# Chapter 2
# Graph Structures Under Neutrosophic Environment

A single-valued neutrosophic graph structure (neutrosophic graph structure, for short) is a generalization of neutrosophic graph. In this chapter, we present the notion of neutrosophic graph structures and explore some properties of neutrosophic graph structures. Moreover, we discuss the concept of $\phi$-complement of neutrosophic graph structure and present certain operations of neutrosophic graph structures elaborated with examples. Further, we discuss some applications of neutrosophic graph structures in decision-making. This chapter is due to [33, 34, 151].

## 2.1 Introduction

Sampathkumar [151] introduced the graph structure which is a generalization of undirected graph and is quite useful in studying some structures like graphs, signed graphs, labelled graphs and edge-coloured graphs.

**Definition 2.1** A *graph structure* $G^* = (X, E_1, \ldots, E_n)$ consists of a nonempty set $X$ together with relations $E_1, E_2, \ldots, E_n$ on $X$ which are mutually disjoint such that each $E_i$, $1 \leq i \leq n$, is symmetric and irreflexive.

One can represent a graph structure $G^* = (X, E_1, \ldots, E_n)$ in the plane just like a graph where each edge is labelled as $E_i$, $1 \leq i \leq n$.

*Example 2.1* Let $X = \{r_1, r_2, r_3, r_4, r_5\}$ and $E_1 = \{(r_1, r_2), (r_3, r_4), (r_1, r_4)\}$, $E_2 = \{(r_1, r_3), (r_1, r_5)\}$, $E_3 = \{(r_2, r_3), (r_4, r_5)\}$ be mutually disjoint, symmetric and irreflexive relations on set $X$. Thus $G = (X, E_1, E_2, E_3)$ is a graph structure and is represented in plane as a graph where each edge is labelled as $E_1$, $E_2$ or $E_3$ (Fig. 2.1).

**Definition 2.2** Let $\phi$ be a permutation on $\{E_1, E_2, \ldots, E_n\}$. Then *$\phi$-complement* of a graph structure $G^*$ denoted by $G^{*\phi c}$ is obtained by replacing $E_i$ by $\phi(E_i)$, $1 \leq i \leq n$.



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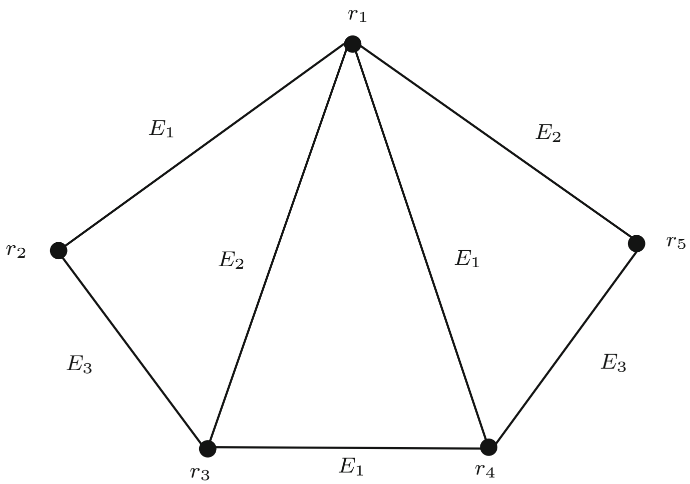


**Fig. 2.1** Graph structure $G^* = (X, E_1, E_2, E_3)$

$G^*$ is *self-complementary* if it is isomorphic to $G^{*\phi c}$, where $\phi$ is not an identity permutation. $G^*$ is *totally strong self-complementary* if it is identical to $G^{*\phi c}$ for all permutations $\phi$ on $\{E_1, E_2, \ldots, E_n\}$.

**Definition 2.3** If graph structure $G^*$ is connected and contains no cycle, in other words, its underlying graph is a tree, then it is called a *tree*. $G^*$ is an $E_i$*-tree* if subgraph structure induced by $E_i$-edges is a tree. Similarly, $G^*$ is an $E_1E_2 \ldots E_n-$*tree* if $G^*$ is an $E_i-$tree for each $i \in \{1, 2, \ldots, n\}$. $G^*$ is an $E_i$*-forest*, if subgraph structure induced by $E_i$-edges is a forest.

**Definition 2.4** Let $G_1^* = (X, E_1, E_2, \ldots, E_n)$ and $G_2^* = (X', E_1', E_2', \ldots, E_n')$ be two graph structures, *Cartesian product* of $G_1^*$ and $G_2^*$ is defined as: $G_1^* \times G_2^* = (X \times X', E_1 \times E_1', E_2 \times E_2', \ldots, E_n \times E_n')$, where $E_i \times E_i' = \{(b_1d, b_2d) \mid d \in X', b_1b_2 \in E_i\} \cup \{(bd_1, bd_2) \mid b \in X, d_1d_2 \in E_i'\}, i = (1, 2, \ldots, n)$.

**Definition 2.5** Let $G_1^* = (X, E_1, E_2, \ldots, E_n)$ and $G_2^* = (X', E_1', E_2', \ldots, E_n')$ be two graph structures, *cross product* of $G_1^*$ and $G_2^*$ is defined as: $G_1^* * G_2^* = (X * X', E_1 * E_1', E_2 * E_2', \ldots, E_n * E_n')$, where $E_i * E_i' = \{(b_1d_1, b_2d_2) \mid b_1b_2 \in E_i, d_1d_2 \in E_i'\}, i = (1, 2, \ldots, n)$.

**Definition 2.6** Let $G_1^* = (X, E_1, E_2, \ldots, E_n)$ and $G_2^* = (X', E_1', E_2', \ldots, E_n')$ be two graph structures, *lexicographic product* of $G_1^*$ and $G_2^*$ is defined as: $G_1^* \bullet G_2^* = (X \bullet X', E_1 \bullet E_1', E_2 \bullet E_2', \ldots, E_n \bullet E_n')$, where $E_i \bullet E_i' = \{(bd_1, bd_2) \mid b \in X, d_1d_2 \in E_i'\} \cup \{(b_1d_1, b_2d_2) \mid b_1b_2 \in E_i, d_1d_2 \in E_i'\}, i = (1, 2, \ldots, n)$.

**Definition 2.7** Let $G_1^* = (X, E_1, E_2, \ldots, E_n)$ and $G_2^* = (X', E_1', E_2', \ldots, E_n')$ be two graph structures, *strong product* of $G_1^*$ and $G_2^*$ is defined as: $G_1^* \boxtimes G_2^* =$

$(X \boxtimes X', E_1 \boxtimes E'_1, E_2 \boxtimes E'_2, \ldots, E_n \boxtimes E'_n)$, where $E_i \boxtimes E'_i = \{(b_1d, b_2d) \mid d \in X', b_1b_2 \in E_i\} \cup \{(bd_1, bd_2) \mid b \in X, d_1d_2 \in E'_i\} \cup \{(b_1d_1, b_2d_2) \mid b_1b_2 \in E_i, \ d_1d_2 \in E'_i\}$, $i = (1, 2, \ldots, n)$.

**Definition 2.8** Let $G_1^* = (X, E_1, E_2, \ldots, E_n)$ and $G_2^* = (X', E'_1, E'_2, \ldots, E'_n)$ be two graph structures, *composition* of $G_1^*$ and $G_2^*$ is defined as: $G_1^* \circ G_2^* = (X \circ X', E_1 \circ E'_1, E_2 \circ E'_2, \ldots, E_n \circ E'_n)$, where $E_i \circ E'_i = \{(b_1d, b_2d) \mid d \in X', b_1b_2 \in E_i\} \cup \{(bd_1, bd_2) \mid b \in X, d_1d_2 \in E'_i\} \cup \{(b_1d_1, b_2d_2) \mid b_1b_2 \in E_i, d_1, d_2 \in X'$ such that $d_1 \neq d_2\}$, $i = (1, 2, \ldots, n)$.

**Definition 2.9** Let $G_1^* = (X, E_1, E_2, \ldots, E_n)$ and $G_2^* = (X', E'_1, E'_2, \ldots, E'_n)$ be two graph structures, *union* of $G_1^*$ and $G_2^*$ is defined as: $G_1^* \cup G_2^* = (X \cup X', E_1 \cup E'_1, E_2 \cup E'_2, \ldots, E_n \cup E'_n)$.

**Definition 2.10** Let $G_1^* = (X, E_1, E_2, \ldots, E_n)$ and $G_2^* = (X', E'_1, E'_2, \ldots, E'_n)$ be two graph structures, *join* of $G_1^*$ and $G_2^*$ is defined as: $G_1^* + G_2^* = (X + X', E_1 + E'_1, E_2 + E'_2, \ldots, E_n + E'_n)$, where $X + X' = X \cup X'$, $E_i + E'_i = E_i \cup E'_i \cup E''_i$ for $i = (1, 2, \ldots, n)$. $E''_i$ contains all those edges, joining the vertices of $E$ and $E'$.

## 2.2 Neutrosophic Graph Structures

**Definition 2.11** Let $X$ be a nonempty set and $E_1, E_2, \ldots, E_n$ relations on $X$. $G = (A, B_1, B_2, \ldots, B_n)$ is called a *single-valued neutrosophic graph structure* if

$$A = \{< n, T_i(n), I_i(n), F_i(n) >: n \in X\}$$

is a single-valued neutrosophic set on $X$ and

$$B_i = \{< (m, n), T(m, n), I(m, n), F(m, n) >: (m, n) \in E_i\}$$

is a single-valued neutrosophic set on $E_i$ such that

$$T_i(m, n) \leq \min\{T(m), T(n)\}, \quad I_i(m, n) \leq \min\{I(m), I(n)\},$$
$$F_i(m, n) \leq \max\{F(m), F(n)\}, \forall m, n \in X.$$

Note that $T_i(m, n) = 0 = I_i(m, n) = F_i(m, n)$ for all $(m, n) \in X \times X - E_i$ and

$$0 \leq T_i(m, n) + I_i(m, n) + F_i(m, n) \leq 3 \text{ for all } (m, n) \in E_i,$$

where $X$ and $E_i$ $(i = 1, 2, \ldots, n)$ are underlying vertex and underlying $i$-edge sets of $G$, respectively.

Throughout this chapter, we will use neutrosophic set, neutrosophic relation and neutrosophic graph structure, for short.

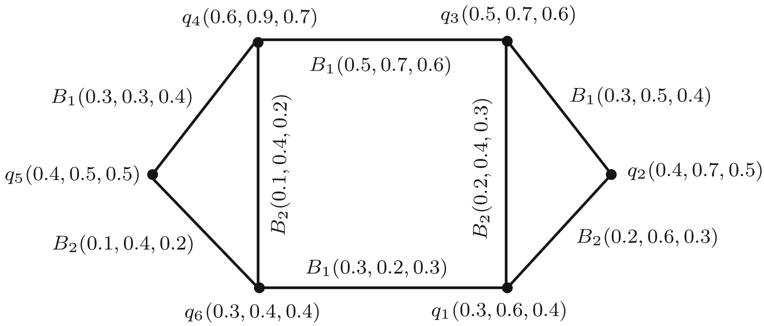


**Fig. 2.2** Single-valued neutrosophic graph structure

**Definition 2.12** Let $G = (A, B_1, B_2, \ldots, B_n)$ be a neutrosophic graph structure of $G^*$. If $H = (A', B_1', B_2', \ldots, B_n')$ is a neutrosophic graph structure of $G^*$ such that

$$T'(n) \leq T(n),\ \ I'(n) \leq I(n),\ F'(n) \geq F(n), \forall n \in X,$$

$$T_i'(m, n) \leq T_i(m, n),\ I_i'(m, n) \leq I_i(m, n) \text{ and } F_i'(m, n) \geq F_i(m, n), \forall m, n \in E_i,$$

where $i = 1, 2, \ldots, n$. Then $H$ is called a *neutrosophic subgraph structure* of neutrosophic graph structure $G$.

*Example 2.2* Let $G^* = (X, E_1, E_2)$ be a graph structure, where $X = \{q_1, q_2, q_3, q_4, q_5, q_6\}$, $E_1 = \{q_1q_6, q_2q_3, q_3q_4, q_4q_5\}$, $E_2 = \{q_1q_2, q_5q_6, q_4q_6, q_1q_3\}$. Now we define neutrosophic sets $A$, $B_1$, $B_2$ on $X$, $E_1$, $E_2$, respectively.

Let $A = \{(q_1, 0.3, 0.6, 0.4), (q_2, 0.4, 0.7, 0.5), (q_3, 0.5, 0.7, 0.6), (q_4, 0.6, 0.9, 0.7), (q_5, 0.4, 0.5, 0.5), (q_6, 0.3, 0.4, 0.4)\}$, $B_1 = \{(q_1q_6, 0.3, 0.2, 0.3), (q_2q_3, 0.3, 0.5, 0.4), (q_3q_4, 0.5, 0.7, 0.6), (q_4q_5, 0.3, 0.3, 0.4)\}$, $B_2 = \{(q_1q_2, 0.2, 0.6, 0.3), (q_5q_6, 0.1, 0.4, 0.2), (q_4q_6, 0.1, 0.4, 0.2), (q_1q_3, 0.2, 0.4, 0.3)\}$. By direct calculations, it is easy to show that $G = (A, B_1, B_2)$ is a neutrosophic graph structure of $G^*$ as shown in Fig. 2.2.

**Definition 2.13** A neutrosophic graph structure $H = (A', B_1', B_2', \ldots, B_n')$ is called an *induced subgraph structure* of $G$ by a subset $R$ of $X$ if

$$T'(n) = T(n), I'(n) = I(n), F'(n) = F(n), \forall n \in E,$$

$$T_i'(m, n) = T_i(m, n), I_i'(m, n) = I_i(m, n) \text{ and } F_i'(m, n) = F_i(m, n), \forall m, n \in E,$$

where $i = 1, 2, \ldots, n$.

**Definition 2.14** A neutrosophic graph structure $H = (A', B_1', B_2', \ldots, B_n')$ is called a *spanning subgraph structure* of $G$ if $A' = A$ and

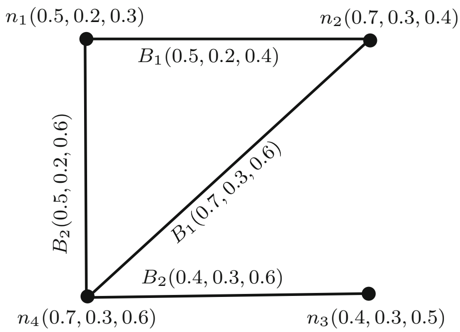


**Fig. 2.3** Neutrosophic graph structure $G$

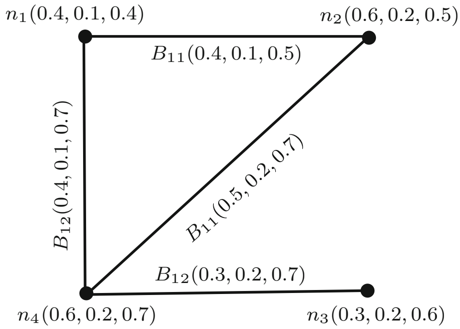


**Fig. 2.4** Neutrosophic subgraph structure

$T_i'(m,n) \leq T_i(m,n)$, $I_i'(m,n) \leq I_i(m,n)$ and $F_i'(m,n) \geq F_i(m,n)$, $i = 1, 2, ..., n$.

*Example 2.3* Consider a graph structure $G^* = (X, E_1, E_2)$ and let $A$, $B_1$, $B_2$ be neutrosophic subsets of $X$, $E_1$, $E_2$, respectively, such that

$$A = \{(n_1, 0.5, 0.2, 0.3), (n_2, 0.7, 0.3, 0.4), (n_3, 0.4, 0.3, 0.5), (n_4, 0.7, 0.3, 0.6)\},$$

$$B_1 = \{(n_1n_2, 0.5, 0.2, 0.4), (n_2n_4, 0.7, 0.3, 0.6)\},$$

$$B_2 = \{(n_3n_4, 0.4, 0.3, 0.6), (n_1n_4, 0.5, 0.2, 0.6)\}.$$

Direct calculations show that $G = (A, B_1, B_2)$ is a neutrosophic graph structure of $G^*$ as shown in Fig. 2.3.

*Example 2.4* A neutrosophic graph structure $K = (A', B_{11}, B_{12})$ shown in Fig. 2.4 is a neutrosophic subgraph structure of $G = (A, B_1, B_2)$ shown in Fig. 2.3.

**Definition 2.15** Let $G = (A, B_1, B_2, \ldots, B_n)$ be a neutrosophic graph structure of $G^*$. Then $mn \in E_i$ is called $B_i$-*edge* or simply $B_i$-edge if $T_i(m,n) > 0$ or $I_i(m,n) > 0$ or $F_i(m,n) > 0$ or all three conditions hold. Consequently, support of $B_i$ is defined as:

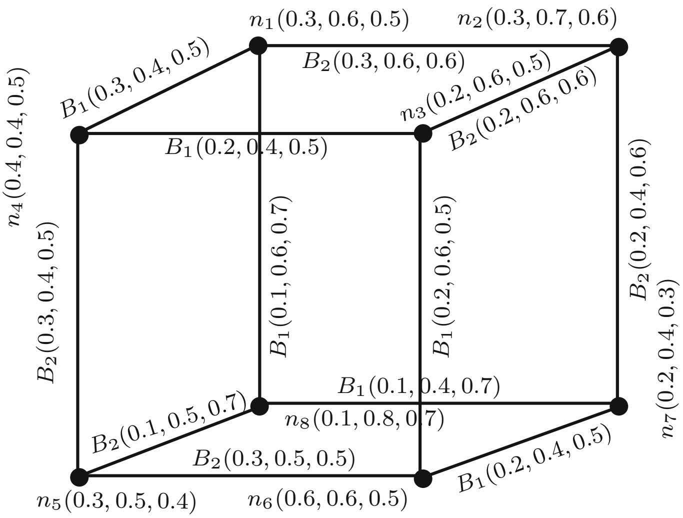


**Fig. 2.5** Strong neutrosophic graph structure $G = (A, B_1, B_2)$

$$supp(B_i) = \{mn \in B_i : T_i(m,n) > 0\} \cup \{mn \in B_i : I_i(m,n) > 0\}$$
$$\cup \{mn \in B_i : F_i(m,n) > 0\}, i = 1, 2, ..., n.$$

**Definition 2.16** $B_i$-path in a neutrosophic graph structure $G = (A, B_1, B_2, \dots, B_n)$ is a sequence of distinct vertices $n_1, n_2, \dots, n_m$ (except choice that $n_m = n_1$) in $X$, such that $n_{j-1}n_j$ is a neutrosophic $B_i$-edge for all $j = 2, \dots, m$.

**Definition 2.17** A neutrosophic graph structure $G = (A, B_1, B_2, \dots, B_n)$ is called *$B_i$-strong* for some $i \in \{1, 2, \dots, n\}$ if

$$T_i(m,n) = \min\{T(m), T(n)\}, I_i(m,n) = \min\{I(m), I(n)\}$$

and

$$F_i(m,n) = \max\{F(m), F(n)\}, \forall mn \in supp(B_i).$$

Furthermore, neutrosophic graph structure $G$ is called strong if it is $B_i$-strong for all $i \in \{1, 2, \dots, n\}$.

*Example 2.5* Consider a neutrosophic graph structure $G = (A, B_1, B_2)$ as shown in Fig. 2.5. Then $G$ is a strong neutrosophic graph structure since it is both $B_1$- and $B_2$-strong.

**Definition 2.18** A neutrosophic graph structure $G = (A, B_1, B_2, \dots, B_n)$ is called *complete* if $G$ is a strong neutrosophic graph structure, $supp(B_i) \neq \phi$ for all $i = 1, 2, \dots, n$ and for every pair of vertices $m, n \in X$, $mn$ is a *$B_i$-edge* for some $i$.

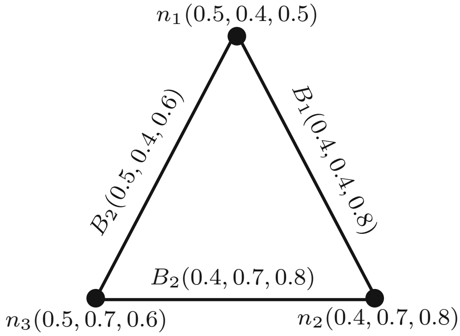


**Fig. 2.6** Complete neutrosophic graph structure

*Example 2.6* Let $G = (A, B_1, B_2)$ be a neutrosophic graph structure of graph structure $G^* = (X, E_1, E_2)$ such that $X = \{n_1, n_2, n_3\}$, $E_1 = \{n_1n_2\}$ and $E_2 = \{n_2n_3, n_1n_3\}$ as shown in Fig. 2.6. By simple calculations, it can be seen that $G$ is a strong neutrosophic graph structure. Moreover, $supp(B_1) \neq \phi$, $supp(B_2) \neq \phi$, and each pair of vertices in $X$ is either a $B_1$-edge or an $B_2$-edge. So $G$ is a complete, i.e. $B_1B_2$-complete neutrosophic graph structure.

**Definition 2.19** Let $G = (A, B_1, B_2, \ldots, B_n)$ be a neutrosophic graph structure. Then truth strength, indeterminacy strength and falsity strength of a $B_i$-path $P_{B_i} = n_1, n_2, \ldots, n_m$ are denoted by $T.P_{B_i}$, $I.P_{B_i}$ and $F.P_{B_i}$, respectively, and defined as

$$T.P_{B_i} = \bigwedge_{j=2}^{m}[T^P_{B_i}(n_{j-1}n_j)]\,,\ I.P_{B_i} = \bigwedge_{j=2}^{m}[I^P_{B_i}(n_{j-1}n_j)]\,,\ F.P_{B_i} = \bigvee_{j=2}^{m}[F^P_{B_i}(n_{j-1}n_j)]\,.$$

*Example 2.7* Consider a neutrosophic graph structure $G = (A, B_1, B_2)$ as shown in Fig. 2.6. We found that $P_{B_2} = n_2, n_3, n_1$ is a $B_2$-path. So $T.P_{B_2} = 0.4$, $I.P_{B_2} = 0.4$ and $F.P_{B_2} = 0.8$.

**Definition 2.20** Let $G = (A, B_1, B_2, \ldots, B_n)$ be a neutrosophic graph structure. Then

(i) $B_i$-truth strength of connectedness between $m$ and $n$ is defined as: $T^{\infty}_{B_i}(mn) = \bigvee_{j \geq 1}\{T^j_{B_i}(mn)\}$ such that $T^j_{B_i}(mn) = (T^{j-1}_{B_i} \circ T^1_{B_i})(mn)$ for $j \geq 2$ and

$$T^2_{B_i}(mn) = (T^1_{B_i} \circ T^1_{B_i})(mn) = \bigvee_z (T^1_{B_i}(mz) \wedge T^1_{B_i}(zn)).$$

(ii) $B_i$-indeterminacy strength of connectedness between $m$ and $n$ is defined as: $I^{\infty}_{B_i}(mn) = \bigvee_{j \geq 1}\{I^j_{B_i}(mn)\}$ such that $I^j_{B_i}(mn) = (I^{j-1}_{B_i} \circ I^1_{B_i})(mn)$ for $j \geq 2$ and

$$I^2_{B_i}(mn) = (I^1_{B_i} \circ I^1_{B_i})(mn) = \bigvee_z (I^1_{B_i}(mz) \wedge I^1_{B_i}(zn)).$$

(iii) $B_i$-falsity strength of connectedness between $m$ and $n$ is defined as: $F^{\infty}_{B_i}(mn) = \bigwedge_{j\geq 1}\{F^j_{B_i}(mn)\}$ such that $F^j_{B_i}(mn) = (F^{j-1}_{B_i} \circ F^1_{B_i})(mn)$ for $j \geq 2$ and

$$F^2_{B_i}(mn) = (F^1_{B_i} \circ F^1_{B_i})(mn) = \bigwedge_z (F^1_{B_i}(mz) \vee F^1_{B_i}(zn)).$$

**Definition 2.21** A neutrosophic graph structure $G = (A, B_1, B_2, \ldots, B_n)$ is a $B_i$-cycle if

$$(supp(A), supp(B_1), supp(B_2), \ldots, supp(B_n)) \text{ is a } B_i\text{-cycle.}$$

**Definition 2.22** A neutrosophic graph structure $G = (A, B_1, B_2, \ldots, B_n)$ is a $B_i$-*cycle* (for some $i$) if $G$ is a $B_i$-cycle, no unique $B_i$-edge $mn$ is in $G$ such that

$$T_{B_i}(mn) = \min\{T_{B_i}(rs) : rs \in E_i = supp(B_i)\},$$

or

$$I_{B_i}(mn) = \min\{I_{B_i}(rs) : rs \in E_i = supp(B_i)\},$$

or

$$F_{B_i}(mn) = \max\{F_{B_i}(rs) : rs \in E_i = supp(B_i)\}.$$

*Example 2.8* Consider a neutrosophic graph structure $G = (A, B_1, B_2)$ as shown in Fig. 2.5. Then $G$ is a $B_1$-cycle and neutrosophic $B_1 - cycle$, since $(supp(A), supp(B_1), supp(B_2))$ is a $B_1$-cycle and there is no unique $B_1$-edge satisfying above condition.

**Definition 2.23** Let $G = (A, B_1, B_2, \ldots, B_n)$ be a neutrosophic graph structure and $p$ be a vertex in $G$. Let $(A', B'_1, B'_2, \ldots, B'_n)$ be a neutrosophic graph structure induced by $X \setminus \{p\}$ such that, for all $v \neq p$, $w \neq p$,

$$T_{A'}(p)=0=I_{A'}(p)=F_{A'}(p),\ T_{B'_i}(pv) = 0 = I_{B'_i}(pv) = F_{B'_i}(pv),\ \forall \text{edges } pv \in G,$$

$$T_{A'}(v) = T_A(v),\ I_{A'}(v) = I_A(v),\ F_{A'}(v) = F_A(v),$$

$$T_{B'_i}(vw) = T_{B_i}(vw),\ I_{B'_i}(vw) = I_{B_i}(vw) \text{ and } F_{B'_i}(vw) = F_{B_i}(vw).$$

Then $p$ is neutrosophic $B_i$-cut vertex for any $i$ if

$$T^{\infty}_{B_i}(vw) > T^{\infty}_{B'_i}(vw),\ I^{\infty}_{B_i}(vw) > I^{\infty}_{B'_i}(vw) \text{ and } F^{\infty}_{B_i}(vw) > F^{\infty}_{B'_i}(vw),$$

for some $v, w \in X \setminus \{p\}$. Note that $p$ is a

- $B_i - T$ neutrosophic cut vertex if $T^{\infty}_{B_i}(vw) > T^{\infty}_{B'_i}(vw)$,
- $B_i - I$ neutrosophic cut vertex if $I^{\infty}_{B_i}(vw) > I^{\infty}_{B'_i}(vw)$,

- $B_i - F$ neutrosophic cut vertex if $F^{\infty}_{B_i}(vw) > F^{\infty}_{B'_i}(vw)$.

*Example 2.9* Consider a neutrosophic graph structure $G = (A, B_1, B_2)$ as shown in Fig. 2.7 and let $G' = (A', B'_1, B'_2)$ be a neutrosophic subgraph structure of neutrosophic graph structure $G$ found by deleting vertex $n_2$. Deleted vertex $n_2$ is a neutrosophic $B_1$-I cut vertex since

$$I^{\infty}_{B_1}(n_2n_5) = 0.4 > 0.3 = I^{\infty}_{B'_1}(n_2n_5),\ I^{\infty}_{B_1}(n_3n_4) = 0.7 = I^{\infty}_{B'_1}(n_3n_4),$$

and

$$I^{\infty}_{B_1}(n_3n_5) = 0.4 > 0.3 = I^{\infty}_{B'_1}(n_3n_5).$$

**Definition 2.24** Suppose $G = (A, B_1, B_2, \ldots, B_n)$ be a neutrosophic graph structure and $mn$ be $B_i$-*edge*. Let $(A', B'_1, B'_2, \ldots, B'_n)$ be a neutrosophic spanning subgraph structure of $G$, such that $\forall$ edges $mn \neq rs$,

$$T_{B'_i}(mn) = 0 = I_{B'_i}(mn) = F_{B'_i}(mn), T_{B'_i}(rs) = T_{B_i}(rs),$$

$$I_{B'_i}(rs) = I_{B_i}(rs) \text{ and } F_{B'_i}(rs) = F_{B_i}(rs).$$

Then $mn$ is a neutrosophic $B_i$-bridge if

$$T^{\infty}_{B_i}(vw) > T^{\infty}_{B'_i}(vw), I^{\infty}_{B_i}(vw) > I^{\infty}_{B'_i}(vw) \text{ and } F^{\infty}_{B_i}(vw) > F^{\infty}_{B'_i}(vw),$$

for some $v, w \in X$. Note that $mn$ is a

- $B_i - T$ neutrosophic bridge if $T^{\infty}_{B_i}(vw) > T^{\infty}_{B'_i}(vw)$,
- $B_i - I$ neutrosophic bridge if $I^{\infty}_{B_i}(vw) > I^{\infty}_{B'_i}(vw)$,
- $B_i - F$ neutrosophic bridge if $F^{\infty}_{B_i}(vw) > F^{\infty}_{B'_i}(vw)$.

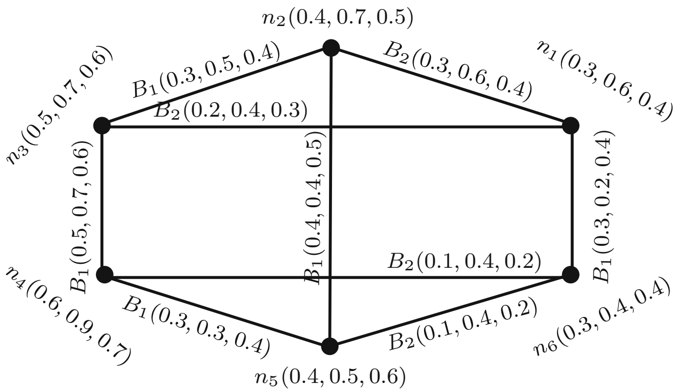


**Fig. 2.7** Neutrosophic graph structure $G = (A, B_1, B_2)$

*Example 2.10* Consider the neutrosophic graph structure $G = (A, B_1, B_2)$ as shown in Fig. 2.7 and $G' = (A', B_1', B_2')$ be a neutrosophic spanning subgraph structure of neutrosophic graph structure $G$ which is found by deleting $B_1$-edge $(n_2n_5)$. Edge $(n_2n_5)$ is a neutrosophic $B_1$-bridge. Since

$$T_{B_1}^{\infty}(n_2n_5) = 0.4 > 0.3 = T_{B_1'}^{\infty}(n_2n_5),$$

$$I_{B_1}^{\infty}(n_2n_5) = 0.4 > 0.3 = I_{B_1'}^{\infty}(n_2n_5)$$

and

$$F_{B_1}^{\infty}(n_2n_5) = 0.5 > 0 = F_{B_1'}^{\infty}(n_2n_5).$$

**Definition 2.25** A neutrosophic graph structure $G = (A, B_1, B_2, \ldots, B_n)$ is a $B_i$-tree if

$$(supp(A), supp(B_1), supp(B_2), \ldots, supp(B_n))$$

is a $B_i$-*tree*. In other words, $G$ is a $B_i$-tree if a subgraph of $G$ induced by $supp(B_i)$ generates a tree.

**Definition 2.26** A neutrosophic graph structure $G = (A, B_1, B_2, \ldots, B_n)$ is $B_i$-*tree* if $G$ has a neutrosophic spanning subgraph structure $H = (A', B_1', B_2', \ldots, B_n')$ such that for all $B_i$-edges $mn$ not in $H$, $H$ is a $B_i'$-tree,

$$T_{B_i}(mn) < T_{B_i'}^{\infty}(mn), I_{B_i}(mn) < I_{B_i'}^{\infty}(mn) \text{ and } F_{B_i}(mn) > F_{B_i'}^{\infty}(mn).$$

In particular, $G$ is a:

- neutrosophic $B_i$-T tree if $T_{B_i}(mn) < T_{B_i'}^{\infty}(mn)$,
- neutrosophic $B_i$-I tree if $I_{B_i}(mn) < I_{B_i'}^{\infty}(mn)$,
- neutrosophic $B_i$-F tree if $F_{B_i}(mn) > F_{B_i'}^{\infty}(mn)$.

*Example 2.11* Consider the neutrosophic graph structure $G = (A, B_1, B_2)$ as shown in Fig. 2.8, which is a $B_2$-tree. It is not a $B_1$-tree but a neutrosophic $B_1$-tree since it has a neutrosophic spanning subgraph $(A', B_1', B_2')$ as a $B_1'$-tree, which is obtained by deleting $B_1$-edge $n_2n_5$ from $G$.

Moreover,

$$T_{B_1}(n_2n_5) = 0.2 < 0.3 = T_{B_1'}^{\infty}(n_2n_5),\ I_{B_1}(n_2n_5) = 0.1 < 0.3 = I_{{B_1}'}^{\infty}(n_2n_5)$$

and

$$F_{B_1}(n_2n_5) = 0.6 > 0.5 = F_{{B_1}'}^{\infty}(n_2n_5).$$

**Definition 2.27** A neutrosophic graph structure $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ of the graph structure $G_1^* = (X_1, E_{11}, E_{12}, \ldots, E_{1n})$ is isomorphic to neutrosophic graph structure $G = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ of the graph structure $G_2^* = (X_2, E_{21},$

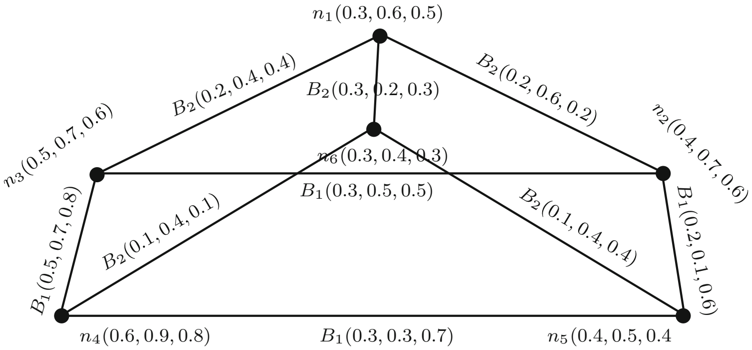


**Fig. 2.8** Neutrosophic $B_1$-tree

$B_{22}, \ldots, E_{2n})$ if we have $(f, \phi)$ where $f : X_1 \to X_2$ is a bijection and $\phi$ is a permutation on set $\{1, 2, \ldots, n\}$ and following relations are satisfied

$$T_{A_1}(m) = T_{A_2}(f(m)),\ I_{A_1}(m) = I_{A_2}(f(m)),\ F_{A_1}(m) = F_{A_2}(f(m)),$$

for all $m \in X_1$ and

$$T_{B_{1i}}(mn) = T_{B_{2\phi(i)}}(f(m)f(n)),\ I_{B_{1i}}(mn) = I_{B_{2\phi(i)}}(f(m)f(n),$$

$$F_{B_{1i}}(mn) = F_{B_{2\phi(i)}}(f(m)f(n)),$$

for all $mn \in E_{1i}$, $i = 1, 2, \ldots, n$.

*Example 2.12* Let $G_1 = (A, B_1, B_2)$ and $G_2 = (A', B_1', B_2')$ be two neutrosophic graph structures as shown in Fig. 2.9. $G_1$ is isomorphic $G_2$ under $(f, \phi)$ where $f : X \to X'$ is a bijection and $\phi$ is a permutation on set $\{1, 2\}$ defined as $\phi(1) = 2$, $\phi(2) = 1$ and following relations are satisfied

$$T_A(n_i) = T_{A'}(f(n_i)), I_A(n_i) = I_{A'}(f(n_i)),\ F_A(n_i) = F_{A'}(f(n_i)),$$

for all $n_i \in X$, and

$$T_{B_i}(n_i n_j) = T_{B'_{\phi(i)}}(f(n_i)f(n_j)), I_{B_i}(n_i n_j) = I_{B'_{\phi(i)}}(f(n_i)f(n_j)),$$

$$F_{B_i}(n_i n_j) = F_{B'_{\phi(i)}}(f(n_i)f(n_j)),$$

$\forall n_i n_j \in E_i$ and $i = 1, 2$.

**Definition 2.28** A neutrosophic graph structure $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ of the graph structure $G_1^* = (X_1, E_{11}, E_{12}, \ldots, E_{1n})$ is identical to neutrosophic graph

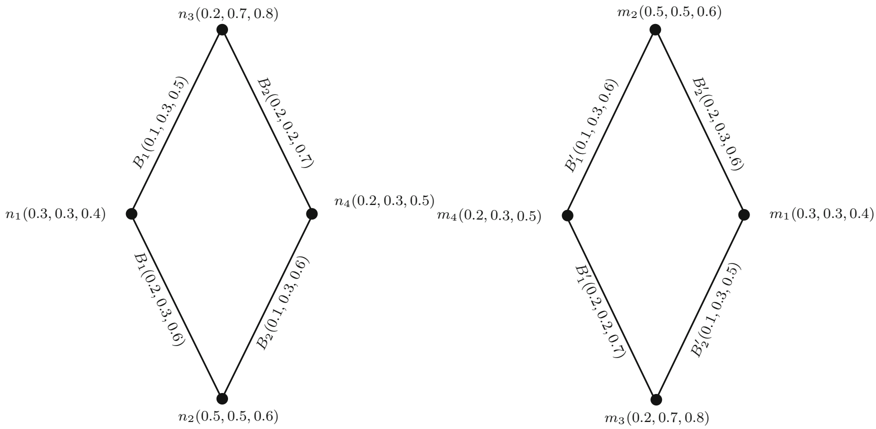


**Fig. 2.9** Isomorphic neutrosophic graph structures

structure $G_2 = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ of graph structure $G_2^* = (X_2, E_{21}, B_{22}, \ldots, E_{2n})$ if $f : X_1 \to X_2$ is a bijection and following relations are satisfied:

$$T_{A_1}(m) = T_{A_2}(f(m)),\ I_{A_1}(m) = I_{A_2}(f(m)),\ F_{A_1}(m) = F_{A_2}(f(m)),$$

for all $m \in X_1$ and

$$T_{B_{1i}}(mn) = T_{B_{2i}}(f(m)f(n)),\ I_{B_{1i}}(mn) = I_{B_{2i}}(f(m)f(n)),$$

$$F_{B_{1i}}(mn) = F_{B_{2i}}(f(m)f(n)),$$

for all $mn \in E_{1i}$ and $i = 1, 2, \ldots, n$.

*Example 2.13* Let $G_1 = (A, B_1, B_2)$ and $G_2 = (A', B_1', B_2')$ be two neutrosophic graph structures of graph structures $G_1^* = (X, E_1, E_2)$ and $G_2^* = (X', E_1', E_2')$, respectively, as shown in Figs. 2.10 and 2.11. Neutrosophic graph structure $G_1$ is identical to $G_2$ under $f : X \to X'$ defined as

$$f(n_1) = m_2,\ f(n_2) = m_1,\ f(n_3) = m_4,\ f(n_4) = m_3,\ f(n_5) = m_5,\ f(n_6) = m_8,$$

$$f(n_7) = m_7,\ f(n_8) = m_6,\ T_A(n_i) = T_{A'}(f(n_i)),$$

$$I_A(n_i) = I_{A'}(f(n_i)),\ F_A(n_i) = F_{A'}(f(n_i)),$$

for all $n_i \in X$ and

$$T_{B_i}(n_i n_j) = T_{B_i'}(f(n_i)f(n_j)),\ I_{B_i}(n_i n_j) = I_{B_i'}(f(n_i)f(n_j)),\ F_{B_i}(n_i n_j) = F_{B_i'}(f(n_i)f(n_j)),$$

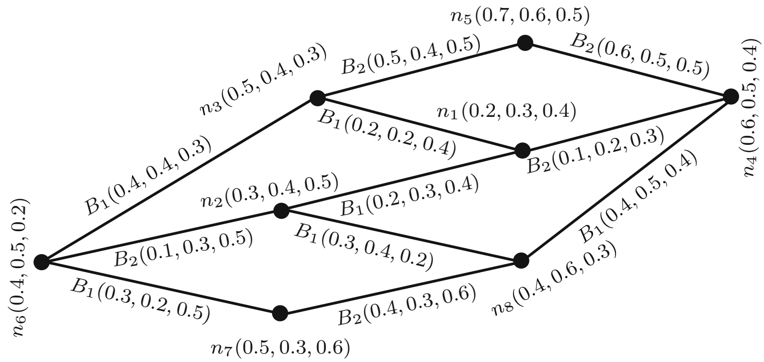


**Fig. 2.10** Neutrosophic graph structure $G_1$

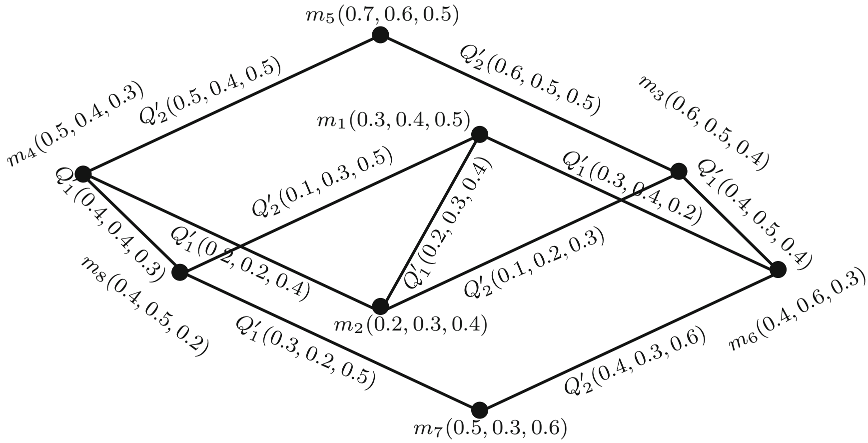


**Fig. 2.11** Neutrosophic graph structure $G_2$

for all $n_i n_j \in E_i$ and $i = 1, 2$.

**Definition 2.29** Let $G = (A, B_1, B_2, \ldots, B_n)$ be a neutrosophic graph structure and $\phi$ be a permutation on $\{B_1, B_2, \ldots, B_n\}$ and on $\{1, 2, \ldots, n\}$ defined by $\phi(B_i) = B_j$ if and only if $\phi(i) = j$ for all $i$. If $mn \in B_i$ for any $i$ and

$$T_{B_i^\phi}(mn) = T_A(m) \wedge T_A(n) - \bigvee_{j \neq i} T_{\phi(B_j)}(mn),\ I_{B_i^\phi}(mn) = I_A(m) \wedge I_A(n) - \bigvee_{j \neq i} I_{\phi(B_j)}(mn),$$

$$F_{B_i^\phi}(mn) = F_A(m) \vee F_A(n) - \bigwedge_{j \neq i} T_{\phi(B_j)}(mn),\ i = 1, 2, \ldots, n,$$

then $mn \in B_k^\phi$, where $k$ is selected such that

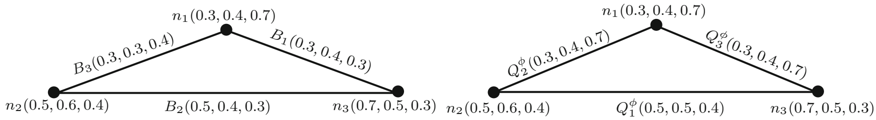


**Fig. 2.12** Neutrosophic graph structures $G$, $G^{\phi c}$

$$T_{B_k^\phi}(mn) \geq T_{B_i^\phi}(mn),\ I_{B_k^\phi}(mn) \geq I_{B_i^\phi}(mn) \text{ and } F_{B_k^\phi}(mn) \geq F_{B_i^\phi}(mn) \text{ for all } i,$$

then neutrosophic graph structure $(A, B_1^\phi, B_2^\phi, \ldots, B_n^\phi)$ is called *$\phi$-complement* of $G$ and denoted by $G^{\phi c}$.

*Example 2.14* Let $G = (A, B_1, B_2, B_3)$ be a neutrosophic graph structure shown in Fig. 2.12 and $\phi$ be a permutation on $\{1, 2, 3\}$ defined as:

$\phi(1) = 2$, $\phi(2) = 3$, $\phi(3) = 1$. By direct calculations, we found that $n_1n_3 \in B_3^\phi, n_2n_3 \in B_1^\phi, n_1n_2 \in B_2^\phi$. So, $G^{\phi c} = (A, B_1^\phi, B_2^\phi, B_3^\phi)$ is $\phi$-complement of neutrosophic graph structure $G$ as shown in Fig. 2.12.

**Proposition 2.1** *$\phi$-complement of a neutrosophic graph structure $G = (A, B_1, B_2, \ldots, B_n)$ is always a strong neutrosophic graph structure. Moreover, if $\phi(i) = k$, where $i, k \in \{1, 2, \ldots, n\}$, then all $B_k$-edges in neutrosophic graph structure $(A, B_1, B_2, \ldots, B_n)$ become $B_i^\phi$-edges in $(A, B_1^\phi, B_2^\phi, \ldots, B_n^\phi)$.*

*Proof* According to the definition of $\phi$-complement,

$$T_{B_i^\phi}(mn) = T_A(m) \wedge T_A(n) - \bigvee_{j \neq i} T_{\phi(B_j)}(mn),$$

$$I_{B_i^\phi}(mn) = I_A(m) \wedge I_A(n) - \bigvee_{j \neq i} I_{\phi(B_j)}(mn),$$

$$F_{B_i^\phi}(mn) = F_A(m) \vee F_A(n) - \bigwedge_{j \neq i} F_{\phi(B_j)}(mn),$$

for $i \in \{1, 2, \ldots, n\}$. For expression of truthness in $\phi$-complement:

Since

$$T_A(m) \wedge T_A(n) \geq 0, \bigvee_{j \neq i} T_{\phi(B_j)}(mn) \geq 0 \text{ and } T_{B_i}(mn) \leq T_A(m) \wedge T_A(n),\ \forall B_i,$$

we see that

$$\bigvee_{j \neq i} T_{\phi(B_j)}(mn) \leq T_A(m) \wedge T_A(n),$$

which implies that

$$T_A(m) \wedge T_A(n) - \bigvee_{j \neq i} T_{\phi(B_j)}(mn) \geq 0.$$

Therefore, $T_{B_i^\phi}(mn) \geq 0 \; \forall i$. Moreover, $T_{B_i^\phi}(mn)$ achieves its maximum value when $\bigvee_{j \neq i} T_{\phi(B_j)}(mn)$ is zero. It is obvious that when $\phi(B_i) = B_k$ and $mn$ is a $B_k$-edge then $\bigvee_{j \neq i} T_{\phi(B_j)}(mn)$ gets zero value. So

$$T_{B_i^\phi}(mn) = T_A(m) \wedge T_A(n), \;\; for \; (mn) \in B_k, \;\; \phi(B_i) = B_k.$$

Similarly, we have

$$I_{B_i^\phi}(mn) = I_A(m) \wedge I_A(n), \;\; for \; (mn) \in B_k, \;\; \phi(B_i) = B_k.$$

In the similar way for expression of falsity in $\phi$-complement:

Since

$$F_A(m) \vee F_A(n) \geq 0, \; \bigwedge_{j \neq i} F_{\phi(B_j)}(mn) \geq 0 \text{ and } F_{B_i}(mn) \leq F_A(m) \vee F_A(n) \forall B_i,$$

we see that

$$\bigwedge_{j \neq i} F_{\phi(B_j)}(mn) \leq F_A(m) \vee F_A(n),$$

which implies that

$$F_A(m) \vee F_A(n) - \bigwedge_{j \neq i} F_{\phi(B_j)}(mn) \geq 0.$$

Therefore, $F_{B_i^\phi}(mn)$ is nonnegative for all $i$. Moreover, $F_{B_i^\phi}(mn)$ attains its maximum value when $\bigwedge_{j \neq i} F_{\phi(B_j)}(mn)$ becomes zero. It is clear that when $\phi(B_i) = B_k$ and $mn$ is a $B_k$-edge then $\bigwedge_{j \neq i} F_{\phi(B_j)}(mn)$ gets zero value. So

$$F_{B_i^\phi}(mn) = F_A(m) \vee F_A(n) \text{ for } (mn) \in B_k, \; \phi(B_i) = B_k.$$

This completes the proof.

**Definition 2.30** Let $G = (A, B_1, B_2, \ldots, B_n)$ be a neutrosophic graph structure and $\phi$ be a permutation on $\{1, 2, \ldots, n\}$. Then

(i) If $G$ is isomorphic to $G^{\phi c}$, then $G$ is said to be *self-complementary*.
(ii) If $G$ is identical to $G^{\phi c}$, then $G$ is said to be *strong self-complementary*.

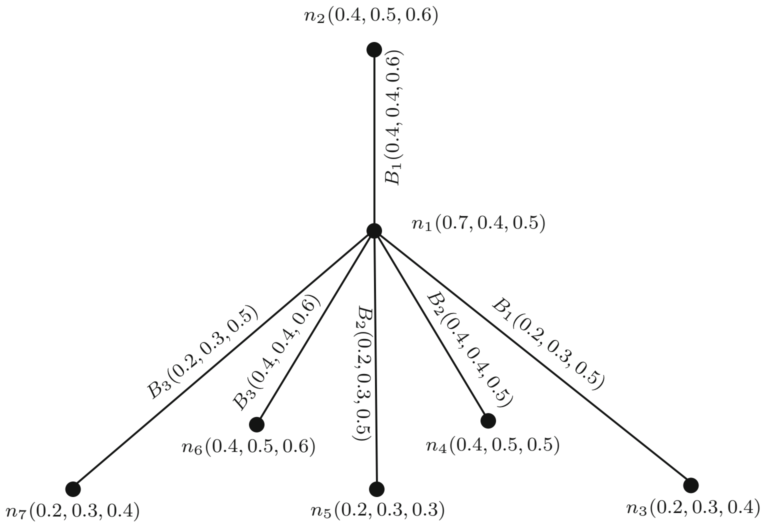


**Fig. 2.13** Totally strong self-complementary neutrosophic graph structure

**Definition 2.31** Suppose $G = (A, B_1, B_2, \ldots, B_n)$ be a neutrosophic graph structure. Then

(i) If $G$ is isomorphic to $G^{\phi c}$, for all permutations $\phi$ on $\{1, 2, \ldots, n\}$, then $G$ is *totally self-complementary*.
(ii) If $G$ is identical to $G^{\phi c}$, for all permutations $\phi$ on $\{1, 2, \ldots, n\}$, then $G$ is *totally strong self-complementary*.

*Remark 2.1* All strong neutrosophic graph structures are self-complementary or totally self-complementary neutrosophic graph structures.

*Example 2.15* A neutrosophic graph structure $G = (A, B_1, B_2, B_3)$ in Fig. 2.13 is a totally strong self-complementary neutrosophic graph structure.

**Theorem 2.1** *A neutrosophic graph structure is totally self-complementary if and only if it is strong neutrosophic graph structure.*

*Proof* Consider a strong neutrosophic graph structure $G$ and a permutation $\phi$ on $\{1, 2, \ldots, n\}$. By Proposition 2.1, $\phi$-complement of a neutrosophic graph structure $G = (A, B_1, B_2, \ldots, B_n)$ is always a strong neutrosophic graph structure. Moreover, if $\phi(i) = k$, where $i, k \in \{1, 2, \ldots, n\}$, then all $B_k$-edges in neutrosophic graph structure $(A, B_1, B_2, \ldots, B_n)$ become $B_i^{\phi}$-edges in $(A, B_1^{\phi}, B_2^{\phi}, \ldots, B_n^{\phi})$. This leads

$$T_{B_k}(mn) = T_A(m) \wedge T_A(n) = T_{B_i^{\phi}}(mn),\ I_{B_k}(mn) = I_A(m) \wedge I_A(n) = I_{B_i^{\phi}}(mn)$$

and

$$F_{B_k}(mn) = F_A(m) \vee F_A(n) = F_{B_i^\phi}(mn).$$

Hence, under the mapping (identity mapping) $f : X \to X$, $G$ and $G^\phi$ are isomorphic such that

$$T_A(m) = T_A(f(m)),\ I_A(m) = I_A(f(m)),\ F_A(m) = F_A(f(m)),$$

$$T_{B_k}(mn) = T_{B_i^\phi}(f(m)f(n)) = T_{B_i^\phi}(mn),\ I_{B_k}(mn) = I_{B_i^\phi}(f(m)f(n)) = I_{B_i^\phi}(mn),$$

$$F_{B_k}(mn) = F_{B_i^\phi}(f(m)f(n)) = F_{B_i^\phi}(mn),$$

for all $mn \in E_k$, $\phi^{-1}(k) = i$ and $k = 1, 2, \ldots, n$. This is satisfied for every permutation $\phi$ on $\{1, 2, \ldots, n\}$. Hence, $G$ is totally self-complementary neutrosophic graph structure. Conversely, let for every permutation $\phi$ on $\{1, 2, \ldots, n\}$, $G$ and $G^\phi$ are isomorphic. Then according to the definition of isomorphism of neutrosophic graph structures and $\phi$-complement of neutrosophic graph structure,

$$T_{B_k}(mn) = T_{B_i^\phi}(f(m)f(n)) = T_A(f(m)) \wedge T_A(f(n)) = T_A(m) \wedge T_A(n),$$

$$I_{B_k}(mn) = I_{B_i^\phi}(f(m)f(n)) = I_A(f(m)) \wedge I_A(f(n)) = T_A(m) \wedge I_A(n),$$

$$F_{B_k}(mn) = F_{B_i^\phi}(f(m)f(n)) = F_A(f(m)) \vee T_A(f(n)) = F_A(m) \wedge T_A(n),$$

for all $mn \in E_k$ and $k = 1, 2, \ldots, n$. Hence, $G$ is strong neutrosophic graph structure.

*Remark 2.2* Every self-complementary neutrosophic graph structure is totally self-complementary.

**Theorem 2.2** *If $G^* = (X, E_1, E_2, \ldots, E_n)$ is a totally strong self-complementary graph structure and $A = (T_A, I_A, F_A)$ is a neutrosophic subset of $X$ where $T_A, I_A, F_A$ are constant valued functions, then a strong neutrosophic graph structure of $G^*$ with neutrosophic vertex set $A$ is always a totally strong self-complementary neutrosophic graph structure.*

*Proof* Consider three constants $p, q, r \in [0, 1]$, such that $T_A(m) = p, I_A(m) = q, F_A(m) = r\ \forall m \in X$. Since $G^*$ is totally self-complementary strong graph structure, so there is a bijection $f : X \to X$ for any permutation $\phi^{-1}$ on $\{1, 2, \ldots, n\}$, such that for any $E_k$-edge $(mn)$, $(f(m)f(n))$ [an $E_i$-edge in $G^*$ ] is an $E_k$-edge in $G^{*\phi^{-1}c}$. Hence, for every $B_k$-edge $(mn)$, $(f(m)f(n))$ [a $B_i$-edge in $G$ ] is a $B_k^\phi$-edge in $G^{\phi^{-1}c}$. Moreover, $G$ is strong neutrosophic graph structure. Thus,

$$T_A(m) = p = T_A(f(m)),\ I_A(m) = q = I_A(f(m)),\ F_A(m) = r = F_A(f(m)),\ \forall m \in X,$$

$$T_{B_k}(mn) = T_A(m) \wedge T_A(n) = T_A(f(m)) \wedge T_A(f(n)) = T_{B_i^\phi}(f(m)f(n)),$$

$$I_{B_k}(mn) = I_A(m) \wedge I_A(n) = I_A(f(m)) \wedge I_A(f(n)) = I_{B_i^\phi}(f(m)f(n)),$$

$$F_{B_k}(mn) = F_A(m) \vee I_A(n) = F_A(f(m)) \vee F_A(f(n)) = F_{B_i^\phi}(f(m)f(n)),$$

for all $mn \in E_i$ and $i = 1, 2, \ldots, n$. This shows that $G$ is self-complementary strong neutrosophic graph structure. Every permutation $\phi$ and $\phi^{-1}$ on $\{1, 2, \ldots, n\}$ satisfy above expressions; thus $G$ is totally strong self-complementary neutrosophic graph structure.

*Remark 2.3* Converse of Theorem 2.2 may not be true, for example a neutrosophic graph structure shown in Fig. 2.13 is a totally strong self-complementary, it is strong and its underlying graph structure is a totally strong self-complementary but $T_A$, $I_A$, $F_A$ are not constant functions.

## 2.3 Operations on Neutrosophic Graph Structures

In this section, we present the operations on neutrosophic graph structures.

**Definition 2.32** Let $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ and $G_2 = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ be neutrosophic graph structures of the graph structures $G_1^* = (X_1, E_{11}, E_{12}, \ldots, E_{1n})$ and $G_2^* = (X_2, E_{21}, E_{22}, \ldots, E_{2n})$, respectively. The *Cartesian product* of $G_1$ and $G_2$, denoted by

$$G_1 \times G_2 = (A_1 \times A_2, B_{11} \times B_{21}, B_{12} \times B_{22}, \ldots, B_{1n} \times B_{2n}),$$

is defined by the following:

(i) $$\begin{cases} T_{(A_1\times A_2)}(qr) = (T_{A_1} \times T_{A_2})(qr) = T_{A_1}(q) \wedge T_{A_2}(r) \\ I_{(A_1\times A_2)}(qr) = (I_{A_1} \times I_{A_2})(qr) = I_{A_1}(q) \wedge I_{A_2}(r) \\ F_{(A_1\times A_2)}(qr) = (F_{A_1} \times F_{A_2})(qr) = F_{A_1}(q) \vee F_{A_2}(r) \end{cases}$$
for all $qr \in E_1 \times E_2$,

(ii) $$\begin{cases} T_{(B_{1i}\times B_{2i})}(qr_1)(qr_2) = (T_{B_{1i}} \times T_{B_{2i}})(qr_1)(qr_2) = T_{A_1}(q) \wedge T_{B_{2i}}(r_1r_2) \\ I_{(B_{1i}\times B_{2i})}(qr_1)(qr_2) = (I_{B_{1i}} \times I_{B_{2i}})(qr_1)(qr_2) = I_{A_1}(q) \wedge I_{B_{2i}}(r_1r_2) \\ F_{(B_{1i}\times B_{2i})}(qr_1)(qr_2) = (F_{B_{1i}} \times F_{B_{2i}})(qr_1)(qr_2) = F_{A_1}(q) \vee F_{B_{2i}}(r_1r_2) \end{cases}$$
for all $q \in X_1$, $r_1r_2 \in E_{2i}$,

(iii) $$\begin{cases} T_{(B_{1i}\times B_{2i})}(q_1r)(q_2r) = (T_{B_{1i}} \times T_{B_{2i}})(q_1r)(q_2r) = T_{A_2}(r) \wedge T_{B_{1i}}(q_1q_2) \\ I_{(B_{1i}\times B_{2i})}(q_1r)(q_2r) = (I_{B_{1i}} \times I_{B_{2i}})(q_1r)(q_2r) = I_{A_2}(r) \wedge I_{B_{1i}}(q_1q_2) \\ F_{(B_{1i}\times B_{2i})}(q_1r)(q_2r) = (F_{B_{1i}} \times F_{B_{2i}})(q_1r)(q_2r) = F_{A_2}(r) \vee F_{B_{1i}}(q_1q_2) \end{cases}$$
for all $r \in X_2$, $q_1q_2 \in E_{1i}$.

*Example 2.16* Consider $G_1 = (A_1, B_{11}, B_{12})$ and $G_2 = (A_2, B_{21}, B_{22})$ are neutrosophic graph structures of graph structures $G_1^* = (X_1, E_{11}, E_{12})$ and $G_2^* = (X_2, E_{21}, E_{22})$, respectively, as shown in Fig. 2.14, where $E_{11} = \{q_1q_2\}$, $E_{12} = \{q_3q_4\}$, $E_{21} = \{r_1r_2\}$, $E_{22} = \{r_2r_3\}$.

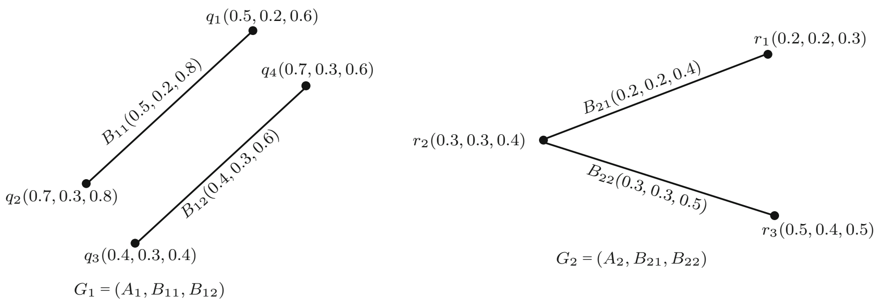


**Fig. 2.14** Neutrosophic graph structures

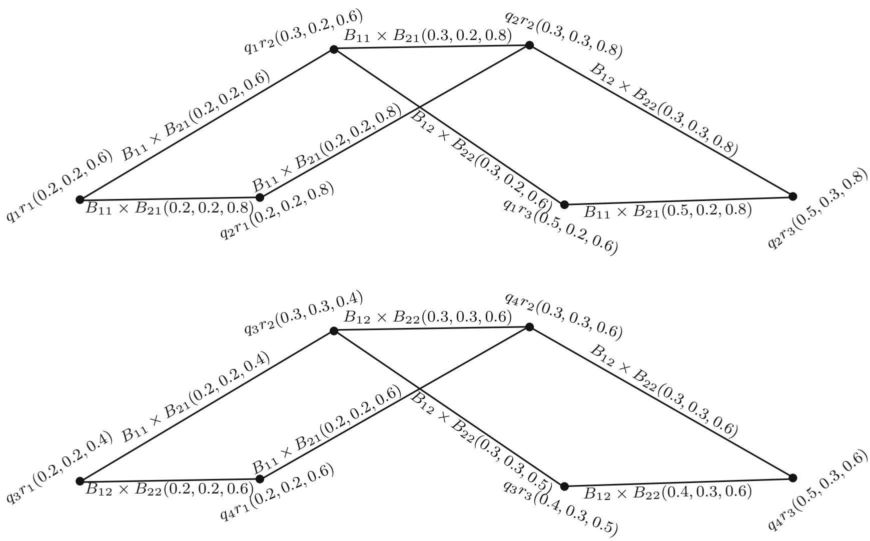


**Fig. 2.15** Cartesian product of two neutrosophic graph structures

Cartesian product of $G_1$ and $G_2$ defined as $G_1 \times G_2 = \{A_1 \times A_2, B_{11} \times B_{21}, B_{12} \times B_{22}\}$ is shown in the Fig. 2.15.

**Theorem 2.3** *The Cartesian product* $G_1 \times G_2 = (A_1 \times A_2, B_{11} \times B_{21}, B_{12} \times B_{22}, \ldots, B_{1n} \times B_{2n})$ *of two neutrosophic graph structures* $G_1$ *and* $G_2$ *of the graph structures* $G_1^*$ *and* $G_2^*$ *is a neutrosophic graph structure of* $G_1^* \times G_2^*$.

*Proof* According to the definition of Cartesian product, there are two cases:

Case 1. When $q \in X_1$, $r_1 r_2 \in E_{2i}$

$$\begin{aligned}T_{(B_{1i}\times B_{2i})}((qr_1)(qr_2)) &= T_{A_1}(q) \wedge T_{B_{2i}}(r_1r_2)\\ &\leq T_{A_1}(q) \wedge [T_{A_2}(r_1) \wedge T_{A_2}(r_2)]\\ &= [T_{A_1}(q) \wedge T_{A_2}(r_1)] \wedge [T_{A_1}(q) \wedge T_{A_2}(r_2)]\\ &= T_{(A_1\times A_2)}(qr_1) \wedge T_{(A_1\times A_2)}(qr_2),\end{aligned}$$

$$\begin{aligned}I_{(B_{1i}\times B_{2i})}((qr_1)(qr_2)) &= I_{A_1}(q) \wedge I_{B_{2i}}(r_1r_2)\\ &\leq I_{A_1}(q) \wedge [I_{A_2}(r_1) \wedge I_{A_2}(r_2)]\\ &= [I_{A_1}(q) \wedge I_{A_2}(r_1)] \wedge [I_{A_1}(q) \wedge I_{A_2}(r_2)]\\ &= I_{(A_1\times A_2)}(qr_1) \wedge I_{(A_1\times A_2)}(qr_2),\end{aligned}$$

$$\begin{aligned}F_{(B_{1i}\times B_{2i})}((qr_1)(qr_2)) &= F_{A_1}(q) \vee F_{B_{2i}}(r_1r_2)\\ &\leq F_{A_1}(q) \vee [F_{A_2}(r_1) \vee F_{A_2}(r_2)]\\ &= [F_{A_1}(q) \vee F_{A_2}(r_1)] \vee [F_{A_1}(q) \vee F_{A_2}(r_2)]\\ &= F_{(A_1\times A_2)}(qr_1) \vee F_{(A_1\times A_2)}(qr_2),\end{aligned}$$

for $qr_1, qr_2 \in X_1 \times X_2$.
Case 2. When $q \in X_2$, $r_1r_2 \in E_{1i}$

$$\begin{aligned}T_{(B_{1i}\times B_{2i})}((r_1q)(r_2q)) &= T_{A_2}(q) \wedge T_{B_{1i}}(r_1r_2)\\ &\leq T_{A_2}(q) \wedge [T_{A_1}(r_1) \wedge T_{A_1}(r_2)]\\ &= [T_{A_2}(q) \wedge T_{A_1}(r_1)] \wedge [T_{A_2}(q) \wedge T_{A_1}(r_2)]\\ &= T_{(A_1\times A_2)}(r_1q) \wedge T_{(A_1\times A_2)}(r_2q),\end{aligned}$$

$$\begin{aligned}I_{(B_{1i}\times B_{2i})}((r_1q)(r_2q)) &= I_{A_2}(q) \wedge I_{B_{1i}}(r_1r_2)\\ &\leq I_{A_2}(q) \wedge [I_{A_1}(r_1) \wedge I_{A_1}(r_2)]\\ &= [I_{A_2}(q) \wedge I_{A_1}(r_1)] \wedge [I_{A_2}(q) \wedge I_{A_1}(r_2)]\\ &= I_{(A_1\times A_2)}(r_1q) \wedge I_{(A_1\times A_2)}(r_2q),\end{aligned}$$

$$\begin{aligned}F_{(B_{1i}\times B_{2i})}((r_1q)(r_2q)) &= F_{A_2}(q) \vee F_{B_{1i}}(r_1r_2)\\ &\leq F_{A_2}(q) \vee [F_{A_1}(r_1) \vee F_{A_1}(r_2)]\\ &= [F_{A_2}(q) \vee F_{A_1}(r_1)] \vee [F_{A_2}(q) \vee F_{A_1}(r_2)]\\ &= F_{(A_1\times A_2)}(r_1q) \vee F_{(A_1\times A_2)}(r_2q),\end{aligned}$$

for $r_1q, r_2q \in X_1 \times X_2$.

Both cases are satisfied $\forall i \in \{1, 2, \ldots, n\}$.

**Definition 2.33** Let $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ and $G_2 = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ be neutrosophic graph structures. The *cross product* of $G_1$ and $G_2$, denoted by

$$G_1 * G_2 = (A_1 * A_2, B_{11} * B_{21}, B_{12} * B_{22}, \ldots, B_{1n} * B_{2n}),$$

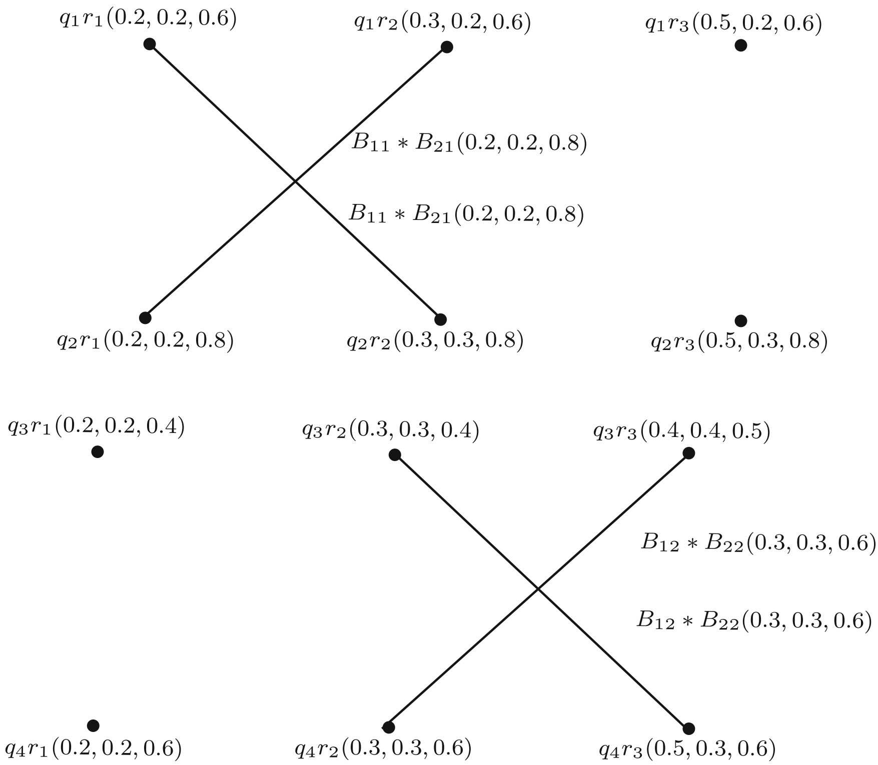


**Fig. 2.16** Cross product of two neutrosophic graph structures

is defined by the following:

(i) $$\begin{cases} T_{(A_1*A_2)}(qr) = (T_{A_1} * T_{A_2})(qr) = T_{A_1}(q) \wedge T_{A_2}(r) \\ I_{(A_1*A_2)}(qr) = (I_{A_1} * I_{A_2})(qr) = I_{A_1}(q) \wedge I_{A_2}(r) \\ F_{(A_1*A_2)}(qr) = (F_{A_1} * F_{A_2})(qr) = F_{A_1}(q) \vee F_{A_2}(r) \end{cases}$$
for all $qr \in X_1 \times X_2$,

(ii) $$\begin{cases} T_{(B_{1i}*B_{2i})}(q_1r_1)(q_2r_2) = (T_{B_{1i}} * T_{B_{2i}})(q_1r_1)(q_2r_2) = T_{B_{1i}}(q_1q_2) \wedge T_{B_{2i}}(r_1r_2) \\ I_{(B_{1i}*B_{2i})}(q_1r_1)(q_2r_2) = (I_{B_{1i}} * I_{B_{2i}})(q_1r_1)(q_2r_2) = I_{B_{1i}}(q_1q_2) \wedge I_{B_{2i}}(r_1r_2) \\ F_{(B_{1i}*B_{2i})}(q_1r_1)(q_2r_2) = (F_{B_{1i}} * F_{B_{2i}})(q_1r_1)(q_2r_2) = F_{B_{1i}}(q_1q_2) \vee F_{B_{2i}}(r_1r_2) \end{cases}$$
for all $q_1q_2 \in E_{1i}, r_1r_2 \in E_{2i}$.

*Example 2.17* Cross product of two neutrosophic graph structures $G_1$ and $G_2$ shown in Fig. 2.14 is defined as $G_1 * G_2 = \{A_1 * A_2, B_{11} * B_{21}, B_{12} * B_{22}\}$ and is shown in the Fig. 2.16.

**Theorem 2.4** *The cross product $G_1 * G_2 = (A_1 * A_2, B_{11} * B_{21}, B_{12} * B_{22}, \ldots, B_{1n} * B_{2n})$ of two neutrosophic graph structures of the graph structures $G*_1$ and $G_2^*$ is a neutrosophic graph structure of $G_1^* * G_2^*$.*

*Proof* For all $q_1r_1, q_2r_2 \in X_1 * X_2$

$$\begin{aligned} T_{(B_{1i}*B_{2i})}((q_1r_1)(q_2r_2)) &= T_{B_{1i}}(q_1q_2) \wedge T_{B_{2i}}(r_1r_2) \\ &\leq [T_{A_1}(q_1) \wedge T_{A_1}(q_2)] \wedge [T_{A_2}(r_1) \wedge T_{A_2}(r_2)] \\ &= [T_{A_1}(q_1) \wedge T_{A_2}(r_1)] \wedge [T_{A_1}(q_2) \wedge T_{A_2}(r_2)] \\ &= T_{(A_1*A_2)}(q_1r_1) \wedge T_{(A_1*A_2)}(q_2r_2), \end{aligned}$$

$$\begin{aligned} I_{(B_{1i}*B_{2i})}((q_1r_1)(q_2r_2)) &= I_{B_{1i}}(q_1q_2) \wedge I_{B_{2i}}(r_1r_2) \\ &\leq [I_{A_1}(q_1) \wedge I_{A_1}(q_2)] \wedge [I_{A_2}(r_1) \wedge I_{A_2}(r_2)] \\ &= [I_{A_1}(q_1) \wedge I_{A_2}(r_1)] \wedge [I_{A_1}(q_2) \wedge I_{A_2}(r_2)] \\ &= I_{(A_1*A_2)}(q_1r_1) \wedge I_{(A_1*A_2)}(q_2r_2), \end{aligned}$$

$$\begin{aligned} F_{(B_{1i}*B_{2i})}((q_1r_1)(q_2r_2)) &= F_{B_{1i}}(q_1q_2) \vee F_{B_{2i}}(r_1r_2) \\ &\leq [F_{A_1}(q_1) \vee F_{A_1}(q_2)] \vee [F_{A_2}(r_1) \vee F_{A_2}(r_2)] \\ &= [F_{A_1}(q_1) \vee F_{A_2}(r_1)] \vee [F_{A_1}(q_2) \vee F_{A_2}(r_2)] \\ &= F_{(A_1*A_2)}(q_1r_1) \vee F_{(A_1*A_2)}(q_2r_2), \end{aligned}$$

for $i \in \{1, 2, \ldots, n\}$.

**Definition 2.34** Let $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ and $G_2 = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ be neutrosophic graph structures. The *lexicographic product* of $G_1$ and $G_2$, denoted by

$$G_1 \bullet G_2 = (A_1 \bullet A_2, B_{11} \bullet B_{21}, B_{12} \bullet B_{22}, \ldots, B_{1n} \bullet B_{2n}),$$

is defined by the following:

(i) $$\begin{cases} T_{(A_1 \bullet A_2)}(qr) = (T_{A_1} \bullet T_{A_2})(qr) = T_{A_1}(q) \wedge T_{A_2}(r) \\ I_{(A_1 \bullet A_2)}(qr) = (I_{A_1} \bullet I_{A_2})(qr) = I_{A_1}(q) \wedge I_{A_2}(r) \\ F_{(A_1 \bullet A_2)}(qr) = (F_{A_1} \bullet F_{A_2})(qr) = F_{A_1}(q) \vee F_{A_2}(r) \end{cases}$$

for all $qr \in X_1 \times X_2$,

(ii) $$\begin{cases} T_{(B_{1i} \bullet B_{2i})}(qr_1)(qr_2) = (T_{B_{1i}} \bullet T_{B_{2i}})(qr_1)(qr_2) = T_{A_1}(q) \wedge T_{B_{2i}}(r_1r_2) \\ I_{(B_{1i} \bullet B_{2i})}(qr_1)(qr_2) = (I_{B_{1i}} \bullet I_{B_{2i}})(qr_1)(qr_2) = I_{A_1}(q) \wedge I_{B_{2i}}(r_1r_2) \\ F_{(B_{1i} \bullet B_{2i})}(qr_1)(qr_2) = (F_{B_{1i}} \bullet F_{B_{2i}})(qr_1)(qr_2) = F_{A_1}(q) \vee F_{B_{2i}}(r_1r_2) \end{cases}$$

for all $q \in X_1, r_1r_2 \in E_{2i}$,

(iii) $$\begin{cases} T_{(B_{1i} \bullet B_{2i})}(q_1r_1)(q_2r_2) = (T_{B_{1i}} \bullet T_{B_{2i}})(q_1r_1)(q_2r_2) = T_{B_{1i}}(q_1q_2) \wedge T_{B_{2i}}(r_1r_2) \\ I_{(B_{1i} \bullet B_{2i})}(q_1r_1)(q_2r_2) = (I_{B_{1i}} \bullet I_{B_{2i}})(q_1r_1)(q_2r_2) = I_{B_{1i}}(q_1q_2) \wedge I_{B_{2i}}(r_1r_2) \\ F_{(B_{1i} \bullet B_{2i})}(q_1r_1)(q_2r_2) = (F_{B_{1i}} \bullet F_{B_{2i}})(q_1r_1)(q_2r_2) = F_{B_{1i}}(q_1q_2) \vee F_{B_{2i}}(r_1r_2) \end{cases}$$

for all $q_1q_2 \in E_{1i}, r_1r_2 \in E_{2i}$.

*Example 2.18 Lexicographic product* of two neutrosophic graph structures $G_1$ and $G_2$ shown in Fig. 2.14 is defined as

$G_1 \bullet G_2 = \{A_1 \bullet A_2, B_{11} \bullet B_{21}, B_{12} \bullet B_{22}\}$ and is shown in the Fig. 2.17.

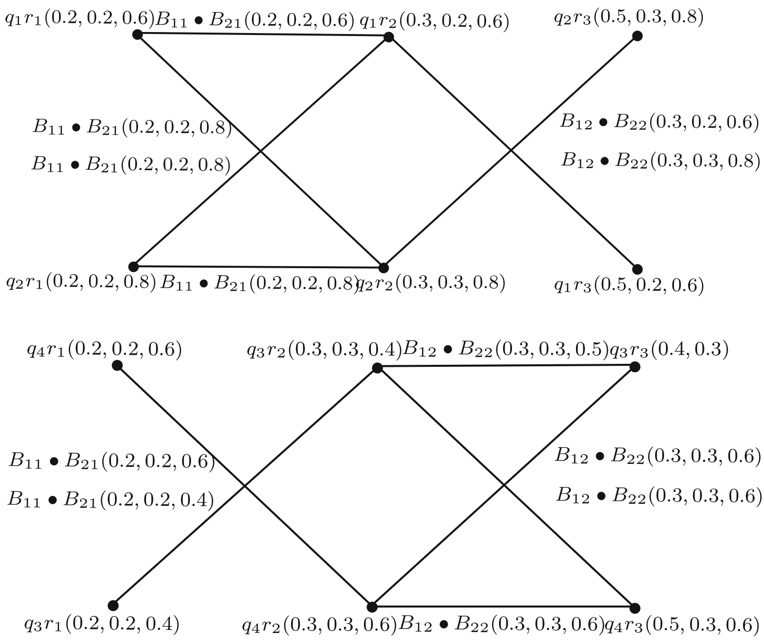


**Fig. 2.17** Lexicographic product of two neutrosophic graph structures

**Theorem 2.5** *The lexicographic product $G_1 \bullet G_2 = (A_1 \bullet A_2, B_{11} \bullet B_{21}, B_{12} \bullet B_{22}, \ldots, B_{1n} \bullet B_{2n})$ of two neutrosophic graph structures of the graph structures $G_1^*$ and $G_2^*$ is a neutrosophic graph structure of $G_1^* \bullet G_2^*$.*

*Proof* According to the definition of lexicographic product, there are two cases:

Case 1. When $q \in X_1$, $r_1r_2 \in E_{2i}$

$$\begin{aligned}
T_{(B_{1i}\bullet B_{2i})}((qr_1)(qr_2)) &= T_{A_1}(q) \wedge T_{B_{2i}}(r_1r_2)\\
&\leq T_{A_1}(q) \wedge [T_{A_2}(r_1) \wedge T_{A_2}(r_2)]\\
&= [T_{A_1}(q) \wedge T_{A_2}(r_1)] \ \wedge [T_{A_1}(q) \wedge T_{A_2}(r_2)]\\
&= T_{(A_1\bullet A_2)}(qr_1) \wedge T_{(A_1\bullet A_2)}(qr_2),
\end{aligned}$$

$$\begin{aligned}
I_{(B_{1i}\bullet B_{2i})}((qr_1)(qr_2)) &= I_{A_1}(q) \wedge I_{B_{2i}}(r_1r_2)\\
&\leq I_{A_1}(q) \wedge [I_{A_2}(r_1) \wedge I_{A_2}(r_2)]\\
&= [I_{A_1}(q) \wedge I_{A_2}(r_1)] \ \wedge [I_{A_1}(q) \wedge I_{A_2}(r_2)]\\
&= I_{(A_1\bullet A_2)}(qr_1) \wedge I_{(A_1\bullet A_2)}(qr_2),
\end{aligned}$$

$$\begin{aligned} F_{(B_{1i}\bullet B_{2i})}((qr_1)(qr_2)) &= F_{A_1}(q) \vee F_{B_{2i}}(r_1r_2) \\ &\leq F_{A_1}(q) \vee [F_{A_2}(r_1) \vee F_{A_2}(r_2)] \\ &= [F_{A_1}(q) \vee F_{A_2}(r_1)] \ \vee [F_{A_1}(q) \vee F_{A_2}(r_2)] \\ &= F_{(A_1\bullet A_2)}(qr_1) \vee F_{(A_1\bullet A_2)}(qr_2), \end{aligned}$$

for $qr_1, qr_2 \in X_1 \bullet X_2$.

Case 2. When $q_1q_2 \in E_{1i}, r_1r_2 \in E_{2i}$

$$\begin{aligned} T_{(B_{1i}\bullet B_{2i})}((q_1r_1)(q_2r_2)) &= T_{B_{1i}}(q_1q_2) \wedge T_{B_{2i}}(r_1r_2) \\ &\leq [T_{A_1}(q_1) \wedge T_{A_1}(q_2)] \ \wedge [T_{A_2}(r_1) \wedge T_{A_2}(r_2)] \\ &= [T_{A_1}(q_1) \wedge T_{A_2}(r_1)] \ \wedge [T_{A_1}(q_2) \wedge T_{A_2}(r_2)] \\ &= T_{(A_1\bullet A_2)}(q_1r_1) \wedge T_{(A_1\bullet A_2)}(q_2r_2), \\ I_{(B_{1i}\bullet B_{2i})}((q_1r_1)(q_2r_2)) &= I_{B_{1i}}(q_1q_2) \wedge I_{B_{2i}}(r_1r_2) \\ &\leq [I_{A_1}(q_1) \wedge I_{A_1}(q_2)] \ \wedge [I_{A_2}(r_1) \wedge I_{A_2}(r_2)] \\ &= [I_{A_1}(q_1) \wedge I_{A_2}(r_1)] \ \wedge [I_{A_1}(q_2) \wedge I_{A_2}(r_2)] \\ &= I_{(A_1\bullet A_2)}(q_1r_1) \wedge I_{(A_1\bullet A_2)}(q_2r_2), \\ F_{(B_{1i}\bullet B_{2i})}((q_1r_1)(q_2r_2)) &= F_{B_{1i}}(q_1q_2) \vee F_{B_{2i}}(r_1r_2) \\ &\leq [F_{A_1}(q_1) \vee F_{A_1}(q_2)] \ \vee [F_{A_2}(r_1) \vee F_{A_2}(r_2)] \\ &= [F_{A_1}(q_1) \vee F_{A_2}(r_1)] \ \vee [F_{A_1}(q_2) \vee F_{A_2}(r_2)] \\ &= F_{(A_1\bullet A_2)}(q_1r_1) \vee F_{(A_1\bullet A_2)}(q_2r_2), \end{aligned}$$

for $q_1r_1, q_2r_2 \in X_1 \bullet X_2$.

Both cases are satisfied for $i \in \{1, 2, \ldots, n\}$.

**Definition 2.35** Let $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ and $G_2 = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ be neutrosophic graph structures. The *strong product* of $G_1$ and $G_2$, denoted by

$$G_1 \boxtimes G_2 = (A_1 \boxtimes A_2, B_{11} \boxtimes B_{21}, B_{12} \boxtimes B_{22}, \ldots, B_{1n} \boxtimes B_{2n}),$$

is defined by the following:

(i) $$\begin{cases} T_{(A_1\boxtimes A_2)}(qr) = (T_{A_1} \boxtimes T_{A_2})(qr) = T_{A_1}(q) \wedge T_{A_2}(r) \\ I_{(A_1\boxtimes A_2)}(qr) = (I_{A_1} \boxtimes I_{A_2})(qr) = I_{A_1}(q) \wedge I_{A_2}(r) \\ F_{(A_1\boxtimes A_2)}(qr) = (F_{A_1} \boxtimes F_{A_2})(qr) = F_{A_1}(q) \vee F_{A_2}(r) \end{cases}$$

for all $qr \in X_1 \times X_2$,

(ii) $$\begin{cases} T_{(B_{1i}\boxtimes B_{2i})}(qr_1)(qr_2) = (T_{B_{1i}} \boxtimes T_{B_{2i}})(qr_1)(qr_2) = T_{A_1}(q) \wedge T_{B_{2i}}(r_1r_2) \\ I_{(B_{1i}\boxtimes B_{2i})}(qr_1)(qr_2) = (I_{B_{1i}} \boxtimes I_{B_{2i}})(qr_1)(qr_2) = I_{A_1}(q) \wedge I_{B_{2i}}(r_1r_2) \\ F_{(B_{1i}\boxtimes B_{2i})}(qr_1)(qr_2) = (F_{B_{1i}} \boxtimes F_{B_{2i}})(qr_1)(qr_2) = F_{A_1}(q) \vee F_{B_{2i}}(r_1r_2) \end{cases}$$

for all $q \in X_1, r_1r_2 \in E_{2i}$,

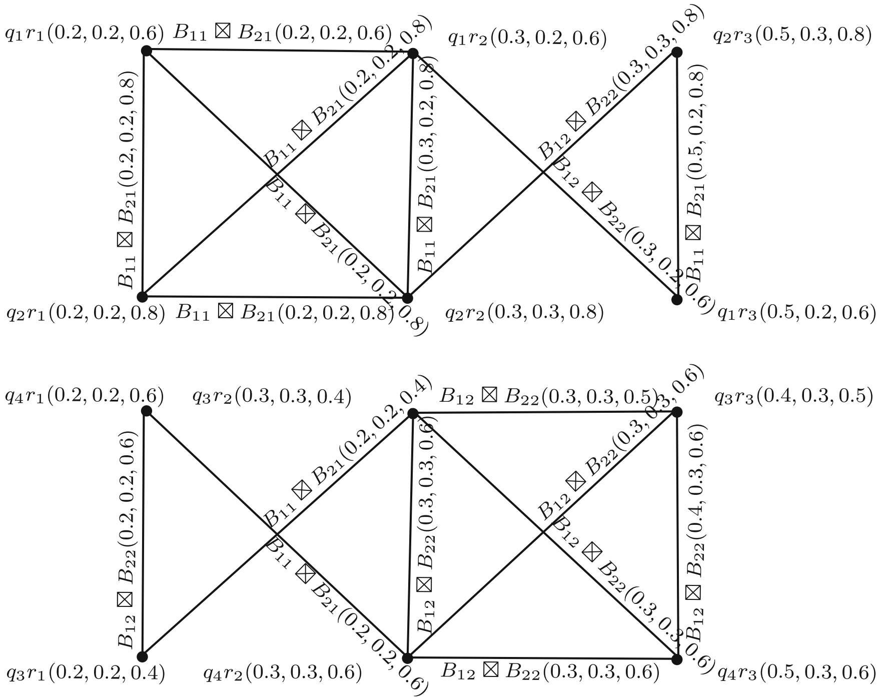


**Fig. 2.18** Strong product of two neutrosophic graph structures

(iii) $$\begin{cases} T_{(B_{1i}\boxtimes B_{2i})}(q_1r)(q_2r) = (T_{B_{1i}} \boxtimes T_{B_{2i}})(q_1r)(q_2r) = T_{A_2}(r) \wedge T_{B_{1i}}(q_1q_2) \\ I_{(B_{1i}\boxtimes B_{2i})}(q_1r)(q_2r) = (I_{B_{1i}} \boxtimes I_{B_{2i}})(q_1r)(q_2r) = I_{A_2}(r) \wedge I_{B_{1i}}(q_1q_2) \\ F_{(B_{1i}\boxtimes B_{2i})}(q_1r)(q_2r) = (F_{B_{1i}} \boxtimes F_{B_{2i}})(q_1r)(q_2r) = F_{A_2}(r) \vee F_{B_{1i}}(q_1q_2) \end{cases}$$

for all $r \in X_2$, $q_1q_2 \in E_{1i}$,

(iv) $$\begin{cases} T_{(B_{1i}\boxtimes B_{2i})}(q_1r_1)(q_2r_2)=(T_{B_{1i}} \boxtimes T_{B_{2i}})(q_1r_1)(q_2r_2)=T_{B_{1i}}(q_1q_2) \wedge T_{B_{2i}}(r_1r_2) \\ I_{(B_{1i}\boxtimes B_{2i})}(q_1r_1)(q_2r_2) = (I_{B_{1i}} \boxtimes I_{B_{2i}})(q_1r_1)(q_2r_2) = I_{B_{1i}}(q_1q_2) \wedge I_{B_{2i}}(r_1r_2) \\ F_{(B_{1i}\boxtimes B_{2i})}(q_1r_1)(q_2r_2)=(F_{B_{1i}} \boxtimes F_{B_{2i}})(q_1r_1)(q_2r_2)=F_{B_{1i}}(q_1q_2) \vee F_{B_{2i}}(r_1r_2) \end{cases}$$

for all $q_1q_2 \in E_{1i}$, $r_1r_2 \in E_{2i}$.

*Example 2.19* Strong product of two neutrosophic graph structures $G_1$ and $G_2$ shown in Fig. 2.14 is defined as $G_1 \boxtimes G_2 = \{A_1 \boxtimes A_2, B_{11} \boxtimes B_{21}, B_{12} \boxtimes B_{22}\}$ and is shown in the Fig. 2.18.

**Theorem 2.6** *The strong product* $G_1 \boxtimes G_2 = (A_1 \boxtimes A_2, B_{11} \boxtimes B_{21}, B_{12} \boxtimes B_{22}, \ldots, B_{1n} \boxtimes B_{2n})$ *of two neutrosophic graph structures of the graph structures* $G_1^*$ *and* $G_2^*$ *is a neutrosophic graph structure of* $G_1^* \boxtimes G_2^*$.

*Proof* According to the definition of strong product, there are three cases:

Case 1. When $q \in X_1$, $r_1r_2 \in E_{2i}$

$$\begin{aligned}
T_{(B_{1i}\boxtimes B_{2i})}((qr_1)(qr_2)) &= T_{A_1}(q)\wedge T_{B_{2i}}(r_1r_2)\\
&\le T_{A_1}(q)\wedge[T_{A_2}(r_1)\wedge T_{A_2}(r_2)]\\
&= [T_{A_1}(q)\wedge T_{A_2}(r_1)]\ \wedge[T_{A_1}(q)\wedge T_{A_2}(r_2)]\\
&= T_{(A_1\boxtimes A_2)}(qr_1)\wedge T_{(A_1\boxtimes A_2)}(qr_2),\\
I_{(B_{1i}\boxtimes B_{2i})}((qr_1)(qr_2)) &= I_{A_1}(q)\wedge I_{B_{2i}}(r_1r_2)\\
&\le I_{A_1}(q)\wedge[I_{A_2}(r_1)\wedge I_{A_2}(r_2)]\\
&= [I_{A_1}(q)\wedge I_{A_2}(r_1)]\ \wedge[I_{A_1}(q)\wedge I_{A_2}(r_2)]\\
&= I_{(A_1\boxtimes A_2)}(qr_1)\wedge I_{(A_1\boxtimes A_2)}(qr_2),\\
F_{(B_{1i}\boxtimes B_{2i})}((qr_1)(qr_2)) &= F_{A_1}(q)\vee F_{B_{2i}}(r_1r_2)\\
&\le F_{A_1}(q)\vee[F_{A_2}(r_1)\vee F_{A_2}(r_2)]\\
&= [F_{A_1}(q)\vee F_{A_2}(r_1)]\ \vee[F_{A_1}(q)\vee F_{A_2}(r_2)]\\
&= F_{(A_1\boxtimes A_2)}(qr_1)\vee F_{(A_1\boxtimes A_2)}(qr_2),
\end{aligned}$$

for $qr_1, qr_2 \in X_1 \boxtimes X_2$.

Case 2. When $q \in X_2$, $r_1r_2 \in E_{1i}$

$$\begin{aligned}
T_{(B_{1i}\boxtimes B_{2i})}((r_1q)(r_2q)) &= T_{A_2}(q)\wedge T_{B_{1i}}(r_1r_2)\\
&\le T_{A_2}(q)\wedge[T_{A_1}(r_1)\wedge T_{A_1}(r_2)]\\
&= [T_{A_2}(q)\wedge T_{A_1}(r_1)]\ \wedge[T_{A_2}(q)\wedge T_{A_1}(r_2)]\\
&= T_{(A_1\boxtimes A_2)}(r_1q)\wedge T_{(A_1\boxtimes A_2)}(r_2q),\\
I_{(B_{1i}\boxtimes B_{2i})}((r_1q)(r_2q)) &= I_{A_2}(q)\wedge I_{B_{1i}}(r_1r_2)\\
&\le I_{A_2}(q)\wedge[I_{A_1}(r_1)\wedge I_{A_1}(r_2)]\\
&= [I_{A_2}(q)\wedge I_{A_1}(r_1)]\ \wedge[I_{A_2}(q)\wedge I_{A_1}(r_2)]\\
&= I_{(A_1\boxtimes A_2)}(r_1q)\wedge I_{(A_1\boxtimes A_2)}(r_2q),\\
F_{(B_{1i}\boxtimes B_{2i})}((r_1q)(r_2q)) &= F_{A_2}(q)\vee F_{B_{1i}}(r_1r_2)\\
&\le F_{A_2}(q)\vee[F_{A_1}(r_1)\vee F_{A_1}(r_2)]\\
&= [F_{A_2}(q)\vee F_{A_1}(r_1)]\ \vee[F_{A_2}(q)\vee F_{A_1}(r_2)]\\
&= F_{(A_1\boxtimes A_2)}(r_1q)\vee F_{(A_1\boxtimes A_2)}(r_2q),
\end{aligned}$$

for $r_1q, r_2q \in X_1 \boxtimes X_2$.

Case 3. For all $q_1q_2 \in E_{1i}$, $r_1r_2 \in E_{2i}$

$$\begin{aligned}
T_{(B_{1i}\boxtimes B_{2i})}((q_1r_1)(q_2r_2)) &= T_{B_{1i}}(q_1q_2)\wedge T_{B_{2i}}(r_1r_2)\\
&\le [T_{A_1}(q_1)\wedge T_{A_1}(q_2)]\ \wedge[T_{A_2}(r_1)\wedge T_{A_2}(r_2)]\\
&= [T_{A_1}(q_1)\wedge T_{A_2}(r_1)]\ \wedge[T_{A_1}(q_2)\wedge T_{A_2}(r_2)]\\
&= T_{(A_1\boxtimes A_2)}(q_1r_1)\wedge T_{(A_1\boxtimes A_2)}(q_2r_2),
\end{aligned}$$

$$
\begin{aligned}
I_{(B_{1i}\boxtimes B_{2i})}((q_1r_1)(q_2r_2)) &= I_{B_{1i}}(q_1q_2) \wedge I_{B_{2i}}(r_1r_2)\\
&\leq [I_{A_1}(q_1) \wedge I_{A_1}(q_2)] \wedge [I_{A_2}(r_1) \wedge I_{A_2}(r_2)]\\
&= [I_{A_1}(q_1) \wedge I_{A_2}(r_1)] \wedge [I_{A_1}(q_2) \wedge I_{A_2}(r_2)]\\
&= I_{(A_1\boxtimes A_2)}(q_1r_1) \wedge I_{(A_1\boxtimes A_2)}(q_2r_2),\\
F_{(B_{1i}\boxtimes B_{2i})}((q_1r_1)(q_2r_2)) &= F_{B_{1i}}(q_1q_2) \vee F_{B_{2i}}(r_1r_2)\\
&\leq [F_{A_1}(q_1) \vee F_{A_1}(q_2)] \vee [F_{A_2}(r_1) \vee F_{A_2}(r_2)]\\
&= [F_{A_1}(q_1) \vee F_{A_2}(r_1)] \vee [F_{A_1}(q_2) \vee F_{A_2}(r_2)]\\
&= F_{(A_1\boxtimes A_2)}(q_1r_1) \vee F_{(A_1\boxtimes A_2)}(q_2r_2),
\end{aligned}
$$

for $q_1r_1, q_2r_2 \in X_1 \boxtimes X_2$.
All cases are satisfied for $i = 1, 2, \ldots, n$.

**Definition 2.36** Let $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ and $G_2 = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ be neutrosophic graph structures. The *composition* of $G_1$ and $G_2$, denoted by

$$G_1 \circ G_2 = (A_1 \circ A_2, B_{11} \circ B_{21}, B_{12} \circ B_{22}, \ldots, B_{1n} \circ B_{2n}),$$

is defined by the following:

(i) $$\begin{cases} T_{(A_1\circ A_2)}(qr) = (T_{A_1} \circ T_{A_2})(qr) = T_{A_1}(q) \wedge T_{A_2}(r)\\ I_{(A_1\circ A_2)}(qr) = (I_{A_1} \circ I_{A_2})(qr) = I_{A_1}(q) \wedge I_{A_2}(r)\\ F_{(A_1\circ A_2)}(qr) = (F_{A_1} \circ F_{A_2})(qr) = F_{A_1}(q) \vee F_{A_2}(r)\end{cases}$$
for all $qr \in X_1 \times X_2$,

(ii) $$\begin{cases} T_{(B_{1i}\circ B_{2i})}(qr_1)(qr_2) = (T_{B_{1i}} \circ T_{B_{2i}})(qr_1)(qr_2) = T_{A_1}(q) \wedge T_{B_{2i}}(r_1r_2)\\ I_{(B_{1i}\circ B_{2i})}(qr_1)(qr_2) = (I_{B_{1i}} \circ I_{B_{2i}})(qr_1)(qr_2) = I_{A_1}(q) \wedge I_{B_{2i}}(r_1r_2)\\ F_{(B_{1i}\circ B_{2i})}(qr_1)(qr_2) = (F_{B_{1i}} \circ F_{B_{2i}})(qr_1)(qr_2) = F_{A_1}(q) \vee F_{B_{2i}}(r_1r_2)\end{cases}$$
for all $q \in X_1, r_1r_2 \in E_{2i}$,

(iii) $$\begin{cases} T_{(B_{1i}\circ B_{2i})}(q_1r)(q_2r) = (T_{B_{1i}} \circ T_{B_{2i}})(q_1r)(q_2r) = T_{A_2}(r) \wedge T_{B_{1i}}(q_1q_2)\\ I_{(B_{1i}\circ B_{2i})}(q_1r)(q_2r) = (I_{B_{1i}} \circ I_{B_{2i}})(q_1r)(q_2r) = I_{A_2}(r) \wedge I_{B_{1i}}(q_1q_2)\\ F_{(B_{1i}\circ B_{2i})}(q_1r)(q_2r) = (F_{B_{1i}} \circ F_{B_{2i}})(q_1r)(q_2r) = F_{A_2}(r) \vee F_{B_{1i}}(q_1q_2)\end{cases}$$
for all $r \in X_2, q_1q_2 \in E_{1i}$,

(iv) $$\begin{cases} T_{(B_{1i}\circ B_{2i})}(q_1r_1)(q_2r_2) = (T_{B_{1i}} \circ T_{B_{2i}})(q_1r_1)(q_2r_2) = T_{B_{1i}}(q_1q_2) \wedge T_{A_2}(r_1) \wedge T_{A_2}(r_2)\\ I_{(B_{1i}\circ B_{2i})}(q_1r_1)(q_2r_2) = (I_{B_{1i}} \circ I_{B_{2i}})(q_1r_1)(q_2r_2) = I_{B_{1i}}(q_1q_2) \wedge I_{A_2}(r_1) \wedge I_{A_2}(r_2)\\ F_{(B_{1i}\circ B_{2i})}(q_1r_1)(q_2r_2) = (F_{B_{1i}} \circ F_{B_{2i}})(q_1r_1)(q_2r_2) = F_{B_{1i}}(q_1q_2) \vee F_{A_2}(r_1) \vee F_{A_2}(r_2)\end{cases}$$
for all $q_1q_2 \in E_{1i}, r_1r_2 \in E_{2i}$ such that $r_1 \neq r_2$.

*Example 2.20* Composition of two neutrosophic graph structures $G_1$ and $G_2$ shown in Fig. 2.14 is defined as $G_1 \circ G_2 = \{A_1 \circ A_2, B_{11} \circ B_{21}, B_{12} \circ B_{22}\}$ and is shown in the Fig. 2.19.

**Theorem 2.7** *The composition* $G_1 \circ G_2 = (A_1 \circ A_2, B_{11} \circ B_{21}, B_{12} \circ B_{22}, \ldots, B_{1n} \circ B_{2n})$ *of two neutrosophic graph structures of the graph structures* $G_1^*$ *and* $G_2^*$ *is a neutrosophic graph structure of* $G_1^* \circ G_2^*$.

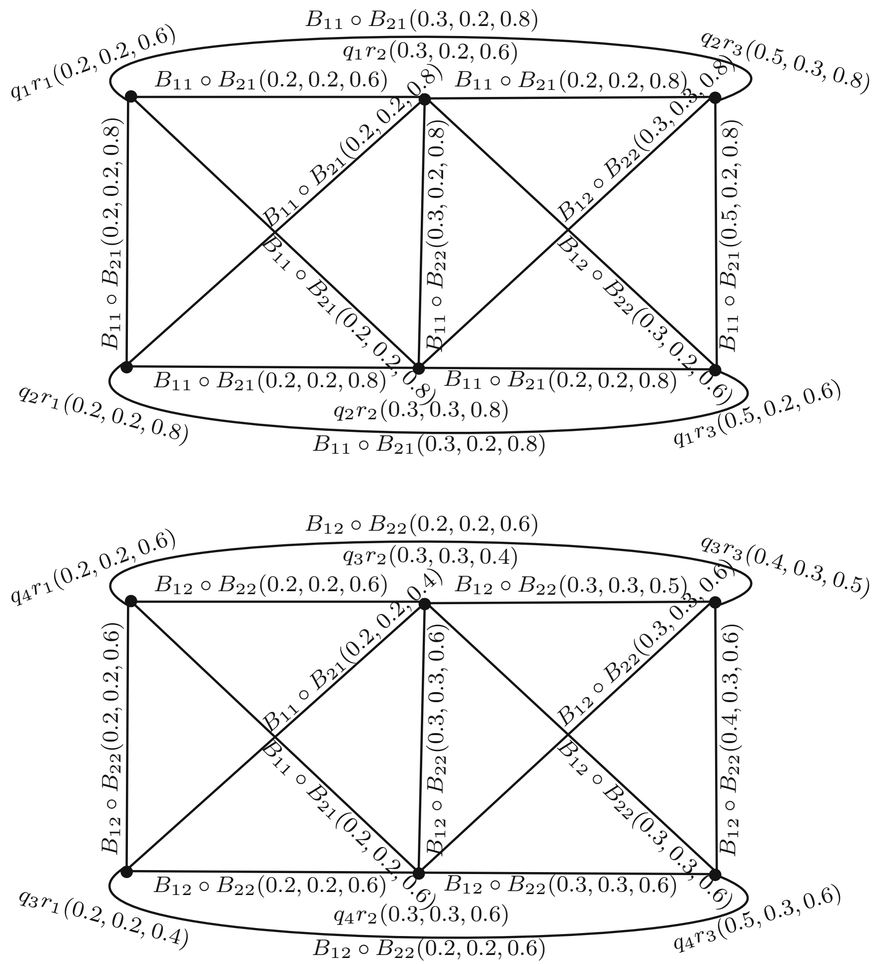


**Fig. 2.19** Composition of two neutrosophic graph structures

*Proof* According to the definition of composition, there are three cases:

Case 1. When $q \in X_1$, $r_1r_2 \in E_{2i}$

$$\begin{aligned} T_{(B_{1i} \circ B_{2i})}((qr_1)(qr_2)) &= T_{A_1}(q) \wedge T_{B_{2i}}(r_1r_2) \\ &\leq T_{A_1}(q) \wedge [T_{A_2}(r_1) \wedge T_{A_2}(r_2)] \\ &= [T_{A_1}(q) \wedge T_{A_2}(r_1)] \wedge [T_{A_1}(q) \wedge T_{A_2}(r_2)] \\ &= T_{(A_1 \circ A_2)}(qr_1) \wedge T_{(A_1 \circ A_2)}(qr_2), \end{aligned}$$

$$\begin{aligned}
I_{(B_{1i}\circ B_{2i})}((qr_1)(qr_2)) &= I_{A_1}(q) \wedge I_{B_{2i}}(r_1r_2)\\
&\leq I_{A_1}(q) \wedge [I_{A_2}(r_1) \wedge I_{A_2}(r_2)]\\
&= [I_{A_1}(q) \wedge I_{A_2}(r_1)] \wedge [I_{A_1}(q) \wedge I_{A_2}(r_2)]\\
&= I_{(A_1\circ A_2)}(qr_1) \wedge I_{(A_1\circ A_2)}(qr_2),\\
F_{(B_{1i}\circ B_{2i})}((qr_1)(qr_2)) &= F_{A_1}(q) \vee F_{B_{2i}}(r_1r_2)\\
&\leq F_{A_1}(q) \vee [F_{A_2}(r_1) \vee F_{A_2}(r_2)]\\
&= [F_{A_1}(q) \vee F_{A_2}(r_1)] \vee [F_{A_1}(q) \vee F_{A_2}(r_2)]\\
&= F_{(A_1\circ A_2)}(qr_1) \vee F_{(A_1\circ A_2)}(qr_2),
\end{aligned}$$

for $qr_1, qr_2 \in X_1 \circ X_2$.
Case 2. When $q \in X_2$, $r_1r_2 \in E_{1i}$

$$\begin{aligned}
T_{(B_{1i}\circ B_{2i})}((r_1q)(r_2q)) &= T_{A_2}(q) \wedge T_{B_{1i}}(r_1r_2)\\
&\leq T_{A_2}(q) \wedge [T_{A_1}(r_1) \wedge T_{A_1}(r_2)]\\
&= [T_{A_2}(q) \wedge T_{A_1}(r_1)] \wedge [T_{A_2}(q) \wedge T_{A_1}(r_2)]\\
&= T_{(A_1\circ A_2)}(r_1q) \wedge T_{(A_1\circ A_2)}(r_2q),\\
I_{(B_{1i}\circ B_{2i})}((r_1q)(r_2q)) &= I_{A_2}(q) \wedge I_{B_{1i}}(r_1r_2)\\
&\leq I_{A_2}(q) \wedge [I_{A_1}(r_1) \wedge I_{A_1}(r_2)]\\
&= [I_{A_2}(q) \wedge I_{A_1}(r_1)] \wedge [I_{A_2}(q) \wedge I_{A_1}(r_2)]\\
&= I_{(A_1\circ A_2)}(r_1q) \wedge I_{(A_1\circ A_2)}(r_2q),\\
F_{(B_{1i}\circ B_{2i})}((r_1q)(r_2q)) &= F_{A_2}(q) \vee F_{B_{1i}}(r_1r_2)\\
&\leq F_{A_2}(q) \vee [F_{A_1}(r_1) \vee F_{A_1}(r_2)]\\
&= [F_{A_2}(q) \vee F_{A_1}(r_1)] \vee [F_{A_2}(q) \vee F_{A_1}(r_2)]\\
&= F_{(A_1\circ A_2)}(r_1q) \vee F_{(A_1\circ A_2)}(r_2q),
\end{aligned}$$

for $r_1q, r_2q \in X_1 \circ X_2$.
Case 3. For all $q_1q_2 \in E_{1i}$, $r_1, r_2 \in X_2$ such that $r_1 \neq r_2$

$$\begin{aligned}
T_{(B_{1i}\circ B_{2i})}((q_1r_1)(q_2r_2)) &= T_{B_{1i}}(q_1q_2) \wedge T_{A_2}(r_1) \wedge T_{A_2}(r_2)\\
&\leq [T_{A_1}(q_1) \wedge T_{A_1}(q_2)] \wedge T_{A_2}(r_1) \wedge T_{A_2}(r_2)\\
&= [T_{A_1}(q_1) \wedge T_{A_2}(r_1)] \wedge [T_{A_1}(q_2) \wedge T_{A_2}(r_2)]\\
&= T_{(A_1\circ A_2)}(q_1r_1) \wedge T_{(A_1\circ A_2)}(q_2r_2),\\
I_{(B_{1i}\circ B_{2i})}((q_1r_1)(q_2r_2)) &= I_{B_{1i}}(q_1q_2) \wedge I_{A_2}(r_1) \wedge I_{A_2}(r_2)\\
&\leq [I_{A_1}(q_1) \wedge I_{A_1}(q_2)] \wedge [I_{A_2}(r_1) \wedge I_{A_2}(r_2)]\\
&= [I_{A_1}(q_1) \wedge I_{A_2}(r_1)] \wedge [I_{A_1}(q_2) \wedge I_{A_2}(r_2)]\\
&= I_{(A_1\circ A_2)}(q_1r_1) \wedge I_{(A_1\circ A_2)}(q_2r_2),
\end{aligned}$$

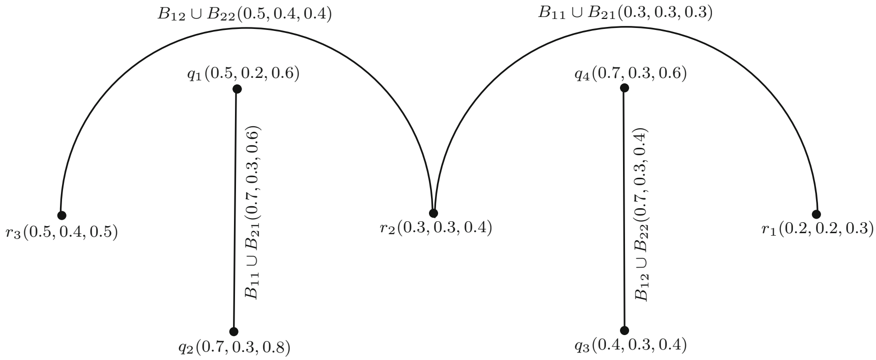


**Fig. 2.20** Union of two neutrosophic graph structures

$$\begin{aligned} F_{(B_{1i} \circ B_{2i})}((q_1r_1)(q_2r_2)) &= F_{B_{1i}}(q_1q_2) \vee F_{A_2}(r_1) \vee F_{A_2}(r_2) \\ &\leq [F_{A_1}(q_1) \vee F_{A_1}(q_2)] \vee [F_{A_2}(r_1) \vee F_{A_2}(r_2)] \\ &= [F_{A_1}(q_1) \vee F_{A_2}(r_1)] \vee [F_{A_1}(q_2) \vee F_{A_2}(r_2)] \\ &= F_{(A_1 \circ A_2)}(q_1r_1) \vee F_{(A_1 \circ A_2)}(q_2r_2), \end{aligned}$$

for $q_1r_1, q_2r_2 \in X_1 \circ X_2$.
All cases are satisfied for $i = 1, 2, \ldots, n$.

**Definition 2.37** Let $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ and $G_2 = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ be neutrosophic graph structures. The *union* of $G_1$ and $G_2$, denoted by

$$G_1 \cup G_2 = (A_1 \cup A_2, B_{11} \cup B_{21}, B_{12} \cup B_{22}, \ldots, B_{1n} \cup B_{2n}),$$

is defined by following:

(i) $\begin{cases} T_{(A_1 \cup A_2)}(q) = (T_{A_1} \cup T_{A_2})(q) = T_{A_1}(q) \vee T_{A_2}(q) \\ I_{(A_1 \cup A_2)}(q) = (I_{A_1} \cup I_{A_2})(q) = I_{A_1}(q) \vee I_{A_2}(q) \\ F_{(A_1 \cup A_2)}(q) = (F_{A_1} \cup F_{A_2})(q) = F_{A_1}(q) \wedge F_{A_2}(q) \end{cases}$

for all $q \in X_1 \cup X_2$,

(ii) $\begin{cases} T_{(B_{1i} \cup B_{2i})}(qr) = (T_{B_{1i}} \cup T_{B_{2i}})(qr) = T_{B_{1i}}(qr) \vee T_{B_{2i}}(qr) \\ I_{(B_{1i} \cup B_{2i})}(qr) = (I_{B_{1i}} \cup I_{B_{2i}})(qr) = I_{B_{1i}}(qr) \vee I_{B_{2i}}(qr) \\ F_{(B_{1i} \cup B_{2i})}(qr) = (F_{B_{1i}} \cup F_{B_{2i}})(qr) = F_{B_{1i}}(qr) \wedge F_{B_{2i}}(qr) \end{cases}$

for all $qr \in E_{1i} \cup E_{2i}$.

*Example 2.21* Union of two neutrosophic graph structures $G_1$ and $G_2$ shown in Fig. 2.14 is defined as $G_1 \cup G_2 = \{A_1 \cup A_2, B_{11} \cup B_{21}, B_{12} \cup B_{22}\}$ and is shown in the Fig. 2.20.

**Theorem 2.8** *The union* $G_1 \cup G_2 = (A_1 \cup A_2, B_{11} \cup B_{21}, B_{12} \cup B_{22}, \ldots, B_{1n} \cup B_{2n})$ *of two neutrosophic graph structures of the graph structures* $G_1^*$ *and* $G_2^*$ *is a neutrosophic graph structure of* $G_1^* \cup G_2^*$.

*Proof* Let $q_1q_2 \in E_{1i} \cup E_{2i}$. Here we consider two cases:

Case 1. When $q_1, q_2 \in X_1$, then according to Definition 2.37, $T_{A_2}(q_1) = T_{A_2}(q_2) = T_{B_{2i}}(q_1q_2) = 0,\ I_{A_2}(q_1) = I_{A_2}(q_2) = I_{B_{2i}}(q_1q_2) = 0,\ F_{A_2}(q_1) = F_{A_2}(q_2) = F_{B_{2i}}(q_1q_2) = 0$, so

$$\begin{aligned}
T_{(B_{1i}\cup B_{2i})}(q_1q_2) &= T_{B_{1i}}(q_1q_2) \vee T_{B_{2i}}(q_1q_2)\\
&= T_{B_{1i}}(q_1q_2) \vee 0\\
&\le [T_{A_1}(q_1) \wedge T_{A_1}(q_2)]\ \vee 0\\
&= [T_{A_1}(q_1) \vee 0] \wedge [T_{A_1}(q_2) \vee 0]\\
&= [T_{A_1}(q_1) \vee T_{A_2}(q_1)] \wedge [T_{A_1}(q_2) \vee T_{A_2}(q_2)]\\
&= T_{(A_1\cup A_2)}(q_1) \wedge T_{(A_1\cup A_2)}(q_2),\\
I_{(B_{1i}\cup B_{2i})}(q_1q_2) &= I_{B_{1i}}(q_1q_2) \vee I_{B_{2i}}(q_1q_2)\\
&= I_{B_{1i}}(q_1q_2) \vee 0\\
&\le [I_{A_1}(q_1) \wedge I_{A_1}(q_2)]\ \vee 0\\
&= [I_{A_1}(q_1) \vee 0] \wedge [I_{A_1}(q_2) \vee 0]\\
&= [I_{A_1}(q_1) \vee I_{A_2}(q_1)] \wedge [I_{A_1}(q_2) \vee I_{A_2}(q_2)]\\
&= I_{(A_1\cup A_2)}(q_1) \wedge I_{(A_1\cup A_2)}(q_2),\\
F_{(B_{1i}\cup B_{2i})}(q_1q_2) &= F_{B_{1i}}(q_1q_2) \wedge F_{B_{2i}}(q_1q_2)\\
&= F_{B_{1i}}(q_1q_2) \wedge 0\\
&\le [F_{A_1}(q_1) \vee F_{A_1}(q_2)]\ \wedge 0\\
&= [F_{A_1}(q_1) \wedge 0] \vee [F_{A_1}(q_2) \wedge 0]\\
&= [F_{A_1}(q_1) \wedge F_{A_2}(q_1)] \vee [F_{A_1}(q_2) \wedge F_{A_2}(q_2)]\\
&= F_{(A_1\cup A_2)}(q_1) \vee F_{(A_1\cup A_2)}(q_2),
\end{aligned}$$

for $q_1, q_2 \in X_1 \cup X_2$.

Case 2. When $q_1, q_2 \in X_2$, then according to Definition 2.37, $T_{A_1}(q_1) = T_{A_1}(q_2) = T_{B_{1i}}(q_1q_2) = 0,\ I_{A_1}(q_1) = I_{A_1}(q_2) = I_{B_{1i}}(q_1q_2) = 0,\ F_{A_1}(q_1) = F_{A_1}(q_2) = F_{B_{1i}}(q_1q_2) = 0$, so

$$\begin{aligned}
T_{(B_{1i}\cup B_{2i})}(q_1q_2) &= T_{B_{1i}}(q_1q_2) \vee T_{B_{2i}}(q_1q_2)\\
&= T_{B_{2i}}(q_1q_2) \vee 0\\
&\le [T_{A_2}(q_1) \wedge T_{A_2}(q_2)]\ \vee 0\\
&= [T_{A_2}(q_1) \vee 0] \wedge [T_{A_2}(q_2) \vee 0]\\
&= [T_{A_1}(q_1) \vee T_{A_2}(q_1)] \wedge [T_{A_1}(q_2) \vee T_{A_2}(q_2)]\\
&= T_{(A_1\cup A_2)}(q_1) \wedge T_{(A_1\cup A_2)}(q_2),
\end{aligned}$$

$$\begin{aligned}
I_{(B_{1i}\cup B_{2i})}(q_1q_2) &= I_{B_{1i}}(q_1q_2) \vee I_{B_{2i}}(q_1q_2) \\
&= I_{B_{2i}}(q_1q_2) \vee 0 \\
&\le [I_{A_2}(q_1) \wedge I_{A_2}(q_2)]\ \vee 0 \\
&= [I_{A_2}(q_1) \vee 0] \wedge [I_{A_2}(q_2) \vee 0] \\
&= [I_{A_1}(q_1) \vee I_{A_2}(q_1)] \wedge [I_{A_1}(q_2) \vee I_{A_2}(q_2)] \\
&= I_{(A_1\cup A_2)}(q_1) \wedge I_{(A_1\cup A_2)}(q_2),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1i}\cup B_{2i})}(q_1q_2) &= F_{B_{1i}}(q_1q_2) \wedge F_{B_{2i}}(q_1q_2) \\
&= F_{B_{2i}}(q_1q_2) \wedge 0 \\
&\le [F_{A_2}(q_1) \vee F_{A_2}(q_2)]\ \wedge 0 \\
&= [F_{A_2}(q_1) \wedge 0] \vee [F_{A_2}(q_2) \wedge 0] \\
&= [F_{A_1}(q_1) \wedge F_{A_2}(q_1)] \vee [F_{A_1}(q_2) \wedge F_{A_2}(q_2)] \\
&= F_{(A_1\cup A_2)}(q_1) \vee F_{(A_1\cup A_2)}(q_2),
\end{aligned}$$

for $q_1, q_2 \in X_1 \cup X_2$.

Both cases are satisfied $\forall i \in \{1, 2, \ldots, n\}$. This completes the proof.

**Theorem 2.9** *Let $G^* = (X_1 \cup X_2, E_{11} \cup E_{21}, E_{12} \cup E_{22}, \ldots, E_{1n} \cup E_{2n})$ be the union of two graph structures $G_1^* = (X_1, E_{11}, E_{12}, \ldots, E_{1n})$ and $G_2^* = (X_2, E_{21}, E_{22}, \ldots, E_{2n})$. Then every neutrosophic graph structure $G = (A, B_1, B_2, \ldots, B_n)$ of $G^*$ is union of two neutrosophic graph structures $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ and $G_2 = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ of graph structures $G_1^*$ and $G_2^*$, respectively.*

*Proof* First we define $A_1$, $A_2$, $B_{1i}$ and $B_{2i}$ for $i \in \{1, 2, \ldots, n\}$ as:
$T_{A_1}(q) = T_A(q), I_{A_1}(q) = I_A(q), F_{A_1}(q) = F_A(q)$, if $q \in X_1$
$T_{A_2}(q) = T_A(q), I_{A_2}(q) = I_A(q), F_{A_2}(q) = F_A(q)$, if $q \in X_2$

$T_{B_{1i}}(q_1q_2) = T_{B_i}(q_1q_2), I_{B_{1i}}(q_1q_2) = I_{B_i}(q_1q_2), F_{B_{1i}}(q_1q_2) = F_{B_i}(q_1q_2)$, if $q_1q_2 \in E_{1i}, T_{B_{2i}}(q_1q_2) = T_{B_i}(q_1q_2), I_{B_{2i}}(q_1q_2) = I_{B_i}(q_1q_2), F_{B_{2i}}(q_1q_2) = F_{B_i}(q_1q_2)$, if $q_1q_2 \in E_{2i}$. Then $A = A_1 \cup A_2$ and $B_i = B_{1i} \cup B_{2i}, i \in \{1, 2, \ldots, n\}$.

Now for $q_1q_2 \in E_{ki}, k = 1, 2, i = 1, 2, \ldots, n$
$T_{B_{ki}}(q_1q_2) = T_{B_i}(q_1q_2) \le T_A(q_1) \wedge T_A(q_2) = T_{A_k}(q_1) \wedge T_{A_k}(q_2)$,
$I_{B_{ki}}(q_1q_2) = I_{B_i}(q_1q_2) \le I_A(q_1) \wedge I_A(q_2) = I_{A_k}(q_1) \wedge I_{A_k}(q_2)$,
$F_{B_{ki}}(q_1q_2) = F_{B_i}(q_1q_2) \le F_A(q_1) \vee F_A(q_2) = F_{A_k}(q_1) \vee F_{A_k}(q_2)$,
i.e.
$G_k = (A_k, B_{k1}, B_{k2}, \ldots, B_{kn})$ is a neutrosophic graph structure of $G_k^*$, $k = 1, 2$.
Thus $G = (A, B_1, B_2, \ldots, B_n)$, a neutrosophic graph structure of $G^* = G_1^* \cup G_2^*$, is union of two neutrosophic graph structures $G_1$ and $G_2$.

**Definition 2.38** Let $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ and $G_2 = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ be neutrosophic graph structures and let $X_1 \cap X_2 = \emptyset$. The *join* of $G_1$ and $G_2$, denoted by

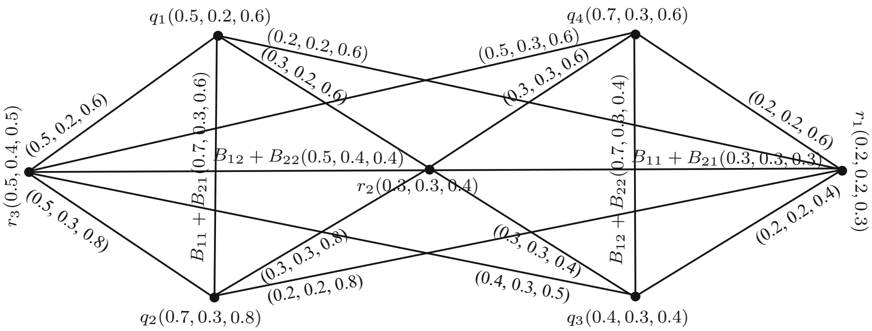


**Fig. 2.21** Join of two neutrosophic graph structures

$$G_1 + G_2 = (A_1 + A_2, B_{11} + B_{21}, B_{12} + B_{22}, \ldots, B_{1n} + B_{2n}),$$

is defined by the following:

(i) $\begin{cases} T_{(A_1+A_2)}(q) = T_{(A_1 \cup A_2)}(q) \\ I_{(A_1+A_2)}(q) = I_{(A_1 \cup A_2)}(q) \\ F_{(A_1+A_2)}(q) = F_{(A_1 \cup A_2)}(q) \end{cases}$

for all $q \in X_1 \cup X_2$,

(ii) $\begin{cases} T_{(B_{1i}+B_{2i})}(qr) = T_{(B_{1i} \cup B_{2i})}(qr) \\ I_{(B_{1i}+B_{2i})}(qr) = I_{(B_{1i} \cup B_{2i})}(qr) \\ F_{(B_{1i}+B_{2i})}(qr) = F_{(B_{1i} \cup B_{2i})}(qr) \end{cases}$

for all $qr \in E_{1i} \cup E_{2i}$,

(iii) $\begin{cases} T_{(B_{1i}+B_{2i})}(qr) = (T_{B_{1i}} + T_{B_{2i}})(qr) = T_{A_1}(q) \wedge T_{A_2}(r) \\ I_{(B_{1i}+B_{2i})}(qr) = (I_{B_{1i}} + I_{B_{2i}})(qr) = I_{A_1}(q) \wedge I_{A_2}(r) \\ F_{(B_{1i}+B_{2i})}(qr) = (F_{B_{1i}} + F_{B_{2i}})(qr) = F_{A_1}(q) \vee F_{A_2}(r) \end{cases}$

for all $q \in X_1, r \in X_2$.

*Example 2.22* Join of two neutrosophic graph structures $G_1$ and $G_2$ shown in Fig. 2.14 is defined as $G_1 + G_2 = \{A_1 + A_2, B_{11} + B_{21}, B_{12} + B_{22}\}$ and is shown in the Fig. 2.21.

**Theorem 2.10** *The join* $G_1 + G_2 = (A_1 + A_2, B_{11} + B_{21}, B_{12} + B_{22}, \ldots, B_{1n} + B_{2n})$ *of two neutrosophic graph structures of the graph structures* $G_1^*$ *and* $G_2^*$ *is a neutrosophic graph structure of* $G_1^* + G_2^*$.

*Proof* Let $q_1q_2 \in E_{1i} + E_{2i}$. Here we consider three cases:

Case 1. When $q_1, q_2 \in X_1$, then according to Definition 2.38, $T_{A_2}(q_1) = T_{A_2}(q_2) = T_{B_{2i}}(q_1q_2) = 0$, $I_{A_2}(q_1) = I_{A_2}(q_2) = I_{B_{2i}}(q_1q_2) = 0$, $F_{A_2}(q_1) = F_{A_2}(q_2) = F_{B_{2i}}(q_1q_2) = 0$, so,

$$T_{(B_{1i}+B_{2i})}(q_1q_2) = T_{B_{1i}}(q_1q_2) \vee T_{B_{2i}}(q_1q_2)$$

$$
\begin{aligned}
&= T_{B_{1i}}(q_1q_2) \vee 0\\
&\leq [T_{A_1}(q_1) \wedge T_{A_1}(q_2)] \ \vee 0\\
&= [T_{A_1}(q_1) \vee 0] \wedge [T_{A_1}(q_2) \vee 0]\\
&= [T_{A_1}(q_1) \vee T_{A_2}(q_1)] \wedge [T_{A_1}(q_2) \vee T_{A_2}(q_2)]\\
&= T_{(A_1+A_2)}(q_1) \wedge T_{(A_1+A_2)}(q_2),
\end{aligned}
$$

$$
\begin{aligned}
I_{(B_{1i}+B_{2i})}(q_1q_2) &= I_{B_{1i}}(q_1q_2) \vee I_{B_{2i}}(q_1q_2)\\
&= I_{B_{1i}}(q_1q_2) \vee 0\\
&\leq [I_{A_1}(q_1) \wedge I_{A_1}(q_2)] \ \vee 0\\
&= [I_{A_1}(q_1) \vee 0] \wedge [I_{A_1}(q_2) \vee 0]\\
&= [I_{A_1}(q_1) \vee I_{A_2}(q_1)] \wedge [I_{A_1}(q_2) \vee I_{A_2}(q_2)]\\
&= I_{(A_1+A_2)}(q_1) \wedge I_{(A_1+A_2)}(q_2),
\end{aligned}
$$

$$
\begin{aligned}
F_{(B_{1i}+B_{2i})}(q_1q_2) &= F_{B_{1i}}(q_1q_2) \wedge F_{B_{2i}}(q_1q_2)\\
&= F_{B_{1i}}(q_1q_2) \wedge 0\\
&\leq [F_{A_1}(q_1) \vee F_{A_1}(q_2)] \ \wedge 0\\
&= [F_{A_1}(q_1) \wedge 0] \vee [F_{A_1}(q_2) \wedge 0]\\
&= [F_{A_1}(q_1) \wedge F_{A_2}(q_1)] \vee [F_{A_1}(q_2) \wedge F_{A_2}(q_2)]\\
&= F_{(A_1+A_2)}(q_1) \vee F_{(A_1+A_2)}(q_2),
\end{aligned}
$$

for $q_1, q_2 \in X_1 + X_2$.

Case 2. When $q_1, q_2 \in X_2$, then according to Definition 2.38, $T_{A_1}(q_1) = T_{A_1}(q_2) = T_{B_{1i}}(q_1q_2) = 0$, $I_{A_1}(q_1) = I_{A_1}(q_2) = I_{B_{1i}}(q_1q_2) = 0$, $F_{A_1}(q_1) = F_{A_1}(q_2) = F_{B_{1i}}(q_1q_2) = 0$, so

$$
\begin{aligned}
T_{(B_{1i}+B_{2i})}(q_1q_2) &= T_{B_{1i}}(q_1q_2) \vee T_{B_{2i}}(q_1q_2)\\
&= T_{B_{2i}}(q_1q_2) \vee 0\\
&\leq [T_{A_2}(q_1) \wedge T_{A_2}(q_2)] \ \vee 0\\
&= [T_{A_2}(q_1) \vee 0] \wedge [T_{A_2}(q_2) \vee 0]\\
&= [T_{A_1}(q_1) \vee T_{A_2}(q_1)] \wedge [T_{A_1}(q_2) \vee T_{A_2}(q_2)]\\
&= T_{(A_1+A_2)}(q_1) \wedge T_{(A_1+A_2)}(q_2),
\end{aligned}
$$

$$
\begin{aligned}
I_{(B_{1i}+B_{2i})}(q_1q_2) &= I_{B_{1i}}(q_1q_2) \vee I_{B_{2i}}(q_1q_2)\\
&= I_{B_{2i}}(q_1q_2) \vee 0\\
&\leq [I_{A_2}(q_1) \wedge I_{A_2}(q_2)] \ \vee 0\\
&= [I_{A_2}(q_1) \vee 0] \wedge [I_{A_2}(q_2) \vee 0]\\
&= [I_{A_1}(q_1) \vee I_{A_2}(q_1)] \wedge [I_{A_1}(q_2) \vee I_{A_2}(q_2)]\\
&= I_{(A_1+A_2)}(q_1) \wedge I_{(A_1+A_2)}(q_2),
\end{aligned}
$$

$$\begin{aligned}
F_{(B_{1i}+B_{2i})}(q_1q_2) &= F_{B_{1i}}(q_1q_2) \wedge F_{B_{2i}}(q_1q_2) \\
&= F_{B_{2i}}(q_1q_2) \wedge 0 \\
&\leq [F_{A_2}(q_1) \vee F_{A_2}(q_2)] \ \wedge 0 \\
&= [F_{A_2}(q_1) \wedge 0] \vee [F_{A_2}(q_2) \wedge 0] \\
&= [F_{A_1}(q_1) \wedge F_{A_2}(q_1)] \vee [F_{A_1}(q_2) \wedge F_{A_2}(q_2)] \\
&= F_{(A_1+A_2)}(q_1) \vee F_{(A_1+A_2)}(q_2),
\end{aligned}$$

for $q_1, q_2 \in X_1 + X_2$.

Case 3. When $q_1 \in X_1, q_2 \in X_2$, then according to Definition 2.38, $T_{A_1}(q_2) = T_{A_2}(q_1) = 0$, $I_{A_1}(q_2) = I_{A_2}(q_1) = 0$, $F_{A_1}(q_2) = F_{A_2}(q_1) = 0$, so

$$\begin{aligned}
T_{(B_{1i}+B_{2i})}(q_1q_2) &= T_{A_1}(q_1) \wedge T_{A_2}(q_2) \\
&= [T_{A_1}(q_1) \vee 0] \wedge [T_{A_2}(q_2) \vee 0] \\
&= [T_{A_1}(q_1) \vee T_{A_2}(q_1)] \wedge [T_{A_2}(q_2) \vee T_{A_1}(q_2)] \\
&= T_{(A_1+A_2)}(q_1) \wedge T_{(A_1+A_2)}(q_2),
\end{aligned}$$

$$\begin{aligned}
I_{(B_{1i}+B_{2i})}(q_1q_2) &= I_{A_1}(q_1) \wedge I_{A_2}(q_2) \\
&= [I_{A_1}(q_1) \vee 0] \wedge [I_{A_2}(q_2) \vee 0] \\
&= [I_{A_1}(q_1) \vee I_{A_2}(q_1)] \wedge [I_{A_2}(q_2) \vee I_{A_1}(q_2)] \\
&= I_{(A_1+A_2)}(q_1) \wedge I_{(A_1+A_2)}(q_2),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1i}+B_{2i})}(q_1q_2) &= F_{A_1}(q_1) \vee F_{A_2}(q_2) \\
&= [F_{A_1}(q_1) \wedge 0] \vee [F_{A_2}(q_2) \wedge 0] \\
&= [F_{A_1}(q_1) \wedge F_{A_2}(q_1)] \vee [F_{A_2}(q_2) \wedge F_{A_1}(q_2)] \\
&= F_{(A_1+A_2)}(q_1) \vee F_{(A_1+A_2)}(q_2),
\end{aligned}$$

for $q_1, q_2 \in X_1 + X_2$.

All cases are satisfied $\forall i \in \{1, 2, \ldots, n\}$.

**Theorem 2.11** *If $G^* = (X_1 + X_2, E_{11} + E_{21}, E_{12} + E_{22}, \ldots, E_{1n} + E_{2n})$ is join of two graph structures $G_1^* = (X_1, E_{11}, E_{12}, \ldots, E_{1n})$ and $G_2^* = (X_2, E_{21}, E_{22}, \ldots, E_{2n})$. Then every strong neutrosophic graph structure $G = (A, B_1, B_2, \ldots, B_n)$ of $G$ is join of two strong neutrosophic graph structures $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ and $G_2 = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ of graph structures $G_1^*$ and $G_2^*$, respectively.*

*Proof* First we define $A_k$ and $B_{ki}$ for $k = 1, 2$ and $i = 1, 2, \ldots, n$ as:
$T_{A_k}(q) = T_A(q), I_{A_k}(q) = I_A(q), F_{A_k}(q) = F_A(q)$, if $q \in X_k$
$T_{B_{ki}}(q_1q_2) = T_{B_i}(q_1q_2), I_{B_{ki}}(q_1q_2) = I_{B_i}(q_1q_2), F_{B_{ki}}(q_1q_2) = F_{B_i}(q_1q_2)$, if $q_1q_2 \in E_{ki}$

Now for $q_1q_2 \in E_{ki}, k = 1, 2, i = 1, 2, \ldots, n$
$T_{B_{ki}}(q_1q_2) = T_{B_i}(q_1q_2) = T_A(q_1) \wedge T_A(q_2) = T_{A_k}(q_1) \wedge T_{A_k}(q_2)$,
$I_{B_{ki}}(q_1q_2) = I_{B_i}(q_1q_2) = I_A(q_1) \wedge I_A(q_2) = I_{A_k}(q_1) \wedge I_{A_k}(q_2)$,
$F_{B_{ki}}(q_1q_2) = F_{B_i}(q_1q_2) = F_A(q_1) \vee T_A(q_2) = T_{A_k}(q_1) \vee T_{A_k}(q_2)$,
i.e.
$G_k = (A_k, B_{k1}, B_{k2}, \ldots, B_{kn})$ is a strong neutrosophic graph structure of $G_k^*$, k = 1,2.

Moreover, $G$ is join of $G_1$ and $G_2$ as shown:

Using Definitions 2.37 and 2.38, $A = A_1 \cup A_2 = A_1 + A_2$ and $B_i = B_{1i} \cup B_{2i} = B_{1i} + B_{2i}, \forall q_1q_2 \in E_{1i} \cup E_{2i}$.

When $q_1q_2 \in E_{1i} + E_{2i}$ $(E_{1i} \cup E_{2i})$, i.e. $q_1 \in X_1$ and $q_2 \in X_2$
$T_{B_i}(q_1q_2) = T_A(q_1) \wedge T_A(q_2) = T_{A_k}(q_1) \wedge T_{A_k}(q_2) = T_{(B_{1i}+B_{2i})}(q_1q_2)$,
$I_{B_i}(q_1q_2) = I_A(q_1) \wedge I_A(q_2) = I_{A_k}(q_1) \wedge I_{A_k}(q_2) = I_{(B_{1i}+B_{2i})}(q_1q_2)$,
$F_{B_i}(q_1q_2) = F_A(q_1) \vee F_A(q_2) = F_{A_k}(q_1) \vee F_{A_k}(q_2) = F_{(B_{1i}+B_{2i})}(q_1q_2)$,
Calculations are similar when $q_1 \in X_2, q_2 \in X_1$. It is true when $i = 1, 2, \ldots, n$. This completes the proof.

## 2.4 Applications of Neutrosophic Graph Structures

Graph structures are amazing source of graph-theoretical notions to represent the most prominent relations between objects. But these graph structures do not represent all real-world relations. Therefore, fuzzy graph structures are important to represent the relations between objects of uncertain systems existing in nature. However, graph structures and fuzzy graph structures are failed to depict the most prominent relations between objects in many real-world phenomenons due to natural existence of indeterminacy or neutrality. It increases the utility of neutrosophic graph structures.

### *2.4.1 Detection of Crucial Crimes During Maritime Trade*

Waters are very important for trade in whole world but crimes through waters are increasing day by day. Crimes held during maritime trade are in abundance but some are very crucial including human trafficking, illegal carrying of weapons, black money transfer, smuggling of precious metals, drug trafficking and smuggling of rare plants and animals. Using neutrosophic graph structure, we can easily investigate the fact that between any two countries which maritime crime is chronic and increasing rapidly with time. Moreover, we can decide which country is most sensitive for particular type of maritime crimes. We consider a set $X$ consisting of eight countries.

$X$={Bangladesh, Malaysia, Singapore, United Arab Emirates, Pakistan, India, Kenya, Italy}. Let $A$ be the neutrosophic set on $X$, defined in Table 2.1.

In Table 2.1, $T$ depicts the importance of that particular country in the world due to its geographic position, $F$ indicates the degree of its nonimportance in the world,

**Table 2.1** Neutrosophic set A of eight countries

| Country | T | I | F |
|---|---|---|---|
| Bangladesh | 0.8 | 0.7 | 0.6 |
| Malaysia | 0.7 | 0.7 | 0.8 |
| Singapore | 0.9 | 0.5 | 0.5 |
| United Arab Emirates | 1.0 | 0.5 | 0.6 |
| Pakistan | 0.9 | 0.5 | 0.5 |
| India | 0.8 | 0.7 | 0.7 |
| Kenya | 0.7 | 0.6 | 0.7 |
| Italy | 0.9 | 0.6 | 0.5 |

**Table 2.2** Neutrosophic set of crimes between Pakistan and other countries during maritime trade

| Type of crime | (P, UAE) | (P, B) | (P, M) | (P, S) |
|---|---|---|---|---|
| Human trafficking | (0.7, 0.4, 0.5) | (0.8, 0.3, 0.4) | (0.7, 0.4, 0.2) | (0.6, 0.4, 0.2) |
| Illegal carrying of weapons | (0.6, 0.3, 0.6) | (0.7, 0.3, 0.4) | (0.4, 0.5, 0.5) | (0.4, 0.3, 0.5) |
| Black money transfer | (0.6, 0.3, 0.2) | (0.7, 0.5, 0.4) | (0.2, 0.4, 0.3) | (0.9, 0.2, 0.2) |
| Smuggling of precious metals | (0.8, 0.3, 0.2) | (0.6, 0.3, 0.3) | (0.2, 0.4, 0.3) | (0.8, 0.5, 0.5) |
| Drug trafficking | (0.7, 0.3, 0.3) | (0.5, 0.4, 0.3) | (0.6, 0.5, 0.6) | (0.8, 0.4, 0.3) |
| Smuggling of rare plants and animals | (0.3, 0.5, 0.5) | (0.4, 0.3, 0.4) | (0.4, 0.4, 0.5) | (0.2, 0.3, 0.3) |

and $I$ expresses, to which extent it is undecided/indeterminate to be beneficial for the world, geographically.

Let Bangladesh = B, Malaysia = M, Singapore = S, United Arab Emirates = UAE, Pakistan = P, India = I, Kenya = K, Italy = IT.

In Tables 2.2, 2.3, 2.4, 2.5, 2.6, 2.7 and 2.8, we have shown the values of $T$, $I$ and $F$ of different crimes for each pair of countries.

Many relations on set $X$ can be defined, let we define six relations on $X$ as: $E_1$ = Human trafficking, $E_2$ = Illegal carrying of weapons, $E_3$ = Black money transfer, $E_4$ = Smuggling of precious metals, $E_5$ = Drug trafficking, $E_6$ = Smuggling of rare plants and animals, such that $(X, E_1, E_2, E_3, E_4, E_5, E_6)$ is a graph structure. An element in a relation detects that kind of crime during maritime trade between those two countries.

As $(X, E_1, E_2, E_3, E_4, E_5, E_6)$ is a graph structure, an element will not be in more than one relations, so it can appear just once. Therefore, we will consider it an element of that relation for which its percentage of truth is high, and percentage of both falsity and indeterminacy is low as compared to other relations.

**Table 2.3** Neutrosophic set of crimes between UAE and other countries during maritime trade

| Type of crime | (UAE, B) | (UAE, M) | (UAE, S) | (UAE, I) |
|---|---|---|---|---|
| Human trafficking | (0.7, 0.3, 0.4) | (0.6, 0.2, 0.5) | (0.3, 0.2, 0.5) | (0.6, 0.4, 0.2) |
| Illegal carrying of weapons | (0.5, 0.2, 0.2) | (0.5, 0.3, 0.2) | (0.4, 0.3, 0.5) | (0.4, 0.3, 0.5) |
| Black money transfer | (0.6, 0.3, 0.3) | (0.6, 0.2, 0.3) | (0.6, 0.2, 0.3) | (0.6, 0.4, 0.5) |
| Smuggling of precious metals | (0.6, 0.2, 0.2) | (0.6, 0.3, 0.3) | (0.6, 0.3, 0.3) | (0.8, 0.3, 0.2) |
| Drug trafficking | (0.6, 0.2, 0.2) | (0.5, 0.4, 0.3) | (0.7, 0.3, 0.2) | (0.7, 0.4, 0.3) |
| Smuggling of rare plants and animals | (0.3, 0.4, 0.4) | (0.4, 0.3, 0.4) | (0.4, 0.2, 0.5) | (0.3, 0.3, 0.3) |

**Table 2.4** Neutrosophic set of crimes between Bangladesh and other countries during maritime trade

| Type of crime | (B, M) | (B, S) | (B, I) | (B, K) |
|---|---|---|---|---|
| Human trafficking | (0.6, 0.3, 0.4) | (0.8, 0.3, 0.2) | (0.5, 0.2, 0.5) | (0.6, 0.4, 0.5) |
| Illegal carrying of weapons | (0.5, 0.2, 0.5) | (0.5, 0.3, 0.2) | (0.7, 0.3, 0.5) | (0.4, 0.3, 0.5) |
| Black money transfer | (0.4, 0.2, 0.2) | (0.7, 0.4, 0.3) | (0.1, 0.1, 0.2) | (0.1, 0.3, 0.4) |
| Smuggling of precious metals | (0.4, 0.2, 0.2) | (0.6, 0.3, 0.3) | (0.2, 0.3, 0.3) | (0.2, 0.2, 0.4) |
| Drug trafficking | (0.6, 0.2, 0.2) | (0.5, 0.4, 0.3) | (0.6, 0.3, 0.5) | (0.5, 0.4, 0.4) |
| Smuggling of rare plants and animals | (0.2, 0.3, 0.3) | (0.3, 0.2, 0.3) | (0.2, 0.1, 0.4) | (0.5, 0.2, 0.2) |

According to given data, we write the elements in relation to their truth, falsity and indeterminacy values, resulting sets are neutrosophic sets on $E_1$, $E_2$, $E_3$, $E_4$, $E_5$, $E_6$, respectively. We can name these sets as $B_1$, $B_2$, $B_3$, $B_4$, $B_5$, $B_6$, respectively. Let

$E_1 = \{(Bangladesh, Pakistan), (Malaysia, Pakistan), (Bangladesh, Singapore)\}$,

$E_2 = \{(Pakistan, India)\}$,

$E_3 = \{(Singapore, Pakistan)\}$,

$E_4 = \{(India, Singapore), (UnitedArabEmirates, India)\}$,

$E_5 = \{(Italy, Pakistan), (India, Italy)\}$,

$E_6 = \{(Kenya, Singapore)\}$.

And corresponding neutrosophic sets are:

**Table 2.5** Neutrosophic set of crimes between Malaysia and other countries during maritime trade

| Type of crime | (M, S) | (M, I) | (M, K) | (M, IT) |
|---|---|---|---|---|
| Human trafficking | (0.5, 0.3, 0.4) | (0.6, 0.2, 0.3) | (0.3, 0.2, 0.5) | (0.6, 0.4, 0.5) |
| Illegal carrying of weapons | (0.6, 0.2, 0.2) | (0.5, 0.3, 0.2) | (0.4, 0.3, 0.5) | (0.4, 0.3, 0.5) |
| Black money transfer | (0.6, 0.3, 0.3) | (0.2, 0.2, 0.3) | (0.2, 0.2, 0.3) | (0.2, 0.4, 0.5) |
| Smuggling of precious metals | (0.6, 0.2, 0.2) | (0.6, 0.3, 0.3) | (0.2, 0.3, 0.3) | (0.2, 0.2, 0.6) |
| Drug trafficking | (0.5, 0.2, 0.2) | (0.5, 0.4, 0.3) | (0.4, 0.3, 0.6) | (0.7, 0.4, 0.2) |
| Smuggling of rare plants and animals | (0.3, 0.4, 0.4) | (0.4, 0.3, 0.4) | (0.6, 0.2, 0.2) | (0.5, 0.3, 0.3) |

**Table 2.6** Neutrosophic set of crimes between Singapore and other countries during maritime trade

| Type of crime | (S, I) | (S, K) | (S, IT) | (P, I) |
|---|---|---|---|---|
| Human trafficking | (0.5, 0.3, 0.4) | (0.3, 0.2, 0.5) | (0.3, 0.2, 0.5) | (0.6, 0.4, 0.6) |
| Illegal carrying of weapons | (0.7, 0.4, 0.5) | (0.5, 0.3, 0.2) | (0.4, 0.3, 0.5) | (0.8, 0.2, 0.4) |
| Black money transfer | (0.5, 0.3, 0.4) | (0.6, 0.2, 0.3) | (0.6, 0.2, 0.3) | (0.7, 0.4, 0.5) |
| Smuggling of precious metals | (0.8, 0.3, 0.7) | (0.6, 0.3, 0.3) | (0.6, 0.3, 0.3) | (0.6, 0.2, 0.4) |
| Drug trafficking | (0.7, 0.3, 0.4) | (0.5, 0.4, 0.3) | (0.6, 0.3, 0.2) | (0.8, 0.4, 0.4) |
| Smuggling of rare plants and animals | (0.7, 0.5, 0.6) | (0.4, 0.3, 0.4) | (0.6, 0.2, 0.5) | (0.7, 0.3, 0.3) |

$B_1 = \{((B, P), 0.8, 0.2, 0.2), ((M, P), 0.7, 0.4, 0.2), ((B, S), 0.8, 0.3, 0.2)\}$,
$B_2 = \{((P, I), 0.8, 0.2, 0.4)\}$,
$B_3 = \{((S, P), 0.9, 0.2, 0.2), \}$,
$B_4 = \{((I, S), 0.8, 0.3, 0.4), ((UAE, I), 0.8, 0.3, 0.2)\}$,
$B_5 = \{((IT, P), 0.9, 0.3, 0.3), ((I, IT), 0.8, 0.3, 0.3)\}$,
$B_6 = \{((K, S), 0.7, 0.2, 0.4)\}$.

Clearly, $(A, B_1, B_2, B_3, B_4, B_5, B_6)$ is a neutrosophic graph structure as shown in Fig. 2.22.

In neutrosophic graph structure shown in Fig. 2.22, every edge detects most frequent crime between adjacent countries during maritime trade. For instance, most frequent maritime crime between Pakistan and Singapore is black money transfer, its strength is 90%, weakness is 20% and indeterminacy is 20%. We can also note that for relation human trafficking, vertex Pakistan has highest vertex degree, it means

**Table 2.7** Neutrosophic set of crimes between Italy and other countries during maritime trade

| Type of crime | (IT, P) | (IT, UAE) | (IT, B) | (IT, I) |
|---|---|---|---|---|
| Human trafficking | (0.5, 0.3, 0.4) | (0.3, 0.2, 0.5) | (0.3, 0.2, 0.5) | (0.6, 0.4, 0.6) |
| Illegal carrying of weapons | (0.8, 0.3, 0.3) | (0.6, 0.3, 0.2) | (0.4, 0.3, 0.5) | (0.7, 0.3, 0.5) |
| Black money transfer | (0.6, 0.3, 0.3) | (0.5, 0.2, 0.3) | (0.2, 0.2, 0.3) | (0.5, 0.4, 0.5) |
| Smuggling of precious metals | (0.7, 0.3, 0.3) | (0.6, 0.3, 0.3) | (0.2, 0.3, 0.3) | (0.7, 0.3, 0.6) |
| Drug trafficking | (0.9, 0.3, 0.3) | (0.6, 0.4, 0.3) | (0.7, 0.3, 0.5) | (0.8, 0.3, 0.3) |
| Smuggling of rare plants and animals | (0.3, 0.4, 0.4) | (0.4, 0.3, 0.4) | (0.6, 0.2, 0.5) | (0.7, 0.3, 0.3) |

**Table 2.8** Neutrosophic set of crimes between Kenya and other countries during maritime trade

| Type of crime | (K, P) | (K, UAE) | (K, I) | (K, IT) |
|---|---|---|---|---|
| Human trafficking | (0.5, 0.3, 0.4) | (0.6, 0.2, 0.5) | (0.5, 0.2, 0.5) | (0.6, 0.4, 0.5) |
| Illegal carrying of weapons | (0.6, 0.2, 0.5) | (0.5, 0.3, 0.4) | (0.5, 0.3, 0.5) | (0.4, 0.3, 0.5) |
| Black money transfer | (0.5, 0.3, 0.3) | (0.5, 0.2, 0.3) | (0.5, 0.2, 0.3) | (0.5, 0.4, 0.5) |
| Smuggling of precious metals | (0.4, 0.2, 0.2) | (0.6, 0.3, 0.3) | (0.6, 0.3, 0.3) | (0.4, 0.2, 0.4) |
| Drug trafficking | (0.7, 0.2, 0.2) | (0.5, 0.4, 0.3) | (0.5, 0.3, 0.5) | (0.8, 0.4, 0.2) |
| Smuggling of rare plants and animals | (0.3, 0.4, 0.4) | (0.7, 0.3, 0.4) | (0.6, 0.2, 0.4) | (0.7, 0.3, 0.3) |

Pakistan is most sensitive country for human trafficking. Moreover, according to our neutrosophic graph structure, most frequent crime is human trafficking. It means that navy and maritime forces of these eight countries should take action to control human trafficking.

### *2.4.2 Decision-Making of Prominent Relationships*

Among the countries of this world, various types of relationships exist, for example friendship, rival or enemy, religious affection, trade, political and military. Between any two countries, all relationships are not of same strength. Some relationships are comparatively stronger than other relationships. In general, it is difficult and time

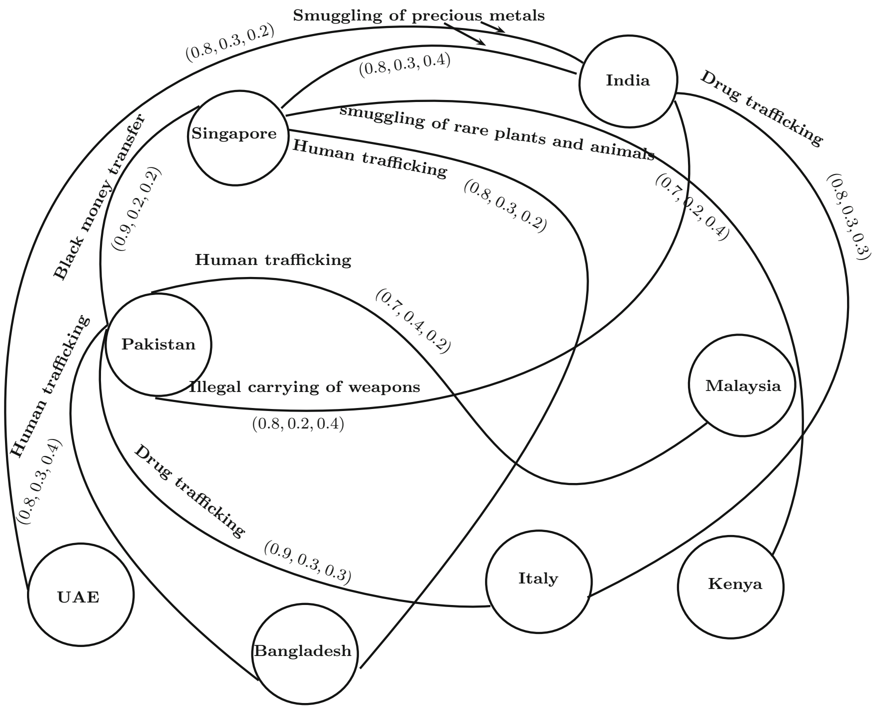


**Fig. 2.22** Neutrosophic graph structure showing most crucial maritime crime between any two countries

consuming to judge all relationships among the countries and to decide the most prominent one. But through neutrosophic graph structure, we can represent all these in easiest way and can be judged even in a single glance on graph. Moreover, we can be aware of the status of relationship, that is, what is percentage of its strength, weakness and in how much percentage it is indeterminate. We can also examine which pair of countries are in same kind of relationship. We consider a set $X$ of eight countries.

$X = \{America, Russia, China, Japan, Pakistan, India, Iran, Saudi\ Arabia\}$. Let $A$ be the neutrosophic set on $X$, defined in Table 2.9.

In Table 2.9, $T$ indicates positive impact (strength) of a particular country for whole world, $F$ indicates negative impact (weakness), and $I$ expresses that in what percentage or magnitude that country's position is undecided or indeterminate for global world. Let we denote the countries with alphabets: A = America, R = Russia, CH = China, J = Japan, P = Pakistan, I = India, IR = Iran, S = Saudi Arabia.

In Tables 2.10, 2.11, 2.12, 2.13, 2.14 and 2.15, we have shown the $T$, $I$ and $F$ values of different relationships for each pair of countries.

**Table 2.9** Neutrosophic set $A$ of a few countries on globe

| Country | T | I | F |
|---|---|---|---|
| America | 0.9 | 0.3 | 0.2 |
| Russia | 0.7 | 0.4 | 0.3 |
| China | 0.8 | 0.4 | 0.4 |
| Japan | 0.8 | 0.5 | 0.4 |
| Pakistan | 0.7 | 0.6 | 0.7 |
| India | 0.7 | 0.8 | 0.6 |
| Iran | 0.7 | 0.7 | 0.6 |
| Saudi Arabia | 0.6 | 0.9 | 0.7 |

**Table 2.10** Neutrosophic set of relationships between America and other countries

| Type of relation | (A, R) | (A, CH) | (A, P) | (A, I) | (A, IR) |
|---|---|---|---|---|---|
| Friendship | (0.0, 0.2, 0.3) | (0.2, 0.3, 0.4) | (0.2, 0.1, 0.4) | (0.5, 0.3, 0.5) | (0.1, 0.3, 0.5) |
| Rival or enemy | (0.7, 0.1, 0.1) | (0.8, 0.2, 0.1) | (0.3, 0.2, 0.4) | (0.3, 0.2, 0.4) | (0.5, 0.2, 0.4) |
| Religious affection | (0.4, 0.2, 0.2) | (0.1, 0.3, 0.2) | (0.1, 0.1, 0.2) | (0.1, 0.3, 0.4) | (0.1, 0.1, 0.2) |
| Trade | (0.3, 0.1, 0.1) | (0.5, 0.2, 0.2) | (0.1, 0.2, 0.2) | (0.1, 0.1, 0.5) | (0.6, 0.1, 0.3) |
| Politics | (0.6, 0.1, 0.1) | (0.4, 0.3, 0.2) | (0.6, 0.1, 0.1) | (0.7, 0.3, 0.2) | (0.7, 0.3, 0.1) |
| Military | (0.2, 0.3, 0.3) | (0.3, 0.2, 0.3) | (0.5, 0.1, 0.4) | (0.6, 0.2, 0.2) | (0.2, 0.3, 0.2) |

**Table 2.11** Neutrosophic set of relationships between Russia and other countries

| Type of relation | (R, CH) | (R, J) | (R, P) | (R, I) | (R, IR) |
|---|---|---|---|---|---|
| Friendship | (0.5, 0.2, 0.3) | (0.5, 0.2, 0.3) | (0.3, 0.3, 0.4) | (0.4, 0.3, 0.3) | (0.1, 0.1, 0.5) |
| Rival or enemy | (0.6, 0.2, 0.2) | (0.6, 0.2, 0.2) | (0.3, 0.3, 0.3) | (0.2, 0.2, 0.4) | (0.4, 0.1, 0.3) |
| Religious affection | (0.1, 0.1, 0.4) | (0.2, 0.1, 0.3) | (0.1, 0.1, 0.4) | (0.4, 0.4, 0.3) | (0.2, 0.1, 0.5) |
| Trade | (0.4, 0.1, 0.3) | (0.4, 0.2, 0.3) | (0.4, 0.1, 0.4) | (0.5, 0.2, 0.3) | (0.4, 0.1, 0.3) |
| Politics | (0.7, 0.3, 0.4) | (0.7, 0.1, 0.3) | (0.4, 0.1, 0.3) | (0.5, 0.2, 0.3) | (0.7, 0.4, 0.5) |
| Military | (0.2, 0.1, 0.4) | (0.4, 0.1, 0.3) | (0.7, 0.1, 0.3) | (0.7, 0.2, 0.4) | (0.2, 0.1, 0.3) |

We can define many relations on set $X$, let we define six relations on $X$ as: $E_1$ = Friendship, $E_2$ = Rival or Enemy, $E_3$ = Religious affection, $E_4$ = Trade, $E_5$ = Politics, $E_6$ = Military, such that $(X, E_1, E_2, E_3, E_4, E_5, E_6)$ is a graph structure. An element in a relation indicates that these two countries have a particular relationship. As $(X, E_1, E_2, E_3, E_4, E_5, E_6)$ is a graph structure, so an element will not be in

**Table 2.12** Neutrosophic set of relationships between China and other countries

| Type of relation | (CH, J) | (CH, P) | (CH, I) | (CH, IR) | (CH, S) |
|---|---|---|---|---|---|
| Friendship | (0.5, 0.2, 0.3) | (0.7, 0.1, 0.1) | (0.2, 0.3, 0.6) | (0.1, 0.4, 0.6) | (0.2, 0.4, 0.6) |
| Rival or enemy | (0.6, 0.2, 0.2) | (0.1, 0.1, 0.7) | (0.7, 0.2, 0.2) | (0.3, 0.3, 0.6) | (0.2, 0.3, 0.5) |
| Religious affection | (0.1, 0.1, 0.4) | (0.3, 0.3, 0.6) | (0.4, 0.4, 0.3) | (0.2, 0.2, 0.5) | (0.1, 0.4, 0.6) |
| Trade | (0.1, 0.1, 0.3) | (0.6, 0.1, 0.1) | (0.4, 0.2, 0.4) | (0.7, 0.1, 0.3) | (0.5, 0.4, 0.2) |
| Politics | (0.8, 0.4, 0.4) | (0.2, 0.4, 0.3) | (0.6, 0.2, 0.2) | (0.7, 0.2, 0.2) | (0.6, 0.4, 0.3) |
| Military | (0.4, 0.2, 0.3) | (0.6, 0.2, 0.3) | (0.1, 0.4, 0.2) | (0.2, 0.4, 0.6) | (0.1, 0.4, 0.6) |

**Table 2.13** Neutrosophic set of relationships between Japan and other countries

| Type of relation | (J, A) | (J, P) | (J, I) | (J, IR) | (J, S) |
|---|---|---|---|---|---|
| Friendship | (0.5, 0.3, 0.4) | (0.2, 0.3, 0.6) | (0.3, 0.4, 0.3) | (0.2, 0.5, 0.6) | (0.1, 0.4, 0.6) |
| Rival or enemy | (0.7, 0.3, 0.3) | (0.3, 0.4, 0.6) | (0.2, 0.3, 0.5) | (0.2, 0.4, 0.4) | (0.3, 0.4, 0.4) |
| Religious affection | (0.1, 0.3, 0.3) | (0.1, 0.4, 0.5) | (0.4, 0.4, 0.5) | ( 0.1, 0.5, 0.6) | ( 0.1, 0.4, 0.6) |
| Trade | (0.1, 0.3, 0.4) | (0.7, 0.3, 0.2) | (0.7, 0.2, 0.1) | (0.6, 0.4, 0.6) | (0.6, 0.5, 0.7) |
| Politics | (0.8, 0.3, 0.3) | (0.6, 0.4, 0.2) | (0.6, 0.5, 0.2) | ( 0.6, 0.3, 0.1) | (0.4, 0.3, 0.4) |
| Military | (0.2, 0.3, 0.3) | (0.4, 0.4, 0.4) | (0.5, 0.4, 0.3) | (0.2, 0.4, 0.6) | (0.1, 0.4, 0.6) |

**Table 2.14** Neutrosophic set of relationships between Saudi Arabia and other countries

| Type of relation | (I, IR) | (S, I) | (S, IR) | (S, A) | (S, R) |
|---|---|---|---|---|---|
| Friendship | (0.2, 0.4, 0.4) | (0.1, 0.7, 0.6) | (0.2, 0.4, 0.6) | (0.4, 0.3, 0.6) | (0.2, 0.2, 0.6) |
| Rival or enemy | (0.6, 0.3, 0.6) | (0.5, 0.4, 0.5) | (0.5, 0.4, 0.4) | (0.4, 0.2, 0.5) | (0.4, 0.2, 0.4) |
| Religious affection | (0.1, 0.4, 0.6) | (0.3, 0.4, 0.6) | (0.6, 0.4, 0.2) | (0.1, 0.1, 0.7) | (0.2, 0.1, 0.6) |
| Trade | (0.4, 0.4, 0.5) | (0.1, 0.4, 0.6) | (0.3, 0.4, 0.6) | (0.2, 0.1, 0.6) | (0.1, 0.1, 0.3) |
| Politics | (0.7, 0.4, 0.2) | (0.3, 0.4, 0.6) | (0.6, 0.4, 0.6) | (0.6, 0.2, 0.3) | (0.6, 0.4, 0.6) |
| Military | (0.2, 0.5, 0.6) | (0.1, 0.4, 0.6) | (0.2, 0.3, 0.7) | (0.1, 0.1, 0.7) | (0.2, 0.1, 0.5) |

more than one relation. So, we will put it in that relation for which percentage of truth is high, percentage of both falsity and indeterminacy is low as compared to other relationships, using above-mentioned data.

We write the elements in relations with their truth, falsity and indeterminacy values according to given data, resulting sets are neutrosophic sets on $E_1$, $E_2$, $E_3$, $E_4$, $E_5$,

**Table 2.15** Neutrosophic set of relationships between Pakistan and other countries

| Type of relation | (P, I) | (P, IR) | (P, S) |
|---|---|---|---|
| Friendship | (0.1, 0.4, 0.6) | (0.5, 0.4, 0.5) | (0.5, 0.1, 0.1) |
| Rival or enemy | (0.7, 0.1, 0.1) | (0.4, 0.4, 0.5) | (0.3, 0.6, 0.6) |
| Religious affection | (0.4, 0.4, 0.6) | (0.7, 0.4, 0.5) | (0.6, 0.1, 0.1) |
| Trade | (0.3, 0.3, 0.6) | (0.4, 0.4, 0.5) | (0.3, 0.2, 0.6) |
| Politics | (0.6, 0.2, 0.2) | (0.5, 0.4, 0.5) | (0.2, 0.4, 0.5) |
| Military | (0.1, 0.2, 0.6) | (0.2, 0.4, 0.6) | (0.1, 0.4, 0.6) |

$E_6$, respectively. We can name these sets as $B_1$, $B_2$, $B_3$, $B_4$, $B_5$, $B_6$, respectively. Let
$B_1 = \{((P, CH), 0.7, 0.1, 0.1)\}$,
$B_2 = \{((P, I), 0.7, 0.1, 0.1), ((A, R), 0.7, 0.1, 0.1), ((A, CH), 0.8, 0.2, 0.1), ((I, CH), 0.7, 0.2, 0.2)\}$,
$B_3 = \{((P, S), 0.6, 0.1, 0.1), ((P, IR), 0.7, 0.4, 0.5)\}$,
$B_4 = \{((P, J), 0.7, 0.3, 0.2), ((I, J), 0.7, 0.2, 0.1)\}$,
$B_5 = \{((P, A), 0.6, 0.1, 0.1), ((A, I), 0.7, 0.3, 0.2), ((A, S), 0.6, 0.2, 0.3), ((A, IR), 0.7, 0.3, 0.1), ((A, J), 0.8, 0.3, 0.3)\}$,
$B_6 = \{((P, R), 0.7, 0.1, 0.3), ((R, I), 0.7, 0.2, 0.4)\}$.

Clearly, $(A, B_1, B_2, B_3, B_4, B_5, B_6)$ is a neutrosophic graph structure as shown in Fig. 2.23.

In neutrosophic graph structure shown in Fig. 2.23, every edge indicates the most prominent relationship of adjacent vertices(countries), for example most prominent relationship between Pakistan and China is friendship, it is 70% strong, 10% weak and 10% indeterminate. It can be noted that for the relation politics, vertex America has highest degree, it shows that America is the most prominent country for having political relationship with other countries in $A$. Further, we can tell that China and India, America and Russia, Pakistan and India have common relationship, that is, they are rival or enemy of each other. Moreover, according to our neutrosophic graph structure most frequent relation is politics, it means that among these eight countries politics is dominating relationship.

This neutrosophic graph structure depicts most prominent relationships among some elements (countries) of $A$. By taking large neutrosophic graph structure, most dominating relationships among all the countries of $A$ can be detected. On the similar basis, we can make a neutrosophic graph structure for all countries across the world, in order to find the status and strength of prominent relationships among them. From neutrosophic graph structure, we can also determine that which pair of countries have common relationships. Further, we can find which country is most prominent for having a particular kind of relationship with other countries. Most frequent relationship in the neutrosophic graph structure will indicate that this relationship is prevailing in the world. So, using neutrosophic graph structure, it is quite easy to judge, in which direction this world is moving? whether it is moving towards peace or war/Cold War.

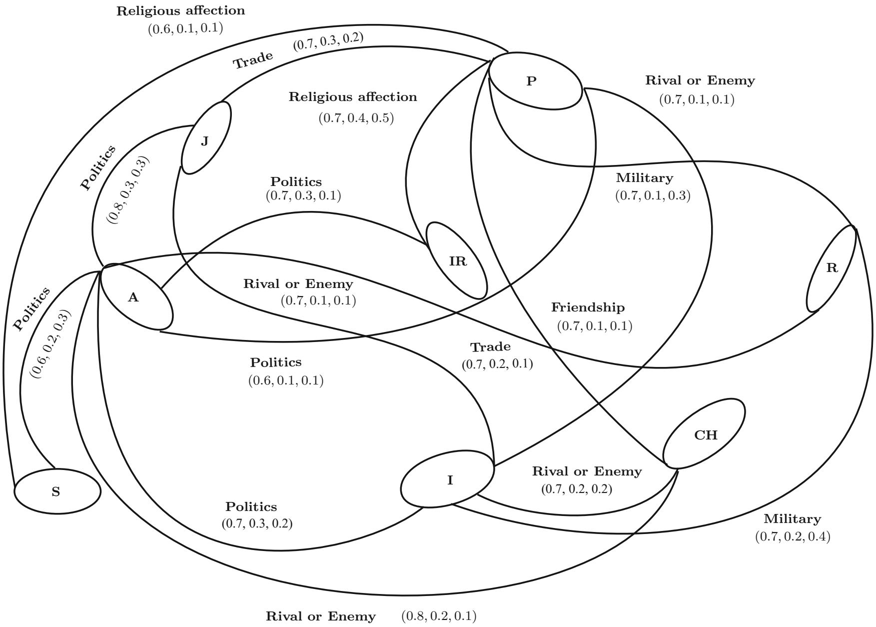


**Fig. 2.23** Neutrosophic graph structure showing most prominent relationship between any two vertices(countries)

## *2.4.3 Detection of Most Frequent Smuggling*

Smuggling on the seaports are increasing rapidly with time. There are 4,764 seaports on Atlantic ocean, Arctic ocean, Indian ocean, Pacific ocean, etc. These seaports are very useful and advantageous for import and export of different types of goods through out the world. Besides, there are also many disadvantages of these seaports. Crimes held on seaports are in abundance, but Smuggling of different kinds like human smuggling, weapons smuggling, black money smuggling, gold and diamond smuggling, smuggling of ivory and drug smuggling are most alarming. A lot of time and labour is required to collect and manipulate the data from all seaports to judge that which type of smuggling is frequent. But using neutrosophic graph structure, we can easily investigate the fact that between any two seaports which type of smuggling is chronic and increasing violently. Moreover, we can decide which seaport is most sensitive for smuggling, globally and need to be focused by security teams. We consider a set $X$ consisting of eight seaports.

$X$= {Chalna, Penang, Singapore, Dubai, Karachi, Mumbai, Mombasa, Gioia Tauro}. Let $A$ be the neutrosophic set on $X$, defined in Table 2.16.

In Table 2.16, $T$ depicts the importance of that particular seaport in the world due to its geographic position, $F$ indicates the degree of its nonimportance in the world,

**Table 2.16** Neutrosophic set A of eight seaports

| Country | T | I | F |
|---|---|---|---|
| Chalna | 0.7 | 0.6 | 0.5 |
| Penang | 0.6 | 0.6 | 0.7 |
| Singapore | 0.8 | 0.4 | 0.4 |
| Dubai | 0.9 | 0.4 | 0.5 |
| Karachi | 0.8 | 0.4 | 0.4 |
| Mumbai | 0.7 | 0.6 | 0.6 |
| Mombasa | 0.6 | 0.5 | 0.6 |
| Gioia Tauro | 0.8 | 0.5 | 0.4 |

**Table 2.17** Neutrosophic set of smuggling between Karachi and other seaports

| Type of smuggling | (K, DU) | (K, C) | (K, P) | (K, S) |
|---|---|---|---|---|
| Human smuggling | (0.6, 0.3, 0.4) | (0.7, 0.2, 0.3) | (0.6, 0.3, 0.1) | (0.5, 0.3, 0.1) |
| Weapons smuggling | (0.5, 0.2, 0.5) | (0.6, 0.2, 0.3) | (0.3, 0.4, 0.4) | (0.3, 0.2, 0.4) |
| Black money smuggling | (0.5, 0.2, 0.1) | (0.6, 0.4, 0.3) | (0.1, 0.3, 0.2) | (0.8, 0.1, 0.1) |
| Gold and diamond smuggling | (0.7, 0.2, 0.1) | (0.5, 0.2, 0.2) | (0.1, 0.3, 0.2) | (0.7, 0.4, 0.4) |
| Drug smuggling | (0.6, 0.2, 0.2) | (0.4, 0.3, 0.2) | (0.5, 0.4, 0.5) | (0.7, 0.3, 0.2) |
| Smuggling of ivory | (0.2, 0.4, 0.4) | (0.3, 0.2, 0.3) | (0.3, 0.3, 0.4) | (0.1, 0.2, 0.2) |

and $I$ expresses, to which extent it is undecided/indeterminate to be beneficial for the world, geographically.

Let Chalna = C, Pengang = P, Singapore = S, Dubai = DU, Karachi = K, Mumbai = MU, Mombasa = MO, Gioia Tauro = GT.

In Tables 2.17, 2.18, 2.19, 2.20, 2.21, 2.22 and 2.23, we have shown the values of $T$, $I$ and $F$ of different smuggling for each pair of seaports.

Many relations on set $X$ can be defined, let we define six relations on $X$ as: $E_1$ = Human smuggling, $E_2$ = Weapons smuggling, $E_3$ = Black money smuggling, $E_4$ = Gold and diamond smuggling, $E_5$ = Drug smuggling, $E_6$ = Smuggling of ivory, such that $(X, E_1, E_2, E_3, E_4, E_5, E_6)$ is a graph structure. An element in a relation detects that kind of smuggling between those two seaports.

As $(X, E_1, E_2, E_3, E_4, E_5, E_6)$ is a graph structure, an element will not be in more than one relations, so it can appear just once. Therefore, we will consider it an element of that relation for which its percentage of truth is high, and percentage of both falsity and indeterminacy is low as compared to other relations.

**Table 2.18** Neutrosophic set of smuggling between Dubai and other seaports

| Type of smuggling | (DU, C) | (DU, P) | (DU, S) | (DU, MU) |
|---|---|---|---|---|
| Human smuggling | (0,6, 0.2, 0.3) | (0.5, 0.1, 0.4) | (0.2, 0.1, 0.4) | (0.5, 0.3, 0.1) |
| Weapons smuggling | (0.4, 0.1, 0.1) | (0.4, 0.2, 0.1) | (0.3, 0.2, 0.4) | (0.3, 0.2, 0.4) |
| Black money smuggling | (0.5, 0.2, 0.2) | (0.5, 0.1, 0.2) | (0.5, 0.1, 0.2) | (0.5, 0.3, 0.4) |
| Gold and diamond smuggling | (0.5, 0.1, 0.1) | (0.5, 0.2, 0.2) | (0.5, 0.2, 0.2) | (0.7, 0.2, 0.1) |
| Drug smuggling | (0.5, 0.1, 0.1) | (0.4, 0.3, 0.2) | (0.6, 0.2, 0.1) | (0.6, 0.3, 0.2) |
| Smuggling of ivory | (0.2, 0.3, 0.3) | (0.3, 0.2, 0.3) | (0.3, 0.1, 0.4) | (0.2, 0.2, 0.2) |

**Table 2.19** Neutrosophic set of smuggling between Chalna and other seaports

| Type of smuggling | (C, P) | (C, S) | (C, MU) | (C, MO) |
|---|---|---|---|---|
| Human smuggling | (0.5, 0.2, 0.3) | (0.7, 0.2, 0.1) | (0.4, 0.1, 0.4) | (0.5, 0.3, 0.4) |
| Weapons smuggling | (0.4, 0.1, 0.4) | (0.4, 0.2, 0.1) | (0.6, 0.2, 0.4) | (0.3, 0.2, 0.4) |
| Black money smuggling | (0.4, 0.2, 0.2) | (0.7, 0.4, 0.3) | (0.1, 0.1, 0.2) | (0.1, 0.3, 0.4) |
| Gold and diamond smuggling | (0.3, 0.1, 0.1) | (0.5, 0.2, 0.2) | (0.1, 0.2, 0.2) | (0.1, 0.1, 0.3) |
| Drug smuggling | (0.5, 0.1, 0.1) | (0.4, 0.3, 0.2) | (0.5, 0.2, 0.4) | (0.4, 0.3, 0.3) |
| Smuggling of ivory | (0.2, 0.3, 0.3) | (0.3, 0.2, 0.3) | (0.2, 0.1, 0.4) | (0.5, 0.2, 0.2) |

According to given data, we write the elements in relations with their truth, falsity and indeterminacy values, so the resulting sets are neutrosophic sets on $E_1$, $E_2$, $E_3$, $E_4$, $E_5$, $E_6$, respectively. We can name these sets as $B_1$, $B_2$, $B_3$, $B_4$, $B_5$, $B_6$, respectively. Let
$E_1 = \{(Chalna, Karachi), (Penang, Karachi), (Chalna, Singapore)\}$,
$E_2 = \{(Karachi, Mumbai)\}$,
$E_3 = \{(Singapore, Karachi)\}$,
$E_4 = \{(Mumbai, Singapore), (Dubai, Mumbai)\}$,
$E_5 = \{(GioiaTauro, Karachi), (Mumbai,$Gioia Tauro$)\}$,
$E_6 = \{(Mombasa, Singapore)\}$.
And corresponding neutrosophic sets are:

**Table 2.20** Neutrosophic set of smuggling between Penang and other seaports

| Type of smuggling | (P, S) | (P, MU) | (P, MO) | (P, GT) |
|---|---|---|---|---|
| Human smuggling | (0.4, 0.2, 0.3) | (0.5, 0.1, 0.2) | (0.2, 0.1, 0.4) | (0.5, 0.3, 0.4) |
| Weapons smuggling | (0.5, 0.1, 0.1) | (0.4, 0.2, 0.1) | (0.3, 0.2, 0.4) | (0.3, 0.2, 0.4) |
| Black money smuggling | (0.5, 0.2, 0.2) | (0.1, 0.1, 0.2) | (0.1, 0.1, 0.2) | (0.1, 0.3, 0.4) |
| Gold and diamond smuggling | (0.5, 0.1, 0.1) | (0.5, 0.2, 0.2) | (0.1, 0.2, 0.2) | (0.1, 0.1, 0.5) |
| Drug smuggling | (0.4, 0.1, 0.1) | (0.4, 0.3, 0.2) | (0.3, 0.2, 0.5) | (0.6, 0.3, 0.1) |
| Smuggling of ivory | (0.2, 0.3, 0.3) | (0.3, 0.2, 0.3) | (0.5, 0.1, 0.1) | (0.4, 0.2, 0.2) |

**Table 2.21** Neutrosophic set of smuggling between Singapore and other seaports

| Type of smuggling | (S, MU) | (S, MO) | (S, GT) | (K, MU) |
|---|---|---|---|---|
| Human smuggling | (0.4, 0.2, 0.3) | (0.2, 0.1, 0.4) | (0.2, 0.1, 0.4) | (0.5, 0.3, 0.5) |
| Weapons smuggling | (0.6, 0.3, 0.4) | (0.4, 0.2, 0.3) | (0.3, 0.2, 0.4) | (0.7, 0.1, 0.3) |
| Black money smuggling | (0.4, 0.2, 0.3) | (0.5, 0.1, 0.3) | (0.5, 0.1, 0.2) | (0.6, 0.3, 0.4) |
| Gold and diamond smuggling | (0.7, 0.2, 0.6) | (0.5, 0.2, 0.4) | (0.5, 0.2, 0.2) | (0.5, 0.1, 0.3) |
| Drug smuggling | (0.6, 0.2, 0.3) | (0.4, 0.3, 0.4) | (0.5, 0.2, 0.1) | (0.7, 0.3, 0.3) |
| Smuggling of ivory | (0.6, 0.4, 0.5) | (0.6, 0.1, 0.3) | (0.5, 0.1, 0.4) | (0.6, 0.2, 0.2) |

$B_1 = \{((C, K), 0.7, 0.2, 0.3), ((P, K), 0.6, 0.3, 0.1), ((C, S), 0.7, 0.2, 0.1)\},$
$B_2 = \{((K, MU), 0.7, 0.1, 0.3)\},$
$B_3 = \{((S, K), 0.8, 0.1, 0.1), \},$
$B_4 = \{((MU, S), 0.7, 0.2, 0.3),\ ((DU, MU), 0.7, 0.2, 0.1)\},$
$B_5 = \{((GT, K), 0.8, 0.2, 0.2),\ ((MU, GT), 0.7, 0.2, 0.2)\},$
$B_6 = \{((MO, S), 0.6, 0.1, 0.3)\}.$

Clearly, $(A, B_1, B_2, B_3, B_4, B_5, B_6)$ is a neutrosophic graph structure as shown in Fig. 2.24.

In neutrosophic graph structure shown in Fig. 2.24, every edge detects most frequent smuggling between adjacent seaports. For instance, most frequent smuggling between Karachi and Singapore is black money smuggling, its strength is 80%, weak-

**Table 2.22** Neutrosophic set of smuggling between Gioia Tauro and other seaports

| Type of smuggling | (GT, K) | (GT, DU) | (GT, C) | (GT, MU) |
|---|---|---|---|---|
| Human smuggling | (0.4, 0.2, 0.3) | (0.2, 0.1, 0.4) | (0.2, 0.1, 0.4) | (0.5, 0.3, 0.5) |
| Weapons smuggling | (0.7, 0.2, 0.2) | (0.5, 0.2, 0.1) | (0.3, 0.2, 0.4) | (0.6, 0.2, 0.4) |
| Black money smuggling | (0.5, 0.2, 0.2) | (0.4, 0.1, 0.2) | (0.1, 0.1, 0.2) | (0.4, 0.3, 0.4) |
| Gold and diamond smuggling | (0.6, 0.2, 0.2) | (0.5, 0.2, 0.2) | (0.1, 0.2, 0.2) | (0.6, 0.2, 0.5) |
| Drug smuggling | (0.8, 0.2, 0.2) | (0.5, 0.3, 0.2) | (0.6, 0.2, 0.4) | (0.7, 0.2, 0.2) |
| Smuggling of ivory | (0.2, 0.3, 0.3) | (0.3, 0.2, 0.3) | (0.5, 0.1, 0.4) | (0.6, 0.2, 0.2) |

**Table 2.23** Neutrosophic set of smuggling between Mombasa and other seaports

| Type of smuggling | (MO, K) | (MO, DU) | (MO, MU) | (MO, GT) |
|---|---|---|---|---|
| Human smuggling | (0.4, 0.2, 0.3) | (0.5, 0.1, 0.4) | (0.4, 0.1, 0.4) | (0.5, 0.3, 0.4) |
| Weapons smuggling | (0.5, 0.1, 0.4) | (0.4, 0.2, 0.3) | (0.4, 0.2, 0.4) | (0.3, 0.2, 0.4) |
| Black money smuggling | (0.4, 0.2, 0.2) | (0.4, 0.1, 0.2) | (0.4, 0.1, 0.2) | (0.4, 0.3, 0.4) |
| Gold and diamond smuggling | (0.3, 0.1, 0.1) | (0.5, 0.2, 0.2) | (0.5, 0.2, 0.2) | (0.3, 0.1, 0.3) |
| Drug smuggling | (0.6, 0.1, 0.1) | (0.4, 0.3, 0.2) | (0.4, 0.2, 0.4) | (0.6, 0.3, 0.1) |
| Smuggling of ivory | (0.2, 0.3, 0.3) | (0.6, 0.2, 0.3) | (0.5, 0.1, 0.3) | (0.6, 0.2, 0.2) |

ness is 10% and indeterminacy is 10%. We can also note that for relation human smuggling, vertex Karachi has highest vertex degree, it means Karachi is most sensitive seaport for human smuggling. Moreover, according to our neutrosophic graph structure most frequent smuggling is human smuggling. It means that at these eight seaports, security forces should take action to control human smuggling.

This neutrosophic graph structure detects most frequent smuggling between some seaports of set $A$. By making a neutrosophic graph structure of all seaports, we can examine between any two seaports, which kind of smuggling is most frequent, we can also tell that which seaport is most sensitive for particular kind of smuggling. Further, we may get information about violently increasing smuggling through seaports in

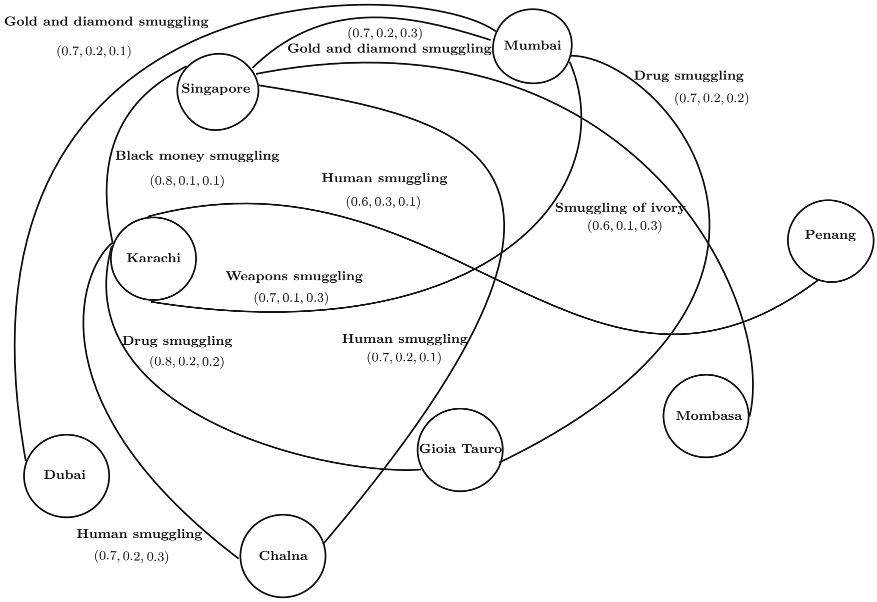


**Fig. 2.24** Neutrosophic graph structure showing most frequent smuggling between any two seaports

the whole world. That is why neutrosophic graph structures can be very helpful for security forces to overcome the smuggling at seaports.

We now elaborate general procedure of our applications in the following Algorithm.

**Algorithm 2.4.1**

Step 1. Input the set $X = \{A_1, A_2, \ldots, A_n\}$ of vertices and the neutrosophic vertex set $A$ defined on $X$.
Step 2. Input neutrosophic set of relationships or smuggling of a vertex with other vertices and compute $T$, $I$ and $F$ of each pair of vertices using:
$T(A_iA_j) \leq \min(T(A_i), T(A_j))$, $I(A_iA_j) \leq \min(I(A_i), I(A_j))$,
$F(A_iA_j) \leq \max(F(A_i), F(A_j))$.
Step 3. Repeat Step 2 for all vertices in $X$.
Step 4. Define relations $E_1, E_2, \ldots, E_n$ on set $X$ such that $(X, E_1, E_2, \ldots, E_n)$ is a graph structure.
Step 5. Put an element in that relation for which value of $T$ is high, and values of $I$ and $F$ are low as compared to other relations.
Step 6. Write all elements of relations with their $T$, $I$ and $F$ values, resulting relations $B_1, B_2, \ldots, B_n$ are neutrosophic sets on $E_1, E_2, E_3, \ldots, E_n$, respectively, and $(A, B_1, B_2, \ldots, B_n)$ is a neutrosophic graph structure.