Chapter 1
Graphs Under Neutrosophic Environment

In this chapter, we first present a concise review of neutrosophic sets. Then we present certain types of single-valued neutrosophic graphs (neutrosophic graphs, for short), including regular neutrosophic graphs, totally regular neutrosophic graphs, edge regular neutrosophic graphs, irregular neutrosophic graphs, highly totally irregular neutrosophic graphs, strongly totally irregular neutrosophic graphs, neighbourly edge irregular neutrosophic graphs and strongly edge irregular neutrosophic graphs. We describe applications of neutrosophic graphs. We also present energy of neutrosophic graphs with applications. This chapter is due to [27, 124, 167, 176].

1.1 Introduction

By a *graph*, we mean an ordered pair $G^* = (X, E)$ such that X is the collection of components taken as *nodes or vertices* and E is a relation on X, called *edges*. It is often convenient to depict the relationships between pairs of elements of a system by means of a graph or a digraph. The vertices of the graph represent the system elements, and its edges or arcs represent the relationships between the elements. This approach is especially useful for transportation, scheduling, sequencing, allocation, assignment and other problems which can be modelled as networks. Such a graph-theoretical model is often useful as an aid in communicating.

Zadeh [194] introduced the degree of membership/truth (T) in 1965 and defined the fuzzy set. Atanassov [47] introduced the degree of nonmembership/falsehood (F) in 1983 and defined the intuitionistic fuzzy set. Smarandache [163] introduced the degree of indeterminacy/neutrality (I) as independent component in 1995 and defined the neutrosophic set on three components $(T, I, F) =$ (Truth, Indeterminacy, Falsity). Fuzzy set theory and intuitionistic fuzzy set theory are useful models for dealing with uncertainty and incomplete information. But they may not be sufficient in modelling of indeterminate and inconsistent information encountered in real world. In order to cope with this issue, neutrosophic (The words "neutrosophy" and "neutrosophic" were invented by Smarandache in 1995. Neutrosophy is a new branch of philosophy

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that studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. It is the base of neutrosophic logic, a multiple-value logic that generalizes the fuzzy logic and deals with paradoxes, contradictions, antitheses, antinomies) set theory was proposed by Smarandache. However, since neutrosophic sets are identified by three functions called truth-membership (T), indeterminacy-membership (I) and falsity-membership (F) whose values are real standard or non-standard subset of unit interval $]^{-}0, 1^{+}[$, where $^{-}0 = 0 - \epsilon$, $1^{+} = 1 + \epsilon$, ϵ is an infinitesimal number. To apply neutrosophic set in real-life problems more conveniently, Smarandache [165] and Wang et al. [172] defined single-valued neutrosophic set which takes the value from the subset of [0, 1]. Thus, a single-valued neutrosophic set is an instance of neutrosophic set and can be used expediently to deal with real-world problems, especially in decision support.

A Geometric Interpretation of the Neutrosophic Set

We describe a geometric interpretation of the neutrosophic set using the neutrosophic cube $A'B'C'D'E'F'G'H'$ as shown in Fig. 1.1. In technical applications only the classical interval [0, 1] is used as range for the neutrosophic parameters T, I and F; we call the cube $ABCDEFGH$ the technical neutrosophic cube and its extension $A'B'C'D'E'F'G'H'$ the neutrosophic cube, used in the field where we need to differentiate between absolute and relative notions. Consider a 3D Cartesian system of coordinates, where T is the truth axis with value range in $]^{-}0, 1^{+}[$, F is the false

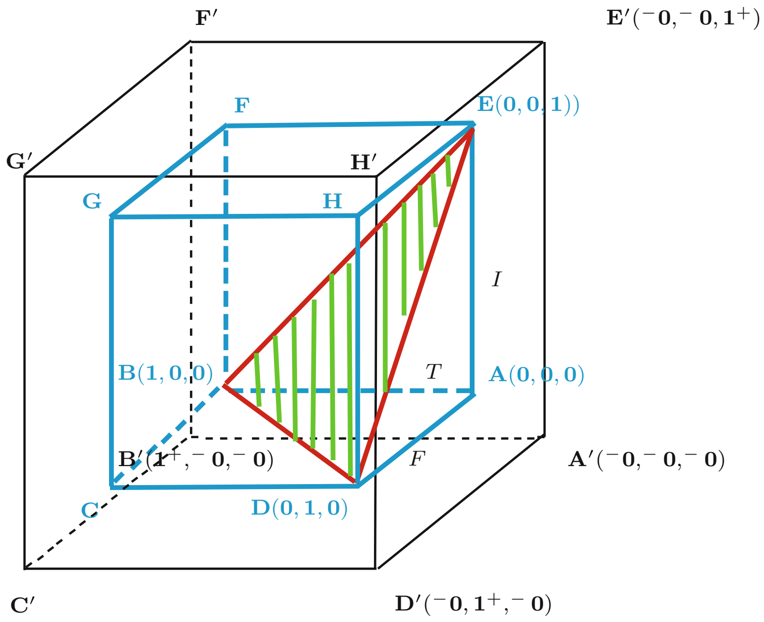


Fig. 1.1 A geometric interpretation of the neutrosophic set

axis with value range in $]^{-}0, 1^{+}[$, and I is the indeterminate axis with value range in $]^{-}0, 1^{+}[$.

We now divide the technical neutrosophic cube $ABCDEFGH$ into three disjoint regions:

1. The equilateral triangle BDE, whose sides are equal to $\sqrt{2}$, which represents the geometrical locus of the points whose sum of the coordinates is 1. If a point Q is situated on the sides of the triangle BDE or inside of it, then $T_Q + I_Q + F_Q = 1$.
2. The pyramid $EABD$ situated in the right side of the ΔEBD, including its faces ΔABD(base), ΔEBA and ΔEDA(lateral faces), but excluding its faces ΔBDE is the locus of the points whose sum of their coordinates is less than 1. If $P \in EABD$, then $T_P + I_P + F_P < 1$.
3. In the left side of ΔBDE in the cube, there is the solid $EFGCDEBD$ (excluding ΔBDE) which is the locus of points whose sum of their coordinates is greater than 1. If a point $R \in EFGCDEBD$, then $T_R + I_R + F_R > 1$.

It is possible to get the sum of coordinates strictly less than 1 or strictly greater than 1. For example:
(1) We have a source which is capable to find only the degree of membership of an element, but it is unable to find the degree of nonmembership.
(2) Another source which is capable to find only the degree of nonmembership of an element.
(3) Or a source which only computes the indeterminacy.
Thus, when we put the results together of these sources, it is possible that their sum is not 1, but smaller or greater.

On the other hand, in information fusion, when dealing with indeterminate models (i.e. elements of the fusion space which are indeterminate/unknown, such as intersections we do not know if they are empty or not since we do not have enough information, similarly for complements of indeterminate elements): if we compute the believe in that element (truth), the disbelieve in that element (falsehood) and the indeterminacy part of that element, then the sum of these three components is strictly less than 1 (the difference to 1 is the missing information).

Definition 1.1 Let X be a space of points (objects). A *single-valued neutrosophic set* A on a nonempty set X is characterized by a truth-membership function $T_A : X \to [0, 1]$, indeterminacy-membership function $I_A : X \to [0, 1]$ and a falsity-membership function $F_A : X \to [0, 1]$. Thus, $A = \{< x, T_A(x), I_A(x), F_A(x) > |x \in X\}$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$ for all $x \in X$.

When X is continuous, a single-valued neutrosophic set A can be written as

$$A = \int_X \langle (T(x), I(x), F(x))/x, x \in X \rangle.$$

When X is discrete, a single-valued neutrosophic set A can be written as

$$A = \sum_{i=1}^{n} \langle (T(x_i), I(x_i), F(x_i))/x_i, x_i \in X \rangle.$$

Example 1.1 Assume that the universe of discourse $X = \{x_1, x_2, x_3\}$, where x_1 describes the capability, x_2 describes the trustworthiness, and x_3 describes the prices of the objects. It may be further assumed that the values of x_1, x_2 and x_3 are in [0, 1] and they are obtained from some questionnaires of some experts. The experts may impose their opinion in three components, namely the degree of goodness, the degree of indeterminacy and that of poorness to explain the characteristics of the objects. Suppose A is a single-valued neutrosophic set of X such that

$$A = \{< x_1, 0.3, 0.5, 0.6 >, < x_2, 0.3, 0.2, 0.3 >, < x_3, 0.3, 0.5, 0.6 >\},$$

where $< x_1, 0.3, 0.5, 0.6 >$ represents that the degree of goodness of capability is 0.3, degree of indeterminacy of capability is 0.5 and degree of falsity of capability is 0.6.

Remark 1.1 When we consider that there are three different experts that are independent (i.e. they do not communicate with each other), so each one focuses on one attribute only (because each one is the best specialist in evaluating a single attribute). Therefore, each expert can assign 1 to his attribute value [for (1, 1, 1)], or each expert can assign 0 to his attribute value [for (0, 0, 0)], respectively.

When we consider a single expert for evaluating all three attributes, then he evaluates each attribute from a different point of view (using a different parameter)and arrives to (1, 1, 1) or (0, 0, 0), respectively.

For example, we examine a student "Muhammad"; for his research in neutrosophic graphs, he deserves 1; for his research in analytical mathematics, he also deserves 1; and for his research in physics, he deserves 1.

Definition 1.2 Let $A = \{< x, T_A(x), I_A(x), F_A(x) > |x \in X\}$ and $B = \{< x, T_B(x), I_B(x), F_B(x) > |x \in X\}$ be two single-valued neutrosophic sets, then operations are defined as follows:

- $A \subseteq B$ if and only if $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$,
- $A = B$ if and only if $T_A(x) = T_B(x)$, $I_A(x) = I_B(x)$ and $F_A(x) = F_B(x)$,
- $A \cap B = \{< x, \min(T_A(x), T_B(x)), \max(I_A(x), I_B(x)), \max(F_A(x), F_A(x)) > |x \in X\}$,
- $A \cup B = \{< x, \max(T_A(x), T_B(x)), \min(I_A(x), I_B(x)), \min(F_A(x), F_A(x)) > |x \in X\}$,
- $A^c = \{< x, F_A(x), 1 - I_A(x), T_A(x) > |x \in X\}$,
- $\mathbf{0} = (0, 1, 1)$ and $\mathbf{1} = (1, 0, 0)$.

Yang et al. [176] introduced the concept of single-valued neutrosophic relations.

Definition 1.3 A *single-valued neutrosophic relation* on a nonempty set X is a single-valued neutrosophic subset of $X \times X$ of the form

$$B = \{(yz, T_B(yz), I_B(yz), F_B(yz)) : yz \in X \times X\},$$

where $T_B : X \times X \to [0, 1]$, $I_B : X \times X \to [0, 1]$, $F_B : X \times X \to [0, 1]$ denote the truth-membership function, indeterminacy-membership function and falsity-membership function of B, respectively.

Definition 1.4 Let B be a single-valued neutrosophic relation in X, the complement and inverse of B are defined as follows, respectively

$$B^c = \{\langle (x, y), T_{R^c}(x, y), I_{R^c}(x, y), F_{R^c}(x, y)\rangle | (x, y) \in X \times X\}, \ \forall (x, y) \in X \times X,$$

where

$$\begin{aligned} T_{R^c}(x, y) &= F_R(x, y), \\ I_{R^c}(x, y) &= 1 - I_R(x, y), \\ F_{R^c}(x, y) &= T_R(x, y). \end{aligned}$$

$$B^{-1} = \{\langle (x, y), T_{R^{-1}}(x, y), I_{R^{-1}}(x, y), F_{R^{-1}}(x, y)\rangle | (x, y) \in X \times X\}, \ \forall (x, y) \in X \times X,$$

where

$$\begin{aligned} T_{R^{-1}}(x, y) &= T_R(y, x), \\ I_{R^{-1}}(x, y) &= I_R(y, x), \\ F_{R^{-1}}(x, y) &= F_R(y, x). \end{aligned}$$

Example 1.2 Let $X = \{x_1, x_2, x_3, x_4, x_5\}$. A single-valued neutrosophic relation B in X is given in Table 1.1. By Definition 1.4, we can compute B^c and B^{-1} which are given in Tables 1.2 and 1.3, respectively.

Definition 1.5 Let R, S be two single-valued neutrosophic relations in X.

1. The union $R \cup S$ of R and S is defined by

$$\begin{aligned} R \cup S = \{\langle (x, y); &\max\{T_R(x, y), T_S(x, y)\}; \min\{I_R(x, y), I_S(x, y)\}; \\ &\min\{F_R(x, y), F_S(x, y)\}\rangle | (x, y) \in X \times X\}. \end{aligned}$$

2. The intersection $R \cap S$ of R and S is defined by

$$\begin{aligned} R \cap S = \{\langle (x, y); &\min\{T_R(x, y), T_S(x, y)\}; \max\{I_R(x, y), I_S(x, y)\}; \\ &\max\{F_R(x, y), F_S(x, y)\}\rangle | (x, y) \in X \times X\}. \end{aligned}$$

Definition 1.6 Let R be a single-valued neutrosophic relation in X.

1. If $\forall x \in X, T_R(x, x) = 1$ and $I_R(x, x) = F_R(x, x) = 0$, then R is called a *reflexive single-valued neutrosophic relation.*

Table 1.1 Single-valued neutrosophic relation B

B	x_1	x_2	x_3	x_4	x_5
x_1	(0.2, 0.6, 0.4)	(0, 0.3, 0.7)	(0.9, 0.2, 0.4)	(0.3, 0.9, 1)	(1, 0.2, 0)
x_2	(0.4, 0.5, 0.1)	(0.1, 0.7, 0)	(1, 1, 1)	(1, 0.3, 0)	(0.5, 0.6, 1)
x_3	(0, 1, 1)	(1, 0.5, 0)	(0, 0, 0)	(0.2, 0.8, 0.1)	(1, 0.8, 1)
x_4	(1, 0, 0)	(0, 0, 1)	(0.5, 0.7, 0.1)	(0.1, 0.4, 1)	(1, 0.8, 0.8)
x_5	(0, 1, 0)	(0.9, 0, 0)	(0, 0.1, 0.7)	(0.8, 0.9, 1)	(0.6, 1, 0)

Table 1.2 Complement B^c of B

B	x_1	x_2	x_3	x_4	x_5
x_1	(0.4, 0.4, 0.2)	(0.7, 0.7, 0)	(0.4, 0.8, 0.9)	(0.1, 0.1, 3)	(0, 0.8, 1)
x_2	(0.1, 0.5, 0.4)	(0, 0.3, 0.1)	(1, 0, 1)	(0, 0.7, 1)	(1, 0.4, 0.5)
x_3	(1, 0, 0)	(0, 0.5, 1)	(0, 1, 0)	(0.1, 0.2, 0.2)	(1, 0.2, 1)
x_4	(0, 1, 1)	(1, 1, 0)	(0.1, 0.3, 0.5)	(1, 0.6, 0.4)	(0.8, 0.2, 1)
x_5	(0, 0, 0)	(0, 1, 0.9)	(0.7, 0.9, 0)	(1, 0.1, 0.8)	(0, 0, 0.6)

Table 1.3 Inverse B^- of B

B	x_1	x_2	x_3	x_4	x_5
x_1	(0.2, 0.6, 0.4)	(0.4, 0.5, 0.1)	(0, 1, 1)	(1, 0, 0)	(0, 1, 0)
x_2	(0, 0.3, 0.7)	(0.1, 0.7, 0)	(1, 0.5, 0)	(0, 0, 1)	(0.9, 0, 0)
x_3	(0.9, 0.2, 0.4)	(1, 1, 1)	(0, 0, 0)	(0.5, 0.7, 0.1)	(0, 0.1, 0.7)
x_4	(0.3, 0.9, 1)	(1, 0.3, 0)	(0.2, 0.8, 0.1)	(0.1, 0.4, 1)	(0.8, 0.9, 1)
x_5	(1, 0.2, 0)	(0.5, 0.6, 1)	(1, 0.8, 1)	(1, 0.8, 0.8)	(0.6, 1, 0)

2. If $\forall\, x, y \in X, T_R(x, y) = T_R(y, x), I_R(x, y) = I_R(y, x)$ and $F_R(y, x) = F_R(x, y)$, then R is called a *symmetric single-valued neutrosophic relation*.
3. If $\forall\, x \in X, T_R(x, x) = 0$ and $I_R(x, x) = F_R(x, x) = 1$, then R is called an *antireflexive single-valued neutrosophic relation*.
4. If $\forall\, x, y, z \in X$,

$$\max_{v \in X} \min\{T_R(x, y), T_R(y, z)\} \leq T_R(x, z),$$
$$\min_{v \in X} \max\{I_R(x, y), I_R(y, z)\} \geq I_R(x, z),$$
$$\min_{v \in X} \max\{F_R(x, y), F_R(y, z)\} \geq F_R(x, z),$$

then R is called a *transitive single-valued neutrosophic relation*.

1.2 Certain Types of Neutrosophic Graphs

Definition 1.7 A *single-valued neutrosophic graph* on a nonempty X is a pair $G = (A, B)$, where A is single-valued neutrosophic set in X and B single-valued neutrosophic relation on X such that

$$T_B(xy) \le \min\{T_A(x), T_A(y)\},$$
$$I_B(xy) \le \min\{I_A(x), I_A(y)\},$$
$$F_B(xy) \le \max\{F_A(x), F_A(y)\}$$

for all $x, y \in X$. A is called *single-valued neutrosophic vertex set* of G and B is called *single-valued neutrosophic edge set* of G, respectively.

Remark 1.2 1. B is called symmetric single-valued neutrosophic relation on A.
2. If B is not symmetric single-valued neutrosophic relation on A, then $G = (A, B)$ is called a *single-valued neutrosophic directed graph (digraph).*
3. X and E are underlying vertex set and underlying edge set of G, respectively.

Throughout this chapter, we will use neutrosophic set, neutrosophic relation and neutrosophic graph, for short.

Example 1.3 Consider a crisp graph $G^* = (X, E)$ such that $X = \{a, b, c, d, e, f\}$, $E = \{ab, ac, bd, cd, be, cf, ef, bc\}$. Let A and B be the neutrosophic sets of X and E, respectively, as shown in Table 1.4. By simple calculations, it is easy to see that $G = (A, B)$ is a neutrosophic graph as shown in Fig. 1.2.

Definition 1.8 A neutrosophic graph $G = (A, B)$ is called *complete* if the following conditions are satisfied:

$$T_B(xy) = \min\{T_A(x), T_A(y)\},$$
$$I_B(xy) = \min\{I_A(x), I_A(y)\},$$

Table 1.4 Neutrosophic sets

A	a	b	c	d	e	f		
T_A	0.2	0.3	0.4	0.3	0.5	0.4		
I_A	0.5	0.4	0.5	0.6	0.5	0.6		
F_A	0.7	0.6	0.4	0.8	0.6	0.6		
B	ab	ac	bd	cd	be	cf	ef	bc
T_B	0.2	0.1	0.2	0.3	0.2	0.1	0.4	0.2
I_B	0.4	0.4	0.2	0.2	0.3	0.4	0.4	0.3
F_B	0.7	0.5	0.6	0.7	0.5	0.5	0.5	0.6

Fig. 1.2 Neutrosophic graph

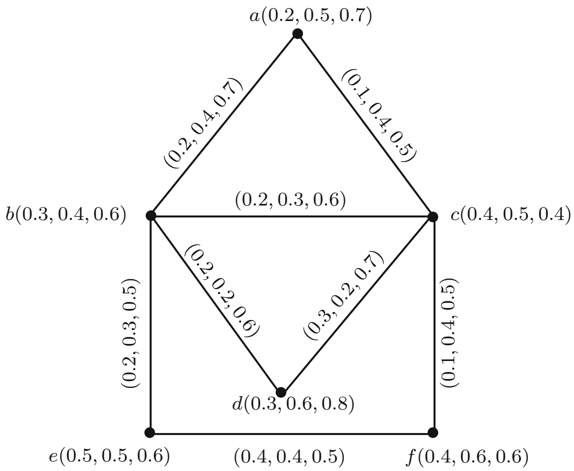
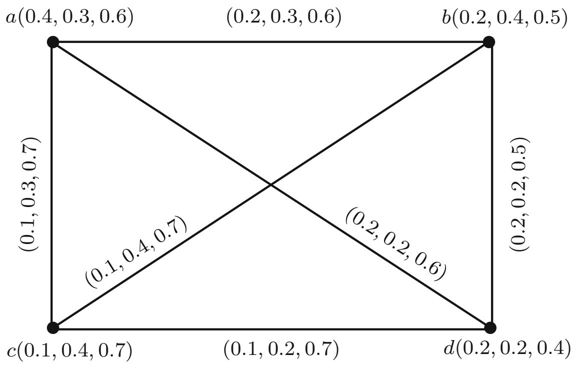


Fig. 1.3 Complete neutrosophic graph



$$F_B(xy) = \max\{F_A(x), F_A(y)\},$$

for all $x, y \in X$.

Example 1.4 Consider a neutrosophic $G=(A, B)$ on the nonempty set $X = \{a, b, c, d\}$ as shown in Fig. 1.3. By direct calculations, it is easy to see that G is a complete.

Definition 1.9 Let $A = \{< x, T_A(x), I_A(x), F_A(x) >, x \in X\}$ be a neutrosophic set of the set X. For $\alpha \in [0, 1]$, the α-cut of A is the crisp set A_α defined by

$$A_\alpha = \{x \in X : \text{either } (T_A(x), I_A(x) \geq \alpha) \text{ or } F_A(x) \leq 1 - \alpha\}.$$

Let $B = \{< xy, T_B(xy), I_B(xy), F_B(xy) >\}$ be a neutrosophic set on $E \subseteq X \times X$. For $\alpha \in [0, 1]$, the α-cut is the crisp set B_α defined by

$$B_\alpha = \{xy \in E : \text{either } (T_B(xy), I_B(xy) \geq \alpha) \text{ or } F_B(xy) \leq 1 - \alpha\}.$$

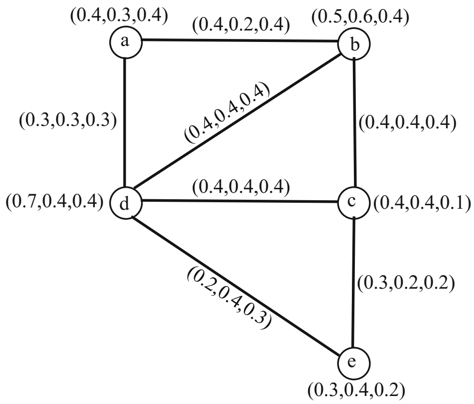


Fig. 1.4 Neutrosophic graph and 0.4-level graph $G_{0.4}$

$G_\alpha = (A_\alpha, B_\alpha)$ is a subgraph of crisp graph G^*.

Example 1.5 Consider a neutrosophic graph G on nonempty set $X = \{a, b, c, d, e\}$ as shown in Fig. 1.4.

For $\alpha = 0.4$, we have

$$A_{0.4} = \{b, c, d\},$$
$$B_{0.4} = \{bc, cd, bd\}.$$

Clearly, the 0.4-level graph $G_{0.4} = (A_{0.4}, B_{0.4})$ is a subgraph of crisp graph G^*.

Definition 1.10 The *order and the size* of a neutrosophic graph G are denoted by $O(G)$ and $S(G)$, respectively, and are defined as

$$O(G) = \left(\sum_{s\in X} T_A(s), \sum_{s\in X} I_A(s), \sum_{s\in X} F_A(s)\right),$$
$$S(G) = \left(\sum_{st\in E} T_B(st), \sum_{st\in E} I_B(st), \sum_{st\in E} F_B(st)\right).$$

Definition 1.11 The *degree and the total degree* of a vertex s of a neutrosophic graph G are denoted by $d_G(s) = (d_T(s), d_I(s), d_F(s))$ and $Td_G(s) = (Td_T(s), Td_I(s), Td_F(s))$, respectively, and are defined as

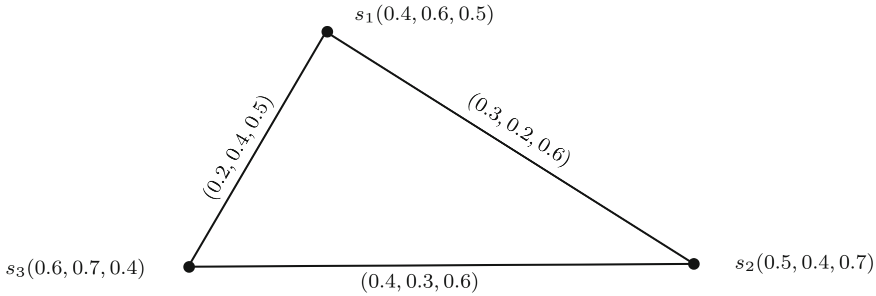


Fig. 1.5 Neutrosophic graph

$$d_G(s) = \left(\sum_{s \neq t} T_B(st), \sum_{s \neq t} I_B(st), \sum_{s \neq t} F_B(st)\right),$$

$$Td_G(s) = \left(\sum_{s \neq t} T_B(st) + T_A(s), \sum_{s \neq t} I_B(st) + I_A(s), \sum_{s \neq t} F_B(st) + F_A(s)\right),$$

for $st \in E$, where $s \in X$.

Example 1.6 Consider a neutrosophic graph G on the nonempty set $X = \{s_1, s_2, s_3\}$ as shown in Fig. 1.5.

By direct calculations, we have O(G) = (1.5, 1.7, 1.6), S(G) = (0.9, 0.9, 1.7),

$$d_G(s_1) = (0.5, 0.6, 1.1),\ d_G(s_2) = (0.7, 0.5, 1.2),\ d_G(s_3) = (0.6, 0.7, 1.1),$$
$$Td_G(s_1) = (0.9, 1.2, 1.6),\ Td_G(s_2) = (1.2, 0.9, 1.9),\ Td_G(s_3) = (1.2, 1.4, 1.5).$$

Definition 1.12 A neutrosophic graph G is called a *regular* if each vertex has same degree, that is,

$$d_G(s) = (m_1, m_2, m_3), \quad \text{for all } s \in X.$$

Example 1.7 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.6.

By direct calculations, we have

$$d_G(s_1) = (0.2, 1.2, 0.8) = d_G(s_2) = d_G(s_3) = d_G(s_4).$$

Hence G is a regular neutrosophic graph.

Definition 1.13 A neutrosophic graph G is called a *totally regular* of degree (n_1, n_2, n_3) if

$$Td_G(s) = (n_1, n_2, n_3), \text{ for all } s \in X.$$

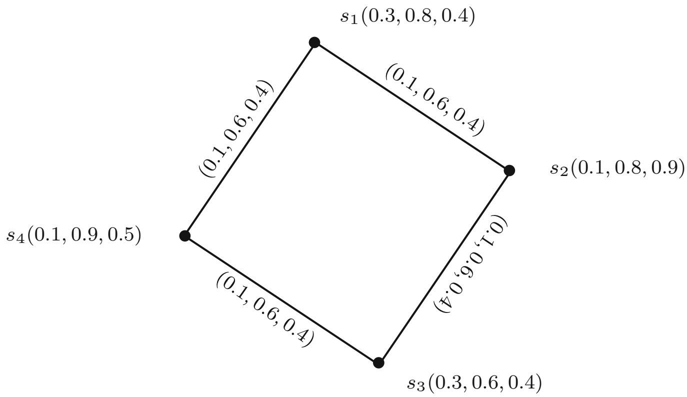


Fig. 1.6 Regular neutrosophic graph

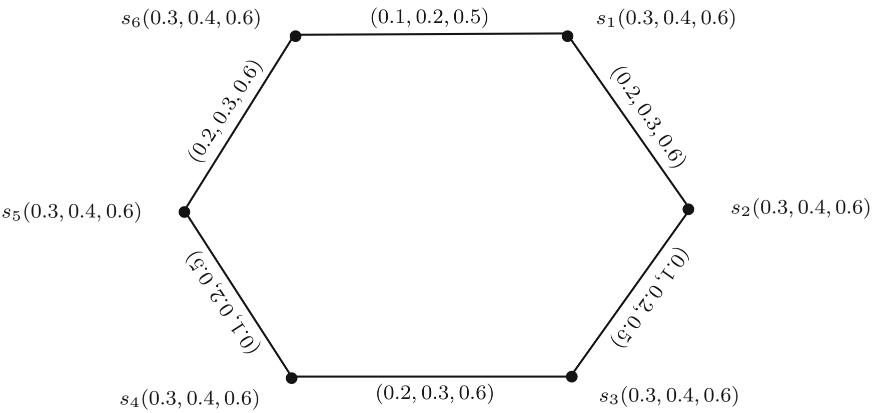


Fig. 1.7 Totally regular neutrosophic graph

Example 1.8 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ as shown in Fig. 1.7.

By direct calculations, we have

$$d_G(s_1) = (0.3, 0.5, 1.1) = d_G(s_2) = d_G(s_3) = d_G(s_4) = d_G(s_5) = d_G(s_6),$$
$$Td_G(s_1) = (0.6, 0.9, 1.7) = Td_G(s_2) = Td_G(s_3) = Td_G(s_4) = Td_G(s_5) = Td_G(s_6).$$

Hence G is a totally regular neutrosophic graph.

Remark 1.3 The above two concepts are independent; that is, it is not necessary that totally regular neutrosophic graph is regular neutrosophic graph and vice versa.

Example 1.9 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.8.

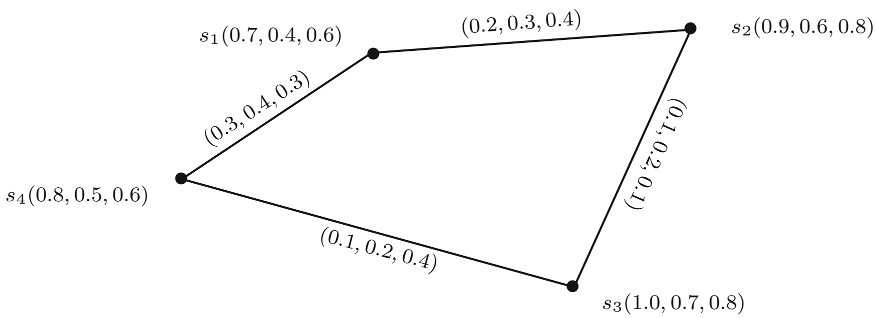


Fig. 1.8 Totally regular but not regular neutrosophic graph

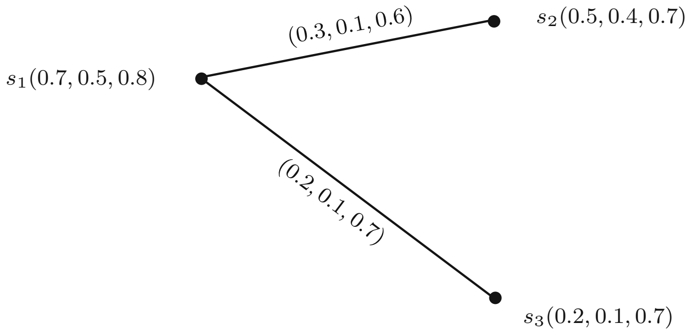


Fig. 1.9 Neutrosophic graph

By direct calculations, we have

$$d_G(s_1) = (0.5, 0.7, 0.7), d_G(s_2) = (0.3, 0.5, 0.5),$$
$$d_G(s_3) = (0.2, 0.4, 0.5), d_G(s_4) = (0.4, 0.6, 0.7),$$
$$Td_G(s_1) = (1.2, 1.1, 1.3) = Td_G(s_2) = Td_G(s_3) = Td_G(s_4).$$

Therefore, G is a totally regular neutrosophic graph but not a regular neutrosophic graph.

Definition 1.14 The degree and the total degree of an edge st of a neutrosophic graph G are denoted by $d_G(st) = (d_T(st), d_I(st), d_F(st))$ and $Td_G(st) = (Td_T(st), Td_I(st), Td_F(st))$, respectively, and are defined as

$$d_G(st) = d_G(s) + d_G(t) - 2(T_B(st), I_B(st), F_B(st)),$$
$$Td_G(st) = d_G(st) + (T_B(st), I_B(st), F_B(st)).$$

Example 1.10 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3\}$ as shown in Fig. 1.9.

By direct calculations, we have

$$d_G(s_1) = (0.5, 0.2, 1.3), d_G(s_2) = (0.3, 0.1, 0.6), d_G(s_3) = (0.2, 0.1, 0.7).$$

- The degree of each edge is given as:

$$\begin{aligned} d_G(s_1s_2) &= d_G(s_1) + d_G(s_2) - 2(T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)), \\ &= (0.7, 0.5, 0.8) + (0.5, 0.4, 0.7) - 2(0.3, 0.1, 0.6), \\ &= (0.2, 0.1, 0.7). \end{aligned}$$

$$\begin{aligned} d_G(s_1s_3) &= d_G(s_1) + d_G(s_3) - 2(T_B(s_1s_3), I_B(s_1s_3), F_B(s_1s_3)), \\ &= (0.7, 0.5, 0.8) + (0.4, 0.2, 0.6) - 2(0.2, 0.1, 0.7), \\ &= (0.3, 0.1, 0.6). \end{aligned}$$

- The total degree of each edge is given as:

$$\begin{aligned} Td_G(s_1s_2) &= d_G(s_1s_2) + (T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)), \\ &= (0.2, 0.1, 0.7) + (0.3, 0.1, 0.6), \\ &= (0.5, 0.2, 1.3). \end{aligned}$$

$$\begin{aligned} Td_G(s_1s_3) &= d_G(s_1s_3) + (T_B(s_1s_3), I_B(s_1s_3), F_B(s_1s_3)), \\ &= (0.3, 0.1, 0.6) + (0.2, 0.1, 0.7), \\ &= (0.5, 0.2, 1.3). \end{aligned}$$

Definition 1.15 The maximum degree of a neutrosophic graph G is defined as $\Delta(G) = (\Delta_T(G), \Delta_I(G), \Delta_F(G))$, where

$$\begin{aligned} \Delta_T(G) &= \max\{d_T(s) : s \in X\}, \\ \Delta_I(G) &= \max\{d_I(s) : s \in X\}, \\ \Delta_F(G) &= \max\{d_F(s) : s \in X\}. \end{aligned}$$

Definition 1.16 The minimum degree of a neutrosophic graph G is defined as $\delta(G) = (\delta_T(G), \delta_I(G), \delta_F(G))$, where

$$\begin{aligned} \delta_T(G) &= \min\{d_T(s) : s \in X\}, \\ \delta_I(G) &= \min\{d_I(s) : s \in X\}, \\ \delta_F(G) &= \min\{d_F(s) : s \in X\}. \end{aligned}$$

Example 1.11 Consider the neutrosophic graph G as shown in Fig. 1.9. By direct calculations, we have

$$\Delta(G) = (0.5, 0.2, 1.3) \text{ and } \delta(G) = (0.2, 0.1, 0.6).$$

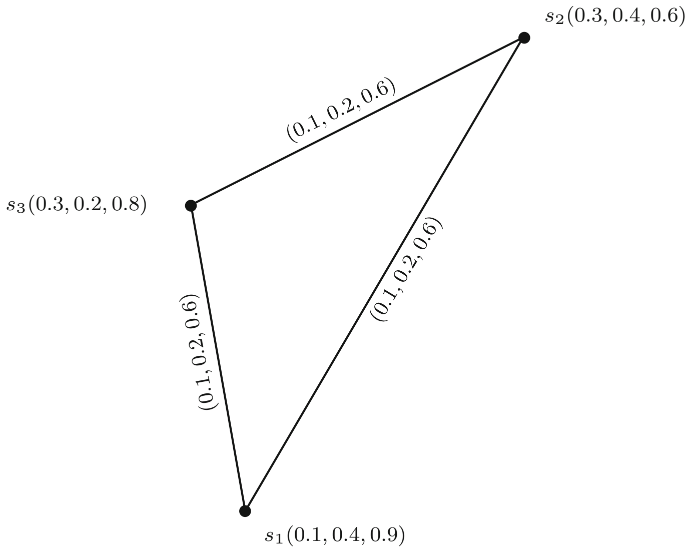


Fig. 1.10 Edge regular neutrosophic graph

Definition 1.17 A neutrosophic graph G on X is called an *edge regular* if every edge in G has the same degree (q_1, q_2, q_3).

Example 1.12 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3\}$ as shown in Fig. 1.10.

By direct calculations, we have

$$d_G(s_1) = (0.2, 0.4, 1.2), \quad d_G(s_2) = (0.2, 0.4, 1.2), \quad d_G(s_3) = (0.2, 0.4, 1.2).$$

The degree of each edge is given below:

$$\begin{aligned} d_G(s_1s_2) &= d_G(s_1) + d_G(s_2) - 2(T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)), \\ &= (0.2, 0.4, 1.2) + (0.2, 0.4, 1.2) - 2(0.1, 0.2, 0.6), \\ &= (0.2, 0.4, 1.2). \end{aligned}$$

$$\begin{aligned} d_G(s_1s_3) &= d_G(s_1) + d_G(s_3) - 2(T_B(s_1s_3), I_B(s_1s_3), F_B(s_1s_3)), \\ &= (0.2, 0.4, 1.2) + (0.2, 0.4, 1.2) - 2(0.1, 0.2, 0.6), \\ &= (0.2, 0.4, 1.2). \end{aligned}$$

$$\begin{aligned} d_G(s_2s_3) &= d_G(s_2) + d_G(s_3) - 2(T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)), \\ &= (0.2, 0.4, 1.2) + (0.2, 0.4, 1.2) - 2(0.1, 0.2, 0.6), \\ &= (0.2, 0.4, 1.2). \end{aligned}$$

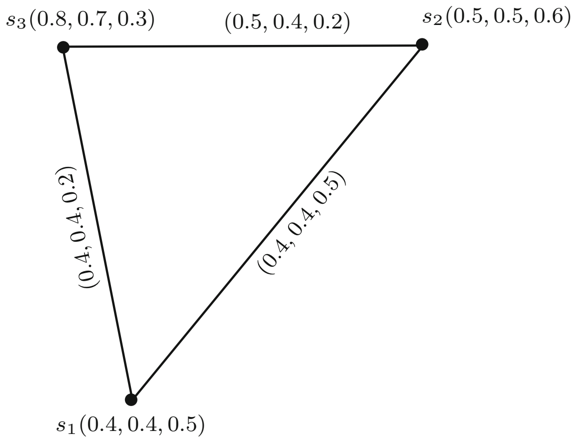


Fig. 1.11 Totally edge regular neutrosophic graph

It is easy to see that each edge of neutrosophic graph G has the same degree. Hence G is an edge regular neutrosophic graph.

Definition 1.18 A neutrosophic graph G on X is called a *totally edge regular* if every edge in G has the same total degree (p_1, p_2, p_3).

Example 1.13 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3\}$ as shown in Fig. 1.11.

By direct calculations, we have

$$d_G(s_1) = (0.8, 0.8, 0.7), \ \ d_G(s_2) = (0.9, 0.8, 0.7), \ \ d_G(s_3) = (0.9, 0.8, 0.4).$$

- The degree of each edge is given below:

$$\begin{aligned} d_G(s_1s_2) &= d_G(s_1) + d_G(s_2) - 2(T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)), \\ &= (0.8, 0.8, 0.7) + (0.9, 0.8, 0.7) - 2(0.4, 0.4, 0.5), \\ &= (0.9, 0.8, 0.4). \end{aligned}$$

$$\begin{aligned} d_G(s_1s_3) &= d_G(s_1) + d_G(s_3) - 2(T_B(s_1s_3), I_B(s_1s_3), F_B(s_1s_3)), \\ &= (0.8, 0.8, 0.7) + (0.9, 0.8, 0.4) - 2(0.4, 0.4, 0.2), \\ &= (0.9, 0.8, 0.7). \end{aligned}$$

$$\begin{aligned} d_G(s_2s_3) &= d_G(s_2) + d_G(s_3) - 2(T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)), \\ &= (0.9, 0.8, 0.7) + (0.9, 0.8, 0.4) - 2(0.5, 0.4, 0.2), \\ &= (0.8, 0.8, 0.7). \end{aligned}$$

It is easy to see that $d_G(s_1s_2) \neq d_G(s_1s_3) \neq d_G(s_2s_3)$. So G is not an edge regular neutrosophic graph.

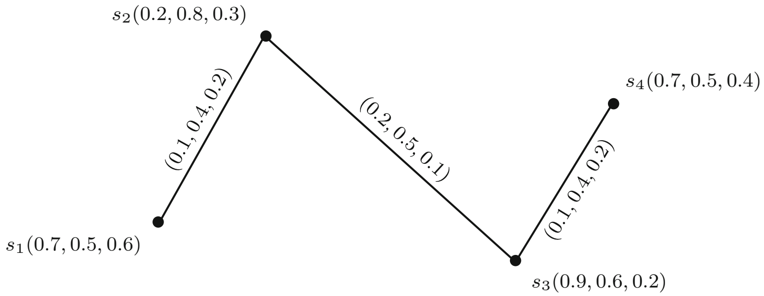


Fig. 1.12 Edge irregular and totally edge irregular neutrosophic graph

- The total degree of each edge is calculated as:

$$\begin{aligned} Td_G(s_1s_2) &= d_G(s_1s_2) + (T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)), \\ &= (1.3, 1.2, 0.9). \end{aligned}$$

$$\begin{aligned} Td_G(s_1s_3) &= d_G(s_1s_3) + (T_B(s_1s_3), I_B(s_1s_3), F_B(s_1s_3)), \\ &= (1.3, 1.2, 0.9). \end{aligned}$$

$$\begin{aligned} Td_G(s_2s_3) &= d_G(s_2s_3) + (T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)), \\ &= (1.3, 1.2, 0.9). \end{aligned}$$

It is easy to see that each edge of neutrosophic graph G has the same total degree. So G is a totally edge regular neutrosophic graph.

Remark 1.4 A neutrosophic graph G is an edge regular neutrosophic graph if and only if $\Delta_d(G) = \delta_d(G) = (q_1, q_2, q_3)$.

Example 1.14 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.12.

By direct calculations, we have

$$d_G(s_1) = (0.1, 0.4, 0.2), \quad d_G(s_2) = (0.3, 0.9, 0.3),$$

$$d_G(s_3) = (0.3, 0.9, 0.3), \quad d_G(s_4) = (0.1, 0.4, 0.2).$$

- The degree of each edge is given below:

$$\begin{aligned} d_G(s_1s_2) &= d_G(s_1) + d_G(s_2) - 2(T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)), \\ &= (0.1, 0.4, 0.2) + (0.3, 0.9, 0.3) - 2(0.1, 0.4, 0.2), \\ &= (0.2, 0.5, 0.1). \end{aligned}$$

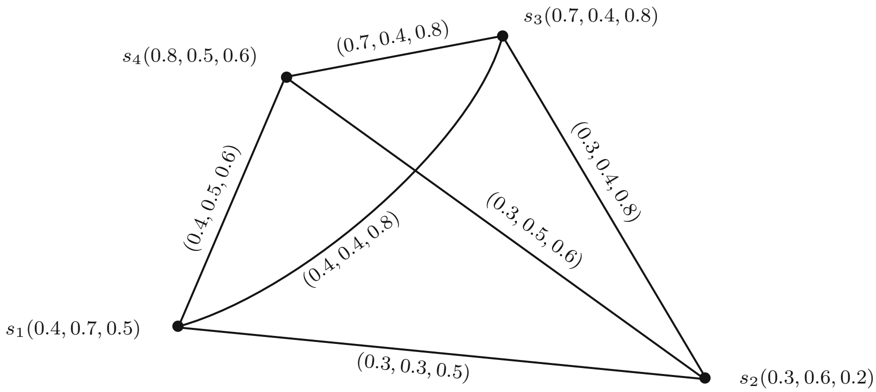


Fig. 1.13 Complete neutrosophic graph

$$\begin{aligned} d_G(s_2s_3) &= d_G(s_2) + d_G(s_3) - 2(T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)), \\ &= (0.3, 0.9, 0.3) + (0.3, 0.9, 0.3) - 2(0.2, 0.5, 0.1), \\ &= (0.2, 0.8, 0.4). \end{aligned}$$

$$\begin{aligned} d_G(s_3s_4) &= d_G(s_3) + d_G(s_4) - 2(T_B(s_3s_4), I_B(s_3s_4), F_B(s_3s_4)), \\ &= (0.3, 0.9, 0.3) + (0.1, 0.4, 0.2) - 2(0.1, 0.4, 0.2), \\ &= (0.2, 0.5, 0.1). \end{aligned}$$

It is easy to see that $d_G(s_1s_2) \neq d_G(s_2s_3)$. So G is not an edge regular neutrosophic graph.

- The total degree of each edge is calculated as:

$$\begin{aligned} Td_G(s_1s_2) &= d_G(s_1s_2) + (T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)), \\ &= (0.3, 0.9, 0.3). \end{aligned}$$

$$\begin{aligned} Td_G(s_2s_3) &= d_G(s_2s_3) + (T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)), \\ &= (0.4, 1.3, 0.5). \end{aligned}$$

$$\begin{aligned} Td_G(s_3s_4) &= d_G(s_3s_4) + (T_B(s_3s_4), I_B(s_3s_4), F_B(s_3s_4)), \\ &= (0.3, 0.9, 0.3). \end{aligned}$$

It is easy to see that $Td_G(s_1s_2) \neq Td_G(s_2s_3)$. So G is not a totally edge regular neutrosophic graph.

Remark 1.5 A complete neutrosophic graph G may not be an edge regular neutrosophic graph.

Example 1.15 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.13.

By direct calculations, we have

$$d_G(s_1) = (1.1, 1.2, 1.9), \quad d_G(s_2) = (0.9, 1.2, 1.9),$$

$$d_G(s_3) = (1.4, 1.2, 2.4), \quad d_G(s_4) = (1.4, 1.4, 2.0).$$

The degree of each edge is given below:

$$\begin{aligned} d_G(s_1s_2) &= d_G(s_1) + d_G(s_2) - 2(T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)),\\ &= (1.1, 1.2, 1.9) + (0.9, 1.2, 1.9) - 2(0.3, 0.3, 0.5),\\ &= (1.4, 2.0, 2.8). \end{aligned}$$

$$\begin{aligned} d_G(s_1s_3) &= d_G(s_1) + d_G(s_3) - 2(T_B(s_1s_3), I_B(s_1s_3), F_B(s_1s_3)),\\ &= (1.1, 1.2, 1.9) + (1.4, 1.2, 2.4) - 2(0.4, 0.4, 0.8),\\ &= (1.7, 1.6, 2.7). \end{aligned}$$

$$\begin{aligned} d_G(s_1s_4) &= d_G(s_1) + d_G(s_4) - 2(T_B(s_1s_4), I_B(s_1s_4), F_B(s_1s_4)),\\ &= (1.1, 1.2, 1.9) + (1.4, 1.4, 2.0) - 2(0.4, 0.5, 0.6),\\ &= (1.7, 1.6, 2.7). \end{aligned}$$

$$\begin{aligned} d_G(s_2s_3) &= d_G(s_2) + d_G(s_3) - 2(T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)),\\ &= (0.9, 1.2, 1.9) + (1.4, 1.2, 2.4) - 2(0.3, 0.4, 0.8),\\ &= (1.7, 1.6, 2.7). \end{aligned}$$

$$\begin{aligned} d_G(s_2s_4) &= d_G(s_2) + d_G(s_4) - 2(T_B(s_2s_4), I_B(s_2s_4), F_B(s_2s_4)),\\ &= (0.9, 1.2, 1.9) + (1.4, 1.4, 2.0) - 2(0.3, 0.5, 0.6),\\ &= (1.7, 1.6, 2.7). \end{aligned}$$

$$\begin{aligned} d_G(s_3s_4) &= d_G(s_3) + d_G(s_4) - 2(T_B(s_3s_4), I_B(s_3s_4), F_B(s_3s_4)),\\ &= (1.4, 1.2, 2.4) + (1.4, 1.4, 2.0) - 2(0.7, 0.4, 0.8),\\ &= (1.4, 1.8, 2.8). \end{aligned}$$

It is easy to see that each edge of neutrosophic graph G has not the same degree. Therefore, G is a complete neutrosophic graph but not an edge regular neutrosophic graph.

Theorem 1.1 *Let G be a neutrosophic graph. Then*

$$\sum_{st\in E} d_G(st) = \sum_{st\in E} d_{G^*}(st)(T_B(st), I_B(st), F_B(st)),$$

where $d_{G^*}(st) = d_{G^*}(s) + d_{G^*}(t) - 2$*, for all* $s, t \in X$.

Theorem 1.2 *Let G be a neutrosophic graph. Then*

$$\sum_{st\in E} Td_G(st) = \sum_{st\in E} d_{G^*}(st)(T_B(st), I_B(st), F_B(st)) + S(G),$$

where $d_{G^*}(st) = d_{G^*}(s) + d_{G^*}(t) - 2$, *for all* $s, t \in X$.

Proof Since the total degree of each edge in a neutrosophic graph G is $Td_G(st) = d_G(st) + (T_B(st), I_B(st), F_B(st))$. Therefore,

$$\sum_{st\in E} Td_G(st) = \sum_{st\in E}(d_G(st) + (T_B(st), I_B(st), F_B(st))),$$
$$\sum_{st\in E} Td_G(st) = \sum_{st\in E} d_G(st) + \sum_{st\in E}(T_B(st), I_B(st), F_B(st)),$$
$$\sum_{st\in E} Td_G(st) = \sum_{st\in E} d_{G^*}(st)(T_B(st), I_B(st), F_B(st)) + S(G).$$

This completes the proof.

Theorem 1.3 *Let* $G^* = (X, E)$ *be an edge regular crisp graph of degree* q *and* G *be an edge regular neutrosophic graph of degree* (q_1, q_2, q_3) *of* G^*. *Then the size of* G *is* $(\frac{mq_1}{q}, \frac{mq_2}{q}, \frac{mq_3}{q})$, *where* $|E| = m$.

Proof Let G be an edge regular neutrosophic graph. Then,

$$d_G(st) = (q_1, q_2, q_3) \text{ and } d_{G^*}(st) = q, \quad \text{for each edge } st \in E.$$

Since,

$$\sum_{st\in E} d_G(st) = \sum_{st\in E} d_{G^*}(st)(T_B(st), I_B(st), F_B(st)),$$
$$\sum_{st\in E}(q_1, q_2, q_3) = q\sum_{st\in E}(T_B(st), I_B(st), F_B(st)),$$
$$m(q_1, q_2, q_3) = qS(G),$$
$$(mq_1, mq_2, mq_3) = qS(G),$$
$$S(G) = \left(\frac{mq_1}{q}, \frac{mq_2}{q}, \frac{mq_3}{q}\right).$$

This completes the proof.

Theorem 1.4 *Let* $G^* = (X, E)$ *be an edge regular crisp graph of degree* q *and* G *be a totally edge regular neutrosophic graph of degree* (p_1, p_2, p_3) *of* G^*. *Then the size of* G *is* $(\frac{mp_1}{q+1}, \frac{mp_2}{q+1}, \frac{mp_3}{q+1})$, *where* $|E| = m$.

Proof Let G be a totally edge regular neutrosophic graph of an edge regular crisp graph $G^* = (X, E)$. Therefore,

$$d_G(st) = (p_1, p_2, p_3) \text{ and } d_{G^*}(st) = q, \quad \text{for each edge } st \in E.$$

Since,

$$\begin{aligned}
\sum_{st\in E} Td_G(st) &= \sum_{st\in E} d_{G^*}(st)(T_B(st), I_B(st), F_B(st)) + S(G),\\
\sum_{st\in E}(p_1, p_2, p_3) &= q\sum_{st\in E}(T_B(st), I_B(st), F_B(st)) + S(G),\\
m(p_1, p_2, p_3) &= qS(G) + S(G),\\
(mp_1, mp_2, mp_3) &= (q+1)S(G),\\
S(G) &= \left(\frac{mp_1}{q+1}, \frac{mp_2}{q+1}, \frac{mp_3}{q+1}\right).
\end{aligned}$$

This completes the proof.

Theorem 1.5 *Suppose that G is an edge regular neutrosophic graph of degree (q_1, q_2, q_3) and a totally edge regular neutrosophic graph of degree (p_1, p_2, p_3) of G^*. Then, the size of G is $m(p_1 - q_1, p_2 - q_2, p_3 - q_3)$, where $|E| = m$.*

Proof Let G be an edge regular neutrosophic graph and a totally edge regular neutrosophic graph of a crisp graph $G^* = (X, E)$. Therefore,

$$d_G(st) = (q_1, q_2, q_3) \text{ and } Td_G(st) = (p_1, p_2, p_3), \quad \text{for each edge } st \in E.$$

$$\begin{aligned}
Td_G(st) &= d_G(st) + (T_B(st), I_B(st), F_B(st)),\\
\sum_{st\in E} Td_G(st) &= \sum_{st\in E} d_G(st) + \sum_{st\in E}(T_B(st), I_B(st), F_B(st)),\\
m(p_1, p_2, p_3) &= m(q_1, q_2, q_3) + S(G),\\
S(G) &= m(p_1 - q_1, p_2 - q_2, p_3 - q_3).
\end{aligned}$$

This completes the proof.

Theorem 1.6 *Let $G^* = (X, E)$ be a crisp graph, which is a cycle on m vertices. Suppose that G be a neutrosophic graph of G^*. Then $\sum_{s_k\in X} d_G(s_k) = \sum_{s_ks_l\in E} d_G(s_ks_l)$.*

Proof Let G be a neutrosophic graph of G^*. Suppose that G^* be a cycle $s_1, s_2, s_3, \dots, s_m, s_1$ on m vertices. Then

$$\begin{aligned}
\sum_{s_ks_l\in E} d_G(s_ks_l) &= d_G(s_1s_2) + d_G(s_2s_3) + \cdots + d_G(s_ms_1),\\
&= [d_G(s_1) + d_G(s_2) - 2(T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2))][d_G(s_2)\\
&\quad + d_G(s_3) - 2(T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3))] + \cdots + [d_G(s_m)\\
&\quad + d_G(s_1) - 2(T_B(s_ms_1), I_B(s_ms_1), F_B(s_ms_1))],\\
&= 2d_G(s_1) + 2d_G(s_2) + \cdots + 2d_G(s_m) - 2(T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)),\\
&\quad -2(T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)) - \cdots - 2(T_B(s_ms_1), I_B(s_ms_1), F_B(s_ms_1)),\\
&= 2\sum_{s_k\in X} d_G(s_k) - 2\sum_{s_ks_l\in E}(T_B(s_ks_l), I_B(s_ks_l), F_B(s_ks_l)),\\
&= \sum_{s_k\in X} d_G(s_k) + \sum_{s_k\in X} d_G(s_k) - 2\sum_{s_ks_l\in E}(T_B(s_ks_l), I_B(s_ks_l), F_B(s_ks_l)),
\end{aligned}$$

$$= \sum_{s_k \in X} d_G(s_k) + 2 \sum_{s_k s_l \in E} (T_B(s_k s_l), I_B(s_k s_l), F_B(s_k s_l))$$
$$-2 \sum_{s_k s_l \in E} (T_B(s_k s_l), I_B(s_k s_l), F_B(s_k s_l)),$$
$$= \sum_{s_k \in X} d_G(s_k).$$

This completes the proof.

Theorem 1.7 *Let G be a neutrosophic graph. Then B is a constant function if and only if the following statements are equivalent:*

(a) G is an edge regular neutrosophic graph.
(b) G is a totally edge regular neutrosophic graph.

Proof Let G be a neutrosophic graph. Suppose that B is a constant function, then

$$T_B(st) = l_1, \quad I_B(st) = l_2, \quad F_B(st) = l_3, \quad \text{for all } st \in E.$$

(a) ⇒ (b): Assume that G is an edge regular neutrosophic graph, i.e.

$$d_G(st) = (q_1, q_2, q_3), \quad \text{for each edge } st \in E.$$

This implies that

$$Td_G(st) = (l_1 + q_1, l_2 + q_2, l_3 + q_3) \text{ for each edge } st \in E.$$

This shows that G is an edge regular neutrosophic graph of degree

$$(l_1 + q_1, l_2 + q_2, l_3 + q_3).$$

(b) ⇒ (a): Suppose that G is a totally edge regular neutrosophic graph, i.e.

$$Td_G(st) = (p_1, p_2, p_3) \text{ for all } st \in E.$$

This implies that

$$d_G(st) + (T_B(st), I_B(st), F_B(st)) = (p_1, p_2, p_3).$$

This implies that

$$d_G(st) = (p_1, p_2, p_3) - 4(T_B(st), I_B(st), F_B(st)).$$

This implies that

$$d_G(st) = (p_1 - l_1, p_2 - l_2, p_3 - l_3) \text{ for each edge } st \in E.$$

Thus G is an edge regular neutrosophic graph of degree

$$(p_1 - l_1, p_2 - l_2, p_3 - l_3).$$

Hence the statements (a) and (b) are equivalent.

Conversely, suppose that (a) and (b) are equivalent. Assume that B is not a constant function. This implies that

$$(T_B(st), I_B(st), F_B(st)) \neq (T_B(uv), I_B(uv), F_B(uv)),$$

for at least one pair of edges $st, uv \in E$.

Assume that G is an edge regular neutrosophic graph. This implies that

$$d_G(st) = d_G(uv) = (q_1, q_2, q_3).$$

This implies that

$$Td_G(st) = d_G(st) + (T_B(st), I_B(st), F_B(st)) = (q_1, q_2, q_3) + (T_B(st), I_B(st), F_B(st)),$$

$$Td_G(uv) = d_G(uv) + (T_B(uv), I_B(uv), F_B(uv)) = (q_1, q_2, q_3) + (T_B(uv), I_B(uv), F_B(uv)).$$

Since

$$(T_B(st), I_B(st), F_B(st)) \neq (T_B(uv), I_B(uv), F_B(uv)).$$

This implies that $Td_G(st) \neq Td_G(uv)$. This shows that G is not a totally edge regular neutrosophic graph, which contradicts our supposition.

Now, suppose that G is a totally edge regular neutrosophic graph, i.e.

$$Td_G(st) = Td_G(uv) = (p_1, p_2, p_3).$$

This implies that

$$Td_G(st) = d_G(st) + (T_B(st), I_B(st), F_B(st)) = d_G(uv) + (T_B(uv), I_B(uv), F_B(uv)).$$

This implies that

$$d_G(st) - d_G(uv) = (T_B(st), I_B(st), F_B(st)) - (T_B(uv), I_B(uv), F_B(uv)).$$

Since

$$(T_B(st), I_B(st), F_B(st)) \neq (T_B(uv), I_B(uv), F_B(uv)).$$

This implies that $d_G(st) - d_G(uv) \neq 0$. This implies that $d_G(st) \neq d_G(uv)$. This shows that G is not an edge regular neutrosophic graph, which contradicts our supposition. Hence B is a constant function.

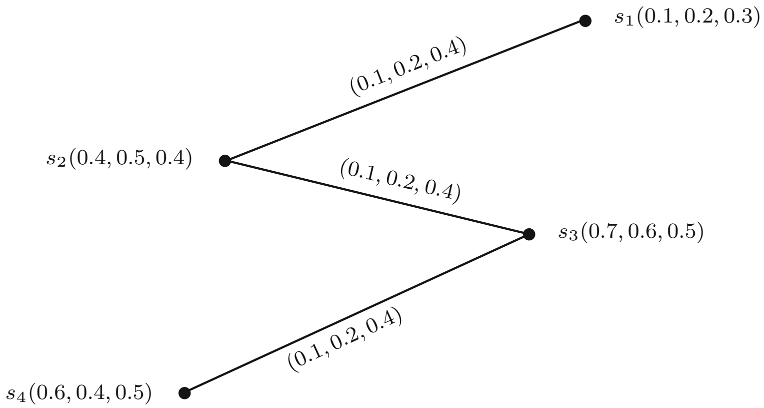


Fig. 1.14 Neutrosophic graph

Theorem 1.8 *Let G be a neutrosophic graph. Assume that G is both edge regular neutrosophic of degree* (q_1, q_2, q_3) *and totally edge regular neutrosophic graph of degree* (p_1, p_2, p_3)*. Then B is a constant function.*

Proof The proof is obvious.

Remark 1.6 The converse of Theorem 1.8 may not be true in general; that is, a neutrosophic graph G, where B is a constant function, may or may not be edge regular and totally edge regular neutrosophic graph.

Example 1.16 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.14.

By direct calculations, we have

$$d_G(s_1) = (0.1, 0.2, 0.4), \quad d_G(s_2) = (0.2, 0.4, 0.8),$$

$$d_G(s_3) = (0.2, 0.4, 0.8), \quad d_G(s_4) = (0.1, 0.2, 0.4).$$

The degree of each edge is

$$d_G(s_1s_2) = (0.1, 0.2, 0.4), \quad d_G(s_2s_3) = (0.2, 0.4, 0.8), \quad d_G(s_3s_4) = (0.1, 0.2, 0.4).$$

The total degree of each edge is

$$Td_G(s_1s_2) = (0.2, 0.4, 0.8), \quad Td_G(s_2s_3) = (0.3, 0.6, 1.2).$$

It is clear from above calculations that G is neither an edge regular nor a totally edge regular neutrosophic graph.

Theorem 1.9 *Let G be a neutrosophic graph of* $G^* = (X, E)$*, where B is a constant function. If G is a regular neutrosophic graph, then G is an edge regular neutrosophic graph.*

Proof Assume that B is a constant function, that is,

$$T_B(st) = l_1, \quad I_B(st) = l_2, \quad F_B(st) = l_3 \text{ for all } st \in E.$$

Suppose that G is a regular neutrosophic graph, that is,

$$d_G(s) = (m_1, m_2, m_3) \text{ for all } s \in X.$$

Now

$$\begin{aligned} d_G(st) &= d_G(s) + d_G(t) - 2(T_B(st), I_B(st), F_B(st)), \\ &= (m_1, m_2, m_3) + (m_1, m_2, m_3) - 2(l_1, l_2, l_3), \\ &= 2(m_1 - l_1, m_2 - l_2, m_3 - l_3), \end{aligned}$$

for all $st \in E$. Hence G is an edge regular neutrosophic graph.

Theorem 1.10 *Let* $G = (A, B)$ *be a neutrosophic graph of* $G^* = (X, E)$*, where B is a constant function. If G is a regular neutrosophic graph, then G is a totally edge regular neutrosophic graph.*

Proof Let B be a constant function, that is,

$$T_B(st) = l_1, \quad I_B(st) = l_2, \quad F_B(st) = l_3 \text{ for all } st \in E.$$

Assume that G is a regular neutrosophic graph, that is,

$$d_G(s) = (m_1, m_2, m_3), \quad \text{for all } s \in X.$$

Then G is an edge regular neutrosophic graph, that is,

$$d_G(st) = (q_1, q_2, q_3).$$

Now

$$\begin{aligned} Td_G(st) &= d_G(st) + (T_B(st), I_B(st), F_B(st)), \\ &= (q_1, q_2, q_3) + (l_1, l_2, l_3), \\ &= 2(q_1 + l_1, q_2 + l_2, q_3 + l_3), \end{aligned}$$

for all $st \in E$. Hence G is a totally edge regular neutrosophic graph.

Theorem 1.11 *Suppose that G is a neutrosophic graph. Then G is both regular and totally edge regular neutrosophic graph if and only if B is a constant function.*

Proof Let $G^* = (X, E)$ be a regular crisp graph. Suppose that G is a neutrosophic graph of G^*. Suppose that G is both regular and totally edge regular neutrosophic graph, that is,

$$d_G(s) = (m_1, m_2, m_3), \quad \text{for all } s \in X,$$
$$Td_G(st) = (p_1, p_2, p_3), \quad \text{for all } st \in E.$$

Now

$$\begin{aligned} Td_G(st) &= d_G(s) + d_G(t) - (T_B(st), I_B(st), F_B(st)), \quad \forall \ st \in E, \\ (p_1, p_2, p_3) &= (m_1, m_2, m_3) + (m_1, m_2, m_3) - (T_B(st), I_B(st), F_B(st)), \\ (T_B(st), I_B(st), F_B(st)) &= (2m_1 - p_1, 2m_2 - p_2, 2m_3 - p_3), \end{aligned}$$

for all $st \in E$. Hence B is a constant function.

Conversely, let B be a constant function, that is,

$$T_B(st) = l_1, \quad I_B(st) = l_2, \quad F_B(st) = l_3, \quad \text{for all } st \in E.$$

So

$$\begin{aligned} d_G(s) &= \sum_{st \in E} (T_B(st), I_B(st), F_B(st)), \quad \forall \ s \in X, \\ &= \sum_{st \in E} (m_1, m_2, m_3), \\ &= (m_1, m_2, m_3) d_{G^*}(s), \\ &= (m_1, m_2, m_3) m. \end{aligned}$$

This implies that

$$d_G(s) = (mm_1, mm_2, mm_3), \quad \text{for all } s \in E.$$

Thus G is a regular neutrosophic graph. Now

$$\begin{aligned} Td_G(st) = &\sum_{sa \in E, s \neq a} (T_B(sa), I_B(sa), F_B(sa)) + \sum_{at \in E, a \neq t} (T_B(at), I_B(at), F_B(at)), \\ &+ (T_B(st), I_B(st), F_B(st)) \quad \forall st \in E, \\ = &\sum_{sa \in E, s \neq a} (l_1, l_2, l_3) + \sum_{at \in E, a \neq t} (l_1, l_2, l_3) + (l_1, l_2, l_3), \\ = &(l_1, l_2, l_3)(d_{G^*}(s) - 1) + (l_1, l_2, l_3)(d_{G^*}(t) - 1) + (l_1, l_2, l_3), \\ = &(l_1, l_2, l_3)(s - 1) + (l_1, l_2, l_3)(t - 1) + (l_1, l_2, l_3), \\ = &(2l_1, 2l_2, 2l_3)(s - 1) + (l_1, l_2, l_3), \end{aligned}$$

for all $st \in E$. Hence G is a totally edge regular neutrosophic graph.

Theorem 1.12 *Let $G^* = (X, E)$ be a crisp graph. Suppose that $G = (A, B)$ is a neutrosophic graph of G^*. Then B is a constant function if and only if G is an edge regular neutrosophic graph.*

Proof Let G be a regular neutrosophic graph, that is,

$$d_G(s) = (m_1, m_2, m_3), \quad \text{for all } \ s \in X.$$

Suppose that B is a constant function, that is,

$$T_B(st) = l_1, \quad I_B(st) = l_2, \quad F_B(st) = l_3, \quad \text{for all } \ st \in E.$$

Now

$$\begin{aligned} d_G(st) &= d_G(s) + d_G(t) - 2(T_B(st), I_B(st), F_B(st)), \quad \forall \ st \in E. \\ &= (m_1, m_2, m_3) + (m_1, m_2, m_3) - 2(l_1, l_2, l_3), \end{aligned}$$

this implies that

$$d_G(st) = 2(m_1, m_2, m_3) - 2(l_1, l_2, l_3), \quad \text{for all } \ st \in E.$$

Hence G is an edge regular neutrosophic graph.

Conversely, assume that G is an edge regular neutrosophic graph, that is,

$$d_G(st) = (q_1, q_2, q_3), \quad \text{for each edge } \ st \in E.$$

Now

$$\begin{aligned} d_G(st) &= d_G(s) + d_G(t) - 2(T_B(st), I_B(st), F_B(st)), \quad \forall \ st \in E, \\ (q_1, q_2, q_3) &= (m_1, m_2, m_3) + (m_1, m_2, m_3) - 2(T_B(st), I_B(st), F_B(st)), \end{aligned}$$

this implies that

$$(T_B(st), I_B(st), F_B(st)) = \frac{(q_1, q_2, q_3) - (2m_1, 2m_2, 2m_3)}{2}, \quad \text{for all } \ st \in E.$$

Thus B is a constant function.

Definition 1.19 Let G^* be an edge regular crisp graph. Then a neutrosophic graph G of G^* is called a *partially edge regular*.

Example 1.17 It can be seen in Example 1.15 that G^* is an edge regular crisp graph. Therefore, G is a partially edge regular neutrosophic graph.

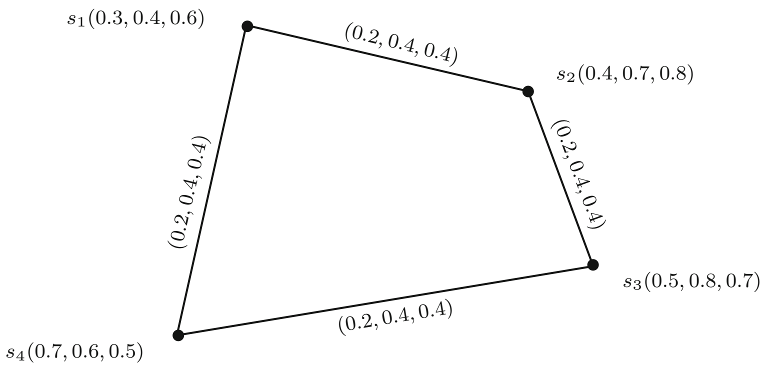


Fig. 1.15 Full edge regular neutrosophic graph

Definition 1.20 A neutrosophic graph G is called a *full edge regular* if it is both edge regular and partially edge regular.

Example 1.18 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.15.

By direct calculations, we have

$$d_G(s_1) = (0.4, 0.8, 0.8), \quad d_G(s_2) = (0.4, 0.8, 0.8),$$

$$d_G(s_3) = (0.4, 0.8, 0.8), \quad d_G(s_4) = (0.4, 0.8, 0.8).$$

The degree of each edge is

$$d_G(s_1s_2) = (0.4, 0.8, 0.8), \quad d_G(s_2s_3) = (0.4, 0.8, 0.8)$$

$$d_G(s_3s_4) = (0.4, 0.8, 0.8), \quad d_G(s_1s_4) = (0.4, 0.8, 0.8).$$

It is clear from calculations that G is full edge regular neutrosophic graph.

Theorem 1.13 *Let G be a neutrosophic graph, where B is a constant function. Then G is full edge regular neutrosophic graph if it is full regular neutrosophic graph.*

Proof Let G be a neutrosophic graph of a crisp graph $G^* = (X, E)$. Suppose that B is a constant function, that is,

$$(T_B(st), I_B(st), F_B(st)) = (l_1, l_2, l_3), \quad \text{for each edge } st \in E.$$

Assume that G is full regular neutrosophic graph. Then G is both regular and partially regular. Therefore,

$$d_G(s) = (m_1, m_2, m_3) \text{ and } d_{G^*}(s) = m, \quad \text{for all } s \in X.$$

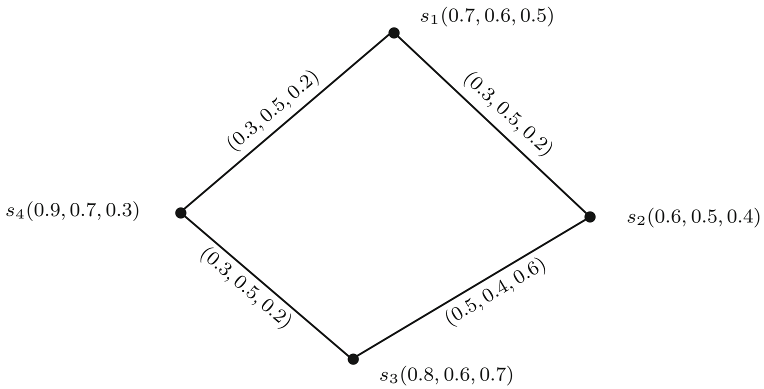


Fig. 1.16 Irregular neutrosophic graph

Since

$$d_{G^*}(st) = d_{G^*}(s) + d_{G^*}(t) - 2, \quad \text{for all } \ st \in E.$$

This shows that $d_{G^*}(st) = 2m - 2$. Therefore, G^* is an edge regular neutrosophic graph. Now

$$\begin{aligned} d_G(st) &= d_G(s) + d_G(t) - 2(T_B(st), I_B(st), F_B(st)), \quad \forall \ st \in E. \\ &= (m_1, m_2, m_3) + (m_1, m_2, m_3) - 2(l_1, l_2, l_3), \end{aligned}$$

this implies that

$$d_G(st) = 2(m_1 - l_1, m_2 - l_2, m_3 - l_3).$$

This shows that G is an edge regular neutrosophic graph. Hence G is a full edge regular neutrosophic graph.

Definition 1.21 A neutrosophic graph G is called an *irregular* if there exists a vertex which is adjacent to vertices with distinct degrees.

Example 1.19 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.16.

By direct calculations, we have

$$d_G(s_1) = (0.6, 1.0, 0.4), \quad d_G(s_2) = (0.8, 0.9, 0.8),$$

$$d_G(s_3) = (0.8, 0.9, 0.8), \quad d_G(s_4) = (0.6, 1.0, 0.4).$$

It is easy to see that s_1 is adjacent to vertices of distinct degrees. Therefore, G is an irregular neutrosophic graph.

Definition 1.22 A neutrosophic graph G is called a *totally irregular* if there exists a vertex which is adjacent to vertices with distinct total degrees.

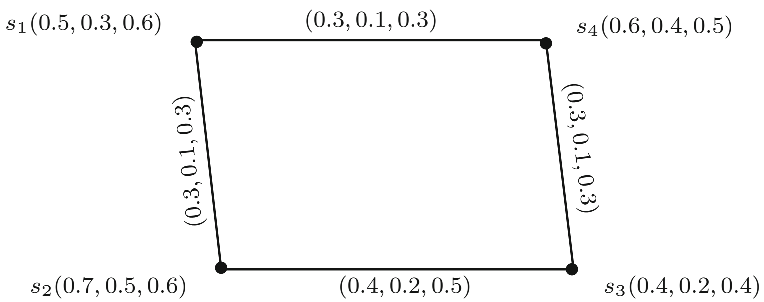


Fig. 1.17 Totally irregular neutrosophic graph

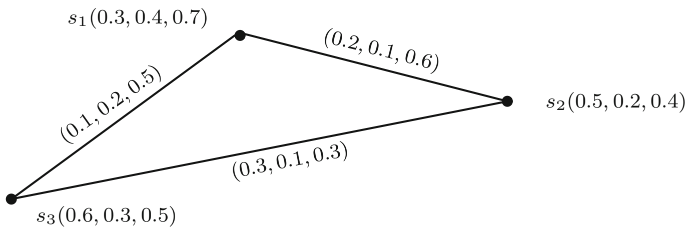


Fig. 1.18 Strongly irregular neutrosophic graph

Example 1.20 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.17.

By direct calculations, we have

$$Td_G(s_1) = (1.1, 0.5, 1.2), \quad Td_G(s_2) = (1.4, 0.8, 1.4),$$

$$Td_G(s_3) = (1.1, 0.5, 1.2), \quad Td_G(s_4) = (1.2, 0.6, 1.1).$$

It is easy to see that s_1 is adjacent to vertices of distinct total degrees. Therefore, G is a totally irregular neutrosophic graph.

Definition 1.23 A neutrosophic graph G is called *strongly irregular* if each vertex has distinct degree.

Example 1.21 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3\}$ as shown in Fig. 1.18.

By direct calculations, we have

$$d_G(s_1) = (0.3, 0.3, 1.1), \quad d_G(s_2) = (0.5, 0.2, 0.9), \quad d_G(s_3) = (0.4, 0.3, 0.8).$$

From Fig. 1.18, it is clear that each vertex has distinct degree. Therefore, G is a strongly irregular neutrosophic graph.

Definition 1.24 A neutrosophic graph G is called *strongly totally irregular neutrosophic graph* if each vertex has distinct total degree.

Example 1.22 Consider the neutrosophic graph G as shown in Fig. 1.18. By direct calculations, we have

$$Td(s_1) = (0.6, 0.7, 1.8), \quad Td(s_2) = (1.0, 0.4, 1.3), \quad Td(s_3) = (1.0, 0.6, 1.3).$$

Since each vertex has distinct total degree, G is a strongly totally irregular neutrosophic graph.

Definition 1.25 A neutrosophic graph G is called *highly irregular* if each vertex in G is adjacent to vertices having distinct degrees.

Example 1.23 Consider the neutrosophic graph G as shown in Fig. 1.16. It is easy to see that each vertex is adjacent to vertices of distinct degree; therefore, G is highly irregular neutrosophic graph.

Definition 1.26 A neutrosophic graph G is called *highly totally irregular* if each vertex in G is adjacent to vertices having distinct total degrees.

Example 1.24 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.19.

By direct calculations, we have

$$Td_G(s_1) = (0.8, 0.8, 0.7), \quad Td_G(s_2) = (0.3, 0.4, 0.7),$$

$$Td_G(s_3) = (0.7, 1.0, 1.1), \quad Td_G(s_4) = (1.1, 1.1, 0.7).$$

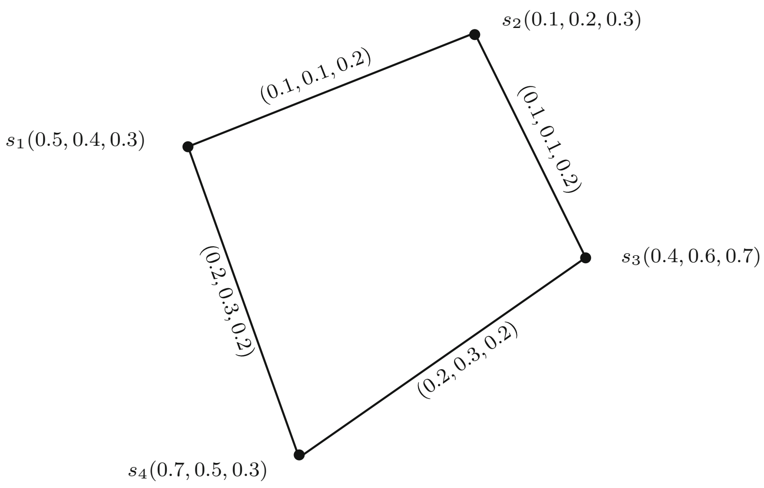


Fig. 1.19 Highly totally irregular neutrosophic graph

From Fig. 1.19, it is clear that each vertex is adjacent to vertices of distinct degrees. Therefore, G is highly totally irregular neutrosophic graph.

Definition 1.27 A connected neutrosophic graph G is called *neighbourly edge irregular* if every two adjacent edges in G have distinct degrees.

Example 1.25 Consider the neutrosophic graph G as shown in Fig. 1.18. It is easy to see that every two adjacent edges in G have distinct degrees; therefore, G is neighbourly edge irregular neutrosophic graph.

Definition 1.28 A connected neutrosophic graph G is called *neighbourly edge totally irregular neutrosophic graph* if every two adjacent edges in G have distinct total degrees.

Example 1.26 Consider the neutrosophic graph G as shown in Fig. 1.18. It is easy to see that every two adjacent edges in G have distinct total degrees; therefore, G is neighbourly edge totally irregular neutrosophic graph.

Definition 1.29 Let G^* be a crisp graph. A neutrosophic graph G of G^* is called a *strongly edge irregular neutrosophic graph* if each edge in G has distinct degree; that is, no two edges in G have the same degree.

Example 1.27 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3\}$ as shown in Fig. 1.20.

By direct calculations, we have

$$d_G(s_1) = (0.8, 0.8, 0.4), \quad d_G(s_2) = (0.6, 0.3, 0.4), \quad d_G(s_3) = (0.8, 0.7, 0.2).$$

- The degree of each edge is given as:

$$\begin{aligned} d_G(s_1s_2) &= d_G(s_1) + d_G(s_2) - 2(T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)), \\ &= (0.8, 0.8, 0.4) + (0.6, 0.3, 0.4) - 2(0.3, 0.2, 0.3), \\ &= (0.8, 0.7, 0.2). \end{aligned}$$

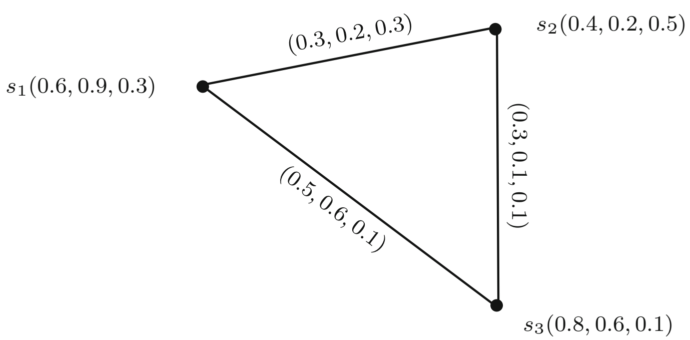


Fig. 1.20 Strongly edge irregular neutrosophic graph

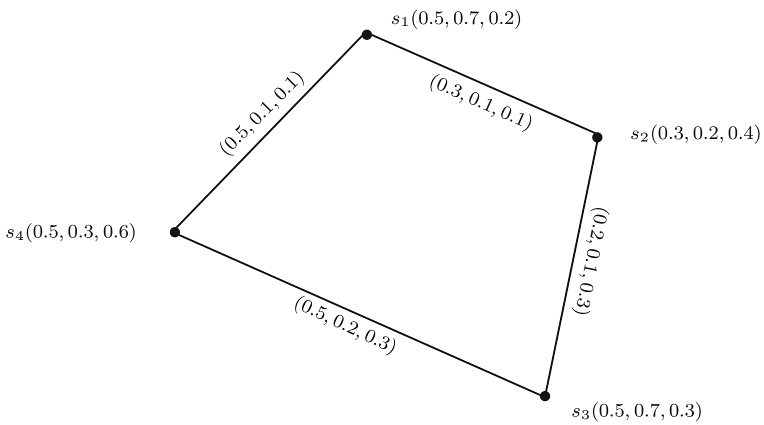


Fig. 1.21 Strongly edge totally irregular neutrosophic graph

$$\begin{aligned} d_G(s_1s_3) &= d_G(s_1) + d_G(s_3) - 2(T_B(s_1s_3), I_B(s_1s_3), F_B(s_1s_3)),\\ &= (0.8, 0.8, 0.4) + (0.8, 0.7, 0.2) - 2(0.5, 0.6, 0.1),\\ &= (0.6, 0.3, 0.4). \end{aligned}$$

$$\begin{aligned} d_G(s_2s_3) &= d_G(s_2) + d_G(s_3) - 2(T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)),\\ &= (0.6, 0.3, 0.4) + (0.8, 0.7, 0.2) - 2(0.3, 0.1, 0.1),\\ &= (0.8, 0.8, 0.4). \end{aligned}$$

Since no two edges in G have the same degree, G is a strongly edge irregular neutrosophic graph.

Definition 1.30 A neutrosophic graph G is called a strongly edge totally irregular neutrosophic graph if each edge in G has distinct total degree; that is, no two edges in G have the same total degree.

Example 1.28 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.21.

By direct calculations, we have

$$d_G(s_1) = (0.8, 0.2, 0.2), \quad d_G(s_2) = (0.5, 0.2, 0.4),$$

$$d_G(s_3) = (0.7, 0.3, 0.6), \quad d_G(s_4) = (1.0, 0.3, 0.4).$$

- The degree of each edge is given as:

$$\begin{aligned} d_G(s_1s_2) &= d_G(s_1) + d_G(s_2) - 2(T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)),\\ &= (0.8, 0.2, 0.2) + (0.5, 0.2, 0.4) - 2(0.3, 0.1, 0.1),\\ &= (0.7, 0.2, 0.4). \end{aligned}$$

$$\begin{aligned} d_G(s_1s_4) &= d_G(s_1) + d_G(s_4) - 2(T_B(s_1s_4), I_B(s_1s_4), F_B(s_1s_4)), \\ &= (0.8, 0.2, 0.2) + (1.0, 0.3, 0.4) - 2(0.5, 0.1, 0.1), \\ &= (0.8, 0.3, 0.4). \end{aligned}$$

$$\begin{aligned} d_G(s_2s_3) &= d_G(s_2) + d_G(s_3) - 2(T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)), \\ &= (0.5, 0.2, 0.4) + (0.7, 0.3, 0.6) - 2(0.2, 0.1, 0.3), \\ &= (0.8, 0.3, 0.4). \end{aligned}$$

$$\begin{aligned} d_G(s_3s_4) &= d_G(s_3) + d_G(s_4) - 2(T_B(s_3s_4), I_B(s_3s_4), F_B(s_3s_4)), \\ &= (0.7, 0.3, 0.6) + (1.0, 0.3, 0.4) - 2(0.5, 0.2, 0.3), \\ &= (0.7, 0.2, 0.4). \end{aligned}$$

- The total degree of each edge is given as:

$$\begin{aligned} Td_G(s_1s_2) &= d_G(s_1s_2) + (T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)), \\ &= (0.7, 0.2, 0.4) + (0.3, 0.1, 0.1), \\ &= (1.0, 0.3, 0.5). \end{aligned}$$

$$\begin{aligned} Td_G(s_1s_4) &= d_G(s_1s_4) + (T_B(s_1s_4), I_B(s_1s_4), F_B(s_1s_4)), \\ &= (0.8, 0.3, 0.4) + (0.5, 0.1, 0.1), \\ &= (1.3, 0.4, 0.5). \end{aligned}$$

$$\begin{aligned} Td_G(s_2s_3) &= d_G(s_2s_3) + (T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)), \\ &= (0.8, 0.3, 0.4) + (0.2, 0.1, 0.3), \\ &= (1.0, 0.4, 0.7). \end{aligned}$$

$$\begin{aligned} Td_G(s_3s_4) &= d_G(s_3s_4) + (T_B(s_3s_4), I_B(s_3s_4), F_B(s_3s_4)), \\ &= (0.7, 0.2, 0.4) + (0.5, 0.2, 0.3), \\ &= (1.2, 0.4, 0.7). \end{aligned}$$

Since no two edges in G have the same total degree, G is a strongly edge totally irregular neutrosophic graph.

Remark 1.7 A strongly edge irregular neutrosophic graph G may not be strongly edge totally irregular neutrosophic graph.

Example 1.29 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3\}$ as shown in Fig. 1.22.

By direct calculations, we have

$$d_G(s_1) = (1.1, 0.5, 0.7), \quad d_G(s_2) = (0.7, 0.4, 0.9), \quad d_G(s_3) = (1.0, 0.3, 0.6).$$

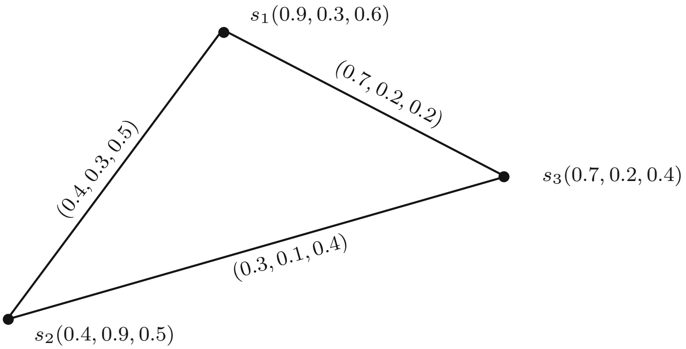


Fig. 1.22 Strongly edge irregular neutrosophic graph

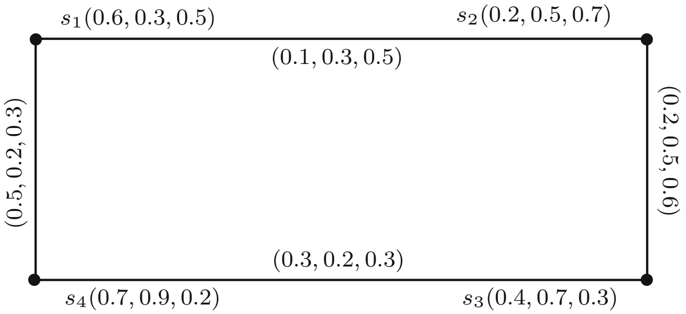


Fig. 1.23 Strongly edge totally irregular neutrosophic graph

The degree of each edge is

$$d_G(s_1s_2) = (1.0, 0.3, 0.6), \quad d_G(s_2s_3) = (1.1, 0.5, 0.7), \quad d_G(s_1s_3) = (0.7, 0.4, 0.9).$$

Since all the edges have distinct degrees, G is a strongly edge irregular neutrosophic graph. The total degree of each edge is

$$Td_G(s_1s_2) = (1.4, 0.6, 1.1) = Td_G(s_2s_3) = Td_G(s_1s_3).$$

Since each edge of G has the same total degree therefore G is not a strongly edge totally irregular neutrosophic graph.

Remark 1.8 A strongly edge totally irregular neutrosophic graph G may not be strongly edge irregular neutrosophic graph.

Example 1.30 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.23.

By direct calculations, we have

$$d_G(s_1) = (0.6, 0.5, 0.8), \quad d_G(s_2) = (0.3, 0.8, 1.1),$$

$$d_G(s_3) = (0.5, 0.7, 0.9), \quad d_G(s_4) = (0.8, 0.4, 0.6).$$

The degree of each edge is

$$d_G(s_1s_2) = (0.7, 0.7, 0.9), \quad d_G(s_2s_3) = (0.4, 0.5, 0.8),$$

$$d_G(s_3s_4) = (0.7, 0.7, 0.9), \quad d_G(s_1s_4) = (0.4, 0.5, 0.8).$$

It is easy to see that $d_G(s_1s_2) = d_G(s_3s_4)$ and $d_G(s_2s_3) = d_G(s_1s_4)$.

Therefore, G is not a strongly edge irregular neutrosophic graph.

The total degree of each edge is

$$Td_G(s_1s_2) = (0.8, 1.0, 1.4), \quad Td_G(s_2s_3) = (0.6, 1.0, 1.4),$$

$$Td_G(s_3s_4) = (1.0, 0.9, 1.2), \quad Td_G(s_1s_4) = (0.9, 0.7, 1.1).$$

Since all the edges have distinct total degrees, G is a strongly edge totally irregular neutrosophic graph.

Theorem 1.14 *If G is a strongly edge irregular connected neutrosophic graph, where B is a constant function, then G is a strongly edge totally irregular neutrosophic graph.*

Proof Let G be a strongly edge irregular connected neutrosophic graph. Assume that B is a constant function. Then

$$T_B(xy) = l_1, \quad I_B(xy) = l_2, \quad F_B(xy) = l_3, \quad \text{for all } xy \in E,$$

where l_1, l_2 and l_3 are constants. Consider a pair of edges xy and uv in E.

Since G is a strongly edge irregular neutrosophic graph,

$$d_G(xy) \neq d_G(uv),$$

where xy and uv are a pair of edges in E. This shows that

$$d_G(xy) + (l_1, l_2, l_3) \neq d_G(uv) + (l_1, l_2, l_3).$$

This implies that

$$d_G(xy) + (T_B(xy), I_B(xy), F_B(xy)) \neq d_G(uv) + (T_B(uv), I_B(uv), F_B(uv)).$$

Thus

$$Td_G(xy) \neq Td_G(uv),$$

where xy and uv are a pair of edges in E. Since the pair of edges xy and uv were taken to be arbitrary, this shows that every pair of edges in G have distinct total degrees.

Hence G is a strongly edge totally irregular neutrosophic graph.

Theorem 1.15 *If G is a strongly edge totally irregular connected neutrosophic graph, where B is a constant function, then G is a strongly edge irregular neutrosophic graph.*

Proof Let G be a strongly edge totally irregular connected neutrosophic graph. Assume that B is a constant function. Then

$$T_B(xy) = l_1, \quad I_B(xy) = l_2 \text{ and } F_B(xy) = l_3, \quad \text{for all } xy \in E,$$

where l_1, l_2 and l_3 are constants. Consider a pair of edges xy and uv in L.

Since G is a strongly edge totally irregular neutrosophic graph,

$$Td_G(xy) \neq Td_G(uv),$$

where xy and uv are a pair of edges in E. This shows that

$$d_G(xy) + (T_B(xy), I_B(xy), F_B(xy)) \neq d_G(uv) + (T_B(uv), I_B(uv), F_B(uv)).$$

This implies that

$$d_G(xy) + (l_1, l_2, l_3) \neq d_G(uv) + (l_1, l_2, l_3).$$

Thus

$$d_G(xy) \neq d_G(uv),$$

where xy and uv are a pair of edges in E. Since the pair of edges xy and uv were taken to be arbitrary, this shows that every pair of edges in G have distinct degrees.

Hence G is a strongly edge irregular neutrosophic graph.

Remark 1.9 If G is both strongly edge irregular neutrosophic graph and strongly edge totally irregular neutrosophic graph, then it is not necessary that B is a constant function.

Example 1.31 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4, s_4\}$ as shown in Fig. 1.24.

By direct calculations, we have

$$d_G(s_1) = (0.6, 0.4, 0.4), \quad d_G(s_2) = (0.3, 0.7, 0.6), \quad d_G(s_3) = (0.3, 0.8, 0.6),$$

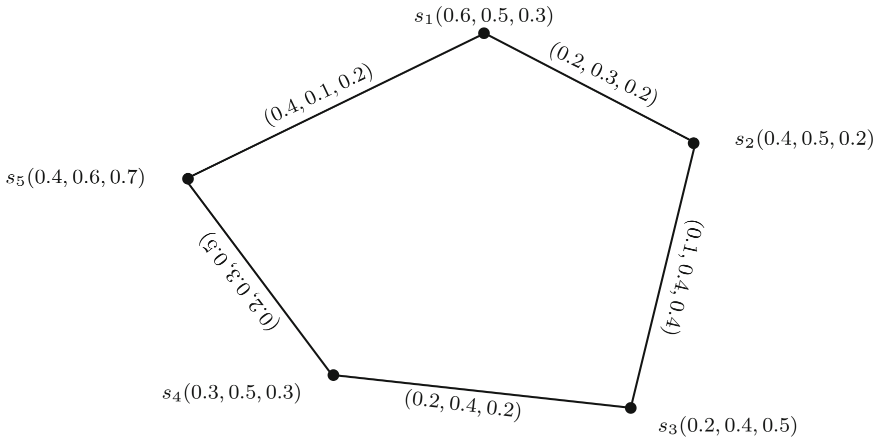


Fig. 1.24 Neutrosophic graph

$$d_G(s_4) = (0.4, 0.7, 0.7), \quad d_G(s_5) = (0.6, 0.4, 0.7).$$

The degree of each edge is

$$d_G(s_1s_2) = (0.5, 0.5, 0.6), \quad d_G(s_2s_3) = (0.4, 0.7, 0.4), \quad d_G(s_3s_4) = (0.3, 0.7, 0.9),$$

$$d_G(s_4s_5) = (0.6, 0.5, 0.4), \quad d_G(s_5s_1) = (0.4, 0.6, 0.7).$$

It is easy to see that all the edges have distinct degrees. Therefore, G is a strongly edge irregular neutrosophic graph.

The total degree of each edge is

$$Td_G(s_1s_2) = (0.7, 0.8, 0.8), \quad Td_G(s_2s_3) = (0.5, 1.1, 0.8), \quad Td_G(s_3s_4) = (0.5, 1.1, 1.1),$$

$$Td_G(s_4s_5) = (0.8, 0.8, 0.9), \quad Td_G(s_5s_1) = (0.8, 0.7, 0.9).$$

Since all the edges have distinct total degrees, G is a strongly edge totally irregular neutrosophic graph. This shows that G is both strongly edge irregular neutrosophic graph and strongly edge totally irregular neutrosophic graph, but B is not a constant function.

Theorem 1.16 *Let G be a strongly edge irregular neutrosophic graph. Then G is a neighbourly edge irregular neutrosophic graph.*

Proof Suppose that G is a strongly edge irregular neutrosophic graph. Then each edge in G has distinct degree. This shows that every pair of edges in G have distinct degrees. Therefore, G is a neighbourly edge irregular neutrosophic graph.

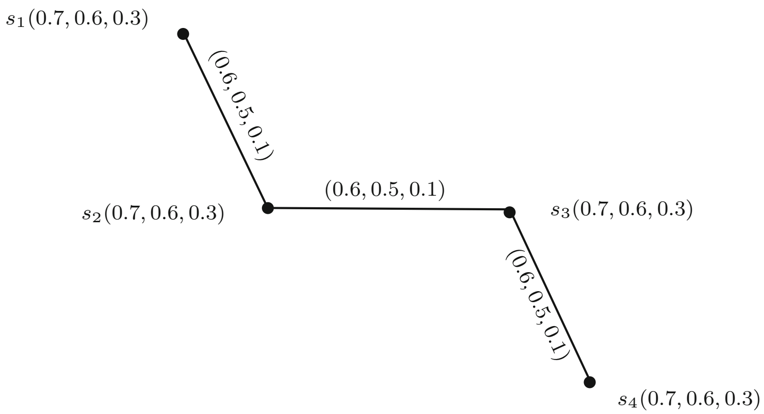


Fig. 1.25 Neutrosophic graph

Theorem 1.17 *Let G be a strongly edge totally irregular neutrosophic graph. Then G is a neighbourly edge totally irregular neutrosophic graph.*

Proof Suppose that G is a strongly edge totally irregular neutrosophic graph. Then each edge in G has distinct total degree. This shows that every pair of edges in G have distinct total degrees. Therefore, G is a neighbourly edge totally irregular neutrosophic graph.

Remark 1.10 If G is a neighbourly edge irregular neutrosophic graph, then it is not necessary that G is a strongly edge irregular neutrosophic graph.

Example 1.32 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.25.

By direct calculations, we have

$$d_G(s_1) = (0.6, 0.5, 0.1), \quad d_G(s_2) = (1.2, 1.0, 0.2),$$

$$d_G(s_3) = (1.2, 1.0, 0.2), \quad d_G(s_4) = (0.6, 0.5, 0.1).$$

The degree of each edge is

$$d_G(s_1s_2) = (0.6, 0.5, 0.1), \quad d_G(s_2s_3) = (1.2, 1.0, 0.2), \quad d_G(s_3s_4) = (0.6, 0.5, 0.1).$$

G is neighbourly edge irregular neutrosophic graph since every two adjacent edges in G have distinct total degrees, that is,

$$d_G(s_1s_2) \neq d_G(s_2s_3) \text{ and } d_G(s_2s_3) \neq d_G(s_3s_4).$$

It is easy to see that $d_G(s_1s_2) = d_G(s_3s_4)$. Therefore, G is not a strongly edge irregular neutrosophic graph.

Remark 1.11 If G is a neighbourly edge totally irregular neutrosophic graph, then it is not necessary that G is a strongly edge totally irregular neutrosophic graph.

Example 1.33 Consider the neutrosophic graph G as shown in Fig. 1.25. The total degree of each edge is

$$Td_G(s_1s_2) = (1.2, 1.0, 0.2), \quad Td_G(s_2s_3) = (1.8, 1.5, 0.3), \quad Td_G(s_1s_2) = (1.2, 1.0, 0.2).$$

It is easy to see that every two adjacent edges in G have distinct total degrees, that is,

$$Td_G(s_1s_2) \neq Td_G(s_2s_3), \quad \text{and} \quad Td_G(s_2s_3) \neq Td_G(s_3s_4).$$

Therefore, G is neighbourly edge totally irregular neutrosophic graph. It is easy to see that $Td_G(s_1s_2) = Td_G(s_3s_4)$. Hence G is not a strongly edge totally irregular neutrosophic graph.

Theorem 1.18 *Let G be a strongly edge irregular connected neutrosophic graph, with B as constant function. Then G is an irregular neutrosophic graph.*

Proof Let G be a strongly edge irregular connected neutrosophic graph, with B as constant function. Then

$$T_B(xy) = l_1, \quad I_B(xy) = l_2, \quad F_B(xy) = l_3, \quad \text{for each edge } xy \in E,$$

where l_1, l_2 and l_3 are constants. Also, every edge in G has distinct degrees, since G is strongly edge irregular neutrosophic graph.

Let xy and yu be any two adjacent edges in G such that

$$d_G(xy) \neq d_G(yu).$$

This implies that

$$d_G(x) + d_G(y) - 2(T_B(xy), I_B(xy), F_B(xy)) \neq d_G(y) + d_G(u) - 2(T_B(yu), I_B(yu), F_B(yu)).$$

This implies that

$$d_G(x) + d_G(y) - 2(l_1, l_2, l_3) \neq d_G(y) + d_G(u) - 2(l_1, l_2, l_3).$$

This shows that

$$d_G(x) \neq d_G(u).$$

Thus there exists a vertex y in G which is adjacent to the vertices with distinct degrees. This shows that G is an irregular neutrosophic graph.

Theorem 1.19 *Let G be a strongly edge totally irregular connected neutrosophic graph, with B as constant function. Then G is an irregular neutrosophic graph.*

Proof Let G be a strongly edge totally irregular connected neutrosophic graph, with B as constant function. Then

$$T_B(xy) = l_1, \quad I_B(xy) = l_2, \quad F_B(xy) = l_3, \quad \text{for each edge } \; xy \in E,$$

where l_1, l_2 and l_3 are constants. Also, every edge in G has distinct total degrees, since G is strongly edge totally irregular neutrosophic graph.

Let xy and yu be any two adjacent edges in G such that

$$Td_G(xy) \neq Td_G(yu).$$

This implies that

$$d_G(xy) + (T_B(xy), I_B(xy), F_B(xy)) \neq d_G(yu) + (T_B(yu), I_B(yu), F_B(yu)).$$

This implies that

$$d_G(x) + d_G(y) - (T_B(xy), I_B(xy), F_B(xy)) \neq d_G(y) + d_G(u) - (T_B(yu), I_B(yu), F_B(yu)).$$

This implies that

$$d_G(x) + d_G(y) - 2(l_1, l_2, l_3) \neq d_G(y) + d_G(u) - 2(l_1, l_2, l_3).$$

This shows that

$$d_G(x) \neq d_G(u).$$

Thus there exists a vertex y in G which is adjacent to the vertices with distinct degrees. This shows that G is an irregular neutrosophic graph.

Remark 1.12 If G is an irregular neutrosophic graph, with B as a constant function. Then it is not necessary that G is a strongly edge irregular neutrosophic graph.

Example 1.34 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.26.

By direct calculations, we have

$$d_G(s_1) = (0.8, 0.2, 0.6), \quad d_G(s_2) = (1.2, 0.3, 0.9),$$

$$d_G(s_3) = (0.8, 0.2, 0.6), \quad d_G(s_4) = (1.2, 0.3, 0.9).$$

The degree of each edge is

$$d_G(s_1s_2) = (1.2, 0.3, 0.9), \quad d_G(s_2s_3) = (1.2, 0.3, 0.9), \quad d_G(s_2s_4) = (1.6, 0.4, 1.2),$$

$$d_G(s_3s_4) = (1.2, 0.3, 0.9), \quad d_G(s_1s_4) = (1.2, 0.3, 0.9).$$

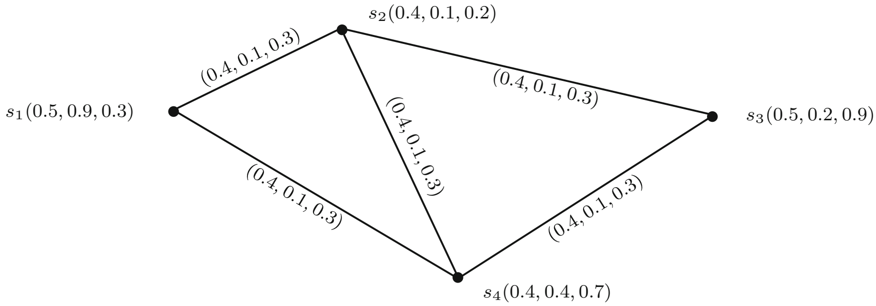


Fig. 1.26 Irregular neutrosophic graph

It is easy to see that all the edges have the same degree except the edge s_2s_4. Therefore, G is not a strongly edge irregular neutrosophic graph.

Remark 1.13 If G is an irregular neutrosophic graph, with B as a constant function. Then it is not necessary that G is a strongly edge totally irregular neutrosophic graph.

Example 1.35 Consider the neutrosophic graph G as shown in Fig. 1.26. The total degree of each edge is

$$Td_G(s_1s_2) = (1.6, 0.4, 1.2),\quad Td_G(s_2s_3) = (1.6, 0.4, 1.2),\quad Td_G(s_2s_4) = (2.0, 0.5, 1.5),$$

$$Td_G(s_3s_4) = (1.6, 0.4, 1.2),\quad Td_G(s_1s_4) = (1.6, 0.4, 1.2).$$

It is easy to see that all the edges have the same total degree except the edge s_2s_4. Therefore, G is not a strongly edge totally irregular neutrosophic graph.

Theorem 1.20 *Let G be a strongly edge irregular connected neutrosophic graph, with B as a constant function. Then G is highly irregular neutrosophic graph.*

Proof Let G be a strongly edge irregular connected neutrosophic graph, with B as a constant function. Then

$$T_B(xy) = l_1,\quad I_B(xy) = l_2,\quad F_B(xy) = l_3,\quad \text{for each edge } xy \in E,$$

where l_1, l_2 and l_3 are constants. Also every pair of adjacent edges in G have distinct degrees.

Let y be any vertex in G which is adjacent to vertices y and u. Since G is strongly edge irregular neutrosophic graph,

$$d_G(xy) \neq d_G(yu).$$

This implies that

$$d_G(x) + d_G(y) - 2(T_B(xy), I_B(xy), F_B(xy)) \neq d_G(y) + d_G(u) - 2(T_B(yu), I_B(yu), F_B(yu)).$$

This implies that

$$d_G(x) + d_G(y) - 2(l_1, l_2, l_3) \neq d_G(y) + d_G(u) - 2(l_1, l_2, l_3).$$

This shows that

$$d_G(x) \neq d_G(u).$$

Thus there exists a vertex y in G which is adjacent to the vertices with distinct degrees. Since y was taken to be an arbitrary vertex in G, all the vertices in G are adjacent to vertices having distinct degrees. Hence G is a highly irregular neutrosophic graph.

Theorem 1.21 *Let G be a strongly edge totally irregular connected neutrosophic graph, with B as a constant function. Then G is highly irregular neutrosophic graph.*

Proof Let G be a strongly edge totally irregular connected neutrosophic graph, with B as a constant function. Then

$$T_B(xy) = l_1, \quad I_B(xy) = l_2, \quad F_B(xy) = l_3, \quad \text{for each edge } xy \in E,$$

where l_1, l_2 and l_3 are constants. Also every pair of adjacent edges in G have distinct total degrees.

Let y be any vertex in G which is adjacent to vertices x and u. Since G is strongly edge totally irregular neutrosophic graph therefore,

$$Td_G(xy) \neq Td_G(yu).$$

This implies that

$$d_G(xy) \neq d_G(yu).$$

This implies that

$$d_G(x) + d_G(y) - 2(T_B(xy), I_B(xy), F_B(xy)) \neq d_G(y) + d_G(u) - 2(T_B(yu), I_B(yu), F_B(yu)).$$

This implies that

$$d_G(x) + d_G(y) - 2(l_1, l_2, l_3) \neq d_G(y) + d_G(u) - 2(l_1, l_2, l_3).$$

This shows that

$$d_G(x) \neq d_G(u).$$

Thus there exists a vertex y in G which is adjacent to the vertices with distinct degrees. Since y was taken to be an arbitrary vertex in G, therefore all the vertices

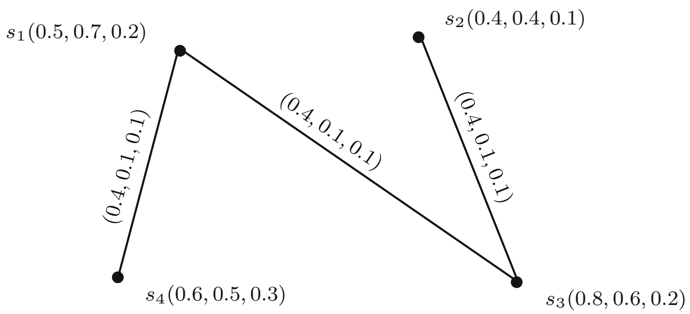


Fig. 1.27 Highly irregular neutrosophic graph

in G are adjacent to vertices having distinct degrees. Hence G is a highly irregular neutrosophic graph.

Remark 1.14 If G is a highly irregular neutrosophic graph, with B as a constant function. Then it is not necessary that G is strongly edge irregular neutrosophic graph.

Example 1.36 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.27.

By direct calculations, we have

$$d_G(s_1) = (0.8, 0.2, 0.2), \quad d_G(s_2) = (0.4, 0.1, 0.1),$$

$$d_G(s_3) = (0.8, 0.2, 0.2), \quad d_G(s_4) = (0.4, 0.1, 0.1).$$

The degree of each edge is

$$d_G(s_1s_3) = (0.8, 0.2, 0.2), \quad d_G(s_1s_4) = (0.4, 0.1, 0.1), \quad d_G(s_2s_3) = (0.4, 0.1, 0.1).$$

Since every vertex is adjacent to vertices with distinct degrees, G is a highly irregular neutrosophic graph. Since the edges s_1s_4 and s_2s_3 in G have the same degree, i.e. $d_G(s_1s_4) = d_G(s_2s_3)$, G is not strongly edge irregular neutrosophic graph.

Remark 1.15 If G is a highly irregular neutrosophic graph, with B as a constant function. Then it is not necessary that G is strongly edge totally irregular neutrosophic graph.

Example 1.37 Consider the neutrosophic graph G as shown in Fig. 1.27. The total degree of each edge is

$$Td_G(s_1s_3) = (1.2, 0.3, 0.3), \quad Td_G(s_1s_4) = (0.8, 0.2, 0.2), \quad Td_G(s_2s_3) = (0.8, 0.2, 0.2).$$

Since the edges s_1s_4 and s_2s_3 in G have the same total degree, G is not a strongly edge totally irregular neutrosophic graph.

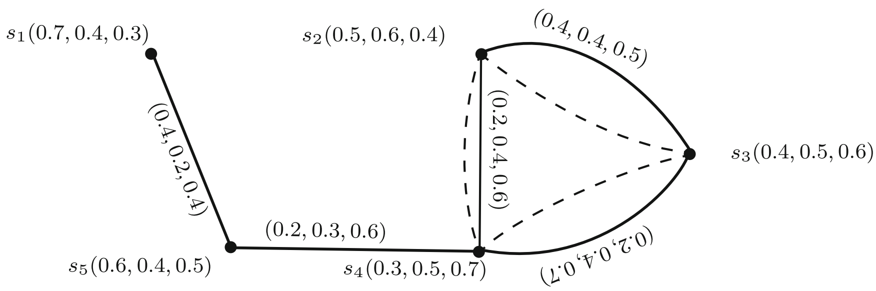


Fig. 1.28 Neutrosophic graph

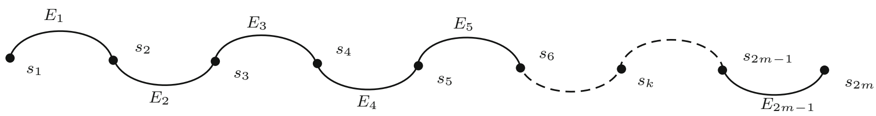


Fig. 1.29 Neutrosophic path P

Definition 1.31 A neutrosophic path is a sequence of distinct vertices $x = x_1, x_2, x_3, \ldots, x_n = y$ such that, for all k, $T_B(x_k x_{k+1}) > 0$, $I_B(x_k x_{k+1}) > 0$ and $F_B(x_k x_{k+1}) > 0$. A neutrosophic path is called a *neutrosophic cycle* if $x = y$.

Example 1.38 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4, s_5\}$ as shown in Fig. 1.28.

The path from s_2 to s_1 is shown with thick lines, and the cycle C from s_2 to s_2 is shown with dashed lines in Fig. 1.28.

Theorem 1.22 *Let $G^* = (X, E)$ be a path as shown in Fig. 1.29 on $2m (m > 1)$ vertices and G be a neutrosophic graph. Let $E_1, E_2, E_3, \ldots, E_{2m-1}$ be the edges in G having $c_1, c_2, c_3, \ldots, c_{2m-1}$ as their membership values, respectively. Assume that $c_1 < c_2 < c_3 < \cdots < c_{2m-1}$, where $c_k = (T_k, I_k, F_k)$, $k = 1, 2, 3, \ldots, 2m-1$. Then G is both strongly edge irregular and strongly edge totally irregular neutrosophic graph.*

Proof Let G be a neutrosophic graph of a crisp graph $G^* = (X, E)$. Assume that G is a neutrosophic path on $2m (m > 1)$ vertices. Suppose that $c_k = (T_k, I_k, F_k)$ be the membership values of the edges L_k in G, where $k = 1, 2, 3, \ldots, 2m-1$. We assume that $c_1 < c_2 < c_3 < \cdots < c_{2m-1}$.

The degree of each vertex in G is calculated as:

$$\begin{aligned}
d_G(s_1) &= c_1 = (T_1, I_1, F_1), \quad for \quad k = 1.\\
d_G(s_k) &= c_{k-1} + c_k = (T_{k-1}, I_{k-1}, F_{k-1}) + (T_k, I_k, F_k),\\
&= (T_{k-1} + T_k, I_{k-1} + I_k, F_{k-1} + F_k), \quad for \quad k = 2, 3, \ldots, 2m-1.\\
d_G(s_{2m}) &= c_{2m-1} = (T_{2m-1}, I_{2m-1}, F_{2m-1}), \quad for \quad k = 2m.
\end{aligned}$$

The degree of each edge in G is calculated as:

$$\begin{aligned}
d_G(E_1) &= c_2 = (T_2, I_2, F_2), \quad for \quad k = 1.\\
d_G(L_k) &= c_{k-1} + c_{k+1} = (T_{k-1}, I_{k-1}, F_{k-1}) + (t_{k+1}, i_{k+1}, f_{k+1}),\\
&= (T_{k-1} + T_{k+1}, I_{k-1} + I_{k+1}, F_{k-1} + F_{k+1}), \quad for \quad k = 2, 3, \ldots, 2m-2.\\
d_G(L_{2m-1}) &= c_{2m-2} = (T_{2m-2}, I_{2m-2}, F_{2m-2}), \quad for \quad k = 2m-1.
\end{aligned}$$

Since each edge in G has distinct degree, G is strongly edge irregular neutrosophic graph. We now calculate the total degree of each edge in G as:

$$\begin{aligned}
Td_G(E_1) &= c_1 + c_2 = (T_1 + T_2, I_1 + I_2, F_1 + F_2), \quad for \quad k = 1.\\
Td_G(L_k) &= c_{k-1} + c_k + c_{k+1} = (T_{k-1}, I_{k-1}, F_{k-1}) + (T_k, I_k, F_k) + (T_{k+1}, I_{k+1}, F_{k+1}),\\
&= (T_{k-1} + T_k + T_{k+1}, I_{k-1} + I_k + I_{k+1}, F_{k-1} + F_k + F_{k+1}),\\
&\quad for \quad k = 2, 3, \ldots, 2m-2.\\
Td_G(L_{2m-1}) &= c_{2m-2} + c_{2m-1} = (T_{2m-2}, I_{2m-2}, F_{2m-2}) + (T_{2m-1}, I_{2m-1}, F_{2m-1}),\\
&= (T_{2m-2} + T_{2m-1}, I_{2m-2} + I_{2m-1}, F_{2m-2} + F_{2m-1}), \quad for \quad k = 2m-1.
\end{aligned}$$

Since each edge in G has distinct total degree, G is strongly edge totally irregular neutrosophic graph. Hence G is both strongly edge irregular and strongly edge totally irregular neutrosophic graph.

Definition 1.32 A complete bipartite graph is a graph whose vertex set can be partitioned into two subsets X_1 and X_2 such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is the part of the graph. A complete bipartite graph with partition of size $|X_1| = m$ and $|X_2| = n$ is denoted by $K_{(m,n)}$. A complete bipartite graph $K_{(1,n)}$ or $K_{(m,1)}$ that is a tree with one internal vertex and n or m leaves is called a star S_n or S_m.

Theorem 1.23 *Let $G^* = (X, E)$ be a star $K_{(m,1)}$ as shown in Fig. 1.30 and G be a neutrosophic graph of G^*. If each edge in G has distinct membership values, then G is strongly edge irregular neutrosophic graph but not strongly edge totally irregular neutrosophic graph.*

Proof Let G be a neutrosophic graph of a crisp graph $G^* = (X, E)$. We assume that G is a star $K_{(m,1)}$. Let $s, s_1, s_2, \ldots, s_m$ be the vertices of the star $K_{(m,1)}$, where s is the centre vertex and $s_1, s_2, \ldots, s_m$ are the vertices adjacent to vertex s as shown in Fig. 1.30. Suppose that $c_k = (T_k, I_k, F_k)$ be the membership values of the edges E_k in G, where $k = 1, 2, \ldots, m$. We assume that $c_1 \neq c_2 \neq c_3 \neq \cdots \neq c_m$. The degree of each edge in G is calculated as:

$$\begin{aligned}
d_G(L_k) &= d_G(x) + d_G(s_k) - 2(T_B(ss_k), I_B(ss_k), F_B(ss_k)),\\
&= (c_1, c_2, \ldots, c_m) + (T_k, I_k, F_k) - 2(T_k, I_k, F_k),\\
&= (T_1, I_1, F_1), (T_2, I_2, F_2), \ldots, (T_m, I_m, F_m) + (T_k, I_k, F_k) - 2(T_k, I_k, F_k),\\
&= (T_1 + T_2 + \cdots + T_m, I_1 + I_2 + \cdots + I_m, F_1 + F_2 + \cdots + F_m) - (T_k, T_k, T_k).
\end{aligned}$$

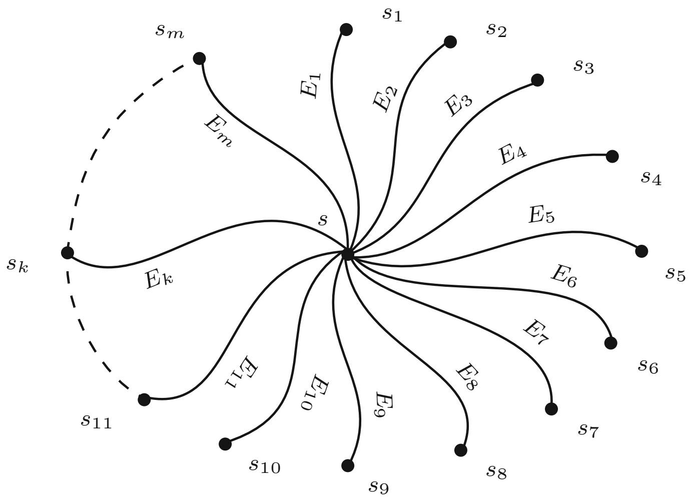


Fig. 1.30 Neutrosophic graph

It is easy to see that each edge in G has distinct degree; therefore, G is strongly edge irregular neutrosophic graph. We now calculate the total degree of each edge in G as:

$$\begin{aligned} Td_G(L_k) &= Td_G(x) + Td_G(s_k) - (T_B(ss_k), I_B(ss_k), F_B(ss_k)), \\ &= (c_1, c_2, \ldots, c_m) + (T_k, I_k, F_k)(T_k, I_k, F_k), \\ &= (T_1, I_1, F_1), (T_2, I_2, F_2), \ldots, (T_m, I_m, F_m), \\ &= (T_1 + T_2 + \cdots + T_m, I_1 + I_2 + \cdots + I_m, F_1 + F_2 + \cdots + F_m). \end{aligned}$$

Since all the edges in G have the same total degree, G is not a strongly edge totally irregular neutrosophic graph

Definition 1.33 The m-barbell graph $B_{(m,m)}$ is the simple graph obtained by connecting two copies of a complete graph K_m by a bridge.

Theorem 1.24 *Let G be a neutrosophic graph of $G^* = (X, E)$, the m-barbell graph $B_{(m,m)}$ as shown in Fig. 1.31. If each edge in G has distinct membership values, then G is a strongly edge irregular neutrosophic graph but not a strongly edge totally irregular neutrosophic graph.*

Proof Let G be a neutrosophic graph of a crisp graph $G^* = (X, E)$. Suppose that G^* is a m-barbell graph, then there exists a bridge, say xy, connecting m new vertices to each of its end vertices x and y. Let $b = (T, I, F)$ be the membership values of the bridge xy. Suppose that $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_m$ are the vertices adjacent to vertices x and y, respectively. Let $c_k = (T_k, I_k, F_k)$ be the membership values of the edges E_k with vertex x, where $k = 1, 2, \ldots, m$ and $a_1 < a_2 < \cdots < a_m$. Let

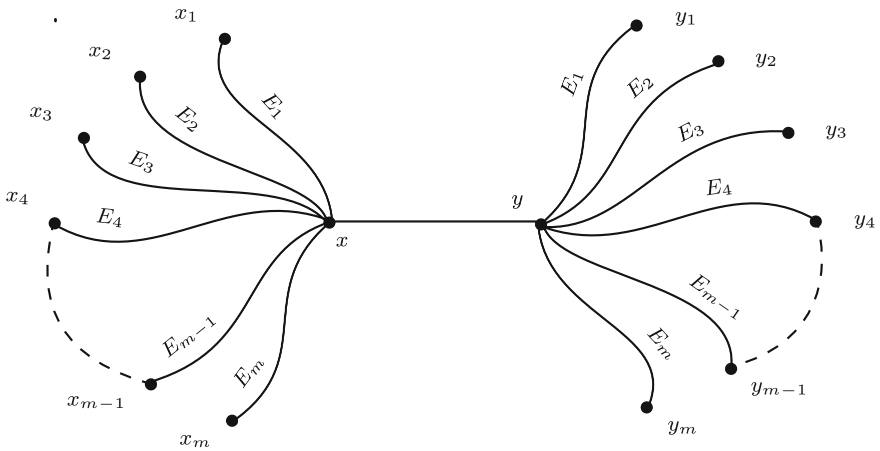


Fig. 1.31 Neutrosophic graph

$c_k^{'} = (T_k^{'}, I_k^{'}, F_k^{'})$ be the membership values of the edges E_k with vertex y, where $k = 1, 2, \ldots, m$ and $c_1 < c_2 < \cdots < c_m$. Assume that $c_1 < c_2 < \cdots < c_m < c_1^{'} < c_2^{'} < \cdots < c_m^{'} < b$. The degree of each edge in G is calculated as:

$$
\begin{aligned}
d_G(xy) &= d_G(x) + d_G(y) - 2b,\\
&= c_1 + c_2 + \cdots + c_m + b + c_1^{'} + c_2^{'} + \cdots + c_m^{'} + b - 2b,\\
&= (T_1, I_1, F_1) + (T_2, I_2, F_2) + \cdots + (T_m, I_m, F_m) + (T_1^{'}, I_1^{'}, F_1^{'}) + (T_2^{'}, I_2^{'}, F_2^{'})\\
&\quad + \cdots + (T_m^{'}, I_m^{'}, F_m^{'}),\\
&= (T_1 + T_2 + \cdots + T_m, I_1 + I_2 + \cdots + I_m, F_1 + F_2 + \cdots + F_m)\\
&\quad +(T_1^{'} + T_2^{'} + \cdots + T_m^{'}, I_1^{'} + I_2^{'} + \cdots + I_m^{'}, F_1^{'} + F_2^{'} + \cdots + F_m^{'}).
\end{aligned}
$$

$$
\begin{aligned}
d_G(L_k) &= d_G(x) + d_G(x_k) - 2c_k, \quad where \quad k = 1, 2, \ldots, m.\\
&= c_1 + c_2 + \cdots + c_m + b + c_k - 2c_k,\\
&= (T_1, I_1, F_1) + (T_2, I_2, F_2) + \cdots + (T_m, I_m, F_m) + (T, I, F) - b_k,\\
&= (T_1 + T_2 + \cdots + T_m + T, I_1 + I_2 + \cdots + I_m + I, F_1 + F_2 + \cdots + F_m + F)\\
&\quad -(T_k, I_k, F_k).
\end{aligned}
$$

$$
\begin{aligned}
d_G(E_k) &= d_G(y) + d_G(y_k) - 2c_k^{'}, \quad where \quad k = 1, 2, \ldots, m.\\
&= c_1^{'} + c_2^{'} + \cdots + c_m^{'} + b + c_k^{'} - 2c_k^{'},\\
&= (T_1^{'}, I_1^{'}, F_1^{'}) + (T_2^{'}, I_2^{'}, F_2^{'}) + \cdots + (T_m^{'}, I_m^{'}, F_m^{'}) + (T, I, F) - c_k^{'},\\
&= (T_1^{'} + T_2^{'} + \cdots + T_m^{'} + t, I_1^{'} + I_2^{'} + \cdots + I_m^{'} + i, F_1^{'} + F_2^{'} + \cdots + F_m^{'} + f)\\
&\quad -(T_k^{'}, I_k^{'}, F_k^{'}).
\end{aligned}
$$

It is easy to see that all the edges in G have distinct degrees; therefore, G is strongly edge irregular neutrosophic graph. The total degree of each edge in G is calculated as:

$$
\begin{aligned}
Td_G(xy) &= d_G(xy) + b,\\
&= c_1 + c_2 + \cdots + c_m + c_1^{'} + c_2^{'} + \cdots + c_m^{'} + b,\\
&= (T_1, I_1, F_1) + (T_2, I_2, F_2) + \cdots + (T_m, I_m, F_m)\\
&\quad +(T_1^{'}, I_1^{'}, F_1^{'}) + (T_2^{'}, I_2^{'}, F_2^{'}) + \cdots + (T_m^{'}, I_m^{'}, F_m^{'}) + (T, I, F),\\
&= (T_1 + T_2 + \cdots + T_m, I_1 + I_2 + \cdots + I_m, F_1 + F_2 + \cdots + F_m)\\
&\quad +(T_1^{'} + T_2^{'} + \cdots + T_m^{'}, I_1^{'} + I_2^{'} + \cdots + I_m^{'}, F_1^{'} + F_2^{'} + \cdots + F_m^{'}) + (T, I, F).\\
Td_G(L_k) &= d_G(L_k) + c_k, \quad where \quad k = 1, 2, \ldots, m.\\
&= c_1 + c_2 + \cdots + c_m + b + c_k - 2c_k + c_k,\\
&= (T_1, I_1, F_1) + (T_2, I_2, F_2) + \cdots + (T_m, I_m, F_m) + (T, I, F),\\
&= (T_1 + T_2 + \cdots + T_m + T, I_1 + I_2 + \cdots + I_m + I, F_1 + F_2 + \cdots + F_m + F).\\
Td_G(E_k) &= d_G(E_k) + c_k^{'}, \quad where \quad k = 1, 2, \ldots, m.\\
&= c_1^{'} + c_2^{'} + \cdots + c_m^{'} + b + c_k^{'} - 2c_k^{'} + c_k^{'},\\
&= (T_1^{'}, I_1^{'}, F_1^{'}) + (T_2^{'}, I_2^{'}, F_2^{'}) + \cdots + (T_m^{'}, I_m^{'}, F_m^{'}) + (T, I, F),\\
&= (T_1^{'} + T_2^{'} + \cdots + T_m^{'} + T, I_1^{'} + I_2^{'} + \cdots + I_m^{'} + I, F_1^{'} + F_2^{'} + \cdots + F_m^{'} + F).
\end{aligned}
$$

Since each edge L_k and E_k in G has the same total degree, where $k = 1, 2, \ldots, m$, G is not a strongly edge totally irregular neutrosophic graph.

1.3 Applications of Neutrosophic Graphs

1.3.1 Social Network Model

Graphical models have many applications in our daily life. Human being is the most adjustable and adapting creature. When human beings interact with each other, more or less they leave an impact(good or bad) on each other. Naturally a human being has influence on others. We can use neutrosophic digraph to examine the influence of the people on each other's thinking in a group. We can investigate a person's good influence and bad influence on the thinking of others. We can examine the percentage of uncertain influence of that person. The neutrosophic digraph will tell us about dominating person and about highly influenced person.

Consider $I = \{$Malik, Haider, Imran, Razi, Ali, Hamza, Aziz$\}$ set of seven persons in a social group on whatsapp. Let $A = \{$(Malik, 0.6, 0.4, 0.5), (Haider, 0.5, 0.6, 0.3), (Imran, 0.4, 0.3, 0.2), (Razi, 0.7, 0.6, 0.4), (Ali, 0.4, 0.1, 0.2), (Hamza, 0.6,

Table 1.5 Neutrosophic set B of edges

Edge	T	I	F
(Hamza, Malik)	0.6	0.4	0.4
(Hamza, Haider)	0.5	0.3	0.3
(Hamza, Razi)	0.3	0.3	0.4
(Hamza, Aziz)	0.3	0.3	0.4
(Malik, Haider)	0.5	0.4	0.5
(Imran, Haider)	0.4	0.3	0.3
(Aziz, Malik)	0.5	0.2	0.5
(Razi, Imran)	0.3	0.3	0.4
(Razi, Ali)	0.4	0.1	0.4
(Ali, Aziz)	0.3	0.1	0.5

0.4, 0.1), (Aziz, 0.7, 0.3, 0.5)} be the neutrosophic set on the set I where truth value of each person represents his good influence on others, falsity value represents his bad influence on others, and indeterminacy value represents uncertainty in his influence. Let $J = \{$(Hamza, Malik), (Hamza, Haider), (Hamza, Razi), (Hamza, Aziz), (Malik, Haider), (Imran, Haider), (Aziz, Malik), (Razi, Imran), (Razi, Ali), (Ali, Aziz)$\}$ be the set of relations on I. Let B be the neutrosophic set on the set J as shown in Table 1.5.

The truth, indeterminacy and falsity values of each edge are calculated using $T_B(xy) \leq T_A(x) \wedge T_A(y), I_B(xy) \leq I_A(x) \wedge I_A(y), F_B(xy) \leq F_A(x) \vee F_A(y)$. The neutrosophic digraph $G = (A, B)$ is shown in Fig. 1.32. This neutrosophic digraph shows that Hamza has influence on Malik, Haider, Razi and Aziz. We can see that Hamza's good influence on Haider is 50%, on Malik is 60%, on Razi is 30% and on Aziz is 30%. His bad influence on Haider, Malik, Razi and Aziz is 30, 40, 40 and 40%, respectively. Similarly his uncertain influence on Haider, Malik, Razi and Aziz is 30, 40, 30 and 30%, respectively. We can investigate that out-degree of vertex

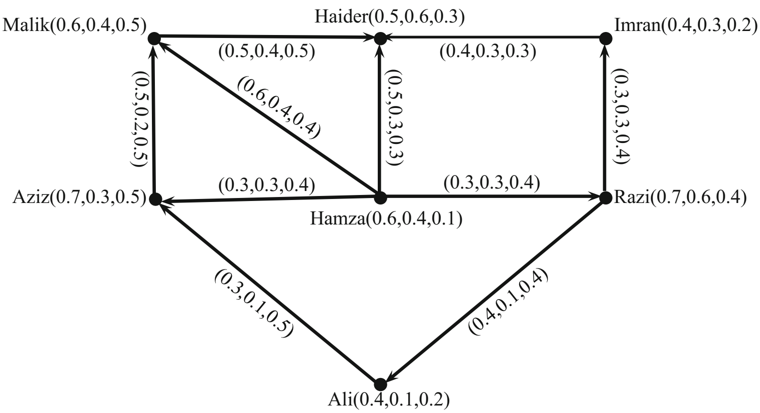


Fig. 1.32 Neutrosophic digraph

Hamza is highest, that is, four. This shows that Hamza is dominating person in this social group. On the other hand, Haider has highest in-degree, that is, three. It tells us that Haider is highly influenced by others in this social group.

We now explain general procedure of this applications through following Algorithm 1.3.1.

Algorithm 1.3.1

Step 1. Input the set of vertices $I = \{I_1, I_2, \ldots, I_n\}$ and a neutrosophic set A which is defined on set I.
Step 2. Input the set of relations $J = \{J_1, J_2, \ldots, J_n\}$.
Step 3. Compute the truth-membership degree, indeterminacy degree and falsity-membership degree of each edge using Definition 1.7.
Step 4. Compute the neutrosophic set B of edges.
Step 5. Obtain a neutrosophic digraph $G = (A, B)$.

1.3.2 Detection of a Safe Root for an Airline Journey

We consider a neutrosophic set of five countries: Germany, China, USA, Brazil and Mexico. Suppose we want to travel between these countries through an airline journey. The airline companies aim to facilitate their passengers with high quality of services. Air traffic controllers have to make sure that company planes must arrive and depart at right time. This task is possible by planning efficient routes for the planes. A neutrosophic graph of airline network among these five countries is shown in Fig. 1.33 in which vertices and edges represent the countries and flights, respectively.

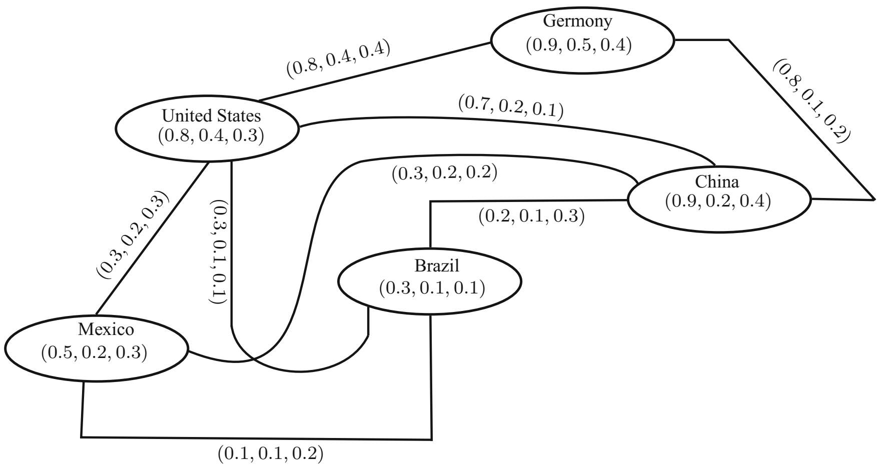


Fig. 1.33 Neutrosophic graph of an airline network

The truth-membership degree of each vertex indicates the strength of that country's airline system. The indeterminacy-membership degree of each vertex demonstrates how much the system is uncertain. The falsity-membership degree of each vertex tells the flaws of that system. The truth-membership degree of each edge interprets that how much the flight is save. The indeterminacy-membership degree of each edge shows the uncertain situations during a flight such as weather conditions, mechanical error and sabotage. The falsity-membership degree of each edge indicates the flaws of that flight. For example, the edge between Germany and China indicates that the flight chosen for this travel is 80% safe, 10% depending on uncertain systems and 20% unsafe. The truth-membership degree, the indeterminacy-membership degree and the falsity-membership degree of each edge are calculated by using the following relations.

$$T_B(xy) \leq \min\{T_A(x), T_A(y)\},$$
$$I_B(xy) \leq \min\{I_A(x), I_A(y)\},$$
$$F_B(xy) \leq \max\{F_A(x), F_A(y)\}, \quad x, y \in X.$$

Sometimes due to weather conditions, technical issues or personal problems, a passenger missed his direct flight between two particular countries. So, if he has to go somewhere urgently, then he has to choose indirect route as there are indirect routes between these countries. For example, if a passenger missed his flight from Germany to USA, then there are four indirect routes given as follows.

P_1: Germany to China then China to USA.
P_2: Germany to China, China to Mexico then Mexico to USA.
P_3: Germany to China, China to Brazil, then Brazil to USA.
P_4: Germany to China, China to Brazil, Brazil to Mexico then Mexico to USA.

We will find the most suitable route by calculating the lengths of all these routes. That route is the most suitable whose truth-membership value is maximum, indeterminacy-membership value is minimum, and falsity-membership value is minimum. After calculating the lengths of all the routes, we get $L(P_1) = (1.5, 0.3, 0.3)$, $L(P_2) = (1.3, 0.5, 0.7)$, $L(P_3) = (1.3, 0.3, 0.6)$ and $L(P_4) = (1.4, 0.5, 1.0)$.

From Fig. 1.33, it looks like travelling through Germany to USA is the most protected route, but after calculating the lengths, we find that the protected route is P_1 because of uncertain conditions. Similarly, one can find the protected route between other countries.

We now present the general procedure of our method which is used in our application from Algorithm 1.3.2.

Algorithm 1.3.2

Step 1. Input the degrees of truth-membership, indeterminacy-membership and falsity-membership of all m vertices(countries).

Step 2. Calculate the degrees of truth-membership, indeterminacy-membership and falsity-membership of all edges using the following relations.

$$\begin{aligned}
T_B(xy) &\leq \min\{T_A(x), T_A(y)\},\\
I_B(xy) &\leq \min\{I_A(x), I_A(y)\},\\
F_B(xy) &\leq \max\{F_A(x), F_A(y)\}, \qquad x, y \in X.
\end{aligned}$$

Step 3. Calculate all the possible routes P_k between the countries.
Step 4. Calculate the lengths of all the routs P_k using the following formula,

$$L(P_k) = \left(\sum_{i=1}^{m-1} T_B(x_i x_{i+1}), \sum_{i=1}^{m-1} I_B(x_i x_{i+1}), \sum_{i=1}^{m-1} F_B(x_i x_{i+1})\right), \quad k = 1, 2, \ldots, n.$$

Step 5. Find the protected route with maximum truth-membership degree, minimum indeterminacy-membership degree and minimum falsity-membership degree.

1.3.3 Selection of Military Weapon

Since in decision-making problems, there is a number of uncertainties, and in some situations, there exist some relations among attributes in a multiple-attribute decision-making problem. So, it is an interesting area of applications in neutrosophic graph theory. A multiple-attribute decision-making problem is solved under the general framework of neutrosophic graphs.

A military unit is planning to purchase new artillery weapons, and there are six feasible artillery weapons (alternatives) $x_i (i = 1, 2, \ldots, 6)$ to be selected. When making a decision, the attributes considered are as follows:
(1) a_1- assault fire capability indices.
(2) a_2- reaction capability indices.
(3) a_3- mobility indices.
(4) a_4- survival ability indices.
Among these four attributes, a_1, a_2, a_4 are of benefit type (beneficial), and a_3 is of cost type (nonbeneficial); the evaluation values are contained in the decision matrix $A = (a_{ij})_{6\times 4}$, listed in Table 1.6.

Normalized values of an attribute assigned to the alternatives are calculated by using the following formula and shown in Table 1.7:

$$r_{ij} = \langle T_{ij}, I_{ij}, F_{ij}\rangle = \begin{cases} a_{ij} & \text{for beneficial attribute,}\\ \bar{a}_{ij} & \text{for nonbeneficial attribute.}\end{cases}$$

$i = 1, 2, \ldots, 6;\ j = 1, 2, 3, 4$, where $\bar{a}_{ij}$ is the complement of a_{ij}, such that $\bar{a}_{ij} = \langle F_{ij}, 1 - I_{ij}, T_{ij}\rangle$.

Relative importance of attributes is also assigned (see table 2 in [136]). Let the decision-maker select the following assignments:

Table 1.6 Neutrosophic decision matrix $A = (a_{ij})_{6\times 4}$

Weapons	a_1	a_2	a_3	a_4
x_1	$\langle 0.5, 0.3, 0.6\rangle$	$\langle 0.6, 0.3, 0.2\rangle$	$\langle 0.4, 0.5, 0.1\rangle$	$\langle 0.1, 0.7, 0.5\rangle$
x_2	$\langle 0.6, 0.1, 0.2\rangle$	$\langle 0.2, 0.1, 0.4\rangle$	$\langle 0.2, 0.3, 0.4\rangle$	$\langle 0.3, 0.4, 0.1\rangle$
x_3	$\langle 0.1, 0.5, 0.3\rangle$	$\langle 0.3, 0.2, 0.5\rangle$	$\langle 0.7, 0.2, 0.1\rangle$	$\langle 0.5, 0.1, 0.2\rangle$
x_4	$\langle 0.3, 0.4, 0.2\rangle$	$\langle 0.4, 0.5, 0.1\rangle$	$\langle 0.3, 0.1, 0.4\rangle$	$\langle 0.5, 0.3, 0.4\rangle$
x_5	$\langle 0.1, 0.2, 0.4\rangle$	$\langle 0.2, 0.7, 0.3\rangle$	$\langle 0.1, 0.3, 0.5\rangle$	$\langle 0.2, 0.1, 0.5\rangle$
x_6	$\langle 0.5, 0.1, 0.7\rangle$	$\langle 0.5, 0.1, 0.4\rangle$	$\langle 0.3, 0.2, 0.6\rangle$	$\langle 0.4, 0.2, 0.6\rangle$

Table 1.7 Neutrosophic decision matrix $R = (r_{ij})_{6\times 4}$ of normalized data

Weapons	a_1	a_2	a_3	a_4
x_1	$\langle 0.5, 0.3, 0.6\rangle$	$\langle 0.6, 0.3, 0.2\rangle$	$\langle 0.1, 0.5, 0.4\rangle$	$\langle 0.1, 0.7, 0.5\rangle$
x_2	$\langle 0.6, 0.1, 0.2\rangle$	$\langle 0.2, 0.1, 0.4\rangle$	$\langle 0.4, 0.7, 0.2\rangle$	$\langle 0.3, 0.4, 0.1\rangle$
x_3	$\langle 0.1, 0.5, 0.3\rangle$	$\langle 0.3, 0.2, 0.5\rangle$	$\langle 0.1, 0.8, 0.7\rangle$	$\langle 0.5, 0.1, 0.2\rangle$
x_4	$\langle 0.3, 0.4, 0.2\rangle$	$\langle 0.4, 0.5, 0.1\rangle$	$\langle 0.4, 0.9, 0.3\rangle$	$\langle 0.5, 0.3, 0.4\rangle$
x_5	$\langle 0.1, 0.2, 0.4\rangle$	$\langle 0.2, 0.7, 0.3\rangle$	$\langle 0.5, 0.7, 0.1\rangle$	$\langle 0.2, 0.1, 0.5\rangle$
x_6	$\langle 0.5, 0.1, 0.7\rangle$	$\langle 0.5, 0.1, 0.4\rangle$	$\langle 0.6, 0.8, 0.3\rangle$	$\langle 0.4, 0.2, 0.6\rangle$

$$\mathcal{R}=\begin{array}{c} \\ a_1\\ a_2\\ a_3\\ a_4\end{array}\begin{array}{c}\begin{array}{cccc} a_1 & a_2 & a_3 & a_4\end{array}\\ \left[\begin{array}{cccc} --- & \langle 0.045, 0.410, 0.865\rangle & \langle 0.665, 0.045, 0.335\rangle & \langle 0.045, 0.590, 0.745\rangle\\ \langle 0.865, 0.590, 0.045\rangle & --- & \langle 0.135, 0.665, 0.335\rangle & \langle 0.590, 0.410, 0.255\rangle\\ \langle 0.335, 0.955, 0.665\rangle & \langle 0.335, 0.335, 0.135\rangle & --- & \langle 0.410, 0.255, 0.135\rangle\\ \langle 0.745, 0.410, 0.045\rangle & \langle 0.255, 0.590, 0.590\rangle & \langle 0.135, 0.745, 0.410\rangle & --- \end{array}\right]\end{array}.$$

The weapon selection attribute neutrosophic digraph given in Fig. 1.34, represents the presence as well as relative importance of four attributes a_1, a_2, a_3 and a_4 which are the vertices of the digraph. The weapon selection index is calculated using the values of A_i and r_{ij} for each alternative weapon, where A_i is the value of ith attribute represented by the weapon x_i and r_{ij} is the relative importance of the ith attribute over jth attribute.

For first weapon x_1, substituting values of A_1, A_2, A_3 and A_4 in above matrix $\mathcal{R}$, we get

$$\mathcal{R}_1=\begin{array}{c} \\ a_1\\ a_2\\ a_3\\ a_4\end{array}\begin{array}{c}\begin{array}{cccc} a_1 & a_2 & a_3 & a_4\end{array}\\ \left[\begin{array}{cccc} \langle 0.5, 0.3, 0.6\rangle & \langle 0.045, 0.410, 0.865\rangle & \langle 0.665, 0.045, 0.335\rangle & \langle 0.045, 0.590, 0.745\rangle\\ \langle 0.865, 0.590, 0.045\rangle & \langle 0.6, 0.3, 0.2\rangle & \langle 0.135, 0.665, 0.335\rangle & \langle 0.590, 0.410, 0.255\rangle\\ \langle 0.335, 0.955, 0.665\rangle & \langle 0.335, 0.335, 0.135\rangle & \langle 0.1, 0.5, 0.4\rangle & \langle 0.410, 0.255, 0.135\rangle\\ \langle 0.745, 0.410, 0.045\rangle & \langle 0.255, 0.590, 0.590\rangle & \langle 0.135, 0.745, 0.410\rangle & \langle 0.1, 0.7, 0.5\rangle \end{array}\right]\end{array}.$$

Now we calculate the permanent function value of above matrix using computer program, that is, per $(\mathcal{R}_1) = \langle 0.4117, 1.3482, 0.4884\rangle$. The permanent function is nothing but the determinant of a matrix but considering all the determinant terms as positive terms [87]. So, the weapon selection index values of different weapons are:

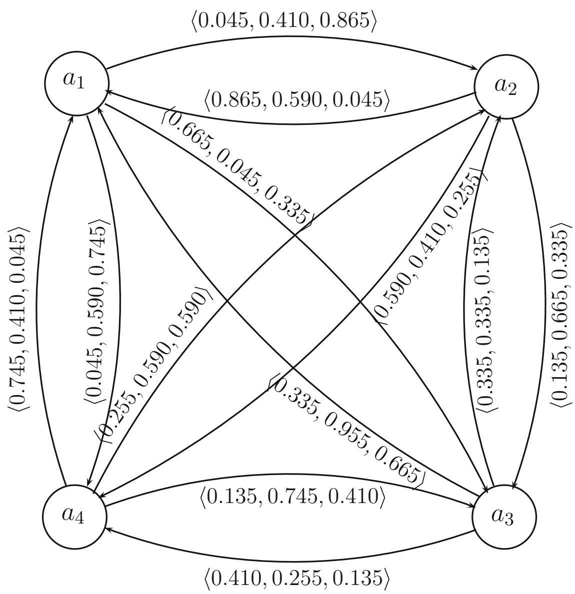


Fig. 1.34 Weapon selection attribute neutrosophic digraph

$$x_1 = \langle 0.4117, 1.3482, 0.4884\rangle,$$
$$x_2 = \langle 0.4224, 1.0522, 0.3415\rangle,$$
$$x_3 = \langle 0.4098, 1.1991, 0.4782\rangle,$$
$$x_4 = \langle 0.5173, 1.5801, 0.3468\rangle,$$
$$x_5 = \langle 0.3272, 1.3426, 0.4429\rangle,$$
$$x_6 = \langle 0.6113, 0.9950, 0.6179\rangle.$$

Calculate the score function $s(x_i) = T_i + 1 - I_i + 1 - F_i$ of the weapons $x_i (i = 1, 2, \ldots, 6)$, respectively: $s(x_1) = 0.5751$, $s(x_2) = 1.0287$, $s(x_3) = 0.7325$, $s(x_4) = 0.5904$, $s(x_5) = 0.5417$, $s(x_6) = 0.9984$. Thus, we can rank the weapons:

$$x_2 \succ x_6 \succ x_3 \succ x_4 \succ x_1 \succ x_5.$$

Therefore, the best choice is the second weapon (x_2).

1.4 Energy of Neutrosophic Graphs

If we change min by max in indeterminacy-membership of Definition 1.7, then we have the following definition of neutrosophic graph.

Definition 1.34 A neutrosophic graph on a nonempty set X is a pair $G = (A, B)$, where A is a neutrosophic set in X and B is a neutrosophic relation on X such that

$$T_B(xy) \leq \min\{T_A(x), T_A(y)\},$$
$$I_B(xy) \leq \max\{I_A(x), I_A(y)\},$$
$$F_B(xy) \leq \max\{F_A(x), F_A(y)\}, \qquad \text{for all } x, y \in X.$$

If B is not symmetric on A, then $D = (A, \overrightarrow{B})$ is called *neutrosophic digraph.*

Example 1.39 Consider a graph $G^* = (X, E)$ where $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ and $E = \{x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1x_5, x_1x_6, x_1x_7, x_3x_5, x_3x_6, x_3x_7, x_2x_5, x_5x_6, x_6x_7, x_4x_7\}$. Let $G = (A, B)$ be a neutrosophic graph on V as shown in Fig. 1.35 defined by

A	x_1	x_2	x_3	x_4	x_5	x_6	x_7
T_A	0.6	0.4	0.5	0.6	0.3	0.2	0.2
I_A	0.5	0.1	0.3	0.4	0.4	0.5	0.4
F_A	0.7	0.3	0.2	0.9	0.5	0.6	0.8

B	x_1x_2	x_2x_3	x_3x_4	x_4x_1	x_1x_5	x_1x_6	x_1x_7	x_3x_5	x_3x_6	x_3x_7	x_2x_5	x_5x_6	x_6x_7	x_4x_7
T_B	0.2	0.3	0.3	0.5	0.2	0.1	0.2	0.2	0.1	0.2	0.2	0.2	0.1	0.2
I_B	0.1	0.1	0.2	0.3	0.4	0.3	0.3	0.3	0.3	0.2	0.1	0.1	0.4	0.3
F_B	0.4	0.3	0.7	0.6	0.6	0.6	0.7	0.4	0.4	0.5	0.4	0.6	0.7	0.7

We now define and investigate the energy of a graph within the framework of neutrosophic set theory.

Definition 1.35 The *adjacency matrix* $\mathcal{A}(G)$ of a neutrosophic graph $G = (A, B)$ is defined as a square matrix $\mathcal{A}(G) = [a_{jk}]$, $a_{jk} = \langle T_B(x_jx_k), I_B(x_jx_k), F_B(x_jx_k)\rangle$, where $T_B(x_jx_k)$, $I_B(x_jx_k)$ and $F_B(x_jx_k)$ represent the strength of relationship, strength of undecided relationship and strength of nonrelationship between x_j and x_k, respectively.

The *adjacency matrix* of a neutrosophic graph can be expressed as three matrices: first matrix contains the entries as truth-membership values, second contains the entries as indeterminacy-membership values, and the third contains the entries as falsity-membership values, i.e., $\mathcal{A}(G) = \langle \mathcal{A}(T_B(x_jx_k)), \mathcal{A}(I_B(x_jx_k)), \mathcal{A}(F_B(x_jx_k))\rangle$.

Definition 1.36 The *spectrum of adjacency matrix* of a neutrosophic graph $\mathcal{A}(G)$ is defined as $\langle M, N, O\rangle$, where M, N and O are the sets of eigenvalues of $\mathcal{A}(T_B(x_jx_k))$, $\mathcal{A}(I_B(x_jx_k))$ and $\mathcal{A}(F_B(x_jx_k))$, respectively.

Example 1.40 The adjacency matrix $\mathcal{A}(G)$ of a neutrosophic graph given in Fig. 1.35 is

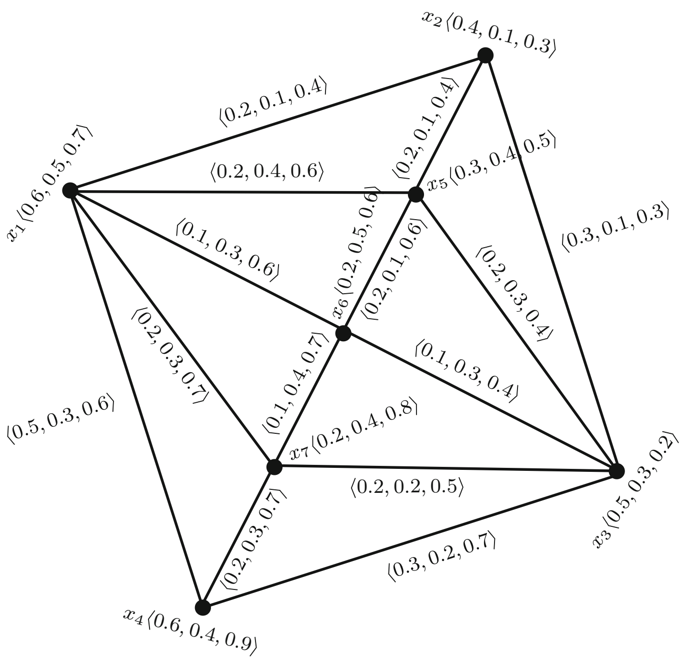


Fig. 1.35 Single-valued neutrosophic graph

$$
\begin{pmatrix}
\langle 0,0,0\rangle & \langle 0.2,0.1,0.4\rangle & \langle 0,0,0\rangle & \langle 0.5,0.3,0.6\rangle & \langle 0.2,0.4,0.6\rangle & \langle 0.1,0.3,0.6\rangle & \langle 0.2,0.3,0.7\rangle \\
\langle 0.2,0.1,0.4\rangle & \langle 0,0,0\rangle & \langle 0.3,0.1,0.3\rangle & \langle 0,0,0\rangle & \langle 0.2,0.1,0.4\rangle & \langle 0,0,0\rangle & \langle 0,0,0\rangle \\
\langle 0,0,0\rangle & \langle 0.3,0.1,0.3\rangle & \langle 0,0,0\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.2,0.3,0.4\rangle & \langle 0.1,0.3,0.4\rangle & \langle 0.2,0.2,0.5\rangle \\
\langle 0.5,0.3,0.6\rangle & \langle 0,0,0\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,0\rangle & \langle 0,0,0\rangle & \langle 0,0,0\rangle & \langle 0.2,0.3,0.7\rangle \\
\langle 0.2,0.4,0.6\rangle & \langle 0.2,0.1,0.4\rangle & \langle 0.2,0.3,0.4\rangle & \langle 0,0,0\rangle & \langle 0,0,0\rangle & \langle 0.2,0.1,0.6\rangle & \langle 0,0,0\rangle \\
\langle 0.1,0.3,0.6\rangle & \langle 0,0,0\rangle & \langle 0.1,0.3,0.4\rangle & \langle 0,0,0\rangle & \langle 0.2,0.1,0.6\rangle & \langle 0,0,0\rangle & \langle 0.1,0.4,0.7\rangle \\
\langle 0.2,0.3,0.7\rangle & \langle 0,0,0\rangle & \langle 0.2,0.2,0.5\rangle & \langle 0.2,0.3,0.7\rangle & \langle 0,0,0\rangle & \langle 0.1,0.4,0.7\rangle & \langle 0,0,0\rangle
\end{pmatrix}.
$$

The spectrum of a neutrosophic graph G given in Fig. 1.35 is as follows:

$$
\begin{aligned}
Spec(T_B(x_j x_k)) &= \{-0.7137, -0.2966, -0.2273, 0.0000, 0.0577, 0.2646, 0.9152\},\\
Spec(I_B(x_j x_k)) &= \{-0.7150, -0.4930, -0.0874, -0.0308, 0.0507, 0.2012, 1.0743\},\\
Spec(F_B(x_j x_k)) &= \{-1.2963, -1.1060, -0.5118, -0.0815, 0.1507, 0.5510, 2.2938\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
Spec(G) = \{&\langle -0.7137, -0.7150, -1.2963\rangle, \langle -0.2966, -0.4930, -1.1060\rangle,\\
&\langle -0.2273, -0.0874, -0.5118\rangle, \langle 0.0000, -0.0308, -0.0815\rangle,\\
&\langle 0.0577, 0.0507, 0.1507\rangle, \langle 0.2646, 0.2012, 0.5510\rangle,\\
&\langle 0.9152, 1.0743, 2.2938\rangle\}.
\end{aligned}
$$

Definition 1.37 The energy of a neutrosophic graph $G = (A, B)$ is defined as,

$$\begin{aligned} E(G) &= \left\langle E(T_B(x_jx_k)), E(I_B(x_jx_k)), E(F_B(x_jx_k)) \right\rangle \\ &= \left\langle \sum_{\substack{j=1 \\ \lambda_j \in M}}^{n} |\lambda_j|, \sum_{\substack{j=1 \\ \zeta_j \in N}}^{n} |\zeta_j|, \sum_{\substack{j=1 \\ \eta_j \in O}}^{n} |\eta_j| \right\rangle. \end{aligned}$$

Definition 1.38 Two neutrosophic graphs with the same number of vertices and the same energy are called *equienergetic*.

Theorem 1.25 *Let $G = (A, B)$ be a neutrosophic graph and $\mathcal{A}(G)$ be its adjacency matrix. If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $\zeta_1 \geq \zeta_2 \geq \cdots \geq \zeta_n$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$ are the eigenvalues of $\mathcal{A}(T_B(x_jx_k))$, $\mathcal{A}(I_B(x_jx_k))$ and $\mathcal{A}(F_B(x_jx_k))$, then*

1. $\sum_{\substack{j=1 \\ \lambda_j \in M}}^{n} \lambda_j = 0$, $\sum_{\substack{j=1 \\ \zeta_j \in N}}^{n} \zeta_j = 0$, $\sum_{\substack{j=1 \\ \eta_j \in O}}^{n} \eta_j = 0$
2. $\sum_{\substack{j=1 \\ \lambda_j \in M}}^{n} \lambda_j^2 = 2\left(\sum_{1 \leq j < k \leq n} (T_B(x_jx_k))^2\right)$,
$\sum_{\substack{j=1 \\ \zeta_j \in N}}^{n} \zeta_j^2 = 2\left(\sum_{1 \leq j < k \leq n} (I_B(x_jx_k))^2\right)$,
$\sum_{\substack{j=1 \\ \eta_j \in O}}^{n} \eta_j^2 = 2\left(\sum_{1 \leq j < k \leq n} (F_B(x_jx_k))^2\right)$.

Proof 1. Since $\mathcal{A}(G)$ is a symmetric matrix whose trace is zero, its eigenvalues are real with zero sum.
2. By matrix trace properties, we have

$$tr((\mathcal{A}(T_B(x_jx_k)))^2) = \sum_{\substack{j=1 \\ \lambda_j \in M}}^{n} \lambda_j^2$$

$$\begin{aligned} tr((\mathcal{A}(T_B(x_jx_k)))^2) &= (0 + T_B^2(x_1x_2) + \cdots + T_B^2(x_1x_n)) + (T_B^2(x_2x_1) + 0 + \cdots \\ &\quad + T_B^2(x_2x_n)) + \cdots + (T_B^2(x_nx_1) + T_B^2(x_nx_2) + \cdots + 0) \\ &= 2\left(\sum_{1 \leq j < k \leq n} (T_B(x_jx_k))^2\right). \end{aligned}$$

Hence $\sum_{\substack{j=1\\ \lambda_j\in M}}^{n} \lambda_j^2 = 2\left(\sum_{1\leq j<k\leq n}(T_B(x_jx_k))^2\right)$. Analogously, we can show that $\sum_{\substack{j=1\\ \zeta_j\in N}}^{n} \zeta_j^2 = 2\left(\sum_{1\leq j<k\leq n}(I_B(x_jx_k))^2\right)$ and $\sum_{\substack{j=1\\ \eta_j\in O}}^{n} \eta_j^2 = 2\left(\sum_{1\leq j<k\leq n}(F_B(x_jx_k))^2\right)$.

We now give upper and lower bounds on energy of a neutrosophic graph G, in terms of the number of vertices and the sum of squares of truth-membership, indeterminacy-membership and falsity-membership values of edges.

Theorem 1.26 *Let $G = (A, B)$ be a neutrosophic graph on n vertices with adjacency matrix $\mathcal{A}(G) = \langle \mathcal{A}(T_B(x_jx_k)), \mathcal{A}(I_B(x_jx_k)), \mathcal{A}(F_B(x_jx_k))\rangle$, then*

1. $$\sqrt{2\sum_{1\leq j<k\leq n}(T_B(x_jx_k))^2 + n(n-1)|T|^{\frac{2}{n}}} \leq E(T_B(x_jx_k)) \leq \sqrt{2n\sum_{1\leq j<k\leq n}(T_B(x_jx_k))^2}$$
2. $$\sqrt{2\sum_{1\leq j<k\leq n}(I_B(x_jx_k))^2 + n(n-1)|I|^{\frac{2}{n}}} \leq E(I_B(x_jx_k)) \leq \sqrt{2n\sum_{1\leq j<k\leq n}(I_B(x_jx_k))^2}$$
3. $$\sqrt{2\sum_{1\leq j<k\leq n}(F_B(x_jx_k))^2 + n(n-1)|F|^{\frac{2}{n}}} \leq E(F_B(x_jx_k)) \leq \sqrt{2n\sum_{1\leq j<k\leq n}(F_B(x_jx_k))^2}.$$

where $|T|, |I|$ and $|F|$ are the determinant of $\mathcal{A}(T_B(x_jx_k)), \mathcal{A}(I_B(x_jx_k))$ and $\mathcal{A}(F_B(x_jx_k))$, respectively.

Proof 1. Upper bound: Apply Cauchy–Schwarz inequality to the n numbers $1, 1, \ldots, 1$ and $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|$, then

$$\sum_{j=1}^{n}|\lambda_j| \leq \sqrt{n}\sqrt{\sum_{j=1}^{n}|\lambda_j|^2} \tag{1.1}$$

$$\left(\sum_{j=1}^{n}\lambda_j\right)^2 = \sum_{j=1}^{n}|\lambda_j|^2 + 2\left(\sum_{1\leq j<k\leq n}\lambda_j\lambda_k\right). \tag{1.2}$$

By comparing the coefficients of λ^{n-2} in the characteristic polynomial $\prod_{j=1}^{n}(\lambda-\lambda_j) = |\mathcal{A}(G)-\lambda I|$, we have

$$\sum_{1\le j<k\le n}\lambda_j\lambda_k = -\sum_{1\le j<k\le n}(T_B(x_jx_k))^2. \tag{1.3}$$

Substituting (1.3) in (1.2), we obtain

$$\sum_{j=1}^{n}|\lambda_j|^2 = 2\sum_{1\le j<k\le n}(T_B(x_jx_k))^2. \tag{1.4}$$

Substituting (1.4) in (1.1), we obtain

$$\sum_{j=1}^{n}|\lambda_j| \le \sqrt{n}\sqrt{2\sum_{1\le j<k\le n}(T_B(x_jx_k))^2} = \sqrt{2n\sum_{1\le j<k\le n}(T_B(x_jx_k))^2}.$$

Therefore,

$$E(T_B(x_jx_k)) \le \sqrt{2n\sum_{1\le j<k\le n}(T_B(x_jx_k))^2}.$$

Lower bound:

$$\begin{aligned}(E(T_B(x_jx_k)))^2 &= \left(\sum_{j=1}^{n}|\lambda_j|\right)^2 = \sum_{j=1}^{n}|\lambda_j|^2 + 2\left(\sum_{1\le j<k\le n}|\lambda_j\lambda_k|\right)\\ &= 2\left(\sum_{1\le j<k\le n}(T_B(x_jx_k))^2\right) + \frac{2n(n-1)}{2}AM\{|\lambda_j\lambda_k|\}.\end{aligned}$$

Since $AM\{|\lambda_j\lambda_k|\} \ge GM\{|\lambda_j\lambda_k|\}$, $1\le j<k\le n$,

$$E(T_B(x_jx_k)) \ge \sqrt{2\left(\sum_{1\le j<k\le n}(T_B(x_jx_k))^2 + n(n-1)GM\{|\lambda_j\lambda_k|\}\right)}.$$

It can also be seen that

$$GM\{|\lambda_j\lambda_k|\} = \left(\prod_{1\le j<k\le n}|\lambda_j\lambda_k|\right)^{\frac{2}{n(n-1)}} = \left(\prod_{j=1}^{n}|\lambda_j|^{n-1}\right)^{\frac{2}{n(n-1)}} = \left(\prod_{j=1}^{n}|\lambda_j|\right)^{\frac{2}{n}} = |T|^{\frac{2}{n}}.$$

Therefore,

$$E(T_B(x_jx_k)) \geq \sqrt{2\sum_{1\leq j<k\leq n}(T_B(x_jx_k))^2 + n(n-1)|T|^{\frac{2}{n}}}.$$

Thus, analogously, we can show that

$$\sqrt{2\left(\sum_{1\leq j<k\leq n}(I_B(x_jx_k))^2 + n(n-1)|I|^{\frac{2}{n}}\right)} \leq E(I_B(x_jx_k)) \leq \sqrt{2n\left(\sum_{1\leq j<k\leq n}(I_B(x_jx_k))^2\right)}$$

$$\sqrt{2\left(\sum_{1\leq j<k\leq n}(F_B(x_jx_k))^2 + n(n-1)|F|^{\frac{2}{n}}\right)} \leq E(F_B(x_jx_k)) \leq \sqrt{2n\left(\sum_{1\leq j<k\leq n}(F_B(x_jx_k))^2\right)}.$$

We now define and investigate the Laplacian energy of a graph under neutrosophic environment and investigate its properties.

Definition 1.39 Let $G = (A, B)$ be a neutrosophic graph on n vertices. The degree matrix, $D(G) = \langle D(T_B(x_jx_k)), D(I_B(x_jx_k)), D(F_B(x_jx_k))\rangle = [d_{jk}]$, of G is a $n \times n$ diagonal matrix defined as,

$$d_{jk} = \begin{cases} d_G(x_j) & \text{if } j = k, \\ 0 & \text{otherwise} \end{cases}$$

Definition 1.40 The *Laplacian matrix* of a neutrosophic graph $G = (A, B)$ is defined as $L(G) = \langle L(T_B(x_jx_k)), L(I_B(x_jx_k)), L(F_B(x_jx_k))\rangle = D(G) - \mathcal{A}(G)$, where $\mathcal{A}(G)$ is an adjacency matrix and $D(G)$ is a degree matrix of a neutrosophic graph G.

Definition 1.41 The spectrum of Laplacian matrix of a neutrosophic graph $L(G)$ is defined as $\langle M_L, N_L, O_L\rangle$, where M_L, N_L and O_L are the sets of Laplacian eigenvalues of $L(T_B(x_jx_k))$, $L(I_B(x_jx_k))$ and $L(F_B(x_jx_k))$, respectively.

Theorem 1.27 *Let $G = (A, B)$ be a neutrosophic graph, and let $L(G) = \langle L(T_B(x_jx_k)), L(I_B(x_jx_k)), L(F_B(x_jx_k))\rangle$ be the Laplacian matrix of G. If $\vartheta_1 \geq \vartheta_2 \geq \cdots \geq \vartheta_n$, $\varphi_1 \geq \varphi_2 \geq \cdots \geq \varphi_n$ and $\psi_1 \geq \psi_2 \geq \cdots \geq \psi_n$ are the eigenvalues of $L(T_B(x_jx_k))$, $L(I_B(x_jx_k))$ and $L(F_B(x_jx_k))$, respectively, then*

1. $\sum_{\substack{j=1\\ \vartheta_j\in M_L}}^{n} \vartheta_j = 2\left(\sum_{1\le j<k\le n} T_B(x_jx_k)\right)$, $\sum_{\substack{j=1\\ \varphi_j\in N_L}}^{n} \varphi_j = 2\left(\sum_{1\le j<k\le n} I_B(x_jx_k)\right)$

   $\sum_{\substack{j=1\\ \psi_j\in O_L}}^{n} \psi_j = 2\left(\sum_{1\le j<k\le n} F_B(x_jx_k)\right)$

2. $\sum_{\substack{j=1\\ \vartheta_j\in M_L}}^{n} \vartheta_j^2 = 2\left(\sum_{1\le j<k\le n} (T_B(x_jx_k))^2\right) + \sum_{j=1}^{n} d^2_{T_B(x_jx_k)}(x_j)$,

   $\sum_{\substack{j=1\\ \varphi_j\in N_L}}^{n} \varphi_j^2 = 2\left(\sum_{1\le j<k\le n} (I_B(x_jx_k))^2\right) + \sum_{j=1}^{n} d^2_{I_B(x_jx_k)}(x_j)$,

   $\sum_{\substack{j=1\\ \psi_j\in O_L}}^{n} \psi_j^2 = 2\left(\sum_{1\le j<k\le n} (F_B(x_jx_k))^2\right) + \sum_{j=1}^{n} d^2_{F_B(x_jx_k)}(x_j)$.

Proof 1. Since $L(G)$ is a symmetric matrix with nonnegative Laplacian eigenvalues,

$$\sum_{\substack{j=1\\ \vartheta_j\in M_L}}^{n} \vartheta_j = tr(L(G)) = \sum_{j=1}^{n} d_{T_B(x_jx_k)}(x_j) = 2\left(\sum_{1\le j<k\le n} T_B(x_jx_k)\right).$$

Similarly, it is easy to show that

$$\sum_{\substack{j=1\\ \varphi_j\in N_L}}^{n} \varphi_j = 2\left(\sum_{1\le j<k\le n} I_B(x_jx_k)\right)$$

$$\sum_{\substack{j=1\\ \psi_j\in O_L}}^{n} \psi_j = 2\left(\sum_{1\le j<k\le n} F_B(x_jx_k)\right).$$

2. By definition of Laplacian matrix, we have

$$L(T_B(x_jx_k)) = \begin{pmatrix} d_{T_B(x_jx_k)}(x_1) & -T_B(x_1x_2) & \dots & -T_B(x_1x_n) \\ -T_B(x_2x_1) & d_{T_B(x_jx_k)}(x_2) & \dots & -T_B(x_2x_n) \\ \vdots & \vdots & \ddots & \vdots \\ -T_B(x_nx_1) & -T_B(x_nx_2) & \dots & d_{T_B(x_jx_k)}(x_n) \end{pmatrix}.$$

By trace properties of a matrix, we have $tr((L(T_B(x_jx_k)))^2) = \sum_{\substack{j=1\\ \vartheta_j\in M_L}}^{n} \vartheta_j^2$ where

$$\begin{aligned} tr((L(T_B(x_jx_k)))^2) &= (d^2_{T_B(x_jx_k)}(x_1) + T_B^2(x_1x_2) + \cdots + T_B^2(x_1x_n)) \\ &\quad + (T_B^2(x_2x_1) + d^2_{T_B(x_jx_k)}(x_2) + \cdots + T_B^2(x_2x_n)) \\ &\quad + \cdots + (T_B^2(x_nx_1) + T_B^2(x_nx_2) + \cdots + d^2_{T_B(x_jx_k)}(x_n)) \\ &= 2\left(\sum_{1\le j<k\le n}(T_B(x_jx_k))^2\right) + \sum_{j=1}^{n} d^2_{T_B(x_jx_k)}(x_j). \end{aligned}$$

Therefore, $\sum\limits_{\substack{j=1 \\ \vartheta_j\in M_L}}^{n} \vartheta_j^2 = 2\left(\sum\limits_{1\le j<k\le n}(T_B(x_jx_k))^2\right) + \sum\limits_{j=1}^{n} d^2_{T_B(x_jx_k)}(x_j)$. Analogously, we can show that

$$\sum_{\substack{j=1 \\ \varphi_j\in N_L}}^{n} \varphi_j^2 = 2\left(\sum_{1\le j<k\le n}(I_B(x_jx_k))^2\right) + \sum_{j=1}^{n} d^2_{I_B(x_jx_k)}(x_j)$$

$$\sum_{\substack{j=1 \\ \psi_j\in O_L}}^{n} \psi_j^2 = 2\left(\sum_{1\le j<k\le n}(F_B(x_jx_k))^2\right) + \sum_{j=1}^{n} d^2_{F_B(x_jx_k)}(x_j).$$

Definition 1.42 The Laplacian energy of a neutrosophic graph $G = (A, B)$ is defined as $LE(G) = \langle LE(T_B(x_jx_k)), LE(I_B(x_jx_k)), LE(F_B(x_jx_k))\rangle = \langle \sum\limits_{j=1}^{n} |\varrho_j|, \sum\limits_{j=1}^{n} |\xi_j|, \sum\limits_{j=1}^{n} |\tau_j|\rangle$ where

$$\varrho_j = \vartheta_j - \frac{2\left(\sum\limits_{1\le j<k\le n} T_B(x_jx_k)\right)}{n},$$

$$\xi_j = \varphi_j - \frac{2\left(\sum\limits_{1\le j<k\le n} I_B(x_jx_k)\right)}{n},$$

$$\tau_j = \psi_j - \frac{2\left(\sum\limits_{1\le j<k\le n} F_B(x_jx_k)\right)}{n}.$$

Theorem 1.28 *Let $G = (A, B)$ be a neutrosophic graph on n vertices and let $L(G) = \langle L(T_B(x_jx_k)), L(I_B(x_jx_k)), L(F_B(x_jx_k))\rangle$ be the Laplacian matrix of G, then*

1. $LE(T_B(x_jx_k))$

$$\leq \sqrt{2n \sum_{1 \leq j < k \leq n} (T_B(x_jx_k))^2 + n \sum_{j=1}^{n} \left(d_{T_B(x_jx_k)}(x_j) - \frac{2 \sum_{1 \leq j < k \leq n} T_B(x_jx_k)}{n} \right)^2},$$

2. $LE(I_B(x_jx_k))$

$$\leq \sqrt{2n \sum_{1 \leq j < k \leq n} (I_B(x_jx_k))^2 + n \sum_{j=1}^{n} \left(d_{I_B(x_jx_k)}(x_j) - \frac{2 \sum_{1 \leq j < k \leq n} I_B(x_jx_k)}{n} \right)^2},$$

3. $LE(F_B(x_jx_k))$

$$\leq \sqrt{2n \sum_{1 \leq j < k \leq n} (F_B(x_jx_k))^2 + n \sum_{j=1}^{n} \left(d_{F_B(x_jx_k)}(x_j) - \frac{2 \sum_{1 \leq j < k \leq n} F_B(x_jx_k)}{n} \right)^2}.$$

Proof Apply Cauchy–Schwarz inequality to the n numbers 1, 1, ..., 1 and $|\varrho_1|$, $|\varrho_2|$, ..., $|\varrho_n|$, and we have $\sum_{j=1}^{n} |\varrho_j| \leq \sqrt{n}\sqrt{\sum_{j=1}^{n} |\varrho_j|^2}$ and $LE(T_B(x_jx_k)) \leq \sqrt{n}\sqrt{2\mathcal{M}_T} = \sqrt{2n\mathcal{M}_T}$. We know that

$\mathcal{M}_T = \sum_{1 \leq j < k \leq n} (T_B(x_jx_k))^2 + \frac{1}{2} \sum_{j=1}^{n} \left(d_{T_B(x_jx_k)}(x_j) - \frac{2 \sum_{1 \leq j < k \leq n} T_B(x_jx_k)}{n} \right)^2,$

Therefore, it can be proved that

$$LE(T_B(x_jx_k)) \leq \sqrt{2n \sum_{1 \leq j < k \leq n} (T_B(x_jx_k))^2 + n \sum_{j=1}^{n} \left(d_{T_B(x_jx_k)}(x_j) - \frac{2 \sum_{1 \leq j < k \leq n} T_B(x_jx_k)}{n} \right)^2},$$

$$LE(I_B(x_jx_k)) \leq \sqrt{2n \sum_{1 \leq j < k \leq n} (I_B(x_jx_k))^2 + n \sum_{j=1}^{n} \left(d_{I_B(x_jx_k)}(x_j) - \frac{2 \sum_{1 \leq j < k \leq n} I_B(x_jx_k)}{n} \right)^2},$$

$$LE(F_B(x_jx_k)) \leq \sqrt{2n \sum_{1 \leq j < k \leq n} (F_B(x_jx_k))^2 + n \sum_{j=1}^{n} \left(d_{F_B(x_jx_k)}(x_j) - \frac{2 \sum_{1 \leq j < k \leq n} F_B(x_jx_k)}{n} \right)^2}.$$

Theorem 1.29 *Let $G = (A, B)$ be a neutrosophic graph on n vertices and let $L(G) = \langle L(T_B(x_jx_k)), L(I_B(x_jx_k)), L(F_B(x_jx_k)) \rangle$ be the Laplacian matrix of G, then*

$$LE(T_B(x_jx_k)) \geq 2\sqrt{\sum_{1\leq j<k\leq n}(T_B(x_jx_k))^2 + \frac{1}{2}\sum_{j=1}^{n}\left(d_{T_B(x_jx_k)}(x_j) - \frac{2\sum_{1\leq j<k\leq n}T_B(x_jx_k)}{n}\right)^2},$$

$$LE(I_B(x_jx_k)) \geq 2\sqrt{\sum_{1\leq j<k\leq n}(I_B(x_jx_k))^2 + \frac{1}{2}\sum_{j=1}^{n}\left(d_{I_B(x_jx_k)}(x_j) - \frac{2\sum_{1\leq j<k\leq n}I_B(x_jx_k)}{n}\right)^2},$$

$$LE(F_B(x_jx_k)) \geq 2\sqrt{\sum_{1\leq j<k\leq n}(F_B(x_jx_k))^2 + \frac{1}{2}\sum_{j=1}^{n}\left(d_{F_B(x_jx_k)}(x_j) - \frac{2\sum_{1\leq j<k\leq n}F_B(x_jx_k)}{n}\right)^2}.$$

Proof Here $\left(\sum_{j=1}^{n}|\varrho_j|\right)^2 = \sum_{j=1}^{n}|\varrho_j|^2 + 2\sum_{1\leq j<k\leq n}|\varrho_j\varrho_k| \geq 4\mathcal{M}_T$ and $LE(T_B(x_jx_k)) \geq 2\sqrt{\mathcal{M}_T}$. Since $\mathcal{M}_T = \sum_{1\leq j<k\leq n}(T_B(x_jx_k))^2 + \frac{1}{2}\sum_{j=1}^{n}\left(d_{T_B(x_jx_k)}(x_j) - \frac{2\sum_{1\leq j<k\leq n}T_B(x_jx_k)}{n}\right)^2$,

$$LE(T_B(x_jx_k)) \geq 2\sqrt{\sum_{1\leq j<k\leq n}(T_B(x_jx_k))^2 + \frac{1}{2}\sum_{j=1}^{n}\left(d_{T_B(x_jx_k)}(x_j) - \frac{2\sum_{1\leq j<k\leq n}T_B(x_jx_k)}{n}\right)^2},$$

$$LE(I_B(x_jx_k)) \geq 2\sqrt{\sum_{1\leq j<k\leq n}(I_B(x_jx_k))^2 + \frac{1}{2}\sum_{j=1}^{n}\left(d_{I_B(x_jx_k)}(x_j) - \frac{2\sum_{1\leq j<k\leq n}I_B(x_jx_k)}{n}\right)^2}$$

$$LE(F_B(x_jx_k)) \geq 2\sqrt{\sum_{1\leq j<k\leq n}(F_B(x_jx_k))^2 + \frac{1}{2}\sum_{j=1}^{n}\left(d_{F_B(x_jx_k)}(x_j) - \frac{2\sum_{1\leq j<k\leq n}F_B(x_jx_k)}{n}\right)^2}.$$

Definition 1.43 The *signless Laplacian matrix* of a neutrosophic graph $G = (A, B)$ is defined as $L^+(G) = \langle L^+(T_B(x_jx_k)), L^+(I_B(x_jx_k)), L^+(F_B(x_jx_k))\rangle = D(G) +$

$\mathcal{A}(G)$, where $D(G)$ and $\mathcal{A}(G)$ are the degree matrix and the adjacency matrix, respectively, of a neutrosophic graph G. The spectrum of signless Laplacian matrix of a neutrosophic graph $L^+(G)$ is defined as $\langle M_{L^+}, N_{L^+}, O_{L^+}\rangle$, where M_{L^+}, N_{L^+} and O_{L^+} are the sets of signless Laplacian eigenvalues of $L^+(T_B(x_jx_k))$, $L^+(I_B(x_jx_k))$ and $L^+(F_B(x_jx_k))$, respectively.

1.5 Application to Group Decision-Making

Group decision-making is a commonly used tool in human activities, which determines the optimal alternative from a given finite set of alternatives using the evaluation information given by a group of decision-makers or experts. With the rapid development of society, group decision-making plays an increasingly important role when dealing with the decision-making problems. Recently, many scholars have investigated the approaches for group decision-making based on different kinds of decision information. However, in order to reflect the relationships among the alternatives, we need to make pairwise comparisons for all the alternatives in the process of decision- making. Preference relation is a powerful quantitative decision technique that supports experts in expressing their preferences over the given alternatives. For a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$, the experts compare each pair of alternatives and construct preference relations, respectively. If every element in the preference relations is a neutrosophic number, then the concept of the neutrosophic preference relation (NPR) can be put forth as follows:

Definition 1.44 A NPR on the set $X = \{x_1, x_2, \ldots, x_n\}$ is represented by a matrix $R = (r_{jk})_{n\times n}$, where $r_{jk}=\langle x_jx_k, T(x_jx_k), I(x_jx_k), F(x_jx_k)\rangle$ for all $j, k=1, 2, \ldots, n$. For convenience, let $r_{jk} = \langle T_{jk}, I_{jk}, F_{jk}\rangle$ where T_{jk} indicates the degree to which the object x_j is preferred to the object x_k, F_{jk} denotes the degree to which the object x_j is not preferred to the object x_k, and I_{jk} is interpreted as an indeterminacy-membership degree, with the conditions: $T_{jk}, I_{jk}, F_{jk} \in [0, 1]$, $T_{jk} = F_{kj}$, $F_{jk} = T_{kj}$, $I_{jk} + I_{kj} = 1$, $T_{jj} = I_{jj} = F_{jj} = 0.5$, for all $j, k = 1, 2, \ldots, n$.

A group decision-making problem concerning the 'Alliance partner selection of a software company' is solved to illustrate the applicability of the proposed concepts of energy of neutrosophic graphs in realistic scenario.

1.5.1 Alliance Partner Selection of a Software Company

Eastsoft is one of the top five software companies in China [77]. It offers a rich portfolio of businesses, including product engineering solutions, industry solutions, and related software products and platform and services. It is dedicated to becoming a globally leading IT solution and service provider through continuous improvement

Table 1.8 NPR of the expert from the engineering management department

R_1	a_1	a_2	a_3	a_4	a_5
a_1	$\langle 0.5, 0.5, 0.5\rangle$	$\langle 0.4, 0.6, 0.3\rangle$	$\langle 0.2, 0.4, 0.6\rangle$	$\langle 0.7, 0.6, 0.3\rangle$	$\langle 0.3, 0.1, 0.6\rangle$
a_2	$\langle 0.3, 0.4, 0.4\rangle$	$\langle 0.5, 0.5, 0.5\rangle$	$\langle 0.7, 0.3, 0.8\rangle$	$\langle 0.4, 0.1, 0.4\rangle$	$\langle 0.1, 0.3, 0.5\rangle$
a_3	$\langle 0.6, 0.6, 0.2\rangle$	$\langle 0.8, 0.7, 0.7\rangle$	$\langle 0.5, 0.5, 0.5\rangle$	$\langle 0.3, 0.6, 0.4\rangle$	$\langle 0.2, 0.3, 0.4\rangle$
a_4	$\langle 0.3, 0.4, 0.7\rangle$	$\langle 0.4, 0.9, 0.4\rangle$	$\langle 0.4, 0.4, 0.3\rangle$	$\langle 0.5, 0.5, 0.5\rangle$	$\langle 0.3, 0.1, 0.3\rangle$
a_5	$\langle 0.6, 0.9, 0.3\rangle$	$\langle 0.5, 0.7, 0.1\rangle$	$\langle 0.4, 0.7, 0.2\rangle$	$\langle 0.3, 0.9, 0.3\rangle$	$\langle 0.5, 0.5, 0.5\rangle$

Table 1.9 NPR of the expert from the human resource department

R_2	a_1	a_2	a_3	a_4	a_5
a_1	$\langle 0.5, 0.5, 0.5\rangle$	$\langle 0.5, 0.3, 0.1\rangle$	$\langle 0.1, 0.7, 0.5\rangle$	$\langle 0.3, 0.9, 0.5\rangle$	$\langle 0.2, 0.7, 0.8\rangle$
a_2	$\langle 0.1, 0.7, 0.5\rangle$	$\langle 0.5, 0.5, 0.5\rangle$	$\langle 0.5, 0.1, 0.6\rangle$	$\langle 0.6, 0.7, 0.1\rangle$	$\langle 0.4, 0.6, 0.8\rangle$
a_3	$\langle 0.5, 0.3, 0.1\rangle$	$\langle 0.6, 0.9, 0.5\rangle$	$\langle 0.5, 0.5, 0.5\rangle$	$\langle 0.9, 0.2, 0.3\rangle$	$\langle 0.1, 0.4, 0.1\rangle$
a_4	$\langle 0.5, 0.1, 0.3\rangle$	$\langle 0.1, 0.3, 0.6\rangle$	$\langle 0.3, 0.8, 0.9\rangle$	$\langle 0.5, 0.5, 0.5\rangle$	$\langle 0.8, 0.4, 0.2\rangle$
a_5	$\langle 0.8, 0.3, 0.2\rangle$	$\langle 0.8, 0.4, 0.4\rangle$	$\langle 0.1, 0.6, 0.1\rangle$	$\langle 0.2, 0.6, 0.8\rangle$	$\langle 0.5, 0.5, 0.5\rangle$

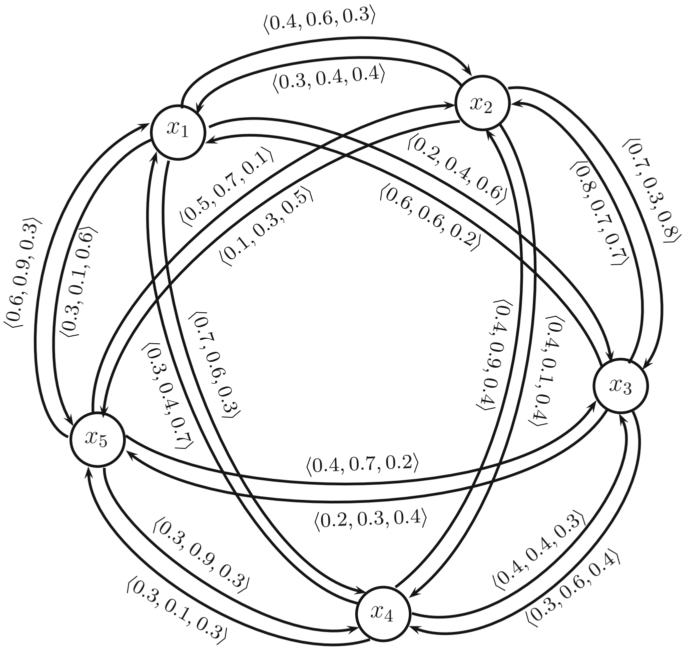
Table 1.10 NPR of the expert from the finance department

R_3	a_1	a_2	a_3	a_4	a_5
a_1	$\langle 0.5, 0.5, 0.5\rangle$	$\langle 0.9, 0.8, 0.7\rangle$	$\langle 0.1, 0.7, 0.2\rangle$	$\langle 0.4, 0.3, 0.1\rangle$	$\langle 0.6, 0.3, 0.6\rangle$
a_2	$\langle 0.7, 0.2, 0.9\rangle$	$\langle 0.5, 0.5, 0.5\rangle$	$\langle 0.4, 0.3, 0.6\rangle$	$\langle 0.6, 0.3, 0.4\rangle$	$\langle 0.7, 0.2, 0.9\rangle$
a_3	$\langle 0.2, 0.3, 0.1\rangle$	$\langle 0.6, 0.7, 0.4\rangle$	$\langle 0.5, 0.5, 0.5\rangle$	$\langle 0.1, 0.2, 0.4\rangle$	$\langle 0.6, 0.2, 0.8\rangle$
a_4	$\langle 0.1, 0.7, 0.4\rangle$	$\langle 0.4, 0.7, 0.6\rangle$	$\langle 0.4, 0.8, 0.1\rangle$	$\langle 0.5, 0.5, 0.5\rangle$	$\langle 0.6, 0.7, 0.3\rangle$
a_5	$\langle 0.6, 0.7, 0.6\rangle$	$\langle 0.9, 0.8, 0.7\rangle$	$\langle 0.8, 0.8, 0.6\rangle$	$\langle 0.3, 0.3, 0.6\rangle$	$\langle 0.5, 0.5, 0.5\rangle$

of organization and process, competence development of leadership and employees, and alliance and open innovation. To improve the operation and competitiveness capability in the global market, Eastsoft plans to establish a strategic alliance with a transnational corporation. After numerous consultations, five transnational corporations would like to establish a strategic alliance with Eastsoft; they are HP a_1, PHILIPS a_2, EMC a_3, SAP a_4 and LK a_5. To select the desirable strategic alliance partner, three experts e_i $(i = 1, 2, 3)$ are invited to participate in the decision analysis, who come from the engineering management department, the human resource department and the finance department of Eastsoft, respectively. Based on their experiences, the experts compare each pair of alternatives and give individual judgments using the following NPRs $R_i = (r^{(i)}_{jk})_{5\times 5}$ $(i = 1, 2, 3)$:

The neutrosophic digraphs D_i corresponding to NPRs R_i $(i = 1, 2, 3)$ given in Tables 1.8, 1.9 and 1.10 are shown in Figs. 1.36, 1.37 and 1.38.

The energy of a neutrosophic digraph is the sum of absolute values of the real part of eigenvalues of D. The energy of each neutrosophic digraph $D_i (i = 1, 2, 3)$ is calculated as $E(D_1) = \langle 3.2419, 3.5861, 3.2419\rangle$, $E(D_2) = \langle 3.2790, 3.9089, 3.2790\rangle$,



D_1

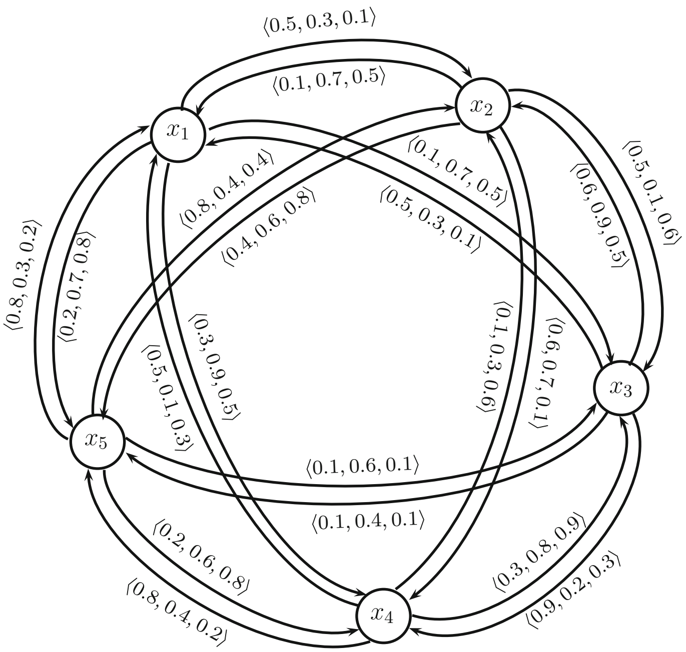
Fig. 1.36 Neutrosophic digraph

$E(D_3) = \langle 4.1587, 3.5618, 4.1587\rangle$. Then the weight of each expert can be determined as,

$$w_i = \left(\frac{E((D_T)_i)}{\sum_{l=1}^{m} E((D_T)_l)}, \frac{E((D_I)_i)}{\sum_{l=1}^{m} E((D_I)_l)}, \frac{E((D_F)_i)}{\sum_{l=1}^{m} E((D_F)_l)} \right), \; 1 \le i \le m.$$

The weights are calculated as $w_1 = \langle 0.3219,\ 0.3561,\ 0.3219\rangle$, $w_2 = \langle 0.3133,\ 0.3735,\ 0.3133\rangle$, $w_3 = \langle 0.3501,\ 0.2998,\ 0.3501\rangle$. Utilize the aggregation operator to fuse all the individual NPRs $R_i = (r_{jk}^{(i)})_{5\times 5}$ $(i = 1, 2, 3)$ into the collective NPR $R = (r_{jk})_{5\times 5}$ as shown in Table 1.11. Here we apply the neutrosophic weighted averaging (NWA) operator [59] to fuse the individual NPR.

$$\mathrm{NWA}(r_{jk}^{(1)}, r_{jk}^{(2)}, \ldots, r_{jk}^{(s)}) = \left\langle 1 - \prod_{i=1}^{s} \left(1 - T_{jk}^{(i)}\right)^{w_i}, \prod_{i=1}^{s} \left(I_{jk}^{(i)}\right)^{w_i}, \prod_{i=1}^{s} \left(F_{jk}^{(i)}\right)^{w_i} \right\rangle$$



D_2

Fig. 1.37 Neutrosophic digraph

Draw a directed network corresponding to a collective NPR above, as shown in Fig. 1.39. Then under the condition $T_{jk} \geq 0.5$ $(j, k = 1,\ 2,\ 3,\ 4,\ 5)$, a partial diagram is drawn, as shown in Fig. 1.40.

Calculate the out-degrees out-d(a_j) $(j{=}1,\ 2,\ 3,\ 4,\ 5)$ of all criteria in a partial directed network as follows: out-d$(a_1) = \langle 0.6951, 0.4973, 0.2912\rangle$, out-d$(a_2) = \langle 1.0813, 0.4608, 0.9258\rangle$, out-d$(a_3) = \langle 1.2580, 1.0430, 0.8911\rangle$, out-d$(a_4) = \langle 0.6093, 0.2811, 0.2689\rangle$, out-d$(a_5) = \langle 1.9907, 1.8177, 0.9005\rangle$. According to membership degrees of out-d(a_j) $(j = 1,\ 2,\ 3,\ 4,\ 5)$, we get the ranking of the factors a_j $(j = 1, 2, 3, 4, 5)$ as $a_5 \succ a_3 \succ a_2 \succ a_1 \succ a_4$. Thus, the best choice is LK a_5. Now elements of the Laplacian matrices of the neutrosophic digraphs $L(D_i) = R_i^L$ $(i = 1, 2, 3)$ shown in Figs. 1.36, 1.37, 1.38 are provided in Tables 1.12, 1.13 and 1.14.

The Laplacian energy of each neutrosophic digraph is calculated as
$LE(D_1) = \langle 3.2800, 4.0000, 3.8893\rangle$,
$LE(D_2) = \langle 3.3600, 4.0000, 3.8798\rangle$,
$LE(D_3) = \langle 4.6806, 4.5858, 4.9687\rangle$. Then the weight of each expert can be determined as

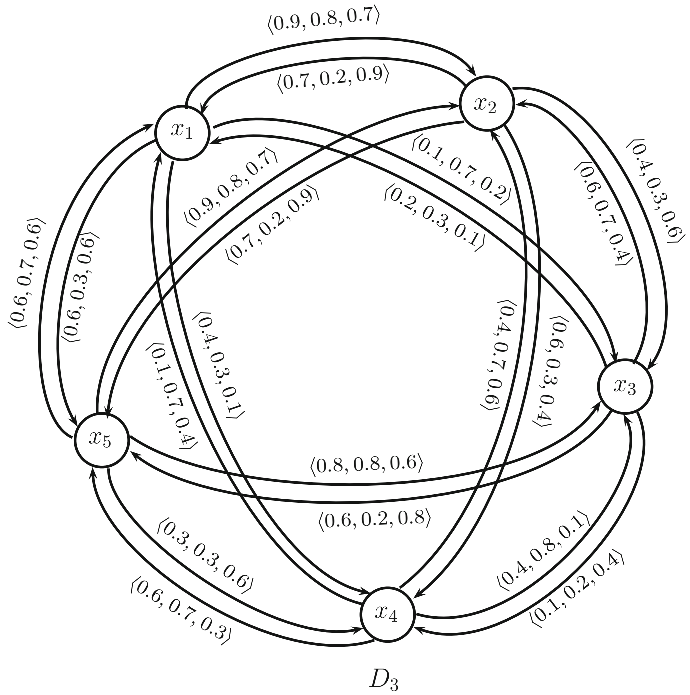


Fig. 1.38 Neutrosophic digraph

$$w_i = \left(\frac{LE((D_T)_i)}{\sum_{l=1}^{m} LE((D_T)_l)}, \frac{LE((D_I)_i)}{\sum_{l=1}^{m} LE((D_I)_l)}, \frac{LE((D_F)_i)}{\sum_{l=1}^{m} LE((D_F)_l)} \right), \quad i = 1, 2, \ldots, m.$$

$w_1 = \langle 0.2937, 0.3581, 0.3482 \rangle$, $w_2 = \langle 0.2989, \quad 0.3559, \quad 0.3452 \rangle$, $w_3 = \langle 0.3288, 0.3221, 0.3490 \rangle$ based on which, using the NWA operator, the fused NPR is determined, as shown in Table 1.15. In the directed network corresponding to a collective NPR above, we select those neutrosophic numbers whose membership degrees $T_{jk} \geq 0.5$ $(j, k = 1,\ 2,\ 3,\ 4,\ 5)$, and resulting partial diagram is shown in Fig. 1.41.

Calculate the out-degrees out-d(a_j) $(j = 1,\ 2,\ 3,\ 4,\ 5)$ of all criteria in a partial directed network as follows out-d$(a_1) = \langle 0.6719, 0.5050, 0.2622 \rangle$, out-d$(a_2) = \langle 1.0333, 0.4563, 0.8874 \rangle$, out-d$(a_3) = \langle 1.2122, 1.0354, 0.8534 \rangle$, out-d$(a_4) = \langle 0.5881, 0.2821, 0.2478 \rangle$, out-d$(a_5)$=$\langle 1.9228, 1.8333, 0.8201 \rangle$. According to membership degrees of out-d(a_j) $(j = 1,\ 2,\ 3,\ 4,\ 5)$, we get the ranking of the factors a_j, $j = 1, 2, 3, 4, 5$ as $a_5 \succ a_3 \succ a_2 \succ a_1 \succ a_4$. Thus, the best choice is LK a_5. Now, the elements of the signless Laplacian matrices of the neutrosophic

Table 1.11 Collective NPR of all the above individual NPRs

R	a_1	a_2	a_3	a_4	a_5
a_1	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.6951, 0.4973, 0.2912\rangle$	$\langle 0.1321, 0.5675, 0.3887\rangle$	$\langle 0.4924, 0.5587, 0.2439\rangle$	$\langle 0.3968, 0.2687, 0.6615\rangle$
a_2	$\langle 0.4341, 0.3898, 0.5775\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.5432, 0.1921, 0.6632\rangle$	$\langle 0.5381, 0.2687, 0.2626\rangle$	$\langle 0.4596, 0.3322, 0.7190\rangle$
a_3	$\langle 0.4458, 0.3706, 0.1293\rangle$	$\langle 0.6757, 0.7609, 0.5206\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.5823, 0.2821, 0.3705\rangle$	$\langle 0.3466, 0.2855, 0.3347\rangle$
a_4	$\langle 0.3085, 0.2744, 0.4436\rangle$	$\langle 0.3136, 0.5520, 0.5306\rangle$	$\langle 0.3656, 0.6209, 0.2933\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.6093, 0.2811, 0.2689\rangle$
a_5	$\langle 0.6737, 0.5520, 0.3428\rangle$	$\langle 0.7842, 0.5850, 0.3156\rangle$	$\langle 0.5328, 0.6807, 0.2421\rangle$	$\langle 0.2663, 0.5547, 0.5292\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$

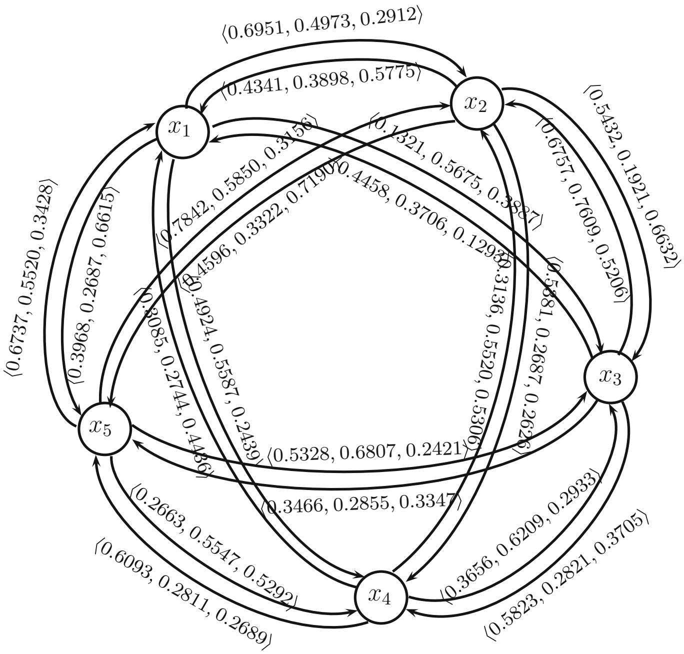


Fig. 1.39 Directed network of the fused NPR

digraphs $L^+(D_i) = R_i^{L^+}$ $(i = 1, 2, 3)$ shown in Figs. 1.36, 1.37 and 1.38 are given in Tables 1.16, 1.17 and 1.18. The signless Laplacian energy of each neutrosophic digraph is calculated as $LE^+(D_1) = \langle 3.3244, 4.7474, 3.5570\rangle$, $LE^+(D_2) = \langle 3.3826, 4.0000, 3.4427\rangle$, $LE^+(D_3) = \langle 4.5859, 4.4103, 4.7228\rangle$. Then the weight of each expert is

$$w_i = \left(\frac{LE^+((D_T)_i)}{\sum_{l=1}^{m} LE^+((D_T)_l)}, \frac{LE^+((D_I)_i)}{\sum_{l=1}^{m} LE^+((D_I)_l)}, \frac{LE^+((D_F)_i)}{\sum_{l=1}^{m} LE^+((D_F)_l)} \right), \quad i = 1, 2, \ldots, m,$$

$w_1 = \langle 0.2859, 0.4082, 0.3059\rangle$, $w_2 = \langle 0.3125, 0.3695, 0.3180\rangle$, $w_3 = \langle 0.3343, 0.3215, 0.3443\rangle$, based on which fuse all the individual NPRs $R_i = (r_{jk}^{(i)})_{5\times 5}$ $(i = 1, 2, 3)$ into the collective NPR $R = (r_{jk})_{5\times 5}$, by using the NWA operator, as shown in Table 1.19. In the directed network corresponding to a collective NPR above, we select those neutrosophic numbers whose membership degrees $T_{jk} \geq 0.5$ $(j, k = 1, 2, 3, 4, 5)$, and resulting partial diagram is shown in Fig. 1.42.

Calculate the out-degrees out-d(a_j) $(j = 1, 2, 3, 4, 5)$ of all criteria in a partial directed network as follows out-d$(a_1) = \langle 0.6777, 0.4843, 0.2943\rangle$, out-d$(a_2) = \langle 1.0412, 0.4099, 0.9309\rangle$, out-d$(a_3) = \langle 1.2265, 1.0084, 0.9005\rangle$, out-d$(a_4) = \langle 0.5980, 0.2483, 0.2740\rangle$, out-d$(a_5) = \langle 1.9395, 1.7873, 0.9212\rangle$. According

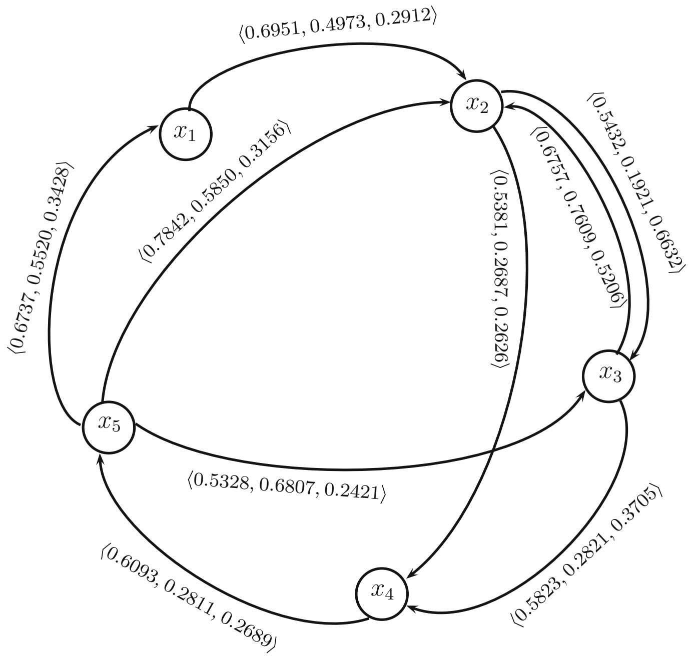


Fig. 1.40 Partial directed network of the fused NPR

to membership degrees of out-d(a_j) ($j = 1, 2, 3, 4, 5$), we get the ranking of the factors a_j ($j = 1, 2, 3, 4, 5$) as $a_5 \succ a_3 \succ a_2 \succ a_1 \succ a_4$. Thus, the best choice is LK a_5.

1.5.2 Real-Time Example

The proposed concepts of energy, Laplacian energy and signless Laplacian energy of a neutrosophic graph are explained here through a real-time example. We have taken the website http://www.pantechsolutions.net modelled as a neutrosophic graph by considering the navigation of the customer. We have taken the four links: 1. microcontroller boards, 2. log-in html, 3. and 4. project kits for our calculation. A neutrosophic graph of this site for four different time periods is considered. The energy, Laplacian energy and signless Laplacian energy of a neutrosophic graph are calculated for each of these periods. The energy, Laplacian energy and signless Laplacian energy are represented in terms of bar graphs. In the website http://www.pantechsolutions.net (accessed on 8 May 2012). The above four links are considered for the period 16 January 2018 to 15 February 2018, and for this graph, as shown in Fig. 1.43, we have

Table 1.12 Elements of the Laplacian matrix of the neutrosophic digraph D_1

R_1^L	a_1	a_2	a_3	a_4	a_5
a_1	$\langle 1.6, 1.7, 1.8\rangle$	$\langle -0.4, -0.6, -0.3\rangle$	$\langle -0.2, -0.4, -0.6\rangle$	$\langle -0.7, -0.6, -0.3\rangle$	$\langle -0.3, -0.1, -0.6\rangle$
a_2	$\langle -0.3, -0.4, -0.4\rangle$	$\langle 1.5, 1.1, 2.1\rangle$	$\langle -0.7, -0.3, -0.8\rangle$	$\langle -0.4, -0.1, -0.4\rangle$	$\langle -0.1, -0.3, -0.5\rangle$
a_3	$\langle -0.6, -0.6, -0.2\rangle$	$\langle -0.8, -0.7, -0.7\rangle$	$\langle 1.9, 2.2, 1.7\rangle$	$\langle -0.3, -0.6, -0.4\rangle$	$\langle -0.2, -0.3, -0.4\rangle$
a_4	$\langle -0.3, -0.4, -0.7\rangle$	$\langle -0.4, -0.9, -0.4\rangle$	$\langle -0.4, -0.4, -0.3\rangle$	$\langle 1.4, 1.8, 1.7\rangle$	$\langle -0.3, -0.1, -0.3\rangle$
a_5	$\langle -0.6, -0.9, -0.3\rangle$	$\langle -0.5, -0.7, -0.1\rangle$	$\langle -0.4, -0.7, -0.2\rangle$	$\langle -0.3, -0.9, -0.3\rangle$	$\langle 1.8, 3.2, -0.9\rangle$

Table 1.13 Elements of the Laplacian matrix of the neutrosophic digraph D_2

R_2^L	a_1	a_2	a_3	a_4	a_5
a_1	$\langle 1.1, 2.6, 1.9\rangle$	$\langle -0.5, -0.3, -0.1\rangle$	$\langle -0.1, -0.7, -0.5\rangle$	$\langle -0.3, -0.9, -0.5\rangle$	$\langle -0.2, -0.7, -0.8\rangle$
a_2	$\langle -0.1, -0.7, -0.5\rangle$	$\langle 1.6, 2.1, 2.0\rangle$	$\langle -0.5, -0.1, -0.6\rangle$	$\langle -0.6, -0.7, -0.1\rangle$	$\langle -0.4, -0.6, -0.8\rangle$
a_3	$\langle -0.5, -0.3, -0.1\rangle$	$\langle -0.6, -0.9, -0.5\rangle$	$\langle 2.1, 1.8, 1.0\rangle$	$\langle -0.9, -0.2, -0.3\rangle$	$\langle -0.1, -0.4, -0.1\rangle$
a_4	$\langle -0.5, -0.1, -0.3\rangle$	$\langle -0.1, -0.3, -0.6\rangle$	$\langle -0.3, -0.8, -0.9\rangle$	$\langle 1.7, 1.6, 2.0\rangle$	$\langle -0.8, -0.4, -0.2\rangle$
a_5	$\langle -0.8, -0.3, -0.2\rangle$	$\langle -0.8, -0.4, -0.4\rangle$	$\langle -0.1, -0.6, -0.1\rangle$	$\langle -0.2, -0.6, -0.8\rangle$	$\langle 1.9, 1.9, 1.5\rangle$

Table 1.14 Elements of the Laplacian matrix of the neutrosophic digraph D_3

R_3^L	a_1	a_2	a_3	a_4	a_5
a_1	$\langle 2.0, 2.1, 1.6\rangle$	$\langle -0.9, -0.8, -0.7\rangle$	$\langle -0.1, -0.7, -0.2\rangle$	$\langle -0.4, -0.3, -0.1\rangle$	$\langle -0.6, -0.3, -0.6\rangle$
a_2	$\langle -0.7, -0.2, -0.9\rangle$	$\langle 2.4, 1.0, 2.8\rangle$	$\langle -0.4, -0.3, -0.6\rangle$	$\langle -0.6, -0.3, -0.4\rangle$	$\langle -0.7, -0.2, -0.9\rangle$
a_3	$\langle -0.2, -0.3, -0.1\rangle$	$\langle -0.6, -0.7, -0.4\rangle$	$\langle 1.5, 1.4, 1.7\rangle$	$\langle -0.1, -0.2, -0.4\rangle$	$\langle -0.6, -0.2, -0.8\rangle$
a_4	$\langle -0.1, -0.7, -0.4\rangle$	$\langle -0.4, -0.7, -0.6\rangle$	$\langle -0.4, -0.8, -0.1\rangle$	$\langle 1.5, 2.9, 1.4\rangle$	$\langle -0.6, -0.7, -0.3\rangle$
a_5	$\langle -0.6, -0.7, -0.6\rangle$	$\langle -0.9, -0.8, -0.7\rangle$	$\langle -0.8, -0.8, -0.6\rangle$	$\langle -0.3, -0.3, -0.6\rangle$	$\langle 2.6, 2.6, 2.5\rangle$

$\text{Spec}(T_Y(x_jx_k)) = \{-0.3442, -0.1000, 0.0066, 0.4376\}$,
$\text{Spec}(I_Y(x_jx_k)) = \{-0.6630, -0.2742, 0.0774, 0.8598\}$,
$\text{Spec}(F_Y(x_jx_k)) = \{-0.6703, -0.3296, 0.0299, 0.9701\}$,
$E(T_Y(x_jx_k)) = 0.8884$, $E(I_Y(x_jx_k)) = 1.8744$, $E(F_Y(x_jx_k)) = 1.9999$.
Therefore, $E(G_1) = \langle 0.8884, 1.8744, 1.9999\rangle$.
Laplacian $\text{Spec}(T_Y(x_jx_k)) = \{0, 0.2492, 0.5244, 0.8264\}$,
Laplacian $\text{Spec}(I_Y(x_jx_k)) = \{0, 0.6975, 1.1757, 1.5269\}$,
Laplacian $\text{Spec}(F_Y(x_jx_k)) = \{0, 0.7605, 1.4139, 1.6256\}$,
$LE(T_Y(x_jx_k)) = 1.1016$, $LE(I_Y(x_jx_k)) = 2.0051$, $LE(F_Y(x_jx_k)) = 2.2790$.
Therefore, $LE(G_1) = \langle 1.1016, 2.0051, 2.2790\rangle$.

Signless Laplacian $\text{Spec}(T_Y(x_jx_k)) = \{-0.3183, -0.1339, -0.0555, 0.5076\}$,
Signless Laplacian $\text{Spec}(I_Y(x_jx_k)) = \{-0.6764, -0.2500, 0.0385, 0.8879\}$,
Signless Laplacian $\text{Spec}(F_Y(x_jx_k)) = \{-0.7056, -0.2572, -0.0582, 1.0211\}$,
$LE^+(T_Y(x_jx_k)) = 1.0153$, $LE^+(I_Y(x_jx_k)) = 1.8529$, $LE^+(F_Y(x_jx_k)) = 2.0421$.
Therefore, $LE^+(G_1) = \langle 1.0153, 1.8529, 2.0421\rangle$.

Table 1.15 Collective NPR of all the above individual NPRs

R	a_1	a_2	a_3	a_4	a_5
a_1	⟨0.5000, 0.5000, 0.5000⟩	⟨0.6719, 0.5050, 0.2622⟩	⟨0.1234, 0.5656, 0.3757⟩	⟨0.4664, 0.5443, 0.2317⟩	⟨0.3767, 0.2620, 0.6484⟩
a_2	⟨0.4126, 0.3778, 0.5515⟩	⟨0.5000, 0.5000, 0.5000⟩	⟨0.5175, 0.1943, 0.6490⟩	⟨0.5158, 0.2620, 0.2384⟩	⟨0.4398, 0.3226, 0.7011⟩
a_3	⟨0.4229, 0.3682, 0.1155⟩	⟨0.6493, 0.7557, 0.5050⟩	⟨0.5000, 0.5000, 0.5000⟩	⟨0.5629, 0.2797, 0.3484⟩	⟨0.3285, 0.2792, 0.3037⟩
a_4	⟨0.2929, 0.2829, 0.4233⟩	⟨0.2949, 0.5593, 0.5098⟩	⟨0.3460, 0.6191, 0.2839⟩	⟨0.5000, 0.5000, 0.5000⟩	⟨0.5881, 0.2821, 0.2478⟩
a_5	⟨0.6506, 0.5593, 0.3157⟩	⟨0.7635, 0.5911, 0.2886⟩	⟨0.5087, 0.6829, 0.2158⟩	⟨0.2508, 0.5448, 0.5094⟩	⟨0.5000, 0.5000, 0.5000⟩

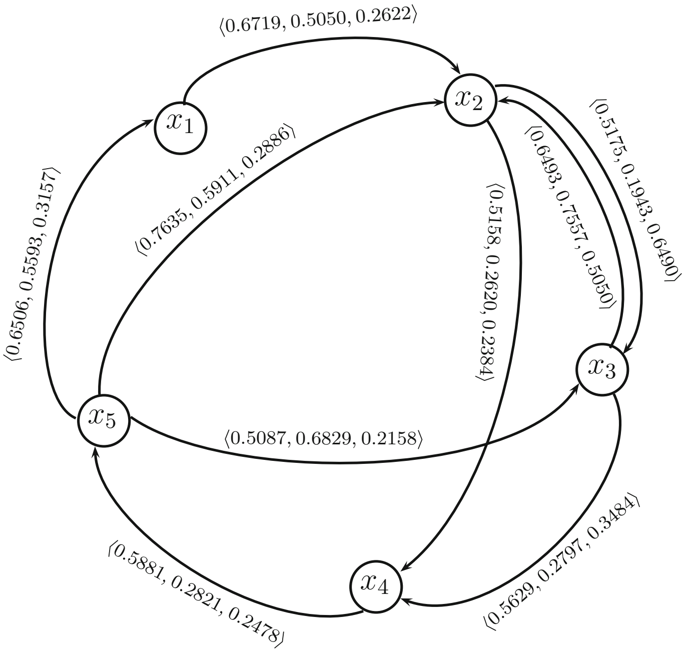


Fig. 1.41 Partial directed network of the fused NPR

Table 1.16 Elements of the signless Laplacian matrix of the neutrosophic digraph D_1

$R_1^{L^+}$	a_1	a_2	a_3	a_4	a_5
a_1	⟨1.6, 1.7, 1.8⟩	⟨0.4, 0.6, 0.3⟩	⟨0.2, 0.4, 0.6⟩	⟨0.7, 0.6, 0.3⟩	⟨0.3, 0.1, 0.6⟩
a_2	⟨0.3, 0.4, 0.4⟩	⟨1.5, 1.1, 2.1⟩	⟨0.7, 0.3, 0.8⟩	⟨0.4, 0.1, 0.4⟩	⟨0.1, 0.3, 0.5⟩
a_3	⟨0.6, 0.6, 0.2⟩	⟨0.8, 0.7, 0.7⟩	⟨1.9, 2.2, 1.7⟩	⟨0.3, 0.6, 0.4⟩	⟨0.2, 0.3, 0.4⟩
a_4	⟨0.3, 0.4, 0.7⟩	⟨0.4, 0.9, 0.4⟩	⟨0.4, 0.4, 0.3⟩	⟨1.4, 1.8, 1.7⟩	⟨0.3, 0.1, 0.3⟩
a_5	⟨0.6, 0.9, 0.3⟩	⟨0.5, 0.7, 0.1⟩	⟨0.4, 0.7, 0.2⟩	⟨0.3, 0.9, 0.3⟩	⟨1.8, 3.2, 0.9⟩

Table 1.17 Elements of the signless Laplacian matrix of the neutrosophic digraph D_2

$R_2^{L^+}$	a_1	a_2	a_3	a_4	a_5
a_1	⟨1.1, 2.6, 1.9⟩	⟨0.5, 0.3, 0.1⟩	⟨0.1, 0.7, 0.5⟩	⟨0.3, 0.9, 0.5⟩	⟨0.2, 0.7, 0.8⟩
a_2	⟨0.1, 0.7, 0.5⟩	⟨1.6, 2.1, 2.0⟩	⟨0.5, 0.1, 0.6⟩	⟨0.6, 0.7, 0.1⟩	⟨0.4, 0.6, 0.8⟩
a_3	⟨0.5, 0.3, 0.1⟩	⟨0.6, 0.9, 0.5⟩	⟨2.1, 1.8, 1.0⟩	⟨0.9, 0.2, 0.3⟩	⟨0.1, 0.4, 0.1⟩
a_4	⟨0.5, 0.1, 0.3⟩	⟨0.1, 0.3, 0.6⟩	⟨0.3, 0.8, 0.9⟩	⟨1.7, 1.6, 2.0⟩	⟨0.8, 0.4, 0.2⟩
a_5	⟨0.8, 0.3, 0.2⟩	⟨0.8, 0.4, 0.4⟩	⟨0.1, 0.6, 0.1⟩	⟨0.2, 0.6, 0.8⟩	⟨1.9, 1.9, 1.5⟩

Table 1.18 Elements of the signless Laplacian matrix of the neutrosophic digraph D_3

$R_3^{L^+}$	a_1	a_2	a_3	a_4	a_5
a_1	⟨2.0, 2.1, 1.6⟩	⟨0.9, 0.8, 0.7⟩	⟨0.1, 0.7, 0.2⟩	⟨0.4, 0.3, 0.1⟩	⟨0.6, 0.3, 0.6⟩
a_2	⟨0.7, 0.2, 0.9⟩	⟨2.4, 1.0, 2.8⟩	⟨0.4, 0.3, 0.6⟩	⟨0.6, 0.3, 0.4⟩	⟨0.7, 0.2, 0.9⟩
a_3	⟨0.2, 0.3, 0.1⟩	⟨0.6, 0.7, 0.4⟩	⟨1.5, 1.4, 1.7⟩	⟨0.1, 0.2, 0.4⟩	⟨0.6, 0.2, 0.8⟩
a_4	⟨0.1, 0.7, 0.4⟩	⟨0.4, 0.7, 0.6⟩	⟨0.4, 0.8, 0.1⟩	⟨1.5, 2.9, 1.4⟩	⟨0.6, 0.7, 0.3⟩
a_5	⟨0.6, 0.7, 0.6⟩	⟨0.9, 0.8, 0.7⟩	⟨0.8, 0.8, 0.6⟩	⟨0.3, 0.3, 0.6⟩	⟨2.6, 2.6, 2.5⟩

For the period 16 February 2018 to 15 March 2018 (see Fig. 1.44), we have

$\text{Spec}(T_Y(x_jx_k)) = \{-0.4245, -0.1714, 0.0215, 0.5744\}$,
$\text{Spec}(I_Y(x_jx_k)) = \{-0.7909, -0.5799, 0.0536, 1.3173\}$,
$\text{Spec}(F_Y(x_jx_k)) = \{-0.5037, -0.3400, 0.0007, 0.8430\}$,
$E(T_Y(x_jx_k)) = 1.1919$, $E(I_Y(x_jx_k)) = 2.7418$, $E(F_Y(x_jx_k)) = 1.6874$.
Therefore, $E(G_2) = \langle 1.1919, 2.7418, 1.6874\rangle$.

Laplacian $\text{Spec}(T_Y(x_jx_k)) = \{0, 0.4200, 0.6908, 1.0892\}$,
Laplacian $\text{Spec}(I_Y(x_jx_k)) = \{0, 0.8716, 1.7656, 2.3629\}$,
Laplacian $\text{Spec}(F_Y(x_jx_k)) = \{0, 0.5672, 1.1546, 1.4783\}$,
$LE(T_Y(x_jx_k)) = 1.36$, $LE(I_Y(x_jx_k)) = 3.2569$, $LE(F_Y(x_jx_k)) = 2.0657$.
Therefore, $LE(G_2) = \langle 1.36, 3.2569, 2.0657\rangle$.

Signless Laplacian $\text{Spec}(T_Y(x_jx_k)) = \{-0.4023, -0.1931, -0.0585, 0.6538\}$,
Signless Laplacian $\text{Spec}(I_Y(x_jx_k)) = \{-0.7962, -0.5500, -0.1538, 1.5000\}$,
Signless Laplacian $\text{Spec}(F_Y(x_jx_k)) = \{-0.5321, -0.2209, -0.2000, 0.9530\}$,
$LE^+(T_Y(x_jx_k)) = 1.3076$, $LE^+(I_Y(x_jx_k)) = 2.9999$, $LE^+(F_Y(x_jx_k)) = 1.9059$.
Therefore, $LE^+(G_2) = \langle 1.3076, 2.9999, 1.9059\rangle$.

For the period 16 March 2018 to 15 April 2018 (see Fig. 1.45), we have

$\text{Spec}(T_Y(x_jx_k)) = \{-0.6287, -0.3884, 0.0004, 1.0168\}$,
$\text{Spec}(I_Y(x_jx_k)) = \{-1.0779, -0.5696, 0.0698, 1.5776\}$,
$\text{Spec}(F_Y(x_jx_k)) = \{-0.8184, -0.4650, 0.0051, 1.2783\}$,
$E(T_Y(x_jx_k)) = 2.0343$, $E(I_Y(x_jx_k)) = 3.2949$, $E(F_Y(x_jx_k)) = 2.5668$.
Therefore, $E(G_3) = \langle 2.0343, 3.2949, 2.5668\rangle$.

Laplacian $\text{Spec}(T_Y(x_jx_k)) = \{0, 0.2604, 1.4221, 1.7175\}$,
Laplacian $\text{Spec}(I_Y(x_jx_k)) = \{0, 1.2472, 2.3360, 2.6168\}$,
Laplacian $\text{Spec}(F_Y(x_jx_k)) = \{0, 0.8182, 1.6721, 2.3097\}$,
$LE(T_Y(x_jx_k)) = 2.8792$, $LE(I_Y(x_jx_k)) = 3.7056$,$LE(F_Y(x_jx_k)) = 3.1636$.
Therefore, $LE(G_3) = \langle 2.8792, 3.7056, 3.1636\rangle$.

Table 1.19 The collective NPR of all the above individual NPRs

R	a_1	a_2	a_3	a_4	a_5
a_1	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.6777, 0.4843, 0.2943\rangle$	$\langle 0.1236, 0.5377, 0.3942\rangle$	$\langle 0.4655, 0.5302, 0.2512\rangle$	$\langle 0.3800, 0.2325, 0.6682\rangle$
a_2	$\langle 0.4157, 0.3594, 0.5845\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.5189, 0.1774, 0.6659\rangle$	$\langle 0.5223, 0.2325, 0.2650\rangle$	$\langle 0.4469, 0.3019, 0.7267\rangle$
a_3	$\langle 0.4249, 0.3533, 0.1330\rangle$	$\langle 0.6510, 0.7414, 0.5247\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.5755, 0.2670, 0.3758\rangle$	$\langle 0.3317, 0.2599, 0.3364\rangle$
a_4	$\langle 0.2980, 0.2620, 0.4460\rangle$	$\langle 0.2951, 0.5474, 0.5387\rangle$	$\langle 0.3484, 0.5897, 0.3028\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.5980, 0.2483, 0.2740\rangle$
a_5	$\langle 0.6574, 0.5474, 0.3479\rangle$	$\langle 0.7703, 0.5736, 0.3268\rangle$	$\langle 0.5118, 0.6663, 0.2465\rangle$	$\langle 0.2524, 0.5386, 0.5406\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$

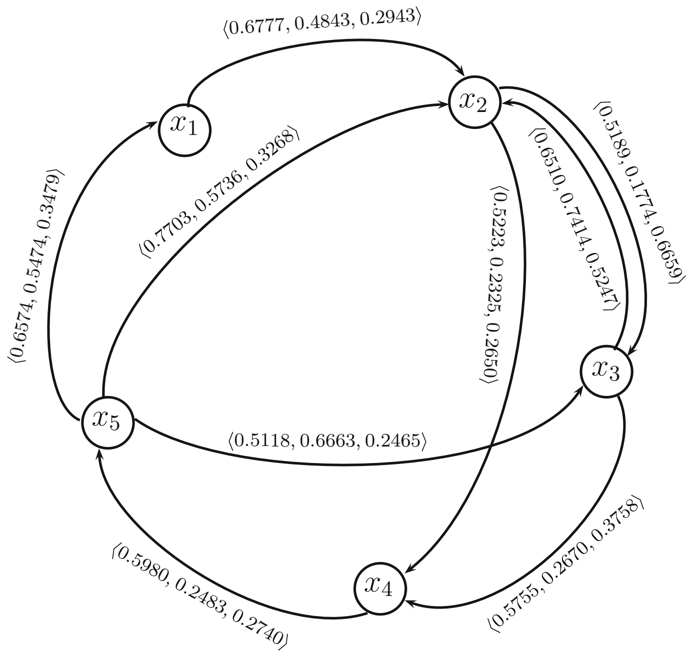


Fig. 1.42 Partial directed network of the fused NPR

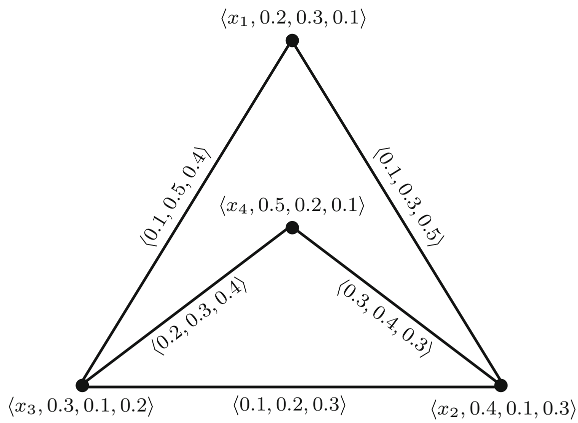


Fig. 1.43 Neutrosophic graph G_1

Signless Laplacian Spec$(T_Y(x_jx_k)) = \{-0.6816, -0.3513, -0.2007, 1.2336\}$,
Signless Laplacian Spec$(I_Y(x_jx_k)) = \{-1.1436, -0.4542, -0.0553, 1.6531\}$,
Signless Laplacian Spec$(F_Y(x_jx_k)) = \{-0.8066, -0.4000, -0.2632, 1.4698\}$,
$LE^+(T_Y(x_jx_k)) = 2.4671$, $LE^+(I_Y(x_jx_k)) = 3.3062$, $LE^+(F_Y(x_jx_k)) = 2.9395$.
Therefore, $LE^+(G_3) = \langle 2.4671, 3.3062, 2.9395\rangle$.

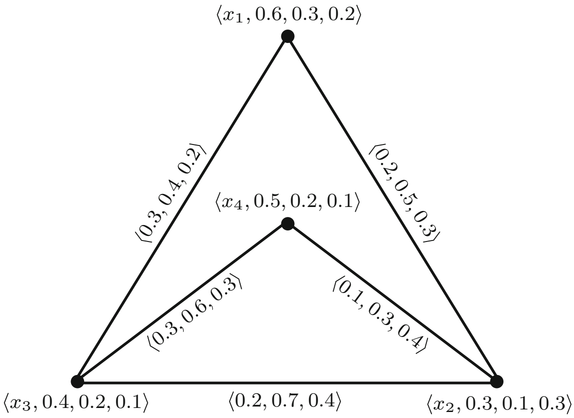


Fig. 1.44 Neutrosophic graph G_2

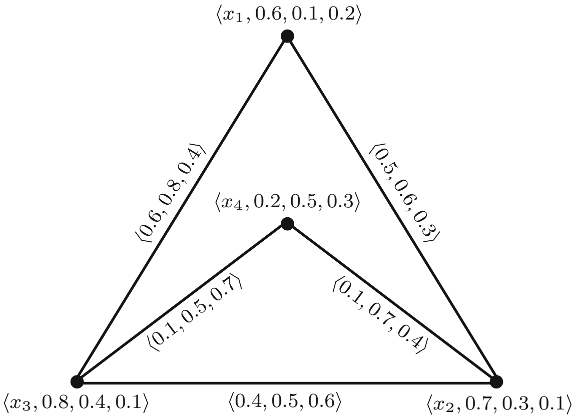


Fig. 1.45 Neutrosophic graph G_3

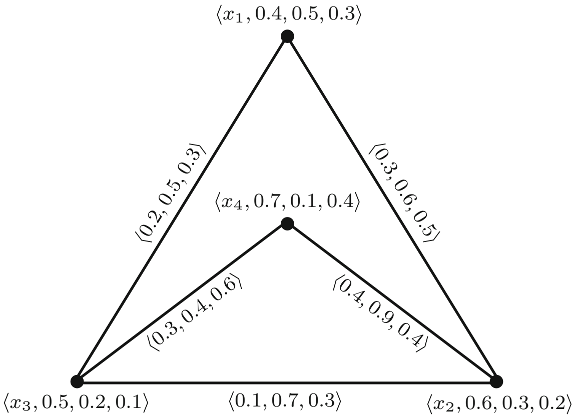


Fig. 1.46 Neutrosophic graph G_4

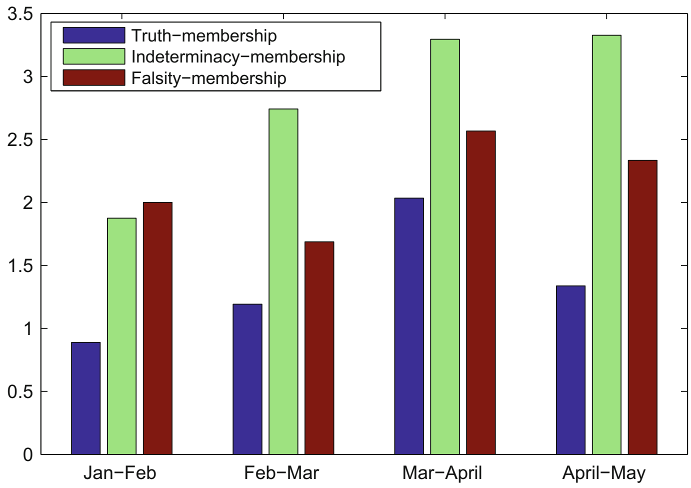


Fig. 1.47 Energy of neutrosophic graphs

Finally, for the period 16 April 2018 to 15 May 2018 (see Fig. 1.46), we have

$\text{Spec}(T_Y(x_jx_k)) = \{-0.5716, -0.0973, 0.0027, 0.6662\}$,
$\text{Spec}(I_Y(x_jx_k)) = \{-1.0878, -0.5755, 0.0435, 1.6198\}$,
$\text{Spec}(F_Y(x_jx_k)) = \{-0.7686, -0.3985, 0.0990, 1.0680\}$,
$E(T_Y(x_jx_k)) = 1.3378, E(I_Y(x_jx_k)) = 3.3265, E(F_Y(x_jx_k)) = 2.3342$.
Therefore, $E(G_4) = \langle 1.3378, 3.3265, 2.3342\rangle$.

Laplacian $\text{Spec}(T_Y(x_jx_k)) = \{0, 0.5637, 0.7641, 1.2721\}$,
Laplacian $\text{Spec}(I_Y(x_jx_k)) = \{0, 1.1660, 2.0643, 2.9697\}$,
Laplacian $\text{Spec}(F_Y(x_jx_k)) = \{0, 0.8207, 1.5544, 1.8249\}$,
$LE(T_Y(x_jx_k)) = 1.4725$, $LE(I_Y(x_jx_k)) = 3.868$, $LE(F_Y(x_jx_k)) = 2.5586$.
Therefore, $LE(G_4) = \langle 1.4725, 3.8680, 2.5586\rangle$.

Signless Laplacian $\text{Spec}(T_Y(x_jx_k)) = \{-0.5588, -0.1017, -0.0500, 0.7105\}$,
Signless Laplacian $\text{Spec}(I_Y(x_jx_k)) = \{-1.0582, -0.5617, -0.2105, 1.8304\}$,
Signless Laplacian $\text{Spec}(F_Y(x_jx_k)) = \{-0.7996, -0.3562, 0.0413, 1.1145\}$,
$LE^+(T_Y(x_jx_k)) = 1.4211$, $LE^+(I_Y(x_jx_k)) = 3.6608$, $LE^+(F_Y(x_jx_k)) = 2.3116$.
Therefore, $LE^+(G_4) = \langle 1.4211, 3.6608, 2.3116\rangle$.

The bar graphs, shown in Figs. 1.47, 1.48 and 1.49, represent the energy, Laplacian energy and signless Laplacian energy of four links for the above four periods corresponding to the truth-membership, indeterminacy-membership and falsity-membership values. From the above bar graphs, the energy, Laplacian energy and signless Laplacian energy of truth-membership for the period March to April are high

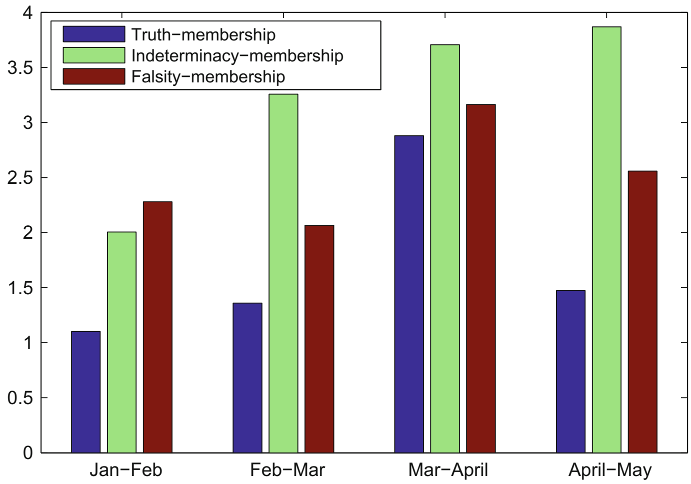


Fig. 1.48 Laplacian energy of neutrosophic graphs

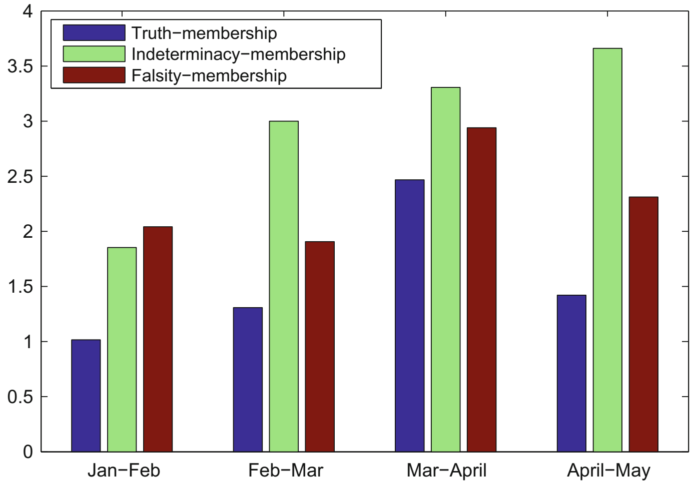


Fig. 1.49 Signless Laplacian energy of neutrosophic graphs

as compared to other periods; the energy, Laplacian energy and signless Laplacian energy of indeterminacy-membership for the period April to May are high; and the energy, Laplacian energy and signless Laplacian energy of falsity-membership for the period March to April are high.