework · Teacher education · Teacher growth model · Teacher professional development

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17.1 Introduction

Quality teachers have long been recognised as key to preparing the future generation for the nation. Indeed, the influential McKinsey report on how the world's best-performing school systems come on top articulated that "the quality of an education system cannot exceed the quality of its teachers" (Barber and Mourshed 2007, p. 16). Well-qualified teachers not only exert a critical impact on student learning (Gopinathan et al. 2008) but also an enduring influence in the lives of future generations. Hence, it is critical that education ministries, policymakers, and teacher educators work together on several fronts to draw interested individuals into teaching and retain them in education service: (a) provide robust teacher education programmes at pre- and in-service levels at various junctures of the teaching career, (b) maintain rigour in teaching certification, (c) allow for career progression in teaching, and (d) support the teaching fraternity to chart their own professional growth. Collaborations between teacher educators and schools in teaching-research projects (see Ng et al. 2015), formations of professional learning communities among teachers (see Hairon and Dimmock 2012) and establishments of sharing platforms (e.g. conferences, seminars) among key stakeholders in education (i.e. schools, teacher educators, curriculum planners, policymakers) are some infrastructure put in place in many education systems around the world.

Yet, as Tan and her colleagues (2017b) put it, educational systems all over the world face at least two challenges in the twenty-first century. Age old constructs or concepts such as "creativity, critical thinking, collaboration, communication, socio-emotional and lifelong learning aptitudes" are now recognised as "new knowledge economy competencies" and they have been given a renewed lease of attention in view of a technologically dominated globally connected world in the twenty-first century (Tan et al. 2017b). Learning is not confined to traditional forms of delivery. Neither is it confined to individual experts. These have implications on the teacher education system. Firstly, how should schools and educators scaffold and assess students' new knowledge economy competencies individually and collectively? Secondly, how would we activate and sustain a cultural and pedagogical shift from traditional modes of education and perception on achievement to a more inclusive, varied form of education? Some answers may lie with a progressive teacher education framework which is aligned with the learning needs of twenty-first-century teachers. Views on what quality teachers in the twenty-first century are may morph from the complex, multidimensional and multifaceted discussions that follow.

17.2 Teacher Education in Singapore: A Brief Understanding of the Current Landscape

In this chapter, *Teacher Education* refers to a broader concept which encompasses the desired outcomes (e.g. philosophical, theoretical, political and economic) to be integrated and balanced among major stakeholders such as policymakers, government bodies, education administrators, universities and funding agencies. Ideally, teacher education should be in a continuum of three seamless stages according to the teaching career progression: (a) initial teacher preparation (referred to as "pre-service" in this chapter where student–teachers are in the process of being accredited), (b) induction (referred to as "beginning teachers" where accredited teachers may still work with school mentors during the first few years of teaching experience) and (c) *Teacher Professional Development* (PD) (referred to as "in-service" where experienced teachers may chart their own growth in teaching repertoire). In this chapter, teacher PD includes those conducted in formal delivery-style settings (e.g. workshops, training sessions, talks, seminars, and conferences) as well as those which involve targeted group-based discussions or sharing sessions (e.g. professional learning communities). This is in line with the Teacher Growth Model for twenty-first-century teachers articulated by the Academy of Singapore Teachers (AST) (see Sect. 17.2.3). In addition, we also recognise that beginning teachers can also participate in PD courses alongside with experienced teachers.

We begin with a discussion of the teacher education landscape in Singapore in this chapter and subsequently focus on teacher PD in Singapore, particularly analysing the structure of PD for mathematics teachers. Lim-Teo (2009) provided an in-depth discussion of the context of teacher education in Singapore and factors of influence prior to 2009. She articulated the synergistic effect between the National Institute of Education (NIE) and the Singapore Ministry of Education (MOE) which alleviated teacher education to higher levels globally. This chapter moves on from Lim-Teo's report after 2009, presenting the complex yet integrated approach Singapore has taken and introducing more factors of influence in its teacher education journey thus far. The next section summarises how teacher PD is offered through four major collaborative avenues in Singapore; namely the NIE, MOE, AST, and professional associations. Following which, we outline the current situation of PD for mathematics teachers at the NIE, analysing the factors of influences and their impact on curriculum, and hence PD. Then, some constraints, issues and challenges to mathematics PD faced by teacher educators at NIE are highlighted. A proposed PD conceptual framework tailored for mathematics teachers in the twenty-first century after an analysis of the broader landscape of teacher education in Singapore and beyond is presented. Finally, future directions for mathematics PD and related research in Singapore are discussed.

17.2.1 Teacher Education at the National Institute of Education: The Journey After 2009

Singapore is progressing towards a transformative system that produces quality teachers equipped to raise a new generation of twenty-first-century learners. In this system, teachers are expected to prepare students for a knowledge-driven global economy by helping them develop higher-order competencies and new technology-based skills, while simultaneously building character and grounding values so that Singapore students still remain rooted in their national identity against the backdrop of a multicultural, globalised world (NIE 2012).

The NIE is Singapore's sole and national teacher education institute (NIE 2017a). As the premier teacher education institute in Singapore and an autonomous institute of the Nanyang Technological University (NTU), Singapore, NIE not only provides teacher accreditation during initial teacher preparation but also designs programmes, courses and workshops to empower accredited teachers in their professional development journey for enhancement of competence and knowledge as they progress in their teaching career. NIE embarked on a Programme Review and Enhancement initiative in 2008 as an institute-wide strategic effort to review and enhance NIE's model of teacher education (NIE 2012). *The Teacher Education 21 Model (TE21)* was inaugurated by NIE in 2009 to cultivate the "thinking teacher" while maintaining "strong partnerships with key stakeholders and the schools to ensure strong clinical practice and to inject the reality of professionalism in teacher development" (NIE 2009). Recommendations in the TE21 Model address the entire initial teacher preparation (i.e. pre-service) to teacher PD (i.e. in-service) continuum (NIE 2017a). It is NIE's mission to provide a curriculum that is "cognizant of nationwide policies and initiatives implemented by the Ministry of Education" (NIE 2017b). TE21 puts major emphasis on teachers' values because values are the "anchor of stability, consistency and centredness in a changing vortex" (Tan 2012, p. 39) in the midst of the rapid changes in curriculum and policy brought about by challenges in the twenty-first century. Three key values are identified: learner-centred values, teacher identity values and the values of service.

Initiatives by the Singapore MOE are put forth to foster students' *new knowledge economy competencies* (see Sect. 17.1) required in the twenty-first century. A crucial recommendation of TE21 is a robust theory-practice nexus in developing teachers' own proficiencies to scaffold students' twenty-first-century competencies, building upon the content and pedagogy associated with different subject disciplines. Hence, NIE works closely with the Singapore MOE, the AST and schools so that policies and initiatives are not only integrated into the pre- and in-service teacher education programmes and courses but also realised in practice among prospective and experienced teachers.

Being in a unique and privileged position, NIE offers a robust system of teacher accreditation at primary, secondary and pre-university levels across various subject-discipline areas in Singapore through various programmes such as the NTU-NIE Teaching Scholars Programme (TSP), the Postgraduate Diploma in Education programme, undergraduate programmes (i.e. Bachelor of Arts (Education), Bachelor of Science (Education) and Diploma programmes. Lim-Teo (2009) summarised the model of pre-service preparation of teachers in Singapore in Fig. 17.1. The model is still applicable to date. Since the implementation of TE21 in 2009, pre-service programmes have been reviewed to address the emphases of TE21. For example, the NTU-NIE TSP is a new undergraduate programme that was launched in August 2014. This programme offers final-year scholars individually supervised research

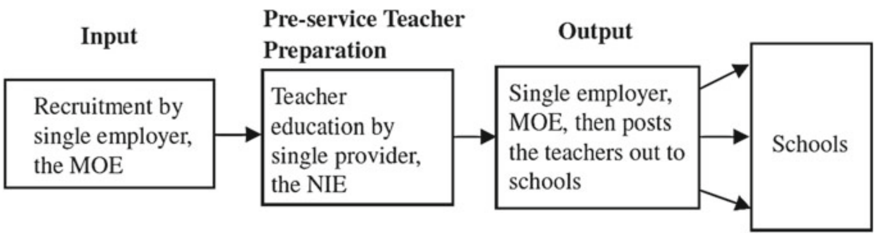

Fig. 17.1 Model of pre-service preparation of teachers in Singapore (Lim-Teo 2009, p. 50)

opportunities with eminent research mentors at NTU or NIE where the scholars and their mentors pursue areas of interest together, immersing in extended content knowledge academic exchanges over a 11-month period. In addition, TSP scholars also undertake an educational research project in their third year of study to build their capacity in the theory-practice nexus alongside their classes in curriculum studies with pedagogical focuses for various subject disciplines (NIE 2017c).

Helmed by the Office of Graduate Studies and Professional Learning at NIE, in-service programmes are crafted for experienced teachers, education officers with the MOE and interested individuals from other educational institutions in Singapore and overseas. Singapore teachers with the MOE are placed in three career tracks: Teaching Track, Leadership Track, Senior Specialist Track (MOE 2017a). NIE in-service and higher degree programmes cater to teachers at various junctures of their careers in these tracks. For example, besides considering a comprehensive list of stand-alone PD courses, primary mathematics teachers can choose to enrol in the Advanced Diploma in Primary Mathematics Education programme should they like to develop their pedagogical content knowledge further and have a deeper conceptual understanding of mathematics content knowledge in the horizon to make connections between primary mathematics topics and beyond (NIE 2017d). In addition, NIE also offers a highly anticipated Teacher Leaders Programme which develops "leaders on the Teaching Track" (i.e. Senior Teachers, Lead Teachers and Master Teachers) through an intricate progression of learning journeys aligned with the *Teacher Growth Model (TGM)* (MOE 2012a; see below for details). The Teacher Leaders Programme aims to "nurture teachers as ethical educators, competent professionals, collaborative learners, transformational leaders and community builders" (NIE 2017a). Separately, working with MOE to develop education officers on the Leadership Track (i.e. school leaders such as Heads of Departments, Vice-Principals, Principals and Cluster Superintendents), NIE provides at least three programmes: Leaders in Education Programme, Management and Leadership in Schools Programme, and Building Educational Bridges—Innovation for School Leaders. MOE education officers who are keen to progress on the Senior Specialist Track can enrol in a myriad of higher degree programmes (e.g. Masters in Education) to enhance their theory-practice nexus for curriculum development. Since 2005, NIE has implemented the *Professional Development Continuum Model (PDCM)* scheme to provide graduate teachers of Singapore MOE with alternative pathways to higher degree certification

(NIE 2017e). Most of the Master's degree programmes at NIE are available under this scheme. A step further into TE21, with effect from August 2012, the *enhanced PDCM* scheme was put in place to allow for more flexibility in structure for candidates with different interests and work commitments. 2014 saw the cross-listing of selected courses in NIE Master's degree programmes with PD. This allows for teachers who meet the entry requirements of the Master's degree programmes to take higher degree courses as PD prior to their admission to the programme under clear accreditation and time frame conditions (NIE 2017f). In this way, teachers can experience the rigour and depth of NIE higher degree courses with a slightly more theoretical and research stance compared to other more practice-orientated PD courses. This also creates opportunities for teachers to design integrated research and practise projects within their own capacities under the tutelage of faculty members, spear-heading innovative pedagogies and curricula approaches in schools, grounded in sound theoretical underpinnings and empirical findings; achieving another TE21 recommendation.

17.2.2 Envisioning Teacher Professional Development for the twenty-first century in Singapore: Ministry of Education

Launched in 2012 by Mr. Heng Swee Keat, then Singapore Minister of Education, the Teacher Growth Model (TGM) is a "professional development model which encourages Singapore teachers to engage in continual learning and become student-centric professionals who take ownership of their growth" (MOE 2012a). Developed by the AST under MOE, the TGM (for in-service teachers) was a result of a collaborative conceptualisation effort among educators of diverse profiles across MOE to construct the learning needs of the twenty-first-century Singapore teacher. Figure 17.2 illustrates the five desired outcomes of the twenty-first-century Singapore teacher in the TGM (The Ethical Educator, The Competent Professional, The Collaborative Learner, The Transformational Leader and The Community Builder). Although the TGM Learning Continuum suggests learning focuses for teachers at various stages of their careers, teachers have the autonomy to plan for their PD based on their needs and interests, bearing in mind alignment to the knowledge and skills needed to nurture students in twenty-first-century competencies (MOE 2012a). Recognising that teachers also have diverse learning needs, the TGM encourages teachers to pursue PD through multiple modes of learning (e.g. face-to-face, ICT-enabled, conferences, mentoring and research-based practice, networked learning, reflective practice and experiential learning). There are seven learning dimensions associated with the TGM and all NIE PD courses for in-service teachers (including those cross-listed with higher degree) are mapped to these learning dimensions (NIE 2017g).

Within the TGM framework, various departments at MOE also conduct PD workshops or sharing sessions for teachers. Curriculum specialists with the curriculum

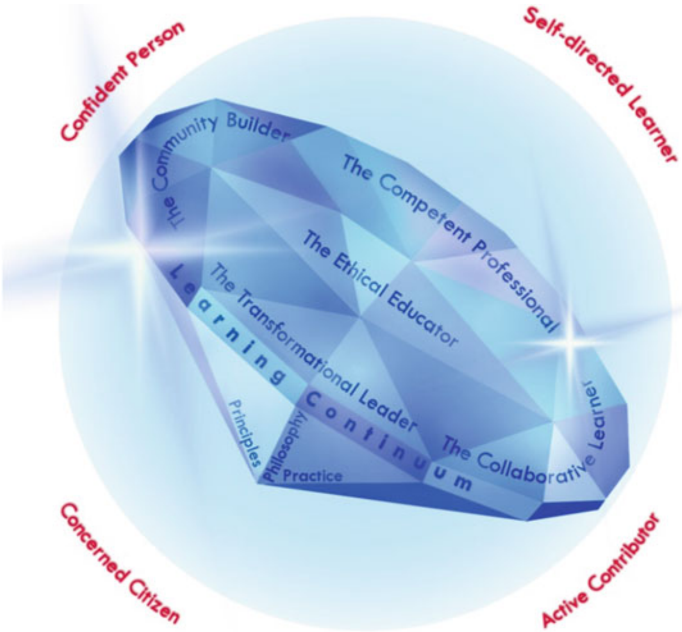

Fig. 17.2 Teacher growth model (Ministry of Education 2012a)

planning and development division at MOE conduct nationwide PD workshops periodically for schools and clusters to communicate and help kick-start the implementation of curriculum initiatives. Senior management in schools send representatives to attend such PD workshops so that they can take leadership in incorporating the initiatives in school-wide programmes. An example is "Fostering Mathematical Reasoning in Classrooms" workshops conducted by the curriculum specialists for secondary school teachers in recent years. Often, curriculum specialists work with teacher educators at NIE to plan complementary workshops for teachers so that the teachers are aware of the background and key messages associated with new curriculum focuses, and have opportunities to draw connections between the curriculum focuses with appropriate pedagogical approaches grounded from theory and research. For instance, mathematics curriculum specialists have conducted introductory workshops on mathematical modelling since 2009 where they show examples of mathematical modelling activities secondary mathematics teachers can use in their classrooms and discuss elements of the modelling cycle (see Balakrishnan et al. 2010; MOE 2012b). However, teachers who are leading mathematical modelling activities in their schools or are interested to have a more in-depth understanding of the design, facilitation and assessment of mathematical outcomes during the full cycle of mathematical modelling are directed to attend NIE PD workshops for a comprehensive hands-on experience (see Ng 2017).

In addition, MOE holds regular meetings and briefing sessions with heads of departments or pedagogical leaders in schools to communicate curriculum and assessment changes. Teachers often attend conferences (e.g. Kwek and Ko 2011), seminars and workshops based on interest or upon the encouragement of school management. Again, within the TGM, many interest-driven professional learning communities (see Hairon and Dimmock 2012) are set up within schools (e.g. among teachers teaching the same level) or within school clusters. These professional learning communities can be steered by teachers, school leaders, mathematics educators, MOE education officers and Master Teachers from the AST. Each professional learning community is set up to achieve explicit objectives and they typically outline PD sessions that are aligned with their progressive implementation of school-based projects or initiatives. Subsequently, the professional learning communities in schools may organise their own in-house PD sessions conducted by the pedagogical leaders in the schools or by invited instructors. For example, between 2014 and 2016, the mathematics department head of one primary school in Singapore held a series of workshops on Talk Moves (see Michaels and O'Connor 2015) with the teachers in the school. There was meticulous planning and mentoring by the mathematics department head to help the mathematics teachers in the school implement what was shared in the workshops in progressive steps, the first of which was lesson observations of the head of department in action with her mathematics class where she used Talk Moves to generate more mathematical productive discourse and encourage students to share their mathematical reasoning. The school finally took on this initiative as a school-wide approach after successful implementation and good reviews from the mathematics department and students (Lee et al. 2016). One advantage of such carefully planned professional learning communities within the school as illustrated above is the close links between theory and practice where teachers engage in iterative cycles of reflective practice among like-minded peers under full support of the school management.

MOE is cognisant of the importance of providing opportunities for serving teachers to extend their professional repertoire in order to help them achieve their academic and professional aspirations. Funding and professional development leave infrastructure are put in place to encourage teachers to engage in lifelong learning. Every teacher in the school system is eligible to 100 h of PD a year, fully funded by MOE directly or through MOE-administered school or cluster budgets (Lim-Teo 2009, p. 66). Teachers can choose from various professional development packages and leave schemes (MOE 2016, 2017a) to participate in PD and higher degree work.

17.2.3 The Role of the Academy of Singapore Teachers in Teacher Professional Development

The Academy of Singapore Teachers (AST) was set up in September 2010 to look into "the development of a teaching fraternity that is characterised by a shared ethos, strong pedagogical expertise and ownership of professional development" (MOE 2012a). AST serves four functions: to (a) champion the ethos of the profession, (b) foster a teacher-led culture of collaborative professionalism, (c) build a culture of continuous learning and improvement and (d) strengthen enablers of professional development (MOE 2017b). Crafted by AST, the TGM guides the planning and implementation of PD activities at AST. AST articulated the TGM as a "representation of a coherent whole of core learning areas of holistic professional growth and development for Singapore teachers" which "facilitates teachers taking ownership of their professional growth to nurture in students the competencies required for the twenty first century" (MOE 2017c).

To date, AST has organised an array of PD opportunities for in-service education officers (i.e. teachers, school leaders, including those seconded to MOE headquarters), executive and administrative staff working in schools, and allied educators (i.e. teaching assistants). Besides workshops, seminars and talks, PD opportunities from AST also come in the form of focused-group discussions and sharing sessions during subject chapter meetings. Materials from these subject chapter meetings and those from follow-up sessions are typically shared on a private portal with exclusive access rights given to MOE staff.

Master Teachers are identified by MOE as "role models of teaching excellence" based on their track records of "strong pedagogical knowledge" demonstrated in schools over many years. Master Teachers are at the pinnacle of the Teaching Track. The main role of a Master Teacher is to "develop and enhance the capacity of teachers through mentoring and demonstrating good teaching practice" (Ng and Foo 2009, p. 150) through working with schools, cluster schools and beyond (e.g. school-based research projects, curriculum reviews). PD opportunities organised by AST are usually conducted by Master Teachers although they too get invited to be course instructors for school- or cluster-led PD sessions for professional learning communities.

While PD courses conducted by MOE curriculum specialists are mainly to communicate curriculum initiatives, those by AST Master Teachers have predominantly pedagogical focuses with a clear practice-oriented stance. On the other hand, NIE PD courses provide theory-practice nexus where participants learn, experience and reflect on research-based theoretically informed pedagogical practices. NIE collaborates with MOE and AST to plan in-service teacher PD stand-alone courses and programmes offered by NIE every year so that teachers receive a wide selection of complementary PD offers meeting different needs and interests. Funding for most NIE teacher PD courses and programmes comes from annual MOE budgets. In essence, there is a synergistic tripartite collaboration between NIE, MOE and AST for a holistic teacher PD in Singapore.

17.2.4 The Role of the Professional Associations in Teacher Professional Development

Local professional bodies such as the Association of Mathematics Educators (AME) and the Singapore Mathematical Society (SMS) also hold conferences and seminars regularly for mathematics teachers to learn from both foreign and local experts. A case in point is the annual Mathematics Teachers Conference co-organised by AME and the Mathematics and Mathematics Education Academic Group at NIE with support from SMS. This one-day programme includes plenary lectures and PD workshops by invited foreign and local experts for primary school, secondary school and pre-university mathematics teachers, as well as sharing sessions by local academics and mathematics educators to showcase their research studies and share findings with implications drawn for teaching and learning. The Mathematics Teachers Conference has been held for over a decade with a different theme each year articulating current educational focuses in Singapore and globally. Participation in each Mathematics Teachers Conference has been enthusiastic throughout the years with numbers ranging between 500 and 800. In addition, AME also produces a newsletter and an academic journal entitled "The Mathematics Educator" for the mathematics teaching and research fraternity.

17.3 Mathematics Teacher Professional Development at the National Institute of Education

PD courses and programmes for mathematics teachers at NIE are mainly offered by educators from the Mathematics and Mathematics Education Academic Group. Nested within larger global expectations of quality teachers in the twenty-first century, several other factors of influence have impact on current and future PD for mathematics teachers in particular.

17.3.1 Factors of Influence

One key factor of influence on the nature and format of PD for mathematics teachers at NIE is research on mathematics teacher education.

17.3.1.1 Research in Mathematics Teacher Education

Teacher education research gained momentum since Shulman (1986) called for the spotlight to be shone on a teacher's knowledge base on teaching. Grossman (1990) proposed four key components of teacher knowledge: general pedagogical knowledge, subject matter knowledge, pedagogical content knowledge and knowledge of context. The foundational concept of pedagogical content knowledge was first coined by Shulman, who in a later publication defines it as "that special amalgam of content and pedagogy which is uniquely the province of the teacher" (1987, p. 8). Researchers recognise the need for teacher knowledge base to be discussed in terms of subject-specific disciplines because teaching requires a professional integration of various components of teacher knowledge with respect to the rigour of the discipline. A domain map of mathematical knowledge for teaching was outlined by Hill et al. (2008) and this unpacks Pedagogical Content Knowledge further and distinguishes it from Subject Matter Knowledge. Pedagogical Content Knowledge for mathematics teachers includes "Knowledge of Content and Students, Knowledge of Content and Teacher, and Knowledge of Curriculum". On the other hand, Subject Matter Knowledge refers to "Specialised Content Knowledge, Common Content Knowledge, and Knowledge at the Mathematical Horizon" (p. 377).

Research on Mathematics Pedagogical Content Knowledge (MPCK) of Teachers began to take root after initial efforts by Ball and her colleagues (Ball 1991; Ball et al. 2001, 2008). In Singapore, an inaugural project on MPCK was conducted by an NIE team of mathematics educators to investigate the development of MPCK in primary school beginning teachers. One main goal of this project was to evaluate the impact of a mathematics methods programme on prospective teachers' MPCK at NIE. As part of this large-scale longitudinal project, the team administered a 16-item instrument to measure the performance of pre-service teachers in the Diploma in Education programme before they started their initial teacher preparation journey at NIE and after they completed the programme (Lim-Teo et al. 2007). The instrument used took the form of a MPCK test where items assessed the teachers' (a) own knowledge of mathematical structure and connections, (b) representations (multiple or alternative) of concepts for the purpose of explanations, (c) perceptions of the cognitive demands of the mathematical tasks on learners and (d) identification of the difficulties faced by learners and learners' misconceptions along with teachers' choice of follow-up actions (p. 257). Quantitative pre- and post-test results suggested the pre-teachers with the Diploma programme had generally made some improvements across (a) to (d) at the end of the programme. However, qualitative analysis of the responses revealed that these pre-service teachers were rather weak in mathematical communication; especially in explaining and developing mathematical ideas alongside their logical reasoning using precise mathematical terms and language (p. 251). The researchers also surfaced challenges faced by the pre-service teachers in composing word problems to illustrate mathematical concepts (e.g. quotitive division) (p. 252). Implications were drawn from these findings on reviewing pre-service mathematics methods courses at NIE. In addition, teacher educators can draw upon the research by Lim-Teo and her colleagues when planning PD focusing on MPCK for mathematics in-service teachers so as to further deepen teachers' understanding of subject matter knowledge and introduce innovative pedagogical approaches to help students overcome mathematical learning difficulties.

Another large-scale project measuring teachers' MPCK came from the international Teacher Education and Development Study in Mathematics (TEDS-M) survey.

The performance of NIE pre-service teachers in Mathematics Content Knowledge and MPCK as assessed by the TEDS-M survey was reported in Wong et al. (2010). TEDS-M Mathematics Content Knowledge framework covers four domains (Number, Geometry, Algebra and Data) and three cognitive domains (Knowing, Applying and Reasoning). TEDS-M MPCK framework includes: mathematical curricular knowledge, knowledge of planning for mathematics teaching and learning, and enacting mathematics for teaching and learning (Tatto et al. 2012). Although pre-service primary mathematics teachers at NIE who participated captured top spots in the TEDS-M survey in terms of Mathematics Content Knowledge and MPCK compared to the other participating countries (Wong et al. 2010, p. 300), gaps were identified. There is a need to provide more opportunities for prospective primary mathematics teachers (and even those teaching mathematics at higher levels) to learn different approaches to rectify students' misconceptions in mathematics. Wong et al. also suggested that teacher educators could use publicly released TEDS-M to "explore strategies to remedy misconceptions, design classroom activities that mirror the scenarios described in the TEDS-M items" so as to work towards "assessment for teacher training" (p. 304) with formative purposes. Interested readers may like to refer to Chap. 6 for more in-depth discussions on the results from Singapore's participation in the TEDS-M study.

17.3.1.2 Research in Professional Development Models for Mathematics Teachers

Research projects by Singapore teacher educators on PD models or structures with respect to different fields in mathematics education research can also have an impact on PD for mathematics teachers provided at NIE. These will be summarised briefly in this section.

On mathematical modelling, Tan and Ang (2015) designed a school-based PD programme using Ang's (2015) framework which scaffolds mathematics teachers in secondary schools through progressive stages of modelling task design. The school-based PD programme consists of three phases where teacher reflections from earlier phases provided inputs for subsequent phases. At the end of the programme, participating teachers would have designed mathematical modelling tasks for their schools, facilitated students through the tasks and reflected on their learning about the mathematical modelling process. In another project on mathematical modelling but at the primary level, Ng and her colleagues (see Chan et al. 2012; Ng et al. 2012, 2015) incorporated a multi-tiered teaching experiment (Lesh and Kelly 2000) with adapted design research methods (Dolk et al. 2010) in their PD structure to scaffold the incorporation of mathematical modelling in primary schools.

On design of learning tasks to engage students in reasoning and communication, Kaur (2012) investigated the impact of a hybrid model of PD that integrates the PD training model from Matos et al. (2009) with "sustained support for teachers to integrate knowledge gained from the PD into their classroom practice" (Kaur 2012, p. 5148). This hybrid model of PD advocated three phases: (a) teachers attending training workshops, (b) teachers enacting what is learnt at the workshops in their schools guided by the PD providers and (c) teachers sustaining what they have learnt from the previous phases through school-based self-directed activities. This PD model took two years to realise, a significantly longer duration compared to other PDs which are constrained by MOE timelines. There are other PD designs implemented for mathematics teachers at NIE. Detailed discussions of another PD design from research involving Replacement Units can be found in Chap. 19 of this book.

17.3.1.3 Curriculum Focuses

Given the widespread implementation across the world, there has been global impact from the results of the Trends in International Mathematics and Science Study (TIMSS) and the Programme for International Student Assessment (PISA) on policymaking as well as mathematics curriculum planning and review (OECD 2017; Stacey 2012). In Singapore, the mathematics curriculum framework articulates the need for students to solve a variety of problems, including open-ended and real-world problems (MOE 2012c, p. 15). Mathematical modelling has been incorporated into the curriculum framework since 2007 (MOE 2012b) and students' ability to solve "Problems in Real-World Contexts" (similar to applications problems) is assessed formally at the high-stakes GCE "O" Levels mathematics examination since 2016 (MOE 2015). Maintaining the rigour and depth of mathematical content and skills in the mathematics syllabi at the various school levels, but also in response to the global focus on students' competencies to solve non-routine, open-ended real-world problems, Singapore teachers are encouraged to develop their pedagogical content knowledge in support of more student-centric approaches. Such approaches require teachers to scaffold student-directed learning, critical thinking, as well as appropriate mathematical communication and reasoning during group collaborative problem-solving. There is also a need for teachers to be confident facilitators during problem-solving while discussing possibilities of alternative solution pathways in view of real-world constraints stipulated in the context of the problem. Contents of Mathematics PD courses at NIE not only address the mathematics curriculum framework, but also bring in the larger global picture, drawing upon research to provide sound theory-practice nexus during the courses.

17.3.2 Professional Development for Mathematics Teachers at the National Institute of Education

There is a comprehensive array of PD courses for mathematics teachers at NIE across four domains: subject matter knowledge, pedagogical content knowledge, school-based curriculum planning and assessment practices. The PD courses can be offered as standalones, as a series of progressive courses for the same topic, or as part of a collection of courses during a programme. Stand-alone PD courses are typically short-term hands-on practice-oriented courses which can range from a three-hour workshop to several sessions of three-hour blocks either taken in a school-day afternoon or in a full day during school holidays. In contrast, PD programmes have stipulated entry requirements and time frame for completion of academic units through a number of courses and a subsequent accreditation process. Teachers can register for these programmes as part-time or full-time participants depending on their commitments. Currently, there are two PD programmes focused on mathematics teaching and learning; both for primary school teachers: the Advanced Diploma in Primary Mathematics Education programme and the Certificate for Primary Mathematics Education programme. Primary school teachers are accredited to teach more than one subject. Many also enrol in other NIE-accredited programmes or PD courses which may take a more generic stance (i.e. non-subject-specific) and apply what they learn from these programmes in the various subject disciplines they teach. Courses in PD programmes are at times offered as standalones should participants prefer taking up selected courses within the programme on an ad hoc basis.

17.3.2.1 Professional Development Through MOE-Commissioned Courses

Mathematics teachers in schools can enrol in PD stand-alone courses or programmes through three main avenues, each comprising of complementary PD lists. Firstly, majority of PD courses and programmes for mathematics teachers are MOE-commissioned. Teachers from MOE schools enrol in these through an online system called "TRAISI" (Training Administration System on Internet) using their MOE-registered email and password. MOE, AST and NIE representatives from various subject disciplines engage in annual discussions of course offers by NIE for the following year. Decisions are made based on needs assessment of teachers for further PD on curriculum initiatives and focuses. This is balanced with the overall allocated MOE budget for PD. Table 17.1 summarises the types of TRAISI stand-alone courses offered by NIE mathematics educators between 2014 and 2019. Higher degree courses (i.e. Master's) which are cross-listed with in-service and offered under TRAISI are not reflected in Table 17.1. The types of PD courses are classified according to Hill et al.'s (2008) domain map of mathematical knowledge for teaching (see Sect. 17.3.1.1). An example of a course under Pedagogical Content Knowledge with a focus on Knowledge of Curriculum is that of "Promoting Metacognition in Primary School Children" where mathematics teachers learn how to foster student's use of metacognitive strategies for problem-solving. A Subject Matter Knowledge course can be illustrated with "Algebra in Secondary Additional Mathematics" where teachers learn algebraic concepts that are Specialised Content Knowledge needed for the additional mathematics syllabus. Tan et al. (2017a) called for the incorporation of teachers' assessment literacy in examining teacher knowledge because assessment is a crucial part of curriculum, teaching and learning. Hence, a third classification,

Table 17.1 Types of TRAISI stand-alone courses offered by NIE mathematics educators

Year	Pedagogical content knowledge			Subject matter knowledge			Knowledge of assessment		
	Pri	Sec	Pre-U	Pri	Sec	Pre-U	Pri	Sec	Pre-U
2014	6	7	1	0	7	0	2	0	0
2015	6	6	8	0	7	0	2	0	0
2016	7	2	8	0	4	0	2	1	0
2017	8	3	8	0	3	0	2	1	0
2018	9	3	7	2	3	1	2	0	0
2019	5	3	3	0	4	0	1	1	1

Knowledge of Assessment, is added to provide a more comprehensive representation of the available courses. One example of a course under this classification is "Problems in Real-World Contexts: Design, Implementation and Assessment" where secondary mathematics teachers learn how to design problems situated in real-world contexts which require students to select and apply appropriate mathematics content and skills, similar in format to those assessed in GCE "O" level mathematics examination.

Data shown in Table 17.1 reveal that most TRAISI stand-alone courses are MPCK-related across primary, secondary and pre-university levels. However, at least two gaps in PD can be noted. The first gap refers to Subject Matter Knowledge. Such PD courses are not offered at primary and pre-university levels. Many educators may agree that it is not easy to untangle MPCK and Subject Matter Knowledge in a PD course because competent teachers are often able to integrate both seamlessly to achieve their lesson objectives. Nonetheless, it may be crucial for experienced primary mathematics teachers to attend PD courses on Subject Matter Knowledge because at least a large majority of them are essentially generalist in training and do not have a mathematics degree. Interestingly, graduates who are on the enhanced Postgraduate Diploma in Primary Education programme for pre-service teacher education since its inception in December 2016 have been attending Subject Matter Knowledge courses. On the other hand, it is understandable why teachers teaching pre-university level mathematics are not provided with Subject Matter Knowledge PD courses. Many of them already have honours with their mathematics degrees or even higher degree certification in mathematics. A second gap in PD shown from Table 17.1 is that of Knowledge of Assessment, particularly for mathematics teachers teaching secondary and pre-university levels. The philosophy, types, purposes of assessment, as well as different modes of assessment are taught at pre-service teacher education programmes in NIE. However, MOE dipstick surveys in schools discovered that experienced teachers in schools need refresher PD courses on assessment literacy or other assessment-related courses in view of the changing GCE "O" and "A" levels mathematics examination question types. One such example is the recent focus on Problems in Real-World Contexts at secondary and pre-university mathematics examinations.

17.3.2.2 Professional Development Through Customisation

Secondly, another PD avenue for mathematics teachers is through customised courses or school-based PD. TRAISI course registration is limited to two teachers from the same school. Customised courses address a need in schools for tailor-made courses by experts to help springboard from the entry levels of specific groups of teachers in a school or cluster. Such courses are particularly popular with professional learning communities and range from a three-hour workshop to a series of consecutive workshops. There are several advantages to a customised approach to PD. As most of the customised courses are conducted in schools, teachers are in their "home ground" working with familiar colleagues, and are hence more willing to engage in an open discussion because of the natural conducive environment. Unlike TRAISI courses which run on standardised timeslots, customised courses can be conducted during periods of time convenient to both the instructor and the participants. Moreover, the customised approach to PD is not bound by the formality of institute-based PD programmes such as tests, examinations, assignments and projects. Generally, there is no prescribed syllabus from MOE for customised courses. These courses are often crafted out of school-based needs analysis where the teacher's voice is heard. In some cases, mathematics educators may offer customised courses to schools in line with their research focuses. In other cases, TRAISI courses can be re-modelled to a customised version should there be a need. Majority of the customised courses for mathematics teachers offered in 2017 are those for pedagogical content knowledge (17 for Primary, 6 for Secondary and 1 for Pre-U). Only one and two courses were offered for subject matter knowledge for Secondary and Pre-U levels respectively in 2017. There were no customised courses on knowledge of assessment in 2017. Similar to what was observed for TRAISI PD courses, it appears that most customised courses centred on MPCK. The two gaps still remain.

17.3.2.3 Research-Based Professional Development

A third PD avenue for mathematics teachers come from research projects. Kaur (2012), Tan and Ang (2015), and Ng et al. (2015) are examples of this. The structure and duration of research-based PD typically follows what is required in the research methodology, not constrained by standardised time frames like TRAISI PD courses. Findings from research-based PD could be used for related TRAISI or customised courses during or after the research project. For example, the contents of a TRAISI course on mathematics modelling by Ng (2017) were reviewed as a result of findings from a research-based PD.

17.4 Some Constraints, Issues and Challenges in Mathematics Teacher Professional Development

The analysis of the TRAISI and customised mathematics PD courses from NIE above seems to show a lack of Subject Matter Knowledge and Knowledge of Assessment PD courses for mathematics teachers. However, a brief scan into the PD course list by MOE and AST reveal that other PD courses on assessments conducted by MOE assessment curriculum specialist are available for teachers. In addition, schools have been known to engage consultants to work with them on their assessment practices. Nevertheless, the case is not the same for Subject Matter courses. PD administrators face constraints and challenges when trying to address this situation. As cautioned by Lim-Teo (2009), teachers prefer to sign up for generic or MPCK courses based on their interests rather than courses on Subject Matter Knowledge which may address their areas of weakness (p. 72). Thus, even when courses on Subject Matter Knowledge are offered, the enrolment for the course may not be sufficient to warrant running it due to high overheads costs. Although such courses have been offered for secondary mathematics teachers, enrolment has declined over the years. In some instances, the courses did not run despite being offered.

The Advanced Diploma in Primary Mathematics Education programme has tried to incorporate courses on Subject Matter Knowledge, MPCK and Knowledge of Assessment. However, this programme has seen a decline in enrolment since 2012. One main reason for this is that the programme was offered on a 13-week full-time immersion basis at NIE during a teaching semester. Because participants needed time off from work to attend the programme, recruitment for the programme was done through top management in schools. There had been challenges mediating between school staff deployment needs and ensuring there were enough minimum cohort size for the programme to be activated. Although teachers have expressed their interest to become full-time participants in the programme so as to focus on their learning journey, it has been very difficult for them to apply for staff development leave to attend this programme because of certain stipulated time frames for leave in order to minimise disruptions to school functioning needs. In response to these challenges, PD administrators presented a revised dual pathway for the programme in 2017 where interested teachers can enrol in the programme on part-time or full-time pathways in a modular stackable structure. Nonetheless, it was a dismay to many that enrolment was still insufficient to meet the minimum class size.

Last but not least, Singapore teachers are offered a wide array of PD courses from NIE, MOE, AST and other organisations; not to mention those from private vendors, professional learning communities and conferences. An abundance of courses to choose from would ensure that the 100-h of PD encouraged by MOE is well-spent or beyond, albeit physically and mentally exhaustive for some teachers who might not be able to make choices as to which courses to attend. Although there have been attempts to streamline course offers from NIE, MOE and AST, more could be done to work out long-term PD plans for teachers, schools and clusters where specific needs are met with discerning choices of PD. This may ensure a greater and perhaps a more sustainable impact from the PD courses.

17.5 A Conceptual Framework for Mathematics Teacher Professional Development for the Twenty-First Century

An analysis of the mathematics PD climate surfaces the need for a conceptual framework to describe the rationale and progression of PD courses. Such a framework presents a strategic overview of the role of PD within the continuum of teacher education and beyond, incorporating factors of influence, and the emphases of NIE TE21 and AST TGM. The framework will also assist in reflections on possible connections or deliberate overlaps between PD courses and the purposes they serve with respect to the domain map of mathematical knowledge for teaching proposed by Hill et al. (2008). Lastly, this conceptual framework can serve as a point of reference during a comprehensive review of PD courses and programmes for mathematics teachers in time to come, so as to streamline efforts in planning future research-practice nexus.

Figure 17.3 illustrates this conceptual framework. NIE TE21 recommendations underpin the conceptual framework. Mathematics PD courses and programmes can provide platforms for teachers to move on to higher degree or lead to further research by mathematics educators at NIE (as shown by the thick arrows representing pathways of further opportunities). Factors of influence have impact on PD designs and focuses which in turn, have impact on Pedagogical Content Knowledge, Subject Matter Knowledge and Knowledge of Assessment (as shown by the thinner arrows). There is mutual impact between PD and TGM goals from AST (represented by the double arrow).

17.6 Conclusion and Future Directions

This chapter began with a discussion of the key stakeholders in the PD landscape of Singapore and highlighted their vision of teacher education in the twenty-first century. There is a synergistic tripartite collaboration between NIE, MOE and AST in providing a holistic teacher PD in Singapore. The chapter further analysed the various factors of influence (i.e. international comparative studies, research) which bring about curriculum initiatives and thereby have impact on mathematics PD. The structure of NIE mathematics PD was outlined in view of TE21 and TGM. Mathematics PD courses were then classified according to aspects of teacher knowledge derived from research and some insights into the content, pedagogical and assessment focuses of current PD offerings were gleaned. Finally, a case is built for a proposed

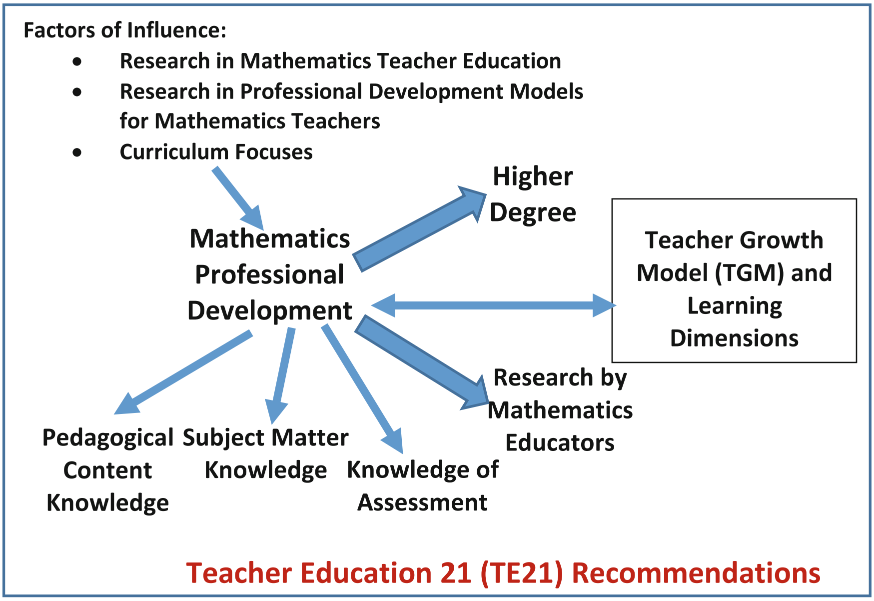

Fig. 17.3 A conceptual framework for mathematics professional development

conceptual framework to describe mathematics teacher professional development for the twenty-first century.

Some implications could be drawn from the conceptual framework and the analysis presented in this chapter. Firstly, a comprehensive review of mathematics PD could follow from this chapter. This review could examine the role of existing mathematics PD within the continuum of teacher education and propose efforts to streamline TRAISI and customised PD courses in view of pre-service teacher programmes. In line with TGM, new and revised PD courses which are connected could be planned in the form of progressive series of PD tailored for schools and clusters.

Secondly, there could be coordinated research into the impact of mathematics PD in schools as well as the sustainability and application of knowledge gleaned from PD. Though ambitious, longitudinal studies could be done to track a cohort of teachers as they advance from pre-service to experienced teachers on their teacher knowledge base expansion pertaining to specific mathematics content topics.

Thirdly, there has been a dearth of research on MPCK since those reported in the chapter. Recruitment requirements of pre-service teachers at NIE have changed much after the time frame of Lim-Teo et al.'s (2009) and Wong et al.'s (2010) research. The time is ripe for more current insights into teacher knowledge base in mathematics from robust research that would be sure to contribute to pre-service and PD course designs.

Lastly, research on effective PD models in the context of Singapore could continue, developing ways to provide impactful PD within the constraints. There could also be more dialogue or collaborations among mathematics educators to share ideas about various PD models. Professional learning communities spearheaded by like-minded mathematics educators working together could be formed with teacher participants from various PD with connected contents, perhaps further extending the impact and sustainability of PD.

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Chapter 18
Models of Teacher Professional Development

Berinderjeet Kaur, Lu Pien Cheng, Lai Fong Wong and Cynthia Seto

Abstract It is a known fact that the usual "deficit" model of teacher professional development (PD) is no longer effective in developing teachers professionally. This chapter presents three models of continuing PD that exemplify a critical development in the professional development of teachers in Singapore and also many parts of the world. This development reflects a shift in the centre of gravity away from the University-based, "supply-side", "off-line" forms of knowledge production by university professors for teachers towards an emergent school-based, demand-side, on-line, in situ forms of knowledge production by teachers with support from university professors. The first model is a hybrid one that integrates the "training model of PD" with sustained support for mathematics teachers to integrate knowledge gained from the PD into their classroom practice. The second model is the laboratory class, a school-based PD programme for primary school mathematics teachers that evolved from a lesson study process. The third model is networked learning communities. In such communities, teachers work and learn collaboratively to examine and reflect on their practice. As teachers learn *from* one another, *with* one another, and *on behalf* of others, they are engaged in purposeful and sustained developmental activities to co-create knowledge and share it with their fraternity.

Keywords Models of teacher professional development · Hybrid model · Laboratory class cycle · Networked learning communities · Singapore mathematics teachers

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18.1 Introduction

Since 1998 the professional development (PD) of all teachers, including mathematics teachers, in Singapore, is guided and supported by the Ministry of Education (MOE) and other professional bodies. Teachers are entitled to 100 h of training and core-upgrading courses each year to keep abreast with current knowledge and skills (MOE 2005). With the adoption of the Professional Learning Communities (PLCs) framework, in 2010, teachers in schools belong to learning teams (Training and Development Division 2010). The framework of the PLCs focusses on three aims—improving student learning; building a culture of teacher collaboration; and addressing four critical aspects of outcomes couched in terms of collective responsibility: What is it we expect students to learn? How will we know when they have learned? How will we respond when they do not learn? How will we respond when they already know it? Teachers work and learn collaboratively through participation in a variety of professional development activities (see, Kaur and Wong 2017).

There are several models of continuing PD (Kennedy 2005) and they may be categorized as transmission (training, award-bearing, deficit and cascade), transitional (standards-based, coaching/mentoring, community of practice) and transformative (action research and transformative). This categorization suggests increasing capacity for teacher autonomy as one moves from transmission through transitional to transformative categories (Fraser et al. 2007). For specific purposes, teachers may be developed through appropriate models of continuing PD. Research has shown that effective continuing PD for mathematics teachers involve experimenting in their classrooms and reporting back to the group, working collaboratively with fellow teachers, having time away from school to think and discuss common issues they faced in their classrooms and in addition to the pedagogical aspects of their deliberation also doing some mathematics (Joubert et al. 2010).

In this chapter, we present three models of continuing PD that exemplify a critical development in the professional development of teachers in Singapore and also many parts of the world. This development reflects a shift in the centre of gravity away from the University-based, "supply-side", "off-line" forms of knowledge production by university professors for teachers towards an emergent school-based, demand-side, on-line, in situ forms of knowledge production by teachers with support from university professors. The first model is a hybrid one that integrates the "training model of PD" with sustained support for teachers to integrate knowledge gained from the PD into their classroom practice. The second model is the laboratory class, a school-based PD programme for primary school mathematics teachers that evolved from a lesson study process. The third model is networked learning communities. In such communities, teachers work and learn collaboratively to examine and reflect on their practice. They are engaged in purposeful and sustained developmental activities together in which they learn *from* one another, *with* one another, and *on behalf* of others.

18.2 The Hybrid Model

In Singapore, in-service mathematics teachers continue to develop themselves through many ways. One of the ways is through participation in a research project facilitated by professors at the National Institute of Education (NIE), the sole institute for teacher education in Singapore. Two such past projects were EPMT-R&C (Enhancing the Pedagogy of Mathematics Teachers to teach for Reasoning and Comunication) (Kaur 2009, 2011) and EPMT-TfM (Enhancing the Pedagogy of Mathematics Teachers to Teach for Metacognition) (Kaur et al. 2017). In Singapore, these projects initiated the shift of PD activities from the "training model" (Matos et al. 2009, p. 167) to the "hybrid model" (Kaur 2011, p. 791).

In the training model of PD, teachers attend courses conducted by specialist officers from the mathematics Curriculum Planning and Development Division of the MOE or professors from the NIE. These courses conducted for about 3 h per day span four to ten consecutive days or days spread over some weeks. Almost always following the completion of such a course, there is no follow up with the teachers about the use of the knowledge acquired and any impact that knowledge may have had on student achievement. Research has shown that such courses are ineffective. This is so as teachers are likely to reject knowledge and skill requirements when

(i) the requirements are imposed or encountered in the context of multiple, contradictory and overwhelming innovations;
(ii) they are excluded from the development of the courses;
(iii) PD is packaged in off-site courses or one-off workshops that are alien to the purposes and contexts of their work; and/or
(iv) they experience them alone and are afraid of being criticized by colleagues or being seen as elevating themselves on pedestals above them (Hargreaves 1995).

The hybrid model of PD (Kaur 2011) integrates the "training model" (Matos et al. 2009) with sustained support for teachers to integrate knowledge gained from the PD into their classroom practice. The model has five significant features. The features are:

- *Content focus*

The PD is focussed on what to teach and how to teach (Stiff 2002; Desimone 2009). It is specific to the pedagogy of mathematics. This focus is similar to that of most in-service courses conducted for mathematics teachers in Singapore as the main objective of such courses is to introduce teachers to new initiatives that arise from curriculum revisions. Teachers participating in the PD work with mathematical content that is appropriate for the grade levels of their students.

- *Coherence*

The PD is coherent with the needs of the teachers. The PD supports the instructional activities of teachers at school, such as the adoption of initiatives (Stiff 2002;

Desimone 2009). Ball and Cohen (1999) have argued that classroom activities can form the basis of constructive professional development, and many other researchers have also determined that effective PD is embedded in teacher work (Clarke 1994; Abdal-Haqq 1995; Hawley and Valli 1999; Carpenter et al. 1999; Elmore 2002).

- *Duration*

The duration of the PD is at least two years and comprises three phases. Teachers attend training workshops for the first part, followed by a period of school-based work guided and monitored by the university professors (PD providers) and followed by another year or more of self-directed school-based work. The duration of the PD is significantly longer than most in-service courses that mathematics teachers usually attend.

- *Active learning*

The PD engages teachers in active learning (Wilson and Berne 1999; Desimone 2009). It includes training, practice and feedback, and follow-up activities (Abdal-Haqq1995), consistent with Stiff (2002), who suggested that teachers learn best when observing, planning for classroom implementation, reviewing student work, and presenting, leading, and writing. As stated earlier, Ball (1996) also claimed that the most effective professional development model includes follow-up activities in the form of long-term support, coaching in teachers' classrooms and on-going interactions with colleagues.

- *Collective participation*

In the PD, there is collective participation at two levels—school and project (all schools participating in the PD belong to a project). At the school level, at least four teachers, with pairs of teachers teaching the same grade year and mathematics programme participate. These teachers work together during the training workshops and also at school when implementing their learning in their classrooms. At the project level, teachers also work together building their knowledge by participating in sessions during which they critique their peers' work and share their experiences and difficulties encountered during the implementation of their newly gained knowledge.

In the next two sub-sections, we describe two PD projects carried out in Singapore that adopted the hybrid model. The two PD projects illustrate a form of PD for mathematics teachers that are gaining momentum in Singapore. This is so as the PD is nestled in the classrooms of the teachers and addresses their needs. The three phases of the PD, namely, ***Learn*** (Acquisition and co-construction of knowledge), ***Apply*** (integrate new knowledge into classroom practice) and ***Teach*** (develop fellow teachers nationally and/or internationally) appear to make the engagement of teachers in PD holistic (Kaur et al. 2017).

18.2.1 Enhancing the Pedagogy of Mathematics Teachers to Teach for Reasoning and Communication Project

The 2006 revision of the Singapore school mathematics curriculum and research findings of Ginsburg et al. (2005) and Kaur et al. (2005) spurred the conceptualization of a PD project—EPMT-R&C (Enahncing the Pedagogy of Mathematics teachers to Teach for Reasoning and Communication) that was carried out in ten Singapore schools for two years. Forty teachers, 22 from five secondary schools and 18 from five primary schools participated in the PD.

The PD was coherent with the needs of the teachers as the teachers relied heavily on textbooks for their daily work and there was a need for teachers to draw on textbook questions as starting points and craft tasks that would engage students in reasoning and communication. During the first phase of the PD, teachers attended training workshops conducted by the university professors. The workshops were organized as two modules, the first centred around crafting of tasks that would engage students in reasoning and communication and the second centred around teaching for understanding. Each workshop began with the university professor introducing the teachers to an idea. In the case of the first module, they were introduced to ideas of how typical textbook questions could be crafted into tasks that would engage students in reasoning and communication. The module introduced the teachers to eight strategies (Kaur 2012; Kaur and Ghani 2011). The second module focussed on the "why, what and how" of teaching for understanding. This module engaged teachers in planning lessons and crafting/selection of appropriate learning and practice tasks.

In the second phase of the PD, teachers were encouraged to infuse in their lessons their learning from the training workshops they participated in during the first phase of the project. Teachers were given specific assignments by the university professors. While teachers were working on their assignments, the university professors' facilitated fortnightly meeting sessions during which teachers shared their work with the others and invited critique. It was during these sessions that teachers' shared with the rest of the project participants their tasks, lessons (through video records), students' work and students' voices. They invited both applause and critique. We must say that after the first few sessions, the activity picked up momentum and teachers became more "welcoming" of critique. It was during these sessions that teachers were meaningfully engaged in the production of pedagogical knowledge, creating and testing their plans, most importantly taking into consideration their students' inputs like what made the lessons enjoyable and meaningful (Kaur 2010, 2013a, b). Using video records of their lessons they watched the performance of their students in class, reflected on their goals and evaluated their lessons. These actions led to revision/modification of plans for subsequent lessons. Towards the end of this phase, teachers submitted their assignments. The assignments submitted by the teachers led to the publication of the resource *Pedagogy for engaged mathematics learning* (Yeap and Kaur 2010).

During the third phase, teachers were left to work with their project mates in their schools to advance the knowledge they had gained from the first two phases.

The university professors facilitated monthly meeting sessions during which project participants were engaged in a variety of activities:

(i) They continued to share their "highs and lows" of lessons that engaged students in reasoning and communication and also lessons that "taught for understanding".
(ii) They prepared exemplars of mathematical tasks that were suitable for engaging students in reasoning and communication (primary and secondary), for publication as print resources (Kaur and Yeap 2009a, b) for mathematics teachers in Singapore schools.
(iii) Teachers participating in national conferences, school-based and cluster level presentations prepared their presentations.

Following participation in the PD teachers from two schools went on to enlarge their community of practice and scaled up the intervention school-wide. The experts, teachers who had participated in the PD, were able to enlarge their school-based community of practice subsequently from four teachers each to 18 in the first school (primary) and 12 in the second school (secondary) (see Kaur 2015 for details).

18.2.2 Enhancing the Pedagogy of Mathematics Teachers to Teach for Metacognition Project

Following the review of the Singapore school mathematics curriculum in 2012, a group of teachers, university professors and curriculum specialists, examined the outcomes of three significant studies related to student achievement in mathematics. The studies are

(i) Programme for International Student Assessment (PISA) of 2009 (OECD 2010) and 2012 (OECD 2013);
(ii) Trends in International Mathematics and Science Study (TIMSS) of 2011 (Mullis et al. 2012; Kaur et al. 2013) and 2007 (Mullis et al. 2008; Kaur et al. 2012);
(iii) CORE 2 research at the National Institute of Education (NIE) by Hogan et al. (2013).

The findings of PISA and TIMSS showed that the majority of Singapore students are very good in applying their knowledge in routine situations and this is definitely a consequence of what teachers do and use during their mathematics lessons. Hogan et al. (2013) found that there was a dominant use of performative tasks compared to knowledge-building tasks in grades 5 and 9 mathematics lessons that they studied. A performative task mainly entails the use of lower order thinking skills such as recall, comprehension and application of knowledge while a knowledge-building task calls for higher order thinking skills such as synthesis, evaluation and creation of knowledge. From the findings of these three studies, the group hypothesized that for students in Singapore to scale greater heights, teachers need to nurture metacognitive learners who are active and confident in constructing mathematical knowledge. Thus, a PD project—Enhancing the Pedagogy of Mathematics Teachers to Teach for Metacognition—was conceptualized, as the greatest source of variance in the learning equation comes from teachers (Hattie 2009), and forty in-service secondary mathematics teachers from seven secondary schools participated in the PD.

The PD was coherent with the needs of the teachers. It focussed on higher order thinking and use of metacognitive strategies for learning acknowledging the place of metacognition as one of the five components of the school mathematics framework that nurtures mathematical problem solvers. In addition, it also addressed a gap in instruction identified by Hogan et al. (2013), i.e. the disproportionate low use of knowledge-building tasks by teachers to engage learners in higher order thinking during mathematics lessons. The PD was facilitated by a professor from the NIE and a lead teacher from a secondary school. In the first phase of the PD, teachers attended seven 3 h long knowledge-building workshops (see Kaur et al. 2015, 2016, 2017 for details). During the second phase, teachers worked in a group at the school level, planned a lesson that used knowledge-building tasks and engaged students in metacognitive strategies for learning. They wrote a detailed lesson plan for the lesson they were carrying out. One teacher from the group taught the lesson to his/her students and the lesson was video recorded. The teachers in a school met and viewed the lesson and prepared their presentation for the PD group sharing meetings. Two PD group meetings were held. During the PD group meetings, the teachers from a school that presented solicited feedback from the PD group. All participants in the PD group except the teachers from the presenting school participated in the feedback session. They used the "four lens noticing" feedback framework to give their feedback (see Kaur et al. 2017 for details).

Following the PD group sessions, the facilitators of the PD organized a meeting with every group of teachers school-wise. Each meeting lasted between 2 and 3 h. During the meetings, the feedback from the PD group was discussed and addressed. The feedback was very helpful as it provided the views of many more pairs of eyes reviewing the lesson. In addition, during the meetings the facilitators of the PD inducted the teachers into a four-step approach to facilitate working and learning collaboratively when integrating their new knowledge into classroom practice (see Kaur et al. 2017 for details). During the third phase of the PD, teachers continued to integrate their new knowledge into classroom practice. They attended periodic PD group meetings and shared their lessons inviting critique and suggestions for improvement. They also organized a national level "Teaching for metacognition" event during which teachers from the PD project presented their lessons. More than 200 secondary school mathematics teachers attended the event. Teachers who attended the event and were keen to learn more about "how to craft knowledge-building tasks and engage learners using metacognitive strategies" were matched with those in the PD project who were keen to develop fellow teachers. Teachers from three schools also presented their work at conferences.

18.3 The Laboratory Class

This section reports on the laboratory class (LC), a school-based PD programme for primary school mathematics teachers that shares some common characteristics with the lesson study process. In Japan, the purpose of Lesson Study (LS) is for teachers and researchers to develop deeper knowledge and the expertise necessary to provide "students with opportunities to understand basic ideas, and support their learning so that the students become independent learners" Takahashi (2017, p. 48). In LS, teachers conduct collaborative "study–plan–do–reflect inquiry cycles designed to improve classroom instruction (Lewis and Hurd 2011; Takahashi 2014; Wang-Iverson and Yoshida 2005)" (as cited in Lewis and Perry 2014).

According to Takahashi (as cited in Fuji 2014), "Lesson study in Japan takes place at three different levels: the individual school level; the district or regional level; and the national level" (p. 67). The well-defined structure of the school research organization for LS (comprising of grade-level group, grade-band teams, research steering committee) is possible because "Grade-level groups typically exist in Japanese elementary schools to facilitate the sharing of responsibilities for running school events and for academic activities" (Takahashi 2017, p. 52). Several characteristics of the Japanese LS are elaborated in Fuji (2014), one of which is the "lengthy period of planning a lesson before it crystallises into a detailed lesson proposal or lesson plan" (Fuji 2014, p. 67). The planning of a lesson and task may take more than half a year. Another characteristic is that it is teacher-led, with teachers taking the initiative. Also, "[in] Japanese lesson study, continuity is a fundamental feature" (Fuji 2014, p. 71).

The concept of LC, reported in this section, is influenced by the 2004 Centre for Proficiency in Teaching Mathematics Summer Institute (see www.cptm.us/Summer_Institute_2004.html). The LC provides a context to study teaching and learning. One of the key features of the LC is that it enables the experimentation of pedagogies and curricular approaches with a community of practice. Another feature is that the LC provides a platform for teachers to engage "directly with practice, not only through observing live teaching, but also co-planning it with the laboratory class teacher" (Naik and Ball 2014, p. 42). It also involves teachers "reflecting on the enactment in collaboration" (Naik and Ball 2014, p. 42). The LC, reported in this section, was mostly employed as the school-based PD structure by Cheng who was both the researcher and professional developer (the university professor). The LC offers a "light" structure for teachers to examine teaching, learning and assessment in the Singapore primary mathematics classrooms. "Light" here refers to the planning of the research lesson that does not require more than half a year. The structure is also light as compared to the highly integrative structure of grade-level group, grade-band teams and research steering committee described by Takahashi (2017) in the school-wide lesson study. The light structure was suitable as teachers who participated in the LC were involved in teaching more than one subject and had to attend to other subject(s)-related level meetings and PD activities.

Table 18.1 One LS cycle for a mathematics topic in School W

Step	Stage	Activities
1 and 2	Planning	• Define the problem and plan the mathematics research lesson
3	Observing	• Teach and observe the research lesson Research lesson taught by Teacher A in the team
4 and 5	Reflecting and revising	• Critique, reflect and discuss the research lesson after classroom observations • Revise the research lesson
6	Observing	• Teach and observe the revised research lesson Revised research lesson taught by Teacher B in the team
7	Reflecting	• Critique, reflect and revise
8	Sharing	• Share the results

In the following sections, we first present the LS we did in School W. The strengths and weaknesses of the LS inspired us to transit the PD to school X. However, the key weakness, being extensive time (common time slots for teachers to meet) spurred us to transit to the LC in School X and the LC evolved mainly as a result of the existing structure and space in schools to support the PD of teachers. The tools to support critical reflection of the research lesson also evolved when the reflecting stage (in Schools X, Y and Z) took place about one to two weeks after the research lesson, and it was a challenge for teachers to recall specific details of the lessons. Due to similarities of the LC employed in schools X, Y and Z, we only present our work in schools W and X.

18.3.1 Lesson Study at School W

18.3.1.1 Structure

The lower primary mathematics teachers employed Stigler and Hiebert's (1999) eight steps for collaborative LS at School W to design and implement the mathematical task that required primary two students to explore and apply mathematical concepts. Table 18.1 illustrates the eight steps that School W employed to plan, observe, reflect and revise mathematics lessons. The school administrator and the mathematics level head participated fully in the activities of the LS so that they could lead the PD when the university professor leaves the school.

18.3.1.2 Strengths and Weakness

The teachers expressed appreciation that LS offered a structured system for PD and development of rich mathematical tasks within the school context. *Planning* stage allowed the teachers to identify a common topic that the team was interested in studying. For example, the team decided to examine the concept of fraction as they found this concept to be challenging for primary two students. Next, the team discussed common errors made by students for the topic identified before unpacking specific strategies to address those errors. Such in-depth discussions were very useful for the teachers as they were assisted to develop in areas that they were interested, for example, explaining and unpacking the concept of unit fractions for young children.

Because the school had a common scheme of work for the primary two cohorts, one LS cycle for a mathematics topic had to be completed within a stipulated period. The *Reflecting* and *Revising* stage was implemented within a few days after the observation of the research lesson so that the LS team could revise and teach the revised lesson in another class within that week. The *Observing* and *Reflecting* stages provided the LS team a common platform to study student thinking. Through this activity of examining and analysing students' thinking and learning, greater appreciation of the multiple perspectives of how children learn was developed. As a result, this enhanced teachers' ability to look beyond their existing perceptions and interpretations of student misconceptions and student learning difficulties.

However, extensive time was required for students to fully explore and investigate the problems. The LS process also required substantial time and commitment from the team members especially when the LS team was required to revise, teach and reflect the revised lesson. In addition, the teachers in the LS team were also teaching subjects other than mathematics, and it was a great challenge to find common timeslots to meet.

18.3.2 Laboratory Class at School X

18.3.2.1 Structure

In School X, two teams (lower primary mathematics team and upper primary mathematics team) participated in the LC to discuss more systematically problems they encountered in teaching specific mathematics topics and to address pedagogical issues related to the topics identified. Similar to School W, one school administrator participated in the entire study so that she could lead the professional development when the university professor leaves the school. Unlike School W, School X did not revise, teach and reflect on the revised lesson. Table 18.2 shows an example of the LC employed by School X.

Table 18.2 One LC for a mathematics topic in School X

Step	Stage	Activities
1 and 2	Planning (about 4 h)	• Define a problem during the first meeting • Plan the mathematics research lesson
3	Observing (1 h)	• Teach and observe the research lesson
4	Reflecting (1 h)	• Critique, reflect and discuss the research lesson after classroom observations

18.3.2.2 Strengths and Weakness

Similar to School W, the teachers found common benefits of the LC. Because each of the teams comprised of members from diverse backgrounds (e.g. for the upper primary mathematics team, teachers teaching standard and/or foundation mathematics for primary 5 and/or 6), planning lessons became very meaningful when team members were able to build ideas (e.g. teaching strategies to cater to the learning needs of their students) from one another. By becoming more critical of the design of their mathematics lessons with a focus on possible students responses to the lesson, the teachers were able to cater to the diverse learners in their mathematics classrooms. The opportunities for teachers to receive feedback about their hypothesis of how students learn during the *Observing* and *Reflecting* stage were important for School X. Because each team comprised of mathematics teachers from different grade levels, the school administrator and "expert from outside the school", feedback from the various groups of people enabled the teams to develop different perspectives of the same issue. Learning and getting feedback from the expert was viewed as a more efficient way of learning. Also, the team's mathematical and pedagogical investigations were more theoretically rooted when a more Knowledgeable Other was around to lead the teams. Different from School W, School X had the opportunity to deal with upper primary mathematics topics. The teachers in the upper primary mathematics team in School X gained deeper understanding of fraction division concept when they observed the research lesson on fraction division.

Overall, School X found the activities of the LC engaging and hands-on because they were involved in planning, observing and reflecting on the mathematics lessons. Despite its advantages, School X still found the LC time-consuming especially the *Planning* stage as it required team effort to conceptualize the research lesson grounded in sound theories and this takes time. Unlike School W, the *Reflecting* stage took place about one to two weeks after the *Observing* stage. Hence, it was difficult for teachers to recall specific details of the research lessons.

Table 18.3 Sample of question prompts in teacher reflection log

Session	Summary/activities	Question prompts
1	Discussion on students' learning difficulties in the topic identified	My thoughts about session 1 …
2	Planning the research lesson	My thoughts about session 2 …
4	Observing the research lesson • Focus on students thinking and learning • Identify and record incidents when unexpected students responses occur • Consider the students' perspectives and try to offer any explanations for the occurrence of those incidents	
5	Reflection of the research lesson	What is one critical moment that I observe? What did the teacher do? Why and how did the teacher do that? What were the students' responses? What could I have done?

18.3.3 Reflection Log

To facilitate teachers' recollection of the activities of the LC, a reflection log was used in Schools Y and Z. Table 18.3 shows samples of the question prompts in the reflection log.

18.3.4 Conclusion

A number of factors contributed to the teachers' engagement in the LC. The LC was useful to the teachers because they were directly assisted in teaching mathematics through the activities of the LC. Teachers had more ownership of the LC when they were both receivers and givers of assistance in the LC. Although the feedback given to the learning teams positively influenced the teachers' continual learning, a certain culture needed to be put in place in the school environment before continual professional learning from and with colleagues could take place using the LC e.g. supportive platforms for all the teachers to share their ideas and not be perceived negatively. Those views and suggestions could then be used to generate and build appropriate teaching ideas so that teachers are directly assisted to grow professionally. A different perspective by a more Knowledgeable Other was also important for knowledge-building and experimentation of innovative pedagogies in the LC. The LC required collaborative teams to identify common shared goals for the team to engage in the inquiry process of defining the problem of investigation, planning, carrying out, observing and reflecting on the research lesson.

18.4 Networked Learning Communities

The Ministry of Education launched the Academy of Singapore Teachers (AST) in 2010. The mission of AST is to build a teacher-led culture of professional excellence, centred on the holistic development of the child. Recognizing the critical role of collaborative professionalism in developing teachers, AST adopts networked learning communities (NLCs) as one of the strategies to raise the level of professional practice in the classroom and teaching expertise across the system (Seto et al. 2018). Through NLCs, teachers have the opportunities to display a higher level of ownership of their professional learning and enlarge their influence by leading and guiding other teachers. In Singapore, an NLC is broadly defined as a team of teachers from different schools working together to learn from one another, learn with one another and on behalf of others, as they share and co-create new knowledge and practices to improve student outcomes (Jackson and Temperley 2007; Katz et al. 2009; MOE 2017).

Central to the notion of professional development of teachers through NLCs is the concept of "Teacher Ownership and Teacher Leadership". Teacher ownership and leadership are attained "when teachers, driven by a sense of mission, individually or collectively, exert intentional influence to achieve an enhanced state of professional excellence within a climate of trusting and supportive relationships" (MOE 2011). Leadership in NLCs can take many forms. Within the NLCs, teachers take on the roles of learners, researchers and leaders to improve their practice. Teachers assume these various roles, as a part of the teaching fraternity to improve classroom practice and enhance student learning.

AST distinguishes between two types of networked learning communities; namely designed networks that are initiated and managed at AST, and emergent networks that are led by teachers, for teachers. In its infancy stage of development, most NLCs are designed and managed by Master Teachers or appointed teachers. With Master Teachers as instructional mentors and advisors, more emergent networks have been formed in recent years. These emergent networks are networks that have evolved organically, through common interests and needs. Every NLC has a learning focus which is the unifying theme for teachers to collaborate (Seto et al. 2011). Most NLCs have about four face-to-face meetings in a year with online platforms for greater professional collaborations among teachers. Networks can also take the form of subject-based networks, role-based (e.g. Senior Teachers-Lead Teachers) or professional interests-based. The following are examples of NLCs which are formed by (1) subject-, (2) interest- and (3) role-based.

18.4.1 Subject-Based NLC

Subject Chapters are subject-based NLCs helmed by Master Teachers (MTTs) to advance the pedagogies and instructional practice among subject teachers. Subject Chapter activities, such as workshops, are open to all teachers who teach that subject.

For example, a teacher teaching primary mathematics is a member of the Primary Mathematics Chapter. The three objectives of Subject Chapters are to:

- deepen the pedagogical content knowledge of teachers for quality student learning;
- build a culture of teacher-led professionalism and pride in the teaching fraternity; and
- champion professional collaboration and networked learning among teachers (MOE 2012).

The Mathematics Chapter champions quality learning in Mathematics by building teacher capacity, encouraging pedagogical innovations and facilitating research-informed practice to impact student learning. It aims to build a culture of professionalism and pride in the teaching of Mathematics. The Mathematics Chapter serves as a focal point for mathematics teachers to collaborate and network, via face-to-face meetings or collaborative workspace in OPAL (One Portal All Learners), a system-wide learning content management system that engages teachers in online learning, collaboration, asynchronized discussions and exchange of resources.

Two of the designed NLCs in the Mathematics Chapter are the Primary Mathematics Core Team and the Secondary Mathematics Core Team. Within each core team, there are two teacher representatives from each of the four zones (North, South, East and West Zones Schools), a representative each from the National Institute of Education (NIE), the Curriculum Planning and Development Division and the Educational Technology Division. Led by Master Teachers from the Mathematics Chapters, members engage with one another to enquire into the teaching and learning of mathematics. With the diverse range of experiences in the core team, teachers capitalize on their expertise and practical wisdom to contribute towards their shared interest in mathematics teaching. During one of the meetings, a member voiced the challenges of using mathematics tasks in the classrooms. A NIE mathematics educator shared the theoretical perspectives and research findings of using mathematics tasks for engaged learning. Drawing on their practitioner knowledge based on these insights, the teachers redesigned and implemented some lessons on mathematics tasks in their respective classrooms. Interactions among members from diverse expertise therefore strengthen the theory-practice nexus. In *learning from* and *with* one another, the teachers in the core team deepened their pedagogical content knowledge. They also *learnt on behalf* of their peers when they led the professional learning for 125 primary and secondary mathematics teachers through a Mathematics Learning Day on "*Fostering Critical and Inventive thinking through Mathematics Tasks*".

18.4.2 Interest-Based NLC

Interest-based NLCs serve as platforms that empower teachers to assume teacher ownership and teacher leadership of their professional development. Bonded by a common interest to engage in a topic or pedagogy of their choice, teachers collaborate to explore issues of teaching. Activities in the NLCs include co-designing lessons or resource packages, lesson observations, trialling and refining research-informed interventions. Adopting an inquiry stance in NLCs encourages teachers to continually inquire into their practice, discover, create and negotiate new meanings that improve their teaching practice, which often also results in enhanced student learning. This is supported by findings from a study which suggests that teachers participating in an NLC showed changes in teaching behaviours which resulted in a more positive classroom learning environment and student outcomes (attitudes to mathematics) as compared to those teachers who did not participate in networked learning (Seto and Fraser 2014).

An example of an interest-based NLC is the Differentiated Instruction (DI) NLC formed in 2012. This learning team comprised secondary mathematics teachers of diverse teaching experiences (from beginning teachers to senior teachers) and of different designations (from classroom teachers to department leaders), coming from at least six different schools from different zones in Singapore. The formation of the team was organic. Through an informal conversation, the teachers decided to form an NLC based on their common goal to make mathematics accessible to the diverse profiles of students in their respective classes. As such, the team decided to collaborate on “Differentiated Instruction” that would benefit the diverse classrooms and make learning of secondary mathematics more accessible to their students.

Members in the NLC met on a regular basis to first understand and learn the what and why of DI, by reading and discussing related literature, such as Carol Tomlinson’s *The Differentiated Classroom* (Tomlinson 2014), building on and refining one another’s understanding of DI. They then decided on a common topic to design a lesson package to be implemented by members in their own school. Members who did not have a suitable class worked with another teacher in their school to implement the lesson package. When these lessons were carried, other members were invited to observe the lessons and to provide feedback. By carrying the same lesson across the various schools of different student profiles, the lesson idea and implementation got better and better each time as the team collaboratively addressed implementation issues and continued to improve the lesson package. These processes facilitated teachers to co-construct knowledge that has practical relevance to the context of their classrooms.

At the end of the first year, they shared their learning journey with fellow teachers in the fraternity at a learning symposium and invitation to join the NLC was extended to the audience. While some teachers retired from the NLC due to other work commitments, new members came aboard in the subsequent year. The team continued to identify common topics to work to prepare lesson packages. They also continued the good practice of inviting teachers into their classrooms when these lesson packages were carried out. With open minds, members gave feedback, shared concerns, made refinements to the lesson packages before implementing in their own school again. They also shared implementation issues, impact on students’ learning and post-implementation survey findings. The benefits and deliverables of this NLC were not confined within the team as the team unselfishly shared their learning, their reflections and the lesson packages with teachers from other schools at different platforms, such as the Academy NLC Symposium and Zone/Cluster Sharing Sessions. One of the members of the NLC remarked

> It was an NLC where all the like-minded, similar interest teachers came together to make things work… so everyone was pretty aligned in terms of goals and things that we want to do for the students, which really inspires and motivate you to do more for the students in the future as well. (Y.Y. Tan, personal communication, December 14, 2017)

Subsequently, the members of the DI NLC moved on to identify another focus to differentiate mathematics instructions and began a new learning journey.

18.4.3 Role-Based NLC

Role-based networks, such as the Primary Mathematics Lead Teacher-Senior Teacher Network, bring together Lead Teachers (LTs) and Senior Teachers (STs) of the same subject to focus on teacher-led efforts in professional development. LTs and STs are pedagogical leaders and instructional mentors at their respective clusters and schools. For greater outreach to schools, there is a need to be more intentional in the capacity-building of LTs and STs in order to develop them to be more effective in their roles. As such, the main objective of the LT-ST Network is to provide a holistic framework of learning for the development of knowledge and skills in mathematics and leadership skills in generic professional development.

In the inaugural meeting of the Primary Mathematics LT-ST Network on 19th May 2010, 38 teacher leaders (36 STs and 2 LTs) responded to an email invitation to start this network. To-date, there are 150 STs and 19 LTs in the Primary Mathematics LT-ST Network. Besides collaborating on mathematics-related and pedagogy-related matters, this role-based NLC also provides opportunities for STs/LTs to develop their teacher leadership and for AST to grow the teacher-leader pipeline.

As the STs/LTs collaborate to create and share knowledge to expand their repertoire of skills, they grow professionally and contribute to the development of their peers. Seto et al. (2011) found that striving for pedagogical excellence is a compelling need to bring the teacher leaders together, and this fosters a sense of commitment and a shared purpose for the network. Their findings also suggest that the learning in the Primary Mathematics ST-LT Network is more tailored to the needs of STs. The STs also expressed that the learning in this NLC is different from other professional development courses. Teacher leaders feel empowered when they are involved in the professional development of their peers. Their influence stem from the respect they command from their colleagues through their expertise and practice. When the STs/LTs engage in such professional collaborations, they build trust within the group and develop new ideas, which in turn energize teacher leaders to engender new NLCs (Seto et al. 2011). Improving professional excellence is therefore the collective responsibility of all.

18.5 Conclusion

The three models of PD presented in this chapter exemplify a critical development in the PD of mathematics teachers in Singapore. This development affirms a shift from the University-based, "supply-side", "off-line" forms of knowledge production by university professors for teachers towards an emergent school-based, demand-side, on-line, in situ forms of knowledge production by teachers with support from university professors and fellow teachers. It is apparent from the three models of PD presented in this chapter that they are transformative in nature. They facilitate the PD of mathematics teachers in ways that allow teachers to acquire new knowledge, integrate the knowledge in their practice and reflect on their practice such that the learning of mathematics is enhanced. Furthermore, the roles teachers take on in their PD allow them to grow their capacities and also contribute towards the development of fellow teachers. The models engage teachers to learn and work collaboratively at several levels.

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Chapter 19
Teaching Simultaneous Linear Equations: A Case of Realistic Ambitious Pedagogy

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Abstract In this chapter, we present a conceptualisation of mathematics teaching and learning which we term realistic ambitious pedagogy. We locate this pedagogy within the domains of teaching goals and teaching enactment, and the interactions between them. We argue that it is a suitable pedagogy for use in teacher development enterprises because it takes into deliberate consideration the realistic constraints within which teachers work while pursuing ambitious goals of mathematics teaching. To illustrate, we provide an example taken from our work of redesigning a curriculum unit on simultaneous linear equations in two variables with some Year 8 mathematics teachers in Singapore.

Keywords Realistic ambitious pedagogy · Teaching goals · Teacher professional development

19.1 Introduction

The research that is reported here is part of an ongoing school-based teacher development work that we began with Singapore secondary schools about a decade ago. Through this rather long process of learning, we formalised and refined certain innovations that we argue would advance the goals of the teacher development enterprise. One such innovation is the Replacement Unit Strategy. We have reported on this

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strategy in other publications (e.g. Leong et al. 2016a, b). In this paper, we highlight another innovation: realistic ambitious pedagogy (RAP).

When we think of "pedagogy", we do not have in mind the rather narrow conceptions of the term as used in popular discourses—such as captured in condensed phrases like "direct teaching", "cooperative learning", or "student-centred pedagogy". We define pedagogy more broadly as a theoretical conceptualisation of teaching that takes into account the main complexities which may hinder or advance the goals of instruction. We think this framing of pedagogy is more suited to our purpose (and the purpose of other researchers who share our stance), which is to provide a theoretical framework of discussion with mathematics teachers across a wide range of schools.

We begin by explicating RAP before reporting empirical findings based on a specific instantiation of the pedagogy in the case of teaching simultaneous linear equations.

19.2 Realistic Ambitious Pedagogy (RAP)

The term "ambitious" in RAP is inspired by current literature on ambitious teaching (e.g. Kazemi et al. 2009; Lampert et al. 2010). These authors conceived of "ambitious" in the sense of teachers striving for ambitious goals of teaching. They can include goals that target the "big ideas" of mathematics (e.g. Charles 2005; Schielack and Chancellor 2010), the use of problem-solving in learning mathematics (e.g. Hiebert et al. 1996; Lester 2003; Schoen 2003), and the attainment of these goals consistently for all students (Kazemi et al. 2009; Lampert et al. 2010).

We think of "ambitious" as that which targets the teaching of disciplinarity in mathematics. This is influenced by a more discipline-based view of mathematics teaching. Following a long tradition (e.g. Lakatos 1976; Lampert 1990) of seeing pedagogy in the mathematics classroom as rooted in how mathematics is done by mathematicians, we share the commitment of helping students learn the disciplinarity of mathematics in actual mathematics classrooms in Singapore. This means the teaching of big ideas of mathematics and of problem-solving, as mentioned in the previous paragraph. But a reflection on "disciplinarity" would reveal that other mathematical dispositions and skills should be considered too. As examples, it includes the key strands of inductive and deductive reasoning (Lakatos 1976) as a process of reaching conclusions in mathematics; it also includes the need for sense-making when connecting mathematical ideas.

In summary, ambitious mathematics teaching to us is a commitment to go beyond presenting mathematics as a set of arbitrary rules to follow—which, sadly to us, remains the popular image of mathematics almost everywhere; it must target the essential elements of disciplinarity relevant to the mathematical topics under study.

But we think that for ambitious teaching to be a sustainable reality in schools, efforts to engage mathematics teachers in PD settings for this purpose must also take into serious considerations the "realistic" constraints of teaching. In fact, we think that much of the failures in actualising the vision of ambitious teaching at scale are due to an inappropriate treatment of the realistic constraints of practice faced by teachers on a day-to-day basis.

A common contextual constraint experienced by teachers is time pressure (e.g. Assude 2005; Jones 2012; Keiser and Lambdin 1996; Leong and Chick 2011; Meek 2003; National Education Commission on Time and Learning 1994/2005). These studies explicated the role time pressure played in teachers' decision-making: the effect was not merely over isolated situations of little consequence to instructional pathways; rather, constraints in time can significantly hinder the fulfilment of worthy (and ambitious) goals of teaching.

Closely related to time pressure is the commitment by many teachers to prepare their students to do well in high-stakes examinations (Amador and Lamberg 2013; Barksdale-Ladd and Thomas 2000; Diamond and Spillane 2004; Plank and Condliffe 2013; Valli and Buese 2007; Wu and Zhang 2006). Many teachers see it as their implicit social responsibility to help their students achieve high examination scores in school mathematics as a way to help them realise their career choices in life.

The contextual constraints of teaching are not usually thought of as significant enough to be brought into play in the theoretical considerations of "pedagogy". This is the point of departure for RAP: For efficacious teacher development that is committed to teachers' buy into (ambitious) pedagogical changes, we argue that realistic constraints of teaching should not be thought of as a theoretical afterthought that we need to put up with; rather, they ought to be incorporated into the theoretical conceptualisation and development of pedagogy. RAP is thus a teacher-sensitive pedagogy that stays true to its mathematics discipline-driven tradition within the contextual givens of teaching in schools. By pedagogy, we are not restricted to a set of specific teaching methods. Instead, it is a useful theoretical framework within the context of teacher development to discuss teaching goals in the classroom.

19.3 Teaching Goals and Teaching Enactment in RAP

The theoretical premise of RAP is that much of teaching actions and decision-making in the classroom is driven by the teacher's instructional goals (Lampert 2001; Leong and Chick 2007/2008; Leong et al. 2007). As such, while pedagogical studies must include the examination of actual classroom teaching, we think that the starting point of the pedagogical inquiry is not teaching enactment; rather it is the set of teaching goals. This is also substantiated from the standpoint of teacher development: In PD work, the aim is to help teachers reflect upon their pedagogy in ways that would challenge existing conceptions and methods of teaching. To facilitate this inquiry, we think that the discourse with teachers should begin with identifying and examining the goals of teaching mathematics.

It is in the examination of instructional goals that both the realistic givens and the disciplinary ideals of RAP can find their places in the theorisation of pedagogy.

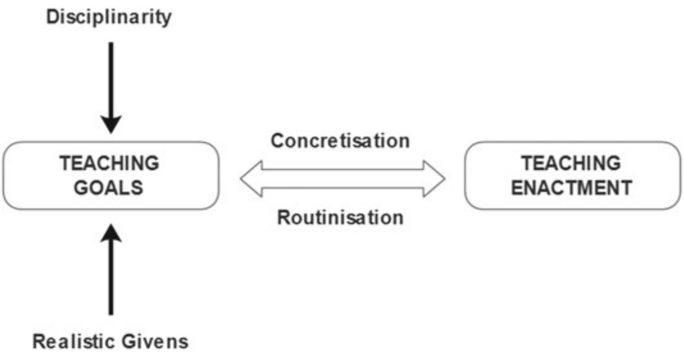

Fig. 19.1 Components and interactions involved in realistic ambitious pedagogy

As discussed in the previous section, some realistic givens discussed in the previous section can be translated into these goal-statements:

R1. Goal-exam: Help students gain fluency with exam-type questions
R2. Goal-time: Balance the use of classroom time judiciously to satisfactorily fulfil all the other instructional goals

Likewise, the ambitious part of the equation can be represented in the form of written instructional goals:

D1. Goal-sense: Help students experience the learning of mathematics as sense-making
D2. Goal-problems: Help students learn mathematics through problem-solving

Thus, in RAP, there is recognition that realistic givens (expressed in the R-goals, R1, R2, …) and aspirations for teaching that mirrors practices in the discipline (expressed in the D-goals, D1, D2, …) are to be included when considering worthy goals of teaching.

In seeking to enact the formalised set of R- and D-goals in the classroom, it is critical that the pedagogical considerations be translated into a form that teachers can use or carry out directly in classroom instruction. For this purpose, we propose the following constructs: concretisation and routinisation. With these components included, the proposed model of RAP is shown in Fig. 19.1.

Concretisation refers to the process of making ideas concrete to teachers. These tangible implements are not merely theoretical conceptualisations of good practices; they are imbued with concreteness so that they can be seen and/or acted on by students and teachers for the purpose of mathematical learning. Examples of these concretised objects are: Activities, including the actual tools for use in the activities and the activity sequences; student tasks, including the task sheets that student work on; and whiteboard working, including the organisation, diagrams, and chronological sequence. Routinisation refers to the process whereby the intended goals are captured in classroom routines in a way that, through repeated rehearsals, they support the fulfilment of these goals. Students' orientations of what school mathematics is and how it is learnt are not easily changed by a one-off innovative task. There is a need for habituation towards this intended instructional goal, and hence the need for routine formation.

The model, as summarised in Fig. 19.1, does not prescribe details as is usually found in recognisable pedagogy. However, for the purpose of teacher development, we think the model contains elements that are essential in the sense that if any of these components are not carefully attended to, ambitious teaching in typical classroom settings is unlikely to be sustainable. RAP is broad enough to encompass a range of different forms of actual classroom enactment yet rooted in worthwhile educational values (as expressed in the teaching goals) and in the commitment to realise these values in class (as expressed in the goals-enactment interaction).

In RAP, we ask these organising questions: (a) How do we bring the R- and D-goals together? (b) How can they be concretised and routinised for classroom use? (c) To what extent is the teaching enactment supportive of the goals? We may further ask the extent in which the goals are realised in the students. But since RAP is developed primarily with teacher PD in view, the focus of inquiry is on teacher enactment. In the next section of the paper, we address these questions in the context of a specific case of unit design that was framed by RAP.

19.4 Context and Method

We are involved in a project with the mathematics department of a secondary school to redesign curricula units for their mathematics classes. This project has been ongoing for more than eight years and we have worked together on a number of secondary level units. The mathematics teachers who participate as co-designers in these units consider the work of planning, developing, and refining the curricula materials a form of professional development. To us, one other goal of the enterprise is to trial and study RAP. We use the intellectual journey of co-designing the unit on simultaneous linear equations in two variables for the two Year 8 classes of the school as a case of RAP.

Chronologically, Questions (a) and (b) relate mainly to activities before the start of the lessons; and Question (c) is aimed at the teaching activities in class during the lessons.

For Questions (a) and (b), the data we drew upon include all the video records of discussion meetings we held with the teachers and the various refinements of teaching materials leading up to the final version for use in class. As the design and PD process is significant in RAP, the focus is on documenting the chronological and intellectual process involved in overcoming the key challenges in the enterprise. Seen in this way, these two questions are not "Research Questions" in the usual sense of requiring a rigorous analytical process to address them. We have nevertheless included them to provide a more complete portrait of RAP.

Question (c) targets the degree to which the teaching goals are realised in teaching enactment. The analysis comes from video records of teaching actions in both classes for each lesson throughout the entire unit. We also draw upon the video records of discussion meetings we held with the teachers and the various refinements of teaching materials during the period of the class enactment. The inquiry is on whether the instructional work of the teachers was supportive of both the R- and D-goals. Since the goals were by design captured—through concretisation and routinisation—into concrete tangibles and routines, the inquiry began with how the teachers utilised these implements to fulfil the intended goals. Further evidence was obtained at other regions of classroom practice where we noticed that teachers appeared to exhibit consciousness in advancing a careful integration of the R- and D-goals. We were able to confirm (or refute) these hunches through the discussion data.

19.5 RAP in the Case of Teaching Simultaneous Linear Equations

We organise this section according to the questions (a)–(c) that we seek to address.

(a) **How do we bring the R- and D-goals together**?

We started by discussing the instructional goals of the topic. As we had worked with the teachers over a number of previous similar curriculum development units, they were comfortable enough to share freely about the realistic demands in the coverage of the topic, summarised as:

r1. To help students develop proficiency with both the technique of "Elimination" and "Substitution"[1]
r2. To inculcate in students the habit of checking by substituting the obtained solution into a suitable equation
r3. To complete the coverage of the topic within five 45-min lessons

In our conception, the topic provides the opportunity to experience something of what learning mathematics within the discipline is like. First, since solving simultaneous linear equations builds on an earlier topic on solving linear equations in one variable, there is an opportunity, in the structuring of the unit, to help students make connections among the mathematical content strands. Second, the concept of "simultaneous" is a recurring idea in Algebra at the secondary school levels. We think it is important that students are not only able to perform the steps in solving the equations but also to develop the disposition of seeking to make sense of critical concepts. In this case, the critical concepts are the idea of "simultaneous" as same

[1] Here, the lower case letters r and d are used to label the goals. This is to make a distinction between this set of goals and the goals listed in the previous section of this paper where they were labelled with capital letters R and D. The relationship between the r- and d-goals and the R- and D-goals respectively is roughly one of subordination. For example, fulfilling r1 within the teaching of this unit supports the fulfilment of some more broad-grained R-goals (such as R1).

variables that are subjected to constraints by both equations, and "solving" as obtaining the solution that satisfies both equations. Third, as the solution steps for this topic tend to be considered long and tedious for many students, it is an opportunity for students to learn to keep track of their thinking and decision-making at critical junctures throughout the solution process. This is an important problem-solving disposition. Thus, through discussions with the teachers and with their agreement, the ambitious set of goals for the topic may be summarised as:

d1. To help students make connections between solving linear equation in one variable and solving simultaneous linear equations in two variables
d2. To make explicit the meaning of "simultaneous" and "solving" as used in this topic
d3. To develop in students the disposition of keeping track of their solution steps and decision-making

It is perhaps important at this juncture to clarify that the r- and d-goals listed do not exhaust the instructional goals for the teaching of this topic. A goal such as helping students correct their errors in algebraic manipulation is certainly necessary for this topic and is indeed included in the teachers' consideration in the planning process. We list only the goals that are directly relevant to this topic under consideration and are overarching across the whole instructional unit. Similarly, it is easy to point out other ambitious goals that are relevant to this topic that are not listed. For example, should we not try to be more ambitious and generalise to the solution of a system of linear equations with more variables? This would certainly be in line with the learning of mathematics within the discipline; but we think that doing so would be more ambitious than is workable. In particular, it would conflict directly with Goal r3 which specifies a fixed number of lessons for the completion of the unit. Thus, the ambitiousness of our goals is also tempered by the realistic situation we work in.

These sets of r- and d-goals were then carefully weighed when we planned the structure of the unit. In the planning process, we discussed possibilities and projected them into the imagined vision of classroom enactment. In particular, we placed heavy emphasis on how the students would respond to certain instructional moves, the difficulties they may face, and how they can be alleviated. The broad-grained trajectory of the unit is summarised in Table 19.1.

By "prominent goals", we mean the goals the teachers should consciously foreground in their classroom instructional moves. This does not mean, however, that no other goals were at play. As an example, though only r1, r2, and d3 are listed as prominent in Lesson 2 as shown in Table 19.1, it does not mean that relating the solution method to earlier methods learnt or the concepts of simultaneity (embodied in Goals d1 and d2, respectively) became unimportant. Rather, it means that these other goals were implicitly embedded in the lesson sequence and were not emphasised to the same degree as the prominent goals. Another example is Goal r3. Although it is not listed in Table 19.1 as prominent in any of the lessons, it tacitly guided the planning of the whole unit in terms of the content to include/exclude and the balance of goals to pursue within the constraint of limited time.

Table 19.1 Overall plan of the unit on solving simultaneous linear equations

Lesson	Key moves	Prominent goals	Main considerations
1	• Recall solution of linear equation in one variable • Consider a single linear equation in two variables—to introduce many solutions • Consider a pair of linear equations in two variables—to introduce the meaning of simultaneous in this context • Motivate a method to solve a pair of simultaneous linear equations in two variables • Introduce method of substitution as a way to reduce the problem to the familiar setting of solving one equation in one variable	d1, d2	This can be considered an introductory lesson that connects prior knowledge with the contents of this topic. But due to r3, we cannot devote the entire lesson to motivational elements. We include in this lesson an overall structure of the method of substitution as a way to deal with the problem presented by the need to solve simultaneous equations
2	• Familiarise students with the method of substitution for simpler equations • Emphasise the overall strategy of reducing the problem into one equation with one variable • Demonstrate the usefulness of substituting the found values into a suitable equation as a way to check that they satisfy both equations—to reinforce simultaneity	r1, r2, d3	This lesson is on the method of substitution. The focus is on helping students take a more zoomed-out view of the process: While solving the equations, they learn to keep track of when and why they apply the required strategies
3	• Use students' work to address key conceptual and methodical errors in their use of the method of substitution • Same focus as Lesson 2, with harder equations that require more careful manipulation	r1, r2, d3	This lesson is a follow-up on the method of substitution. The focus shifts more to procedural fluency at the more fine-grained level as they practise the details of the method over a range of suitably gradated examples
4	• Motivate the learning of method of elimination by contrasting it against the method of substitution—to show relative ease for certain types of simultaneous linear equations • Reinforce the same overall strategy of reducing the problem into one equation with one variable • Familiarise students with the method of elimination for simpler equations	d1, d2, r1, d3	The introduction of another method provides the motivation to revisit and re-emphasise the disciplinary goals The main considerations for Lessons 4 and 5 mirror that of Lessons 2 and 3 respectively, but for the method of elimination instead

(continued)

Table 19.1 (continued)

Lesson	Key moves	Prominent goals	Main considerations
5	• Use students' work to address key conceptual and methodical errors in their use of the method of elimination • Same focus as Lesson 4, with harder equations that require more careful manipulation	r1, r2, d3	

We have also learnt the need to keep the number of listed goals in each lesson small. Beyond a certain number of ostensible goals, it becomes unproductive both for the planning and discussion of teaching. The more goals we load into a lesson, the greater the tendency during enactment for the emphasis on each of the goals to thin out, lessening the success of their fulfilment in the lesson. For the purpose of teacher professional development, it is also helpful to limit the number of goals being emphasised so that teachers can focus their attention on instructional innovations in response to a few goals per lesson instead of having their attention diffused over many instructional goals.

(b) **How can the goals be concretised and routinised**?

The next stage is the concretisation of the goals and plan into a form that would enhance the potential of fulfilment of these goals during classroom enactment. We carried this out together with the teachers by designing student task sheets that embodied the intended learning trajectory of the students. Each lesson in the module was supported by a task sheet. In some lessons, homework task sheets were also developed. Since space is limited, we only discuss selected sections of some task sheets to illustrate how the goals were concretised.

Figure 19.2 shows a section extracted from the task sheet in Lesson 1. The blanks in the extract were meant to draw students' attention to how the solution of simultaneous equations differs from their prior experience of solving linear equations in one variable. The ellipsis "…" signalled an opportunity for students to conjecture and discuss what they think "simultaneous" in this context could possibly mean. This section exemplifies a concretisation of Goal d1.

A follow-up section within the same task sheet illustrates the link to an explicit consideration of what "simultaneous" means (Goal d2), as shown in Fig. 19.3.

In subsequent lessons, the task sheets take on a more familiar form—consisting of exercise questions in which students were required to develop fluency (Goal r1). A typical item in these task sheets features a pair of simultaneous equations, which students are to solve. The item shown in Fig. 19.4 is taken from a task sheet used in Lesson 2.

The space given below the equations in Fig. 19.4 were for students to write the usual steps involved in solving simultaneous equations; the column under "What's

Solving "***simultaneous***" linear equations

What does it mean to solve this pair of simultaneous linear equations?

$$x + y = 7$$
$$x - y = 3$$

I notice: (a) each equation has _____ unknowns; (b) it is now a ***pair*** of equations instead of a single equation; and (c) "simultaneous" here means _____________

To "solve" this pair of simultaneous equations means …

Fig. 19.2 An extract from the task sheet prepared for Lesson 1

An illustration:

Values of x and y in each table satisfy the respective equations

$x + y = 7$	
x	y
7	0
6	1

$x - y = 3$	
x	y
3	0
4	1

…

The same pair of x and y values that I circle in both tables is the solution to the simultaneous equations: _______________

Fig. 19.3 Another extract from the task sheet prepared for Lesson 1

Example 1 Solve the following simultaneous equations $5x - 9y = 17$ $3x - 8y = 5$	What's going on?

Fig. 19.4 An extract from the task sheet prepared for Lesson 2

going on?" was a provision for students to make visible the rationale or prompts corresponding to suitable junctures of the working on the left. This is in line with Goal d3. Under this column, students were expected to write phrases such as "Label equations", "Form one equation, one unknown", "Obtain both x and y values", and "Check" at suitable points of the solution process. Clearly, the task sheet design was not merely to facilitate opportunities to practise (Goal r1), to examine one's solutions steps (Goal d3), and to check the provisional answer by substitution (Goal r2) as ostensible goals. There was also an implicit intent to reiterate the meaning of

Example 1	What's going on?
Solve the following simultaneous equations $x + y = 13$ $x - y = 5$	

Fig. 19.5 An extract from the task sheet prepared for Lesson 4

"solving" "simultaneous" equations as finding the same solution for both equations (Goal d2) and to reiterate the connection between this process of solving to the process of finding the solution of one linear equation with one unknown (Goal d1). Thus, within one exercise item in the student's task sheet was a concretisation and an integration of the goals intended for this module.

The item as presented in this layout was not a one-off rarity. In fact, an item of this nature was a standard feature in most of the in-class task sheets as well as homework task sheets. Figure 19.5 shows an extract[2] of the task sheet used in Lesson 4. In other words, routinisation was also built into design of the task sheets.

(c) **To what extent is the teaching enactment supportive of the goals**?

We base our analysis on 14 video recordings of two classes (seven each), transcript of a meeting where all teachers and researchers reviewed the simultaneous equations unit, and a student perception survey of the 80 students in the two classes that was conducted after the unit was taught. We shall focus only on the enactment of the "What's going on" column as a concretisation of r1, r2, and d3.

The two classes A and B were taught by Teacher X and Teacher Z, respectively. The videos were full recordings of seven 40-min lessons for each class. Teacher X started the first lesson with the "What's going on" column already prominently written on the whiteboard (see Fig. 19.6). He consistently used the column from the beginning to the end for all of his lessons (see Fig. 19.7), except for Lesson 5, when students did board work for the first half of the class. Teacher Z started using the column about 18 min into the first lesson. In Lesson 2, he waited until the student's board work was completed before writing the column on the board. Lessons 3, 5, and 6 were similar to Lesson 1—he started using the column a quarter way into the lesson. He did not use the column at all in Lesson 4. In the final lesson, he had the column written on the board from the beginning of class.

We can see that the two teachers enacted the concretisation of the "What's going on" column faithfully. We now turn to the summary meeting to see what all the teachers, including those who observed the lessons in a lesson study context, have to say about the use of this concretisation towards achieving the desired R- and D-goals.

[2]The task may appear at first look to be a repetition of contents taught in earlier lessons. In Lesson 4, the teacher has moved to teaching the method of elimination. Here, students were asked to use this newly learnt method to solve equations they would have solved in earlier lessons using the substitution method.

Fig. 19.6 Beginning of Lesson 1 in Class A

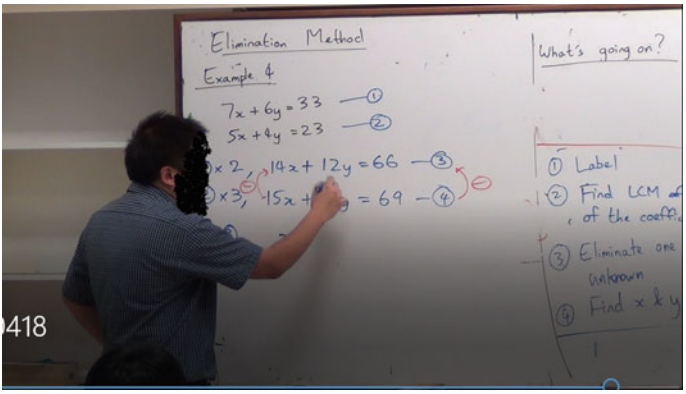

Fig. 19.7 End of Lesson 7 in Class A

They observed that the students treated the column initially as space in which to copy verbatim what the teacher wrote in the same column on the board; then some appreciated its use; and towards the end, some because they had internalised the procedure and others because they had given up, stopped filling in the column:

Teacher Z: *Basically, I think … it is a trend whereby starting there are more people who will write down and follow what you write as you go on, I think once they grasp the concept that column is as good as blank especially for the homework.*
Teacher A: *I observed two lessons. One is Teacher X's first lesson but I think because it's starting so it was getting them to know what that column is about and so when he asked them to write, they did ask them to write down a few things but it's mostly what you wrote on the white board. So they just copied it down. After that I went on to Teacher Z's third lesson. So his third lesson, I think as he said, it's nearing the back, so the two that I observed they didn't write anything down even when he wrote*

it, even when Teacher Z wrote it on the column on the white board, they didn't copy it down.

Teacher B: *I only observed the first two of Teacher Z's lesson. That was the substitution method. Generally the students they just copy and even when … okay, first lesson they basically copied everything that Teacher Z mentioned. Second lesson when Teacher Z wrote things, then for the first example they also copy, but once you moved on to the second example, I saw them flipping because the questions are similar so I saw them flipping to the previous example but their focus is on the steps. How to solve and not so much on "What's going on". So what I gather was the note that they take actually at the column was not very helpful to them.*

Teacher C: *I went for Teacher X's lesson. The first lesson and the second last lesson. The first lesson was Teacher X introducing it. Then the two students that I observed, they were copying down whatever Teacher X is writing down. Then I just asked, "Why are you copying down?" Then the girl said, "It's is very useful for me." And she even do it in different colours. But subsequently, the next lesson that I went for Teacher X's class right, the faster … I noticed that those boys at the back, those two boys, they didn't take down any notes but they were able to do it very fast. They didn't fill up the "What's going on" column at all.*

Teacher D: *I went in for Teacher X's second class. And what basically the two of them were doing was, they were just listing the steps they need to do. So step number 1, "label", step number 2, "write 'x' in terms of 'y'", and step number 3, "solve for 'x' and 'y'" and step number 4, "check". So before they even attempt the question, they will quickly fill up what's going on first. So every single page they will fill up that four steps first and then they will start doing. So I don't know whether maybe by doing this, the four steps went into their heads. So they will remember to do these four steps every time they do. I don't know because I didn't observe them after that lesson so I don't know how useful … But when I went in for Teacher Z's last lesson, almost none around me were writing what's going on anymore. They were already done with this "What's going on" thing.*

Teacher E: *Okay, I observed quite a number … three or four … I actually move down the row, so I managed to observe random students, many random students. Mostly when Teacher X asked them to write down, they will write down, whatever, most of them will write down. Well I'm actually very impressed with one of the students but I don't know his name. The Indonesian one. So that student… he suddenly understand … then when I go to the second example, "What's going on", he write, "Step number one", then he do, then "Step number two", then he could do, "Step number three", then he could do. After he finished ah, he checked you know. After that, you can feel and say that, he is happy. Very happy. But only one student is like that. Then … I also see the two boys behind. These two boys were very fast. They already finished already but nothing there lah. I observed one boy like that. Then I … I mean … It's the front boy … You can see really happiness. He followed everything you know. After that he re-checked. When his answer is correct, so he was so happy. But I do feel that they do benefit some of the students. Maybe not all, maybe very few number but I feel that it is good and I see some writing summary.*

Teacher Z felt that the "What's going on" column helped make his thinking visible to the students and allowed for some permanence unlike if he relied solely on "talking" out the steps: *Okay, I mean, like err, it is the same consensus as everyone that I don't really know whether my students have benefitted from it or not but I do buy Researcher L's idea that it has benefitted the teacher meaning myself in the sense that, I think I'd spoken about this during one of the post meetings as well—that when my thought is being portrayed on the board, it makes my explanation clearer to myself even in that sense too so that I have that consciousness that I need to explain certain things. And to make it visible for the students whether … and also to help students whereby they have those one, two seconds just switched off and suddenly they missed out that bit that I was trying to say but once they switched back on and see the thing that is on the board, they are able to somewhat connect back to that lesson. I thought that was something that is useful. So whether or not they have written it down, and it helps them or not I'm not too sure but at least by seeing something that is on the board, I think that will help them. If it's one word that I am trying to go by audio with my voice, there's another mode whereby you can see the visual of whatever is going on in my mind is actually on the board. That's one thing. The other thing is I think as I look through let's say for example, in a particular lesson after you do example one, you can refer to the steps to do example two, maybe tomorrow if I ask them to do again, they will forget some steps here and there.*

Teacher Z also realised that he had used the column to write down procedural steps rather than exploit its full potential for giving reason to the steps: *Ultimately I think, for my class, what I could focus on more is the whole essence behind solving simultaneous equation. Like what most of you correctly pointed out is that I focused quite a bit on procedural steps. So even my "What's going on" column right is ninety per cent of the procedures—"do this", "do this", "do this" … So procedures without exactly the reason behind it in that sense, I may or may not have said it in class but it wasn't on the "What's going on" column. I can say lah, about ninety per cent about the "What's going on". So in that sense, it ends up, even if students right now, it will be a case of they will have to memorise the step one, step two, step four in order to get the answer instead of let's say if I were to throw in the idea that, "Hey, we are doing elimination because we are trying to eliminate one of the variable whether it is 'x' or 'y'." If they go back with the idea they internalised the idea then when they see that question again, then they will know, "Oh, I need to somehow eliminate one of the variables. How am I going to do it? Err, okay, let's remember the steps." Rather than straight away come in and "What is the first step? 'Label'." As I reflect I thought. The first step shouldn't be 'label'. The first step should be, "Hey I've got two equations, I want to eliminate one of the variables in order to solve the other one." That sort of thing instead of just very procedural, first one, "label", second one …*

Teacher Z felt that teaching this way was better than what he did in the previous year: *My gut feel is that they did better in that sense. I mean, they may not be good. But with my limited teaching capability, what I taught last year and what I taught this year, with the addition of the "What's going on" column, I felt that my students of this batch, they learn better. In the sense that, at least, because I focused a lot on*

Table 19.2 Responses to two items from student perception survey

Type	Description	Class 1	Class 2
1	Writing down notes and my thoughts in the "What's going on" column has helped in my understanding	34/38	35/42
		89.5%	83.3%
2	Writing down notes and my thoughts in the "What's going on" column has helped in my homework	34/38	35/41
		89.5%	85.4%

procedure, so at least, they know what are the procedures to do as compared to my batch of students last year who were not very sure even when to begin with. So they mix up all the steps in that sense. So to put it in that way, I don't know it helps or not to make the procedures clearer in that sense. Step one, step two, step three, step four, rather than just one whole chunk of working, and I don't know what the hell is going on.

Teacher X was the chair of the meeting and gave his own views at the end. He felt that the column was important to show the students the reasoning behind the procedures:

Most of the things he mentioned already but the thing I added was the, I think the reasoning part to tell them the reason behind what we are doing is I guess, very important. Not just teach them the steps.

Teacher E agreed: *I find your lesson quite good. I find the first few lessons, you were very procedural, then after that you talked more on their understanding.*

The discussion above suggests strongly the two teachers were conscientiously trying to teach better with the affordance of the "What's going on" column. The encouragement to students to write down their thoughts in the column both as the teacher was teaching and in their homework was intended to develop in students the disposition of keeping track of their solution steps and decision-making (d3). What seems interesting also is that the teachers also realised the affordance of the column to make their thinking clear, both procedurally and more importantly, the reasons for the procedures.

With the addition of other parts of the transcript that we have not deemed necessary to add, we think it is clear that both the teachers were focused on achieving the R-goals of helping students develop proficiency with both the technique of "Elimination" and "Substitution" (r1) and inculcating in students the habit of checking by substituting the obtained solution into a suitable equation (r2).

We finally refer to the results of the student perception survey. Table 19.2 shows the number of positive responses (Strongly Agree and Agree) to two related items out of the total class number for the two classes. In spite of the generally pessimistic view of the teachers regarding the use of the column by the students, the results suggest that at least from the students' point of view, the "What's going on" column has helped their understanding and in their homework.

19.6 Conclusion

Ambitious instructional goals provide a vision for teacher educators and teachers to strive towards desirable educational outcomes for students. However, goals that are too far removed from the real constraints within which teachers work on a regular basis can result in a disconnect between teacher educators and teachers—the former being driven by theoretical ideals and the latter by immediate solutions to practical problems of teaching.

In this chapter, we present an alternative approach—a frame of thinking and acting upon teaching goals and enactment—that would draw both teacher educators and teachers into a common ground for teacher development enterprises. This requires careful consideration of the realistic demands of teaching while aiming for goals that would still be considered ambitious for teachers. RAP is a way of conceiving instructional work that would both engage and challenge teachers' existing teaching practices. As such, it is a suitable pedagogy to ground sustainable teacher development programmes, the success of which is dependent on strong teacher motivations and buy-in.

It may be argued that the pedagogy that we advocate here lacks a matching classroom image—unlike the case, for example, of the "pedagogy" of cooperative learning, where one can easily imagine a class of students engaging in collaborative discussions as the main mode of learning. As such, it can be further contended that it is hard for teachers to conceive of how RAP looks like in their classrooms. Indeed, RAP does not map narrowly into certain fixed instructional forms in the classroom. In other words, teachers who purportedly adhere to this pedagogy can enact their classroom work in a variety of forms that nevertheless fulfil both the ambitious and the realistic goals of teaching. In fact, this feature of broad interpretation is intentional so that teachers participate in genuine pedagogical inquiry as co-designers of their lessons. It allows for teachers with different pedagogical starting points to benefit from a reconsideration of their existing ways of teaching in terms of ambitious yet realistic instructional goals.

Realistic ambitious pedagogy is not constructed as an immediately transformative pedagogy. It does not require teachers to make a big leap into the unknown. It allows teachers to take a small but decisive step into familiar and reconstructed territory. If that small step brings satisfaction in terms of better fulfilment of instructional goals, then it is a beginning of a sustained teacher development journey into the elusive but worthwhile target of ambitious teaching.

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Chapter 20
Productive Teacher Noticing: Implications for Improving Teaching

Ban Heng Choy and Jaguthsing Dindyal

Abstract Although there have been calls to focus on teacher-inquiry approaches to teacher professional development, simply putting teachers together in a professional learning team is not sufficient for improving teaching. What matters is what, and how, teachers notice when they learn from their teaching practices. In this chapter, we first give an overview of the crucial role of teacher noticing in professional development by drawing on relevant literature. Next, we explain the notion of productive teacher noticing by highlighting what and how teachers notice as they attempt to enact teaching practices, aimed at enhancing students' reasoning. Following this, we describe two studies on productive teacher noticing in Singapore before we highlight some implications for improving practice and suggest possible future trajectories of research into teacher noticing.

Keywords Learning from teaching · Mathematics teacher noticing · Teacher education · Teacher professional development

20.1 Introduction

There has been a shift towards professional development activities that involve some form of job-embedded collaborative teacher inquiry with teachers learning from their own teaching (Lave 1996; Timperley et al. 2007). Examples of this kind of professional development activities include video clubs where teachers examine and reflect on practices using video recordings of lessons (van Es 2012); study groups where teachers examine classroom artefacts (Goldsmith and Seago 2013) or analyse lesson plans (Santagata 2011); and lesson study (Lewis et al. 2006). However, participating in these activities alone does not necessarily help teachers to change or improve their

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teaching. The contention is that what matters is, not the kind of professional development activities, but what teachers focus on and how they engage with the activities within the contexts of learning communities (Lampert 2009). Although teachers in Singapore generally take an active role in their professional development, what they learn from their participation in learning communities, and how this learning actually helps improve their quality of mathematics instruction have not been well understood (Chua 2009).

Recent literature in mathematics education has positioned mathematics teacher noticing—a component of teaching expertise and a form of professional vision (Goodwin 1994)—to be critical for examining one's teaching practices, and for improving instruction (Barnhart and van Es 2015; Choy 2013; Goodwin 1994; Mason and Davis 2013; Schoenfeld 2011). From the perspective of professional vision, mathematics teacher noticing is conceptualised as the process of attending to students' mathematical ideas, and making sense of the information to make instructional decisions (Jacobs et al. 2010; van Es and Sherin 2008). On the other hand, Mason (2002) views noticing as a set of practices, aimed at raising one's awareness to have a different act in mind. Although all teachers notice instructional details to some extent, not all noticing is productive (Choy et al. 2017). As highlighted by Choy (2015), productive noticing not only empowers teachers to shift their foci of attention (Mason 2002), but more importantly, decide and implement instructional decisions that potentially enhance students' mathematical thinking. But the ability to "notice productively" during mathematics teaching is both difficult to master, and complex to study (Jacobs et al. 2011, p. xxvii). In this chapter, we first give an overview of the critical role of teacher noticing in professional development by drawing on relevant literature. We then describe two studies on teacher noticing in Singapore to provide a characterisation of productive noticing. Following this, we illustrate the notion of productive teacher noticing through a series of snapshots what and how teachers notice as they attempted to enact productive teaching practices. Lastly, we highlight some implications for improving practice and suggest possible future trajectories of research into teacher noticing.

20.2 Learning from Professional Development: The Role of Teacher Noticing

People notice all the time, but they are "sensitised to notice certain things" (Mason 2002, p. xi). This sensitised noticing is closely associated with the idea of "professional vision, which consists of socially organised ways of seeing and understanding events that are answerable to the distinctive interests of a particular social group" (Goodwin 1994, p. 606). Following Goodwin's (1994) idea of professional vision, teacher noticing has also been understood in terms of three inter-related processes: attending, interpreting, and deciding to respond in teaching contexts (Sherin et al. 2011). But how does noticing enhance teacher development? To this end, Mason

(2002) sees noticing as a discipline and highlights noticing as “the heart of all practice” (p. 1) and offers it as a means by which teachers can “do something about it [teaching] in a practical and disciplined manner” (p. 1). Hence, the practical aim of noticing is to improve teachers’ sensitivity to act differently during instructional situations (Mason 2002) and is distinguished from spontaneous or impromptu noticing by adopting a more disciplined approach (Mason 2002, p. 61):

> The idea is simply to work on becoming more sensitive to notice opportunities in the moment; to be methodical without being mechanical. This is the difference between ‘finding opportunities’ and ‘making them’. Instead of being caught up in moment by moment flow of events according to habits and pre-established patterns, the idea is to have the opportunity to respond freshly and creatively yet appropriately, every so often.

As argued by Mason (2002), teacher noticing is critical for improving teaching practices—teachers who notice can learn to respond freshly or have a different act in mind for the future. The potential for changing practices places noticing at the centre of any professional development efforts. But how does noticing support teacher professional development?

With the aim of making opportunities to improve instruction, Mason (2002) highlighted two important ways to enhance noticing: advance preparation and using past experience. According to him, professional learning takes place in the world of personal experience and is supported by one’s colleagues while drawing on the world of theories, which informs how noticing can take place. Therefore, noticing does not necessarily occur at an individual level, but rather, the practices of disciplined noticing lie in the “merging” of three worlds of experience (see Fig. 20.1)—“the world of personal experience, the world of one’s colleagues’ experience and the world of observations, accounts, and theories” (Mason 2002, p. 93).

As seen in Fig. 20.1, the ability to recognise possibilities to act differently lies at the intersection of the three worlds of experience, which underscore the importance of collaborative professional development. By reflecting on their colleagues’ and their own experiences systematically, teachers can prepare to notice by developing sensitivity to the common observations, which emerge from their discussion.

Fig. 20.1 Noticing in the three worlds of experience (Mason 2002, p. 94)

They then interpret these observations in light of the theories and observations; and sometimes validate their observations with other people, in order to distinguish and recognise the possibility to act differently. While these actions usually happen retrospectively during post-lesson reflections, the essence of noticing is to bring these "moments of noticing from the retrospective to the spective" (Mason 2002, p. 87) so that the teachers are better prepared to notice in the moment during lessons.

Our perspective of noticing adopted in this chapter sees noticing as a form of professional vision view (Jacobs et al. 2011; van Es 2011) and at the same time, as a discipline or practice (Mason 2002, 2011). First, we examine teachers' noticing in terms of the events or details they attend to, how they interpret these events, and the instructional decisions they make based on their interpretations. Next, we use Mason's (2011) notion of noticing as a shift in attention to investigate what and how teachers notice in terms of the following micro-level structure of their attention: holding wholes; discerning details; recognising relationships, and perceiving properties.

Not all noticing, however, is productive with regard to improving instruction, and it is challenging for teachers to move their "moments of noticing from the retrospective to the spective". Erickson (2011) highlights that teacher noticing is very selective in its focus, and while different teachers can notice various aspects of the classrooms, what they notice may not always be helpful, and at times, direct students' attention away from the mathematical issues. It is common for teachers, for example, to see students' active participation in the tasks, or their enthusiastic raising of hands to answer questions, as indicators of students' understanding (Erickson 2011; Star et al. 2011; Star and Strickland 2008). Furthermore, what teachers notice depends on their knowledge (Kazemi et al. 2011; Schifter 2011) and beliefs or philosophical stance towards teaching (Erickson 2011; Schoenfeld 2011). This diversity of knowledge and orientations has the potential for both "insight" and "misperception" in noticing (Erickson 2011, p. 32; Miller 2011). This begs the question, what makes teacher noticing productive?

20.3 What Makes Teacher Noticing Productive? Findings from Two Studies

The construct of teacher noticing is still relatively new to mathematics teachers and mathematics educators in Singapore. Many of the studies, based overseas, investigated pre-service teachers' development of noticing expertise within the contexts of video clubs, or video-based professional development (Miller 2011; Seidel et al. 2011; Star et al. 2011; van Es 2011). In Singapore, however, most of the studies were situated within the contexts of lesson study discussions (Choy 2014a, b, 2015, 2016; Lee and Choy 2017), or other related professional development activities, where in-service teachers had opportunities to plan, observe, and discuss lessons (Seto and Loh 2015). These studies, together with a few others (Chia 2017; Choy and Dindyal

2017a, b), have begun to expand the contexts of research on teacher noticing beyond video-related professional development activities and pre-service teacher education. In this section, we focus on two studies on teacher noticing situated in Singapore to highlight the notion of productive noticing. In the first study (see Sect. 20.3.1), the notion of productive noticing goes beyond the three inter-related processes—attending, interpreting, and deciding to respond—by including the *foci* for teachers to notice, and extending the notion of noticing beyond the walls of the classrooms to include planning during lesson study. In the second study (see Sect. 20.3.2), we built on Choy's (2015) notion of productive noticing and looked at how teachers planned day-to-day lessons around rapid cycles of simple tasks instead of a single rich task, for example, in the context of lesson study.

20.3.1 Study 1: The FOCUS Framework

The FOCUS Framework was developed from a doctoral study (Choy 2015), which used a design-based research approach (Cobb et al. 2003) to address the twin challenges of theoretical development and practical application (Zawojewski et al. 2008). Data collection for the doctoral study, which consisted of three phases, took place in Singapore over a period of eight months in 2012 and 2013. A total of 36 teachers from three schools, a primary school and two secondary schools, volunteered to participate in the study. All three schools had actively supported learning communities by providing teachers time and space to discuss curriculum-related matters. The teacher participants had used lesson study as a professional development activity, and they were familiar with the lesson study protocol.

The voice recordings of the lesson study discussions and video recordings of the observed lessons formed the bulk of the data collected. The recordings were marked for segments that focused on the key tasks of lesson study and notable episodes involving mathematically significant moments. Segments related to logistics, administrative matters, and other irrelevant incidents were discarded. The selected segments were reviewed and initially classified according to the framework for noticing students' thinking (van Es 2011), shown in Table 20.1. The reviewed segments were then transcribed before they were coded using a "thematic approach" (Bryman 2012, p. 578).

The FOCUS framework, developed from Choy's (2015) study, reflects the following two characteristics of productive mathematical noticing:

1. The focus—what to notice: (a) Specific mathematically significant aspects of learning and teaching, such as the three points; mathematics-learner-teacher milieu; or simply the concept, confusion, and course of action. (b) The alignment between the teaching approach and students' learning difficulties associated with mathematical concepts; and
2. The focusing—how to notice: The central role of sense-making or reasoning as a mediator between seeing and responding. It is the analysis of the observations

Table 20.1 Framework for noticing students' thinking adapted from van Es (2011, p. 139)

	What teachers notice	How teachers notice
Level 1 Baseline	Attend to generic aspects of teaching and learning, e.g. seating arrangement and student behaviour	Provide general descriptive comments with little or no evidence from observations
Level 2 Mixed	Begin to attend to particular instances of students' mathematical thinking and behaviours	Provide mostly evaluative comments with a few references to specific instances or interactions as evidence
Level 3 Focused	Attend to particular students' mathematical thinking	Provide elaborate and interpretive comments by drawing upon specific instances and interactions from observations as evidence
Level 4 Extended	Attend to the relationships between particular students' mathematical thinking, mathematical concepts, and teaching approaches	Provide elaborate and interpretive comments by drawing upon specific instances and interactions from observations as evidence, make connections to principles of teaching and learning and propose alternative pedagogical solutions

that provide the evidence or justification for making an instructional response that promotes student reasoning.

Choy (2015) positions teachers' noticing as productive when teachers' noticing leads to one or more of the following teaching practices for enhancing students' mathematical thinking: designing or planning tasks to reveal students' thinking; listening and responding to students' thinking, and analysing students' thinking. This stance extends the study of teacher noticing from examining teachers' practices during and after lesson to the planning processes before instruction. In particular, Choy (2015) focuses on what and how a teacher anticipates students' responses to a task during planning. This is similar to the practice of anticipation as highlighted by Smith and Stein (2011). Moreover, Choy (2015) also finds it necessary to focus teacher's attention on some specified aspects of teaching and learning so that teachers can cope with the enormous amount of information encountered during real-time teaching.

Building on the work by Yang and Ricks (2012), Choy (2015) highlights that an explicit focus is an essential characteristic of productive noticing, especially in the context of professional development. More specifically, he proposed that teachers could focus on the following focal points to promote productive noticing:

1. Mathematics Concept. The key mathematical ideas, themes, or constructs that are of interest to the lesson, discussion, or teaching episode;
2. Students' Confusion. The mathematical difficulties, cognitive obstacles, errors, misconceptions, or uncertainties demonstrated by students; and
3. Teachers' Course of Action. The instructional decision or response by teachers during the planning, teaching, or reviewing of the lesson.

Besides noticing specific aspects of these three focal points, it is also crucial for teachers to notice the alignment between these three points. That is, whether the teacher's course of action addresses students' confusion when learning the concept. Focusing on the alignment between the three focal points, or more generally, the mathematics-learner-teacher milieu (Brousseau 1997) is related to what researchers like van Es (2011), and Barnhart and van Es (2015), had highlighted about responding with instructional decisions that are based on teachers' observations. However, these researchers were more concerned about the issue of alignment during the responding component of noticing, whereas Choy's (2015) study demonstrates that the alignment of the milieu is crucial even during the attending and making sense stages of noticing.

Seeing the alignment between the three focal points is challenging, even for experienced teachers. For example, it is possible for teachers to give a highly detailed description of what they notice about the three focal points, and not generate a pedagogically productive instructional decision (Choy 2014a, b). In other words, it is possible for teachers to discern the details, but not to recognise relationships between the three points, and thus fail to learn from their teaching practices. Therefore, the second key characteristic of productive noticing is the need for pedagogical reasoning. According to Choy (2015), in order to coordinate their instructional decisions with what they observe, it is necessary for teachers to make sense of their students' difficulties when learning the key mathematical ideas. Based on their interpretation of students' errors, teachers can then make a reasoned decision about a potential approach or strategy targeted at the mistakes observed. This highlights the central role of teacher pedagogical reasoning (Shulman 1987) in aligning what teachers see to their instructional responses.

There are two aspects of this alignment to be considered for productive teacher noticing. Firstly, to see whether there is an alignment between the three points; and secondly, to ensure that a teacher's decision to respond is aligned to what he or she has seen and interpreted with regard to the concept and confusion. For each of these aspects, Choy (2015) argues that teacher reasoning is essential for focusing one's noticing. Although other similar research suggests the importance of analysing or interpreting instructional details during teacher reflection (Barnhart and van Es 2015; Berliner 2001; Timperley et al. 2007), Choy's (2015) study extends their findings by addressing the object of this teacher reasoning process. In a way, this "completes" the micro-level structure of attention—reasoning (Mason 2011). For teachers to notice productively, it is necessary for them to hold the wholes (see the big ideas involved in the concept) and discern the details of the concept and students' confusion. They have to recognise the relationships or connections between the three focal points and perceive the affordances or properties of their instructional strategies before deciding on the course of action based on their pedagogical reasoning.

A theoretical model of noticing was developed from the FOCUS Framework to describe what, and how, a teacher can notice productively when learning from practice (see Choy 2015 p. 178.). It maps a teacher's noticing processes (attending, making sense, and responding) through three stages of learning from practice (planning, teaching, and reviewing) to the three key productive practices for mathematical reasoning (designing lesson to reveal thinking; listening and responding to student thinking, and analysing student thinking). In other words, the model describes a theoretical process of productive noticing, which highlights explicitly the three crucial focal points, and how the alignment between these three points can be achieved. A teacher's noticing can be analysed and then compared against this theoretical model to highlight the similarities and differences, so that specific actions can be pinpointed as part of a teacher's reflection on his/her instruction (Choy 2015, 2016).

To summarise, Choy's (2015) study introduced the notion of productive noticing as a means to distinguish the kind of noticing expertise that sets expert teachers apart from less accomplished ones. Furthermore, noticing expertise is not necessarily a function of experience. A comparison of noticing expertise in pre-service teachers in a US teacher preparatory course and in-service teachers in a Singapore school highlights that both groups of teachers faced the same challenges in noticing the relevant instructional details (Lee and Choy 2017). Since then, several other studies have explored the notion of productive teacher noticing in different contexts. For example, Seto and Loh (2015) examined how a teacher mentor can direct a teacher's noticing to focus on relevant instructional details related to the concept of decimals during mentoring conversations. Their study highlights the importance of using appropriate questions to support teachers in shifting their focus on their own thinking to how students think. Similarly, Choy (2017) details how the knowledgeable other (Watanabe and Wang-Iverson 2005) in lesson study can redirect teachers' attention to mathematically significant details during the kyouzai kenkyuu stage of lesson study. Research on teacher noticing has also begun to explore the use of the construct to reflect on teaching at the micro-level. For instance, Chia (2017) investigates what and how a teacher may notice about her use of multiple representations when teaching percentage from a commognitive perspective (Sfard 2008).

20.3.2 Study 2: Noticing and Orchestrating Learning Experiences

We now turn our attention to the second study on teacher noticing, which is still ongoing at the time of writing. This project extends the investigation of what teachers notice beyond professional development activities such as lesson study, to explore the role of noticing in the context of day-to-day teaching. In this section, we give a preliminary report on an ongoing project, which examines what, and how, three experienced mathematics teachers in Singapore notice as they orchestrate learning experiences in their own classrooms. Learning experiences refer to "the interaction between the learner and the external conditions in the environment to which he can react" (Tyler 1949, p. 63). In this project, we refer to students' learning experiences as their engagement with mathematical tasks selected or designed by the teachers, through which the students develop their mathematical processes. Learning experiences have been incorporated into the current mathematics syllabus and may involve teachers providing "opportunities for students to discover mathematical results on their own", or "work together on a problem and present their ideas using appropriate mathematical language and methods" (Ministry of Education-Singapore 2012). This project is therefore situated within the context of everyday teaching activities and not within the context of a particular professional development activity as in Choy's (2015) study.

This ongoing project adopted a design-based research paradigm (Design-Based Research Collective 2003), similar to Choy (2015), to develop a toolkit to support teachers in noticing and a theory to describe their noticing when orchestrating learning experiences. We engaged in three iterative cycles of theory-driven design, classroom-based field testing, and data-driven revision of the Mathematical Learning Experience Toolkit (MATHLET) to provide a theoretical justification for the analytical frameworks on which the toolkit is based. By engaging with our teacher participants in designing, implementing, and reviewing learning experiences using the MATHLET, we aimed to develop a deeper theoretical understanding of how teachers orchestrate mathematically meaningful learning experiences.

Four experienced mathematics teachers from three secondary schools, with different achievement bands and demographic factors, participated in this study. Each teacher designed and implemented a lesson of their choice during each design-cycle phase using the MATHLET. This would result in 12 design cycles at the end of the project across three different schools. Data were generated through voice recordings of planning discussions, pre-lesson discussions, post-lesson discussions, video recordings of lessons, and lesson artefacts. Findings were then developed using a "thematic approach" (Bryman 2012, p. 578) together with the two characteristics of productive noticing as proposed by the FOCUS Framework (Choy 2015).

In this project, we focus on what the teacher attended in relation to the interactions between students, content, and the task. These interactions can be visualised as a socio-didactical tetrahedron as shown in Fig. 20.2 (Rezat and Sträßer 2012). We follow Rezat and Sträßer (2012) in seeing each face of the tetrahedron as an instantiation of the interactions between task, students, teacher, and mathematics. For example, the task–students–teacher face represents the interactions that occur amongst teacher, students, and the task. Given our emphasis on teacher noticing, we put "teacher" as the apex of the tetrahedron as seen in Fig. 20.2 to reflect our focus on how the teacher managed these interactions.

Preliminary findings from this study go beyond Choy's (2016) work on productive noticing involving a single task and extend the study of noticing into the realm of using a sequence of typical problems to bring about mathematically productive learning experiences for students. Typical problems are examination-type or textbook-type questions, often used by teachers to develop procedural skills. What we have found is that experienced teachers notice the affordances of typical problems and modify them to develop both procedural skills and conceptual understanding. By affordances, we refer to what typical problems have to offer to develop conceptual understanding beyond their usual usage. In particular, we focus on how teachers are able to notice the characteristics of the task in relation to the particular understandings of the related concepts, and see how these problems are deployed in the classrooms.

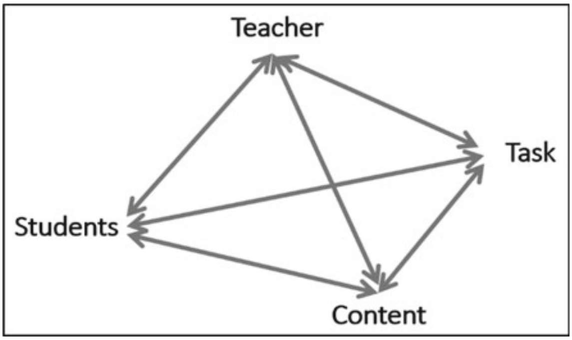
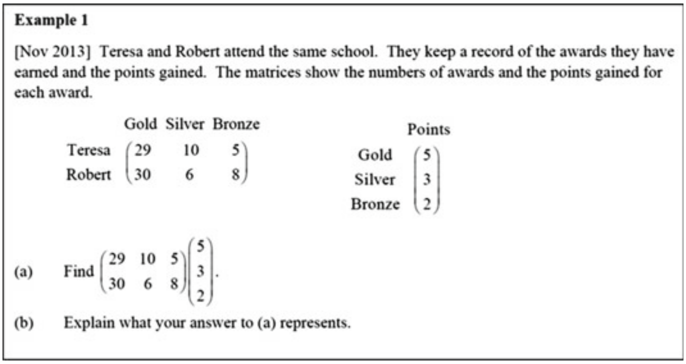


Fig. 20.2 Socio-didactical tetrahedron for using the task to orchestrate learning experiences



Example 1

[Nov 2013] Teresa and Robert attend the same school. They keep a record of the awards they have earned and the points gained. The matrices show the numbers of awards and the points gained for each award.

	Gold	Silver	Bronze
Teresa	29	10	5
Robert	30	6	8

	Points
Gold	5
Silver	3
Bronze	2

(a) Find $\begin{pmatrix} 29 & 10 & 5 \\ 30 & 6 & 8 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$.

(b) Explain what your answer to (a) represents.

Fig. 20.3 An example of a typical problem used by Alice

We see that experienced teachers who harnessed the affordance of typical problems to enhance students' learning experiences do so in two ways. First, they are able to see, within the typical problem, opportunities for developing mathematical ideas. For example, as described in the case of Alice (Choy and Dindyal 2017a, b), she modified a typical matrix calculation problem (see Fig. 20.3) by opening up its solution space, which provided opportunities for students to use different methods to solve the problem. Alice used students' responses to the typical problems to develop relational understanding by connecting their responses to different key mathematical ideas in the same topic.

Another teacher, John, exploited a sequence of typical problems through the idea of *bianshi* (Wong et al. 2013), which is similar to Marton and Pang's idea of variations (see Marton and Pang 2006), by making deliberate modifications to typical problems for broadening and deepening students' understanding of the skills and concepts. He tried to guide students in making connections between the procedural skills and the concepts they had learned. His use of typical problems was characterised by deliberate changes to the structure of the chosen problems to highlight specific aspects of the

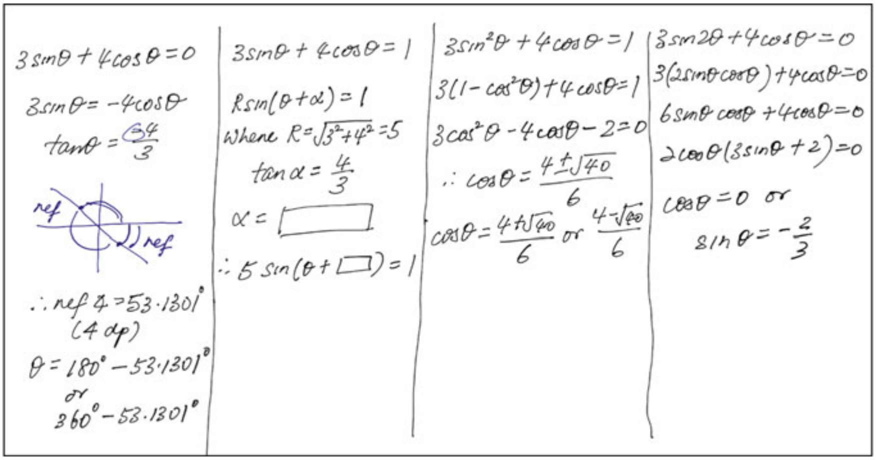


Fig. 20.4 A sequence of four typical problems used by John

concept or skill. Referring to Fig. 20.4, we see that John used four trigonometric equations, which looked similar but are structurally different.

For example, the difference between the first equation ($3\sin\theta + 4\cos\theta = 0$) and the second equation ($3\sin\theta + 4\cos\theta = 1$) lies in the number on the right-hand side. This variation in the number changes the structure and solution method of the first equation. In the first equation, we see that students can divide both sides by $\cos\theta$ to obtain an equation containing only the tangent function. The second equation, however, requires students to transform the equation into the form $R\sin(\theta + \alpha) = 1$. By harnessing variations, John was able to enhance his students' understanding of the solution methods and highlight the key considerations when solving such equations. Therefore, both Alice and John noticed the mathematical opportunities embedded within typical problems and planned how they could be used in the classroom (teacher-mathematics-task face of the tetrahedron, see Fig. 20.3).

Secondly, experienced teachers harnessed typical problems by orchestrating discussions (teacher-student-task face) about such problems to bring out key mathematical ideas (teacher-student-mathematics face). Smith and Stein (2011) highlight the importance of a good task in orchestrating mathematically productive discussions. In their model, they suggest an instructional sequence which centres about a single rich task in which students attempt, present, and discuss the mathematics under the orchestration of a skilful mathematics teacher. However, Alice's lesson differed from that envisioned by Smith and Stein (2011) in the plurality of tasks within the same lesson, punctuated by several more rapid successions of the same discussion moves: monitoring, selecting, sequencing, and connecting. This structure was made feasible by the use of typical problems which generally take a shorter time to complete. Similarly, in John's case, he orchestrated a series of short discussions about the four trigonometric equations.

Both Alice and John's noticing were productive because they were able to enact the productive practices of using tasks to reveal or enhance students' thinking, and at the same time, listened and used students' responses to the tasks to further students' understanding of the concepts. From our interviews with them, we realised that they were able to hold the whole curriculum in their minds, while attending to the details of each task. They recognised the relationships of each task to the topic taught and perceived the potential affordances of these problems. Although more work is needed before we can theorise further about their noticing, these findings suggest these teachers were able to recognise possibilities and have a different act in mind (Mason 2002). This sets up the stage to improve one's practices of using typical problems to enhance students' learning experiences.

20.4 Implications for Improving Teaching

These findings beg the question: whether such noticing is trainable, and if so, how? We believe so. Mason (2002) persuasively argues that the object of noticing is to recognise possibilities and generate alternatives from our habitual responses. This, according to Mason (2002), is achieved through noticing from the three worlds of experiences: one's experiences, other colleagues' experiences, and the world of theory and observations (p. 94). To support teachers in seeing the mathematical connections afforded by typical problems, there is a need to empower teachers to do this work through professional development, which focus on noticing the mathematics envisioned in the curriculum. This has important implications for mathematics educators in how we conceptualise professional development.

20.4.1 Noticing the Mathematics: Going Beyond the Surface

As seen from Alice's case, it is equally important to notice mathematical possibilities during the planning of the lesson, and not just to notice in the moment during the lesson. We realised that it is not so much of learning new content, but rather using teachers' existing knowledge to delve deeper into school mathematics. It is more about supporting teachers to use what they know and guiding them to see new connections between different aspects of the mathematics they are teaching. In a way, it is about guiding them to see the "forest and the trees". Teachers need to have opportunities to zoom in and zoom out of the curriculum, and notice systematically about the details of the curriculum (Mason 2011). In particular, they have to learn how to attend to the whole curriculum (holding wholes); discern the details of the concept; seeing the teaching of this concept in a sequence of lessons; conceptualising a lesson as a sequence of tasks, and encapsulating the mathematics within the tasks, paying careful attention to inter typical problem differences (see Atkinson et al. 2000). Although both Alice and John had thought of using their respective sequence of tasks

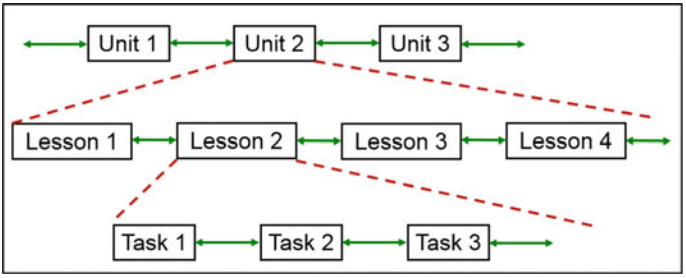


Fig. 20.5 Embeddedness of tasks–lessons–units

in different ways, both teachers highlight the importance of noticing and drawing on the connections between the sequence of tasks embedded within a sequence of lessons, which in turn are embedded within a sequence of units (see Fig. 20.5).

20.4.2 Listening to and not Listening for

Much of the research on noticing focused on what teachers see as a proxy to analyse what teachers attend to. As the two studies have demonstrated, it is crucial for teachers to listen and respond to students' thinking, instead of focusing on what teachers themselves are thinking for noticing to be productive. According to Davis (1997), this stance of listening is the difference between listening to and listening for. In other words, teachers who listen for certain responses in students' answers are less likely to orchestrate mathematically meaningful conversations because the teachers' attention is placed on getting students to utter the "correct responses". Instead, when teachers truly attend to students' thinking by listening to their students in order to make sense of the mathematics embedded in students' responses, they are more likely to move beyond the standard initiate–response–evaluate (IRE) discourse patterns, to a more interactive discourse pattern. The *listening to* stance positions teachers to notice other possibilities in orchestrating discussions, by focusing on what students say, instead of what students ought to say. This stance is a critical shift towards enhancing teachers' formative assessment of their students' understanding.

Moreover, the teachers in our studies demonstrate their competencies in orchestrating discussions by supporting students to make connections between the different ideas presented. For example, in Alice's case, she demonstrated her productive noticing through her attempts to make connections between the ideas presented by different students:

1	Alice:	(Walks around the class and comes to Student S1.) Can you write this for me on the board?
2	S1:	Ok. (Walks to the whiteboard and writes the following: $T = 5 \times 29 + 3 \times 10 + 2 \times 5 = 185$ $R = 5 \times 30 + 3 \times 6 + 2 \times 8 = 184$)
3	Alice:	(Walks around while waiting for Student S1 to finish writing.) Ok. Most of you have written what [Student S1] has written. 5 points for 29 gold, 3 points for 10 silver and 2 points for 5 bronze. Most of you have written in this manner. The last few days, we have been talking about matrices, right? Would you like to convert this to a matrix problem? (Looks at Student S2) Have you written it in matrix form? (Student S2 nods and Alice goes over to look at his answers.) Okay. Can you write your answer on the board?
4	S2:	(Walks to the board and writes the following) $T = \begin{pmatrix} 29 & 10 & 5 \end{pmatrix} \times \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} = 185$ and $R = \begin{pmatrix} 30 & 6 & 8 \end{pmatrix} \times \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} = 184$
5	Alice:	Any other answers from [Student S2's] answer? (Walks around the class and selects Student S3's answer) Can you write this on the board?
6	S3:	(Walks to the board and writes the following) $\begin{pmatrix} 29 & 10 & 5 \\ 30 & 6 & 8 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 29 \times 5 + 10 \times 3 + 5 \times 2 \\ 30 \times 5 + 6 \times 3 + 8 \times 2 \end{pmatrix} = \begin{pmatrix} 185 \\ 184 \end{pmatrix}$
7	Alice:	Thank you all three of you. [Student S1] has written using an arithmetic method. Most of you have written in this manner. This one comes very naturally to you, ok? [Student S2] has written Robert and Theresa's award separately. He has tried to use the matrix method, (points to Student S1's solution.) Something like this, ok? Let's check whether the order of matrix is correct or not (Alice goes through the method of matrix multiplication and gets the class to check the order of Student S2's matrices) … Ok. Student S3 has written Robert's and Theresa's together so that you only write this matrix once (points to the column matrix [5 3 2]). Don't need to write two times, correct or not? See. Over here. You have to write two times but here, [Student S3] only has to write it once. Let's check the order again…
8	Alice:	(After a short time) I would like to bring this problem a little bit further. Notice that Student S3 presented the information this way. Is there another way to represent the same information? (After some time, Student S4 highlights another possible way)

Here, we see how Alice orchestrated a mathematically productive discussion (Smith and Stein 2011). Alice carefully attended to students' answers before she asked for volunteers during the whole class discussion. However, it can be inferred that she was deliberate in her selection and sequencing of students' responses (see Lines 1 to 6). By beginning with an arithmetic solution, Alice connected Student S1's arithmetic operations to matrix multiplications through the sequencing of Student S2's and Student S3's matrix solutions. The reason for using a single matrix multiplication (Student S3's solution) was also made explicit when Alice moved from Student S2's solution to Student S3's using a matrix approach (Line 7) before she highlighted the different ways to express the given information as matrices (Line 8), which was an important idea for the lesson. Hence, we see how Alice connected the different ideas together during the discourse. This practice of connecting (Smith and Stein 2011) is predicated on Alice's ability to *listen to* her students (Davis 1997). This stance is challenging for teachers to adopt, even experienced ones. Developing and supporting teachers in listening to students during classroom discussion will hence be an important area for investigation.

20.4.3 Support Needed by Teachers

As discussed, teacher noticing is critical but can be a difficult skill to hone. Hence, we propose supporting teachers in (a) thinking about the use of tasks, both mathematically rich tasks and typical problems, through guided exploration of the curriculum materials to see the mathematical potential during planning; (b) empowering teachers to orchestrate discussions through discussion prompts during the lesson, and (c) encouraging them to relook at the mathematical connections from students' work after the lesson. Accordingly, we have developed the MATHLET (toolkit) to address three aspects of incorporating and modifying different tasks for orchestrating learning experiences. First, we provide a protocol for teachers to unpack the big ideas of the unit and specify the learning outcomes in terms of concepts, conventions, techniques, results, and processes for a lesson (Backhouse et al. 1992). Second, we support teachers to select tasks, including typical problems, and suggest how they can modify the structure of such problems to facilitate productive mathematics discussions. Last, we suggest prompts and questions enrich classroom discussions around typical problems that will meet the dual objectives of developing skills and concepts. The MATHLET is still under development at this stage, but we believe that such a toolkit will be useful for teachers as they engage in practices that are aimed at bringing a different act to mind.

20.5 Possible Future Trajectories for Research

Teacher noticing matters. As we have described in this chapter, teachers who notice productively about instructional and curricular details have a higher likelihood of improving their teaching practices. As a critical component of teaching expertise, it is important for mathematics educators to continue their research into this construct. We see at least three potential trajectories for this research:

1. *Unlocking the "black box" of teacher noticing*. As argued by Scheiner (2016), the processes of attending and interpreting instructional events are still unclear

and hidden within a black box. In particular, it is crucial to investigate how and why teachers attend to particular events and other instructional details in the background simultaneously. This may help to answer how and why teachers may attend to something and yet not perceive the object, and vice versa. A promising way is to tap into cognitive sciences and human factor research. This poses several methodological issues, which require more thought. Nevertheless, such investigations may potentially unlock the black box of noticing and offer us insights into the underlying mechanisms of productive teacher noticing;

2. *Overcoming methodological challenges in researching teacher noticing*. Currently, data collected on teacher noticing come from records of teachers' conversations during video clubs (Santagata and Yeh 2013; van Es 2012), lesson study sessions (Choy et al. 2017; Lee and Choy 2017), and other professional development settings (Fernandez et al. 2012; Goldsmith and Seago 2013; Seto and Loh 2015). While there were some attempts to use technology, such as wearable cameras (Sherin et al. 2011), to capture what teachers see, it remains unclear how this information can be connected to what teachers attend to, and how they interpret classroom details. For example, it is possible now to record and analyse gaze plots of teachers using wearables, but the link between the gaze plots and teaching actions remains tenuous at best; and
3. *Developing teachers' expertise in noticing*. We believe that teacher noticing is trainable (Choy and Dindyal 2017b), but it remains to be seen how this can be implemented across teachers of varying experience levels, and across different schooling years. Research into teacher noticing in Singapore is still in its infancy, and developmental research into novel professional development programmes will be a fertile area to look into. While researchers have used video clubs and lesson study sessions to develop teaching expertise, there may be a need to develop programmes to hone teachers' noticing skills in Singapore. This is one key area in our current research and there is definitely room to examine more closely, the features of, and the effectiveness of such a programme.

To conclude, we argue for the critical role that productive noticing plays in teacher education and professional development and highlight how this component of teaching expertise may be developed and honed. More importantly, as Mason and Johnston-Wilder (2006, p. 127) put it: "The heart of teaching is interaction with learners; the rest is preparation to make this interaction useful", it is important for us to see noticing as a deliberate practice, which needs preparation, and not something that is impromptu. Many questions about teacher noticing and challenges in research and development of teachers' noticing expertise remain. Nevertheless, we find it exciting that to see research in this area gaining momentum in Singapore, and we look forward to see how future research trajectories of productive teacher noticing in Singapore can contribute towards a more comprehensive understanding of this important construct in the teaching and learning of mathematics.

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Part IV
Conclusion

Chapter 21
Reviewing the Past, Striving in the Present and Moving Towards a Future-Ready Mathematics Education

Tin Lam Toh, Berinderjeet Kaur and Eng Guan Tay

Abstract This book serves the purpose of discussing the state of mathematics education in Singapore at the time it is written, and an update of a previous edition of the book on Singapore mathematics education published about a decade ago. This chapter identifies the central message that runs across the chapters in this book: Singapore is striving for excellence in mathematics education while addressing her imperfections; Singapore is also learning from good practices of other countries and at the same time bases her practices on sound educational theory. In addition, Singapore places great emphasis on teacher professional development. Further, this chapter discusses the future trends of mathematics education in Singapore: an increasing emphasis on big ideas, big data and computational thinking.

Keywords Mathematics education · Big ideas · Big data · Computational thinking · Early childhood · STEM

21.1 High-Quality Education in Singapore

According to the latest report (Save The Children 2018) released on 31 May 2018 by the non-governmental organization Save The Children, Singapore is ranked first among all the countries in the world for children to grow up. Among the many reasons, equitable high-quality education is one of the factors.

> Singapore is a great place for children to grow up with *good access* to *high quality education* [emphasis added] and medical care services, while also being one of the safest countries in the world… (The Straits Times, 1 Jun 2018)

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Being internationally recognized for her "high-quality education" accessible by all children, it is indeed not surprising that Singapore has performed well in mathematics, science and language in various international comparative studies as mentioned in Chap. 1.

In one of his recent visits to NIE, the Deputy Prime Minister of Singapore, Mr. Teo Chee Hean, stressed the importance of Singapore having a high level of quality education. In his closing address at the visit, Mr. Teo also highlighted the importance for Singapore to understand the reasons why the country had performed well in the various areas of education. Understanding the reasons why Singapore was performing well would enable us to better adapt to the constantly changing world, thereby allowing us to sustain the good performance in the future. He also emphasized on three aspects of education that Singapore would be focusing on—mathematics, science and language.

This concluding chapter summarizes the earlier chapters of this book in taking stock of what Singapore has been doing in mathematics education and the future directions that it will be taking.

21.2 A Sense of Singapore Mathematics Education from the Earlier Chapters in This Book

What are some key impressions that the various chapters of this book coherently convey? It is not difficult for the reader, as he or she reads through all the chapters in this book, to get a sense that Singapore is not resting on her laurels despite having been recognized as a country having one of the best mathematics education systems in the world. As the various chapters unfold, the Singapore mathematics educators are self-critical in their approach to mathematics education. On the one hand, the educators are evidently striving for excellence in the various fields of mathematics education of their specialization. On the other hand, they are continuing to tease out the unresolved imperfections in order to approach excellence in mathematics education.

A good illustration of Singapore mathematics educators' effort in regularly attempting to address the imperfections in the midst of good general performance is in Kaur's (2009) paper on international comparative studies. It was identified that Singaporean students were still relatively unfamiliar with solving unseen problems, amidst the relatively good performance of Singapore students in the international comparative studies. This paper led researchers to deliberate on the true spirit of mathematical problem-solving beyond the mere performance in the usual paper-and-pencil test (Chap. 7). Metacognition, one of the five important dimensions of problem-solving in the Singapore mathematics curriculum, continues to receive the attention of mathematics educators and researchers (Chap. 8), as metacognition is recognized as one of the five attributes of mathematical problem-solving in the Singapore mathematics curriculum (Chap. 3).

We also recognize that Singapore mathematics education is not flawless. For example, Chap. 13 reports that Singapore has a significant number of low-attaining students who are not performing well in comparison with the low-attaining students in several other high-performing East Asian countries. As a result, efforts from a spectrum of concerned educators that range across researchers in NIE, education and policy officers from the Ministry of Education and school teachers were taken to address the learning needs of the low-attaining students in the education system. In the process, there was much collaborative effort among the educators in addressing the imperfection in the system.

Any reader familiar with Singapore history will know that the Singapore education system evolved from the inherited colonial British education system. However, the various chapters in this book also show that Singapore has learnt from various other countries, such as the USA, Australia, New Zealand, and even non-English speaking countries such as Finland and Japan (Wong et al. 2009). Singapore has boarded the bandwagon of the latest trends in mathematics education, but with caution. To cite two examples of Singapore learning from other countries and testing them out in our local education system: (1) information and communications technology (ICT) is being used in Singapore schools in teaching and learning mathematics in various degrees across all levels (Chap. 14); (2) mathematical modelling and problem-solving in the real-world context are being implemented in the Singapore schools at the primary and secondary levels (Chap. 9).

It is also important to note that Singapore does not take ideas wholesale from overseas. Rather, Singapore adapted these ideas to fit the unique Singapore context and to reinterpret and reconfigure rather than replace the existing mathematics curriculum with the new trends. Corresponding to the two examples cited in the preceding paragraph: (1) technology is introduced in the Singapore schools not to replace students' computation, but to stretch students' higher-order thinking skills in order to enhance their problem-solving ability. (2) The introduction of mathematical modelling and problems in the real-world context into the Singapore mathematics curriculum was not meant to displace the centrality of mathematical problem-solving in the mathematics curriculum. Instead, mathematical modelling is used to enrich the problem-solving emphasis of the curriculum: modelling and application of mathematics (in real-world context) have now become important *processes* of mathematical problem-solving, one of the five important attributes of problem-solving.

Singapore builds from established international education theories found in education literature as the basis to address the local education needs in order to solve local problems or to stretch for excellence. As an example, algebra learning has always been difficult for students worldwide. Piagetian theory suggests the futility of teaching algebra to young children who are not yet ready cognitively to acquire abstract mathematical concepts. The well-known Singapore model method in solving word problems (or more affectionately known as "Singapore Math" in the west), and the Algebra Manipulatives (AlgeDiscs™ and AlgeCards described in Chaps. 8 and 13), are founded on Bruner's theory of three modes of representation and subsequently the more recently developed Concrete-Pictorial-Abstract (CPA) theory. These practices used in the Singapore schools were tested in some Singapore classrooms to ensure its efficacy before its large-scale implementation in the classrooms through the national mathematics curriculum document. As a side note, in the chapters in which all the various teaching practices are discussed, one can easily find a long list of references on the studies carried out to test these practices in Singapore schools. Perhaps, this could be the reason why most practicing Singapore mathematics teachers will wholeheartedly use these innovative practices in their classrooms, as confidence has been built in the teachers that these tested practices will improve their students' learning.

As one reads through the various chapters of Part III of this book, it is evident that Singapore places much emphasis on teacher professional development, as we believe that the most important single factor for the quality of education is the quality of the teachers' training as discussed in Chap. 15. In mathematical jargon, we believe that the quality of the teachers is bounded by the quality of the teacher education programme. It is also clear that teacher professional development in Singapore has transcended the traditional workshop deficit model of professional development. Teacher professional development has moved to the strong partnership between the practicing teachers and the researchers in the Singapore National Institute of Education (NIE), and/or education and policy officers from the Singapore Ministry of Education over a sustained period of time. This close partnership ensures the validity and sustainability of the professional development, which is both practice-oriented and theory-based.

The reader will also get a sense from reading this book that teaching practice, theory and research are strongly intertwined in the Singapore mathematics education landscape. In each of the three parts of this book (curriculum, practice and professional development), the theme discussed in each chapter revolves around practices which are based on sound education theory and informed by research carried out locally and internationally. In addition to these practices, how Singapore teachers are prepared are also discussed in detail.

21.3 Some Other Areas of Mathematics Education

Are there other aspects of mathematics education that deserve our attention? Wong et al. (2009) asserted that there is a "huge gap in our knowledge about how best to help young children make the transition from less-structured acquisition of mathematics at kindergartens to more formal instruction at Primary 1…" (p. 526). They lamented that there was only one chapter on early childhood education in their book (Wong et al. 2009). We find the same situation in this book that is published about one decade later. There is little mention of research or teaching practice in early childhood years.

Following the recent announcement by the Singapore Ministry of Education placing its emphasis on early childhood education and channelling much resource into this area of education, more mathematics education research will be carried out at the early childhood level. Studies on early numeracy and other aspects of mathematics education on early childhood are in the pipeline with this new initiative. We resonate with the government's decision in focusing on early childhood education, as this is the first step of education in an individual. We believe that there will be more chapters on early childhood mathematics education if there is a next volume on Singapore mathematics education published several years after the publication of this book.

Singapore has been perceived by the world to have provided her students with a strong STEM education. STEM education has received the world's attention in recent years. It is noted that countries such as Thailand and South Korea have looked towards Singapore, and the NIE in particular, for STEM education. There seems to be a lack of attention on STEM education (only technology is mentioned in Chap. 14, not as an integrated interdisciplinary approach). This might not be too surprising since there is no isolated STEM subject in the Singapore curriculum either in primary or secondary level. Singapore does not offer a unique STEM subject in their school curriculum, a subject with explicit integration across all the four disciplines. In the context of Singapore, the concept of STEM is perhaps well integrated into the system implicitly through the connections and applications taught in the individual subjects of science and mathematics. With the increasing emphasis on STEM education, the prominent role of mathematics in an interdisciplinary context might spur some interdisciplinary research related to STEM—mathematics will no longer be perceived as playing merely a supporting role, but as leading in other systems of thought such as design thinking and scientific thought.

21.4 Looking Forward in the Journey of Mathematics Education

In addition to striving towards excellence from what Singapore has achieved, we see at least two emerging trends in Singapore that we shall discuss below.

21.4.1 Big Ideas in Mathematics

We see that the next leap in mathematics education in Singapore is the incorporating of big ideas into the mainstream discussion in the design of the school mathematics curriculum. As early as 2000, the notion of big ideas has been discussed in the NCTM document.

> Teachers need to understand the big ideas of mathematics and be able to represent mathematics as a coherent and connected enterprise. (NCTM 2000, p. 17)

The definition of a big idea was not made explicit then, although it was briefly discussed in the NCTM document. It seems that it was only after the seminal paper by Charles (2005) that the notion of big ideas began to attract much attention internationally and was subsequently introduced into the "conversations about mathematics standards, curriculum, teaching, learning, and assessment" (p. 9). According to Charles (2005), a big idea is "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" (p. 10).

Understanding big ideas allows an individual to develop a deep and robust understanding of mathematics. After all, the notion of big ideas shows the connection across the many topics of mathematics. The Singapore Ministry of Education (MOE) has since the latest curriculum review incorporated the use of big ideas in the new school mathematics curriculum. This development of the school mathematics curriculum was supported by the new series of textbooks that support the approach of big ideas in the curriculum. To support the effort of MOE in her implementation of big ideas in the mathematics curriculum, the theme of the annual Mathematics Teachers' Conference 2018 held in Singapore was "Big Ideas in Mathematics". This platform gathered a group of international and local mathematics educators and researchers to share their knowledge and experience on the theory and practice of big ideas in mathematics classrooms. Subsequently, it raised an awareness among Singapore mathematics teachers on the concept of big ideas.

It seems that research projects on big ideas are in the pipeline with the rising awareness among educators and teachers. The next big step of mathematics education in Singapore taken in the latest school mathematics curriculum revision is a re-examination of the mathematics curriculum with a focus on big ideas in mathematics using a threefold approach: (1) incorporation in the mathematics curriculum; (2) alignment with the school textbooks and (3) increasing awareness among the Singapore mathematics teachers and conducting teacher professional development on big ideas.

21.4.2 Big Data and Computational Thinking

With the exponential increase in the amount of information every moment, it is not surprising that now we are faced with extremely large volumes of data that are almost impossible to process using the traditional data processing techniques. The advent of highly sophisticated technology and powerful computers today makes it possible to analyse large data in order to identify patterns and trends which could be crucial information for day-to-day business. This challenges the traditional method of analysing small sample data sets in order to obtain information about the whole population. As mathematics is closely linked to computer science due to the nature of its logical deductive reasoning approach, it will indeed not be surprising that mathematics education in Singapore could be paying much attention to big data in preparing our students to be future-ready.

In order to be program computers act on large amounts of information, the notion of computational thinking could become a new trend in education and, in particular, mathematics education. The term *computational thinking* was first used by Papert (1980). According to Papert (1980, 1996), *computational thinking* is defined as the thought processes in formulating a problem and expressing its solution in a way that computers (and human beings) can work on it. This is reminiscent of mathematical problem-solving as it shares the similarity in formulating the problem and looking for solutions to the problems. Indeed, computational thinking shares similar thought processes with problem-solving, and in fact, the possibility of formulating a problem is expressing a problem into its equivalent form—one big idea in mathematics (Charles 2005).

21.5 Conclusion

This book reports the current state of mathematics education in Singapore in 2018, and it is a timely update of the book by Wong et al. (2009). The issues discussed in this book include a broad spectrum of issues related to mathematics education faced in Singapore. To sum up the state of the art of mathematics education in Singapore, we would say it is encapsulated in the phrase "*Reviewing the past, striving in the present and moving towards a future-ready mathematics education*".

Interested readers may consider examining many of the ideas presented in this book, which is written in the Singapore context, bearing in mind the contexts of their own country. We sincerely hope that this book will be of use to readers who are interested in mathematics education. Hopefully, the ideas expressed here can be transferred to another context.

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