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Muhammad Akram

Single-Valued Neutrosophic Graphs

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To my lovely parents and worthy teachers!

Foreword

The Königsberg bridge problem originated in the city of Königsberg, located on the river Pregel. The city had seven bridges, which connected two islands with the mainland. People staying there always wondered whether there was any way to walk over all the bridges once and only once and return to the same place where they started the walk. In 1736, Euler came out with the solution in terms of graph theory. He proved that it was not possible to walk through the seven bridges exactly one time. In coming to this conclusion, Euler formulated the problem in terms of graph theory. Each landmark was represented as a point (node) and every bridge as an edge. This led to the formation of graph theory. Graph theory is a beautiful part of mathematics. Not only computer science is heavily based on graph theory, but there are a lot of applications of graph theory in operational research, combinatorial optimization and bioinformatics.

Neutrosophy was introduced by Smarandache in 1995, as a new branch of philosophy, which is a generalization of dialectics. Neutrosophy is the base of neutrosophic set, neutrosophic logic, neutrosophic probability and statistics, and neutrosophic calculus that have many real applications. A single-valued neutrosophic set is a special neutrosophic set and can be used expediently to deal with the real-world problems, especially in decision support.

This book presents readers with fundamental concepts, including single-valued neutrosophic, neutrosophic graph structures, bipolar neutrosophic graphs, domination in bipolar neutrosophic graphs, bipolar neutrosophic planar graphs, interval-valued neutrosophic graphs, interval-valued neutrosophic graph structures, rough neutrosophic digraphs, neutrosophic rough digraphs, neutrosophic soft graphs and intuitionistic neutrosophic soft graphs. This book also presents practical applications of the concepts in real world. Therefore, the book presents a valuable contribution for students and researchers in neutrosophic graphs and their applications.

The author, Muhammad Akram, is a well-known international researcher in the field of neutrosophic graphs and he manifests a great enthusiasm and strong potential in developingthe neutrosophic environment and applying it to practical problems.

Gallup, USA

Florentin Smarandache
University of New Mexico

Preface

The concept of fuzzy sets was introduced by Zadeh in 1965. Since then, fuzzy sets and fuzzy logic have been applied in many real applications to handle uncertainty. The traditional fuzzy set uses one real value from the unit interval [0, 1] to represent the grade of membership of fuzzy set defined on the universe. In some applications, including an expert system, belief system and information fusion, we should consider not only the truth-membership supported by the evident but also the falsity-membership against by the evident. That is beyond the scope of fuzzy sets. In 1983, Atanassov introduced the intuitionistic fuzzy sets which are a generalization of fuzzy sets. The intuitionistic fuzzy sets consider both truth-membership ($T_A(x)$) and falsity-membership ($F_A(x)$) with $T_A(x)$, $F_A(x) \in [0, 1]$ and $T_A(x) + F_A(x) \leq 1$. Intuitionistic fuzzy sets can only handle incomplete information and not the indeterminate information and inconsistent information which exist commonly in the belief system. In intuitionistic fuzzy sets, hesitancy is $1 - T_A(x) - F_A(x)$ by default. In a neutrosophic set [163], indeterminacy is quantified explicitly and truth-membership, indeterminacy-membership and falsity-membership are independent. This assumption is very important in a lot of situations such as information fusion when we try to combine the data from different sensors. Neutrosophy was introduced by Smarandache in 1995. "It is a branch of philosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra". Neutrosophy is the base of neutrosophic set, neutrosophic logic, neutrosophic probability and statistics, and neutrosophic calculus. A single-valued neutrosophic set is a special neutrosophic set and can be used expediently to deal with the real-world problems, especially in decision support. Thus, a single-valued neutrosophic set is a powerful general formal framework which generalizes the concept of fuzzy set and intuitionistic fuzzy set. The work presented here intends to overcome the lack of a mathematical approach towards indeterminate information and inconsistent information. This monograph deals with single-valued neutrosophic graphs and their applications. It is based on a number of papers by the author, which have been published in various scientific journals. This book may be useful for researchers in mathematics, computer scientists and social scientists alike.

In Chap. 1, a concise review of the single-valued neutrosophic sets is presented. Certain types of single-valued neutrosophic (neutrosophic, for short) graphs are discussed. Applications of neutrosophic graphs are described. Moreover, the energy of neutrosophic graphs with applications is presented.

In Chap. 2, certain concepts of neutrosophic graph structures and some of their properties are presented. Moreover, some interesting applications of neutrosophic graph structures are discussed.

In Chap. 3, certain bipolar neutrosophic graphs are studied. Domination in bipolar neutrosophic graphs is presented. Bipolar neutrosophic planar graphs and bipolar neutrosophic line graphs are discussed. Further, some applications of bipolar neutrosophic graphs are described.

In Chap. 4, the concept of interval-valued neutrosophic graphs is presented. Certain types including k-competition interval-valued neutrosophic graphs, p-competition interval-valued neutrosophic graphs and m-step interval-valued neutrosophic competition graphs are discussed.

In Chap. 5, certain notions of interval-valued neutrosophic graph structures are presented. The concepts of interval-valued neutrosophic graph structures with examples are elaborated. Moreover, the concept of φ-complement of an interval-valued neutrosophic graph structure is discussed. Finally, some related properties, including φ-complement, totally self-complementary and totally strong self-complementary, of interval-valued neutrosophic graph structures are described.

In Chap. 6, the concepts of rough neutrosophic digraphs and neutrosophic rough digraphs are presented. Further, applications of rough neutrosophic digraphs and neutrosophic rough digraphs in decision-making problems are described. Moreover, comparative analysis of rough neutrosophic digraphs and neutrosophic rough digraphs is given.

In Chap. 7, the notions of neutrosophic soft graphs and intuitionistic neutrosophic soft graphs are presented. Further, applications of neutrosophic soft graphs and intuitionistic neutrosophic soft graphs are discussed. Moreover, the notion of neutrosophic soft rough graphs is described. Finally, in Chap. 8, applications of neutrosophic soft rough graphs are considered.

Acknowledgements

I am grateful to the administration of the University of the Punjab, who provided the facilities which were required for the successful completion of this monograph. I would like to express my gratitude to the researchers worldwide whose contributions are referenced in this book, especially L. A. Zadeh, K. T. Atanassov, Florentin Smarandache, Jun Ye and John Mordeson. I would like to acknowledge the assistance of my students Nabeela Ishfaq, Sidra Sayed, Musavarah Sarwar and Muzzamal Sitara.

Lahore, Pakistan Muhammad Akram

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About the Author

Muhammad Akram is Professor at the Department of Mathematics, University of the Punjab, Pakistan. He earned his Ph.D. in fuzzy mathematics from the Government College University, Pakistan. His research interests include numerical solutions of parabolic PDEs, fuzzy graphs, fuzzy algebras and new trends in fuzzy set theory. He has published six monographs and over 265 research articles in international peer-reviewed journals. He has been an editorial board member of 10 international academic journals and a reviewer/referee for 120 international journals, including *Mathematical Reviews* (USA) and *Zentralblatt MATH* (Germany). Seven students have successfully completed their Ph.D. research work under his supervision. Currently, he is supervising six Ph.D. students.

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Chapter 1
Graphs Under Neutrosophic Environment

In this chapter, we first present a concise review of neutrosophic sets. Then we present certain types of single-valued neutrosophic graphs (neutrosophic graphs, for short), including regular neutrosophic graphs, totally regular neutrosophic graphs, edge regular neutrosophic graphs, irregular neutrosophic graphs, highly totally irregular neutrosophic graphs, strongly totally irregular neutrosophic graphs, neighbourly edge irregular neutrosophic graphs and strongly edge irregular neutrosophic graphs. We describe applications of neutrosophic graphs. We also present energy of neutrosophic graphs with applications. This chapter is due to [27, 124, 167, 176].

1.1 Introduction

By a *graph*, we mean an ordered pair $G^* = (X, E)$ such that X is the collection of components taken as *nodes or vertices* and E is a relation on X, called *edges*. It is often convenient to depict the relationships between pairs of elements of a system by means of a graph or a digraph. The vertices of the graph represent the system elements, and its edges or arcs represent the relationships between the elements. This approach is especially useful for transportation, scheduling, sequencing, allocation, assignment and other problems which can be modelled as networks. Such a graph-theoretical model is often useful as an aid in communicating.

Zadeh [194] introduced the degree of membership/truth (T) in 1965 and defined the fuzzy set. Atanassov [47] introduced the degree of nonmembership/falsehood (F) in 1983 and defined the intuitionistic fuzzy set. Smarandache [163] introduced the degree of indeterminacy/neutrality (I) as independent component in 1995 and defined the neutrosophic set on three components (T, I, F) = (Truth, Indeterminacy, Falsity). Fuzzy set theory and intuitionistic fuzzy set theory are useful models for dealing with uncertainty and incomplete information. But they may not be sufficient in modelling of indeterminate and inconsistent information encountered in real world. In order to cope with this issue, neutrosophic (The words "neutrosophy" and "neutrosophic" were invented by Smarandache in 1995. Neutrosophy is a new branch of philosophy

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that studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. It is the base of neutrosophic logic, a multiple-value logic that generalizes the fuzzy logic and deals with paradoxes, contradictions, antitheses, antinomies) set theory was proposed by Smarandache. However, since neutrosophic sets are identified by three functions called truth-membership (T), indeterminacy-membership (I) and falsity-membership (F) whose values are real standard or non-standard subset of unit interval $]^-0, 1^+[$, where $^-0 = 0 - \epsilon$, $1^+ = 1 + \epsilon$, ϵ is an infinitesimal number. To apply neutrosophic set in real-life problems more conveniently, Smarandache [165] and Wang et al. [172] defined single-valued neutrosophic set which takes the value from the subset of [0, 1]. Thus, a single-valued neutrosophic set is an instance of neutrosophic set and can be used expediently to deal with real-world problems, especially in decision support.

A Geometric Interpretation of the Neutrosophic Set

We describe a geometric interpretation of the neutrosophic set using the neutrosophic cube $A'B'C'D'E'F'G'H'$ as shown in Fig. 1.1. In technical applications only the classical interval [0, 1] is used as range for the neutrosophic parameters T, I and F; we call the cube $ABCDEFGH$ the technical neutrosophic cube and its extension $A'B'C'D'E'F'G'H'$ the neutrosophic cube, used in the field where we need to differentiate between absolute and relative notions. Consider a 3D Cartesian system of coordinates, where T is the truth axis with value range in $]^-0, 1^+[$, F is the false

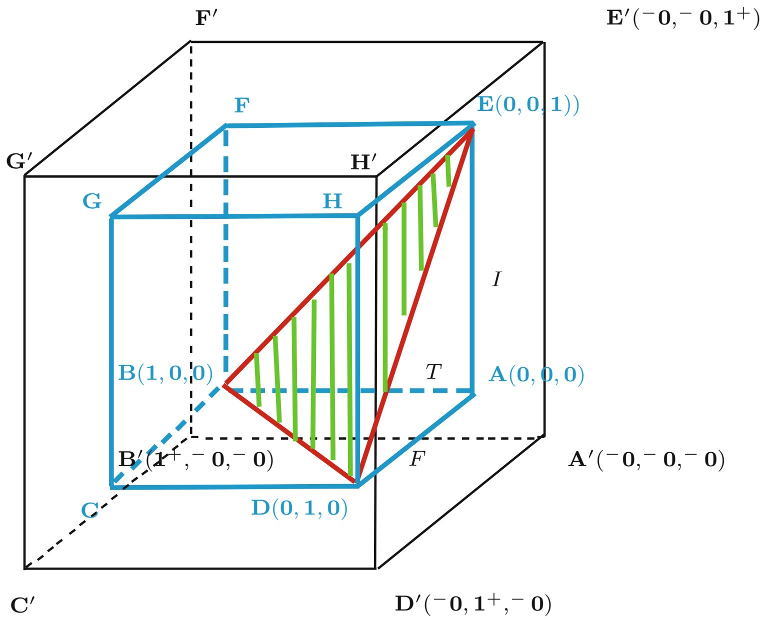


Fig. 1.1 A geometric interpretation of the neutrosophic set

axis with value range in $]^{-}0, 1^{+}[$, and I is the indeterminate axis with value range in $]^{-}0, 1^{+}[$.

We now divide the technical neutrosophic cube $ABCDEFGH$ into three disjoint regions:

1. The equilateral triangle BDE, whose sides are equal to $\sqrt{2}$, which represents the geometrical locus of the points whose sum of the coordinates is 1. If a point Q is situated on the sides of the triangle BDE or inside of it, then $T_Q + I_Q + F_Q = 1$.
2. The pyramid $EABD$ situated in the right side of the ΔEBD, including its faces ΔABD(base), ΔEBA and ΔEDA(lateral faces), but excluding its faces ΔBDE is the locus of the points whose sum of their coordinates is less than 1. If $P \in EABD$, then $T_P + I_P + F_P < 1$.
3. In the left side of ΔBDE in the cube, there is the solid $EFGCDEBD$ (excluding ΔBDE) which is the locus of points whose sum of their coordinates is greater than 1. If a point $R \in EFGCDEBD$, then $T_R + I_R + F_R > 1$.

It is possible to get the sum of coordinates strictly less than 1 or strictly greater than 1. For example:
(1) We have a source which is capable to find only the degree of membership of an element, but it is unable to find the degree of nonmembership.
(2) Another source which is capable to find only the degree of nonmembership of an element.
(3) Or a source which only computes the indeterminacy.
Thus, when we put the results together of these sources, it is possible that their sum is not 1, but smaller or greater.

On the other hand, in information fusion, when dealing with indeterminate models (i.e. elements of the fusion space which are indeterminate/unknown, such as intersections we do not know if they are empty or not since we do not have enough information, similarly for complements of indeterminate elements): if we compute the believe in that element (truth), the disbelieve in that element (falsehood) and the indeterminacy part of that element, then the sum of these three components is strictly less than 1 (the difference to 1 is the missing information).

Definition 1.1 Let X be a space of points (objects). A *single-valued neutrosophic set* A on a nonempty set X is characterized by a truth-membership function $T_A : X \to [0, 1]$, indeterminacy-membership function $I_A : X \to [0, 1]$ and a falsity-membership function $F_A : X \to [0, 1]$. Thus, $A = \{< x, T_A(x), I_A(x), F_A(x) > | x \in X\}$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$ for all $x \in X$.

When X is continuous, a single-valued neutrosophic set A can be written as

$$A = \int_X \langle (T(x), I(x), F(x))/x, x \in X \rangle.$$

When X is discrete, a single-valued neutrosophic set A can be written as

$$A = \sum_{i=1}^{n} \langle (T(x_i), I(x_i), F(x_i))/x_i, x_i \in X \rangle.$$

Example 1.1 Assume that the universe of discourse $X = \{x_1, x_2, x_3\}$, where x_1 describes the capability, x_2 describes the trustworthiness, and x_3 describes the prices of the objects. It may be further assumed that the values of x_1, x_2 and x_3 are in [0, 1] and they are obtained from some questionnaires of some experts. The experts may impose their opinion in three components, namely the degree of goodness, the degree of indeterminacy and that of poorness to explain the characteristics of the objects. Suppose A is a single-valued neutrosophic set of X such that

$$A = \{< x_1, 0.3, 0.5, 0.6 >, < x_2, 0.3, 0.2, 0.3 >, < x_3, 0.3, 0.5, 0.6 >\},$$

where $< x_1, 0.3, 0.5, 0.6 >$ represents that the degree of goodness of capability is 0.3, degree of indeterminacy of capability is 0.5 and degree of falsity of capability is 0.6.

Remark 1.1 When we consider that there are three different experts that are independent (i.e. they do not communicate with each other), so each one focuses on one attribute only (because each one is the best specialist in evaluating a single attribute). Therefore, each expert can assign 1 to his attribute value [for (1, 1, 1)], or each expert can assign 0 to his attribute value [for (0, 0, 0)], respectively.

When we consider a single expert for evaluating all three attributes, then he evaluates each attribute from a different point of view (using a different parameter)and arrives to (1, 1, 1) or (0, 0, 0), respectively.

For example, we examine a student "Muhammad"; for his research in neutrosophic graphs, he deserves 1; for his research in analytical mathematics, he also deserves 1; and for his research in physics, he deserves 1.

Definition 1.2 Let $A = \{< x, T_A(x), I_A(x), F_A(x) > |x \in X\}$ and $B = \{< x, T_B(x), I_B(x), F_B(x) > |x \in X\}$ be two single-valued neutrosophic sets, then operations are defined as follows:

- $A \subseteq B$ if and only if $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$,
- $A = B$ if and only if $T_A(x) = T_B(x)$, $I_A(x) = I_B(x)$ and $F_A(x) = F_B(x)$,
- $A \cap B = \{< x, \min(T_A(x), T_B(x)), \max(I_A(x), I_B(x)), \max(F_A(x), F_A(x)) > |x \in X\}$,
- $A \cup B = \{< x, \max(T_A(x), T_B(x)), \min(I_A(x), I_B(x)), \min(F_A(x), F_A(x)) > |x \in X\}$,
- $A^c = \{< x, F_A(x), 1 - I_A(x), T_A(x) > |x \in X\}$,
- $\mathbf{0} = (0, 1, 1)$ and $\mathbf{1} = (1, 0, 0)$.

Yang et al. [176] introduced the concept of single-valued neutrosophic relations.

Definition 1.3 A *single-valued neutrosophic relation* on a nonempty set X is a single-valued neutrosophic subset of $X \times X$ of the form

$$B = \{(yz, T_B(yz), I_B(yz), F_B(yz)) : yz \in X \times X\},$$

where $T_B : X \times X \to [0, 1]$, $I_B : X \times X \to [0, 1]$, $F_B : X \times X \to [0, 1]$ denote the truth-membership function, indeterminacy-membership function and falsity-membership function of B, respectively.

Definition 1.4 Let B be a single-valued neutrosophic relation in X, the complement and inverse of B are defined as follows, respectively

$$B^c = \{\langle (x, y), T_{R^c}(x, y), I_{R^c}(x, y), F_{R^c}(x, y)\rangle | (x, y) \in X \times X\}, \ \forall (x, y) \in X \times X,$$

where

$$T_{R^c}(x, y) = F_R(x, y),$$
$$I_{R^c}(x, y) = 1 - I_R(x, y),$$
$$F_{R^c}(x, y) = T_R(x, y).$$

$$B^{-1} = \{\langle (x, y), T_{R^{-1}}(x, y), I_{R^{-1}}(x, y), F_{R^{-1}}(x, y)\rangle | (x, y) \in X \times X\}, \ \forall (x, y) \in X \times X,$$

where

$$T_{R^{-1}}(x, y) = T_R(y, x),$$
$$I_{R^{-1}}(x, y) = I_R(y, x),$$
$$F_{R^{-1}}(x, y) = F_R(y, x).$$

Example 1.2 Let $X = \{x_1, x_2, x_3, x_4, x_5\}$. A single-valued neutrosophic relation B in X is given in Table 1.1. By Definition 1.4, we can compute B^c and B^{-1} which are given in Tables 1.2 and 1.3, respectively.

Definition 1.5 Let R, S be two single-valued neutrosophic relations in X.

1. The union $R \cup S$ of R and S is defined by

$$R \cup S = \{\langle (x, y); \max\{T_R(x, y), T_S(x, y)\}; \min\{I_R(x, y), I_S(x, y)\}; \min\{F_R(x, y), F_S(x, y)\}\rangle | (x, y) \in X \times X\}.$$

2. The intersection $R \cap S$ of R and S is defined by

$$R \cap S = \{\langle (x, y); \min\{T_R(x, y), T_S(x, y)\}; \max\{I_R(x, y), I_S(x, y)\}; \max\{F_R(x, y), F_S(x, y)\}\rangle | (x, y) \in X \times X\}.$$

Definition 1.6 Let R be a single-valued neutrosophic relation in X.

1. If $\forall x \in X, T_R(x, x) = 1$ and $I_R(x, x) = F_R(x, x) = 0$, then R is called a *reflexive single-valued neutrosophic relation*.

Table 1.1 Single-valued neutrosophic relation B

B	x_1	x_2	x_3	x_4	x_5
x_1	(0.2, 0.6, 0.4)	(0, 0.3, 0.7)	(0.9, 0.2, 0.4)	(0.3, 0.9, 1)	(1, 0.2, 0)
x_2	(0.4, 0.5, 0.1)	(0.1, 0.7, 0)	(1, 1, 1)	(1, 0.3, 0)	(0.5, 0.6, 1)
x_3	(0, 1, 1)	(1, 0.5, 0)	(0, 0, 0)	(0.2, 0.8, 0.1)	(1, 0.8, 1)
x_4	(1, 0, 0)	(0, 0, 1)	(0.5, 0.7, 0.1)	(0.1, 0.4, 1)	(1, 0.8, 0.8)
x_5	(0, 1, 0)	(0.9, 0, 0)	(0, 0.1, 0.7)	(0.8, 0.9, 1)	(0.6, 1, 0)

Table 1.2 Complement B^c of B

B	x_1	x_2	x_3	x_4	x_5
x_1	(0.4, 0.4, 0.2)	(0.7, 0.7, 0)	(0.4, 0.8, 0.9)	(0.1, 0.1, 3)	(0, 0.8, 1)
x_2	(0.1, 0.5, 0.4)	(0, 0.3, 0.1)	(1, 0, 1)	(0, 0.7, 1)	(1, 0.4, 0.5)
x_3	(1, 0, 0)	(0, 0.5, 1)	(0, 1, 0)	(0.1, 0.2, 0.2)	(1, 0.2, 1)
x_4	(0, 1, 1)	(1, 1, 0)	(0.1, 0.3, 0.5)	(1, 0.6, 0.4)	(0.8, 0.2, 1)
x_5	(0, 0, 0)	(0, 1, 0.9)	(0.7, 0.9, 0)	(1, 0.1, 0.8)	(0, 0, 0.6)

Table 1.3 Inverse B^- of B

B	x_1	x_2	x_3	x_4	x_5
x_1	(0.2, 0.6, 0.4)	(0.4, 0.5, 0.1)	(0, 1, 1)	(1, 0, 0)	(0, 1, 0)
x_2	(0, 0.3, 0.7)	(0.1, 0.7, 0)	(1, 0.5, 0)	(0, 0, 1)	(0.9, 0, 0)
x_3	(0.9, 0.2, 0.4)	(1, 1, 1)	(0, 0, 0)	(0.5, 0.7, 0.1)	(0, 0.1, 0.7)
x_4	(0.3, 0.9, 1)	(1, 0.3, 0)	(0.2, 0.8, 0.1)	(0.1, 0.4, 1)	(0.8, 0.9, 1)
x_5	(1, 0.2, 0)	(0.5, 0.6, 1)	(1, 0.8, 1)	(1, 0.8, 0.8)	(0.6, 1, 0)

2. If $\forall\, x, y \in X, T_R(x, y) = T_R(y, x), I_R(x, y) = I_R(y, x)$ and $F_R(y, x) = F_R(x, y)$, then R is called a *symmetric single-valued neutrosophic relation*.
3. If $\forall\, x \in X, T_R(x, x) = 0$ and $I_R(x, x) = F_R(x, x) = 1$, then R is called an *antireflexive single-valued neutrosophic relation*.
4. If $\forall\, x, y, z \in X$,

$$\max_{v \in X} \min\{T_R(x, y), T_R(y, z)\} \leq T_R(x, z),$$
$$\min_{v \in X} \max\{I_R(x, y), I_R(y, z)\} \geq I_R(x, z),$$
$$\min_{v \in X} \max\{F_R(x, y), F_R(y, z)\} \geq F_R(x, z),$$

then R is called a *transitive single-valued neutrosophic relation*.

1.2 Certain Types of Neutrosophic Graphs

Definition 1.7 A *single-valued neutrosophic graph* on a nonempty X is a pair $G = (A, B)$, where A is single-valued neutrosophic set in X and B single-valued neutrosophic relation on X such that

$$T_B(xy) \leq \min\{T_A(x), T_A(y)\},$$
$$I_B(xy) \leq \min\{I_A(x), I_A(y)\},$$
$$F_B(xy) \leq \max\{F_A(x), F_A(y)\}$$

for all $x, y \in X$. A is called *single-valued neutrosophic vertex set* of G and B is called *single-valued neutrosophic edge set* of G, respectively.

Remark 1.2
1. B is called symmetric single-valued neutrosophic relation on A.
2. If B is not symmetric single-valued neutrosophic relation on A, then $G = (A, B)$ is called a *single-valued neutrosophic directed graph (digraph).*
3. X and E are underlying vertex set and underlying edge set of G, respectively.

Throughout this chapter, we will use neutrosophic set, neutrosophic relation and neutrosophic graph, for short.

Example 1.3 Consider a crisp graph $G^* = (X, E)$ such that $X = \{a, b, c, d, e, f\}$, $E = \{ab, ac, bd, cd, be, cf, ef, bc\}$. Let A and B be the neutrosophic sets of X and E, respectively, as shown in Table 1.4. By simple calculations, it is easy to see that $G = (A, B)$ is a neutrosophic graph as shown in Fig. 1.2.

Definition 1.8 A neutrosophic graph $G = (A, B)$ is called *complete* if the following conditions are satisfied:

$$T_B(xy) = \min\{T_A(x), T_A(y)\},$$
$$I_B(xy) = \min\{I_A(x), I_A(y)\},$$

Table 1.4 Neutrosophic sets

A	a	b	c	d	e	f		
T_A	0.2	0.3	0.4	0.3	0.5	0.4		
I_A	0.5	0.4	0.5	0.6	0.5	0.6		
F_A	0.7	0.6	0.4	0.8	0.6	0.6		
B	ab	ac	bd	cd	be	cf	ef	bc
T_B	0.2	0.1	0.2	0.3	0.2	0.1	0.4	0.2
I_B	0.4	0.4	0.2	0.2	0.3	0.4	0.4	0.3
F_B	0.7	0.5	0.6	0.7	0.5	0.5	0.5	0.6

Fig. 1.2 Neutrosophic graph

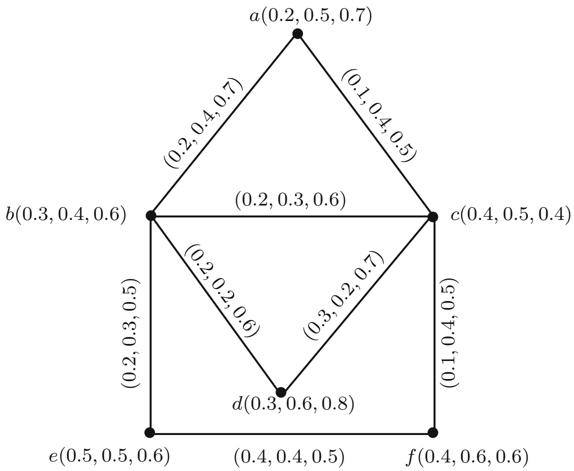
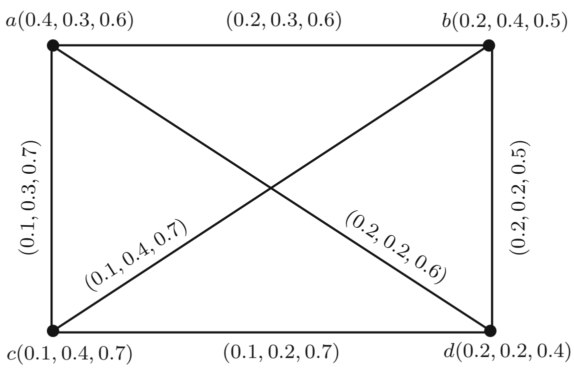


Fig. 1.3 Complete neutrosophic graph



$$F_B(xy) = \max\{F_A(x), F_A(y)\},$$

for all $x, y \in X$.

Example 1.4 Consider a neutrosophic $G=(A, B)$ on the nonempty set $X = \{a, b, c, d\}$ as shown in Fig. 1.3. By direct calculations, it is easy to see that G is a complete.

Definition 1.9 Let $A = \{< x, T_A(x), I_A(x), F_A(x) >, x \in X\}$ be a neutrosophic set of the set X. For $\alpha \in [0, 1]$, the α-cut of A is the crisp set A_α defined by

$$A_\alpha = \{x \in X : \text{either } (T_A(x), I_A(x) \geq \alpha) \text{ or } F_A(x) \leq 1 - \alpha\}.$$

Let $B = \{< xy, T_B(xy), I_B(xy), F_B(xy) >\}$ be a neutrosophic set on $E \subseteq X \times X$. For $\alpha \in [0, 1]$, the α-cut is the crisp set B_α defined by

$$B_\alpha = \{xy \in E : \text{either } (T_B(xy), I_B(xy) \geq \alpha) \text{ or } F_B(xy) \leq 1 - \alpha\}.$$

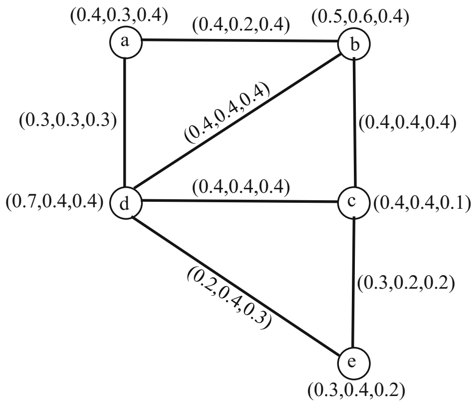


Fig. 1.4 Neutrosophic graph and 0.4-level graph $G_{0.4}$

$G_\alpha = (A_\alpha, B_\alpha)$ is a subgraph of crisp graph G^*.

Example 1.5 Consider a neutrosophic graph G on nonempty set $X = \{a, b, c, d, e\}$ as shown in Fig. 1.4.

For $\alpha = 0.4$, we have

$$A_{0.4} = \{b, c, d\},$$
$$B_{0.4} = \{bc, cd, bd\}.$$

Clearly, the 0.4-level graph $G_{0.4} = (A_{0.4}, B_{0.4})$ is a subgraph of crisp graph G^*.

Definition 1.10 The *order and the size* of a neutrosophic graph G are denoted by $O(G)$ and $S(G)$, respectively, and are defined as

$$O(G) = \left(\sum_{s\in X} T_A(s), \sum_{s\in X} I_A(s), \sum_{s\in X} F_A(s)\right),$$
$$S(G) = \left(\sum_{st\in E} T_B(st), \sum_{st\in E} I_B(st), \sum_{st\in E} F_B(st)\right).$$

Definition 1.11 The *degree and the total degree* of a vertex s of a neutrosophic graph G are denoted by $d_G(s) = (d_T(s), d_I(s), d_F(s))$ and $Td_G(s) = (Td_T(s), Td_I(s), Td_F(s))$, respectively, and are defined as

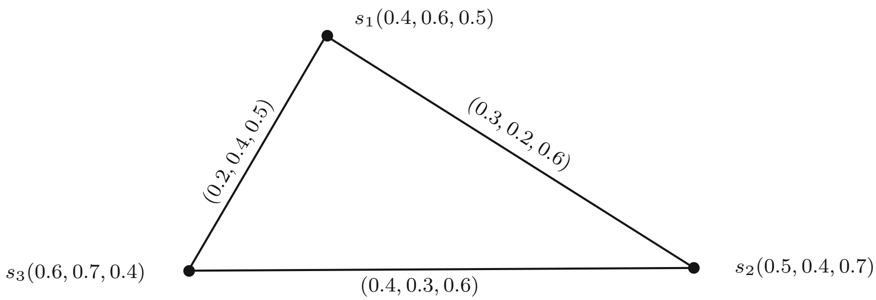


Fig. 1.5 Neutrosophic graph

$$d_G(s) = \left(\sum_{s \neq t} T_B(st), \sum_{s \neq t} I_B(st), \sum_{s \neq t} F_B(st) \right),$$

$$Td_G(s) = \left(\sum_{s \neq t} T_B(st) + T_A(s), \sum_{s \neq t} I_B(st) + I_A(s), \sum_{s \neq t} F_B(st) + F_A(s) \right),$$

for $st \in E$, where $s \in X$.

Example 1.6 Consider a neutrosophic graph G on the nonempty set $X = \{s_1, s_2, s_3\}$ as shown in Fig. 1.5.

By direct calculations, we have O(G) = (1.5, 1.7, 1.6), S(G) = (0.9, 0.9, 1.7),

$$d_G(s_1) = (0.5, 0.6, 1.1),\ d_G(s_2) = (0.7, 0.5, 1.2),\ d_G(s_3) = (0.6, 0.7, 1.1),$$
$$Td_G(s_1) = (0.9, 1.2, 1.6),\ Td_G(s_2) = (1.2, 0.9, 1.9),\ Td_G(s_3) = (1.2, 1.4, 1.5).$$

Definition 1.12 A neutrosophic graph G is called a *regular* if each vertex has same degree, that is,

$$d_G(s) = (m_1, m_2, m_3), \quad \text{for all } s \in X.$$

Example 1.7 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.6.

By direct calculations, we have

$$d_G(s_1) = (0.2, 1.2, 0.8) = d_G(s_2) = d_G(s_3) = d_G(s_4).$$

Hence G is a regular neutrosophic graph.

Definition 1.13 A neutrosophic graph G is called a *totally regular* of degree (n_1, n_2, n_3) if

$$Td_G(s) = (n_1, n_2, n_3), \text{ for all } s \in X.$$

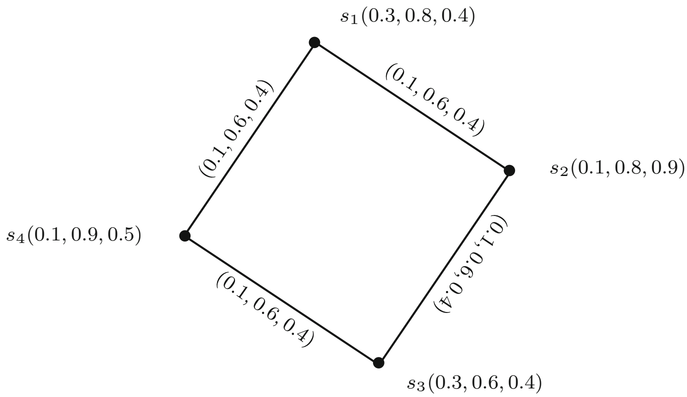


Fig. 1.6 Regular neutrosophic graph

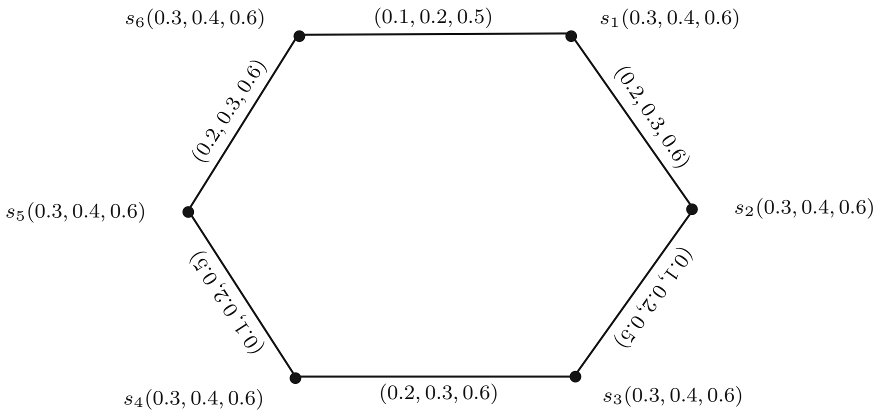


Fig. 1.7 Totally regular neutrosophic graph

Example 1.8 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ as shown in Fig. 1.7.

By direct calculations, we have

$$d_G(s_1) = (0.3, 0.5, 1.1) = d_G(s_2) = d_G(s_3) = d_G(s_4) = d_G(s_5) = d_G(s_6),$$
$$Td_G(s_1) = (0.6, 0.9, 1.7) = Td_G(s_2) = Td_G(s_3) = Td_G(s_4) = Td_G(s_5) = Td_G(s_6).$$

Hence G is a totally regular neutrosophic graph.

Remark 1.3 The above two concepts are independent; that is, it is not necessary that totally regular neutrosophic graph is regular neutrosophic graph and vice versa.

Example 1.9 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.8.

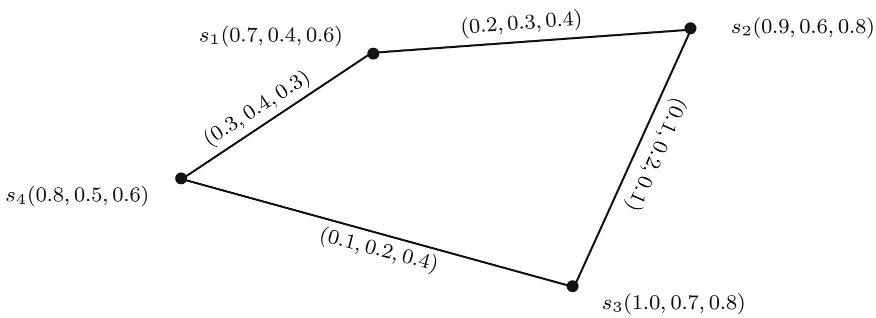


Fig. 1.8 Totally regular but not regular neutrosophic graph

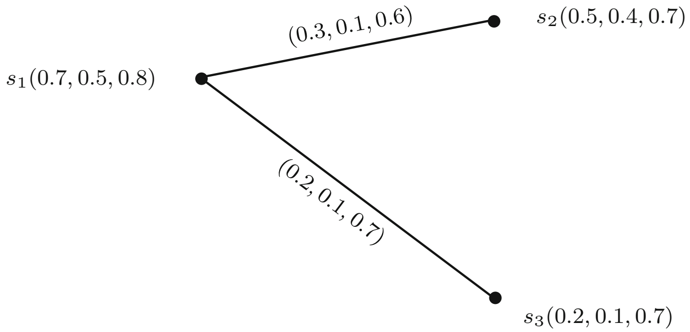


Fig. 1.9 Neutrosophic graph

By direct calculations, we have

$$\begin{aligned} d_G(s_1) &= (0.5, 0.7, 0.7), d_G(s_2) = (0.3, 0.5, 0.5), \\ d_G(s_3) &= (0.2, 0.4, 0.5), d_G(s_4) = (0.4, 0.6, 0.7), \\ Td_G(s_1) &= (1.2, 1.1, 1.3) = Td_G(s_2) = Td_G(s_3) = Td_G(s_4). \end{aligned}$$

Therefore, G is a totally regular neutrosophic graph but not a regular neutrosophic graph.

Definition 1.14 The degree and the total degree of an edge st of a neutrosophic graph G are denoted by $d_G(st) = (d_T(st), d_I(st), d_F(st))$ and $Td_G(st) = (Td_T(st), Td_I(st), Td_F(st))$, respectively, and are defined as

$$\begin{aligned} d_G(st) &= d_G(s) + d_G(t) - 2(T_B(st), I_B(st), F_B(st)), \\ Td_G(st) &= d_G(st) + (T_B(st), I_B(st), F_B(st)). \end{aligned}$$

Example 1.10 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3\}$ as shown in Fig. 1.9.

By direct calculations, we have

$$d_G(s_1) = (0.5, 0.2, 1.3), d_G(s_2) = (0.3, 0.1, 0.6), d_G(s_3) = (0.2, 0.1, 0.7).$$

- The degree of each edge is given as:

$$\begin{aligned} d_G(s_1s_2) &= d_G(s_1) + d_G(s_2) - 2(T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)), \\ &= (0.7, 0.5, 0.8) + (0.5, 0.4, 0.7) - 2(0.3, 0.1, 0.6), \\ &= (0.2, 0.1, 0.7). \end{aligned}$$

$$\begin{aligned} d_G(s_1s_3) &= d_G(s_1) + d_G(s_3) - 2(T_B(s_1s_3), I_B(s_1s_3), F_B(s_1s_3)), \\ &= (0.7, 0.5, 0.8) + (0.4, 0.2, 0.6) - 2(0.2, 0.1, 0.7), \\ &= (0.3, 0.1, 0.6). \end{aligned}$$

- The total degree of each edge is given as:

$$\begin{aligned} Td_G(s_1s_2) &= d_G(s_1s_2) + (T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)), \\ &= (0.2, 0.1, 0.7) + (0.3, 0.1, 0.6), \\ &= (0.5, 0.2, 1.3). \end{aligned}$$

$$\begin{aligned} Td_G(s_1s_3) &= d_G(s_1s_3) + (T_B(s_1s_3), I_B(s_1s_3), F_B(s_1s_3)), \\ &= (0.3, 0.1, 0.6) + (0.2, 0.1, 0.7), \\ &= (0.5, 0.2, 1.3). \end{aligned}$$

Definition 1.15 The maximum degree of a neutrosophic graph G is defined as $\Delta(G) = (\Delta_T(G), \Delta_I(G), \Delta_F(G))$, where

$$\begin{aligned} \Delta_T(G) &= \max\{d_T(s) : s \in X\}, \\ \Delta_I(G) &= \max\{d_I(s) : s \in X\}, \\ \Delta_F(G) &= \max\{d_F(s) : s \in X\}. \end{aligned}$$

Definition 1.16 The minimum degree of a neutrosophic graph G is defined as $\delta(G) = (\delta_T(G), \delta_I(G), \delta_F(G))$, where

$$\begin{aligned} \delta_T(G) &= \min\{d_T(s) : s \in X\}, \\ \delta_I(G) &= \min\{d_I(s) : s \in X\}, \\ \delta_F(G) &= \min\{d_F(s) : s \in X\}. \end{aligned}$$

Example 1.11 Consider the neutrosophic graph G as shown in Fig. 1.9. By direct calculations, we have

$$\Delta(G) = (0.5, 0.2, 1.3) \text{ and } \delta(G) = (0.2, 0.1, 0.6).$$

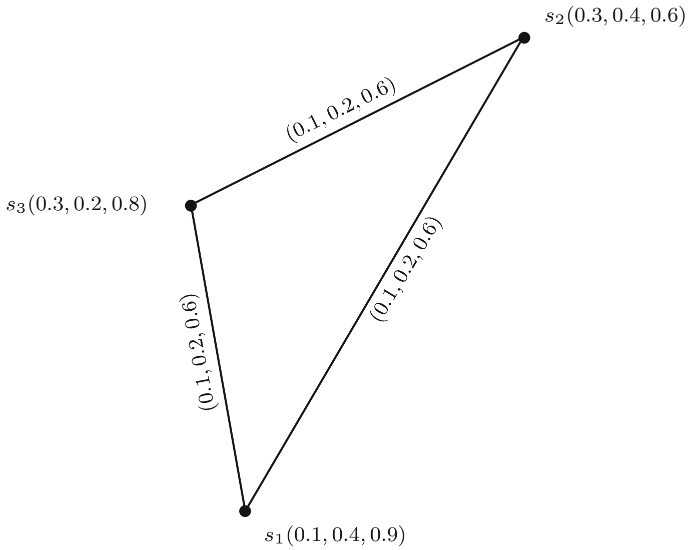


Fig. 1.10 Edge regular neutrosophic graph

Definition 1.17 A neutrosophic graph G on X is called an *edge regular* if every edge in G has the same degree (q_1, q_2, q_3).

Example 1.12 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3\}$ as shown in Fig. 1.10.

By direct calculations, we have

$$d_G(s_1) = (0.2, 0.4, 1.2), \quad d_G(s_2) = (0.2, 0.4, 1.2), \quad d_G(s_3) = (0.2, 0.4, 1.2).$$

The degree of each edge is given below:

$$\begin{aligned} d_G(s_1s_2) &= d_G(s_1) + d_G(s_2) - 2(T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)), \\ &= (0.2, 0.4, 1.2) + (0.2, 0.4, 1.2) - 2(0.1, 0.2, 0.6), \\ &= (0.2, 0.4, 1.2). \end{aligned}$$

$$\begin{aligned} d_G(s_1s_3) &= d_G(s_1) + d_G(s_3) - 2(T_B(s_1s_3), I_B(s_1s_3), F_B(s_1s_3)), \\ &= (0.2, 0.4, 1.2) + (0.2, 0.4, 1.2) - 2(0.1, 0.2, 0.6), \\ &= (0.2, 0.4, 1.2). \end{aligned}$$

$$\begin{aligned} d_G(s_2s_3) &= d_G(s_2) + d_G(s_3) - 2(T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)), \\ &= (0.2, 0.4, 1.2) + (0.2, 0.4, 1.2) - 2(0.1, 0.2, 0.6), \\ &= (0.2, 0.4, 1.2). \end{aligned}$$

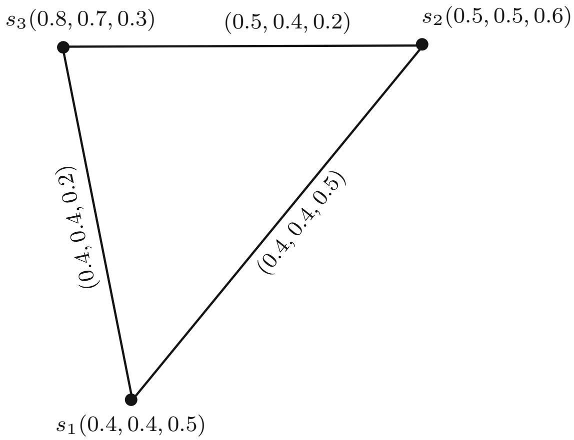


Fig. 1.11 Totally edge regular neutrosophic graph

It is easy to see that each edge of neutrosophic graph G has the same degree. Hence G is an edge regular neutrosophic graph.

Definition 1.18 A neutrosophic graph G on X is called a *totally edge regular* if every edge in G has the same total degree (p_1, p_2, p_3).

Example 1.13 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3\}$ as shown in Fig. 1.11.

By direct calculations, we have

$$d_G(s_1) = (0.8, 0.8, 0.7), \quad d_G(s_2) = (0.9, 0.8, 0.7), \quad d_G(s_3) = (0.9, 0.8, 0.4).$$

- The degree of each edge is given below:

$$\begin{aligned} d_G(s_1s_2) &= d_G(s_1) + d_G(s_2) - 2(T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)), \\ &= (0.8, 0.8, 0.7) + (0.9, 0.8, 0.7) - 2(0.4, 0.4, 0.5), \\ &= (0.9, 0.8, 0.4). \end{aligned}$$

$$\begin{aligned} d_G(s_1s_3) &= d_G(s_1) + d_G(s_3) - 2(T_B(s_1s_3), I_B(s_1s_3), F_B(s_1s_3)), \\ &= (0.8, 0.8, 0.7) + (0.9, 0.8, 0.4) - 2(0.4, 0.4, 0.2), \\ &= (0.9, 0.8, 0.7). \end{aligned}$$

$$\begin{aligned} d_G(s_2s_3) &= d_G(s_2) + d_G(s_3) - 2(T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)), \\ &= (0.9, 0.8, 0.7) + (0.9, 0.8, 0.4) - 2(0.5, 0.4, 0.2), \\ &= (0.8, 0.8, 0.7). \end{aligned}$$

It is easy to see that $d_G(s_1s_2) \neq d_G(s_1s_3) \neq d_G(s_2s_3)$. So G is not an edge regular neutrosophic graph.

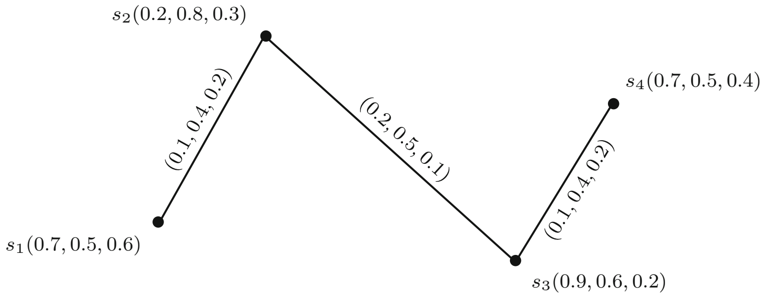


Fig. 1.12 Edge irregular and totally edge irregular neutrosophic graph

- The total degree of each edge is calculated as:

$$\begin{aligned} Td_G(s_1s_2) &= d_G(s_1s_2) + (T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)), \\ &= (1.3, 1.2, 0.9). \end{aligned}$$

$$\begin{aligned} Td_G(s_1s_3) &= d_G(s_1s_3) + (T_B(s_1s_3), I_B(s_1s_3), F_B(s_1s_3)), \\ &= (1.3, 1.2, 0.9). \end{aligned}$$

$$\begin{aligned} Td_G(s_2s_3) &= d_G(s_2s_3) + (T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)), \\ &= (1.3, 1.2, 0.9). \end{aligned}$$

It is easy to see that each edge of neutrosophic graph G has the same total degree. So G is a totally edge regular neutrosophic graph.

Remark 1.4 A neutrosophic graph G is an edge regular neutrosophic graph if and only if $\Delta_d(G) = \delta_d(G) = (q_1, q_2, q_3)$.

Example 1.14 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.12.

By direct calculations, we have

$$d_G(s_1) = (0.1, 0.4, 0.2), \quad d_G(s_2) = (0.3, 0.9, 0.3),$$

$$d_G(s_3) = (0.3, 0.9, 0.3), \quad d_G(s_4) = (0.1, 0.4, 0.2).$$

- The degree of each edge is given below:

$$\begin{aligned} d_G(s_1s_2) &= d_G(s_1) + d_G(s_2) - 2(T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)), \\ &= (0.1, 0.4, 0.2) + (0.3, 0.9, 0.3) - 2(0.1, 0.4, 0.2), \\ &= (0.2, 0.5, 0.1). \end{aligned}$$

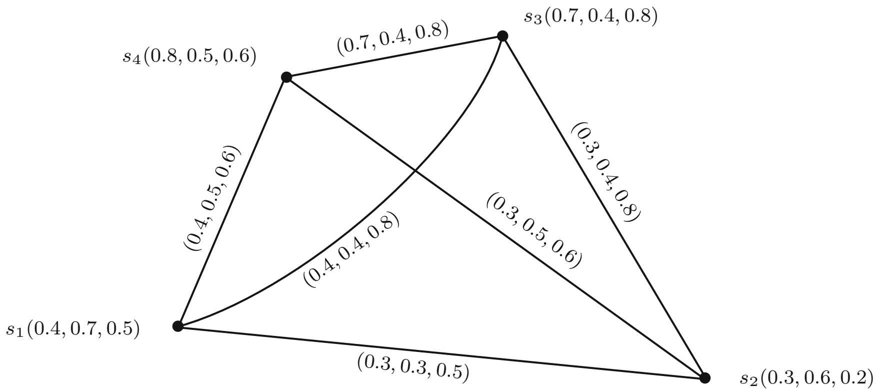


Fig. 1.13 Complete neutrosophic graph

$$\begin{aligned} d_G(s_2s_3) &= d_G(s_2) + d_G(s_3) - 2(T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)), \\ &= (0.3, 0.9, 0.3) + (0.3, 0.9, 0.3) - 2(0.2, 0.5, 0.1), \\ &= (0.2, 0.8, 0.4). \end{aligned}$$

$$\begin{aligned} d_G(s_3s_4) &= d_G(s_3) + d_G(s_4) - 2(T_B(s_3s_4), I_B(s_3s_4), F_B(s_3s_4)), \\ &= (0.3, 0.9, 0.3) + (0.1, 0.4, 0.2) - 2(0.1, 0.4, 0.2), \\ &= (0.2, 0.5, 0.1). \end{aligned}$$

It is easy to see that $d_G(s_1s_2) \neq d_G(s_2s_3)$. So G is not an edge regular neutrosophic graph.

- The total degree of each edge is calculated as:

$$\begin{aligned} Td_G(s_1s_2) &= d_G(s_1s_2) + (T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)), \\ &= (0.3, 0.9, 0.3). \end{aligned}$$

$$\begin{aligned} Td_G(s_2s_3) &= d_G(s_2s_3) + (T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)), \\ &= (0.4, 1.3, 0.5). \end{aligned}$$

$$\begin{aligned} Td_G(s_3s_4) &= d_G(s_3s_4) + (T_B(s_3s_4), I_B(s_3s_4), F_B(s_3s_4)), \\ &= (0.3, 0.9, 0.3). \end{aligned}$$

It is easy to see that $Td_G(s_1s_2) \neq Td_G(s_2s_3)$. So G is not a totally edge regular neutrosophic graph.

Remark 1.5 A complete neutrosophic graph G may not be an edge regular neutrosophic graph.

Example 1.15 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.13.

By direct calculations, we have

$$d_G(s_1) = (1.1, 1.2, 1.9), \quad d_G(s_2) = (0.9, 1.2, 1.9),$$

$$d_G(s_3) = (1.4, 1.2, 2.4), \quad d_G(s_4) = (1.4, 1.4, 2.0).$$

The degree of each edge is given below:

$$\begin{aligned} d_G(s_1s_2) &= d_G(s_1) + d_G(s_2) - 2(T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)),\\ &= (1.1, 1.2, 1.9) + (0.9, 1.2, 1.9) - 2(0.3, 0.3, 0.5),\\ &= (1.4, 2.0, 2.8). \end{aligned}$$

$$\begin{aligned} d_G(s_1s_3) &= d_G(s_1) + d_G(s_3) - 2(T_B(s_1s_3), I_B(s_1s_3), F_B(s_1s_3)),\\ &= (1.1, 1.2, 1.9) + (1.4, 1.2, 2.4) - 2(0.4, 0.4, 0.8),\\ &= (1.7, 1.6, 2.7). \end{aligned}$$

$$\begin{aligned} d_G(s_1s_4) &= d_G(s_1) + d_G(s_4) - 2(T_B(s_1s_4), I_B(s_1s_4), F_B(s_1s_4)),\\ &= (1.1, 1.2, 1.9) + (1.4, 1.4, 2.0) - 2(0.4, 0.5, 0.6),\\ &= (1.7, 1.6, 2.7). \end{aligned}$$

$$\begin{aligned} d_G(s_2s_3) &= d_G(s_2) + d_G(s_3) - 2(T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)),\\ &= (0.9, 1.2, 1.9) + (1.4, 1.2, 2.4) - 2(0.3, 0.4, 0.8),\\ &= (1.7, 1.6, 2.7). \end{aligned}$$

$$\begin{aligned} d_G(s_2s_4) &= d_G(s_2) + d_G(s_4) - 2(T_B(s_2s_4), I_B(s_2s_4), F_B(s_2s_4)),\\ &= (0.9, 1.2, 1.9) + (1.4, 1.4, 2.0) - 2(0.3, 0.5, 0.6),\\ &= (1.7, 1.6, 2.7). \end{aligned}$$

$$\begin{aligned} d_G(s_3s_4) &= d_G(s_3) + d_G(s_4) - 2(T_B(s_3s_4), I_B(s_3s_4), F_B(s_3s_4)),\\ &= (1.4, 1.2, 2.4) + (1.4, 1.4, 2.0) - 2(0.7, 0.4, 0.8),\\ &= (1.4, 1.8, 2.8). \end{aligned}$$

It is easy to see that each edge of neutrosophic graph G has not the same degree. Therefore, G is a complete neutrosophic graph but not an edge regular neutrosophic graph.

Theorem 1.1 *Let G be a neutrosophic graph. Then*

$$\sum_{st\in E} d_G(st) = \sum_{st\in E} d_{G^*}(st)(T_B(st), I_B(st), F_B(st)),$$

where $d_{G^*}(st) = d_{G^*}(s) + d_{G^*}(t) - 2$, *for all* $s, t \in X$.

Theorem 1.2 *Let G be a neutrosophic graph. Then*

$$\sum_{st\in E} Td_G(st) = \sum_{st\in E} d_{G^*}(st)(T_B(st), I_B(st), F_B(st)) + S(G),$$

where $d_{G^*}(st) = d_{G^*}(s) + d_{G^*}(t) - 2$*, for all* $s, t \in X$.

Proof Since the total degree of each edge in a neutrosophic graph G is $Td_G(st) = d_G(st) + (T_B(st), I_B(st), F_B(st))$. Therefore,

$$\begin{aligned}
\sum_{st\in E} Td_G(st) &= \sum_{st\in E}(d_G(st) + (T_B(st), I_B(st), F_B(st))),\\
\sum_{st\in E} Td_G(st) &= \sum_{st\in E} d_G(st) + \sum_{st\in E}(T_B(st), I_B(st), F_B(st)),\\
\sum_{st\in E} Td_G(st) &= \sum_{st\in E} d_{G^*}(st)(T_B(st), I_B(st), F_B(st)) + S(G).
\end{aligned}$$

This completes the proof.

Theorem 1.3 *Let* $G^* = (X, E)$ *be an edge regular crisp graph of degree* q *and* G *be an edge regular neutrosophic graph of degree* (q_1, q_2, q_3) *of* G^**. Then the size of* G *is* $(\frac{mq_1}{q}, \frac{mq_2}{q}, \frac{mq_3}{q})$*, where* $|E| = m$.

Proof Let G be an edge regular neutrosophic graph. Then,

$$d_G(st) = (q_1, q_2, q_3) \text{ and } d_{G^*}(st) = q, \quad \text{for each edge } st \in E.$$

Since,

$$\begin{aligned}
\sum_{st\in E} d_G(st) &= \sum_{st\in E} d_{G^*}(st)(T_B(st), I_B(st), F_B(st)),\\
\sum_{st\in E}(q_1, q_2, q_3) &= q\sum_{st\in E}(T_B(st), I_B(st), F_B(st)),\\
m(q_1, q_2, q_3) &= qS(G),\\
(mq_1, mq_2, mq_3) &= qS(G),\\
S(G) &= \left(\frac{mq_1}{q}, \frac{mq_2}{q}, \frac{mq_3}{q}\right).
\end{aligned}$$

This completes the proof.

Theorem 1.4 *Let* $G^* = (X, E)$ *be an edge regular crisp graph of degree* q *and* G *be a totally edge regular neutrosophic graph of degree* (p_1, p_2, p_3) *of* G^**. Then the size of* G *is* $(\frac{mp_1}{q+1}, \frac{mp_2}{q+1}, \frac{mp_3}{q+1})$*, where* $|E| = m$.

Proof Let G be a totally edge regular neutrosophic graph of an edge regular crisp graph $G^* = (X, E)$. Therefore,

$$d_G(st) = (p_1, p_2, p_3) \text{ and } d_{G^*}(st) = q, \quad \text{for each edge } st \in E.$$

Since,

$$\begin{aligned}
\sum_{st\in E} Td_G(st) &= \sum_{st\in E} d_{G^*}(st)(T_B(st), I_B(st), F_B(st)) + S(G),\\
\sum_{st\in E}(p_1, p_2, p_3) &= q\sum_{st\in E}(T_B(st), I_B(st), F_B(st)) + S(G),\\
m(p_1, p_2, p_3) &= qS(G) + S(G),\\
(mp_1, mp_2, mp_3) &= (q+1)S(G),\\
S(G) &= \left(\frac{mp_1}{q+1}, \frac{mp_2}{q+1}, \frac{mp_3}{q+1}\right).
\end{aligned}$$

This completes the proof.

Theorem 1.5 *Suppose that G is an edge regular neutrosophic graph of degree (q_1, q_2, q_3) and a totally edge regular neutrosophic graph of degree (p_1, p_2, p_3) of G^*. Then, the size of G is $m(p_1 - q_1, p_2 - q_2, p_3 - q_3)$, where $|E| = m$.*

Proof Let G be an edge regular neutrosophic graph and a totally edge regular neutrosophic graph of a crisp graph $G^* = (X, E)$. Therefore,

$$d_G(st) = (q_1, q_2, q_3) \text{ and } Td_G(st) = (p_1, p_2, p_3), \quad \text{for each edge } st \in E.$$

$$\begin{aligned}
Td_G(st) &= d_G(st) + (T_B(st), I_B(st), F_B(st)),\\
\sum_{st\in E} Td_G(st) &= \sum_{st\in E} d_G(st) + \sum_{st\in E}(T_B(st), I_B(st), F_B(st)),\\
m(p_1, p_2, p_3) &= m(q_1, q_2, q_3) + S(G),\\
S(G) &= m(p_1 - q_1, p_2 - q_2, p_3 - q_3).
\end{aligned}$$

This completes the proof.

Theorem 1.6 *Let $G^* = (X, E)$ be a crisp graph, which is a cycle on m vertices. Suppose that G be a neutrosophic graph of G^*. Then $\sum_{s_k\in X} d_G(s_k) = \sum_{s_ks_l\in E} d_G(s_ks_l)$.*

Proof Let G be a neutrosophic graph of G^*. Suppose that G^* be a cycle $s_1, s_2, s_3, \ldots, s_m, s_1$ on m vertices. Then

$$\begin{aligned}
\sum_{s_ks_l\in E} d_G(s_ks_l) &= d_G(s_1s_2) + d_G(s_2s_3) + \cdots + d_G(s_ms_1),\\
&= [d_G(s_1) + d_G(s_2) - 2(T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2))][d_G(s_2)\\
&\quad + d_G(s_3) - 2(T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3))] + \cdots + [d_G(s_m)\\
&\quad + d_G(s_1) - 2(T_B(s_ms_1), I_B(s_ms_1), F_B(s_ms_1))],\\
&= 2d_G(s_1) + 2d_G(s_2) + \cdots + 2d_G(s_m) - 2(T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)),\\
&\quad -2(T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)) - \cdots - 2(T_B(s_ms_1), I_B(s_ms_1), F_B(s_ms_1)),\\
&= 2\sum_{s_k\in X} d_G(s_k) - 2\sum_{s_ks_l\in E}(T_B(s_ks_l), I_B(s_ks_l), F_B(s_ks_l)),\\
&= \sum_{s_k\in X} d_G(s_k) + \sum_{s_k\in X} d_G(s_k) - 2\sum_{s_ks_l\in E}(T_B(s_ks_l), I_B(s_ks_l), F_B(s_ks_l)),
\end{aligned}$$

$$= \sum_{s_k \in X} d_G(s_k) + 2 \sum_{s_k s_l \in E} (T_B(s_k s_l), I_B(s_k s_l), F_B(s_k s_l))$$
$$-2 \sum_{s_k s_l \in E} (T_B(s_k s_l), I_B(s_k s_l), F_B(s_k s_l)),$$
$$= \sum_{s_k \in X} d_G(s_k).$$

This completes the proof.

Theorem 1.7 *Let G be a neutrosophic graph. Then B is a constant function if and only if the following statements are equivalent:*

(a) G is an edge regular neutrosophic graph.
(b) G is a totally edge regular neutrosophic graph.

Proof Let G be a neutrosophic graph. Suppose that B is a constant function, then

$$T_B(st) = l_1, \quad I_B(st) = l_2, \quad F_B(st) = l_3, \quad \text{for all } st \in E.$$

(a) $\Rightarrow$ (b): Assume that G is an edge regular neutrosophic graph, i.e.

$$d_G(st) = (q_1, q_2, q_3), \quad \text{for each edge } st \in E.$$

This implies that

$$T d_G(st) = (l_1 + q_1, l_2 + q_2, l_3 + q_3) \text{ for each edge } st \in E.$$

This shows that G is an edge regular neutrosophic graph of degree

$$(l_1 + q_1, l_2 + q_2, l_3 + q_3).$$

(b) $\Rightarrow$ (a): Suppose that G is a totally edge regular neutrosophic graph, i.e.

$$T d_G(st) = (p_1, p_2, p_3) \text{ for all } st \in E.$$

This implies that

$$d_G(st) + (T_B(st), I_B(st), F_B(st)) = (p_1, p_2, p_3).$$

This implies that

$$d_G(st) = (p_1, p_2, p_3) - 4(T_B(st), I_B(st), F_B(st)).$$

This implies that

$$d_G(st) = (p_1 - l_1, p_2 - l_2, p_3 - l_3) \text{ for each edge } st \in E.$$

Thus G is an edge regular neutrosophic graph of degree

$$(p_1 - l_1, p_2 - l_2, p_3 - l_3).$$

Hence the statements (a) and (b) are equivalent.

Conversely, suppose that (a) and (b) are equivalent. Assume that B is not a constant function. This implies that

$$(T_B(st), I_B(st), F_B(st)) \neq (T_B(uv), I_B(uv), F_B(uv)),$$

for at least one pair of edges $st, uv \in E$.

Assume that G is an edge regular neutrosophic graph. This implies that

$$d_G(st) = d_G(uv) = (q_1, q_2, q_3).$$

This implies that

$$Td_G(st) = d_G(st) + (T_B(st), I_B(st), F_B(st)) = (q_1, q_2, q_3) + (T_B(st), I_B(st), F_B(st)),$$

$$Td_G(uv) = d_G(uv) + (T_B(uv), I_B(uv), F_B(uv)) = (q_1, q_2, q_3) + (T_B(uv), I_B(uv), F_B(uv)).$$

Since

$$(T_B(st), I_B(st), F_B(st)) \neq (T_B(uv), I_B(uv), F_B(uv)).$$

This implies that $Td_G(st) \neq Td_G(uv)$. This shows that G is not a totally edge regular neutrosophic graph, which contradicts our supposition.

Now, suppose that G is a totally edge regular neutrosophic graph, i.e.

$$Td_G(st) = Td_G(uv) = (p_1, p_2, p_3).$$

This implies that

$$Td_G(st) = d_G(st) + (T_B(st), I_B(st), F_B(st)) = d_G(uv) + (T_B(uv), I_B(uv), F_B(uv)).$$

This implies that

$$d_G(st) - d_G(uv) = (T_B(st), I_B(st), F_B(st)) - (T_B(uv), I_B(uv), F_B(uv)).$$

Since

$$(T_B(st), I_B(st), F_B(st)) \neq (T_B(uv), I_B(uv), F_B(uv)).$$

This implies that $d_G(st) - d_G(uv) \neq 0$. This implies that $d_G(st) \neq d_G(uv)$. This shows that G is not an edge regular neutrosophic graph, which contradicts our supposition. Hence B is a constant function.

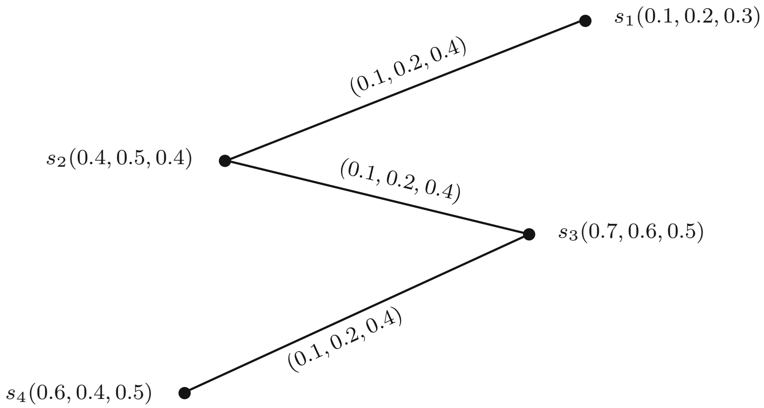


Fig. 1.14 Neutrosophic graph

Theorem 1.8 *Let G be a neutrosophic graph. Assume that G is both edge regular neutrosophic of degree* (q_1, q_2, q_3) *and totally edge regular neutrosophic graph of degree* (p_1, p_2, p_3). *Then B is a constant function.*

Proof The proof is obvious.

Remark 1.6 The converse of Theorem 1.8 may not be true in general; that is, a neutrosophic graph G, where B is a constant function, may or may not be edge regular and totally edge regular neutrosophic graph.

Example 1.16 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.14.

By direct calculations, we have

$$d_G(s_1) = (0.1, 0.2, 0.4), \quad d_G(s_2) = (0.2, 0.4, 0.8),$$

$$d_G(s_3) = (0.2, 0.4, 0.8), \quad d_G(s_4) = (0.1, 0.2, 0.4).$$

The degree of each edge is

$$d_G(s_1s_2) = (0.1, 0.2, 0.4), \quad d_G(s_2s_3) = (0.2, 0.4, 0.8), \quad d_G(s_3s_4) = (0.1, 0.2, 0.4).$$

The total degree of each edge is

$$Td_G(s_1s_2) = (0.2, 0.4, 0.8), \quad Td_G(s_2s_3) = (0.3, 0.6, 1.2).$$

It is clear from above calculations that G is neither an edge regular nor a totally edge regular neutrosophic graph.

Theorem 1.9 *Let G be a neutrosophic graph of* $G^* = (X, E)$*, where B is a constant function. If G is a regular neutrosophic graph, then G is an edge regular neutrosophic graph.*

Proof Assume that B is a constant function, that is,

$$T_B(st) = l_1, \quad I_B(st) = l_2, \quad F_B(st) = l_3 \text{ for all } st \in E.$$

Suppose that G is a regular neutrosophic graph, that is,

$$d_G(s) = (m_1, m_2, m_3) \text{ for all } s \in X.$$

Now

$$\begin{aligned} d_G(st) &= d_G(s) + d_G(t) - 2(T_B(st), I_B(st), F_B(st)), \\ &= (m_1, m_2, m_3) + (m_1, m_2, m_3) - 2(l_1, l_2, l_3), \\ &= 2(m_1 - l_1, m_2 - l_2, m_3 - l_3), \end{aligned}$$

for all $st \in E$. Hence G is an edge regular neutrosophic graph.

Theorem 1.10 *Let* $G = (A, B)$ *be a neutrosophic graph of* $G^* = (X, E)$*, where B is a constant function. If G is a regular neutrosophic graph, then G is a totally edge regular neutrosophic graph.*

Proof Let B be a constant function, that is,

$$T_B(st) = l_1, \quad I_B(st) = l_2, \quad F_B(st) = l_3 \text{ for all } st \in E.$$

Assume that G is a regular neutrosophic graph, that is,

$$d_G(s) = (m_1, m_2, m_3), \quad \text{for all } s \in X.$$

Then G is an edge regular neutrosophic graph, that is,

$$d_G(st) = (q_1, q_2, q_3).$$

Now

$$\begin{aligned} Td_G(st) &= d_G(st) + (T_B(st), I_B(st), F_B(st)), \\ &= (q_1, q_2, q_3) + (l_1, l_2, l_3), \\ &= 2(q_1 + l_1, q_2 + l_2, q_3 + l_3), \end{aligned}$$

for all $st \in E$. Hence G is a totally edge regular neutrosophic graph.

Theorem 1.11 *Suppose that G is a neutrosophic graph. Then G is both regular and totally edge regular neutrosophic graph if and only if B is a constant function.*

Proof Let $G^* = (X, E)$ be a regular crisp graph. Suppose that G is a neutrosophic graph of G^*. Suppose that G is both regular and totally edge regular neutrosophic graph, that is,

$$d_G(s) = (m_1, m_2, m_3), \quad \text{for all } \; s \in X,$$
$$Td_G(st) = (p_1, p_2, p_3), \quad \text{for all } \; st \in E.$$

Now

$$\begin{aligned}
Td_G(st) &= d_G(s) + d_G(t) - (T_B(st), I_B(st), F_B(st)), \quad \forall \; st \in E,\\
(p_1, p_2, p_3) &= (m_1, m_2, m_3) + (m_1, m_2, m_3) - (T_B(st), I_B(st), F_B(st)),\\
(T_B(st), I_B(st), F_B(st)) &= (2m_1 - p_1, 2m_2 - p_2, 2m_3 - p_3),
\end{aligned}$$

for all $st \in E$. Hence B is a constant function.

Conversely, let B be a constant function, that is,

$$T_B(st) = l_1, \quad I_B(st) = l_2, \quad F_B(st) = l_3, \quad \text{for all } \; st \in E.$$

So

$$\begin{aligned}
d_G(s) &= \sum_{st \in E} (T_B(st), I_B(st), F_B(st)), \quad \forall \; s \in X,\\
&= \sum_{st \in E} (m_1, m_2, m_3),\\
&= (m_1, m_2, m_3) d_{G^*}(s),\\
&= (m_1, m_2, m_3) m.
\end{aligned}$$

This implies that

$$d_G(s) = (mm_1, mm_2, mm_3), \quad \text{for all } \; s \in E.$$

Thus G is a regular neutrosophic graph. Now

$$\begin{aligned}
Td_G(st) = & \sum_{sa \in E, s \neq a} (T_B(sa), I_B(sa), F_B(sa)) + \sum_{at \in E, a \neq t} (T_B(at), I_B(at), F_B(at)),\\
& + (T_B(st), I_B(st), F_B(st)) \quad \forall st \in E,\\
= & \sum_{sa \in E, s \neq a} (l_1, l_2, l_3) + \sum_{at \in E, a \neq t} (l_1, l_2, l_3) + (l_1, l_2, l_3),\\
= & (l_1, l_2, l_3)(d_{G^*}(s) - 1) + (l_1, l_2, l_3)(d_{G^*}(t) - 1) + (l_1, l_2, l_3),\\
= & (l_1, l_2, l_3)(s - 1) + (l_1, l_2, l_3)(t - 1) + (l_1, l_2, l_3),\\
= & (2l_1, 2l_2, 2l_3)(s - 1) + (l_1, l_2, l_3),
\end{aligned}$$

for all $st \in E$. Hence G is a totally edge regular neutrosophic graph.

Theorem 1.12 *Let $G^* = (X, E)$ be a crisp graph. Suppose that $G = (A, B)$ is a neutrosophic graph of G^*. Then B is a constant function if and only if G is an edge regular neutrosophic graph.*

Proof Let G be a regular neutrosophic graph, that is,

$$d_G(s) = (m_1, m_2, m_3), \quad \text{for all } \ s \in X.$$

Suppose that B is a constant function, that is,

$$T_B(st) = l_1, \quad I_B(st) = l_2, \quad F_B(st) = l_3, \quad \text{for all } \ st \in E.$$

Now

$$\begin{aligned} d_G(st) &= d_G(s) + d_G(t) - 2(T_B(st), I_B(st), F_B(st)), \quad \forall \ st \in E. \\ &= (m_1, m_2, m_3) + (m_1, m_2, m_3) - 2(l_1, l_2, l_3), \end{aligned}$$

this implies that

$$d_G(st) = 2(m_1, m_2, m_3) - 2(l_1, l_2, l_3), \quad \text{for all } \ st \in E.$$

Hence G is an edge regular neutrosophic graph.

Conversely, assume that G is an edge regular neutrosophic graph, that is,

$$d_G(st) = (q_1, q_2, q_3), \quad \text{for each edge } \ st \in E.$$

Now

$$\begin{aligned} d_G(st) &= d_G(s) + d_G(t) - 2(T_B(st), I_B(st), F_B(st)), \quad \forall \ st \in E, \\ (q_1, q_2, q_3) &= (m_1, m_2, m_3) + (m_1, m_2, m_3) - 2(T_B(st), I_B(st), F_B(st)), \end{aligned}$$

this implies that

$$(T_B(st), I_B(st), F_B(st)) = \frac{(q_1, q_2, q_3) - (2m_1, 2m_2, 2m_3)}{2}, \quad \text{for all } \ st \in E.$$

Thus B is a constant function.

Definition 1.19 Let G^* be an edge regular crisp graph. Then a neutrosophic graph G of G^* is called a *partially edge regular*.

Example 1.17 It can be seen in Example 1.15 that G^* is an edge regular crisp graph. Therefore, G is a partially edge regular neutrosophic graph.

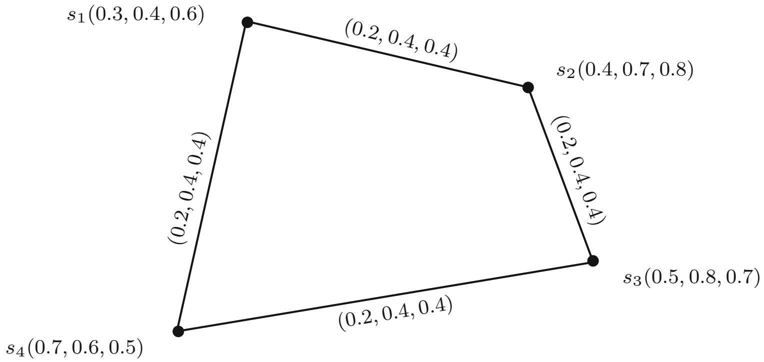


Fig. 1.15 Full edge regular neutrosophic graph

Definition 1.20 A neutrosophic graph G is called a *full edge regular* if it is both edge regular and partially edge regular.

Example 1.18 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.15.

By direct calculations, we have

$$d_G(s_1) = (0.4, 0.8, 0.8), \quad d_G(s_2) = (0.4, 0.8, 0.8),$$

$$d_G(s_3) = (0.4, 0.8, 0.8), \quad d_G(s_4) = (0.4, 0.8, 0.8).$$

The degree of each edge is

$$d_G(s_1s_2) = (0.4, 0.8, 0.8), \quad d_G(s_2s_3) = (0.4, 0.8, 0.8)$$

$$d_G(s_3s_4) = (0.4, 0.8, 0.8), \quad d_G(s_1s_4) = (0.4, 0.8, 0.8).$$

It is clear from calculations that G is full edge regular neutrosophic graph.

Theorem 1.13 *Let G be a neutrosophic graph, where B is a constant function. Then G is full edge regular neutrosophic graph if it is full regular neutrosophic graph.*

Proof Let G be a neutrosophic graph of a crisp graph $G^* = (X, E)$. Suppose that B is a constant function, that is,

$$(T_B(st), I_B(st), F_B(st)) = (l_1, l_2, l_3), \quad \text{for each edge } st \in E.$$

Assume that G is full regular neutrosophic graph. Then G is both regular and partially regular. Therefore,

$$d_G(s) = (m_1, m_2, m_3) \text{ and } d_{G^*}(s) = m, \quad \text{for all } s \in X.$$

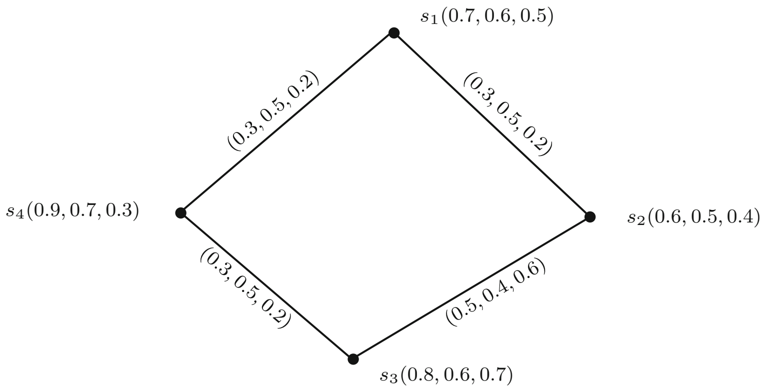


Fig. 1.16 Irregular neutrosophic graph

Since

$$d_{G^*}(st) = d_{G^*}(s) + d_{G^*}(t) - 2, \quad \text{for all } st \in E.$$

This shows that $d_{G^*}(st) = 2m - 2$. Therefore, G^* is an edge regular neutrosophic graph. Now

$$\begin{aligned} d_G(st) &= d_G(s) + d_G(t) - 2(T_B(st), I_B(st), F_B(st)), \quad \forall \ st \in E. \\ &= (m_1, m_2, m_3) + (m_1, m_2, m_3) - 2(l_1, l_2, l_3), \end{aligned}$$

this implies that

$$d_G(st) = 2(m_1 - l_1, m_2 - l_2, m_3 - l_3).$$

This shows that G is an edge regular neutrosophic graph. Hence G is a full edge regular neutrosophic graph.

Definition 1.21 A neutrosophic graph G is called an *irregular* if there exists a vertex which is adjacent to vertices with distinct degrees.

Example 1.19 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.16.

By direct calculations, we have

$$d_G(s_1) = (0.6, 1.0, 0.4), \quad d_G(s_2) = (0.8, 0.9, 0.8),$$

$$d_G(s_3) = (0.8, 0.9, 0.8), \quad d_G(s_4) = (0.6, 1.0, 0.4).$$

It is easy to see that s_1 is adjacent to vertices of distinct degrees. Therefore, G is an irregular neutrosophic graph.

Definition 1.22 A neutrosophic graph G is called a *totally irregular* if there exists a vertex which is adjacent to vertices with distinct total degrees.

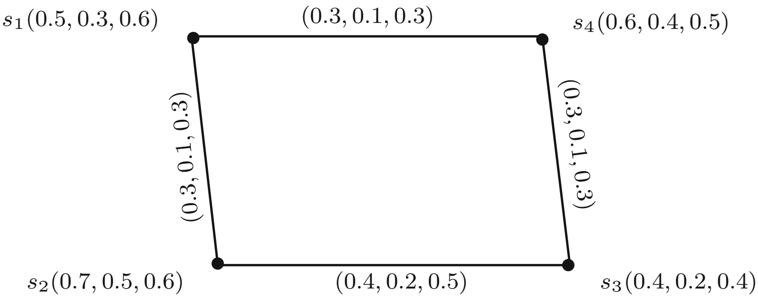


Fig. 1.17 Totally irregular neutrosophic graph

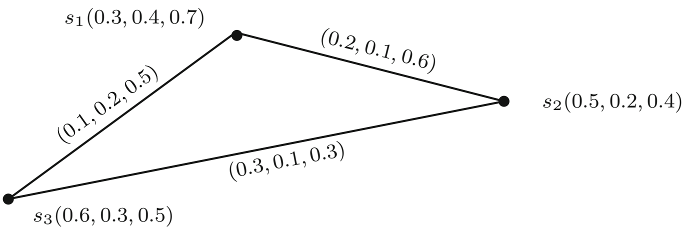


Fig. 1.18 Strongly irregular neutrosophic graph

Example 1.20 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.17.

By direct calculations, we have

$$Td_G(s_1) = (1.1, 0.5, 1.2), \quad Td_G(s_2) = (1.4, 0.8, 1.4),$$

$$Td_G(s_3) = (1.1, 0.5, 1.2), \quad Td_G(s_4) = (1.2, 0.6, 1.1).$$

It is easy to see that s_1 is adjacent to vertices of distinct total degrees. Therefore, G is a totally irregular neutrosophic graph.

Definition 1.23 A neutrosophic graph G is called *strongly irregular* if each vertex has distinct degree.

Example 1.21 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3\}$ as shown in Fig. 1.18.

By direct calculations, we have

$$d_G(s_1) = (0.3, 0.3, 1.1), \quad d_G(s_2) = (0.5, 0.2, 0.9), \quad d_G(s_3) = (0.4, 0.3, 0.8).$$

From Fig. 1.18, it is clear that each vertex has distinct degree. Therefore, G is a strongly irregular neutrosophic graph.

Definition 1.24 A neutrosophic graph G is called *strongly totally irregular neutrosophic graph* if each vertex has distinct total degree.

Example 1.22 Consider the neutrosophic graph G as shown in Fig. 1.18. By direct calculations, we have

$$Td(s_1) = (0.6, 0.7, 1.8),\quad Td(s_2) = (1.0, 0.4, 1.3),\quad Td(s_3) = (1.0, 0.6, 1.3).$$

Since each vertex has distinct total degree, G is a strongly totally irregular neutrosophic graph.

Definition 1.25 A neutrosophic graph G is called *highly irregular* if each vertex in G is adjacent to vertices having distinct degrees.

Example 1.23 Consider the neutrosophic graph G as shown in Fig. 1.16. It is easy to see that each vertex is adjacent to vertices of distinct degree; therefore, G is highly irregular neutrosophic graph.

Definition 1.26 A neutrosophic graph G is called *highly totally irregular* if each vertex in G is adjacent to vertices having distinct total degrees.

Example 1.24 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.19.

By direct calculations, we have

$$Td_G(s_1) = (0.8, 0.8, 0.7),\quad Td_G(s_2) = (0.3, 0.4, 0.7),$$

$$Td_G(s_3) = (0.7, 1.0, 1.1),\quad Td_G(s_4) = (1.1, 1.1, 0.7).$$

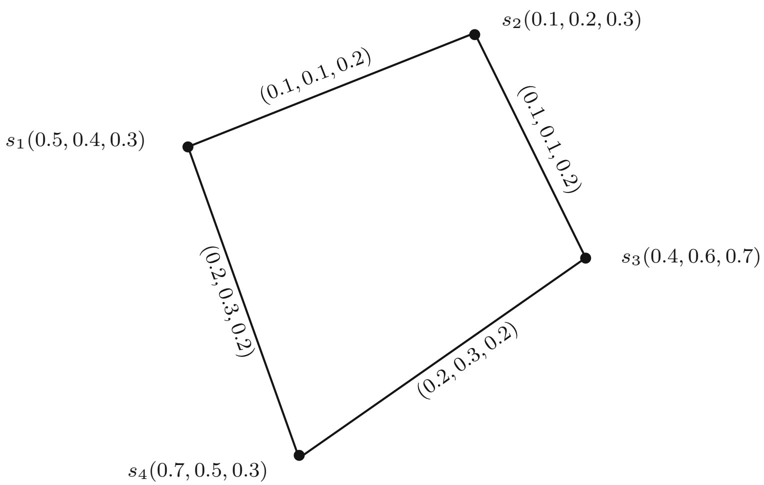


Fig. 1.19 Highly totally irregular neutrosophic graph

From Fig. 1.19, it is clear that each vertex is adjacent to vertices of distinct degrees. Therefore, G is highly totally irregular neutrosophic graph.

Definition 1.27 A connected neutrosophic graph G is called *neighbourly edge irregular* if every two adjacent edges in G have distinct degrees.

Example 1.25 Consider the neutrosophic graph G as shown in Fig. 1.18. It is easy to see that every two adjacent edges in G have distinct degrees; therefore, G is neighbourly edge irregular neutrosophic graph.

Definition 1.28 A connected neutrosophic graph G is called *neighbourly edge totally irregular neutrosophic graph* if every two adjacent edges in G have distinct total degrees.

Example 1.26 Consider the neutrosophic graph G as shown in Fig. 1.18. It is easy to see that every two adjacent edges in G have distinct total degrees; therefore, G is neighbourly edge totally irregular neutrosophic graph.

Definition 1.29 Let G^* be a crisp graph. A neutrosophic graph G of G^* is called a *strongly edge irregular neutrosophic graph* if each edge in G has distinct degree; that is, no two edges in G have the same degree.

Example 1.27 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3\}$ as shown in Fig. 1.20.

By direct calculations, we have

$$d_G(s_1) = (0.8, 0.8, 0.4), \quad d_G(s_2) = (0.6, 0.3, 0.4), \quad d_G(s_3) = (0.8, 0.7, 0.2).$$

- The degree of each edge is given as:

$$\begin{aligned} d_G(s_1s_2) &= d_G(s_1) + d_G(s_2) - 2(T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)), \\ &= (0.8, 0.8, 0.4) + (0.6, 0.3, 0.4) - 2(0.3, 0.2, 0.3), \\ &= (0.8, 0.7, 0.2). \end{aligned}$$

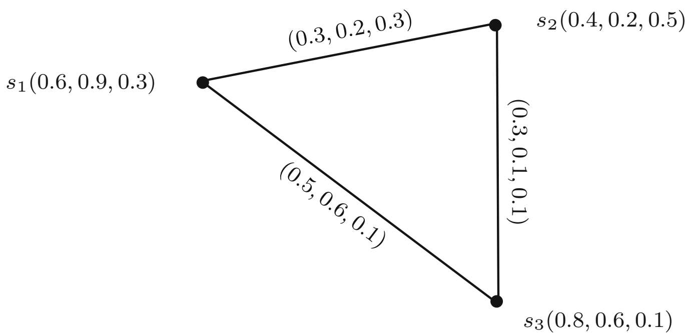


Fig. 1.20 Strongly edge irregular neutrosophic graph

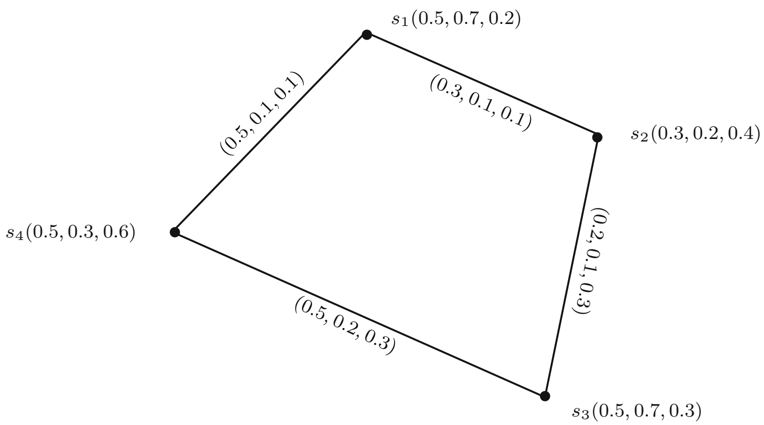


Fig. 1.21 Strongly edge totally irregular neutrosophic graph

$$\begin{aligned} d_G(s_1s_3) &= d_G(s_1) + d_G(s_3) - 2(T_B(s_1s_3), I_B(s_1s_3), F_B(s_1s_3)),\\ &= (0.8, 0.8, 0.4) + (0.8, 0.7, 0.2) - 2(0.5, 0.6, 0.1),\\ &= (0.6, 0.3, 0.4). \end{aligned}$$

$$\begin{aligned} d_G(s_2s_3) &= d_G(s_2) + d_G(s_3) - 2(T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)),\\ &= (0.6, 0.3, 0.4) + (0.8, 0.7, 0.2) - 2(0.3, 0.1, 0.1),\\ &= (0.8, 0.8, 0.4). \end{aligned}$$

Since no two edges in G have the same degree, G is a strongly edge irregular neutrosophic graph.

Definition 1.30 A neutrosophic graph G is called a strongly edge totally irregular neutrosophic graph if each edge in G has distinct total degree; that is, no two edges in G have the same total degree.

Example 1.28 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.21.

By direct calculations, we have

$$d_G(s_1) = (0.8, 0.2, 0.2), \quad d_G(s_2) = (0.5, 0.2, 0.4),$$

$$d_G(s_3) = (0.7, 0.3, 0.6), \quad d_G(s_4) = (1.0, 0.3, 0.4).$$

- The degree of each edge is given as:

$$\begin{aligned} d_G(s_1s_2) &= d_G(s_1) + d_G(s_2) - 2(T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)),\\ &= (0.8, 0.2, 0.2) + (0.5, 0.2, 0.4) - 2(0.3, 0.1, 0.1),\\ &= (0.7, 0.2, 0.4). \end{aligned}$$

$$\begin{aligned} d_G(s_1s_4) &= d_G(s_1) + d_G(s_4) - 2(T_B(s_1s_4), I_B(s_1s_4), F_B(s_1s_4)),\\ &= (0.8, 0.2, 0.2) + (1.0, 0.3, 0.4) - 2(0.5, 0.1, 0.1),\\ &= (0.8, 0.3, 0.4). \end{aligned}$$

$$\begin{aligned} d_G(s_2s_3) &= d_G(s_2) + d_G(s_3) - 2(T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)),\\ &= (0.5, 0.2, 0.4) + (0.7, 0.3, 0.6) - 2(0.2, 0.1, 0.3),\\ &= (0.8, 0.3, 0.4). \end{aligned}$$

$$\begin{aligned} d_G(s_3s_4) &= d_G(s_3) + d_G(s_4) - 2(T_B(s_3s_4), I_B(s_3s_4), F_B(s_3s_4)),\\ &= (0.7, 0.3, 0.6) + (1.0, 0.3, 0.4) - 2(0.5, 0.2, 0.3),\\ &= (0.7, 0.2, 0.4). \end{aligned}$$

- The total degree of each edge is given as:

$$\begin{aligned} Td_G(s_1s_2) &= d_G(s_1s_2) + (T_B(s_1s_2), I_B(s_1s_2), F_B(s_1s_2)),\\ &= (0.7, 0.2, 0.4) + (0.3, 0.1, 0.1),\\ &= (1.0, 0.3, 0.5). \end{aligned}$$

$$\begin{aligned} Td_G(s_1s_4) &= d_G(s_1s_4) + (T_B(s_1s_4), I_B(s_1s_4), F_B(s_1s_4)),\\ &= (0.8, 0.3, 0.4) + (0.5, 0.1, 0.1),\\ &= (1.3, 0.4, 0.5). \end{aligned}$$

$$\begin{aligned} Td_G(s_2s_3) &= d_G(s_2s_3) + (T_B(s_2s_3), I_B(s_2s_3), F_B(s_2s_3)),\\ &= (0.8, 0.3, 0.4) + (0.2, 0.1, 0.3),\\ &= (1.0, 0.4, 0.7). \end{aligned}$$

$$\begin{aligned} Td_G(s_3s_4) &= d_G(s_3s_4) + (T_B(s_3s_4), I_B(s_3s_4), F_B(s_3s_4)),\\ &= (0.7, 0.2, 0.4) + (0.5, 0.2, 0.3),\\ &= (1.2, 0.4, 0.7). \end{aligned}$$

Since no two edges in G have the same total degree, G is a strongly edge totally irregular neutrosophic graph.

Remark 1.7 A strongly edge irregular neutrosophic graph G may not be strongly edge totally irregular neutrosophic graph.

Example 1.29 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3\}$ as shown in Fig. 1.22.

By direct calculations, we have

$$d_G(s_1) = (1.1, 0.5, 0.7),\quad d_G(s_2) = (0.7, 0.4, 0.9),\quad d_G(s_3) = (1.0, 0.3, 0.6).$$

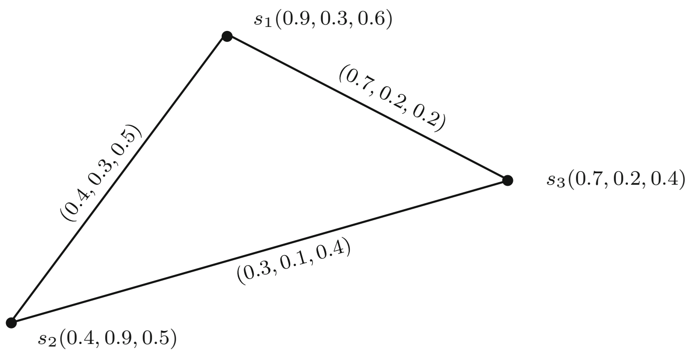


Fig. 1.22 Strongly edge irregular neutrosophic graph

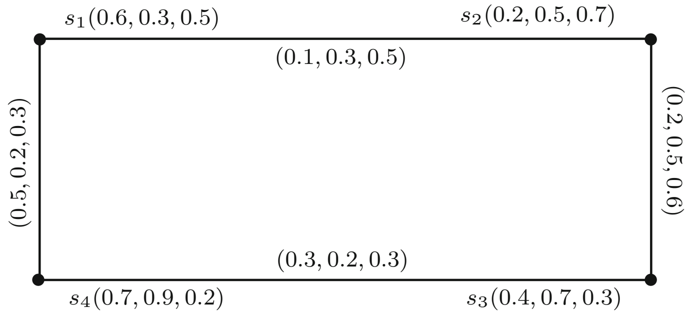


Fig. 1.23 Strongly edge totally irregular neutrosophic graph

The degree of each edge is

$$d_G(s_1s_2) = (1.0, 0.3, 0.6), \ \ d_G(s_2s_3) = (1.1, 0.5, 0.7), \ \ d_G(s_1s_3) = (0.7, 0.4, 0.9).$$

Since all the edges have distinct degrees, G is a strongly edge irregular neutrosophic graph. The total degree of each edge is

$$Td_G(s_1s_2) = (1.4, 0.6, 1.1) = Td_G(s_2s_3) = Td_G(s_1s_3).$$

Since each edge of G has the same total degree therefore G is not a strongly edge totally irregular neutrosophic graph.

Remark 1.8 A strongly edge totally irregular neutrosophic graph G may not be strongly edge irregular neutrosophic graph.

Example 1.30 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.23.

By direct calculations, we have

$$d_G(s_1) = (0.6, 0.5, 0.8), \quad d_G(s_2) = (0.3, 0.8, 1.1),$$

$$d_G(s_3) = (0.5, 0.7, 0.9), \quad d_G(s_4) = (0.8, 0.4, 0.6).$$

The degree of each edge is

$$d_G(s_1s_2) = (0.7, 0.7, 0.9), \quad d_G(s_2s_3) = (0.4, 0.5, 0.8),$$

$$d_G(s_3s_4) = (0.7, 0.7, 0.9), \quad d_G(s_1s_4) = (0.4, 0.5, 0.8).$$

It is easy to see that $d_G(s_1s_2) = d_G(s_3s_4)$ and $d_G(s_2s_3) = d_G(s_1s_4)$.

Therefore, G is not a strongly edge irregular neutrosophic graph.

The total degree of each edge is

$$Td_G(s_1s_2) = (0.8, 1.0, 1.4), \quad Td_G(s_2s_3) = (0.6, 1.0, 1.4),$$

$$Td_G(s_3s_4) = (1.0, 0.9, 1.2), \quad Td_G(s_1s_4) = (0.9, 0.7, 1.1).$$

Since all the edges have distinct total degrees, G is a strongly edge totally irregular neutrosophic graph.

Theorem 1.14 *If G is a strongly edge irregular connected neutrosophic graph, where B is a constant function, then G is a strongly edge totally irregular neutrosophic graph.*

Proof Let G be a strongly edge irregular connected neutrosophic graph. Assume that B is a constant function. Then

$$T_B(xy) = l_1, \quad I_B(xy) = l_2, \quad F_B(xy) = l_3, \quad \text{for all } xy \in E,$$

where l_1, l_2 and l_3 are constants. Consider a pair of edges xy and uv in E.

Since G is a strongly edge irregular neutrosophic graph,

$$d_G(xy) \neq d_G(uv),$$

where xy and uv are a pair of edges in E. This shows that

$$d_G(xy) + (l_1, l_2, l_3) \neq d_G(uv) + (l_1, l_2, l_3).$$

This implies that

$$d_G(xy) + (T_B(xy), I_B(xy), F_B(xy)) \neq d_G(uv) + (T_B(uv), I_B(uv), F_B(uv)).$$

Thus

$$Td_G(xy) \neq Td_G(uv),$$

where xy and uv are a pair of edges in E. Since the pair of edges xy and uv were taken to be arbitrary, this shows that every pair of edges in G have distinct total degrees.

Hence G is a strongly edge totally irregular neutrosophic graph.

Theorem 1.15 *If G is a strongly edge totally irregular connected neutrosophic graph, where B is a constant function, then G is a strongly edge irregular neutrosophic graph.*

Proof Let G be a strongly edge totally irregular connected neutrosophic graph. Assume that B is a constant function. Then

$$T_B(xy) = l_1, \quad I_B(xy) = l_2 \text{ and } F_B(xy) = l_3, \quad \text{for all } xy \in E,$$

where l_1, l_2 and l_3 are constants. Consider a pair of edges xy and uv in L.

Since G is a strongly edge totally irregular neutrosophic graph,

$$Td_G(xy) \neq Td_G(uv),$$

where xy and uv are a pair of edges in E. This shows that

$$d_G(xy) + (T_B(xy), I_B(xy), F_B(xy)) \neq d_G(uv) + (T_B(uv), I_B(uv), F_B(uv)).$$

This implies that

$$d_G(xy) + (l_1, l_2, l_3) \neq d_G(uv) + (l_1, l_2, l_3).$$

Thus

$$d_G(xy) \neq d_G(uv),$$

where xy and uv are a pair of edges in E. Since the pair of edges xy and uv were taken to be arbitrary, this shows that every pair of edges in G have distinct degrees.

Hence G is a strongly edge irregular neutrosophic graph.

Remark 1.9 If G is both strongly edge irregular neutrosophic graph and strongly edge totally irregular neutrosophic graph, then it is not necessary that B is a constant function.

Example 1.31 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4, s_4\}$ as shown in Fig. 1.24.

By direct calculations, we have

$$d_G(s_1) = (0.6, 0.4, 0.4), \quad d_G(s_2) = (0.3, 0.7, 0.6), \quad d_G(s_3) = (0.3, 0.8, 0.6),$$

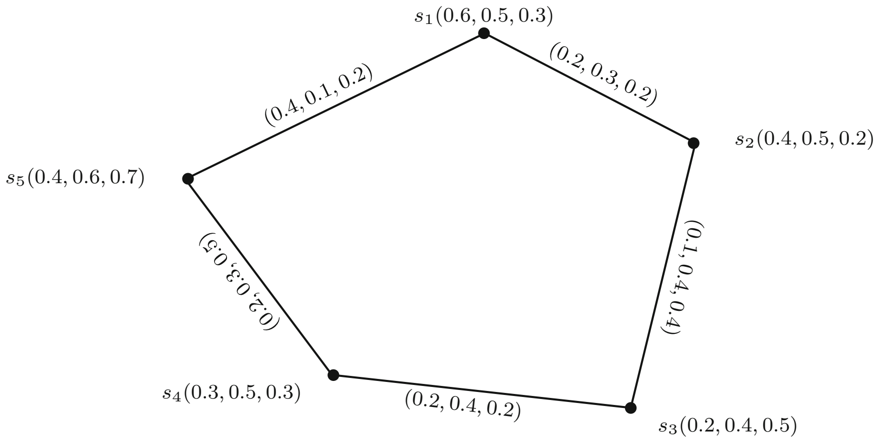


Fig. 1.24 Neutrosophic graph

$$d_G(s_4) = (0.4, 0.7, 0.7), \quad d_G(s_5) = (0.6, 0.4, 0.7).$$

The degree of each edge is

$$d_G(s_1s_2) = (0.5, 0.5, 0.6), \quad d_G(s_2s_3) = (0.4, 0.7, 0.4), \quad d_G(s_3s_4) = (0.3, 0.7, 0.9),$$

$$d_G(s_4s_5) = (0.6, 0.5, 0.4), \quad d_G(s_5s_1) = (0.4, 0.6, 0.7).$$

It is easy to see that all the edges have distinct degrees. Therefore, G is a strongly edge irregular neutrosophic graph.

The total degree of each edge is

$$Td_G(s_1s_2) = (0.7, 0.8, 0.8), \quad Td_G(s_2s_3) = (0.5, 1.1, 0.8), \quad Td_G(s_3s_4) = (0.5, 1.1, 1.1),$$

$$Td_G(s_4s_5) = (0.8, 0.8, 0.9), \quad Td_G(s_5s_1) = (0.8, 0.7, 0.9).$$

Since all the edges have distinct total degrees, G is a strongly edge totally irregular neutrosophic graph. This shows that G is both strongly edge irregular neutrosophic graph and strongly edge totally irregular neutrosophic graph, but B is not a constant function.

Theorem 1.16 *Let G be a strongly edge irregular neutrosophic graph. Then G is a neighbourly edge irregular neutrosophic graph.*

Proof Suppose that G is a strongly edge irregular neutrosophic graph. Then each edge in G has distinct degree. This shows that every pair of edges in G have distinct degrees. Therefore, G is a neighbourly edge irregular neutrosophic graph.

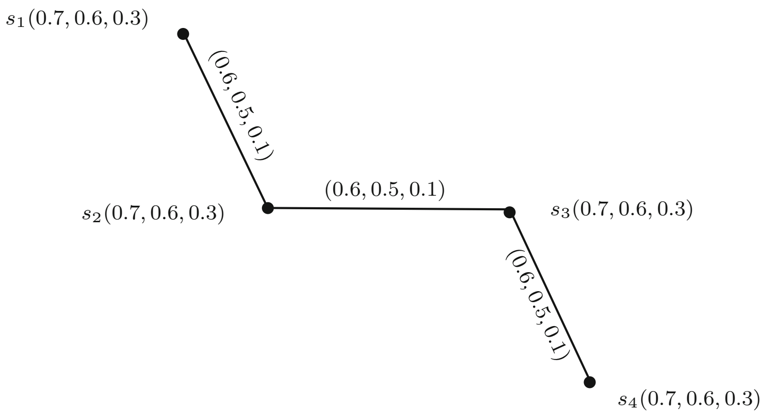


Fig. 1.25 Neutrosophic graph

Theorem 1.17 *Let G be a strongly edge totally irregular neutrosophic graph. Then G is a neighbourly edge totally irregular neutrosophic graph.*

Proof Suppose that G is a strongly edge totally irregular neutrosophic graph. Then each edge in G has distinct total degree. This shows that every pair of edges in G have distinct total degrees. Therefore, G is a neighbourly edge totally irregular neutrosophic graph.

Remark 1.10 If G is a neighbourly edge irregular neutrosophic graph, then it is not necessary that G is a strongly edge irregular neutrosophic graph.

Example 1.32 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.25.

By direct calculations, we have

$$d_G(s_1) = (0.6, 0.5, 0.1), \quad d_G(s_2) = (1.2, 1.0, 0.2),$$

$$d_G(s_3) = (1.2, 1.0, 0.2), \quad d_G(s_4) = (0.6, 0.5, 0.1).$$

The degree of each edge is

$$d_G(s_1s_2) = (0.6, 0.5, 0.1), \quad d_G(s_2s_3) = (1.2, 1.0, 0.2), \quad d_G(s_3s_4) = (0.6, 0.5, 0.1).$$

G is neighbourly edge irregular neutrosophic graph since every two adjacent edges in G have distinct total degrees, that is,

$$d_G(s_1s_2) \neq d_G(s_2s_3) \text{ and } d_G(s_2s_3) \neq d_G(s_3s_4).$$

It is easy to see that $d_G(s_1s_2) = d_G(s_3s_4)$. Therefore, G is not a strongly edge irregular neutrosophic graph.

Remark 1.11 If G is a neighbourly edge totally irregular neutrosophic graph, then it is not necessary that G is a strongly edge totally irregular neutrosophic graph.

Example 1.33 Consider the neutrosophic graph G as shown in Fig. 1.25. The total degree of each edge is

$$Td_G(s_1s_2) = (1.2, 1.0, 0.2),\quad Td_G(s_2s_3) = (1.8, 1.5, 0.3),\quad Td_G(s_1s_2) = (1.2, 1.0, 0.2).$$

It is easy to see that every two adjacent edges in G have distinct total degrees, that is,

$$Td_G(s_1s_2) \neq Td_G(s_2s_3),\quad \text{and}\quad Td_G(s_2s_3) \neq Td_G(s_3s_4).$$

Therefore, G is neighbourly edge totally irregular neutrosophic graph. It is easy to see that $Td_G(s_1s_2) = Td_G(s_3s_4)$. Hence G is not a strongly edge totally irregular neutrosophic graph.

Theorem 1.18 *Let G be a strongly edge irregular connected neutrosophic graph, with B as constant function. Then G is an irregular neutrosophic graph.*

Proof Let G be a strongly edge irregular connected neutrosophic graph, with B as constant function. Then

$$T_B(xy) = l_1,\quad I_B(xy) = l_2,\quad F_B(xy) = l_3,\quad \text{for each edge } xy \in E,$$

where l_1, l_2 and l_3 are constants. Also, every edge in G has distinct degrees, since G is strongly edge irregular neutrosophic graph.

Let xy and yu be any two adjacent edges in G such that

$$d_G(xy) \neq d_G(yu).$$

This implies that

$$d_G(x) + d_G(y) - 2(T_B(xy), I_B(xy), F_B(xy)) \neq d_G(y) + d_G(u) - 2(T_B(yu), I_B(yu), F_B(yu)).$$

This implies that

$$d_G(x) + d_G(y) - 2(l_1, l_2, l_3) \neq d_G(y) + d_G(u) - 2(l_1, l_2, l_3).$$

This shows that

$$d_G(x) \neq d_G(u).$$

Thus there exists a vertex y in G which is adjacent to the vertices with distinct degrees. This shows that G is an irregular neutrosophic graph.

Theorem 1.19 *Let G be a strongly edge totally irregular connected neutrosophic graph, with B as constant function. Then G is an irregular neutrosophic graph.*

Proof Let G be a strongly edge totally irregular connected neutrosophic graph, with B as constant function. Then

$$T_B(xy) = l_1, \quad I_B(xy) = l_2, \quad F_B(xy) = l_3, \quad \text{for each edge } \; xy \in E,$$

where l_1, l_2 and l_3 are constants. Also, every edge in G has distinct total degrees, since G is strongly edge totally irregular neutrosophic graph.

Let xy and yu be any two adjacent edges in G such that

$$Td_G(xy) \neq Td_G(yu).$$

This implies that

$$d_G(xy) + (T_B(xy), I_B(xy), F_B(xy)) \neq d_G(yu) + (T_B(yu), I_B(yu), F_B(yu)).$$

This implies that

$$d_G(x) + d_G(y) - (T_B(xy), I_B(xy), F_B(xy)) \neq d_G(y) + d_G(u) - (T_B(yu), I_B(yu), F_B(yu)).$$

This implies that

$$d_G(x) + d_G(y) - 2(l_1, l_2, l_3) \neq d_G(y) + d_G(u) - 2(l_1, l_2, l_3).$$

This shows that

$$d_G(x) \neq d_G(u).$$

Thus there exists a vertex y in G which is adjacent to the vertices with distinct degrees. This shows that G is an irregular neutrosophic graph.

Remark 1.12 If G is an irregular neutrosophic graph, with B as a constant function. Then it is not necessary that G is a strongly edge irregular neutrosophic graph.

Example 1.34 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.26.

By direct calculations, we have

$$d_G(s_1) = (0.8, 0.2, 0.6), \quad d_G(s_2) = (1.2, 0.3, 0.9),$$

$$d_G(s_3) = (0.8, 0.2, 0.6), \quad d_G(s_4) = (1.2, 0.3, 0.9).$$

The degree of each edge is

$$d_G(s_1s_2) = (1.2, 0.3, 0.9), \quad d_G(s_2s_3) = (1.2, 0.3, 0.9), \quad d_G(s_2s_4) = (1.6, 0.4, 1.2),$$

$$d_G(s_3s_4) = (1.2, 0.3, 0.9), \quad d_G(s_1s_4) = (1.2, 0.3, 0.9).$$

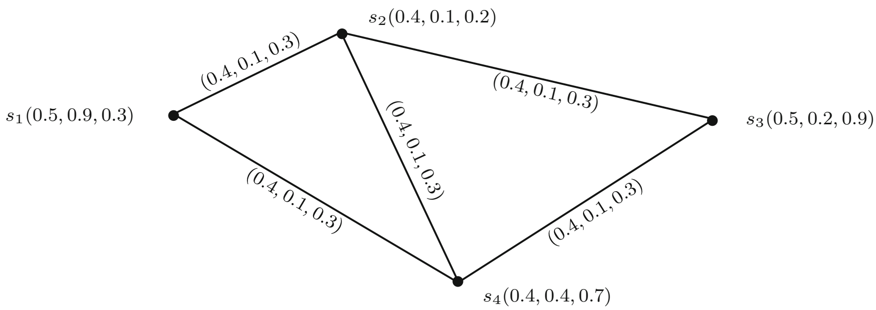


Fig. 1.26 Irregular neutrosophic graph

It is easy to see that all the edges have the same degree except the edge s_2s_4. Therefore, G is not a strongly edge irregular neutrosophic graph.

Remark 1.13 If G is an irregular neutrosophic graph, with B as a constant function. Then it is not necessary that G is a strongly edge totally irregular neutrosophic graph.

Example 1.35 Consider the neutrosophic graph G as shown in Fig. 1.26. The total degree of each edge is

$$TdG(s_1s_2) = (1.6, 0.4, 1.2),\quad TdG(s_2s_3) = (1.6, 0.4, 1.2),\quad TdG(s_2s_4) = (2.0, 0.5, 1.5),$$

$$TdG(s_3s_4) = (1.6, 0.4, 1.2),\quad TdG(s_1s_4) = (1.6, 0.4, 1.2).$$

It is easy to see that all the edges have the same total degree except the edge s_2s_4. Therefore, G is not a strongly edge totally irregular neutrosophic graph.

Theorem 1.20 *Let G be a strongly edge irregular connected neutrosophic graph, with B as a constant function. Then G is highly irregular neutrosophic graph.*

Proof Let G be a strongly edge irregular connected neutrosophic graph, with B as a constant function. Then

$$T_B(xy) = l_1,\quad I_B(xy) = l_2,\quad F_B(xy) = l_3,\quad \text{for each edge } xy \in E,$$

where l_1, l_2 and l_3 are constants. Also every pair of adjacent edges in G have distinct degrees.

Let y be any vertex in G which is adjacent to vertices y and u. Since G is strongly edge irregular neutrosophic graph,

$$d_G(xy) \neq d_G(yu).$$

This implies that

$$d_G(x) + d_G(y) - 2(T_B(xy), I_B(xy), F_B(xy)) \neq d_G(y) + d_G(u) - 2(T_B(yu), I_B(yu), F_B(yu)).$$

This implies that

$$d_G(x) + d_G(y) - 2(l_1, l_2, l_3) \neq d_G(y) + d_G(u) - 2(l_1, l_2, l_3).$$

This shows that

$$d_G(x) \neq d_G(u).$$

Thus there exists a vertex y in G which is adjacent to the vertices with distinct degrees. Since y was taken to be an arbitrary vertex in G, all the vertices in G are adjacent to vertices having distinct degrees. Hence G is a highly irregular neutrosophic graph.

Theorem 1.21 *Let G be a strongly edge totally irregular connected neutrosophic graph, with B as a constant function. Then G is highly irregular neutrosophic graph.*

Proof Let G be a strongly edge totally irregular connected neutrosophic graph, with B as a constant function. Then

$$T_B(xy) = l_1, \quad I_B(xy) = l_2, \quad F_B(xy) = l_3, \quad \text{for each edge } xy \in E,$$

where l_1, l_2 and l_3 are constants. Also every pair of adjacent edges in G have distinct total degrees.

Let y be any vertex in G which is adjacent to vertices x and u. Since G is strongly edge totally irregular neutrosophic graph therefore,

$$Td_G(xy) \neq Td_G(yu).$$

This implies that

$$d_G(xy) \neq d_G(yu).$$

This implies that

$$d_G(x) + d_G(y) - 2(T_B(xy), I_B(xy), F_B(xy)) \neq d_G(y) + d_G(u) - 2(T_B(yu), I_B(yu), F_B(yu)).$$

This implies that

$$d_G(x) + d_G(y) - 2(l_1, l_2, l_3) \neq d_G(y) + d_G(u) - 2(l_1, l_2, l_3).$$

This shows that

$$d_G(x) \neq d_G(u).$$

Thus there exists a vertex y in G which is adjacent to the vertices with distinct degrees. Since y was taken to be an arbitrary vertex in G, therefore all the vertices

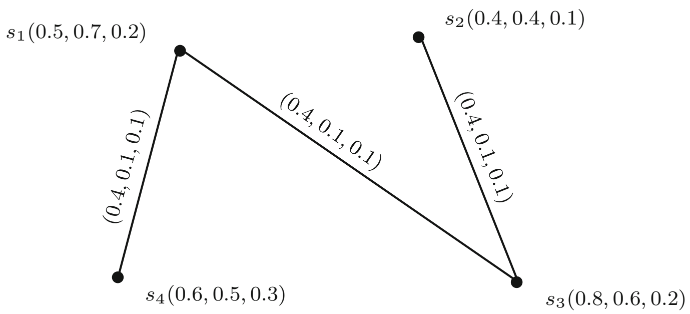


Fig. 1.27 Highly irregular neutrosophic graph

in G are adjacent to vertices having distinct degrees. Hence G is a highly irregular neutrosophic graph.

Remark 1.14 If G is a highly irregular neutrosophic graph, with B as a constant function. Then it is not necessary that G is strongly edge irregular neutrosophic graph.

Example 1.36 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 1.27.

By direct calculations, we have

$$d_G(s_1) = (0.8, 0.2, 0.2), \quad d_G(s_2) = (0.4, 0.1, 0.1),$$

$$d_G(s_3) = (0.8, 0.2, 0.2), \quad d_G(s_4) = (0.4, 0.1, 0.1).$$

The degree of each edge is

$$d_G(s_1s_3) = (0.8, 0.2, 0.2), \quad d_G(s_1s_4) = (0.4, 0.1, 0.1), \quad d_G(s_2s_3) = (0.4, 0.1, 0.1).$$

Since every vertex is adjacent to vertices with distinct degrees, G is a highly irregular neutrosophic graph. Since the edges s_1s_4 and s_2s_3 in G have the same degree, i.e. $d_G(s_1s_4) = d_G(s_2s_3)$, G is not strongly edge irregular neutrosophic graph.

Remark 1.15 If G is a highly irregular neutrosophic graph, with B as a constant function. Then it is not necessary that G is strongly edge totally irregular neutrosophic graph.

Example 1.37 Consider the neutrosophic graph G as shown in Fig. 1.27. The total degree of each edge is

$$Td_G(s_1s_3) = (1.2, 0.3, 0.3), \quad Td_G(s_1s_4) = (0.8, 0.2, 0.2), \quad Td_G(s_2s_3) = (0.8, 0.2, 0.2).$$

Since the edges s_1s_4 and s_2s_3 in G have the same total degree, G is not a strongly edge totally irregular neutrosophic graph.

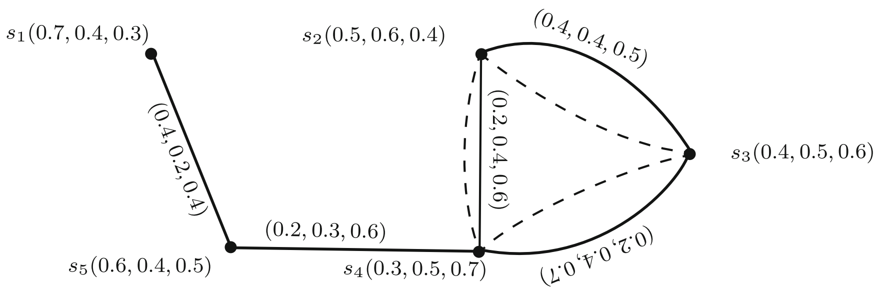


Fig. 1.28 Neutrosophic graph

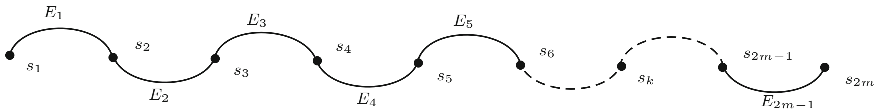


Fig. 1.29 Neutrosophic path P

Definition 1.31 A neutrosophic path is a sequence of distinct vertices $x = x_1, x_2, x_3, \ldots, x_n = y$ such that, for all k, $T_B(x_k x_{k+1}) > 0$, $I_B(x_k x_{k+1}) > 0$ and $F_B(x_k x_{k+1}) > 0$. A neutrosophic path is called a *neutrosophic cycle* if $x = y$.

Example 1.38 Consider a neutrosophic graph G on $X = \{s_1, s_2, s_3, s_4, s_5\}$ as shown in Fig. 1.28.

The path from s_2 to s_1 is shown with thick lines, and the cycle C from s_2 to s_2 is shown with dashed lines in Fig. 1.28.

Theorem 1.22 *Let $G^* = (X, E)$ be a path as shown in Fig. 1.29 on $2m(m > 1)$ vertices and G be a neutrosophic graph. Let $E_1, E_2, E_3, \ldots, E_{2m-1}$ be the edges in G having $c_1, c_2, c_3, \ldots, c_{2m-1}$ as their membership values, respectively. Assume that $c_1 < c_2 < c_3 < \cdots < c_{2m-1}$, where $c_k = (T_k, I_k, F_k)$, $k = 1, 2, 3, \ldots, 2m-1$. Then G is both strongly edge irregular and strongly edge totally irregular neutrosophic graph.*

Proof Let G be a neutrosophic graph of a crisp graph $G^* = (X, E)$. Assume that G is a neutrosophic path on $2m(m > 1)$ vertices. Suppose that $c_k = (T_k, I_k, F_k)$ be the membership values of the edges L_k in G, where $k = 1, 2, 3, \ldots, 2m-1$. We assume that $c_1 < c_2 < c_3 < \cdots < c_{2m-1}$.

The degree of each vertex in G is calculated as:

$$\begin{aligned}
d_G(s_1) &= c_1 = (T_1, I_1, F_1), \quad for \quad k = 1. \\
d_G(s_k) &= c_{k-1} + c_k = (T_{k-1}, I_{k-1}, F_{k-1}) + (T_k, I_k, F_k), \\
&= (T_{k-1} + T_k, I_{k-1} + I_k, F_{k-1} + F_k), \quad for \quad k = 2, 3, \ldots, 2m-1. \\
d_G(s_{2m}) &= c_{2m-1} = (T_{2m-1}, I_{2m-1}, F_{2m-1}), \quad for \quad k = 2m.
\end{aligned}$$

The degree of each edge in G is calculated as:

$$\begin{aligned} d_G(E_1) &= c_2 = (T_2, I_2, F_2), \quad for \quad k = 1.\\ d_G(L_k) &= c_{k-1} + c_{k+1} = (T_{k-1}, I_{k-1}, F_{k-1}) + (t_{k+1}, i_{k+1}, f_{k+1}),\\ &= (T_{k-1} + T_{k+1}, I_{k-1} + I_{k+1}, F_{k-1} + F_{k+1}), \quad for \quad k = 2, 3, \ldots, 2m-2.\\ d_G(L_{2m-1}) &= c_{2m-2} = (T_{2m-2}, I_{2m-2}, F_{2m-2}), \quad for \quad k = 2m-1. \end{aligned}$$

Since each edge in G has distinct degree, G is strongly edge irregular neutrosophic graph. We now calculate the total degree of each edge in G as:

$$\begin{aligned} Td_G(E_1) &= c_1 + c_2 = (T_1 + T_2, I_1 + I_2, F_1 + F_2), \quad for \quad k = 1.\\ Td_G(L_k) &= c_{k-1} + c_k + c_{k+1} = (T_{k-1}, I_{k-1}, F_{k-1}) + (T_k, I_k, F_k) + (T_{k+1}, I_{k+1}, F_{k+1}),\\ &= (T_{k-1} + T_k + T_{k+1}, I_{k-1} + I_k + I_{k+1}, F_{k-1} + F_k + F_{k+1}),\\ &\quad for \quad k = 2, 3, \ldots, 2m-2.\\ Td_G(L_{2m-1}) &= c_{2m-2} + c_{2m-1} = (T_{2m-2}, I_{2m-2}, F_{2m-2}) + (T_{2m-1}, I_{2m-1}, F_{2m-1}),\\ &= (T_{2m-2} + T_{2m-1}, I_{2m-2} + I_{2m-1}, F_{2m-2} + F_{2m-1}), \quad for \quad k = 2m-1. \end{aligned}$$

Since each edge in G has distinct total degree, G is strongly edge totally irregular neutrosophic graph. Hence G is both strongly edge irregular and strongly edge totally irregular neutrosophic graph.

Definition 1.32 A complete bipartite graph is a graph whose vertex set can be partitioned into two subsets X_1 and X_2 such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is the part of the graph. A complete bipartite graph with partition of size $|X_1| = m$ and $|X_2| = n$ is denoted by $K_{(m,n)}$. A complete bipartite graph $K_{(1,n)}$ or $K_{(m,1)}$ that is a tree with one internal vertex and n or m leaves is called a star S_n or S_m.

Theorem 1.23 *Let $G^* = (X, E)$ be a star $K_{(m,1)}$ as shown in Fig. 1.30 and G be a neutrosophic graph of G^*. If each edge in G has distinct membership values, then G is strongly edge irregular neutrosophic graph but not strongly edge totally irregular neutrosophic graph.*

Proof Let G be a neutrosophic graph of a crisp graph $G^* = (X, E)$. We assume that G is a star $K_{(m,1)}$. Let $s, s_1, s_2, \ldots, s_m$ be the vertices of the star $K_{(m,1)}$, where s is the centre vertex and $s_1, s_2, \ldots, s_m$ are the vertices adjacent to vertex s as shown in Fig. 1.30. Suppose that $c_k = (T_k, I_k, F_k)$ be the membership values of the edges E_k in G, where $k = 1, 2, \ldots, m$. We assume that $c_1 \neq c_2 \neq c_3 \neq \cdots \neq c_m$. The degree of each edge in G is calculated as:

$$\begin{aligned} d_G(L_k) &= d_G(x) + d_G(s_k) - 2(T_B(ss_k), I_B(ss_k), F_B(ss_k)),\\ &= (c_1, c_2, \ldots, c_m) + (T_k, I_k, F_k) - 2(T_k, I_k, F_k),\\ &= (T_1, I_1, F_1), (T_2, I_2, F_2), \ldots, (T_m, I_m, F_m) + (T_k, I_k, F_k) - 2(T_k, I_k, F_k),\\ &= (T_1 + T_2 + \cdots + T_m, I_1 + I_2 + \cdots + I_m, F_1 + F_2 + \cdots + F_m) - (T_k, T_k, T_k). \end{aligned}$$

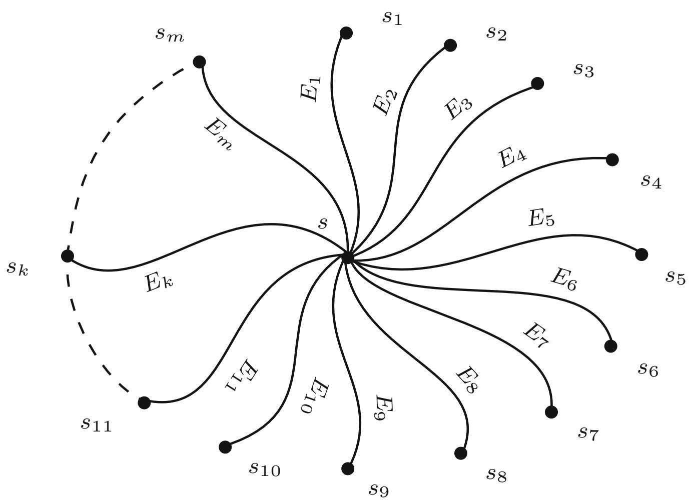


Fig. 1.30 Neutrosophic graph

It is easy to see that each edge in G has distinct degree; therefore, G is strongly edge irregular neutrosophic graph. We now calculate the total degree of each edge in G as:

$$\begin{aligned}
Td_G(L_k) &= Td_G(x) + Td_G(s_k) - (T_B(ss_k), I_B(ss_k), F_B(ss_k)),\\
&= (c_1, c_2, \ldots, c_m) + (T_k, I_k, F_k)(T_k, I_k, F_k),\\
&= (T_1, I_1, F_1), (T_2, I_2, F_2), \ldots, (T_m, I_m, F_m),\\
&= (T_1 + T_2 + \cdots + T_m, I_1 + I_2 + \cdots + I_m, F_1 + F_2 + \cdots + F_m).
\end{aligned}$$

Since all the edges in G have the same total degree, G is not a strongly edge totally irregular neutrosophic graph

Definition 1.33 The m-barbell graph $B_{(m,m)}$ is the simple graph obtained by connecting two copies of a complete graph K_m by a bridge.

Theorem 1.24 *Let G be a neutrosophic graph of $G^* = (X, E)$, the m-barbell graph $B_{(m,m)}$ as shown in Fig. 1.31. If each edge in G has distinct membership values, then G is a strongly edge irregular neutrosophic graph but not a strongly edge totally irregular neutrosophic graph.*

Proof Let G be a neutrosophic graph of a crisp graph $G^* = (X, E)$. Suppose that G^* is a m-barbell graph, then there exists a bridge, say xy, connecting m new vertices to each of its end vertices x and y. Let $b = (T, I, F)$ be the membership values of the bridge xy. Suppose that $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_m$ are the vertices adjacent to vertices x and y, respectively. Let $c_k = (T_k, I_k, F_k)$ be the membership values of the edges E_k with vertex x, where $k = 1, 2, \ldots, m$ and $a_1 < a_2 < \cdots < a_m$. Let

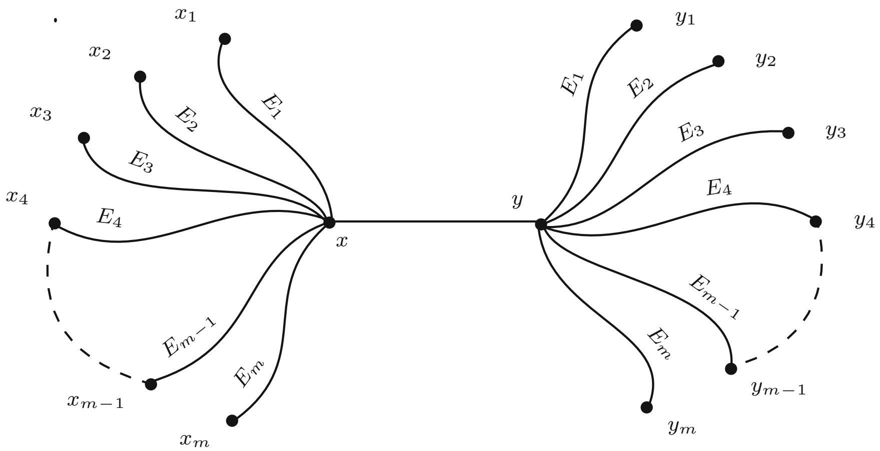


Fig. 1.31 Neutrosophic graph

$c_k^{'} = (T_k^{'}, I_k^{'}, F_k^{'})$ be the membership values of the edges E_k with vertex y, where $k = 1, 2, \ldots, m$ and $c_1 < c_2 < \cdots < c_m$. Assume that $c_1 < c_2 < \cdots < c_m < c_1^{'} < c_2^{'} < \cdots < c_m^{'} < b$. The degree of each edge in G is calculated as:

$$
\begin{aligned}
d_G(xy) &= d_G(x) + d_G(y) - 2b,\\
&= c_1 + c_2 + \cdots + c_m + b + c_1^{'} + c_2^{'} + \cdots + c_m^{'} + b - 2b,\\
&= (T_1, I_1, F_1) + (T_2, I_2, F_2) + \cdots + (T_m, I_m, F_m) + (T_1^{'}, I_1^{'}, F_1^{'}) + (T_2^{'}, I_2^{'}, F_2^{'})\\
&\quad + \cdots + (T_m^{'}, I_m^{'}, F_m^{'}),\\
&= (T_1 + T_2 + \cdots + T_m, I_1 + I_2 + \cdots + I_m, F_1 + F_2 + \cdots + F_m)\\
&\quad +(T_1^{'} + T_2^{'} + \cdots + T_m^{'}, I_1^{'} + I_2^{'} + \cdots + I_m^{'}, F_1^{'} + F_2^{'} + \cdots + F_m^{'}).
\end{aligned}
$$

$$
\begin{aligned}
d_G(L_k) &= d_G(x) + d_G(x_k) - 2c_k, \quad where \quad k = 1, 2, \ldots, m.\\
&= c_1 + c_2 + \cdots + c_m + b + c_k - 2c_k,\\
&= (T_1, I_1, F_1) + (T_2, I_2, F_2) + \cdots + (T_m, I_m, F_m) + (T, I, F) - b_k,\\
&= (T_1 + T_2 + \cdots + T_m + T, I_1 + I_2 + \cdots + I_m + I, F_1 + F_2 + \cdots + F_m + F)\\
&\quad -(T_k, I_k, F_k).
\end{aligned}
$$

$$
\begin{aligned}
d_G(E_k) &= d_G(y) + d_G(y_k) - 2c_k^{'}, \quad where \quad k = 1, 2, \ldots, m.\\
&= c_1^{'} + c_2^{'} + \cdots + c_m^{'} + b + c_k^{'} - 2c_k^{'},\\
&= (T_1^{'}, I_1^{'}, F_1^{'}) + (T_2^{'}, I_2^{'}, F_2^{'}) + \cdots + (T_m^{'}, I_m^{'}, F_m^{'}) + (T, I, F) - c_k^{'},\\
&= (T_1^{'} + T_2^{'} + \cdots + T_m^{'} + t, I_1^{'} + I_2^{'} + \cdots + I_m^{'} + i, F_1^{'} + F_2^{'} + \cdots + F_m^{'} + f)\\
&\quad -(T_k^{'}, I_k^{'}, F_k^{'}).
\end{aligned}
$$

It is easy to see that all the edges in G have distinct degrees; therefore, G is strongly edge irregular neutrosophic graph. The total degree of each edge in G is calculated as:

$$\begin{aligned}
Td_G(xy) &= d_G(xy) + b,\\
&= c_1 + c_2 + \cdots + c_m + c_1' + c_2' + \cdots + c_m' + b,\\
&= (T_1, I_1, F_1) + (T_2, I_2, F_2) + \cdots + (T_m, I_m, F_m)\\
&\quad + (T_1', I_1', F_1') + (T_2', I_2', F_2') + \cdots + (T_m', I_m', F_m') + (T, I, F),\\
&= (T_1 + T_2 + \cdots + T_m, I_1 + I_2 + \cdots + I_m, F_1 + F_2 + \cdots + F_m)\\
&\quad + (T_1' + T_2' + \cdots + T_m', I_1' + I_2' + \cdots + I_m', F_1' + F_2' + \cdots + F_m') + (T, I, F).\\
Td_G(L_k) &= d_G(L_k) + c_k, \quad where \quad k = 1, 2, \ldots, m.\\
&= c_1 + c_2 + \cdots + c_m + b + c_k - 2c_k + c_k,\\
&= (T_1, I_1, F_1) + (T_2, I_2, F_2) + \cdots + (T_m, I_m, F_m) + (T, I, F),\\
&= (T_1 + T_2 + \cdots + T_m + T, I_1 + I_2 + \cdots + I_m + I, F_1 + F_2 + \cdots + F_m + F).\\
Td_G(E_k) &= d_G(E_k) + c_k', \quad where \quad k = 1, 2, \ldots, m.\\
&= c_1' + c_2' + \cdots + c_m' + b + c_k' - 2c_k' + c_k',\\
&= (T_1', I_1', F_1') + (T_2', I_2', F_2') + \cdots + (T_m', I_m', F_m') + (T, I, F),\\
&= (T_1' + T_2' + \cdots + T_m' + T, I_1' + I_2' + \cdots + I_m' + I, F_1' + F_2' + \cdots + F_m' + F).
\end{aligned}$$

Since each edge L_k and E_k in G has the same total degree, where $k = 1, 2, \ldots, m$, G is not a strongly edge totally irregular neutrosophic graph.

1.3 Applications of Neutrosophic Graphs

1.3.1 Social Network Model

Graphical models have many applications in our daily life. Human being is the most adjustable and adapting creature. When human beings interact with each other, more or less they leave an impact(good or bad) on each other. Naturally a human being has influence on others. We can use neutrosophic digraph to examine the influence of the people on each other's thinking in a group. We can investigate a person's good influence and bad influence on the thinking of others. We can examine the percentage of uncertain influence of that person. The neutrosophic digraph will tell us about dominating person and about highly influenced person.

Consider $I = \{$Malik, Haider, Imran, Razi, Ali, Hamza, Aziz$\}$ set of seven persons in a social group on whatsapp. Let $A = \{$(Malik, 0.6, 0.4, 0.5), (Haider, 0.5, 0.6, 0.3), (Imran, 0.4, 0.3, 0.2), (Razi, 0.7, 0.6, 0.4), (Ali, 0.4, 0.1, 0.2), (Hamza, 0.6,

Table 1.5 Neutrosophic set B of edges

Edge	T	I	F
(Hamza, Malik)	0.6	0.4	0.4
(Hamza, Haider)	0.5	0.3	0.3
(Hamza, Razi)	0.3	0.3	0.4
(Hamza, Aziz)	0.3	0.3	0.4
(Malik, Haider)	0.5	0.4	0.5
(Imran, Haider)	0.4	0.3	0.3
(Aziz, Malik)	0.5	0.2	0.5
(Razi, Imran)	0.3	0.3	0.4
(Razi, Ali)	0.4	0.1	0.4
(Ali, Aziz)	0.3	0.1	0.5

0.4, 0.1), (Aziz, 0.7, 0.3, 0.5)} be the neutrosophic set on the set I where truth value of each person represents his good influence on others, falsity value represents his bad influence on others, and indeterminacy value represents uncertainty in his influence. Let $J = \{$(Hamza, Malik), (Hamza, Haider), (Hamza, Razi), (Hamza, Aziz), (Malik, Haider), (Imran, Haider), (Aziz, Malik), (Razi, Imran), (Razi, Ali), (Ali, Aziz)$\}$ be the set of relations on I. Let B be the neutrosophic set on the set J as shown in Table 1.5.

The truth, indeterminacy and falsity values of each edge are calculated using $T_B(xy) \leq T_A(x) \wedge T_A(y), I_B(xy) \leq I_A(x) \wedge I_A(y), F_B(xy) \leq F_A(x) \vee F_A(y)$. The neutrosophic digraph $G = (A, B)$ is shown in Fig. 1.32. This neutrosophic digraph shows that Hamza has influence on Malik, Haider, Razi and Aziz. We can see that Hamza's good influence on Haider is 50%, on Malik is 60%, on Razi is 30% and on Aziz is 30%. His bad influence on Haider, Malik, Razi and Aziz is 30, 40, 40 and 40%, respectively. Similarly his uncertain influence on Haider, Malik, Razi and Aziz is 30, 40, 30 and 30%, respectively. We can investigate that out-degree of vertex

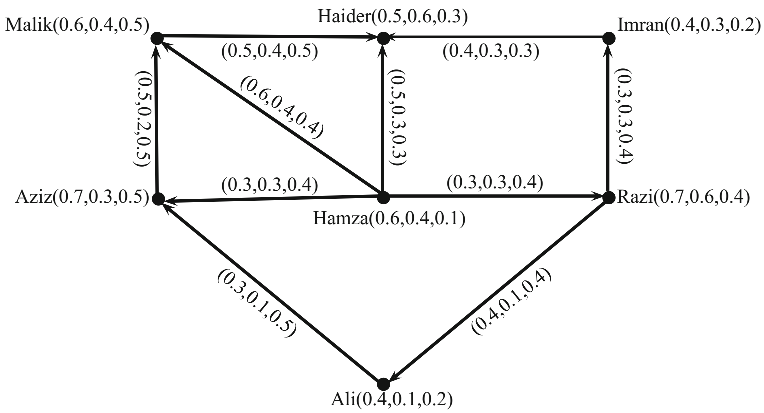


Fig. 1.32 Neutrosophic digraph

Hamza is highest, that is, four. This shows that Hamza is dominating person in this social group. On the other hand, Haider has highest in-degree, that is, three. It tells us that Haider is highly influenced by others in this social group.

We now explain general procedure of this applications through following Algorithm 1.3.1.

Algorithm 1.3.1

Step 1. Input the set of vertices $I = \{I_1, I_2, \ldots, I_n\}$ and a neutrosophic set A which is defined on set I.
Step 2. Input the set of relations $J = \{J_1, J_2, \ldots, J_n\}$.
Step 3. Compute the truth-membership degree, indeterminacy degree and falsity-membership degree of each edge using Definition 1.7.
Step 4. Compute the neutrosophic set B of edges.
Step 5. Obtain a neutrosophic digraph $G = (A, B)$.

1.3.2 Detection of a Safe Root for an Airline Journey

We consider a neutrosophic set of five countries: Germany, China, USA, Brazil and Mexico. Suppose we want to travel between these countries through an airline journey. The airline companies aim to facilitate their passengers with high quality of services. Air traffic controllers have to make sure that company planes must arrive and depart at right time. This task is possible by planning efficient routes for the planes. A neutrosophic graph of airline network among these five countries is shown in Fig. 1.33 in which vertices and edges represent the countries and flights, respectively.

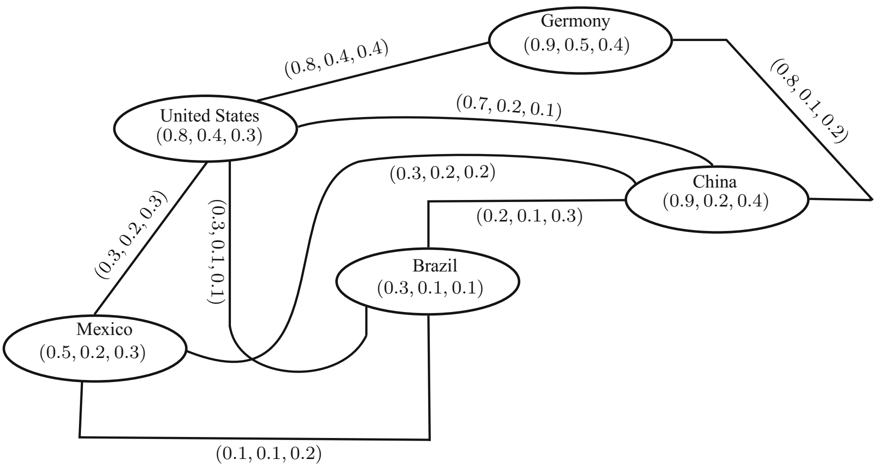


Fig. 1.33 Neutrosophic graph of an airline network

The truth-membership degree of each vertex indicates the strength of that country's airline system. The indeterminacy-membership degree of each vertex demonstrates how much the system is uncertain. The falsity-membership degree of each vertex tells the flaws of that system. The truth-membership degree of each edge interprets that how much the flight is save. The indeterminacy-membership degree of each edge shows the uncertain situations during a flight such as weather conditions, mechanical error and sabotage. The falsity-membership degree of each edge indicates the flaws of that flight. For example, the edge between Germany and China indicates that the flight chosen for this travel is 80% safe, 10% depending on uncertain systems and 20% unsafe. The truth-membership degree, the indeterminacy-membership degree and the falsity-membership degree of each edge are calculated by using the following relations.

$$\begin{aligned}
T_B(xy) &\leq \min\{T_A(x), T_A(y)\},\\
I_B(xy) &\leq \min\{I_A(x), I_A(y)\},\\
F_B(xy) &\leq \max\{F_A(x), F_A(y)\}, \quad x, y \in X.
\end{aligned}$$

Sometimes due to weather conditions, technical issues or personal problems, a passenger missed his direct flight between two particular countries. So, if he has to go somewhere urgently, then he has to choose indirect route as there are indirect routes between these countries. For example, if a passenger missed his flight from Germany to USA, then there are four indirect routes given as follows.

P_1: Germany to China then China to USA.
P_2: Germany to China, China to Mexico then Mexico to USA.
P_3: Germany to China, China to Brazil, then Brazil to USA.
P_4: Germany to China, China to Brazil, Brazil to Mexico then Mexico to USA.

We will find the most suitable route by calculating the lengths of all these routes. That route is the most suitable whose truth-membership value is maximum, indeterminacy-membership value is minimum, and falsity-membership value is minimum. After calculating the lengths of all the routes, we get $L(P_1) = (1.5, 0.3, 0.3)$, $L(P_2) = (1.3, 0.5, 0.7)$, $L(P_3) = (1.3, 0.3, 0.6)$ and $L(P_4) = (1.4, 0.5, 1.0)$.

From Fig. 1.33, it looks like travelling through Germany to USA is the most protected route, but after calculating the lengths, we find that the protected route is P_1 because of uncertain conditions. Similarly, one can find the protected route between other countries.

We now present the general procedure of our method which is used in our application from Algorithm 1.3.2.

Algorithm 1.3.2

Step 1. Input the degrees of truth-membership, indeterminacy-membership and falsity-membership of all m vertices(countries).

Step 2. Calculate the degrees of truth-membership, indeterminacy-membership and falsity-membership of all edges using the following relations.

$$T_B(xy) \leq \min\{T_A(x), T_A(y)\},$$
$$I_B(xy) \leq \min\{I_A(x), I_A(y)\},$$
$$F_B(xy) \leq \max\{F_A(x), F_A(y)\}, \qquad x, y \in X.$$

Step 3. Calculate all the possible routes P_k between the countries.
Step 4. Calculate the lengths of all the routs P_k using the following formula,

$$L(P_k) = \left(\sum_{i=1}^{m-1} T_B(x_i x_{i+1}), \sum_{i=1}^{m-1} I_B(x_i x_{i+1}), \sum_{i=1}^{m-1} F_B(x_i x_{i+1})\right), \quad k = 1, 2, \ldots, n.$$

Step 5. Find the protected route with maximum truth-membership degree, minimum indeterminacy-membership degree and minimum falsity-membership degree.

1.3.3 Selection of Military Weapon

Since in decision-making problems, there is a number of uncertainties, and in some situations, there exist some relations among attributes in a multiple-attribute decision-making problem. So, it is an interesting area of applications in neutrosophic graph theory. A multiple-attribute decision-making problem is solved under the general framework of neutrosophic graphs.

A military unit is planning to purchase new artillery weapons, and there are six feasible artillery weapons (alternatives) $x_i (i = 1, 2, \ldots, 6)$ to be selected. When making a decision, the attributes considered are as follows:
(1) a_1- assault fire capability indices.
(2) a_2- reaction capability indices.
(3) a_3- mobility indices.
(4) a_4- survival ability indices.
Among these four attributes, a_1, a_2, a_4 are of benefit type (beneficial), and a_3 is of cost type (nonbeneficial); the evaluation values are contained in the decision matrix $A = (a_{ij})_{6\times 4}$, listed in Table 1.6.

Normalized values of an attribute assigned to the alternatives are calculated by using the following formula and shown in Table 1.7:

$$r_{ij} = \langle T_{ij}, I_{ij}, F_{ij}\rangle = \begin{cases} a_{ij} & \text{for beneficial attribute,} \\ \bar{a}_{ij} & \text{for nonbeneficial attribute.} \end{cases}$$

$i = 1, 2, \ldots, 6;\ j = 1, 2, 3, 4$, where $\bar{a}_{ij}$ is the complement of a_{ij}, such that $\bar{a}_{ij} = \langle F_{ij}, 1 - I_{ij}, T_{ij}\rangle$.

Relative importance of attributes is also assigned (see table 2 in [136]). Let the decision-maker select the following assignments:

Table 1.6 Neutrosophic decision matrix $A = (a_{ij})_{6\times4}$

Weapons	a_1	a_2	a_3	a_4
x_1	$\langle 0.5, 0.3, 0.6\rangle$	$\langle 0.6, 0.3, 0.2\rangle$	$\langle 0.4, 0.5, 0.1\rangle$	$\langle 0.1, 0.7, 0.5\rangle$
x_2	$\langle 0.6, 0.1, 0.2\rangle$	$\langle 0.2, 0.1, 0.4\rangle$	$\langle 0.2, 0.3, 0.4\rangle$	$\langle 0.3, 0.4, 0.1\rangle$
x_3	$\langle 0.1, 0.5, 0.3\rangle$	$\langle 0.3, 0.2, 0.5\rangle$	$\langle 0.7, 0.2, 0.1\rangle$	$\langle 0.5, 0.1, 0.2\rangle$
x_4	$\langle 0.3, 0.4, 0.2\rangle$	$\langle 0.4, 0.5, 0.1\rangle$	$\langle 0.3, 0.1, 0.4\rangle$	$\langle 0.5, 0.3, 0.4\rangle$
x_5	$\langle 0.1, 0.2, 0.4\rangle$	$\langle 0.2, 0.7, 0.3\rangle$	$\langle 0.1, 0.3, 0.5\rangle$	$\langle 0.2, 0.1, 0.5\rangle$
x_6	$\langle 0.5, 0.1, 0.7\rangle$	$\langle 0.5, 0.1, 0.4\rangle$	$\langle 0.3, 0.2, 0.6\rangle$	$\langle 0.4, 0.2, 0.6\rangle$

Table 1.7 Neutrosophic decision matrix $R = (r_{ij})_{6\times4}$ of normalized data

Weapons	a_1	a_2	a_3	a_4
x_1	$\langle 0.5, 0.3, 0.6\rangle$	$\langle 0.6, 0.3, 0.2\rangle$	$\langle 0.1, 0.5, 0.4\rangle$	$\langle 0.1, 0.7, 0.5\rangle$
x_2	$\langle 0.6, 0.1, 0.2\rangle$	$\langle 0.2, 0.1, 0.4\rangle$	$\langle 0.4, 0.7, 0.2\rangle$	$\langle 0.3, 0.4, 0.1\rangle$
x_3	$\langle 0.1, 0.5, 0.3\rangle$	$\langle 0.3, 0.2, 0.5\rangle$	$\langle 0.1, 0.8, 0.7\rangle$	$\langle 0.5, 0.1, 0.2\rangle$
x_4	$\langle 0.3, 0.4, 0.2\rangle$	$\langle 0.4, 0.5, 0.1\rangle$	$\langle 0.4, 0.9, 0.3\rangle$	$\langle 0.5, 0.3, 0.4\rangle$
x_5	$\langle 0.1, 0.2, 0.4\rangle$	$\langle 0.2, 0.7, 0.3\rangle$	$\langle 0.5, 0.7, 0.1\rangle$	$\langle 0.2, 0.1, 0.5\rangle$
x_6	$\langle 0.5, 0.1, 0.7\rangle$	$\langle 0.5, 0.1, 0.4\rangle$	$\langle 0.6, 0.8, 0.3\rangle$	$\langle 0.4, 0.2, 0.6\rangle$

$$\mathcal{R}=\begin{array}{c} \\ a_1\\ a_2\\ a_3\\ a_4\end{array}\begin{array}{c}\begin{array}{cccc} a_1 & a_2 & a_3 & a_4\end{array}\\ \left[\begin{array}{cccc} --- & \langle 0.045, 0.410, 0.865\rangle & \langle 0.665, 0.045, 0.335\rangle & \langle 0.045, 0.590, 0.745\rangle\\ \langle 0.865, 0.590, 0.045\rangle & --- & \langle 0.135, 0.665, 0.335\rangle & \langle 0.590, 0.410, 0.255\rangle\\ \langle 0.335, 0.955, 0.665\rangle & \langle 0.335, 0.335, 0.135\rangle & --- & \langle 0.410, 0.255, 0.135\rangle\\ \langle 0.745, 0.410, 0.045\rangle & \langle 0.255, 0.590, 0.590\rangle & \langle 0.135, 0.745, 0.410\rangle & ---\end{array}\right]\end{array}.$$

The weapon selection attribute neutrosophic digraph given in Fig. 1.34, represents the presence as well as relative importance of four attributes a_1, a_2, a_3 and a_4 which are the vertices of the digraph. The weapon selection index is calculated using the values of A_i and r_{ij} for each alternative weapon, where A_i is the value of ith attribute represented by the weapon x_i and r_{ij} is the relative importance of the ith attribute over jth attribute.

For first weapon x_1, substituting values of A_1, A_2, A_3 and A_4 in above matrix $\mathcal{R}$, we get

$$\mathcal{R}_1=\begin{array}{c} \\ a_1\\ a_2\\ a_3\\ a_4\end{array}\begin{array}{c}\begin{array}{cccc} a_1 & a_2 & a_3 & a_4\end{array}\\ \left[\begin{array}{cccc} \langle 0.5, 0.3, 0.6\rangle & \langle 0.045, 0.410, 0.865\rangle & \langle 0.665, 0.045, 0.335\rangle & \langle 0.045, 0.590, 0.745\rangle\\ \langle 0.865, 0.590, 0.045\rangle & \langle 0.6, 0.3, 0.2\rangle & \langle 0.135, 0.665, 0.335\rangle & \langle 0.590, 0.410, 0.255\rangle\\ \langle 0.335, 0.955, 0.665\rangle & \langle 0.335, 0.335, 0.135\rangle & \langle 0.1, 0.5, 0.4\rangle & \langle 0.410, 0.255, 0.135\rangle\\ \langle 0.745, 0.410, 0.045\rangle & \langle 0.255, 0.590, 0.590\rangle & \langle 0.135, 0.745, 0.410\rangle & \langle 0.1, 0.7, 0.5\rangle\end{array}\right]\end{array}.$$

Now we calculate the permanent function value of above matrix using computer program, that is, per $(\mathcal{R}_1) = \langle 0.4117, 1.3482, 0.4884\rangle$. The permanent function is nothing but the determinant of a matrix but considering all the determinant terms as positive terms [87]. So, the weapon selection index values of different weapons are:

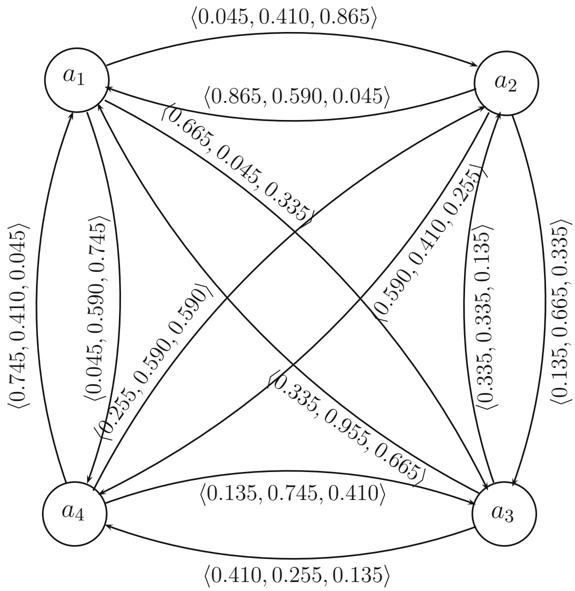


Fig. 1.34 Weapon selection attribute neutrosophic digraph

$$\begin{aligned}
x_1 &= \langle 0.4117, 1.3482, 0.4884\rangle, \\
x_2 &= \langle 0.4224, 1.0522, 0.3415\rangle, \\
x_3 &= \langle 0.4098, 1.1991, 0.4782\rangle, \\
x_4 &= \langle 0.5173, 1.5801, 0.3468\rangle, \\
x_5 &= \langle 0.3272, 1.3426, 0.4429\rangle, \\
x_6 &= \langle 0.6113, 0.9950, 0.6179\rangle.
\end{aligned}$$

Calculate the score function $s(x_i) = T_i + 1 - I_i + 1 - F_i$ of the weapons $x_i (i = 1, 2, \ldots, 6)$, respectively: $s(x_1) = 0.5751$, $s(x_2) = 1.0287$, $s(x_3) = 0.7325$, $s(x_4) = 0.5904$, $s(x_5) = 0.5417$, $s(x_6) = 0.9984$. Thus, we can rank the weapons:

$$x_2 \succ x_6 \succ x_3 \succ x_4 \succ x_1 \succ x_5.$$

Therefore, the best choice is the second weapon (x_2).

1.4 Energy of Neutrosophic Graphs

If we change min by max in indeterminacy-membership of Definition 1.7, then we have the following definition of neutrosophic graph.

Definition 1.34 A neutrosophic graph on a nonempty set X is a pair $G = (A, B)$, where A is a neutrosophic set in X and B is a neutrosophic relation on X such that

$$T_B(xy) \leq \min\{T_A(x), T_A(y)\},$$
$$I_B(xy) \leq \max\{I_A(x), I_A(y)\},$$
$$F_B(xy) \leq \max\{F_A(x), F_A(y)\}, \qquad \text{for all } x, y \in X.$$

If B is not symmetric on A, then $D = (A, \overrightarrow{B})$ is called *neutrosophic digraph.*

Example 1.39 Consider a graph $G^* = (X, E)$ where $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ and $E = \{x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1x_5, x_1x_6, x_1x_7, x_3x_5, x_3x_6, x_3x_7, x_2x_5, x_5x_6, x_6x_7, x_4x_7\}$. Let $G = (A, B)$ be a neutrosophic graph on V as shown in Fig. 1.35 defined by

A	x_1	x_2	x_3	x_4	x_5	x_6	x_7
T_A	0.6	0.4	0.5	0.6	0.3	0.2	0.2
I_A	0.5	0.1	0.3	0.4	0.4	0.5	0.4
F_A	0.7	0.3	0.2	0.9	0.5	0.6	0.8

B	x_1x_2	x_2x_3	x_3x_4	x_4x_1	x_1x_5	x_1x_6	x_1x_7	x_3x_5	x_3x_6	x_3x_7	x_2x_5	x_5x_6	x_6x_7	x_4x_7
T_B	0.2	0.3	0.3	0.5	0.2	0.1	0.2	0.2	0.1	0.2	0.2	0.2	0.1	0.2
I_B	0.1	0.1	0.2	0.3	0.4	0.3	0.3	0.3	0.3	0.2	0.1	0.1	0.4	0.3
F_B	0.4	0.3	0.7	0.6	0.6	0.6	0.7	0.4	0.4	0.5	0.4	0.6	0.7	0.7

We now define and investigate the energy of a graph within the framework of neutrosophic set theory.

Definition 1.35 The *adjacency matrix* $\mathcal{A}(G)$ of a neutrosophic graph $G = (A, B)$ is defined as a square matrix $\mathcal{A}(G) = [a_{jk}]$, $a_{jk} = \langle T_B(x_jx_k), I_B(x_jx_k), F_B(x_jx_k)\rangle$, where $T_B(x_jx_k)$, $I_B(x_jx_k)$ and $F_B(x_jx_k)$ represent the strength of relationship, strength of undecided relationship and strength of nonrelationship between x_j and x_k, respectively.

The *adjacency matrix* of a neutrosophic graph can be expressed as three matrices: first matrix contains the entries as truth-membership values, second contains the entries as indeterminacy-membership values, and the third contains the entries as falsity-membership values, i.e., $\mathcal{A}(G) = \langle \mathcal{A}(T_B(x_jx_k)), \mathcal{A}(I_B(x_jx_k)), \mathcal{A}(F_B(x_jx_k))\rangle$.

Definition 1.36 The *spectrum of adjacency matrix* of a neutrosophic graph $\mathcal{A}(G)$ is defined as $\langle M, N, O\rangle$, where M, N and O are the sets of eigenvalues of $\mathcal{A}(T_B(x_jx_k))$, $\mathcal{A}(I_B(x_jx_k))$ and $\mathcal{A}(F_B(x_jx_k))$, respectively.

Example 1.40 The adjacency matrix $\mathcal{A}(G)$ of a neutrosophic graph given in Fig. 1.35 is

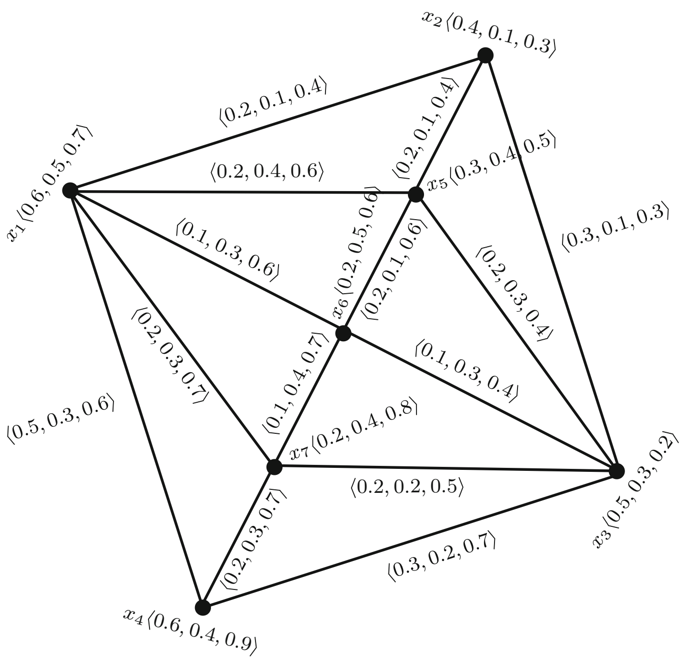


Fig. 1.35 Single-valued neutrosophic graph

$$\begin{pmatrix}
\langle 0,0,0\rangle & \langle 0.2,0.1,0.4\rangle & \langle 0,0,0\rangle & \langle 0.5,0.3,0.6\rangle & \langle 0.2,0.4,0.6\rangle & \langle 0.1,0.3,0.6\rangle & \langle 0.2,0.3,0.7\rangle \\
\langle 0.2,0.1,0.4\rangle & \langle 0,0,0\rangle & \langle 0.3,0.1,0.3\rangle & \langle 0,0,0\rangle & \langle 0.2,0.1,0.4\rangle & \langle 0,0,0\rangle & \langle 0,0,0\rangle \\
\langle 0,0,0\rangle & \langle 0.3,0.1,0.3\rangle & \langle 0,0,0\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.2,0.3,0.4\rangle & \langle 0.1,0.3,0.4\rangle & \langle 0.2,0.2,0.5\rangle \\
\langle 0.5,0.3,0.6\rangle & \langle 0,0,0\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,0\rangle & \langle 0,0,0\rangle & \langle 0,0,0\rangle & \langle 0.2,0.3,0.7\rangle \\
\langle 0.2,0.4,0.6\rangle & \langle 0.2,0.1,0.4\rangle & \langle 0.2,0.3,0.4\rangle & \langle 0,0,0\rangle & \langle 0,0,0\rangle & \langle 0.2,0.1,0.6\rangle & \langle 0,0,0\rangle \\
\langle 0.1,0.3,0.6\rangle & \langle 0,0,0\rangle & \langle 0.1,0.3,0.4\rangle & \langle 0,0,0\rangle & \langle 0.2,0.1,0.6\rangle & \langle 0,0,0\rangle & \langle 0.1,0.4,0.7\rangle \\
\langle 0.2,0.3,0.7\rangle & \langle 0,0,0\rangle & \langle 0.2,0.2,0.5\rangle & \langle 0.2,0.3,0.7\rangle & \langle 0,0,0\rangle & \langle 0.1,0.4,0.7\rangle & \langle 0,0,0\rangle
\end{pmatrix}.$$

The spectrum of a neutrosophic graph G given in Fig. 1.35 is as follows:

$$\begin{aligned}
Spec(T_B(x_jx_k)) &= \{-0.7137, -0.2966, -0.2273, 0.0000, 0.0577, 0.2646, 0.9152\},\\
Spec(I_B(x_jx_k)) &= \{-0.7150, -0.4930, -0.0874, -0.0308, 0.0507, 0.2012, 1.0743\},\\
Spec(F_B(x_jx_k)) &= \{-1.2963, -1.1060, -0.5118, -0.0815, 0.1507, 0.5510, 2.2938\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
Spec(G) = \{&\langle -0.7137, -0.7150, -1.2963\rangle, \langle -0.2966, -0.4930, -1.1060\rangle,\\
&\langle -0.2273, -0.0874, -0.5118\rangle, \langle 0.0000, -0.0308, -0.0815\rangle,\\
&\langle 0.0577, 0.0507, 0.1507\rangle, \langle 0.2646, 0.2012, 0.5510\rangle,\\
&\langle 0.9152, 1.0743, 2.2938\rangle\}.
\end{aligned}$$

Definition 1.37 The energy of a neutrosophic graph $G = (A, B)$ is defined as,

$$E(G) = \left\langle E(T_B(x_jx_k)), E(I_B(x_jx_k)), E(F_B(x_jx_k)) \right\rangle$$
$$= \left\langle \sum_{\substack{j=1 \\ \lambda_j \in M}}^{n} |\lambda_j|, \sum_{\substack{j=1 \\ \zeta_j \in N}}^{n} |\zeta_j|, \sum_{\substack{j=1 \\ \eta_j \in O}}^{n} |\eta_j| \right\rangle.$$

Definition 1.38 Two neutrosophic graphs with the same number of vertices and the same energy are called *equienergetic*.

Theorem 1.25 *Let $G = (A, B)$ be a neutrosophic graph and $\mathcal{A}(G)$ be its adjacency matrix. If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $\zeta_1 \geq \zeta_2 \geq \cdots \geq \zeta_n$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$ are the eigenvalues of $\mathcal{A}(T_B(x_jx_k))$, $\mathcal{A}(I_B(x_jx_k))$ and $\mathcal{A}(F_B(x_jx_k))$, then*

1. $\sum_{\substack{j=1 \\ \lambda_j \in M}}^{n} \lambda_j = 0, \sum_{\substack{j=1 \\ \zeta_j \in N}}^{n} \zeta_j = 0, \sum_{\substack{j=1 \\ \eta_j \in O}}^{n} \eta_j = 0$
2. $\sum_{\substack{j=1 \\ \lambda_j \in M}}^{n} \lambda_j^2 = 2\left(\sum_{1 \leq j < k \leq n} (T_B(x_jx_k))^2\right),$

   $\sum_{\substack{j=1 \\ \zeta_j \in N}}^{n} \zeta_j^2 = 2\left(\sum_{1 \leq j < k \leq n} (I_B(x_jx_k))^2\right),$

   $\sum_{\substack{j=1 \\ \eta_j \in O}}^{n} \eta_j^2 = 2\left(\sum_{1 \leq j < k \leq n} (F_B(x_jx_k))^2\right).$

Proof 1. Since $\mathcal{A}(G)$ is a symmetric matrix whose trace is zero, its eigenvalues are real with zero sum.
2. By matrix trace properties, we have

$$tr((\mathcal{A}(T_B(x_jx_k)))^2) = \sum_{\substack{j=1 \\ \lambda_j \in M}}^{n} \lambda_j^2$$

$$\begin{aligned} tr((\mathcal{A}(T_B(x_jx_k)))^2) &= (0 + T_B^2(x_1x_2) + \cdots + T_B^2(x_1x_n)) + (T_B^2(x_2x_1) + 0 + \cdots \\ &\quad + T_B^2(x_2x_n)) + \cdots + (T_B^2(x_nx_1) + T_B^2(x_nx_2) + \cdots + 0) \\ &= 2\left(\sum_{1 \leq j < k \leq n} (T_B(x_jx_k))^2\right). \end{aligned}$$

Hence $\sum\limits_{\substack{j=1\\ \lambda_j \in M}}^{n} \lambda_j^2 = 2\left(\sum\limits_{1\leq j<k\leq n} (T_B(x_jx_k))^2\right)$. Analogously, we can show that $\sum\limits_{\substack{j=1\\ \zeta_j \in N}}^{n} \zeta_j^2 = 2\left(\sum\limits_{1\leq j<k\leq n} (I_B(x_jx_k))^2\right)$ and $\sum\limits_{\substack{j=1\\ \eta_j \in O}}^{n} \eta_j^2 = 2\left(\sum\limits_{1\leq j<k\leq n} (F_B(x_jx_k))^2\right)$.

We now give upper and lower bounds on energy of a neutrosophic graph G, in terms of the number of vertices and the sum of squares of truth-membership, indeterminacy-membership and falsity-membership values of edges.

Theorem 1.26 *Let $G = (A, B)$ be a neutrosophic graph on n vertices with adjacency matrix $\mathcal{A}(G) = \langle \mathcal{A}(T_B(x_jx_k)), \mathcal{A}(I_B(x_jx_k)), \mathcal{A}(F_B(x_jx_k))\rangle$, then*

1. $$\sqrt{2\sum_{1\leq j<k\leq n} (T_B(x_jx_k))^2 + n(n-1)|T|^{\frac{2}{n}}} \leq E(T_B(x_jx_k)) \leq \sqrt{2n\sum_{1\leq j<k\leq n} (T_B(x_jx_k))^2}$$
2. $$\sqrt{2\sum_{1\leq j<k\leq n} (I_B(x_jx_k))^2 + n(n-1)|I|^{\frac{2}{n}}} \leq E(I_B(x_jx_k)) \leq \sqrt{2n\sum_{1\leq j<k\leq n} (I_B(x_jx_k))^2}$$
3. $$\sqrt{2\sum_{1\leq j<k\leq n} (F_B(x_jx_k))^2 + n(n-1)|F|^{\frac{2}{n}}} \leq E(F_B(x_jx_k)) \leq \sqrt{2n\sum_{1\leq j<k\leq n} (F_B(x_jx_k))^2}.$$

where $|T|, |I|$ and $|F|$ are the determinant of $\mathcal{A}(T_B(x_jx_k)), \mathcal{A}(I_B(x_jx_k))$ and $\mathcal{A}(F_B(x_jx_k))$, respectively.

Proof 1. Upper bound: Apply Cauchy–Schwarz inequality to the n numbers $1, 1, \ldots, 1$ and $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|$, then

$$\sum_{j=1}^{n} |\lambda_j| \leq \sqrt{n}\sqrt{\sum_{j=1}^{n} |\lambda_j|^2} \tag{1.1}$$

$$\left(\sum_{j=1}^{n} \lambda_j\right)^2 = \sum_{j=1}^{n} |\lambda_j|^2 + 2\left(\sum_{1\leq j<k\leq n} \lambda_j\lambda_k\right). \tag{1.2}$$

By comparing the coefficients of λ^{n-2} in the characteristic polynomial $\prod_{j=1}^{n}(\lambda-\lambda_j) = |\mathcal{A}(G)-\lambda I|$, we have

$$\sum_{1\le j<k\le n}\lambda_j\lambda_k = -\sum_{1\le j<k\le n}(T_B(x_jx_k))^2. \tag{1.3}$$

Substituting (1.3) in (1.2), we obtain

$$\sum_{j=1}^{n}|\lambda_j|^2 = 2\sum_{1\le j<k\le n}(T_B(x_jx_k))^2. \tag{1.4}$$

Substituting (1.4) in (1.1), we obtain

$$\sum_{j=1}^{n}|\lambda_j| \le \sqrt{n}\sqrt{2\sum_{1\le j<k\le n}(T_B(x_jx_k))^2} = \sqrt{2n\sum_{1\le j<k\le n}(T_B(x_jx_k))^2}.$$

Therefore,

$$E(T_B(x_jx_k)) \le \sqrt{2n\sum_{1\le j<k\le n}(T_B(x_jx_k))^2}.$$

Lower bound:

$$\begin{aligned}(E(T_B(x_jx_k)))^2 &= \left(\sum_{j=1}^{n}|\lambda_j|\right)^2 = \sum_{j=1}^{n}|\lambda_j|^2 + 2\left(\sum_{1\le j<k\le n}|\lambda_j\lambda_k|\right)\\ &= 2\left(\sum_{1\le j<k\le n}(T_B(x_jx_k))^2\right) + \frac{2n(n-1)}{2}AM\{|\lambda_j\lambda_k|\}.\end{aligned}$$

Since $AM\{|\lambda_j\lambda_k|\} \ge GM\{|\lambda_j\lambda_k|\}$, $1\le j<k\le n$,

$$E(T_B(x_jx_k)) \ge \sqrt{2\left(\sum_{1\le j<k\le n}(T_B(x_jx_k))^2 + n(n-1)GM\{|\lambda_j\lambda_k|\}\right)}.$$

It can also be seen that

$$GM\{|\lambda_j\lambda_k|\} = \left(\prod_{1\le j<k\le n}|\lambda_j\lambda_k|\right)^{\frac{2}{n(n-1)}} = \left(\prod_{j=1}^{n}|\lambda_j|^{n-1}\right)^{\frac{2}{n(n-1)}} = \left(\prod_{j=1}^{n}|\lambda_j|\right)^{\frac{2}{n}} = |T|^{\frac{2}{n}}.$$

Therefore,

$$E(T_B(x_jx_k)) \geq \sqrt{2\sum_{1\leq j<k\leq n}(T_B(x_jx_k))^2 + n(n-1)|T|^{\frac{2}{n}}}.$$

Thus, analogously, we can show that

$$\begin{aligned}\sqrt{2\left(\sum_{1\leq j<k\leq n}(I_B(x_jx_k))^2 + n(n-1)|I|^{\frac{2}{n}}\right)} &\leq E(I_B(x_jx_k))\\ &\leq \sqrt{2n\left(\sum_{1\leq j<k\leq n}(I_B(x_jx_k))^2\right)}\end{aligned}$$

$$\begin{aligned}\sqrt{2\left(\sum_{1\leq j<k\leq n}(F_B(x_jx_k))^2 + n(n-1)|F|^{\frac{2}{n}}\right)} &\leq E(F_B(x_jx_k))\\ &\leq \sqrt{2n\left(\sum_{1\leq j<k\leq n}(F_B(x_jx_k))^2\right)}.\end{aligned}$$

We now define and investigate the Laplacian energy of a graph under neutrosophic environment and investigate its properties.

Definition 1.39 Let $G = (A, B)$ be a neutrosophic graph on n vertices. The degree matrix, $D(G) = \langle D(T_B(x_jx_k)), D(I_B(x_jx_k)), D(F_B(x_jx_k))\rangle = [d_{jk}]$, of G is a $n \times n$ diagonal matrix defined as,

$$d_{jk} = \begin{cases} d_G(x_j) & \text{if } j = k,\\ 0 & \text{otherwise}\end{cases}$$

Definition 1.40 The *Laplacian matrix* of a neutrosophic graph $G = (A, B)$ is defined as $L(G) = \langle L(T_B(x_jx_k)), L(I_B(x_jx_k)), L(F_B(x_jx_k))\rangle = D(G) - \mathcal{A}(G)$, where $\mathcal{A}(G)$ is an adjacency matrix and $D(G)$ is a degree matrix of a neutrosophic graph G.

Definition 1.41 The spectrum of Laplacian matrix of a neutrosophic graph $L(G)$ is defined as $\langle M_L, N_L, O_L\rangle$, where M_L, N_L and O_L are the sets of Laplacian eigenvalues of $L(T_B(x_jx_k))$, $L(I_B(x_jx_k))$ and $L(F_B(x_jx_k))$, respectively.

Theorem 1.27 *Let $G = (A, B)$ be a neutrosophic graph, and let $L(G) = \langle L(T_B(x_jx_k)), L(I_B(x_jx_k)), L(F_B(x_jx_k))\rangle$ be the Laplacian matrix of G. If $\vartheta_1 \geq \vartheta_2 \geq \cdots \geq \vartheta_n$, $\varphi_1 \geq \varphi_2 \geq \cdots \geq \varphi_n$ and $\psi_1 \geq \psi_2 \geq \cdots \geq \psi_n$ are the eigenvalues of $L(T_B(x_jx_k))$, $L(I_B(x_jx_k))$ and $L(F_B(x_jx_k))$, respectively, then*

1. $\sum_{\substack{j=1\\ \vartheta_j\in M_L}}^{n} \vartheta_j = 2\left(\sum_{1\le j<k\le n} T_B(x_jx_k)\right)$, $\sum_{\substack{j=1\\ \varphi_j\in N_L}}^{n} \varphi_j = 2\left(\sum_{1\le j<k\le n} I_B(x_jx_k)\right)$

   $\sum_{\substack{j=1\\ \psi_j\in O_L}}^{n} \psi_j = 2\left(\sum_{1\le j<k\le n} F_B(x_jx_k)\right)$

2. $\sum_{\substack{j=1\\ \vartheta_j\in M_L}}^{n} \vartheta_j^2 = 2\left(\sum_{1\le j<k\le n} (T_B(x_jx_k))^2\right) + \sum_{j=1}^{n} d^2_{T_B(x_jx_k)}(x_j)$,

   $\sum_{\substack{j=1\\ \varphi_j\in N_L}}^{n} \varphi_j^2 = 2\left(\sum_{1\le j<k\le n} (I_B(x_jx_k))^2\right) + \sum_{j=1}^{n} d^2_{I_B(x_jx_k)}(x_j)$,

   $\sum_{\substack{j=1\\ \psi_j\in O_L}}^{n} \psi_j^2 = 2\left(\sum_{1\le j<k\le n} (F_B(x_jx_k))^2\right) + \sum_{j=1}^{n} d^2_{F_B(x_jx_k)}(x_j)$.

Proof 1. Since $L(G)$ is a symmetric matrix with nonnegative Laplacian eigenvalues,

$$\sum_{\substack{j=1\\ \vartheta_j\in M_L}}^{n} \vartheta_j = tr(L(G)) = \sum_{j=1}^{n} d_{T_B(x_jx_k)}(x_j) = 2\left(\sum_{1\le j<k\le n} T_B(x_jx_k)\right).$$

Similarly, it is easy to show that

$$\sum_{\substack{j=1\\ \varphi_j\in N_L}}^{n} \varphi_j = 2\left(\sum_{1\le j<k\le n} I_B(x_jx_k)\right)$$

$$\sum_{\substack{j=1\\ \psi_j\in O_L}}^{n} \psi_j = 2\left(\sum_{1\le j<k\le n} F_B(x_jx_k)\right).$$

2. By definition of Laplacian matrix, we have

$$L(T_B(x_jx_k)) = \begin{pmatrix} d_{T_B(x_jx_k)}(x_1) & -T_B(x_1x_2) & \dots & -T_B(x_1x_n) \\ -T_B(x_2x_1) & d_{T_B(x_jx_k)}(x_2) & \dots & -T_B(x_2x_n) \\ \vdots & \vdots & \ddots & \vdots \\ -T_B(x_nx_1) & -T_B(x_nx_2) & \dots & d_{T_B(x_jx_k)}(x_n) \end{pmatrix}.$$

By trace properties of a matrix, we have $tr((L(T_B(x_jx_k)))^2) = \sum_{\substack{j=1\\ \vartheta_j\in M_L}}^{n} \vartheta_j^2$ where

$$\begin{aligned} tr((L(T_B(x_jx_k)))^2) &= (d^2_{T_B(x_jx_k)}(x_1) + T_B^2(x_1x_2) + \cdots + T_B^2(x_1x_n)) \\ &\quad + (T_B^2(x_2x_1) + d^2_{T_B(x_jx_k)}(x_2) + \cdots + T_B^2(x_2x_n)) \\ &\quad + \cdots + (T_B^2(x_nx_1) + T_B^2(x_nx_2) + \cdots + d^2_{T_B(x_jx_k)}(x_n)) \\ &= 2\left(\sum_{1\le j<k\le n} (T_B(x_jx_k))^2\right) + \sum_{j=1}^{n} d^2_{T_B(x_jx_k)}(x_j). \end{aligned}$$

Therefore, $\sum\limits_{\substack{j=1 \\ \vartheta_j \in M_L}}^{n} \vartheta_j^2 = 2\left(\sum\limits_{1\le j<k\le n} (T_B(x_jx_k))^2\right) + \sum\limits_{j=1}^{n} d^2_{T_B(x_jx_k)}(x_j)$. Analogously, we can show that

$$\sum_{\substack{j=1 \\ \varphi_j \in N_L}}^{n} \varphi_j^2 = 2\left(\sum_{1\le j<k\le n} (I_B(x_jx_k))^2\right) + \sum_{j=1}^{n} d^2_{I_B(x_jx_k)}(x_j)$$

$$\sum_{\substack{j=1 \\ \psi_j \in O_L}}^{n} \psi_j^2 = 2\left(\sum_{1\le j<k\le n} (F_B(x_jx_k))^2\right) + \sum_{j=1}^{n} d^2_{F_B(x_jx_k)}(x_j).$$

Definition 1.42 The Laplacian energy of a neutrosophic graph $G = (A, B)$ is defined as $LE(G) = \big\langle LE(T_B(x_jx_k)), LE(I_B(x_jx_k)), LE(F_B(x_jx_k))\big\rangle = \big\langle \sum\limits_{j=1}^{n} |\varrho_j|, \sum\limits_{j=1}^{n} |\xi_j|, \sum\limits_{j=1}^{n} |\tau_j|\big\rangle$ where

$$\varrho_j = \vartheta_j - \frac{2\left(\sum\limits_{1\le j<k\le n} T_B(x_jx_k)\right)}{n},$$

$$\xi_j = \varphi_j - \frac{2\left(\sum\limits_{1\le j<k\le n} I_B(x_jx_k)\right)}{n},$$

$$\tau_j = \psi_j - \frac{2\left(\sum\limits_{1\le j<k\le n} F_B(x_jx_k)\right)}{n}.$$

Theorem 1.28 *Let $G = (A, B)$ be a neutrosophic graph on n vertices and let $L(G) = \langle L(T_B(x_jx_k)), L(I_B(x_jx_k)), L(F_B(x_jx_k))\rangle$ be the Laplacian matrix of G, then*

1. $LE(T_B(x_jx_k))$

$$\leq \sqrt{2n \sum_{1\leq j<k\leq n} (T_B(x_j x_k))^2 + n \sum_{j=1}^{n} \left(d_{T_B(x_j x_k)}(x_j) - \frac{2 \sum_{1\leq j<k\leq n} T_B(x_j x_k)}{n} \right)^2},$$

2. $LE(I_B(x_j x_k))$

$$\leq \sqrt{2n \sum_{1\leq j<k\leq n} (I_B(x_j x_k))^2 + n \sum_{j=1}^{n} \left(d_{I_B(x_j x_k)}(x_j) - \frac{2 \sum_{1\leq j<k\leq n} I_B(x_j x_k)}{n} \right)^2},$$

3. $LE(F_B(x_j x_k))$

$$\leq \sqrt{2n \sum_{1\leq j<k\leq n} (F_B(x_j x_k))^2 + n \sum_{j=1}^{n} \left(d_{F_B(x_j x_k)}(x_j) - \frac{2 \sum_{1\leq j<k\leq n} F_B(x_j x_k)}{n} \right)^2}.$$

Proof Apply Cauchy–Schwarz inequality to the n numbers $1, 1, \ldots, 1$ and $|\varrho_1|, |\varrho_2|, \ldots, |\varrho_n|$, and we have $\sum_{j=1}^{n} |\varrho_j| \leq \sqrt{n}\sqrt{\sum_{j=1}^{n} |\varrho_j|^2}$ and $LE(T_B(x_j x_k)) \leq \sqrt{n}\sqrt{2\mathcal{M}_T} = \sqrt{2n\mathcal{M}_T}$. We know that
$\mathcal{M}_T = \sum_{1\leq j<k\leq n} (T_B(x_j x_k))^2 + \frac{1}{2} \sum_{j=1}^{n} \left(d_{T_B(x_j x_k)}(x_j) - \frac{2 \sum_{1\leq j<k\leq n} T_B(x_j x_k)}{n} \right)^2,$
Therefore, it can be proved that

$$LE(T_B(x_j x_k)) \leq \sqrt{2n \sum_{1\leq j<k\leq n} (T_B(x_j x_k))^2 + n \sum_{j=1}^{n} \left(d_{T_B(x_j x_k)}(x_j) - \frac{2 \sum_{1\leq j<k\leq n} T_B(x_j x_k)}{n} \right)^2},$$

$$LE(I_B(x_j x_k)) \leq \sqrt{2n \sum_{1\leq j<k\leq n} (I_B(x_j x_k))^2 + n \sum_{j=1}^{n} \left(d_{I_B(x_j x_k)}(x_j) - \frac{2 \sum_{1\leq j<k\leq n} I_B(x_j x_k)}{n} \right)^2},$$

$$LE(F_B(x_j x_k)) \leq \sqrt{2n \sum_{1\leq j<k\leq n} (F_B(x_j x_k))^2 + n \sum_{j=1}^{n} \left(d_{F_B(x_j x_k)}(x_j) - \frac{2 \sum_{1\leq j<k\leq n} F_B(x_j x_k)}{n} \right)^2}.$$

Theorem 1.29 *Let $G = (A, B)$ be a neutrosophic graph on n vertices and let $L(G) = \langle L(T_B(x_j x_k)), L(I_B(x_j x_k)), L(F_B(x_j x_k))\rangle$ be the Laplacian matrix of G, then*

$$LE(T_B(x_jx_k)) \geq 2\sqrt{\sum_{1\leq j<k\leq n}(T_B(x_jx_k))^2+\frac{1}{2}\sum_{j=1}^{n}\left(d_{T_B(x_jx_k)}(x_j)-\frac{2\sum_{1\leq j<k\leq n}T_B(x_jx_k)}{n}\right)^2},$$

$$LE(I_B(x_jx_k)) \geq 2\sqrt{\sum_{1\leq j<k\leq n}(I_B(x_jx_k))^2+\frac{1}{2}\sum_{j=1}^{n}\left(d_{I_B(x_jx_k)}(x_j)-\frac{2\sum_{1\leq j<k\leq n}I_B(x_jx_k)}{n}\right)^2},$$

$$LE(F_B(x_jx_k)) \geq 2\sqrt{\sum_{1\leq j<k\leq n}(F_B(x_jx_k))^2+\frac{1}{2}\sum_{j=1}^{n}\left(d_{F_B(x_jx_k)}(x_j)-\frac{2\sum_{1\leq j<k\leq n}F_B(x_jx_k)}{n}\right)^2}.$$

Proof Here $\left(\sum_{j=1}^{n}|\varrho_j|\right)^2=\sum_{j=1}^{n}|\varrho_j|^2+2\sum_{1\leq j<k\leq n}|\varrho_j\varrho_k|\geq 4\mathcal{M}_T$ and $LE(T_B(x_jx_k))\geq 2\sqrt{\mathcal{M}_T}$. Since $\mathcal{M}_T=\sum_{1\leq j<k\leq n}(T_B(x_jx_k))^2+\frac{1}{2}\sum_{j=1}^{n}\left(d_{T_B(x_jx_k)}(x_j)-\frac{2\sum_{1\leq j<k\leq n}T_B(x_jx_k)}{n}\right)^2$,

$$LE(T_B(x_jx_k)) \geq 2\sqrt{\sum_{1\leq j<k\leq n}(T_B(x_jx_k))^2+\frac{1}{2}\sum_{j=1}^{n}\left(d_{T_B(x_jx_k)}(x_j)-\frac{2\sum_{1\leq j<k\leq n}T_B(x_jx_k)}{n}\right)^2},$$

$$LE(I_B(x_jx_k)) \geq 2\sqrt{\sum_{1\leq j<k\leq n}(I_B(x_jx_k))^2+\frac{1}{2}\sum_{j=1}^{n}\left(d_{I_B(x_jx_k)}(x_j)-\frac{2\sum_{1\leq j<k\leq n}I_B(x_jx_k)}{n}\right)^2}$$

$$LE(F_B(x_jx_k)) \geq 2\sqrt{\sum_{1\leq j<k\leq n}(F_B(x_jx_k))^2+\frac{1}{2}\sum_{j=1}^{n}\left(d_{F_B(x_jx_k)}(x_j)-\frac{2\sum_{1\leq j<k\leq n}F_B(x_jx_k)}{n}\right)^2}.$$

Definition 1.43 The *signless Laplacian matrix* of a neutrosophic graph $G=(A,B)$ is defined as $L^+(G)=\langle L^+(T_B(x_jx_k)), L^+(I_B(x_jx_k)), L^+(F_B(x_jx_k))\rangle = D(G)+$

$\mathcal{A}(G)$, where $D(G)$ and $\mathcal{A}(G)$ are the degree matrix and the adjacency matrix, respectively, of a neutrosophic graph G. The spectrum of signless Laplacian matrix of a neutrosophic graph $L^+(G)$ is defined as $\langle M_{L^+}, N_{L^+}, O_{L^+}\rangle$, where M_{L^+}, N_{L^+} and O_{L^+} are the sets of signless Laplacian eigenvalues of $L^+(T_B(x_jx_k))$, $L^+(I_B(x_jx_k))$ and $L^+(F_B(x_jx_k))$, respectively.

1.5 Application to Group Decision-Making

Group decision-making is a commonly used tool in human activities, which determines the optimal alternative from a given finite set of alternatives using the evaluation information given by a group of decision-makers or experts. With the rapid development of society, group decision-making plays an increasingly important role when dealing with the decision-making problems. Recently, many scholars have investigated the approaches for group decision-making based on different kinds of decision information. However, in order to reflect the relationships among the alternatives, we need to make pairwise comparisons for all the alternatives in the process of decision- making. Preference relation is a powerful quantitative decision technique that supports experts in expressing their preferences over the given alternatives. For a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$, the experts compare each pair of alternatives and construct preference relations, respectively. If every element in the preference relations is a neutrosophic number, then the concept of the neutrosophic preference relation (NPR) can be put forth as follows:

Definition 1.44 A NPR on the set $X = \{x_1, x_2, \ldots, x_n\}$ is represented by a matrix $R = (r_{jk})_{n\times n}$, where $r_{jk}=\langle x_jx_k, T(x_jx_k), I(x_jx_k), F(x_jx_k)\rangle$ for all $j, k=1, 2, \ldots, n$. For convenience, let $r_{jk} = \langle T_{jk}, I_{jk}, F_{jk}\rangle$ where T_{jk} indicates the degree to which the object x_j is preferred to the object x_k, F_{jk} denotes the degree to which the object x_j is not preferred to the object x_k, and I_{jk} is interpreted as an indeterminacy-membership degree, with the conditions: $T_{jk}, I_{jk}, F_{jk} \in [0, 1]$, $T_{jk} = F_{kj}$, $F_{jk} = T_{kj}$, $I_{jk} + I_{kj} = 1$, $T_{jj} = I_{jj} = F_{jj} = 0.5$, for all $j, k = 1, 2, \ldots, n$.

A group decision-making problem concerning the ‘Alliance partner selection of a software company’ is solved to illustrate the applicability of the proposed concepts of energy of neutrosophic graphs in realistic scenario.

1.5.1 Alliance Partner Selection of a Software Company

Eastsoft is one of the top five software companies in China [77]. It offers a rich portfolio of businesses, including product engineering solutions, industry solutions, and related software products and platform and services. It is dedicated to becoming a globally leading IT solution and service provider through continuous improvement

Table 1.8 NPR of the expert from the engineering management department

R_1	a_1	a_2	a_3	a_4	a_5
a_1	⟨0.5, 0.5, 0.5⟩	⟨0.4, 0.6, 0.3⟩	⟨0.2, 0.4, 0.6⟩	⟨0.7, 0.6, 0.3⟩	⟨0.3, 0.1, 0.6⟩
a_2	⟨0.3, 0.4, 0.4⟩	⟨0.5, 0.5, 0.5⟩	⟨0.7, 0.3, 0.8⟩	⟨0.4, 0.1, 0.4⟩	⟨0.1, 0.3, 0.5⟩
a_3	⟨0.6, 0.6, 0.2⟩	⟨0.8, 0.7, 0.7⟩	⟨0.5, 0.5, 0.5⟩	⟨0.3, 0.6, 0.4⟩	⟨0.2, 0.3, 0.4⟩
a_4	⟨0.3, 0.4, 0.7⟩	⟨0.4, 0.9, 0.4⟩	⟨0.4, 0.4, 0.3⟩	⟨0.5, 0.5, 0.5⟩	⟨0.3, 0.1, 0.3⟩
a_5	⟨0.6, 0.9, 0.3⟩	⟨0.5, 0.7, 0.1⟩	⟨0.4, 0.7, 0.2⟩	⟨0.3, 0.9, 0.3⟩	⟨0.5, 0.5, 0.5⟩

Table 1.9 NPR of the expert from the human resource department

R_2	a_1	a_2	a_3	a_4	a_5
a_1	⟨0.5, 0.5, 0.5⟩	⟨0.5, 0.3, 0.1⟩	⟨0.1, 0.7, 0.5⟩	⟨0.3, 0.9, 0.5⟩	⟨0.2, 0.7, 0.8⟩
a_2	⟨0.1, 0.7, 0.5⟩	⟨0.5, 0.5, 0.5⟩	⟨0.5, 0.1, 0.6⟩	⟨0.6, 0.7, 0.1⟩	⟨0.4, 0.6, 0.8⟩
a_3	⟨0.5, 0.3, 0.1⟩	⟨0.6, 0.9, 0.5⟩	⟨0.5, 0.5, 0.5⟩	⟨0.9, 0.2, 0.3⟩	⟨0.1, 0.4, 0.1⟩
a_4	⟨0.5, 0.1, 0.3⟩	⟨0.1, 0.3, 0.6⟩	⟨0.3, 0.8, 0.9⟩	⟨0.5, 0.5, 0.5⟩	⟨0.8, 0.4, 0.2⟩
a_5	⟨0.8, 0.3, 0.2⟩	⟨0.8, 0.4, 0.4⟩	⟨0.1, 0.6, 0.1⟩	⟨0.2, 0.6, 0.8⟩	⟨0.5, 0.5, 0.5⟩

Table 1.10 NPR of the expert from the finance department

R_3	a_1	a_2	a_3	a_4	a_5
a_1	⟨0.5, 0.5, 0.5⟩	⟨0.9, 0.8, 0.7⟩	⟨0.1, 0.7, 0.2⟩	⟨0.4, 0.3, 0.1⟩	⟨0.6, 0.3, 0.6⟩
a_2	⟨0.7, 0.2, 0.9⟩	⟨0.5, 0.5, 0.5⟩	⟨0.4, 0.3, 0.6⟩	⟨0.6, 0.3, 0.4⟩	⟨0.7, 0.2, 0.9⟩
a_3	⟨0.2, 0.3, 0.1⟩	⟨0.6, 0.7, 0.4⟩	⟨0.5, 0.5, 0.5⟩	⟨0.1, 0.2, 0.4⟩	⟨0.6, 0.2, 0.8⟩
a_4	⟨0.1, 0.7, 0.4⟩	⟨0.4, 0.7, 0.6⟩	⟨0.4, 0.8, 0.1⟩	⟨0.5, 0.5, 0.5⟩	⟨0.6, 0.7, 0.3⟩
a_5	⟨0.6, 0.7, 0.6⟩	⟨0.9, 0.8, 0.7⟩	⟨0.8, 0.8, 0.6⟩	⟨0.3, 0.3, 0.6⟩	⟨0.5, 0.5, 0.5⟩

of organization and process, competence development of leadership and employees, and alliance and open innovation. To improve the operation and competitiveness capability in the global market, Eastsoft plans to establish a strategic alliance with a transnational corporation. After numerous consultations, five transnational corporations would like to establish a strategic alliance with Eastsoft; they are HP a_1, PHILIPS a_2, EMC a_3, SAP a_4 and LK a_5. To select the desirable strategic alliance partner, three experts e_i $(i = 1, 2, 3)$ are invited to participate in the decision analysis, who come from the engineering management department, the human resource department and the finance department of Eastsoft, respectively. Based on their experiences, the experts compare each pair of alternatives and give individual judgments using the following NPRs $R_i = (r_{jk}^{(i)})_{5\times 5}$ $(i = 1, 2, 3)$:

The neutrosophic digraphs D_i corresponding to NPRs R_i $(i = 1, 2, 3)$ given in Tables 1.8, 1.9 and 1.10 are shown in Figs. 1.36, 1.37 and 1.38.

The energy of a neutrosophic digraph is the sum of absolute values of the real part of eigenvalues of D. The energy of each neutrosophic digraph $D_i (i = 1, 2, 3)$ is calculated as $E(D_1) = \langle 3.2419, 3.5861, 3.2419\rangle$, $E(D_2) = \langle 3.2790, 3.9089, 3.2790\rangle$,

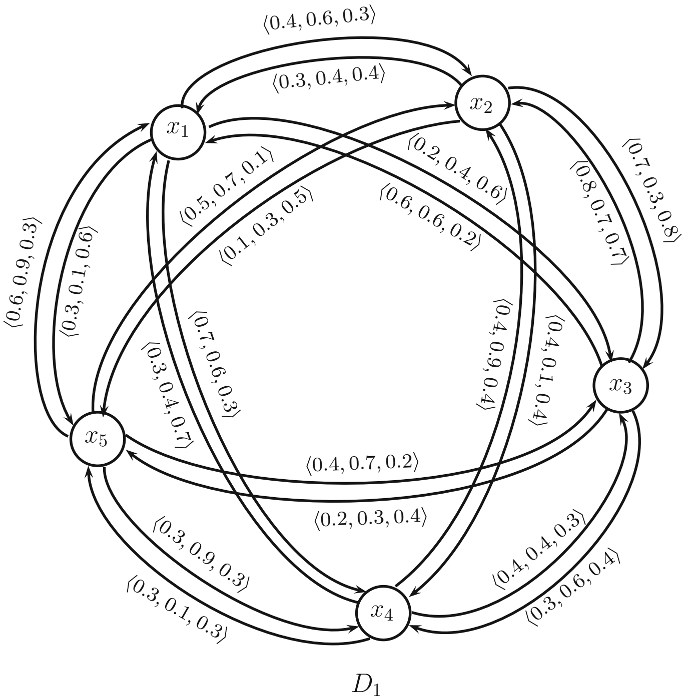


Fig. 1.36 Neutrosophic digraph

$E(D_3) = \langle 4.1587, 3.5618, 4.1587\rangle$. Then the weight of each expert can be determined as,

$$w_i = \left(\frac{E((D_T)_i)}{\sum_{l=1}^{m} E((D_T)_l)}, \frac{E((D_I)_i)}{\sum_{l=1}^{m} E((D_I)_l)}, \frac{E((D_F)_i)}{\sum_{l=1}^{m} E((D_F)_l)} \right), \ 1 \le i \le m.$$

The weights are calculated as $w_1 = \langle 0.3219, 0.3561, 0.3219\rangle$, $w_2 = \langle 0.3133, 0.3735, 0.3133\rangle$, $w_3 = \langle 0.3501, 0.2998, 0.3501\rangle$. Utilize the aggregation operator to fuse all the individual NPRs $R_i = (r_{jk}^{(i)})_{5\times 5}$ $(i = 1, 2, 3)$ into the collective NPR $R = (r_{jk})_{5\times 5}$ as shown in Table 1.11. Here we apply the neutrosophic weighted averaging (NWA) operator [59] to fuse the individual NPR.

$$\mathrm{NWA}(r_{jk}^{(1)}, r_{jk}^{(2)}, \ldots, r_{jk}^{(s)}) = \left\langle 1 - \prod_{i=1}^{s} \left(1 - T_{jk}^{(i)}\right)^{w_i}, \prod_{i=1}^{s} \left(I_{jk}^{(i)}\right)^{w_i}, \prod_{i=1}^{s} \left(F_{jk}^{(i)}\right)^{w_i} \right\rangle$$

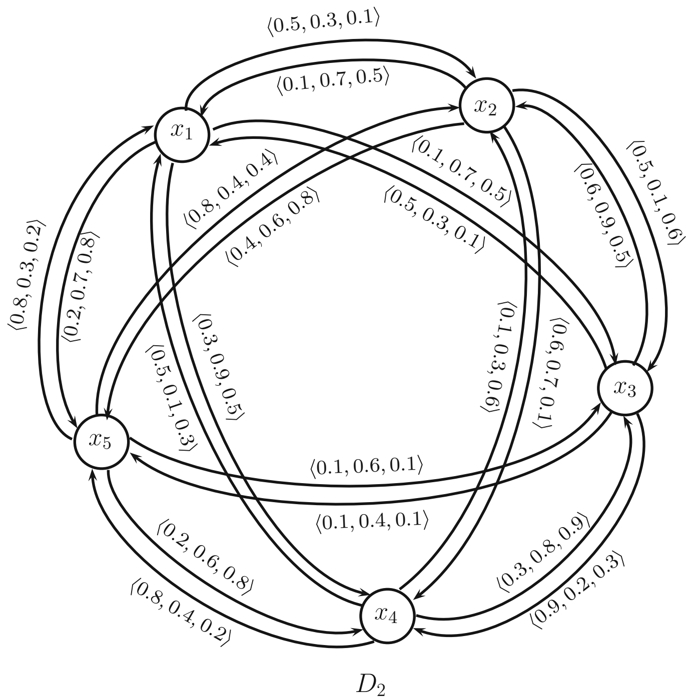


Fig. 1.37 Neutrosophic digraph

Draw a directed network corresponding to a collective NPR above, as shown in Fig. 1.39. Then under the condition $T_{jk} \geq 0.5$ $(j, k = 1,\ 2,\ 3,\ 4,\ 5)$, a partial diagram is drawn, as shown in Fig. 1.40.

Calculate the out-degrees out-d(a_j) $(j$=1, 2, 3, 4, 5) of all criteria in a partial directed network as follows: out-d$(a_1) = \langle 0.6951, 0.4973, 0.2912\rangle$, out-d$(a_2) = \langle 1.0813, 0.4608, 0.9258\rangle$, out-d$(a_3) = \langle 1.2580, 1.0430, 0.8911\rangle$, out-d$(a_4) = \langle 0.6093, 0.2811, 0.2689\rangle$, out-d$(a_5) = \langle 1.9907, 1.8177, 0.9005\rangle$. According to membership degrees of out-d(a_j) $(j = 1,\ 2,\ 3,\ 4,\ 5)$, we get the ranking of the factors a_j $(j = 1, 2, 3, 4, 5)$ as $a_5 \succ a_3 \succ a_2 \succ a_1 \succ a_4$. Thus, the best choice is LK a_5. Now elements of the Laplacian matrices of the neutrosophic digraphs $L(D_i) = R_i^L$ $(i = 1, 2, 3)$ shown in Figs. 1.36, 1.37, 1.38 are provided in Tables 1.12, 1.13 and 1.14.

The Laplacian energy of each neutrosophic digraph is calculated as
$LE(D_1) = \langle 3.2800, 4.0000, 3.8893\rangle$,
$LE(D_2) = \langle 3.3600, 4.0000, 3.8798\rangle$,
$LE(D_3) = \langle 4.6806, 4.5858, 4.9687\rangle$. Then the weight of each expert can be determined as

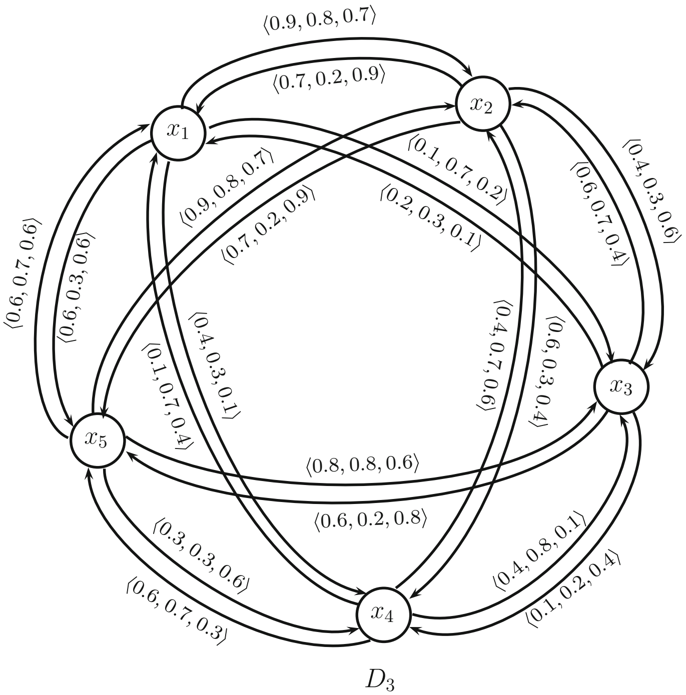


Fig. 1.38 Neutrosophic digraph

$$w_i = \left(\frac{LE((D_T)_i)}{\sum_{l=1}^{m} LE((D_T)_l)}, \frac{LE((D_I)_i)}{\sum_{l=1}^{m} LE((D_I)_l)}, \frac{LE((D_F)_i)}{\sum_{l=1}^{m} LE((D_F)_l)} \right), \quad i = 1, 2, \ldots, m.$$

$w_1 = \langle 0.2937, 0.3581, 0.3482 \rangle$, $w_2 = \langle 0.2989, \;\; 0.3559, \;\; 0.3452 \rangle$, $w_3 = \langle 0.3288, 0.3221, 0.3490 \rangle$ based on which, using the NWA operator, the fused NPR is determined, as shown in Table 1.15. In the directed network corresponding to a collective NPR above, we select those neutrosophic numbers whose membership degrees $T_{jk} \geq 0.5$ $(j, k = 1,\ 2,\ 3,\ 4,\ 5)$, and resulting partial diagram is shown in Fig. 1.41.

Calculate the out-degrees out-d(a_j) $(j = 1,\ 2,\ 3,\ 4,\ 5)$ of all criteria in a partial directed network as follows out-d$(a_1) = \langle 0.6719, 0.5050, 0.2622 \rangle$, out-d$(a_2) = \langle 1.0333, 0.4563, 0.8874 \rangle$, out-d$(a_3) = \langle 1.2122, 1.0354, 0.8534 \rangle$, out-d$(a_4) = \langle 0.5881, 0.2821, 0.2478 \rangle$, out-d$(a_5)$=$\langle 1.9228, 1.8333, 0.8201 \rangle$. According to membership degrees of out-d(a_j) $(j = 1,\ 2,\ 3,\ 4,\ 5)$, we get the ranking of the factors a_j, $j = 1, 2, 3, 4, 5$ as $a_5 \succ a_3 \succ a_2 \succ a_1 \succ a_4$. Thus, the best choice is LK a_5. Now, the elements of the signless Laplacian matrices of the neutrosophic

Table 1.11 Collective NPR of all the above individual NPRs

R	a_1	a_2	a_3	a_4	a_5
a_1	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.6951, 0.4973, 0.2912\rangle$	$\langle 0.1321, 0.5675, 0.3887\rangle$	$\langle 0.4924, 0.5587, 0.2439\rangle$	$\langle 0.3968, 0.2687, 0.6615\rangle$
a_2	$\langle 0.4341, 0.3898, 0.5775\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.5432, 0.1921, 0.6632\rangle$	$\langle 0.5381, 0.2687, 0.2626\rangle$	$\langle 0.4596, 0.3322, 0.7190\rangle$
a_3	$\langle 0.4458, 0.3706, 0.1293\rangle$	$\langle 0.6757, 0.7609, 0.5206\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.5823, 0.2821, 0.3705\rangle$	$\langle 0.3466, 0.2855, 0.3347\rangle$
a_4	$\langle 0.3085, 0.2744, 0.4436\rangle$	$\langle 0.3136, 0.5520, 0.5306\rangle$	$\langle 0.3656, 0.6209, 0.2933\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.6093, 0.2811, 0.2689\rangle$
a_5	$\langle 0.6737, 0.5520, 0.3428\rangle$	$\langle 0.7842, 0.5850, 0.3156\rangle$	$\langle 0.5328, 0.6807, 0.2421\rangle$	$\langle 0.2663, 0.5547, 0.5292\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$

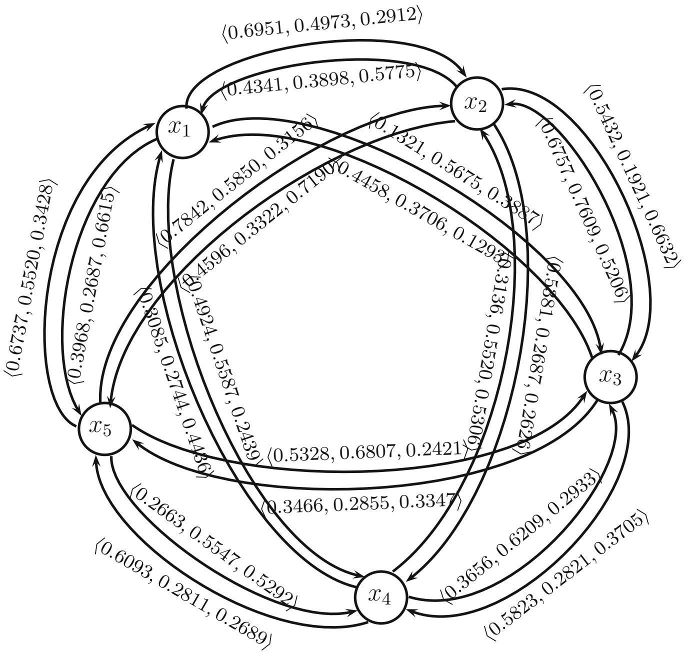


Fig. 1.39 Directed network of the fused NPR

digraphs $L^+(D_i) = R_i^{L^+}$ $(i = 1, 2, 3)$ shown in Figs. 1.36, 1.37 and 1.38 are given in Tables 1.16, 1.17 and 1.18. The signless Laplacian energy of each neutrosophic digraph is calculated as $LE^+(D_1) = \langle 3.3244, 4.7474, 3.5570\rangle$, $LE^+(D_2) = \langle 3.3826, 4.0000, 3.4427\rangle$, $LE^+(D_3) = \langle 4.5859, 4.4103, 4.7228\rangle$. Then the weight of each expert is

$$w_i = \left(\frac{LE^+((D_T)_i)}{\sum\limits_{l=1}^{m} LE^+((D_T)_l)}, \frac{LE^+((D_I)_i)}{\sum\limits_{l=1}^{m} LE^+((D_I)_l)}, \frac{LE^+((D_F)_i)}{\sum\limits_{l=1}^{m} LE^+((D_F)_l)} \right), \quad i = 1, 2, \ldots, m,$$

$w_1 = \langle 0.2859, 0.4082, 0.3059\rangle$, $w_2 = \langle 0.3125, 0.3695, 0.3180\rangle$, $w_3 = \langle 0.3343, 0.3215, 0.3443\rangle$, based on which fuse all the individual NPRs $R_i = (r_{jk}^{(i)})_{5\times 5}$ $(i = 1, 2, 3)$ into the collective NPR $R = (r_{jk})_{5\times 5}$, by using the NWA operator, as shown in Table 1.19. In the directed network corresponding to a collective NPR above, we select those neutrosophic numbers whose membership degrees $T_{jk} \geq 0.5$ $(j, k = 1, 2, 3, 4, 5)$, and resulting partial diagram is shown in Fig. 1.42.

Calculate the out-degrees out-d(a_j) $(j = 1, 2, 3, 4, 5)$ of all criteria in a partial directed network as follows out-d$(a_1) = \langle 0.6777, 0.4843, 0.2943\rangle$, out-d$(a_2) = \langle 1.0412, 0.4099, 0.9309\rangle$, out-d$(a_3) = \langle 1.2265, 1.0084, 0.9005\rangle$, out-d$(a_4) = \langle 0.5980, 0.2483, 0.2740\rangle$, out-d$(a_5) = \langle 1.9395, 1.7873, 0.9212\rangle$. According

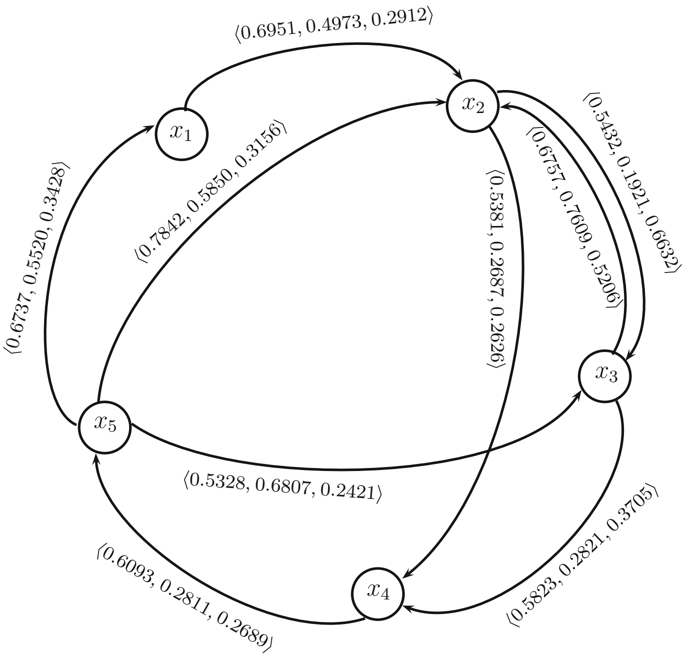


Fig. 1.40 Partial directed network of the fused NPR

to membership degrees of out-d(a_j) $(j = 1,\ 2,\ 3,\ 4,\ 5)$, we get the ranking of the factors a_j $(j = 1, 2, 3, 4, 5)$ as $a_5 \succ a_3 \succ a_2 \succ a_1 \succ a_4$. Thus, the best choice is LK a_5.

1.5.2 Real-Time Example

The proposed concepts of energy, Laplacian energy and signless Laplacian energy of a neutrosophic graph are explained here through a real-time example. We have taken the website http://www.pantechsolutions.net modelled as a neutrosophic graph by considering the navigation of the customer. We have taken the four links: 1. microcontroller boards, 2. log-in html, 3. and 4. project kits for our calculation. A neutrosophic graph of this site for four different time periods is considered. The energy, Laplacian energy and signless Laplacian energy of a neutrosophic graph are calculated for each of these periods. The energy, Laplacian energy and signless Laplacian energy are represented in terms of bar graphs. In the website http://www.pantechsolutions.net (accessed on 8 May 2012). The above four links are considered for the period 16 January 2018 to 15 February 2018, and for this graph, as shown in Fig. 1.43, we have

Table 1.12 Elements of the Laplacian matrix of the neutrosophic digraph D_1

R_1^L	a_1	a_2	a_3	a_4	a_5
a_1	$\langle 1.6, 1.7, 1.8\rangle$	$\langle -0.4, -0.6, -0.3\rangle$	$\langle -0.2, -0.4, -0.6\rangle$	$\langle -0.7, -0.6, -0.3\rangle$	$\langle -0.3, -0.1, -0.6\rangle$
a_2	$\langle -0.3, -0.4, -0.4\rangle$	$\langle 1.5, 1.1, 2.1\rangle$	$\langle -0.7, -0.3, -0.8\rangle$	$\langle -0.4, -0.1, -0.4\rangle$	$\langle -0.1, -0.3, -0.5\rangle$
a_3	$\langle -0.6, -0.6, -0.2\rangle$	$\langle -0.8, -0.7, -0.7\rangle$	$\langle 1.9, 2.2, 1.7\rangle$	$\langle -0.3, -0.6, -0.4\rangle$	$\langle -0.2, -0.3, -0.4\rangle$
a_4	$\langle -0.3, -0.4, -0.7\rangle$	$\langle -0.4, -0.9, -0.4\rangle$	$\langle -0.4, -0.4, -0.3\rangle$	$\langle 1.4, 1.8, 1.7\rangle$	$\langle -0.3, -0.1, -0.3\rangle$
a_5	$\langle -0.6, -0.9, -0.3\rangle$	$\langle -0.5, -0.7, -0.1\rangle$	$\langle -0.4, -0.7, -0.2\rangle$	$\langle -0.3, -0.9, -0.3\rangle$	$\langle 1.8, 3.2, -0.9\rangle$

Table 1.13 Elements of the Laplacian matrix of the neutrosophic digraph D_2

R_2^L	a_1	a_2	a_3	a_4	a_5
a_1	$\langle 1.1, 2.6, 1.9\rangle$	$\langle -0.5, -0.3, -0.1\rangle$	$\langle -0.1, -0.7, -0.5\rangle$	$\langle -0.3, -0.9, -0.5\rangle$	$\langle -0.2, -0.7, -0.8\rangle$
a_2	$\langle -0.1, -0.7, -0.5\rangle$	$\langle 1.6, 2.1, 2.0\rangle$	$\langle -0.5, -0.1, -0.6\rangle$	$\langle -0.6, -0.7, -0.1\rangle$	$\langle -0.4, -0.6, -0.8\rangle$
a_3	$\langle -0.5, -0.3, -0.1\rangle$	$\langle -0.6, -0.9, -0.5\rangle$	$\langle 2.1, 1.8, 1.0\rangle$	$\langle -0.9, -0.2, -0.3\rangle$	$\langle -0.1, -0.4, -0.1\rangle$
a_4	$\langle -0.5, -0.1, -0.3\rangle$	$\langle -0.1, -0.3, -0.6\rangle$	$\langle -0.3, -0.8, -0.9\rangle$	$\langle 1.7, 1.6, 2.0\rangle$	$\langle -0.8, -0.4, -0.2\rangle$
a_5	$\langle -0.8, -0.3, -0.2\rangle$	$\langle -0.8, -0.4, -0.4\rangle$	$\langle -0.1, -0.6, -0.1\rangle$	$\langle -0.2, -0.6, -0.8\rangle$	$\langle 1.9, 1.9, 1.5\rangle$

Table 1.14 Elements of the Laplacian matrix of the neutrosophic digraph D_3

R_3^L	a_1	a_2	a_3	a_4	a_5
a_1	$\langle 2.0, 2.1, 1.6\rangle$	$\langle -0.9, -0.8, -0.7\rangle$	$\langle -0.1, -0.7, -0.2\rangle$	$\langle -0.4, -0.3, -0.1\rangle$	$\langle -0.6, -0.3, -0.6\rangle$
a_2	$\langle -0.7, -0.2, -0.9\rangle$	$\langle 2.4, 1.0, 2.8\rangle$	$\langle -0.4, -0.3, -0.6\rangle$	$\langle -0.6, -0.3, -0.4\rangle$	$\langle -0.7, -0.2, -0.9\rangle$
a_3	$\langle -0.2, -0.3, -0.1\rangle$	$\langle -0.6, -0.7, -0.4\rangle$	$\langle 1.5, 1.4, 1.7\rangle$	$\langle -0.1, -0.2, -0.4\rangle$	$\langle -0.6, -0.2, -0.8\rangle$
a_4	$\langle -0.1, -0.7, -0.4\rangle$	$\langle -0.4, -0.7, -0.6\rangle$	$\langle -0.4, -0.8, -0.1\rangle$	$\langle 1.5, 2.9, 1.4\rangle$	$\langle -0.6, -0.7, -0.3\rangle$
a_5	$\langle -0.6, -0.7, -0.6\rangle$	$\langle -0.9, -0.8, -0.7\rangle$	$\langle -0.8, -0.8, -0.6\rangle$	$\langle -0.3, -0.3, -0.6\rangle$	$\langle 2.6, 2.6, 2.5\rangle$

$\text{Spec}(T_Y(x_jx_k)) = \{-0.3442, -0.1000, 0.0066, 0.4376\}$,
$\text{Spec}(I_Y(x_jx_k)) = \{-0.6630, -0.2742, 0.0774, 0.8598\}$,
$\text{Spec}(F_Y(x_jx_k)) = \{-0.6703, -0.3296, 0.0299, 0.9701\}$,
$E(T_Y(x_jx_k)) = 0.8884$, $E(I_Y(x_jx_k)) = 1.8744$, $E(F_Y(x_jx_k)) = 1.9999$.
Therefore, $E(G_1) = \langle 0.8884, 1.8744, 1.9999\rangle$.
Laplacian $\text{Spec}(T_Y(x_jx_k)) = \{0, 0.2492, 0.5244, 0.8264\}$,
Laplacian $\text{Spec}(I_Y(x_jx_k)) = \{0, 0.6975, 1.1757, 1.5269\}$,
Laplacian $\text{Spec}(F_Y(x_jx_k)) = \{0, 0.7605, 1.4139, 1.6256\}$,
$LE(T_Y(x_jx_k)) = 1.1016$, $LE(I_Y(x_jx_k)) = 2.0051$, $LE(F_Y(x_jx_k)) = 2.2790$.
Therefore, $LE(G_1) = \langle 1.1016, 2.0051, 2.2790\rangle$.

Signless Laplacian $\text{Spec}(T_Y(x_jx_k)) = \{-0.3183, -0.1339, -0.0555, 0.5076\}$,
Signless Laplacian $\text{Spec}(I_Y(x_jx_k)) = \{-0.6764, -0.2500, 0.0385, 0.8879\}$,
Signless Laplacian $\text{Spec}(F_Y(x_jx_k)) = \{-0.7056, -0.2572, -0.0582, 1.0211\}$,
$LE^+(T_Y(x_jx_k)) = 1.0153$, $LE^+(I_Y(x_jx_k)) = 1.8529$, $LE^+(F_Y(x_jx_k)) = 2.0421$.
Therefore, $LE^+(G_1) = \langle 1.0153, 1.8529, 2.0421\rangle$.

Table 1.15 Collective NPR of all the above individual NPRs

R	a_1	a_2	a_3	a_4	a_5
a_1	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.6719, 0.5050, 0.2622\rangle$	$\langle 0.1234, 0.5656, 0.3757\rangle$	$\langle 0.4664, 0.5443, 0.2317\rangle$	$\langle 0.3767, 0.2620, 0.6484\rangle$
a_2	$\langle 0.4126, 0.3778, 0.5515\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.5175, 0.1943, 0.6490\rangle$	$\langle 0.5158, 0.2620, 0.2384\rangle$	$\langle 0.4398, 0.3226, 0.7011\rangle$
a_3	$\langle 0.4229, 0.3682, 0.1155\rangle$	$\langle 0.6493, 0.7557, 0.5050\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.5629, 0.2797, 0.3484\rangle$	$\langle 0.3285, 0.2792, 0.3037\rangle$
a_4	$\langle 0.2929, 0.2829, 0.4233\rangle$	$\langle 0.2949, 0.5593, 0.5098\rangle$	$\langle 0.3460, 0.6191, 0.2839\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.5881, 0.2821, 0.2478\rangle$
a_5	$\langle 0.6506, 0.5593, 0.3157\rangle$	$\langle 0.7635, 0.5911, 0.2886\rangle$	$\langle 0.5087, 0.6829, 0.2158\rangle$	$\langle 0.2508, 0.5448, 0.5094\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$

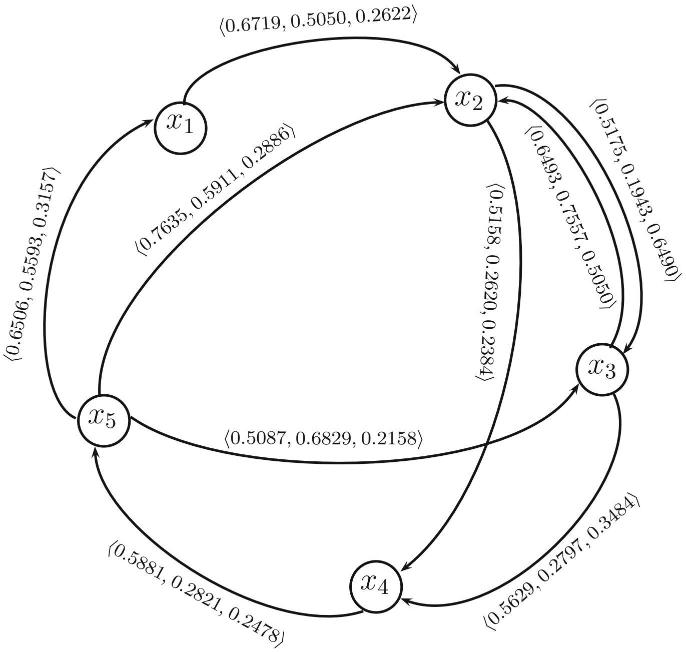


Fig. 1.41 Partial directed network of the fused NPR

Table 1.16 Elements of the signless Laplacian matrix of the neutrosophic digraph D_1

$R_1^{L^+}$	a_1	a_2	a_3	a_4	a_5
a_1	⟨1.6, 1.7, 1.8⟩	⟨0.4, 0.6, 0.3⟩	⟨0.2, 0.4, 0.6⟩	⟨0.7, 0.6, 0.3⟩	⟨0.3, 0.1, 0.6⟩
a_2	⟨0.3, 0.4, 0.4⟩	⟨1.5, 1.1, 2.1⟩	⟨0.7, 0.3, 0.8⟩	⟨0.4, 0.1, 0.4⟩	⟨0.1, 0.3, 0.5⟩
a_3	⟨0.6, 0.6, 0.2⟩	⟨0.8, 0.7, 0.7⟩	⟨1.9, 2.2, 1.7⟩	⟨0.3, 0.6, 0.4⟩	⟨0.2, 0.3, 0.4⟩
a_4	⟨0.3, 0.4, 0.7⟩	⟨0.4, 0.9, 0.4⟩	⟨0.4, 0.4, 0.3⟩	⟨1.4, 1.8, 1.7⟩	⟨0.3, 0.1, 0.3⟩
a_5	⟨0.6, 0.9, 0.3⟩	⟨0.5, 0.7, 0.1⟩	⟨0.4, 0.7, 0.2⟩	⟨0.3, 0.9, 0.3⟩	⟨1.8, 3.2, 0.9⟩

Table 1.17 Elements of the signless Laplacian matrix of the neutrosophic digraph D_2

$R_2^{L^+}$	a_1	a_2	a_3	a_4	a_5
a_1	⟨1.1, 2.6, 1.9⟩	⟨0.5, 0.3, 0.1⟩	⟨0.1, 0.7, 0.5⟩	⟨0.3, 0.9, 0.5⟩	⟨0.2, 0.7, 0.8⟩
a_2	⟨0.1, 0.7, 0.5⟩	⟨1.6, 2.1, 2.0⟩	⟨0.5, 0.1, 0.6⟩	⟨0.6, 0.7, 0.1⟩	⟨0.4, 0.6, 0.8⟩
a_3	⟨0.5, 0.3, 0.1⟩	⟨0.6, 0.9, 0.5⟩	⟨2.1, 1.8, 1.0⟩	⟨0.9, 0.2, 0.3⟩	⟨0.1, 0.4, 0.1⟩
a_4	⟨0.5, 0.1, 0.3⟩	⟨0.1, 0.3, 0.6⟩	⟨0.3, 0.8, 0.9⟩	⟨1.7, 1.6, 2.0⟩	⟨0.8, 0.4, 0.2⟩
a_5	⟨0.8, 0.3, 0.2⟩	⟨0.8, 0.4, 0.4⟩	⟨0.1, 0.6, 0.1⟩	⟨0.2, 0.6, 0.8⟩	⟨1.9, 1.9, 1.5⟩

Table 1.18 Elements of the signless Laplacian matrix of the neutrosophic digraph D_3

$R_3^{L^+}$	a_1	a_2	a_3	a_4	a_5
a_1	$\langle 2.0, 2.1, 1.6\rangle$	$\langle 0.9, 0.8, 0.7\rangle$	$\langle 0.1, 0.7, 0.2\rangle$	$\langle 0.4, 0.3, 0.1\rangle$	$\langle 0.6, 0.3, 0.6\rangle$
a_2	$\langle 0.7, 0.2, 0.9\rangle$	$\langle 2.4, 1.0, 2.8\rangle$	$\langle 0.4, 0.3, 0.6\rangle$	$\langle 0.6, 0.3, 0.4\rangle$	$\langle 0.7, 0.2, 0.9\rangle$
a_3	$\langle 0.2, 0.3, 0.1\rangle$	$\langle 0.6, 0.7, 0.4\rangle$	$\langle 1.5, 1.4, 1.7\rangle$	$\langle 0.1, 0.2, 0.4\rangle$	$\langle 0.6, 0.2, 0.8\rangle$
a_4	$\langle 0.1, 0.7, 0.4\rangle$	$\langle 0.4, 0.7, 0.6\rangle$	$\langle 0.4, 0.8, 0.1\rangle$	$\langle 1.5, 2.9, 1.4\rangle$	$\langle 0.6, 0.7, 0.3\rangle$
a_5	$\langle 0.6, 0.7, 0.6\rangle$	$\langle 0.9, 0.8, 0.7\rangle$	$\langle 0.8, 0.8, 0.6\rangle$	$\langle 0.3, 0.3, 0.6\rangle$	$\langle 2.6, 2.6, 2.5\rangle$

For the period 16 February 2018 to 15 March 2018 (see Fig. 1.44), we have

$\mathrm{Spec}(T_Y(x_jx_k)) = \{-0.4245, -0.1714, 0.0215, 0.5744\}$,
$\mathrm{Spec}(I_Y(x_jx_k)) = \{-0.7909, -0.5799, 0.0536, 1.3173\}$,
$\mathrm{Spec}(F_Y(x_jx_k)) = \{-0.5037, -0.3400, 0.0007, 0.8430\}$,
$E(T_Y(x_jx_k)) = 1.1919$, $E(I_Y(x_jx_k)) = 2.7418$, $E(F_Y(x_jx_k)) = 1.6874$.
Therefore, $E(G_2) = \langle 1.1919, 2.7418, 1.6874\rangle$.

Laplacian $\mathrm{Spec}(T_Y(x_jx_k)) = \{0, 0.4200, 0.6908, 1.0892\}$,
Laplacian $\mathrm{Spec}(I_Y(x_jx_k)) = \{0, 0.8716, 1.7656, 2.3629\}$,
Laplacian $\mathrm{Spec}(F_Y(x_jx_k)) = \{0, 0.5672, 1.1546, 1.4783\}$,
$LE(T_Y(x_jx_k)) = 1.36$, $LE(I_Y(x_jx_k)) = 3.2569$, $LE(F_Y(x_jx_k)) = 2.0657$.
Therefore, $LE(G_2) = \langle 1.36, 3.2569, 2.0657\rangle$.

Signless Laplacian $\mathrm{Spec}(T_Y(x_jx_k)) = \{-0.4023, -0.1931, -0.0585, 0.6538\}$,
Signless Laplacian $\mathrm{Spec}(I_Y(x_jx_k)) = \{-0.7962, -0.5500, -0.1538, 1.5000\}$,
Signless Laplacian $\mathrm{Spec}(F_Y(x_jx_k)) = \{-0.5321, -0.2209, -0.2000, 0.9530\}$,
$LE^+(T_Y(x_jx_k)) = 1.3076$, $LE^+(I_Y(x_jx_k)) = 2.9999$, $LE^+(F_Y(x_jx_k)) = 1.9059$.
Therefore, $LE^+(G_2) = \langle 1.3076, 2.9999, 1.9059\rangle$.

For the period 16 March 2018 to 15 April 2018 (see Fig. 1.45), we have

$\mathrm{Spec}(T_Y(x_jx_k)) = \{-0.6287, -0.3884, 0.0004, 1.0168\}$,
$\mathrm{Spec}(I_Y(x_jx_k)) = \{-1.0779, -0.5696, 0.0698, 1.5776\}$,
$\mathrm{Spec}(F_Y(x_jx_k)) = \{-0.8184, -0.4650, 0.0051, 1.2783\}$,
$E(T_Y(x_jx_k)) = 2.0343$, $E(I_Y(x_jx_k)) = 3.2949$, $E(F_Y(x_jx_k)) = 2.5668$.
Therefore, $E(G_3) = \langle 2.0343, 3.2949, 2.5668\rangle$.

Laplacian $\mathrm{Spec}(T_Y(x_jx_k)) = \{0, 0.2604, 1.4221, 1.7175\}$,
Laplacian $\mathrm{Spec}(I_Y(x_jx_k)) = \{0, 1.2472, 2.3360, 2.6168\}$,
Laplacian $\mathrm{Spec}(F_Y(x_jx_k)) = \{0, 0.8182, 1.6721, 2.3097\}$,
$LE(T_Y(x_jx_k)) = 2.8792$, $LE(I_Y(x_jx_k)) = 3.7056$,$LE(F_Y(x_jx_k)) = 3.1636$.
Therefore, $LE(G_3) = \langle 2.8792, 3.7056, 3.1636\rangle$.

Table 1.19 The collective NPR of all the above individual NPRs

R	a_1	a_2	a_3	a_4	a_5
a_1	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.6777, 0.4843, 0.2943\rangle$	$\langle 0.1236, 0.5377, 0.3942\rangle$	$\langle 0.4655, 0.5302, 0.2512\rangle$	$\langle 0.3800, 0.2325, 0.6682\rangle$
a_2	$\langle 0.4157, 0.3594, 0.5845\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.5189, 0.1774, 0.6659\rangle$	$\langle 0.5223, 0.2325, 0.2650\rangle$	$\langle 0.4469, 0.3019, 0.7267\rangle$
a_3	$\langle 0.4249, 0.3533, 0.1330\rangle$	$\langle 0.6510, 0.7414, 0.5247\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.5755, 0.2670, 0.3758\rangle$	$\langle 0.3317, 0.2599, 0.3364\rangle$
a_4	$\langle 0.2980, 0.2620, 0.4460\rangle$	$\langle 0.2951, 0.5474, 0.5387\rangle$	$\langle 0.3484, 0.5897, 0.3028\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$	$\langle 0.5980, 0.2483, 0.2740\rangle$
a_5	$\langle 0.6574, 0.5474, 0.3479\rangle$	$\langle 0.7703, 0.5736, 0.3268\rangle$	$\langle 0.5118, 0.6663, 0.2465\rangle$	$\langle 0.2524, 0.5386, 0.5406\rangle$	$\langle 0.5000, 0.5000, 0.5000\rangle$

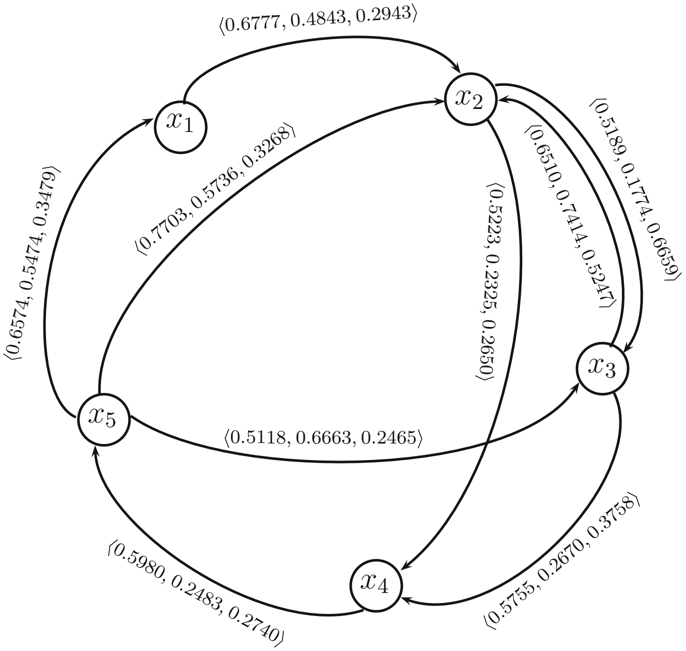
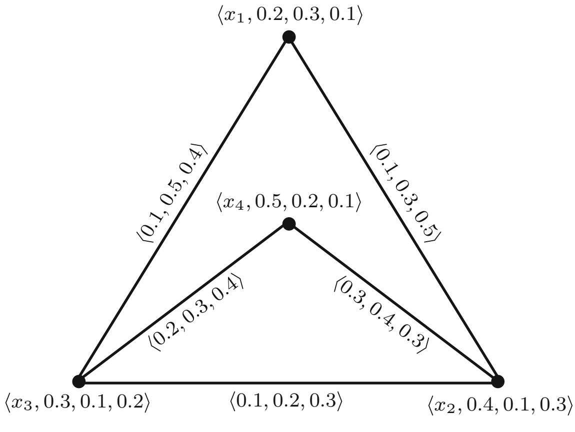


Fig. 1.42 Partial directed network of the fused NPR

Fig. 1.43 Neutrosophic graph G_1



Signless Laplacian Spec$(T_Y(x_jx_k)) = \{-0.6816, -0.3513, -0.2007, 1.2336\}$,
Signless Laplacian Spec$(I_Y(x_jx_k)) = \{-1.1436, -0.4542, -0.0553, 1.6531\}$,
Signless Laplacian Spec$(F_Y(x_jx_k)) = \{-0.8066, -0.4000, -0.2632, 1.4698\}$,
$LE^+(T_Y(x_jx_k)) = 2.4671$, $LE^+(I_Y(x_jx_k)) = 3.3062$, $LE^+(F_Y(x_jx_k)) = 2.9395$.
Therefore, $LE^+(G_3) = \langle 2.4671, 3.3062, 2.9395\rangle$.

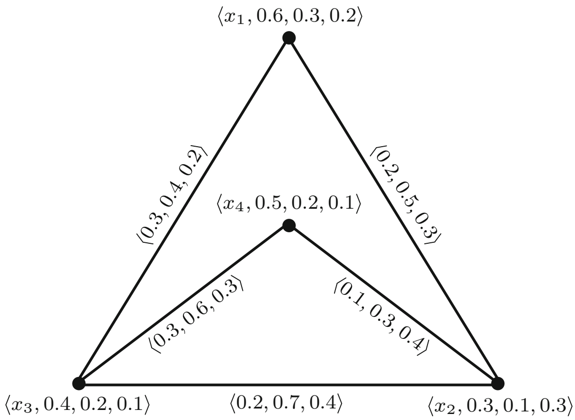


Fig. 1.44 Neutrosophic graph G_2

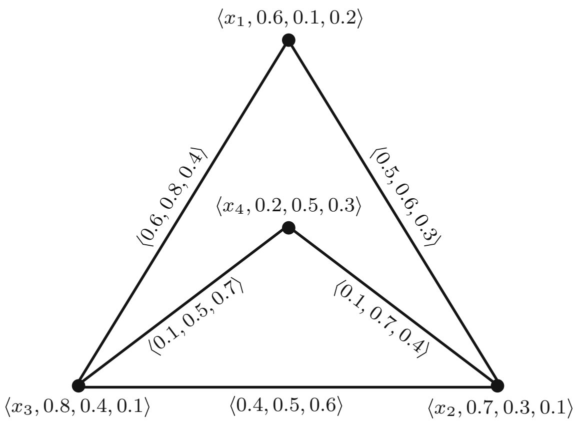


Fig. 1.45 Neutrosophic graph G_3

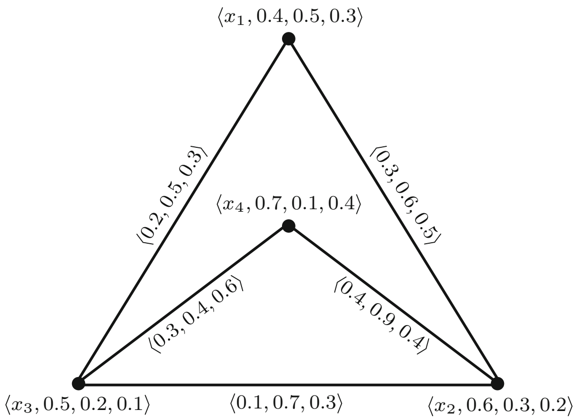


Fig. 1.46 Neutrosophic graph G_4

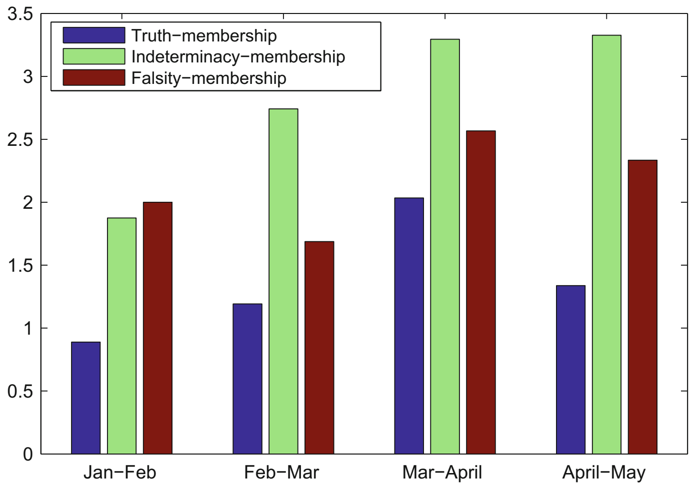


Fig. 1.47 Energy of neutrosophic graphs

Finally, for the period 16 April 2018 to 15 May 2018 (see Fig. 1.46), we have

$\text{Spec}(T_Y(x_jx_k)) = \{-0.5716, -0.0973, 0.0027, 0.6662\}$,
$\text{Spec}(I_Y(x_jx_k)) = \{-1.0878, -0.5755, 0.0435, 1.6198\}$,
$\text{Spec}(F_Y(x_jx_k)) = \{-0.7686, -0.3985, 0.0990, 1.0680\}$,
$E(T_Y(x_jx_k)) = 1.3378, E(I_Y(x_jx_k)) = 3.3265, E(F_Y(x_jx_k)) = 2.3342$.
Therefore, $E(G_4) = \langle 1.3378, 3.3265, 2.3342 \rangle$.

Laplacian $\text{Spec}(T_Y(x_jx_k)) = \{0, 0.5637, 0.7641, 1.2721\}$,
Laplacian $\text{Spec}(I_Y(x_jx_k)) = \{0, 1.1660, 2.0643, 2.9697\}$,
Laplacian $\text{Spec}(F_Y(x_jx_k)) = \{0, 0.8207, 1.5544, 1.8249\}$,
$LE(T_Y(x_jx_k)) = 1.4725$, $LE(I_Y(x_jx_k)) = 3.868$, $LE(F_Y(x_jx_k)) = 2.5586$.
Therefore, $LE(G_4) = \langle 1.4725, 3.8680, 2.5586 \rangle$.

Signless Laplacian $\text{Spec}(T_Y(x_jx_k)) = \{-0.5588, -0.1017, -0.0500, 0.7105\}$,
Signless Laplacian $\text{Spec}(I_Y(x_jx_k)) = \{-1.0582, -0.5617, -0.2105, 1.8304\}$,
Signless Laplacian $\text{Spec}(F_Y(x_jx_k)) = \{-0.7996, -0.3562, 0.0413, 1.1145\}$,
$LE^+(T_Y(x_jx_k)) = 1.4211$, $LE^+(I_Y(x_jx_k)) = 3.6608$, $LE^+(F_Y(x_jx_k)) = 2.3116$.
Therefore, $LE^+(G_4) = \langle 1.4211, 3.6608, 2.3116 \rangle$.

The bar graphs, shown in Figs. 1.47, 1.48 and 1.49, represent the energy, Laplacian energy and signless Laplacian energy of four links for the above four periods corresponding to the truth-membership, indeterminacy-membership and falsity-membership values. From the above bar graphs, the energy, Laplacian energy and signless Laplacian energy of truth-membership for the period March to April are high

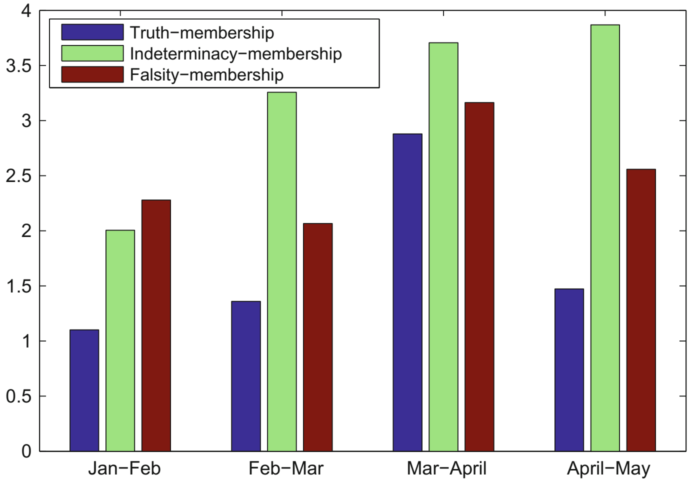


Fig. 1.48 Laplacian energy of neutrosophic graphs

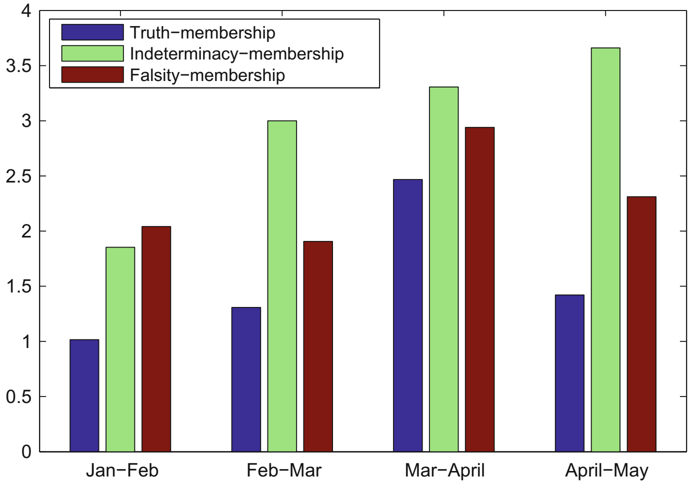


Fig. 1.49 Signless Laplacian energy of neutrosophic graphs

as compared to other periods; the energy, Laplacian energy and signless Laplacian energy of indeterminacy-membership for the period April to May are high; and the energy, Laplacian energy and signless Laplacian energy of falsity-membership for the period March to April are high.

Chapter 2
Graph Structures Under Neutrosophic Environment

A single-valued neutrosophic graph structure (neutrosophic graph structure, for short) is a generalization of neutrosophic graph. In this chapter, we present the notion of neutrosophic graph structures and explore some properties of neutrosophic graph structures. Moreover, we discuss the concept of ϕ-complement of neutrosophic graph structure and present certain operations of neutrosophic graph structures elaborated with examples. Further, we discuss some applications of neutrosophic graph structures in decision-making. This chapter is due to [33, 34, 151].

2.1 Introduction

Sampathkumar [151] introduced the graph structure which is a generalization of undirected graph and is quite useful in studying some structures like graphs, signed graphs, labelled graphs and edge-coloured graphs.

Definition 2.1 A *graph structure* $G^* = (X, E_1, \ldots, E_n)$ consists of a nonempty set X together with relations $E_1, E_2, \ldots, E_n$ on X which are mutually disjoint such that each E_i, $1 \leq i \leq n$, is symmetric and irreflexive.

One can represent a graph structure $G^* = (X, E_1, \ldots, E_n)$ in the plane just like a graph where each edge is labelled as E_i, $1 \leq i \leq n$.

Example 2.1 Let $X = \{r_1, r_2, r_3, r_4, r_5\}$ and $E_1 = \{(r_1, r_2), (r_3, r_4), (r_1, r_4)\}$, $E_2 = \{(r_1, r_3), (r_1, r_5)\}$, $E_3 = \{(r_2, r_3), (r_4, r_5)\}$ be mutually disjoint, symmetric and irreflexive relations on set X. Thus $G = (X, E_1, E_2, E_3)$ is a graph structure and is represented in plane as a graph where each edge is labelled as E_1, E_2 or E_3 (Fig. 2.1).

Definition 2.2 Let ϕ be a permutation on $\{E_1, E_2, \ldots, E_n\}$. Then *$\phi$-complement* of a graph structure G^* denoted by $G^{*\phi c}$ is obtained by replacing E_i by $\phi(E_i)$, $1 \leq i \leq n$.

M. Akram, *Single-Valued Neutrosophic Graphs*, Infosys Science Foundation Series, https://doi.org/10.1007/978-981-13-3522-8_2

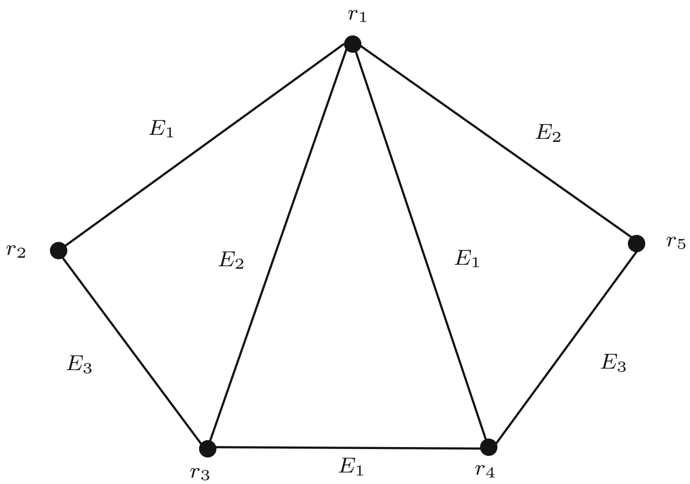


Fig. 2.1 Graph structure $G^* = (X, E_1, E_2, E_3)$

G^* is *self-complementary* if it is isomorphic to $G^{*\phi c}$, where ϕ is not an identity permutation. G^* is *totally strong self-complementary* if it is identical to $G^{*\phi c}$ for all permutations ϕ on $\{E_1, E_2, \ldots, E_n\}$.

Definition 2.3 If graph structure G^* is connected and contains no cycle, in other words, its underlying graph is a tree, then it is called a *tree*. G^* is an E_i*-tree* if subgraph structure induced by E_i-edges is a tree. Similarly, G^* is an $E_1E_2\ldots E_n-$ *tree* if G^* is an E_i-tree for each $i \in \{1, 2, \ldots, n\}$. G^* is an E_i*-forest*, if subgraph structure induced by E_i-edges is a forest.

Definition 2.4 Let $G_1^* = (X, E_1, E_2, \ldots, E_n)$ and $G_2^* = (X', E_1', E_2', \ldots, E_n')$ be two graph structures, *Cartesian product* of G_1^* and G_2^* is defined as: $G_1^* \times G_2^* = (X \times X', E_1 \times E_1', E_2 \times E_2', \ldots, E_n \times E_n')$, where $E_i \times E_i' = \{(b_1d, b_2d) \mid d \in X', b_1b_2 \in E_i\} \cup \{(bd_1, bd_2) \mid b \in X, d_1d_2 \in E_i'\}, i = (1, 2, \ldots, n)$.

Definition 2.5 Let $G_1^* = (X, E_1, E_2, \ldots, E_n)$ and $G_2^* = (X', E_1', E_2', \ldots, E_n')$ be two graph structures, *cross product* of G_1^* and G_2^* is defined as: $G_1^* * G_2^* = (X * X', E_1 * E_1', E_2 * E_2', \ldots, E_n * E_n')$, where $E_i * E_i' = \{(b_1d_1, b_2d_2) \mid b_1b_2 \in E_i, d_1d_2 \in E_i'\}, i = (1, 2, \ldots, n)$.

Definition 2.6 Let $G_1^* = (X, E_1, E_2, \ldots, E_n)$ and $G_2^* = (X', E_1', E_2', \ldots, E_n')$ be two graph structures, *lexicographic product* of G_1^* and G_2^* is defined as: $G_1^* \bullet G_2^* = (X \bullet X', E_1 \bullet E_1', E_2 \bullet E_2', \ldots, E_n \bullet E_n')$, where $E_i \bullet E_i' = \{(bd_1, bd_2) \mid b \in X, d_1d_2 \in E_i'\} \cup \{(b_1d_1, b_2d_2) \mid b_1b_2 \in E_i, d_1d_2 \in E_i'\}, i = (1, 2, \ldots, n)$.

Definition 2.7 Let $G_1^* = (X, E_1, E_2, \ldots, E_n)$ and $G_2^* = (X', E_1', E_2', \ldots, E_n')$ be two graph structures, *strong product* of G_1^* and G_2^* is defined as: $G_1^* \boxtimes G_2^* =$

$(X \boxtimes X', E_1 \boxtimes E'_1, E_2 \boxtimes E'_2, \ldots, E_n \boxtimes E'_n)$, where $E_i \boxtimes E'_i = \{(b_1d, b_2d) \mid d \in X', b_1b_2 \in E_i\} \cup \{(bd_1, bd_2) \mid b \in X, d_1d_2 \in E'_i\} \cup \{(b_1d_1, b_2d_2) \mid b_1b_2 \in E_i, \ d_1d_2 \in E'_i\}$, $i = (1, 2, \ldots, n)$.

Definition 2.8 Let $G_1^* = (X, E_1, E_2, \ldots, E_n)$ and $G_2^* = (X', E'_1, E'_2, \ldots, E'_n)$ be two graph structures, *composition* of G_1^* and G_2^* is defined as: $G_1^* \circ G_2^* = (X \circ X', E_1 \circ E'_1, E_2 \circ E'_2, \ldots, E_n \circ E'_n)$, where $E_i \circ E'_i = \{(b_1d, b_2d) \mid d \in X', b_1b_2 \in E_i\} \cup \{(bd_1, bd_2) \mid b \in X, d_1d_2 \in E'_i\} \cup \{(b_1d_1, b_2d_2) \mid b_1b_2 \in E_i, d_1, d_2 \in X'$ such that $d_1 \neq d_2\}$, $i = (1, 2, \ldots, n)$.

Definition 2.9 Let $G_1^* = (X, E_1, E_2, \ldots, E_n)$ and $G_2^* = (X', E'_1, E'_2, \ldots, E'_n)$ be two graph structures, *union* of G_1^* and G_2^* is defined as: $G_1^* \cup G_2^* = (X \cup X', E_1 \cup E'_1, E_2 \cup E'_2, \ldots, E_n \cup E'_n)$.

Definition 2.10 Let $G_1^* = (X, E_1, E_2, \ldots, E_n)$ and $G_2^* = (X', E'_1, E'_2, \ldots, E'_n)$ be two graph structures, *join* of G_1^* and G_2^* is defined as: $G_1^* + G_2^* = (X + X', E_1 + E'_1, E_2 + E'_2, \ldots, E_n + E'_n)$, where $X + X' = X \cup X'$, $E_i + E'_i = E_i \cup E'_i \cup E''_i$ for $i = (1, 2, \ldots, n)$. E''_i contains all those edges, joining the vertices of E and E'.

2.2 Neutrosophic Graph Structures

Definition 2.11 Let X be a nonempty set and $E_1, E_2, \ldots, E_n$ relations on X. $G = (A, B_1, B_2, \ldots, B_n)$ is called a *single-valued neutrosophic graph structure* if

$$A = \{< n, T_i(n), I_i(n), F_i(n) >: n \in X\}$$

is a single-valued neutrosophic set on X and

$$B_i = \{< (m, n), T(m, n), I(m, n), F(m, n) >: (m, n) \in E_i\}$$

is a single-valued neutrosophic set on E_i such that

$$T_i(m, n) \leq \min\{T(m), T(n)\}, \ \ I_i(m, n) \leq \min\{I(m), I(n)\},$$
$$F_i(m, n) \leq \max\{F(m), F(n)\}, \forall m, n \in X.$$

Note that $T_i(m, n) = 0 = I_i(m, n) = F_i(m, n)$ for all $(m, n) \in X \times X - E_i$ and

$$0 \leq T_i(m, n) + I_i(m, n) + F_i(m, n) \leq 3 \text{ for all } (m, n) \in E_i,$$

where X and E_i $(i = 1, 2, \ldots, n)$ are underlying vertex and underlying i-edge sets of G, respectively.

Throughout this chapter, we will use neutrosophic set, neutrosophic relation and neutrosophic graph structure, for short.

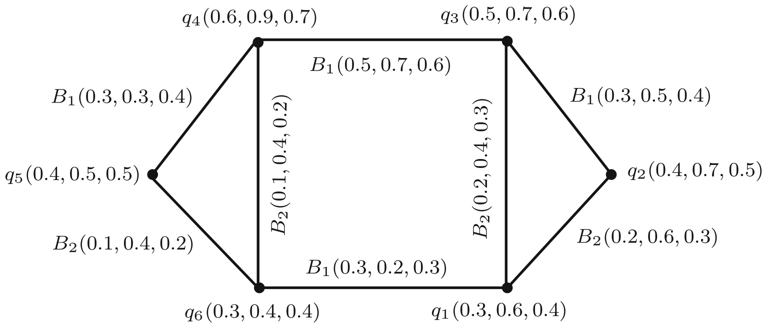


Fig. 2.2 Single-valued neutrosophic graph structure

Definition 2.12 Let $G = (A, B_1, B_2, \ldots, B_n)$ be a neutrosophic graph structure of G^*. If $H = (A', B_1', B_2', \ldots, B_n')$ is a neutrosophic graph structure of G^* such that

$$T'(n) \leq T(n), \quad I'(n) \leq I(n), \, F'(n) \geq F(n), \forall n \in X,$$

$$T_i'(m, n) \leq T_i(m, n), I_i'(m, n) \leq I_i(m, n) \text{ and } F_i'(m, n) \geq F_i(m, n), \forall m, n \in E_i,$$

where $i = 1, 2, \ldots, n$. Then H is called a *neutrosophic subgraph structure* of neutrosophic graph structure G.

Example 2.2 Let $G^* = (X, E_1, E_2)$ be a graph structure, where $X = \{q_1, q_2, q_3, q_4, q_5, q_6\}$, $E_1 = \{q_1q_6, q_2q_3, q_3q_4, q_4q_5\}$, $E_2 = \{q_1q_2, q_5q_6, q_4q_6, q_1q_3\}$. Now we define neutrosophic sets A, B_1, B_2 on X, E_1, E_2, respectively.

Let $A = \{(q_1, 0.3, 0.6, 0.4), (q_2, 0.4, 0.7, 0.5), (q_3, 0.5, 0.7, 0.6), (q_4, 0.6, 0.9, 0.7), (q_5, 0.4, 0.5, 0.5), (q_6, 0.3, 0.4, 0.4)\}$, $B_1 = \{(q_1q_6, 0.3, 0.2, 0.3), (q_2q_3, 0.3, 0.5, 0.4), (q_3q_4, 0.5, 0.7, 0.6), (q_4q_5, 0.3, 0.3, 0.4)\}$, $B_2 = \{(q_1q_2, 0.2, 0.6, 0.3), (q_5q_6, 0.1, 0.4, 0.2), (q_4q_6, 0.1, 0.4, 0.2), (q_1q_3, 0.2, 0.4, 0.3)\}$. By direct calculations, it is easy to show that $G = (A, B_1, B_2)$ is a neutrosophic graph structure of G^* as shown in Fig. 2.2.

Definition 2.13 A neutrosophic graph structure $H = (A', B_1', B_2', \ldots, B_n')$ is called an *induced subgraph structure* of G by a subset R of X if

$$T'(n) = T(n), I'(n) = I(n), F'(n) = F(n), \forall n \in E,$$

$$T_i'(m, n) = T_i(m, n), I_i'(m, n) = I_i(m, n) \text{ and } F_i'(m, n) = F_i(m, n), \forall m, n \in E,$$

where $i = 1, 2, \ldots, n$.

Definition 2.14 A neutrosophic graph structure $H = (A', B_1', B_2', \ldots, B_n')$ is called a *spanning subgraph structure* of G if $A' = A$ and

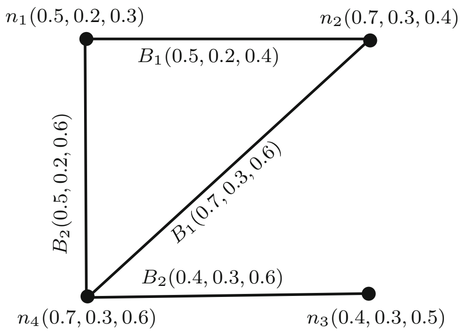


Fig. 2.3 Neutrosophic graph structure G

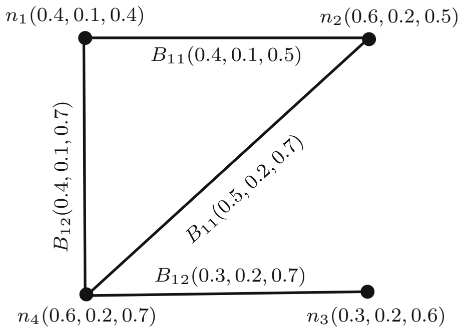


Fig. 2.4 Neutrosophic subgraph structure

$T_i'(m,n) \leq T_i(m,n)$, $I_i'(m,n) \leq I_i(m,n)$ and $F_i'(m,n) \geq F_i(m,n)$, $i = 1, 2, ..., n$.

Example 2.3 Consider a graph structure $G^* = (X, E_1, E_2)$ and let A, B_1, B_2 be neutrosophic subsets of X, E_1, E_2, respectively, such that

$$A = \{(n_1, 0.5, 0.2, 0.3), (n_2, 0.7, 0.3, 0.4), (n_3, 0.4, 0.3, 0.5), (n_4, 0.7, 0.3, 0.6)\},$$

$$B_1 = \{(n_1n_2, 0.5, 0.2, 0.4), (n_2n_4, 0.7, 0.3, 0.6)\},$$

$$B_2 = \{(n_3n_4, 0.4, 0.3, 0.6), (n_1n_4, 0.5, 0.2, 0.6)\}.$$

Direct calculations show that $G = (A, B_1, B_2)$ is a neutrosophic graph structure of G^* as shown in Fig. 2.3.

Example 2.4 A neutrosophic graph structure $K = (A', B_{11}, B_{12})$ shown in Fig. 2.4 is a neutrosophic subgraph structure of $G = (A, B_1, B_2)$ shown in Fig. 2.3.

Definition 2.15 Let $G = (A, B_1, B_2, \ldots, B_n)$ be a neutrosophic graph structure of G^*. Then $mn \in E_i$ is called B_i-*edge* or simply B_i-edge if $T_i(m,n) > 0$ or $I_i(m,n) > 0$ or $F_i(m,n) > 0$ or all three conditions hold. Consequently, support of B_i is defined as:

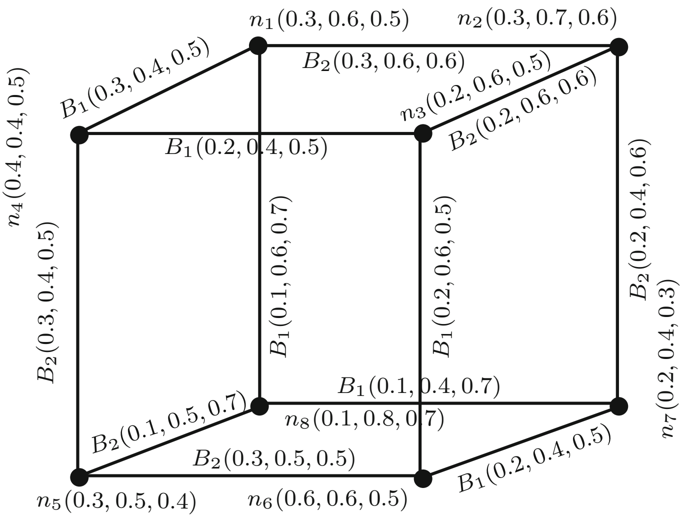


Fig. 2.5 Strong neutrosophic graph structure $G = (A, B_1, B_2)$

$$supp(B_i) = \{mn \in B_i : T_i(m, n) > 0\} \cup \{mn \in B_i : I_i(m, n) > 0\}$$
$$\cup\{mn \in B_i : F_i(m, n) > 0\}, i = 1, 2, ..., n.$$

Definition 2.16 B_i-path in a neutrosophic graph structure $G = (A, B_1, B_2, \ldots, B_n)$ is a sequence of distinct vertices $n_1, n_2, \ldots, n_m$ (except choice that $n_m = n_1$) in X, such that $n_{j-1}n_j$ is a neutrosophic B_i-edge for all $j = 2, \ldots, m$.

Definition 2.17 A neutrosophic graph structure $G = (A, B_1, B_2, \ldots, B_n)$ is called *B_i-strong* for some $i \in \{1, 2, \ldots, n\}$ if

$$T_i(m, n) = \min\{T(m), T(n)\}, I_i(m, n) = \min\{I(m), I(n)\}$$

and

$$F_i(m, n) = \max\{F(m), F(n)\}, \forall mn \in supp(B_i).$$

Furthermore, neutrosophic graph structure G is called strong if it is B_i-strong for all $i \in \{1, 2, \ldots, n\}$.

Example 2.5 Consider a neutrosophic graph structure $G = (A, B_1, B_2)$ as shown in Fig. 2.5. Then G is a strong neutrosophic graph structure since it is both B_1- and B_2-strong.

Definition 2.18 A neutrosophic graph structure $G = (A, B_1, B_2, \ldots, B_n)$ is called *complete* if G is a strong neutrosophic graph structure, $supp(B_i) \neq \phi$ for all $i = 1, 2, \ldots, n$ and for every pair of vertices $m, n \in X$, mn is a *B_i-edge* for some i.

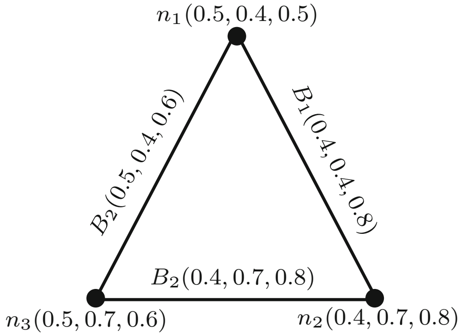


Fig. 2.6 Complete neutrosophic graph structure

Example 2.6 Let $G = (A, B_1, B_2)$ be a neutrosophic graph structure of graph structure $G^* = (X, E_1, E_2)$ such that $X = \{n_1, n_2, n_3\}$, $E_1 = \{n_1n_2\}$ and $E_2 = \{n_2n_3, n_1n_3\}$ as shown in Fig. 2.6. By simple calculations, it can be seen that G is a strong neutrosophic graph structure. Moreover, $supp(B_1) \neq \phi$, $supp(B_2) \neq \phi$, and each pair of vertices in X is either a B_1-edge or an B_2-edge. So G is a complete, i.e. B_1B_2-complete neutrosophic graph structure.

Definition 2.19 Let $G = (A, B_1, B_2, \ldots, B_n)$ be a neutrosophic graph structure. Then truth strength, indeterminacy strength and falsity strength of a B_i-path $P_{B_i} = n_1, n_2, \ldots, n_m$ are denoted by $T.P_{B_i}$, $I.P_{B_i}$ and $F.P_{B_i}$, respectively, and defined as

$$T.P_{B_i} = \bigwedge_{j=2}^{m}[T^P_{B_i}(n_{j-1}n_j)]\,, I.P_{B_i} = \bigwedge_{j=2}^{m}[I^P_{B_i}(n_{j-1}n_j)]\,, F.P_{B_i} = \bigvee_{j=2}^{m}[F^P_{B_i}(n_{j-1}n_j)]\,.$$

Example 2.7 Consider a neutrosophic graph structure $G = (A, B_1, B_2)$ as shown in Fig. 2.6. We found that $P_{B_2} = n_2, n_3, n_1$ is a B_2-path. So $T.P_{B_2} = 0.4$, $I.P_{B_2} = 0.4$ and $F.P_{B_2} = 0.8$.

Definition 2.20 Let $G = (A, B_1, B_2, \ldots, B_n)$ be a neutrosophic graph structure. Then

(i) B_i-truth strength of connectedness between m and n is defined as: $T^{\infty}_{B_i}(mn) = \bigvee_{j\geq 1}\{T^j_{B_i}(mn)\}$ such that $T^j_{B_i}(mn) = (T^{j-1}_{B_i} \circ T^1_{B_i})(mn)$ for $j \geq 2$ and
$$T^2_{B_i}(mn) = (T^1_{B_i} \circ T^1_{B_i})(mn) = \bigvee_z (T^1_{B_i}(mz) \wedge T^1_{B_i}(zn)).$$

(ii) B_i-indeterminacy strength of connectedness between m and n is defined as: $I^{\infty}_{B_i}(mn) = \bigvee_{j\geq 1}\{I^j_{B_i}(mn)\}$ such that $I^j_{B_i}(mn) = (I^{j-1}_{B_i} \circ I^1_{B_i})(mn)$ for $j \geq 2$ and
$$I^2_{B_i}(mn) = (I^1_{B_i} \circ I^1_{B_i})(mn) = \bigvee_z (I^1_{B_i}(mz) \wedge I^1_{B_i}(zn)).$$

(iii) B_i-falsity strength of connectedness between m and n is defined as: $F^{\infty}_{B_i}(mn) = \bigwedge_{j\geq 1}\{F^j_{B_i}(mn)\}$ such that $F^j_{B_i}(mn) = (F^{j-1}_{B_i} \circ F^1_{B_i})(mn)$ for $j \geq 2$ and

$$F^2_{B_i}(mn) = (F^1_{B_i} \circ F^1_{B_i})(mn) = \bigwedge_z (F^1_{B_i}(mz) \vee F^1_{B_i}(zn)).$$

Definition 2.21 A neutrosophic graph structure $G = (A, B_1, B_2, \ldots, B_n)$ is a B_i-cycle if

$$(supp(A), supp(B_1), supp(B_2), \ldots, supp(B_n)) \text{ is a } B_i\text{-cycle.}$$

Definition 2.22 A neutrosophic graph structure $G = (A, B_1, B_2, \ldots, B_n)$ is a B_i-*cycle* (for some i) if G is a B_i-cycle, no unique B_i-edge mn is in G such that

$$T_{B_i}(mn) = \min\{T_{B_i}(rs) : rs \in E_i = supp(B_i)\},$$

or

$$I_{B_i}(mn) = \min\{I_{B_i}(rs) : rs \in E_i = supp(B_i)\},$$

or

$$F_{B_i}(mn) = \max\{F_{B_i}(rs) : rs \in E_i = supp(B_i)\}.$$

Example 2.8 Consider a neutrosophic graph structure $G = (A, B_1, B_2)$ as shown in Fig. 2.5. Then G is a B_1-cycle and neutrosophic $B_1 - cycle$, since $(supp(A), supp(B_1), supp(B_2))$ is a B_1-cycle and there is no unique B_1-edge satisfying above condition.

Definition 2.23 Let $G = (A, B_1, B_2, \ldots, B_n)$ be a neutrosophic graph structure and p be a vertex in G. Let $(A', B'_1, B'_2, \ldots, B'_n)$ be a neutrosophic graph structure induced by $X \setminus \{p\}$ such that, for all $v \neq p$, $w \neq p$,

$$T_{A'}(p)=0=I_{A'}(p)=F_{A'}(p),\ T_{B'_i}(pv) = 0 = I_{B'_i}(pv) = F_{B'_i}(pv),\ \forall \text{edges } pv \in G,$$

$$T_{A'}(v) = T_A(v),\ I_{A'}(v) = I_A(v),\ F_{A'}(v) = F_A(v),$$

$$T_{B'_i}(vw) = T_{B_i}(vw),\ I_{B'_i}(vw) = I_{B_i}(vw) \text{ and } F_{B'_i}(vw) = F_{B_i}(vw).$$

Then p is neutrosophic B_i-cut vertex for any i if

$$T^{\infty}_{B_i}(vw) > T^{\infty}_{B'_i}(vw),\ I^{\infty}_{B_i}(vw) > I^{\infty}_{B'_i}(vw) \text{ and } F^{\infty}_{B_i}(vw) > F^{\infty}_{B'_i}(vw),$$

for some $v, w \in X \setminus \{p\}$. Note that p is a

- $B_i - T$ neutrosophic cut vertex if $T^{\infty}_{B_i}(vw) > T^{\infty}_{B'_i}(vw)$,
- $B_i - I$ neutrosophic cut vertex if $I^{\infty}_{B_i}(vw) > I^{\infty}_{B'_i}(vw)$,

- $B_i - F$ neutrosophic cut vertex if $F^{\infty}_{B_i}(vw) > F^{\infty}_{B'_i}(vw)$.

Example 2.9 Consider a neutrosophic graph structure $G = (A, B_1, B_2)$ as shown in Fig. 2.7 and let $G' = (A', B'_1, B'_2)$ be a neutrosophic subgraph structure of neutrosophic graph structure G found by deleting vertex n_2. Deleted vertex n_2 is a neutrosophic B_1-I cut vertex since

$$I^{\infty}_{B_1}(n_2n_5) = 0.4 > 0.3 = I^{\infty}_{B'_1}(n_2n_5),\ I^{\infty}_{B_1}(n_3n_4) = 0.7 = I^{\infty}_{B'_1}(n_3n_4),$$

and

$$I^{\infty}_{B_1}(n_3n_5) = 0.4 > 0.3 = I^{\infty}_{B'_1}(n_3n_5).$$

Definition 2.24 Suppose $G = (A, B_1, B_2, \ldots, B_n)$ be a neutrosophic graph structure and mn be B_i-*edge*. Let $(A', B'_1, B'_2, \ldots, B'_n)$ be a neutrosophic spanning subgraph structure of G, such that $\forall$ edges $mn \neq rs$,

$$T_{B'_i}(mn) = 0 = I_{B'_i}(mn) = F_{B'_i}(mn), T_{B'_i}(rs) = T_{B_i}(rs),$$

$$I_{B'_i}(rs) = I_{B_i}(rs) \text{ and } F_{B'_i}(rs) = F_{B_i}(rs).$$

Then mn is a neutrosophic B_i-bridge if

$$T^{\infty}_{B_i}(vw) > T^{\infty}_{B'_i}(vw), I^{\infty}_{B_i}(vw) > I^{\infty}_{B'_i}(vw) \text{ and } F^{\infty}_{B_i}(vw) > F^{\infty}_{B'_i}(vw),$$

for some $v, w \in X$. Note that mn is a

- $B_i - T$ neutrosophic bridge if $T^{\infty}_{B_i}(vw) > T^{\infty}_{B'_i}(vw)$,
- $B_i - I$ neutrosophic bridge if $I^{\infty}_{B_i}(vw) > I^{\infty}_{B'_i}(vw)$,
- $B_i - F$ neutrosophic bridge if $F^{\infty}_{B_i}(vw) > F^{\infty}_{B'_i}(vw)$.

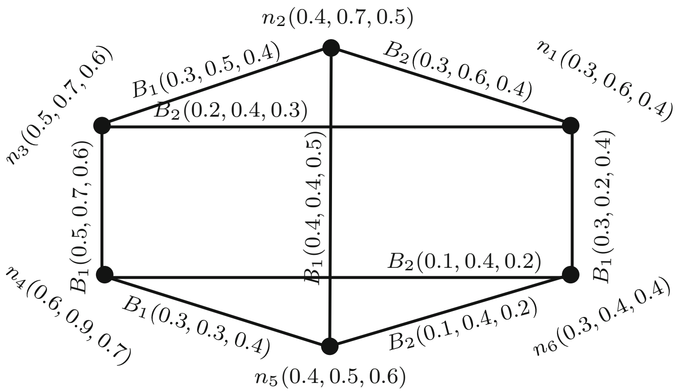


Fig. 2.7 Neutrosophic graph structure $G = (A, B_1, B_2)$

Example 2.10 Consider the neutrosophic graph structure $G = (A, B_1, B_2)$ as shown in Fig. 2.7 and $G' = (A', B_1', B_2')$ be a neutrosophic spanning subgraph structure of neutrosophic graph structure G which is found by deleting B_1-edge (n_2n_5). Edge (n_2n_5) is a neutrosophic B_1-bridge. Since

$$T^{\infty}_{B_1}(n_2n_5) = 0.4 > 0.3 = T^{\infty}_{B_1'}(n_2n_5),$$

$$I^{\infty}_{B_1}(n_2n_5) = 0.4 > 0.3 = I^{\infty}_{B_1'}(n_2n_5)$$

and

$$F^{\infty}_{B_1}(n_2n_5) = 0.5 > 0 = F^{\infty}_{B_1'}(n_2n_5).$$

Definition 2.25 A neutrosophic graph structure $G = (A, B_1, B_2, \ldots, B_n)$ is a B_i-tree if

$$(supp(A), supp(B_1), supp(B_2), \ldots, supp(B_n))$$

is a B_i-*tree*. In other words, G is a B_i-tree if a subgraph of G induced by $supp(B_i)$ generates a tree.

Definition 2.26 A neutrosophic graph structure $G = (A, B_1, B_2, \ldots, B_n)$ is B_i-*tree* if G has a neutrosophic spanning subgraph structure $H = (A', B_1', B_2', \ldots, B_n')$ such that for all B_i-edges mn not in H, H is a B_i'-tree,

$$T_{B_i}(mn) < T^{\infty}_{B_i'}(mn),\ I_{B_i}(mn) < I^{\infty}_{B_i'}(mn) \text{ and } F_{B_i}(mn) > F^{\infty}_{B_i'}(mn).$$

In particular, G is a:

- neutrosophic B_i-T tree if $T_{B_i}(mn) < T^{\infty}_{B_i'}(mn)$,
- neutrosophic B_i-I tree if $I_{B_i}(mn) < I^{\infty}_{B_i'}(mn)$,
- neutrosophic B_i-F tree if $F_{B_i}(mn) > F^{\infty}_{B_i'}(mn)$.

Example 2.11 Consider the neutrosophic graph structure $G = (A, B_1, B_2)$ as shown in Fig. 2.8, which is a B_2-tree. It is not a B_1-tree but a neutrosophic B_1-tree since it has a neutrosophic spanning subgraph (A', B_1', B_2') as a B_1'-tree, which is obtained by deleting B_1-edge n_2n_5 from G.

Moreover,

$$T_{B_1}(n_2n_5) = 0.2 < 0.3 = T^{\infty}_{B_1'}(n_2n_5),\ I_{B_1}(n_2n_5) = 0.1 < 0.3 = I^{\infty}_{B_1'}(n_2n_5)$$

and

$$F_{B_1}(n_2n_5) = 0.6 > 0.5 = F^{\infty}_{B_1'}(n_2n_5).$$

Definition 2.27 A neutrosophic graph structure $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ of the graph structure $G_1^* = (X_1, E_{11}, E_{12}, \ldots, E_{1n})$ is isomorphic to neutrosophic graph structure $G = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ of the graph structure $G_2^* = (X_2, E_{21},$

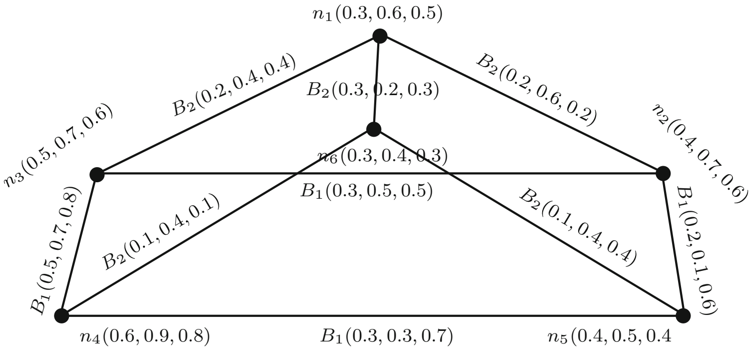


Fig. 2.8 Neutrosophic B_1-tree

$B_{22}, \ldots, E_{2n})$ if we have (f, ϕ) where $f : X_1 \to X_2$ is a bijection and ϕ is a permutation on set $\{1, 2, \ldots, n\}$ and following relations are satisfied

$$T_{A_1}(m) = T_{A_2}(f(m)),\ I_{A_1}(m) = I_{A_2}(f(m)),\ F_{A_1}(m) = F_{A_2}(f(m)),$$

for all $m \in X_1$ and

$$T_{B_{1i}}(mn) = T_{B_{2\phi(i)}}(f(m)f(n)),\ I_{B_{1i}}(mn) = I_{B_{2\phi(i)}}(f(m)f(n),$$

$$F_{B_{1i}}(mn) = F_{B_{2\phi(i)}}(f(m)f(n)),$$

for all $mn \in E_{1i}$, $i = 1, 2, \ldots, n$.

Example 2.12 Let $G_1 = (A, B_1, B_2)$ and $G_2 = (A', B_1', B_2')$ be two neutrosophic graph structures as shown in Fig. 2.9. G_1 is isomorphic G_2 under (f, ϕ) where $f : X \to X'$ is a bijection and ϕ is a permutation on set $\{1, 2\}$ defined as $\phi(1) = 2$, $\phi(2) = 1$ and following relations are satisfied

$$T_A(n_i) = T_{A'}(f(n_i)), I_A(n_i) = I_{A'}(f(n_i)),\ F_A(n_i) = F_{A'}(f(n_i)),$$

for all $n_i \in X$, and

$$T_{B_i}(n_in_j) = T_{B'_{\phi(i)}}(f(n_i)f(n_j)), I_{B_i}(n_in_j) = I_{B'_{\phi(i)}}(f(n_i)f(n_j)),$$

$$F_{B_i}(n_in_j) = F_{B'_{\phi(i)}}(f(n_i)f(n_j)),$$

$\forall n_in_j \in E_i$ and $i = 1, 2$.

Definition 2.28 A neutrosophic graph structure $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ of the graph structure $G_1^* = (X_1, E_{11}, E_{12}, \ldots, E_{1n})$ is identical to neutrosophic graph

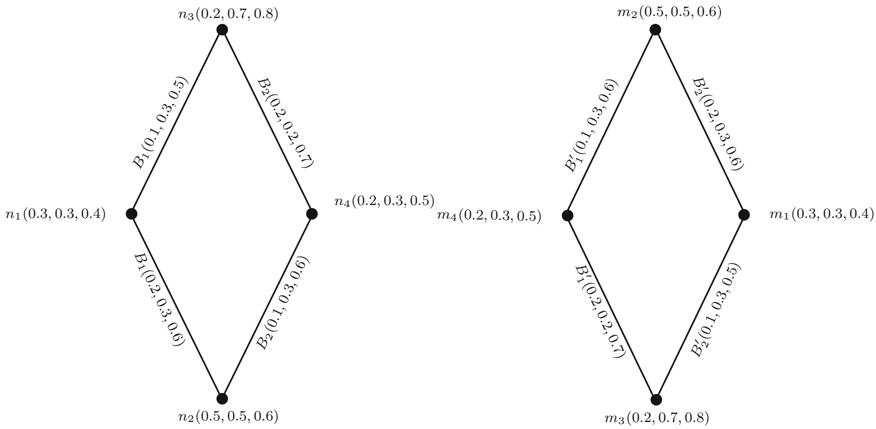


Fig. 2.9 Isomorphic neutrosophic graph structures

structure $G_2 = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ of graph structure $G_2^* = (X_2, E_{21}, B_{22}, \ldots, E_{2n})$ if $f : X_1 \to X_2$ is a bijection and following relations are satisfied:

$$T_{A_1}(m) = T_{A_2}(f(m)),\ I_{A_1}(m) = I_{A_2}(f(m)),\ F_{A_1}(m) = F_{A_2}(f(m)),$$

for all $m \in X_1$ and

$$T_{B_{1i}}(mn) = T_{B_{2i}}(f(m)f(n)),\ I_{B_{1i}}(mn) = I_{B_{2i}}(f(m)f(n)),$$

$$F_{B_{1i}}(mn) = F_{B_{2i}}(f(m)f(n)),$$

for all $mn \in E_{1i}$ and $i = 1, 2, \ldots, n$.

Example 2.13 Let $G_1 = (A, B_1, B_2)$ and $G_2 = (A', B_1', B_2')$ be two neutrosophic graph structures of graph structures $G_1^* = (X, E_1, E_2)$ and $G_2^* = (X', E_1', E_2')$, respectively, as shown in Figs. 2.10 and 2.11. Neutrosophic graph structure G_1 is identical to G_2 under $f : X \to X'$ defined as

$$f(n_1) = m_2,\ f(n_2) = m_1,\ f(n_3) = m_4,\ f(n_4) = m_3,\ f(n_5) = m_5,\ f(n_6) = m_8,$$

$$f(n_7) = m_7,\ f(n_8) = m_6,\ T_A(n_i) = T_{A'}(f(n_i)),$$

$$I_A(n_i) = I_{A'}(f(n_i)),\ F_A(n_i) = F_{A'}(f(n_i)),$$

for all $n_i \in X$ and

$$T_{B_i}(n_i n_j) = T_{B_i'}(f(n_i)f(n_j)),\ I_{B_i}(n_i n_j) = I_{B_i'}(f(n_i)f(n_j)),\ F_{B_i}(n_i n_j) = F_{B_i'}(f(n_i)f(n_j)),$$

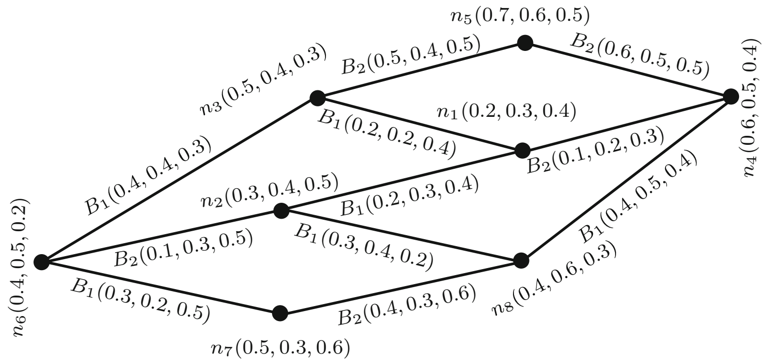


Fig. 2.10 Neutrosophic graph structure G_1

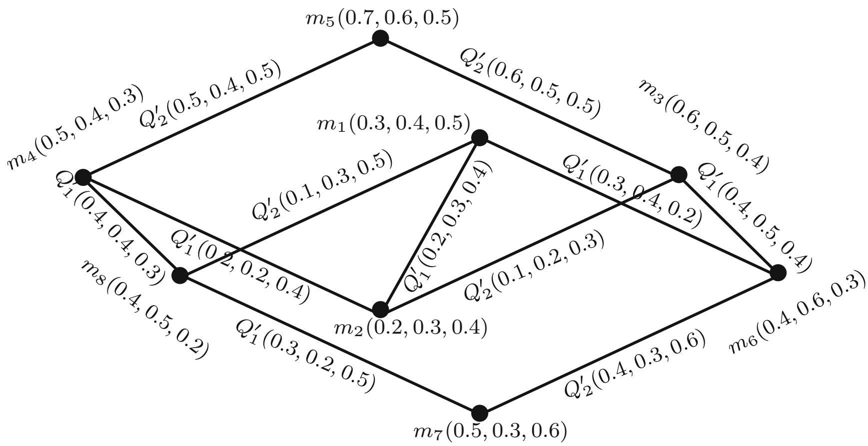


Fig. 2.11 Neutrosophic graph structure G_2

for all $n_in_j \in E_i$ and $i = 1, 2$.

Definition 2.29 Let $G = (A, B_1, B_2, \ldots, B_n)$ be a neutrosophic graph structure and ϕ be a permutation on $\{B_1, B_2, \ldots, B_n\}$ and on $\{1, 2, \ldots, n\}$ defined by $\phi(B_i) = B_j$ if and only if $\phi(i) = j$ for all i. If $mn \in B_i$ for any i and

$$T_{B_i^\phi}(mn) = T_A(m) \wedge T_A(n) - \bigvee_{j \neq i} T_{\phi(B_j)}(mn),\ I_{B_i^\phi}(mn) = I_A(m) \wedge I_A(n) - \bigvee_{j \neq i} I_{\phi(B_j)}(mn),$$

$$F_{B_i^\phi}(mn) = F_A(m) \vee F_A(n) - \bigwedge_{j \neq i} T_{\phi(B_j)}(mn),\ i = 1, 2, \ldots, n,$$

then $mn \in B_k^\phi$, where k is selected such that

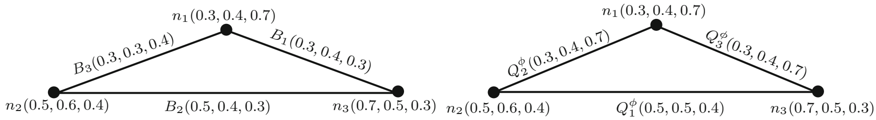


Fig. 2.12 Neutrosophic graph structures G, $G^{\phi c}$

$$T_{B_k^\phi}(mn) \geq T_{B_i^\phi}(mn),\ I_{B_k^\phi}(mn) \geq I_{B_i^\phi}(mn) \text{ and } F_{B_k^\phi}(mn) \geq F_{B_i^\phi}(mn) \text{ for all } i,$$

then neutrosophic graph structure $(A, B_1^\phi, B_2^\phi, \ldots, B_n^\phi)$ is called *ϕ-complement* of G and denoted by $G^{\phi c}$.

Example 2.14 Let $G = (A, B_1, B_2, B_3)$ be a neutrosophic graph structure shown in Fig. 2.12 and ϕ be a permutation on $\{1, 2, 3\}$ defined as:

$\phi(1) = 2$, $\phi(2) = 3$, $\phi(3) = 1$. By direct calculations, we found that $n_1n_3 \in B_3^\phi, n_2n_3 \in B_1^\phi, n_1n_2 \in B_2^\phi$. So, $G^{\phi c} = (A, B_1^\phi, B_2^\phi, B_3^\phi)$ is ϕ-complement of neutrosophic graph structure G as shown in Fig. 2.12.

Proposition 2.1 *ϕ-complement of a neutrosophic graph structure $G = (A, B_1, B_2, \ldots, B_n)$ is always a strong neutrosophic graph structure. Moreover, if $\phi(i) = k$, where $i, k \in \{1, 2, \ldots, n\}$, then all B_k-edges in neutrosophic graph structure $(A, B_1, B_2, \ldots, B_n)$ become B_i^ϕ-edges in $(A, B_1^\phi, B_2^\phi, \ldots, B_n^\phi)$.*

Proof According to the definition of ϕ-complement,

$$T_{B_i^\phi}(mn) = T_A(m) \wedge T_A(n) - \bigvee_{j \neq i} T_{\phi(B_j)}(mn),$$

$$I_{B_i^\phi}(mn) = I_A(m) \wedge I_A(n) - \bigvee_{j \neq i} I_{\phi(B_j)}(mn),$$

$$F_{B_i^\phi}(mn) = F_A(m) \vee F_A(n) - \bigwedge_{j \neq i} F_{\phi(B_j)}(mn),$$

for $i \in \{1, 2, \ldots, n\}$. For expression of truthness in ϕ-complement:

Since

$$T_A(m) \wedge T_A(n) \geq 0, \bigvee_{j \neq i} T_{\phi(B_j)}(mn) \geq 0 \text{ and } T_{B_i}(mn) \leq T_A(m) \wedge T_A(n),\ \forall B_i,$$

we see that

$$\bigvee_{j \neq i} T_{\phi(B_j)}(mn) \leq T_A(m) \wedge T_A(n),$$

which implies that

$$T_A(m) \wedge T_A(n) - \bigvee_{j \neq i} T_{\phi(B_j)}(mn) \geq 0.$$

Therefore, $T_{B_i^{\phi}}(mn) \geq 0 \; \forall i$. Moreover, $T_{B_i^{\phi}}(mn)$ achieves its maximum value when $\bigvee_{j \neq i} T_{\phi(B_j)}(mn)$ is zero. It is obvious that when $\phi(B_i) = B_k$ and mn is a B_k-edge then $\bigvee_{j \neq i} T_{\phi(B_j)}(mn)$ gets zero value. So

$$T_{B_i^{\phi}}(mn) = T_A(m) \wedge T_A(n), \;\; for \; (mn) \in B_k, \;\; \phi(B_i) = B_k.$$

Similarly, we have

$$I_{B_i^{\phi}}(mn) = I_A(m) \wedge I_A(n), \;\; for \; (mn) \in B_k, \;\; \phi(B_i) = B_k.$$

In the similar way for expression of falsity in ϕ-complement:

Since

$$F_A(m) \vee F_A(n) \geq 0, \; \bigwedge_{j \neq i} F_{\phi(B_j)}(mn) \geq 0 \text{ and } F_{B_i}(mn) \leq F_A(m) \vee F_A(n) \forall B_i,$$

we see that

$$\bigwedge_{j \neq i} F_{\phi(B_j)}(mn) \leq F_A(m) \vee F_A(n),$$

which implies that

$$F_A(m) \vee F_A(n) - \bigwedge_{j \neq i} F_{\phi(B_j)}(mn) \geq 0.$$

Therefore, $F_{B_i^{\phi}}(mn)$ is nonnegative for all i. Moreover, $F_{B_i^{\phi}}(mn)$ attains its maximum value when $\bigwedge_{j \neq i} F_{\phi(B_j)}(mn)$ becomes zero. It is clear that when $\phi(B_i) = B_k$ and mn is a B_k-edge then $\bigwedge_{j \neq i} F_{\phi(B_j)}(mn)$ gets zero value. So

$$F_{B_i^{\phi}}(mn) = F_A(m) \vee F_A(n) \text{ for } (mn) \in B_k, \; \phi(B_i) = B_k.$$

This completes the proof.

Definition 2.30 Let $G = (A, B_1, B_2, \ldots, B_n)$ be a neutrosophic graph structure and ϕ be a permutation on $\{1, 2, \ldots, n\}$. Then

(i) If G is isomorphic to $G^{\phi c}$, then G is said to be *self-complementary*.
(ii) If G is identical to $G^{\phi c}$, then G is said to be *strong self-complementary*.

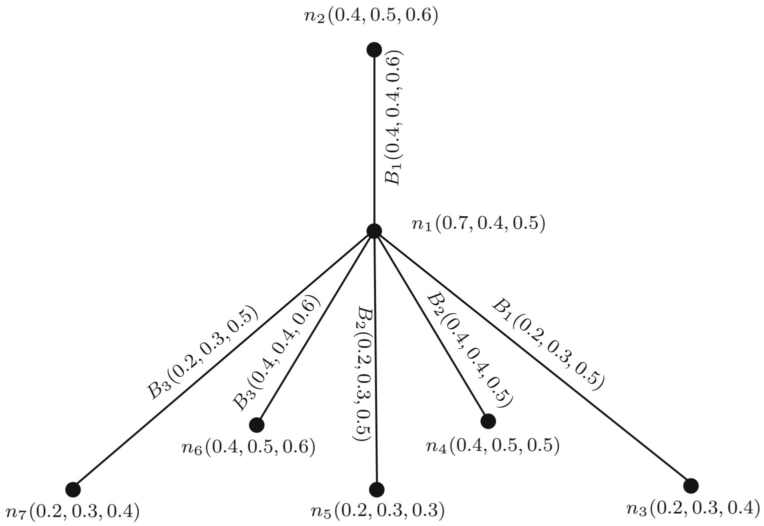


Fig. 2.13 Totally strong self-complementary neutrosophic graph structure

Definition 2.31 Suppose $G = (A, B_1, B_2, \ldots, B_n)$ be a neutrosophic graph structure. Then

(i) If G is isomorphic to $G^{\phi c}$, for all permutations ϕ on $\{1, 2, \ldots, n\}$, then G is *totally self-complementary*.
(ii) If G is identical to $G^{\phi c}$, for all permutations ϕ on $\{1, 2, \ldots, n\}$, then G is *totally strong self-complementary*.

Remark 2.1 All strong neutrosophic graph structures are self-complementary or totally self-complementary neutrosophic graph structures.

Example 2.15 A neutrosophic graph structure $G = (A, B_1, B_2, B_3)$ in Fig. 2.13 is a totally strong self-complementary neutrosophic graph structure.

Theorem 2.1 *A neutrosophic graph structure is totally self-complementary if and only if it is strong neutrosophic graph structure.*

Proof Consider a strong neutrosophic graph structure G and a permutation ϕ on $\{1, 2, \ldots, n\}$. By Proposition 2.1, ϕ-complement of a neutrosophic graph structure $G = (A, B_1, B_2, \ldots, B_n)$ is always a strong neutrosophic graph structure. Moreover, if $\phi(i) = k$, where $i, k \in \{1, 2, \ldots, n\}$, then all B_k-edges in neutrosophic graph structure $(A, B_1, B_2, \ldots, B_n)$ become B_i^{ϕ}-edges in $(A, B_1^{\phi}, B_2^{\phi}, \ldots, B_n^{\phi})$. This leads

$$T_{B_k}(mn) = T_A(m) \wedge T_A(n) = T_{B_i^{\phi}}(mn),\ I_{B_k}(mn) = I_A(m) \wedge I_A(n) = I_{B_i^{\phi}}(mn)$$

and

$$F_{B_k}(mn) = F_A(m) \vee F_A(n) = F_{B_i^\phi}(mn).$$

Hence, under the mapping (identity mapping) $f : X \to X$, G and G^ϕ are isomorphic such that

$$T_A(m) = T_A(f(m)),\ I_A(m) = I_A(f(m)),\ F_A(m) = F_A(f(m)),$$

$$T_{B_k}(mn) = T_{B_i^\phi}(f(m)f(n)) = T_{B_i^\phi}(mn),\ I_{B_k}(mn) = I_{B_i^\phi}(f(m)f(n)) = I_{B_i^\phi}(mn),$$

$$F_{B_k}(mn) = F_{B_i^\phi}(f(m)f(n)) = F_{B_i^\phi}(mn),$$

for all $mn \in E_k$, $\phi^{-1}(k) = i$ and $k = 1, 2, \ldots, n$. This is satisfied for every permutation ϕ on $\{1, 2, \ldots, n\}$. Hence, G is totally self-complementary neutrosophic graph structure. Conversely, let for every permutation ϕ on $\{1, 2, \ldots, n\}$, G and G^ϕ are isomorphic. Then according to the definition of isomorphism of neutrosophic graph structures and ϕ-complement of neutrosophic graph structure,

$$T_{B_k}(mn) = T_{B_i^\phi}(f(m)f(n)) = T_A(f(m)) \wedge T_A(f(n)) = T_A(m) \wedge T_A(n),$$

$$I_{B_k}(mn) = I_{B_i^\phi}(f(m)f(n)) = I_A(f(m)) \wedge I_A(f(n)) = T_A(m) \wedge I_A(n),$$

$$F_{B_k}(mn) = F_{B_i^\phi}(f(m)f(n)) = F_A(f(m)) \vee T_A(f(n)) = F_A(m) \wedge T_A(n),$$

for all $mn \in E_k$ and $k = 1, 2, \ldots, n$. Hence, G is strong neutrosophic graph structure.

Remark 2.2 Every self-complementary neutrosophic graph structure is totally self-complementary.

Theorem 2.2 *If $G^* = (X, E_1, E_2, \ldots, E_n)$ is a totally strong self-complementary graph structure and $A = (T_A, I_A, F_A)$ is a neutrosophic subset of X where T_A, I_A, F_A are constant valued functions, then a strong neutrosophic graph structure of G^* with neutrosophic vertex set A is always a totally strong self-complementary neutrosophic graph structure.*

Proof Consider three constants $p, q, r \in [0, 1]$, such that $T_A(m) = p$, $I_A(m) = q$, $F_A(m) = r$ $\forall m \in X$. Since G^* is totally self-complementary strong graph structure, so there is a bijection $f : X \to X$ for any permutation ϕ^{-1} on $\{1, 2, \ldots, n\}$, such that for any E_k-edge (mn), $(f(m)f(n))$ [an E_i-edge in G^*] is an E_k-edge in $G^{*\phi^{-1}c}$. Hence, for every B_k-edge (mn), $(f(m)f(n))$ [a B_i-edge in G] is a B_k^ϕ-edge in $G^{\phi^{-1}c}$. Moreover, G is strong neutrosophic graph structure. Thus,

$$T_A(m) = p = T_A(f(m)),\ I_A(m) = q = I_A(f(m)),\ F_A(m) = r = F_A(f(m)),\ \forall m \in X,$$

$$T_{B_k}(mn) = T_A(m) \wedge T_A(n) = T_A(f(m)) \wedge T_A(f(n)) = T_{B_i^\phi}(f(m)f(n)),$$

$$I_{B_k}(mn) = I_A(m) \wedge I_A(n) = I_A(f(m)) \wedge I_A(f(n)) = I_{B_i^{\phi}}(f(m)f(n)),$$

$$F_{B_k}(mn) = F_A(m) \vee I_A(n) = F_A(f(m)) \vee F_A(f(n)) = F_{B_i^{\phi}}(f(m)f(n)),$$

for all $mn \in E_i$ and $i = 1, 2, \ldots, n$. This shows that G is self-complementary strong neutrosophic graph structure. Every permutation ϕ and ϕ^{-1} on $\{1, 2, \ldots, n\}$ satisfy above expressions; thus G is totally strong self-complementary neutrosophic graph structure.

Remark 2.3 Converse of Theorem 2.2 may not be true, for example a neutrosophic graph structure shown in Fig. 2.13 is a totally strong self-complementary, it is strong and its underlying graph structure is a totally strong self-complementary but T_A, I_A, F_A are not constant functions.

2.3 Operations on Neutrosophic Graph Structures

In this section, we present the operations on neutrosophic graph structures.

Definition 2.32 Let $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ and $G_2 = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ be neutrosophic graph structures of the graph structures $G_1^* = (X_1, E_{11}, E_{12}, \ldots, E_{1n})$ and $G_2^* = (X_2, E_{21}, E_{22}, \ldots, E_{2n})$, respectively. The *Cartesian product* of G_1 and G_2, denoted by

$$G_1 \times G_2 = (A_1 \times A_2, B_{11} \times B_{21}, B_{12} \times B_{22}, \ldots, B_{1n} \times B_{2n}),$$

is defined by the following:

(i) $$\begin{cases} T_{(A_1 \times A_2)}(qr) = (T_{A_1} \times T_{A_2})(qr) = T_{A_1}(q) \wedge T_{A_2}(r) \\ I_{(A_1 \times A_2)}(qr) = (I_{A_1} \times I_{A_2})(qr) = I_{A_1}(q) \wedge I_{A_2}(r) \\ F_{(A_1 \times A_2)}(qr) = (F_{A_1} \times F_{A_2})(qr) = F_{A_1}(q) \vee F_{A_2}(r) \end{cases}$$
for all $qr \in E_1 \times E_2$,

(ii) $$\begin{cases} T_{(B_{1i} \times B_{2i})}(qr_1)(qr_2) = (T_{B_{1i}} \times T_{B_{2i}})(qr_1)(qr_2) = T_{A_1}(q) \wedge T_{B_{2i}}(r_1r_2) \\ I_{(B_{1i} \times B_{2i})}(qr_1)(qr_2) = (I_{B_{1i}} \times I_{B_{2i}})(qr_1)(qr_2) = I_{A_1}(q) \wedge I_{B_{2i}}(r_1r_2) \\ F_{(B_{1i} \times B_{2i})}(qr_1)(qr_2) = (F_{B_{1i}} \times F_{B_{2i}})(qr_1)(qr_2) = F_{A_1}(q) \vee F_{B_{2i}}(r_1r_2) \end{cases}$$
for all $q \in X_1$, $r_1r_2 \in E_{2i}$,

(iii) $$\begin{cases} T_{(B_{1i} \times B_{2i})}(q_1r)(q_2r) = (T_{B_{1i}} \times T_{B_{2i}})(q_1r)(q_2r) = T_{A_2}(r) \wedge T_{B_{1i}}(q_1q_2) \\ I_{(B_{1i} \times B_{2i})}(q_1r)(q_2r) = (I_{B_{1i}} \times I_{B_{2i}})(q_1r)(q_2r) = I_{A_2}(r) \wedge I_{B_{1i}}(q_1q_2) \\ F_{(B_{1i} \times B_{2i})}(q_1r)(q_2r) = (F_{B_{1i}} \times F_{B_{2i}})(q_1r)(q_2r) = F_{A_2}(r) \vee F_{B_{1i}}(q_1q_2) \end{cases}$$
for all $r \in X_2$, $q_1q_2 \in E_{1i}$.

Example 2.16 Consider $G_1 = (A_1, B_{11}, B_{12})$ and $G_2 = (A_2, B_{21}, B_{22})$ are neutrosophic graph structures of graph structures $G_1^* = (X_1, E_{11}, E_{12})$ and $G_2^* = (X_2, E_{21}, E_{22})$, respectively, as shown in Fig. 2.14, where $E_{11} = \{q_1q_2\}$, $E_{12} = \{q_3q_4\}$, $E_{21} = \{r_1r_2\}$, $E_{22} = \{r_2r_3\}$.

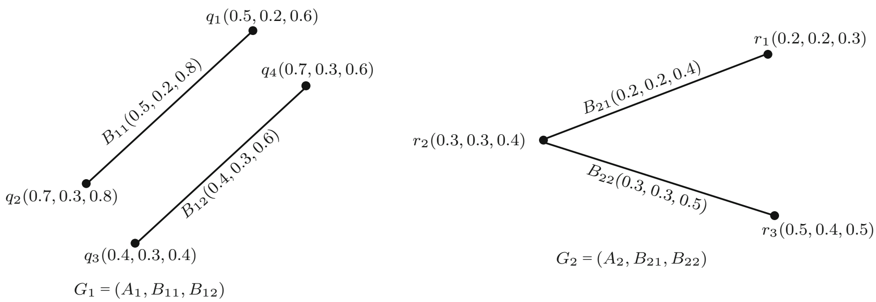


Fig. 2.14 Neutrosophic graph structures

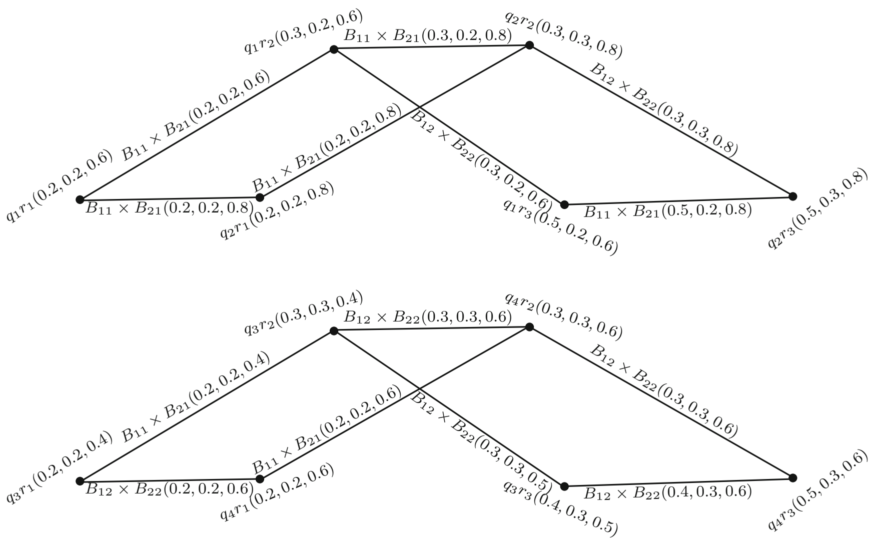


Fig. 2.15 Cartesian product of two neutrosophic graph structures

Cartesian product of G_1 and G_2 defined as $G_1 \times G_2 = \{A_1 \times A_2, B_{11} \times B_{21}, B_{12} \times B_{22}\}$ is shown in the Fig. 2.15.

Theorem 2.3 *The Cartesian product* $G_1 \times G_2 = (A_1 \times A_2, B_{11} \times B_{21}, B_{12} \times B_{22}, \ldots, B_{1n} \times B_{2n})$ *of two neutrosophic graph structures* G_1 *and* G_2 *of the graph structures* G_1^* *and* G_2^* *is a neutrosophic graph structure of* $G_1^* \times G_2^*$.

Proof According to the definition of Cartesian product, there are two cases:

Case 1. When $q \in X_1$, $r_1 r_2 \in E_{2i}$

$$\begin{aligned}
T_{(B_{1i}\times B_{2i})}((qr_1)(qr_2)) &= T_{A_1}(q)\wedge T_{B_{2i}}(r_1r_2)\\
&\le T_{A_1}(q)\wedge [T_{A_2}(r_1)\wedge T_{A_2}(r_2)]\\
&= [T_{A_1}(q)\wedge T_{A_2}(r_1)]\ \wedge [T_{A_1}(q)\wedge T_{A_2}(r_2)]\\
&= T_{(A_1\times A_2)}(qr_1)\wedge T_{(A_1\times A_2)}(qr_2),
\end{aligned}$$

$$\begin{aligned}
I_{(B_{1i}\times B_{2i})}((qr_1)(qr_2)) &= I_{A_1}(q)\wedge I_{B_{2i}}(r_1r_2)\\
&\le I_{A_1}(q)\wedge [I_{A_2}(r_1)\wedge I_{A_2}(r_2)]\\
&= [I_{A_1}(q)\wedge I_{A_2}(r_1)]\ \wedge [I_{A_1}(q)\wedge I_{A_2}(r_2)]\\
&= I_{(A_1\times A_2)}(qr_1)\wedge I_{(A_1\times A_2)}(qr_2),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1i}\times B_{2i})}((qr_1)(qr_2)) &= F_{A_1}(q)\vee F_{B_{2i}}(r_1r_2)\\
&\le F_{A_1}(q)\vee [F_{A_2}(r_1)\vee F_{A_2}(r_2)]\\
&= [F_{A_1}(q)\vee F_{A_2}(r_1)]\ \vee [F_{A_1}(q)\vee F_{A_2}(r_2)]\\
&= F_{(A_1\times A_2)}(qr_1)\vee F_{(A_1\times A_2)}(qr_2),
\end{aligned}$$

for $qr_1, qr_2 \in X_1 \times X_2$.

Case 2. When $q \in X_2, r_1r_2 \in E_{1i}$

$$\begin{aligned}
T_{(B_{1i}\times B_{2i})}((r_1q)(r_2q)) &= T_{A_2}(q)\wedge T_{B_{1i}}(r_1r_2)\\
&\le T_{A_2}(q)\wedge [T_{A_1}(r_1)\wedge T_{A_1}(r_2)]\\
&= [T_{A_2}(q)\wedge T_{A_1}(r_1)]\ \wedge [T_{A_2}(q)\wedge T_{A_1}(r_2)]\\
&= T_{(A_1\times A_2)}(r_1q)\wedge T_{(A_1\times A_2)}(r_2q),
\end{aligned}$$

$$\begin{aligned}
I_{(B_{1i}\times B_{2i})}((r_1q)(r_2q)) &= I_{A_2}(q)\wedge I_{B_{1i}}(r_1r_2)\\
&\le I_{A_2}(q)\wedge [I_{A_1}(r_1)\wedge I_{A_1}(r_2)]\\
&= [I_{A_2}(q)\wedge I_{A_1}(r_1)]\ \wedge [I_{A_2}(q)\wedge I_{A_1}(r_2)]\\
&= I_{(A_1\times A_2)}(r_1q)\wedge I_{(A_1\times A_2)}(r_2q),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1i}\times B_{2i})}((r_1q)(r_2q)) &= F_{A_2}(q)\vee F_{B_{1i}}(r_1r_2)\\
&\le F_{A_2}(q)\vee [F_{A_1}(r_1)\vee F_{A_1}(r_2)]\\
&= [F_{A_2}(q)\vee F_{A_1}(r_1)]\ \vee [F_{A_2}(q)\vee F_{A_1}(r_2)]\\
&= F_{(A_1\times A_2)}(r_1q)\vee F_{(A_1\times A_2)}(r_2q),
\end{aligned}$$

for $r_1q, r_2q \in X_1 \times X_2$.

Both cases are satisfied $\forall i \in \{1, 2, \ldots, n\}$.

Definition 2.33 Let $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ and $G_2 = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ be neutrosophic graph structures. The *cross product* of G_1 and G_2, denoted by

$$G_1 * G_2 = (A_1 * A_2, B_{11} * B_{21}, B_{12} * B_{22}, \ldots, B_{1n} * B_{2n}),$$

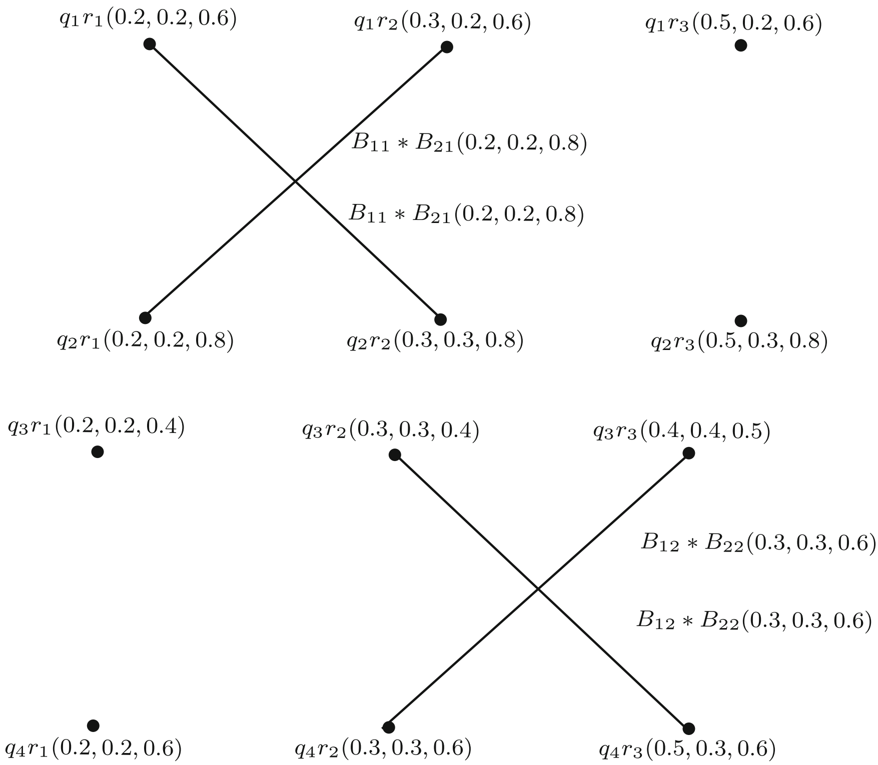


Fig. 2.16 Cross product of two neutrosophic graph structures

is defined by the following:

(i) $$\begin{cases} T_{(A_1*A_2)}(qr) = (T_{A_1} * T_{A_2})(qr) = T_{A_1}(q) \wedge T_{A_2}(r) \\ I_{(A_1*A_2)}(qr) = (I_{A_1} * I_{A_2})(qr) = I_{A_1}(q) \wedge I_{A_2}(r) \\ F_{(A_1*A_2)}(qr) = (F_{A_1} * F_{A_2})(qr) = F_{A_1}(q) \vee F_{A_2}(r) \end{cases}$$
for all $qr \in X_1 \times X_2$,

(ii) $$\begin{cases} T_{(B_{1i}*B_{2i})}(q_1r_1)(q_2r_2) = (T_{B_{1i}} * T_{B_{2i}})(q_1r_1)(q_2r_2) = T_{B_{1i}}(q_1q_2) \wedge T_{B_{2i}}(r_1r_2) \\ I_{(B_{1i}*B_{2i})}(q_1r_1)(q_2r_2) = (I_{B_{1i}} * I_{B_{2i}})(q_1r_1)(q_2r_2) = I_{B_{1i}}(q_1q_2) \wedge I_{B_{2i}}(r_1r_2) \\ F_{(B_{1i}*B_{2i})}(q_1r_1)(q_2r_2) = (F_{B_{1i}} * F_{B_{2i}})(q_1r_1)(q_2r_2) = F_{B_{1i}}(q_1q_2) \vee F_{B_{2i}}(r_1r_2) \end{cases}$$
for all $q_1q_2 \in E_{1i}$, $r_1r_2 \in E_{2i}$.

Example 2.17 Cross product of two neutrosophic graph structures G_1 and G_2 shown in Fig. 2.14 is defined as $G_1 * G_2 = \{A_1 * A_2, B_{11} * B_{21}, B_{12} * B_{22}\}$ and is shown in the Fig. 2.16.

Theorem 2.4 *The cross product $G_1 * G_2 = (A_1 * A_2, B_{11} * B_{21}, B_{12} * B_{22}, \ldots, B_{1n} * B_{2n})$ of two neutrosophic graph structures of the graph structures $G*_1$ and G_2^* is a neutrosophic graph structure of $G_1^* * G_2^*$.*

Proof For all $q_1r_1, q_2r_2 \in X_1 * X_2$

$$\begin{aligned}
T_{(B_{1i}*B_{2i})}((q_1r_1)(q_2r_2)) &= T_{B_{1i}}(q_1q_2) \wedge T_{B_{2i}}(r_1r_2)\\
&\leq [T_{A_1}(q_1) \wedge T_{A_1}(q_2)] \ \wedge [T_{A_2}(r_1) \wedge T_{A_2}(r_2)]\\
&= [T_{A_1}(q_1) \wedge T_{A_2}(r_1)] \ \wedge [T_{A_1}(q_2) \wedge T_{A_2}(r_2)]\\
&= T_{(A_1*A_2)}(q_1r_1) \wedge T_{(A_1*A_2)}(q_2r_2),
\end{aligned}$$

$$\begin{aligned}
I_{(B_{1i}*B_{2i})}((q_1r_1)(q_2r_2)) &= I_{B_{1i}}(q_1q_2) \wedge I_{B_{2i}}(r_1r_2)\\
&\leq [I_{A_1}(q_1) \wedge I_{A_1}(q_2)] \ \wedge [I_{A_2}(r_1) \wedge I_{A_2}(r_2)]\\
&= [I_{A_1}(q_1) \wedge I_{A_2}(r_1)] \ \wedge [I_{A_1}(q_2) \wedge I_{A_2}(r_2)]\\
&= I_{(A_1*A_2)}(q_1r_1) \wedge I_{(A_1*A_2)}(q_2r_2),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1i}*B_{2i})}((q_1r_1)(q_2r_2)) &= F_{B_{1i}}(q_1q_2) \vee F_{B_{2i}}(r_1r_2)\\
&\leq [F_{A_1}(q_1) \vee F_{A_1}(q_2)] \ \vee [F_{A_2}(r_1) \vee F_{A_2}(r_2)]\\
&= [F_{A_1}(q_1) \vee F_{A_2}(r_1)] \ \vee [F_{A_1}(q_2) \vee F_{A_2}(r_2)]\\
&= F_{(A_1*A_2)}(q_1r_1) \vee F_{(A_1*A_2)}(q_2r_2),
\end{aligned}$$

for $i \in \{1, 2, \ldots, n\}$.

Definition 2.34 Let $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ and $G_2 = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ be neutrosophic graph structures. The *lexicographic product* of G_1 and G_2, denoted by

$$G_1 \bullet G_2 = (A_1 \bullet A_2, B_{11} \bullet B_{21}, B_{12} \bullet B_{22}, \ldots, B_{1n} \bullet B_{2n}),$$

is defined by the following:

(i) $$\begin{cases} T_{(A_1\bullet A_2)}(qr) = (T_{A_1} \bullet T_{A_2})(qr) = T_{A_1}(q) \wedge T_{A_2}(r)\\ I_{(A_1\bullet A_2)}(qr) = (I_{A_1} \bullet I_{A_2})(qr) = I_{A_1}(q) \wedge I_{A_2}(r)\\ F_{(A_1\bullet A_2)}(qr) = (F_{A_1} \bullet F_{A_2})(qr) = F_{A_1}(q) \vee F_{A_2}(r)\end{cases}$$
for all $qr \in X_1 \times X_2$,

(ii) $$\begin{cases} T_{(B_{1i}\bullet B_{2i})}(qr_1)(qr_2) = (T_{B_{1i}} \bullet T_{B_{2i}})(qr_1)(qr_2) = T_{A_1}(q) \wedge T_{B_{2i}}(r_1r_2)\\ I_{(B_{1i}\bullet B_{2i})}(qr_1)(qr_2) = (I_{B_{1i}} \bullet I_{B_{2i}})(qr_1)(qr_2) = I_{A_1}(q) \wedge I_{B_{2i}}(r_1r_2)\\ F_{(B_{1i}\bullet B_{2i})}(qr_1)(qr_2) = (F_{B_{1i}} \bullet F_{B_{2i}})(qr_1)(qr_2) = F_{A_1}(q) \vee F_{B_{2i}}(r_1r_2)\end{cases}$$
for all $q \in X_1$, $r_1r_2 \in E_{2i}$,

(iii) $$\begin{cases} T_{(B_{1i}\bullet B_{2i})}(q_1r_1)(q_2r_2) = (T_{B_{1i}} \bullet T_{B_{2i}})(q_1r_1)(q_2r_2) = T_{B_{1i}}(q_1q_2) \wedge T_{B_{2i}}(r_1r_2)\\ I_{(B_{1i}\bullet B_{2i})}(q_1r_1)(q_2r_2) = (I_{B_{1i}} \bullet I_{B_{2i}})(q_1r_1)(q_2r_2) = I_{B_{1i}}(q_1q_2) \wedge I_{B_{2i}}(r_1r_2)\\ F_{(B_{1i}\bullet B_{2i})}(q_1r_1)(q_2r_2) = (F_{B_{1i}} \bullet F_{B_{2i}})(q_1r_1)(q_2r_2) = F_{B_{1i}}(q_1q_2) \vee F_{B_{2i}}(r_1r_2)\end{cases}$$
for all $q_1q_2 \in E_{1i}$, $r_1r_2 \in E_{2i}$.

Example 2.18 Lexicographic product of two neutrosophic graph structures G_1 and G_2 shown in Fig. 2.14 is defined as

$G_1 \bullet G_2 = \{A_1 \bullet A_2, B_{11} \bullet B_{21}, B_{12} \bullet B_{22}\}$ and is shown in the Fig. 2.17.

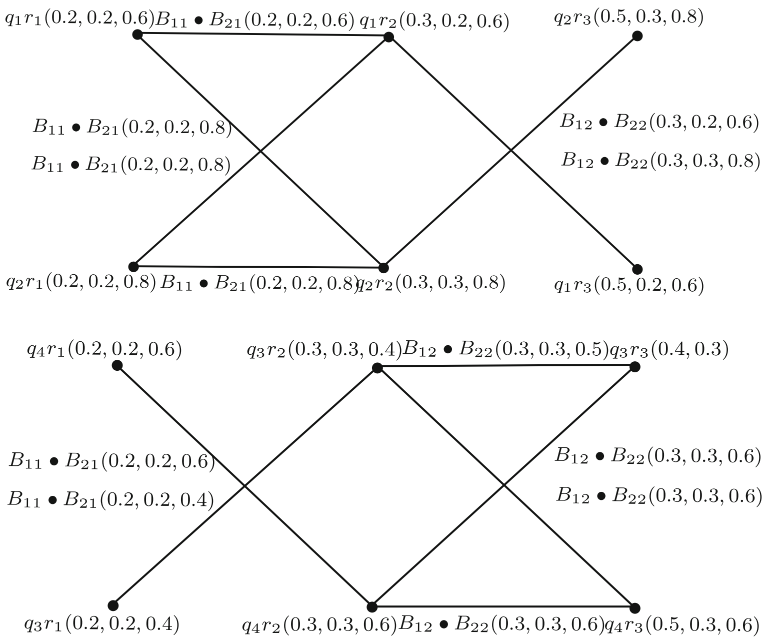


Fig. 2.17 Lexicographic product of two neutrosophic graph structures

Theorem 2.5 *The lexicographic product* $G_1 \bullet G_2 = (A_1 \bullet A_2, B_{11} \bullet B_{21}, B_{12} \bullet B_{22}, \ldots, B_{1n} \bullet B_{2n})$ *of two neutrosophic graph structures of the graph structures* G_1^* *and* G_2^* *is a neutrosophic graph structure of* $G_1^* \bullet G_2^*$.

Proof According to the definition of lexicographic product, there are two cases:

Case 1. When $q \in X_1$, $r_1r_2 \in E_{2i}$

$$\begin{aligned}
T_{(B_{1i}\bullet B_{2i})}((qr_1)(qr_2)) &= T_{A_1}(q) \wedge T_{B_{2i}}(r_1r_2)\\
&\leq T_{A_1}(q) \wedge [T_{A_2}(r_1) \wedge T_{A_2}(r_2)]\\
&= [T_{A_1}(q) \wedge T_{A_2}(r_1)] \ \wedge [T_{A_1}(q) \wedge T_{A_2}(r_2)]\\
&= T_{(A_1\bullet A_2)}(qr_1) \wedge T_{(A_1\bullet A_2)}(qr_2),
\end{aligned}$$

$$\begin{aligned}
I_{(B_{1i}\bullet B_{2i})}((qr_1)(qr_2)) &= I_{A_1}(q) \wedge I_{B_{2i}}(r_1r_2)\\
&\leq I_{A_1}(q) \wedge [I_{A_2}(r_1) \wedge I_{A_2}(r_2)]\\
&= [I_{A_1}(q) \wedge I_{A_2}(r_1)] \ \wedge [I_{A_1}(q) \wedge I_{A_2}(r_2)]\\
&= I_{(A_1\bullet A_2)}(qr_1) \wedge I_{(A_1\bullet A_2)}(qr_2),
\end{aligned}$$

$$\begin{aligned}F_{(B_{1i}\bullet B_{2i})}((qr_1)(qr_2)) &= F_{A_1}(q) \vee F_{B_{2i}}(r_1r_2)\\&\leq F_{A_1}(q) \vee [F_{A_2}(r_1) \vee F_{A_2}(r_2)]\\&= [F_{A_1}(q) \vee F_{A_2}(r_1)] \vee [F_{A_1}(q) \vee F_{A_2}(r_2)]\\&= F_{(A_1\bullet A_2)}(qr_1) \vee F_{(A_1\bullet A_2)}(qr_2),\end{aligned}$$

for $qr_1, qr_2 \in X_1 \bullet X_2$.

Case 2. When $q_1q_2 \in E_{1i}, r_1r_2 \in E_{2i}$

$$\begin{aligned}T_{(B_{1i}\bullet B_{2i})}((q_1r_1)(q_2r_2)) &= T_{B_{1i}}(q_1q_2) \wedge T_{B_{2i}}(r_1r_2)\\&\leq [T_{A_1}(q_1) \wedge T_{A_1}(q_2)] \wedge [T_{A_2}(r_1) \wedge T_{A_2}(r_2)]\\&= [T_{A_1}(q_1) \wedge T_{A_2}(r_1)] \wedge [T_{A_1}(q_2) \wedge T_{A_2}(r_2)]\\&= T_{(A_1\bullet A_2)}(q_1r_1) \wedge T_{(A_1\bullet A_2)}(q_2r_2),\\I_{(B_{1i}\bullet B_{2i})}((q_1r_1)(q_2r_2)) &= I_{B_{1i}}(q_1q_2) \wedge I_{B_{2i}}(r_1r_2)\\&\leq [I_{A_1}(q_1) \wedge I_{A_1}(q_2)] \wedge [I_{A_2}(r_1) \wedge I_{A_2}(r_2)]\\&= [I_{A_1}(q_1) \wedge I_{A_2}(r_1)] \wedge [I_{A_1}(q_2) \wedge I_{A_2}(r_2)]\\&= I_{(A_1\bullet A_2)}(q_1r_1) \wedge I_{(A_1\bullet A_2)}(q_2r_2),\\F_{(B_{1i}\bullet B_{2i})}((q_1r_1)(q_2r_2)) &= F_{B_{1i}}(q_1q_2) \vee F_{B_{2i}}(r_1r_2)\\&\leq [F_{A_1}(q_1) \vee F_{A_1}(q_2)] \vee [F_{A_2}(r_1) \vee F_{A_2}(r_2)]\\&= [F_{A_1}(q_1) \vee F_{A_2}(r_1)] \vee [F_{A_1}(q_2) \vee F_{A_2}(r_2)]\\&= F_{(A_1\bullet A_2)}(q_1r_1) \vee F_{(A_1\bullet A_2)}(q_2r_2),\end{aligned}$$

for $q_1r_1, q_2r_2 \in X_1 \bullet X_2$.

Both cases are satisfied for $i \in \{1, 2, \ldots, n\}$.

Definition 2.35 Let $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ and $G_2 = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ be neutrosophic graph structures. The *strong product* of G_1 and G_2, denoted by

$$G_1 \boxtimes G_2 = (A_1 \boxtimes A_2, B_{11} \boxtimes B_{21}, B_{12} \boxtimes B_{22}, \ldots, B_{1n} \boxtimes B_{2n}),$$

is defined by the following:

(i) $$\begin{cases}T_{(A_1\boxtimes A_2)}(qr) = (T_{A_1} \boxtimes T_{A_2})(qr) = T_{A_1}(q) \wedge T_{A_2}(r)\\I_{(A_1\boxtimes A_2)}(qr) = (I_{A_1} \boxtimes I_{A_2})(qr) = I_{A_1}(q) \wedge I_{A_2}(r)\\F_{(A_1\boxtimes A_2)}(qr) = (F_{A_1} \boxtimes F_{A_2})(qr) = F_{A_1}(q) \vee F_{A_2}(r)\end{cases}$$

for all $qr \in X_1 \times X_2$,

(ii) $$\begin{cases}T_{(B_{1i}\boxtimes B_{2i})}(qr_1)(qr_2) = (T_{B_{1i}} \boxtimes T_{B_{2i}})(qr_1)(qr_2) = T_{A_1}(q) \wedge T_{B_{2i}}(r_1r_2)\\I_{(B_{1i}\boxtimes B_{2i})}(qr_1)(qr_2) = (I_{B_{1i}} \boxtimes I_{B_{2i}})(qr_1)(qr_2) = I_{A_1}(q) \wedge I_{B_{2i}}(r_1r_2)\\F_{(B_{1i}\boxtimes B_{2i})}(qr_1)(qr_2) = (F_{B_{1i}} \boxtimes F_{B_{2i}})(qr_1)(qr_2) = F_{A_1}(q) \vee F_{B_{2i}}(r_1r_2)\end{cases}$$

for all $q \in X_1, r_1r_2 \in E_{2i}$,

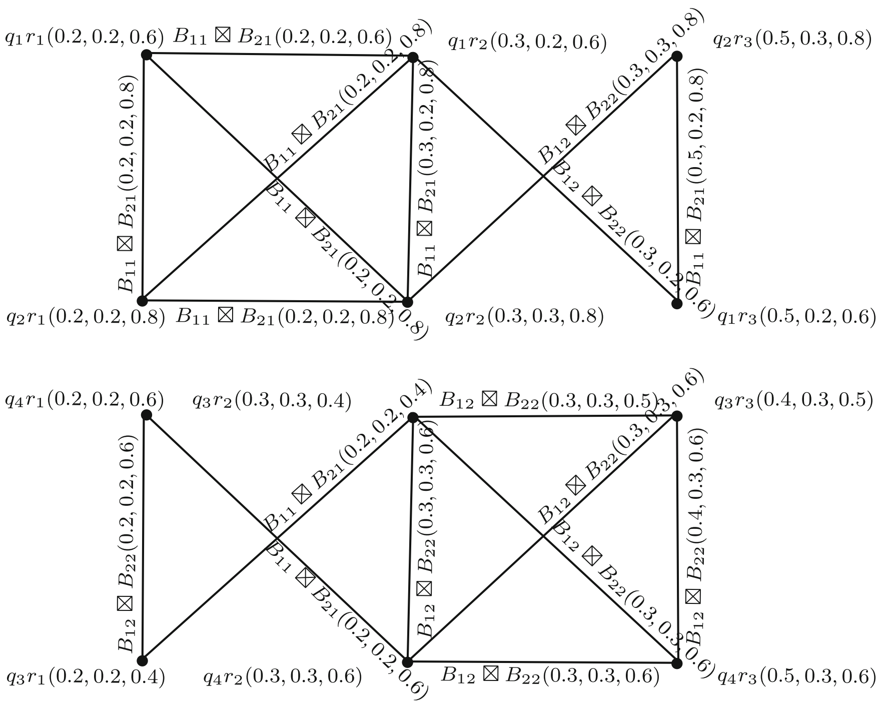


Fig. 2.18 Strong product of two neutrosophic graph structures

(iii) $\begin{cases} T_{(B_{1i} \boxtimes B_{2i})}(q_1r)(q_2r) = (T_{B_{1i}} \boxtimes T_{B_{2i}})(q_1r)(q_2r) = T_{A_2}(r) \wedge T_{B_{1i}}(q_1q_2) \\ I_{(B_{1i} \boxtimes B_{2i})}(q_1r)(q_2r) = (I_{B_{1i}} \boxtimes I_{B_{2i}})(q_1r)(q_2r) = I_{A_2}(r) \wedge I_{B_{1i}}(q_1q_2) \\ F_{(B_{1i} \boxtimes B_{2i})}(q_1r)(q_2r) = (F_{B_{1i}} \boxtimes F_{B_{2i}})(q_1r)(q_2r) = F_{A_2}(r) \vee F_{B_{1i}}(q_1q_2) \end{cases}$

for all $r \in X_2$, $q_1q_2 \in E_{1i}$,

(iv) $\begin{cases} T_{(B_{1i} \boxtimes B_{2i})}(q_1r_1)(q_2r_2) = (T_{B_{1i}} \boxtimes T_{B_{2i}})(q_1r_1)(q_2r_2) = T_{B_{1i}}(q_1q_2) \wedge T_{B_{2i}}(r_1r_2) \\ I_{(B_{1i} \boxtimes B_{2i})}(q_1r_1)(q_2r_2) = (I_{B_{1i}} \boxtimes I_{B_{2i}})(q_1r_1)(q_2r_2) = I_{B_{1i}}(q_1q_2) \wedge I_{B_{2i}}(r_1r_2) \\ F_{(B_{1i} \boxtimes B_{2i})}(q_1r_1)(q_2r_2) = (F_{B_{1i}} \boxtimes F_{B_{2i}})(q_1r_1)(q_2r_2) = F_{B_{1i}}(q_1q_2) \vee F_{B_{2i}}(r_1r_2) \end{cases}$

for all $q_1q_2 \in E_{1i}$, $r_1r_2 \in E_{2i}$.

Example 2.19 Strong product of two neutrosophic graph structures G_1 and G_2 shown in Fig. 2.14 is defined as $G_1 \boxtimes G_2 = \{A_1 \boxtimes A_2, B_{11} \boxtimes B_{21}, B_{12} \boxtimes B_{22}\}$ and is shown in the Fig. 2.18.

Theorem 2.6 *The strong product* $G_1 \boxtimes G_2 = (A_1 \boxtimes A_2, B_{11} \boxtimes B_{21}, B_{12} \boxtimes B_{22}, \ldots, B_{1n} \boxtimes B_{2n})$ *of two neutrosophic graph structures of the graph structures* G_1^* *and* G_2^* *is a neutrosophic graph structure of* $G_1^* \boxtimes G_2^*$.

Proof According to the definition of strong product, there are three cases:

Case 1. When $q \in X_1$, $r_1r_2 \in E_{2i}$

$$\begin{aligned} T_{(B_{1i}\boxtimes B_{2i})}((qr_1)(qr_2)) &= T_{A_1}(q) \wedge T_{B_{2i}}(r_1r_2) \\ &\leq T_{A_1}(q) \wedge [T_{A_2}(r_1) \wedge T_{A_2}(r_2)] \\ &= [T_{A_1}(q) \wedge T_{A_2}(r_1)] \wedge [T_{A_1}(q) \wedge T_{A_2}(r_2)] \\ &= T_{(A_1\boxtimes A_2)}(qr_1) \wedge T_{(A_1\boxtimes A_2)}(qr_2), \\ I_{(B_{1i}\boxtimes B_{2i})}((qr_1)(qr_2)) &= I_{A_1}(q) \wedge I_{B_{2i}}(r_1r_2) \\ &\leq I_{A_1}(q) \wedge [I_{A_2}(r_1) \wedge I_{A_2}(r_2)] \\ &= [I_{A_1}(q) \wedge I_{A_2}(r_1)] \wedge [I_{A_1}(q) \wedge I_{A_2}(r_2)] \\ &= I_{(A_1\boxtimes A_2)}(qr_1) \wedge I_{(A_1\boxtimes A_2)}(qr_2), \\ F_{(B_{1i}\boxtimes B_{2i})}((qr_1)(qr_2)) &= F_{A_1}(q) \vee F_{B_{2i}}(r_1r_2) \\ &\leq F_{A_1}(q) \vee [F_{A_2}(r_1) \vee F_{A_2}(r_2)] \\ &= [F_{A_1}(q) \vee F_{A_2}(r_1)] \vee [F_{A_1}(q) \vee F_{A_2}(r_2)] \\ &= F_{(A_1\boxtimes A_2)}(qr_1) \vee F_{(A_1\boxtimes A_2)}(qr_2), \end{aligned}$$

for $qr_1, qr_2 \in X_1 \boxtimes X_2$.

Case 2. When $q \in X_2, r_1r_2 \in E_{1i}$

$$\begin{aligned} T_{(B_{1i}\boxtimes B_{2i})}((r_1q)(r_2q)) &= T_{A_2}(q) \wedge T_{B_{1i}}(r_1r_2) \\ &\leq T_{A_2}(q) \wedge [T_{A_1}(r_1) \wedge T_{A_1}(r_2)] \\ &= [T_{A_2}(q) \wedge T_{A_1}(r_1)] \wedge [T_{A_2}(q) \wedge T_{A_1}(r_2)] \\ &= T_{(A_1\boxtimes A_2)}(r_1q) \wedge T_{(A_1\boxtimes A_2)}(r_2q), \\ I_{(B_{1i}\boxtimes B_{2i})}((r_1q)(r_2q)) &= I_{A_2}(q) \wedge I_{B_{1i}}(r_1r_2) \\ &\leq I_{A_2}(q) \wedge [I_{A_1}(r_1) \wedge I_{A_1}(r_2)] \\ &= [I_{A_2}(q) \wedge I_{A_1}(r_1)] \wedge [I_{A_2}(q) \wedge I_{A_1}(r_2)] \\ &= I_{(A_1\boxtimes A_2)}(r_1q) \wedge I_{(A_1\boxtimes A_2)}(r_2q), \\ F_{(B_{1i}\boxtimes B_{2i})}((r_1q)(r_2q)) &= F_{A_2}(q) \vee F_{B_{1i}}(r_1r_2) \\ &\leq F_{A_2}(q) \vee [F_{A_1}(r_1) \vee F_{A_1}(r_2)] \\ &= [F_{A_2}(q) \vee F_{A_1}(r_1)] \vee [F_{A_2}(q) \vee F_{A_1}(r_2)] \\ &= F_{(A_1\boxtimes A_2)}(r_1q) \vee F_{(A_1\boxtimes A_2)}(r_2q), \end{aligned}$$

for $r_1q, r_2q \in X_1 \boxtimes X_2$.

Case 3. For all $q_1q_2 \in E_{1i}, r_1r_2 \in E_{2i}$

$$\begin{aligned} T_{(B_{1i}\boxtimes B_{2i})}((q_1r_1)(q_2r_2)) &= T_{B_{1i}}(q_1q_2) \wedge T_{B_{2i}}(r_1r_2) \\ &\leq [T_{A_1}(q_1) \wedge T_{A_1}(q_2)] \wedge [T_{A_2}(r_1) \wedge T_{A_2}(r_2)] \\ &= [T_{A_1}(q_1) \wedge T_{A_2}(r_1)] \wedge [T_{A_1}(q_2) \wedge T_{A_2}(r_2)] \\ &= T_{(A_1\boxtimes A_2)}(q_1r_1) \wedge T_{(A_1\boxtimes A_2)}(q_2r_2), \end{aligned}$$

$$\begin{aligned}
I_{(B_{1i}\boxtimes B_{2i})}((q_1r_1)(q_2r_2)) &= I_{B_{1i}}(q_1q_2) \wedge I_{B_{2i}}(r_1r_2)\\
&\leq [I_{A_1}(q_1) \wedge I_{A_1}(q_2)] \wedge [I_{A_2}(r_1) \wedge I_{A_2}(r_2)]\\
&= [I_{A_1}(q_1) \wedge I_{A_2}(r_1)] \wedge [I_{A_1}(q_2) \wedge I_{A_2}(r_2)]\\
&= I_{(A_1\boxtimes A_2)}(q_1r_1) \wedge I_{(A_1\boxtimes A_2)}(q_2r_2),\\
F_{(B_{1i}\boxtimes B_{2i})}((q_1r_1)(q_2r_2)) &= F_{B_{1i}}(q_1q_2) \vee F_{B_{2i}}(r_1r_2)\\
&\leq [F_{A_1}(q_1) \vee F_{A_1}(q_2)] \vee [F_{A_2}(r_1) \vee F_{A_2}(r_2)]\\
&= [F_{A_1}(q_1) \vee F_{A_2}(r_1)] \vee [F_{A_1}(q_2) \vee F_{A_2}(r_2)]\\
&= F_{(A_1\boxtimes A_2)}(q_1r_1) \vee F_{(A_1\boxtimes A_2)}(q_2r_2),
\end{aligned}$$

for $q_1r_1, q_2r_2 \in X_1 \boxtimes X_2$.
All cases are satisfied for $i = 1, 2, \ldots, n$.

Definition 2.36 Let $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ and $G_2 = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ be neutrosophic graph structures. The *composition* of G_1 and G_2, denoted by

$$G_1 \circ G_2 = (A_1 \circ A_2, B_{11} \circ B_{21}, B_{12} \circ B_{22}, \ldots, B_{1n} \circ B_{2n}),$$

is defined by the following:

(i) $$\begin{cases} T_{(A_1\circ A_2)}(qr) = (T_{A_1} \circ T_{A_2})(qr) = T_{A_1}(q) \wedge T_{A_2}(r)\\ I_{(A_1\circ A_2)}(qr) = (I_{A_1} \circ I_{A_2})(qr) = I_{A_1}(q) \wedge I_{A_2}(r)\\ F_{(A_1\circ A_2)}(qr) = (F_{A_1} \circ F_{A_2})(qr) = F_{A_1}(q) \vee F_{A_2}(r)\end{cases}$$
for all $qr \in X_1 \times X_2$,

(ii) $$\begin{cases} T_{(B_{1i}\circ B_{2i})}(qr_1)(qr_2) = (T_{B_{1i}} \circ T_{B_{2i}})(qr_1)(qr_2) = T_{A_1}(q) \wedge T_{B_{2i}}(r_1r_2)\\ I_{(B_{1i}\circ B_{2i})}(qr_1)(qr_2) = (I_{B_{1i}} \circ I_{B_{2i}})(qr_1)(qr_2) = I_{A_1}(q) \wedge I_{B_{2i}}(r_1r_2)\\ F_{(B_{1i}\circ B_{2i})}(qr_1)(qr_2) = (F_{B_{1i}} \circ F_{B_{2i}})(qr_1)(qr_2) = F_{A_1}(q) \vee F_{B_{2i}}(r_1r_2)\end{cases}$$
for all $q \in X_1, r_1r_2 \in E_{2i}$,

(iii) $$\begin{cases} T_{(B_{1i}\circ B_{2i})}(q_1r)(q_2r) = (T_{B_{1i}} \circ T_{B_{2i}})(q_1r)(q_2r) = T_{A_2}(r) \wedge T_{B_{1i}}(q_1q_2)\\ I_{(B_{1i}\circ B_{2i})}(q_1r)(q_2r) = (I_{B_{1i}} \circ I_{B_{2i}})(q_1r)(q_2r) = I_{A_2}(r) \wedge I_{B_{1i}}(q_1q_2)\\ F_{(B_{1i}\circ B_{2i})}(q_1r)(q_2r) = (F_{B_{1i}} \circ F_{B_{2i}})(q_1r)(q_2r) = F_{A_2}(r) \vee F_{B_{1i}}(q_1q_2)\end{cases}$$
for all $r \in X_2, q_1q_2 \in E_{1i}$,

(iv) $$\begin{cases} T_{(B_{1i}\circ B_{2i})}(q_1r_1)(q_2r_2) = (T_{B_{1i}} \circ T_{B_{2i}})(q_1r_1)(q_2r_2) = T_{B_{1i}}(q_1q_2) \wedge T_{A_2}(r_1) \wedge T_{A_2}(r_2)\\ I_{(B_{1i}\circ B_{2i})}(q_1r_1)(q_2r_2) = (I_{B_{1i}} \circ I_{B_{2i}})(q_1r_1)(q_2r_2) = I_{B_{1i}}(q_1q_2) \wedge I_{A_2}(r_1) \wedge I_{A_2}(r_2)\\ F_{(B_{1i}\circ B_{2i})}(q_1r_1)(q_2r_2) = (F_{B_{1i}} \circ F_{B_{2i}})(q_1r_1)(q_2r_2) = F_{B_{1i}}(q_1q_2) \vee F_{A_2}(r_1) \vee F_{A_2}(r_2)\end{cases}$$
for all $q_1q_2 \in E_{1i}, r_1r_2 \in E_{2i}$ such that $r_1 \neq r_2$.

Example 2.20 Composition of two neutrosophic graph structures G_1 and G_2 shown in Fig. 2.14 is defined as $G_1 \circ G_2 = \{A_1 \circ A_2, B_{11} \circ B_{21}, B_{12} \circ B_{22}\}$ and is shown in the Fig. 2.19.

Theorem 2.7 *The composition* $G_1 \circ G_2 = (A_1 \circ A_2, B_{11} \circ B_{21}, B_{12} \circ B_{22}, \ldots, B_{1n} \circ B_{2n})$ *of two neutrosophic graph structures of the graph structures* G_1^* *and* G_2^* *is a neutrosophic graph structure of* $G_1^* \circ G_2^*$.

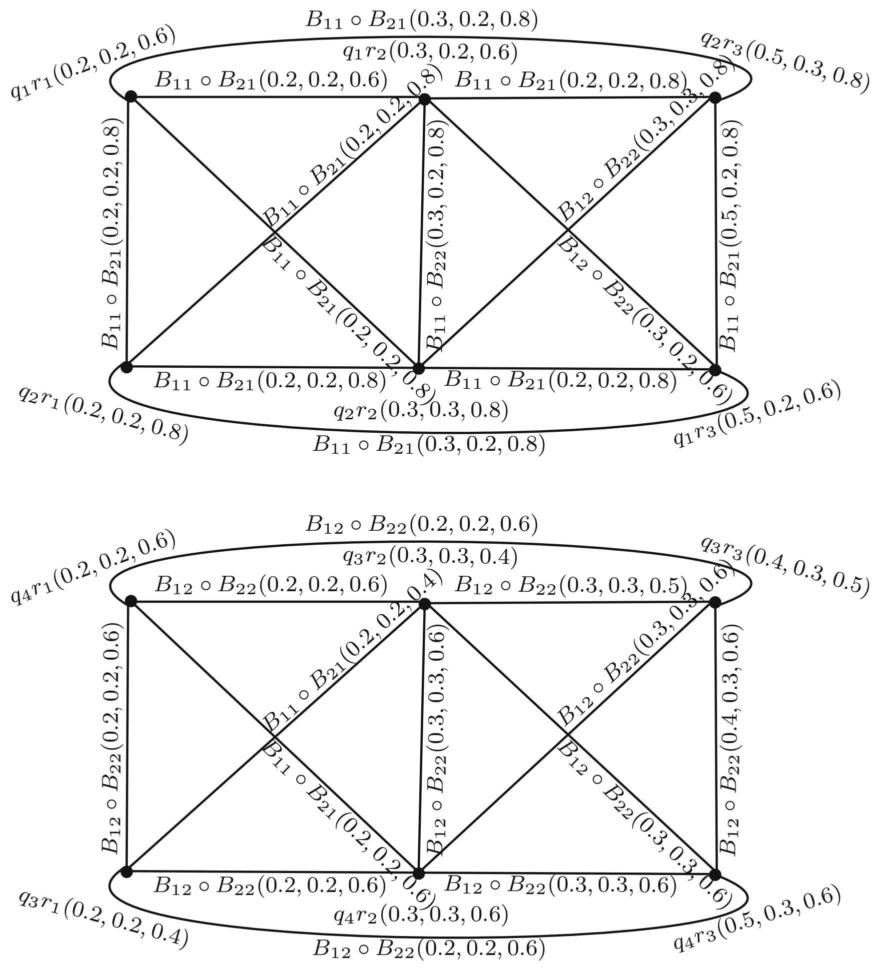


Fig. 2.19 Composition of two neutrosophic graph structures

Proof According to the definition of composition, there are three cases:

Case 1. When $q \in X_1, r_1r_2 \in E_{2i}$

$$\begin{aligned} T_{(B_{1i} \circ B_{2i})}((qr_1)(qr_2)) &= T_{A_1}(q) \wedge T_{B_{2i}}(r_1r_2) \\ &\leq T_{A_1}(q) \wedge [T_{A_2}(r_1) \wedge T_{A_2}(r_2)] \\ &= [T_{A_1}(q) \wedge T_{A_2}(r_1)] \wedge [T_{A_1}(q) \wedge T_{A_2}(r_2)] \\ &= T_{(A_1 \circ A_2)}(qr_1) \wedge T_{(A_1 \circ A_2)}(qr_2), \end{aligned}$$

$$\begin{aligned}
I_{(B_{1i}\circ B_{2i})}((qr_1)(qr_2)) &= I_{A_1}(q) \wedge I_{B_{2i}}(r_1r_2) \\
&\leq I_{A_1}(q) \wedge [I_{A_2}(r_1) \wedge I_{A_2}(r_2)] \\
&= [I_{A_1}(q) \wedge I_{A_2}(r_1)] \ \wedge [I_{A_1}(q) \wedge I_{A_2}(r_2)] \\
&= I_{(A_1\circ A_2)}(qr_1) \wedge I_{(A_1\circ A_2)}(qr_2), \\
F_{(B_{1i}\circ B_{2i})}((qr_1)(qr_2)) &= F_{A_1}(q) \vee F_{B_{2i}}(r_1r_2) \\
&\leq F_{A_1}(q) \vee [F_{A_2}(r_1) \vee F_{A_2}(r_2)] \\
&= [F_{A_1}(q) \vee F_{A_2}(r_1)] \ \vee [F_{A_1}(q) \vee F_{A_2}(r_2)] \\
&= F_{(A_1\circ A_2)}(qr_1) \vee F_{(A_1\circ A_2)}(qr_2),
\end{aligned}$$

for $qr_1, qr_2 \in X_1 \circ X_2$.

Case 2. When $q \in X_2$, $r_1r_2 \in E_{1i}$

$$\begin{aligned}
T_{(B_{1i}\circ B_{2i})}((r_1q)(r_2q)) &= T_{A_2}(q) \wedge T_{B_{1i}}(r_1r_2) \\
&\leq T_{A_2}(q) \wedge [T_{A_1}(r_1) \wedge T_{A_1}(r_2)] \\
&= [T_{A_2}(q) \wedge T_{A_1}(r_1)] \ \wedge [T_{A_2}(q) \wedge T_{A_1}(r_2)] \\
&= T_{(A_1\circ A_2)}(r_1q) \wedge T_{(A_1\circ A_2)}(r_2q), \\
I_{(B_{1i}\circ B_{2i})}((r_1q)(r_2q)) &= I_{A_2}(q) \wedge I_{B_{1i}}(r_1r_2) \\
&\leq I_{A_2}(q) \wedge [I_{A_1}(r_1) \wedge I_{A_1}(r_2)] \\
&= [I_{A_2}(q) \wedge I_{A_1}(r_1)] \ \wedge [I_{A_2}(q) \wedge I_{A_1}(r_2)] \\
&= I_{(A_1\circ A_2)}(r_1q) \wedge I_{(A_1\circ A_2)}(r_2q), \\
F_{(B_{1i}\circ B_{2i})}((r_1q)(r_2q)) &= F_{A_2}(q) \vee F_{B_{1i}}(r_1r_2) \\
&\leq F_{A_2}(q) \vee [F_{A_1}(r_1) \vee F_{A_1}(r_2)] \\
&= [F_{A_2}(q) \vee F_{A_1}(r_1)] \ \vee [F_{A_2}(q) \vee F_{A_1}(r_2)] \\
&= F_{(A_1\circ A_2)}(r_1q) \vee F_{(A_1\circ A_2)}(r_2q),
\end{aligned}$$

for $r_1q, r_2q \in X_1 \circ X_2$.

Case 3. For all $q_1q_2 \in E_{1i}$, $r_1, r_2 \in X_2$ such that $r_1 \neq r_2$

$$\begin{aligned}
T_{(B_{1i}\circ B_{2i})}((q_1r_1)(q_2r_2)) &= T_{B_{1i}}(q_1q_2) \wedge T_{A_2}(r_1) \wedge T_{A_2}(r_2) \\
&\leq [T_{A_1}(q_1) \wedge T_{A_1}(q_2)] \ \wedge T_{A_2}(r_1) \wedge T_{A_2}(r_2) \\
&= [T_{A_1}(q_1) \wedge T_{A_2}(r_1)] \ \wedge [T_{A_1}(q_2) \wedge T_{A_2}(r_2)] \\
&= T_{(A_1\circ A_2)}(q_1r_1) \wedge T_{(A_1\circ A_2)}(q_2r_2), \\
I_{(B_{1i}\circ B_{2i})}((q_1r_1)(q_2r_2)) &= I_{B_{1i}}(q_1q_2) \wedge I_{A_2}(r_1) \wedge I_{A_2}(r_2) \\
&\leq [I_{A_1}(q_1) \wedge I_{A_1}(q_2)] \ \wedge [I_{A_2}(r_1) \wedge I_{A_2}(r_2)] \\
&= [I_{A_1}(q_1) \wedge I_{A_2}(r_1)] \ \wedge [I_{A_1}(q_2) \wedge I_{A_2}(r_2)] \\
&= I_{(A_1\circ A_2)}(q_1r_1) \wedge I_{(A_1\circ A_2)}(q_2r_2),
\end{aligned}$$

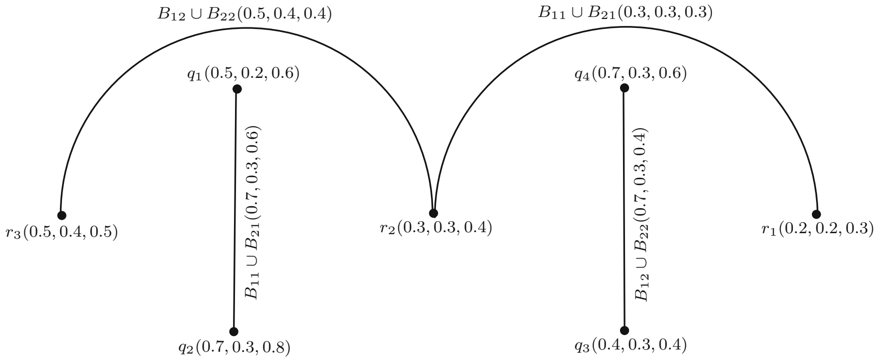


Fig. 2.20 Union of two neutrosophic graph structures

$$\begin{aligned} F_{(B_{1i} \circ B_{2i})}((q_1r_1)(q_2r_2)) &= F_{B_{1i}}(q_1q_2) \vee F_{A_2}(r_1) \vee F_{A_2}(r_2) \\ &\leq [F_{A_1}(q_1) \vee F_{A_1}(q_2)] \ \vee [F_{A_2}(r_1) \vee F_{A_2}(r_2)] \\ &= [F_{A_1}(q_1) \vee F_{A_2}(r_1)] \ \vee [F_{A_1}(q_2) \vee F_{A_2}(r_2)] \\ &= F_{(A_1 \circ A_2)}(q_1r_1) \vee F_{(A_1 \circ A_2)}(q_2r_2), \end{aligned}$$

for $q_1r_1, q_2r_2 \in X_1 \circ X_2$.
All cases are satisfied for $i = 1, 2, \ldots, n$.

Definition 2.37 Let $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ and $G_2 = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ be neutrosophic graph structures. The *union* of G_1 and G_2, denoted by

$$G_1 \cup G_2 = (A_1 \cup A_2, B_{11} \cup B_{21}, B_{12} \cup B_{22}, \ldots, B_{1n} \cup B_{2n}),$$

is defined by following:

(i) $$\begin{cases} T_{(A_1 \cup A_2)}(q) = (T_{A_1} \cup T_{A_2})(q) = T_{A_1}(q) \vee T_{A_2}(q) \\ I_{(A_1 \cup A_2)}(q) = (I_{A_1} \cup I_{A_2})(q) = I_{A_1}(q) \vee I_{A_2}(q) \\ F_{(A_1 \cup A_2)}(q) = (F_{A_1} \cup F_{A_2})(q) = F_{A_1}(q) \wedge F_{A_2}(q) \end{cases}$$
for all $q \in X_1 \cup X_2$,

(ii) $$\begin{cases} T_{(B_{1i} \cup B_{2i})}(qr) = (T_{B_{1i}} \cup T_{B_{2i}})(qr) = T_{B_{1i}}(qr) \vee T_{B_{2i}}(qr) \\ I_{(B_{1i} \cup B_{2i})}(qr) = (I_{B_{1i}} \cup I_{B_{2i}})(qr) = I_{B_{1i}}(qr) \vee I_{B_{2i}}(qr) \\ F_{(B_{1i} \cup B_{2i})}(qr) = (F_{B_{1i}} \cup F_{B_{2i}})(qr) = F_{B_{1i}}(qr) \wedge F_{B_{2i}}(qr) \end{cases}$$
for all $qr \in E_{1i} \cup E_{2i}$.

Example 2.21 Union of two neutrosophic graph structures G_1 and G_2 shown in Fig. 2.14 is defined as $G_1 \cup G_2 = \{A_1 \cup A_2, B_{11} \cup B_{21}, B_{12} \cup B_{22}\}$ and is shown in the Fig. 2.20.

Theorem 2.8 *The union $G_1 \cup G_2 = (A_1 \cup A_2, B_{11} \cup B_{21}, B_{12} \cup B_{22}, \ldots, B_{1n} \cup B_{2n})$ of two neutrosophic graph structures of the graph structures G_1^* and G_2^* is a neutrosophic graph structure of $G_1^* \cup G_2^*$.*

Proof Let $q_1q_2 \in E_{1i} \cup E_{2i}$. Here we consider two cases:

Case 1. When $q_1, q_2 \in X_1$, then according to Definition 2.37, $T_{A_2}(q_1) = T_{A_2}(q_2) = T_{B_{2i}}(q_1q_2) = 0,\ I_{A_2}(q_1) = I_{A_2}(q_2) = I_{B_{2i}}(q_1q_2) = 0,\ F_{A_2}(q_1) = F_{A_2}(q_2) = F_{B_{2i}}(q_1q_2) = 0$, so

$$\begin{aligned}
T_{(B_{1i}\cup B_{2i})}(q_1q_2) &= T_{B_{1i}}(q_1q_2) \vee T_{B_{2i}}(q_1q_2)\\
&= T_{B_{1i}}(q_1q_2) \vee 0\\
&\le [T_{A_1}(q_1) \wedge T_{A_1}(q_2)]\ \vee 0\\
&= [T_{A_1}(q_1) \vee 0] \wedge [T_{A_1}(q_2) \vee 0]\\
&= [T_{A_1}(q_1) \vee T_{A_2}(q_1)] \wedge [T_{A_1}(q_2) \vee T_{A_2}(q_2)]\\
&= T_{(A_1\cup A_2)}(q_1) \wedge T_{(A_1\cup A_2)}(q_2),\\
I_{(B_{1i}\cup B_{2i})}(q_1q_2) &= I_{B_{1i}}(q_1q_2) \vee I_{B_{2i}}(q_1q_2)\\
&= I_{B_{1i}}(q_1q_2) \vee 0\\
&\le [I_{A_1}(q_1) \wedge I_{A_1}(q_2)]\ \vee 0\\
&= [I_{A_1}(q_1) \vee 0] \wedge [I_{A_1}(q_2) \vee 0]\\
&= [I_{A_1}(q_1) \vee I_{A_2}(q_1)] \wedge [I_{A_1}(q_2) \vee I_{A_2}(q_2)]\\
&= I_{(A_1\cup A_2)}(q_1) \wedge I_{(A_1\cup A_2)}(q_2),\\
F_{(B_{1i}\cup B_{2i})}(q_1q_2) &= F_{B_{1i}}(q_1q_2) \wedge F_{B_{2i}}(q_1q_2)\\
&= F_{B_{1i}}(q_1q_2) \wedge 0\\
&\le [F_{A_1}(q_1) \vee F_{A_1}(q_2)]\ \wedge 0\\
&= [F_{A_1}(q_1) \wedge 0] \vee [F_{A_1}(q_2) \wedge 0]\\
&= [F_{A_1}(q_1) \wedge F_{A_2}(q_1)] \vee [F_{A_1}(q_2) \wedge F_{A_2}(q_2)]\\
&= F_{(A_1\cup A_2)}(q_1) \vee F_{(A_1\cup A_2)}(q_2),
\end{aligned}$$

for $q_1, q_2 \in X_1 \cup X_2$.

Case 2. When $q_1, q_2 \in X_2$, then according to Definition 2.37, $T_{A_1}(q_1) = T_{A_1}(q_2) = T_{B_{1i}}(q_1q_2) = 0,\ I_{A_1}(q_1) = I_{A_1}(q_2) = I_{B_{1i}}(q_1q_2) = 0,\ F_{A_1}(q_1) = F_{A_1}(q_2) = F_{B_{1i}}(q_1q_2) = 0$, so

$$\begin{aligned}
T_{(B_{1i}\cup B_{2i})}(q_1q_2) &= T_{B_{1i}}(q_1q_2) \vee T_{B_{2i}}(q_1q_2)\\
&= T_{B_{2i}}(q_1q_2) \vee 0\\
&\le [T_{A_2}(q_1) \wedge T_{A_2}(q_2)]\ \vee 0\\
&= [T_{A_2}(q_1) \vee 0] \wedge [T_{A_2}(q_2) \vee 0]\\
&= [T_{A_1}(q_1) \vee T_{A_2}(q_1)] \wedge [T_{A_1}(q_2) \vee T_{A_2}(q_2)]\\
&= T_{(A_1\cup A_2)}(q_1) \wedge T_{(A_1\cup A_2)}(q_2),
\end{aligned}$$

$$\begin{aligned} I_{(B_{1i}\cup B_{2i})}(q_1q_2) &= I_{B_{1i}}(q_1q_2) \vee I_{B_{2i}}(q_1q_2) \\ &= I_{B_{2i}}(q_1q_2) \vee 0 \\ &\leq [I_{A_2}(q_1) \wedge I_{A_2}(q_2)] \;\vee 0 \\ &= [I_{A_2}(q_1) \vee 0] \wedge [I_{A_2}(q_2) \vee 0] \\ &= [I_{A_1}(q_1) \vee I_{A_2}(q_1)] \wedge [I_{A_1}(q_2) \vee I_{A_2}(q_2)] \\ &= I_{(A_1\cup A_2)}(q_1) \wedge I_{(A_1\cup A_2)}(q_2), \end{aligned}$$

$$\begin{aligned} F_{(B_{1i}\cup B_{2i})}(q_1q_2) &= F_{B_{1i}}(q_1q_2) \wedge F_{B_{2i}}(q_1q_2) \\ &= F_{B_{2i}}(q_1q_2) \wedge 0 \\ &\leq [F_{A_2}(q_1) \vee F_{A_2}(q_2)] \;\wedge 0 \\ &= [F_{A_2}(q_1) \wedge 0] \vee [F_{A_2}(q_2) \wedge 0] \\ &= [F_{A_1}(q_1) \wedge F_{A_2}(q_1)] \vee [F_{A_1}(q_2) \wedge F_{A_2}(q_2)] \\ &= F_{(A_1\cup A_2)}(q_1) \vee F_{(A_1\cup A_2)}(q_2), \end{aligned}$$

for $q_1, q_2 \in X_1 \cup X_2$.

Both cases are satisfied $\forall i \in \{1, 2, \dots, n\}$. This completes the proof.

Theorem 2.9 *Let $G^* = (X_1 \cup X_2, E_{11} \cup E_{21}, E_{12} \cup E_{22}, \dots, E_{1n} \cup E_{2n})$ be the union of two graph structures $G_1^* = (X_1, E_{11}, E_{12}, \dots, E_{1n})$ and $G_2^* = (X_2, E_{21}, E_{22}, \dots, E_{2n})$. Then every neutrosophic graph structure $G = (A, B_1, B_2, \dots, B_n)$ of G^* is union of two neutrosophic graph structures $G_1 = (A_1, B_{11}, B_{12}, \dots, B_{1n})$ and $G_2 = (A_2, B_{21}, B_{22}, \dots, B_{2n})$ of graph structures G_1^* and G_2^*, respectively.*

Proof First we define A_1, A_2, B_{1i} and B_{2i} for $i \in \{1, 2, \dots, n\}$ as:
$T_{A_1}(q) = T_A(q)$, $I_{A_1}(q) = I_A(q)$, $F_{A_1}(q) = F_A(q)$, if $q \in X_1$
$T_{A_2}(q) = T_A(q)$, $I_{A_2}(q) = I_A(q)$, $F_{A_2}(q) = F_A(q)$, if $q \in X_2$

$T_{B_{1i}}(q_1q_2) = T_{B_i}(q_1q_2)$, $I_{B_{1i}}(q_1q_2) = I_{B_i}(q_1q_2)$, $F_{B_{1i}}(q_1q_2) = F_{B_i}(q_1q_2)$, if $q_1q_2 \in E_{1i}$, $T_{B_{2i}}(q_1q_2) = T_{B_i}(q_1q_2)$, $I_{B_{2i}}(q_1q_2) = I_{B_i}(q_1q_2)$, $F_{B_{2i}}(q_1q_2) = F_{B_i}(q_1q_2)$, if $q_1q_2 \in E_{2i}$. Then $A = A_1 \cup A_2$ and $B_i = B_{1i} \cup B_{2i}$, $i \in \{1, 2, \dots, n\}$.

Now for $q_1q_2 \in E_{ki}$, $k = 1, 2$, $i = 1, 2, \dots, n$
$T_{B_{ki}}(q_1q_2) = T_{B_i}(q_1q_2) \leq T_A(q_1) \wedge T_A(q_2) = T_{A_k}(q_1) \wedge T_{A_k}(q_2)$,
$I_{B_{ki}}(q_1q_2) = I_{B_i}(q_1q_2) \leq I_A(q_1) \wedge I_A(q_2) = I_{A_k}(q_1) \wedge I_{A_k}(q_2)$,
$F_{B_{ki}}(q_1q_2) = F_{B_i}(q_1q_2) \leq F_A(q_1) \vee F_A(q_2) = F_{A_k}(q_1) \vee F_{A_k}(q_2)$,
i.e.
$G_k = (A_k, B_{k1}, B_{k2}, \dots, B_{kn})$ is a neutrosophic graph structure of G_k^*, $k = 1, 2$.
Thus $G = (A, B_1, B_2, \dots, B_n)$, a neutrosophic graph structure of $G^* = G_1^* \cup G_2^*$, is union of two neutrosophic graph structures G_1 and G_2.

Definition 2.38 Let $G_1 = (A_1, B_{11}, B_{12}, \dots, B_{1n})$ and $G_2 = (A_2, B_{21}, B_{22}, \dots, B_{2n})$ be neutrosophic graph structures and let $X_1 \cap X_2 = \emptyset$. The *join* of G_1 and G_2, denoted by

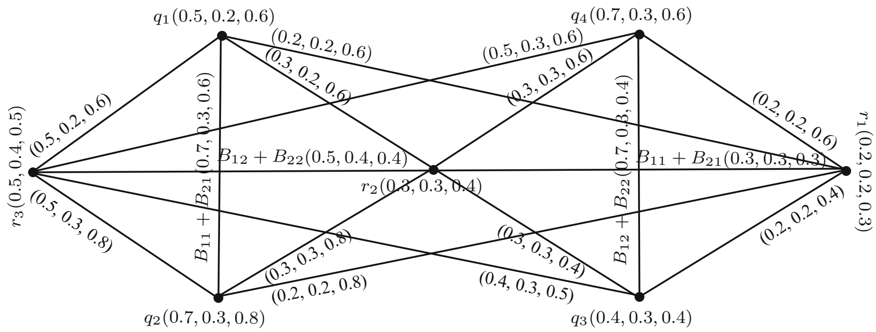


Fig. 2.21 Join of two neutrosophic graph structures

$$G_1 + G_2 = (A_1 + A_2, B_{11} + B_{21}, B_{12} + B_{22}, \ldots, B_{1n} + B_{2n}),$$

is defined by the following:

(i) $\begin{cases} T_{(A_1+A_2)}(q) = T_{(A_1 \cup A_2)}(q) \\ I_{(A_1+A_2)}(q) = I_{(A_1 \cup A_2)}(q) \\ F_{(A_1+A_2)}(q) = F_{(A_1 \cup A_2)}(q) \end{cases}$

for all $q \in X_1 \cup X_2$,

(ii) $\begin{cases} T_{(B_{1i}+B_{2i})}(qr) = T_{(B_{1i} \cup B_{2i})}(qr) \\ I_{(B_{1i}+B_{2i})}(qr) = I_{(B_{1i} \cup B_{2i})}(qr) \\ F_{(B_{1i}+B_{2i})}(qr) = F_{(B_{1i} \cup B_{2i})}(qr) \end{cases}$

for all $qr \in E_{1i} \cup E_{2i}$,

(iii) $\begin{cases} T_{(B_{1i}+B_{2i})}(qr) = (T_{B_{1i}} + T_{B_{2i}})(qr) = T_{A_1}(q) \wedge T_{A_2}(r) \\ I_{(B_{1i}+B_{2i})}(qr) = (I_{B_{1i}} + I_{B_{2i}})(qr) = I_{A_1}(q) \wedge I_{A_2}(r) \\ F_{(B_{1i}+B_{2i})}(qr) = (F_{B_{1i}} + F_{B_{2i}})(qr) = F_{A_1}(q) \vee F_{A_2}(r) \end{cases}$

for all $q \in X_1, r \in X_2$.

Example 2.22 Join of two neutrosophic graph structures G_1 and G_2 shown in Fig. 2.14 is defined as $G_1 + G_2 = \{A_1 + A_2, B_{11} + B_{21}, B_{12} + B_{22}\}$ and is shown in the Fig. 2.21.

Theorem 2.10 *The join $G_1 + G_2 = (A_1 + A_2, B_{11} + B_{21}, B_{12} + B_{22}, \ldots, B_{1n} + B_{2n})$ of two neutrosophic graph structures of the graph structures G_1^* and G_2^* is a neutrosophic graph structure of $G_1^* + G_2^*$.*

Proof Let $q_1q_2 \in E_{1i} + E_{2i}$. Here we consider three cases:

Case 1. When $q_1, q_2 \in X_1$, then according to Definition 2.38, $T_{A_2}(q_1) = T_{A_2}(q_2) = T_{B_{2i}}(q_1q_2) = 0$, $I_{A_2}(q_1) = I_{A_2}(q_2) = I_{B_{2i}}(q_1q_2) = 0$, $F_{A_2}(q_1) = F_{A_2}(q_2) = F_{B_{2i}}(q_1q_2) = 0$, so,

$$T_{(B_{1i}+B_{2i})}(q_1q_2) = T_{B_{1i}}(q_1q_2) \vee T_{B_{2i}}(q_1q_2)$$

$$
\begin{aligned}
&= T_{B_{1i}}(q_1q_2) \vee 0\\
&\leq [T_{A_1}(q_1) \wedge T_{A_1}(q_2)] \ \vee 0\\
&= [T_{A_1}(q_1) \vee 0] \wedge [T_{A_1}(q_2) \vee 0]\\
&= [T_{A_1}(q_1) \vee T_{A_2}(q_1)] \wedge [T_{A_1}(q_2) \vee T_{A_2}(q_2)]\\
&= T_{(A_1+A_2)}(q_1) \wedge T_{(A_1+A_2)}(q_2),
\end{aligned}
$$

$$
\begin{aligned}
I_{(B_{1i}+B_{2i})}(q_1q_2) &= I_{B_{1i}}(q_1q_2) \vee I_{B_{2i}}(q_1q_2)\\
&= I_{B_{1i}}(q_1q_2) \vee 0\\
&\leq [I_{A_1}(q_1) \wedge I_{A_1}(q_2)] \ \vee 0\\
&= [I_{A_1}(q_1) \vee 0] \wedge [I_{A_1}(q_2) \vee 0]\\
&= [I_{A_1}(q_1) \vee I_{A_2}(q_1)] \wedge [I_{A_1}(q_2) \vee I_{A_2}(q_2)]\\
&= I_{(A_1+A_2)}(q_1) \wedge I_{(A_1+A_2)}(q_2),
\end{aligned}
$$

$$
\begin{aligned}
F_{(B_{1i}+B_{2i})}(q_1q_2) &= F_{B_{1i}}(q_1q_2) \wedge F_{B_{2i}}(q_1q_2)\\
&= F_{B_{1i}}(q_1q_2) \wedge 0\\
&\leq [F_{A_1}(q_1) \vee F_{A_1}(q_2)] \ \wedge 0\\
&= [F_{A_1}(q_1) \wedge 0] \vee [F_{A_1}(q_2) \wedge 0]\\
&= [F_{A_1}(q_1) \wedge F_{A_2}(q_1)] \vee [F_{A_1}(q_2) \wedge F_{A_2}(q_2)]\\
&= F_{(A_1+A_2)}(q_1) \vee F_{(A_1+A_2)}(q_2),
\end{aligned}
$$

for $q_1, q_2 \in X_1 + X_2$.

Case 2. When $q_1, q_2 \in X_2$, then according to Definition 2.38, $T_{A_1}(q_1) = T_{A_1}(q_2) = T_{B_{1i}}(q_1q_2) = 0$, $I_{A_1}(q_1) = I_{A_1}(q_2) = I_{B_{1i}}(q_1q_2) = 0$, $F_{A_1}(q_1) = F_{A_1}(q_2) = F_{B_{1i}}(q_1q_2) = 0$, so

$$
\begin{aligned}
T_{(B_{1i}+B_{2i})}(q_1q_2) &= T_{B_{1i}}(q_1q_2) \vee T_{B_{2i}}(q_1q_2)\\
&= T_{B_{2i}}(q_1q_2) \vee 0\\
&\leq [T_{A_2}(q_1) \wedge T_{A_2}(q_2)] \ \vee 0\\
&= [T_{A_2}(q_1) \vee 0] \wedge [T_{A_2}(q_2) \vee 0]\\
&= [T_{A_1}(q_1) \vee T_{A_2}(q_1)] \wedge [T_{A_1}(q_2) \vee T_{A_2}(q_2)]\\
&= T_{(A_1+A_2)}(q_1) \wedge T_{(A_1+A_2)}(q_2),
\end{aligned}
$$

$$
\begin{aligned}
I_{(B_{1i}+B_{2i})}(q_1q_2) &= I_{B_{1i}}(q_1q_2) \vee I_{B_{2i}}(q_1q_2)\\
&= I_{B_{2i}}(q_1q_2) \vee 0\\
&\leq [I_{A_2}(q_1) \wedge I_{A_2}(q_2)] \ \vee 0\\
&= [I_{A_2}(q_1) \vee 0] \wedge [I_{A_2}(q_2) \vee 0]\\
&= [I_{A_1}(q_1) \vee I_{A_2}(q_1)] \wedge [I_{A_1}(q_2) \vee I_{A_2}(q_2)]\\
&= I_{(A_1+A_2)}(q_1) \wedge I_{(A_1+A_2)}(q_2),
\end{aligned}
$$

$$\begin{aligned}
F_{(B_{1i}+B_{2i})}(q_1q_2) &= F_{B_{1i}}(q_1q_2) \wedge F_{B_{2i}}(q_1q_2) \\
&= F_{B_{2i}}(q_1q_2) \wedge 0 \\
&\leq [F_{A_2}(q_1) \vee F_{A_2}(q_2)] \wedge 0 \\
&= [F_{A_2}(q_1) \wedge 0] \vee [F_{A_2}(q_2) \wedge 0] \\
&= [F_{A_1}(q_1) \wedge F_{A_2}(q_1)] \vee [F_{A_1}(q_2) \wedge F_{A_2}(q_2)] \\
&= F_{(A_1+A_2)}(q_1) \vee F_{(A_1+A_2)}(q_2),
\end{aligned}$$

for $q_1, q_2 \in X_1 + X_2$.

Case 3. When $q_1 \in X_1, q_2 \in X_2$, then according to Definition 2.38, $T_{A_1}(q_2) = T_{A_2}(q_1) = 0$, $I_{A_1}(q_2) = I_{A_2}(q_1) = 0$, $F_{A_1}(q_2) = F_{A_2}(q_1) = 0$, so

$$\begin{aligned}
T_{(B_{1i}+B_{2i})}(q_1q_2) &= T_{A_1}(q_1) \wedge T_{A_2}(q_2) \\
&= [T_{A_1}(q_1) \vee 0] \wedge [T_{A_2}(q_2) \vee 0] \\
&= [T_{A_1}(q_1) \vee T_{A_2}(q_1)] \wedge [T_{A_2}(q_2) \vee T_{A_1}(q_2)] \\
&= T_{(A_1+A_2)}(q_1) \wedge T_{(A_1+A_2)}(q_2),
\end{aligned}$$

$$\begin{aligned}
I_{(B_{1i}+B_{2i})}(q_1q_2) &= I_{A_1}(q_1) \wedge I_{A_2}(q_2) \\
&= [I_{A_1}(q_1) \vee 0] \wedge [I_{A_2}(q_2) \vee 0] \\
&= [I_{A_1}(q_1) \vee I_{A_2}(q_1)] \wedge [I_{A_2}(q_2) \vee I_{A_1}(q_2)] \\
&= I_{(A_1+A_2)}(q_1) \wedge I_{(A_1+A_2)}(q_2),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1i}+B_{2i})}(q_1q_2) &= F_{A_1}(q_1) \vee F_{A_2}(q_2) \\
&= [F_{A_1}(q_1) \wedge 0] \vee [F_{A_2}(q_2) \wedge 0] \\
&= [F_{A_1}(q_1) \wedge F_{A_2}(q_1)] \vee [F_{A_2}(q_2) \wedge F_{A_1}(q_2)] \\
&= F_{(A_1+A_2)}(q_1) \vee F_{(A_1+A_2)}(q_2),
\end{aligned}$$

for $q_1, q_2 \in X_1 + X_2$.

All cases are satisfied $\forall i \in \{1, 2, \ldots, n\}$.

Theorem 2.11 *If $G^* = (X_1 + X_2, E_{11} + E_{21}, E_{12} + E_{22}, \ldots, E_{1n} + E_{2n})$ is join of two graph structures $G_1^* = (X_1, E_{11}, E_{12}, \ldots, E_{1n})$ and $G_2^* = (X_2, E_{21}, E_{22}, \ldots, E_{2n})$. Then every strong neutrosophic graph structure $G = (A, B_1, B_2, \ldots, B_n)$ of G is join of two strong neutrosophic graph structures $G_1 = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ and $G_2 = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ of graph structures G_1^* and G_2^*, respectively.*

Proof First we define A_k and B_{ki} for $k = 1, 2$ and $i = 1, 2, \ldots, n$ as:
$T_{A_k}(q) = T_A(q)$, $I_{A_k}(q) = I_A(q)$, $F_{A_k}(q) = F_A(q)$, if $q \in X_k$
$T_{B_{ki}}(q_1q_2) = T_{B_i}(q_1q_2)$, $I_{B_{ki}}(q_1q_2) = I_{B_i}(q_1q_2)$, $F_{B_{ki}}(q_1q_2) = F_{B_i}(q_1q_2)$, if $q_1q_2 \in E_{ki}$

Now for $q_1q_2 \in E_{ki}, k = 1, 2, i = 1, 2, \ldots, n$
$T_{B_{ki}}(q_1q_2) = T_{B_i}(q_1q_2) = T_A(q_1) \wedge T_A(q_2) = T_{A_k}(q_1) \wedge T_{A_k}(q_2)$,
$I_{B_{ki}}(q_1q_2) = I_{B_i}(q_1q_2) = I_A(q_1) \wedge I_A(q_2) = I_{A_k}(q_1) \wedge I_{A_k}(q_2)$,
$F_{B_{ki}}(q_1q_2) = F_{B_i}(q_1q_2) = F_A(q_1) \vee T_A(q_2) = T_{A_k}(q_1) \vee T_{A_k}(q_2)$,
i.e.
$G_k = (A_k, B_{k1}, B_{k2}, \ldots, B_{kn})$ is a strong neutrosophic graph structure of G_k^*, k = 1,2.

Moreover, G is join of G_1 and G_2 as shown:

Using Definitions 2.37 and 2.38, $A = A_1 \cup A_2 = A_1 + A_2$ and $B_i = B_{1i} \cup B_{2i} = B_{1i} + B_{2i}, \forall q_1q_2 \in E_{1i} \cup E_{2i}$.

When $q_1q_2 \in E_{1i} + E_{2i}$ $(E_{1i} \cup E_{2i})$, i.e. $q_1 \in X_1$ and $q_2 \in X_2$
$T_{B_i}(q_1q_2) = T_A(q_1) \wedge T_A(q_2) = T_{A_k}(q_1) \wedge T_{A_k}(q_2) = T_{(B_{1i}+B_{2i})}(q_1q_2)$,
$I_{B_i}(q_1q_2) = I_A(q_1) \wedge I_A(q_2) = I_{A_k}(q_1) \wedge I_{A_k}(q_2) = I_{(B_{1i}+B_{2i})}(q_1q_2)$,
$F_{B_i}(q_1q_2) = F_A(q_1) \vee F_A(q_2) = F_{A_k}(q_1) \vee F_{A_k}(q_2) = F_{(B_{1i}+B_{2i})}(q_1q_2)$,
Calculations are similar when $q_1 \in X_2, q_2 \in X_1$. It is true when $i = 1, 2, \ldots, n$. This completes the proof.

2.4 Applications of Neutrosophic Graph Structures

Graph structures are amazing source of graph-theoretical notions to represent the most prominent relations between objects. But these graph structures do not represent all real-world relations. Therefore, fuzzy graph structures are important to represent the relations between objects of uncertain systems existing in nature. However, graph structures and fuzzy graph structures are failed to depict the most prominent relations between objects in many real-world phenomenons due to natural existence of indeterminacy or neutrality. It increases the utility of neutrosophic graph structures.

2.4.1 Detection of Crucial Crimes During Maritime Trade

Waters are very important for trade in whole world but crimes through waters are increasing day by day. Crimes held during maritime trade are in abundance but some are very crucial including human trafficking, illegal carrying of weapons, black money transfer, smuggling of precious metals, drug trafficking and smuggling of rare plants and animals. Using neutrosophic graph structure, we can easily investigate the fact that between any two countries which maritime crime is chronic and increasing rapidly with time. Moreover, we can decide which country is most sensitive for particular type of maritime crimes. We consider a set X consisting of eight countries.

X={Bangladesh, Malaysia, Singapore, United Arab Emirates, Pakistan, India, Kenya, Italy}. Let A be the neutrosophic set on X, defined in Table 2.1.

In Table 2.1, T depicts the importance of that particular country in the world due to its geographic position, F indicates the degree of its nonimportance in the world,

Table 2.1 Neutrosophic set A of eight countries

Country	T	I	F
Bangladesh	0.8	0.7	0.6
Malaysia	0.7	0.7	0.8
Singapore	0.9	0.5	0.5
United Arab Emirates	1.0	0.5	0.6
Pakistan	0.9	0.5	0.5
India	0.8	0.7	0.7
Kenya	0.7	0.6	0.7
Italy	0.9	0.6	0.5

Table 2.2 Neutrosophic set of crimes between Pakistan and other countries during maritime trade

Type of crime	(P, UAE)	(P, B)	(P, M)	(P, S)
Human trafficking	(0.7, 0.4, 0.5)	(0.8, 0.3, 0.4)	(0.7, 0.4, 0.2)	(0.6, 0.4, 0.2)
Illegal carrying of weapons	(0.6, 0.3, 0.6)	(0.7, 0.3, 0.4)	(0.4, 0.5, 0.5)	(0.4, 0.3, 0.5)
Black money transfer	(0.6, 0.3, 0.2)	(0.7, 0.5, 0.4)	(0.2, 0.4, 0.3)	(0.9, 0.2, 0.2)
Smuggling of precious metals	(0.8, 0.3, 0.2)	(0.6, 0.3, 0.3)	(0.2, 0.4, 0.3)	(0.8, 0.5, 0.5)
Drug trafficking	(0.7, 0.3, 0.3)	(0.5, 0.4, 0.3)	(0.6, 0.5, 0.6)	(0.8, 0.4, 0.3)
Smuggling of rare plants and animals	(0.3, 0.5, 0.5)	(0.4, 0.3, 0.4)	(0.4, 0.4, 0.5)	(0.2, 0.3, 0.3)

and I expresses, to which extent it is undecided/indeterminate to be beneficial for the world, geographically.

Let Bangladesh = B, Malaysia = M, Singapore = S, United Arab Emirates = UAE, Pakistan = P, India = I, Kenya = K, Italy = IT.

In Tables 2.2, 2.3, 2.4, 2.5, 2.6, 2.7 and 2.8, we have shown the values of T, I and F of different crimes for each pair of countries.

Many relations on set X can be defined, let we define six relations on X as: E_1 = Human trafficking, E_2 = Illegal carrying of weapons, E_3 = Black money transfer, E_4 = Smuggling of precious metals, E_5 = Drug trafficking, E_6 = Smuggling of rare plants and animals, such that $(X, E_1, E_2, E_3, E_4, E_5, E_6)$ is a graph structure. An element in a relation detects that kind of crime during maritime trade between those two countries.

As $(X, E_1, E_2, E_3, E_4, E_5, E_6)$ is a graph structure, an element will not be in more than one relations, so it can appear just once. Therefore, we will consider it an element of that relation for which its percentage of truth is high, and percentage of both falsity and indeterminacy is low as compared to other relations.

Table 2.3 Neutrosophic set of crimes between UAE and other countries during maritime trade

Type of crime	(UAE, B)	(UAE, M)	(UAE, S)	(UAE, I)
Human trafficking	(0.7, 0.3, 0.4)	(0.6, 0.2, 0.5)	(0.3, 0.2, 0.5)	(0.6, 0.4, 0.2)
Illegal carrying of weapons	(0.5, 0.2, 0.2)	(0.5, 0.3, 0.2)	(0.4, 0.3, 0.5)	(0.4, 0.3, 0.5)
Black money transfer	(0.6, 0.3, 0.3)	(0.6, 0.2, 0.3)	(0.6, 0.2, 0.3)	(0.6, 0.4, 0.5)
Smuggling of precious metals	(0.6, 0.2, 0.2)	(0.6, 0.3, 0.3)	(0.6, 0.3, 0.3)	(0.8, 0.3, 0.2)
Drug trafficking	(0.6, 0.2, 0.2)	(0.5, 0.4, 0.3)	(0.7, 0.3, 0.2)	(0.7, 0.4, 0.3)
Smuggling of rare plants and animals	(0.3, 0.4, 0.4)	(0.4, 0.3, 0.4)	(0.4, 0.2, 0.5)	(0.3, 0.3, 0.3)

Table 2.4 Neutrosophic set of crimes between Bangladesh and other countries during maritime trade

Type of crime	(B, M)	(B, S)	(B, I)	(B, K)
Human trafficking	(0.6, 0.3, 0.4)	(0.8, 0.3, 0.2)	(0.5, 0.2, 0.5)	(0.6, 0.4, 0.5)
Illegal carrying of weapons	(0.5, 0.2, 0.5)	(0.5, 0.3, 0.2)	(0.7, 0.3, 0.5)	(0.4, 0.3, 0.5)
Black money transfer	(0.4, 0.2, 0.2)	(0.7, 0.4, 0.3)	(0.1, 0.1, 0.2)	(0.1, 0.3, 0.4)
Smuggling of precious metals	(0.4, 0.2, 0.2)	(0.6, 0.3, 0.3)	(0.2, 0.3, 0.3)	(0.2, 0.2, 0.4)
Drug trafficking	(0.6, 0.2, 0.2)	(0.5, 0.4, 0.3)	(0.6, 0.3, 0.5)	(0.5, 0.4, 0.4)
Smuggling of rare plants and animals	(0.2, 0.3, 0.3)	(0.3, 0.2, 0.3)	(0.2, 0.1, 0.4)	(0.5, 0.2, 0.2)

According to given data, we write the elements in relation to their truth, falsity and indeterminacy values, resulting sets are neutrosophic sets on E_1, E_2, E_3, E_4, E_5, E_6, respectively. We can name these sets as B_1, B_2, B_3, B_4, B_5, B_6, respectively. Let
$E_1 = \{(Bangladesh, Pakistan), (Malaysia, Pakistan), (Bangladesh, Singapore)\}$,
$E_2 = \{(Pakistan, India)\}$,
$E_3 = \{(Singapore, Pakistan)\}$,
$E_4 = \{(India, Singapore), (UnitedArabEmirates, India)\}$,
$E_5 = \{(Italy, Pakistan), (India, Italy)\}$,
$E_6 = \{(Kenya, Singapore)\}$.
And corresponding neutrosophic sets are:

Table 2.5 Neutrosophic set of crimes between Malaysia and other countries during maritime trade

Type of crime	(M, S)	(M, I)	(M, K)	(M, IT)
Human trafficking	(0.5, 0.3, 0.4)	(0.6, 0.2, 0.3)	(0.3, 0.2, 0.5)	(0.6, 0.4, 0.5)
Illegal carrying of weapons	(0.6, 0.2, 0.2)	(0.5, 0.3, 0.2)	(0.4, 0.3, 0.5)	(0.4, 0.3, 0.5)
Black money transfer	(0.6, 0.3, 0.3)	(0.2, 0.2, 0.3)	(0.2, 0.2, 0.3)	(0.2, 0.4, 0.5)
Smuggling of precious metals	(0.6, 0.2, 0.2)	(0.6, 0.3, 0.3)	(0.2, 0.3, 0.3)	(0.2, 0.2, 0.6)
Drug trafficking	(0.5, 0.2, 0.2)	(0.5, 0.4, 0.3)	(0.4, 0.3, 0.6)	(0.7, 0.4, 0.2)
Smuggling of rare plants and animals	(0.3, 0.4, 0.4)	(0.4, 0.3, 0.4)	(0.6, 0.2, 0.2)	(0.5, 0.3, 0.3)

Table 2.6 Neutrosophic set of crimes between Singapore and other countries during maritime trade

Type of crime	(S, I)	(S, K)	(S, IT)	(P, I)
Human trafficking	(0.5, 0.3, 0.4)	(0.3, 0.2, 0.5)	(0.3, 0.2, 0.5)	(0.6, 0.4, 0.6)
Illegal carrying of weapons	(0.7, 0.4, 0.5)	(0.5, 0.3, 0.2)	(0.4, 0.3, 0.5)	(0.8, 0.2, 0.4)
Black money transfer	(0.5, 0.3, 0.4)	(0.6, 0.2, 0.3)	(0.6, 0.2, 0.3)	(0.7, 0.4, 0.5)
Smuggling of precious metals	(0.8, 0.3, 0.7)	(0.6, 0.3, 0.3)	(0.6, 0.3, 0.3)	(0.6, 0.2, 0.4)
Drug trafficking	(0.7, 0.3, 0.4)	(0.5, 0.4, 0.3)	(0.6, 0.3, 0.2)	(0.8, 0.4, 0.4)
Smuggling of rare plants and animals	(0.7, 0.5, 0.6)	(0.4, 0.3, 0.4)	(0.6, 0.2, 0.5)	(0.7, 0.3, 0.3)

$B_1 = \{((B, P), 0.8, 0.2, 0.2), ((M, P), 0.7, 0.4, 0.2), ((B, S), 0.8, 0.3, 0.2)\}$,
$B_2 = \{((P, I), 0.8, 0.2, 0.4)\}$,
$B_3 = \{((S, P), 0.9, 0.2, 0.2), \}$,
$B_4 = \{((I, S), 0.8, 0.3, 0.4), ((UAE, I), 0.8, 0.3, 0.2)\}$,
$B_5 = \{((IT, P), 0.9, 0.3, 0.3), ((I, IT), 0.8, 0.3, 0.3)\}$,
$B_6 = \{((K, S), 0.7, 0.2, 0.4)\}$.

Clearly, $(A, B_1, B_2, B_3, B_4, B_5, B_6)$ is a neutrosophic graph structure as shown in Fig. 2.22.

In neutrosophic graph structure shown in Fig. 2.22, every edge detects most frequent crime between adjacent countries during maritime trade. For instance, most frequent maritime crime between Pakistan and Singapore is black money transfer, its strength is 90%, weakness is 20% and indeterminacy is 20%. We can also note that for relation human trafficking, vertex Pakistan has highest vertex degree, it means

Table 2.7 Neutrosophic set of crimes between Italy and other countries during maritime trade

Type of crime	(IT, P)	(IT, UAE)	(IT, B)	(IT, I)
Human trafficking	(0.5, 0.3, 0.4)	(0.3, 0.2, 0.5)	(0.3, 0.2, 0.5)	(0.6, 0.4, 0.6)
Illegal carrying of weapons	(0.8, 0.3, 0.3)	(0.6, 0.3, 0.2)	(0.4, 0.3, 0.5)	(0.7, 0.3, 0.5)
Black money transfer	(0.6, 0.3, 0.3)	(0.5, 0.2, 0.3)	(0.2, 0.2, 0.3)	(0.5, 0.4, 0.5)
Smuggling of precious metals	(0.7, 0.3, 0.3)	(0.6, 0.3, 0.3)	(0.2, 0.3, 0.3)	(0.7, 0.3, 0.6)
Drug trafficking	(0.9, 0.3, 0.3)	(0.6, 0.4, 0.3)	(0.7, 0.3, 0.5)	(0.8, 0.3, 0.3)
Smuggling of rare plants and animals	(0.3, 0.4, 0.4)	(0.4, 0.3, 0.4)	(0.6, 0.2, 0.5)	(0.7, 0.3, 0.3)

Table 2.8 Neutrosophic set of crimes between Kenya and other countries during maritime trade

Type of crime	(K, P)	(K, UAE)	(K, I)	(K, IT)
Human trafficking	(0.5, 0.3, 0.4)	(0.6, 0.2, 0.5)	(0.5, 0.2, 0.5)	(0.6, 0.4, 0.5)
Illegal carrying of weapons	(0.6, 0.2, 0.5)	(0.5, 0.3, 0.4)	(0.5, 0.3, 0.5)	(0.4, 0.3, 0.5)
Black money transfer	(0.5, 0.3, 0.3)	(0.5, 0.2, 0.3)	(0.5, 0.2, 0.3)	(0.5, 0.4, 0.5)
Smuggling of precious metals	(0.4, 0.2, 0.2)	(0.6, 0.3, 0.3)	(0.6, 0.3, 0.3)	(0.4, 0.2, 0.4)
Drug trafficking	(0.7, 0.2, 0.2)	(0.5, 0.4, 0.3)	(0.5, 0.3, 0.5)	(0.8, 0.4, 0.2)
Smuggling of rare plants and animals	(0.3, 0.4, 0.4)	(0.7, 0.3, 0.4)	(0.6, 0.2, 0.4)	(0.7, 0.3, 0.3)

Pakistan is most sensitive country for human trafficking. Moreover, according to our neutrosophic graph structure, most frequent crime is human trafficking. It means that navy and maritime forces of these eight countries should take action to control human trafficking.

2.4.2 *Decision-Making of Prominent Relationships*

Among the countries of this world, various types of relationships exist, for example friendship, rival or enemy, religious affection, trade, political and military. Between any two countries, all relationships are not of same strength. Some relationships are comparatively stronger than other relationships. In general, it is difficult and time

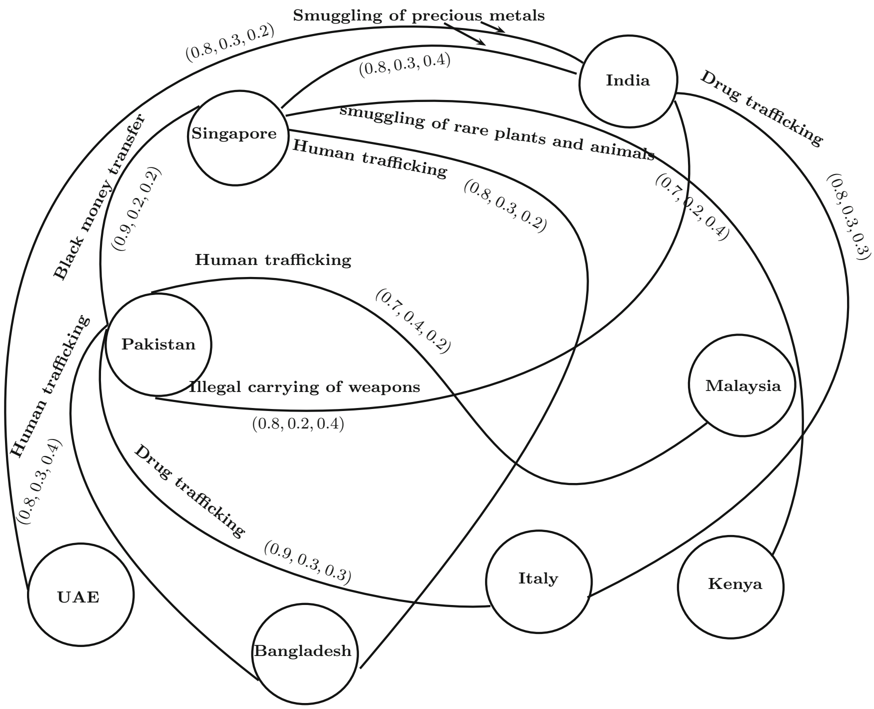


Fig. 2.22 Neutrosophic graph structure showing most crucial maritime crime between any two countries

consuming to judge all relationships among the countries and to decide the most prominent one. But through neutrosophic graph structure, we can represent all these in easiest way and can be judged even in a single glance on graph. Moreover, we can be aware of the status of relationship, that is, what is percentage of its strength, weakness and in how much percentage it is indeterminate. We can also examine which pair of countries are in same kind of relationship. We consider a set X of eight countries.

$X = \{America, Russia, China, Japan, Pakistan, India, Iran, Saudi\ Arabia\}$. Let A be the neutrosophic set on X, defined in Table 2.9.

In Table 2.9, T indicates positive impact (strength) of a particular country for whole world, F indicates negative impact (weakness), and I expresses that in what percentage or magnitude that country's position is undecided or indeterminate for global world. Let we denote the countries with alphabets: A = America, R = Russia, CH = China, J = Japan, P = Pakistan, I = India, IR = Iran, S = Saudi Arabia.

In Tables 2.10, 2.11, 2.12, 2.13, 2.14 and 2.15, we have shown the T, I and F values of different relationships for each pair of countries.

Table 2.9 Neutrosophic set A of a few countries on globe

Country	T	I	F
America	0.9	0.3	0.2
Russia	0.7	0.4	0.3
China	0.8	0.4	0.4
Japan	0.8	0.5	0.4
Pakistan	0.7	0.6	0.7
India	0.7	0.8	0.6
Iran	0.7	0.7	0.6
Saudi Arabia	0.6	0.9	0.7

Table 2.10 Neutrosophic set of relationships between America and other countries

Type of relation	(A, R)	(A, CH)	(A, P)	(A, I)	(A, IR)
Friendship	(0.0, 0.2, 0.3)	(0.2, 0.3, 0.4)	(0.2, 0.1, 0.4)	(0.5, 0.3, 0.5)	(0.1, 0.3, 0.5)
Rival or enemy	(0.7, 0.1, 0.1)	(0.8, 0.2, 0.1)	(0.3, 0.2, 0.4)	(0.3, 0.2, 0.4)	(0.5, 0.2, 0.4)
Religious affection	(0.4, 0.2, 0.2)	(0.1, 0.3, 0.2)	(0.1, 0.1, 0.2)	(0.1, 0.3, 0.4)	(0.1, 0.1, 0.2)
Trade	(0.3, 0.1, 0.1)	(0.5, 0.2, 0.2)	(0.1, 0.2, 0.2)	(0.1, 0.1, 0.5)	(0.6, 0.1, 0.3)
Politics	(0.6, 0.1, 0.1)	(0.4, 0.3, 0.2)	(0.6, 0.1, 0.1)	(0.7, 0.3, 0.2)	(0.7, 0.3, 0.1)
Military	(0.2, 0.3, 0.3)	(0.3, 0.2, 0.3)	(0.5, 0.1, 0.4)	(0.6, 0.2, 0.2)	(0.2, 0.3, 0.2)

Table 2.11 Neutrosophic set of relationships between Russia and other countries

Type of relation	(R, CH)	(R, J)	(R, P)	(R, I)	(R, IR)
Friendship	(0.5, 0.2, 0.3)	(0.5, 0.2, 0.3)	(0.3, 0.3, 0.4)	(0.4, 0.3, 0.3)	(0.1, 0.1, 0.5)
Rival or enemy	(0.6, 0.2, 0.2)	(0.6, 0.2, 0.2)	(0.3, 0.3, 0.3)	(0.2, 0.2, 0.4)	(0.4, 0.1, 0.3)
Religious affection	(0.1, 0.1, 0.4)	(0.2, 0.1, 0.3)	(0.1, 0.1, 0.4)	(0.4, 0.4, 0.3)	(0.2, 0.1, 0.5)
Trade	(0.4, 0.1, 0.3)	(0.4, 0.2, 0.3)	(0.4, 0.1, 0.4)	(0.5, 0.2, 0.3)	(0.4, 0.1, 0.3)
Politics	(0.7, 0.3, 0.4)	(0.7, 0.1, 0.3)	(0.4, 0.1, 0.3)	(0.5, 0.2, 0.3)	(0.7, 0.4, 0.5)
Military	(0.2, 0.1, 0.4)	(0.4, 0.1, 0.3)	(0.7, 0.1, 0.3)	(0.7, 0.2, 0.4)	(0.2, 0.1, 0.3)

We can define many relations on set X, let we define six relations on X as: E_1 = Friendship, E_2 = Rival or Enemy, E_3 = Religious affection, E_4 = Trade, E_5 = Politics, E_6 = Military, such that $(X, E_1, E_2, E_3, E_4, E_5, E_6)$ is a graph structure. An element in a relation indicates that these two countries have a particular relationship. As $(X, E_1, E_2, E_3, E_4, E_5, E_6)$ is a graph structure, so an element will not be in

Table 2.12 Neutrosophic set of relationships between China and other countries

Type of relation	(CH, J)	(CH, P)	(CH, I)	(CH, IR)	(CH, S)
Friendship	(0.5, 0.2, 0.3)	(0.7, 0.1, 0.1)	(0.2, 0.3, 0.6)	(0.1, 0.4, 0.6)	(0.2, 0.4, 0.6)
Rival or enemy	(0.6, 0.2, 0.2)	(0.1, 0.1, 0.7)	(0.7, 0.2, 0.2)	(0.3, 0.3, 0.6)	(0.2, 0.3, 0.5)
Religious affection	(0.1, 0.1, 0.4)	(0.3, 0.3, 0.6)	(0.4, 0.4, 0.3)	(0.2, 0.2, 0.5)	(0.1, 0.4, 0.6)
Trade	(0.1, 0.1, 0.3)	(0.6, 0.1, 0.1)	(0.4, 0.2, 0.4)	(0.7, 0.1, 0.3)	(0.5, 0.4, 0.2)
Politics	(0.8, 0.4, 0.4)	(0.2, 0.4, 0.3)	(0.6, 0.2, 0.2)	(0.7, 0.2, 0.2)	(0.6, 0.4, 0.3)
Military	(0.4, 0.2, 0.3)	(0.6, 0.2, 0.3)	(0.1, 0.4, 0.2)	(0.2, 0.4, 0.6)	(0.1, 0.4, 0.6)

Table 2.13 Neutrosophic set of relationships between Japan and other countries

Type of relation	(J, A)	(J, P)	(J, I)	(J, IR)	(J, S)
Friendship	(0.5, 0.3, 0.4)	(0.2, 0.3, 0.6)	(0.3, 0.4, 0.3)	(0.2, 0.5, 0.6)	(0.1, 0.4, 0.6)
Rival or enemy	(0.7, 0.3, 0.3)	(0.3, 0.4, 0.6)	(0.2, 0.3, 0.5)	(0.2, 0.4, 0.4)	(0.3, 0.4, 0.4)
Religious affection	(0.1, 0.3, 0.3)	(0.1, 0.4, 0.5)	(0.4, 0.4, 0.5)	(0.1, 0.5, 0.6)	(0.1, 0.4, 0.6)
Trade	(0.1, 0.3, 0.4)	(0.7, 0.3, 0.2)	(0.7, 0.2, 0.1)	(0.6, 0.4, 0.6)	(0.6, 0.5, 0.7)
Politics	(0.8, 0.3, 0.3)	(0.6, 0.4, 0.2)	(0.6, 0.5, 0.2)	(0.6, 0.3, 0.1)	(0.4, 0.3, 0.4)
Military	(0.2, 0.3, 0.3)	(0.4, 0.4, 0.4)	(0.5, 0.4, 0.3)	(0.2, 0.4, 0.6)	(0.1, 0.4, 0.6)

Table 2.14 Neutrosophic set of relationships between Saudi Arabia and other countries

Type of relation	(I, IR)	(S, I)	(S, IR)	(S, A)	(S, R)
Friendship	(0.2, 0.4, 0.4)	(0.1, 0.7, 0.6)	(0.2, 0.4, 0.6)	(0.4, 0.3, 0.6)	(0.2, 0.2, 0.6)
Rival or enemy	(0.6, 0.3, 0.6)	(0.5, 0.4, 0.5)	(0.5, 0.4, 0.4)	(0.4, 0.2, 0.5)	(0.4, 0.2, 0.4)
Religious affection	(0.1, 0.4, 0.6)	(0.3, 0.4, 0.6)	(0.6, 0.4, 0.2)	(0.1, 0.1, 0.7)	(0.2, 0.1, 0.6)
Trade	(0.4, 0.4, 0.5)	(0.1, 0.4, 0.6)	(0.3, 0.4, 0.6)	(0.2, 0.1, 0.6)	(0.1, 0.1, 0.3)
Politics	(0.7, 0.4, 0.2)	(0.3, 0.4, 0.6)	(0.6, 0.4, 0.6)	(0.6, 0.2, 0.3)	(0.6, 0.4, 0.6)
Military	(0.2, 0.5, 0.6)	(0.1, 0.4, 0.6)	(0.2, 0.3, 0.7)	(0.1, 0.1, 0.7)	(0.2, 0.1, 0.5)

more than one relation. So, we will put it in that relation for which percentage of truth is high, percentage of both falsity and indeterminacy is low as compared to other relationships, using above-mentioned data.

We write the elements in relations with their truth, falsity and indeterminacy values according to given data, resulting sets are neutrosophic sets on E_1, E_2, E_3, E_4, E_5,

Table 2.15 Neutrosophic set of relationships between Pakistan and other countries

Type of relation	(P, I)	(P, IR)	(P, S)
Friendship	(0.1, 0.4, 0.6)	(0.5, 0.4, 0.5)	(0.5, 0.1, 0.1)
Rival or enemy	(0.7, 0.1, 0.1)	(0.4, 0.4, 0.5)	(0.3, 0.6, 0.6)
Religious affection	(0.4, 0.4, 0.6)	(0.7, 0.4, 0.5)	(0.6, 0.1, 0.1)
Trade	(0.3, 0.3, 0.6)	(0.4, 0.4, 0.5)	(0.3, 0.2, 0.6)
Politics	(0.6, 0.2, 0.2)	(0.5, 0.4, 0.5)	(0.2, 0.4, 0.5)
Military	(0.1, 0.2, 0.6)	(0.2, 0.4, 0.6)	(0.1, 0.4, 0.6)

E_6, respectively. We can name these sets as B_1, B_2, B_3, B_4, B_5, B_6, respectively. Let
$B_1 = \{((P, CH), 0.7, 0.1, 0.1)\}$,
$B_2 = \{((P, I), 0.7, 0.1, 0.1), ((A, R), 0.7, 0.1, 0.1), ((A, CH), 0.8, 0.2, 0.1), ((I, CH), 0.7, 0.2, 0.2)\}$,
$B_3 = \{((P, S), 0.6, 0.1, 0.1), ((P, IR), 0.7, 0.4, 0.5)\}$,
$B_4 = \{((P, J), 0.7, 0.3, 0.2), ((I, J), 0.7, 0.2, 0.1)\}$,
$B_5 = \{((P, A), 0.6, 0.1, 0.1), ((A, I), 0.7, 0.3, 0.2), ((A, S), 0.6, 0.2, 0.3), ((A, IR), 0.7, 0.3, 0.1), ((A, J), 0.8, 0.3, 0.3)\}$,
$B_6 = \{((P, R), 0.7, 0.1, 0.3), ((R, I), 0.7, 0.2, 0.4)\}$.

Clearly, $(A, B_1, B_2, B_3, B_4, B_5, B_6)$ is a neutrosophic graph structure as shown in Fig. 2.23.

In neutrosophic graph structure shown in Fig. 2.23, every edge indicates the most prominent relationship of adjacent vertices(countries), for example most prominent relationship between Pakistan and China is friendship, it is 70% strong, 10% weak and 10% indeterminate. It can be noted that for the relation politics, vertex America has highest degree, it shows that America is the most prominent country for having political relationship with other countries in A. Further, we can tell that China and India, America and Russia, Pakistan and India have common relationship, that is, they are rival or enemy of each other. Moreover, according to our neutrosophic graph structure most frequent relation is politics, it means that among these eight countries politics is dominating relationship.

This neutrosophic graph structure depicts most prominent relationships among some elements (countries) of A. By taking large neutrosophic graph structure, most dominating relationships among all the countries of A can be detected. On the similar basis, we can make a neutrosophic graph structure for all countries across the world, in order to find the status and strength of prominent relationships among them. From neutrosophic graph structure, we can also determine that which pair of countries have common relationships. Further, we can find which country is most prominent for having a particular kind of relationship with other countries. Most frequent relationship in the neutrosophic graph structure will indicate that this relationship is prevailing in the world. So, using neutrosophic graph structure, it is quite easy to judge, in which direction this world is moving? whether it is moving towards peace or war/Cold War.

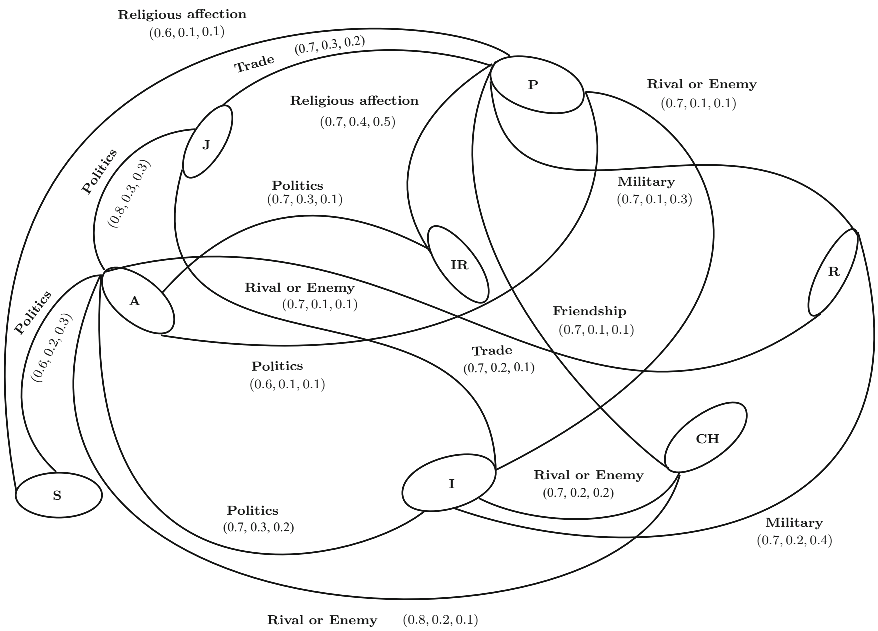


Fig. 2.23 Neutrosophic graph structure showing most prominent relationship between any two vertices(countries)

2.4.3 Detection of Most Frequent Smuggling

Smuggling on the seaports are increasing rapidly with time. There are 4,764 seaports on Atlantic ocean, Arctic ocean, Indian ocean, Pacific ocean, etc. These seaports are very useful and advantageous for import and export of different types of goods through out the world. Besides, there are also many disadvantages of these seaports. Crimes held on seaports are in abundance, but Smuggling of different kinds like human smuggling, weapons smuggling, black money smuggling, gold and diamond smuggling, smuggling of ivory and drug smuggling are most alarming. A lot of time and labour is required to collect and manipulate the data from all seaports to judge that which type of smuggling is frequent. But using neutrosophic graph structure, we can easily investigate the fact that between any two seaports which type of smuggling is chronic and increasing violently. Moreover, we can decide which seaport is most sensitive for smuggling, globally and need to be focused by security teams. We consider a set X consisting of eight seaports.

X= {Chalna, Penang, Singapore, Dubai, Karachi, Mumbai, Mombasa, Gioia Tauro}. Let A be the neutrosophic set on X, defined in Table 2.16.

In Table 2.16, T depicts the importance of that particular seaport in the world due to its geographic position, F indicates the degree of its nonimportance in the world,

Table 2.16 Neutrosophic set A of eight seaports

Country	T	I	F
Chalna	0.7	0.6	0.5
Penang	0.6	0.6	0.7
Singapore	0.8	0.4	0.4
Dubai	0.9	0.4	0.5
Karachi	0.8	0.4	0.4
Mumbai	0.7	0.6	0.6
Mombasa	0.6	0.5	0.6
Gioia Tauro	0.8	0.5	0.4

Table 2.17 Neutrosophic set of smuggling between Karachi and other seaports

Type of smuggling	(K, DU)	(K, C)	(K, P)	(K, S)
Human smuggling	(0.6, 0.3, 0.4)	(0.7, 0.2, 0.3)	(0.6, 0.3, 0.1)	(0.5, 0.3, 0.1)
Weapons smuggling	(0.5, 0.2, 0.5)	(0.6, 0.2, 0.3)	(0.3, 0.4, 0.4)	(0.3, 0.2, 0.4)
Black money smuggling	(0.5, 0.2, 0.1)	(0.6, 0.4, 0.3)	(0.1, 0.3, 0.2)	(0.8, 0.1, 0.1)
Gold and diamond smuggling	(0.7, 0.2, 0.1)	(0.5, 0.2, 0.2)	(0.1, 0.3, 0.2)	(0.7, 0.4, 0.4)
Drug smuggling	(0.6, 0.2, 0.2)	(0.4, 0.3, 0.2)	(0.5, 0.4, 0.5)	(0.7, 0.3, 0.2)
Smuggling of ivory	(0.2, 0.4, 0.4)	(0.3, 0.2, 0.3)	(0.3, 0.3, 0.4)	(0.1, 0.2, 0.2)

and I expresses, to which extent it is undecided/indeterminate to be beneficial for the world, geographically.

Let Chalna = C, Pengang = P, Singapore = S, Dubai = DU, Karachi = K, Mumbai = MU, Mombasa = MO, Gioia Tauro = GT.

In Tables 2.17, 2.18, 2.19, 2.20, 2.21, 2.22 and 2.23, we have shown the values of T, I and F of different smuggling for each pair of seaports.

Many relations on set X can be defined, let we define six relations on X as: E_1 = Human smuggling, E_2 = Weapons smuggling, E_3 = Black money smuggling, E_4 = Gold and diamond smuggling, E_5 = Drug smuggling, E_6 = Smuggling of ivory, such that $(X, E_1, E_2, E_3, E_4, E_5, E_6)$ is a graph structure. An element in a relation detects that kind of smuggling between those two seaports.

As $(X, E_1, E_2, E_3, E_4, E_5, E_6)$ is a graph structure, an element will not be in more than one relations, so it can appear just once. Therefore, we will consider it an element of that relation for which its percentage of truth is high, and percentage of both falsity and indeterminacy is low as compared to other relations.

Table 2.18 Neutrosophic set of smuggling between Dubai and other seaports

Type of smuggling	(DU, C)	(DU, P)	(DU, S)	(DU, MU)
Human smuggling	(0,6, 0.2, 0.3)	(0.5, 0.1, 0.4)	(0.2, 0.1, 0.4)	(0.5, 0.3, 0.1)
Weapons smuggling	(0.4, 0.1, 0.1)	(0.4, 0.2, 0.1)	(0.3, 0.2, 0.4)	(0.3, 0.2, 0.4)
Black money smuggling	(0.5, 0.2, 0.2)	(0.5, 0.1, 0.2)	(0.5, 0.1, 0.2)	(0.5, 0.3, 0.4)
Gold and diamond smuggling	(0.5, 0.1, 0.1)	(0.5, 0.2, 0.2)	(0.5, 0.2, 0.2)	(0.7, 0.2, 0.1)
Drug smuggling	(0.5, 0.1, 0.1)	(0.4, 0.3, 0.2)	(0.6, 0.2, 0.1)	(0.6, 0.3, 0.2)
Smuggling of ivory	(0.2, 0.3, 0.3)	(0.3, 0.2, 0.3)	(0.3, 0.1, 0.4)	(0.2, 0.2, 0.2)

Table 2.19 Neutrosophic set of smuggling between Chalna and other seaports

Type of smuggling	(C, P)	(C, S)	(C, MU)	(C, MO)
Human smuggling	(0.5, 0.2, 0.3)	(0.7, 0.2, 0.1)	(0.4, 0.1, 0.4)	(0.5, 0.3, 0.4)
Weapons smuggling	(0.4, 0.1, 0.4)	(0.4, 0.2, 0.1)	(0.6, 0.2, 0.4)	(0.3, 0.2, 0.4)
Black money smuggling	(0.4, 0.2, 0.2)	(0.7, 0.4, 0.3)	(0.1, 0.1, 0.2)	(0.1, 0.3, 0.4)
Gold and diamond smuggling	(0.3, 0.1, 0.1)	(0.5, 0.2, 0.2)	(0.1, 0.2, 0.2)	(0.1, 0.1, 0.3)
Drug smuggling	(0.5, 0.1, 0.1)	(0.4, 0.3, 0.2)	(0.5, 0.2, 0.4)	(0.4, 0.3, 0.3)
Smuggling of ivory	(0.2, 0.3, 0.3)	(0.3, 0.2, 0.3)	(0.2, 0.1, 0.4)	(0.5, 0.2, 0.2)

According to given data, we write the elements in relations with their truth, falsity and indeterminacy values, so the resulting sets are neutrosophic sets on E_1, E_2, E_3, E_4, E_5, E_6, respectively. We can name these sets as B_1, B_2, B_3, B_4, B_5, B_6, respectively. Let
$E_1 = \{(Chalna, Karachi), (Penang, Karachi), (Chalna, Singapore)\}$,
$E_2 = \{(Karachi, Mumbai)\}$,
$E_3 = \{(Singapore, Karachi)\}$,
$E_4 = \{(Mumbai, Singapore), (Dubai, Mumbai)\}$,
$E_5 = \{(GioiaTauro, Karachi), (Mumbai,$Gioia Tauro$)\}$,
$E_6 = \{(Mombasa, Singapore)\}$.
And corresponding neutrosophic sets are:

Table 2.20 Neutrosophic set of smuggling between Penang and other seaports

Type of smuggling	(P, S)	(P, MU)	(P, MO)	(P, GT)
Human smuggling	(0.4, 0.2, 0.3)	(0.5, 0.1, 0.2)	(0.2, 0.1, 0.4)	(0.5, 0.3, 0.4)
Weapons smuggling	(0.5, 0.1, 0.1)	(0.4, 0.2, 0.1)	(0.3, 0.2, 0.4)	(0.3, 0.2, 0.4)
Black money smuggling	(0.5, 0.2, 0.2)	(0.1, 0.1, 0.2)	(0.1, 0.1, 0.2)	(0.1, 0.3, 0.4)
Gold and diamond smuggling	(0.5, 0.1, 0.1)	(0.5, 0.2, 0.2)	(0.1, 0.2, 0.2)	(0.1, 0.1, 0.5)
Drug smuggling	(0.4, 0.1, 0.1)	(0.4, 0.3, 0.2)	(0.3, 0.2, 0.5)	(0.6, 0.3, 0.1)
Smuggling of ivory	(0.2, 0.3, 0.3)	(0.3, 0.2, 0.3)	(0.5, 0.1, 0.1)	(0.4, 0.2, 0.2)

Table 2.21 Neutrosophic set of smuggling between Singapore and other seaports

Type of smuggling	(S, MU)	(S, MO)	(S, GT)	(K, MU)
Human smuggling	(0.4, 0.2, 0.3)	(0.2, 0.1, 0.4)	(0.2, 0.1, 0.4)	(0.5, 0.3, 0.5)
Weapons smuggling	(0.6, 0.3, 0.4)	(0.4, 0.2, 0.3)	(0.3, 0.2, 0.4)	(0.7, 0.1, 0.3)
Black money smuggling	(0.4, 0.2, 0.3)	(0.5, 0.1, 0.3)	(0.5, 0.1, 0.2)	(0.6, 0.3, 0.4)
Gold and diamond smuggling	(0.7, 0.2, 0.6)	(0.5, 0.2, 0.4)	(0.5, 0.2, 0.2)	(0.5, 0.1, 0.3)
Drug smuggling	(0.6, 0.2, 0.3)	(0.4, 0.3, 0.4)	(0.5, 0.2, 0.1)	(0.7, 0.3, 0.3)
Smuggling of ivory	(0.6, 0.4, 0.5)	(0.6, 0.1, 0.3)	(0.5, 0.1, 0.4)	(0.6, 0.2, 0.2)

$B_1 = \{((C, K), 0.7, 0.2, 0.3), ((P, K), 0.6, 0.3, 0.1), ((C, S), 0.7, 0.2, 0.1)\}$,
$B_2 = \{((K, MU), 0.7, 0.1, 0.3)\}$,
$B_3 = \{((S, K), 0.8, 0.1, 0.1), \}$,
$B_4 = \{((MU, S), 0.7, 0.2, 0.3),\ ((DU, MU), 0.7, 0.2, 0.1)\}$,
$B_5 = \{((GT, K), 0.8, 0.2, 0.2),\ ((MU, GT), 0.7, 0.2, 0.2)\}$,
$B_6 = \{((MO, S), 0.6, 0.1, 0.3)\}$.

Clearly, $(A, B_1, B_2, B_3, B_4, B_5, B_6)$ is a neutrosophic graph structure as shown in Fig. 2.24.

In neutrosophic graph structure shown in Fig. 2.24, every edge detects most frequent smuggling between adjacent seaports. For instance, most frequent smuggling between Karachi and Singapore is black money smuggling, its strength is 80%, weak-

Table 2.22 Neutrosophic set of smuggling between Gioia Tauro and other seaports

Type of smuggling	(GT, K)	(GT, DU)	(GT, C)	(GT, MU)
Human smuggling	(0.4, 0.2, 0.3)	(0.2, 0.1, 0.4)	(0.2, 0.1, 0.4)	(0.5, 0.3, 0.5)
Weapons smuggling	(0.7, 0.2, 0.2)	(0.5, 0.2, 0.1)	(0.3, 0.2, 0.4)	(0.6, 0.2, 0.4)
Black money smuggling	(0.5, 0.2, 0.2)	(0.4, 0.1, 0.2)	(0.1, 0.1, 0.2)	(0.4, 0.3, 0.4)
Gold and diamond smuggling	(0.6, 0.2, 0.2)	(0.5, 0.2, 0.2)	(0.1, 0.2, 0.2)	(0.6, 0.2, 0.5)
Drug smuggling	(0.8, 0.2, 0.2)	(0.5, 0.3, 0.2)	(0.6, 0.2, 0.4)	(0.7, 0.2, 0.2)
Smuggling of ivory	(0.2, 0.3, 0.3)	(0.3, 0.2, 0.3)	(0.5, 0.1, 0.4)	(0.6, 0.2, 0.2)

Table 2.23 Neutrosophic set of smuggling between Mombasa and other seaports

Type of smuggling	(MO, K)	(MO, DU)	(MO, MU)	(MO, GT)
Human smuggling	(0.4, 0.2, 0.3)	(0.5, 0.1, 0.4)	(0.4, 0.1, 0.4)	(0.5, 0.3, 0.4)
Weapons smuggling	(0.5, 0.1, 0.4)	(0.4, 0.2, 0.3)	(0.4, 0.2, 0.4)	(0.3, 0.2, 0.4)
Black money smuggling	(0.4, 0.2, 0.2)	(0.4, 0.1, 0.2)	(0.4, 0.1, 0.2)	(0.4, 0.3, 0.4)
Gold and diamond smuggling	(0.3, 0.1, 0.1)	(0.5, 0.2, 0.2)	(0.5, 0.2, 0.2)	(0.3, 0.1, 0.3)
Drug smuggling	(0.6, 0.1, 0.1)	(0.4, 0.3, 0.2)	(0.4, 0.2, 0.4)	(0.6, 0.3, 0.1)
Smuggling of ivory	(0.2, 0.3, 0.3)	(0.6, 0.2, 0.3)	(0.5, 0.1, 0.3)	(0.6, 0.2, 0.2)

ness is 10% and indeterminacy is 10%. We can also note that for relation human smuggling, vertex Karachi has highest vertex degree, it means Karachi is most sensitive seaport for human smuggling. Moreover, according to our neutrosophic graph structure most frequent smuggling is human smuggling. It means that at these eight seaports, security forces should take action to control human smuggling.

This neutrosophic graph structure detects most frequent smuggling between some seaports of set A. By making a neutrosophic graph structure of all seaports, we can examine between any two seaports, which kind of smuggling is most frequent, we can also tell that which seaport is most sensitive for particular kind of smuggling. Further, we may get information about violently increasing smuggling through seaports in

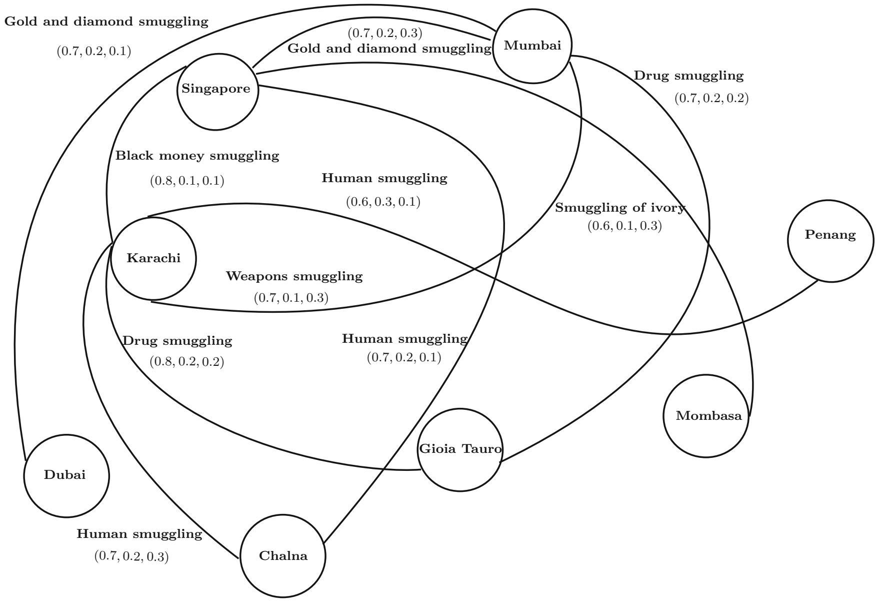


Fig. 2.24 Neutrosophic graph structure showing most frequent smuggling between any two seaports

the whole world. That is why neutrosophic graph structures can be very helpful for security forces to overcome the smuggling at seaports.

We now elaborate general procedure of our applications in the following Algorithm.

Algorithm 2.4.1

Step 1. Input the set $X = \{A_1, A_2, \ldots, A_n\}$ of vertices and the neutrosophic vertex set A defined on X.
Step 2. Input neutrosophic set of relationships or smuggling of a vertex with other vertices and compute T, I and F of each pair of vertices using:
$T(A_iA_j) \leq \min(T(A_i), T(A_j)),\ \ I(A_iA_j) \leq \min(I(A_i), I(A_j)),$
$F(A_iA_j) \leq \max(F(A_i), F(A_j)).$
Step 3. Repeat Step 2 for all vertices in X.
Step 4. Define relations $E_1, E_2, \ldots, E_n$ on set X such that $(X, E_1, E_2, \ldots, E_n)$ is a graph structure.
Step 5. Put an element in that relation for which value of T is high, and values of I and F are low as compared to other relations.
Step 6. Write all elements of relations with their T, I and F values, resulting relations $B_1, B_2, \ldots, B_n$ are neutrosophic sets on $E_1, E_2, E_3, \ldots, E_n$, respectively, and $(A, B_1, B_2, \ldots, B_n)$ is a neutrosophic graph structure.

Chapter 3
Certain Bipolar Neutrosophic Graphs

In this chapter, we present a concise review of bipolar neutrosophic sets. We present operations on bipolar single-valued neutrosophic graphs (bipolar neutrosophic graphs, for short). We discuss certain bipolar neutrosophic graphs, including totally regular bipolar neutrosophic graphs, totally irregular bipolar single-valued neutrosophic graphs and edge regular bipolar neutrosophic graphs. We study domination in bipolar neutrosophic graphs. We present bipolar neutrosophic planar graphs and bipolar neutrosophic line graphs. We also describe some applications of bipolar neutrosophic graphs. This chapter is due to [19, 25].

3.1 Introduction

In 1994, Zhang [201] introduced the notion of bipolar fuzzy sets (YinYang bipolar fuzzy sets, Yin represents the negative side while yang represents the positive side in a system) and relations. Bipolar fuzzy sets are extension of fuzzy sets whose membership degree ranges $[-1, 1]$. In a bipolar fuzzy set, if the degree of membership is zero, then we say the element is unrelated to the corresponding property; membership degree $(0, 1]$ indicates that the object satisfies a certain property, whereas the membership degree $[-1, 0)$ indicates that the element satisfies the implicit counter property. Positive information represents what is considered to be possible, and negative information represents what is granted to be impossible. Actually, a variety of decision-making problems are based on two-sided bipolar judgements on a positive side and a negative side. Smarandache [163] incorporated indeterminacy-membership function as independent component and defined neutrosophic set on three components truth, indeterminacy and falsehood. However, from practical point of view, Wang et al. [172] defined single-valued neutrosophic sets where degree of truth-membership, indeterminacy-membership and falsity-membership belong to $[0, 1]$. Deli et al. [74] extended the ideas of bipolar fuzzy sets and neutrosophic sets to bipolar neutrosophic sets (bipolar single-valued neutrosophic sets) and studied its operations and applications in decision-making problems.

M. Akram, *Single-Valued Neutrosophic Graphs*, Infosys Science Foundation Series, https://doi.org/10.1007/978-981-13-3522-8_3

Definition 3.1 A *bipolar fuzzy set* on a nonempty set X has the form C=$\{(y, \mu^+(y), \mu^-(y)) : y \in X\}$ where $\mu^+ : X \to [0, 1]$ and $\mu^- : X \to [-1, 0]$ are mappings. The positive membership value $\mu^+(y)$ represents the strength of truth or satisfaction of an element y to a certain property corresponding to bipolar fuzzy set C, and $\mu^-(y)$ denotes the strength of satisfaction of an element y to some counter property of bipolar fuzzy set C. If $\mu^+(y) \neq 0$ and $\mu^-(y) = 0$, it is the situation when y has only truth satisfaction degree for property C. If $\mu^-(y) \neq 0$ and $\mu^+(y) = 0$, it is the case that y is not satisfying the property of C but satisfying the counter property to C. It is possible for y that $\mu^+(y) \neq 0$ and $\mu^-(y) \neq 0$ when y satisfies the property of C as well as its counter property in some part of X.

Definition 3.2 A *bipolar single-valued neutrosophic set* on a nonempty set X is an object of the form

$$C = \{(y, T_C^+(y), I_C^+(y), F_C^+(y), T_C^-(y), I_C^-(y), F_C^-(y)) : y \in X\}$$

where $T_C^+, I_C^+, F_C^+ : X \to [0, 1]$ and $T_C^-, I_C^-, F_C^- : X \to [-1, 0]$ are mappings. The positive values $T_C^+(y)$, $I_C^+(y)$, $F_C^+(y)$ denote respectively the truth-, indeterminacy- and falsity-membership degrees of an element $y \in X$, whereas $T_C^-(y)$, $I_C^-(y)$, $F_C^-(y)$ denote the implicit counter property of the truth-, indeterminacy- and falsity-membership degrees of the element $y \in X$ corresponding to the bipolar neutrosophic set C.

Definition 3.3 A *bipolar single-valued neutrosophic relation* on a nonempty set X is a bipolar neutrosophic subset of $X \times X$ of the form

$$D = \{(yz, T_D^+(yz), I_D^+(yz), F_D^+(yz), T_D^-(yz), I_D^-(yz), F_D^-(yz)) : yz \in X \times X\}$$

where $T_D^+, I_D^+, F_D^+, T_D^-, I_D^-, F_D^-$ are defined by the mappings $T_D^+,\ I_D^+,\ F_D^+ : X \times X \to [0, 1]$ and $T_D^-,\ I_D^-,\ F_D^- : X \times X \to [-1, 0]$.

3.2 Bipolar Neutrosophic Graphs

Definition 3.4 A *bipolar single-valued neutrosophic graph* on a nonempty set X is a pair $G = (C, D)$, where C is a bipolar single-valued neutrosophic set on X and D is a bipolar single-valued neutrosophic relation in X such that

$$\begin{aligned} &T_D^+(yz) \leq T_C^+(y) \wedge T_C^+(z), \quad &&I_D^+(yz) \leq I_C^+(y) \wedge I_C^+(z),\\ &F_D^+(yz) \leq F_C^+(y) \vee F_C^+(z), \quad &&T_D^-(yz) \geq T_C^-(y) \vee T_C^-(z),\\ &I_D^-(yz) \geq I_C^-(y) \vee I_C^-(z), \quad &&F_D^-(yz) \geq F_C^-(y) \wedge F_C^-(z) \end{aligned}$$

for all $y, z \in X$. Note that $D(yz) = (0, 0, 1, 0, 0, -1)$ for all $yz \in X \times X \setminus E$.

Table 3.1 Bipolar neutrosophic set C

C	x	y	z
T_C^+	0.3	0.5	0.4
I_C^+	0.4	0.4	0.3
F_C^+	0.5	0.2	0.2
T_C^-	−0.6	−0.1	−0.5
I_C^-	−0.5	−0.8	−0.5
F_C^-	−0.2	−0.2	−0.5

Table 3.2 Bipolar neutrosophic relation D

D	xy	yz	xz
T_D^+	0.3	0.3	0.3
I_D^+	0.4	0.4	0.4
F_D^+	0.5	0.2	0.5
T_D^-	−0.1	−0.1	−0.5
I_D^-	−0.8	−0.8	−0.5
F_D^-	−0.2	−0.5	−0.5

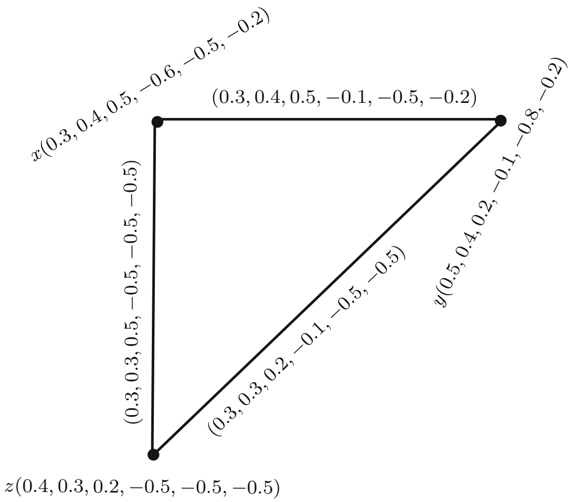


Fig. 3.1 Bipolar neutrosophic graph G

Throughout this chapter, we will use bipolar neutrosophic set, bipolar neutrosophic relation and bipolar neutrosophic graph, for short.

Example 3.1 Consider a bipolar neutrosophic graph on set $X = \{x, y, z\}$. Let C be a bipolar neutrosophic set on X given in Table 3.1 and D be a bipolar single-valued neutrosophic relation in X given in Table 3.2. Routine calculations show that $G = (C, D)$ is a bipolar neutrosophic graph. The bipolar neutrosophic graph G is shown in Fig. 3.1.

Definition 3.5 A bipolar neutrosophic graph $G = (C, D)$ is called *strong bipolar neutrosophic graph* if
$T^+_D(yz) = T^+_C(y) \wedge T^+_C(z)$, $I^+_D(yz) = I^+_C(y) \wedge I^+_C(z)$, $F^+_D(yz) = F^+_C(y) \vee F^+_C(z)$,
$T^-_D(yz) = T^-_C(y) \vee T^-_C(z)$, $I^-_D(yz) = I^-_C(y) \vee I^-_C(z)$, $F^-_D(yz) = F^-_C(y) \wedge F^-_C(z)$,
for all $yz \in E$, E is the set of edges.

Definition 3.6 A bipolar neutrosophic graph $G = (C, D)$ is called *complete bipolar neutrosophic graph* if
$T^+_D(yz) = T^+_C(y) \wedge T^+_C(z)$, $I^+_D(yz) = I^+_C(y) \wedge I^+_C(z)$, $F^+_D(yz) = F^+_C(y) \vee F^+_C(z)$,
$T^-_D(yz) = T^-_C(y) \vee T^-_C(z)$, $I^-_D(yz) = I^-_C(y) \vee I^-_C(z)$, $F^-_D(yz) = F^-_C(y) \wedge F^-_C(z)$,
for all $y, z \in X$.

Definition 3.7 The *Cartesian product* of two bipolar neutrosophic graphs G_1 and G_2 is denoted by the pair $G_1 \times G_2 = (C_1 \times C_2, D_1 \times D_2)$ and defined as,

$$T^+_{C_1\times C_2}(y) = T^+_{C_1}(y) \wedge T^+_{D_2}(y), \qquad I^+_{C_1\times C_2}(y) = I^+_{C_1}(y) \wedge I^+_{C_2}(y),$$
$$F^+_{C_1\times C_2}(y) = F^+_{C_1}(y) \vee F^+_{C_2}(y), \qquad T^-_{C_1\times D_2}(y) = T^-_{C_1}(y) \vee T^-_{C_2}(y),$$
$$I^-_{C_1\times C_2}(y) = I^-_{C_1}(y) \vee I^-_{C_2}(y), \qquad F^-_{C_1\times C_2}(y) = F^-_{C_1}(y) \wedge F^-_{C_2}(y).$$

for all $y \in X_1 \times X_2$. The membership values of the edges in $G_1 \times G_2$ can be calculated as,

1. $T^+_{D_1\times D_2}((y_1, y_2)(y_1, z_2)) = T^+_{C_1}(y_1) \wedge T^+_{D_2}(y_2z_2)$, $T^-_{D_1\times D_2}((y_1, y_2)(y_1, z_2)) = T^-_{C_1}(y_1) \vee T^-_{D_2}(y_2z_2)$, for all $y_1 \in X_1$, $y_2z_2 \in E_2$,
2. $T^+_{D_1\times D_2}((y_1, y_2)(z_1, y_2)) = T^+_{D_1}(y_1z_1) \wedge T^+_{C_2}(y_2)$, $T^-_{D_1\times D_2}((y_1, y_2)(z_1, y_2)) = T^-_{D_1}(y_1z_1) \vee T^-_{C_2}(y_2)$, for all $y_1z_1 \in E_1$, $y_2 \in X_2$,
3. $I^+_{D_1\times D_2}((y_1, y_2)(y_1, z_2)) = I^+_{C_1}(y_1) \wedge I^+_{D_2}(y_2z_2)$, $I^-_{D_1\times D_2}((y_1, y_2)(y_1, z_2)) = I^-_{C_1}(y_1) \vee I^-_{D_2}(y_2z_2)$, for all $y_1 \in X_1$, $y_2z_2 \in E_2$,
4. $I^+_{D_1\times D_2}((y_1, y_2)(z_1, y_2)) = I^+_{D_1}(y_1z_1) \wedge I^+_{C_2}(y_2)$, $I^-_{D_1\times D_2}((y_1, y_2)(z_1, y_2)) = I^-_{D_1}(y_1z_1) \vee I^-_{C_2}(y_2)$, for all $y_1z_1 \in E_1$, $y_2 \in X_2$,
5. $F^+_{D_1\times D_2}((y_1, y_2)(y_1, z_2)) = F^+_{C_1}(y_1) \vee F^+_{D_2}(y_2z_2)$, $F^-_{D_1\times D_2}((y_1, y_2)(y_1, z_2)) = F^-_{C_1}(y_1) \wedge F^-_{D_2}(y_2z_2)$, for all $y_1 \in X_1$, $y_2z_2 \in E_2$,
6. $F^+_{D_1\times D_2}((y_1, y_2)(z_1, y_2)) = F^+_{D_1}(y_1z_1) \vee F^+_{C_2}(y_2)$, $F^-_{D_1\times D_2}((y_1, y_2)(z_1, y_2)) = F^-_{D_1}(y_1z_1) \wedge F^-_{C_2}(y_2)$, for all $y_1z_1 \in E_1$, $y_2 \in X_2$.

Example 3.2 Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two bipolar neutrosophic graphs as shown in Fig. 3.2.

The Cartesian product of G_1 and G_2 is shown in Fig. 3.3.

Proposition 3.1 *The Cartesian product of bipolar neutrosophic graphs is a bipolar neutrosophic graph.*

Definition 3.8 Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two bipolar neutrosophic graphs of $G^*_1 = (X_1, E_1)$ and $G^*_2 = (X_2, E_2)$, respectively, where C_1 and C_2 are

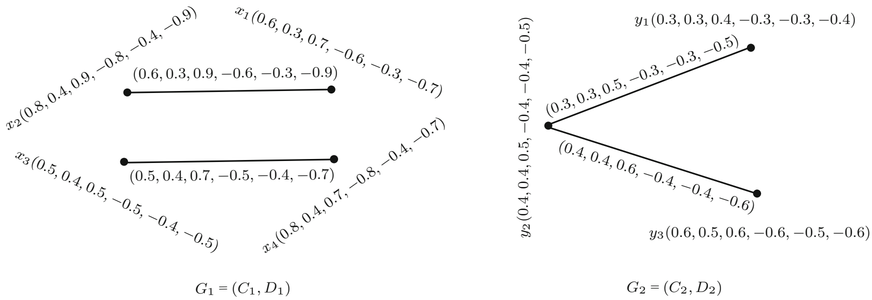


Fig. 3.2 Two bipolar neutrosophic graphs

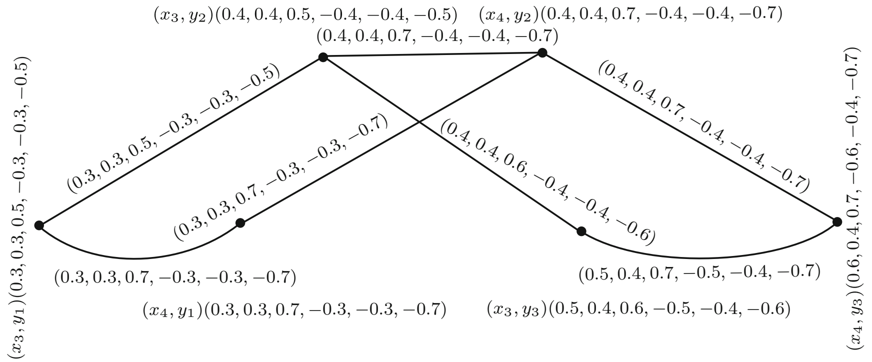


Fig. 3.3 Cartesian product of two bipolar neutrosophic graphs

bipolar neutrosophic sets on X_1 and X_2, and D_1 and D_2 are bipolar neutrosophic relations in X_1 and X_2, respectively. The *union* of G_1 and G_2 is a pair $G_1 \cup G_2 = (C_1 \cup C_2, D_1 \cup D_2)$ such that for all $x, y \in X$,

1. If $x \in X_1, x \notin X_2,$ then $(C_1 \cup C_2)(x) = C_1(x)$.
2. If $x \in X_2, x \notin X_1,$ then $(C_1 \cup C_2)(x) = C_2(x)$.
3. If $x \in X_1 \cap X_2,$ then

$$(C_1 \cup C_2)(x) = \big(T^+_{C_1}(x) \vee T^+_{C_2}(x), \frac{I^+_{C_1}(x) + I^+_{C_2}(x)}{2}, F^+_{C_1}(x) \wedge F^+_{C_2}(x),$$
$$T^-_{C_1}(x) \wedge T^-_{C_2}(x), \frac{I^-_{C_1}(x) + I^-_{C_2}(x)}{2}, F^-_{C_1}(x) \vee F^-_{C_2}(x)\big).$$

If E_1 and E_2 are the sets of edges in G_1 and G_2, then $D_1 \cup D_2$ can be defined as:

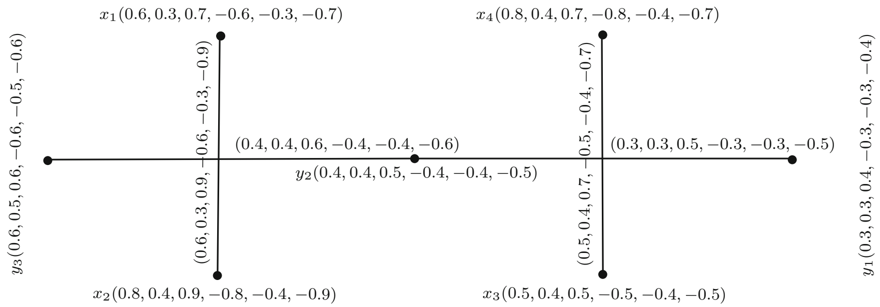


Fig. 3.4 $G_1 \cup G_2$

1. If $xy \in E_1, xy \notin E_2,$ then $(D_1 \cup D_2)(xy) = D_1(xy)$.
2. If $xy \in E_2, xy \notin E_1,$ then $(D_1 \cup D_2)(xy) = D_2(xy)$.
3. If $xy \in E_1 \cap E_2,$ then

$$(D_1 \cup D_2)(xy) = \big(T^+_{D_1}(xy) \vee T^+_{D_2}(xy), \frac{I^+_{D_1}(xy) + I^+_{D_2}(xy)}{2}, F^+_{D_1}(xy) \wedge F^+_{D_2}(xy),$$
$$T^-_{D_1}(xy) \wedge T^-_{D_2}(xy), \frac{I^-_{D_1}(xy) + I^-_{D_2}(xy)}{2}, F^-_{D_1}(xy) \vee F^-_{D_2}(xy)\big).$$

Example 3.3 The union of two bipolar neutrosophic graphs G_1 and G_2 shown in Fig. 3.2 is defined as $G_1 \cup G_2 = \{C_1 \cup C_2, D_1 \cup D_2\}$ and is represented in Fig. 3.4.

Proposition 3.2 *The union of bipolar neutrosophic graphs is a bipolar neutrosophic graph.*

Definition 3.9 The *intersection* of two bipolar neutrosophic graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ is a pair $G_1 \cap G_2 = (C_1 \cap C_2, D_1 \cap D_2)$ where C_1, C_2, D_1 and D_2 are given in Definition 3.8. The membership values of vertices and edges in $G_1 \cap G_2$ is defined such that for all $y \in X_1 \cap X_2$,

$$(C_1 \cap C_2)(y) = \big(T^+_{C_1}(y) \wedge T^+_{C_2}(y), \frac{I^+_{C_1}(y) + I^+_{C_2}(y)}{2}, F^+_{C_1}(y) \vee F^+_{C_2}(y),$$
$$T^-_{C_1}(y) \vee T^-_{C_2}(y), \frac{I^-_{C_1}(y) + I^-_{C_2}(y)}{2}, F^-_{C_1}(y) \wedge F^-_{C_2}(y)\big).$$

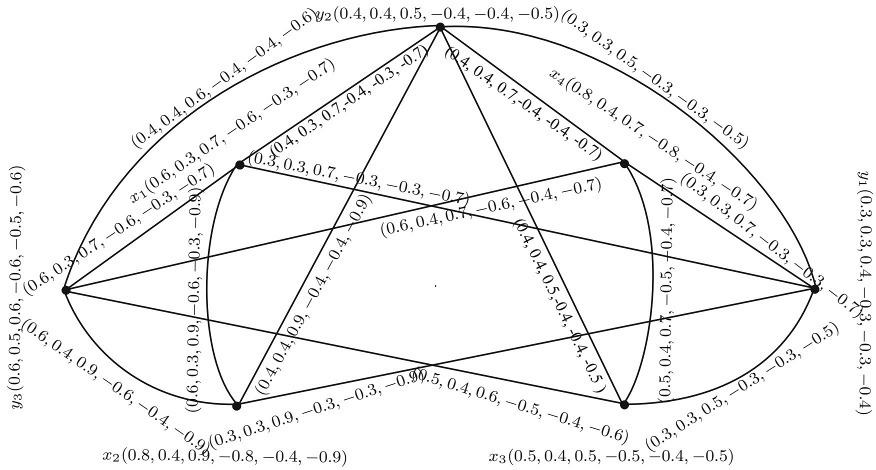


Fig. 3.5 $G_1 + G_2$

$$(D_1 \cap D_2)(yz) = \Big(T^+_{D_1}(yz) \wedge T^+_{D_2}(yz), \frac{I^+_{D_1}(yz) + I^+_{D_2}(yz)}{2}, F^+_{D_1}(yz) \vee F^+_{D_2}(yz),$$
$$T^-_{D_1}(yz) \vee T^-_{D_2}(yz), \frac{I^-_{D_1}(yz) + I^-_{D_2}(yz)}{2}, F^-_{D_1}(yz) \wedge F^-_{D_2}(yz)\Big),$$
for all $yz \in E_1 \cap E_2$.

Definition 3.10 The *join* of two bipolar neutrosophic graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ is defined by the pair $G_1 + G_2 = (C_1 + C_2, D_1 + D_2)$ such that $C_1 + C_2 = C_1 \cup C_2$, for all $x \in X_1 \cup X_2$, and the membership values of the edges in $G_1 + G_2$ are defined as,

1. $D_1 + D_2 = D_1 \cup D_2$, for all $xy \in E_1 \cup E_2$.
2. Let $E^{'}$ be the set of all edges joining the vertices of G_1 and G_2; then for all $xy \in E^{'}$, where $x \in X_1$ and $y \in X_2$,

$$(D_1 + D_2)(xy) = \big(T^+_{D_1}(xy) \wedge T^+_{D_2}(xy), I^+_{D_1}(xy) \wedge I^+_{D_2}(xy), F^+_{D_1}(xy) \vee F^+_{D_2}(xy),$$
$$T^-_{D_1}(xy) \vee T^-_{D_2}(xy), I^-_{D_1}(xy) \vee I^-_{D_2}(xy), F^-_{D_1}(xy) \wedge F^-_{D_2}(xy)\big).$$

Example 3.4 Join of two bipolar neutrosophic graphs G_1 and G_2 shown in Fig. 3.2 is defined as $G_1 + G_2 = \{C_1 + C_2, D_1 + D_2\}$ and is represented in Fig. 3.5.

Proposition 3.3 *The join of bipolar neutrosophic graphs is a bipolar neutrosophic graph.*

Definition 3.11 The *cross product* of two bipolar neutrosophic graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ is denoted by the pair $G_1 * G_2 = (C_1 * C_2, D_1 * D_2)$ such that

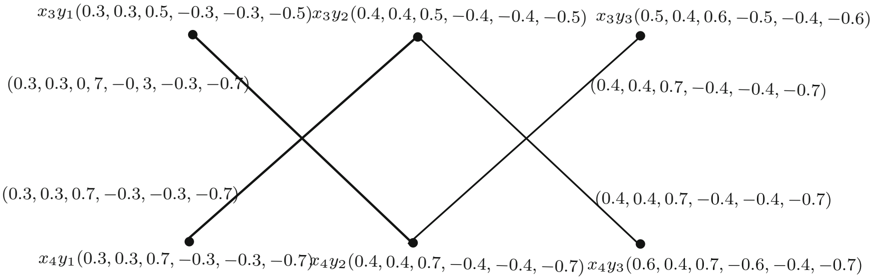


Fig. 3.6 Cross product of two bipolar neutrosophic graphs

$$T^+_{C_1*C_2}(y) = T^+_{C_1}(y) \wedge T^+_{D_2}(y), \qquad I^+_{C_1*C_2}(y) = I^+_{C_1}(y) \wedge I^+_{C_2}(y),$$
$$F^+_{C_1*C_2}(y) = F^+_{C_1}(y) \vee F^+_{C_2}(y), \qquad T^-_{C_1*C_2}(y) = T^-_{C_1}(y) \vee T^-_{C_2}(y),$$
$$I^-_{C_1*C_2}(y) = I^-_{C_1}(y) \vee I^-_{C_2}(y), \qquad F^-_{C_1*C_2}(y) = F^-_{C_1}(y) \wedge F^-_{C_2}(y),$$

for all $y \in X_1 \times X_2$.

1. $T^+_{D_1*D_2}((y_1, y_2)(z_1, z_2)) = T^+_{D_1}(y_1z_1) \wedge T^+_{D_2}(y_2z_2)$, $T^-_{D_1*D_2}((y_1, y_2)(z_1, z_2)) = T^-_{D_1}(y_1z_1) \vee T^-_{D_2}(y_2z_2)$, for all $y_1z_1 \in E_1$, $y_2z_2 \in E_2$,
2. $I^+_{D_1*D_2}((y_1, y_2)(z_1, z_2)) = I^+_{D_1}(y_1z_1) \wedge I^+_{D_2}(y_2z_2)$, $I^-_{D_1*D_2}((y_1, y_2)(z_1, z_2)) = I^-_{D_1}(y_1z_1) \vee I^-_{D_2}(y_2z_2)$, for all $y_1z_1 \in E_1$, $y_2z_2 \in E_2$,
3. $F^+_{D_1*D_2}((y_1, y_2)(z_1, z_2)) = F^+_{D_1}(y_1z_1) \vee F^+_{D_2}(y_2z_2)$, $F^-_{D_1*D_2}((y_1, y_2)(z_1, z_2)) = F^-_{D_1}(y_1z_1) \wedge F^-_{D_2}(y_2z_2)$, for all $y_1z_1 \in E_1$, $y_2z_2 \in E_2$.

Proposition 3.4 *The cross product of bipolar neutrosophic graphs is a bipolar neutrosophic graph.*

Example 3.5 The cross product of two bipolar neutrosophic graphs G_1 and G_2 shown in Fig. 3.2 is defined as $G_1 * G_2 = \{C_1 * C_2, D_1 * D_2\}$ and is shown in Fig. 3.6.

Definition 3.12 The *lexicographic product* of two bipolar neutrosophic graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ is denoted by $G_1 \bullet G_2$ and defined as a pair $(C_1 \bullet C_2, D_1 \bullet D_2)$ such that

$$T^+_{C_1\bullet C_2}(y) = T^+_{C_1}(y) \wedge T^+_{D_2}(y), \qquad I^+_{C_1\bullet C_2}(y) = I^+_{C_1}(y) \wedge I^+_{C_2}(y),$$
$$F^+_{C_1\bullet C_2}(y) = F^+_{C_1}(y) \vee F^+_{C_2}(y), \qquad T^-_{C_1\bullet C_2}(y) = T^-_{C_1}(y) \vee T^-_{C_2}(y),$$
$$I^-_{C_1\bullet C_2}(y) = I^-_{C_1}(y) \vee I^-_{C_2}(y), \qquad F^-_{C_1\bullet C_2}(y) = F^-_{C_1}(y) \wedge F^-_{C_2}(y),$$

for all $y \in X_1 \times X_2$.

1. $T^+_{D_1\bullet D_2}((y, y_2)(y, z_2)) = T^+_{C_1}(y) \wedge T^+_{D_2}(y_2z_2)$, $T^-_{D_1\bullet D_2}((y, y_2)(y, z_2)) = T^-_{C_1}(y) \vee T^-_{D_2}(y_2z_2)$, for all $y \in X_1$, $y_2z_2 \in E_2$,
2. $I^+_{D_1\bullet D_2}((y, y_2)(y, z_2)) = I^+_{C_1}(y) \wedge I^+_{D_2}(y_2z_2)$, $I^-_{D_1\bullet D_2}((y, y_2)(y, z_2)) = I^-_{C_1}(y) \vee I^-_{D_2}(y_2z_2)$, for all $y \in X_1$, $y_2z_2 \in E_2$,

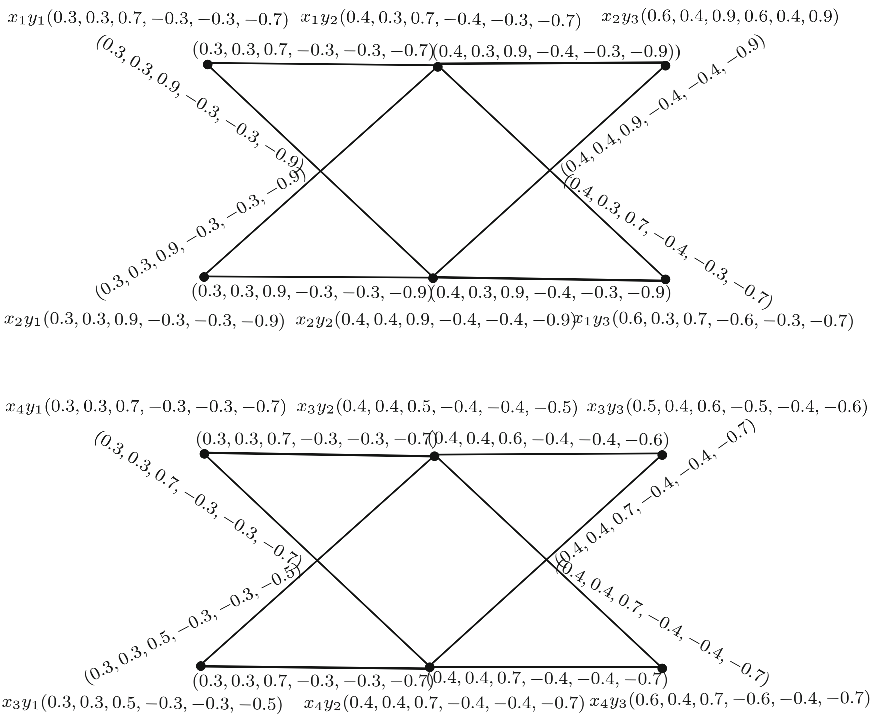


Fig. 3.7 Lexicographic product of two bipolar neutrosophic graphs

3. $F^+_{D_1 \bullet D_2}((y, y_2)(y, z_2)) = F^+_{C_1}(y) \vee F^+_{D_2}(y_2z_2)$, $\quad F^-_{D_1 \bullet D_2}((y, y_2)(y, z_2)) = F^-_{C_1}(y) \wedge F^-_{D_2}(y_2z_2)$, for all $y \in X_1$, $y_2z_2 \in E_2$.
4. $T^+_{D_1 \bullet D_2}((y_1, y_2)(z_1, z_2)) = T^+_{D_1}(y_1z_1) \wedge T^+_{D_2}(y_2z_2)$, $\quad T^-_{D_1 \bullet D_2}((y_1, y_2)(z_1, z_2)) = T^-_{D_1}(y_1z_1) \vee T^-_{D_2}(y_2z_2)$, for all $y_1z_1 \in E_1$, $y_2z_2 \in E_2$,
5. $I^+_{D_1 \bullet D_2}((y_1, y_2)(z_1, z_2)) = I^+_{D_1}(y_1z_1) \wedge I^+_{D_2}(y_2z_2)$, $\quad I^-_{D_1 \bullet D_2}((y_1, y_2)(z_1, z_2)) = I^-_{D_1}(y_1z_1) \vee I^-_{D_2}(y_2z_2)$, for all $y_1z_1 \in E_1$, $y_2z_2 \in E_2$,
6. $F^+_{D_1 \bullet D_2}((y_1, y_2)(z_1, z_2)) = F^+_{D_1}(y_1z_1) \vee F^+_{D_2}(y_2z_2)$, $\quad F^-_{D_1 \bullet D_2}((y_1, y_2)(z_1, z_2)) = F^-_{D_1}(y_1z_1) \wedge F^-_{D_2}(y_2z_2)$, for all $y_1z_1 \in E_1$, $y_2z_2 \in E_2$.

Proposition 3.5 *The lexicographic product of bipolar neutrosophic graphs is a bipolar neutrosophic graph.*

Example 3.6 The *lexicographic product* of two bipolar neutrosophic graphs G_1 and G_2, shown in Fig. 3.2, is given in Fig. 3.7.

Definition 3.13 The *strong product* of two bipolar neutrosophic graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ is denoted by $G_1 \boxtimes G_2$ and defined as a pair $(C_1 \boxtimes C_2, D_1 \boxtimes D_2)$ such that

$$T^+_{C_1\boxtimes C_2}(y) = T^+_{C_1}(y) \wedge T^+_{D_2}(y), \qquad I^+_{C_1\boxtimes C_2}(y) = I^+_{C_1}(y) \wedge I^+_{C_2}(y),$$
$$F^+_{C_1\boxtimes C_2}(y) = F^+_{C_1}(y) \vee F^+_{C_2}(y), \qquad T^-_{C_1\boxtimes C_2}(y) = T^-_{C_1}(y) \vee T^-_{C_2}(y),$$
$$I^-_{C_1\boxtimes C_2}(y) = I^-_{C_1}(y) \vee I^-_{C_2}(y), \qquad F^-_{C_1\boxtimes C_2}(y) = F^-_{C_1}(y) \wedge F^-_{C_2}(y),$$

for all $y \in X_1 \times X_2$.

1. $T^+_{D_1\boxtimes D_2}((y, y_2)(y, z_2)) = T^+_{C_1}(y) \wedge T^+_{D_2}(y_2z_2)$, $T^-_{D_1\boxtimes D_2}((y, y_2)(y, z_2)) = T^-_{C_1}(y) \vee T^-_{D_2}(y_2z_2)$, for all $y \in X_1$, $y_2z_2 \in E_2$,
2. $I^+_{D_1\boxtimes D_2}((y, y_2)(y, z_2)) = I^+_{C_1}(y) \wedge I^+_{D_2}(y_2z_2)$, $I^-_{D_1\boxtimes D_2}((y, y_2)(y, z_2)) = I^-_{C_1}(y) \vee I^-_{D_2}(y_2z_2)$, for all $y \in X_1$, $y_2z_2 \in E_2$,
3. $F^+_{D_1\boxtimes D_2}((y, y_2)(y, z_2)) = F^+_{C_1}(y) \vee F^+_{D_2}(y_2z_2)$, $F^-_{D_1\boxtimes D_2}((y, y_2)(y, z_2)) = F^-_{C_1}(y) \wedge F^-_{D_2}(y_2z_2)$, for all $y \in X_1$, $y_2z_2 \in E_2$,
4. $T^+_{D_1\boxtimes D_2}((y_1, z)(z_1, z)) = T^+_{D_1}(y_1z_1) \wedge T^+_{C_2}(z)$, $T^-_{D_1\boxtimes D_2}((y_1, z)(z_1, z)) = T^-_{D_1}(y_1z_1) \vee T^-_{C_2}(z)$, for all $y_1z_1 \in E_1$, $z \in X_2$,
5. $I^+_{D_1\boxtimes D_2}((y_1, z)(z_1, z)) = I^+_{D_1}(y_1z_1) \wedge I^+_{C_2}(z)$, $I^-_{D_1\boxtimes D_2}((y_1, z)(z_1, z)) = I^-_{D_1}(y_1z_1) \vee I^-_{C_2}(z)$, for all $y_1z_1 \in E_1$, $z \in X_2$,
6. $F^+_{D_1\boxtimes D_2}((y_1, z)(z_1, z)) = F^+_{D_1}(y_1z_1) \vee F^+_{C_2}(z)$, $F^-_{D_1\boxtimes D_2}((y_1, z)(z_1, z)) = F^-_{D_1}(y_1z_1) \wedge F^-_{C_2}(z)$, for all $y_1z_1 \in E_1$, $z \in X_2$,
7. $T^+_{D_1\boxtimes D_2}((y_1, y_2)(z_1, z_2)) = T^+_{D_1}(y_1z_1) \wedge T^+_{D_2}(y_2z_2)$, $T^-_{D_1\boxtimes D_2}((y_1, y_2)(z_1, z_2)) = T^-_{D_1}(y_1z_1) \vee T^-_{D_2}(y_2z_2)$, for all $y_1z_1 \in E_1$, $y_2z_2 \in E_2$,
8. $I^+_{D_1\boxtimes D_2}((y_1, y_2)(z_1, z_2)) = I^+_{D_1}(y_1z_1) \wedge I^+_{D_2}(y_2z_2)$, $I^-_{D_1\boxtimes D_2}((y_1, y_2)(z_1, z_2)) = I^-_{D_1}(y_1z_1) \vee I^-_{D_2}(y_2z_2)$, for all $y_1z_1 \in E_1$, $y_2z_2 \in E_2$,
9. $F^+_{D_1\boxtimes D_2}((y_1, y_2)(z_1, z_2)) = F^+_{D_1}(y_1z_1) \vee F^+_{D_2}(y_2z_2)$, $F^-_{D_1\boxtimes D_2}((y_1, y_2)(z_1, z_2)) = F^-_{D_1}(y_1z_1) \wedge F^-_{D_2}(y_2z_2)$, for all $y_1z_1 \in E_1$, $y_2z_2 \in E_2$.

Example 3.7 The strong product $G_1 \boxtimes G_2$ of two bipolar neutrosophic graphs G_1 and G_2, shown in Fig. 3.2, is given in Fig. 3.8.

Proposition 3.6 *The strong product of bipolar neutrosophic graphs is a bipolar neutrosophic graph.*

Definition 3.14 The *complement* of a bipolar neutrosophic graph $G = (C, D)$ is defined as a pair $G^c = (C^c, D^c)$ such that for all $y \in X$ and $yz \in \widetilde{Y}^2$,

$$T^+_{C^c}(y) = T^+_C(y), I^+_{C^c}(y) = I^+_C(y), F^+_{C^c}(y) = F^+_C(y),$$

$$T^-_{C^c}(y) = T^-_C(y), I^-_{C^c}(y) = I^-_C(y), F^+_{C^c}(y) = F^+_C(y).$$

$$T^+_{D^c}(yz) = T^+_C(y) \wedge T^+_C(z) - T^+_D(yz), \quad T^-_{D^c}(yz) = T^-_C(y) \vee T^-_C(z) - T^-_D(yz),$$
$$I^+_{D^c}(yz) = I^+_C(y) \wedge I^+_C(z) - I^+_D(yz), \quad I^-_{D^c}(yz) = I^-_C(y) \vee I^-_C(z) - I^-_D(yz),$$
$$F^+_{D^c}(yz) = F^+_C(y) \vee F^+_C(z) - F^+_D(yz), \quad F^-_{D^c}(yz) = F^-_C(y) \wedge F^-_C(z) - F^-_D(yz).$$

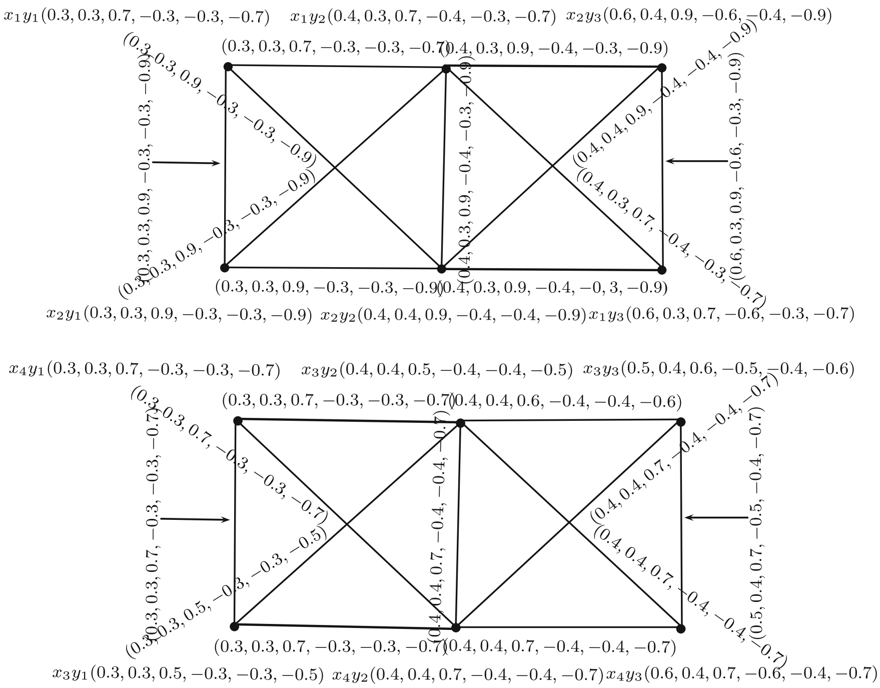


Fig. 3.8 Strong product of two bipolar neutrosophic graphs

Remark 3.1 A bipolar neutrosophic graph G is said to be *self-complementary* if $G \approx G^c$.

Theorem 3.1 *Let G be a self-complementary bipolar neutrosophic graph, then*

$$\sum_{y\neq z} T_D^+(yz) = \frac{1}{2}\sum_{y\neq z} T_C^+(y) \wedge T_C^+(z), \qquad \sum_{y\neq z} I_D^+(yz) = \frac{1}{2}\sum_{y\neq z} I_C^+(y) \wedge I_C^+(z),$$
$$\sum_{y\neq z} F_D^+(yz) = \frac{1}{2}\sum_{y\neq z} F_C^+(y) \vee F_C^+(z), \qquad \sum_{y\neq z} T_D^-(yz) = \frac{1}{2}\sum_{y\neq z} T_C^-(y) \vee T_C^-(z),$$
$$\sum_{y\neq z} I_D^-(yz) = \frac{1}{2}\sum_{y\neq z} I_C^-(y) \vee I_C^-(z), \qquad \sum_{y\neq z} F_D^-(yz) = \frac{1}{2}\sum_{y\neq z} F_C^-(y) \wedge F_C^-(z).$$

Theorem 3.2 *Let $G = (C, D)$ be a bipolar neutrosophic graph such that for all $y, z \in X$,*

$$T^+_{D^c}(yz) = \frac{1}{2}T^+_C(y) \wedge T^+_C(z), \quad T^-_{D^c}(yz) = \frac{1}{2}T^-_C(y) \vee T^-_C(z),$$
$$I^+_{D^c}(yz) = \frac{1}{2}I^+_C(y) \wedge I^+_C(z), \quad I^-_{D^c}(yz) = \frac{1}{2}I^-_C(y) \vee I^-_C(z),$$
$$F^+_{D^c}(yz) = \frac{1}{2}F^+_C(y) \vee F^+_C(z), \quad F^-_{D^c}(yz) = \frac{1}{2}F^-_C(y) \wedge F^-_C(z).$$

Then G is self-complementary bipolar neutrosophic graph.

Proof Let $G^c = (C^c, D^c)$ be the complement of bipolar neutrosophic graph $G = (C, D)$, then by Definition 3.14,

$$T^+_{D^c}(yz) = T^+_C(y) \wedge T^+_C(z) - T^+_D(yz)$$
$$T^+_{D^c}(yz) = T^+_C(y) \wedge T^+_C(z) - \frac{1}{2}T^+_C(y) \wedge T^+_C(z)$$
$$T^+_{D^c}(yz) = \frac{1}{2}T^+_C(y) \wedge T^+_C(z)$$
$$T^+_{D^c}(yz) = T^+_D(yz)$$

$$T^-_{D^c}(yz) = T^-_C(y) \vee T^-_C(z) - T^-_D(yz)$$
$$T^-_{D^c}(yz) = T^-_C(y) \vee T^-_C(z) - \frac{1}{2}T^-_C(y) \vee T^-_C(z)$$
$$T^-_{D^c}(yz) = \frac{1}{2}T^-_C(y) \vee T^-_C(z)$$
$$T^-_{D^c}(yz) = T^-_D(yz)$$

Similarly, it can be shown that

$$I^+_{D^c}(yz) = I^+_D(yz), \quad I^-_{D^c}(yz) = I^-_D(yz),$$
$$F^+_{D^c}(yz) = F^+_D(yz), \quad F^-_{D^c}(yz) = F^-_D(yz).$$

Hence, G is self-complementary.

Definition 3.15 The *degree* of a vertex y in a bipolar neutrosophic graph $G = (C, D)$ is denoted by $\deg(y)$ and defined by the 6−tuple as,
$\deg(y) = (\deg^+_T(y),\ \deg^+_I(y),\ \deg^+_F(y),\ \deg^-_T(y),\ \deg^-_I(y),\ \deg^-_F(y)), =$
$(\sum\limits_{yz\in E} T^+_D(yz),\ \sum\limits_{yz\in E} I^+_D(yz),\ \sum\limits_{yz\in E} F^+_D(yz),\ \sum\limits_{yz\in E} T^-_D(yz),\ \sum\limits_{yz\in E} I^-_D(yz),\ \sum\limits_{yz\in E} F^-_D(yz)).$
The term degree is also referred as *neighbourhood degree*.

Definition 3.16 The *closed neighbourhood* degree of a vertex y in a bipolar neutrosophic graph is denoted by $\deg[y]$ and defined as,

$$\begin{aligned}\deg[y] &= (\deg^+_T[y],\ \deg^+_I[y],\ \deg^+_F[y],\ \deg^-_T[y],\ \deg^-_I[y],\ \deg^-_F[y],\\ &= (\deg^+_T(y) + T^+_C(y),\ \deg^+_I(y) + I^+_C(y),\ \deg^+_F(y) + F^+_C(y),\\ &\quad \deg^-_T(y) + T^-_C(y),\ \deg^-_I(y) + T^-_C(y), \deg^-_F(y) + F^+_C(y)).\end{aligned}$$

Definition 3.17 A bipolar neutrosophic graph G is known as a *regular bipolar neutrosophic graph* if all vertices of G have same degree. A bipolar neutrosophic graph G is known as a *totally regular bipolar neutrosophic graph* if all vertices of G have same closed neighbourhood degree.

Theorem 3.3 *A complete bipolar neutrosophic graph is totally regular.*

Theorem 3.4 *Let $G = (C, D)$ be a bipolar neutrosophic graph, then $C = (T^+, I^+, F^+, T^-, I^-, F^-)$ is a constant function if and only if the following statements are equivalent:*
1. G is a regular bipolar neutrosophic graph,
2. G is totally regular bipolar neutrosophic graph.

Proof Assume that C is a constant function and for all $y \in X$,

$$T_C^+(y) = k_T,\ I_C^+(y) = k_I,\ F_C^+(y) = k_F,$$
$$T_C^-(y) = k_T^{'},\ I_C^-(y) = k_I^{'},\ F_C^-(y) = k_F^{'}$$

where $k_T,\ k_I,\ k_F,\ k_T^{'},\ k_I^{'},\ k_F^{'}$ are constants.
(1) $\Rightarrow$ (2) Suppose that G is a regular bipolar neutrosophic graph and

$$\deg(y) = (p_T,\ p_I,\ p_F,\ n_T,\ n_I,\ n_F),\ \text{for all } y \in X.$$

Now consider,

$$\begin{aligned}\deg[y] = {} & (\deg_T^+(y) + T_C^+(y), \deg_I^+(y) + I_C^+(y), \deg_F^+(y) + F_C^+(y),\\ & \deg_T^-(y) + T_C^-(y), \deg_I^-(y) + T_C^-(y), \deg_F^-(y) + F_C^+(y))\\ = {} & (p_T + k_T, p_I + k_I, p_F + k_F, n_T + k_T^{'}, n_I + k_I^{'}, n_F + k_F^{'})\end{aligned}$$

for all $y \in X$. It is proved that G is totally regular bipolar neutrosophic graph.
(2) $\Rightarrow$ (1) Suppose that G is totally regular bipolar neutrosophic graph and for all $y \in X$

$$\begin{aligned}\deg[y] &= (p_T^{'},\ p_I^{'},\ p_F^{'},\ n_T^{'},\ n_I^{'},\ n_F^{'})\\ &= (\deg_T^+(y) + k_T, \deg_I^+(y) + k_I, \deg_F^+(y) + k_F, \deg_T^-(y) + k_T^{'},\\ &\quad \deg_I^-(y) + k_I^{'}, \deg_F^-(y) + k_F^{'})\\ \Rightarrow \deg(y) &= (p_T^{'} - k_T,\ p_I^{'} - k_I,\ p_F^{'} - k_F,\ n_T^{'} - k_t^{'},\ n_I^{'} - k_I^{'},\ n_F^{'} - k_F^{'}).\end{aligned}$$

for all $y \in X$. Thus, G is a regular bipolar neutrosophic graph. Conversely, assume that the conditions are equivalent. Let

$$\deg(y) = (c_T, c_I, c_F, d_T, d_I, d_F), \qquad \deg[y] = (c_T^{'}, c_I^{'}, c_F^{'}, d_T^{'}, d_I^{'}, d_F^{'}).$$

By Definition 3.16 for all $y \in X$,

$$\begin{aligned} \deg[y] &= \deg(y) + (T_C^+(y), I_C^+(y), F_C^+(y), T_C^-(y), I_C^-(y), F_C^+(y)), \\ \deg[y] - \deg(y) &= (T_C^+(y), I_C^+(y), F_C^+(y), T_C^-(y), I_C^-(y), F_C^+(y)) \\ &= (T_C^+(y), I_C^+(y), F_C^+(y), T_C^-(y), I_C^-(y), F_C^+(y)) \\ &= (c_T' - c_T, c_I' - c_I, c_F' - c_F, d_T' - d_T, d_I' - d_I, d_F' - d_F). \end{aligned}$$

Hence, $C = (c_T' - c_T, c_I' - c_I, c_F' - c_F, d_T' - d_T, d_I' - d_I, d_F' - d_F)$ is a constant function which completes the proof.

Definition 3.18 A bipolar neutrosophic graph G is said to be *irregular* if at least two vertices have distinct degrees. If all vertices do not have same closed neighbourhood degrees, then G is known as *totally irregular bipolar neutrosophic graph.*

Theorem 3.5 *Let $G = (C, D)$ be a bipolar neutrosophic graph and C=$(T_C^+, I_C^+, F_C^+, T_C^-, I_C^-, F_C^-)$ be a constant function, then G is an irregular bipolar neutrosophic graph if and only if G is a totally irregular bipolar neutrosophic graph.*

Proof Assume that G is an irregular bipolar neutrosophic graph, then at least two vertices of G have distinct degrees. Let y and z be two vertices such that $\deg(y) = r_1, r_2, r_3, s_1, s_2, s_3)$, $\deg(z) = (r_1', r_2', r_3', s_1', s_2', s_3')$ where $r_i \neq r_i'$, for some $i = 1, 2, 3$. Since C is a constant function, assume that $C = (k_1, k_2, k_3, l_1, l_2, l_3)$. Thus,

$$\begin{aligned} \deg[y] &= \deg(y) + (k_1,\ k_2,\ k_3,\ l_1,\ l_2,\ l_3) \\ \deg[y] &= (r_1 + k_1, r_2 + k_2, r_3 + k_3, s_1 + l_1, s_2 + l_2, s_3 + l_3) \\ \text{and}\quad \deg[z] &= (r_1' + k_1, r_2' + k_2, r_3' + k_3, s_1' + l_1, s_2' + l_2, s_3' + l_3). \end{aligned}$$

Clearly $r_i + k_i \neq r_i' + k_i$, for some $i = 1, 2, 3$; therefore, y and z have distinct closed neighbourhood degrees. Hence, G is a totally irregular bipolar neutrosophic graph. The converse part is similar.

Definition 3.19 If $G = (C, D)$ be a bipolar neutrosophic graph and y, z are two vertices in G, then we say that y *dominates* z if

$$\begin{aligned} T_D^+(yz) &= T_C^+(y) \wedge T_C^+(z), \quad T_D^-(yz) = T_C^-(y) \vee T_C^-(z), \\ I_D^+(yz) &= I_C^+(y) \wedge I_C^+(z), \quad I_D^-(yz) = I_C^-(y) \vee I_C^-(z), \\ F_D^+(yz) &= F_C^+(y) \vee F_C^+(z), \quad F_D^-(yz) = F_C^-(y) \wedge F_C^-(z). \end{aligned}$$

A subset $D' \subseteq Y$ is a *dominating set* if for each $z \in X \setminus D'$, there exists $y \in D'$ such that y dominates z. A dominating set D' is minimal if for every $y \in D'$, $D' \setminus \{y\}$ is not a dominating set. The *domination number* of G is the minimum cardinality among all minimal dominating sets of G, denoted by $\lambda(G)$.

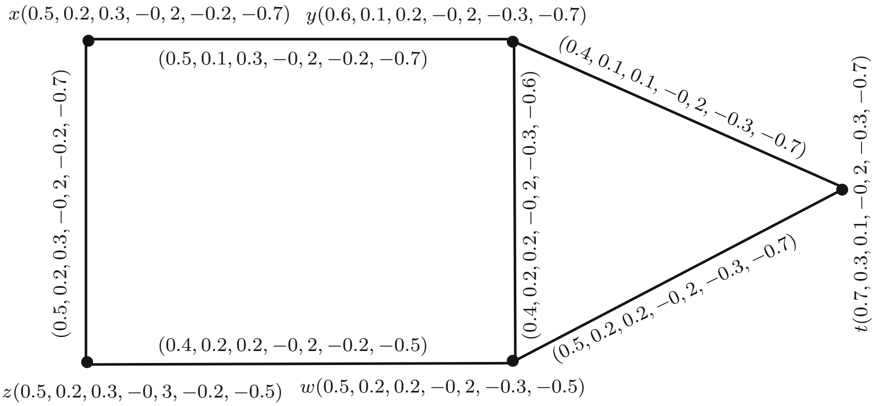


Fig. 3.9 Bipolar neutrosophic graph G

Example 3.8 Consider a bipolar neutrosophic graph as shown in Fig. 3.9. The set $\{x, w\}$ is a minimal dominating set, and $\lambda(G) = 2$

Theorem 3.6 *If G_1 and G_2 are two bipolar neutrosophic graphs with $D_1^{'}$ and $D_2^{'}$ as dominating sets, then*

$$\lambda(G_1 \cup G_2) = \lambda(G_1) + \lambda(G_2) - |D_1^{'} \cap D_2^{'}|$$

Proof Since $D_1^{'}$ and $D_2^{'}$ are dominating sets of G_1 and G_2, $D_1^{'} \cup D_2^{'}$ is a dominating set of $G_1 \cup G_2$. Therefore, $\lambda(G_1 \cup G_2) \leq |D_1^{'} \cup D_2^{'}|$. It only remains to show that $D_1^{'} \cup D_2^{'}$ is the minimal dominating set. On contrary, assume that $D^{'} = D_1^{'} \cup D_2^{'} \setminus \{y\}$ is a minimal dominating set of $G_1 \cup G_2$. There are two cases.
Case 1. If $y \in D_1^{'}$ and $y \notin D_2^{'}$, then $D_1^{'} \setminus \{y\}$ is not a dominating set of G_1 which implies that $D_1^{'} \cup D_2^{'} \setminus \{y\} = D^{'}$ is not a dominating set of $G_1 \cup G_2$. A contradiction, hence, $D_1^{'} \cup D_2^{'}$ is a minimal dominating set and

$$\begin{aligned} &\lambda(G_1 \cup G_2) = |D_1^{'} \cup D_2^{'}|, \\ \Rightarrow \quad &\lambda(G_1 \cup G_2) = \lambda(G_1) + \lambda(G_2) - |D_1^{'} \cap D_2^{'}|. \end{aligned}$$

Case 2. If $y \in D_2^{'}$ and $y \notin D_1^{'}$, same contradiction can be obtained.

Theorem 3.7 *If G_1 and G_2 are two bipolar neutrosophic graphs with $X_1 \cap X_2 \neq \emptyset$, then*

$$\lambda(G_1 + G_2) = \min\{\lambda(G_1), \lambda(G_2), 2\}.$$

Proof Let $y_1 \in X_1$ and $y_2 \in X_2$, since $G_1 + G_2$ is a bipolar neutrosophic graph, we have

$$
\begin{aligned}
T^{+}_{D_1+D_2}(y_1y_2) &= T^{+}_{C_1+C_2}(y_1) \wedge T^{+}_{C_1+C_2}(y_2)),\\
T^{-}_{D_1+D_2}(y_1y_2) &= T^{-}_{C_1+C_2}(y_1) \vee T^{-}_{C_1+C_2}(y_2)),\\
I^{+}_{D_1+D_2}(y_1y_2) &= I^{+}_{C_1+C_2}(y_1) \wedge I^{+}_{C_1+C_2}(y_2)),\\
I^{-}_{D_1+D_2}(y_1y_2) &= I^{-}_{C_1+C_2}(y_1) \vee I^{-}_{C_1+C_2}(y_2)),\\
F^{+}_{D_1+D_2}(y_1y_2) &= F^{+}_{C_1+C_2}(y_1) \vee F^{+}_{C_1+C_2}(y_2)),\\
F^{-}_{D_1+D_2}(y_1y_2) &= F^{-}_{C_1+C_2}(y_1) \wedge F^{-}_{C_1+C_2}(y_2)).
\end{aligned}
$$

Hence, any vertex of G_1 dominates all vertices of G_2, and similarly, any vertex of G_2 dominates all vertices of G_1. So, $\{y_1, y_2\}$ is a dominating set of $G_1 + G_2$. If D is a minimum dominating set of $G_1 + G_2$, then D is one of the following forms,

1. $D = D_1$ where, $\lambda(G_1) = |D_1|$,
2. $D = D_2$ where, $\lambda(G_2) = |D_2|$,
3. $D = \{y_1, y_2\}$ where $y_1 \in X_1$ and $y_2 \in X_2$. $\{y_1\}$ and $\{y_2\}$ are not dominating sets of G_1 or G_2, respectively.

Hence, $\lambda(G_1 + G_2) = \min\{\lambda(G_1), \lambda(G_2), 2\}$.

Theorem 3.8 *Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two bipolar neutrosophic graphs. If for $y_1 \in X_1$, $T^{+}_{C_1}(y_1) > 0$ and y_2 dominates z_2 in G_2, then (y_1, y_2) dominates (y_1, z_2) in $G_1 \times G_2$.*

Proof Since y_2 dominates z_2,

$$
\begin{aligned}
T^{+}_{D_2}(y_2z_2) &= T^{+}_{C_2}(y_2) \wedge T^{+}_{C_2}(z_2), & T^{-}_{D_2}(y_2z_2) &= T^{-}_{C_2}(y_2) \vee T^{-}_{C_2}(z_2),\\
I^{+}_{D_2}(y_2z_2) &= I^{+}_{C_2}(y_2) \wedge I^{+}_{C_2}(z_2), & I^{-}_{D_2}(y_2z_2) &= I^{-}_{C_2}(y_2) \vee I^{-}_{C_2}(z_2),\\
F^{+}_{D_2}(y_2z_2) &= F^{+}_{C_2}(y_2) \vee F^{+}_{C_2}(z_2), & F^{-}_{D_2}(y_2z_2) &= F^{-}_{C_2}(y_2) \wedge F^{-}_{C_2}(z_2).
\end{aligned}
$$

For $y_1 \in X_1$, take $(y_1, z_2) \in X_1 \times X_2$. By Definition 3.7,

$$
\begin{aligned}
T^{+}_{D_1\times D_2}((y_1, y_2)(y_1, z_2)) &= T^{+}_{C_1}(y_1) \wedge T^{+}_{D_2}(y_2z_2),\\
&= T^{+}_{C_1}(y_1) \wedge (T^{+}_{C_2}(y_2) \wedge T^{+}_{C_2}(z_2)),\\
&= (T^{+}_{C_1}(y_1) \wedge T^{+}_{C_2}(y_2)) \wedge (T^{+}_{C_1}(y_1) \wedge T^{+}_{C_2}(z_2)),\\
&= T^{+}_{C_1\times C_2}(y_1, y_2) \wedge T^{+}_{C_1\times C_2}(y_1, z_2).
\end{aligned}
$$

$$
\begin{aligned}
T^{-}_{D_1\times D_2}((y_1, y_2)(y_1, z_2)) &= T^{-}_{C_1}(y_1) \vee T^{-}_{D_2}(y_2z_2),\\
&= T^{-}_{C_1}(y_1) \vee (T^{-}_{C_2}(y_2) \vee T^{-}_{C_2}(z_2))),\\
&= (T^{-}_{C_1}(y_1) \vee T^{-}_{C_2}(y_2)) \vee (T^{-}_{C_1}(y_1) \vee T^{-}_{C_2}(z_2)),\\
&= T^{-}_{C_1\times C_2}(y_1, y_2) \vee T^{-}_{C_1\times C_2}(y_1, z_2).
\end{aligned}
$$

Similarly, it can be proved that

$$
\begin{aligned}
I^+_{D_1\times D_2}((y_1,y_2)(y_1,z_2)) &= I^+_{C_1\times C_2}(y_1,y_2)\wedge I^+_{C_1\times C_2}(y_1,z_2),\\
I^-_{D_1\times D_2}((y_1,y_2)(y_1,z_2)) &= I^-_{C_1\times C_2}(y_1,y_2)\vee I^-_{C_1\times C_2}(y_1,z_2),\\
F^+_{D_1\times D_2}((y_1,y_2)(y_1,z_2)) &= F^+_{C_1\times C_2}(y_1,y_2)\vee F^+_{C_1\times C_2}(y_1,z_2),\\
F^-_{D_1\times D_2}((y_1,y_2)(y_1,z_2)) &= F^-_{C_1\times C_2}(y_1y_2)\wedge F^-_{C_1\times C_2}(y_1,z_2).
\end{aligned}
$$

Hence, (y_1, y_2) dominates (y_1, z_2) and the proof is complete.

Proposition 3.7 *If G_1 and G_2 are bipolar neutrosophic graphs and for $z_2 \in X_2$, $T^+_{C_2}(z_2) > 0$ and y_1 dominate z_1 in G_1, then (y_1, z_2) dominates (z_1, z_2) in $G_1 \times G_2$.*

Theorem 3.9 *If $D_1^{'}$ and $D_2^{'}$ are minimal dominating sets of $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$, respectively. Then $D_1^{'} \times X_2$ and $X_1 \times D_2^{'}$ are dominating sets of $G_1 \times G_2$ and*

$$\lambda(G_1 \times G_2) \leq \min(|D_1^{'} \times X_2|, |X_1 \times D_2^{'}|). \tag{3.1}$$

Proof To prove inequality Eq. 3.1, we need to show that $D_1^{'} \times X_2$ and $X_1 \times D_2^{'}$ are dominating sets of $G_1 \times G_2$. Let $(z_1, z_2) \notin D_1^{'} \times X_2$, then $z_1 \notin D_1^{'}$. Since $D_1^{'}$ is a dominating set of G_1, there exists $y_1 \in D_1^{'}$ that dominates z_1. By Proposition 3.7, (y_1, z_2) dominates (z_1, z_2) in $G_1 \times G_2$. Since (z_1, z_2) was taken to be arbitrary, $D_1^{'} \times X_2$ is a dominating set of $G_1 \times G_2$. Similarly, $X_1 \times D_2^{'}$ is a dominating set if $G_1 \times G_2$. Hence, the proof.

Theorem 3.10 *Let $D_1^{'}$ and $D_2^{'}$ be the dominating sets of $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$, respectively. Then $D_1^{'} \times D_2^{'}$ is a dominating set of the cross product $G_1 * G_2$ and*

$$\lambda(G_1 \times G_2) = |D_1^{'} \times D_2^{'}|. \tag{3.2}$$

Proof Let $(z_1, z_2) \in X_1 \times X_2 \setminus D_1^{'} \times D_2^{'}$, then $z_1 \in X_1 \setminus D_1^{'}$ and $z_2 \in X_2 \setminus D_2^{'}$. Since $D_1^{'}$ and $D_2^{'}$ are dominating sets, there exist $y_1 \in D_1^{'}$ and $y_2 \in D_2^{'}$ such that y_1 dominates z_1 and y_2 dominates z_2. Consider,

$$
\begin{aligned}
T^+_{D_1*D_2}((y_1,y_2)(z_1,z_2)) &= T^+_{D_1}(y_1z_1)\wedge T^+_{D_2}(y_2z_2),\\
&= (T^+_{C_1}(y_1)\wedge T^+_{C_1}(z_1))\wedge(T^+_{C_2}(y_2)\wedge T^+_{C_2}(z_2)),\\
&= (T^+_{C_1}(y_1)\wedge T^+_{C_2}(y_2))\wedge(T^+_{C_1}(z_1)\wedge T^+_{C_2}(z_2)),\\
&= T^+_{C_1*C_2}(y_1,y_2)\wedge T^+_{C_1*C_2}(z_1,z_2).
\end{aligned}
$$

It shows that (y_1, y_2) dominates (z_1, z_2). Since (y_1, y_2) was taken to be arbitrary, every element of $X_1 \times X_2 \setminus D_1^{'} \times D_2^{'}$ is dominated by some element of $D_1^{'} \times D_2^{'}$. It only remains to show that $D_1^{'} \times D_2^{'}$ is a minimal dominating set.

On contrary, assume that $D^{'}$ is a minimal dominating set of $G_1 * G_2$ such that $|D^{'}| < |D_1^{'} \times D_2^{'}|$.

Let $(t_1, t_2) \in D_1^{'} \times D_2^{'}$ such that $(t_1, t_2) \notin D^{'}$, i.e. $t_1 \in D_1^{'}$ and $t_2 \in D_2^{'}$, then there exist $t_1^{'} \in X_1 \setminus D_1^{'}$ and $t_2^{'} \in X_2 \setminus D_2^{'}$ which are only dominated by t_1 and t_2, respectively. Hence, no element other than (t_1, t_2) dominates $(t_1^{'}, t_2^{'})$; so $(t_1, t_2) \in D^{'}$. A contradiction, thus $\lambda(G_1 * G_2) = |D_1^{'} \times D_2^{'}|$.

Corollary 3.1 *If G_1 and G_2 are two bipolar neutrosophic graphs, y_1 dominates z_1 in G_1 and y_2 dominates z_2 in G_2, then (y_1, z_1) dominates (y_2, z_2) in $G_1 * G_2$.*

Definition 3.20 In a bipolar neutrosophic graph two vertices y and z are *independent* if

$$\begin{aligned} T_D^+(yz) &< T_C^+(y) \wedge T_C^+(z), \quad T_D^-(yz) > T_C^-(y) \vee T_C^-(z), \\ I_D^+(yz) &< I_C^+(y) \wedge I_C^+(z), \quad I_D^-(yz) > I_C^-(y) \vee I_C^-(z), \\ F_D^+(yz) &< F_C^+(y) \vee F_C^+(z), \quad F_D^-(yz) > F_C^-(y) \wedge F_C^-(z). \end{aligned} \tag{3.3}$$

An *independent set* N of a bipolar neutrosophic graph is a subset N of A such that for all $y, z \in N$ Eq. 3.3 are satisfied. An independent set is *maximal* if for every $t \in X \setminus N$, $N \cup \{t\}$ is not an independent set. An *independent number* is the maximal cardinality among all maximal independent sets of a bipolar neutrosophic graph. It is denoted by $\alpha(G)$.

Theorem 3.11 *If G_1 and G_2 are bipolar neutrosophic graphs on X_1 and X_2, respectively, such that $X_1 \cap X_2 = \emptyset$, then $\alpha(G_1 \cup G_2) = \alpha(G_1) + \alpha(G_2)$.*

Proof Let N_1 and N_2 be maximal independent sets of G_1 and G_2. Since $N_1 \cap N_2 = \emptyset$, $N_1 \cup N_2$ is a maximal independent set of $G_1 \cup G_2$. Hence, $\alpha(G_1 \cup G_2) = \alpha(G_1) + \alpha(G_2)$.

Theorem 3.12 *Let G_1 and G_2 be two bipolar neutrosophic graphs, then $\alpha(G_1 + G_2) = \alpha(G_1) \vee \alpha(G_2)$.*

Proof Let N_1 and N_2 be maximal independent sets. Since every vertex of G_1 dominates every vertex of G_2 in $G_1 + G_1$. Hence, maximal independent set of $G_1 + G_2$ is either N_1 or N_2. Thus, $\alpha(G_1 + G_2) = \alpha(G_1) \vee \alpha(G_2)$.

Theorem 3.13 *If N_1 and N_2 are maximal independent sets of G_1 and G_2, respectively, and $X_1 \cap X_2 = \emptyset$. Then $\alpha(G_1 \times G_2) = |N_1 \times N_2| + |N|$ where*

$$N = \{(y_i, z_i) : y_i \in X_1 \setminus N_1, z_i \in X_2 \setminus N_2, y_i y_{i+1} \in E_1, z_i z_{i+1} \in E_2, \quad i = 1, 2, 3, \ldots\}.$$

Proof N_1 and N_2 are maximal independent sets of G_1 and G_2, respectively. Clearly, $N_1 \times N_2$ is an independent set of $G_1 \times G_2$ as no vertex of $N_1 \times N_2$ dominates any other vertex of $N_1 \times N_2$. Consider the set of vertices

$$N = \{(y_i, z_i) : y_i \in X_1 \setminus N_1, z_i \in X_2 \setminus N_2, y_i y_{i+1} \in E_1, z_i z_{i+1} \in E_2\}.$$

It can be seen that no vertex $(y_i, z_i) \in N$, for each $i = 1, 2, 3, \ldots$, dominates $(y_{i+1}, z_{i+1}) \in N$. Hence, $N^{'} = (N_1 \times N_2) \cup N$ is an independent set of $G_1 \times G_2$.

Assume that $S = N^{'} \cup \{(y_i, z_j)\}$, for some $i \neq j$, $y_i \in X_1 \setminus N_1$ and $z_j \in X_2 \setminus N_2$, is a maximal independent set. Without loss of generality, assume that $j = i + 1$, then (y_i, z_j) is dominated by (y_i, z_i). A contradiction, hence $N^{'}$ is a maximal independent set and $\alpha(G_1 \times G_2) = |N^{'}| = |N_1 \times N_2| + |N|$.

Theorem 3.14 *If $D_1^{'}$ and $D_2^{'}$ are minimal dominating sets of G_1 and G_2, then $X_1 \times X_2 \setminus D_1^{'} \times D_2^{'}$ is a maximal independent set of $G_1 * G_2$ and $\alpha(G_1 * G_2) = n_1 n_2 - \lambda(G_1 * G_2)$ where n_1 and n_2 are the number of vertices in G_1 and G_2, respectively.*

The proof is obvious.

Theorem 3.15 *An independent set of a bipolar neutrosophic graph $G = (C, D)$ is maximal if and only if it is independent and dominating.*

Proof If N is a maximal independent set of G, then for every $y \in X \setminus N$, $N \cup \{y\}$ is not an independent set. For every vertex $y \in X \setminus N$, there exists some $z \in N$ such that

$$\begin{aligned} T_D^+(yz) &= T_C^+(y) \wedge T_C^+(z), & T_D^-(yz) &= T_C^-(y) \vee T_C^-(z), \\ I_D^+(yz) &= I_C^+(y) \wedge I_C^+(z), & I_D^-(yz) &= I_C^-(y) \vee I_C^-(z), \\ F_D^+(yz) &= F_C^+(y) \vee F_C^+(z), & F_D^-(yz) &= F_C^-(y) \wedge F_C^-(z). \end{aligned}$$

Thus, y dominates x, and hence, N is both independent and dominating set.

Conversely, assume that D is both independent and dominating set but not maximal independent set. So there exists a vertex $y \in X \setminus N$ such that $N \cup \{y\}$ is an independent set, i.e. no vertex in N dominates y, a contradiction to the fact that N is a dominating set. Hence, N is maximal.

Theorem 3.16 *Any maximal independent set of a bipolar neutrosophic graph is a minimal dominating set.*

Proof If N is a maximal independent set of a bipolar neutrosophic graph, then by Theorem 3.15, N is a dominating set. Assume that N is not a minimal dominating set, then there always exist at least one $z \in N$ for which $N \setminus \{z\}$ is a dominating set. On the other hand if $N \setminus \{z\}$ dominates $Y \setminus \{N \setminus \{z\}\}$, at least one vertex in $N \setminus \{z\}$ dominates z. A contradiction to the fact that N is an independent set of bipolar neutrosophic graph G. Hence, N is a minimal dominating set.

3.3 Applications to Multiple Criteria Decision-Making

Multiple criteria decision-making refers to making decisions in the presence of multiple, usually conflicting, criteria. Multiple criteria decision-making problems are common in everyday life. We present multiple criteria decision-making method for the identification of risk in decision support systems. The method is explained by an example for prevention of accidental hazards in chemical industry. The proposed methodology can be implemented in various fields in different ways, for instance, multicriteria decision-making problems with bipolar neutrosophic information. However, our main focus is the identification of risk assessments in industry which is described in the following steps. The bipolar neutrosophic information consists of a group of risks/alternatives $R = \{r_1, r_2, \ldots, r_n\}$ evaluated on the basis of criteria $C = \{c_1, c_2, \ldots, c_m\}$. Here $r_i,\ i = 1, 2, \ldots, n$ is the possibility for the criteria $c_k,\ k = 1, 2, \ldots, m$ and r_{ik} are in the form of bipolar neutrosophic values. This method is suitable if we have a small set of data and experts are able to evaluate the data in the form of bipolar neutrosophic information. Take the values of r_{ik} as $r_{ik} = (T^+_{ik}, I^+_{ik}, F^+_{ik}, T^-_{ik}, I^-_{ik}, F^-_{ik})$.

Step 1. Construct the table of the given data.

Step 2. Determine the average values using the following bipolar neutrosophic average operator, $A_i = \frac{1}{n}(\sum\limits_{j=1}^{m} T^+_{ij} - \prod\limits_{j=1}^{m} T^+_{ij}, \prod\limits_{j=1}^{m} I^+_{ij}, \prod\limits_{j=1}^{m} F^+_{ij}, \prod\limits_{j=1}^{m} T^-_{ij}, \sum\limits_{j=1}^{m} I^-_{ij} - \prod\limits_{j=1}^{m} I^-_{ij}, \sum\limits_{j=1}^{m} F^-_{ij} - \prod\limits_{j=1}^{m} F^-_{ij})$, for each $i = 1, 2, \ldots, n$.

Step 3. Construct the weighted average matrix.

Choose the weight vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. According to the weights for each alternative, the weighted average table can be calculated by multiplying each average value with the corresponding weight as:

$$\beta_i = A_i w_i, \qquad i = 1, 2, \ldots, n.$$

Step 4. Calculate the normalized value for each alternative/risk β_i using the formula,

$$\alpha_i = \sqrt{(T_i^+)^2 + (I_i^+)^2 + (F_i^+)^2 + (1 - T_i^-)^2 + (-1 + I_i^-)^2 + (-1 + F_i^-)^2}, \tag{3.4}$$

for each $i = 1, 2, \ldots, n$. The resulting table indicates the preference ordering of the alternatives/risks. The alternative/risk with maximum α_i value is most dangerous or more preferable.

Example 3.9 Chemical industry is a very important part of human society. These industries contain large amount of organic and inorganic chemicals and materials. Many chemical products have a high risk of fire due to flammable materials, large explosions, oxygen deficiency, etc. These accidents can cause the death of employs, damages to building, destruction of machines and transports, economical losses, etc. Therefore, it is very important to prevent these accidental losses by identifying the major risks of fire, explosions and oxygen deficiency.

A manager of a chemical industry Y wants to prevent such types of accidents that caused the major loss to company in the past. He collected data from witness reports, investigation teams and nearby chemical industries and found that the major causes could be the chemical reactions, oxidizing materials, formation of toxic substances, electric hazards, oil spill, hydrocarbon gas leakage and energy systems. The witness reports, investigation teams and industries have different opinions. There is

Table 3.3 Bipolar neutrosophic data

	Fire	Oxygen deficiency	Large explosion
Chemical exposures	(0.5, 0.7, 0.2, −0.6, −0.3, −0.7)	(0.1, 0.5, 0.7, −0.5, −0.2, −0.8)	(0.6, 0.2, 0.3, −0.4, 0.0, −0.1)
Oxidizing materials	(0.9, 0.7, 0.2, −0.8, −0.6, −0.1)	(0.3, 0.5, 0.2, −0.5, −0.5, −0.2)	(0.9, 0.5, 0.5, −0.6, −0.5, −0.2)
Toxic vapour cloud	(0.7, 0.3, 0.1, −0.4, −0.1, −0.3)	(0.6, 0.3, 0.2, −0.5, −0.3, −0.3)	(0.5, 0.1, 0.2, −0.6, −0.2, −0.2)
Electric hazard	(0.3, 0.4, 0.2, −0.6, −0.3, −0.7)	(0.9, 0.4, 0.6, −0.1, −0.7, −0.5)	(0.7, 0.6, 0.8, −0.7, −0.5, −0.1)
Oil spill	(0.7, 0.5, 0.3, −0.4, −0.2, −0.2)	(0.2, 0.2, 0.2, −0.7, −0.4, −0.4)	(0.9, 0.2, 0.7, −0.1, −0.6, −0.8)
Hydrocarbon gas leakage	(0.5, 0.3, 0.2, −0.5, −0.2, −0.2)	(0.3, 0.2, 0.3,−0.7, −0.4, −0.3)	(0.8, 0.2, 0.1, −0.1, −0.9, −0.2)
Ammonium nitrate	(0.3, 0.2, 0.3, −0.5, −0.6, −0.5)	(0.9, 0.2, 0.1, 0.0, −0.6, −0.5)	(0.6, 0.2, 0.1, −0.2, −0.3, −0.5)

bipolarity in people's thinking and judgement. The data can be considered as bipolar neutrosophic information. The bipolar neutrosophic information about company Y old accidents is given in Table 3.3.

By applying bipolar neutrosophic average operator on Table 3.3, the average values are given in Table 3.4.

With regard to the weight vector (0.35, 0.80, 0.30, 0.275, 0.65, 0.75, 0.50) associated to each cause of accident, the weighted average values are obtained by multiplying each average value with corresponding weight and are given in Table 3.5.

Using Eq. 3.4, the resulting normalized values are shown in Table 3.6.

The accident possibilities can be placed in the following order: toxic vapour cloud $\succ$ electric hazard $\succ$ hydrocarbon gas leakage $\succ$ chemical exposures $\succ$ ammonium nitrate $\succ$ oxidizing materials $\succ$ oil spill where the symbol $\succ$ represents partial ordering of objects. It can be easily seen that the formation of toxic vapour clouds, electrical and energy systems and hydrocarbon gas leakage are the major dangers to the chemical industry. There is a very little danger due to oil spill. Chemical exposures, oxidizing materials and ammonium nitrate have an average accidental danger. Therefore, industry needs special precautions to prevent the major hazards that could happen due the formation of toxic vapour clouds.

Graph theory is considered an important part of Mathematics for solving countless real-world problems in information technology, psychology, engineering, combinatorics and medical sciences. Everything in this world is connected, for instance, cities and countries are connected by roads, railways are linked by railway lines, flight networks are connected by air, electrical devices are connected by wires, pages on internet by hyperlinks, components of electric circuits by various paths. Scientists, analysts and engineers are trying to optimize these networks to find a way to save millions of lives by reducing traffic accidents, plane crashes, circuit shots and

Table 3.4 Bipolar neutrosophic average values

	Average value
Chemical exposures	(0.39, 0.023, 0.014, −0.04, −0.167, −0.515)
Oxidizing materials	(0.619, 0.032, 0.001, −0.08, −0.483, −0.165)
Toxic vapour cloud	(0.53, 0.003, 0.001, −0.04, −0.198, −0.261)
Electric hazard	(0.570, 0.032, 0.032, −0.014, −0.465, −0.422)
Oil spill	(0.558, 0.007, 0.014, −0.009, −0.384, −0.445)
Hydrocarbon gas leakage	(0.493, 0.004, 0.002, −0.011, −0.543, −0.229)
Ammonium nitrate	(0.546, 0.003, 0.001, 0.0, −0.464, −0.417)

Table 3.5 Bipolar neutrosophic weighted average table

	Average value
Chemical exposures	(0.1365, 0.0081, 0.0049, −0.0140, −0.0585, −0.1803)
Oxidizing materials	(0.4952, 0.0256, 0.0008, −0.0640, −0.3864, −0.1320)
Toxic vapour cloud	(0.1590, 0.0009, 0.0003, −0.012, −0.0594, −0.0783)
Electric zard	(0.2850, 0.0160, 0.0160, −0.0070, −0.2325, −0.2110)
Oil spill	(0.1535, 0.0019, 0.0039, −0.0025, −0.1056, −0.1224)
Hydrocarbon gas leakage	(0.3205, 0.0026, 0.0013, −0.0072, −0.3530, −0.1489)
Ammonium nitrate	(0.4095, 0.0023, 0.0008, 0.0, −0.3480, −0.2110)

Table 3.6 Normalized values

	Normalized value
Chemical exposures	1.5966
Oxidizing materials	1.5006
Toxic vapour cloud	1.6540
Electric hazard	1.6090
Oil spill	1.4938
Hydrocarbon gas leakage	1.6036
Ammonium nitrate	1.5089

pollution. Graphs are used to find such graphical representations of networks. But there is always an uncertainty and degree of indeterminacy in data which can be dealt using bipolar neutrosophic graphs.

3.3.1 Bipolar Neutrosophic Graphs for the Reduction of Pollution

Major living organisms on the Earth are human beings, plants and animals. Their survival is strongly dependent on air, water and land. The interaction between living

Table 3.7 Bipolar neutrosophic set C of living organisms and life elements

Elements	T_C^+	I_C^+	F_C^-	T_C^-	I_C^-	F_C^N
Human beings	0.7	0.3	0.8	−0.9	−0.1	−0.9
Animals	0.8	0.4	0.7	−0.8	−0.3	−0.8
Plants	0.9	0.3	0.6	−0.8	−0.2	−0.8
Air	0.9	0.3	0.6	−0.7	−0.3	−0.8
Water	0.8	0.2	0.6	−0.7	−0.4	−0.8
Land	0.8	0.3	0.7	−0.8	−0.4	−0.9

Table 3.8 Bipolar neutrosophic set D of pairs of living organisms and life elements

Elements	T_D^+	I_D^+	F_D^-	T_D^-	I_D^-	F_D^N
(Human beings, animals)	0.7	0.3	0.6	−0.8	−0.1	−0.7
(Human beings, plants)	0.7	0.3	0.6	−0.8	−0.1	−0.5
(Human beings, air)	0.7	0.3	0.8	−0.7	−0.1	−0.9
(Human beings, water)	0.6	0.2	0.7	−0.6	−0.1	−0.8
(Human beings, land)	0.7	0.2	0.7	−0.8	−0.1	−0.7
(Animals, air)	0.6	0.3	0.6	−0.7	−0.2	−0.7
(Animals, water)	0.8	0.2	0.6	−0.7	−0.3	−0.8
(Animals, land)	0.8	0.3	0.7	−0.7	−0.3	−0.6
(Plants, air)	0.9	0.2	0.5	−0.7	−0.2	−0.6
(Plants, water)	0.8	0.2	0.6	−0.7	−0.2	−0.7
(Plants, land)	0.8	0.1	0.7	−0.8	−0.2	−0.6
(Water, land)	0.8	0.2	0.6	−0.7	−0.3	−0.8

organisms and life elements has good, bad or indeterminable effects. We can show this effecting processes using a bipolar neutrosophic graph. We consider a set A of living organisms and life elements in the realm of nature as: $A = \{$human beings, animals, plants, air, water, land$\}$. Further we consider a bipolar neutrosophic set C on set A, as shown in Table 3.7.

In Table 3.7, T_C^+, F_C^- of a living organism or life element shows its positive and negative impacts on nature and I_C^+ show indeterminacy/ambiguity of its impact. Whereas T_C^-, F_C^N denote nature's negative impact on living organism or life element and I_C^- is the percentage of negative ambiguous impact. We now consider a set $E \subseteq X \times X = \{$(human beings, animals), (human beings,plants), (human beings, air), (human beings, water), (human beings, land), (animals, air), (animals, water), (animals, land), (plants, air), (plants, water), (plants, land), (water, land)$\}$. Moreover, we define a bipolar neutrosophic set D on set A as shown in Table 3.8.

In Table 3.8, T_D^+, T_D^- of a pair denote the percentage of positive and negative impacts on each other. Similarly F_D^-, F_D^N and I_D^+, I_D^- represent the percentage of positive and negative false and intermediate effects. A bipolar neutrosophic graph $G = (C, D)$ is shown in Fig. 3.10.

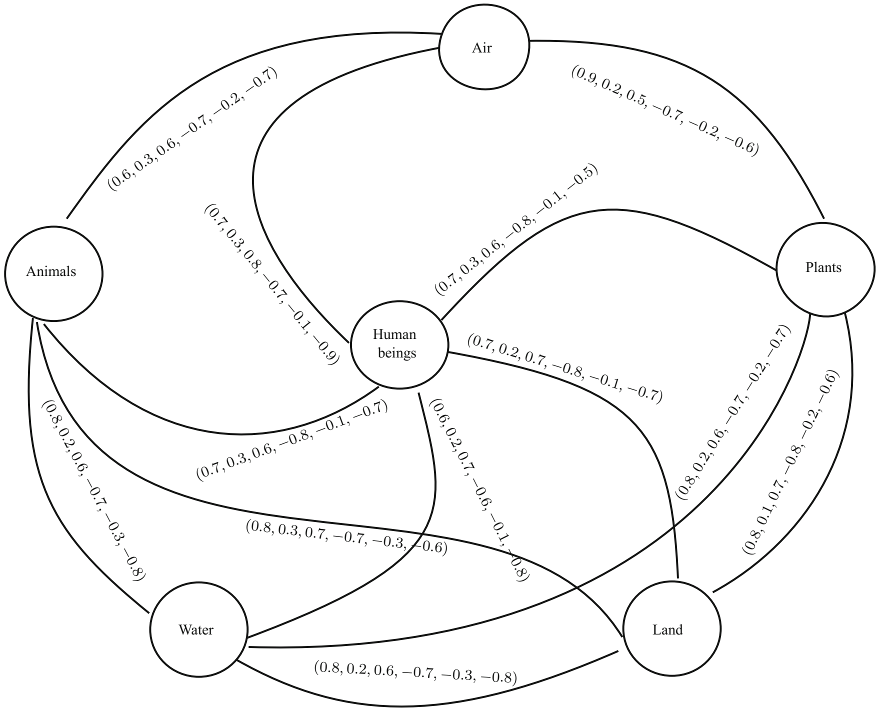


Fig. 3.10 Bipolar neutrosophic graph identifying highly responsible factors for pollution

In this bipolar neutrosophic graph, vertex human being has highest F_D^- value, that is, 0.8 and lowest F_D^N value, that is -0.9 for air which shows that human beings are highly responsible for air pollution and also highly effected by it as compared to other pollution types. Animals and plants have highest F_D^- values for land, 0,7, 0.7, respectively; it shows that they have major contribution in land pollution as compared to other types of pollution. Moreover, animals and plants have lowest F_D^N values for water -0.8, -0.7, respectively; it indicates they are strongly effected by water pollution. This bipolar neutrosophic graph can be a guideline for ENGOs and other pollution control and health organizations that they should prevail awareness and try to take steps to increase positive interaction of human beings with air and take preventive measures to save animals and plants from water pollution. Further, it emphasizes to minimize land pollution by animals and plants. The method for the construction of a structure among living things is given in Algorithm 3.3.1.

Algorithm 3.3.1 Structure among living things

1. Input the n number of objects $L_1, L_2, \ldots, L_n$.
2. Input the bipolar neutrosophic set C of objects.
3. **do** i from $1 \to n$
4.   **do** j from $1 \to n$

read*, ξ_{ij}
if$(i < j, \xi_{ij} \neq (0, 0, 1, 0, 0, -1))$**then**
Draw an edge between L_i and L_j.
$D(L_i L_j) = \xi_{ij}$
end if
end do
end do

3.3.2 *Domination in Bipolar Neutrosophic Graphs*

Domination has a wide variety of applications in communication networks, coding theory, fixing surveillance cameras, detecting biological proteins and social networks, etc. Consider the example of a TV channel that wants to set up transmission stations in a number of cities such that every city in the country gets access to the channel signals from at least one of the stations. To reduce the cost for building large stations it is required to set up minimum number of stations. This problem can be represented by a bipolar neutrosophic graph in which vertices represent the cities and there is an edge between two cities if they can communicate directly with each other. Consider the network of ten cities $\{C_1, C_2, \ldots, C_{10}\}$. In the bipolar neutrosophic graph, the degree of each vertex represents the level of signals it can transmit to other cities and the bipolar neutrosophic value of each edge represents the degree of communication between the cities. The graph is shown in Fig. 3.11. $D = \{C_8, C_{10}\}$ is the minimum dominating set. It is concluded that by building only two large transmitting stations in C_8 and C_{10}, a high economical benefit can be achieved. The method of calculating the minimum number of stations is described in the following Algorithm 3.3.2.

Algorithm 3.3.2 Finding minimum number of stations

Enter the total number of possible locations n.
Input the adjacency matrix $[C_{ij}]_{n\times n}$ of transmission stations $C_1, C_2, \ldots, C_n$.
$k = 0, D = \emptyset$
do i from $1 \to n$
do j from $i + 1 \to n$
if $(T^+, I^+, F^+, T^-, I^-, F^-)(C_i C_j) =$
$(T^+, I^+, F^+, T^-, I^-, F^-)(C_i) \cap (T^+, I^+, F^+, T^-, I^-, F^-)(C_j)$ **then**
$C_i \in D, k = k + 1, x_k = C_i$
end if
end do
end do
Arrange $X \setminus D = \{x_{k+1}, x_{k+2}, \ldots, x_n\} = J$, $p = 0, q = 1$
do i from $1 \to k$
$D' = D \setminus x_{k-i+1}, x_{k-i+1} = x_{n+1}$
do j from $k \to n + 1$

do m from $1 \to k-1$
  if $(T^+, I^+, F^+, T^-, I^-, F^-)(x_m x_j) =$
   $(T^+, I^+, F^+, T^-, I^-, F^-)(x_m) \cap (T^+, I^+, F^+, T^-, I-, F^-)$
   (x_j) **then**
   $D = D^{'}, p = p+1, k = k-1, d_q = x_i,$
   $q = q+1$ stop the loop
  else if $(m = k-1)$ **then**
   $D = D, D^{'} = \emptyset$
  end if
 end do
end do
end do
if $(D \cup (\cup_{i=1}^{q} d_i) \cup J = X)$ **then**
 D is a minimal dominating set.
else
 There is no dominating set.
end if

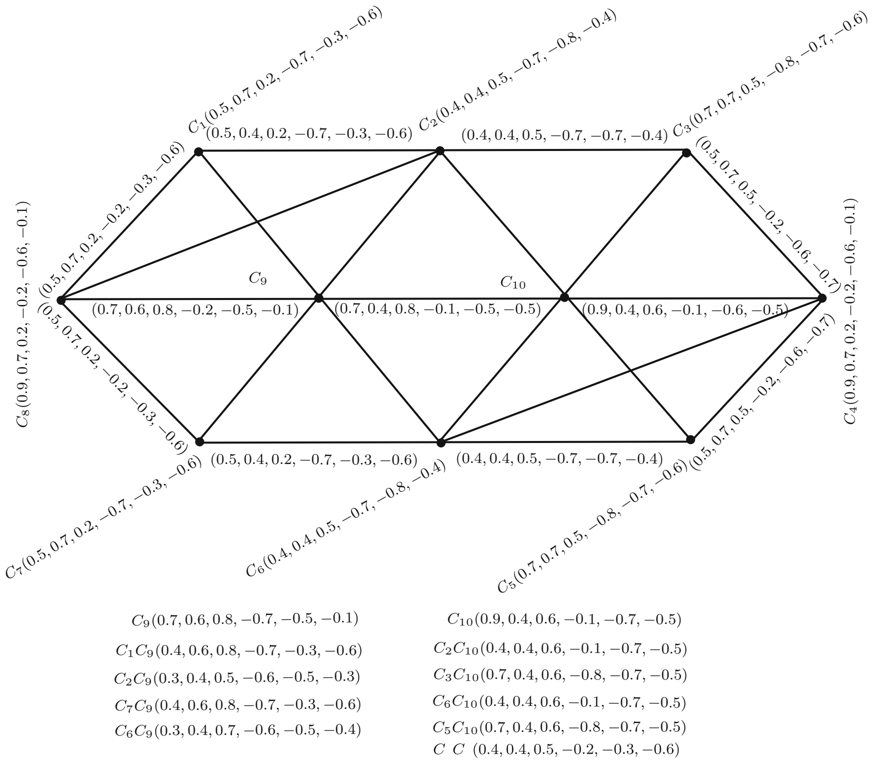


Fig. 3.11 Domination in bipolar neutrosophic graph

3.4 Bipolar Neutrosophic Planar Graphs

Definition 3.21 Let A be a nonempty set with generic elements in A denoted by x. A *bipolar neutrosophic multiset* C drawn from A is characterized by the three positive functions: count truth-membership of CT_C^+, count indeterminacy-membership of CI_C^+ and count falsity-membership of CF_C^+ such that

$$CT_C^+(x) : X \to R^+,$$

$$CI_C^+(x) : X \to R^+,$$

$$CF_C^+(x) : X \to R^+,$$

for $x \in X$, where R^+ is the set of all real number multisets in the real unit interval [0, 1]. The three negative functions: count truth-membership of CT_C^-, count indeterminacy-membership of CI_C^- and count falsity-membership of CF_C^- such that

$$CT_C^-(x) : X \to R^-,$$

$$CI_C^-(x) : X \to R^-,$$

$$CF_C^-(x) : X \to R^-,$$

for $x \in X$, where R^- is the set of all real number multisets in the real unit interval $[-1, 0]$. Then, a bipolar single-valued neutrosophic multiset A is defined as follows.

$$\begin{aligned}
A = \{\langle x, &((T^1)_C^+(x), (T^2)_C^+(x), \ldots, (T^q)_C^+(x)),\\
&((I^1)_C^+(x), (I^2)_C^+(x), \ldots, (I^q)_C^+(x)),\\
&((F^1)_C^+(x), (F^2)_C^+(x), \ldots, (F^q)_C^+(x)),\\
&(T^1)_C^-(x), (T^2)_C^-(x), \ldots, (T^q)_C^-(x)),\\
&((I^1)_C^-(x), (I^2)_C^-(x), \ldots, (I^q)_C^-(x)),\\
&((F^1)_C^-(x), (F^2)_C^-(x), \ldots, (F^q)_C^-(x))\rangle | x \in X\},
\end{aligned}$$

where the positive truth-, indeterminacy- and falsity-membership sequences are given as,

$$\begin{aligned}
&((T^1)_C^+(x), (T^2)_C^+(x), \ldots, (T^q)_C^+(x)),\\
&((I^1)_C^+(x), (I^2)_C^+(x), \ldots, (I^q)_C^+(x)),\\
&((F^1)_C^+(x), (F^2)_C^+(x), \ldots, (F^q)_C^+(x)).
\end{aligned}$$

These sequences may be in decreasing or increasing order. The sum of $(T_C^i)^+(x)$, $(I^i)_C^+(x)$, $(F^i)_C^+(x) \in [0, 1]$ satisfies the following condition: $0 \leq \sup(T^i)_C^+(x) +$

$\sup(I^i)^+_C(x) + \sup(F^i)^+_C(x) \leq 3$, for $x \in X$, $1 \leq i \leq q$. The negative truth-, indeterminacy- and falsity-membership sequences,

$$((T^1)^+_C(x), (T^2)^+_C(x), \ldots, (T^q)^+_C(x)),$$
$$((I^1)^-_C(x), (I^2)^-_C(x), \ldots, (I^q)^-_C(x)),$$
$$((F^1)^-_C(x), (F^2)^-_C(x), \ldots, (F^q)^-_C(x)),$$

may be in decreasing or increasing order. The sum of $(T^i_C)^-(x), (I^i)^-_C(x), (F^i)^-_C(x) \in [-1, 0]$ satisfies the condition: $-3 \leq \inf(T^i)^-_C(x) + \inf(I^i)^-_C(x) + \inf(F^i)^-_C(x) \leq 0$ for $x \in X$ and $1 \leq i \leq q$. For convenience, a bipolar neutrosophic multiset C can be denoted by the simplified form: $C = \{\langle x, (T)^+_C(x)_i, (I)^+_C(x)_i, (F)^+_C(x)_i, (T)^-_C(x)_i, (I)^-_C(x)_i, (F)^-_C(x)_i\rangle | x \in X, 1 \leq i \leq q\}$.

Definition 3.22 Let $C = (T^+_C, I^+_C, F^+_C, T^-_C, I^-_C, F^-_C)$ be a bipolar neutrosophic set on A and $D = \{(xy, T^+_D(xy)_i, I^+_D(xy)_i, F^+_D(xy)_i, T^-_D(xy)_i, I^-_D(xy)_i, F^-_D(xy)_i), 1 \leq i \leq m | xy \in X \times X\}$ be a bipolar neutrosophic multiset of $X \times X$ such that

1. $T^+_D(xy)_i \leq T^+_C(x) \wedge T^+_C(y)$,
2. $T^-_D(xy)_i \geq T^-_C(x) \vee T^-_C(y)$,
3. $I^+_D(xy)_i \leq I^+_C(x) \wedge I^+_C(y)$,
4. $I^-_D(xy)_i \geq I^-_C(x) \vee I^-_C(y)$,
5. $F^+_D(xy)_i \leq F^+_C(x) \vee F^+_C(y)$,
6. $F^-_D(xy)_i \geq F^-_C(x) \wedge F^-_C(y)$,

for all $1 \leq i \leq m$. Then, $G = (C, D)$ is called *a bipolar neutrosophic multigraph.*

There may be more than one edge between the vertices x and y. The positive values $T^+_D(xy)_i, I^+_D(xy)_i, F^+_D(xy)_i$ represent truth, indeterminacy and falsity of the edge xy in G, whereas the negative values $T^-_D(xy)_i, I^-_D(xy)_i, F^-_D(xy)_i$ represent the implicit counter property of the truth-, indeterminacy- and falsity-membership degrees of the edge xy in G. m denotes the number of edges between the vertices. In bipolar neutrosophic multigraph G, D is said to be bipolar neutrosophic multiedge set.

Example 3.10 Let $C = (T^+_C, I^+_C, F^+_C, T^-_C, I^-_C, F^-_C)$ be a bipolar neutrosophic set on $X = \{a, b, c, d\}$, given in Table 3.9, and $D = (T^+_D, I^+_D, F^+_D, T^-_D, I^-_D, F^-_D)$ be a bipolar neutrosophic multiedge set on $\{ab, ab, ab, bc, bd\} = E \subseteq X \times X$ defined in Table 3.10.

By direct calculations, it can be seen from Fig. 3.12 that it is a bipolar neutrosophic multigraph.

Definition 3.23 Let $D = \{(xy, T^+_D(xy)_i, I^+_D(xy)_i, F^+_D(xy)_i, T^-_D(xy)_i, I^-_D(xy)_i, F^-_D(xy)_i), 1 \leq i \leq m | xy \in X \times X\}$ be a bipolar neutrosophic multiedge set in bipolar neutrosophic multigraph G. The degree of a vertex $x \in X$, denoted by $\deg(x)$, is defined by $\deg(x) = (\sum_{i=1}^{m} T^+_D(xy)_i, \sum_{i=1}^{m} I^+_D(xy)_i, \sum_{i=1}^{m} F^+_D(xy)_i, \sum_{i=1}^{m} T^-_D(xy)_i, \sum_{i=1}^{m} I^-_D(xy)_i, \sum_{i=1}^{m} F^-_D(xy)_i)$.

Table 3.9 Bipolar neutrosophic set C

C	a	b	c	d
T_C^+	0.5	0.4	0.5	0.4
I_C^+	0.3	0.2	0.4	0.3
F_C^+	0.3	0.4	0.3	0.4
T_C^-	−0.5	−0.4	−0.5	−0.4
I_C^-	−0.3	−0.2	−0.4	−0.3
F_C^-	−0.3	−0.4	−0.3	−0.4

Table 3.10 Bipolar neutrosophic multiedge set D

D	ab	ab	ab	bc	bd
T_D^+	0.2	0.1	0.2	0.3	0.1
I_D^+	0.2	0.1	0.2	0.1	0.2
F_D^+	0.2	0	0.2	0.3	0.2
T_D^-	−0.2	−0.1	−0.2	−0.3	−0.1
I_D^-	−0.2	−0.1	−0.2	−0.1	−0.2
F_D^-	−0.2	−0	−0.2	−0.3	−0.2

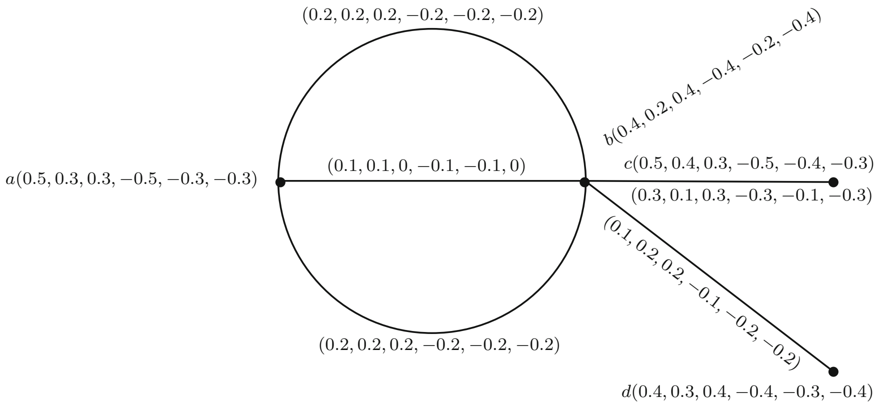


Fig. 3.12 Bipolar neutrosophic multigraph

Example 3.11 In Example 3.10, the degree of vertices a, b, c, d are calculated as,

$$deg(a) = (0.5, 0.5, 0.4, -0.5, -0.5, -0.4),$$

$$deg(b) = (0.9, 0.8, 0.9, -0.9, -0.8, -0.9),$$

$$deg(c) = (0.3, 0.1, 0.3, -0.3, -0.1, -0.3),$$

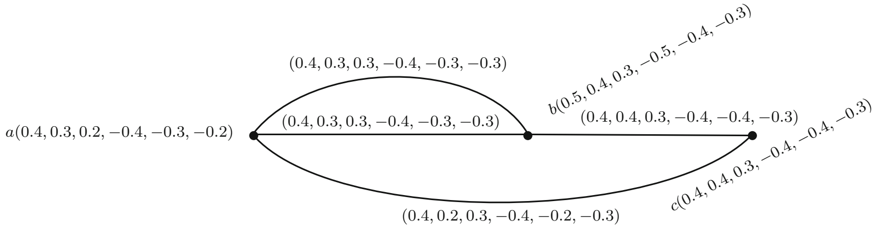


Fig. 3.13 Bipolar neutrosophic complete multigraph

$$deg(d) = (0.1, 0.2, 0.2, -0.1, -0.2, -0.2).$$

Definition 3.24 Let $D = \{(xy, T_D^+(xy)_i, I_D^+(xy)_i, F_D^+(xy)_i, T_D^-(xy)_i, I_D^-(xy)_i, F_D^-(xy)_i), 1 \leq i \leq m | xy \in X \times X\}$ be a bipolar neutrosophic multiedge set in bipolar neutrosophic multigraph G. A multiedge xy of G is *strong* if the following conditions are satisfied,

1. $\frac{1}{2}T_C^+(x) \wedge T_C^+(y)\} \leq T_D^+(xy)_i$,
2. $\frac{1}{2}T_C^-(x) \vee T_C^-(y)\} \geq T_D^-(xy)_i$,
3. $\frac{1}{2}I_C^+(x) \wedge I_C^+(y)\} \leq I_D^+(xy)_i$,
4. $\frac{1}{2}I_C^-(x) \vee I_C^-(y)\} \geq I_D^-(xy)_i$,
5. $\frac{1}{2}F_C^+(x) \vee F_C^+(y)\} \geq F_D^+(xy)_i$,
6. $\frac{1}{2}F_C^-(x) \wedge F_C^-(y)\} \leq F_D^-(xy)_i$, for all $1 \leq i \leq m$.

Definition 3.25 Let $D = \{(xy, T_D^+(xy)_i, I_D^+(xy)_i, F_D^+(xy)_i, T_D^-(xy)_i, I_D^-(xy)_i, F_D^-(xy)_i), 1 \leq i \leq m | xy \in X \times X\}$ be a bipolar neutrosophic multiedge set in bipolar neutrosophic multigraph G. A bipolar neutrosophic multigraph G is *complete* if the following conditions are satisfied.

1. $T_C^+(x) \wedge T_C^+(y)\} = T_D^+(xy)_i$,
2. $T_C^-(x) \vee T_C^-(y)\} = T_D^-(xy)_i$,
3. $I_C^+(x) \wedge I_C^+(y)\} = I_D^+(xy)_i$,
4. $I_C^-(x) \vee I_C^-(y)\} = I_D^-(xy)_i$,
5. $F_C^+(x) \vee F_C^+(y)\} = F_D^+(xy)_i$,
6. $F_C^-(x) \wedge F_C^-(y)\} = F_D^-(xy)_i$, for all $x, y \in X$, $1 \leq i \leq m$.

Example 3.12 Consider a bipolar neutrosophic multigraph G as shown in Fig. 3.13. By routine calculations, it is easy to see that Fig. 3.13 is a bipolar neutrosophic complete multigraph.

Suppose that geometric insight for bipolar neutrosophic graphs has only one crossing between single bipolar valued neutrosophic edges,
$(ab, T_D^+(ab)_i, I_D^+(ab)_i, F_D^+(ab)_i, T_D^-(ab)_i, I_D^-(ab)_i, F_D^-(ab)_i)$ and
$(cd, T_D^+(cd)_i, I_D^+(cd)_i, F_D^+(cd)_i, T_D^-(cd)_i, I_D^-(cd)_i, F_D^-(cd)_i)$. We note that:

$$\begin{aligned}
\text{If } & (ab, T_D^+(ab)_i, I_D^+(ab)_i, F_D^+(ab)_i, T_D^-(ab)_i, I_D^-(ab)_i, F_D^-(ab)_i) \\
&= (1, 1, 1, -1, -1, -1), \\
& (cd, T_D^+(cd)_i, I_D^+(cd)_i, F_D^+(cd)_i, T_D^-(cd)_i, I_D^-(cd)_i, F_D^-(cd)_i) \\
&= (0, 0, 0, 0, 0, 0), \\
\text{or } & (ab, T_D^+(ab)_i, I_D^+(ab)_i, F_D^+(ab)_i, T_D^-(ab)_i, I_D^-(ab)_i, F_D^-(ab)_i) \\
&= (0, 0, 0, 0, 0, 0), \\
& (cd, T_D^+(cd)_i, I_D^+(cd)_i, F_D^+(cd)_i, T_D^-(cd)_i, I_D^-(cd)_i, F_D^-(cd)_i) \\
&= (1, 1, 1, -1, -1, -1),
\end{aligned}$$

then bipolar neutrosophic graph has no crossing,

$$\begin{aligned}
\text{If } & (ab, T_D^+(ab)_i, I_D^+(ab)_i, F_D^+(ab)_i, T_D^-(ab)_i, I_D^-(ab)_i, F_D^-(ab)_i) \\
&= (1, 1, 1, -1, -1, -1), \\
& (cd, T_D^+(cd)_i, I_D^+(cd)_i, F_D^+(cd)_i, T_D^-(cd)_i, I_D^-(cd)_i, F_D^-(cd)_i) \\
&= (1, 1, 1, -1, -1, -1),
\end{aligned}$$

then there exists a crossing for the representation of the graph.

Definition 3.26 The *strength of the bipolar neutrosophic edge ab* can be measured by the following value,

$$\begin{aligned}
S_{ab} &= ((S_{T^+})_{ab}, (S_{I^+})_{ab}, (S_{F^+})_{ab}, (S_{T^-})_{ab}, (S_{I^-})_{ab}, (S_{F^-})_{ab}) \\
&= \Bigg(\frac{T_D^+(ab)_i}{T_C^+(a) \wedge T_C^+(b)}, \frac{I_D^+(ab)_i}{I_C^+(a) \wedge I_C^+(b)}, \frac{F_D^+(ab)_i}{F_C^+(a) \vee F_C^+(b)}, \\
&\quad \frac{T_D^-(ab)_i}{T_C^-(a) \vee T_C^-(b)}, \frac{I_D^-(ab)_i}{I_C^-(a) \vee I_C^-(b)}, \frac{F_D^-(ab)_i}{F_C^-(a) \wedge F_C^-(b)} \Bigg).
\end{aligned}$$

Definition 3.27 Let G be a bipolar neutrosophic multigraph. An edge ab is said to be a *strong* if

$$\begin{aligned}
&(S_{T^+})_{ab} \geq 0.5, && (S_{I^+})_{ab} \geq 0.5, && (S_{F^+})_{ab} \geq 0.5, \\
&(S_{T^-})_{ab} \leq -0.5, && (S_{I^-})_{ab} \leq -0.5, && (S_{F^-})_{ab} \leq -0.5.
\end{aligned}$$

Otherwise, it is called a *weak edge*.

Definition 3.28 Let $G = (C, D)$ be a bipolar neutrosophic multigraph such that D contains two edges as,

$$(ab, T_D^+(ab)_i, I_D^+(ab)_i, F_D^+(ab)_i, T_D^-(ab)_i, I_D^-(ab)_i, F_D^-(ab)_i)$$

and

$$(cd, T_D^+(cd)_j, I_D^+(cd)_j, F_D^+(cd)_j, T_D^-(cd)_j, I_D^-(cd)_j, F_D^-(cd)_j),$$

intersected at a point P, where i and j are fixed integers. We define the intersecting value at the point Q as,

$$\begin{aligned}\mathcal{S}_Q &= ((\mathcal{S}_{\mathcal{T}^+})_Q, (\mathcal{S}_{\mathcal{I}^+})_Q, (\mathcal{S}_{\mathcal{F}^+})_Q, (\mathcal{S}_{\mathcal{T}^-})_Q, (\mathcal{S}_{\mathcal{I}^-})_Q, (\mathcal{S}_{\mathcal{F}^-})_Q)\\ &= \Bigg(\frac{(S_{T^+})_{ab} + (S_{T^+})_{cd}}{2}, \frac{(S_{I^+})_{ab} + (S_{I^+})_{cd}}{2}, \frac{(S_{F^+})_{ab} + (S_{F^+})_{cd}}{2},\\ &\quad \frac{(S_{T^-})_{ab} + (S_{T^-})_{cd}}{2}, \frac{(S_{I^-})_{ab} + (S_{I^-})_{cd}}{2}, \frac{(S_{F^-})_{ab} + (S_{F^-})_{cd}}{2}\Bigg).\end{aligned}$$

If the number of point of intersections in a bipolar neutrosophic multigraph increases, planarity decreases. Thus for bipolar neutrosophic multigraph, $\mathcal{S}_Q$ is inversely proportional to the planarity. We now introduce the concept of a bipolar neutrosophic planar graph.

Definition 3.29 Let G be a bipolar neutrosophic multigraph and $Q_1, Q_2, \ldots, Q_z$ be the points of intersection between the edges for a certain geometrical representation, G is said to be a bipolar neutrosophic planar graph with bipolar neutrosophic planarity value $f = (f_{T^+}, f_{I^+}, f_{F^+}, f_{T^-}, f_{I^-}, f_{F^-})$ where

$$\begin{aligned}f &= (f_{T^+}, f_{I^+}, f_{F^+}, f_{T^-}, f_{I^-}, f_{F^-}),\\ &= (\frac{1}{1 + \{(S_{T^+})_{Q_1} + (S_{T^+})_{Q_2} + \cdots + (S_{T^+})_{Q_z}\}},\\ &\quad \frac{1}{1 + \{(S_{I^+})_{Q_1} + (S_{I^+})_{Q_2} + \cdots + (S_{I^+})_{Q_z}\}}\\ &\quad \frac{1}{1 + \{(S_{F^+})_{Q_1} + (S_{F^+})_{Q_2} + \cdots + (S_{F^+})_{Q_z}\}},\\ &\quad \frac{1}{-1 - \{(S_{T^-})_{Q_1} + (S_{T^-})_{Q_2} + \cdots + (S_{T^-})_{Q_z}\}},\\ &\quad \frac{1}{-1 - \{(S_{I^-})_{Q_1} + (S_{I^-})_{Q_2} + \cdots + (S_{I^-})_{Q_z}\}},\\ &\quad \frac{1}{-1 - \{(S_{F^-})_{Q_1} + (S_{F^-})_{Q_2} + \cdots + (S_{F^-})_{Q_z}\}}).\end{aligned}$$

Clearly, $f = (f_{T^+}, f_{I^+}, f_{F^+}, f_{T^-}, f_{I^-}, f_{F^-})$ is bounded and

$$\begin{aligned}&0 < f_{T^+} \leq 1, &&0 < f_{I^+} \leq 1, &&0 < f_{F^+} \leq 1,\\ &-1 < f_{T^-} \leq 0, &&-1 < f_{I^-} \leq 0, &&-1 < f_{F^-} \leq 0.\end{aligned}$$

If there is no point of intersection for a certain geometrical representation of a bipolar neutrosophic planar graph, then its bipolar neutrosophic planarity value is $(1, 1, 1, -1, -1, -1)$. We conclude that every bipolar neutrosophic graph is a bipolar neutrosophic planar graph with certain bipolar neutrosophic planarity value.

Example 3.13 Let $X = \{a, b, c, d, e\}$ and $E = \{ab, ac, ad, ad, bc, bd, cd, ce, ae, de, be\}$. Let $C = (T_C^+, I_C^+, F_C^+, T_C^-, I_C^-, F_C^-)$ be a bipolar neutrosophic set on A and $D = (T_D^+, I_D^+, F_D^+, T_D^-, I_D^-, F_D^-)$ be a bipolar neutrosophic multiedge set on $X \times X$ defined in Tables 3.11 and 3.12.

The bipolar neutrosophic multigraph as shown in Fig. 3.14 has two point of intersections P_1 and P_2. P_1 is a point between the edges

$$(ad, 0.2, 0.2, 0.1, -0.2, -0.2, -0.1),$$

$$(bc, 0.2, 0.2, 0.1, -0.2, -0.2, -0.1),$$

and P_2 is between

$$(ad, 0.3, 0.3, 0.1, -0.3, -0.3, -0.1),$$

$$(bc, 0.2, 0.2, 0.1, -0.2, -0.2, -0.1).$$

For the edge $(ad, 0.2, 0.2, 0.1, -0.2, -0.2, -0.1)$,

$$S_{ad} = (0.4, 0.4, 0.5, -0.4, -0.4, -0.5).$$

For the edge $(ad, 0.3, 0.3, 0.1, -0.3, -0.3, -0.1)$,

$$S_{ad} = (0.6, 0.6, 0.5, -0.6, -0.6, -0.5),$$

Table 3.11 Bipolar neutrosophic set C

A	a	b	c	d	e
T_C^+	0.5	0.4	0.3	0.6	0.6
I_C^+	0.5	0.4	0.3	0.6	0.6
F_C^+	0.2	0.1	0.1	0.2	0.1
T_C^-	−0.5	−0.4	−0.3	−0.6	−0.6
I_C^-	−0.5	−0.4	−0.3	−0.6	−0.6
F_C^-	−0.2	−0.1	−0.1	−0.2	−0.1

Table 3.12 Bipolar neutrosophic multiedge set D

B	ab	ac	ad	ad	bc	bd	cd	ae	ce	de	be
T_D^+	0.2	0.2	0.2	0.3	0.2	0.2	0.2	0.2	0.2	0.2	0.2
I_D^+	0.2	0.2	0.2	0.3	0.2	0.2	0.2	0.2	0.2	0.2	0.2
F_D^+	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
T_D^-	−0.2	−0.2	−0.2	−0.3	−0.2	−0.2	−0.2	−0.2	−0.2	−0.2	−0.2
I_D^-	−0.2	−0.2	−0.2	−0.3	−0.2	−0.2	−0.2	−0.2	−0.2	−0.2	−0.2
F_D^-	−0.1	−0.1	−0.1	−0.1	−0.1	−0.1	−0.1	−0.1	−0.1	−0.1	−0.1

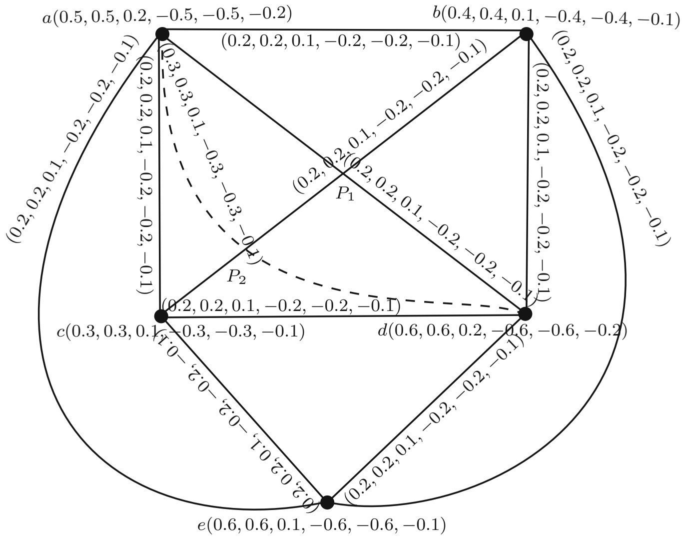


Fig. 3.14 Neutrosophic planar graph

and for the edge $(bc, 0.2, 0.2, 0.1, -0.2, -0.2, -0.1)$,

$$S_{bc} = (0.6667, 0.6667, 1, -0.6667, -0.6667, -1).$$

For the first point of intersection P_1, intersecting value $\mathcal{S}_{P_1}$ is (0.5334, 0.5334, 0.75, −0.5334, −0.5334, −0.75). For the second point of intersection P_2, $\mathcal{S}_{P_2}$, the intersecting value is (0.63335, 0.63335, 0.75, −0.63335, −0.63335, −0.75). Therefore, the bipolar neutrosophic planarity value for the bipolar neutrosophic multigraph shown in Fig. 3.14 is (0.461, 0.461, 0.4, −0.461, −0.461,−0.4).

Theorem 3.17 *Let G be a bipolar neutrosophic complete multigraph. The planarity value, $f = (f_{T^+}, f_{I^+}, f_{F^+}, f_{T^-}, f_{I^-}, f_{F^-})$, of G is given by*

$$f_{T^+} = \frac{1}{1+n_Q}, f_{I^+} = \frac{1}{1+n_Q}, f_{F^+} = \frac{1}{1+n_Q}, 0 \leq f_{T^+} + f_{I^+} + f_{F^+} \leq 3,$$

$$f_{T^-} = \frac{1}{-1-n_Q}, f_{I^-} = \frac{1}{-1-n_Q}, f_{F^-} = \frac{1}{-1-n_Q}, -3 \leq f_{T^-} + f_{I^-} + f_{F^-} \leq 0,$$

where n_Q is the number of point of intersections between the edges in G.

Definition 3.30 A bipolar neutrosophic planar graph G is called *strong bipolar neutrosophic planar graph* if the bipolar neutrosophic planarity value $f=(f_{T^+}, f_{I^+}, f_{F^+}, f_{T^-}, f_{I^-}, f_{F^-})$ of G satisfies the following conditions,

$$f_{T^+} \geq 0.5, \qquad f_{I^+} \geq 0.5, \qquad f_{F^+} \leq 0.5,$$
$$f_{T^-} \leq -0.5, \qquad f_{I^-} \leq -0.5, \qquad f_{F^+} \geq -0.5.$$

Theorem 3.18 *Let G be a strong bipolar neutrosophic planar graph. The number of points of intersections between strong edges in G is at most one.*

Proof Let G be a strong bipolar neutrosophic planar graph. Assume that G has at least two point of intersections P_1 and P_2 between two strong bipolar neutrosophic edges in G. For any strong edge
$(ab, T_D^+(ab)_i, I_D^+(ab)_i, F_D^+(ab)_i, T_D^-(ab)_i, I_D^-(ab)_i, F_D^-(ab)_i)$,

$$T_D^+(ab)_i \geq \frac{1}{2}T_C^+(a) \wedge T_C^+(b), \qquad T_D^-(ab)_i \leq \frac{1}{2}T_C^-(a) \vee T_C^-(b),$$
$$I_D^+(ab)_i \geq \frac{1}{2}I_C^+(a) \wedge I_C^+(b), \qquad I_D^-(ab)_i \leq \frac{1}{2}I_C^-(a) \vee I_C^-(b),$$
$$F_D^+(ab)_i \leq \frac{1}{2}F_C^+(a) \vee F_C^+(b), \qquad F_D^-(ab)_i \geq \frac{1}{2}F_C^-(a) \wedge F_C^-(b).$$

It shows that

$$(S_{T^+})_{ab} \geq 0.5, \quad (S_{I^+})_{ab} \geq 0.5, \quad (S_{F^+})_{ab} \leq 0.5,$$
$$(S_{T^-})_{ab} \leq -0.5, \quad (S_{I^-})_{ab} \leq -0.5, \quad (S_{F^-})_{ab} \geq -0.5.$$

Thus for two intersecting strong bipolar neutrosophic edges, we have

$$(ab, T_D^+(ab)_i, I_D^+(ab)_i, F_D^+(ab)_i, T_D^-(ab)_i, I_D^-(ab)_i, F_D^-(ab)_i),$$

$$(cd, T_D^+(cd)_j, I_D^+(cd)_j, F_D^+(cd)_j, T_D^-(cd)_j, I_D^-(cd)_j, F_D^-(cd)_j).$$

$$\frac{(S_{T^+})_{ab} + (S_{T^+})_{cd}}{2} \geq 0.5, \qquad \frac{(S_{I^+})_{ab} + (S_{I^+})_{cd}}{2} \geq 0.5,$$
$$\frac{(S_{F^+})_{ab} + (S_{F^+})_{cd}}{2} \leq 0.5, \qquad \frac{(S_{T^-})_{ab} + (S_{T^-})_{cd}}{2} \leq -0.5,$$
$$\frac{(S_{I^-})_{ab} + (S_{I^-})_{cd}}{2} \leq -0.5, \qquad \frac{(S_{F^-})_{ab} + (S_{F^-})_{cd}}{2} \geq -0.5.$$

That is,

$$(S_{T^+})_{Q_1} \geq 0.5, \quad (S_{I^+})_{Q_1} \geq 0.5, \quad (S_{F^+})_{Q_1} \leq 0.5,$$
$$(S_{T^-})_{Q_1} \leq -0.5, \quad (S_{I^-})_{Q_1} \leq -0.5, \quad (S_{F^-})_{Q_1} \geq -0.5.$$

Similarly, we can prove that

$$(S_{T^+})_{Q_2} \geq 0.5, \quad (S_{I^+})_{Q_2} \geq 0.5, \quad (S_{F^+})_{Q_2} \leq 0.5,$$
$$(S_{T^-})_{Q_2} \leq -0.5, \quad (S_{I^-})_{Q_2} \leq -0.5, \quad (S_{F^-})_{Q_2} \geq -0.5.$$

$$\Rightarrow 1 + (S_{T^+})_{Q_1} + (S_{T^+})_{Q_2} \geq 2, \quad -1 + (S_{T^-})_{Q_1} + (S_{T^-})_{Q_2} \leq -2,$$
$$1 + (S_{I^+})_{Q_1} + (S_{I^+})_{Q_2} \geq 2, \quad -1 + (S_{I^-})_{Q_1} + (S_{I^-})_{Q_2} \leq -2,$$
$$1 + (S_{F^+})_{Q_1} + (S_{F^+})_{Q_2} \leq 2, \quad -1 + (S_{F^-})_{Q_1} + (S_{F^-})_{Q_2} \geq -2.$$

Therefore,

$$f_{T^+} = \frac{1}{1 + (S_{T^+})_{Q_1} + (S_{T^+})_{Q_2}} \leq 0.5, \quad f_{T^-} = \frac{1}{-1 + (S_{T^-})_{Q_1} + (S_{T^-})_{Q_2}} \geq -0.5,$$
$$f_{I^+} = \frac{1}{1 + (S_{I^+})_{Q_1} + (S_{I^+})_{Q_2}} \leq 0.5, \quad f_{I^-} = \frac{1}{-1 + (S_{I^-})_{Q_1} + (S_{I^-})_{Q_2}} \geq -0.5,$$
$$f_{F^+} = \frac{1}{1 + (S_{F^+})_{Q_1} + (S_{F^+})_{Q_2}} \geq 0.5, \quad f_{F^-} = \frac{1}{-1 + (S_{F^-})_{Q_1} + (S_{F^-})_{Q_2}} \leq -0.5.$$

It contradicts the fact that the bipolar neutrosophic graph is a strong bipolar neutrosophic planar graph. Thus, number of point of intersections between strong edges cannot be two. Obviously, if the number of point of intersections of strong bipolar neutrosophic edges increases, the bipolar neutrosophic planarity value decreases. Similarly, if the number of point of intersection of strong edges is one, then the bipolar neutrosophic planarity value

$$f_{T^+} > 0.5, \quad f_{I^+} > 0.5, \quad f_{I^+} > 0.5,$$
$$f_{T^-} < -0.5, \quad f_{I^-} < -0.5, \quad f_{I^-} < -0.5.$$

Any bipolar neutrosophic planar graph without any crossing between edges is a strong bipolar neutrosophic planar graph. Thus, we conclude that the maximum number of point of intersections between the strong edges in G is one.

Face of a bipolar neutrosophic planar graph is an important parameter. Face of a bipolar neutrosophic graph is a region bounded by bipolar neutrosophic edges. Every bipolar neutrosophic face is characterized by bipolar neutrosophic edges in its boundary. If all the edges in the boundary of a bipolar neutrosophic face have T^+, I^+, F^+, T^-, I^- and F^- values $(1, 1, 1, -1, -1, -1)$ and $(0, 0, 0, 0, 0, 0)$, respectively, it becomes crisp face. If one of such edges is removed or has T^+, I^+, F^+, T^-, I^- and F^- values $(0, 0, 0, 0, 0, 0)$ and $(1, 1, 1, -1, -1, -1)$, respectively, the bipolar neutrosophic face does not exist. So the existence of a bipolar neutrosophic face depends on the minimum value of strength of bipolar neutrosophic edges in its boundary. A bipolar neutrosophic face and its T^+, I^+, F^+, T^-, I^-, and F^- values of a bipolar neutrosophic graph are defined below.

Definition 3.31 Let G be a bipolar neutrosophic planar graph and
$D = \{(xy,\ T^+_D(xy)_i,\ I^+_D(xy)_i,\ F^+_D(xy)_i, T^-_D(xy)_i,\ I^-_D(xy)_i,\ F^-_D(xy)_i),\ i = 1,\ 2,$

$\ldots, m|xy \in X \times X\}$. A bipolar neutrosophic face of G is a region, bounded by the set of bipolar neutrosophic edges $E' \subset E$, of a geometric representation of G. The truth, indeterminacy and falsity values of the bipolar neutrosophic face are:

1. $\min\left\{\dfrac{T_D^+(xy)_i}{T_C^+(x) \wedge T_C^+(y)}, 1 \le i \le m|\, xy \in E'\right\}$,
2. $\max\left\{\dfrac{T_D^-(xy)_i}{T_C^-(x) \vee T_C^-(y)}, 1 \le i \le m|\, xy \in E'\right\}$,
3. $\min\left\{\dfrac{I_D^+(xy)_i}{I_C^+(x) \wedge I_C^+(y)}, 1 \le i \le m|\, xy \in E'\right\}$,
4. $\max\left\{\dfrac{I_D^-(xy)_i}{I_C^-(x) \vee I_C^-(y)}, 1 \le i \le m|\, xy \in E'\right\}$,
5. $\max\left\{\dfrac{F_D^+(xy)_i}{F_C^+(x) \vee F_C^+(y)}, 1 \le i \le m|\, xy \in E'\right\}$,
6. $\min\left\{\dfrac{F_D^-(xy)_i}{F_C^-(x) \wedge F_C^-(y)}, 1 \le i \le m|\, xy \in E'\right\}$.

Definition 3.32 A bipolar neutrosophic face is called *strong bipolar neutrosophic face* if its positive true and indeterminacy value is greater than 0.5 but false value is lesser than 0.5, and negative true and indeterminacy value is less than −0.5 but false value is greater than −0.5. Otherwise, face is weak. Every bipolar neutrosophic planar graph has an infinite region which is called *outer bipolar neutrosophic face*. Other faces are called *inner bipolar neutrosophic faces*.

Example 3.14 Consider a bipolar neutrosophic planar graph as shown in Fig. 3.15. The bipolar neutrosophic planar graph has the following faces.

- Bipolar neutrosophic face F_1 is bounded by the edges $(v_1v_2, 0.5, 0.5, 0.1, -0.5, -0.5, -0.1)$, $(v_2v_3, 0.6, 0.6, 0.1, -0.6, -0.6, -0.1)$, $(v_1v_3, 0.5, 0.5, 0.1, -0.5, -0.5, -0.1)$.
- Outer bipolar neutrosophic face F_2 surrounded by edges $(v_1v_3, 0.5, 0.5, 0.1, -0.5, -0.5, -0.1)$, $(v_1v_4, 0.5, 0.5, 0.1, -0.5, -0.5, -0.1)$, $(v_2v_4, 0.6, 0.6, 0.1, -0.6, -0.6, -0.1)$, $(v_2v_3, 0.6, 0.6, 0.1, -0.6, -0.6, -0.1)$.
- Bipolar neutrosophic face F_3 is bounded by the edges $(v_1v_2, 0.5, 0.5, 0.1, -0.5, -0.5, -0.1)$, $(v_2v_4, 0.6, 0.6, 0.1, -0.6, -0.6, -0.1)$, $(v_1v_4, 0.5, 0.5, 0.1, -0.5, -0.5, -0.1)$.

Clearly, the positive truth, indeterminacy and falsity values of a bipolar neutrosophic face F_1 are 0.833, 0.833 and 0.333, respectively, and the negative truth, indeterminacy and falsity values of a bipolar neutrosophic face F_1 are −0.833, −0.833 and −0.333, respectively. The positive truth, indeterminacy and falsity values of a bipolar neutrosophic face F_3 are 0.833, 0.833 and 0.333, respectively, and the negative truth, indeterminacy and falsity values of a bipolar neutrosophic face F_3 are −0.833, −0.833 and −0.333, respectively. Thus, F_1 and F_3 are strong bipolar neutrosophic faces.

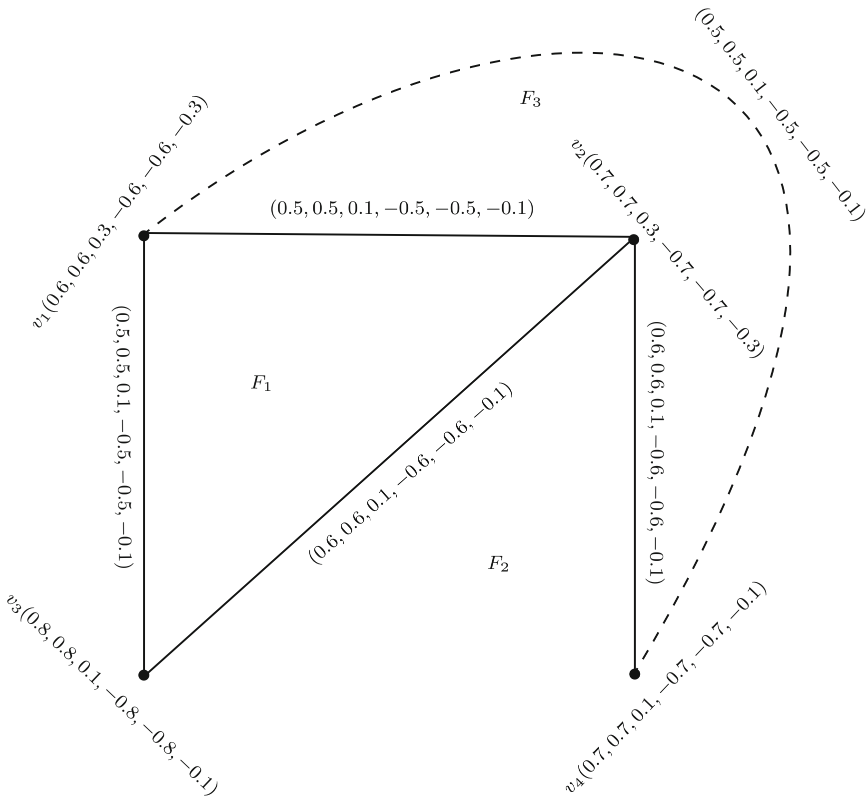


Fig. 3.15 Faces in bipolar neutrosophic planar graph

We now introduce dual of bipolar neutrosophic planar graph. In bipolar neutrosophic dual graph, vertices are corresponding to the strong bipolar neutrosophic faces of the bipolar neutrosophic planar graph and each bipolar neutrosophic edge between two vertices is corresponding to each edge in the boundary between two faces of bipolar neutrosophic planar graph. The formal definition is given below.

Definition 3.33 Let G be a bipolar neutrosophic planar graph, and let $D = \{(xy, T_D^+(xy)_i, I_D^+(xy)_i, F_D^+(xy)_i,\ T_D^-(xy)_i,\ I_D^-(xy)_i,\ F_D^-(xy)_i),\ i = 1,\ 2,\ \ldots, m | xy \in X \times X\}$. Let $F_1, F_2, \ldots, F_k$ be the strong bipolar neutrosophic faces of G. The bipolar neutrosophic dual graph of G is a bipolar neutrosophic planar graph $G' = (X', C', D')$, where $X' = \{x_i, i = 1, 2, \ldots, k\}$, and the vertex x_i of G' is considered for the face F_i of G. The truth-membership, indeterminacy and falsetruth-membership values of vertices are given by the mapping $C' = (T_{C'}^+, I_{C'}^+, F_{C'}^+, T_{C'}^-, I_{C'}^-, F_{C'}^-) : X' \to [0, 1] \times [0, 1] \times [0, 1] \times [-1, 0] \times [-1, 0] \times [-1, 0]$ such that

$T_{C'}^+(x_i) = \max\{T_{D'}^+(uv)_i, 1 \le i \le p | uv$ is an edge of the boundary of the strong bipolar neutrosophic face $F_i\}$,

$T^-_{C'}(x_i) = \min\{T^-_{D'}(uv)_i, 1 \leq i \leq p | uv$ is an edge of the boundary of the strong bipolar neutrosophic face $F_i\}$,
$I^+_{C'}(x_i) = \max\{I^+_{D'}(uv)_i, 1 \leq i \leq p | uv$ is an edge of the boundary of the strong bipolar neutrosophic face $F_i\}$,
$I^-_{C'}(x_i) = \min\{I^-_{D'}(uv)_i, 1 \leq i \leq p | uv$ is an edge of the boundary of the strong bipolar neutrosophic face $F_i\}$,
$F^+_{C'}(x_i) = \min\{F^+_{D'}(uv)_i, 1 \leq i \leq p | uv$ is an edge of the boundary of the strong bipolar neutrosophic face $F_i\}$,
$F^-_{C'}(x_i) = \max\{F^-_{D'}(uv)_i, 1 \leq i \leq p | uv$ is an edge of the boundary of the strong bipolar neutrosophic face $F_i\}$.

There may exist more than one common edges between two faces F_i and F_j of G. Thus, there may be more than one edges between two vertices x_i and x_j in bipolar neutrosophic dual graph G'. Let $(T^+)^l_D(x_i x_j)$, $(I^+)^l_D(x_i x_j)$ and $(F^+)^l_D(x_i x_j)$ denote the positive truth-, indeterminacy- and falsity-membership values of the lth edge between x_i and x_j, and let $(T^-)^l_D(x_i x_j)$, $(I^-)^l_D(x_i x_j)$ and $(F^-)^l_D(x_i x_j)$ denote the negative truth-, indeterminacy- and falsity-membership values of the lth edge between x_i and x_j. The positive and negative truth, indeterminacy and falsity values of the bipolar neutrosophic edges of the bipolar neutrosophic dual graph are given as

$$T^+_{D'}(x_i x_j)_l = (T^+)^l_D(uv)_j, \qquad I^+_{D'}(x_i x_j)_l = (I^+)^l_D(uv)_j,$$
$$F^+_{D'}(x_i x_j)_l = (F^+)^l_D(uv)_j, \qquad T^-_{D'}(x_i x_j)_l = (T^-)^l_D(uv)_j,$$
$$I^-_{D'}(x_i x_j)_l = (I^-)^l_D(uv)_j, \qquad F^-_{D'}(x_i x_j)_l = (F^-)^l_D(uv)_j.$$

where $(uv)_l$ is an edge in the boundary between two strong bipolar neutrosophic faces F_i and F_j and $1 \leq l \leq s$, where s is the number of common edges in the boundary between F_i and F_j or the number of edges between x_i and x_j. If there be any strong pendant edge in the bipolar neutrosophic planar graph, then there will be a self-loop in G' corresponding to this pendant edge. The edge truth-membership, indeterminacy-membership and falsity-membership values of the self-loop are equal to the truth-membership, indeterminacy-membership and falsity-membership values of the pendant edge. Single-valued neutrosophic dual graph of bipolar neutrosophic planar graph does not contain point of intersection of edges for a certain representation, so it is bipolar neutrosophic planar graph with planarity value $(1, 1, 1, -1, -1, -1)$. Thus, the bipolar neutrosophic face of bipolar neutrosophic dual graph can be similarly described as in bipolar neutrosophic planar graphs.

Example 3.15 Consider a bipolar neutrosophic planar graph $G = (X, A, B)$ as shown in Fig. 3.16 such that $A = \{a, b, c, d\}$,

$$C = \{(a, 0.6, 0.6, 0.2, -0.6, -0.6, -0.2), \quad (b, 0.7, 0.7, 0.2, -0.7, -0.7, -0.2),$$
$$(c, 0.8, 0.8, 0.2, -0.8, -0.8, -0.2), \quad (d, 0.9, 0.9, 0.1, -0.9, -0.9, -0.1)\},$$

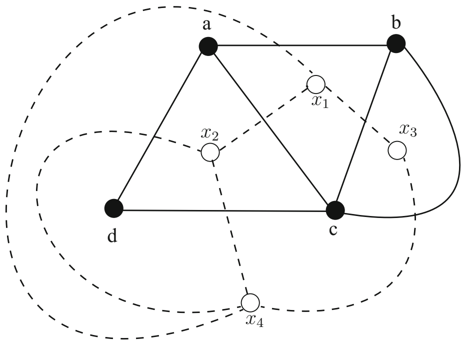


Fig. 3.16 Neutrosophic dual graph

$$D = \{(ab, 0.5, 0.5, 0.01, -0.5, -0.5, -0.01), (ac, 0.4, 0.4, 0.01, -0.4, -0.4, -0.01),$$
$$(bc, 0.6, 0.6, 0.01, -0.6, -0.6, -0.01), (cd, 0.7, 0.7, 0.01, -0.7, -0.7, -0.01),$$
$$(ad, 0.55, 0.55, 0.01, -0.55, -0.55, -0.01), (bc, 0.45, 0.45, 0.01, -0.45,$$
$$-0.45, -0.01)\}.$$

The bipolar neutrosophic planar graph has the following faces.

- Bipolar neutrosophic face F_1 is bounded by $(ab, 0.5, 0.5, 0.01, -0.5, -0.5, -0.01)$, $(ac, 0.4, 0.4, 0.01, -0.4, -0.4, -0.01)$, $(bc, 0.45, 0.45, 0.01, -0.45, -0.45, -0.01)$.
- Bipolar neutrosophic face F_2 is bounded by $(ad, 0.55, 0.55, 0.01, -0.55, -0.55, -0.01)$, $(cd, 0.7, 0.7, 0.01, -0.7, -0.7, -0.01)$, $(ac, 0.4, 0.4, 0.01, -0.4, -0.4, -0.01)$.
- Bipolar neutrosophic face F_3 is bounded by $(bc, 0.45, 0.45, 0.01, -0.45, -0.45, -0.01)$, $(bc, 0.6, 0.6, 0.01, -0.6, -0.6, -0.01)$.
- Outer bipolar neutrosophic face F_4 is surrounded by $(ab, 0.5, 0.5, 0.01, -0.5, -0.5, -0.01)$, $(bc, 0.6, 0.6, 0.01, -0.6, -0.6, -0.01)$, $(cd, 0.7, 0.7, 0.01, -0.7, -0.7, -0.01)$, $(ad, 0.55, 0.55, 0.01, -0.55, -0.55, -0.01)$.

Routine calculations show that all faces are strong bipolar neutrosophic faces. For each strong bipolar neutrosophic face, we consider a vertex for the bipolar neutrosophic dual graph. So the vertex set $X' = \{x_1, x_2, x_3, x_4\}$, where the vertex x_i is taken corresponding to the strong bipolar neutrosophic face F_i, $i = 1, 2, 3, 4$. Thus,

$$T^+_{C'}(x_1) = \max\{0.5, 0.4, 0.45\} = 0.5, T^+_{C'}(x_2) = \max\{0.55, 0.7, 0.4\} = 0.7,$$
$$T^-_{C'}(x_1) = \min\{-0.5, -0.4, -0.45\} = -0.5,$$
$$T^-_{C'}(x_2) = \min\{-0.55, -0.7, -0.4\} = -0.7,$$
$$I^+_{C'}(x_1) = \max\{0.5, 0.4, 0.45\} = 0.5, I^+_{C'}(x_2) = \max\{0.55, 0.7, 0.4\} = 0.7,$$
$$I^-_{C'}(x_1) = \min\{-0.5, -0.4, -0.45\} = -0.5,$$

$$I^-_{C'}(x_2) = \min\{-0.55, -0.7, -0.4\} = -0.7,$$
$$F^+_{C'}(x_1) = \min\{0.01, 0.01, 0.01\} = 0.01,\ F^+_{C'}(x_2) = \min\{0.01, 0.01, 0.01\} = 0.01,$$
$$F^-_{C'}(x_1) = \max\{-0.01, -0.01, -0.01\} = -0.01,$$
$$F^-_{C'}(x_2) = \max\{-0.01, -0.01, -0.01\} = -0.01,$$
$$T^+_{C'}(x_3) = \max\{0.45, 0.6\} = 0.6,\ T^+_{C'}(x_4) = \max\{0.5, 0.6, 0.7, 0.55\} = 0.7,$$
$$T^-_{C'}(x_3) = \min\{-0.45, -0.6\} = -0.6,$$
$$T^-_{C'}(x_4) = \min\{-0.5, -0.6, -0.7, -0.55\} = -0.7,$$
$$I^+_{C'}(x_3) = \max\{0.45, 0.6\} = 0.6,\ I^+_{C'}(x_4) = \max\{0.5, 0.6, 0.7, 0.55\} = 0.7,$$
$$F^+_{C'}(x_3) = \min\{0.01, 0.01\} = 0.01,\ F^+_{C'}(x_4) = \min\{0.01, 0.01, 0.01, 0.01\} = 0.01.$$
$$F^-_{C'}(x_3) = \max\{-0.01, -0.01\} = -0.01,$$
$$F^-_{C'}(x_4) = \max\{-0.01, -0.01, -0.01, -0.01\} = -0.01.$$

There are two common edges ad and cd between the faces F_2 and F_4 in G. Hence between the vertices x_2 and x_4, there exist two edges in the bipolar neutrosophic dual graph of G. Truth-membership, indeterminacy-membership and falsity-membership values of these edges are given as

$$\begin{array}{llll}
T^+_{D'}(x_2x_4) = T^+_D(cd) = 0.7, & T^+_{D'}(x_2x_4) = T^+_D(ad) = 0.55,\\
I^+_{D'}(x_2x_4) = I^+_D(cd) = 0.7, & I^+_{D'}(x_2x_4) = I^+_D(ad) = 0.55,\\
F^+_{D'}(x_2x_4) = F^+_D(cd) = 0.01, & F^+_{D'}(x_2x_4) = F^+_D(ad) = 0.01,\\
T^-_{D'}(x_2x_4) = T^-_D(cd) = -0.7, & T^-_{D'}(x_2x_4) = T^-_D(ad) = -0.55,\\
I^-_{D'}(x_2x_4) = I^-_D(cd) = -0.7, & I^-_{D'}(x_2x_4) = I^-_D(ad) = -0.55,\\
F^-_{D'}(x_2x_4) = F^-_D(cd) = -0.01, & F^-_{D'}(x_2x_4) = F^-_D(ad) = -0.01.
\end{array}$$

The truth-membership, indeterminacy-membership and falsity-membership values of other edges of the bipolar neutrosophic dual graph are calculated as

$$\begin{array}{ll}
T^+_{D'}(x_1x_3) = T^+_D(bc) = 0.45, & T^+_{D'}(x_1x_2) = T^+_D(ac) = 0.4,\\
T^+_{D'}(x_1x_4) = T^+_D(ab) = 0.5, & T^+_{D'}(x_3x_4) = T^+_{D'}(bc) = 0.6,\\
T^-_{D'}(x_1x_3) = T^-_D(bc) = -0.45, & T^-_{D'}(x_1x_2) = T^-_D(ac) = -0.4,\\
T^-_{D'}(x_1x_4) = T^-_D(ab) = -0.5, & T^-_{D'}(x_3x_4) = T^-_{D'}(bc) = -0.6,\\
I^+_{D'}(x_1x_3) = I^+_D(bc) = 0.45, & I^+_{D'}(x_1x_2) = I^+_D(ac) = 0.4,\\
I^+_{D'}(x_1x_4) = I^+_D(ab) = 0.5, & I^+_{D'}(x_3x_4) = I^+_{D'}(bc) = 0.6,\\
I^-_{D'}(x_1x_3) = I^-_D(bc) = -0.45, & I^-_{D'}(x_1x_2) = I^-_D(ac) = -0.4,\\
I^-_{D'}(x_1x_4) = I^-_D(ab) = -0.5, & I^-_{D'}(x_3x_4) = I^-_{D'}(bc) = -0.6,\\
F^+_{D'}(x_1x_3) = T^+_D(bc) = 0.01, & F^+_{D'}(x_1x_2) = F^+_D(ac) = 0.01,\\
F^+_{D'}(x_1x_4) = F^+_D(ab) = 0.01, & F^+_{D'}(x_3x_4) = F^+_D(bc) = 0.01,\\
F^-_{D'}(x_1x_3) = T^-_D(bc) = 0.01, & F^-_{D'}(x_1x_2) = F^-_D(ac) = 0.01,\\
F^-_{D'}(x_1x_4) = F^-_D(ab) = 0.01, & F^-_{D'}(x_3x_4) = F^-_D(bc) = 0.01.
\end{array}$$

Thus, the bipolar neutrosophic edge set of bipolar neutrosophic dual graph is computed as

$$\begin{aligned} D' = \{&(x_1x_3, 0.45, 0.45, 0.01, -0.45, -0.45, -0.01),\\ &(x_1x_2, 0.4, 0.4, 0.01, -0.4, -0.4, -0.01),\\ &(x_1x_4, 0.5, 0.5, 0.01, -0.5, -0.5, -0.01),\\ &(x_3x_4, 0.6, 0.6, 0.01, -0.6, -0.6, -0.01),\\ &(x_2x_4, 0.7, 0.7, 0.01, -0.7, -0.7, -0.01),\\ &(x_2x_4, 0.55, 0.55, 0.01, -0.55, -0.55, -0.01)\}. \end{aligned}$$

In Fig. 3.16, the bipolar neutrosophic dual graph $G' = (X', C', D')$ of G is drawn by dotted line.

Weak edges in planar graphs are not considered for any calculation in bipolar neutrosophic dual graphs.

Theorem 3.19 *Let $G = (X, C, D)$ be a bipolar neutrosophic planar graph without weak edges and the bipolar neutrosophic dual graph of G be $G' = (X', C', D')$. The truth-membership, indeterminacy-membership and falsity-membership values of bipolar neutrosophic edges of G' are equal to truth-membership, indeterminacy-membership and falsity-membership values of the bipolar neutrosophic edges of G.*

3.5 Applications of Neutrosophic Planar Graphs

Graph is considered an important part of Mathematics for solving countless real-world problems in information technology, psychology, engineering, combinatorics and medical sciences. Everything in this world is connected, for instance, cities and countries are connected by roads, railways are linked by railway lines, flight networks are connected by air, electrical devices are connected by wires, pages on internet by hyperlinks, components of electric circuits by various paths. Scientists, analysts and engineers are trying to optimize these networks to find a way to save millions of lives by reducing traffic accidents, plane crashes and circuit shots. Planar graphs are used to find such graphical representations of networks without any crossing or minimum number of crossings. But there is always an uncertainty and degree of indeterminacy in data which can be dealt using bipolar neutrosophic graphs. We now present applications of bipolar neutrosophic graphs in road networks.

3.5.1 Road Network Model to Monitor Traffic

Roads are a mean of frequent and unacceptable number of fatalities every year. Road accidents are increasing due to dense traffic, negligence of drivers and speed of vehicles. Traffic accidents can be minimized by modelling road networks to monitor the traffic, apply quick emergency services and to take action against the speedily going vehicles quickly. The practical approach of bipolar neutrosophic planar graphs can be applied to construct road networks, as these are the combination of vertices and edges along with the degree of truth, indeterminacy and falsity. The method for the construction of road network is given in Algorithm 3.5.1.

Algorithm 3.5.1 Construction of a road network

1. Input: The n number of location $L_1, L_2, \ldots, L_n$.
2. Input: The bipolar neutrosophic set of cities.
3. Input: The adjacency matrix of $\xi = [\xi_{ij}]_{n\times n}$ of cities.
4. **do** i from $1 \to n$
5.   **do** j from $1 \to n$
6.   **if**$(i < j, \xi_{ij} \neq (0, 0, 1, 0, 0, -1))$**then**
7.   Draw an edge between L_i and L_j.
8. $B(L_iL_j) = \xi_{ij}$
9.   **end if**
10.   **end do**
11. **end do**

Consider the problem of road networks between six locations $L_1, L_2, L_3, L_4, L_5, L_6$. The degree of memberships of cities and roads between cities is given in Tables 3.13 and 3.14. The positive degree of membership $T^+(x)$ of each vertex x represents the

Table 3.13 Bipolar neutrosophic set of cities

A	L_1	L_2	L_3	L_4	L_5	L_6
T_A^p	0.7	0.5	0.8	0.6	0.5	0.4
I_A^p	0.4	0.4	0.2	0.1	0.4	0.5
F_A^p	0.2	0.3	0.2	0.1	0.4	0.5
T_A^n	−0.2	−0.3	−0.2	−0.1	−0.4	−0.5
I_A^n	−0.4	−0.4	−0.2	−0.1	−0.4	−0.5
F_A^n	−0.7	−0.5	−0.8	−0.6	−0.5	−0.4

Table 3.14 Bipolar neutrosophic set of roads

A	L_1L_3	L_1L_6	L_2L_3	L_2L_4	L_3L_5	L_5L_6	L_2L_5	L_3L_6	L_4L_6
T_B^p	0.4	0.4	0.5	0.5	0.5	0.4	0.5	0.4	0.4
I_B^p	0.2	0.4	0.2	0.1	0.2	0.4	0.4	0.2	0.1
F_B^p	0.2	0.5	0.3	0.1	0.4	0.4	0.3	0.5	0.5
T_B^n	−0.2	−0.2	−0.3	−0.1	−0.2	−0.4	−0.3	−0.2	−0.1
I_B^n	−0.4	−0.4	−0.2	−0.1	−0.2	−0.4	−0.4	−0.2	−0.1
F_B^n	−0.7	−0.4	−0.8	−0.6	−0.8	−0.4	−0.5	−0.8	−0.6

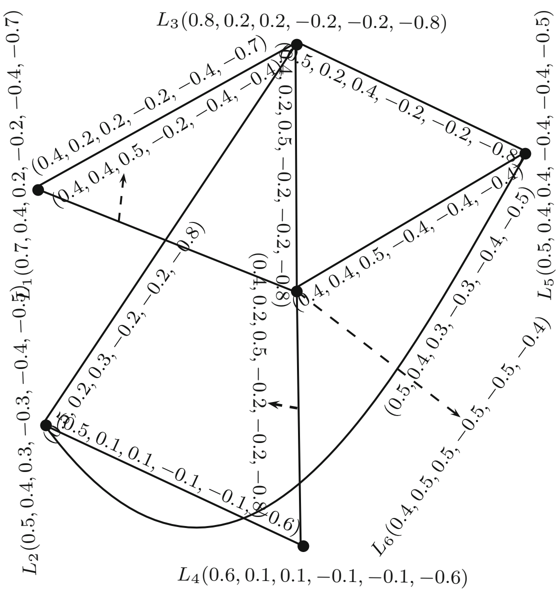


Fig. 3.17 Bipolar neutrosophic road model

percentage that vehicles travelling to or from this city are dense, $I^+(x)$ and $F^+(x)$ represent the indeterminacy and falsity in this percentage. The negative degree of membership $T^-(x)$ represents the percentage that traffic is not dense, $I^-(x)$ and $F^-(x)$ represent the indeterminacy and falsity in this percentage. The positive degree of memberships of each edge xy indicates the percentage of truth, indeterminacy and falsity of road accidents through this road. The negative degree of memberships of xy shows the percentage of truth, indeterminacy and falsity that the road is safer. The bipolar neutrosophic model of road connections between the cities is shown in Fig. 3.17. This bipolar neutrosophic model can be used to check and monitor the percentage of annual accidents. Also, by monitoring and taking special security actions, the total number of accidents can be minimized.

3.5.2 *Electrical Connections*

Graph theory is extensively used in designing circuit connections and installation of wires in order to prevent crossing which can cause dangerous electrical hazards. The twisted and crossing wires are a serious safety risk to human life. There is a need to install electrical wires to reduce crossing. Bipolar neutrosophic planar graphs can be used to model electrical connections and to study the degree of damage that can cause due to the connection.

Consider the problem of setting electrical wires between five electrical utilities and power plugs E_1, E_2, E_3, E_4, E_5 in a factory as shown in Fig. 3.18. The positive

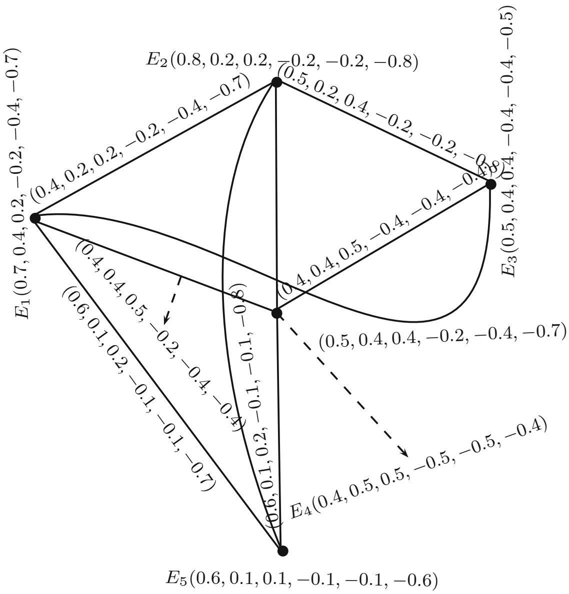


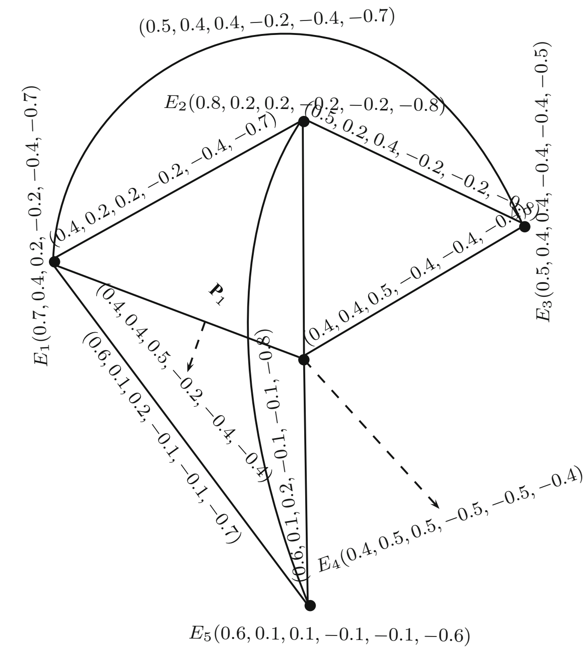
Fig. 3.18 Electrical connections

degree of membership $T^+(E_i)$ of each vertex E_i represents the percentage of faults and electrical sparks of utility or power plug E_i ; $I^+(E_i)$ and $F^p(E_i)$ represent the indeterminacy and falsity in this percentage. The negative degree of membership $T^-(E_i)$ represents the percentage that E_i is updated and safer; $I^-(x)$ and $F^-(x)$ represent the indeterminacy and falsity in this percentage. The positive degree of memberships of each edge E_iE_j indicates the percentage of truth, indeterminacy and falsity of electrical hazards through this connection. The negative degree of memberships of E_iE_j shows the percentage of truth, indeterminacy and falsity that the connection is safer. The crossing of wires can be reduced if we change the geometrical representation of Fig. 3.18. The other representation is shown in Fig. 3.19 which has only one crossing, at point P_1, between the edges E_1E_4 and E_2E_5. The electrical damage at crossing point P_1 can be reduced by using better electrical wires between E_1 and E_4, E_2 and E_5. The method for the construction of bipolar neutrosophic planar graph is given in Algorithm 3.5.2.

Algorithm 3.5.2 Construction of bipolar neutrosophic planar graph

1. Input: The n number of utilities $E_1, E_2, \ldots, E_n$ and p number of connections $e_1, e_2, \ldots, e_p$.
2. Input: The bipolar neutrosophic set of utilities.
3. Input: The points of intersection $P_1, P_2, \ldots, P_r$.
4. **do** i from $1 \to r$
5. P_i is a point of intersection between e_j and e_k.
6.   Change the graphical representation of one of the edges e_j and e_k.

Fig. 3.19 Bipolar neutrosophic planar graph



if There is no new point of intersection in this representation **then**
    Keep this graphical representation.
else
    Keep the previous graphical representation.
end if
end do

3.6 Bipolar Neutrosophic Line Graphs

Definition 3.34 Let $L(G^*) = (Y, Z)$ be line graph of the crisp graph $G^* = (X, E)$. Let A_1=$(T^+_{A_1}, I^+_{A_1}, F^-_{A_1}, T^-_{A_1}, I^-_{A_1}, F^-_{A_1})$ and $B_1 = (T^+_{B_1}, I^+_{B_1}, F^-_{B_1}, T^-_{B_1}, I^-_{B_1}, F^-_{B_1})$ be bipolar neutrosophic sets on A and E, respectively. $A_2 = (T^+_{A_2}, I^+_{A_2}, F^-_{A_2}, T^-_{A_2}, I^-_{A_2}, F^-_{A_2})$ and $B_2 = (T^+_{B_2}, I^+_{B_2}, F^-_{B_2}, T^-_{B_2}, I^-_{B_2}, F^-_{B_2})$ are bipolar neutrosophic sets on Y and Z, respectively. Then, a *bipolar neutrosophic line graph* of the bipolar neutrosophic graph $G = (A_1, B_1)$ is a bipolar neutrosophic graph $L(G) = (A_2, B_2)$ such that

1. $T^+_{A_2}(S_x) = T^+_{B_1}(x) = T^+_{B_1}(u_x v_x)$, $T^-_{A_2}(S_x) = T^-_{B_1}(x) = T^-_{B_1}(u_x v_x)$,
2. $I^+_{A_2}(S_x) = I^+_{B_1}(x) = I^+_{B_1}(u_x v_x)$, $I^-_{A_2}(S_x) = I^-_{B_1}(x) = I^-_{B_1}(u_x v_x)$,
3. $F^-_{A_2}(S_x) = F^-_{B_1}(x) = F^-_{B_1}(u_x v_x)$, $F^-_{A_2}(S_x) = F^-_{B_1}(x) = F^-_{B_1}(u_x v_x)$,
4. $T^+_{B_2}(S_x S_y) = T^+_{B_1}(x) \wedge T^+_{B_1}(y)$, $T^-_{B_2}(S_x S_y) = T^-_{B_1}(x) \vee T^-_{B_1}(y)$,

Table 3.15 Bipolar neutrosophic set A

$x \in X$	$A(x)$
a	(0.7, 0.4, 0.4, −0.4, −0.4, −0.7)
b	(0.8, 0.5, 0.5, −0.5, −0.7, −0.8)
c	(0.9, 0.6, 0.6, −0.6, −0.5, −0.7)
d	(0.6, 0.6, 0.4, −0.4, −0.5, −0.5)
e	(0.7, 0.4, 0.2, −0.3, −0.3, −0.6)

Table 3.16 Bipolar neutrosophic relation B

$xy \in X \times X$	$B(xy)$
ab	(0.7, 0.4, 0.4, −0.4, −0.4, −0.7)
ac	(0.6, 0.3, 0.2, −0.2, −0.3, −0.6)
be	(0.5, 0.2, 0.2, −0.2, −0.3, −0.6)
bd	(0.5, 0.5, 0.4, −0.4, −0.5, −0.5)
cd	(0.3, 0.4, 0.4, −0.3, −0.5, −0.5)
de	(0.6, 0.3, 0.2, −0.2, −0.3, −0.6)

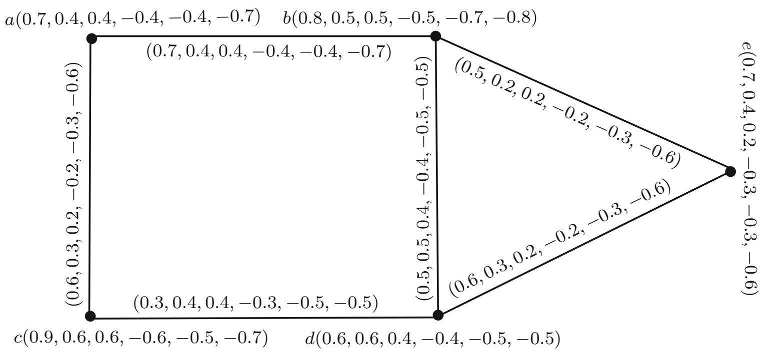


Fig. 3.20 Bipolar neutrosophic graph G

5. $I^+_{B_2}(S_xS_y) = I^+_{B_1}(x) \wedge I^+_{B_1}(y)$, $I^-_{B_2}(S_xS_y) = I^-_{B_1}(x) \vee I^-_{B_1}(y)$,
6. $F^-_{B_2}(S_xS_y) = F^-_{B_1}(x) \vee F^-_{B_1}(y)$, $F^-_{B_2}(S_xS_y) = F^-_{B_1}(x) \wedge F^-_{B_1}(y)$,

$\forall\ S_x, S_y \in Y,\ S_xS_y \in Z.$

Example 3.16 Let A be a bipolar neutrosophic set on $X = \{a, b, c, d, e\}$, given in Table 3.15, and B be a bipolar neutrosophic relation on X, given in Table 3.16. It can be seen that $G = (A, B)$ as shown in Fig. 3.20 is a bipolar neutrosophic graph. The bipolar neutrosophic line graph of Fig. 3.20 is shown in Fig. 3.21.

Proposition 3.8 *$L(G) = (A_2, B_2)$ is a bipolar neutrosophic line graph of some bipolar neutrosophic graph $G = (A_1, B_1)$ if and only if*

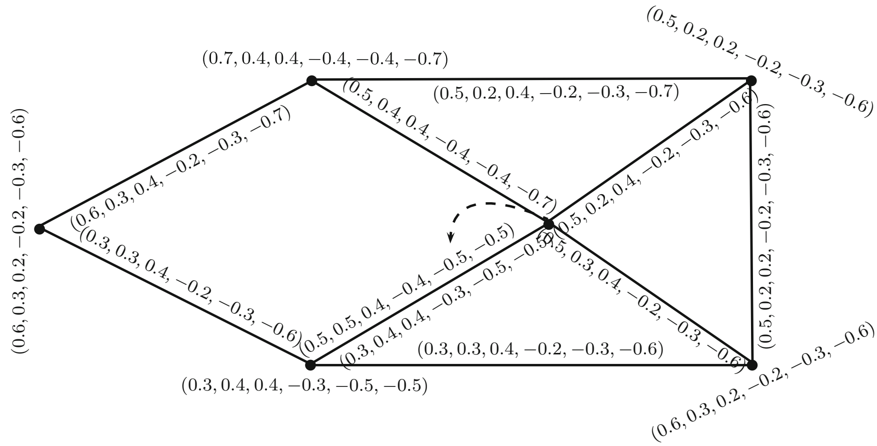


Fig. 3.21 Bipolar neutrosophic line graph

$$T^+_{B_2}(S_xS_y) = T^+_{A_2}(S_x) \wedge T^+_{A_2}(S_y), \;\; T^N_{B_2}(S_xS_y) = T^-_{A_2}(S_x) \vee T^-_{A_2}(S_y),$$
$$I^+_{B_2}(S_xS_y) = I^+_{A_2}(S_x) \wedge I^+_{A_2}(S_y), \;\; I^-_{B_2}(S_xS_y) = I^-_{A_2}(S_x) \vee I^-_{A_2}(S_y)),$$
$$F^+_{B_2}(S_xS_y) = F^+_{A_2}(S_x) \vee F^+_{A_2}(S_y), \;\; F^-_{B_2}(S_xS_y) = F^-_{A_2}(S_x) \wedge F^-_{A_2}(S_y),$$

for all $S_x, S_y \in Y$.

Definition 3.35 Consider two bipolar neutrosophic graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$. A mapping $\psi : X_1 \to X_2$ is called *homomorphism* $\psi : G_1 \to G_2$ if

$$\text{(a)} \quad \begin{cases} T^+_{A_1}(x_1) \le T^+_{A_2}(\psi(x_1)), \;\; I^+_{A_1}(x_1) \le I^+_{A_2}(\psi(x_1)), \;\; F^+_{A_1}(x_1) \le F^+_{A_2}(\psi(x_1)), \\ T^-_{A_1}(x_1) \ge T^-_{A_2}(\psi(x_1)), \;\; I^-_{A_1}(x_1) \ge I^-_{A_2}(\psi(x_1)), \;\; F^-_{A_1}(x_1) \ge F^+_{A_2}(\psi(x_1)), \end{cases}$$

$$\text{(b)} \quad \begin{cases} T^+_{B_1}(x_1y_1) \le T^+_{B_2}(\psi(x_1)\psi(y_1)), \;\; T^-_{B_1}(x_1y_1) \ge T^-_{B_2}(\psi(x_1)\psi(y_1)), \\ I^+_{B_1}(x_1y_1) \le I^+_{B_2}(\psi(x_1)\psi(y_1)), \;\; I^-_{B_1}(x_1y_1) \ge I^-_{B_2}(\psi(x_1)\psi(y_1)), \\ F^+_{B_1}(x_1y_1) \le F^+_{B_2}(\psi(x_1)\psi(y_1)), \;\; F^-_{B_1}(x_1y_1) \ge F^-_{B_2}(\psi(x_1)\psi(y_1)), \end{cases}$$

for all $x_1 \in X_1$, $x_1y_1 \in E_1$. The *weak vertex isomorphism* of bipolar neutrosophic graphs is a bijective homomorphism $\psi : G_1 \to G_2$, such that

$$\text{(c)} \quad \begin{cases} T^+_{A_1}(x_1) = T^+_{A_2}(\psi(x_1)), \;\; I^+_{A_1}(x_1) = I^+_{A_2}(\psi(x_1)), \;\; F^+_{A_1}(x_1) = F^+_{A_2}(\psi(x_1)), \\ T^-_{A_1}(x_1) = T^-_{A_2}(\psi(x_1)), \;\; I^-_{A_1}(x_1) = I^-_{A_2}(\psi(x_1)), \;\; F^-_{A_1}(x_1) = F^-_{A_2}(\psi(x_1)), \end{cases}$$

for all $x_1 \in X_1$ and $\psi : G_1 \to G_2$ is called *weak line isomorphism* if

$$\text{(d)} \quad \begin{cases} T^+_{B_1}(x_1y_1) = T^+_{B_2}(\psi(x_1)\psi(y_1)), \;\; T^-_{B_1}(x_1y_1) = T^-_{B_2}(\psi(x_1)\psi(y_1)), \\ I^+_{B_1}(x_1y_1) = I^+_{B_2}(\psi(x_1)\psi(y_1)), \;\; I^-_{B_1}(x_1y_1) = I^-_{B_2}(\psi(x_1)\psi(y_1)), \\ F^+_{B_1}(x_1y_1) = F^+_{B_2}(\psi(x_1)\psi(y_1)), \;\; F^-_{B_1}(x_1y_1) = F^-_{B_2}(\psi(x_1)\psi(y_1)), \end{cases}$$

for all $x_1y_1 \in E_1$. The *weak isomorphism* $\psi : G_1 \to G_2$ of two bipolar neutrosophic graphs G_1 and G_2 is bijective homomorphism and satisfies (c) and (d). The weak isomorphism may not preserve the weights of the edges but preserves the weights of vertices.

Proposition 3.9 *The weak isomorphism of two bipolar neutrosophic graphs G_1 and G_2 is an isomorphism between their crisp graphs $\hat{G}_1$ and $\hat{G}_2$.*

Theorem 3.20 *Let $L(G) = (A_2, B_2)$ be a bipolar neutrosophic line graph corresponding to a bipolar neutrosophic graph $G = (A_1, B_1)$. Then,*

(i) *there is a week isomorphism between G and $L(G)$ if and only if G^* is a cyclic graph and $\forall\ v \in X$, $x \in E$,*

$$T^+_{A_1}(v) = T^+_{B_1}(x),\ \ I^+_{A_1}(v) = I^+_{B_1}(x),\ \ F^+_{A_1}(v) = F^+_{B_1}(x),$$
$$T^-_{A_1}(v) = T^-_{B_1}(x),\ \ I^-_{A_1}(v) = I^-_{B_1}(x),\ \ F^-_{A_1}(v) = F^-_{B_1}(x),$$

i.e. $A_1 = (T^+_{A_1}, I^+_{A_1}, F^+_{A_1}, T^-_{A_1}, I^-_{A_1}, F^-_{A_1})$ and $B_1 = (T^+_{B_1}, I^+_{B_1}, F^+_{B_1}, T^-_{B_1}, I^-_{B_1}, F^-_{B_1})$ are constant functions on the sets A and E, respectively, taking on same value.

(ii) *If ψ is a weak isomorphism between G and $L(G)$, then ψ is an isomorphism.*

Proof Consider a weak isomorphism $\psi : G \Rightarrow L(G)$. By Proposition 3.8, $G^* = (V, E)$ is a cycle. Let $X = \{v_1, v_2, \ldots, v_n\}$ and $E = \{x_1 = v_1v_2,\ x_2 = v_2v_3, \ldots, x_n = v_nv_1\}$, where $v_1v_2v_3\ldots v_n$ is a cycle. Define bipolar neutrosophic sets

$$T^+_{A_1}(v_i) = s_i,\ \ I^+_{A_1}(v_i) = s'_i,\ \ F^+_{A_1}(v_i) = s''_i,$$

$$T^-_{A_1}(v_i) = t_i,\ \ I^-_{A_1}(v_i) = t'_i,\ \ F^-_{A_1}(v_i) = t''_i$$

$$T^+_{B_1}(x_i) = T^+_{B_1}(v_iv_{i+1}) = r_i,\ \ I^+_{B_1}(x_i) = I^+_{B_1}(v_iv_{i+1}) = r'_i,\ \ F^+_{B_1}(x_i) = F^+_{B_1}(v_iv_{i+1}) = r''_i,$$

$$T^-_{B_1}(x_i) = T^-_{B_1}(v_iv_{i+1}) = q_i,\ \ I^-_{B_1}(x_i) = I^-_{B_1}(v_iv_{i+1}) = q'_i,\ \ F^-_{B_1}(x_i) = F^-_{B_1}(v_iv_{i+1}) = q''_i,$$

$i = 1, 2, \ldots, n$, $v_{n+1} = v_1$. For $s''_{n+1} = s''_1$, $s'_{n+1} = s'_1$, $s_{n+1} = s_1$, $t''_{n+1} = t''_1$, $t'_{n+1} = t'_1$, $t_{n+1} = t_1$, we have

$$r_i \le s_i \wedge s_{i+1},\quad r'_i \le s'_i \wedge s'_{i+1},\quad r''_i \le s''_i \vee s''_{i+1}, \tag{3.5}$$

$$q_i \ge t_i \vee t_{i+1},\quad q'_i \ge t'_i \vee t'_{i+1},\quad q''_i \ge t''_i \wedge t''_{i+1}, \tag{3.6}$$

$1 \le i \le n$. Now

$$X = \{S_{x_1}, S_{x_2}, S_{x_3}, \ldots, S_{x_n}\},\quad Y = \{S_{x_1}S_{x_2},\ S_{x_2}S_{x_3}, \ldots, S_{x_n}S_{x_1}\}.$$

Thus, for $r_{n+1} = r_1$, we obtain

$$T^+_{A_2}(S_{x_i}) = T^+_{B_1}(x_i) = r_i,\ \ I^+_{A_2}(S_{x_i}) = I^+_{B_1}(x_i) = r'_i,\ \ F^+_{A_2}(S_{x_i}) = F^+_{B_1}(x_i) = r''_i,$$

$$T^+_{B_2}(S_{x_i}S_{x_{i+1}}) = T^+_{B_1}(x_i) \wedge T^+_{B_1}(x_{i+1}) = r_i \wedge r_{i+1},$$

$$I^+_{B_2}(S_{x_i}S_{x_{i+1}}) = I^+_{B_1}(x_i) \wedge I^+_{B_1}(x_{i+1}) = r'_i \wedge r'_{i+1},$$

$$F^+_{B_2}(S_{x_i}S_{x_{i+1}}) = F^+_{B_1}(x_i) \vee F^+_{B_1}(x_{i+1}) = r''_i \vee r''_{i+1}.$$

For $q_{n+1} = q_1$, we obtain

$$T^-_{A_2}(S_{x_i}) = T^-_{B_1}(x_i) = q_i, \quad I^-_{A_2}(S_{x_i}) = I^-_{B_1}(x_i) = q'_i, \quad F^-_{A_2}(S_{x_i}) = F^-_{B_1}(x_i) = q''_i,$$

$$T^-_{B_2}(S_{x_i}S_{x_{i+1}}) = T^-_{B_1}(x_i) \vee T^-_{B_1}(x_{i+1}) = q_i \vee q_{i+1},$$

$$I^-_{B_2}(S_{x_i}S_{x_{i+1}}) = I^-_{B_1}(x_i) \vee I^-_{B_1}(x_{i+1}) = q'_i \vee q'_{i+1},$$

$$F^-_{B_2}(S_{x_i}S_{x_{i+1}}) = F^-_{B_1}(x_i) \wedge F^-_{B_1}(x_{i+1}) = q''_i \wedge q''_{i+1}$$

for $1 \le i \le n$, $v_{n+1} = v_1$. Since ψ is isomorphism of $\hat{G}$ onto $L(G^*)$, ψ is a bijection of A onto Y. Also ψ preserves the adjacency. Hence, ψ induces a permutation π' of $\{1, 2, \ldots, n\}$ such that

$$\psi(v_i) = S_{v_{\pi'(i)}v_{\pi'(i)+1}}$$

$$v_iv_{i+1} \to \psi(v_i)\psi(v_{i+1}) = S_{v_{\pi'(i)}v_{\pi'(i)+1}}S_{v_{\pi'(i+1)}v_{\pi'(i+1)+1}}, \quad 1 \le i \le n-1.$$

Thus, we conclude that

$$s_i = T^+_{A_1}(v_i) \le T^+_{A_2}(\psi(v_i)) = T^+_{A_2}(S_{v_{\pi'(i)}v_{\pi'(i)+1}}) = T^+_{B_1}(v_{\pi'(i)}v_{\pi'(i)+1}) = r_{\pi'(i)},$$

$$s'_i = I^+_{A_1}(v_i) \le I^+_{A_2}(\psi(v_i)) = I^+_{A_2}(S_{v_{\pi'(i)}v_{\pi'(i)+1}}) = I^+_{B_1}(v_{\pi'(i)}v_{\pi'(i)+1}) = r'_{\pi'(i)},$$

$$s''_i = F^+_{A_1}(v_i) \le F^+_{A_2}(\psi(v_i)) = F^+_{A_2}(S_{v_{\pi'(i)}v_{\pi'(i)+1}}) = F^+_{B_1}(v_{\pi'(i)}v_{\pi'(i)+1}) = r''_{\pi'(i)},$$

$$t_i = T^-_{A_1}(v_i) \ge T^-_{A_2}(\psi(v_i)) = T^-_{A_2}(S_{v_{\pi'(i)}v_{\pi'(i)+1}}) = T^-_{B_1}(v_{\pi'(i)}v_{\pi'(i)+1}) = q_{\pi'(i)},$$

$$t'_i = I^-_{A_1}(v_i) \ge I^-_{A_2}(\psi(v_i)) = I^-_{A_2}(S_{v_{\pi'(i)}v_{\pi'(i)+1}}) = I^+_{B_1}(v_{\pi'(i)}v_{\pi'(i)+1}) = q'_{\pi'(i)},$$

$$t''_i = F^-_{A_1}(v_i) \ge F^-_{A_2}(\psi(v_i)) = F^-_{A_2}(S_{v_{\pi'(i)}v_{\pi'(i)+1}}) = F^-_{B_1}(v_{\pi'(i)}v_{\pi'(i)+1}) = q''_{\pi'(i)},$$

$$\begin{aligned} r_i = T^+_{B_1}(v_iv_{i+1}) &\le T^+_{B_2}(\psi(v_i)\psi(v_{i+1})) \\ &= T^+_{B_2}(S_{v_{\pi'(i)}v_{\pi'(i)+1}}S_{v_{\pi'(i+1)}v_{\pi'(i+1)+1}}) \\ &= T^+_{B_1}(v_{\pi'(i)}v_{\pi'(i)+1}) \wedge T^+_{B_1}(v_{\pi'(i+1)}v_{\pi'(i+1)+1}) \\ &= r_{\pi'(i)} \wedge r_{\pi'(i+1)}. \end{aligned}$$

$$\begin{aligned} r'_i = I^+_{B_1}(v_i v_{i+1}) &\le I^+_{B_2}(\psi(v_i)\psi(v_{i+1})) \\ &= I^+_{B_2}(S_{v_{\pi'(i)}v_{\pi'(i)+1}} S_{v_{\pi'(i+1)}v_{\pi'(i+1)+1}}) \\ &= I^+_{B_1}(v_{\pi'(i)}v_{\pi'(i)+1}) \wedge I^+_{B_1}(v_{\pi'(i+1)}v_{\pi'(i+1)+1}) \\ &= r'_{\pi'(i)} \wedge r'_{\pi'(i+1)}. \end{aligned}$$

Similarly,

$$\begin{aligned} r''_i = F^+_{B_1}(v_i v_{i+1}) &\le F^+_{B_2}(\psi(v_i)\psi(v_{i+1})) \\ &= F^+_{B_2}(S_{v_{\pi'(i)}v_{\pi'(i)+1}} S_{v_{\pi'(i+1)}v_{\pi'(i+1)+1}}) \\ &= F^+_{B_1}(v_{\pi'(i)}v_{\pi'(i)+1}) \vee F^+_{B_1}(v_{\pi'(i+1)}v_{\pi'(i+1)+1}) \\ &= r''_{\pi'(i)} \vee r''_{\pi'(i+1)} \end{aligned}$$

$$\begin{aligned} q_i = T^-_{B_1}(v_i v_{i+1}) &\ge T^-_{B_2}(\psi(v_i)\psi(v_{i+1})) \\ &= T^-_{B_2}(S_{v_{\pi'(i)}v_{\pi'(i)+1}} S_{v_{\pi'(i+1)}v_{\pi'(i+1)+1}}) \\ &= T^-_{B_1}(v_{\pi'(i)}v_{\pi'(i)+1}) \vee T^-_{B_1}(v_{\pi'(i+1)}v_{\pi'(i+1)+1}) \\ &= q_{\pi'(i)} \vee q_{\pi'(i+1)}. \end{aligned}$$

$$\begin{aligned} q'_i = I^-_{B_1}(v_i v_{i+1}) &\ge I^-_{B_2}(\psi(v_i)\psi(v_{i+1})) \\ &= I^-_{B_2}(S_{v_{\pi'(i)}v_{\pi'(i)+1}} S_{v_{\pi'(i+1)}v_{\pi'(i+1)+1}}) \\ &= I^-_{B_1}(v_{\pi'(i)}v_{\pi'(i)+1}) \vee I^-_{B_1}(v_{\pi'(i+1)}v_{\pi'(i+1)+1}) \\ &= q'_{\pi'(i)} \vee q'_{\pi'(i+1)}. \end{aligned}$$

Similarly,

$$\begin{aligned} q''_i = F^-_{B_1}(v_i v_{i+1}) &\ge F^-_{B_2}(\psi(v_i)\psi(v_{i+1})) \\ &= F^-_{B_2}(S_{v_{\pi'(i)}v_{\pi'(i)+1}} S_{v_{\pi'(i+1)}v_{\pi'(i+1)+1}}) \\ &= F^-_{B_1}(v_{\pi'(i)}v_{\pi'(i)+1}) \wedge F^-_{B_1}(v_{\pi'(i+1)}v_{\pi'(i+1)+1}) \\ &= q''_{\pi'(i)} \wedge q''_{\pi'(i+1)} \end{aligned}$$

for $1 \le i \le n$. That is,

$$s_i \le r_{\pi'(i)}, \quad s'_i \le r'_{\pi'(i)}, \quad s''_i \le r''_{\pi'(i)}, \tag{3.7}$$

$$t_i \ge q_{\pi'(i)}, \quad t'_i \ge q'_{\pi'(i)}, \quad t''_i \ge q''_{\pi'(i)} \tag{3.8}$$

$$r_i \le r_{\pi'(i)} \wedge r_{\pi'(i+1)}, \quad r'_i \le r'_{\pi'(i)} \wedge r'_{\pi'(i+1)}, \quad r''_i \le r''_{\pi'(i)} \vee r''_{\pi'(i+1)}, \tag{3.9}$$

$$q_i \ge q_{\pi'(i)} \vee q_{\pi'(i+1)}, \quad q'_i \ge q'_{\pi'(i)} \vee q'_{\pi'(i+1)}, \quad q''_i \ge q''_{\pi'(i)} \wedge q''_{\pi'(i+1)}. \tag{3.10}$$

Thus, $r_i \le r_{\pi'(i)},\ r'_i \le r'_{\pi'(i)},\ r''_i \le r''_{\pi'(i)}, q_i \ge q_{\pi'(i)},\ q'_i \ge q'_{\pi'(i)},\ q''_i \ge q''_{\pi'(i)}$, and so $r_{\pi'(i)} \le r_{\pi'(\pi'(i))}, r'_{\pi'(i)} \le r'_{\pi'(\pi'(i))}, r''_{\pi'(i)} \le r''_{\pi'(\pi'(i))}, q_{\pi'(i)} \ge q_{\pi'(\pi'(i))}, q'_{\pi'(i)} \ge q'_{\pi'(\pi'(i))}, q''_{\pi'(i)} \ge q''_{\pi'(\pi'(i))}$ for all $1 \le i \le n$. Continuing, we obtain

$$r_i \le r_{\pi'(i)} \le \cdots \le r_{\pi'^j(i)} \le r_i,$$

$$r'_i \le r'_{\pi'(i)} \le \cdots \le r'_{\pi'^j(i)} \le r'_i,$$

$$r''_i \le r''_{\pi'(i)} \le \cdots \le r''_{\pi'^j(i)} \le r''_i,$$

$$q_i \ge q_{\pi'(i)} \ge \cdots \ge q_{\pi'^j(i)} \ge q_i,$$

$$q'_i \ge q'_{\pi'(i)} \ge \cdots \ge q'_{\pi'^j(i)} \ge q'_i,$$

$$q''_i \ge q''_{\pi'(i)} \ge \cdots \ge q''_{\pi'^j(i)} \ge q''_i,$$

where π'^{j+1} is identity map. So, $r_i = r_{\pi'(i)},\ r'_i = r'_{\pi'(i)}, r''_i = r''_{\pi'(i)}, q_i = q_{\pi'(i)},\ q'_i = q'_{\pi'(i)}, q''_i = q''_{\pi'(i)}$ for all $1 \le i \le n$. But, by (3.9), (3.10) we also have $r_i \le r_{\pi'(i+1)} = r_{i+1}, r'_i \le r'_{\pi'(i+1)} = r'_{i+1}$ and $r''_i \le r''_{\pi(i+1)} = r''_{i+1}, q_i \ge q_{\pi'(i+1)} = q_{i+1}, q'_i \ge q'_{\pi'(i+1)} = q'_{i+1}$ and $q''_i \ge q''_{\pi(i+1)} = q''_{i+1}$, which together with $r_{n+1} = r_1,\ r'_{n+1} = r'_1,\ r''_{n+1} = r''_1, q_{n+1} = q_1,\ q'_{n+1} = q'_1,\ q''_{n+1} = q''_1$, implies $r_i = r_1,\ r'_i = r'_1,\ r''_i = r''_1,\ q_i = q_1,\ q'_i = q'_1, q''_i = q''_1$. for all $i = 1, 2, \ldots, n$. Hence by (3.5)–(3.8), we get

$$r_1 = \cdots = r_n = s_1 = \cdots = s_n,$$

$$r'_1 = \cdots = r'_n = s'_1 = \cdots = s'_n,$$

$$r''_1 = \cdots = r''_n = s''_1 = \cdots = s''_n.$$

$$q_1 = \cdots = q_n = t_1 = \cdots = t_n,$$

$$q'_1 = \cdots = q'_n = t'_1 = \cdots = t'_n,$$

$$q''_1 = \cdots = q''_n = t''_1 = \cdots = t''_n.$$

Thus, we proved the conclusion about A_1 and B_1 being constant function, but we have also shown that (ii) holds. The converse part of (i) is obvious.

3.7 Application of Bipolar Neutrosophic Line Graphs

Child kidnapping is an illegal removal of children from the guardians for the sake of ransom and profit. According to a US estimate, about 800,000 children are missing every year. These type of criminal activities threaten the parents and have huge impact on society. Child kidnappers are in common practise to spread their network. It always remain a difficult task for the security agencies to detect and expose such networks. The telephone network of criminals can be used to detect the people involved in child kidnapping. Graphs are a key tool to study such networks. As the data structures in such cases contain only observations about the suspect, there is always uncertainty in data. Bipolar neutrosophic graphs can be used to reduce uncertainty in data and to detect the involvement of suspect in child kidnapping. If there are n number of suspects under investigation, then the procedure for the detection of suspects involved in kidnappers network is given in Algorithm 3.7.1.

Algorithm 3.7.1 Detection of suspects involved in child kidnapping

1. Enter the number of suspects n.
2. Enter the membership value $\mu(s_i) = (T^+(s_i), I^+(s_i), F^+(s_i), T^-(s_i), I^-(s_i), F^-(s_i))$ of each suspect s_i, $1 \leq i \leq n$.
3. Enter the adjacency matrix of the suspects' network $\xi = [s_{ij}]_{n\times n}$.
4. **do** i from 1 to n
5.   **do** j from 1 to n
6. $R(s_i) = (T^+(s_i), I^+(s_i), F^+(s_i), T^-(s_i), I^-(s_i), F^-(s_i))$
7.   **if**($T^+(s_i) > 0$ or $I^+(s_i) > 0$ or $T^-(s_i) < 0$ or $I^-(s_i) < 0$)**then**
8. $R(s_i) = R(s_i) + \mu(s_i)$
9.   **end if**
10.   **end do**
11. **end do**
12. **do** i from 1 to n
13. $T(s_i) = 2 + T^+(R(s_i)) - I^+(R(s_i)) - F^+(R(s_i))$
14. $N(s_i) = 2 - T^-(R(s_i)) + I^+(R(s_i)) + F^+(R(s_i))$
15. $S(s_i) = T(s_i) - N(s_i)$
16. **end do**
17. $A = 0$
18. **do** i from 1 to n
19. $A = \max\{A, S(s_i)\}$
20. **end do**
21. **do** i from 1 to n
22.   **if**($A = S(s_i)$)**then**
23.   **print***, s_i is the most suspicious person.
24.   **end if**
25. **end do**

Description and time complexity: The algorithms start by taking the input of membership values and adjacency matrix; therefore, the time complexity of lines 1–3 is $O(n^2)$. The loops from lines 4–11 calculate the sum values for each s_i so, the time complexity of these loops is $O(n^2)$. The do loop from lines 12–15 calculate

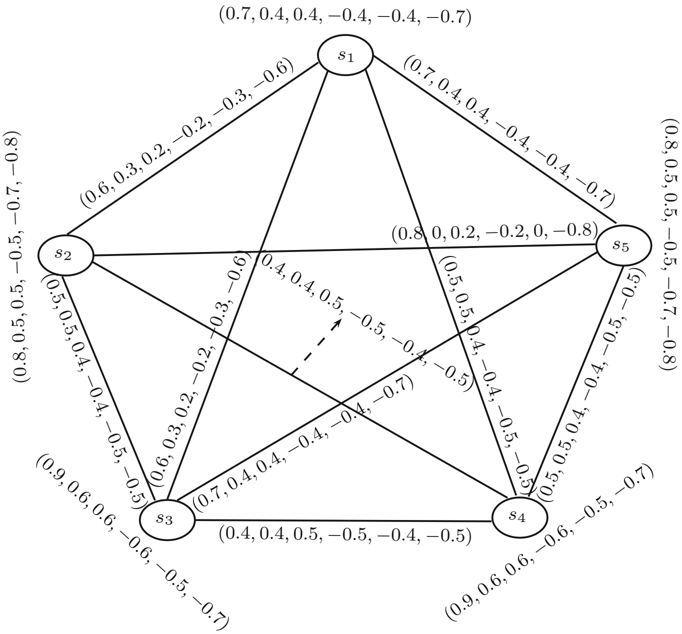


Fig. 3.22 Telephone connection among suspects

the strength of exactness of observations against each s_i, $1 \leq i \leq n$; the time complexity is $O(n)$. Lines 17–25 calculate and print the suspect with maximum strength of involving in criminal activities; therefore, lines 17–25 has time complexity $O(n)$. Thus, the net time complexity of the Algorithm 3.7.1 is $O(n^2)$.

An example of a bipolar neutrosophic graph with five suspects s_1, s_2, s_3, s_4, s_5 is shown in Fig. 3.22. The positive degree of membership (T^+, I^+, F^+) of each suspect shows the strength of truth, indeterminacy and falsity of observation to be involved in criminal network. The negative degree of membership (T^-, I^-, F^-) of each suspect shows the strength of truth, indeterminacy and falsity of observation that he/she is innocent. The positive degree of membership (T^+, I^+, F^+) of each edge shows the strength of truth, indeterminacy and falsity that the two suspects are in contact for criminal activities. The negative degree of membership (T^-, I^-, F^-) of each edge shows the strength of truth, indeterminacy and falsity that the two suspects are in contact for some other purpose. Using Algorithm 3.7.1, sum values sum (s_i) and strength of each suspect $S(s_i)$, $1 \leq i \leq 5$, are shown in Table 1.17. For each i, sum(s_i) can be obtained by taking the sum of membership value of each vertex and membership values of the incident edges. Also, $T(s_i) = 2 + T^+(R(s_i)) - I^+(R(s_i)) - F^+(R(s_i))$, $N(s_i) = 2 - T^-(R(s_i)) + I^+(R(s_i)) + F^+(R(s_i))$ and $S(s_i) = T(s_i) - N(s_i)$. In Table 3.17, column 4 indicates the strength of correctness of observations against

Table 3.17 Strength of exactness of observations

Suspects s_i	$R(s_i)$	$(T(s_i), N(s_i))$	$S(s_i)$
s_1	(3.1, 2.3, 2.2, −2.0, −2.3, −3.1)	(0.6,-1.4)	2.0
s_2	(3.1, 1.7, 1.8, −1.8, −1.9, −3.2)	(0.6, −1.8)	2.4
s_3	(3.1, 2.2, 2.1, −1.9, −2.1, −3.0)	(0.8,-1.2)	2.0
s_4	(2.7, 2.4, 2.8, −2.4, −2.3, −2.7)	(−0.5, −0.6)	0.1
s_5	(3.5, 1.8, 1.9, −1.9, −2.0, −3.5)	(1.8, −1.6)	3.4

the suspect in the investigation. For example, the strength of s_5 shows the greatest exactness of the investigation report against s_5, whereas the strength of s_4 shows the least exactness of observations against s_4. $S(s_4)$ indicates that s_4 may be innocent; therefore, the security agency should take it into consideration from the beginning.

Chapter 4
Graphs Under Interval-Valued Neutrosophic Environment

In this chapter, we present the concept of interval-valued neutrosophic competition graphs. We then discuss certain types, including k-competition interval-valued neutrosophic graphs, p-competition interval-valued neutrosophic graphs and m-step interval-valued neutrosophic competition graphs. Moreover, we present the concept of m-step interval-valued neutrosophic neighbourhood graphs. This chapter is due to [12].

4.1 Introduction

In 1975, Zadeh [199] introduced the notion of interval-valued fuzzy sets as an extension of fuzzy sets [194] in which the values of the membership degrees are intervals of numbers instead of the numbers. Interval-valued fuzzy sets provide a more adequate description of uncertainty than traditional fuzzy sets. It is therefore important to use interval-valued fuzzy sets in applications, such as fuzzy control. One of the computationally most intensive parts of fuzzy control is defuzzification. Smarandache [165] and Wang et al. [172] presented the notion of single-valued neutrosophic sets to apply neutrosophic sets in real-life problems more conveniently. In single-valued neutrosophic sets, three components are independent and their values are taken from the standard unit interval [0, 1]. Wang et al. [170] presented the concept of interval-valued neutrosophic sets, which is more precise and more flexible than the single-valued neutrosophic set. An interval-valued neutrosophic set is a generalization of the concept of single-valued neutrosophic set, in which three membership (T, I, F) functions are independent, and their values belong to the unit interval [0, 1].

Definition 4.1 An interval-valued fuzzy set I in X is defined by

$$\mathrm{I} = \{(s, [T_{\mathrm{I}}^{l}(s), T_{\mathrm{I}}^{u}(s)]) : s \in X\},$$

M. Akram, *Single-Valued Neutrosophic Graphs*, Infosys Science Foundation Series, https://doi.org/10.1007/978-981-13-3522-8_4

where $T^l_{\mathrm{I}}(s)$ and $T^u_{\mathrm{I}}(s)$ are fuzzy subsets of X such that $T^l_{\mathrm{I}}(s) \leq T^u_{\mathrm{I}}(s)$ for all $x \in X$. An interval-valued fuzzy relation on X is an interval-valued fuzzy set J in $X \times X$.

Definition 4.2 For any two interval-valued neutrosophic sets

$$\mathrm{I} = ([T^l_{\mathrm{I}}(x), T^u_{\mathrm{I}}(x)], [I^l_{\mathrm{I}}(x), I^u_{\mathrm{I}}(x)], [F^l_{\mathrm{I}}(x), F^u_{\mathrm{I}}(x)])$$

and

$$\mathrm{J} = ([T^l_{\mathrm{J}}(x), T^u_{\mathrm{J}}(x)], [I^l_{\mathrm{J}}(x), I^u_{\mathrm{J}}(x)], [F^l_{\mathrm{J}}(x), F^u_{\mathrm{J}}(x)])$$

in X, we define:

1.

$$\mathrm{I} \cup \mathrm{J} = \{(x, \max(T^l_{\mathrm{I}}(x), T^l_{\mathrm{J}}(x)), \max(T^u_{\mathrm{I}}(x), T^u_{\mathrm{J}}(x)), \max(I^l_{\mathrm{I}}(x), I^l_{\mathrm{J}}(x)), \max(I^u_{\mathrm{I}}(x), I^u_{\mathrm{J}}(x)), \min(F^l_{\mathrm{I}}(x), F^l_{\mathrm{J}}(x)), \min(F^u_{\mathrm{I}}(x), F^u_{\mathrm{J}}(x))) : x \in X\}.$$

2.

$$\mathrm{I} \cap \mathrm{J} = \{(x, \min(T^l_{\mathrm{I}}(x), T^l_{\mathrm{J}}(x)), \min(T^u_{\mathrm{I}}(x), T^u_{\mathrm{J}}(x)), \min(I^l_{\mathrm{I}}(x), I^l_{\mathrm{J}}(x)), \min(I^u_{\mathrm{I}}(x), I^u_{\mathrm{J}}(x)), \max(F^l_{\mathrm{I}}(x), F^l_{\mathrm{J}}(x)), \max(F^u_{\mathrm{I}}(x), F^u_{\mathrm{J}}(x))) : x \in X\}.$$

4.2 Interval-Valued Neutrosophic Graphs

Definition 4.3 An *interval-valued neutrosophic graph* on a nonempty set X is a pair $G = (A, B)$, where A is an interval-valued neutrosophic set on X and B is an interval-valued neutrosophic relation on X such that

1. $T^l_B(xy) \leq \min(T^l_A(x), T^l_A(y))$, $T^u_B(xy) \leq \min(T^u_A(x), T^u_A(y))$,
2. $I^l_B(xy) \leq \min(I^l_A(x), I^l_A(y))$, $I^u_B(xy) \leq \min(I^u_A(x), I^u_A(y))$,
3. $F^l_B(xy) \leq \min(F^l_A(x), F^l_A(y))$, $F^u_B(xy) \leq \min(F^u_A(x), F^u_A(y))$, for all $x, y \in X$.

Note that B is called symmetric relation on A.

Example 4.1 Consider a graph G^* such that $X = \{a, b, c\}$, $E = \{ab, bc, ac\}$. Let A be an interval-valued neutrosophic subset of X and let B be an interval-valued neutrosophic subset of $E \subseteq X \times X$, as shown in the following tables.

By routine calculations, it can be observed that the graph shown in Fig. 4.1 is an interval-valued neutrosophic graph.

Definition 4.4 An *interval-valued neutrosophic digraph* on a nonempty set X is a pair $G = (A, \overrightarrow{B})$, (in short, G), where $A = ([T^l_A, T^u_A], [I^l_A, I^u_A], [F^l_A, F^u_A])$ is an interval-valued neutrosophic set on X and $B = ([T^l_B, T^u_B], [I^l_B, I^u_B], [F^l_B, F^u_B])$ is an interval-valued neutrosophic relation on X, such that:

A	a	b	c
T_A^l	0.2	0.2	0.2
T_A^u	0.4	0.5	0.8
I_A^l	0.3	0.3	0.3
I_A^u	0.7	0.4	0.8
F_A^l	0.4	0.2	0.2
F_A^u	0.5	0.9	0.7

B	ab	bc	ac
T_B^l	0.1	0.1	0.1
T_B^u	0.3	0.3	0.3
I_B^l	0.2	0.2	0.2
I_B^u	0.3	0.3	0.3
F_B^l	0.2	0.2	0.2
F_B^u	0.5	0.7	0.5

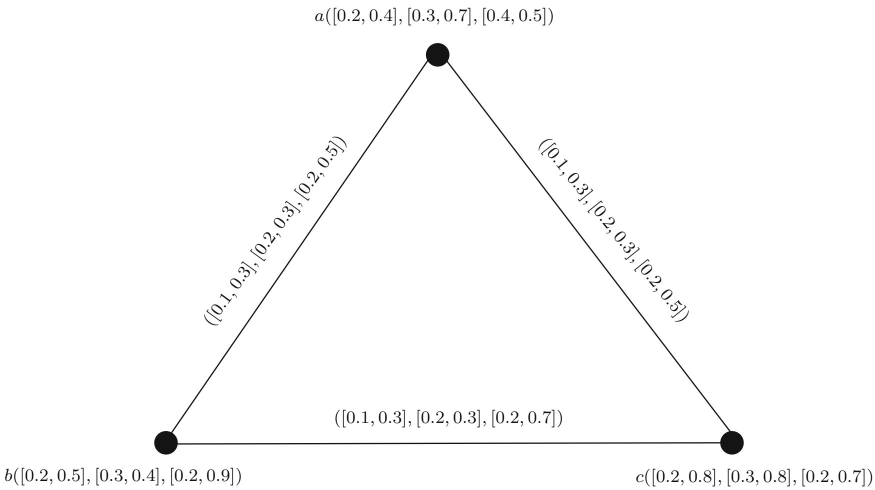


Fig. 4.1 Interval-valued neutrosophic graph

1. $T_B^l\overrightarrow{(s,w)} \leq T_A^l(s) \wedge T_A^l(w)$, $\quad T_B^u\overrightarrow{(s,w)} \leq T_A^u(s) \wedge T_A^u(w)$,
2. $I_B^l\overrightarrow{(s,w)} \leq I_A^l(s) \wedge I_A^l(w)$, $\quad I_B^u\overrightarrow{(s,w)} \leq I_A^u(s) \wedge I_A^u(w)$,
3. $F_B^l\overrightarrow{(s,w)} \leq F_A^l(s) \wedge F_A^l(w)$, $\quad F_B^u\overrightarrow{(s,w)} \leq F_A^u(s) \wedge F_A^u(w)$, for all $s, w \in X$.

Example 4.2 We construct an interval-valued neutrosophic digraph $G = (A, \overrightarrow{B})$ on $X = \{a, b, c\}$ as shown in Fig. 4.2.

Definition 4.5 Let $\overrightarrow{G}$ be an interval-valued neutrosophic digraph; then interval-valued neutrosophic out-neighbourhoods of a vertex s is an interval-valued neutrosophic set

$$\mathbb{N}^+(s) = (X_s^+, [T_s^{(l)^+}, T_s^{(u)^+}], [I_s^{(l)^+}, I_s^{(u)^+}], [F_s^{(l)^+}, T_s^{(u)^+}]),$$

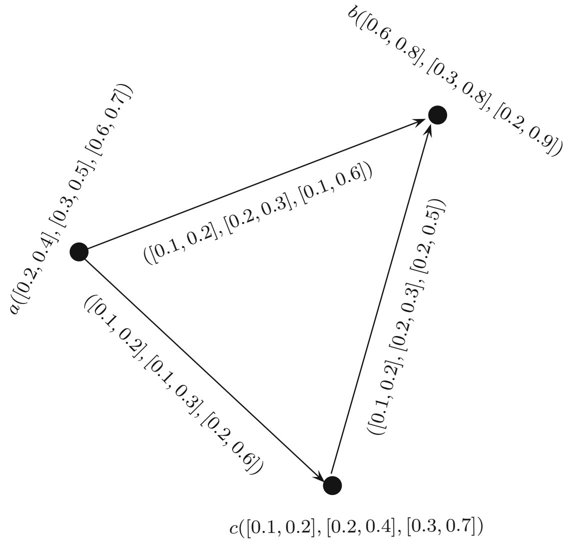


Fig. 4.2 Interval-valued neutrosophic digraph

where

$$X_s^+ = \{w | [T_B^l\overrightarrow{(s,w)} > 0, T_B^u\overrightarrow{(s,w)} > 0], [I_B^l\overrightarrow{(s,w)} > 0, I_B^u\overrightarrow{(s,w)} > 0], [F_B^l\overrightarrow{(s,w)} > 0, F_B^u\overrightarrow{(s,w)} > 0]\},$$

such that $T_s^{(l)^+} : X_s^+ \to [0,1]$, defined by $T_s^{(l)^+}(w) = T_B^l\overrightarrow{(s,w)}$, $T_s^{(u)^+} : X_s^+ \to [0,1]$, defined by $T_s^{(u)^+}(w) = T_B^u\overrightarrow{(s,w)}$, $I_s^{(l)^+} : X_s^+ \to [0,1]$, defined by $I_s^{(l)^+}(w) = I_B^l\overrightarrow{(s,w)}$, $I_s^{(u)^+} : X_s^+ \to [0,1]$, defined by $I_s^{(u)^+}(w) = I_B^u\overrightarrow{(s,w)}$, $F_s^{(l)^+} : X_s^+ \to [0,1]$, defined by $F_s^{(l)^+}(w) = F_B^l\overrightarrow{(s,w)}$, $F_s^{(u)^+} : X_s^+ \to [0,1]$, defined by $F_s^{(u)^+}(w) = F_B^u\overrightarrow{(s,w)}$.

Definition 4.6 Let $\overrightarrow{G}$ be an interval-valued neutrosophic digraph; then interval-valued neutrosophic in-neighbourhoods of a vertex s is an interval-valued neutrosophic set

$$\mathbb{N}^-(s) = (X_s^-, [T_s^{(l)^-}, T_s^{(u)^-}], [I_s^{(l)^-}, I_s^{(u)^-}], [F_s^{(l)^-}, T_s^{(u)^-}]),$$

where

$$X_s^- = \{w | [T_B^l\overrightarrow{(w,s)} > 0, T_B^u\overrightarrow{(w,s)} > 0], [I_B^l\overrightarrow{(w,s)} > 0, I_B^u\overrightarrow{(w,s)} > 0], [F_B^l\overrightarrow{(w,s)} > 0, F_B^u\overrightarrow{(w,s)} > 0]\},$$

such that $T_s^{(l)^-} : X_s^- \to [0,1]$, defined by $T_s^{(l)^-}(w) = T_B^l\overrightarrow{(w,s)}$, $T_s^{(u)^-} : X_s^- \to [0,1]$, defined by $T_s^{(u)^-}(w) = T_B^u\overrightarrow{(w,s)}$, $I_s^{(l)^-} : X_s^- \to [0,1]$, defined by $I_s^{(l)^-}(w) = I_B^l\overrightarrow{(w,s)}$, $I_s^{(u)^-} : X_s^- \to [0,1]$, defined by $I_s^{(u)^-}(w) = I_B^u\overrightarrow{(w,s)}$, $F_s^{(l)^-} : X_s^- \to [0,1]$,

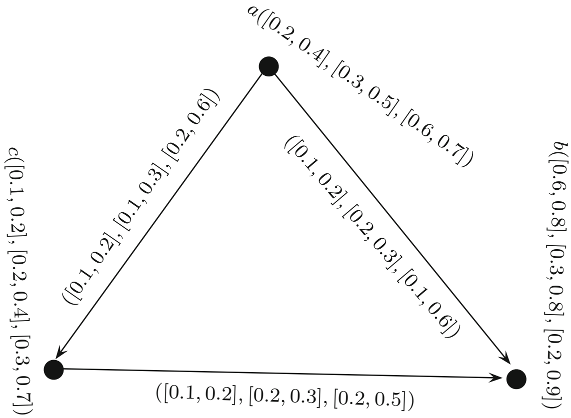


Fig. 4.3 Interval-valued neutrosophic digraph

Table 4.1 Interval-valued neutrosophic out-neighbourhoods

s	$\mathbb{N}^+(s)$
a	{(b, [0.1, 0.2], [0.2, 0.3], [0.1, 0.6]), (c, [0.1, 0.2], [0.1, 0.3], [0.2, 0.6])}
b	Ø
c	{(b, [0.1, 0.2], [0.2, 0.3], [0.2, 0.5])}

Table 4.2 Interval-valued neutrosophic in-neighbourhoods

s	$\mathbb{N}^-(s)$
a	Ø
b	{(a, [0.1, 0.2], [0.2, 0.3], [0.1, 0.6]), (c, [0.1, 0.2], [0.2, 0.3], [0.2, 0.5])}
c	{(a, [0.1, 0.2], [0.1, 0.3], [0.2, 0.6])}

defined by $F_s^{(l)^-}(w) = F_B^l\overrightarrow{(w, s)}$, $F_s^{(u)^-} : X_s^- \to [0, 1]$, defined by $F_s^{(u)^-}(w) = F_B^u\overrightarrow{(w, s)}$.

Example 4.3 Consider an interval-valued neutrosophic digraph $G = (A, \overrightarrow{B})$ on $X = \{a, b, c\}$ as shown in Fig. 4.3.

We have Tables 4.1 and 4.2 representing interval-valued neutrosophic out- and in-neighbourhoods, respectively.

Definition 4.7 The *height* of interval-valued neutrosophic set $A = (s, [T_A^l, T_A^u], [I_A^l, I_A^u], [F_A^l, F_A^u])$ in universe of discourse X is defined as,

$$\begin{aligned} h(A) &= ([h_1^l(A), h_1^u(A)], [h_2^l(A), h_2^u(A)], [h_3^l(A), h_3^u(A)]), \\ &= ([\sup_{s\in X} T_A^l(s), \sup_{s\in X} T_A^u(s)], [\sup_{s\in X} I_A^l(s), \sup_{s\in X} I_A^u(s)], [\inf_{s\in X} F_A^l(s), \inf_{s\in X} F_A^u(s)]), \quad \text{for all} \\ &\quad s \in X. \end{aligned}$$

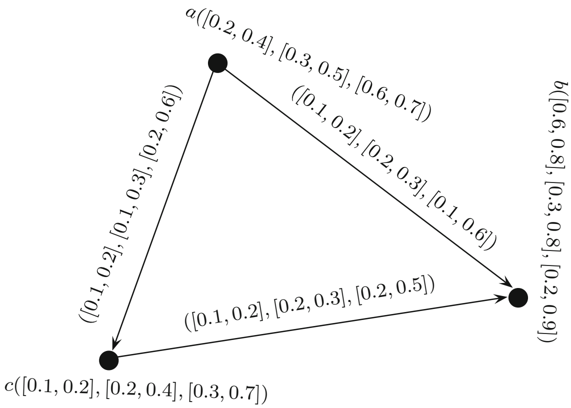


Fig. 4.4 Interval-valued neutrosophic digraph

Table 4.3 Interval-valued neutrosophic out-neighbourhoods

s	$\mathbb{N}^+(s)$
a	{(b, [0.1, 0.2], [0.2, 0.3], [0.1, 0.6]), (c, [0.1, 0.2], [0.1, 0.3],[0.2, 0.6])}
b	Ø
c	{(b, [0.1, 0.2], [0.2, 0.3], [0.2, 0.5])}

Definition 4.8 An interval-valued neutrosophic competition graph of an interval-valued neutrosophic graph $\overrightarrow{G} = (A, \overrightarrow{B})$ is an undirected interval-valued neutrosophic graph $\mathbb{C}(\overrightarrow{G}) = (A, W)$ which has the same vertex set as in $\overrightarrow{G}$ and there is an edge between two vertices s and w if and only if $\mathbb{N}^+(s) \cap \mathbb{N}^+(w) \neq \emptyset$. The truth-membership, indeterminacy-membership and falsity-membership values of the edge (s, w) are defined as,

1. $T^l_W(s, w) = (T^l_A(s) \wedge T^l_A(w))h^l_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w), T^u_W(s, w) = (T^u_A(s) \wedge T^u_A(w))$ $h^u_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)$,
2. $I^l_W(s, w) = (I^l_A(s) \wedge I^l_A(w))h^l_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w), \quad I^u_W(s, w) = (I^u_A(s) \wedge I^u_A(w))$ $h^u_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)$,
3. $F^l_W(s, w) = (F^l_A(s) \wedge F^l_A(w))h^l_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w), F^u_W(s, w) = (F^u_A(s) \wedge F^u_A(w))$ $h^u_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)$,

for all $s, w \in X$.

Example 4.4 Consider an interval-valued neutrosophic digraph $G = (A, \overrightarrow{B})$ on $X = \{a, b, c\}$ as shown in Fig. 4.4.

We have Tables 4.3 and 4.4 representing interval-valued neutrosophic out- and in-neighbourhoods, respectively.

Then interval-valued neutrosophic competition graph of Fig. 4.4 is shown in Fig. 4.5.

Table 4.4 Interval-valued neutrosophic in-neighbourhoods

s	$\mathbb{N}^-(s)$
a	∅
b	{(a, [0.1, 0.2], [0.2, 0.3], [0.1, 0.6]), (c, [0.1, 0.2], [0.2, 0.3], [0.2, 0.5])}
c	{(a, [0.1, 0.2], [0.1, 0.3], [0.2, 0.6])}

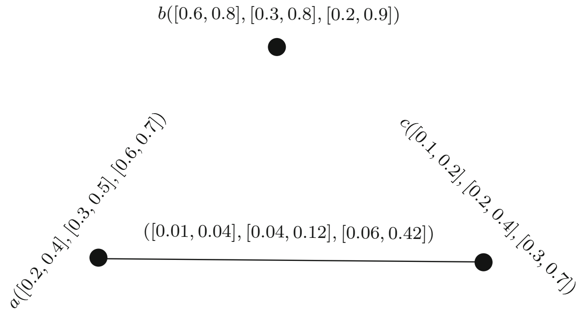


Fig. 4.5 Interval-valued neutrosophic competition graph

Definition 4.9 Consider an interval-valued neutrosophic graph $G = (A, B)$, where $A = ([A_1^l, A_1^u], [A_2^l, A_2^u], [A_3^l, A_3^u])$ and $B = ([B_1^l, B_1^u], [B_2^l, B_2^u], [B_3^l, B_3^u])$; then an edge (s, w), $s, w \in X$ is called *independent strong* if

$$\frac{1}{2}[A_1^l(s) \wedge A_1^l(w)] < B_1^l(s, w), \quad \frac{1}{2}[A_1^u(s) \wedge A_1^u(w)] < B_1^u(s, w),$$
$$\frac{1}{2}[A_2^l(s) \wedge A_2^l(w)] < B_2^l(s, w), \quad \frac{1}{2}[A_2^u(s) \wedge A_2^u(w)] < B_2^u(s, w),$$
$$\frac{1}{2}[A_3^l(s) \wedge A_3^l(w)] > B_3^l(s, w), \quad \frac{1}{2}[A_3^u(s) \wedge A_3^u(w)] > B_3^u(s, w).$$

Otherwise, it is called weak.

We state the following theorems without their proofs.

Theorem 4.1 *Suppose $\overrightarrow{G}$ is an interval-valued neutrosophic digraph. If $\mathbb{N}^+(s) \cap \mathbb{N}^+(w)$ contains only one element of $\overrightarrow{G}$, then the edge (s, w) of $\mathbb{C}(\overrightarrow{G})$ is independent strong if and only if*

$$|[\mathbb{N}^+(s) \cap \mathbb{N}^+(w)]|_{t^l} > 0.5, \quad |[\mathbb{N}^+(s) \cap \mathbb{N}^+(w)]|_{t^u} > 0.5,$$
$$|[\mathbb{N}^+(s) \cap \mathbb{N}^+(w)]|_{i^l} > 0.5, \quad |[\mathbb{N}^+(s) \cap \mathbb{N}^+(w)]|_{i^u} > 0.5,$$
$$|[\mathbb{N}^+(s) \cap \mathbb{N}^+(w)]|_{f^l} < 0.5, \quad |[\mathbb{N}^+(s) \cap \mathbb{N}^+(w)]|_{f^u} < 0.5.$$

Theorem 4.2 *If all the edges of an interval-valued neutrosophic digraph $\overrightarrow{G}$ are independent strong, then*

$$\frac{B_1^l(s,w)}{(A_1^l(s)\wedge A_1^l(w))^2} > 0.5, \quad \frac{B_1^u(s,w)}{(A_1^u(s)\wedge A_1^u(w))^2} > 0.5,$$
$$\frac{B_2^l(s,w)}{(A_2^l(s)\wedge A_2^l(w))^2} > 0.5, \quad \frac{B_2^u(s,w)}{(A_2^u(s)\wedge A_2^u(w))^2} > 0.5,$$
$$\frac{B_3^l(s,w)}{(A_3^l(s)\wedge A_3^l(w))^2} < 0.5, \quad \frac{B_3^u(s,w)}{(A_3^u(s)\wedge A_3^u(w))^2} < 0.5,$$

for all edges (s, w) *in* $\mathbb{C}(\overrightarrow{G})$.

Definition 4.10 The interval-valued neutrosophic open-neighbourhood (interval-valued neutrosophic open-neighbourhood) of a vertex s of an interval-valued neutrosophic graph $G = (A, B)$ is interval-valued neutrosophic set $\mathbb{N}(s) = (X_s, [T_s^l, T_s^u], [I_s^l, I_s^u], [F_s^l, F_s^u])$, where

$$X_s = \{w|[B_1^l(s,w) > 0, B_1^u(s,w) > 0], [B_2^l(s,w) > 0, B_2^u(s,w) > 0], [B_3^l(s,w) > 0, B_3^u(s,w) > 0]\},$$

and $T_s^l : X_s \to [0,1]$ defined by $T_s^l(w) = B_1^l(s,w)$, $T_s^u : X_s \to [0,1]$ defined by $T_s^u(w) = B_1^u(s,w)$, $I_s^l : X_s \to [0,1]$ defined by $I_s^l(w) = B_2^l(s,w)$, $I_s^u : X_s \to [0,1]$ defined by $I_s^u(w) = B_2^u(s,w)$, $F_s^l : X_s \to [0,1]$ defined by $F_s^l(w) = B_3^l(s,w)$, $F_s^u : X_s \to [0,1]$ defined by $F_s^u(w) = B_3^u(s,w)$. For every vertex $s \in X$, the interval-valued neutrosophic singleton set, $\breve{A}_s = (s, [A_1^{l\prime}, A_1^{u\prime}], [A_2^{l\prime}, A_2^{u\prime}], [A_3^{l\prime}, A_3^{u\prime})$ such that: $A_1^{l\prime} : \{s\} \to [0,1]$, $A_1^{u\prime} : \{s\} \to [0,1]$, $A_2^{l\prime} : \{s\} \to [0,1]$, $A_2^{u\prime} : \{s\} \to [0,1]$, $A_3^{l\prime} : \{s\} \to [0,1]$, $A_3^{u\prime} : \{s\} \to [0,1]$, defined by $A_1^{l\prime}(s) = A_1^l(s)$, $A_1^{u\prime}(s) = A_1^u(s)$, $A_2^{l\prime}(s) = A_2^l(s)$, $A_2^{u\prime}(s) = A_2^u(s)$, $A_3^{l\prime}(s) = A_3^l(s)$ and $A_3^{u\prime}(s) = A_3^u(s)$, respectively. The interval-valued neutrosophic closed-neighbourhood (interval-valued neutrosophic closed-neighbourhood) of a vertex s is $\mathbb{N}[s] = \mathbb{N}(s) \cup A_s$.

Definition 4.11 Suppose $G = (A, B)$ is an interval-valued neutrosophic graph. Interval-valued neutrosophic open-neighbourhood graph (interval-valued neutrosophic open-neighbourhood-graph) of G is an interval-valued neutrosophic graph $\mathbb{N}(G) = (A, B')$ which has the same interval-valued neutrosophic set of vertices in G and has an interval-valued neutrosophic edge between two vertices $s, w \in X$ in $\mathbb{N}(G)$ if and only if $\mathbb{N}(s) \cap \mathbb{N}(w)$ is a nonempty interval-valued neutrosophic set in G. The truth-membership, indeterminacy-membership, falsity-membership values of the edge (s, w) are given by:

$$B_1^{l\prime}(s,w) = [A_1^l(s) \wedge A_1^l(w)]h_1^l(\mathbb{N}(s) \cap \mathbb{N}(w)),$$
$$B_2^{l\prime}(s,w) = [A_2^l(s) \wedge A_2^l(w)]h_2^l(\mathbb{N}(s) \cap \mathbb{N}(w)),$$
$$B_3^{l\prime}(s,w) = [A_3^l(s) \wedge A_3^l(w)]h_3^l(\mathbb{N}(s) \cap \mathbb{N}(w)),$$
$$B_1^{u\prime}(s,w) = [A_1^u(s) \wedge A_1^u(w)]h_1^u(\mathbb{N}(s) \cap \mathbb{N}(w)),$$
$$B_2^{u\prime}(s,w) = [A_2^u(s) \wedge A_2^u(w)]h_2^u(\mathbb{N}(s) \cap \mathbb{N}(w)),$$

$$B_3^{u'}(s, w) = [A_3^u(s) \wedge A_3^u(w)]h_3^u(\mathbb{N}(s) \cap \mathbb{N}(w)), \text{ respectively.}$$

Definition 4.12 Suppose $G = (A, B)$ is an interval-valued neutrosophic graph. Interval-valued neutrosophic closed-neighbourhood graph (interval-valued neutrosophic closed-neighbourhood-graph) of G is an interval-valued neutrosophic graph $\mathbb{N}(G) = (A, B')$ which has the same interval-valued neutrosophic set of vertices in G and has an interval-valued neutrosophic edge between two vertices $s, w \in X$ in $\mathbb{N}[G]$ if and only if $\mathbb{N}[s] \cap \mathbb{N}[w]$ is a nonempty interval-valued neutrosophic set in G. The truth-membership, indeterminacy-membership, falsity-membership values of the edge (s, w) are given by:

$$\begin{aligned}
B_1^{l'}(s, w) &= [A_1^l(s) \wedge A_1^l(w)]h_1^l(\mathbb{N}[s] \cap \mathbb{N}[w]),\\
B_2^{l'}(s, w) &= [A_2^l(s) \wedge A_2^l(w)]h_2^l(\mathbb{N}[s] \cap \mathbb{N}[w]),\\
B_3^{l'}(s, w) &= [A_3^l(s) \wedge A_3^l(w)]h_3^l(\mathbb{N}[s] \cap \mathbb{N}[w]),\\
B_1^{u'}(s, w) &= [A_1^u(s) \wedge A_1^u(w)]h_1^u(\mathbb{N}[s] \cap \mathbb{N}[w]),\\
B_2^{u'}(s, w) &= [A_2^u(s) \wedge A_2^u(w)]h_2^u(\mathbb{N}[s] \cap \mathbb{N}[w]),\\
B_3^{u'}(s, w) &= [A_3^u(s) \wedge A_3^u(w)]h_3^u(\mathbb{N}[s] \cap \mathbb{N}[w]), \text{ respectively.}
\end{aligned}$$

We now discuss the method of construction of interval-valued neutrosophic competition graph of the Cartesian product of interval-valued neutrosophic digraph in following theorem.

Theorem 4.3 *Let $\mathbb{C}(\overrightarrow{G_1}) = (A_1, B_1)$ and $\mathbb{C}(\overrightarrow{G_2}) = (A_2, B_2)$ be two interval-valued neutrosophic competition graphs of interval-valued neutrosophic digraphs $\overrightarrow{G_1} = (A_1, \overrightarrow{L_1})$ and $\overrightarrow{G_2} = (A_2, \overrightarrow{L_2})$, respectively. Then $\mathbb{C}(\overrightarrow{G_1} \square \overrightarrow{G_2}) = G_{\mathbb{C}(\overrightarrow{G_1})^* \square \mathbb{C}(\overrightarrow{G_2})^*} \cup G^{\square}$ where $G_{\mathbb{C}(\overrightarrow{G_1})^* \square \mathbb{C}(\overrightarrow{G_2})^*}$ is an interval-valued neutrosophic graph on the crisp graph $(X_1 \times X_2, E_{\mathbb{C}(\overrightarrow{G_1})^*} \square E_{\mathbb{C}(\overrightarrow{G_2})^*})$, $\mathbb{C}(\overrightarrow{G_1})^*$ and $\mathbb{C}(\overrightarrow{G_2})^*$ are the crisp competition graphs of $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$, respectively. $G^{\square}$ is an interval-valued neutrosophic graph on $(X_1 \times X_2, E^{\square})$ such that:*

1. $E^{\square} = \{(s_1, s_2)(w_1, w_2) : w_1 \in \mathbb{N}^-(s_1)^*, w_2 \in \mathbb{N}^+(s_2)^*\}$
   $E_{\mathbb{C}(\overrightarrow{G_1})^*} \square E_{\mathbb{C}(\overrightarrow{G_2})^*} = \{(s_1, s_2)(s_1, w_2) : s_1 \in X_1, s_2 w_2 \in E_{\mathbb{C}(\overrightarrow{G_2})^*}\}$
   $\cup \{(s_1, s_2)(w_1, s_2) : s_2 \in X_2, s_1 w_1 \in E_{\mathbb{C}(\overrightarrow{G_1})^*}\}$.
2. $T^l_{A_1 \square A_2} = T^l_{A_1}(s_1) \wedge T^l_{A_2}(s_2)$, $I^l_{A_1 \square A_2} = I^l_{A_1}(s_1) \wedge I^l_{A_2}(s_2)$, $F^l_{A_1 \square A_2} = F^l_{A_1}(s_1) \wedge F^l_{A_2}(s_2)$,
   $T^u_{A_1 \square A_2} = T^u_{A_1}(s_1) \wedge T^u_{A_2}(s_2)$, $I^u_{A_1 \square A_2} = I^u_{A_1}(s_1) \wedge I^u_{A_2}(s_2)$, $F^u_{A_1 \square A_2} = F^u_{A_1}(s_1) \wedge F^u_{A_2}(s_2)$.
3. $T^l_B((s_1, s_2)(s_1, w_2)) = [T^l_{A_1}(s_1) \wedge T^l_{A_2}(s_2) \wedge T^l_{A_2}(w_2)] \times \vee_{a_2} \{T^l_{A_1}(s_1) \wedge T^l_{\overrightarrow{L_2}}(s_2 a_2) \wedge T^l_{\overrightarrow{L_2}}(w_2 a_2)\}$,
   $(s_1, s_2)(s_1, w_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*} \square E_{\mathbb{C}(\overrightarrow{G_2})^*}$, $a_2 \in (\mathbb{N}^+(s_2) \cap \mathbb{N}^+(w_2))^*$.

4. $I^l_B((s_1,s_2)(s_1,w_2)) = [I^l_{A_1}(s_1) \wedge I^l_{A_2}(s_2) \wedge I^l_{A_2}(w_2)] \times \vee_{a_2}\{I^l_{A_1}(s_1) \wedge I^l_{\overrightarrow{L_2}}(s_2a_2) \wedge I^l_{\overrightarrow{L_2}}(w_2a_2)\}$,
$(s_1,s_2)(s_1,w_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_2 \in (\mathbb{N}^+(s_2)\cap\mathbb{N}^+(w_2))^*$.
5. $F^l_B((s_1,s_2)(s_1,w_2)) = [F^l_{A_1}(s_1) \wedge F^l_{A_2}(s_2) \wedge F^l_{A_2}(w_2)] \times \vee_{a_2}\{F^l_{A_1}(s_1) \wedge F^l_{\overrightarrow{L_2}}(s_2a_2) \wedge F^l_{\overrightarrow{L_2}}(w_2a_2)\}$,
$(s_1,s_2)(s_1,w_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_2 \in (\mathbb{N}^+(s_2)\cap\mathbb{N}^+(w_2))^*$.
6. $T^u_B((s_1,s_2)(s_1,w_2)) = [T^u_{A_1}(s_1) \wedge T^u_{A_2}(s_2) \wedge T^u_{A_2}(w_2)] \times \vee_{a_2}\{T^u_{A_1}(s_1) \wedge T^u_{\overrightarrow{L_2}}(s_2a_2) \wedge T^u_{\overrightarrow{L_2}}(w_2a_2)\}$,
$(s_1,s_2)(s_1,w_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_2 \in (\mathbb{N}^+(s_2)\cap\mathbb{N}^+(w_2))^*$.
7. $I^u_B((s_1,s_2)(s_1,w_2)) = [I^u_{A_1}(s_1) \wedge I^u_{A_2}(s_2) \wedge I^u_{A_2}(w_2)] \times \vee_{a_2}\{I^u_{A_1}(s_1) \wedge I^u_{\overrightarrow{L_2}}(s_2a_2) \wedge I^u_{\overrightarrow{L_2}}(w_2a_2)\}$,
$(s_1,s_2)(s_1,w_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_2 \in (\mathbb{N}^+(s_2)\cap\mathbb{N}^+(w_2))^*$.
8. $F^u_B((s_1,s_2)(s_1,w_2)) = [F^u_{A_1}(s_1) \wedge F^u_{A_2}(s_2) \wedge F^u_{A_2}(w_2)] \times \vee_{a_2}\{F^u_{A_1}(s_1) \wedge F^u_{\overrightarrow{L_2}}(s_2a_2) \wedge F^u_{\overrightarrow{L_2}}(w_2a_2)\}$,
$(s_1,s_2)(s_1,w_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_2 \in (\mathbb{N}^+(s_2)\cap\mathbb{N}^+(w_2))^*$.
9. $T^l_B((s_1,s_2)(w_1,s_2)) = [T^l_{A_1}(s_1) \wedge T^l_{A_1}(w_1) \wedge T^l_{A_2}(s_2)] \times \vee_{a_1}\{T^l_{A_2}(s_2) \wedge T^l_{\overrightarrow{L_1}}(s_1a_1) \wedge T^l_{\overrightarrow{L_1}}(w_1a_1)\}$,
$(s_1,s_2)(w_1,s_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_1 \in (\mathbb{N}^+(s_1)\cap\mathbb{N}^+(w_1))^*$.
10. $I^l_B((s_1,s_2)(w_1,s_2)) = [I^l_{A_1}(s_1) \wedge I^l_{A_1}(w_1) \wedge I^l_{A_2}(s_2)] \times \vee_{a_1}\{I^l_{A_2}(s_2) \wedge I^l_{\overrightarrow{L_1}}(s_1a_1) \wedge I^l_{\overrightarrow{L_1}}(w_1a_1)\}$,
$(s_1,s_2)(w_1,s_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_1 \in (\mathbb{N}^+(s_1)\cap\mathbb{N}^+(w_1))^*$.
11. $F^l_B((s_1,s_2)(w_1,s_2)) = [F^l_{A_1}(s_1) \wedge F^l_{A_1}(w_1) \wedge F^l_{A_2}(s_2)] \times \vee_{a_1}\{T^l_{A_2}(s_2) \wedge F^l_{\overrightarrow{L_1}}(s_1a_1) \wedge F^l_{\overrightarrow{L_1}}(w_1a_1)\}$,
$(s_1,s_2)(w_1,s_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_1 \in (\mathbb{N}^+(s_1)\cap\mathbb{N}^+(w_1))^*$.
12. $T^u_B((s_1,s_2)(w_1,s_2)) = [T^u_{A_1}(s_1) \wedge T^u_{A_1}(w_1) \wedge T^u_{A_2}(s_2)] \times \vee_{a_1}\{T^u_{A_2}(s_2) \wedge T^u_{\overrightarrow{L_1}}(s_1a_1) \wedge T^u_{\overrightarrow{L_1}}(w_1a_1)\}$,
$(s_1,s_2)(w_1,s_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_1 \in (\mathbb{N}^+(s_1)\cap\mathbb{N}^+(w_1))^*$.
13. $I^u_B((s_1,s_2)(w_1,s_2)) = [I^u_{A_1}(s_1) \wedge I^u_{A_1}(w_1) \wedge I^u_{A_2}(s_2)] \times \vee_{a_1}\{I^u_{A_2}(s_2) \wedge I^u_{\overrightarrow{L_1}}(s_1a_1) \wedge I^u_{\overrightarrow{L_1}}(w_1a_1)\}$,
$(s_1,s_2)(w_1,s_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_1 \in (\mathbb{N}^+(s_1)\cap\mathbb{N}^+(w_1))^*$.
14. $F^u_B((s_1,s_2)(w_1,s_2)) = [F^u_{A_1}(s_1) \wedge F^u_{A_1}(w_1) \wedge F^u_{A_2}(s_2)] \times \vee_{a_1}\{T^u_{A_2}(s_2) \wedge F^u_{\overrightarrow{L_1}}(s_1a_1) \wedge F^u_{\overrightarrow{L_1}}(w_1a_1)\}$,
$(s_1,s_2)(w_1,s_2) \in E_{\mathbb{C}(\overrightarrow{G_1})^*}\Box E_{\mathbb{C}(\overrightarrow{G_2})^*}, \quad a_1 \in (\mathbb{N}^+(s_1)\cap\mathbb{N}^+(w_1))^*$.
15. $T^l_B((s_1,s_2)(w_1,w_2)) = [T^l_{A_1}(s_1) \wedge T^l_{A_1}(w_1) \wedge T^l_{A_2}(s_2) \wedge T^l_{A_2}(w_2)] \times [T^l_{A_1}(s_1) \wedge T^l_{\overrightarrow{L_1}}(w_1s_1) \wedge T^l_{A_2}(w_2) \wedge T^l_{\overrightarrow{L_2}}(s_2w_2)]$,
$(s_1,w_1)(s_2,w_2) \in E^{\Box}$.

16. $I^l_B((s_1, s_2)(w_1, w_2)) = [I^l_{A_1}(s_1) \wedge I^l_{A_1}(w_1) \wedge I^l_{A_2}(s_2) \wedge I^l_{A_2}(w_2)] \times [I^l_{A_1}(s_1) \wedge I^l_{\overrightarrow{L_1}}(w_1 s_1) \wedge I^l_{A_2}(w_2) \wedge I^l_{\overrightarrow{L_2}}(s_2 w_2)]$,
$(s_1, w_1)(s_2, w_2) \in E^{\square}$.
17. $F^l_B((s_1, s_2)(w_1, w_2)) = [F^l_{A_1}(s_1) \wedge F^l_{A_1}(w_1) \wedge F^l_{A_2}(s_2) \wedge F^l_{A_2}(w_2)] \times [F^l_{A_1}(s_1) \wedge F^l_{\overrightarrow{L_1}}(w_1 s_1) \wedge F^l_{A_2}(w_2) \wedge F^l_{\overrightarrow{L_2}}(s_2 w_2)]$,
$(s_1, w_1)(s_2, w_2) \in E^{\square}$.
18. $T^u_B((s_1, s_2)(w_1, w_2)) = [T^u_{A_1}(s_1) \wedge T^u_{A_1}(w_1) \wedge T^u_{A_2}(s_2) \wedge T^u_{A_2}(w_2)] \times [T^u_{A_1}(s_1) \wedge T^u_{\overrightarrow{L_1}}(w_1 s_1) \wedge T^u_{A_2}(w_2) \wedge T^u_{\overrightarrow{L_2}}(s_2 w_2)]$,
$(s_1, w_1)(s_2, w_2) \in E^{\square}$.
19. $I^u_B((s_1, s_2)(w_1, w_2)) = [I^u_{A_1}(s_1) \wedge I^u_{A_1}(w_1) \wedge I^u_{A_2}(s_2) \wedge I^u_{A_2}(w_2)] \times [I^u_{A_1}(s_1) \wedge I^u_{\overrightarrow{L_1}}(w_1 s_1) \wedge I^u_{A_2}(w_2) \wedge I^u_{\overrightarrow{L_2}}(s_2 w_2)]$,
$(s_1, w_1)(s_2, w_2) \in E^{\square}$.
20. $F^u_B((s_1, s_2)(w_1, w_2)) = [F^u_{A_1}(s_1) \wedge F^u_{A_1}(w_1) \wedge F^u_{A_2}(s_2) \wedge F^u_{A_2}(w_2)] \times [F^u_{A_1}(s_1) \wedge F^u_{\overrightarrow{L_1}}(w_1 s_1) \wedge F^u_{A_2}(w_2) \wedge F^u_{\overrightarrow{L_2}}(s_2 w_2)]$,
$(s_1, w_1)(s_2, w_2) \in E^{\square}$.

4.3 *k*-Competition Interval-Valued Neutrosophic Graphs

In this section, we discuss an extension of interval-valued neutrosophic competition graphs, called *k-competition interval-valued neutrosophic graphs*.

Definition 4.13 The cardinality of an interval-valued neutrosophic set A is denoted by

$$|A| = \left(\left[|A|_{t^l}, |A|_{t^u}\right], \left[|A|_{i^l}, |A|_{i^u}\right], \left[|A|_{f^l}, |A|_{f^u}\right]\right).$$

where $\left[|A|_{t^l}, |A|_{t^u}\right]$, $\left[|A|_{i^l}, |A|_{i^u}\right]$ and $\left[|A|_{f^l}, |A|_{f^u}\right]$ represent the sum of truth-membership values, indeterminacy-membership values and falsity-membership values, respectively, of all the elements of A.

Example 4.5 The cardinality of an interval-valued neutrosophic set $A = \{(a$, [0.5, 0.7], [0.2, 0.8], [0.1, 0.3]), (b, [0.1, 0.2], [0.1, 0.5], [0.7, 0.9]), (c, [0.3, 0.5], [0.3, 0.8], [0.6, 0.9])} in $X = \{a, b, c\}$ is

$$\begin{aligned}|A| &= \left(\left[|A|_{t^l}, |A|_{t^u}\right], \left[|A|_{i^l}, |A|_{i^u}\right], \left[|A|_{f^l}, |A|_{f^u}\right]\right)\\ &= ([0.9, 1.4], [0.6, 2.1], [1.4, 2.1]).\end{aligned}$$

We now discuss k-competition interval-valued neutrosophic graphs.

Definition 4.14 Let k be a nonnegative number. Then k-competition interval-valued neutrosophic graph $\mathbb{C}_k(\overrightarrow{G})$ of an interval-valued neutrosophic digraph $\overrightarrow{G} = (A, \overrightarrow{B})$ is an undirected interval-valued neutrosophic graph $G = (A, B)$ which has same interval-valued neutrosophic set of vertices as in $\overrightarrow{G}$ and has an interval-valued neutrosophic edge between two vertices $s,\ w \in X$ in $\mathbb{C}_k(\overrightarrow{G})$ if and only if $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^l} > k$, $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^u} > k$, $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^l} > k$, $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^u} > k$, $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^l} > k$ and $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^u} > k$. The interval-valued truth-membership value of edge (s, w) in $\mathbb{C}_k(\overrightarrow{G})$ is $t^l_B(s, w) = \frac{k^l_1 - k}{k^l_1}[t^l_A(s) \wedge t^l_A(w)]h^l_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $k^l_1 = |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^l}$ and $t^u_B(s,\ w) = \frac{k^u_1 - k}{k^u_1}[t^u_A(s) \wedge t^u_A(w)]h^u_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $k^u_1 = |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^u}$, the interval-valued indeterminacy-membership value of edge $(s,\ w)$ in $\mathbb{C}_k(\overrightarrow{G})$ is $i^l_B(s,\ w) = \frac{k^l_2 - k}{k^l_2}[i^l_A(s) \wedge i^l_A(w)]h^l_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $k^l_2 = |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^l}$, and $i^u_B(s, w) = \frac{k^u_2 - k}{k^u_2}[i^u_A(s) \wedge i^u_A(w)]h^u_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $k^u_2 = |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^u}$, the interval-valued falsity-membership value of edge $(s,\ w)$ in $\mathbb{C}_k(\overrightarrow{G})$ is $f^l_B(s,\ w) = \frac{k^l_3 - k}{k^l_3}[f^l_A(s) \wedge f^l_A(w)]h^l_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $k^l_3 = |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^l}$, and $f^u_B(s, w) = \frac{k^u_3 - k}{k^u_3}[f^u_A(s) \wedge f^u_A(w)]h^u_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $k^u_3 = |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^u}$.

Example 4.6 Consider an interval-valued neutrosophic digraph $G = (A,\ \overrightarrow{B})$ on $X = \{s, w, a, b, c\}$, such that $A = \{(s, [0.4, 0.5], [0.5, 0.7], [0.8, 0.9]), (w, [0.6, 0.7], [0.4, 0.6], [0.2, 0.3]), (a, [0.2, 0.6], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.6]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.6])\}$, and $B = \{(\overrightarrow{(s, a)}, [0.1, 0.4], [0.3, 0.6], [0.2, 0.6]), (\overrightarrow{(s, b)}, [0.2, 0.4], [0.1, 0.5], [0.2, 0.6]), (\overrightarrow{(s, c)}, [0.2, 0.5], [0.3, 0.5], [0.2, 0.6]), (\overrightarrow{(w, a)}, [0.2, 0.5], [0.2, 0.5], [0.2, 0.3]), (\overrightarrow{(w, b)}, [0.2, 0.6], [0.1, 0.6], [0.2, 0.3]), (\overrightarrow{(w, c)}, [0.2, 0.7], [0.3, 0.5], [0.2, 0.3])\}$, as shown in Fig. 4.6.

We calculate $\mathbb{N}^+(s) = \{(a, [0.1, 0.4], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.4], [0.1, 0.5], [0.2, 0.6]), (c, [0.2, 0.5], [0.3, 0.5], [0.2, 0.6])\}$ and $\mathbb{N}^+(w) = \{(a, [0.2, 0.5], [0.2, 0.5], [0.2, 0.3]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.3]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.3])\}$. Therefore, $\mathbb{N}^+(s) \cap \mathbb{N}^+(w) = \{(a, [0.1, 0.4], [0.2, 0.5], [0.2, 0.3]), (b, [0.2, 0.4], [0.1, 0.5], [0.2, 0.3]), (c, [0.2, 0.5], [0.3, 0.5], [0.2, 0.3)\}$. So, $k^l_1 = 0.5$, $k^u_1 = 1.3$, $k^l_2 = 0.6$, $k^u_2 = 1.5$, $k^l_3 = 0.6$ and $k^u_3 = 0.9$. Let $k = 0.4$, then, $t^l_B(s, w) = 0.02$, $t^u_B(s, w) = 0.56$, $i^l_B(s, w) = 0.06$, $i^u_B(s, w) = 0.82$, $f^l_B(s, w) = 0.02$ and $f^u_B(s, w) = 0.11$. This graph is depicted in Fig. 4.7.

Theorem 4.4 *Let $\overrightarrow{G} = (A,\ \overrightarrow{B})$ be an interval-valued neutrosophic digraph. If*

$$h^l_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h^l_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h^l_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1,$$
$$h^u_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h^u_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h^u_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1,$$

and

$$|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^l} > 2k, \quad |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^l} > 2k, \quad |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^l} < 2k,$$
$$|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^u} > 2k, \quad |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^u} > 2k, \quad |(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^u} < 2k,$$

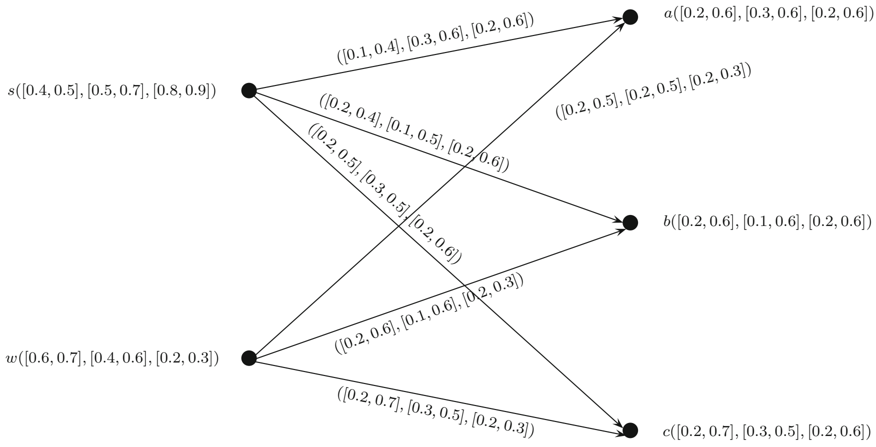


Fig. 4.6 Interval-valued neutrosophic digraph

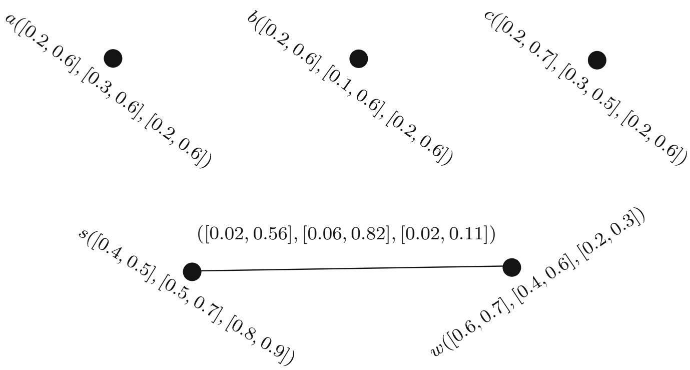


Fig. 4.7 0.4-competition interval-valued neutrosophic graph

Then the edge (s, w) *is independent strong in* $\mathbb{C}_k(\overrightarrow{G})$.

Proof Let $\overrightarrow{G} = (A, \overrightarrow{B})$ be an interval-valued neutrosophic digraph. Let $\mathbb{C}_k(\overrightarrow{G})$ be the corresponding k-competition interval-valued neutrosophic graph. If $h_1^l(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1$ and $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^l} > 2k$, then $k_1^l > 2k$ and therefore,

$$t_B^l(s, w) = \frac{k_1^l - k}{k_1^l}[t_A^l(s) \wedge t_A^l(w)]h_1^l(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$$

$$\text{or,} \quad t_B^l(s, w) = \frac{k_1^l - k}{k_1^l}[t_A^l(s) \wedge t_A^l(w)]$$

$$\frac{t_B^l(s, w)}{[t_A^l(s) \wedge t_A^l(w)]} = \frac{k_1^l - k}{k_1^l} > 0.5.$$

If $h_1^u(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1$ and $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{t^u} > 2k$, then $k_1^u > 2k$ and therefore,

$$\begin{aligned} t_B^u(s,w) &= \frac{k_1^u - k}{k_1^u}[t_A^u(s) \wedge t_A^u(w)]h_1^u(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) \\ \text{or,} \quad t_B^u(s,w) &= \frac{k_1^u - k}{k_1^u}[t_A^u(s) \wedge t_A^u(w)] \\ \frac{t_B^u(s,w)}{[t_A^u(s) \wedge t_A^u(w)]} &= \frac{k_1^u - k}{k_1^u} > 0.5. \end{aligned}$$

If $h_2^l(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1$ and $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^l} > 2k$, then $k_2^l > 2k$ and therefore,

$$\begin{aligned} i_B^l(s,w) &= \frac{k_2^l - k}{k_2^l}[i_A^l(s) \wedge i_A^l(w)]h_2^l(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) \\ \text{or,} \quad i_B^l(s,w) &= \frac{k_2^l - k}{k_2^l}[i_A^l(s) \wedge i_A^l(w)] \\ \frac{i_B^l(s,w)}{[i_A^l(s) \wedge i_A^l(w)]} &= \frac{k_2^l - k}{k_2^l} > 0.5. \end{aligned}$$

If $h_2^u(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1$ and $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{i^u} > 2k$, then $k_2^u > 2k$ and therefore,

$$\begin{aligned} i_B^u(s,w) &= \frac{k_2^u - k}{k_2^u}[i_A^u(s) \wedge i_A^u(w)]h_2^u(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) \\ \text{or,} \quad i_B^u(s,w) &= \frac{k_2^u - k}{k_2^u}[i_A^u(s) \wedge i_A^u(w)] \\ \frac{i_B^u(s,w)}{[i_A^u(s) \wedge i_A^u(w)]} &= \frac{k_2^u - k}{k_2^u} > 0.5. \end{aligned}$$

If $h_3^l(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1$ and $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^l} < 2k$, then $k_3^l < 2k$ and therefore,

$$\begin{aligned} f_B^l(s,w) &= \frac{k_3^l - k}{k_3^l}[f_A^l(s) \wedge f_A^l(w)]h_3^l(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) \\ \text{or,} \quad f_B^l(s,w) &= \frac{k_3^l - k}{k_3^l}[f_A^l(s) \wedge f_A^l(w)] \\ \frac{f_B^l(s,w)}{[f_A^l(s) \wedge f_A^l(w)]} &= \frac{k_3^l - k}{k_3^l} < 0.5. \end{aligned}$$

If $h_3^u(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1$ and $|(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|_{f^u} < 2k$, then $k_3^u < 2k$ and therefore,

$$f_B^u(s, w) = \frac{k_3^u - k}{k_3^u}[f_A^u(s) \wedge f_A^u(w)]h_3^u(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$$
$$\text{or,} \quad f_B^u(s, w) = \frac{k_3^u - k}{k_3^u}[f_A^u(s) \wedge f_A^u(w)]$$
$$\frac{f_B^u(s, w)}{[f_A^u(s) \wedge f_A^u(w)]} = \frac{k_3^u - k}{k_3^u} < 0.5.$$

Hence, the edge (s, w) is independent strong in $\mathbb{C}_k(\overrightarrow{G})$.

4.4 p-Competition Interval-Valued Neutrosophic Graphs

In this section, we define another extension of interval-valued neutrosophic competition graphs, called *p-competition interval-valued neutrosophic graphs*.

Definition 4.15 The *support* of an interval-valued neutrosophic set $A = (s, [T_A^l, T_A^u], [I_A^l, I_A^u], [F_A^l, F_A^u])$ in X is the subset of X defined by

$$supp(A) = \{s \in X : [T_A^l(s) \neq 0, T_A^u(s) \neq 0], [I_A^l(s) \neq 0, I_A^u(s) \neq 0], [F_A^l(s) \neq 1, F_A^u(s) \neq 1]\}$$

and $|supp(A)|$ is the number of elements in the set.

Example 4.7 The support of an interval-valued neutrosophic set $A = \{(a, [0.5, 0.7], [0.2, 0.8], [0.1, 0.3]), (b, [0.1, 0.2], [0.1, 0.5], [0.7, 0.9]), (c, [0.3, 0.5], [0.3, 0.8], [0.6, 0.9]), (d, [0, 0], [0, 0], [1, 1])\}$ in $X = \{a, b, c, d\}$ is $supp(A) = \{a, b, c\}$ and $|supp(A)| = 3$.

We now define p-competition interval-valued neutrosophic graphs.

Definition 4.16 Let p be a positive integer. Then p-competition interval-valued neutrosophic graph $\mathbb{C}^p(\overrightarrow{G})$ of the interval-valued neutrosophic digraph $\overrightarrow{G} = (A, \overrightarrow{B})$ is an undirected interval-valued neutrosophic graph $G = (A, B)$ which has same interval-valued neutrosophic set of vertices as in $\overrightarrow{G}$ and has an interval-valued neutrosophic edge between two vertices $s, w \in X$ in $\mathbb{C}^p(\overrightarrow{G})$ if and only if $|supp(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))| \geq p$. The interval-valued truth-membership value of edge (s, w) in $\mathbb{C}^p(\overrightarrow{G})$ is $t_B^l(s, w) = \frac{(i-p)+1}{i}[t_A^l(s) \wedge t_A^l(w)]h_1^l(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, and $t_B^u(s, w) = \frac{(i-p)+1}{i}[t_A^u(s) \wedge t_A^u(w)]h_1^u(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, the interval-valued indeterminacy-membership value of edge (s, w) in $\mathbb{C}^p(\overrightarrow{G})$ is $i_B^l(s, w) =$

$\frac{(i-p)+1}{i}[i^l_A(s) \wedge i^l_A(w)]h^l_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, and $i^u_B(s, w) = \frac{(i-p)+1}{i}[i^u_A(s) \wedge i^u_A(w)]h^u_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, the interval-valued falsity-membership value of edge (s, w) in $\mathbb{C}^p(\overrightarrow{G})$ is $f^l_B(s, w) = \frac{(i-p)+1}{i}[f^l_A(s) \wedge f^l_A(w)]h^l_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, and $f^u_B(s, w) = \frac{(i-p)+1}{i}[f^u_A(s) \wedge f^u_A(w)]h^u_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))$, where $i = |supp(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|$.

Example 4.8 Consider an interval-valued neutrosophic digraph $G = (A, \overrightarrow{B})$ on $X = \{s, w, a, b, c\}$, such that $A = \{(s, [0.4, 0.5], [0.5, 0.7], [0.8, 0.9]), (w, [0.6, 0.7], [0.4, 0.6], [0.2, 0.3]), (a, [0.2, 0.6], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.6]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.6])\}$, and $B = \{(\overrightarrow{(s, a)}, [0.1, 0.4], [0.3, 0.6], [0.2, 0.6]), (\overrightarrow{(s, b)}, [0.2, 0.4], [0.1, 0.5], [0.2, 0.6]), (\overrightarrow{(s, c)}, [0.2, 0.5], [0.3, 0.5], [0.2, 0.6]), (\overrightarrow{(w, a)}, [0.2, 0.5], [0.2, 0.5], [0.2, 0.3]), (\overrightarrow{(w, b)}, [0.2, 0.6], [0.1, 0.6], [0.2, 0.3]), (\overrightarrow{(w, c)}, [0.2, 0.7], [0.3, 0.5], [0.2, 0.3])\}$, as shown in Fig. 4.8.

We calculate $\mathbb{N}^+(s) = \{(a, [0.1, 0.4], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.4], [0.1, 0.5], [0.2, 0.6]), (c, [0.2, 0.5], [0.3, 0.5], [0.2, 0.6])\}$ and $\mathbb{N}^+(w) = \{(a, [0.2, 0.5], [0.2, 0.5], [0.2, 0.3]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.3]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.3])\}$. Therefore, $\mathbb{N}^+(s) \cap \mathbb{N}^+(w) = \{(a, [0.1, 0.4], [0.2, 0.5], [0.2, 0.3]), (b, [0.2, 0.4], [0.1, 0.5], [0.2, 0.3]), (c, [0.2, 0.5], [0.3, 0.5], [0.2, 0.3)\}$. Now, $i = |supp(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))| = 3$. For $p = 3$, we have, $t^l_B(s, w) = 0.02$, $t^u_B(s, w) = 0.08$, $i^l_B(s, w) = 0.04$, $i^u_B(s, w) = 0.1$, $f^l_B(s, w) = 0.01$ and $f^u_B(s, w) = 0.03$. This graph is depicted in Fig. 4.9.

We state the following theorem without its proof.

Theorem 4.5 *Let $\overrightarrow{G} = (A, \overrightarrow{B})$ be an interval-valued neutrosophic digraph. If*

$$h^l_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h^l_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h^l_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 0,$$
$$h^u_1(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h^u_2(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 1, \quad h^u_3(\mathbb{N}^+(s) \cap \mathbb{N}^+(w)) = 0,$$

in $\mathbb{C}^{[\frac{i}{2}]}(\overrightarrow{G})$, then the edge (s, w) is strong, where $i = |supp(\mathbb{N}^+(s) \cap \mathbb{N}^+(w))|$. (Note that for any real number s, $[s]$=greatest integer not exceeding s.)

4.5 *m*-Step Interval-Valued Neutrosophic Competition Graphs

We define here another extension of interval-valued neutrosophic competition graph known as *m-step interval-valued neutrosophic competition graph*. We will use the following notations:

$P^m_{s,w}$: An interval-valued neutrosophic path of length m from s to w,
$\overrightarrow{P}^m_{s,w}$: A directed interval-valued neutrosophic path of length m from s to w,
$\mathbb{N}^+_m(s)$: m-step interval-valued neutrosophic out-neighbourhood of vertex s,
$\mathbb{N}^-_m(s)$: m-step interval-valued neutrosophic in-neighbourhood of vertex s,

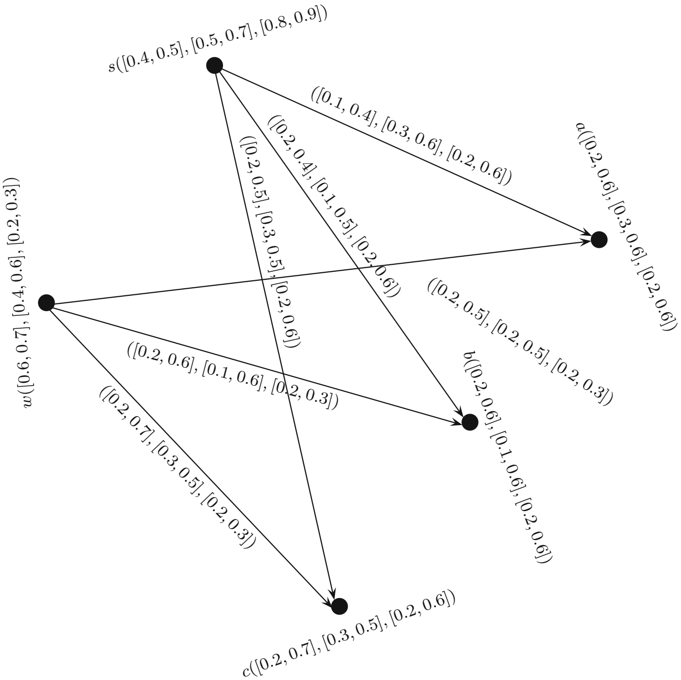


Fig. 4.8 Interval-valued neutrosophic digraph

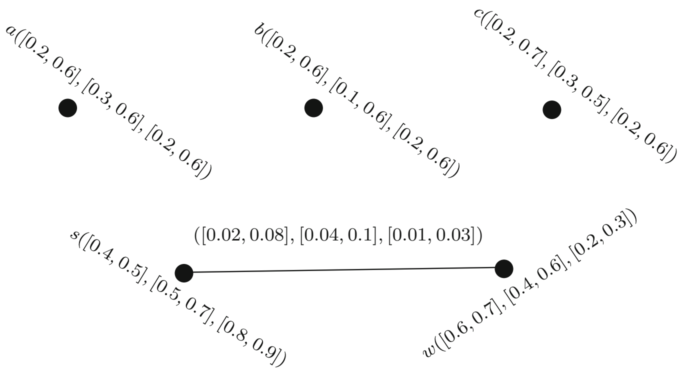


Fig. 4.9 3-competition interval-valued neutrosophic graph

$\mathbb{N}_m(s)$: m-step interval-valued neutrosophic neighbourhood of vertex s,
$\mathbb{N}_m(G)$: m-step interval-valued neutrosophic neighbourhood graph of the interval-valued neutrosophic graph G,
$\mathbb{C}_m(\overrightarrow{G})$: m-step interval-valued neutrosophic competition graph of the interval-valued neutrosophic digraph $\overrightarrow{G}$.

Definition 4.17 Suppose $\overrightarrow{G} = (A, \overrightarrow{B})$ is an interval-valued neutrosophic digraph. The m-step interval-valued neutrosophic digraph of $\overrightarrow{G}$ is denoted by $\overrightarrow{G}_m = (A, B)$, where interval-valued neutrosophic set of vertices of $\overrightarrow{G}$ is same with interval-valued neutrosophic set of vertices of $\overrightarrow{G}_m$ and has an edge between s and w in $\overrightarrow{G}_m$ if and only if there exists an interval-valued neutrosophic directed path $\overrightarrow{P}^m_{s,w}$ in $\overrightarrow{G}$.

Definition 4.18 The m-step interval-valued neutrosophic out-neighbourhood of vertex s of an interval-valued neutrosophic digraph $\overrightarrow{G} = (A, \overrightarrow{B})$ is interval-valued neutrosophic set

$$\mathbb{N}^+_m(s) = (X^+_s, [t^{(l)^+}_s, t^{(u)^+}_s], [i^{(l)^+}_s, i^{(u)^+}_s], [f^{(l)^+}_s, f^{(u)^+}_s]), \quad where$$

$X^+_s = \{w|$ there exists a directed interval-valued neutrosophic path of length m from s to w, $\overrightarrow{P}^m_{s,w}\}$, $t^{(l)^+}_s : X^+_s \to [0, 1]$, $t^{(u)^+}_s : X^+_s \to [0, 1]$, $i^{(l)^+}_s : X^+_s \to [0, 1]$, $i^{(u)^+}_s : X^+_s \to [0, 1]$, $f^{(l)^+}_s : X^+_s \to [0, 1]$ $f^{(u)^+}_s : X^+_s \to [0, 1]$ are defined by $t^{(l)^+}_s = \min\{t^l\overrightarrow{(s_1, s_2)}, (s_1, s_2)$ is an edge of $\overrightarrow{P}^m_{s,w}\}$, $t^{(u)^+}_s = \min\{t^u\overrightarrow{(s_1, s_2)}, (s_1, s_2)$ is an edge of $\overrightarrow{P}^m_{s,w}\}$, $i^{(l)^+}_s = \min\{i^l\overrightarrow{(s_1, s_2)}, (s_1, s_2)$ is an edge of $\overrightarrow{P}^m_{s,w}\}$, $i^{(u)^+}_s = \min\{i^u\overrightarrow{(s_1, s_2)}$, (s_1, s_2) is an edge of $\overrightarrow{P}^m_{s,w}\}$, $f^{(l)^+}_s = \min\{f^l\overrightarrow{(s_1, s_2)}, (s_1, s_2)$ is an edge of $\overrightarrow{P}^m_{s,w}\}$, $f^{(u)^+}_s = \min\{f^u\overrightarrow{(s_1, s_2)}, (s_1, s_2)$ is an edge of $\overrightarrow{P}^m_{s,w}\}$, respectively.

Example 4.9 Consider an interval-valued neutrosophic digraph $G = (A, \overrightarrow{B})$ on $X = \{s, w, a, b, c, d\}$, such that $A = \{(s, [0.4, 0.5], [0.5, 0.7], [0.8, 0.9]), (w, [0.6, 0.7], [0.4, 0.6], [0.2, 0.3]), (a, [0.2, 0.6], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.6]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.6]), d([0.2, 0.6], [0.3, 0.6], [0.2, 0.6])\}$, and $B = \{(\overrightarrow{(s, a)}, [0.1, 0.4], [0.3, 0.6], [0.2, 0.6]), (\overrightarrow{(a, c)}, [0.2, 0.6], [0.3, 0.5], [0.2, 0.6]), (\overrightarrow{(a, d)}, [0.2, 0.6], [0.3, 0.5], [0.2, 0.4]), (\overrightarrow{(w, b)}, [0.2, 0.6], [0.1, 0.6], [0.2, 0.3]), (\overrightarrow{(b, c)}, [0.2, 0.4], [0.1, 0.2], [0.1, 0.3]), (\overrightarrow{(b, d)}, [0.1, 0.3], [0.1, 0.2], [0.2, 0.4])\}$, as shown in Fig. 4.10.

We calculate 2-step interval-valued neutrosophic out-neighbourhoods as, $\mathbb{N}^+_2(s) = \{(c, [0.1, 0.4], [0.3, 0.5], [0.2, 0.6]), (d, [0.1, 0.4], [0.3, 0.5], [0.2, 0.4])\}$ and $\mathbb{N}^+_2(w) = \{(c, [0.2, 0.4], [0.1, 0.2], [0.1, 0.3]), (d, [0.1, 0.3], [0.1, 0.2], [0.2, 0.3])\}$.

Definition 4.19 The m-step interval-valued neutrosophic in-neighbourhood of vertex s of an interval-valued neutrosophic digraph $\overrightarrow{G} = (A, \overrightarrow{B})$ is interval-valued neutrosophic set

$$\mathbb{N}^-_m(s) = (X^-_s, [t^{(l)^-}_s, t^{(u)^-}_s], [i^{(l)^-}_s, i^{(u)^-}_s], [f^{(l)^-}_s, f^{(u)^-}_s]), \quad \text{where}$$

$X^-_s = \{w|$ there exists a directed interval-valued neutrosophic path of length m from w to s, $\overrightarrow{P}^m_{w,s}\}$, $t^{(l)^-}_s : X^-_s \to [0, 1]$, $t^{(u)^-}_s : X^-_s \to [0, 1]$, $i^{(l)^-}_s : X^-_s \to [0, 1]$, $i^{(u)^-}_s : X^-_s \to [0, 1]$, $f^{(l)^-}_s : X^-_s \to [0, 1]$ $f^{(u)^-}_s : X^-_s \to [0, 1]$ are defined by $t^{(l)^-}_s = \min\{t^l\overrightarrow{(s_1, s_2)}, (s_1, s_2)$ is an edge of $\overrightarrow{P}^m_{w,s}\}$, $t^{(u)^-}_s = \min\{t^u\overrightarrow{(s_1, s_2)}, (s_1, s_2)$ is an edge

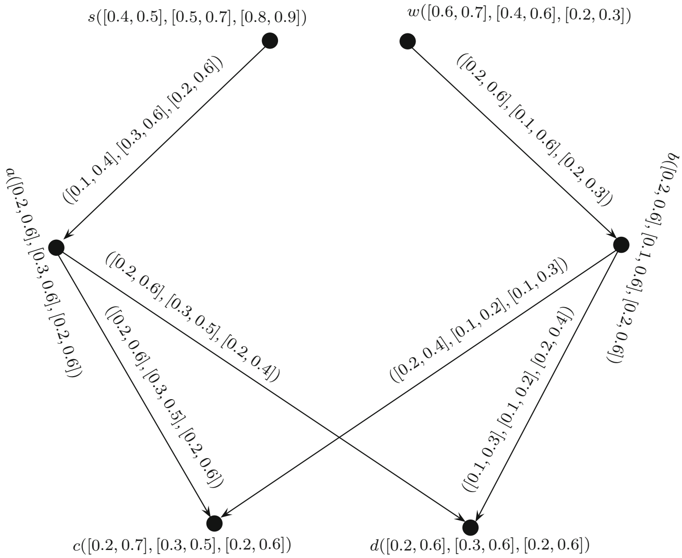


Fig. 4.10 Interval-valued neutrosophic digraph

of $\overrightarrow{P}^m_{w,s}\}$, $i_s^{(l)^-} = \min\{i^l\overrightarrow{(s_1, s_2)}$, (s_1, s_2) is an edge of $\overrightarrow{P}^m_{w,s}\}$, $i_s^{(u)^-} = \min\{i^u\overrightarrow{(s_1, s_2)}$, (s_1, s_2) is an edge of $\overrightarrow{P}^m_{w,s}\}$, $f_s^{(l)^-} = \min\{f^l\overrightarrow{(s_1, s_2)}$, (s_1, s_2) is an edge of $\overrightarrow{P}^m_{w,s}\}$, $f_s^{(u)^-} = \min\{f^u\overrightarrow{(s_1, s_2)}$, (s_1, s_2) is an edge of $\overrightarrow{P}^m_{w,s}\}$, respectively.

Example 4.10 Consider an interval-valued neutrosophic digraph $G = (A, \overrightarrow{B})$ on $X = \{s, w, a, b, c, d\}$, such that $A = \{(s, [0.4, 0.5], [0.5, 0.7], [0.8, 0.9]), (w, [0.6, 0.7], [0.4, 0.6], [0.2, 0.3]), (a, [0.2, 0.6], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.6]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.6]), d([0.2, 0.6], [0.3, 0.6], [0.2, 0.6])\}$, and $B = \{(\overrightarrow{(s, a)}, [0.1, 0.4], [0.3, 0.6], [0.2, 0.6]), (\overrightarrow{(a, c)}, [0.2, 0.6], [0.3, 0.5], [0.2, 0.6]), (\overrightarrow{(a, d)}, [0.2, 0.6], [0.3, 0.5], [0.2, 0.4]), (\overrightarrow{(w, b)}, [0.2, 0.6], [0.1, 0.6], [0.2, 0.3]), (\overrightarrow{(b, c)}, [0.2, 0.4], [0.1, 0.2], [0.1, 0.3]), (\overrightarrow{(b, d)}, [0.1, 0.3], [0.1, 0.2], [0.2, 0.4])\}$, as shown in Fig. 4.11.

We calculate 2-step interval-valued neutrosophic in-neighbourhoods as, $\mathbb{N}_2^-(s) = \{(c, [0.1,0.4], [0.3,0.5], [0.2,0.6]), (d, [0.1,0.4], [0.3,0.5], [0.2,0.4])\}$ and $\mathbb{N}_2^-(w) = \{(c, [0.2,0.4], [0.1,0.2], [0.1,0.3]), (d, [0.1,0.3], [0.1,0.2], [0.2,0.3])\}$.

Definition 4.20 Suppose $\overrightarrow{G} = (A, \overrightarrow{B})$ is an interval-valued neutrosophic digraph. The m-step interval-valued neutrosophic competition graph of interval-valued neutrosophic digraph $\overrightarrow{G}$ is denoted by $\mathbb{C}_m(\overrightarrow{G}) = (A, B)$ which has same interval-valued neutrosophic set of vertices as in $\overrightarrow{G}$ and has an edge between two vertices $s, w \in X$

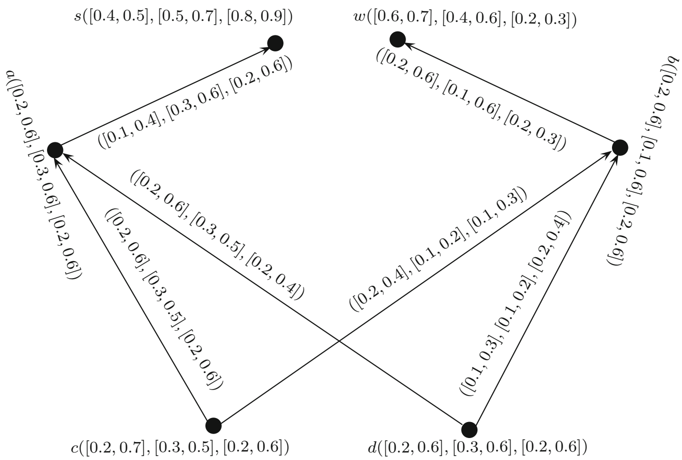


Fig. 4.11 Interval-valued neutrosophic digraph

in $\mathbb{C}_m(\overrightarrow{G})$ if and only if $(\mathbb{N}^+_m(s) \cap \mathbb{N}^+_m(w))$ is a nonempty interval-valued neutrosophic set in $\overrightarrow{G}$. The interval-valued truth-membership value of edge (s, w) in $\mathbb{C}_m(\overrightarrow{G})$ is $t^l_B(s, w) = [t^l_A(s) \wedge t^l_A(w)]h^l_1(\mathbb{N}^+_m(s) \cap \mathbb{N}^+_m(w))$, and $t^u_B(s, w) = [t^u_A(s) \wedge t^u_A(w)]h^u_1(\mathbb{N}^+_m(s) \cap \mathbb{N}^+_m(w))$, the interval-valued indeterminacy-membership value of edge (s, w) in $\mathbb{C}_m(\overrightarrow{G})$ is $i^l_B(s, w) = [i^l_A(s) \wedge i^l_A(w)]h^l_2(\mathbb{N}^+_m(s) \cap \mathbb{N}^+_m(w))$, and $i^u_B(s, w) = [i^u_A(s) \wedge i^u_A(w)]h^u_2(\mathbb{N}^+_m(s) \cap \mathbb{N}^+_m(w))$, the interval-valued falsity-membership value of edge (s, w) in $\mathbb{C}_m(\overrightarrow{G})$ is $f^l_B(s, w) = [f^l_A(s) \wedge f^l_A(w)]h^l_3(\mathbb{N}^+_m(s) \cap \mathbb{N}^+_m(w))$, and $f^u_B(s, w) = [f^u_A(s) \wedge f^u_A(w)]h^u_3(\mathbb{N}^+_m(s) \cap \mathbb{N}^+_m(w))$.

The 2-step interval-valued neutrosophic competition graph is illustrated by the following example.

Example 4.11 Consider an interval-valued neutrosophic digraph $G = (A, \overrightarrow{B})$ on $X = \{s, w, a, b, c, d\}$, such that $A = \{(s, [0.4, 0.5], [0.5, 0.7], [0.8, 0.9]), (w, [0.6, 0.7], [0.4, 0.6], [0.2, 0.3]), (a, [0.2, 0.6], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.6]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.6]), d([0.2, 0.6], [0.3, 0.6], [0.2, 0.6])\}$, and $B = \{(\overrightarrow{(s, a)}, [0.1, 0.4], [0.3, 0.6], [0.2, 0.6]), (\overrightarrow{(a, c)}, [0.2, 0.6], [0.3, 0.5], [0.2, 0.6]), (\overrightarrow{(a, d)}, [0.2, 0.6], [0.3, 0.5], [0.2, 0.4]), (\overrightarrow{(w, b)}, [0.2, 0.6], [0.1, 0.6], [0.2, 0.3]), (\overrightarrow{(b, c)}, [0.2, 0.4], [0.1, 0.2], [0.1, 0.3]), (\overrightarrow{(b, d)}, [0.1, 0.3], [0.1, 0.2], [0.2, 0.4])\}$, as shown in Fig. 4.12.

We calculate $\mathbb{N}^+_2(s) = \{(c, [0.1, 0.4], [0.3, 0.5], [0.2, 0.6]), (d, [0.1, 0.4], [0.3, 0.5], [0.2, 0.4])\}$ and $\mathbb{N}^+_2(w) = \{(c, [0.2, 0.4], [0.1, 0.2], [0.1, 0.3]), (d, [0.1, 0.3], [0.1, 0.2], [0.2, 0.3])\}$. Therefore, $\mathbb{N}^+_2(s) \cap \mathbb{N}^+_2(w) = \{(c, [0.1, 0.4], [0.1, 0.2], [0.2, 0.6]), (d, [0.1, 0.3], [0.1, 0.2], [0.2, 0.4])\}$. Thus, $t^l_B(s, w) = 0.04$, $t^u_B(s, w) = 0.20$, $i^l_B(s, w) = 0.04$, $i^u_B(s, w) = 0.12$, $f^l_B(s, w) = 0.04$ and $f^u_B(s, w) = 0.12$. This graph is depicted in Fig. 4.13.

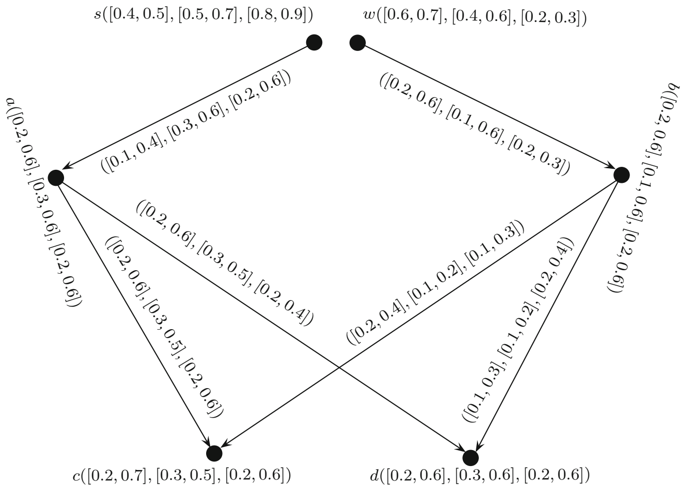


Fig. 4.12 Interval-valued neutrosophic digraph

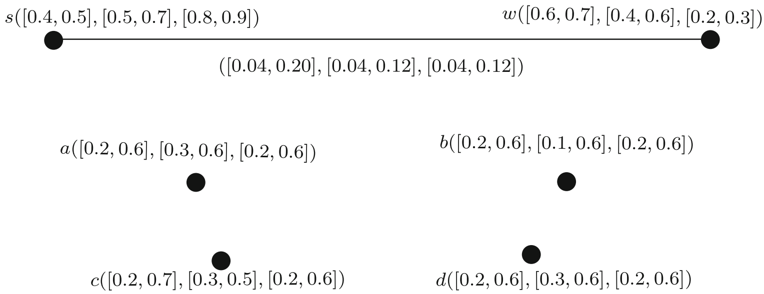


Fig. 4.13 2-Step interval-valued neutrosophic competition graph

If a predator s attacks one prey w, then the linkage is shown by an edge $\overrightarrow{(s, w)}$ in an interval-valued neutrosophic digraph. But, if predator needs help of many other mediators $s_1, s_2, \ldots, s_{m-1}$, then linkage among them is shown by interval-valued neutrosophic directed path $\overrightarrow{P}^{m}_{s,w}$ in an interval-valued neutrosophic digraph. So, m-step prey in an interval-valued neutrosophic digraph is represented by a vertex which is the m-step out-neighbourhood of some vertices. Now, the strength of an interval-valued neutrosophic competition graphs is defined below.

Definition 4.21 Let $\overrightarrow{G} = (A, \overrightarrow{B})$ be an interval-valued neutrosophic digraph. Let w be a common vertex of m-step out-neighbourhoods of vertices $s_1, s_2, \ldots, s_l$. Also, let $\overrightarrow{B}^l_1(u_1, v_1)$, $\overrightarrow{B}^l_1(u_2, v_2)$, …, $\overrightarrow{B}^l_1(u_r, v_r)$ and $\overrightarrow{B}^u_1(u_1, v_1)$, $\overrightarrow{B}^u_1(u_2, v_2)$, …, $\overrightarrow{B}^u_1(u_r, v_r)$ be the minimum interval-valued truth-membership values, $\overrightarrow{B}^l_2(u_1, v_1)$, $\overrightarrow{B}^l_2(u_2, v_2)$,…, $\overrightarrow{B}^l_2(u_r, v_r)$ and $\overrightarrow{B}^u_2(u_1, v_1)$, $\overrightarrow{B}^u_2(u_2, v_2)$, …, $\overrightarrow{B}^u_2(u_r, v_r)$ be the minimum indeterminacy-membership values, $\overrightarrow{B_3^l}(u_1, v_1)$, $\overrightarrow{B_3^l}(u_2, v_2)$, ..., $\overrightarrow{B_3^l}(u_r, v_r)$ and $\overrightarrow{B_3^u}(u_1, v_1)$, $\overrightarrow{B_3^u}(u_2, v_2)$, ..., $\overrightarrow{B_3^u}(u_r, v_r)$ be the maximum false-membership values, of edges of the paths $\overrightarrow{P}^m_{s_1,w}$, $\overrightarrow{P}^m_{s_2,w}$, ..., $\overrightarrow{P}^m_{s_r,w}$, respectively. The m-step prey $w \in X$ is strong prey if

$$\begin{aligned}&\overrightarrow{B_1^l}(u_i, v_i) > 0.5, \quad \overrightarrow{B_2^l}(u_i, v_i) > 0.5, \quad \overrightarrow{B_3^l}(u_i, v_i) < 0.5,\\&\overrightarrow{B_1^u}(u_i, v_i) > 0.5, \quad \overrightarrow{B_2^u}(u_i, v_i) > 0.5, \quad \overrightarrow{B_3^u}(u_i, v_i) < 0.5, \text{ for all } i = 1, 2, \ldots, r.\end{aligned}$$

The strength of the prey w can be measured by the mapping $S : X \to [0, 1]$, such that:

$$\begin{aligned}S(w) =& \frac{1}{r}\Bigg\{\sum_{i=1}^{r}[\overrightarrow{B_1^l}(u_i, v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_1^u}(u_i, v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_2^l}(u_i, v_i)]\\&+ \sum_{i=1}^{r}[\overrightarrow{B_2^u}(u_i, v_i)] - \sum_{i=1}^{r}[\overrightarrow{B_3^l}(u_i, v_i)] - \sum_{i=1}^{r}[\overrightarrow{B_3^u}(u_i, v_i)]\Bigg\}.\end{aligned}$$

Example 4.12 Consider an interval-valued neutrosophic digraph $\overrightarrow{G} = (A, \overrightarrow{B})$ as shown in Fig. 4.12, the strength of the prey c is equal to

$$\begin{aligned}\frac{(0.2 + 0.2) + (0.6 + 0.4) + (0.1 + 0.1) + (0.6 + 0.2) - (0.2 + 0.1) - (0.3 + 0.3)}{2} &= 1.5\\&> 0.5.\end{aligned}$$

Hence, c is strong 2-step prey.

We state the following theorem without its proof.

Theorem 4.6 *If a prey w of $\overrightarrow{G} = (A, \overrightarrow{B})$ is strong, then the strength of w, $S(w) >$ 0.5.*

Remark: The converse of the above theorem is not true; i.e., if $S(w) > 0.5$, then all preys may not be strong. This can be explained as: Let $S(w) > 0.5$ for a prey w in $\overrightarrow{G}$. So,

$$\begin{aligned}S(w) =& \frac{1}{r}\Bigg\{\sum_{i=1}^{r}[\overrightarrow{B_1^l}(u_i, v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_1^u}(u_i, v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_2^l}(u_i, v_i)]\\&+ \sum_{i=1}^{r}[\overrightarrow{B_2^u}(u_i, v_i)] - \sum_{i=1}^{r}[\overrightarrow{B_3^l}(u_i, v_i)] - \sum_{i=1}^{r}[\overrightarrow{B_3^u}(u_i, v_i)]\Bigg\}.\end{aligned}$$

Hence,

$$\left\{ \sum_{i=1}^{r}[\overrightarrow{B_1^l}(u_i, v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_1^u}(u_i, v_i)] + \sum_{i=1}^{r}[\overrightarrow{B_2^l}(u_i, v_i)] \right.$$
$$\left. + \sum_{i=1}^{r}[\overrightarrow{B_2^u}(u_i, v_i)] - \sum_{i=1}^{r}[\overrightarrow{B_3^l}(u_i, v_i)] - \sum_{i=1}^{r}[\overrightarrow{B_3^u}(u_i, v_i)] \right\} > \frac{r}{2}.$$

This result does not necessarily imply that

$$\overrightarrow{B_1^l}(u_i, v_i) > 0.5, \quad \overrightarrow{B_2^l}(u_i, v_i) > 0.5, \quad \overrightarrow{B_3^l}(u_i, v_i) < 0.5,$$
$$\overrightarrow{B_1^u}(u_i, v_i) > 0.5, \quad \overrightarrow{B_2^u}(u_i, v_i) > 0.5, \quad \overrightarrow{B_3^u}(u_i, v_i) < 0.5, \text{ for all } i = 1, 2, \ldots, r.$$

Since, all edges of the directed paths $\overrightarrow{P}^m_{s_1,w}, \overrightarrow{P}^m_{s_2,w}, \ldots, \overrightarrow{P}^m_{s_r,w}$ are not strong. So, the converse of the above statement is not true; i.e., if $S(w) > 0.5$, the prey w of $\overrightarrow{G}$ may not be strong. Now, m-step interval-valued neutrosophic neighbourhood graphs are defines below.

Definition 4.22 The m-step interval-valued neutrosophic out-neighbourhood of vertex s of an interval-valued neutrosophic digraph $\overrightarrow{G} = (A, \overrightarrow{B})$ is interval-valued neutrosophic set

$$\mathbb{N}_m(s) = (X_s, [T_s^l, T_s^u], [I_s^l, I_s^u], [F_s^l, F_s^u]), \quad \text{where}$$

$X_s = \{w|$ there exists a directed interval-valued neutrosophic path of length m from s to $w, \mathbb{P}^m_{s,w}\}$, $T_s^l : X_s \to [0, 1]$, $T_s^u : X_s \to [0, 1]$, $I_s^l : X_s \to [0, 1]$, $I_s^u : X_s \to [0, 1]$, $F_s^l : X_s \to [0, 1]$, $F_s^u : X_s \to [0, 1]$, are defined by $T_s^l = \min\{t^l(s_1, s_2), (s_1, s_2)$ is an edge of $\mathbb{P}^m_{s,w}\}$, $T_s^u = \min\{t^u(s_1, s_2), (s_1, s_2)$ is an edge of $\mathbb{P}^m_{s,w}\}$, $I_s^l = \min\{i^l(s_1, s_2), (s_1, s_2)$ is an edge of $\mathbb{P}^m_{s,w}\}$, $I_s^u = \min\{i^u(s_1, s_2), (s_1, s_2)$ is an edge of $\mathbb{P}^m_{s,w}\}$, $F_s^l = \min\{f^l(s_1, s_2), (s_1, s_2)$ is an edge of $\mathbb{P}^m_{s,w}\}$, $F_s^u = \min\{f^u(s_1, s_2), (s_1, s_2)$ is an edge of $\mathbb{P}^m_{s,w}\}$, respectively.

Definition 4.23 Suppose $G = (A, B)$ is an interval-valued neutrosophic graph. Then m-step interval-valued neutrosophic neighbourhood graph $\mathbb{N}_m(G)$ is defined by $\mathbb{N}_m(G) = (A, \acute{B})$ where $A = ([A_1^l, A_1^u], [A_2^l, A_2^u], [A_3^l, A_3^u])$, $\acute{B} = ([\acute{B}_1^l, \acute{B}_1^u], [\acute{B}_2^l, \acute{B}_2^u], [\acute{B}_3^l, \acute{B}_3^u])$, $\acute{B}_1^l : X \times X \to [0, 1]$, $\acute{B}_1^u : X \times X \to [0, 1]$, $\acute{B}_2^l : X \times X \to [0, 1]$, $\acute{B}_2^u : X \times X \to [0, 1]$, $\acute{B}_3^l : X \times X \to [0, 1]$, and $\acute{B}_3^u : X \times X \to [0, -1]$ are such that:

$$\acute{B}_1^l(s, w) = A_1^l(s) \wedge A_1^l(w) h_1^l(\mathbb{N}_m(s) \cap \mathbb{N}_m(w)),$$
$$\acute{B}_2^l(s, w) = A_2^l(s) \wedge A_2^l(w) h_2^l(\mathbb{N}_m(s) \cap \mathbb{N}_m(w)),$$
$$\acute{B}_3^l(s, w) = A_3^l(s) \wedge A_3^l(w) h_3^l(\mathbb{N}_m(s) \cap \mathbb{N}_m(w)),$$

$$\acute{B}_1^u(s,w) = A_1^u(s) \wedge A_1^u(w) h_1^u(\mathbb{N}_m(s) \cap \mathbb{N}_m(w)),$$
$$\acute{B}_2^u(s,w) = A_2^u(s) \wedge A_2^u(w) h_2^u(\mathbb{N}_m(s) \cap \mathbb{N}_m(w)),$$
$$\acute{B}_3^u(s,w) = A_3^u(s) \wedge A_3^u(w) h_3^u(\mathbb{N}_m(s) \cap \mathbb{N}_m(w)), \text{ respectively.}$$

We state the following theorems without their proofs.

Theorem 4.7 *If all preys of $\overrightarrow{G} = (A, \overrightarrow{B})$ are strong, then all edges of $\mathbb{C}_m(\overrightarrow{G}) = (A, B)$ are strong.*

A relation is established between m-step interval-valued neutrosophic competition graph of an interval-valued neutrosophic digraph and interval-valued neutrosophic competition graph of m-step interval-valued neutrosophic digraph.

Theorem 4.8 *If $\overrightarrow{G}$ is an interval-valued neutrosophic digraph and $\overrightarrow{G_m}$ is the m-step interval-valued neutrosophic digraph of $\overrightarrow{G}$, then $\mathbb{C}(\overrightarrow{G}_m) = \mathbb{C}_m(\overrightarrow{G})$.*

Theorem 4.9 *Let $\overrightarrow{G} = (A, \overrightarrow{B})$ be an interval-valued neutrosophic digraph. If $m > |X|$, then $\mathbb{C}_m(\overrightarrow{G}) = (A, B)$ has no edge.*

Theorem 4.10 *If all the edges of interval-valued neutrosophic digraph $\overrightarrow{G} = (A, \overrightarrow{B})$ are independent strong, then all the edges of $\mathbb{C}_m(\overrightarrow{G})$ are independent strong.*

Chapter 5
Interval-Valued Neutrosophic Graph Structures

In this chapter, we present certain notions of interval-valued neutrosophic graph structures. We elaborate the concepts of interval-valued neutrosophic graph structures with examples. Moreover, we discuss the concept of φ-complement of an interval-valued neutrosophic graph structure. Finally, we describe some related properties, including φ-complement, totally self-complementary and totally strong self-complementary, of interval-valued neutrosophic graph structures. This chapter is due to [35].

5.1 Introduction

Zadeh [199] introduced interval-valued fuzzy set theory which is an extension of fuzzy set theory [194]. Membership degrees in an interval-valued fuzzy set are intervals rather than numbers, and uncertainty is reflected by length of interval membership degree. Interval-valued fuzzy set theory has numerous applications in various fields of science and technology, including fuzzy control, artificial intelligence, operations research and decision-making. An interval-valued neutrosophic graph constitutes a generalization of the notion interval-valued fuzzy graph. Atanassov [47] proposed an extension of fuzzy sets by adding a new component, called intuitionistic fuzzy sets. The concept of intuitionistic fuzzy sets is more meaningful and inventive due to the presence of degree of truth, indeterminacy and falsity-membership. The intuitionistic fuzzy sets have more describing possibilities as compared to fuzzy sets. The hesitation margin of an intuitionistic fuzzy set is its uncertainty by default, and sum of truth-membership degree and falsity-membership degree does not exceed unity. In many phenomenons, including information fusion, uncertainty and indeterminacy is doubtlessly quantified. Smarandache [165, 166] proposed the idea of neutrosophic sets, and he mingled tricomponent logic, nonstandard analysis and philosophy. For convenient and advantageous usage of neutrosophic sets in science and engineering, Wang et al. [169] proposed the notion of single-valued neutrosophic sets, whose three independent components have values in standard unit interval [0, 1].

M. Akram, *Single-Valued Neutrosophic Graphs*, Infosys Science Foundation Series, https://doi.org/10.1007/978-981-13-3522-8_5

Neutrosophic set theory being a generalization of fuzzy set theory and intuitionistic fuzzy set theory is more practical, advantageous and applicable in various fields, including medical diagnosis, control theory, topology, decision-making problems and in many more real-life problems. Wang et al. [170] proposed the notion of interval-valued neutrosophic sets, which is more precise and flexible than the single-valued neutrosophic sets. An interval-valued neutrosophic set is a generalization of the notion of single-valued neutrosophic set, in which three independent components (t, i, f) are intervals which are subsets of standard unit interval [0, 1].

Definition 5.1 A *graph structure* $G^* = (X, E_1, \ldots, E_t)$ consists of a nonempty set X together with relations $E_1, E_2, \ldots, E_t$ on X which are mutually disjoint such that each E_j, $1 \leq j \leq t$, is symmetric and irreflexive.

Definition 5.2 The *interval-valued neutrosophic set* I on set X is defined by $I = \{(r, [t^-(r), t^+(r)], [i^-(r), i^+(r)], [f^-(r), f^+(r)]) : r \in X\}$, where t^-, t^+, i^-, i^+, f^-, and f^+ are functions from U to [0, 1] such that:
$t^-(r) \leq t^+(r), i^-(r) \leq i^+(r)$ and $f^-(r) \leq f^+(r)$ for all $r \in X$.

5.2 Notions of Interval-Valued Neutrosophic Graph Structures

Definition 5.3 $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is called an *interval-valued neutrosophic graph structure* of graph structure $G^* = (X, E_1, E_2, \ldots, E_t)$ if
$I = \{(r, [t^-(r), t^+(r)], [i^-(r), i^+(r)], [f^-(r), f^+(r)]) : r \in X\}$
and $I_j = \{ ((r, s), [t_j^-(r, s), t_j^+(r, s)], [i_j^-(r, s), i_j^+(r, s)], [f_j^-(r, s), f_j^+(r, s)]) : (r, s) \in E_j \}$ are interval-valued neutrosophic sets on X and E_j, respectively, such that:

1. $t_j^-(r, s) \leq \min\{t^-(r), t^-(s)\}, \quad t_j^+(r, s) \leq \min\{t^+(r), t^+(s)\}$,
2. $i_j^-(r, s) \leq \min\{i^-(r), i^-(s)\}, \quad i_j^+(r, s) \leq \min\{i^+(r), i^+(s)\}$,
3. $f_j^-(r, s) \leq \min\{f^-(r), f^-(s)\}, \quad f_j^+(r, s) \leq \min\{f^+(r), f^+(s)\}$,

where t_j^-, t_j^+, i_j^-, i_j^+, f_j^-, and f_j^+ are functions from E_j to [0, 1] such that
$t_j^-(r, s) \leq t_j^+(r, s), i_j^-(r, s) \leq i_j^+(r, s)$ and $f_j^-(r, s) \leq f_j^+(r, s)$ for all $(r, s) \in E_j$.

In this paper we will use rs in place of ordered pair (r, s) which represents an edge between vertices r and s.

Example 5.1 Consider the graph structure $G^* = (X, E_1, E_2)$ such that $X = \{r_1, r_2, r_3, r_4\}$, $E_1 = \{r_1r_3, r_1r_2, r_3r_4\}$, $E_2 = \{r_1r_4, r_2r_3\}$. By defining interval-valued neutrosophic sets I, I_1 and I_2 on X, E_1 and E_2, respectively, we draw an interval-valued neutrosophic graph structure as shown in Fig. 5.1.

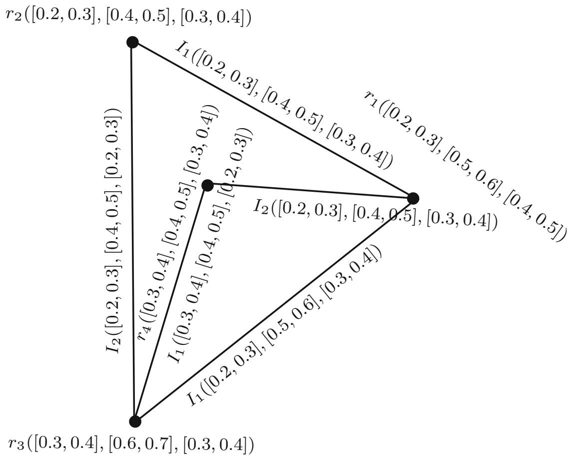


Fig. 5.1 Interval-valued neutrosophic graph structure

Definition 5.4 Let $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ be an interval-valued neutrosophic graph structure of graph structure (GS) $G^* = (X, E_1, E_2, \ldots, E_t)$. If $\check{H}_{iv} = (I', I_1', I_2', \ldots, I_t')$ is an interval-valued neutrosophic graph structure of G^* such that

$$t'^-(r) \leq t^-(r), i'^-(r) \leq i^-(r), f'^-(r) \leq f^-(r),$$
$$t'^+(r) \leq t^+(r), i'^+(r) \leq i^+(r), f'^+(r) \leq f^+(r),$$

$$t_j'^-(rs) \leq t_j^-(rs), i_j'^-(rs) \leq i_j^-(rs), f_j'^-(rs) \leq f_j^-(rs),$$
$$t_j'^+(rs) \leq t_j^+(rs), i_j'^+(rs) \leq i_j^+(rs), f_j'^+(rs) \leq f_j^+(rs),$$
$$\text{for all } r \in X \text{ and } rs \in E_j, j = 1, 2, \ldots, t.$$

Then $\check{H}_{iv}$ is called an *interval-valued neutrosophic subgraph structure* of interval-valued neutrosophic graph structure $\check{G}_{iv}$.

Example 5.2 Consider an interval-valued neutrosophic graph structure $\check{H}_{iv} = (I', I_1', I_2')$ of graph structure $G^* = (X, E_1, E_2)$ as illustrated in Fig. 5.2. Through direct calculations, it is shown that $\check{H}_{iv}$ is an interval-valued neutrosophic subgraph structure of interval-valued neutrosophic graph structure $\check{G}_{iv}$ shown in Fig. 5.1.

Definition 5.5 An interval-valued neutrosophic graph structure $\check{H}_{iv} = (I', I_1', I_2', \ldots, I_t')$ is called an *induced subgraph structure* of interval-valued neutrosophic graph structure $\check{G}_{iv}$ by $Q \subseteq X$ if

$$t'^-(r) = t^-(r), i'^-(r) = i^-(r), f'^-(r) = f^-(r),$$
$$t'^+(r) = t^+(r), i'^+(r) = i^+(r), f'^+(r) = f^+(r),$$

$$t_j'^-(rs) = t_j^-(rs), i_j'^-(rs) = i_j^-(rs), f_j'^-(rs) = f_j^-(rs), t_j'^+(rs) = t_j^+(rs),$$
$$i_j'^+(rs) = i_j^+(rs), f_j'^+(rs) = f_j^+(rs), \text{for all } r, s \in Q, j = 1, 2, \ldots, t.$$

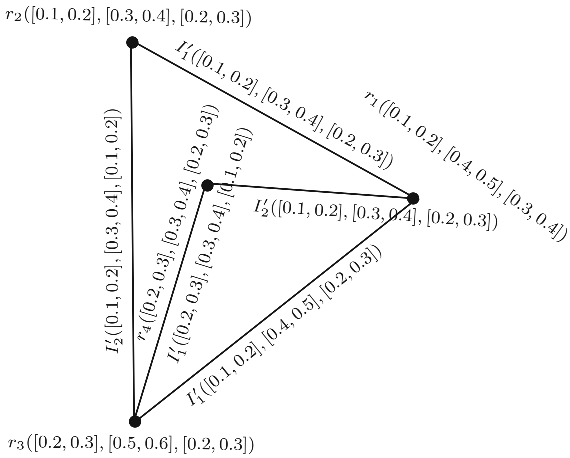


Fig. 5.2 Interval-valued neutrosophic subgraph structure

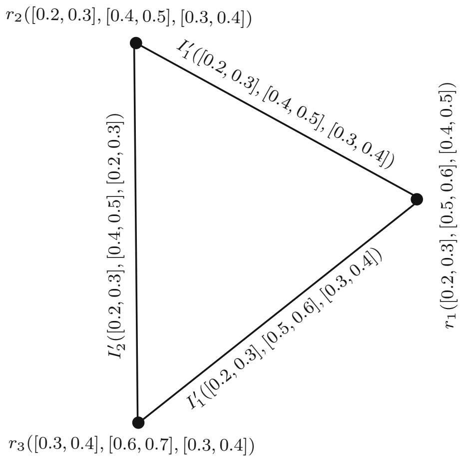


Fig. 5.3 Interval-valued neutrosophic-induced subgraph structure

Example 5.3 An interval-valued neutrosophic graph structure $\check{H}_{iv} = (I', I_1', I_2')$ of graph structure $G^* = (X, E_1, E_2)$ shown in Fig. 5.3 is an interval-valued neutrosophic-induced subgraph structure of interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2)$ represented in Fig. 5.1.

Definition 5.6 An interval-valued neutrosophic graph structure $\check{H}_{iv} = (I', I_1', I_2', \ldots, I_t')$ is called *spanning subgraph structure* of interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ if $I' = I$ and

$$t_j'^-(rs) \leq t_j^-(rs), \;\; i_j'^-(rs) \leq i_j^-(rs), \;\; f_j'^-(rs) \leq f_j^-(rs),$$
$$t_j'^+(rs) \leq t_j^+(rs), \;\; i_j'^+(rs) \leq i_j^+(rs), \;\; f_j'^+(rs) \leq f_j^+(rs), \; j = 1, 2, \ldots, t.$$

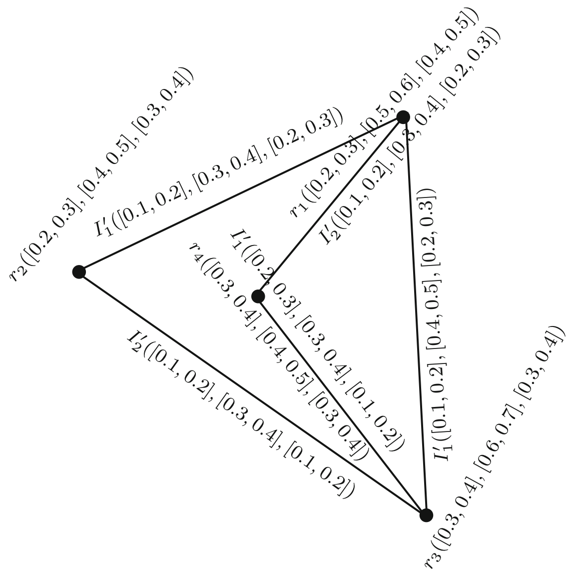


Fig. 5.4 Interval-valued neutrosophic spanning subgraph structure

Example 5.4 An interval-valued neutrosophic graph structure $\check{H}_{iv} = (I', I'_1, I'_2)$ shown in Fig. 5.4 is an interval-valued neutrosophic spanning subgraph structure of interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2)$ represented in Fig. 5.1.

Definition 5.7 Let $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ be an interval-valued neutrosophic graph structure. Then edge $rs \in I_j$ is called an *interval-valued neutrosophic I_j-edge* or in short an *I_j-edge* if

$$t_j^-(rs) > 0 \text{ or } i_j^-(rs) > 0 \text{ or } f_j^-(rs) > 0 \text{ or } t_j^+(rs) > 0 \text{ or } i_j^+(rs) > 0 \text{ or } f_j^+(rs) > 0$$

or all of conditions are satisfied. Hence support of I_j is defined as;

$$supp(I_j) = \{rs \in I_j : t_j^-(rs) > 0\} \cup \{rs \in I_j : i_j^-(rs) > 0\} \cup \{rs \in I_j : f_j^-(rs) > 0\} \cup \{rs \in I_j : t_j^+(rs) > 0\} \cup \{rs \in I_j : i_j^+(rs) > 0\} \cup \{rs \in I_j : f_j^+(rs) > 0\},\quad j = 1, 2, \ldots, t.$$

Definition 5.8 An *I_j-path* in an interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is a sequence $r_1, r_2, \ldots, r_t$ of distinct vertices (except $r_t = r_1$) in X such that $r_{j-1}r_j$ is an interval-valued neutrosophic I_j-edge for all $j = 2, 3, \ldots, t$.

Definition 5.9 An interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is *I_j-strong* for any $j \in \{1, 2, \ldots, t\}$ if

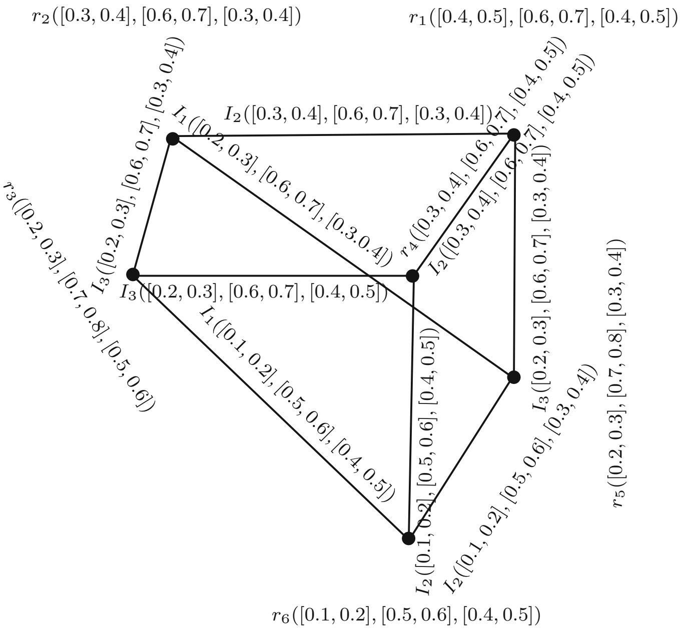


Fig. 5.5 Strong interval-valued neutrosophic graph structure

$$t_j^-(rs) = \min\{t^-(r), t^-(s)\}, \;\; i_j^-(rs) = \min\{i^-(r), i^-(s)\},$$
$$f_j^-(rs) = \min\{f^-(r), f^-(s)\}, \;\; t_j^+(rs) = \min\{t^+(r), t^+(s)\},$$
$$i_j^+(rs) = \min\{i^+(r), i^+(s)\}, \;\; f_j^+(rs) = \min\{f^+(r), f^+(s)\},$$

for all $rs \in supp(I_j)$. If $\check{G}_{iv}$ is I_j-strong for all $j \in \{1, 2, \ldots, t\}$, then $\check{G}_{iv}$ is called a *strong interval-valued neutrosophic graph structure*.

Example 5.5 Consider an interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2, I_3)$ as shown in Fig. 5.5. $\check{G}_{iv}$ is a strong interval-valued neutrosophic graph structure, since it is I_1, I_2 and I_3 strong.

Definition 5.10 An interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is called *complete* , if

1. $\check{G}_{iv}$ is a strong interval-valued neutrosophic graph structure.
2. $Supp(I_j) \neq \emptyset$, for all $j = 1, 2, \ldots, t$.
3. For all $r, s \in X$, rs is an $I_j - edge$ for some j.

Example 5.6 Let $\check{G}_{iv} = (I, I_1, I_2, I_3)$ be an interval-valued neutrosophic graph structure of graph structure $G^* = (X, E_1, E_2, E_3)$, and it is shown in Fig. 5.6, where $X = \{r_1, r_2, r_3, r_4, r_5, r_6\}$, $E_1 = \{r_1r_6, r_1r_2, r_2r_4, r_2r_5, r_2r_6, r_4r_5\}$, $E_2 = \{r_4r_3, r_5r_6, r_1r_4\}$, and $E_3 = \{r_1r_5, r_5r_3, r_2r_3, r_1r_3, r_4r_6\}$. By direct calculations, we can show that $\check{G}_{iv}$ is a strong interval-valued neutrosophic graph structure. Moreover, $supp(I_1) \neq$

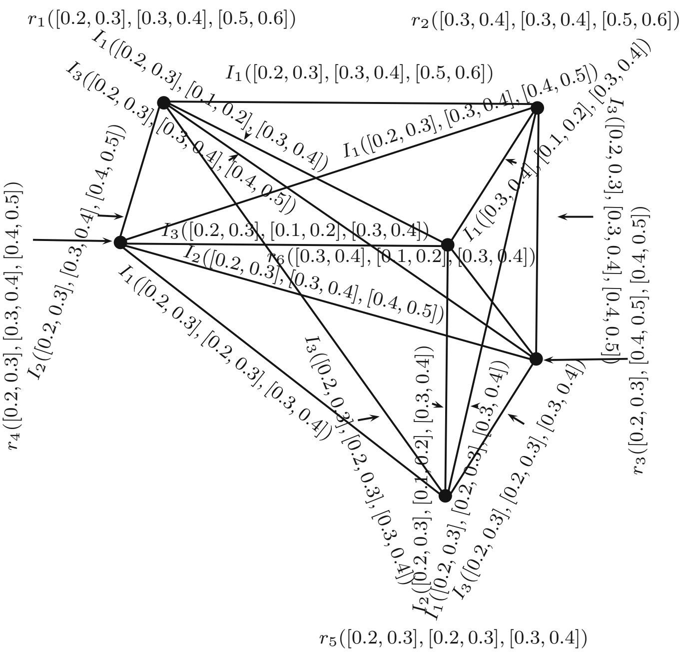


Fig. 5.6 Complete interval-valued neutrosophic graph structure

$\emptyset$, $supp(I_2) \neq \emptyset$, $supp(I_3) \neq \emptyset$, and each pair r_jr_k of nodes in X is either an I_1- edge or I_2-edge or I_3- edge. Hence $\check{G}_{iv}$ is a complete interval-valued neutrosophic graph structure, that is, $I_1I_2I_3-$complete interval-valued neutrosophic graph structure.

Definition 5.11 Let $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ be an interval-valued neutrosophic graph structure. The *truth strength* $[t^-.P_{I_j}, t^+.P_{I_j}]$, *indeterminacy strength* $[i^-.P_{I_j}, i^+.P_{I_j}]$ and *falsity strength* $[f^-.P_{I_j}, f^+.P_{I_j}]$ of an I_j-path, $P_{I_j} = r_1, r_2, \ldots, r_n$ are defined as:

$$[t^-.P_{I_j},\ t^+.P_{I_j}] = \left[\bigwedge_{k=2}^{n}[t^-_{I_j}(r_{k-1}r_k)],\ \bigwedge_{k=2}^{n}[t^+_{I_j}(r_{k-1}r_k)]\right],$$

$$[i^-.P_{I_j},\ i^+.P_{I_j}] = \left[\bigwedge_{k=2}^{n}[i^-_{I_j}(r_{k-1}r_k)],\ \bigwedge_{k=2}^{n}[i^+_{I_j}(r_{k-1}r_k)]\right],$$

$$[f^-.P_{I_j},\ f^+.P_{I_j}] = \left[\bigwedge_{k=2}^{n}[f^-_{I_j}(r_{k-1}r_k)],\ \bigwedge_{k=2}^{n}[f^+_{I_j}(r_{k-1}r_k)]\right].$$

Example 5.7 Consider an interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2)$ of graph structure $G^* = (X, E_1, E_2)$ as shown in Fig. 5.7. For I_2-path $P_{I_2} =$

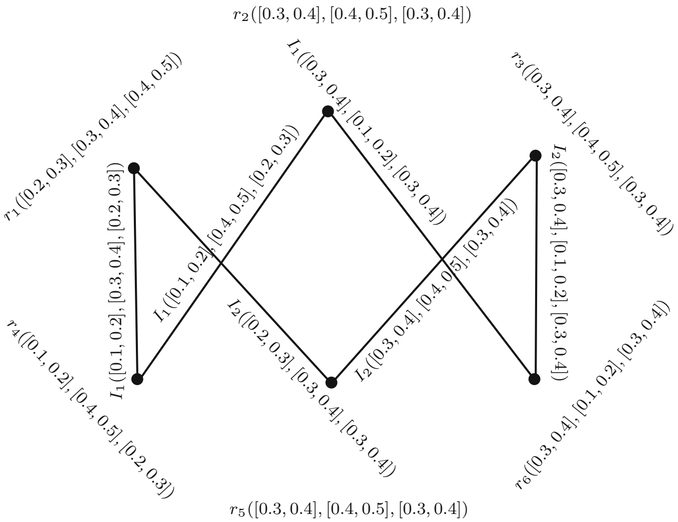


Fig. 5.7 Interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2)$

r_1, r_5, r_3, r_6, $[t^-.P_{I_2},\ t^+.P_{I_2}] = [0.2, 0.3]$, $[i^-.P_{I_2},\ i^+.P_{I_2}] = [0.1, 0.2]$ and $[f^-.P_{I_2}, f^+.P_{I_2}] = [0.3, 0.4]$.

Definition 5.12 Let $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ be an interval-valued neutrosophic graph structure. Then

- I_j—*Truth strength of connectedness* between two nodes r and s is defined by:
  $[t_{I_j}^{-\infty}(rs), t_{I_j}^{+\infty}(rs)] = [\bigvee_{i\geq 1}\{t_{I_j}^{-i}(rs)\}, \bigvee_{i\geq 1}\{t_{I_j}^{+i}(rs)\}]$ such that
  $[t_{I_j}^{-i}(rs), t_{I_j}^{+i}(rs)] = [(t_{I_j}^{-(i-1)} \circ t_{I_j}^{-(1)})(rs), (t_{I_j}^{+(i-1)} \circ t_{I_j}^{+(1)})(rs)]$ for $i \geq 2$ and
  $[t_{I_j}^{-2}(rs), t_{I_j}^{+2}(rs)] = [(t_{I_j}^{-1} \circ t_{I_j}^{-1})(rs), (t_{I_j}^{+1} \circ t_{I_j}^{+1})(rs)]$
  $= [\bigvee_{y}(t_{I_j}^{-1}(ry) \wedge t_{I_j}^{-1}(ys)), \bigvee_{y}(t_{I_j}^{+1}(ry) \wedge t_{I_j}^{+1}(ys))]$.
- I_j—*Indeterminacy strength of connectedness* between two nodes r and s is defined by:
  $[i_{I_j}^{-\infty}(rs), i_{I_j}^{+\infty}(rs)] = [\bigvee_{i\geq 1}\{i_{I_j}^{-i}(rs)\}, \bigvee_{i\geq 1}\{i_{I_j}^{+i}(rs)\}]$ such that
  $[i_{I_j}^{-i}(rs), i_{I_j}^{+i}(rs)] = [(i_{I_j}^{-(i-1)} \circ i_{I_j}^{-(1)})(rs), (i_{I_j}^{+(i-1)} \circ i_{I_j}^{+(1)})(rs)]$ for $i \geq 2$ and
  $[i_{I_j}^{-2}(rs), i_{I_j}^{+2}(rs)] = [(i_{I_j}^{-1} \circ i_{I_j}^{-1})(rs), (i_{I_j}^{+1} \circ i_{I_j}^{+1})(rs)]$
  $= [\bigvee_{y}(i_{I_j}^{-1}(ry) \wedge i_{I_j}^{-1}(ys)), \bigvee_{y}(i_{I_j}^{+1}(ry) \wedge i_{I_j}^{+1}(ys))]$.
- I_j—*Falsity strength of connectedness* between two nodes r and s is defined by:
  $[f_{I_j}^{-\infty}(rs), f_{I_j}^{+\infty}(rs)] = [\bigvee_{i\geq 1}\{f_{I_j}^{-i}(rs)\}, \bigvee_{i\geq 1}\{f_{I_j}^{+i}(rs)\}]$ such that
  $[f_{I_j}^{-i}(rs), f_{I_j}^{+i}(rs)] = [(f_{I_j}^{-(i-1)} \circ f_{I_j}^{-(1)})(rs), (f_{I_j}^{+(i-1)} \circ f_{I_j}^{+(1)})(rs)]$ for $i \geq 2$ and

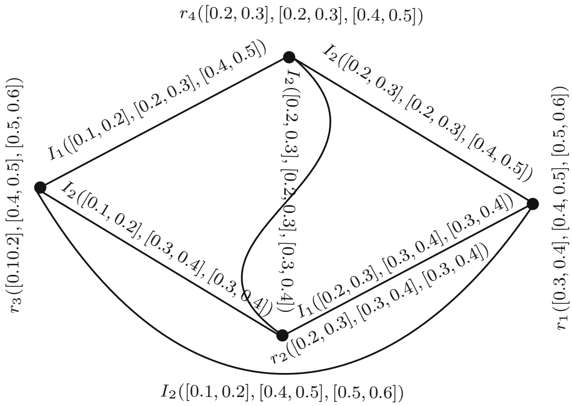


Fig. 5.8 Interval-valued neutrosophic I_2-cycle

$$[f_{I_j}^{-2}(rs), f_{I_j}^{+2}(rs)] = [(f_{I_j}^{-1} \circ f_{I_j}^{-1})(rs), (f_{I_j}^{+1} \circ f_{I_j}^{+1})(rs)]$$
$$= [\bigvee_y (f_{I_j}^{-1}(ry) \wedge f_{I_j}^{-1}(ys)), \bigvee_y (f_{I_j}^{+1}(ry) \wedge f_{I_j}^{+1}(ys))].$$

Definition 5.13 An interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is called an I_j-*cycle* if $(supp(I), supp(I_1), supp(I_2), \ldots, supp(I_t))$ is an $I_j - cycle$.

Definition 5.14 An interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is an interval-valued neutrosophic I_j-cycle (for some j) if $\check{G}_{iv}$ is an I_j-cycle and no unique I_j-edge rs exists in $\check{G}_{iv}$ such that:
$[t_{I_j}^-(rs), t_{I_j}^+(rs)] = [\min\{t_{I_j}^-(uv) : uv \in I_j = supp(I_j)\}$,
$\min\{t_{I_j}^+(uv) : uv \in I_j = supp(I_j)\}]$ or
$[i_{I_j}^-(rs), i_{I_j}^+(rs)] = [\min\{i_{I_j}^-(uv) : uv \in I_j = supp(I_j)\}$,
$\min\{i_{I_j}^+(uv) : uv \in I_j = supp(I_j)\}]$ or
$[f_{I_j}^-(rs), f_{I_j}^+(rs)] = [\min\{f_{I_j}^-(uv) : uv \in I_j = supp(I_j)\}$,
$\min\{f_{I_j}^+(uv) : uv \in I_j = supp(I_j)\}]$.

Example 5.8 Consider an interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2)$ of graph structure $G^* = (X, E_1, E_2)$ as shown in Fig. 5.8. This interval-valued neutrosophic graph structure $\check{G}_{iv}$ is an I_2-cycle, that is, $r_1 - r_4 - r_2 - r_3 - r_1$, and no unique I_2-edge rs exists in $\check{G}_{iv}$ satisfying following condition:
$[t_{I_2}^-(rs), t_{I_2}^+(rs)] = [\min\{t_{I_2}^-(uv) : uv \in I_2 = supp(I_2)\}$,
$\min\{t_{I_2}^+(uv) : uv \in I_2 = supp(I_2)\}]$ or
$[i_{I_2}^-(rs), i_{I_2}^+(rs)] = [\min\{i_{I_2}^-(uv) : uv \in I_2 = supp(I_2)\}$,
$\min\{i_{I_2}^+(uv) : uv \in I_2 = supp(I_2)\}]$ or
$[f_{I_2}^-(rs), f_{I_2}^+(rs)] = [\min\{f_{I_2}^-(uv) : uv \in I_2 = supp(I_2)\}$,
$\min\{f_{I_2}^+(uv) : uv \in I_2 = supp(I_2)\}]$.

Definition 5.15 Let $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ be an interval-valued neutrosophic graph structure and 'r' be a vertex of $\check{G}_{iv}$. If $(I', I'_1, I'_2, \ldots, I'_t)$ is an interval-valued neutrosophic subgraph structure of $\check{G}_{iv}$ induced by $U \setminus \{r\}$ such that for all $u \neq r, v \neq r$

$$t^-_{I'}(r) = i^-_{I'}(r) = f^-_{I'}(r) = t^-_{I'_j}(ru) = i^-_{I'_j}(ru) = f^-_{I'_j}(ru) = 0,$$
$$t^+_{I'}(r) = i^+_{I'}(r) = f^+_{I'}(r) = t^+_{I'_j}(ru) = i^+_{I'_j}(ru) = f^+_{I'_j}(ru) = 0,$$
$$[t^-_{I'}(u), t^+_{I'}(u)] = [t^-_I(u), t^+_I(u)], \quad [i^-_{I'}(u), i^+_{I'}(u)] = [i^-_I(u), i^+_I(u)],$$
$$[f^-_{I'}(u), f^+_{I'}(u)] = [f^-_I(u), f^+_I(u)],$$
$$[t^-_{I'_j}(uv), t^+_{I'_j}(uv)] = [t^-_{I_j}(uv), t^+_{I_j}(uv)], \quad [i^-_{I'_j}(uv), i^+_{I'_j}(uv)] = [i^-_{I_j}(uv), i^+_{I_j}(uv)],$$
$$[f^-_{I'_j}(uv), f^+_{I'_j}(uv)] = [f^-_{I_j}(uv), f^+_{I_j}(uv)].$$

for all edges $ru, uv \in \check{G}_{iv}$, then vertex r is an *interval-valued neutrosophic I_j cut-vertex*, if

1. $t^{-\infty}_{I_j}(uv) > t^{-\infty}_{I'_j}(uv), t^{+\infty}_{I_j}(uv) > t^{+\infty}_{I'_j}(uv), [t^{-\infty}_{I_j}(uv), t^{+\infty}_{I_j}(uv)] \cap [t^{-\infty}_{I'_j}(uv), t^{+\infty}_{I'_j}(uv)] = \emptyset$
2. $i^{-\infty}_{I_j}(uv) > i^{-\infty}_{I'_j}(uv), i^{+\infty}_{I_j}(uv) > i^{+\infty}_{I'_j}(uv), [i^{-\infty}_{I_j}(uv), i^{+\infty}_{I_j}(uv)] \cap [i^{-\infty}_{I'_j}(uv), i^{+\infty}_{I'_j}(uv)] = \emptyset$
3. $f^{-\infty}_{I_j}(uv) > f^{-\infty}_{I'_j}(uv), f^{+\infty}_{I_j}(uv) > f^{+\infty}_{I'_j}(uv), [f^{-\infty}_{I_j}(uv), f^{+\infty}_{I_j}(uv)] \cap [f^{-\infty}_{I'_j}(uv), f^{+\infty}_{I'_j}(uv)] = \emptyset$

for some $u, v \in X \setminus \{r\}$. Note that vertex r is an

- *interval-valued neutrosophic $I_j - t$ cut-vertex*, if $t^{-\infty}_{I_j}(uv) > t^{-\infty}_{I'_j}(uv)$, $t^{+\infty}_{I_j}(uv) > t^{+\infty}_{I'_j}(uv), [t^{-\infty}_{I_j}(uv), t^{+\infty}_{I_j}(uv)] \cap [t^{-\infty}_{I'_j}(uv), t^{+\infty}_{I'_j}(uv)] = \emptyset$
- *interval-valued neutrosophic $I_j - i$ cut-vertex*, if $i^{-\infty}_{I_j}(uv) > i^{-\infty}_{I'_j}(uv)$, $i^{+\infty}_{I_j}(uv) > i^{+\infty}_{I'_j}(uv), [i^{-\infty}_{I_j}(uv), i^{+\infty}_{I_j}(uv)] \cap [i^{-\infty}_{I'_j}(uv), i^{+\infty}_{I'_j}(uv)] = \emptyset$
- *interval-valued neutrosophic $I_j - f$ cut-vertex*, if $f^{-\infty}_{I_j}(uv) > f^{-\infty}_{I'_j}(uv)$, $f^{+\infty}_{I_j}(uv) > f^{+\infty}_{I'_j}(uv), [f^{-\infty}_{I_j}(uv), f^{+\infty}_{I_j}(uv)] \cap [f^{-\infty}_{I'_j}(uv), f^{+\infty}_{I'_j}(uv)] = \emptyset$

Example 5.9 Consider an interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2)$ of graph structure $G^* = (X, E_1, E_2)$ as represented in Fig. 5.9. $\check{H}_{iv} = (I', I'_1, I'_2)$ is an interval-valued neutrosophic subgraph structure of interval-valued neutrosophic graph structure $\check{G}_{iv}$, which is obtained by deleting vertex r_2 and shown in Fig. 5.10.

The vertex r_2 is an interval-valued neutrosophic $I_1 - i$ cut-vertex. Since $i^{-\infty}_{I'_1}(r_4r_5) = 0.3$, $i^{-\infty}_{I_1}(r_4r_5) = 0.5$, $i^{+\infty}_{I'_1}(r_4r_5) = 0.4$, $i^{+\infty}_{I_1}(r_4r_5) = 0.6$. Clearly $i^{-\infty}_{I_1}(r_4r_5) = 0.5 > 0.3 = i^{-\infty}_{I'_1}(r_4r_5)$, $i^{+\infty}_{I_1}(r_4r_5) = 0.6 > 0.4 = i^{+\infty}_{I'_1}(r_4r_5)$, $[i^{-\infty}_{I_1}(r_4r_5), i^{+\infty}_{I_1}(r_4r_5)] \cap [i^{-\infty}_{I'_1}(r_4r_5), i^{+\infty}_{I'_1}(r_4r_5)] = [0.5, 0.6] \cap [0.3, 0.4] = \emptyset$.

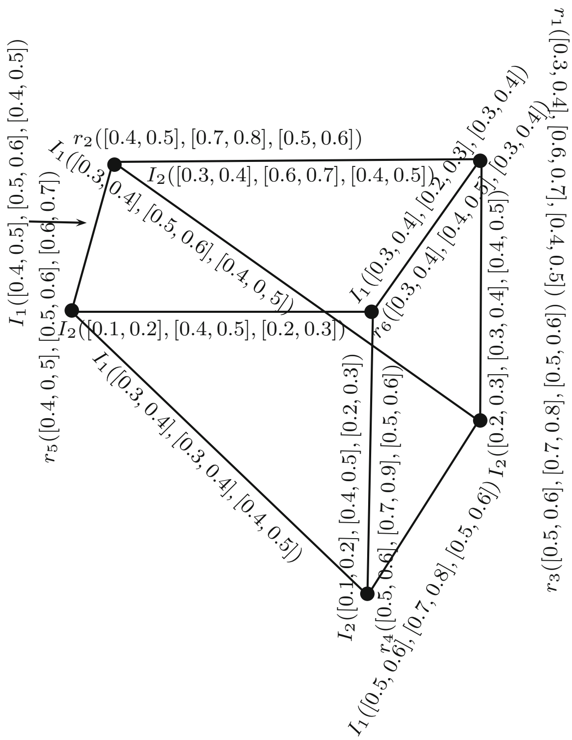


Fig. 5.9 Interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2)$

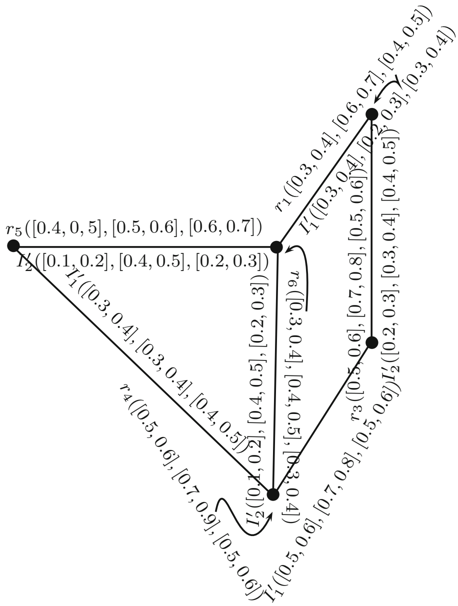


Fig. 5.10 Interval-valued neutrosophic graph structure $\check{H}_{iv} = (I', I'_1, I'_2)$

Definition 5.16 Let $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ be an interval-valued neutrosophic graph structure and rs be an $I_j - edge$. If $(I', I_1', I_2', \ldots, I_t')$ is an interval-valued neutrosophic spanning subgraph structure of $\check{G}_{iv}$, such that

$$t^-_{I'_j}(rs) = i^-_{I'_j}(rs) = f^-_{I'_j}(rs) = 0, \quad t^+_{I'_j}(rs) = i^+_{I'_j}(rs) = f^+_{I'_j}(rs) = 0,$$

$$[t^-_{I'_j}(wx), t^+_{I'_j}(wx)] = [t^-_{I_j}(wx), t^+_{I_j}(wx)], \ [i^-_{I'_j}(wx), i^+_{I'_j}(wx)] = [i^-_{I_j}(wx), i^+_{I_j}(wx)],$$

$$[f^-_{I'_j}(wx), f^+_{I'_j}(wx)] = [f^-_{I_j}(wx), f^+_{I_j}(wx)],$$

for all edges $wx \neq rs$, then edge rs is an interval-valued neutrosophic I_j-bridge if

1. $t^{-\infty}_{I_j}(uv) > t^{-\infty}_{I'_j}(uv), t^{+\infty}_{I_j}(uv) > t^{+\infty}_{I'_j}(uv), [t^{-\infty}_{I_j}(uv), t^{+\infty}_{I_j}(uv)] \cap [t^{-\infty}_{I'_j}(uv), t^{+\infty}_{I'_j}(uv)] = \emptyset$
2. $i^{-\infty}_{I_j}(uv) > i^{-\infty}_{I'_j}(uv), i^{+\infty}_{I_j}(uv) > i^{+\infty}_{I'_j}(uv), [i^{-\infty}_{I_j}(uv), i^{+\infty}_{I_j}(uv)] \cap [i^{-\infty}_{I'_j}(uv), i^{+\infty}_{I'_j}(uv)] = \emptyset$
3. $f^{-\infty}_{I_j}(uv) > f^{-\infty}_{I'_j}(uv), f^{+\infty}_{I_j}(uv) > f^{+\infty}_{I'_j}(uv), [f^{-\infty}_{I_j}(uv), f^{+\infty}_{I_j}(uv)] \cap [f^{-\infty}_{I'_j}(uv), f^{+\infty}_{I'_j}(uv)] = \emptyset$

for some $u, v \in X$. Note that edge rs is an

- *interval-valued neutrosophic $I_j - t$ bridge*, if $t^{-\infty}_{I_j}(uv) > t^{-\infty}_{I'_j}(uv), t^{+\infty}_{I_j}(uv) > t^{+\infty}_{I'_j}(uv), [t^{-\infty}_{I_j}(uv), t^{+\infty}_{I_j}(uv)] \cap [t^{-\infty}_{I'_j}(uv), t^{+\infty}_{I'_j}(uv)] = \emptyset$
- *interval-valued neutrosophic $I_j - i$ bridge*, if $i^{-\infty}_{I_j}(uv) > i^{-\infty}_{I'_j}(uv), i^{+\infty}_{I_j}(uv) > i^{+\infty}_{I'_j}(uv), [i^{-\infty}_{I_j}(uv), i^{+\infty}_{I_j}(uv)] \cap [i^{-\infty}_{I'_j}(uv), i^{+\infty}_{I'_j}(uv)] = \emptyset$
- *interval-valued neutrosophic $I_j - f$ bridge*, if $f^{-\infty}_{I_j}(uv) > f^{-\infty}_{I'_j}(uv), f^{+\infty}_{I_j}(uv) > f^{+\infty}_{I'_j}(uv), [f^{-\infty}_{I_j}(uv), f^{+\infty}_{I_j}(uv)] \cap [f^{-\infty}_{I'_j}(uv), f^{+\infty}_{I'_j}(uv)] = \emptyset$

Example 5.10 Consider an interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2)$ of graph structure $G^* = (X, E_1, E_2)$ as shown in Fig. 5.11. $\check{H}_{iv} = (I', I_1', I_2')$ is an interval-valued neutrosophic spanning subgraph structure of interval-valued neutrosophic graph structure $\check{G}_{iv}$ obtained by deleting an I_1-edge r_2r_5 and shown in Fig. 5.12. The edge r_2r_5 is an interval-valued neutrosophic $I_1 - bridge$ since

- $t^{-\infty}_{I'_1}(r_2r_5) = 0.2, t^{-\infty}_{I_1}(r_2r_5) = 0.7, t^{+\infty}_{I'_1}(r_2r_5) = 0.3, t^{+\infty}_{I_1}(r_2r_5) = 0.8. t^{-\infty}_{I_1}(r_2r_5) = 0.7 > 0.2 = t^{-\infty}_{I'_1}(r_2r_5), \ t^{+\infty}_{I_1}(r_2r_5) = 0.8 > 0.3 = t^{+\infty}_{I'_1}(r_2r_5), \ [t^{-\infty}_{I_1}(r_2r_5), t^{+\infty}_{I_1}(r_2r_5)] \cap [t^{-\infty}_{I'_1}(r_2r_5), t^{+\infty}_{I'_1}(r_2r_5)] = [0.7, 0.8] \cap [0.2, 0.3] = \emptyset$.

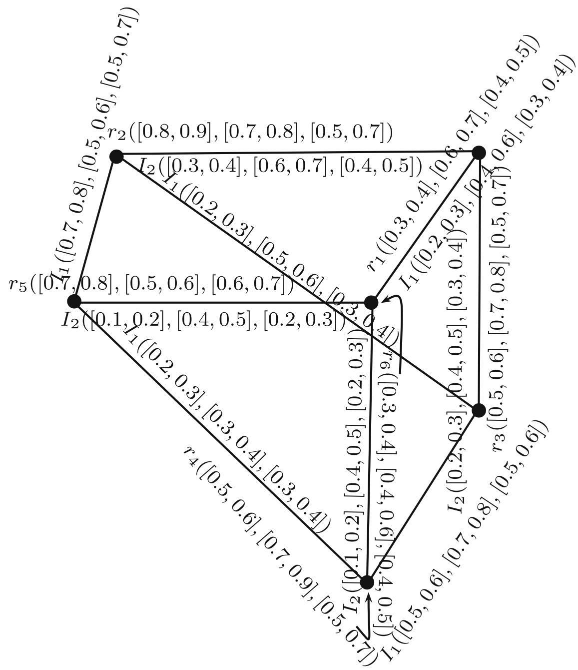


Fig. 5.11 Interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2)$

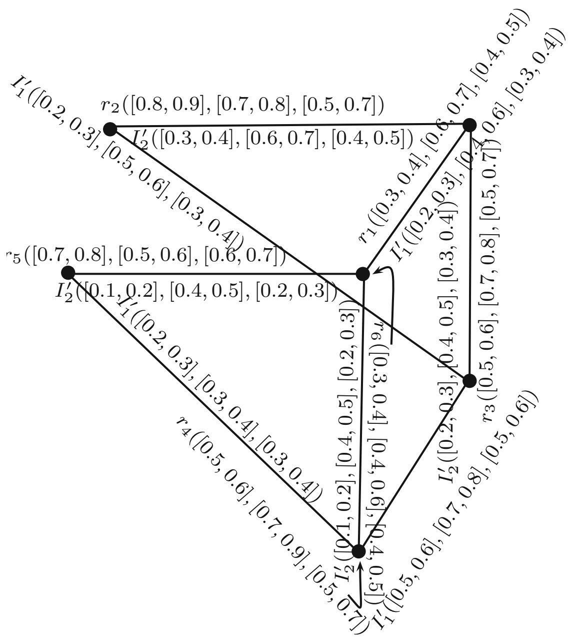


Fig. 5.12 Interval-valued neutrosophic graph structure $\check{H}_{iv} = (I', I'_1, I'_2)$

- $i^{-\infty}_{I'_1}(r_2r_5) = 0.3$, $i^{-\infty}_{I_1}(r_2r_5) = 0.5$, $i^{+\infty}_{I'_1}(r_2r_5) = 0.4$, $i^{+\infty}_{I_1}(r_2r_5) = 0.6$. $i^{-\infty}_{I_1}(r_2r_5) = 0.5 > 0.3 = i^{-\infty}_{I'_1}(r_2r_5)$, $i^{+\infty}_{I_1}(r_2r_5) = 0.6 > 0.4 = i^{+\infty}_{I'_1}(r_2r_5)$, $[i^{-\infty}_{I_1}(r_2r_5), i^{+\infty}_{I_1}(r_2r_5)] \cap [i^{-\infty}_{I'_1}(r_2r_5), i^{+\infty}_{I'_1}(r_2r_5)] = [0.5, 0.6] \cap [0.3, 0.4] = \emptyset$.
- $f^{-\infty}_{I'_1}(r_2r_5) = 0.3$, $f^{-\infty}_{I_1}(r_2r_5) = 0.5$, $f^{+\infty}_{I'_1}(r_2r_5) = 0.4$, $f^{+\infty}_{I_1}(r_2r_5) = 0.7$. $f^{-\infty}_{I_1}(r_2r_5) = 0.5 > 0.3 = f^{-\infty}_{I'_1}(r_2r_5)$, $f^{+\infty}_{I_1}(r_2r_5) = 0.7 > 0.4 = f^{+\infty}_{I'_1}(r_2r_5)$, $[f^{-\infty}_{I_1}(r_2r_5), f^{+\infty}_{I_1}(r_2r_5)] \cap [f^{-\infty}_{I'_1}(r_2r_5), f^{+\infty}_{I'_1}(r_2r_5)] =$ $[0.5, 0.7] \cap [0.3, 0.4] = \emptyset$.

Definition 5.17 An interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is an I_j*-tree* if $(supp(I), supp(I_1), supp(I_2), \ldots, supp(I_t))$ is an $I_j -$ *tree*. Alternatively, $\check{G}_{iv}$ is an I_j-tree, if $\check{G}_{iv}$ has a subgraph induced by $supp(I_j)$ that forms a tree.

Definition 5.18 An interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is an *interval-valued neutrosophic* I_j*-tree* if $\check{G}_{iv}$ has an interval-valued neutrosophic spanning subgraph structure $\check{H}_{iv} = (I'', I''_1, I''_2, \ldots, I''_t)$ such that for all I_j-edges rs not in $\check{H}_{iv}$,
$\check{H}_{iv}$ is an I''_j-tree and

1. $t^-_{I_j}(rs) < t^{-\infty}_{I''_j}(rs)$, $t^+_{I_j}(rs) < t^{+\infty}_{I''_j}(rs)$, $[t^-_{I_j}(rs), t^+_{I_j}(rs)] \cap [t^{-\infty}_{I''_j}(rs), t^{+\infty}_{I''_j}(rs)] = \emptyset$
2. $i^-_{I_j}(rs) < i^{-\infty}_{I''_j}(rs)$, $i^+_{I_j}(rs) < i^{+\infty}_{I''_j}(rs)$, $[i^-_{I_j}(rs), i^+_{I_j}(rs)] \cap [i^{-\infty}_{I''_j}(rs), i^{+\infty}_{I''_j}(rs)] = \emptyset$
3. $f^-_{I_j}(rs) < f^{-\infty}_{I''_j}(rs)$, $f^+_{I_j}(rs) < f^{+\infty}_{I''_j}(rs)$, $[f^-_{I_j}(rs), f^+_{I_j}(rs)] \cap [f^{-\infty}_{I''_j}(rs), f^{+\infty}_{I''_j}(rs)] = \emptyset$

In particular,

- $\check{G}_{iv}$ is an *interval-valued neutrosophic* $I_j - t$ *tree* if $t^-_{I_j}(rs) < t^{-\infty}_{I''_j}(rs)$, $t^+_{I_j}(rs) < t^{+\infty}_{I''_j}(rs)$, $[t^-_{I_j}(rs), t^+_{I_j}(rs)] \cap [t^{-\infty}_{I''_j}(rs), t^{+\infty}_{I''_j}(rs)] = \emptyset$
- $\check{G}_{iv}$ is an *interval-valued neutrosophic* $I_j - i$ *tree* if $i^-_{I_j}(rs) < i^{-\infty}_{I''_j}(rs)$, $i^+_{I_j}(rs) < i^{+\infty}_{I''_j}(rs)$, $[i^-_{I_j}(rs), i^+_{I_j}(rs)] \cap [i^{-\infty}_{I''_j}(rs), i^{+\infty}_{I''_j}(rs)] = \emptyset$
- $\check{G}_{iv}$ is an *interval-valued neutrosophic* $I_j - f$ *tree* if $f^-_{I_j}(rs) < f^{-\infty}_{I''_j}(rs)$, $f^+_{I_j}(rs) < f^{+\infty}_{I''_j}(rs)$, $[f^-_{I_j}(rs), f^+_{I_j}(rs)] \cap [f^{-\infty}_{I''_j}(uv), f^{+\infty}_{I''_j}(uv)] = \emptyset$

Example 5.11 Consider an interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2)$ of graph structure $G^* = (X, E_1, E_2)$ as shown in Fig. 5.13. This interval-valued neutrosophic graph structure is I_2-tree, not I_1-tree. But it is interval-valued neutrosophic $I_1 - t$ tree, since it has an interval-valued neutrosophic spanning subgraph structure $\check{H}_{iv} = (I'', I''_1, I''_2)$ as an I''_1-tree, which is obtained by deleting I_1-edge r_2r_5 from $\check{G}_{iv}$ and shown in Fig. 5.14. By direct calculations, we found that

Fig. 5.13 $\check{G}_{iv} = (I, I_1, I_2)$

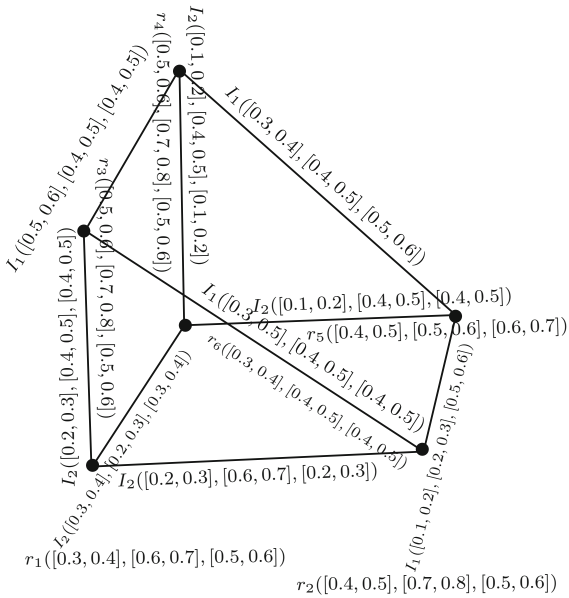
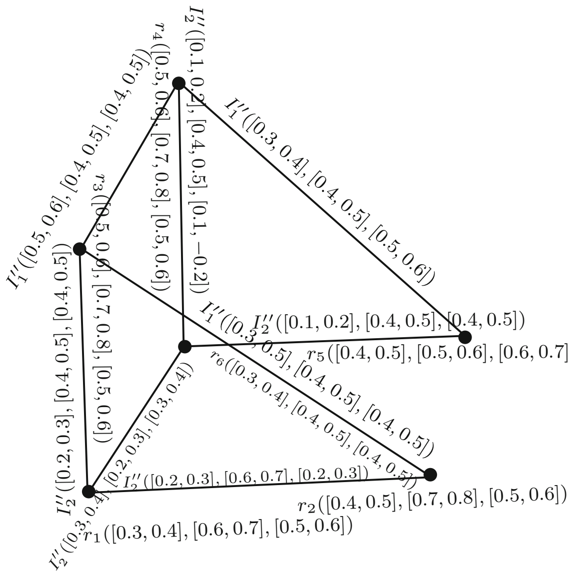


Fig. 5.14 $\check{H}_{iv} = (I'', I_1'', I_2'')$



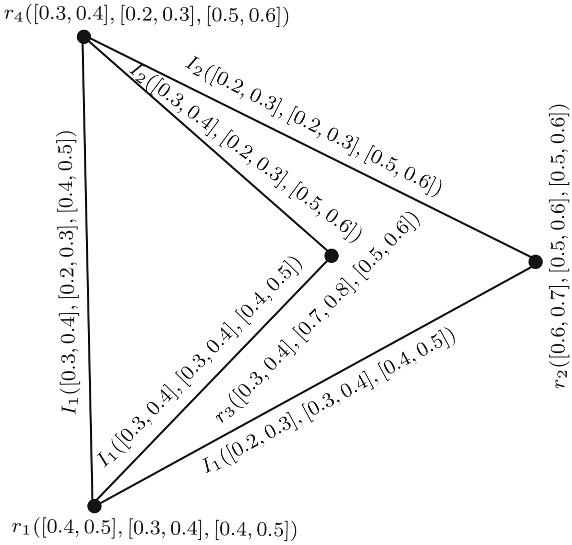


Fig. 5.15 Interval-valued neutrosophic graph structure $\check{G}_{iv1} = (I, I_1, I_2)$

$t^{-\infty}_{I''_1}(r_2r_5) = 0.3,\ t^{+\infty}_{I''_1}(r_2r_5) = 0.5,\ t^{-}_{I_1}(r_2r_5) = 0.1,\ t^{+}_{I_1}(r_2r_5) = 0.2,$
$t^{-}_{I_1}(r_2r_5) = 0.1 < 0.3 = t^{-\infty}_{I''_1}(r_2r_5),\ t^{+}_{I_1}(r_2r_5) = 0.2 < 0.5 = t^{+\infty}_{I''_1}(r_2r_5),$
$[t^{-\infty}_{I''_1}(r_2r_5), t^{+\infty}_{I''_1}(r_2r_5)] \cap [t^{-}_{I_1}(r_2r_5), t^{+}_{I_1}(r_2r_5)] = [0.3, 0.5] \cap [0.1, 0.2] = \emptyset.$

Definition 5.19 An interval-valued neutrosophic graph structure $\check{G}_{iv1} = (I_1, I_{11}, I_{12}, \ldots, I_{1t})$ of graph structure $G^*_1 = (X_1, E_{11}, E_{12}, \ldots, E_{1t})$ is isomorphic to interval-valued neutrosophic graph structure $\check{G}_{iv2} = (I_2, I_{21}, I_{22}, \ldots, I_{2t})$ of graph structure $G^*_2 = (X_2, E_{21}, E_{22}, \ldots, E_{2t})$, if there is a pair (f, φ), where $f : U_1 \to U_2$ is bijection and φ is a permutation on set $\{1, 2, \ldots, t\}$ such that:

$$[t^{-}_{I_1}(r), t^{+}_{I_1}(r)] = [t^{-}_{I_2}(f(r)), t^{+}_{I_2}(f(r))],\ \ [i^{-}_{I_1}(r), i^{+}_{I_1}(r)] = [i^{-}_{I_2}(f(r)), i^{+}_{I_2}(f(r))],$$
$$[f^{-}_{I_1}(r), f^{+}_{I_1}(r)] = [f^{-}_{I_2}(f(r)), f^{+}_{I_2}(f(r))],$$

$$[t^{-}_{I_{1j}}(rs), t^{+}_{I_{1j}}(rs)] = [t^{-}_{I_{2\varphi(j)}}(f(r)f(s)), t^{+}_{I_{2\varphi(j)}}(f(r)f(s))],$$
$$[i^{-}_{I_{1j}}(rs), i^{+}_{I_{1j}}(rs)] = [i^{-}_{I_{2\varphi(j)}}(f(r)f(s)), i^{+}_{I_{2\varphi(j)}}(f(r)f(s))],$$
$$[f^{-}_{I_{1j}}(rs), f^{+}_{I_{1j}}(rs)] = [f^{-}_{I_{2\varphi(j)}}(f(r)f(s)), f^{+}_{I_{2\varphi(j)}}(f(r)f(s))],$$

for all $r \in X_1$, $rs \in I_{1j}$, $j \in \{1, 2, \ldots, t\}$.

Example 5.12 Let $\check{G}_{iv1} = (I, I_1, I_2)$ and $\check{G}_{iv2} = (I', I'_1, I'_2)$ be two interval-valued neutrosophic graph structures of two GSs $G_1 = (X, E_1, E_2)$ and $G_2 = (X', E'_1, E'_2)$ as shown in Figs. 5.15 and 5.16, respectively.

$\check{G}_{iv1}$ and $\check{G}_{iv2}$ are isomorphic under (f, φ), where $f : U \to U'$ is bijection and φ is permutation on set $\{1, 2\}$ defined as $\varphi(1) = 2$, $\varphi(2) = 1$, such that:

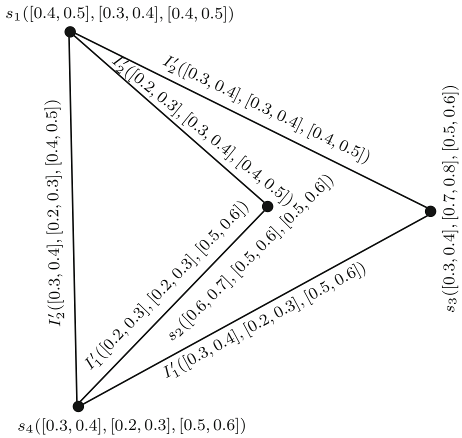


Fig. 5.16 Interval-valued neutrosophic graph structure $\check{G}_{iv2} = (I', I'_1, I'_2)$

$$[t_I^-(r_i), t_I^+(r_i)] = [t_{I'}^-(f(r_i)), t_{I'}^+(f(r_i))],$$
$$[i_I^-(r_i), i_I^+(r_i)] = [i_{I'}^-(f(r_i)), i_{I'}^+(f(r_i))],$$
$$[f_I^-(r_i), f_I^+(r_i)] = [f_{I'}^-(f(r_i)), f_{I'}^+(f(r_i))],$$

$$[t_{I_j}^-(r_ir_k), t_{I_j}^+(r_ir_k)] = [t_{I_{\varphi(j)}}^-(f(r_i)f(r_k)), t_{I_{\varphi(j)}}^+(f(r_i)f(r_k))],$$
$$[i_{I_j}^-(r_ir_k), i_{I_j}^+(r_ir_k)] = [i_{I_{\varphi(j)}}^-(f(r_i)f(r_k)), i_{I_{\varphi(j)}}^+(f(r_i)f(r_k))],$$
$$[f_{I_j}^-(r_ir_k), f_{I_j}^+(r_ir_k)] = [f_{I_{\varphi(j)}}^-(f(r_i)f(r_k)), f_{I_{\varphi(j)}}^+(f(r_i)f(r_k))],$$

for all $r_i \in X$, $r_ir_k \in I_j$, $j \in \{1, 2\}$ and $i, k \in \{1, 2, 3, 4\}$.

Definition 5.20 An interval-valued neutrosophic graph structure $\check{G}_{iv1} = (I_1, I_{11}, I_{12}, \ldots, I_{1t})$ of graph structure $G_1^* = (X_1, E_{11}, E_{12}, \ldots, E_{1t})$ is identical to interval-valued neutrosophic graph structure $\check{G}_{iv2} = (I_2, I_{21}, I_{22}, \ldots, I_{2t})$ of graph structure $G_2^* = (X_2, E_{21}, E_{22}, \ldots, E_{2t})$ if $f : U_1 \to U_2$ is a bijection, such that

$$[t_{I_1}^-(r), t_{I_1}^+(r)] = [t_{I_2}^-(f(r)), t_{I_2}^+(f(r))], [i_{I_1}^-(r), i_{I_1}^+(r)] = [i_{I_2}^-(f(r)), i_{I_2}^+(f(r))],$$
$$[f_{I_1}^-(r), f_{I_1}^+(r)] = [f_{I_2}^-(f(r)), f_{I_2}^+(f(r))],$$

$$[t_{I_{1j}}^-(rs), t_{I_{1j}}^+(rs)] = [t_{I_{2j}}^-(f(r)f(s)), t_{I_{2j}}^+(f(r)f(s))],$$
$$[i_{I_{1j}}^-(rs), i_{I_{1j}}^+(rs)] = [i_{I_{2j}}^-(f(r)f(s)), i_{I_{2j}}^+(f(r)f(s))],$$
$$[f_{I_{1j}}^-(rs), f_{I_{1j}}^+(rs)] = [f_{I_{2j}}^-(f(r)f(s)), f_{I_{2j}}^+(f(r)f(s))],$$

for all $r \in X_1$, $rs \in X_{1j}$, $j \in \{1, 2, \ldots, t\}$.

Example 5.13 Let $\check{G}_{iv1} = (I, I_1, I_2)$ and $\check{G}_{iv2} = (I', I'_1, I'_2)$ be two interval-valued neutrosophic graph structures of the graph structures $G_1^* = (X, E_1, E_2)$ and $G_2^* = (X', E'_1, E'_2)$, respectively, as shown in Figs. 5.17 and 5.18, respectively.

Interval-valued neutrosophic graph structure $\check{G}_{iv1}$ is identical to $\check{G}_{iv2}$ under $f : X \to X'$ defined as :

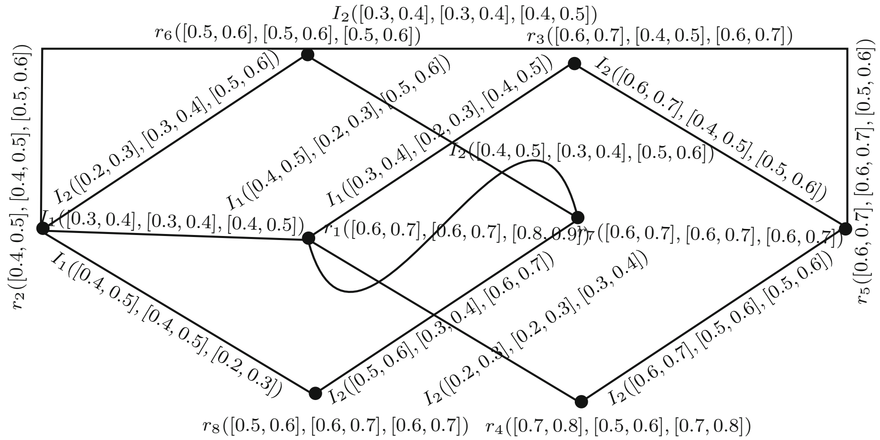


Fig. 5.17 Interval-valued neutrosophic graph structure $\check{G}_{iv1}$

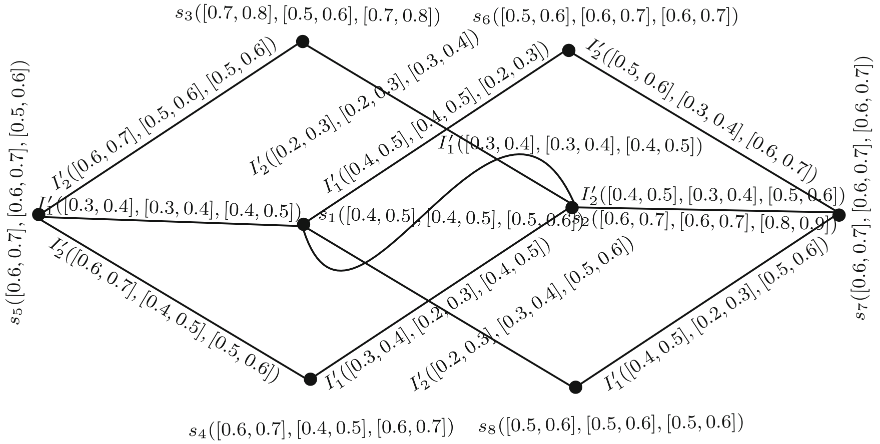


Fig. 5.18 Interval-valued neutrosophic graph structure $\check{G}_{iv2}$

$f(r_1) = s_2$, $f(r_2) = s_1$, $f(r_3) = s_4$, $f(r_4) = s_3$, $f(r_5) = s_5$, $f(r_6) = s_8$, $f(r_7) = s_7$, $f(r_8) = s_6$. Moreover,

$$[t_I^-(r_i), t_I^+(r_i)] = [t_{I'}^-(f(r_i)), t_{I'}^+(f(r_i))],$$
$$[i_I^-(r_i), i_I^+(r_i)] = [i_{I'}^-(f(r_i)), i_{I'}^+(f(r_i))],$$
$$[f_I^-(r_i), f_I^+(r_i)] = [f_{I'}^-(f(r_i)), f_{I'}^+(f(r_i))],$$

$$[t^-_{I_j}(r_ir_k), t^+_{I_j}(r_ir_k)] = [t^-_{I'_j}(f(r_i)f(r_k)), t^+_{I'_j}(f(r_i)f(r_k))],$$
$$[i^-_{I_j}(r_ir_k), i^+_{I_j}(r_ir_k)] = [i^-_{I'_j}(f(r_i)f(r_k)), i^+_{I'_j}(f(r_i)f(r_k))],$$
$$[f^-_{I_j}(r_ir_k), f^+_{I_j}(r_ir_k)] = [f^-_{I'_j}(f(r_i)f(r_k)), f^+_{I'_j}(f(r_i)f(r_k))],$$

for all $r_i \in X$, $r_ir_k \in E_j$, $j \in \{1, 2\}$, $i, k \in \{1, 2, \ldots, 8\}$.

5.3 φ-Complement of Interval-Valued Neutrosophic Graph Structure

Definition 5.21 Let $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ be an interval-valued neutrosophic graph structure and φ be a permutation on $\{I_1, I_2, \ldots, I_t\}$ on the set $\{1, 2, \ldots, t\}$, that is, $\varphi(I_j) = I_l$ if and only if $\varphi(j) = l$ for all j. If $rs \in I_j$ and

$$[t^-_{I^\varphi_j}(rs), t^+_{I^\varphi_j}(rs)] = [t^-_I(r) \wedge t^-_I(s) - \bigvee_{l \neq j} t^-_{\varphi(I_l)}(rs),$$
$$t^+_I(r) \wedge t^+_I(s) - \bigvee_{l \neq j} t^+_{\varphi(I_l)}(rs)], [i^-_{I^\varphi_j}(rs), i^+_{I^\varphi_j}(rs)] =$$
$$[i^-_I(r) \wedge i^-_I(s) - \bigvee_{l \neq j} i^-_{\varphi(I_l)}(rs), i^+_I(r) \wedge i^+_I(s) - \bigvee_{l \neq j} i^+_{\varphi(I_l)}(rs)],$$
$$[f^-_{I^\varphi_j}(rs), f^+_{I^\varphi_j}(rs)] =$$
$$[f^-_I(r) \wedge f^-_I(s) - \bigvee_{l \neq j} f^-_{\varphi(I_l)}(rs), f^+_I(r) \wedge f^+_I(s) - \bigvee_{l \neq j} f^+_{\varphi(I_l)}(rs)],$$

$j = 1, 2, \ldots, t$, then $rs \in I^\varphi_u$, where u is selected, such that

- $t^-_{I^\varphi_u}(rs) \geq t^-_{I^\varphi_j}(rs)$, $t^+_{I^\varphi_u}(rs) \geq t^+_{I^\varphi_j}(rs)$, $[t^-_{I^\varphi_u}(rs), t^+_{I^\varphi_u}(rs)] \cap [t^-_{I^\varphi_j}(rs), t^+_{I^\varphi_j}(rs)] = \emptyset$
- $i^-_{I^\varphi_u}(rs) \geq i^-_{I^\varphi_j}(rs)$, $i^+_{I^\varphi_u}(rs) \geq i^+_{I^\varphi_j}(rs)$, $[i^-_{I^\varphi_u}(rs), i^+_{I^\varphi_u}(rs)] \cap [i^-_{I^\varphi_j}(rs), i^+_{I^\varphi_j}(rs)] = \emptyset$
- $f^-_{I^\varphi_u}(rs) \geq f^-_{I^\varphi_j}(rs)$, $f^+_{I^\varphi_u}(rs) \geq f^+_{I^\varphi_j}(rs)$, $[f^-_{I^\varphi_u}(rs), f^+_{I^\varphi_u}(rs)] \cap [f^-_{I^\varphi_j}(rs), f^+_{I^\varphi_j}(rs)] = \emptyset$

for all j. Then interval-valued neutrosophic graph structure $(I, I^\varphi_1, I^\varphi_2, \ldots, I^\varphi_t)$ is said to be φ-complement of interval-valued neutrosophic graph structure $\check{G}_{iv}$ and denoted by $\check{G}^{\varphi c}_{iv}$.

Example 5.14 Let $I = \{(r_1, [0.4, 0.5], [0.4, 0.5], [0.7, 0.8]), (r_2, [0.6, 0.7], [0.6, 0.7], [0.4, 0.5]), (r_3, [0.8, 0.9], [0.5, 0.6], [0.3, 0.4])\}$, $I_1 = \{(r_1r_3, [0.4, 0.5], [0.4, 0.5], [0.3, 0.4])\}$, $I_2 = \{(r_2r_3, [0.6, 0.7], [0.4, 0.5], [0.3, 0.4])\}$, $I_3 = \{(r_1r_2, [0.4, 0.5], [0.3, 0.4], [0.4, 0.5])\}$ be interval-valued neutrosophic subsets of $U = \{r_1, r_2, r_3\}$, $E_1 = \{r_1r_3\}$, $E_2 = \{r_2r_3\}$, $E_3 = \{r_1r_2\}$, respectively. Obviously, $\check{G}_{iv} = (I, I_1, I_2, I_3)$ is an interval-valued neutrosophic graph structure of GS G^* = (X, E_1, E_2, E_3) as shown in Fig. 5.19.

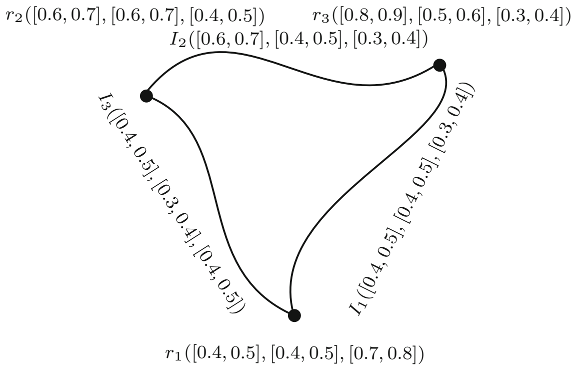


Fig. 5.19 $\check{G}_{iv} = (I, I_1, I_2, I_3)$

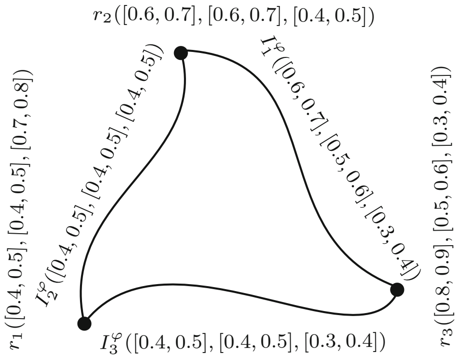


Fig. 5.20 $\check{G}_{iv} = (I, I_1^{\varphi}, I_2^{\varphi}, I_3^{\varphi})$

Simple calculations of edges $r_1r_3, r_2r_3, r_1r_2 \in I_1, I_2, I_3$, respectively, show that $r_1r_3 \in I_3^{\varphi}, r_2r_3 \in I_1^{\varphi}, r_1r_2 \in I_2^{\varphi}$. So, $\check{G}_{iv}^{\varphi c} = (I, I_1^{\varphi}, I_2^{\varphi}, I_3^{\varphi})$ is φ-complement of interval-valued neutrosophic graph structure $\check{G}_{iv}$ as shown in Fig. 5.20.

Proposition 5.1 *φ-complement of an interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is a strong interval-valued neutrosophic graph structure. Moreover, if $\varphi(j) = u$, where $j, u \in \{1, 2, \ldots, t\}$, then all I_u-edges in interval-valued neutrosophic graph structure $(I, I_1, I_2, \ldots, I_t)$ become I_j^{φ}-edges in $(I, I_1^{\varphi}, I_2^{\varphi}, \ldots, I_t^{\varphi})$.*

Proof By definition of φ-complement,

$$[t^-_{I_j^{\varphi}}(rs),\ t^+_{I_j^{\varphi}}(rs)] = [t_I^-(r) \wedge t_I^-(s) - \bigvee_{l \neq j} t^-_{\varphi(I_l)}(rs),\ t_I^+(r) \wedge t_I^+(s) - \bigvee_{l \neq j} t^+_{\varphi(I_l)}(rs)], \quad (5.1)$$

$$[i^-_{I_j^{\varphi}}(rs),\ i^+_{I_j^{\varphi}}(rs)] = [i_I^-(r) \wedge i_I^-(s) - \bigvee_{l \neq j} i^-_{\varphi(I_l)}(rs),\ i_I^+(r) \wedge i_I^+(s) - \bigvee_{l \neq j} i^+_{\varphi(I_l)}(rs)], \quad (5.2)$$

$$[f^-_{I_j^{\varphi}}(rs),\ f^+_{I_j^{\varphi}}(rs)] = [f_I^-(r) \wedge f_I^-(s) - \bigvee_{l \neq j} f^-_{\varphi(I_l)}(rs),\ f_I^+(r) \wedge f_I^+(s) - \bigvee_{l \neq j} f^+_{\varphi(I_l)}(rs)], \quad (5.3)$$

for $j \in \{1, 2, \dots, t\}$. For expression of truth-membership value:
As $t_I^-(r) \wedge t_I^-(s) \geq 0$, $t_I^+(r) \wedge t_I^+(s) \geq 0$ and $\bigvee_{l \neq j} t^-_{\varphi(I_l)}(rs) \geq 0$, $\bigvee_{l \neq j} t^+_{\varphi(I_l)}(rs) \geq 0$. Since $t^-_{I_j}(rs) \leq t_I^-(r) \wedge t_I^-(s)$, $t^+_{I_j}(rs) \leq t_I^+(r) \wedge t_I^+(s)$, for all I_j. This implies $\bigvee_{l \neq j} t^-_{\varphi(I_l)}(rs) \leq t_I^-(r) \wedge t_I^-(s)$ and $\bigvee_{l \neq j} t^+_{\varphi(I_l)}(rs) \leq t_I^+(r) \wedge t_I^+(s)$. It shows that $t_I^-(r) \wedge t_I^-(r) - \bigvee_{l \neq j} t^-_{\varphi(I_l)}(rs) \geq 0$, $t_I^+(r) - \bigvee_{l \neq j} t^+_{\varphi(I_l)}(rs) \geq 0$. Hence $t^-_{I_j^\varphi}(rs) \geq 0$ and $t^+_{I_j^\varphi}(rs) \geq 0$, for all j. Furthermore, $t^-_{I_j^\varphi}(rs)$ and $t^+_{I_j^\varphi}(rs)$ obtain maximum value when $\bigvee_{l \neq j} t^-_{\varphi(I_l)}(rs)$ and $\bigvee_{l \neq j} t^+_{\varphi(I_l)}(rs)$ are zero. Obviously, when $\varphi(I_j) = I_u$ and rs is an I_u-edge then $\bigvee_{l \neq j} t^-_{\varphi(I_l)}(rs)$ and $\bigvee_{l \neq j} t^+_{\varphi(I_l)}(rs)$ acquire zero value. Hence

$$[t^-_{I_j^\varphi}(rs),\ t^+_{I_j^\varphi}(rs)] = [t_I^-(r) \wedge t_I^-(s),\ t_I^+(r) \wedge t_I^+(s)],\ \ for\ \ (rs) \in I_u,\ \varphi(I_j) = I_u. \tag{5.4}$$

For expression of indeterminacy-membership value:
As $i_I^-(r) \wedge i_I^-(s) \geq 0$, $i_I^+(r) \wedge i_I^+(s) \geq 0$ and $\bigvee_{l \neq j} i^-_{\varphi(I_l)}(rs) \geq 0$, $\bigvee_{l \neq j} i^+_{\varphi(I_l)}(rs) \geq 0$. Since $i^-_{I_j}(rs) \leq i_I^-(r) \wedge i_I^-(s)$, $i^+_{I_j}(rs) \leq i_I^+(r) \wedge i_I^+(s)$, for all I_j. This implies $\bigvee_{l \neq j} i^-_{\varphi(I_l)}(rs) \leq i_I^-(r) \wedge i_I^-(s)$ and $\bigvee_{l \neq j} i^+_{\varphi(I_l)}(rs) \leq i_I^+(r) \wedge i_I^+(s)$. It shows that $i_I^-(r) \wedge i_I^-(r) - \bigvee_{l \neq j} i^-_{\varphi(I_l)}(rs) \geq 0$, $i_I^+(r) - \bigvee_{l \neq j} i^+_{\varphi(I_l)}(rs) \geq 0$. Hence $i^-_{I_j^\varphi}(rs) \geq 0$ and $i^+_{I_j^\varphi}(rs) \geq 0$, for all j. Furthermore, $i^-_{I_j^\varphi}(rs)$ and $i^+_{I_j^\varphi}(rs)$ achieve maximum value when $\bigvee_{l \neq j} i^-_{\varphi(I_l)}(rs)$ and $\bigvee_{l \neq j} i^+_{\varphi(I_l)}(rs)$ are zero. Obviously, when $\varphi(I_j) = I_u$ and rs is an I_u-edge then $\bigvee_{l \neq j} i^-_{\varphi(I_l)}(rs)$ and $\bigvee_{l \neq j} i^+_{\varphi(I_l)}(rs)$ get zero value. Hence

$$[i^-_{I_j^\varphi}(rs),\ i^+_{I_j^\varphi}(rs)] = [i_I^-(r) \wedge i_I^-(s),\ i_I^+(r) \wedge i_I^+(s)],\ \ for\ \ (rs) \in I_u,\ \varphi(I_j) = I_u. \tag{5.5}$$

For expression of falsity-membership value:
As $f_I^-(r) \wedge f_I^-(s) \geq 0$, $f_I^+(r) \wedge f_I^+(s) \geq 0$ and $\bigvee_{l \neq j} f^-_{\varphi(I_l)}(rs) \geq 0$, $\bigvee_{l \neq j} f^+_{\varphi(I_l)}(rs) \geq 0$. Since $f^-_{I_j}(rs) \leq f_I^-(r) \wedge f_I^-(s)$, $f^+_{I_j}(rs) \leq f_I^+(r) \wedge f_I^+(s)$, for all I_j. This implies $\bigvee_{l \neq j} f^-_{\varphi(I_l)}(rs) \leq f_I^-(r) \wedge f_I^-(s)$ and $\bigvee_{l \neq j} f^+_{\varphi(I_l)}(rs) \leq f_I^+(r) \wedge f_I^+(s)$. It shows that $f_I^-(r) \wedge f_I^-(r) - \bigvee_{l \neq j} f^-_{\varphi(I_l)}(rs) \geq 0$, $f_I^+(r) - \bigvee_{l \neq j} f^+_{\varphi(I_l)}(rs) \geq 0$. Hence $f^-_{I_j^\varphi}(rs) \geq 0$ and $f^+_{I_j^\varphi}(rs) \geq 0$, for all j. Furthermore, $f^-_{I_j^\varphi}(rs)$ and $f^+_{I_j^\varphi}(rs)$ obtain maximum value when $\bigvee_{l \neq j} f^-_{\varphi(I_l)}(rs)$ and $\bigvee_{l \neq j} f^+_{\varphi(I_l)}(rs)$ are zero. Obviously, when $\varphi(I_j) = I_u$ and rs is an I_u-edge then $\bigvee_{l \neq j} f^-_{\varphi(I_l)}(rs)$ and $\bigvee_{l \neq j} f^+_{\varphi(I_l)}(rs)$ acquire zero value. Hence

$$[f^-_{I_j^\varphi}(rs),\ f^+_{I_j^\varphi}(rs)] = [f_I^-(r) \wedge f_I^-(s),\ f_I^+(r) \wedge f_I^+(s)],\ \ for\ \ (rs) \in I_u,\ \varphi(I_j) = I_u. \tag{5.6}$$

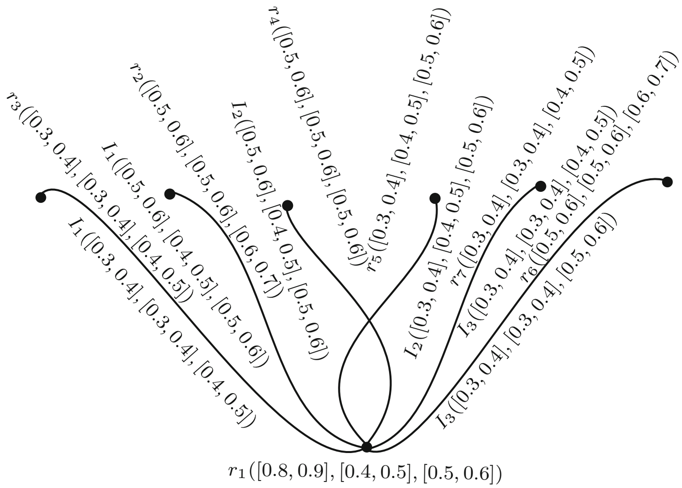


Fig. 5.21 Totally strong self-complementary interval-valued neutrosophic graph structure

From expressions (4), (5) and (6), it is clear that

$$t_j^-(rs) = \min\{t^-(r), t^-(d)\}, i_j^-(rs) = \min\{i^-(r), i^-(s)\},$$
$$f_j^-(rs) = \min\{f^-(-), f^-(s)\}, t_j^+(rs) = \min\{t^+(r), t^+(s)\},$$
$$i_j^+(rs) = \min\{i^+(r), i^+(s)\}, f_j^+(rs) = \min\{f^+(r), f^+(s)\},$$

Hence $\check{G}_{iv}$ is a strong $interval-valuedneutrosophicgraphstructure$ and all I_u-edges in interval-valued neutrosophic graph structure $(I, I_1, I_2, \ldots, I_t)$ become I_j^{φ}-edges in $(I, I_1^{\varphi}, I_2^{\varphi}, \ldots, I_t^{\varphi})$.

Definition 5.22 Let $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ be an interval-valued neutrosophic graph structure and φ be a permutation on $\{1, 2, \ldots, t\}$. Then

(i) $\check{G}_{iv}$ is self-complementary interval-valued neutrosophic graph structure if $\check{G}_{iv}$ is isomorphic to $\check{G}_{iv}^{\varphi c}$.
(ii) $\check{G}_{iv}$ is strong self-complementary interval-valued neutrosophic graph structure if $\check{G}_{iv}$ is identical to $\check{G}_{iv}^{\varphi c}$.

Definition 5.23 Let $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ be an interval-valued neutrosophic graph structure. Then

(i) $\check{G}_{iv}$ is totally self-complementary interval-valued neutrosophic graph structure if $\check{G}_{iv}$ is isomorphic to $\check{G}_{iv}^{\varphi c}$, for all permutations φ on $\{1, 2, \ldots, t\}$.
(ii) $\check{G}_{iv}$ is totally strong self-complementary interval-valued neutrosophic graph structure if $\check{G}_{iv}$ is identical to $\check{G}_{iv}^{\varphi c}$, for all permutations φ on $\{1, 2, \ldots, t\}$.

Example 5.15 An interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2, I_3)$ shown in Fig. 5.21 is identical to $\varphi-$complement for all permutations φ on set $\{1, 2, 3\}$. Hence it is totally strong self-complementary interval-valued neutrosophic graph structure.

Theorem 5.1 *An interval-valued neutrosophic graph structure is totally self-complementary if and only if it is a strong interval-valued neutrosophic graph structure.*

Proof Consider a strong interval-valued neutrosophic graph structure $\check{G}_{iv}$ and permutation φ on {1,2, ..., t}. By Proposition 5.1, φ-complement of interval-valued neutrosophic graph structure $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is a strong interval-valued neutrosophic graph structure. Moreover, if $\varphi^{-1}(u) = j$, where $j, u \in \{1, 2, \ldots, t\}$, then all I_u-edges in interval-valued neutrosophic graph structure $(I, I_1, I_2, \ldots, I_t)$ become I_j^{φ}-edges in $(I, I_1^{\varphi}, I_2^{\varphi}, \ldots, I_t^{\varphi})$, this leads

$$t^-_{I_u}(rs) = t^-_I(r) \wedge t^-_I(s) = t^-_{I_j^{\varphi}}(rs), i^-_{I_u}(rs) = i^-_I(r) \wedge i^-_I(s) = i^-_{I_j^{\varphi}}(rs),$$
$$f^-_{I_u}(rs) = f^-_I(r) \wedge f^-_I(s) = f^-_{I_j^{\varphi}}(rs), t^+_{I_u}(rs) = t^+_I(r) \wedge t^+_I(s) = t^+_{I_j^{\varphi}}(rs),$$
$$i^+_{I_u}(rs) = i^+_I(r) \wedge i^+_I(s) = i^+_{I_j^{\varphi}}(rs), f^+_{I_u}(rs) = f^+_I(r) \wedge f^+_I(s) = f^+_{I_j^{\varphi}}(rs).$$

Therefore, under $f : U \to U$ (identity mapping), $\check{G}_{iv}$ and $\check{G}^{\varphi}_{iv}$ are isomorphic such that:

$$t^-_I(r) = t^-_I(f(r)), i^-_I(r) = i^-_I(f(r)), f^-_I(r) = f^-_I(f(r)),$$
$$t^+_I(r) = t^+_I(f(r)), i^+_I(r) = i^+_I(f(r)), f^+_I(r) = f^+_I(f(r)).$$

$$t^-_{I_u}(rs) = t^-_{I_j^{\varphi}}(f(r)f(s)) = t^-_{I_j^{\varphi}}(rs), \;\; t^+_{I_u}(rs) = t^+_{I_j^{\varphi}}(f(r)f(s)) = t^+_{I_j^{\varphi}}(rs),$$

$$i^-_{I_u}(rs) = i^-_{I_j^{\varphi}}(f(r)f(s)) = i^-_{I_j^{\varphi}}(rs), \;\; i^+_{I_u}(rs) = i^+_{I_j^{\varphi}}(f(r)f(s)) = i^+_{I_j^{\varphi}}(rs),$$

$$f^-_{I_u}(rs) = f^-_{I_j^{\varphi}}(f(r)f(s)) = f^-_{I_j^{\varphi}}(rs), \;\; f^+_{I_u}(rs) = f^+_{I_j^{\varphi}}(f(r)f(s)) = f^+_{I_j^{\varphi}}(rs),$$

for all $rs \in I_u$, for $\varphi^{-1}(u) = j$; $j, u = 1, 2, \ldots, t$.

This holds for every permutation φ on $\{1, 2, \ldots, t\}$. Hence $\check{G}_{iv}$ is totally self-complementary interval-valued neutrosophic graph structure. Conversely, let $\check{G}_{iv}$ be isomorphic to $\check{G}^{\varphi}_{iv}$ for each permutation φ on $\{1, 2, \ldots, t\}$. Moreover, according to the definitions of isomorphism of interval-valued neutrosophic graph structures and φ-complement of an interval-valued neutrosophic graph structure

$$t^-_{I_u}(rs) = t^-_{I_j^{\varphi}}(f(r)f(s)) = t^-_I(f(r)) \wedge t^-_I(f(s)) = t^-_I(r) \wedge t^-_I(s),$$
$$t^+_{I_u}(rs) = t^+_{I_j^{\varphi}}(f(r)f(s)) = t^+_I(f(r)) \wedge t^+_I(f(s)) = t^+_I(r) \wedge t^+_I(s),$$

$$i^-_{I_u}(rs) = i^-_{I_j^{\varphi}}(f(r)f(s)) = i^-_I(f(r)) \wedge i^-_I(f(s)) = i^-_I(r) \wedge i^-_I(s),$$

$$i^+_{I_u}(rs) = i^+_{I_j^{\varphi}}(f(r)f(s)) = i^+_I(f(r)) \wedge i^+_I(f(s)) = i^+_I(r) \wedge i^+_I(s),$$

$$f^-_{I_u}(rs) = f^-_{I^\varphi_j}(f(r)f(s)) = f^-_I(f(r)) \wedge f^-_I(f(s)) = f^-_I(r) \wedge f^-_I(s),$$
$$f^+_{I_u}(rs) = f^+_{I^\varphi_j}(f(r)f(s)) = f^+_I(f(r)) \wedge f^+_I(f(s)) = f^+_I(r) \wedge f^+_I(s),$$

for all $rs \in I_u$, $u = 1, 2, \ldots, t$. Hence $\check{G}_{iv}$ is a strong interval-valued neutrosophic graph structure.

Remark 5.1 Every self-complementary interval-valued neutrosophic graph structure is totally self-complementary.

Theorem 5.2 *If $G^* = (X, E_1, E_2, \ldots, E_t)$ is a totally strong self-complementary graph structure and $I = ([t^-_I, t^+_I], [i^-_I, i^+_I], [f^-_I, f^+_I])$ is an interval-valued neutrosophic subset of X, where $t^-_I, i^-_I, f^-_I, t^+_I, i^+_I, f^+_I$ are constant functions, then every strong interval-valued neutrosophic graph structure of G^* with interval-valued neutrosophic vertex set I is a totally strong self-complementary interval-valued neutrosophic graph structure.*

Proof Let $a, a' \in [0, 1]$, $b, b' \in [0, 1]$ and $c, c' \in [0, 1]$ be six constants and

$$t^-_I(r) = a, i^-_I(r) = b, f^-_I(r) = c, t^+_I(r) = a', i^+_I(r) = b', f^+_I(r) = c', \text{ for all } r \in X.$$

Since G^* is a totally strong self-complementary GS, so for every permutation φ^{-1} on $\{1, 2, \ldots, t\}$ there is a bijection $f : X \to U$, such that for every I_u−edge (rs), (f(r)f(s)) [an I_j-edge in G^*] is an I_u-edge in $G_s{}^{\varphi^{-1}c}$. Thus for every I_u-edge (rs), (f(r)f(s)) [an I_j-edge in $\check{G}_{iv}$] is an I^φ_u-edge in $\check{G}_{iv}{}^{\varphi^{-1}c}$.

Moreover, $\check{G}_{iv}$ is a strong interval-valued neutrosophic graph structure, so

$$t^-_I(r) = a = t^-_I(f(r)),\ \ i^-_I(r) = b = i^-_I(f(r)),\ \ f^-_I(r) = c = f^-_I(f(r)),$$
$$t^+_I(r) = a' = t^+_I(f(r)),\ \ i^+_I(r) = b' = i^+_I(f(r)),\ \ f^+_I(r) = c' = f^+_I(f(r)),$$

for all $r \in X$, and

$$t^-_{I_u}(rs) = t^-_I(r) \wedge t^-_I(s) = t^-_I(f(r)) \wedge t^-_I(f(s)) = t^-_{I^\varphi_j}(f(r)f(s)),$$
$$i^-_{I_u}(rs) = i^-_I(r) \wedge i^-_I(s) = i^-_I(f(r)) \wedge i^-_I(f(s)) = i^-_{I^\varphi_j}(f(r)f(s)),$$
$$f^-_{I_u}(rs) = f^-_I(r) \wedge i^-_I(s) = f^-_I(f(r)) \wedge f^-_I(f(s)) = f^-_{I^\varphi_j}(f(r)f(s)),$$
$$t^+_{I_u}(rs) = t^+_I(r) \wedge t^+_I(s) = t^+_I(f(r)) \wedge t^+_I(f(s)) = t^+_{I^\varphi_j}(f(r)f(s)),$$
$$i^+_{I_u}(rs) = i^+_I(r) \wedge i^+_I(s) = i^+_I(f(r)) \wedge i^+_I(f(s)) = i^+_{I^\varphi_j}(f(r)f(s)),$$
$$f^+_{I_u}(rs) = f^+_I(r) \wedge i^+_I(s) = f^+_I(f(r)) \wedge f^+_I(f(s)) = f^+_{I^\varphi_j}(f(r)f(s)),$$

for all $rs \in I_j$, $j = 1, 2, \ldots, t$.

This shows $\check{G}_{iv}$ is a strong self-complementary interval-valued neutrosophic graph structure. This satisfies for each permutation φ and φ^{-1} on set $\{1, 2, \ldots, t\}$, thus $\check{G}_{iv}$ is a totally strong self-complementary interval-valued neutrosophic graph structure. This completes the proof.

Remark 5.2 Converse of Theorem 5.2 may not true; for example, a interval-valued neutrosophic graph structure depicted in Fig. 5.21 is totally strong self-complementary interval-valued neutrosophic graph structure, and it is also strong interval-valued neutrosophic graph structure with a totally strong self-complementary underlying graph structure, but t_I^-, i_I^-, f_I^-,t_I^+, i_I^+, f_I^+ are not the constant functions.

Chapter 6
Graphs Under Neutrosophic Hybrid Models

Rough sets and single-valued neutrosophic sets are mathematical models to deal with incomplete and vague information. These two models can be combined into two frameworks for modelling and processing incomplete information in information systems. Thus, single-valued neutrosophic rough set model and rough single-valued neutrosophic set model are hybrid models, which give more precision, flexibility and compatibility to the system as compared to the classic and fuzzy models. In this chapter, we present rough single-valued neutrosophic digraphs (rough neutrosophic digraphs, for short) and neutrosophic rough digraphs and describe methods of their construction. We consider the concept of self-complementary rough neutrosophic digraphs. We discuss regular neutrosophic rough digraphs. We also give a comparative analysis of rough neutrosophic digraphs and neutrosophic rough digraphs. This chapter is due to [16, 123, 162].

6.1 Introduction

Pawlak [142] introduced the concept of rough set. He was a Polish mathematician (citizen of Poland) and computer scientist. Rough means approximate or inexact. Rough set theory expresses vagueness in terms of a boundary region of a set not in terms of membership function as in fuzzy set. The idea of rough set theory is a generalization of classical set theory to study the intelligence systems containing inexact, uncertain or incomplete information. It is an effective drive for bestowal with uncertain or incomplete information. Rough set theory is a novel mathematical approach to imprecise knowledge. Rough set theory expresses vagueness by means of a boundary region of a set. The emptiness of boundary region of a set shows that this is a crisp set, and nonemptiness shows that this is a rough set. Nonemptiness of boundary region also describes the deficiency of our knowledge about a set. A subset of a universe in rough set theory is expressed by two approximations which are known as lower and upper approximations. Equivalence classes are the basic building blocks in rough set theory, for upper and lower approximations.

M. Akram, *Single-Valued Neutrosophic Graphs*, Infosys Science Foundation Series, https://doi.org/10.1007/978-981-13-3522-8_6

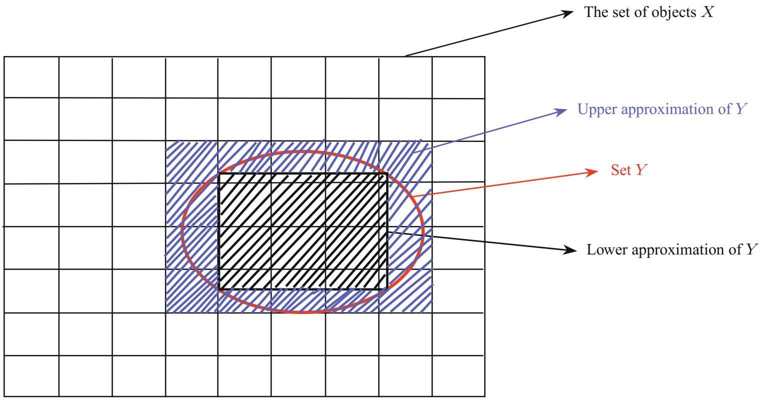


Fig. 6.1 Diagram of a rough set

Neutrosophic set and rough set are two different theories to deal with uncertain, imprecise and incomplete information. Due to the limitation of human knowledge to understand the complex problems, it is very difficult to apply only a single type of uncertainty method to deal with such problems. Therefore, it is necessary to develop hybrid models by incorporating the advantages of many other different mathematical models dealing the uncertainty. Thus, by combining these two mathematical tools, Broumi et al. [61] introduced the concept of rough neutrosophic sets. Yang et al. [177] proposed single-valued neutrosophic rough sets by combining single-valued neutrosophic sets and rough sets, and established an algorithm for decision-making based on single-valued neutrosophic rough sets on two universes.

Definition 6.1 Let X be a nonempty finite universe and R an equivalence relation on X. A pair (X, R) is called a *Pawlak approximation space*. Let Y be a subset of X, then the lower and upper approximations of Y are defined as follows:

$$\underline{R}(Y) =\{x \in X : [x]_R \subseteq Y\},$$
$$\overline{R}(Y) =\{x \in X : [x]_R \cap Y \neq \phi\},$$

where

$$[x]_R = \{y \in X : (x, y) \in R\}$$

denotes equivalence class of R containing x. $\underline{R}$ and $\overline{R}$ are called the *lower and upper approximations operators*, respectively. The pair $(\underline{R}(Y), \overline{R}(Y))$ is called a *Pawlak rough set*.

The graphical representation of rough set is shown in Fig. 6.1

Example 6.1 Let $X = \{1, 2, 3, 4, 5, 6\}$ be a universe and $R = \{\{1, 5\}, \{2, 3\}, \{4, 6\}\}$ an equivalence relation on X. Let $Y = \{2, 3, 5\}$. Then

$[1]_R = \{1, 5\} = [5]_R \nsubseteq Y$ but $[1]_R \cap Y \neq \emptyset \neq [5]_R \cap Y$
$[2]_R = \{2, 3\} = [3]_R \subseteq Y$ but $[3]_R \cap Y \neq \emptyset \neq [2]_R \cap Y$
$[4]_R = \{4, 6\} = [6]_R \nsubseteq Y$ but $[4]_R \cap Y = \emptyset$
Hence $\underline{R}(Y) = \{2, 3\}$ and
$\overline{R}(Y) = \{1, 2, 3, 5\}$
$bd(X) = \overline{R}(Y) - \underline{R}(Y) = \{1, 5\} \neq \emptyset$
Thus, $(\underline{R}(Y), \overline{R}(Y))$ is a rough set w.r.t. R.

Definition 6.2 Let X be a nonempty universe and R an equivalence relation on X. Let A be a neutrosophic set on X, defined as

$$A = \{< x, T_A(x), I_A(x), F_A(x) >: x \in X\}.$$

Then lower and upper approximations of A in the approximation space (X, R) denoted by $\underline{R}A$ and $\overline{R}A$, respectively, are defined as follows:

$$\underline{R}A = \{< x, T_{\underline{R}(A)}(x), I_{\underline{R}(A)}(x), F_{\underline{R}(A)}(x) >: y \in [x]_R, x \in X\},$$
$$\overline{R}A = \{< x, T_{\overline{R}(A)}(x), I_{\overline{R}(A)}(x), F_{\overline{R}(A)}(x) >: y \in [x]_R, x \in X\},$$

where

$$T_{\underline{R}(A)}(x) = \bigwedge_{y \in [x]_R} T_A(y), \quad T_{\overline{R}(A)}(x) = \bigvee_{y \in [x]_R} T_A(y),$$
$$I_{\underline{R}(A)}(x) = \bigwedge_{y \in [x]_R} I_A(y), \quad I_{\overline{R}(A)}(x) = \bigvee_{y \in [x]_R} I_A(y),$$
$$F_{\underline{R}(A)}(x) = \bigvee_{y \in [x]_R} F_A(y), \quad F_{\overline{R}(A)}(x) = \bigwedge_{y \in [x]_R} F_A(y).$$

A pair $(\underline{R}A, \overline{R}A)$ is called *rough neutrosophic set*.

6.2 Rough Neutrosophic Digraphs

Definition 6.3 Let X be a nonempty set and R an equivalence relation on X. Let A be a single-valued neutrosophic set on X, defined as

$$A = \{< x, T_A(x), I_A(x), F_A(x) >: x \in X\}.$$

Then the lower and upper approximations of A represented by $\underline{R}A$ and $\overline{R}A$, respectively, are characterized as single-valued neutrosophic sets in X such that $\forall\ x \in X$,

$$\underline{R}(A) = \{< x, T_{\underline{R}(A)}(x), I_{\underline{R}(A)}(x), F_{\underline{R}(A)}(x) >: y \in [x]_R\},$$
$$\overline{R}(A) = \{< x, T_{\overline{R}(A)}(x), I_{\overline{R}(A)}(x), F_{\overline{R}(A)}(x) >: y \in [x]_R\},$$

where

$$T_{\underline{R}(A)}(x) = \bigwedge_{y \in [x]_R} T_A(y), \quad T_{\overline{R}(A)}(x) = \bigvee_{y \in [x]_R} T_A(y),$$
$$I_{\underline{R}(A)}(x) = \bigwedge_{y \in [x]_R} I_A(y), \quad I_{\overline{R}(A)}(x) = \bigvee_{y \in [x]_R} I_A(y),$$
$$F_{\underline{R}(A)}(x) = \bigvee_{y \in [x]_R} F_A(y), \quad F_{\overline{R}(A)}(x) = \bigwedge_{y \in [x]_R} F_A(y).$$

Let $E \subseteq X \times X$ and S be an equivalence relation on E such that

$$((x_1, x_2), (y_1, y_2)) \in S \Leftrightarrow (x_1, y_1), (x_2, y_2) \in R.$$

Let B be a single-valued neutrosophic set on $E \subseteq X \times X$ defined as

$$B = \{< xy, T_B(xy), I_B(xy), F_B(xy) >: xy \in X \times X\},$$

such that

$$T_B(xy) \leq \min\{T_{\underline{R}A}(x), T_{\underline{R}A}(y)\},$$
$$I_B(xy) \leq \min\{I_{\underline{R}A}(x), I_{\underline{R}A}(y)\},$$
$$F_B(xy) \leq \max\{F_{\overline{R}A}(x), F_{\overline{R}A}(y)\}, \quad \forall x, y \in X.$$

Then the lower and upper approximations of B represented by $\underline{S}B$ and $\overline{S}B$ are defined as follows

$$\underline{S}B = \{< xy, T_{\underline{S}B}(xy), I_{\underline{S}B}(xy), F_{\underline{S}B}(xy) >: wz \in [xy]_S, xy \in X \times X\},$$
$$\overline{S}B = \{< xy, T_{\overline{S}B}(xy), I_{\overline{S}B}(xy), F_{\overline{S}B}(xy) >: wz \in [xy]_S, xy \in X \times X\},$$

where,

$$T_{\underline{S}(B)}(xy) = \bigwedge_{wz \in [xy]_S} T_B(wz), \quad T_{\overline{S}(B)}(xy) = \bigvee_{wz \in [xy]_S} T_B(wz),$$
$$I_{\underline{S}(B)}(xy) = \bigwedge_{wz \in [xy]_S} I_B(wz), \quad I_{\overline{S}(B)}(xy) = \bigvee_{wz \in [xy]_S} I_B(wz),$$
$$F_{\underline{S}(B)}(xy) = \bigvee_{wz \in [xy]_S} F_B(wz), \quad F_{\overline{S}(B)}(xy) = \bigwedge_{wz \in [xy]_S} F_B(wz).$$

A pair $SB = (\underline{S}B, \overline{S}B)$ is called a *rough single-valued neutrosophic relation*.

Definition 6.4 A *rough single-valued neutrosophic digraph* on a nonempty set X is a four-ordered tuple $G = (R, RA, S, SB)$ such that

(a) R is an equivalence relation on X.
(b) S is an equivalence relation on $E \subseteq X \times X$.
(c) $RA = (\underline{R}A, \overline{R}A)$ is a rough single-valued neutrosophic set on X.
(d) $SB = (\underline{S}B, \overline{S}B)$ is a rough single-valued neutrosophic relation on X.
(e) (RA, SB) is a rough single-valued neutrosophic digraph where $\underline{G} = (\underline{R}A, \underline{S}B)$ and $\overline{G} = (\overline{R}A, \overline{S}B)$ are lower and upper approximate single-valued neutrosophic digraphs of G such that

$$\begin{aligned}
T_{\underline{S}B}(x, y) &\leq \min\{T_{\underline{R}A}(x), T_{\underline{R}A}(y)\},\\
I_{\underline{S}B}(x, y) &\leq \min\{I_{\underline{R}A}(x), I_{\underline{R}A}(y)\},\\
F_{\underline{S}B}(x, y) &\leq \max\{F_{\underline{R}A}(x), F_{\underline{R}A}(y)\},
\end{aligned}$$

and

$$\begin{aligned}
T_{\overline{S}B}(x, y) &\leq \min\{T_{\overline{R}A}(x), T_{\overline{R}A}(y)\},\\
I_{\overline{S}B}(x, y) &\leq \min\{I_{\overline{R}A}(x), I_{\overline{R}A}(y)\},\\
F_{\overline{S}B}(x, y) &\leq \max\{F_{\overline{R}A}(x), F_{\overline{R}A}(y)\}, \quad \forall\, x, y \in X.
\end{aligned}$$

Throughout this chapter, we will use a rough neutrosophic set, rough neutrosophic relation and rough neutrosophic digraph, for short.

Example 6.2 Let $X = \{a, b, c, d\}$ be a set and R an equivalence relation on X defined as:

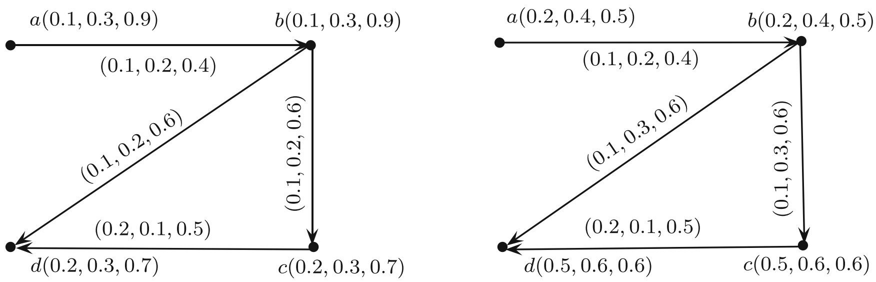
$$R = \begin{bmatrix} 1 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Let $A_1=\{(a, 0.2, 0.4, 0.9), (b, 0.1, 0.3, 0.5), (c, 0.2, 0.3, 0.6), (d, 0.5, 0.6, 0.7)\}$ be a neutrosophic set on X. The lower and upper approximations of A_1 are given by

$$\begin{aligned}
\underline{R}A_1 &= \{(a, 0.1, 0.3, 0.9), (b, 0.1, 0.3, 0.9), (c, 0.2, 0.3, 0.7), (d, 0.2, 0.3, 0.7)\},\\
\overline{R}A_1 &= \{(a, 0.2, 0.4, 0.5), (b, 0.2, 0.4, 0.5), (c, 0.5, 0.6, 0.6), (d, 0.5, 0.6, 0.6)\}.
\end{aligned}$$

Let $E = \{(a, b), (b, c), (b, d), (c, d)\} \subseteq X \times X$ and S be an equivalence relation on E defined as:

$$S = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}.$$



$\underline{G}_1 = (\underline{R}A_1, \underline{S}B_1)$ $\overline{G}_1 = (\overline{R}A_1, \overline{S}B_1)$

Fig. 6.2 Rough neutrosophic digraph $G_1 = (\underline{G}_1, \overline{G}_1)$

Let $B_1 = \{((a, b), 0.1, 0.2, 0.4), ((b, c), 0.1, 0.3, 0.6), ((b, d), 0.1, 0.2, 0.6), ((c, d), 0.2, 0.1, 0.5)\}$ be a neutrosophic set on E and $SB_1 = (\underline{S}B_1, \overline{S}B_1)$ a rough neutrosophic relation, where $\underline{S}B_1$ and $\overline{S}B_1$ are given as:

$$\underline{S}B_1 = \{((a, b), 0.1, 0.2, 0.4), ((b, c), 0.1, 0.2, 0.6), ((b, d), 0.1, 0.2, 0.6), ((c, d), 0.2, 0.1, 0.5)\},$$
$$\overline{S}B_1 = \{((a, b), 0.1, 0.2, 0.4), ((b, c), 0.1, 0.3, 0.6), ((b, d), 0.1, 0.3, 0.6), ((c, d), 0.2, 0.1, 0.5)\}.$$

Thus, $\underline{G_1} = (\underline{R}A_1, \underline{S}B_1)$ and $\overline{G_1} = (\overline{R}A_1, \overline{S}B_1)$ are neutrosophic digraphs as shown in Fig. 6.2.

Example 6.3 Let $X = \{a, b, c\}$ be a crisp set and R an equivalence relation on X defined as:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Let $A_2 = \{(a, 0.1, 0.7, 0.8), (b, 0.9, 0.6, 0.5), (c, 0.2, 0.4, 0.3)\}$ be a neutrosophic set on X and $RA_2 = (\underline{R}A_2, \overline{R}A_2)$ a rough neutrosophic set, where $\underline{R}A_2$ and $\overline{R}A_2$ are given as:

$$\underline{R}A_2 = \{(a, 0.1, 0.7, 0.8), (b, 0.2, 0.4, 0.5), (c, 0.2, 0.4, 0.5)\},$$
$$\overline{R}A_2 = \{(a, 0.1, 0.7, 0.8), (b, 0.9, 0.6, 0.3), (c, 0.9, 0.6, 0.3)\}.$$

Let $E = \{(a, b), (b, c)\} \subseteq X \times X$ and S be an equivalence relation on E defined as:

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $B_2 = \{((a, b), 0.1, 0.4, 0.7), ((b, c), 0.2, 0.3, 0.2)\}$ be a neutrosophic set on E, then by definition we have

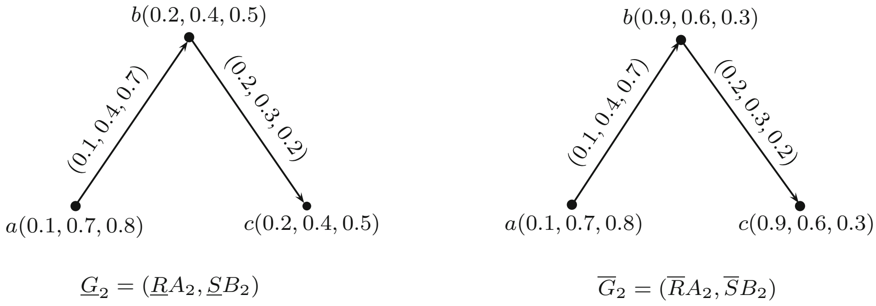


Fig. 6.3 Rough neutrosophic digraph $G_2 = (\underline{G}_2, \overline{G}_2)$

$$\underline{S}B_2 = \{((a, b), 0.1, 0.4, 0.7), ((b, c), 0.2, 0.3, 0.2)\},$$
$$\overline{S}B_2 = \{((a, b), 0.1, 0.4, 0.7), ((b, c), 0.2, 0.3, 0.2)\}.$$

Thus, $\underline{G_2} = (\underline{R}A_2, \underline{S}B_2)$ and $\overline{G_2} = (\overline{R}A_2, \overline{S}B_2)$ are neutrosophic digraphs as shown in Fig. 6.3.

Definition 6.5 Let $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ be two rough neutrosophic digraphs on a set X. Then the *lexicographic product* of G_1 and G_2 is a rough neutrosophic digraph $G = G_1 \odot G_2 = (\underline{G}_1 \odot \underline{G}_2, \overline{G}_1 \odot \overline{G}_2)$, where $\underline{G}_1 \odot \underline{G}_2 = (\underline{R}A_1 \odot \underline{R}A_2, \underline{S}B_1 \odot \underline{S}B_2)$ and $\overline{G}_1 \odot \overline{G}_2 = (\overline{R}A_1 \odot \overline{R}A_2, \overline{S}B_1 \odot \overline{S}B_2)$ are neutrosophic digraphs, respectively, such that

(1)
$$\begin{aligned}
T_{\underline{R}A_1 \odot \underline{R}A_2}(x_1, x_2) &= \min\{T_{\underline{R}A_1}(x_1), T_{\underline{R}A_2}(x_2)\},\\
I_{\underline{R}A_1 \odot \underline{R}A_2}(x_1, x_2) &= \min\{I_{\underline{R}A_1}(x_1), T_{\underline{R}A_2}(x_2)\},\\
F_{\underline{R}A_1 \odot \underline{R}A_2}(x_1, x_2) &= \max\{F_{\underline{R}A_1}(x_1), T_{\underline{R}A_2}(x_2)\}, \quad \forall\,(x_1, x_2) \in \underline{R}A_1 \ltimes \underline{R}A_2,\\
T_{\underline{S}B_1 \odot \underline{S}B_2}((x, x_2), (x, y_2)) &= \min\{T_{\underline{R}A_1}(x), T_{\underline{S}B_2}(x_2, y_2)\},\\
I_{\underline{S}B_1 \odot \underline{S}B_2}((x, x_2), (x, y_2)) &= \min\{I_{\underline{R}A_1}(x), I_{\underline{S}B_2}(x_2, y_2)\},\\
F_{\underline{S}B_1 \odot \underline{S}B_2}((x, x_2), (x, y_2)) &= \max\{F_{\underline{R}A_1}(x), F_{\underline{S}B_2}(x_2, y_2)\}, \quad \forall\, x \in \underline{R}A_1, (x_2, y_2) \in \underline{S}B_2,\\
T_{\underline{S}B_1 \odot \underline{S}B_2}((x_1, x_2), (y_1, y_2)) &= \min\{T_{\underline{S}B_1}(x_1, y_1), T_{\underline{S}B_2}(x_2, y_2)\},\\
I_{\underline{S}B_1 \odot \underline{S}B_2}((x_1, x_2), (y_1, y_2)) &= \min\{I_{\underline{S}B_1}(x_1, y_1), I_{\underline{S}B_2}(x_2, y_2)\},\\
F_{\underline{S}B_1 \odot \underline{S}B_2}((x_1, x_2), (y_1, y_2)) &= \max\{F_{\underline{S}B_1}(x_1, y_1), F_{\underline{S}B_2}(x_2, y_2)\}, \quad \forall\,(x_1, y_1) \in \underline{S}B_1, (x_2, y_2) \in \underline{S}B_2.
\end{aligned}$$

(2)
$$\begin{aligned}
T_{\overline{R}A_1 \odot \overline{R}A_2}(x_1, x_2) &= \min\{T_{\overline{R}A_1}(x_1), T_{\overline{R}A_2}(x_2)\},\\
I_{\overline{R}A_1 \odot \overline{R}A_2}(x_1, x_2) &= \min\{I_{\overline{R}A_1}(x_1), T_{\overline{R}A_2}(x_2)\},\\
F_{\overline{R}A_1 \odot \overline{R}A_2}(x_1, x_2) &= \max\{F_{\overline{R}A_1}(x_1), T_{\overline{R}A_2}(x_2)\}, \quad \forall\,(x_1, x_2) \in \overline{R}A_1 \ltimes \overline{R}A_2,\\
T_{\overline{S}B_1 \odot \overline{S}B_2}((x, x_2), (x, y_2)) &= \min\{T_{\overline{R}A_1}(x), T_{\overline{S}B_2}(x_2, y_2)\},\\
I_{\overline{S}B_1 \odot \overline{S}B_2}((x, x_2), (x, y_2)) &= \min\{I_{\overline{R}A_1}(x), I_{\overline{S}B_2}(x_2, y_2)\},\\
F_{\overline{S}B_1 \odot \overline{S}B_2}((x, x_2), (x, y_2)) &= \max\{F_{\overline{R}A_1}(x), F_{\overline{S}B_2}(x_2, y_2)\}, \quad \forall\, x \in \overline{R}A_1, (x_2, y_2) \in \overline{S}B_2,\\
T_{\overline{S}B_1 \odot \overline{S}B_2}((x_1, x_2), (y_1, y_2)) &= \min\{T_{\overline{S}B_1}(x_1, y_1), T_{\overline{S}B_2}(x_2, y_2)\},\\
I_{\overline{S}B_1 \odot \overline{S}B_2}((x_1, x_2), (y_1, y_2)) &= \min\{I_{\overline{S}B_1}(x_1, y_1), I_{\overline{S}B_2}(x_2, y_2)\},\\
F_{\overline{S}B_1 \odot \overline{S}B_2}((x_1, x_2), (y_1, y_2)) &= \max\{F_{\overline{S}B_1}(x_1, y_1), F_{\overline{S}B_2}(x_2, y_2)\}, \quad \forall\,(x_1, y_1) \in \overline{S}B_1, (x_2, y_2) \in \overline{S}B_2.
\end{aligned}$$

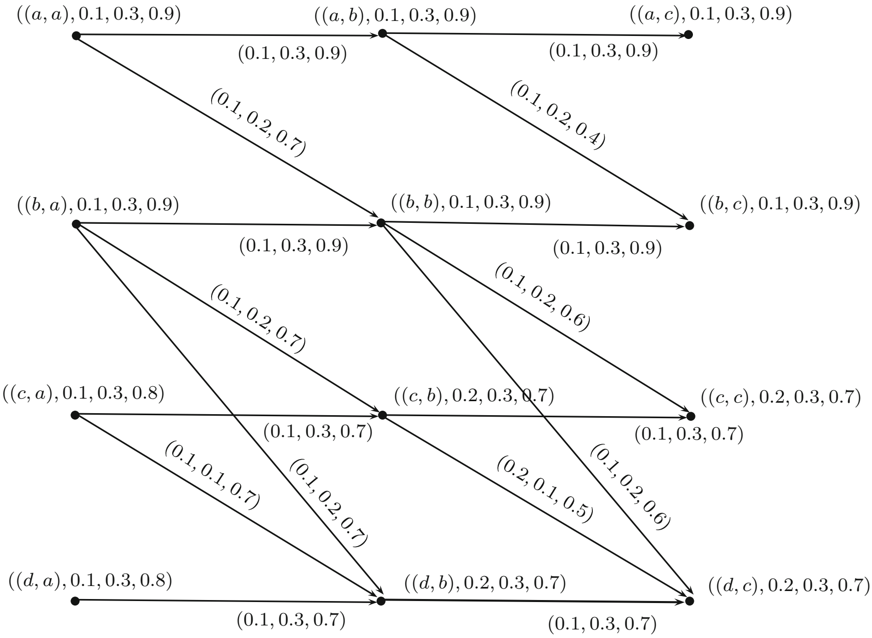


Fig. 6.4 $\underline{G}_1 \odot \underline{G}_2 = (\underline{R}A_1 \odot \underline{R}A_2, \underline{S}B_1 \odot \underline{S}B_2)$

Example 6.4 Consider the two rough neutrosophic digraphs G_1 and G_2 as shown in Figs. 6.2 and 6.3. The lexicographic product of G_1 and G_2 is $G = G_1 \odot G_2 = (\underline{G}_1 \odot \underline{G}_2, \overline{G}_1 \odot \overline{G}_2)$, where $\underline{G}_1 \odot \underline{G}_2 = (\underline{R}A_1 \odot \underline{R}A_2, \underline{S}B_1 \odot \underline{S}B_2)$ and $\overline{G}_1 \odot \overline{G}_2 = (\overline{R}A_1 \odot \overline{R}A_2, \overline{S}B_1 \odot \overline{S}B_2)$ are neutrosophic digraphs as shown in Figs. 6.4 and 6.5.

Definition 6.6 The *strong product* of two rough neutrosophic digraphs G_1 and G_2 is a rough neutrosophic digraph $G = G_1 \boxtimes G_2 = (\underline{G}_1 \boxtimes \underline{G}_2, \overline{G}_1 \boxtimes \overline{G}_2)$, where $\underline{G}_1 \boxtimes \underline{G}_2 = (\underline{R}A_1 \boxtimes \underline{R}A_2, \underline{S}B_1 \boxtimes \underline{S}B_2)$ and $\overline{G}_1 \boxtimes \overline{G}_2 = (\overline{R}A_1 \boxtimes \overline{R}A_2, \overline{S}B_1 \boxtimes \overline{S}B_2)$ are neutrosophic digraphs, respectively, such that

$$
\begin{aligned}
\textbf{(1)}\quad & T_{\underline{R}A_1 \boxtimes \underline{R}A_2}(x, y) = \min\{T_{\underline{R}A_1}(x), T_{\underline{R}A_2}(y)\},\\
& I_{\underline{R}A_1 \boxtimes \underline{R}A_2}(x, y) = \min\{I_{\underline{R}A_1}(x), I_{\underline{R}A_2}(y)\},\\
& F_{\underline{R}A_1 \boxtimes \underline{R}A_2}(x, y) = \max\{F_{\underline{R}A_1}(x), F_{\underline{R}A_2}(y)\}, \quad \forall\, (x, y) \in \underline{R}A_1 \times \underline{R}A_2,\\
& T_{\underline{S}B_1 \boxtimes \underline{S}B_2}((x, x_2), (x, y_2)) = \min\{T_{\underline{R}A_1}(x), T_{\underline{S}B_2}(x_2, y_2)\},\\
& I_{\underline{S}B_1 \boxtimes \underline{S}B_2}((x, x_2), (x, y_2)) = \min\{I_{\underline{R}A_1}(x), I_{\underline{S}B_2}(x_2, y_2)\},\\
& F_{\underline{S}B_1 \boxtimes \underline{S}B_2}((x, x_2), (x, y_2)) = \max\{F_{\underline{R}A_1}(x), F_{\underline{S}B_2}(x_2, y_2)\}, \quad \forall\, x \in \underline{R}A_1, (x_2, y_2) \in \underline{S}B_2,\\
& T_{\underline{S}B_1 \boxtimes \underline{S}B_2}((x_1, y), (y_1, y)) = \min\{T_{\underline{S}B_1}(x_1, y_1), T_{\underline{R}A_2}(y)\},\\
& I_{\underline{S}B_1 \boxtimes \underline{S}B_2}((x_1, y), (y_1, y)) = \min\{I_{\underline{S}B_1}(x_1, y_1), I_{\underline{R}A_2}(y)\},\\
& F_{\underline{S}B_1 \boxtimes \underline{S}B_2}((x_1, y), (y_1, y)) = \max\{F_{\underline{S}B_1}(x_1, y_1), F_{\underline{R}A_2}(y)\}, \quad \forall\, (x_1, y_1) \in \underline{S}B_1, y \in \underline{R}A_2,\\
& T_{\underline{S}B_1 \boxtimes \underline{S}B_2}((x_1, x_2), (y_1, y_2)) = \min\{T_{\underline{S}B_1}(x_1, y_1), T_{\underline{S}B_2}(x_2, y_2)\},
\end{aligned}
$$

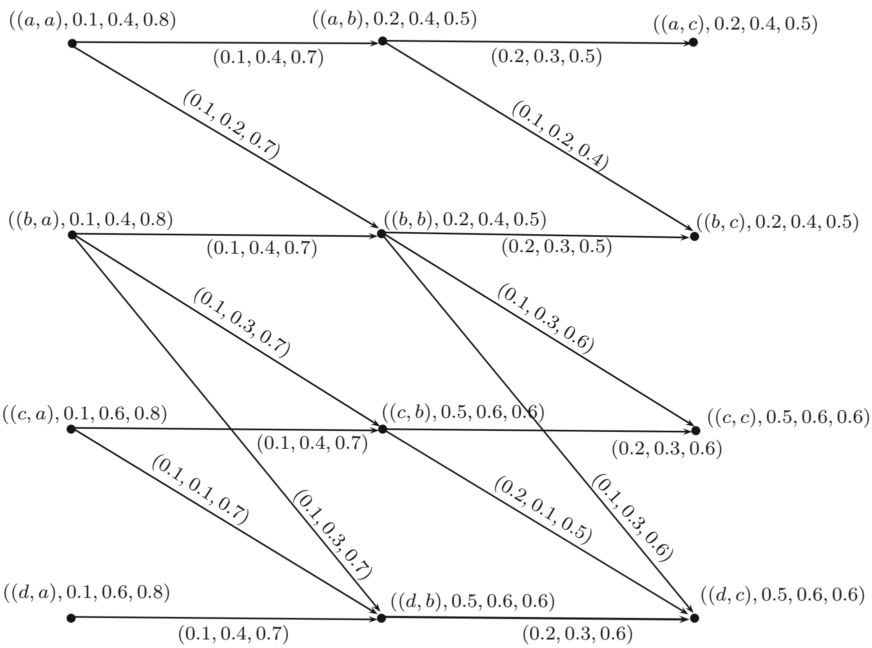


Fig. 6.5 $\overline{G}_1 \odot \overline{G}_2 = (\overline{R}A_1 \odot \overline{R}A_2, \overline{S}B_1 \odot \overline{S}B_2)$

$$I_{\underline{S}B_1 \boxtimes \underline{S}B_2}((x_1, x_2), (y_1, y_2)) = \min\{I_{\underline{S}B_1}(x_1, y_1), I_{\underline{S}B_2}(x_2, y_2)\},$$
$$F_{\underline{S}B_1 \boxtimes \underline{S}B_2}((x_1, x_2), (y_1, y_2)) = \max\{F_{\underline{S}B_1}(x_1, y_1), F_{\underline{S}B_2}(x_2, y_2)\}, \quad \forall\, (x_1, y_1) \in \underline{S}B_1, (x_2, y_2) \in \underline{S}B_2.$$

(2)
$$T_{\overline{R}A_1 \boxtimes \overline{R}A_2}(x, y) = \min\{T_{\overline{R}A_1}(x), T_{\overline{R}A_2}(y)\},$$
$$I_{\overline{R}A_1 \boxtimes \overline{R}A_2}(x, y) = \min\{I_{\overline{R}A_1}(x), I_{\overline{R}A_2}(y)\},$$
$$F_{\overline{R}A_1 \boxtimes \overline{R}A_2}(x, y) = \max\{F_{\overline{R}A_1}(x), F_{\overline{R}A_2}(y)\}, \quad \forall\, (x, y) \in \overline{R}A_1 \times \overline{R}A_2,$$
$$T_{\overline{S}B_1 \boxtimes \overline{S}B_2}((x, x_2), (x, y_2)) = \min\{T_{\overline{R}A_1}(x), T_{\overline{S}B_2}(x_2, y_2)\},$$
$$I_{\overline{S}B_1 \boxtimes \overline{S}B_2}((x, x_2), (x, y_2)) = \min\{I_{\overline{R}A_1}(x), I_{\overline{S}B_2}(x_2, y_2)\},$$
$$F_{\overline{S}B_1 \boxtimes \overline{S}B_2}((x, x_2), (x, y_2)) = \max\{F_{\overline{R}A_1}(x), F_{\overline{S}B_2}(x_2, y_2)\}, \quad \forall\, x \in \overline{R}A_1, (x_2, y_2) \in \overline{S}B_2,$$
$$T_{\overline{S}B_1 \boxtimes \overline{S}B_2}((x_1, y), (y_1, y)) = \min\{T_{\overline{S}B_1}(x_1, y_1), T_{\overline{R}A_2}(y)\},$$
$$I_{\overline{S}B_1 \boxtimes \overline{S}B_2}((x_1, y), (y_1, y)) = \min\{I_{\overline{S}B_1}(x_1, y_1), I_{\overline{R}A_2}(y)\},$$
$$F_{\overline{S}B_1 \boxtimes \overline{S}B_2}((x_1, y), (y_1, y)) = \max\{F_{\overline{S}B_1}(x_1, y_1), F_{\overline{R}A_2}(y)\}, \quad \forall\, (x_1, y_1) \in \overline{S}B_1, y \in \overline{R}A_2,$$
$$T_{\overline{S}B_1 \boxtimes \overline{S}B_2}((x_1, x_2), (y_1, y_2)) = \min\{T_{\overline{S}B_1}(x_1, y_1), T_{\overline{S}B_2}(x_2, y_2)\},$$
$$I_{\overline{S}B_1 \boxtimes \overline{S}B_2}((x_1, x_2), (y_1, y_2)) = \min\{I_{\overline{S}B_1}(x_1, y_1), I_{\overline{S}B_2}(x_2, y_2)\},$$
$$F_{\overline{S}B_1 \boxtimes \overline{S}B_2}((x_1, x_2), (y_1, y_2)) = \max\{F_{\overline{S}B_1}(x_1, y_1), F_{\overline{S}B_2}(x_2, y_2)\}, \quad \forall\, (x_1, y_1) \in \overline{S}B_1, (x_2, y_2) \in \overline{S}B_2.$$

Example 6.5 Consider the two rough neutrosophic digraphs G_1 and G_2 as shown in Figs. 6.2 and 6.3. The strong product of G_1 and G_2 is $G = G_1 \boxtimes G_2 = (\underline{G}_1 \boxtimes \underline{G}_2, \overline{G}_1 \boxtimes \overline{G}_2)$, where $\underline{G}_1 \boxtimes \underline{G}_2 = (\underline{R}A_1 \boxtimes \underline{R}A_2, \underline{S}B_1 \boxtimes \underline{S}B_2)$ and $\overline{G}_1 \boxtimes \overline{G}_2 = (\overline{R}A_1 \boxtimes \overline{R}A_2, \overline{S}B_1 \boxtimes \overline{S}B_2)$ are neutrosophic digraphs as shown in Figs. 6.6 and 6.7.

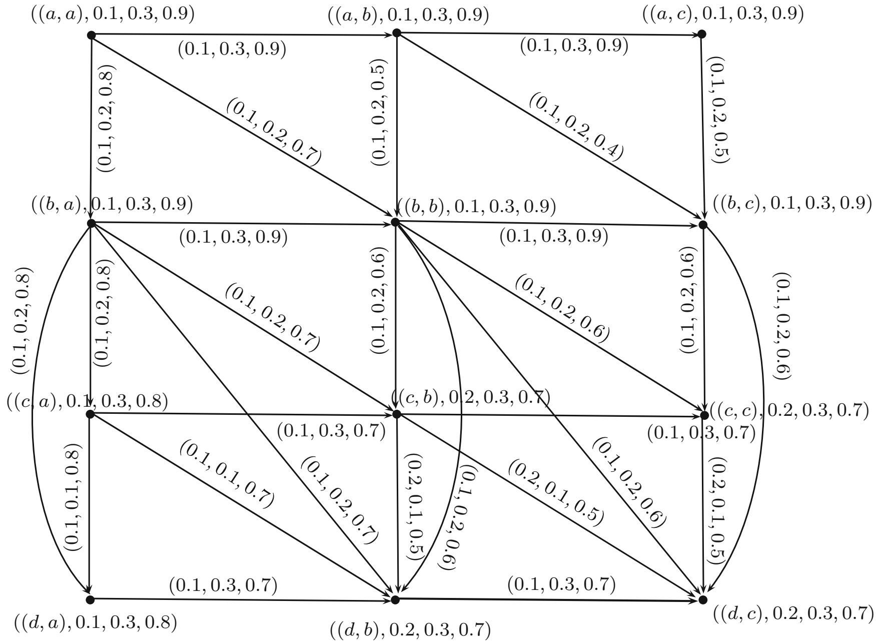


Fig. 6.6 Rough neutrosophic digraph $\underline{G}_1 \boxtimes \underline{G}_2$

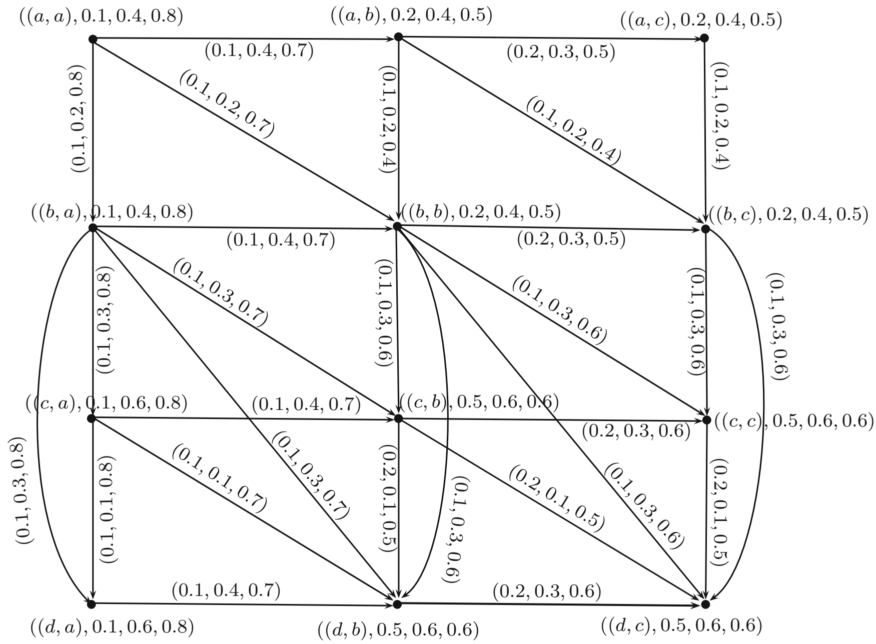


Fig. 6.7 Rough neutrosophic digraph $\overline{G}_1 \boxtimes \overline{G}_2$

Definition 6.7 Let $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ be two rough neutrosophic digraphs on a set X. Then the *rejection* of G_1 and G_2 is a rough neutrosophic digraph $G = G_1|G_2 = (\underline{G}_1|\underline{G}_2, \overline{G}_1|\overline{G}_2)$, where $\underline{G}_1|\underline{G}_2 = (\underline{R}A_1|\underline{R}A_2, \underline{S}B_1|\underline{S}B_2)$ and $\overline{G}_1|\overline{G}_2 = (\overline{R}A_1|\overline{R}A_2, \overline{S}B_1|\overline{S}B_2)$ are neutrosophic digraphs, respectively, such that

(1)
$$T_{\underline{R}A_1|\underline{R}A_2}(x_1, x_2) = \min\{T_{\underline{R}A_1}(x_1), T_{\underline{R}A_2}(x_2)\},$$
$$I_{\underline{R}A_1|\underline{R}A_2}(x_1, x_2) = \min\{I_{\underline{R}A_1}(x_1), T_{\underline{R}A_2}(x_2)\},$$
$$F_{\underline{R}A_1|\underline{R}A_2}(x_1, x_2) = \max\{F_{\underline{R}A_1}(x_1), T_{\underline{R}A_2}(x_2)\}, \forall\, (x_1, x_2) \in \underline{R}A_1 \ltimes \underline{R}A_2,$$
$$T_{\underline{S}B_1|\underline{S}B_2}((x, x_2), (x, y_2)) = \min\{T_{\underline{R}A_1}(x), T_{\underline{R}A_2}(x_2), T_{\underline{R}A_2}(y_2)\},$$
$$I_{\underline{S}B_1|\underline{S}B_2}((x, x_2), (x, y_2)) = \min\{I_{\underline{R}A_1}(x), I_{\underline{R}A_2}(x_2), I_{\underline{R}A_2}(y_2)\},$$
$$F_{\underline{S}B_1|\underline{S}B_2}((x, x_2), (x, y_2)) = \max\{F_{\underline{R}A_1}(x), F_{\underline{R}A_2}(x_2), F_{\underline{R}A_2}(y_2)\}, \forall\, x \in \underline{R}A_1, (x_2, y_2) \notin \underline{S}B_2,$$
$$T_{\underline{S}B_1|\underline{S}B_2}((x_1, z), (y_1, z)) = \min\{T_{\underline{R}A_1}(x_1), T_{\underline{R}A_1}(y_1), T_{\underline{R}A_2}(z)\},$$
$$I_{\underline{S}B_1|\underline{S}B_2}((x_1, z), (y_1, z)) = \min\{I_{\underline{R}A_1}(x_1), I_{\underline{R}A_1}(y_1), I_{\underline{R}A_2}(z)\},$$
$$F_{\underline{S}B_1|\underline{S}B_2}((x_1, z), (y_1, z)) = \max\{F_{\underline{R}A_1}(x_1), F_{\underline{R}A_1}(y_1), F_{\underline{R}A_2}(z)\}, \forall\, (x_1, y_1) \notin \underline{S}B_1, z \in \underline{R}A_2,$$
$$T_{\underline{S}B_1|\underline{S}B_2}((x_1, x_2), (y_1, y_2)) = \min\{T_{\underline{R}A_1}(x_1), T_{\underline{R}A_1}(y_1), T_{\underline{R}A_2}(x_2), T_{\underline{R}A_2}(y_2)\},$$
$$I_{\underline{S}B_1|\underline{S}B_2}((x_1, x_2), (y_1, y_2)) = \min\{I_{\underline{R}A_1}(x_1), I_{\underline{R}A_1}(y_1), I_{\underline{R}A_2}(x_2), I_{\underline{R}A_2}(y_2)\},$$
$$F_{\underline{S}B_1|\underline{S}B_2}((x_1, x_2), (y_1, y_2)) = \max\{F_{\underline{R}A_1}(x_1), F_{\underline{R}A_1}(y_1), F_{\underline{R}A_2}(x_2), F_{\underline{R}A_2}(y_2)\}, \forall\, (x_1, y_1) \notin \underline{S}B_1, (x_2, y_2) \notin \underline{S}B_2.$$

(2)
$$T_{\overline{R}A_1|\overline{R}A_2}(x_1, x_2) = \min\{T_{\overline{R}A_1}(x_1), T_{\overline{R}A_2}(x_2)\},$$
$$I_{\overline{R}A_1|\overline{R}A_2}(x_1, x_2) = \min\{I_{\overline{R}A_1}(x_1), T_{\overline{R}A_2}(x_2)\},$$
$$F_{\overline{R}A_1|\overline{R}A_2}(x_1, x_2) = \max\{F_{\overline{R}A_1}(x_1), T_{\overline{R}A_2}(x_2)\}, \forall\, (x_1, x_2) \in \overline{R}A_1 \ltimes \overline{R}A_2,$$
$$T_{\overline{S}B_1|\overline{S}B_2}((x, x_2), (x, y_2)) = \min\{T_{\overline{R}A_1}(x), T_{\overline{R}A_2}(x_2), T_{\overline{R}A_2}(y_2)\},$$
$$I_{\overline{S}B_1|\overline{S}B_2}((x, x_2), (x, y_2)) = \min\{I_{\overline{R}A_1}(x), I_{\overline{R}A_2}(x_2), I_{\overline{R}A_2}(y_2)\},$$
$$F_{\overline{S}B_1|\overline{S}B_2}((x, x_2), (x, y_2)) = \max\{F_{\overline{R}A_1}(x), F_{\overline{R}A_2}(x_2), F_{\overline{R}A_2}(y_2)\}, \forall\, x \in \overline{R}A_1, (x_2, y_2) \notin \overline{S}B_2,$$
$$T_{\overline{S}B_1|\overline{S}B_2}((x_1, z), (y_1, z)) = \min\{T_{\overline{R}A_1}(x_1), T_{\overline{R}A_1}(y_1), T_{\overline{R}A_2}(z)\},$$
$$I_{\overline{S}B_1|\overline{S}B_2}((x_1, z), (y_1, z)) = \min\{I_{\overline{R}A_1}(x_1), I_{\overline{R}A_1}(y_1), I_{\overline{R}A_2}(z)\},$$
$$F_{\overline{S}B_1|\overline{S}B_2}((x_1, z), (y_1, z)) = \max\{F_{\overline{R}A_1}(x_1), F_{\overline{R}A_1}(y_1), F_{\overline{R}A_2}(z)\}, \forall (x_1, y_1) \notin \overline{S}B_1, z \in \overline{R}A_2,$$
$$T_{\overline{S}B_1|\overline{S}B_2}((x_1, x_2), (y_1, y_2)) = \min\{T_{\overline{R}A_1}(x_1), T_{\overline{R}A_1}(y_1), T_{\overline{R}A_2}(x_2), T_{\overline{R}A_2}(y_2)\},$$
$$I_{\overline{S}B_1|\overline{S}B_2}((x_1, x_2), (y_1, y_2)) = \min\{I_{\overline{R}A_1}(x_1), I_{\overline{R}A_1}(y_1), I_{\overline{R}A_2}(x_2), I_{\overline{R}A_2}(y_2)\},$$
$$F_{\overline{S}B_1|\overline{S}B_2}((x_1, x_2), (y_1, y_2)) = \max\{F_{\overline{R}A_1}(x_1), F_{\overline{R}A_1}(y_1), F_{\overline{R}A_2}(x_2), F_{\overline{R}A_2}(y_2)\}, \forall\, (x_1, y_1) \notin \overline{S}B_1, (x_2, y_2) \notin \overline{S}B_2.$$

Example 6.6 Consider the two rough neutrosophic digraphs G_1 and G_2 as shown in Figs. 6.8 and 6.9. The rejection of G_1 and G_2 is $G = G_1|G_2 = (\underline{G}_1|\underline{G}_2, \overline{G}_1|\overline{G}_2)$, where $\underline{G}_1|\underline{G}_2 = (\underline{R}A_1|\underline{R}A_2, \underline{S}B_1|\underline{S}B_2)$ and $\overline{G}_1|\overline{G}_2 = (\overline{R}A_1|\overline{R}A_2, \overline{S}B_1|\overline{S}B_2)$ are neutrosophic digraphs as shown in Figs. 6.10 and 6.11.

Definition 6.8 The *tensor product* of two rough neutrosophic digraphs G_1 and G_2 is a rough neutrosophic digraph $G = (\underline{G}_1 \star \underline{G}_2, \overline{G}_1 \star \overline{G}_2)$, where $\underline{G}_1 \star \underline{G}_2 = (\underline{R}A_1 \star \underline{R}A_2, \underline{S}B_1 \star \underline{S}B_2)$ and $\overline{G}_1 \star \overline{G}_2 = (\overline{R}A_1 \star \overline{R}A_2, \overline{S}B_1 \star \overline{S}B_2)$ are neutrosophic digraphs, respectively, such that

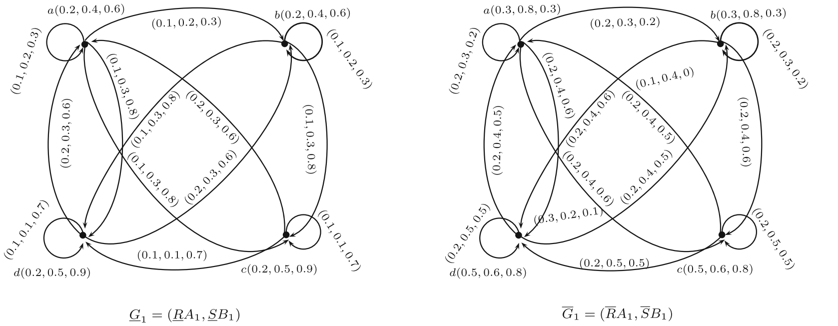


Fig. 6.8 Rough neutrosophic digraph $G_1 = (\underline{G}_1, \overline{G}_1)$

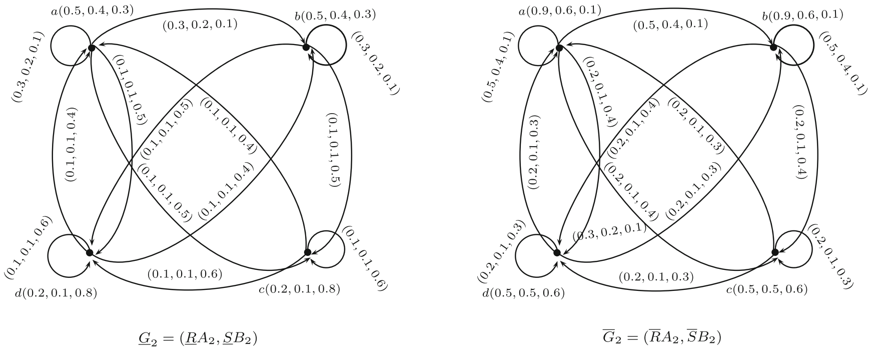


Fig. 6.9 Rough neutrosophic digraph $G_2 = (\underline{G}_2, \overline{G}_2)$

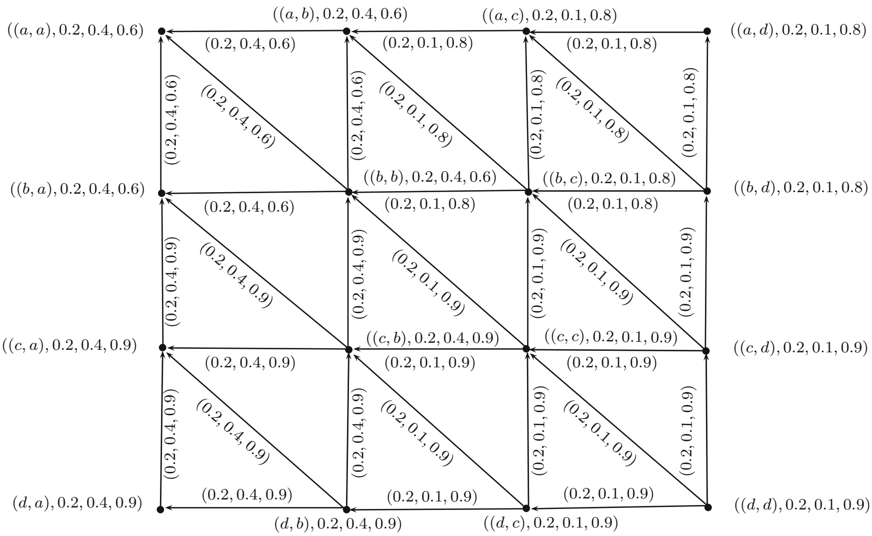


Fig. 6.10 $\underline{G}_1|\underline{G}_2 = (\underline{R}A_1|\underline{R}A_2, \underline{S}B_1|\underline{S}B_2)$

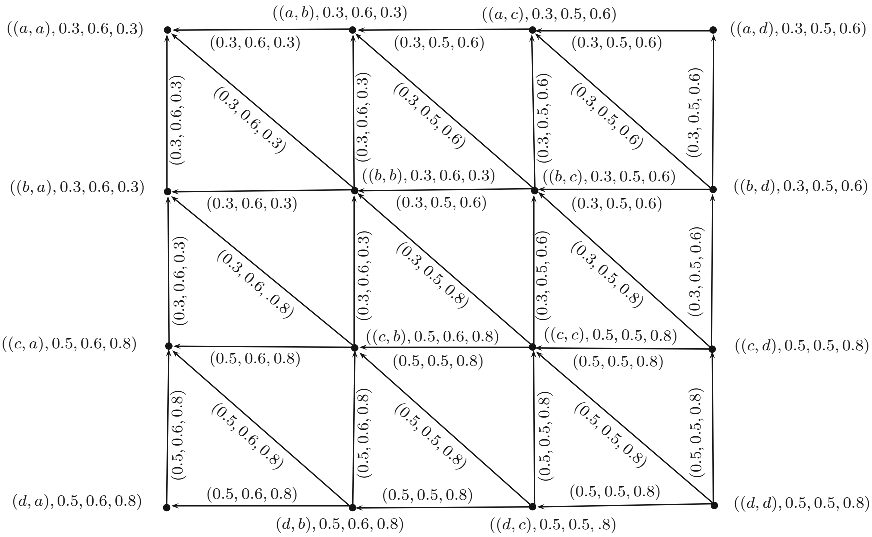


Fig. 6.11 $\overline{G}_1|\overline{G}_2 = (\overline{R}A_1|\overline{R}A_2, \overline{S}B_1|\overline{S}B_2)$

(1)
$$T_{\underline{R}A_1\star\underline{R}A_2}(x,y) = \min\{T_{\underline{R}A_1}(x), T_{\underline{R}A_2}(y)\},$$
$$I_{\underline{R}A_1\star\underline{R}A_2}(x,y) = \min\{I_{\underline{R}A_1}(x), I_{\underline{R}A_2}(y)\},$$
$$F_{\underline{R}A_1\star\underline{R}A_2}(x,y) = \max\{F_{\underline{R}A_1(x)}, F_{\underline{R}A_2}(y)\}, \quad \forall\,(x,y)\in \underline{R}A_1\times\underline{R}A_2,$$

$$T_{\underline{R}A_1\star\underline{R}A_2}((x_1,x_2),(y_1,y_2)) = \min\{T_{\underline{S}B_1}(x_1,y_1), T_{\underline{S}B_2}(x_2,y_2)\},$$
$$I_{\underline{R}A_1\star\underline{R}A_2}((x_1,x_2),(y_1,y_2)) = \min\{I_{\underline{S}B_1}(x_1,y_1), I_{\underline{S}B_2}(x_2,y_2)\},$$
$$F_{\underline{R}A_1\star\underline{R}A_2}((x_1,x_2),(y_1,y_2)) = \max\{F_{\underline{S}B_1}(x_1,y_1), F_{\underline{S}B_2}(x_2,y_2)\}, \quad \forall\,(x_1,y_1)\in\underline{S}B_1,\,(x_2,y_2)\in\underline{S}B_2.$$

(2)
$$T_{\overline{R}A_1\star\overline{R}A_2}(x,y) = \min\{T_{\overline{R}A_1}(x), T_{\overline{R}A_2}(y)\},$$
$$I_{\overline{R}A_1\star\overline{R}A_2}(x,y) = \min\{I_{\overline{R}A_1}(x), I_{\overline{R}A_2}(y)\},$$
$$F_{\overline{R}A_1\star\overline{R}A_2}(x,y) = \max\{F_{\overline{R}A_1(x)}, F_{\overline{R}A_2}(y)\}, \quad \forall\,(x,y)\in \overline{R}A_1\times\overline{R}A_2,$$
$$T_{\overline{R}A_1\star\overline{R}A_2}((x_1,x_2),(y_1,y_2)) = \min\{T_{\overline{S}B_1}(x_1,y_1), T_{\overline{S}B_2}(x_2,y_2)\},$$
$$I_{\overline{R}A_1\star\overline{R}A_2}((x_1,x_2),(y_1,y_2)) = \min\{I_{\overline{S}B_1}(x_1,y_1), I_{\overline{S}B_2}(x_2,y_2)\},$$
$$F_{\overline{R}A_1\star\overline{R}A_2}((x_1,x_2),(y_1,y_2)) = \max\{F_{\overline{S}B_1}(x_1,y_1), F_{\overline{S}B_2}(x_2,y_2)\}, \quad \forall\,(x_1,y_1)\in\overline{S}B_1,\,(x_2,y_2)\in\overline{S}B_2.$$

Example 6.7 Let $X_1 = \{a, b, c\}$ and $X_2 = \{w, x, y, z\}$ be two crisp sets. Let $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ be two rough neutrosophic digraphs on X_1 and X_2, respectively, where $\underline{G}_1 = (\underline{R}A_1, \underline{S}B_1)$ and $\overline{G}_1 = (\overline{R}A_1, \overline{S}B_1)$ are neutrosophic digraphs as shown in Fig. 6.12.

$\underline{G}_2 = (\underline{R}A_2, \underline{S}B_2)$ and $\overline{G}_2 = (\overline{R}A_2, \overline{S}B_2)$ are also neutrosophic digraphs as shown in Fig. 6.13.

The tensor product of G_1 and G_2 is $G = G_1 \star G_2 = (\underline{G}_1 \star \underline{G}_2, \overline{G}_1 \star \overline{G}_2)$, where $\underline{G}_1 \star \underline{G}_2 = (\underline{R}A_1 \star \underline{R}A_2, \underline{S}B_1 \star \underline{S}B_2)$ and $\overline{G}_1 \star \overline{G}_2 = (\overline{R}A_1 \star \overline{R}A_2, \overline{S}B_1 \star \overline{S}B_2)$ are neutrosophic digraphs as shown in Figs. 6.14 and 6.15, respectively.

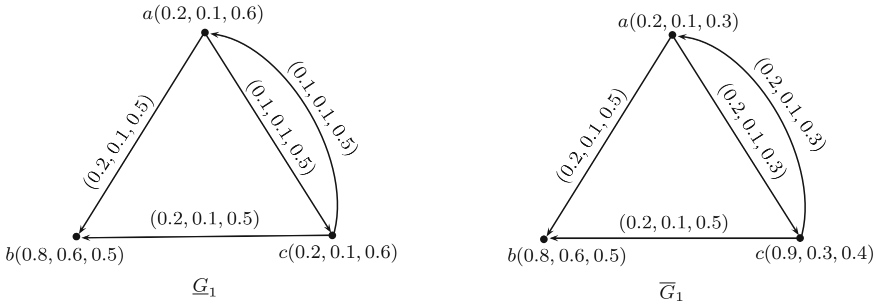


Fig. 6.12 Rough neutrosophic digraph $G_1 = (\underline{G}_1, \overline{G}_1)$

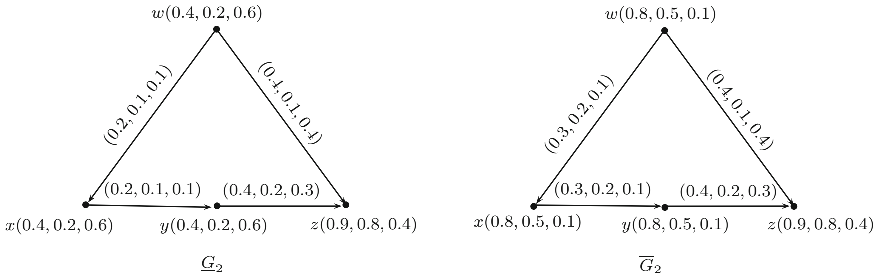


Fig. 6.13 Rough neutrosophic digraph $G_2 = (\underline{G}_2, \overline{G}_2)$

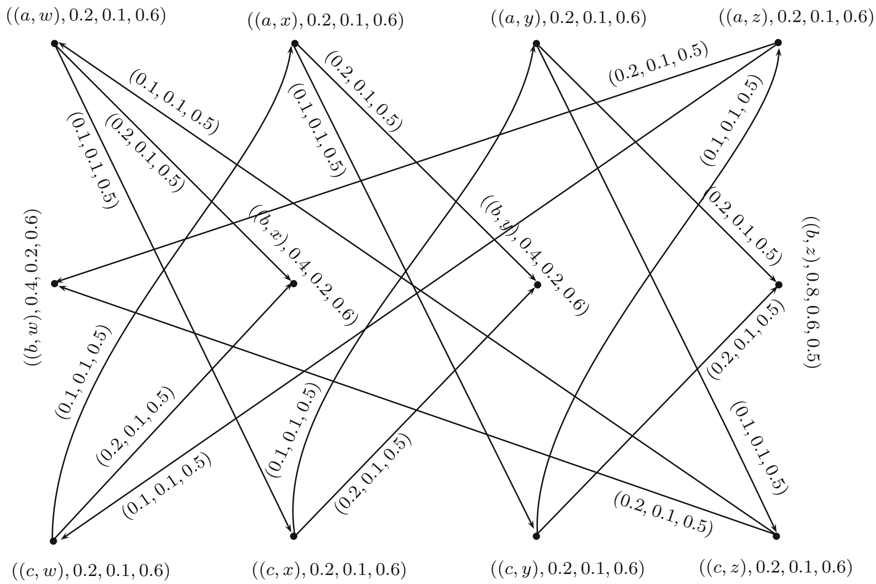


Fig. 6.14 $\underline{G}_1 \star \underline{G}_2 = (\underline{R}A_1 \star \underline{R}A_2, \underline{S}B_1 \star \underline{S}B_2)$

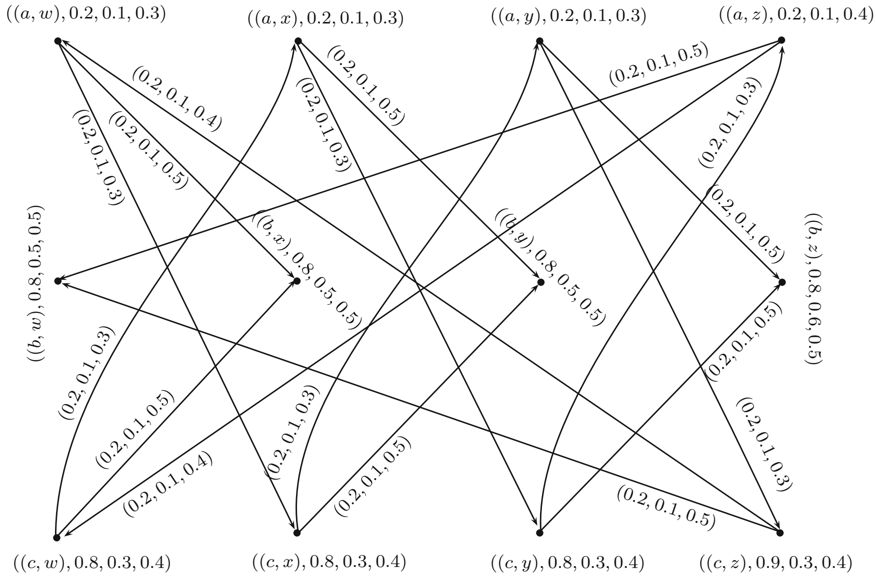


Fig. 6.15 $\overline{G}_1 \star \overline{G}_2 = (\overline{R}A_1 \star \overline{R}A_2, \overline{S}B_1 \star \overline{S}B_2)$

Definition 6.9 A rough neutrosophic digraph $G = (\underline{G}, \overline{G})$ is *self-complementary* if G and G' are isomorphic, that is, $\underline{G} \cong \underline{G'}$ and $\overline{G} \cong \overline{G}'$.

Example 6.8 Let $X = \{a, b, c\}$ be a set and R an equivalence relation on X defined as:

$$R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Let $A = \{(a, 0.2, 0.4, 0.8), (b, 0.2, 0.4, 0.8), (c, 0.4, 0.6, 0.4)\}$ be a neutrosophic set on X. The lower and upper approximations of A are given as
$\underline{R}A = \{(a, 0.2, 0.4, 0.8), (b, 0.2, 0.4, 0.8), (c, 0.2, 0.4, 0.8)\}$,
$\overline{R}A = \{(a, 0.4, 0.6, 0.4), (b, 0.2, 0.4, 0.8), (c, 0.4, 0.6, 0.4)\}$.

Let $E = \{aa, ab, ac, ba\} \subseteq X \times X$ and S be an equivalence relation on E defined as

$$S = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $B = \{(aa, 0.1, 0.3, 0.2), (ab, 0.1, 0.2, 0.4), (ac, 0.2, 0.2, 0.4), (ba, 0.1, 0.2, 0.4)\}$ be a neutrosophic set on E and $SB = (\underline{S}B, \overline{S}B)$ a rough neutrosophic relation, where $\underline{S}B$ and $\overline{S}B$ are given as
$\underline{S}B = \{(aa, 0.1, 0.2, 0.4), (ab, 0.1, 0.2, 0.4), (ac, 0.1, 0.2, 0.4), (ba, 0.1, 0.2, 0.4)\}$,
$\overline{S}B = \{(aa, 0.2, 0.3, 0.2), (ab, 0.1, 0.2, 0.4), (ac, 0.2, 0.3, 0.2), (ba, 0.1, 0.2, 0.4)\}$.

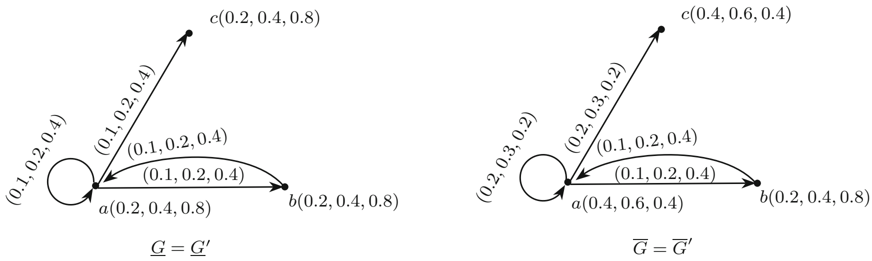


Fig. 6.16 Self-complementary rough neutrosophic digraph $G = (\underline{G}, \overline{G})$

Thus, $\underline{G} = (\underline{R}A, \underline{S}B)$ and $\overline{G} = (\overline{R}A, \overline{S}B)$ are neutrosophic digraphs as shown in Fig. 6.16. The complement of G is $G' = (\underline{G}', \overline{G}')$, where $\underline{G}' = \underline{G}$ and $\overline{G}' = \overline{G}$ are neutrosophic digraphs as shown in Fig. 6.16, and it can be easily shown that G and G' are isomorphic. Hence, $G = (\underline{G}, \overline{G})$ is a self-complementary rough neutrosophic digraph.

Theorem 6.1 *Let $G = (\underline{G}, \overline{G})$ be a self-complementary rough neutrosophic digraph. Then*

$$\sum_{w,z\in X} T_{\underline{S}B}(wz) = \frac{1}{2}\sum_{w,z\in X} (T_{\underline{R}A}(w) \wedge T_{\underline{R}A}(z)),$$
$$\sum_{w,z\in X} I_{\underline{S}B}(wz) = \frac{1}{2}\sum_{w,z\in X} (I_{\underline{R}A}(w) \wedge I_{\underline{R}A}(z)),$$
$$\sum_{w,z\in X} F_{\underline{S}B}(wz) = \frac{1}{2}\sum_{w,z\in X} (F_{\underline{R}A}(w) \vee F_{\underline{R}A}(z)),$$
$$\sum_{w,z\in X} T_{\overline{S}B}(wz) = \frac{1}{2}\sum_{w,z\in X} (T_{\overline{R}A}(w) \wedge T_{\overline{R}A}(z)),$$
$$\sum_{w,z\in X} I_{\overline{S}B}(wz) = \frac{1}{2}\sum_{w,z\in X} (I_{\overline{R}A}(w) \wedge I_{\overline{R}A}(z)),$$
$$\sum_{w,z\in X} F_{\overline{S}B}(wz) = \frac{1}{2}\sum_{w,z\in X} (F_{\overline{R}A}(w) \vee F_{\overline{R}A}(z)).$$

Proof Let $G = (\underline{G}, \overline{G})$ be a self-complementary rough neutrosophic digraph. Then there exist two isomorphisms $\underline{g} : X \longrightarrow X$ and $\overline{g} : X \longrightarrow X$, respectively, such that

$$\begin{aligned}
T_{(\underline{R}A)'}(\underline{g}(w)) &= T_{\underline{R}A}(w),\\
I_{(\underline{R}A)'}(\underline{g}(w)) &= I_{\underline{R}A}(w),\\
F_{(\underline{R}A)'}(\underline{g}(w)) &= F_{\underline{R}A}(w), \quad \forall\, w \in X,\\
T_{(\underline{S}B)'}(\underline{g}(w)\underline{g}(z)) &= T_{\underline{S}B}(wz),
\end{aligned}$$

$$I_{(\underline{S}B)'}(\underline{g}(w)\underline{g}(z)) = I_{\underline{S}B}(wz),$$
$$F_{(\underline{S}B)'}(\underline{g}(w)\underline{g}(z)) = F_{\underline{S}B}(wz), \quad \forall\, w, z \in X.$$

and

$$\begin{aligned}
T_{(\overline{R}A)'}(\overline{g}(w)) &= T_{\overline{R}A}(w),\\
I_{(\overline{R}A)'}(\overline{g}(w)) &= I_{\overline{R}A}(w),\\
F_{(\overline{R}A)'}(\overline{g}(w)) &= F_{\overline{R}A}(w), \quad \forall\, w \in X,\\
T_{(\overline{S}B)'}(\overline{g}(w)\overline{g}(z)) &= T_{\overline{S}B}(wz),\\
I_{(\overline{S}B)'}(\overline{g}(w)\overline{g}(z)) &= I_{\overline{S}B}(wz),\\
F_{(\overline{S}B)'}(\overline{g}(w)\overline{g}(z)) &= F_{\overline{S}B}(wz), \quad \forall\, w, z \in X.
\end{aligned}$$

By Definition of compliment, we have

$$\begin{aligned}
T_{(\underline{S}B)'}(\underline{g}(w)\underline{g}(z)) &= (T_{\underline{R}A}(w) \wedge T_{\underline{R}A}(z)) - T_{\underline{S}B}(wz)\\
T_{\underline{S}B}(wz) &= (T_{\underline{R}A}(w) \wedge T_{\underline{R}A}(z)) - T_{\underline{S}B}(wz)\\
\sum_{w,z\in X} T_{\underline{S}B}(wz) &= \sum_{w,z\in X} (T_{\underline{R}A}(w) \wedge T_{\underline{R}A}(z)) - \sum_{w,z\in X} T_{\underline{S}B}(wz)\\
2\sum_{w,z\in X} T_{\underline{S}B}(wz) &= \sum_{w,z\in X} (T_{\underline{R}A}(w) \wedge T_{\underline{R}A}(z))\\
\sum_{w,z\in X} T_{\underline{S}B}(wz) &= \frac{1}{2}\sum_{w,z\in X} (T_{\underline{R}A}(w) \wedge T_{\underline{R}A}(z))
\end{aligned}$$

$$\begin{aligned}
I_{(\underline{S}B)'}(\underline{g}(w)\underline{g}(z)) &= (I_{\underline{R}A}(w) \wedge I_{\underline{R}A}(z)) - I_{\underline{S}B}(wz)\\
I_{\underline{S}B}(wz) &= (I_{\underline{R}A}(w) \wedge I_{\underline{R}A}(z)) - I_{\underline{S}B}(wz)\\
\sum_{w,z\in X} I_{\underline{S}B}(wz) &= \sum_{w,z\in X} (I_{\underline{R}A}(w) \wedge I_{\underline{R}A}(z)) - \sum_{w,z\in X} I_{\underline{S}B}(wz)\\
2\sum_{w,z\in X} I_{\underline{S}B}(wz) &= \sum_{w,z\in X} (I_{\underline{R}A}(w) \wedge I_{\underline{R}A}(z))\\
\sum_{w,z\in X} I_{\underline{S}B}(wz) &= \frac{1}{2}\sum_{w,z\in X} (I_{\underline{R}A}(w) \wedge I_{\underline{R}A}(z))
\end{aligned}$$

$$\begin{aligned}
F_{(\underline{S}B)'}(\underline{g}(w)\underline{g}(z)) &= (F_{\underline{R}A}(w) \vee F_{\underline{R}A}(z)) - F_{\underline{S}B}(wz)\\
F_{\underline{S}B}(wz) &= (F_{\underline{R}A}(w) \vee F_{\underline{R}A}(z)) - F_{\underline{S}B}(wz)\\
\sum_{w,z\in X} F_{\underline{S}B}(wz) &= \sum_{w,z\in X} (F_{\underline{R}A}(w) \vee F_{\underline{R}A}(z)) - \sum_{w,z\in X} F_{\underline{S}B}(wz)\\
2\sum_{w,z\in X} F_{\underline{S}B}(wz) &= \sum_{w,z\in X} (F_{\underline{R}A}(w) \vee F_{\underline{R}A}(z))
\end{aligned}$$

$$\sum_{w,z\in X} F_{\underline{S}B}(wz) = \frac{1}{2}\sum_{w,z\in X}(F_{\underline{R}A}(w) \vee F_{\underline{R}A}(z))$$

Similarly, it can be shown that

$$\sum_{w,z\in X} T_{\overline{S}B}(wz) = \frac{1}{2}\sum_{w,z\in X}(T_{\overline{R}A}(w) \wedge T_{\overline{R}A}(z))$$
$$\sum_{w,z\in X} I_{\overline{S}B}(wz) = \frac{1}{2}\sum_{w,z\in X}(I_{\overline{R}A}(w) \wedge I_{\overline{R}A}(z))$$
$$\sum_{w,z\in X} F_{\overline{S}B}(wz) = \frac{1}{2}\sum_{w,z\in X}(F_{\overline{R}A}(w) \vee F_{\overline{R}A}(z)).$$

This completes the proof.

6.3 Applications of Rough Neutrosophic Digraphs

6.3.1 Optimal Flight Path for Weather Emergency Landing

In this application, we use the concept of rough neutrosophic digraph for decision-making in real-life problems. To obtain the optimal decision, we use the following formula:

$$S_{ij} = (T_{S_{ij}}, I_{S_{ij}}, F_{S_{ij}}),$$

where

$$\text{(i)} \begin{cases} T_{S_{ij}} = T_{\underline{S}B} \oplus T_{\overline{S}B}(v_i, v_j) = \frac{T_{\underline{R}A}(v_i)*T_{\overline{R}A}(v_j)}{3-\left(T_{\underline{S}B}(v_i,v_j)+T_{\overline{S}B}(v_i,v_j)-T_{\underline{S}B}(v_i,v_j)*T_{\overline{S}B}(v_i,v_j)\right)}, \\ I_{S_{ij}} = I_{\underline{S}B} \oplus I_{\overline{S}B}(v_i, v_j) = \frac{I_{\underline{R}A}(v_i)*I_{\overline{R}A}(v_j)}{3-\left(I_{\underline{S}B}(v_i,v_j)+I_{\overline{S}B}(v_i,v_j)-I_{\underline{S}B}(v_i,v_j)*I_{\overline{S}B}(v_i,v_j)\right)}, \\ F_{S_{ij}} = F_{\underline{S}B} \oplus F_{\overline{S}B}(v_i, v_j) = \frac{F_{\underline{R}A}(v_i)*F_{\overline{R}A}(v_j)}{3-\left(F_{\underline{S}B}(v_i,v_j)+F_{\overline{S}B}(v_i,v_j)-F_{\underline{S}B}(v_i,v_j)*F_{\overline{S}B}(v_i,v_j)\right)}. \end{cases}$$

Flight planning is the process of producing a flight plan to describe a proposed aeroplane flight. Flight plan generally includes basic information such as departure and arrival points, estimated time en route, alternate airports in case of bad weather. The presented application provides alternate airports for a plane in case of bad weather.

Suppose $X = \{$Chicago(CHI), Beijing(BJ), Lahore(LHR), Paris(PAR), Istanbul (IST)$\}$ be the set of cities under consideration and R an equivalence relation on X, where equivalence classes represent cities having same characteristics.

$$R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Assume that a flight *Boeing* 747 of Pakistan International Airways (PIA) travels to these cities. In case of bad weather, the flight will be directed to the city with good weather condition among the cities under consideration.

Let A={(CHI, 0.1, 0.2, 0.8), (BJ, 0.9, 0.7, 0.5), (LHR, 0.8, 0.4, 0.3), (PAR, 0.6, 0.5, 0.4), (IST, 0.2, 0.4, 0.6)}

be a neutrosophic set on X which describe the characteristic of each city, and $RA = (\underline{R}A, \overline{R}A)$ a rough neutrosophic set, where $\underline{R}A$ and $\overline{R}A$ are lower and upper approximations of A, respectively, as follows:

$$\underline{R}A = \{(CHI, 0.1, 0.2, 0.8), (BJ, 0.2, 0.4, 0.6), (LHR, 0.2, 0.4, 0.6), (PAR, 0.1, 0.2, 0.8), (IST, 0.2, 0.4, 0.6)\}$$
$$\overline{R}A = \{(CHI, 0.6, 0.5, 0.4), (BJ, 0.9, 0.7, 0.3).(LHR, 0.9, 0.7, 0.3), (PAR, 0.6, 0.5, 0.4), (IST, o.9, 0.7, 0.3)\}.$$

Let $\text{E} = \{(BJ, CHI), (LHR, CHI), (BJ, LHR), (IST, BJ), (PAR, BJ), (PAR, LHR)\}$

be a subset of $X \times X$ and S an equivalence relation on E defined as:

$$S = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where S represents the equivalence classes of "weather between different cities". For example the relationships (BJ,CHI), (IST,BJ) and (PAR,BJ) belong to the same equivalence class. This means that weather between Beijing and Chicago is the same as the weather between Paris and Beijing.

Let B = {((BJ, CHI), 0.1, 0.1, 0.3), ((LHR, CHI), 0.1, 0.2, 0.3), ((BJ, LHR), 0.1, 0.3, 0.2), $((IST, BJ), 0.2, 0.1, 0.1), ((PAR, BJ), 0.1, 0.1, 0.4), ((PAR, LHR), 0.2, 0.2, 0.3))\}$

be a neutrosophic set on E which describes the comparison of weathers of the cities under consideration. Let $SB = (\underline{S}B, \overline{S}B)$ be a rough neutrosophic set, where $\underline{S}B$ and $\overline{S}B$ are lower and upper approximations of B, respectively, as follows:

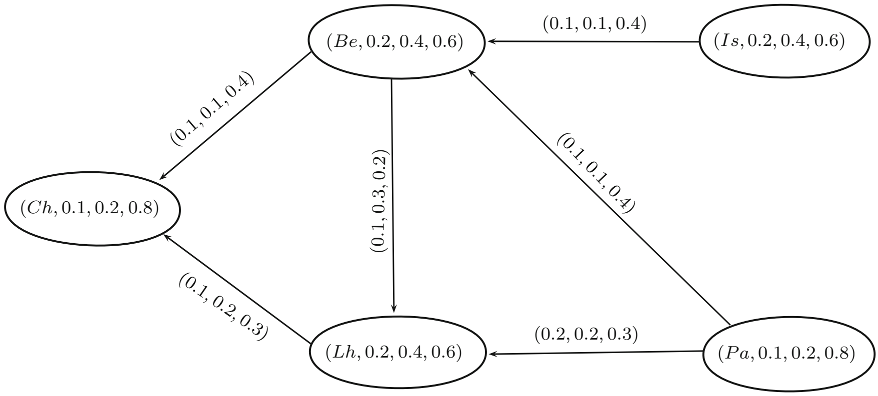


Fig. 6.17 $\underline{G} = (\underline{R}A, \underline{S}B)$

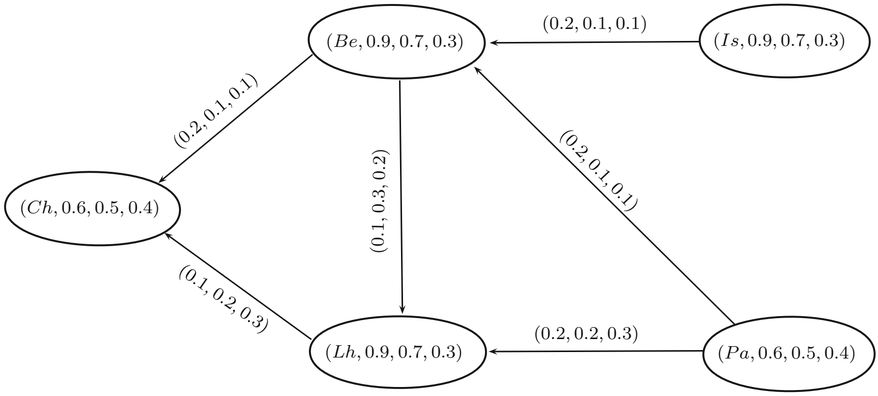


Fig. 6.18 $\overline{G} = (\overline{R}A, \overline{S}B)$

$$\underline{S}B = \{((BJ, CHI), 0.1, 0.1, 0.4), ((LHR, CHI), 0.1, 0.2, 0.3), ((BJ, LHR)0.1, 0.3, 0.2),\\ ((IST, BJ), 0.1, 0.1, 0.4), ((PAR, BJ), 0.1, 0.1, 0.4), ((PAR, LHR), 0.2, 0.2, 0.3))\},$$
$$\overline{S}B = \{((BJ, CHI), 0.2, 0.1, 0.1), ((LHR, CHI), 0.1, 0.2, 0.3), ((BJ, LHR)0.1, 0.3, 0.2),\\ ((IST, BJ), 0.2, 0.1, 0.1), ((PJ, BJ), 0.2, 0.1, 0.1), ((PAR, LHR), 0.2, 0.2, 0.3))\}.$$

Thus, $\underline{G} = (\underline{R}A, \underline{S}B)$ and $\overline{G} = (\overline{R}A, \overline{S}B)$ are neutrosophic digraphs as shown in Figs. 6.17 and 6.18.

To find the city with good weather condition, we use the formula which we mentioned in equation (i).

Our decision is e_k if $e_k = \max_i (T_{\underline{S}B} \oplus T_{\overline{S}B})(e_i)$, where $e_i = (v_i, v_j)$. By direct calculations, we have
$T_{\underline{S}B} \oplus T_{\overline{S}B}(BJ, CHI) = 0.044$, $I_{\underline{S}B} \oplus I_{\overline{S}B}(BJ, CHI) = 0.071$,
$F_{\underline{S}B} \oplus F_{\overline{S}B}(BJ, CHI) = 0.094$.

$T_{\underline{S}B} \oplus T_{\overline{S}B}(LHR, CHI) = 0.043$, $I_{\underline{S}B} \oplus I_{\overline{S}B}(LHR, CHI) = 0.076$,
$F_{\underline{S}B} \oplus F_{\overline{S}B}(LHR, CHI) = 0.096$.
$T_{\underline{S}B} \oplus T_{\overline{S}B}(BJ, LHR) = 0.064$, $I_{\underline{S}B} \oplus I_{\overline{S}B}(BJ, LHR) = 0.112$,
$F_{\underline{S}B} \oplus F_{\overline{S}B}(BJ, LHR) = 0.068$.
$T_{\underline{S}B} \oplus T_{\overline{S}B}(IST, BJ) = 0.066$, $I_{\underline{S}B} \oplus I_{\overline{S}B}(IST, BJ) = 0.100$,
$F_{\underline{S}B} \oplus F_{\overline{S}B}(IST, BJ) = 0.070$.
$T_{\underline{S}B} \oplus T_{\overline{S}B}(PAR, BJ) = 0.033$, $I_{\underline{S}B} \oplus I_{\overline{S}B}(PAR, BJ) = 0.050$,
$F_{\underline{S}B} \oplus F_{\overline{S}B}(PAR, BJ) = 0.094$.
$T_{\underline{S}B} \oplus T_{\overline{S}B}(PAR, LHR) = 0.034$, $I_{\underline{S}B} \oplus I_{\overline{S}B}(PAR, LHR) = 0.155$,
$F_{\underline{S}B} \oplus F_{\overline{S}B}(PAR, LHR) = 0.096$.
Hence the weather condition between Istanbul and Beijing is good, and *Boeing* 747 can use this path in case of weather emergency.

We present an algorithm for the above-mentioned application. The presented algorithm can be applied to avoid lengthy calculations when dealing with a large number of objects.

Algorithm 6.3.1 1. Input the vertex set X.
2. Construct an equivalence relation R on the set X.
3. Calculate the approximation sets $\underline{R}A$ and $\overline{R}A$.
4. Input the edge set $E \subseteq X \times X$.
5. Construct an equivalence relation S on E.
6. Calculate the approximation sets $\underline{S}B$ and $\overline{S}B$.
7. Calculate the score value, by using the formula

$$T_{\underline{S}B} \oplus T_{\overline{S}B}(v_i, v_j) = \frac{T_{\underline{R}A}(v_i) * T_{\overline{R}A}(v_j)}{3 - \big(T_{\underline{S}B}(v_i, v_j) + T_{\overline{S}B}(v_i, v_j) - T_{\underline{S}B}(v_i, v_j) * T_{\overline{S}B}(v_i, v_j)\big)},$$
$$I_{\underline{S}B} \oplus I_{\overline{S}B}(v_i, v_j) = \frac{I_{\underline{R}A}(v_i) * I_{\overline{R}A}(v_j)}{3 - \big(I_{\underline{S}B}(v_i, v_j) + I_{\overline{S}B}(v_i, v_j) - I_{\underline{S}B}(v_i, v_j) * I_{\overline{S}B}(v_i, v_j)\big)},$$
$$F_{\underline{S}B} \oplus F_{\overline{S}B}(v_i, v_j) = \frac{F_{\underline{R}A}(v_i) * F_{\overline{R}A}(v_j)}{3 - \big(F_{\underline{S}B}(v_i, v_j) + F_{\overline{S}B}(v_i, v_j) - F_{\underline{S}B}(v_i, v_j) * F_{\overline{S}B}(v_i, v_j)\big)}.$$

8. Decision is e_k if $e_k = \max_i (T_{\underline{S}B} \oplus T_{\overline{S}B})(e_i)$, where $e_i = (v_i, v_j)$.
9. If e_k has more than one value, then any one of $S(v_k)$ may be chosen.

6.3.2 Suitable Investment Company

Investment is a very good way of getting profit, and wisely invested money surely gives certain profit. The most important factors that influence individual investment decision are: company's reputation, corporate earnings and prices per share. In this application, we combine these factors into one factor: company's status in industry, to describe overall performance of the company. Let us consider an individual Mr. Shahid who wants to invest his money. For this purpose, he considers some private companies which are telecommunication company (TC), carpenter company (CC), real estate (RE) business, vehicle leasing (VL) company, advertising (AD) company, textile testing (TT) company. Let $X = \{$TC, CC, RE, VL, AD, TT $\}$ be a set. Let R be an equivalence relation defined on X as follows:

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Let $A = \{(TC, 0.3, 0.4, 0.1), (CC, 0.8, 0.1, 0.5), (RE, 0.1, 0.2, 0.6), (VL, 0.9, 0.6, 0.1), (AD, 0.2, 0.5, 0.2), (TT, 0.8, 0.6, 0.5)\}$ be a neutrosophic set on X with three components corresponding to each company, which represents its status in the industry and $RA = (\underline{R}A, \overline{R}A)$ a rough neutrosophic set, where $\underline{R}A$ and $\overline{R}A$ are lower and upper approximations of A, respectively, as follows:

$$\begin{aligned} \underline{R}A &= \{(TC, 0.1, 0.2, 0.6), (CC, 0.8, 0.1, 0.5), (RE, 0.1, 0.2, 0.6), (VL, 0.8, 0.6, 0.5), (AD, \\ &\quad 0.1, 0.2, 0.6), (TT, 0.8, 0.6, 0.5)\}, \\ \overline{R}A &= \{(TC, 0.3, 0.5, 0.1), (CC, 0.8, 0.1, 0.5), (RE, 0.3, 0.5, 0.1), (VL, 0.9, 0.6, 0.1), (AD, \\ &\quad 0.3, 0.5, 0.1), (TT, 0.9, 0.6, 0.1)\}. \\ Let\ \ E &= \{(TC, CC), (TC, AD), (TC, RE), (CC, VL), (CC, TT), (AD, RE), (TT, VL)\}, \end{aligned}$$

be the set of edges and S an equivalence relation on E defined as follows:

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\begin{aligned} \text{Let } B = \{&\big((TC, CC), 0.1, 0.1, 01\big), \big((TC, AD), 0.1, 0.2, 0.1\big), \big((TC, RE), 0.1, 0.2, 0.1\big), \\ &\big((CC, VL), 0.8, 0.1, 0.5\big), \big((CC, TT), 0.8, 0.1, 0.5\big), \big((AD, RE), 0.1, 0.2, 0.1\big), \\ &\big((TT, VL), 0.8, 0.6, 0.1\big)\} \end{aligned}$$

be a neutrosophic set on E which represents relationship between companies and $SB = (\underline{S}B, \overline{S}B)$ a rough neutrosophic relation, where $\underline{S}B$ and $\overline{S}B$ are lower and upper approximations of B, respectively, as follows:

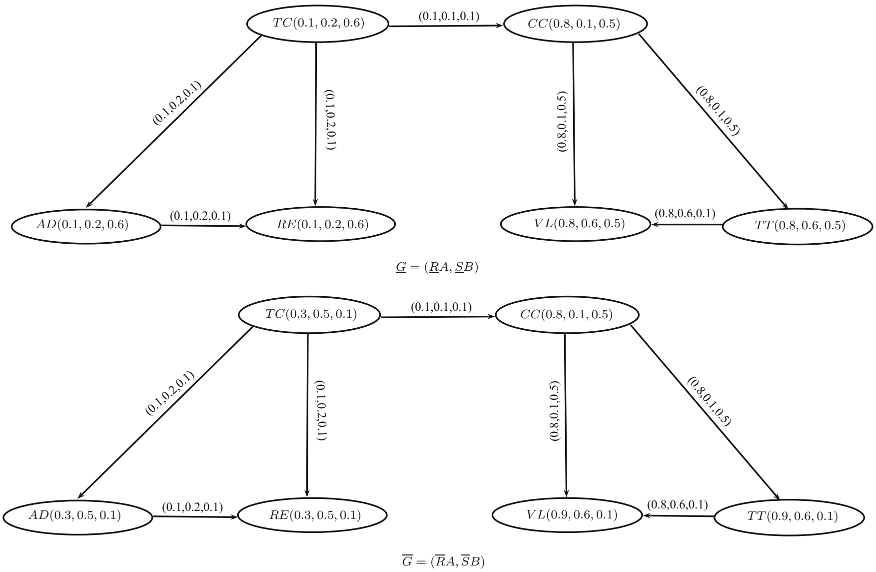


Fig. 6.19 Rough neutrosophic digraph $G = (\underline{G}, \overline{G})$

$$\begin{aligned}\underline{S}B = \{&\big((TC, CC), 0.1, 0.1, 0.1\big), \big((TC, AD), 0.1, 0.2, 0.1\big), \big((TC, RE), 0.1, 0.2, 0.1\big),\\ &\big((CC, VL), 0.8, 0.1, 0.5\big), \big((CC, TT), 0.8, 0.1, 0.5\big), \big((AD, RE), 0.1, 0.2, 0.1\big),\\ &\big((TT, VL), 0.8, 0.6, 0.1\big)\},\\ \overline{S}B = \{&\big((TC, CC), 0.1, 0.1, 0.1\big), \big((TC, AD), 0.1, 0.2, 0.1\big), \big((TC, RE), 0.1, 0.2, 0.1\big),\\ &\big((CC, VL), 0.8, 0.1, 0.5\big), \big((CC, TT), 0.8, 0.1, 0.5\big), \big((AD, RE)0.1, 0.2, 0.1\big),\\ &\big((TT, VL), 0.8, 0.6, 0.1\big)\}.\end{aligned}$$

Thus, $\underline{G} = (\underline{R}A, \underline{S}B)$ and $\overline{G} = (\overline{R}A, \overline{S}B)$ are rough neutrosophic digraphs as shown in Fig. 6.19.

In order to find out the most suitable investment company, we define the score values

$$S(v_i) = \sum_{v_i v_j \in E} \frac{T(v_j) + I(v_j) - F(v_j)}{3 - (T(v_i v_j) + I(v_i v_j) - F(v_i v_j))},$$

where

$$T(v_j) = \frac{\underline{T}(v_j) + \overline{T}(v_j)}{2},$$
$$I(v_j) = \frac{\underline{I}(v_j) + \overline{I}(v_j)}{2},$$
$$F(v_j) = \frac{\underline{F}(v_j) + \overline{F}(v_j)}{2},$$

and

$$T(v_iv_j) = \frac{\underline{T}(v_iv_j) + \overline{T}(v_iv_j)}{2},$$
$$I(v_iv_j) = \frac{\underline{I}(v_iv_j) + \overline{I}(v_iv_j)}{2},$$
$$F(v_iv_j) = \frac{\underline{F}(v_iv_j) + \overline{F}(v_iv_j)}{2}.$$

of each selected company and industry decision is v_k if $v_k = \max_i S(v_i)$. By calculation, we have
$S(TC) = 0.4926$, $S(CC) = 1.4038$, $S(RE) = 0.0667$, $S(VL) = 0.3833$, $S(AD) = 0.1429$ and $S(TT) = 1.3529$. Clearly, CC is the optimal decision. Therefore, the carpenter company is selected to get maximum possible profit. We present our proposed method as an algorithm. This algorithm returns the optimal solution for the investment problem.

Algorithm 6.3.2 1. Input the vertex set X.
2. Construct an equivalence relation R on the set X.
3. Calculate the approximation sets $\underline{R}A$ and $\overline{R}A$.
4. Input the edge set $E \subseteq X \times X$.
5. Construct an equivalence relation S on E.
6. Calculate the approximation sets $\underline{S}B$ and $\overline{S}B$.
7. Calculate the score value, by using the formula

$$S(v_i) = \sum_{v_iv_j \in E} \frac{T(v_j) + I(v_j) - F(v_j)}{3 - (T(v_iv_j) + I(v_iv_j) - F(v_iv_j))}.$$

8. The decision is $S(v_k) = \max_{v_i \in X} S(v_i)$.
9. If v_k has more than one value, then any one of $S(v_k)$ may be chosen.

6.4 Neutrosophic Rough Digraphs

Definition 6.10 Let X be a nonempty universe and $\hat{R}$ a single-valued neutrosophic relation on X. Let A be a single-valued neutrosophic set on X, defined as

$$A = \{< x, T_A(x), I_A(x), F_A(x) >: x \in X\}.$$

Then the lower and upper approximations of A represented by $\underline{\hat{R}}A$ and $\overline{\hat{R}}A$, respectively, are characterized as single-valued neutrosophic sets in X such that $\forall x \in X$

$$\underline{\hat{R}}A = \{< x, T_{\underline{\hat{R}}(A)}(x), I_{\underline{\hat{R}}(A)}(x), F_{\underline{\hat{R}}(A)}(x) >: y \in X\},$$
$$\overline{\hat{R}}A = \{< x, T_{\overline{\hat{R}}(A)}(x), I_{\overline{\hat{R}}(A)}(x), F_{\overline{\hat{R}}(A)}(x) >: y \in X\},$$

where

$$T_{\underline{\hat{R}}A}(x) = \bigwedge_{y \in X} \big(F_{\hat{R}}(x, y) \vee T_A(y)\big),$$
$$I_{\underline{\hat{R}}A}(x) = \bigvee_{y \in X} \big(1 - I_{\hat{R}}(x, y) \wedge I_A(y)\big),$$
$$F_{\underline{\hat{R}}A}(x) = \bigvee_{y \in X} \big(T_{\hat{R}}(x, y) \wedge F_A(y)\big),$$

and

$$T_{\overline{\hat{R}}A}(x) = \bigvee_{y \in X} \big(T_{\hat{R}}(x, y) \wedge T_A(y)\big),$$
$$I_{\overline{\hat{R}}A}(x) = \bigwedge_{y \in X} \big(I_{\hat{R}}(x, y) \vee I_A(y)\big),$$
$$F_{\overline{\hat{R}}A}(x) = \bigwedge_{y \in X} \big(F_{\hat{R}}(x, y) \vee F_A(y)\big).$$

A pair $(\underline{\hat{R}}A, \overline{\hat{R}}A)$ is called a *single-valued neutrosophic rough set.*

Definition 6.11 Let X be a nonempty set and $\hat{R}$ a single-valued neutrosophic tolerance relation on X. Let A be a neutrosophic set on X defined as:

$$A = \{< x, T_A(x), I_A(x), F_A(x) >: x \in X\}.$$

Then the lower and upper approximations of A represented by $\underline{\hat{R}}A$ and $\overline{\hat{R}}A$, respectively, are characterized as single-valued neutrosophic sets in X such that $\forall\, x \in X$

$$\underline{\hat{R}}A = \{< x, T_{\underline{\hat{R}}A}(x), I_{\underline{\hat{R}}A}(x), F_{\underline{\hat{R}}A}(x) >: y \in X\},$$
$$\overline{\hat{R}}A = \{< x, T_{\overline{\hat{R}}A}(x), I_{\overline{\hat{R}}A}(x), F_{\overline{\hat{R}}A}(x) >: y \in X\},$$

where

$$T_{\underline{\hat{R}}A}(x) = \bigwedge_{y \in X} \big(F_{\hat{R}}(x, y) \vee T_A(y)\big),$$
$$I_{\underline{\hat{R}}A}(x) = \bigwedge_{y \in X} \big(1 - I_{\hat{R}}(x, y) \vee I_A(y)\big),$$

$$F_{\underline{\hat{R}}A}(x) = \bigvee_{y\in X} \big(T_{\hat{R}}(x, y) \wedge F_A(y)\big),$$

and

$$T_{\overline{\hat{R}}A}(x) = \bigvee_{y\in X} \big(T_{\hat{R}}(x, y) \wedge T_A(y)\big),$$
$$I_{\overline{\hat{R}}A}(x) = \bigvee_{y\in X} \big(I_{\hat{R}}(x, y) \wedge I_A(y)\big),$$
$$F_{\overline{\hat{R}}A}(x) = \bigwedge_{y\in X} \big(F_{\hat{R}}(x, y) \vee F_A(y)\big).$$

Let $E \subseteq X \times X$ and $\hat{S}$ be a single-valued neutrosophic tolerance relation on E such that

$$T_{\hat{S}}((x_1, x_2)(y_1, y_2)) = \min\{T_{\hat{R}}(x_1, y_1), T_{\hat{R}}(x_2, y_2)\},$$
$$I_{\hat{S}}((x_1, x_2)(y_1, y_2)) = \min\{I_{\hat{R}}(x_1, y_1), I_{\hat{R}}(x_2, y_2)\},$$
$$F_{\hat{S}}((x_1, x_2)(y_1, y_2)) = \max\{F_{\hat{R}}(x_1, y_1), F_{\hat{R}}(x_2, y_2)\}.$$

Let B be a neutrosophic set on E defined as:

$$B = \{< xy, T_B(xy), I_B(xy), F_B(xy) >: xy \in E\},$$

such that

$$T_B(xy) \le \min\{T_{\underline{\hat{R}}A}(x), T_{\underline{\hat{R}}A}(y)\},$$
$$I_B(xy) \le \min\{I_{\underline{\hat{R}}A}(x), I_{\underline{\hat{R}}A}(y)\},$$
$$F_B(xy) \le \max\{F_{\overline{\hat{R}}A}(x), F_{\overline{\hat{R}}A}(y)\} \quad \forall\, x, y \in X.$$

Then the lower and the upper approximations of B represented by $\underline{\hat{S}}B$ and $\overline{\hat{S}}B$ are defined as follows:

$$\underline{\hat{S}}B = \{< xy, T_{\underline{\hat{S}}B}(xy), I_{\underline{\hat{S}}B}(xy), F_{\underline{\hat{S}}B}(xy) >: xy \in E\},$$
$$\overline{\hat{S}}B = \{< xy, T_{\overline{\hat{S}}B}(xy), I_{\overline{\hat{S}}B}(xy), F_{\overline{\hat{S}}B}(xy) >: xy \in E\},$$

where

$$T_{\underline{\hat{S}}B}(xy) = \bigwedge_{wz\in E} \big(F_{\hat{S}}((xy), (wz)) \vee T_B(wz)\big),$$
$$I_{\underline{\hat{S}}B}(xy) = \bigwedge_{wz\in E} \big((1 - I_{\hat{S}}((xy), (wz))) \vee I_B(wz)\big),$$

$$F_{\underline{\hat{S}B}}(xy) = \bigvee_{wz \in E} \big(T_{\hat{S}}((xy), (wz)) \wedge F_B(wz)\big),$$

and

$$T_{\overline{\hat{S}B}}(xy) = \bigvee_{wz \in E} \big(T_{\hat{S}}((xy), (wz)) \wedge T_B(wz)\big),$$
$$I_{\overline{\hat{S}B}}(xy) = \bigvee_{wz \in E} \big(I_{\hat{S}}((xy), (wz)) \wedge I_B(wz)\big),$$
$$F_{\overline{\hat{S}B}}(xy) = \bigwedge_{wz \in E} \big(F_{\hat{S}}((xy), (wz)) \vee F_B(wz)\big).$$

A pair $\hat{S}B = (\underline{\hat{S}}B, \overline{\hat{S}}B)$ is called *single-valued neutrosophic rough relation.*

Definition 6.12 A *single-valued neutrosophic rough digraph* on a nonempty set X is a four-ordered tuple $G = (\hat{R}, \hat{R}A, \hat{S}, \hat{S}B)$ such that

(a) $\hat{R}$ is a single-valued neutrosophic tolerance relation on X.
(b) $\hat{S}$ is a single-valued neutrosophic tolerance relation on $E \subseteq X \times X$.
(c) $\hat{R}A = (\underline{\hat{R}}A, \overline{\hat{R}}A)$ is a single-valued neutrosophic rough set on X.
(d) $\hat{S}B = (\underline{\hat{S}}B, \overline{\hat{S}}B)$ is a single-valued neutrosophic rough relation on X.
(e) $(\hat{R}A, \hat{S}B)$ is a neutrosophic rough digraph, where $\underline{G} = (\underline{\hat{R}}A, \underline{\hat{S}}B)$ and $\overline{G} = (\overline{\hat{R}}A, \overline{\hat{S}}B)$ are lower and upper approximate single-valued neutrosophic digraphs of G such that

$$T_{\underline{\hat{S}B}}(xy) \leq \min\{T_{\underline{\hat{R}A}}(x), T_{\underline{\hat{R}A}}(y)\},$$
$$I_{\underline{\hat{S}B}}(xy) \leq \min\{I_{\underline{\hat{R}A}}(x), I_{\underline{\hat{R}A}}(y)\},$$
$$F_{\underline{\hat{S}B}}(xy) \leq \max\{F_{\underline{\hat{R}A}}(x), F_{\underline{\hat{R}A}}(y)\},$$

$$T_{\overline{\hat{S}B}}(xy) \leq \min\{T_{\overline{\hat{R}A}}(x), T_{\overline{\hat{R}A}}(y)\},$$
$$I_{\overline{\hat{S}B}}(xy) \leq \min\{I_{\overline{\hat{R}A}}(x), I_{\overline{\hat{R}A}}(y)\},$$
$$F_{\overline{\hat{S}B}}(xy) \leq \max\{F_{\overline{\hat{R}A}}(x), F_{\overline{\hat{R}A}}(y)\}, \quad \forall\, x, y \in X.$$

Throughout this chapter, we will use neutrosophic rough set, neutrosophic rough relation and neutrosophic rough digraph, for short.

Example 6.9 Let $X = \{p, q, r, s, t\}$ be a nonempty set and $\hat{R}$ a neutrosophic tolerance relation on X which is given as:

$\hat{R}$	p	q	r	s	t
p	(1, 1, 0)	(0.5, 0.2, 0.3)	(0.1, 0.9, 0.4)	(0.6, 0.5, 0.2)	(0.2, 0.1, 0.8)
q	(0.5, 0.2, 0.3)	(1, 1, 0)	(0.3, 0.7, 0.5)	(0.1, 0.9, 0.6)	(0.6, 0.5, 0.1)
r	(0.1, 0.9, 0.4)	(0.3, 0.7, 0.5)	(1, 1, 0)	(0.2, 0.8, 0.7)	(0.1, 0.9, 0.6)
s	(0.6, 0.5, 0.2)	(0.1, 0.9, 0.6)	(0.2, 0.8, 0.7)	(1, 1, 0)	(0.2, 0.3, 0.1)
t	(0.2, 0.1, 0.8)	(0.6, 0.5, 0.1)	(0.1, 0.9, 0.6)	(0.2, 0.3, 0.1)	(1, 1, 0)

Let $A_1 = \{(p, 0.2, 0.1, 0.7), (q, 0.4, 0.5, 0.6), (r, 0.7, 0.8, 0.9), (s, 0.2, 0.9, 0.1), (t, 0.6, 0.8, 0.4)\}$ be a neutrosophic set on X. The lower and upper approximations of A_1 are given as:

$$\underline{\hat{R}}A_1 = \{(p, 0.2, 0.1, 0.7), (q, 0.3, 0.5, 0.6), (r, 0.4, 0.1, 0.9), (s, 0.2, 0.5, 0.6), (t, 0.2, 0.5, 0.6)\},$$
$$\overline{\hat{R}}A_1 = \{(p, 0.4, 0.2, 0.8), (q, 0.6, 0.9, 0.4), (r, 0.7, 0.8, 0.6), (s, 0.2, 0.9, 0.1), (t, 0.6, 0.8, 0.1)\}.$$

Let $E = \{pr, qs, rt, sp, tq\} \subseteq X \times X$ and $\hat{S}$ be a neutrosophic tolerance relation which is given as:

$\hat{S}$	pr	qs	rt	sp	tq
pr	(1, 1, 0)	(0.2, 0.2, 0.7)	(0.1, 0.9, 0.6)	(0.1, 0.5, 0.4)	(0.2, 0.1, 0.8)
qs	(0.2, 0.2, 0.7)	(1, 1, 0)	(0.2, 0.3, 0.5)	(0.1, 0.5, 0.6)	(0.1, 0.5, 0.6)
rt	(0.1, 0.9, 0.6)	(0.2, 0.3, 0.5)	(1, 1, 0)	(0.2, 0.1, 0.8)	(0.1, 0.5, 0.6)
sp	(0.1, 0.5, 0.4)	(0.1, 0.5, 0.6)	(0.2, 0.1, 0.8)	(1, 1, 0)	(0.2, 0.2, 0.3)
tq	(0.2, 0.1, 0.8)	(0.1, 0.5, 0.6)	(0.1, 0.5, 0.6)	(0.2, 0.2, 0.3)	(1, 1, 0)

Let $B_1 = \{(pr, 0.2, 0.1, 0.5), (qs, 0.1, 0.3, 0.3), (rt, 0.2, 0.1, 0.4), (sp, 0.1, 0.1, 0.2), (tq, 0.1, 0.4, 0.3)\}$ be a neutrosophic set on E. The lower and upper approximations of B_1 are given as:

$$\underline{\hat{S}}B_1 = \{(pr, 0.2, 0.1, 0.5), (qs, 0.1, 0.3, 0.3), (rt, 0.2, 0.1, 0.4), (sp, 0.1, 0.1, 0.2), (tq, 0.1, 0.4, 0.3)\},$$
$$\overline{\hat{S}}B_1 = \{(pr, 0.2, 0.2, 0.4), (qs, 0.2, 0.4, 0.3), (rt, 0.2, 0.4, 0.4), (sp, 0.2, 0.3, 0.2), (tq, 0.2, 0.4, 0.3)\}.$$

Thus, $\underline{G} = (\underline{\hat{R}}A_1, \underline{\hat{S}}B_1)$ and $\overline{G} = (\overline{\hat{R}}A_1, \overline{\hat{S}}B_1)$ are neutrosophic digraphs as shown in Fig. 6.20.

Example 6.10 Let $X = \{u, v, w, x, y, z\}$ be a crisp set and $\hat{R}$ a neutrosophic tolerance relation on X given by

$\hat{R}$	u	v	w	x	y	z
u	(1, 1, 0)	(0.2, 0.3, 0.5)	(0.5, 0.6, 0.9)	(0.3, 0.8, 0.3)	(0.3, 0.2, 0.1)	(0.1, 0.1, 0.5)
v	(0.2, 0.3, 0.5)	(1, 1, 0)	(0.9, 0.5, 0.6)	(0.1, 0.5, 0.7)	(0.8, 0.9, 0.1)	(0.8, 0.9, 0.1)
w	(0.5, 0.6, 0.9)	(0.9, 0.5, 0.6)	(1, 1, 0)	(0.3, 0.6, 0.8)	(0.2, 0.3, 0.6)	(0.7, 0.6, 0.6)
x	(0.3, 0.8, 0.3)	(0.1, 0.5, 0.7)	(0.3, 0.6, 0.8)	(1, 1, 0)	(0.5, 0.1, 0.9)	(0.8, 0.7, 0.2)
y	(0.3, 0.2, 0.1)	(0.8, 0.9, 0.1)	(0.2, 0.3, 0.6)	(0.5, 0.1, 0.9)	(1, 1, 0)	(0.6, 0.5, 0.9)
z	(0.1, 0.1, 0.5)	(0.8, 0.9, 0.1)	(0.7, 0.6, 0.6)	(0.8, 0.7, 0.2)	(0.6, 0.5, 0.9)	(1, 1, 0)

Let $A = \{(u, 0.9, 0.3, 0.1), (v, 0.5, 0.6, 0.2), (w, 0.8, 0.5, 0.3), (x, 0.7, 0.6, 0.9), (y, 0.5, 0.2, 0.1), (z, 0.9, 0.7, 0.3)\}$ be a neutrosophic set on X. Then the lower and upper approximations of A are given as follows:

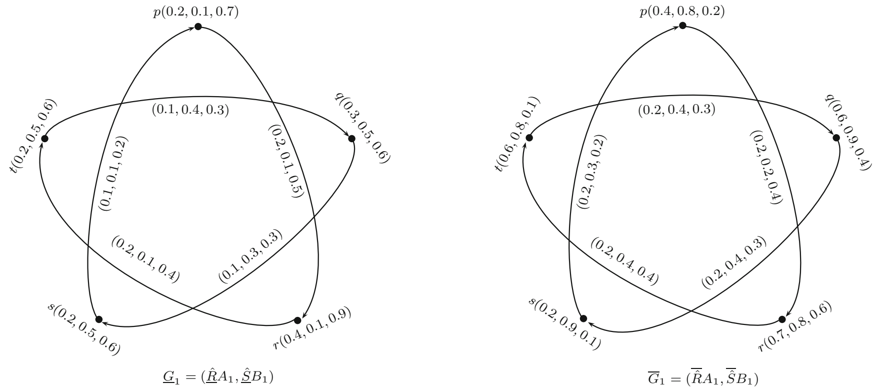


Fig. 6.20 Neutrosophic rough digraph $G_1 = (\underline{G}_1, \overline{G}_1)$

$$\underline{\hat{R}}A = \{(u, 0.5, 0.3, 0.3), (v, 0.5, 0.2, 0.3), (w, 0.6, 0.4, 0.3), (x, 0.7, 0.3, 0.9), (y, 0.5, 0.2, 0.5), (z, 0.5, 0.5, 0.8)\},$$

$$\overline{\hat{R}}A = \{(u, 0.9, 0.6, 0.1), (v, 0.8, 0.7, 0.1), (w, 0.8, 0.6, 0.3), (x, 0.8, 0.7, 0.3), (y, 0.6, 0.6, 0.1), (z, 0.9, 0.7, 0.2)\}.$$

Let $E = \{uv, vw, wx, xy, yz, zu, zw, vy\} \subseteq X \times X$ and $\hat{S}$ be a neutrosophic tolerance relation on E given as

$\hat{S}$	uv	vw	wx	xy	yz	zu	zw	vy
uv	(1, 1, 0)	(0.2, 0.3, 0.6)	(0.1, 0.5, 0.9)	(0.3, 0.8, 0.3)	(0.3, 0.2, 0.1)	(0.1, 0.1, 0.5)	(0.1, 0.1, 0.6)	(0.2, 0.3, 0.5)
vw	(0.2, 0.3, 0.6)	(1, 1, 0)	(0.3, 0.5, 0.8)	(0.1, 0.3, 0.7)	(0.7, 0.6, 0.6)	(0.5, 0.6, 0.9)	(0.8, 0.9, 0.1)	(0.2, 0.3, 0.6)
wx	(0.1, 0.5, 0.9)	(0.3, 0.5, 0.8)	(1, 1, 0)	(0.3, 0.1, 0.9)	(0.2, 0.3, 0.6)	(0.3, 0.6, 0.6)	(0.3, 0.6, 0.8)	(0.5, 0.1, 0.9)
xy	(0.3, 0.8, 0.3)	(0.1, 0.3, 0.7)	(0.3, 0.1, 0.9)	(1, 1, 0)	(0.5, 0.1, 0.9)	(0.3, 0.2, 0.2)	(0.2, 0.3, 0.6)	(0.1, 0.5, 0.7)
yz	(0.3, 0.2, 0.1)	(0.7, 0.6, 0.6)	(0.2, 0.3, 0.6)	(0.5 ,0.1, 0.9)	(1, 1, 0)	(0.1, 0.1, 0.9)	(0.6, 0.5, 0.9)	(0.6, 0.5, 0.9)
zu	(0.1, 0.1, 0.5)	(0.5, 0.6, 0.9)	(0.3, 0.6, 0.6)	(0.3, 0.2, 0.2)	(0.1, 0.1, 0.9)	(1, 1, 0)	(0.5, 0.6, 0.9)	(0.3, 0.3, 0.1)
zw	(0.1, 0.1, 0.6)	(0.8, 0.9, 0.1)	(0.3, 0.6, 0.8)	(0.2, 0.3, 0.6)	(0.6, 0.5, 0.9)	(0.5, 0.6, 0.9)	(1, 1, 0)	(0.2, 0.3, 0.6)
vy	(0.2, 0.3, 0.5)	(0.2, 0.3, 0.6)	(0.5, 0.1, 0.9)	(0.1, 0.5, 0.7)	(0.6, 0.5, 0.9)	(0.3, 0.2, 0.1)	(0.2, 0.3, 0.6)	(1, 1, 0)

Let B be a neutrosophic set on E defined as
$B = \{(uv, 0.5, 0.2, 0.1), (vw, 0.5, 0.2, 0.3), (wx, 0.5, 0.3, 0.3), (xy, 0.5, 0.2, 0.3), (yz, 0.5, 0.2, 0.2), (zu, 0.5, 0.3, 0.2), (zw, 0.5, 0.4, 0.3), (vy, 0.5, 0.2, 0.1)\}$.
Then the lower and upper approximations of B are given as

$$\underline{\hat{S}}B = \{(uv, 0.5, 0.2, 0.3), (vw, 0.5, 0.2, 0.3), (wx, 0.5, 0.3, 0.3), (xy, 0.5, 0.2, 0.3), (yz, 0.5, 0.2, 0.3), (zu, 0.5, 0.3, 0.3), (zw, 0.5, 0.2, 0.3), (vy, 0.5, 0.2, 0.3)\},$$

$$\overline{\hat{S}}B = \{(uv, 0.5, 0.3, 0.1), (vw, 0.5, 0.4, 0.3), (wx, 0.5, 0.4, 0.3), (xy, 0.5, 0.3, 0.3), (yz, 0.5, 0.4, 0.1), (zu, 0.5, 0.4, 0.1), (zw, 0.5, 0.4, 0.3), (vy, 0.5, 0.3, 0.1)\}.$$

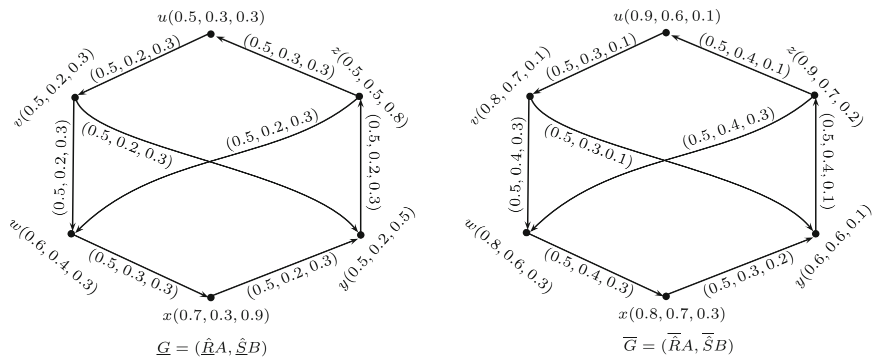


Fig. 6.21 Neutrosophic rough digraph $G = (\underline{G}, \overline{G})$

Thus, $\underline{G} = (\underline{\hat{R}}A, \underline{\hat{S}}B)$ and $\overline{G} = (\overline{\hat{R}}A, \overline{\hat{S}}B)$ are the neutrosophic digraphs as shown in Fig. 6.21.

We now discuss regular neutrosophic rough digraphs.

Definition 6.13 Let $G = (\underline{G}, \overline{G})$ be a neutrosophic rough digraph on a nonempty set X. The *indegree* of a vertex $x \in G$ is the sum of membership degree, indeterminacy and falsity of all edges towards x from other vertices in $\underline{G}$ and $\overline{G}$, respectively, represented by $id_G(x)$ and defined by

$$id_G(x) = id_{\underline{G}}(x) + id_{\overline{G}}(x),$$

where

$$id_{\underline{G}}(x) = \left(\sum_{x,y \in \underline{\hat{S}}B} T_{\underline{G}}(yx), \sum_{x,y \in \underline{\hat{S}}B} I_{\underline{G}}(yx), \sum_{x,y \in \underline{\hat{S}}B} F_{\underline{G}}(yx) \right),$$
$$id_{\overline{G}}(x) = \left(\sum_{x,y \in \overline{\hat{S}}B} T_{\overline{G}}(yx), \sum_{x,y \in \overline{\hat{S}}B} I_{\overline{G}}(yx), \sum_{x,y \in \overline{\hat{S}}B} F_{\overline{G}}(yx) \right).$$

The *outdegree* of a vertex $x \in G$ is the sum of membership degree, indeterminacy and falsity of all edges outwards from x to other vertices in $\underline{G}$ and $\overline{G}$, respectively, represented by $od_G(x)$ and defined by

$$od_G(x) = od_{\underline{G}}(x) + od_{\overline{G}}(x),$$

where

$$od_{\underline{G}}(x) = \left(\sum_{x,y \in \underline{\hat{S}}B} T_{\underline{G}}(xy), \sum_{x,y \in \underline{\hat{S}}B} I_{\underline{G}}(xy), \sum_{x,y \in \underline{\hat{S}}B} F_{\underline{G}}(xy) \right),$$

$$od_{\overline{G}}(x) = \left(\sum_{x,y \in \overline{\hat{S}}B} T_{\overline{G}}(xy), \sum_{x,y \in \overline{\hat{S}}B} I_{\overline{G}}(xy), \sum_{x,y \in \overline{\hat{S}}B} F_{\overline{G}}(xy) \right).$$

$d_G(x) = id_G(x) + od_G(x)$ is called degree of vertex x.

Definition 6.14 A neutrosophic rough digraph is called a *regular neutrosophic rough digraph* of degree (m_1, m_2, m_3) if

$$d_G(x) = (m_1, m_2, m_3), \forall x \in X.$$

Example 6.11 Let $X = \{p, q, r, s\}$ be a nonempty set and $\hat{R}$ a neutrosophic tolerance relation on X which is given as:

$\hat{R}$	p	q	r	s
p	(1, 1, 0)	(0.1, 0.9, 0.8)	(0.7, 0.5, 0.8)	(0.1, 0.9, 0.8)
q	(0.9, 0.8, 0.1)	(1, 1, 0)	(0.1, 0.9, 0.8)	(0.4, 0.3, 0.9)
r	(0.7, 0.5, 0.8)	(0.1, 0.9, 0.8)	(1, 1, 0)	(0.1, 0.9, 0.8)
s	(0.1, 0.9, 0.8)	(0.4, 0.3, 0.9)	(0.1, 0.9, 0.8)	(1, 1, 0)

Let $A_1 = \{(p, 0.1, 0.4, 0.8), (q, 0.2, 0.3, 0.9), (r, 0.1, 0.6, 0.8), (s, 0.9, 0.6, 0.3)\}$ be a neutrosophic set on X. Then the lower and upper approximations of A_1 are given as:

$$\underline{\hat{R}}A_1 = \{(p, 0.1, 0.3, 0.8), (q, 0.2, 0.3, 0.9), (r, 0.1, 0.3, 0.8), (s, 0.8, 0.4, 0.4)\},$$

$$\overline{\hat{R}}A_1 = \{(p, 0.1, 0.6, 0.8), (q, 0.4, 0.6, 0.8), (r, 0.1, 0.6, 0.8), (s, 0.9, 0.6, 0.3)\}.$$

Let $E = \{pq, qr, rs, sp\} \subseteq X \times X$ and $\hat{S}$ be a neutrosophic tolerance relation on E which is given as:

$\hat{S}$	pq	qr	rs	sp
pq	(1, 1, 0)	(0.1, 0.9, 0.8)	(0.4, 0.3, 0.9)	(0.1, 0.9, 0.8)
qr	(0.1, 0.9, 0.8)	(1, 1, 0)	(0.1, 0.9, 0.8)	(0.4, 0.3, 0.9)
rs	(0.4, 0.3, 0.9)	(0.1, 0.9, 0.8)	(1, 1, 0)	(0.1, 0.9, 0.8)
sp	(0.1, 0.9, 0.8)	(0.4, 0.3, 0.9)	(0.1, 0.9, 0.8)	(1, 1, 0)

Let $B_1 = \{(pq, 0.1, 0.3, 0.8), (qr, 0.1, 0.3, 0.3), (rs, 0.1, 0.3, 0.8), (sp, 0.1, 0.3, 0.8)\}$ be a neutrosophic set on E. Then the lower and upper approximations of B_1 are given as:

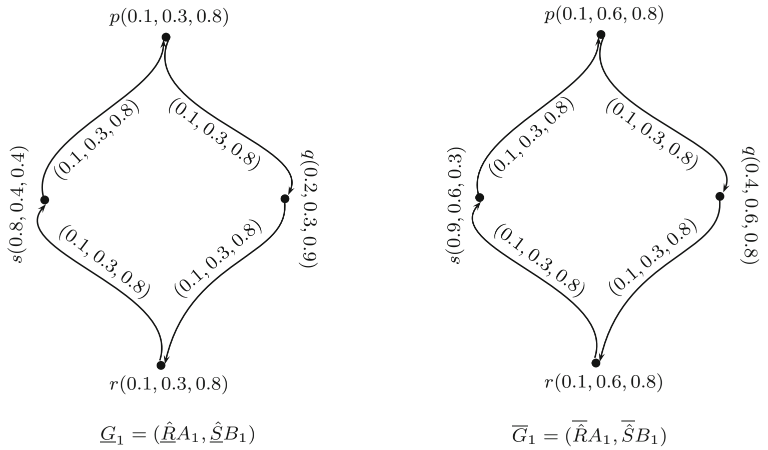


Fig. 6.22 Regular neutrosophic rough digraph $G_1 = (\underline{G}_1, \overline{G}_1)$

$$\underline{\hat{S}}B_1 = \{(pq, 0.1, 0.3, 0.8), (qr, 0.1, 0.3, 0.3), (rs, 0.1, 0.3, 0.8), (sp, 0.1, 0.3, 0.8)\},$$
$$\overline{\hat{S}}B_1 = \{(pq, 0.1, 0.3, 0.8), (qr, 0.1, 0.3, 0.3), (rs, 0.1, 0.3, 0.8), (sp, 0.1, 0.3, 0.8)\}.$$

Thus, $G_1 = (\underline{G}_1, \overline{G}_1)$ is a regular neutrosophic rough digraph as shown in Fig. 6.22.

Definition 6.15 Let $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ be two neutrosophic rough digraphs. Then the *direct sum* of G_1 and G_2 is a neutrosophic rough digraph $G = G_1 \oplus G_2 = (\underline{G}_1 \oplus \underline{G}_2, \overline{G}_1 \oplus \overline{G}_2)$, where $\underline{G}_1 \oplus \underline{G}_2 = (\underline{\hat{R}}A_1 \oplus \underline{\hat{R}}A_2, \underline{\hat{S}}B_1 \oplus \underline{\hat{S}}B_2)$ and $\overline{G}_1 \oplus \overline{G}_2 = (\overline{\hat{R}}A_1 \oplus \overline{\hat{R}}A_2, \overline{\hat{S}}B_1 \oplus \overline{\hat{S}}B_2)$ are neutrosophic digraphs such that

(1)

$$T_{\underline{\hat{R}}A_1 \oplus \underline{\hat{R}}A_2}(x) = \begin{cases} T_{\underline{\hat{R}}A_1}(x), & \text{if } x \in \underline{\hat{R}}A_1 - \underline{\hat{R}}A_2 \\ T_{\underline{\hat{R}}A_2}(x), & \text{if } x \in \underline{\hat{R}}A_2 - \underline{\hat{R}}A_1 \\ \max(T_{\underline{\hat{R}}A_1}(x), T_{\underline{\hat{R}}A_2}(x)), & \text{if } x \in \underline{\hat{R}}A_1 \cap \underline{\hat{R}}A_2 \end{cases}$$

$$I_{\underline{\hat{R}}A_1 \oplus \underline{\hat{R}}A_2}(x) = \begin{cases} I_{\underline{\hat{R}}A_1}(x), & \text{if } x \in \underline{\hat{R}}A_1 - \underline{\hat{R}}A_2 \\ I_{\underline{\hat{R}}A_2}(x), & \text{if } x \in \underline{\hat{R}}A_2 - \underline{\hat{R}}A_1 \\ \max(I_{\underline{\hat{R}}A_1}(x), I_{\underline{\hat{R}}A_2}(x)), & \text{if } x \in \underline{\hat{R}}A_1 \cap \underline{\hat{R}}A_2 \end{cases}$$

$$F_{\underline{\hat{R}}A_1 \oplus \underline{\hat{R}}A_2}(x) = \begin{cases} F_{\underline{\hat{R}}A_1}(x), & \text{if } x \in \underline{\hat{R}}A_1 - \underline{\hat{R}}A_2 \\ F_{\underline{\hat{R}}A_2}(x), & \text{if } x \in \underline{\hat{R}}A_2 - \underline{\hat{R}}A_1 \\ \min(F_{\underline{\hat{R}}A_1}(x), F_{\underline{\hat{R}}A_2}(x)), & \text{if } x \in \underline{\hat{R}}A_1 \cap \underline{\hat{R}}A_2 \end{cases}$$

$$T_{\underline{\hat{S}}B_1 \oplus \underline{\hat{S}}B_2}(x, y) = \begin{cases} T_{\underline{\hat{S}}B_1}(x, y), \text{ if } (x, y) \in \underline{\hat{S}}B_1 \\ T_{\underline{\hat{S}}B_2}(x, y), \text{ if } (x, y) \in \underline{\hat{S}}B_2 \end{cases}$$

$$I_{\underline{\hat{S}}B_1 \oplus \underline{\hat{S}}B_2}(x, y) = \begin{cases} I_{\underline{\hat{S}}B_1}(x, y), \text{ if } (x, y) \in \underline{\hat{S}}B_1 \\ I_{\underline{\hat{S}}B_2}(x, y), \text{ if } (x, y) \in \underline{\hat{S}}B_2 \end{cases}$$

$$F_{\underline{\hat{S}}B_1 \oplus \underline{\hat{S}}B_2}(x, y) = \begin{cases} F_{\underline{\hat{S}}B_1}(x, y), \text{ if } (x, y) \in \underline{\hat{S}}B_1 \\ F_{\underline{\hat{S}}B_2}(x, y), \text{ if } (x, y) \in \underline{\hat{S}}B_2 \end{cases}$$

(2)

$$T_{\overline{\hat{R}}A_1 \oplus \overline{\hat{R}}A_2}(x) = \begin{cases} T_{\overline{\hat{R}}A_1}(x), & \text{if } x \in \overline{\hat{R}}A_1 - \overline{\hat{R}}A_2 \\ T_{\overline{\hat{R}}A_2}(x), & \text{if } x \in \overline{\hat{R}}A_2 - \overline{\hat{R}}A_1 \\ \max(T_{\overline{\hat{R}}A_1}(x), T_{\overline{\hat{R}}A_2}(x)), & \text{if } x \in \overline{\hat{R}}A_1 \cap \overline{\hat{R}}A_2 \end{cases}$$

$$I_{\overline{\hat{R}}A_1 \oplus \overline{\hat{R}}A_2}(x) = \begin{cases} I_{\overline{\hat{R}}A_1}(x), & \text{if } x \in \overline{\hat{R}}A_1 - \overline{\hat{R}}A_2 \\ I_{\overline{\hat{R}}A_2}(x), & \text{if } x \in \overline{\hat{R}}A_2 - \overline{\hat{R}}A_1 \\ \max(I_{\overline{\hat{R}}A_1}(x), I_{\overline{\hat{R}}A_2}(x)), & \text{if } x \in \overline{\hat{R}}A_1 \cap \overline{\hat{R}}A_2 \end{cases}$$

$$F_{\overline{\hat{R}}A_1 \oplus \overline{\hat{R}}A_2}(x) = \begin{cases} F_{\overline{\hat{R}}A_1}(x), & \text{if } x \in \overline{\hat{R}}A_1 - \overline{\hat{R}}A_2 \\ F_{\overline{\hat{R}}A_2}(x), & \text{if } x \in \overline{\hat{R}}A_2 - \overline{\hat{R}}A_1 \\ \min(F_{\overline{\hat{R}}A_1}(x), F_{\overline{\hat{R}}A_2}(x)), & \text{if } x \in \overline{\hat{R}}A_1 \cap \overline{\hat{R}}A_2 \end{cases}$$

$$T_{\overline{\hat{S}}B_1 \oplus \overline{\hat{S}}B_2}(x, y) = \begin{cases} T_{\overline{\hat{S}}B_1}(x, y), \text{ if } (x, y) \in \overline{\hat{S}}B_1 \\ T_{\overline{\hat{S}}B_2}(x, y), \text{ if } (x, y) \in \overline{\hat{S}}B_2 \end{cases}$$

$$I_{\overline{\hat{S}}B_1 \oplus \overline{\hat{S}}B_2}(x, y) = \begin{cases} I_{\overline{\hat{S}}B_1}(x, y), \text{ if } (x, y) \in \overline{\hat{S}}B_1 \\ I_{\overline{\hat{S}}B_2}(x, y), \text{ if } (x, y) \in \overline{\hat{S}}B_2 \end{cases}$$

$$F_{\overline{\hat{S}}B_1 \oplus \overline{\hat{S}}B_2}(x, y) = \begin{cases} F_{\overline{\hat{S}}B_1}(x, y), \text{ if } (x, y) \in \overline{\hat{S}}B_1 \\ F_{\overline{\hat{S}}B_2}(x, y), \text{ if } (x, y) \in \overline{\hat{S}}B_2 \end{cases}$$

Example 6.12 Let $X = \{p, q, r, s, t\}$ be a set. Let $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ be two neutrosophic rough digraphs on X as shown in Figs. 6.20 and 6.23. The direct sum of G_1 and G_2 is $G = (\underline{G}_1 \oplus \underline{G}_2, \overline{G}_1 \oplus \overline{G}_2)$, where $\underline{G}_1 \oplus \underline{G}_2 = (\underline{\hat{R}}A_1 \oplus$

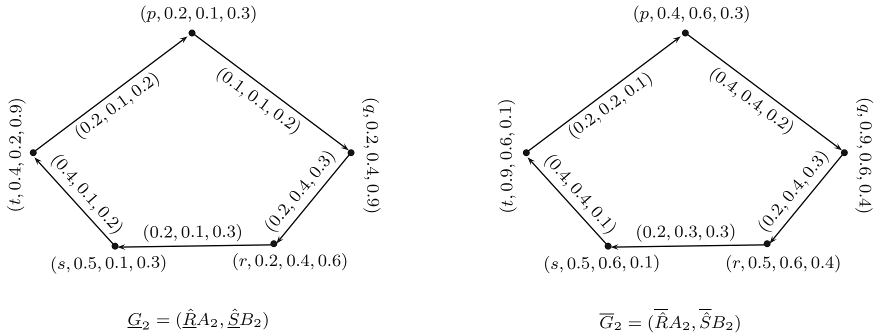


Fig. 6.23 Neutrosophic rough digraph $G_2 = (\underline{G}_2, \overline{G}_2)$

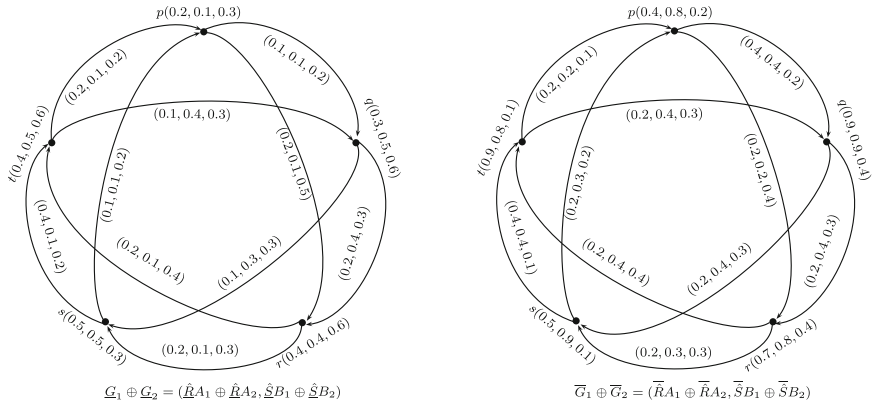


Fig. 6.24 Neutrosophic rough digraph $G = (\underline{G}_1 \oplus \underline{G}_2, \overline{G}_1 \oplus \overline{G}_2)$

$\underline{\hat{R}}A_2, \underline{\hat{S}}B_1 \oplus \underline{\hat{S}}B_2)$ and $\overline{G}_1 \oplus \overline{G}_2 = (\overline{\hat{R}}A_1 \oplus \overline{\hat{R}}A_2, \overline{\hat{S}}B_1 \oplus \overline{\hat{S}}B_2)$ are neutrosophic digraphs as shown in Fig. 6.24.

Remark 6.1 The direct sum of two regular neutrosophic digraphs may not be a regular neutrosophic digraph as it can be seen in the following example.

Example 6.13 Consider the two regular neutrosophic digraphs $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ as shown in Figs. 6.22 and 6.25, respectively; then the direct sum $G = (\underline{G}_1 \oplus \underline{G}_2, \overline{G}_1 \oplus \overline{G}_2)$ of G_1 and G_2 as shown in Fig. 6.26 is not a regular neutrosophic rough digraph.

Definition 6.16 Let $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ be two neutrosophic rough digraphs on crisp sets X_1 and X_2, respectively. The *residue product* of G_1 and G_2 is a neutrosophic rough digraph $G = G_1 * G_2 = (\underline{G}_1 * \underline{G}_2, \overline{G}_1 * \overline{G}_2)$, where $\underline{G}_1 * \underline{G}_2 = (\underline{\hat{R}}A_1 * \underline{\hat{R}}A_2, \underline{\hat{S}}B_1 * \underline{\hat{S}}B_2)$ and $\overline{G}_1 * \overline{G}_2 = (\overline{\hat{R}}A_1 * \overline{\hat{R}}A_2, \overline{\hat{S}}B_1 * \overline{\hat{S}}B_2)$ are neutrosophic digraphs, respectively, such that

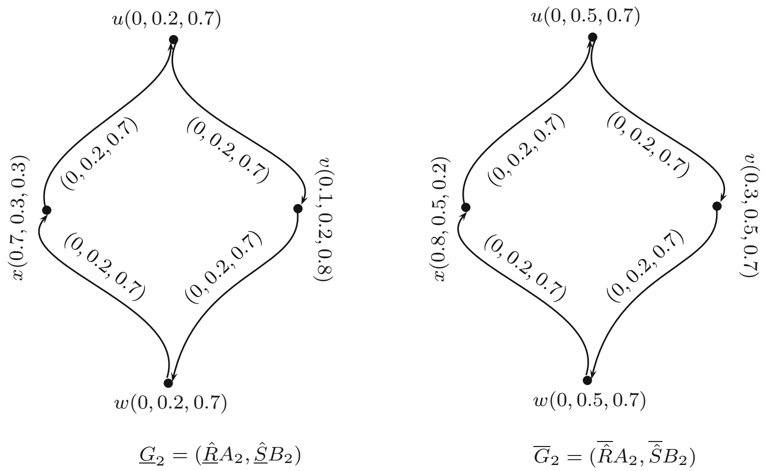


Fig. 6.25 Regular neutrosophic rough digraph $G_2 = (\underline{G}_2, \overline{G}_2)$

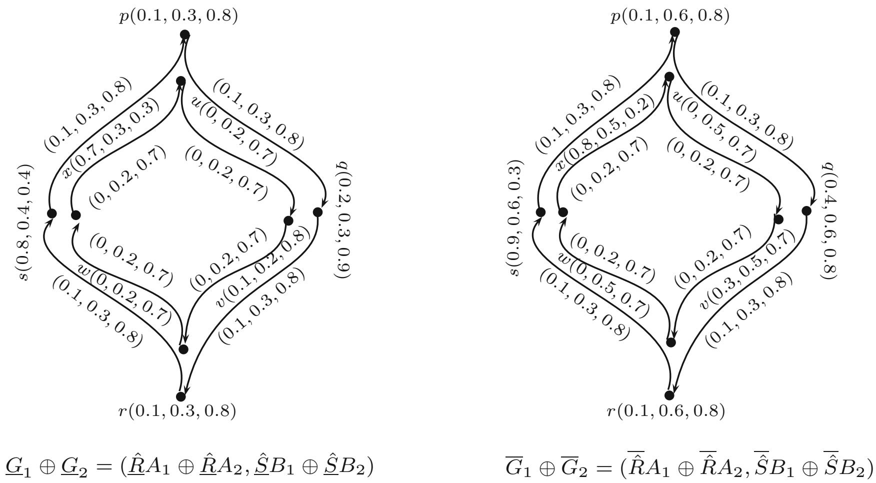


Fig. 6.26 Neutrosophic rough digraph $G = (\underline{G}_1 \oplus \underline{G}_2, \overline{G}_1 \oplus \overline{G}_2)$

(1)

$$\begin{aligned}
T_{\underline{\hat{R}}A_1 * \underline{\hat{R}}A_2}(x_1, x_2) &= \max\{T_{\underline{\hat{R}}A_1}(x_1), T_{\underline{\hat{R}}A_2}(x_2)\},\\
I_{\underline{\hat{R}}A_1 * \underline{\hat{R}}A_2}(x_1, x_2) &= \max\{I_{\underline{\hat{R}}A_1}(x_1), I_{\underline{\hat{R}}A_2}(x_2)\},\\
F_{\underline{\hat{R}}A_1 * \underline{\hat{R}}A_2}(x_1, x_2) &= \min\{F_{\underline{\hat{R}}A_1}(x_1), F_{\underline{\hat{R}}A_2}(x_2)\}, \quad \forall (x_1, x_2) \in \underline{\hat{R}}A_1 \times \underline{\hat{R}}A_2
\end{aligned}$$

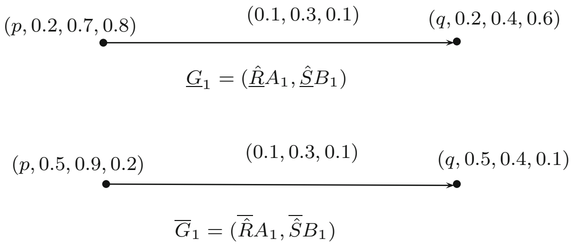


Fig. 6.27 Neutrosophic rough digraph $G_1 = (\underline{G}_1, \overline{G}_1)$

$$\begin{aligned}
T_{\hat{\underline{S}}B_1 * \hat{\underline{S}}B_2}(x_1, x_2)(y_1, y_2) &= T_{\hat{\underline{S}}B_1}(x_1, y_1),\\
I_{\hat{\underline{S}}B_1 * \hat{\underline{S}}B_2}(x_1, x_2)(y_1, y_2) &= I_{\hat{\underline{S}}B_1}(x_1, y_1),\\
F_{\hat{\underline{S}}B_1 * \hat{\underline{S}}B_2}(x_1, x_2)(y_1, y_2) &= F_{\hat{\underline{S}}B_1}(x_1, y_1), \quad \forall (x_1, y_1) \in \hat{\underline{S}}B_1, x_1 \neq y_2
\end{aligned}$$

(2)

$$\begin{aligned}
T_{\overline{\hat{R}}A_1 * \overline{\hat{R}}A_2}(x_1, x_2) &= \max\{T_{\overline{\hat{R}}A_1}(x_1), T_{\overline{\hat{R}}A_2}(x_2)\},\\
I_{\overline{\hat{R}}A_1 * \overline{\hat{R}}A_2}(x_1, x_2) &= \max\{I_{\overline{\hat{R}}A_1}(x_1), I_{\overline{\hat{R}}A_2}(x_2)\},\\
F_{\overline{\hat{R}}A_1 * \overline{\hat{R}}A_2}(x_1, x_2) &= \min\{F_{\overline{\hat{R}}A_1}(x_1), F_{\overline{\hat{R}}A_2}(x_2)\}, \quad \forall (x_1, x_2) \in \overline{\hat{R}}A_1 \times \overline{\hat{R}}A_2
\end{aligned}$$

$$\begin{aligned}
T_{\overline{\hat{S}}B_1 * \overline{\hat{S}}B_2}(x_1, x_2)(y_1, y_2) &= T_{\overline{\hat{S}}B_1}(x_1, y_1),\\
I_{\overline{\hat{S}}B_1 * \overline{\hat{S}}B_2}(x_1, x_2)(y_1, y_2) &= I_{\overline{\hat{S}}B_1}(x_1, y_1),\\
F_{\overline{\hat{S}}B_1 * \overline{\hat{S}}B_2}(x_1, x_2)(y_1, y_2) &= F_{\overline{\hat{S}}B_1}(x_1, y_1), \quad \forall (x_1, y_1) \in \overline{\hat{S}}B_1, x_1 \neq y_2
\end{aligned}$$

Example 6.14 Let $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ be two neutrosophic rough digraphs on the two crisp sets $X_1 = \{p, q\}$ and $X_2 = \{u, v, w, x\}$ as shown in Figs. 6.27 and 6.28. Then the residue product of G_1 and G_2 is a neutrosophic rough digraph $G = G_1 * G_2 = (\underline{G}_1 * \underline{G}_2, \overline{G}_1 * \overline{G}_2)$ where $\underline{G}_1 * \underline{G}_2 = (\hat{\underline{R}}A_1 * \hat{\underline{R}}A_2, \hat{\underline{S}}B_1 * \hat{\underline{S}}B_2)$ and $\overline{G}_1 * \overline{G}_2 = (\overline{\hat{R}}A_1 * \overline{\hat{R}}A_2, \overline{\hat{S}}B_1 \overline{\hat{S}}B_2)$ and the respective figure is shown in Fig. 6.29.

Theorem 6.2 *If $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ are two neutrosophic rough digraphs such that $|X_2| > 1$, then their residue product is regular if and only if G_1 is regular.*

Proof Let $G_1 * G_2$ be a regular neutrosophic rough digraph.
Then, for any two vertices (x_1, x_2) and (y_1, y_2) in $X_1 \times X_2$,

$$\begin{gathered}
d_{G_1 * G_2}(x_1, x_2) = d_{G_1 * G_2}(y_1, y_2)\\
\Rightarrow d_{G_1}(x_1) = d_{G_1}(y_1)
\end{gathered}$$

This is true for all vertices in X_1. Hence G_1 is a regular neutrosophic rough digraph.

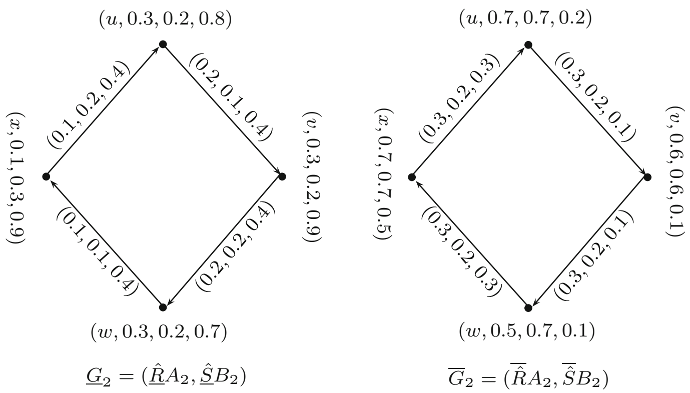


Fig. 6.28 Neutrosophic rough digraph $G_2 = (\underline{G}_2, \overline{G}_2)$

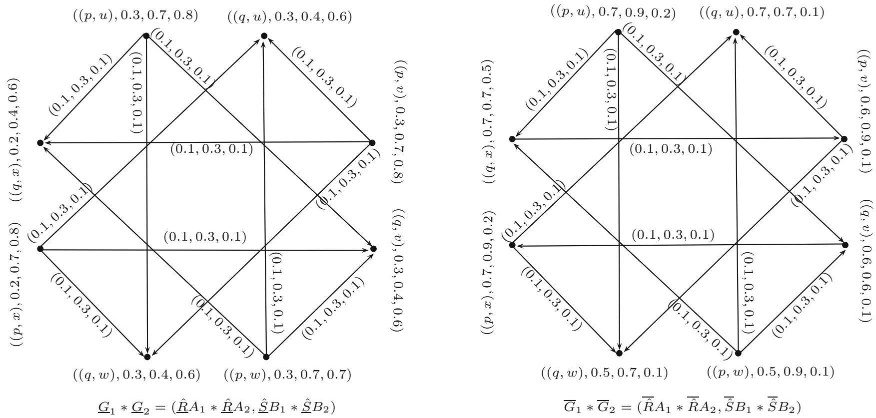


Fig. 6.29 Neutrosophic rough digraph $G = (\underline{G}_1 * \underline{G}_2, \overline{G}_1 * \overline{G}_2)$

Conversely suppose that $G_1 = (\underline{G}_1, \overline{G}_1)$ is a (m_1, m_2, m_3)-regular neutrosophic rough digraph and $G_2 = (\underline{G}_2, \overline{G}_2)$ is any neutrosophic rough digraph with $|X_2| > 1$. If $|X_2| > 1$, then $d_{G_1 * G_2}(x_1, x_2) = d_{G_1}(x_1) = (m_1, m_2, m_3)$. This is a constant ordered triplet for all vertices in $X_1 \times X_2$. Hence $G_1 * G_2$ ia regular neutrosophic rough digraph.

6.5 Applications of Neutrosophic Rough Digraphs

In this section, we present some real-life applications of neutrosophic rough digraphs in decision-making. In decision-making, the selection is facilitated by evaluating each choice on the set of criteria. The criteria must be measurable, and their outcomes must be measured for every decision alternatives.

Table 6.1 Companies and their ratings

X	Good reviews (%)	Neutral (%)	Bad reviews (%)
PEL	45	29	37
Dawlance	52	25	49
Haier	51	43	45
Waves	47	41	38
Orient	51	35	48

6.5.1 Online Reviews and Ratings

Customer reviews are increasingly available online for a wide range of products and services. As customers search online for product information and to evaluate product alternatives, they often have access to dozens or hundreds of product review from other customers. These reviews are very helpful in product selection. But only considering the good reviews about a product is not very helpful in decision-making. The customer should keep in mind bad and neutral reviews as well. We use percentages of good reviews, bad reviews and neutral reviews of a product as truth-membership, false-membership and indeterminacy, respectively.

Mrs. Sadia wants to purchase a refrigerator. For this purpose she visits websites of different refrigerator companies. The refrigerator companies and their ratings by other customers are shown in Table. 6.1

Here $X = \{$Pel(P), Dawlance(D), Haier(H), Waves(W), Orient(O)$\}$ and the neutrosophic set on X according to the reviews will be

$$A = \{(P, 0.45, 0.29, 0.37), (D, 0.52, 0.25, 0.49), (H, 0.51, 0.43, 0.45), (W, 0.47, 0.41, 0.38)(O, 0.51, 0.35, 0.48)\}.$$

The neutrosophic tolerance relation on X is given below

$\hat{R}$	P	D	H	W	O
P	(1, 1, 0)	(0.5, 0.6, 0.9)	(0.2, 0.3, 0.6)	(0.1, 0.2, 0.3)	(0.4, 0.6, 0.8)
D	(0.5, 0.6, 0.9)	(1, 1, 0)	(0.1, 0.6, 0.9)	(0.4, 0.5, 0.9)	(0.9, 0.8, 0.2)
H	(0.2, 0.3, 0.6)	(0.1, 0.6, 0.9)	(1, 1, 0)	(0.2, 0.9, 0.6)	(0.1, 0.9, 0.7)
W	(0.1, 0.2, 0.3)	(0.4, 0.5, 0.9)	(0.2, 0.9, 0.6)	(1, 1, 0)	(0.2, 0.5, 0.9)
O	(0.4, 0.6, 0.8)	(0.9, 0.8, 0.2)	(0.1, 0.9, 0.7)	(0.2, 0.5, 0.9)	(1, 1, 0)

The lower and upper approximations of A are as follows:

$$\underline{\hat{R}}A = \{(P, 0.45, 0.29, 0.49), (D, 0.51, 0.25, 0.49), (H, 0.51, 0.35, 0.45), (W, 0.45, 0.41, 0.40), (O, 0.51, , 0.25, 0.49)\},$$

$$\overline{\hat{R}}A = \{(P, 0.50, 0.35, 0.37), (D, 0.52, 0.43, 0.48), (H, 0.51, 0.43, 0.45), (W, 0.47, 0.43, 0.37), (O, 0.52, 0.43, 0.48)\}.$$

Let E={(P, D), (P, H), (D, H), (D, W), (H, W), (H, O), (W, P), (W, O), (O, P), (O, D)} be the subset of $X \times X$, and the neutrosophic tolerance relation $\hat{S}$ on E is given as follows:

$\hat{S}$	(P,D)	(P,H)	(D,H)	(D,W)	(H,W)
(P,D)	(1, 1, 0)	(0.1, 0.6, 0.9)	(0.1, 0.6, 0.9)	(0.4, 0.5, 0.9)	(0.2, 0.3, 0.9)
(P,H)	(0.1, 0.6, 0.9)	(1, 1, 0)	(0.5, 0.6, 0.9)	(0.2, 0.6, 0.9)	(0.2, 0.3, 0.6)
(D,H)	(0.1, 0.6, 0.9)	(0.5, 0.6, 0.9)	(1, 1, 0)	(0.2, 0.9, 0.6)	(0.1, 0.6, 0.9)
(D,W)	(0.4, 0.5, 0.9)	(0.2, 0.6, 0.9)	(0.2, 0.6, 0.9)	(1, 1, 0)	(0.1, 0.6, 0.9)
(H,W)	(0.2, 0.3, 0.9)	(0.2, 0.3, 0.6)	(0.1, 0.6, 0.9)	(0.1, 0.6, 0.9)	(1, 1, 0)
(H,O)	(0.2, 0.3, 0.6)	(0.1, 0.3, 0.7)	(0.1, 0.6, 0.9)	(0.1, 0.5, 0.9)	(0.2, 0.5, 0.9)
(W,P)	(0.1, 0.2, 0.9)	(0.1, 0.2, 0.6)	(0.2, 0.3, 0.9)	(0.1, 0.2, 0.9)	(0.1, 0.2, 0.6)
(W,O)	(0.1, 0.2, 0.3)	(0.1, 0.2, 0.7)	(0.1, 0.5, 0.9)	(0.2, 0.5, 0.9)	(0.2, 0.5, 0.9)
(O,P)	(0.4, 0.6, 0.9)	(0.2, 0.3, 0.8)	(0.2, 0.3, 0.6)	(0.1, 0.2, 0.3)	(0.1, 0.2, 0.7)
(O,D)	(0.4, 0.6, 0.8)	(0.1, 0.6, 0.9)	(0.1, 0.6, 0.9)	(0.4, 0.5, 0.9)	(0.1, 0.5, 0.9)

$\hat{S}$	(H,O)	(W,P)	(W,O)	(O,P)	(O,D)
(P,D)	(0.2, 0.3, 0.6)	(0.1, 0.2, 0.9)	(0.1, 0.2, 0.3)	(0.4, 0.6, 0.9)	(0.4, 0.6, 0.8)
(P,H)	(0.1, 0.3, 0.7)	(0.1, 0.2, 0.6)	(0.1, 0.2, 0.7)	(0.2, 0.3, 0.8)	(0.1, 0.6, 0.9)
(D,H)	(0.2, 0.3, 0.9)	(0.1, 0.5, 0.9)	(0.2, 0.3, 0.6)	(0.1, 0.6, 0.9)	(0.1, 0.6, 0.9)
(D,W)	(0.1, 0.2, 0.9)	(0.2, 0.5, 0.9)	(0.1, 0.2, 0.3)	(0.4, 0.5, 0.9)	(0.1, 0.5, 0.9)
(H,W)	(0.1, 0.2, 0.6)	(0.2, 0.5, 0.9)	(0.1, 0.2, 0.7)	(0.1, 0.5, 0.9)	(0.2, 0.5, 0.9)
(H,O)	(1, 1, 0)	(0.2, 0.6, 0.8)	(0.2, 0.9, 0.6)	(0.1, 0.6, 0.8)	(0.1, 0.8, 0.7)
(W,P)	(0.2, 0.6, 0.8)	(1, 1, 0)	(0.4, 0.6, 0.8)	(0.2, 0.5, 0.9)	(0.2, 0.5, 0.9)
(W,O)	(0.2, 0.9, 0.6)	(0.4, 0.6, 0.8)	(1, 1, 0)	(0.2, 0.5, 0.9)	(0.2, 0.5, 0.9)
(O,P)	(0.1, 0.6, 0.8)	(0.2, 0.5, 0.9)	(0.2, 0.5, 0.9)	(1, 1, 0)	(0.5, 0.6, 0.9)
(O,D)	(0.1, 0.8, 0.7)	(0.2, 0.5, 0.9)	(0.2, 0.5, 0.9)	(0.5, 0.6, 0.9)	(1, 1, 0)

Thus, the lower and upper approximations of B are calculated as follows:

$$\underline{\hat{S}}B = \{((P, D), 0.42, 0.23, 0.47), ((P, H), 0.45, 0.28, 0.45), ((D, H), 0.50, 0.21, 0.45), ((D, W), 0.43, 0.22, 0.45), ((H, W), 0.41, 0.30, 0.44), ((H, O), 0.51, 0.22, 0.46), ((W, P), 0.42, 0.26, 0.40), ((W, O), 0.42, 0.23, 0.44), ((O, P), 0.43, 0.25, 0.48), ((O, D), 0.50, 0.22, 0.48)\},$$

$$\overline{\hat{S}}B = \{((P, D), 0.42, 0.30, 0.44), ((P, H), 0.50, 0.30, 0.41), ((D, H), 0.50, 0.30, 0.45), ((D, W), 0.43, 0.30, 0.45), ((H, W), 0.41, 0.30, 0.44), ((H, O), 0.51, 0.30, 0.46), ((W, P), 0.42, 0.26, 0.37), ((W, O), 0.45, 0.30, 0.44), ((O, P), 0.50, 0.28, 0.45), ((O, D), 0.50, 0.30, 0.47)\}.$$

Thus, $\underline{G} = (\underline{\hat{R}}A, \underline{\hat{S}}B)$ and $\overline{G} = (\overline{\hat{R}}A, \overline{\hat{S}}B)$ are the neutrosophic digraphs as shown in Fig. 6.30. To find the best company, we use the following formula:

$$S(v_i) = \sum_{v_i \in X} \frac{(T_{\underline{\hat{R}}A}(v_i) \times T_{\overline{\hat{R}}A}(v_i)) + (I_{\underline{\hat{R}}A}(v_i) \times I_{\overline{\hat{R}}A}(v_i)) - (F_{\underline{\hat{R}}A}(v_i) \times F_{\overline{\hat{R}}A}(v_i))}{1 - \{T(v_i v_j) + I(v_i v_j) - F(v_i v_j)\}},$$

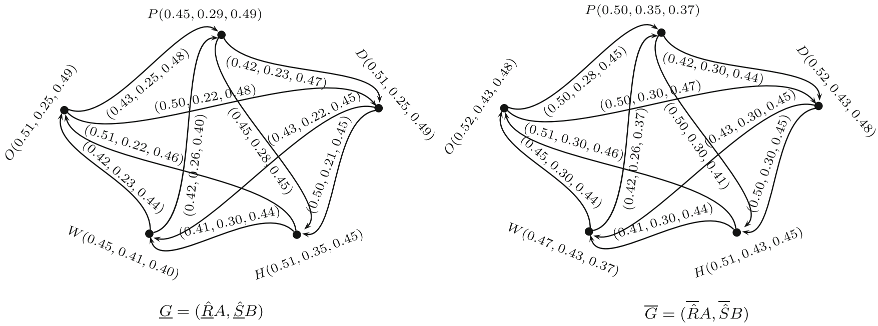


Fig. 6.30 $G = (\underline{G}, \overline{G})$

where

$$T(v_i v_j) = \max_{v_j \in X} T_{\underline{\hat{S}}B}(v_i v_j) \times \max_{v_j \in X} T_{\overline{\hat{S}}B}(v_i v_j),$$
$$I(v_i v_j) = \max_{v_j \in X} I_{\underline{\hat{S}}B}(v_i v_j) \times \max_{v_j \in X} I_{\overline{\hat{S}}B}(v_i v_j),$$
$$F(v_i v_j) = \min_{v_j \in X} F_{\underline{\hat{S}}B}(v_i v_j) \times \min_{v_j \in X} F_{\overline{\hat{S}}B}(v_i v_j).$$

By direct calculations we have

$$S(P) = 0.167, S(D) = 0.156, S(H) = 0.268, S(W) = 0.272, S(O) = 0.155.$$

From the above calculations it is clear that Waves is the best company for refrigerator.

6.5.2 *Context of Recruitment*

Choosing the right candidate for the position available is not something that should be left to chance or guesswork.

How to Choose the Right Candidate

In any recruitment process the ability of the candidate is weighed up against the suitability of the candidate. Pakistan Telecommunication Company Limited (PTCL) wants to recruit a person for the post of administrator. To keep the procedure simple the company wants to appoint their employee on the basis of education(Edu) and experience (Exp). Let $X = \{(C1, Edu), (C1, Exp), (C2, Edu), (C2, Exp), (C3, Edu), (C3, Exp)\}$ be the set of candidates who applied to the post and their corresponding attributes. Let $\hat{R}$ be a neutrosophic tolerance relation on X given as follows:

$\hat{R}$	(C1,Edu)	(C1,Exp)	(C2,Edu)	(C2,Exp)	(C3,Edu)	(C3,Exp)
(C1,Edu)	(1, 1, 0)	(0.3, 0.6, 0.1)	(0.6, 0.7, 0.2)	(0.6, 0.5, 0.8)	(0.3, 0.2, 0.1)	(0.9, 0.1, 0.1)
(C1,Exp)	(0.3, 0.6, 0.1)	(1, 1, 0)	(0.9, 0.9, 0.3)	(0.8, 0.7, 0.6)	(0.4, 0.5, 0.9)	(0.3, 0.1, 0.1)
(C2,Edu)	(0.6, 0.7, 0.2)	(0.9, 0.9, 0.3)	(1, 1, 0)	(0.6, 0.5, 0.1)	(0.3, 0.2, 0.1)	(0.4, 0.8, 0.7)
(C2,Exp)	(0.6, 0.5, 0.8)	(0.8, 0.7, 0.6)	(0.6, 0.5, 0.1)	(1, 1, 0)	(0.1, 0.1, 0.2)	(0.5, 0.6, 0.7)
(C3,Edu)	(0.3, 0.2, 0.1)	(0.4, 0.5, 0.9)	(0.3, 0.2, 0.1)	(0.1, 0.1, 0.2)	(1, 1, 0)	(0.2, 0.1, 0.2)
(C3,Exp)	(0.9, 0.1, 0.1)	(0.3, 0.1, 0.1)	(0.4, 0.8, 0.7)	(0.5, 0.6, 0.7)	(0.2, 0.1, 0.2)	(1, 1, 0)

Let $A = \{((C1, Edu), 0.9, 0.1, 0.5), ((C1, Exp), 0.2, 0.6, 0.5), ((C2, Edu), 0.7, 0.2, 0.3), ((C2, Exp), 0.1, 0.3, 0.9), ((C3, Edu), 0.4, 0.6, 0.8), ((C3, Exp), 0.8, 0.1, 0.2)\}$ be a neutrosophic set defined on X. Then the lower and upper approximations of A are given as:

$$\begin{aligned}\underline{\hat{R}}A = \{&((C1, Edu), 0.2, 0.1, 0.6), ((C1, Exp), 0.2, 0.2, 0.8), ((C2, Edu), 0.1, 0.2, 0.6),\\ &((C2, Exp), 0.1, 0.3, 0.9), ((C3, Edu), 0.2, 0.6, 0.8), ((C3, Exp), 0.2, 0.1, 0.5)\},\\ \overline{\hat{R}}A = \{&((C1, Edu), 0.9, 0.6, 0.2), ((C1, Exp), 0.7, 0.6, 0.2), ((C2, Edu), 0.7, 0.6, 0.3),\\ &((C2, Exp), 0.6, 0.6, 0.3), ((C3, Edu), 0.4, 0.6, 0.2), ((C3, Exp, 0.9, 0.3, 0.2)\}.\end{aligned}$$

Let $E = \{(C1, Edu)(C1, Exp), (C1, Exp)(C2, Edu), (C1, Edu)(C3, Exp), (C3, Exp)(C1, Exp), (C1, Exp)(C2, Exp), (C2, Exp)(C2, Edu), (C3, Exp)(C3, Edu), (C3, Edu)(C2, Exp), (C3, Exp)(C2, Exp)\} \subseteq X \times X$ and $\hat{S}$ be a neutrosophic tolerance relation on E given as follows:

$\hat{S}$	(C1,Edu)(C1,Exp)	(C1,Exp)(C2,Edu)	(C1,Edu)(C3,Exp)	(C3,Exp)(C1,Exp)	(C1,Exp)(C2,Exp)
(C1,Edu)(C1,Exp)	(1, 1, 0)	(0.3, 0.6, 0.3)	(0.3, 0.1, 0.1)	(0.9, 0.1, 0.1)	(0.3, 0.6, 0.6)
(C1,Exp)(C2,Edu)	(0.3, 0.6, 0.3)	(1, 1, 0)	(0.3, 0.6, 0.7)	(0.3, 0.1, 0.3)	(0.6, 0.5, 0.1)
(C1,Edu)(C3,Exp)	(0.3, 0.1, 0.1)	(0.3, 0.6, 0.7)	(1, 1, 0)	(0.3, 0.1, 0.1)	(0.3, 0.6, 0.7)
(C3,Exp)(C1,Exp)	(0.9, 0.1, 0.1)	(0.3, 0.1, 0.3)	(0.3, 0.1, 0.1)	(1, 1, 0)	(0.3, 0.1, 0.6)
(C1,Exp)(C2,Exp)	(0.3, 0.6, 0.6)	(0.6, 0.5, 0.1)	(0.3, 0.6, 0.7)	(0.3, 0.1, 0.6)	(1, 1, 0)
(C2,Exp)(C2,Edu)	(0.6, 0.5, 0.8)	(0.8, 0.7, 0.6)	(0.4, 0.5, 0.8)	(0.5, 0.6, 0.7)	(0.6, 0.5, 0.6)
(C3,Exp)(C2,Exp)	(0.8, 0.1, 0.6)	(0.3, 0.1, 0.1)	(0.5, 0.1, 0.7)	(0.8, 0.7, 0.6)	(0.3, 0.1, 0.1)
(C3,Exp)(C3,Edu)	(0.4, 0.1, 0.9)	(0.3, 0.1, 0.1)	(0.2, 0.1, 0.2)	(0.4, 0.5, 0.9)	(0.1, 0.1, 0.2)
(C3,Edu)(C2,Exp)	(0.3, 0.2, 0.6)	(0.4, 0.5, 0.9)	(0.3, 0.2, 0.7)	(0.2, 0.1, 0.6)	(0.4, 0.5, 0.9)

$\hat{S}$	(C2,Exp)(C2,Edu)	(C3,Exp)(C2,Exp)	(C3,Exp)(C3,Edu)	(C3,Edu)(C2,Exp)
(C1,Edu)(C1,Exp)	(0.6, 0.5, 0.8)	(0.8, 0.1, 0.6)	(0.4, 0.1, 0.9)	(0.3, 0.2, 0.6)
(C1,Exp)(C2,Edu)	(0.8, 0.7, 0.6)	(0.3, 0.1, 0.1)	(0.3, 0.1, 0.1)	(0.4, 0.5, 0.9)
(C1,Edu)(C3,Exp)	(0.4, 0.5, 0.8)	(0.5, 0.1, 0.7)	(0.2, 0.1, 0.2)	(0.3, 0.2, 0.7)
(C3,Exp)(C1,Exp)	(0.5, 0.6, 0.7)	(0.8, 0.7, 0.6)	(0.4, 0.5, 0.9)	(0.2, 0.1, 0.6)
(C1,Exp)(C2,Exp)	(0.6, 0.5, 0.6)	(0.3, 0.1, 0.1)	(0.1, 0.1, 0.2)	(0.4, 0.5, 0.9)
(C2,Exp)(C2,Edu)	(1, 1, 0)	(0.5, 0.5, 0.7)	(0.3, 0.2, 0.7)	(0.1, 0.1, 0.2)
(C3,Exp)(C2,Exp)	(0.5, 0.5, 0.7)	(1, 1, 0)	(0.1, 0.1, 0.2)	(0.2, 0.1, 0.2)
(C3,Exp)(C3,Edu)	(0.3, 0.2, 0.7)	(0.1, 0.1, 0.2)	(1, 1, 0)	(0.1, 0.1, 0.2)
(C3,Edu)(C2,Exp)	(0.1, 0.1, 0.2)	(0.2, 0.1, 0.2)	(0.1, 0.1, 0.2)	(1, 1, 0)

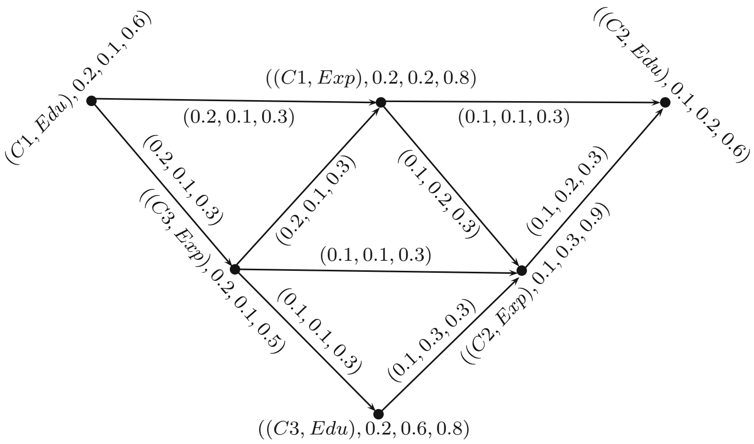


Fig. 6.31 Neutrosophic digraph $\underline{G} = (\underline{\hat{R}}A, \underline{\hat{S}}B)$

Let B={$((C1, Edu)(C1, Exp), 0.2, 0.1, 0.1)$, $((C1, Exp)(C2, Edu), 0.1, 0.1, 0.3)$, $((C1, Edu)(C3, Exp)$,0.2, 0.1, 0.2), $((C3, Exp)(C1, Exp), 0.2, 0.1, 0.2)$, $((C1, Exp)(C2, Exp), 0.1, 0.2, 0.3)$, $((C2, Exp)(C2, Edu), 0.1, 0.2, 0.3))$, $((C3, Exp)(C2, Exp), 0.1, 0.1, 0.3)$, $((C3, Exp)(C3, Edu), 0.2, 0.1, 0.2)$, $((C3, Edu)(C2, Exp), 0.1, 0.3, 0.3)$} be neutrosophic rough set on E. Then the lower and upper approximations of B are given as follows:

$$\begin{aligned}\underline{\hat{S}}B = \{&((C1, Edu)(C1, Exp), 0.2, 0.1, 0.3), ((C1, Exp)(C2, Edu), 0.1, 0.1, 0.3),\\ &((C1, Edu)(C3, Exp), 0.2, 0.1, 0.3), ((C3, Exp)(C1, Exp), 0.2, 0.1, 0.3),\\ &((C1, Exp)(C2, Exp), 0.1, 0.2, 0.3), ((C2, Exp)(C2, Edu, 0.1, 0.2, 0.3)),\\ &((C3, Exp)(C2, Exp), 0.1, 0.1, 0.3), ((C3, Exp)(C3, Edu), 0.1, 0.1, 0.3),\\ &((C3, Edu)(C2, Exp), 0.1, 0.3, 0.3)\},\end{aligned}$$

$$\begin{aligned}\overline{\hat{S}}B = \{&((C1, Edu)(C1, Exp), 0.2, 0.2, 0.1), ((C1, Exp)(C2, Edu), 0.2, 0.3, 0.2),\\ &((C1, Edu)(C3, Exp), 0.2, 0.2, 0.1), ((C3, Exp)(C1, Exp), 0.2, 0.2, 0.1),\\ &((C1, Exp)(C2, Exp), 0.2, 0.2, 0.1), ((C2, Exp)(C2, Edu, 0.2, 0.2, 0.3)),\\ &((C3, Exp)(C2, Exp), 0.2, 0.2, 0.2), ((C3, Exp)(C3, Edu), 0.2, 0.2, 0.2),\\ &((C3, Edu)(C2, Exp), 0.2, 0.3, 0.2)\}.\end{aligned}$$

Thus, $\underline{G} = (\underline{\hat{R}}A, \underline{\hat{S}}B)$ and $\overline{G} = (\overline{\hat{R}}A, \overline{\hat{S}}B)$ are the neutrosophic digraphs as shown in Figs. 6.31 and 6.32.

To find the best employee using the following calculations, we have

$$I_{\overline{\hat{R}}A}(C1) = \frac{I_{\overline{\hat{R}}A}(C1, Edu) + I_{\overline{\hat{R}}A}(C1, Exp)}{2} = \frac{0.9 + 0.7}{2} = 0.8$$

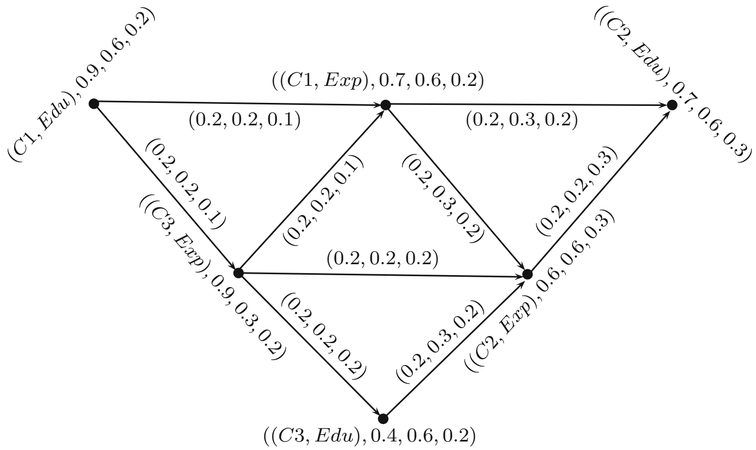


Fig. 6.32 Neutrosophic digraph $\overline{G} = (\overline{\hat{R}A}, \overline{\hat{S}B})$

$$I_{\overline{\hat{R}A}}(C2) = \frac{I_{\overline{\hat{R}A}}(C2, Edu) + I_{\overline{\hat{R}A}}(C2, Exp)}{2} = \frac{0.7 + 0.6}{2} = 0.65$$

$$I_{\overline{\hat{R}A}}(C3) = \frac{I_{\overline{\hat{R}A}}(C3, Edu) + I_{\overline{\hat{R}A}}(C3, Exp)}{2} = \frac{0.4 + 0.9}{2} = 0.65$$

$$\max\{I_{\overline{\hat{R}A}}(C1), I_{\overline{\hat{R}A}}(C2), I_{\overline{\hat{R}A}}(C3)\} = \max\{0.8, 0.65, 0.65\} = 0.8.$$

Thus, $C1$ is best employee for the post under consideration. So PTCL can hire $C1$ for the post of administrator.

6.6 Comparative Analysis of Hybrid Models

Rough neutrosophic digraphs and neutrosophic rough digraphs are two different notions on the basis of their construction and approach. In rough neutrosophic digraphs, the relation defined on the universe of discourse is crisp equivalence relation that only classifies the elements which are related. On the other hand, in neutrosophic rough digraphs the relation defined on the set is neutrosophic tolerance relation. The neutrosophic tolerance relation not just classifies the elements of the set which are related but also expresses their relation in terms of three components, namely truth-membership (T), indeterminacy (I) and falsity (F). This approach leaves a room for indeterminacy and incompleteness.

Below we apply the method of rough neutrosophic digraphs to the above-presented application (online reviews and ratings).

Here $X = \{Pel(P), Dawlance(D), Haier(H), Waves(W), Orient(O)\}$ and the neutrosophic set on X according to the reviews will be

A={(P, 0.45, 0.29, 0.37), (D, 0.52, 0.25, 0.49), (H, 0.51, 0.43, 0.45)}(W, 0.47, 0.41, 0.38), (O, 0.51, 0.35, 0.48)}. The equivalence relation on X is given below

$\hat{R}$	P	D	H	W	O
P	1	0	1	0	1
D	0	1	0	0	0
H	1	0	1	0	1
W	0	0	0	1	0
O	1	0	1	0	1

The lower and upper approximations of A are as follows:

$$\underline{\hat{R}}A = \{(P, 0.45, 0.29, 0.48), (D, 0.52, 0.25, 0.49), (H, 0.45, 0.29, 0.48), (W, 0.47, 0.41, 0.38), (O, 0.45, , 0.29, 0.48)\},$$

$$\overline{\hat{R}}A = \{(P, 0.51, 0.43, 0.37), (D, 0.52, 0.25, 0.49), (H, 0.51, 0.43, 0.37), (W, 0.47, 0.41, 0.38), (O, 0.51, 0.43, 0.37)\}.$$

Let E={(P, D), (P, H), (D, H), (D, W), (H, W), (H, O), (W, P), (W, O), (O, P), (O, D)} be the subset of $X \times X$, and the equivalence relation $\hat{S}$ on E is given as follows:

$\hat{S}$	(P,D)	(P,H)	(D,H)	(D,W)	(H,W)	(H,O)	(W,P)	(W,O)	(O,P)	(O,D)
(P,D)	1	0	0	0	0	0	0	0	0	0
(P,H)	0	1	0	0	0	1	0	0	1	1
(D,H)	0	0	1	0	0	0	0	0	0	0
(D,W)	0	0	0	1	0	0	0	0	0	0
(H,W)	0	0	0	0	1	0	0	0	0	0
(H,O)	0	1	0	0	0	1	0	0	1	1
(W,P)	0	0	0	0	0	0	1	1	0	0
(W,O)	0	0	0	0	0	0	1	1	0	0
(O,P)	0	1	0	0	0	1	0	0	1	1
(O,D)	0	1	0	0	0	1	0	0	1	1

Thus, the lower and upper approximations of B are calculated as follows:

$$\underline{\hat{S}}B = \{((P, D), 0.45, 0.25, 0.48), ((P, H), 0.42, 0.24, 0.37), ((D, H), 0.45, 0.25, 0.47), ((D, W), 0.45, 0.24, 0.48), ((H, W), 0.45, 0.29, 0.38), ((H, O), 0.42, 0.24, 0.37), ((W, P), 0.42, 0.22, 0.37), ((W, O), 0.42, 0.22, 0.37), ((O, P), 0.42, 0.24, 0.37), ((O, D), 0.42, 0.24, 0.37)\},$$

$$\overline{\hat{S}}B = \{((P, D), 0.45, 0.25, 0.48), ((P, H), 0.45, 0.29, 0.37), ((D, H), 0.45, 0.25, 0.47), ((D, W), 0.45, 0.24, 0.48), ((H, W), 0.45, 0.29, 0.38), ((H, O), 0.45, 0.29, 0.37), ((W, P), 0.45, 0.29, 0.35), ((W, O), 0.45, 0.29, 0.35), ((O, P), 0.45, 0.29, 0.37), ((O, D), 0.45, 0.29, 0.37)\}.$$

To find the best company, we use the following formula:

$$S(v_i) = \sum_{v_i \in X} \frac{(T_{\underline{\hat{R}}A}(v_i) \times T_{\overline{\hat{R}}A}(v_i)) + (I_{\underline{\hat{R}}A}(v_i) \times I_{\overline{\hat{R}}A}(v_i)) - (F_{\underline{\hat{R}}A}(v_i) \times F_{\overline{\hat{R}}A}(v_i))}{1 - \{T(v_i v_j) + I(v_i v_j) - F(v_i v_j)\}},$$

where

$$T(v_i v_j) = \max_{v_j \in X} T_{\underline{\hat{S}}B}(v_i v_j) \times \max_{v_j \in X} T_{\overline{\hat{S}}B}(v_i v_j),$$
$$I(v_i v_j) = \max_{v_j \in X} I_{\underline{\hat{S}}B}(v_i v_j) \times \max_{v_j \in X} I_{\overline{\hat{S}}B}(v_i v_j),$$
$$F(v_i v_j) = \min_{v_j \in X} F_{\underline{\hat{S}}B}(v_i v_j) \times \min_{v_j \in X} F_{\overline{\hat{S}}B}(v_i v_j).$$

By direct calculations, we have

$$S(P) = 0.20, S(D) = 0.0971, S(H) = 0.2077, S(W) = 0.2790, S(O) = 0.2011.$$

From the above calculations, we have Waves as the best choice and Dawlance as the least choice for refrigerator; this is because the relation applied in this method is crisp equivalence relation which does not consider the uncertainty between the companies of same equivalence class, whereas in our proposed method least choice for refrigerator is different. So, the results may vary when we apply the method of rough neutrosophic digraphs and neutrosophic rough digraphs to the same application. It means that rough neutrosophic digraphs and neutrosophic rough digraphs have a different approach.

Chapter 7
Graphs Under Neutrosophic Soft Environment

In this chapter, we present concepts of neutrosophic soft graphs and intuitionistic neutrosophic soft graphs. We describe methods of their construction. We consider applications of neutrosophic soft graphs and intuitionistic neutrosophic soft graphs. This chapter is due to [22, 23].

7.1 Introduction

In 1999, Molodtsov [116] initiated soft set theory as a new approach for modelling uncertainties. Later on, Maji et al. [112] expanded this theory to fuzzy soft set theory. Based on the idea of parametrization, a soft set gives a series of approximate descriptions of a complicate object from various different aspects. Each approximate description has two parts, namely predicate and approximate value set. A soft set can be determined by a set-valued mapping assigning to each parameter exactly one crisp subset of the universe. More specifically, we can define the notion of soft set in the following way: let X be the universe of discourse and P be the universe of all possible parameters related to the objects in X. Each parameter is a word or a sentence. In most cases, parameters are considered to be attributes, characteristics or properties of objects in X. The pair (X, P) is also known as a *soft universe*. The power set of X is denoted by $\mathcal{P}(X)$.

Definition 7.1 A pair $F_M = (F, M)$ is called *soft set* over X, where $M \subseteq P$, and F is a set-valued function $F : M \to \mathcal{P}(X)$. In other words, a soft set over X is a parameterized family of subsets of X. For any $e \in M$, $F(e)$ may be considered as set of e-approximate elements of soft set (F, M). A soft set F_M over the universe X can be represented by the set of ordered pairs

$$F_M = \{(e, F_M(e)) \mid e \in M, F_M(e) \in \mathcal{P}(X)\}.$$

M. Akram, *Single-Valued Neutrosophic Graphs*, Infosys Science Foundation Series, https://doi.org/10.1007/978-981-13-3522-8_7

Table 7.1 Tabular arrangement of the soft set

Parameters	1	2	3	4	5	6	7	8	9	10
e_1	0	1	0	1	0	1	0	1	0	1
e_2	0	0	1	0	0	1	0	0	1	0
e_3	0	0	0	1	0	0	0	1	0	0
e_4	0	0	0	0	1	0	0	0	0	1

By means of parametrization, a soft set produces a series of approximate descriptions of a complicated object being perceived from various points of view. It is apparent that a soft set (F, M) over a universe X can be viewed as a parameterized family of subsets of X. For any parameter $e \in M$, the subset $F(e) \subseteq X$ may be interpreted as the set of *e-approximate elements.*

Example 7.1 Let $X = \{1, 2, 3, \ldots, 10\}$ be a set of first ten positive integers and $P = \{e_1, e_2, e_3, e_4, e_5\}$ be the set of parameters, where
e_1 stands for the parameter "divisibility by 2"
e_2 stands for the parameter "divisibility by 3"
e_3 stands for the parameter "divisibility by 4"
e_4 stands for the parameter "divisibility by 5"
e_5 stands for the parameter "divisibility by prime numbers".
If $M = \{e_1, e_2, e_3, e_4\}$, then the soft set (F, M) is given by

$$S = \{F(e_1), F(e_2), F(e_3), F(e_4)\},$$

where

$$F(e_1) = \{2, 4, 6, 8, 10\}, F(e_2) = \{3, 6, 9\}, F(e_3) = \{4, 8\}, F(e_4) = \{5, 10\}.$$

Thus the soft set (F, M) is a parameterized family of subsets of X. The tabular arrangement of the soft set (F, M) is shown in Table 7.1.

Example 7.2 Suppose a soft set (F, M) describes attractiveness of the shirts which the authors are going to wear. Here

$$X = \text{the set of all shirts under consideration} = \{x_1, x_2, x_3, x_4, x_5\},$$
$$M = \{\text{colorful, bright, cheap, warm}\} = \{e_1, e_2, e_3, e_4\},$$
$$F(e_1) = \{x_1, x_5\}, F(e_2) = \{x_2, x_4\}, F(e_3) = \{x_2, x_5\}, F(e_4) = \{x_1, x_2, x_5\}.$$

So, the soft set (F, M) is a subfamily $\{F(e_1), F(e_2), F(e_3), F(e_4)\}$ of $\mathcal{P}(X)$, which represents the attractiveness of shirts w. r. t the parameters given.

In 2013, Maji [111] introduced the concept of neutrosophic soft sets and Deli and Broumi [66] introduced the notion of neutrosophic soft relations.

Definition 7.2 Let X be an initial universe. Let P be a set of parameters and $M \subset P$. Let $\mathcal{P}(X)$ denote the set of all neutrosophic sets of X. The collection (F, M) is termed to be the *neutrosophic soft set* over X, where F is a mapping given by $F : M \to \mathcal{P}(X)$.

Definition 7.3 Let (F, M) and (G, N) be two neutrosophic soft sets over the common universe X. (F, M) is said to be neutrosophic soft subset of (G, N) if $M \subset N$, and

$$\begin{aligned}
&T_{F(e)}(x) \le T_{G(e)}(x),\\
&I_{F(e)}(x) \le I_{G(e)}(x),\\
&F_{F(e)}(x) \ge F_{G(e)}(x) \text{ for all } e \in M,\ x \in X.
\end{aligned}$$

Definition 7.4 Let (H, M) and (G, N) be two neutrosophic soft sets over the common universe X. The *union of two neutrosophic soft sets* (H, M) and (G, N) is neutrosophic soft set $(K, C) = (H, M) \cup (G, N)$, where $C = M \cup N$ and the truth-membership, indeterminacy-membership and falsity-membership of (K, C) are defined by

$$T_{K(e)}(x) = \begin{cases} T_{H(e)}(x), & \text{if } e \in M - N,\\ T_{G(e)}(x), & \text{if } e \in N - M,\\ \max(T_{H(e)}(x), T_{G(e)}(x)) & \text{if } e \in M \cap N. \end{cases}$$

$$I_{K(e)}(x) = \begin{cases} I_{H(e)}(x), & \text{if } e \in M - N,\\ I_{G(e)}(x), & \text{if } e \in N - M,\\ \max(I_{H(e)}(x), I_{G(e)}(x)) & \text{if } e \in M \cap N. \end{cases}$$

$$F_{K(e)}(x) = \begin{cases} F_{H(e)}(x), & \text{if } e \in M - N,\\ F_{G(e)}(x), & \text{if } e \in N - M,\\ \min(F_{H(e)}(x), F_{G(e)}(x)) & \text{if } e \in M \cap N. \end{cases}$$

Definition 7.5 Let (H, M) and (G, N) be two neutrosophic soft sets over the common universe X. The *intersection of two neutrosophic soft sets* (H, M) and (G, N) is neutrosophic soft set $(K, C) = (H, M) \cap (G, N)$, where $C = M \cap N$ and the truth-membership, indeterminacy-membership and falsity-membership of (K, C) are defined by

$$T_{K(e)}(x) = \begin{cases} T_{H(e)}(x), & \text{if } e \in M - N,\\ T_{G(e)}(x), & \text{if } e \in N - M,\\ \min(T_{H(e)}(x), T_{G(e)}(x)) & \text{if } e \in M \cap N. \end{cases}$$

$$I_{K(e)}(x) = \begin{cases} I_{H(e)}(x), & \text{if } e \in M - N,\\ I_{G(e)}(x), & \text{if } e \in N - M,\\ \min(I_{H(e)}(x), I_{G(e)}(x)) & \text{if } e \in M \cap N. \end{cases}$$

$$F_{K(e)}(x) = \begin{cases} F_{H(e)}(x), & \text{if } e \in M - N, \\ F_{G(e)}(x), & \text{if } e \in N - M, \\ \max(F_{H(e)}(x), F_{G(e)}(x)) & \text{if } e \in M \cap N. \end{cases}$$

Definition 7.6 Let (H, M) and (G, N) be two neutrosophic soft sets over the same universe X. The *Cartesian product* of (H, M) and (G, N) is denoted by $(H, M) \times (G, N) = (K, M \times N)$, and the truth-membership, indeterminacy-membership and falsity-membership functions of $(K, M \times N)$ are defined by

$$T_{K(a,b)}(x) = \min\{T_{H(a)}(x), T_{G(b)}(x)\},$$
$$I_{K(a,b)}(x) = \min\{I_{H(a)}(x), I_{G(b)}(x)\},$$
$$F_{K(a,b)}(x) = \max\{F_{H(a)}(x), F_{G(b)}(x)\}.$$

Definition 7.7 Let (H, M) and (G, N) be two neutrosophic soft sets over the same universe X. A *neutrosophic soft relation* from (H, M) to (G, N) is of the form (R, C), where $C \subset M \times N$ and $R(x, y) \subset (H, M) \times (G, N)$ for all $(e_1, e_2) \in C$.

7.2 Neutrosophic Soft Graphs

Definition 7.8 A *single-valued neutrosophic soft graph* on a nonempty set X is an three-ordered tuple $G = (F, K, M)$ if it satisfies the following conditions:
(i) M is a nonempty set of parameters.
(ii) (F, M) is a single-valued neutrosophic soft set over X.
(iii) (K, M) is a single-valued neutrosophic soft set over $E \subseteq X \times X$.
(iv) $(F(e), K(e))$ is a single-valued neutrosophic graph, that is,

$$T_{K(e)}(xy) \le \min\{T_{F(e)}(x), T_{F(e)}(y)\},$$
$$I_{K(e)}(xy) \le \min\{I_{F(e)}(x), I_{F(e)}(y)\},$$
$$F_{K(e)}(xy) \le \max\{F_{F(e)}(x), F_{F(e)}(y)\}$$

such that

$$0 \le T_{K(e)}(xy) + I_{K(e)}(xy) + F_{K(e)}(xy) \le 3 \quad \forall\, e \in M,\ x, y \in X.$$

The neutrosophic graph $(F(e), K(e))$ is denoted by $H(e)$ for convenience. A single-valued neutrosophic soft graph is a parameterized family of single-valued neutrosophic graphs. The class of all single-valued neutrosophic soft graphs is denoted by $\mathcal{NS}(G^*)$. Note that

$$T_{K(e)}(xy) = I_{K(e)}(xy) = 0,\ F_{K(e)}(xy) = 1, \quad \forall\, xy \in X \times X - E,\ e \notin M.$$

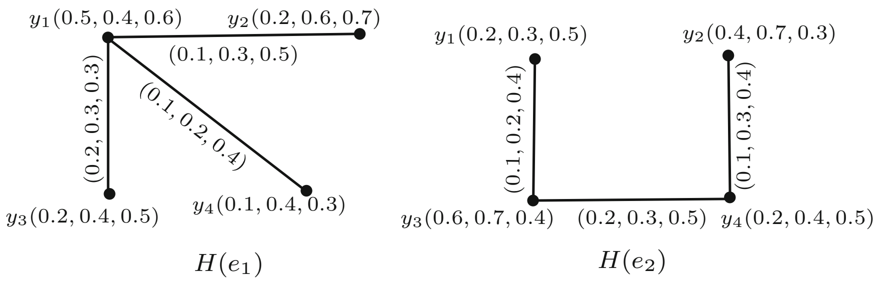


Fig. 7.1 Neutrosophic soft graph $G = \{H(e_1), H(e_2)\}$

Definition 7.9 Let $G_1 = (F_1, K_1, M)$ and $G_2 = (F_2, K_2, N)$ be two neutrosophic soft graphs of G^*. Then G_1 is neutrosophic soft subgraph of G_2 if

(i) $M \subseteq N$,
(ii) $H_1(e)$ is a partial subgraph of $H_2(e)$ for all $e \in M$.

Example 7.3 Consider a simple graph G^* such that $X = \{y_1, y_2, y_3, y_4\}$ and $E = \{y_1y_2, y_1y_3, y_1y_4, y_2y_4, y_3y_4\}$. Let $M = \{e_1, e_2\}$ be a set of parameters, and let (F, M) be a neutrosophic soft set over X with neutrosophic approximation function
$F : M \to \mathcal{P}(X)$ defined by

$$F(e_1) = \{(y_1, 0.5, 0.4, 0.6), (y_2, 0.2, 0.6, 0.7), (y_3, 0.2, 0.4, 0.5), (y_4, 0.1, 0.4, 0.3)\},$$

$$F(e_2) = \{(y_1, 0.2, 0.3, 0.5), (y_2, 0.4, 0.7, 0.3), (y_3, 0.6, 0.7, 0.4), (y_4, 0.2, 0.4, 0.5)\}.$$

Let (K, M) be a neutrosophic soft set over E with neutrosophic approximation function
$K : M \to \mathcal{P}(E)$ defined by

$$K(e_1) = \{(y_1y_2, 0.1, 0.3, 0.5), (y_1y_3, 0.2, 0.3, 0.3), (y_1y_4, 0.1, 0.2, 0.4)\},$$

$$K(e_2) = \{(y_1y_3, 0.1, 0.2, 0.4), (y_2y_4, 0.1, 0.3, 0.4), (y_3y_4, 0.2, 0.3, 0.5)\}.$$

Clearly, $H(e_1) = (F(e_1), K(e_1))$ and $H(e_2) = (F(e_2), K(e_2))$ are neutrosophic graphs corresponding to the parameters e_1 and e_2, respectively, as shown in Fig. 7.1.

Hence $G = \{H(e_1), H(e_2)\}$ is a neutrosophic soft graph of G^*.

Tabular representation of a neutrosophic soft graph is given in Table 7.2.

Definition 7.10 The neutrosophic soft graph $G_1 = (G^*, F_1, K_1, N)$ is called *spanning neutrosophic soft subgraph* of $G = (G^*, F, K, M)$ if
(i) $N \subseteq M$,
(ii) $T_{F_1(e)}(y) = T_{F(e)}(y)$,
$I_{F_1(e)}(y) = I_{F(e)}(y)$,
$F_{F_1(e)}(y) = F_{F(e)}(y)$, for all $e \in M$, $y \in X$.

Table 7.2 Tabular representation of neutrosophic soft graph

F	y_1	y_2	y_3	y_4
e_1	(0.5, 0.4, 0.6)	(0.2, 0.6, 0.7)	(0.2, 0.4, 0.5)	(0.1, 0.4, 0.3)
e_2	(0.2, 0.3, 0.5)	(0.4, 0.7, 0.3)	(0.6, 0.7, 0.4)	(0.2, 0.4, 0.5)

K	y_1y_2	y_2y_3	y_1y_3	y_1y_4	y_2y_4	y_3y_4
e_1	(0.1, 0.3, 0.5)	(0.0, 0.0, 1.0)	(0.2, 0.3, 0.3)	(0.1, 0.2, 0.4)	(0.0, 0.0, 1.0)	(0.0, 0.0, 1.0)
e_2	(0.0, 0.0, 1.0)	(0.0, 0.0, 1.0)	(0.1, 0.2, 0.4)	(0.0, 0.0, 1.0)	(0.1, 0.3, 0.4)	(0.2, 0.3, 0.5)

Definition 7.11 Let $G_1 = (F_1, K_1, M)$ and $G_2 = (F_2, K_2, N)$ be two neutrosophic soft graphs of $G_1^* = (X_1, E_1)$ and $G_2^* = (X_2, E_2)$, respectively. The *Cartesian product* of G_1 and G_2 is a neutrosophic soft graph $G = G_1 \times G_2 = (F, K, M \times N)$, where
$(F = F_1 \times F_2, M \times N)$ is a neutrosophic soft set over

$$X = X_1 \times X_2,$$

$(K = K_1 \times K_2, M \times N)$ is a neutrosophic soft set over

$$E = \{((x, y_1), (x, y_2)) : x \in X_1, (y_1, y_2) \in E_2\} \cup \{((x_1, y), (x_2, y)) : y \in X_2, (x_1, x_2) \in E_1\}$$

and $(F, K, M \times N)$ are neutrosophic soft graphs such that
(i)

$$\begin{aligned} T_{F(a,b)}(x, y) &= T_{F_1(a)}(x) \wedge T_{F_2(b)}(y), \\ I_{F(a,b)}(x, y) &= I_{F_1(a)}(x) \wedge I_{F_2(b)}(y), \\ F_{F(a,b)}(x, y) &= F_{F_1(a)}(x) \vee F_{F_2(b)}(y), \quad \forall\, (x, y) \in X,\ (a, b) \in M \times N, \end{aligned}$$

(ii)

$$\begin{aligned} T_{K(a,b)}\big((x, y_1), (x, y_2)\big) &= T_{F_1(a)}(x) \wedge T_{K_2(b)}(y_1, y_2), \\ I_{K(a,b)}\big((x, y_1), (x, y_2)\big) &= I_{F_1(a)}(x) \wedge I_{K_2(b)}(y_1, y_2), \\ F_{K(a,b)}\big((x, y_1), (x, y_2)\big) &= F_{F_1(a)}(x) \vee F_{K_2(b)}(y_1, y_2), \quad \forall\, x \in X_1,\ (y_1, y_2) \in E_2, \end{aligned}$$

(iii)

$$\begin{aligned} T_{K(a,b)}\big((x_1, y), (x_2, y)\big) &= T_{F_2(b)}(y) \wedge T_{K_1(a)}(x_1, x_2), \\ I_{K(a,b)}\big((x_1, y), (x_2, y)\big) &= I_{F_2(b)}(y) \wedge I_{K_1(a)}(x_1, x_2), \\ F_{K(a,b)}\big((x_1, y), (x_2, y)\big) &= F_{F_2(b)}(y) \vee F_{K_1(a)}(x_1, x_2), \quad \forall\, y \in X_2,\ (x_1, x_2) \in E_1. \end{aligned}$$

$H(a, b) = H_1(a) \times H_2(b)$ for all $(a, b) \in M \times N$ are neutrosophic graphs of G.

Example 7.4 Let $M = \{e_1, e_2\}$ and $N = \{e_3, e_4\}$ be a set of parameters. Consider two neutrosophic soft graphs $G_1 = (H_1, M) = \{H_1(e_1), H_1(e_2)\}$ and $G_2 = (H_2, N) = \{H_2(e_3), H_2(e_4)\}$ such that

$$\begin{aligned}
H_1(e_1) &= \big(\{(x_1, 0.2, 0.4, 0.6), (x_2, 0.4, 0.5, 0.7), (x_3, 0.4, 0.5, 0.7)\},\\
&\quad \{(x_1x_2, 0.2, 0.3, 0.4), (x_2x_3, 0.2, 0.3, 0.4), (x_1x_3, 0.1, 0.2, 0.5)\}\big),\\
H_1(e_2) &= \big(\{(x_1, 0.3, 0.5, 0.7), (x_2, 0.4, 0.5, 0.6), (x_3, 0.5, 0.4, 0.3)\},\\
&\quad \{(x_1x_2, 0.2, 0.4, 0.5), (x_1x_3, 0.2, 0.3, 0.4)\}\big),\\
H_2(e_3) &= \big(\{(y_1, 0.40.5, 0.3), (y_2, 0.3, 0.4, 0.1), (y_3, 0.3, 0.5, 0.8), (y_4, 0.5, 0.3, 0.4)\},\\
&\quad \{(y_1y_2, 0.2, 0.3, 0.3), (y_1y_3, 0.2, 0.3, 0.5), (y_3y_4, 0.2, 0.2, 0.5)\}\big),\\
H_2(e_4) &= \big(\{(y_1, 0.4, 0.5, 0.8), (y_2, 0.6, 0.3, 0.7), (y_3, 0.4, 0.4, 0.5), (y_4, 0.7, 0.2, 0.6)\},\\
&\quad \{(y_1y_2, 0.3, 0.4, 0.6), (y_1y_3, 0.2, 0.3, 0.5), (y_1y_4, 0.3, 0.2, 0.5)\}\big) \quad .
\end{aligned}$$

The Cartesian product of G_1 and G_2 is $G_1 \times G_2 = G = (H, M \times N)$, where

$$\begin{aligned}
M \times N &= \{(e_1, e_3), (e_1, e_4), (e_2, e_3), (e_2, e_4)\},\\
H(e_1, e_3) &= H_1(e_1) \times H_2(e_3),\\
H(e_1, e_4) &= H_1(e_1) \times H_2(e_4),\\
H(e_2, e_3) &= H_1(e_2) \times H_2(e_3),\\
H(e_2, e_4) &= H_1(e_2) \times H_2(e_4),
\end{aligned}$$

are neutrosophic graphs of $G = G_1 \times G_2$.

$H(e_1, e_3) = H_1(e_1) \times H_2(e_3)$ is shown in Fig. 7.2.

In the similar way, Cartesian product of

$$\begin{aligned}
H(e_1, e_4) &= H_1(e_1) \times H_2(e_4),\\
H(e_2, e_3) &= H_1(e_2) \times H_2(e_3),\\
H(e_2, e_4) &= H_1(e_2) \times H_2(e_4)
\end{aligned}$$

can be drawn. Hence $G = G_1 \times G_2 = \{H(e_1, e_3), H(e_1, e_4), H(e_2, e_3), H(e_2, e_4)\}$ is a neutrosophic soft graph.

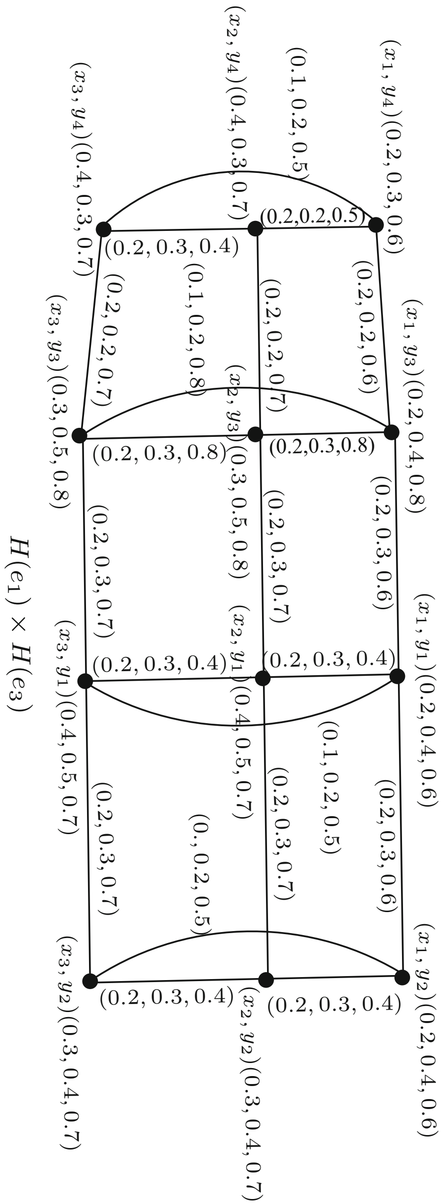
Theorem 7.1 *The Cartesian product of two neutrosophic soft graphs is a neutrosophic soft graph.*

Proof Let $G_1 = (F_1, K_1, M)$ and $G_2 = (F_2, K_2, N)$ be two neutrosophic soft graphs of $G_1^* = (X_1, E_1)$ and $G_2^* = (X_2, E_2)$, respectively. Let $G = G_1 \times G_2 = (F, K, M \times N)$ be the Cartesian product of G_1 and G_2. We claim that $G = (F, K, M \times N)$ is a neutrosophic soft graph and

$$(H, M \times N) = \{F_1 \times F_2(a_i, b_j), K_1 \times K_2(a_i, b_j)\}, \;\; \forall\, a_i \in M, \; b_j \in N$$

for $i = 1, 2, \ldots, m, \;\; j = 1, 2, \ldots, n$ are neutrosophic graphs of G.

Fig. 7.2 Cartesian product: $H_1(e_1) \times H_2(e_3)$



Consider,

$$
\begin{aligned}
T_{K_{(a_i,b_j)}}\big((x,y_1),(x,y_2)\big) &= \min\{T_{F_1(a_i)}(x), T_{K_2(b_j)}(y_1,y_2)\}\\
&\leq \min\{T_{F_1(a_i)}(x), \min\{T_{F_2(b_j)}(y_1), T_{F_2(b_j)}(y_2)\}\}\\
&= \min\{\min\{T_{F_1(a_i)}(x), T_{F_2(b_j)}(y_1)\}, \min\{T_{F_1(a_i)}(x), T_{F_2(b_j)}(y_2)\}\}
\end{aligned}
$$

$$T_{K_{(a_i,b_j)}}\big((x,y_1),(x,y_2)\big) \le \min\{(T_{F_1(a_i)} \times T_{F_2(b_j)})(x,y_1), (T_{F_1(a_i)} \times T_{F_2(b_j)})(x,y_2)\},$$
$$\text{for } i=1,2,\ldots,m,\ j=1,2,\ldots,n,$$

$$\begin{aligned}
I_{K_{(a_i,b_j)}}\big((x,y_1),(x,y_2)\big) &= \min\{I_{F_1(a_i)}(x), I_{K_2(b_j)}(y_1,y_2)\}\\
&\le \min\{I_{F_1(a_i)}(x), \min\{I_{F_2(b_j)}(y_1), I_{F_2(b_j)}(y_2)\}\}\\
&= \min\{\min\{I_{F_1(a_i)}(x), I_{F_2(b_j)}(y_1)\}, \min\{I_{F_1(a_i)}(x), I_{F_2(b_j)}(y_2)\}\}\\
I_{K_{(a_i,b_j)}}\big((x,y_1),(x,y_2)\big) &\le \min\{(I_{F_1(a_i)} \times I_{F_2(b_j)})(x,y_1), (I_{F_1(a_i)} \times I_{F_2(b_j)})(x,y_2)\},\\
&\quad \text{for } i=1,2,\ldots,m,\ j=1,2,\ldots,n
\end{aligned}$$

$$\begin{aligned}
F_{K_{(a_i,b_j)}}\big((x,y_1),(x,y_2)\big) &= \max\{F_{F_1(a_i)}(x), F_{K_2(b_j)}(y_1,y_2)\}\\
&\quad for\ i=1,2,\ldots,m,\ j=1,2,\ldots,n\\
&\le \max\{F_{F_1(a_i)}(x), \max\{F_{F_2(b_j)}(y_1), F_{F_2(b_j)}(y_2)\}\}\\
&= \max\{\max\{F_{F_1(a_i)}(x), F_{F_2(b_j)}(y_1)\}, \max\{F_{F_1(a_i)}(x), F_{F_2(b_j)}(y_2)\}\}\\
F_{K_{(a_i,b_j)}}\big((x,y_1),(x,y_2)\big) &\le \max\{(F_{F_1(a_i)} \times F_{F_2(b_j)})(x,y_1), (F_{F_1(a_i)} \times F_{F_2(b_j)})(x,y_2)\},\\
&\quad \text{for } i=1,2,\ldots,m,\ j=1,2,\ldots,n.
\end{aligned}$$

Similarly,

$$\begin{aligned}
T_{K_{(a_i,b_j)}}((x_1,y),(x_2,y)) &\le \min\{(T_{F_1(a_i)} \times T_{F_2(b_j)})(x_1,y), (T_{F_1(a_i)} \times T_{F_2(b_j)})(x_2,y)\},\\
&\quad \text{for } i=1,2,\ldots,m,\ j=1,2,\ldots,n,\\
I_{K_{(a_i,b_j)}}((x_1,y),(x_2,y)) &\le \min\{(I_{F_1(a_i)} \times I_{F_2(b_j)})(x_1,y), (I_{F_1(a_i)} \times I_{F_2(b_j)})(x_2,y)\},\\
&\quad \text{for } i=1,2,\ldots,m,\ j=1,2,\ldots,n,\\
F_{K_{(a_i,b_j)}}((x_1,y),(x_2,y)) &\le \max\{(F_{F_1(a_i)} \times F_{F_2(b_j)})(x_1,y), (F_{F_1(a_i)} \times F_{F_2(b_j)})(x_2,y)\},\\
&\quad \text{for } i=1,2,\ldots,m,\ j=1,2,\ldots,n.
\end{aligned}$$

Hence $G=(F,K,M\times N)$ is a neutrosophic soft graph.

Definition 7.12 The *cross product* of G_1 and G_2 is a neutrosophic soft graph $G=G_1 \circledcirc G_2=(F,K,M\times N)$, where $(F,M\times N)$ is a neutrosophic soft set over $X=X_1\times X_2$, $(K,M\times N)$ is a neutrosophic soft set over $E=\{((x_1,y_1),(x_2,y_2)):(x_1,x_2)\in E_1,(y_1,y_2)\in E_2\}$, and $(F,K,M\times N)$ are neutrosophic soft graphs such that
(i)

$$\begin{aligned}
T_{F(a,b)}(x,y) &= T_{F_1(a)}(x) \wedge T_{F_2(b)}(y),\\
I_{F(a,b)}(x,y) &= I_{F_1(a)}(x) \wedge I_{F_2(b)}(y),\\
F_{F(a,b)}(x,y) &= F_{F_1(a)}(x) \vee F_{F_2(b)}(y),\ \ \forall\,(x,y)\in X,\,(a,b)\in M\times N
\end{aligned}$$

(ii)

$$\begin{aligned}
T_{K(a,b)}\big((x_1,y_1),(x_2,y_2)\big) &= T_{K_1(a)}(x_1,x_2) \wedge T_{K_2(b)}(y_1,y_2),\\
I_{K(a,b)}\big((x_1,y_1),(x_2,y_2)\big) &= I_{K_1(a)}(x_1,x_2) \wedge I_{K_2(b)}(y_1,y_2),\\
F_{K(a,b)}\big((x_1,y_1),(x_2,y_2)\big) &= F_{K_1(a)}(x_1,x_2) \vee F_{K_2(b)}(y_1,y_2),\ \forall\,(x_1,x_2)\in E_1,(y_1,y_2)\in E_2.
\end{aligned}$$

$H(a,b) = H_1(a) \circledcirc H_2(b)$ for all $(a,b) \in M \times N$ are neutrosophic graphs of G.

Theorem 7.2 *The cross product of two neutrosophic soft graphs is a neutrosophic soft graph.*

Proof Let $G_1 = (F_1, K_1, M)$ and $G_2 = (F_2, K_2, N)$ be two neutrosophic soft graphs of $G_1^* = (X_1, E_1)$ and $G_2^* = (X_2, E_2)$, respectively. Let $G = G_1 \circledcirc G_2 = (F, K, M \times N)$ be the cross product of G_1 and G_2. We claim that $G = (F, K, M \times N)$ is a neutrosophic soft graph and

$$(H, M \times N) = \{F_1 \circledcirc F_2(a_i, b_j), K_1 \circledcirc K_2(a_i, b_j)\}\ \forall\, a_i \in M,\ b_j \in N$$

for $i = 1, 2, \ldots, m,\ \ j = 1, 2, \ldots, n$ are neutrosophic graphs of G.
Consider,

$$\begin{aligned}
T_{K(a_i,b_j)}\big((x_1,y_1),(x_2,y_2)\big) &= \min\{T_{K_1(a_i)}(x_1,x_2), T_{K_2(b_j)}(y_1,y_2)\},\\
&\quad \text{for } i = 1,2,\ldots,m,\ \ j = 1,2,\ldots,n\\
&\le \min\{\min\{T_{F_1(a_i)}(x_1), T_{F_1(a_i)}(x_2)\}, \min\{T_{F_2(b_j)}(y_1), T_{F_2(b_j)}(y_2)\}\}\\
&= \min\{\min\{T_{F_1(a_i)}(x_1), T_{F_2(b_j)})(y_1)\}, \min\{T_{F_1(a_i)}(x_2), T_{F_2(b_j)}(y_2)\}\}\\
T_{K(a_i,b_j)}\big((x_1,y_1),(x_2,y_2)\big) &\le \min\{T_{F_1(a_i)} \circledcirc T_{F_2(b_j)}(x_1,y_1), T_{F_1(a_i)} \circledcirc T_{F_2(b_j)}(x_2,y_2)\},\\
&\quad \text{for } i = 1,2,\ldots,m,\ \ j = 1,2,\ldots,n,
\end{aligned}$$

$$\begin{aligned}
I_{K(a_i,b_j)}\big((x_1,y_1),(x_2,y_2)\big) &= \min\{I_{K_1(a_i)}(x_1,x_2), I_{K_2(b_j)}(y_1,y_2)\},\\
&\quad \text{for } i = 1,2,\ldots,m,\ \ j = 1,2,\ldots,n\\
&\le \min\{\min\{I_{F_1(a_i)}(x_1), I_{F_1(a_i)}(x_2)\}, \min\{I_{F_2(b_j)}(y_1), I_{F_2(b_j)}(y_2)\}\}\\
&= \min\{\min\{I_{F_1(a_i)}(x_1), I_{F_2(b_j)})(y_1)\}, \min\{I_{F_1(a_i)}(x_2), I_{F_2(b_j)}(y_2)\}\}\\
I_{K(a_i,b_j)}\big((x_1,y_1),(x_2,y_2)\big) &\le \min\{I_{F_1(a_i)} \circledcirc I_{F_2(b_j)}(x_1,y_1), I_{F_1(a_i)} \circledcirc I_{F_2(b_j)}(x_2,y_2)\},\\
&\quad \text{for } i = 1,2,\ldots,m,\ \ j = 1,2,\ldots,n,
\end{aligned}$$

$$\begin{aligned}
F_{K(a_i,b_j)}\big((x_1,y_1),(x_2,y_2)\big) &= \max\{F_{K_1(a_i)}(x_1,x_2), F_{K_2(b_j)}(y_1,y_2)\},\\
&\quad \text{for } i = 1,2,\ldots,m,\ \ j = 1,2,\ldots,n\\
&\le \max\{\max\{F_{F_1(a_i)}(x_1), F_{F_1(a_i)}(x_2)\}, \max\{F_{F_2(b_j)}(y_1), F_{F_2(b_j)}(y_2)\}\}\\
&= \max\{\max\{F_{F_1(a_i)}(x_1), F_{F_2(b_j)})(y_1)\}, \max\{F_{F_1(a_i)}(x_2), F_{F_2(b_j)}(y_2)\}\}\\
F_{K(a_i,b_j)}\big((x_1,y_1),(x_2,y_2)\big) &\le \max\{F_{F_1(a_i)} \circledcirc F_{F_2(b_j)}(x_1,y_1), F_{F_1(a_i)} \circledcirc F_{F_2(b_j)}(x_2,y_2)\},\\
&\quad \text{for } i = 1,2,\ldots,m,\ \ j = 1,2,\ldots,n.
\end{aligned}$$

Hence $G = (F, K, M \times N)$ is a neutrosophic soft graph.

Definition 7.13 The *lexicographic product* of G_1 and G_2 is a neutrosophic soft graph $G = G_1 \odot G_2 = (F, K, M \times N)$, where $(F, M \times N)$ is a neutrosophic soft set over $X = X_1 \times X_2$, $(K, M \times N)$ is a neutrosophic soft set over $E = \{((x, y_1), (x, y_2)) : u \in X_1, (y_1, y_2) \in E_2\} \cup \{((x_1, y_1), (x_2, y_2)) : (x_1, x_2) \in E_1, (y_1, y_2) \in E_2\}$, and $(F, K, M \times N)$ are neutrosophic soft graphs such that
(i)

$$\begin{aligned} T_{F(a,b)}(x, y) &= T_{F_1(a)}(x) \wedge T_{F_2(b)}(y), \\ I_{F(a,b)}(x, y) &= I_{F_1(a)}(x) \wedge I_{F_2(b)}(y), \\ F_{F(a,b)}(x, y) &= F_{F_1(a)}(x) \vee F_{F_2(b)}(y), \ \forall (x, y) \in X, \ (a, b) \in M \times N, \end{aligned}$$

(ii)

$$\begin{aligned} T_{K(a,b)}\big((x, y_1), (x, y_2)\big) &= T_{F_1(a)}(x) \wedge T_{K_2(b)}(y_1, y_2), \\ I_{K(a,b)}\big((x, y_1), (x, y_2)\big) &= I_{F_1(a)}(x) \wedge I_{K_2(b)}(y_1, y_2), \\ F_{K(a,b)}\big((x, y_1), (x, y_2)\big) &= F_{F_1(a)}(x) \vee F_{K_2(b)}(y_1, y_2), \ \forall x \in X_1, \ (y_1, y_2) \in E_2, \end{aligned}$$

(iii)

$$\begin{aligned} T_{K(a,b)}\big((x_1, y_1), (x_2, y_2)\big) &= T_{K_1(a)}(x_1, x_2) \wedge T_{K_2(b)}(y_1, y_2), \\ I_{K(a,b)}\big((x_1, y_1), (x_2, y_2)\big) &= I_{K_1(a)}(x_1, x_2) \wedge I_{K_2(b)}(y_1, y_2), \\ F_{K(a,b)}\big((x_1, y_1), (x_2, y_2)\big) &= F_{K_1(a)}(x_1, x_2) \vee F_{K_2(b)}(y_1, y_2), \ \forall (x_1, x_2) \in E_1, \ (y_1, y_2) \in E_2. \end{aligned}$$

$H(a, b) = H_1(a) \odot H_2(b)$ for all $(a, b) \in M \times N$ are neutrosophic graphs of G.

Theorem 7.3 *The lexicographic product of two neutrosophic soft graphs is a neutrosophic soft graph.*

Definition 7.14 The *strong product* of G_1 and G_2 is a neutrosophic soft graph $G = G_1 \otimes G_2 = (F, K, M \times N)$, where $(F, M \times N)$ is a neutrosophic soft set over $X = X_1 \times X_2$, $(K, M \times N)$ is a neutrosophic soft set over $E = \{((x, y_1), (x, y_2)) : u \in X_1, (y_1, y_2) \in E_2\} \cup \{((x_1, y), (x_2, y)) : y \in X_2, (x_1, x_2) \in E_1\} \cup \{((x_1, y_1), (x_2, y_2)) : (x_1, x_2) \in E_1, (y_1, y_2) \in E_2\}$, and $(F, K, M \times N)$ are neutrosophic soft graphs such that
(i)

$$\begin{aligned} T_{F(a,b)}(x, y) &= T_{F_1(a)}(x) \wedge T_{F_2(b)}(y), \\ I_{F(a,b)}(x, y) &= I_{F_1(a)}(x) \wedge I_{F_2(b)}(y), \\ F_{F(a,b)}(x, y) &= F_{F_1(a)}(x) \vee F_{F_2(b)}(y), \ \forall (x, y) \in X, \ (a, b) \in M \times N, \end{aligned}$$

(ii)

$$\begin{aligned} T_{K(a,b)}\big((x, y_1), (x, y_2)\big) &= T_{F_1(a)}(x) \wedge T_{K_2(b)}(y_1, y_2), \\ I_{K(a,b)}\big((x, y_1), (x, y_2)\big) &= I_{F_1(a)}(x) \wedge I_{K_2(b)}(y_1, y_2), \\ F_{K(a,b)}\big((x, y_1), (x, y_2)\big) &= F_{F_1(a)}(x) \vee F_{K_2(b)}(y_1, y_2), \ \forall x \in X_1, \ (y_1, y_2) \in E_2, \end{aligned}$$

(iii)

$$\begin{aligned}
T_{K(a,b)}\big((x_1,y),(x_2,y)\big) &= T_{F_2(b)}(y) \wedge T_{K_1(a)}(x_1,x_2),\\
I_{K(a,b)}\big((x_1,y),(x_2,y)\big) &= I_{F_2(b)}(y) \wedge I_{K_1(a)}(x_1,x_2),\\
F_{K(a,b)}\big((x_1,y),(x_2,y)\big) &= F_{F_2(b)}(y) \vee F_{K_1(a)}(x_1,x_2),\ \forall\, y\in X_2,\ (x_1,x_2)\in E_1,
\end{aligned}$$

(iv)

$$\begin{aligned}
T_{K(a,b)}\big((x_1,y_1),(x_2,y_2)\big) &= T_{K_1(a)}(x_1,x_2) \wedge T_{K_2(b)}(y_1,y_2),\\
I_{K(a,b)}\big((x_1,y_1),(x_2,y_2)\big) &= I_{K_1(a)}(x_1,x_2) \wedge I_{K_2(b)}(y_1,y_2),\\
F_{K(a,b)}\big((x_1,y_1),(x_2,y_2)\big) &= F_{K_1(a)}(x_1,x_2) \vee F_{K_2(b)}(y_1,y_2),\ \forall\,(x_1,x_2)\in E_1,\ (y_1,y_2)\in E_2.
\end{aligned}$$

$H(a,b) = H_1(a) \otimes H_2(b)$ for all $(a,b) \in M \times N$ are neutrosophic graphs of G.

Theorem 7.4 *The strong product of two neutrosophic soft graphs is a neutrosophic soft graph.*

Definition 7.15 The *composition* of G_1 and G_2 is a neutrosophic soft graph $G = G_1[G_2] = (F, K, M \times N)$, where
$(F, M \times N)$ is a neutrosophic soft set over

$$X = X_1 \times X_2,$$

$(K, M \times N)$ is a neutrosophic soft set over

$$\begin{aligned}
E = &\{((x,y_1),(x,y_2)) : u \in X_1, (y_1,y_2)\in E_2\} \cup\\
&\{((x_1,y),(x_2,y)) : v \in X_2, (x_1,x_2)\in E_1\} \cup\\
&\{((x_1,y_1),(x_2,y_2)) : (x_1,x_2)\in E_1, y_1 \neq y_2\}
\end{aligned}$$

and $(F, K, M \times N)$ are neutrosophic soft graphs such that
(i)

$$\begin{aligned}
T_{F(a,b)}(x,y) &= T_{F_1(a)}(x) \wedge T_{F_2(b)}(y),\\
I_{F(a,b)}(x,y) &= I_{F_1(a)}(x) \wedge I_{F_2(b)}(y),\\
F_{F(a,b)}(x,y) &= F_{F_1(a)}(x) \vee F_{F_2(b)}(y),\ \forall\,(x,y)\in X,\ (a,b)\in M\times N,
\end{aligned}$$

(ii)

$$\begin{aligned}
T_{K(a,b)}((x,y_1),(x,y_2)) &= T_{F_1(a)}(x) \wedge T_{K_2(b)}(y_1,y_2),\\
I_{K(a,b)}((x,y_1),(x,y_2)) &= I_{F_1(a)}(x) \wedge I_{K_2(b)}(y_1,y_2),\\
F_{K(a,b)}((x,y_1),(x,y_2)) &= F_{F_1(a)}(x) \vee F_{K_2(b)}(y_1,y_2),\ \forall\, x\in X_1,\ (y_1,y_2)\in E_2,
\end{aligned}$$

(iii)

$$\begin{aligned}
T_{K(a,b)}\big((x_1,y),(x_2,y)\big) &= T_{F_2(b)}(y) \wedge T_{K_1(a)}(x_1,x_2),\\
I_{K(a,b)}\big((x_1,y),(x_2,y)\big) &= I_{F_2(b)}(y) \wedge I_{K_1(a)}(x_1,x_2),\\
F_{K(a,b)}\big((x_1,y),(x_2,y)\big) &= F_{F_2(b)}(y) \vee F_{K_1(a)}(x_1,x_2),\ \forall\, y\in X_2,\ (x_1,x_2)\in E_1,
\end{aligned}$$

(iv)

$$\begin{aligned}
T_{K(a,b)}\big((x_1, y_1), (x_2, y_2)\big) &= T_{K_1(a)}(x_1, x_2) \ \wedge\ T_{F_2(b)}(y_1) \ \wedge\ T_{F_2(b)}(y_2),\\
I_{K(a,b)}\big((x_1, y_1), (x_2, y_2)\big) &= I_{K_1(a)}(x_1, x_2) \ \wedge\ I_{F_2(b)}(y_1) \ \wedge\ I_{F_2(b)}(y_2),\\
F_{K(a,b)}\big((x_1, y_1), (x_2, y_2)\big) &= F_{K_1(a)}(x_1, x_2) \ \vee\ F_{F_2(b)}(y_1) \ \vee\ F_{F_2(b)}(y_2),\ \forall\, (x_1, x_2) \in E_1,
\end{aligned}$$
where $y_1 \neq y_2$.

$H(a, b) = H_1(a)[H_2(b)]$ for all $(a, b) \in M \times N$ are neutrosophic graphs of G.

Example 7.5 Let $M = \{e_1\}$ and $N = \{e_2, e_3\}$ be the parameter sets. Let G_1 and G_2 be the two neutrosophic soft graphs defined as follows:

$$G_1 = \{H_1(e_1)\} = \{(\{(x_1, 0.3, 0.4, 0.6), (x_2, 0.4, 0.5, 0.7)\}, \{(x_1x_2, 0.3, 0.4, 0.6)\})\},$$

$$\begin{aligned}
G_2 = \{H_2(e_2), H_2(e_3)\} = \{&(\{(y_1, 0.4, 0.5, 0.3), (y_2, 0.7, 0.2, 0.4), (y_3, 0.5, 0.6, 0.3)\},\\
&\{(y_1y_3, 0.4, 0.5, 0.2), (y_2y_3, 0.5, 0.2, 0.4)\}),\\
&(\{(y_1, 0.3, 0.4, 0.4), (y_2, 0.2, 0.4, 0.8), (y_3, 0.6, 0.5, 0.7)\},\\
&\{(y_1y_2, 0.2, 0.3, 0.7), (y_1y_3, 0.1, 0.3, 0.6)\})\}.
\end{aligned}$$

The composition of G_1 and G_2 is $G = G_1[G_2] = (H, M \times N)$, where

$$\begin{aligned}
M \times N &= \{(e_1, e_2), (e_1, e_3)\},\\
H(e_1, e_2) &= H_1(e_1)[H_2(e_2)],\\
H(e_1, e_3) &= H_1(e_1)[H_2(e_3)]
\end{aligned}$$

are neutrosophic graphs of $G_1[G_2]$. $H_1(e_1)[H_2(e_2)]$ is shown in Fig. 7.3.

Similarly, composition of neutrosophic graphs $H_1(e_1)$ and $H_2(e_3)$ of G_1 and G_2, respectively, can be drawn.

Hence $G = G_1[G_2] = \{H_1(e_1)[H_2(e_2)], H_1(e_1)[H_2(e_3)]\}$ is a neutrosophic soft graph.

Theorem 7.5 *If G_1 and G_2 are neutrosophic soft graphs, then $G_1[G_2]$ is a neutrosophic soft graph.*

Proof $G_1 = (F_1, K_1, M)$ and $G_2 = (F_2, K_2, N)$ are two neutrosophic soft graphs of $G_1^* = (X_1, E_1)$ and $G_2^* = (X_2, E_2)$, respectively. Let $G_1[G_2] = G = (F, K, M \times N)$ be the composition of G_1 and G_2. We claim that $G_1[G_2] = G = (F, K, M \times N)$ is a neutrosophic soft graph and

$$(H, M \times N) = \{F_1(a_i)[F_2(b_j)], K_1(a_i)[K_2(b_j)]\},\ \forall\, a_i \in M, b_j \in N,$$

for $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n$ are neutrosophic graphs of G.

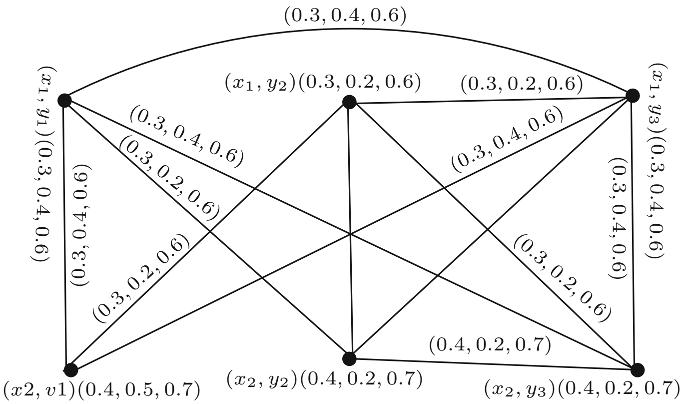


Fig. 7.3 Composition: $H_1(e_1)[H_2(e_2)]$

Let $u \in X_1$ and $(y_1, y_2) \in E_2$, and we have

$$
\begin{aligned}
T_{K(a_i,b_j)}\big((x, y_1), (x, y_2)\big) &= \min\{T_{F_1(a_i)}(x), T_{K_2(b_j)}(y_1, y_2)\}, \text{ for } i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n \\
T_{K(a_i,b_j)}\big((x, y_1), (x, y_2)\big) &\le \min\{T_{F_1(a_i)}(x), \min\{T_{F_2(b_j)}(y_1), T_{F_2(b_j)}(y_2)\}\} \\
&= \min\{\min\{T_{F_1(a_i)}(x), T_{F_2(b_j)}(y_1)\} \min\{T_{F_1(a_i)}(x), T_{F_2(b_j)}(y_2)\}\} \\
&= \min\{(T_{F_1(a_i)} \times T_{F_2(b_j)})(x, y_1), (T_{F_1(a_i)} \times T_{F_2(b_j)})(x, y_2)\} \\
T_{K(a_i,b_j)}\big((x, y_1), (x, y_2)\big) &\le \min\{T_{F(a_i,b_j)}(x, y_1), T_{F(a_i,b_j)}(x, y_2)\},
\end{aligned}
$$

$$
\begin{aligned}
I_{K(a_i,b_j)}\big((x, y_1), (x, y_2)\big) &= \min\{I_{F_1(a_i)}(x), I_{K_2(b_j)}(y_1, y_2)\}, \text{ for } i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n \\
I_{K(a_i,b_j)}\big((x, y_1), (x, y_2)\big) &\le \min\{I_{F_1(a_i)}(x), \min\{I_{F_2(b_j)}(y_1), I_{F_2(b_j)}(y_2)\}\} \\
&= \min\{\min\{I_{F_1(a_i)}(x), I_{F_2(b_j)}(y_1)\}, \min\{I_{F_1(a_i)}(x), I_{F_2(b_j)}(y_2)\}\} \\
&= \min\{(I_{F_1(a_i)} \times I_{F_2(b_j)})(x, y_1), (I_{F_1(a_i)} \times I_{F_2(b_j)})(x, y_2)\} \\
I_{K(a_i,b_j)}\big((x, y_1), (x, y_2)\big) &\le \min\{I_{F(a_i,b_j)}(x, y_1), I_{F(a_i,b_j)}(x, y_2)\},
\end{aligned}
$$

$$
\begin{aligned}
F_{K(a_i,b_j)}\big((x, y_1), (x, y_2)\big) &= \max\{F_{F_1(a_i)}(x), F_{K_2(b_j)}(y_1, y_2)\}, \text{ for } i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n \\
F_{K(a_i,b_j)}\big((x, y_1), (x, y_2)\big) &\le \max\{F_{F_1(a_i)}(x), \max\{F_{F_2(b_j)}(y_1), F_{F_2(b_j)}(y_2)\}\} \\
&= \max\{\max\{F_{F_1(a_i)}(x), F_{F_2(b_j)}(y_1)\}, \max\{F_{F_1(a_i)}(x), F_{F_2(b_j)}(y_2)\}\} \\
&= \max\{(F_{F_1(a_i)} \times F_{F_2(b_j)})(x, y_1), (F_{F_1(a_i)} \times F_{F_2(b_j)})(x, y_2)\} \\
F_{K(a_i,b_j)}\big((x, y_1), (x, y_2)\big) &\le \max\{F_{F(a_i,b_j)}(x, y_1), F_{F(a_i,b_j)}(x, y_2)\}.
\end{aligned}
$$

Similarly, for any $y \in X_2$ and $(x_1, x_2) \in E_1$, we have

$$
\begin{aligned}
T_{K(a_i,b_j)}\big((x_1, y), (x_2, y)\big) &\le \min\{T_{F(a_i,b_j)}(x_1, y), T_{F(a_i,b_j)}(x_2, y)\}, \\
I_{K(a_i,b_j)}\big((x_1, y), (x_2, y)\big) &\le \min\{I_{F(a_i,b_j)}(x_1, y), I_{F(a_i,b_j)}(x_2, y)\}, \\
F_{K(a_i,b_j)}\big((x_1, y), (x_2, y)\big) &\le \max\{F_{F(a_i,b_j)}(x_1, y), F_{F(a_i,b_j)}(x_2, y)\}.
\end{aligned}
$$

Let $(x_1, y_1)(x_2, y_2) \in E$, $(x_1, x_2) \in E_1$, and $y_1 \neq y_2$. Then, we have

$$\begin{aligned}
T_{K(a_i,b_j)}\big((x_1, y_1), (x_2, y_2)\big) &= \min\{T_{K_1(a_i)}(x_1, x_2), T_{F_2(b_j)}(y_1), T_{F_2(b_j)}(y_2)\}\\
&\leq \min\{\min\{T_{F_1(a_i)}(x_1), T_{F_1(a_i)}(x_2)\}, T_{F_2(b_j)}(y_1), T_{F_2(b_j)}(y_2)\}\\
&= \min\{\min\{T_{F_1(a_i)}(x_1), T_{F_2(b_j)}(y_1)\}, \min\{T_{F_1(a_i)}(x_2), T_{F_2(b_j)}(y_2)\}\}\\
T_{K(a_i,b_j)}\big((x_1, y_1), (x_2, y_2)\big) &\leq \min\{T_{F(a_i,b_j)}(x_1, y_1), T_{F(a_i,b_j)}(x_2, y_2)\},
\end{aligned}$$

$$\begin{aligned}
I_{K(a_i,b_j)}\big((x_1, y_1), (x_2, y_2)\big) &= \min\{I_{K_1(a_i)}(x_1, x_2), I_{F_2(b_j)}(y_1), I_{F_2(b_j)}(y_2)\}\\
&\leq \min\{\min\{I_{F_1(a_i)}(x_1), I_{F_1(a_i)}(x_2)\}, I_{F_2(b_j)}(y_1), I_{F_2(b_j)}(y_2)\}\\
&= \min\{\min\{I_{F_1(a_i)}(x_1), I_{F_2(b_j)}(y_1)\}, \min\{I_{F_1(a_i)}(x_2), I_{F_2(b_j)}(y_2)\}\}\\
I_{K(a_i,b_j)}\big((x_1, y_1), (x_2, y_2)\big) &\leq \min\{I_{F(a_i,b_j)}(x_1, y_1), I_{F(a_i,b_j)}(x_2, y_2)\},
\end{aligned}$$

$$\begin{aligned}
F_{K(a_i,b_j)}\big((x_1, y_1), (x_2, y_2)\big) &= \max\{F_{K_1(a_i)}(x_1, x_2), F_{F_2(b_j)}(y_1), F_{F_2(b_j)}(y_2)\}\\
&\leq \max\{\max\{F_{F_1(a_i)}(x_1), F_{F_1(a_i)}(x_2)\}, F_{F_2(b_j)}(y_1), F_{F_2(b_j)}(y_2)\}\\
&= \max\{\max\{F_{F_1(a_i)}(x_1), F_{F_2(b_j)}(y_1)\}, \max\{F_{F_1(a_i)}(x_2), F_{F_2(b_j)}(y_2)\}\}\\
F_{K(a_i,b_j)}\big((x_1, y_1), (x_2, y_2)\big) &\leq \max\{F_{F(a_i,b_j)}(x_1, y_1), F_{F(a_i,b_j)}(x_2, y_2)\}.
\end{aligned}$$

Hence $G = (F, K, M \times N)$ is a neutrosophic soft graph.

Definition 7.16 The *complement* of a neutrosophic soft graph $G = (F, K, M)$ denoted by $G^c = (F^c, K^c, M^c)$ is defined as follows:

(i) $M^c = M$,

(ii) $F^c(e) = F(e)$,

(iii) $T_{K^c(e)}(x, y) = T_{F(e)}(x) \wedge T_{F(e)}(y) - T_{K(e)(x,y)}$,
$I_{K^c(e)}(x, y) = I_{F(e)}(x) \wedge I_{F(e)}(y) - I_{K(e)(x,y)}$,
$F_{K^c(e)}(x, y) = F_{F(e)}(x) \vee F_{F(e)}(y) - F_{K(e)(x,y)}$, for all $x, y \in X$, $e \in M$.

Example 7.6 Consider an undirected graph G^*, where $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{x_1x_2, x_2x_4, x_3x_4\}$. Let $M = \{e_1, e_2\}$ and let (F, M) be a neutrosophic soft set over X with its approximate function $F : M \to \mathcal{P}(X)$ given by

$$\begin{aligned}
F(e_1) &= \{(x_1, 0.5, 0.6, 0.7), (x_2, 0.4, 0.5, 0.3), (x_3, 0.7, 0.5, 0.8), (x_4, 0.4, 0.9, 0.5)\},\\
F(e_2) &= \{(x_1, 0.4, 0.5, 0.2), (x_2, 0.3, 0.6, 0.8), (x_3, 0.3, 0.4, 0.5), (x_4, 0.7, 0.8, 0.5)\}.
\end{aligned}$$

Let (K, M) be a neutrosophic soft set over E with its approximate function $K : M \to \mathcal{P}(E)$ given by

$$\begin{aligned}
K(e_1) &= \{(x_1x_2, 0.3, 0.4, 0.5), (x_2x_4, 0.3, 0.4, 0.4), (x_1x_3, 0.4, 0.3, 0.6)\},\\
K(e_2) &= \{(x_1x_2, 0.2, 0.3, 0.5), (x_2x_3, 0.1, 0.3, 0.4), (x_3x_4, 0.2, 0.2, 0.4)\}.
\end{aligned}$$

By routine calculations, it is easy to see that $H(e_1)$ and $H(e_2)$ are neutrosophic graphs corresponding to the parameters e_1 and e_2, respectively, as shown in Fig. 7.4.

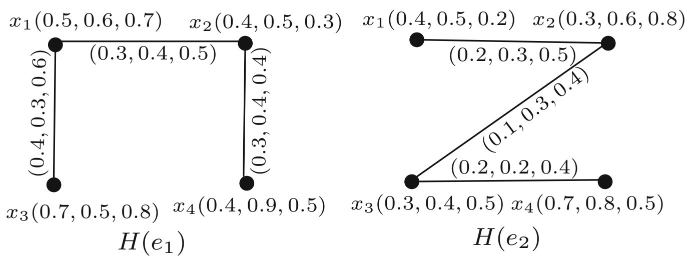


Fig. 7.4 $G = \{H(e_1) = (F(e_1), K(e_1)), H(e_2) = (F(e_2), K(e_2))\}$

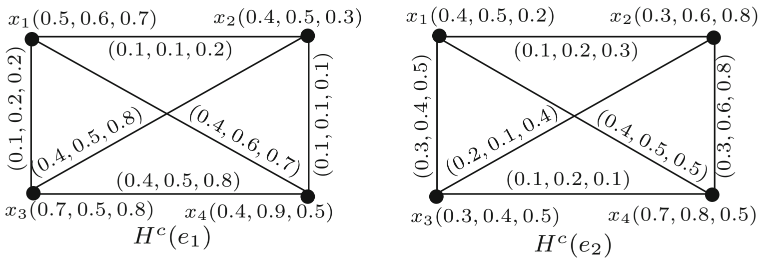


Fig. 7.5 $G^c = \{H^c(e_1) = (F^c(e_1), K^c(e_1)), H^c(e_2) = (F^c(e_2), K^c(e_2))\}$

By the complement of neutrosophic soft graph G is the complement of neutrosophic graphs $H(e_1)$ and $H(e_2)$ which are shown in Fig. 7.5.

Definition 7.17 A neutrosophic soft graph G is *self-complementary* if $G \approx G^c$.

Definition 7.18 A neutrosophic soft graph G is a *complete neutrosophic soft graph* if $H(e)$ is a complete neutrosophic graph of G for all $e \in M$, i.e.,

$$\begin{aligned}
T_{K(e)}(xy) &= \min\{T_{F(e)}(x), T_{F(e)}(y)\},\\
I_{K(e)}(xy) &= \min\{I_{F(e)}(x), I_{F(e)}(y)\},\\
F_{K(e)}(xy) &= \max\{F_{F(e)}(x), F_{F(e)}(y)\}, \ \forall\, x, y \in X,\ e \in M.
\end{aligned}$$

Example 7.7 Consider the simple graph $G^* = (X, E)$ where $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{x_1x_2, x_2x_3, x_3x_4, x_1x_3, x_1x_4, x_2x_4\}$. Let $M = \{e_1, e_2, e_3\}$. Let (F, M) be a neutrosophic soft set over X with its approximation function $F : M \to \mathcal{P}(X)$ defined by

$$\begin{aligned}
&F(e_1) = \{(x_1, 0.5, 0.7, 0.7), (x_2, 0.3, 0.4, 0.6), (x_3, 0.5, 0.4, 0.6)\},\\
&F(e_2) = \{(x_1, 0.8, 0.5, 0.4), (x_2, 0.4, 0.6, 0.8), (x_3, 0.4, 0.5, 0.6), (x_4, 0.7, 0.8, 0.3)\},\\
&F(e_3) = \{(x_1, 0.6, 0.7, 0.4), (x_2, 0.7, 0.4, 0.9), (x_3, 0.8, 0.5, 0.9), (x_4, 0.5, 0.7, 0.7)\}.
\end{aligned}$$

Let (K, M) be a neutrosophic soft set over E with its approximation function $K : M \to \mathcal{P}(E)$ defined by

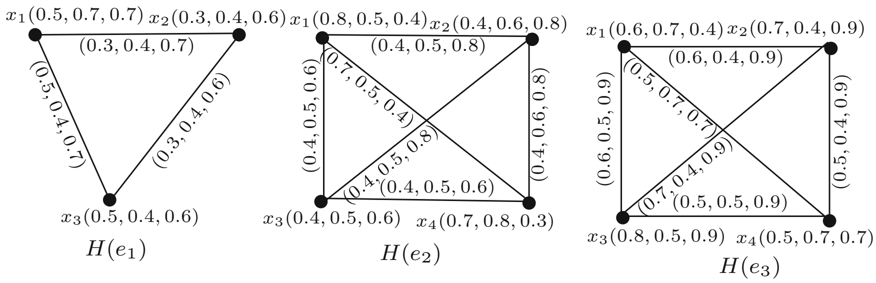


Fig. 7.6 Complete neutrosophic soft graph $G = \{H(e_1), H(e_2), H(e_3)\}$

$$\begin{aligned}
K(e_1) &= \{(x_1x_2, 0.3, 0.4, 0.7), (x_1x_3, 0.5, 0.4, 0.7), (x_2x_3, 0.3, 0.4, 0.6)\},\\
K(e_2) &= \{(x_1x_2, 0.4, 0.5, 0.8), (x_2x_3, 0.4, 0.5, 0.8), (x_3x_4, 0.4, 0.5, 0.6),\\
&\qquad (x_1x_3, 0.4, 0.5, 0.6), (x_1x_4, 0.7, 0.5, 0.4), (x_2x_4, 0.4, 0.6, 0.8)\},\\
K(e_3) &= \{(x_1x_2, 0.6, 0.4, 0.9), (x_2x_3, 0.7, 0.4, 0.9), (x_3x_4, 0.5, 0.5, 0.9),\\
&\qquad (x_1x_3, 0.6, 0.5, 0.9), (x_1x_4, 0.5, 0.7, 0.7), (x_2x_4, 0.5, 0.4, 0.9)\}.
\end{aligned}$$

It is easy to see that $H(e_1)$, $H(e_2)$ and $H(e_3)$ are complete neutrosophic graphs of G corresponding to the parameters e_1, e_2 and e_3, respectively, as shown in Fig. 7.6.

Definition 7.19 A neutrosophic soft graph G is a *strong neutrosophic soft graph* if $H(e)$ is a strong neutrosophic graph for all $e \in M$.

Example 7.8 Consider the simple graph G^* where $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{x_1x_2, x_2x_3, x_3x_4, x_1x_3, x_1x_4, x_2x_4\}$. Let $M = \{e_1, e_2, e_3\}$. Let (F, M) be a neutrosophic soft set over X with its approximation function $F : M \to \mathcal{P}(X)$ defined by

$$\begin{aligned}
F(e_1) &= \{(x_1, 0.5, 0.7, 0.7), (x_2, 0.3, 0.4, 0.6), (x_3, 0.5, 0.4, 0.6)\},\\
F(e_2) &= \{(x_1, 0.8, 0.5, 0.4), (x_2, 0.4, 0.6, 0.8), (x_3, 0.4, 0.5, 0.6), (x_4, 0.7, 0.8, 0.3)\},\\
F(e_3) &= \{(x_1, 0.6, 0.7, 0.4), (x_2, 0.7, 0.4, 0.9), (x_3, 0.8, 0.5, 0.9), (x_4, 0.5, 0.7, 0.7)\}.
\end{aligned}$$

Let (K, M) be a neutrosophic soft set over E with its approximation function $K : M \to \mathcal{P}(E)$ defined by

$$\begin{aligned}
K(e_1) &= \{(x_1x_2, 0.3, 0.4, 0.7), (x_1x_3, 0.5, 0.4, 0.7), (x_2x_3, 0.3, 0.4, 0.6)\},\\
K(e_2) &= \{(x_2x_3, 0.4, 0.5, 0.8), (x_1x_4, 0.7, 0.5, 0.4)\},\\
K(e_3) &= \{(x_1x_2, 0.6, 0.4, 0.9), (x_1x_3, 0.6, 0.5, 0.9), (x_2x_4, 0.5, 0.4, 0.9)\}.\\
H(e_1) &= (F(e_1), K(e_1)),\ H(e_2) = (F(e_2), K(e_2)),\ H(e_3) = (F(e_3), K(e_3))
\end{aligned}$$

are strong neutrosophic graphs of G corresponding to the parameters e_1, e_2 and e_3, respectively, as shown in Fig. 7.7.

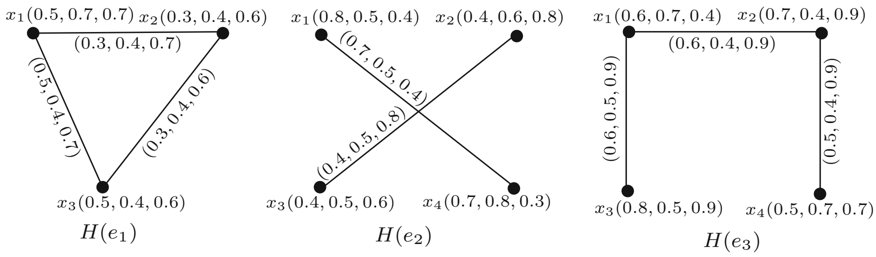


Fig. 7.7 Strong neutrosophic soft graph $G = \{H(e_1), H(e_2), H(e_3)\}$

Proposition 7.1 *If G_1 and G_2 are strong neutrosophic soft graphs, then $G_1 \times G_2$, $G_1[G_2]$ and $G_1 \tilde{+} G_2$ are strong neutrosophic soft graphs.*

Definition 7.20 The complement of a strong neutrosophic soft graph $G = (F, K, M)$ is a neutrosophic soft graph $G^c = (F^c, K^c, M^c)$ defined by

(i) $M^c = M$,

(ii) $F^c(e)(x) = F(e)(x)$, for all $e \in M$ and $x \in X$,

(iii) $T_{K^c(e)}(x, y) = \begin{cases} 0 & \text{if } T_{K(e)}(x, y) > 0, \\ \min\{T_{F(e)}(x), T_{F(e)}(y)\}, & \text{if } T_{K(e)}(x, y) = 0, \end{cases}$

$$I_{K^c(e)}(x, y) = \begin{cases} 0 & \text{if } I_{K(e)}(x, y) > 0, \\ \min\{I_{F(e)}(x), I_{F(e)}(y)\}, & \text{if } I_{K(e)}(x, y) = 0, \end{cases}$$

$$F_{K^c(e)}(x, y) = \begin{cases} 0 & \text{if } F_{K(e)}(x, y) > 0, \\ \max\{F_{F(e)}(x), F_{F(e)}(y)\}, & \text{if } F_{K(e)}(x, y) = 0, \end{cases}$$

We state the following propositions without their proofs.

Proposition 7.2 *If G is a strong neutrosophic soft graph over G^*, then G^c is also a strong neutrosophic soft graph.*

Proposition 7.3 *If G and G^c are strong neutrosophic soft graphs of G^*, then $G \cup G^c$ is a complete neutrosophic soft graph.*

7.3 Application of Neutrosophic Soft Graphs

In this section, we apply the concept of neutrosophic soft graphs to a decision-making problem and then we describe an algorithm for the selection of optimal object based on given set of information. Suppose that $X = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ is the set of six houses under consideration which Mr. Aslam is going to buy a house on the basis of wishing parameters or attributes set $M = \{e_1 = \text{large}, e_2 = \text{beautiful}, e_3 = \text{green surrounding}\}$. So (F, M) is the neutrosophic soft set on X which describes the value of the houses based on the given parameters e_1 = large, e_2 = beautiful, e_3 = green surrounding, respectively.

$$\begin{aligned}
F(e_1) = \{&(h_1, 0.3, 0.5, 0.8), (h_2, 0.2, 0.8, 0.5), (h_3, 0.4, 0.5, 0.2),\\
&(h_4, 0.5, 0.2, 0.7), (h_5, 0.4, 0.7, 0.6), (h_6, 0.2, 0.5, 0.8)\},\\
F(e_2) = \{&(h_1, 0.6, 0.7, 0.4), (h_2, 0.6, 0.2, 0.9), (h_3, 0.2, 0.6, 0.3),\\
&(h_4, 0.7, 0.4, 0.2), (h_5, 0.0, 0.0, 0.0), (h_6, 0.6, 0.2, 0.6)\},\\
F(e_3) = \{&(h_1, 0.6, 0.3, 0.5), (h_2, 0.5, 0.2, 0.8), (h_3, 0.4, 0.4, 0.8),\\
&(h_4, 0.5, 0.6, 0.4), (h_5, 0.6, 0.4, 0.2), (h_6, 0.4, 0.7, 0.8)\}.
\end{aligned}$$

(K, M) is the neutrosophic soft set on

$$E=\{h_1h_2, h_1h_3, h_1h_5, h_1h_6, h_2h_4, h_2h_6, h_2h_3, h_2h_5, h_3h_4, h_3h_5, h_4h_5, h_4h_6, h_5h_6\}$$

which describes the value of two houses corresponding to the given parameters e_1 = large, e_2 = beautiful, e_3 = green surrounding, respectively.

$$\begin{aligned}
K(e_1) = \{&(h_1h_2, 0.1, 0.3, 0.6), (h_1h_4, 0.2, 0.1, 0.4), (h_2h_3, 0.2, 0.4, 0.3),\\
&(h_2h_4, 0.1, 0.1, 0.6), (h_2h_5, 0.2, 0.2, 0.4), (h_3h_5, 0.3, 0.4, 0.5),\\
&(h_3h_6, 0.1, 0.3, 0.6), (h_4h_5, 0.3, 0.1, 0.2), (h_5h_6, 0.2, 0.4, 0.7)\},\\
K(e_2) = \{&(h_1h_2, 0.5, 0.1, 0.6), (h_1h_3, 0.1, 0.5, 0.3), (h_1h_4, 0.4, 0.3, 0.3),\\
&(h_2h_4, 0.5, 0.1, 0.7), (h_2h_6, 0.4, 0.1, 0.7), (h_3h_4, 0.1, 0.3, 0.3),\\
&(h_3h_6, 0.2, 0.1, 0.4)\},\\
K(e_3) = \{&(h_1h_2, 0.4, 0.1, 0.7), (h_1h_5, 0.4, 0.2, 0.3), (h_2h_3, 0.3, 0.1, 0.6),\\
&(h_2h_4, 0.3, 0.1, 0.5), (h_3h_5, 0.3, 0.2, 0.7), (h_3h_6, 0.3, 0.2, 0.6),\\
&(h_4h_5, 0.4, 0.3, 0.1), (h_5h_6, 0.2, 0.3, 0.5), (h_4h_5, 0.3, 0.1, 0.2),\\
&(h_5h_6, 0.2, 0.4, 0.7)\}.
\end{aligned}$$

The neutrosophic graphs $H(e_i)$ $(i = 1, 2, 3)$ of neutrosophic soft graph $G = (F, K, M)$ corresponding to the parameters e_i for $i = 1, 2, 3$ are shown in Fig. 7.8.

The neutrosophic graphs $H(e_1)$, $H(e_2)$ and $H(e_3)$ corresponding to the parameters "large", "beautiful" and "green surrounding", respectively, are represented by the following incidence matrices

$$H(e_1) = \begin{pmatrix}
(0,0,0) & (0.1,0.3,0.6) & (0,0,0) & (0.2,0.1,0.4) & (0,0,0) & (0,0,0)\\
(0.1,0.3,0.6) & (0,0,0) & (0.2,0.4,0.3) & (0.1,0.1,0.6) & (0.2,0.2,0.4) & (0,0,0)\\
(0,0,0) & (0.2,0.4,0.3) & (0,0,0) & (0,0,0) & (0.3,0.4,0.5) & (0.1,0.3,0.6)\\
(0.2,0.1,0.4) & (0.1,0.1,0.6) & (0,0,0) & (0,0,0) & (0.3,0.1,0.2) & (0,0,0)\\
(0,0,0) & (0,0,0) & (0.1,0.3,0.6) & (0,0,0) & (0.2,0.4,0.7) & (0,0,0)\\
(0,0,0) & (0,0,0) & (0.1,0.3,0.6) & (0,0,0) & (0.2,0.4,0.7) & (0,0,0)
\end{pmatrix},$$

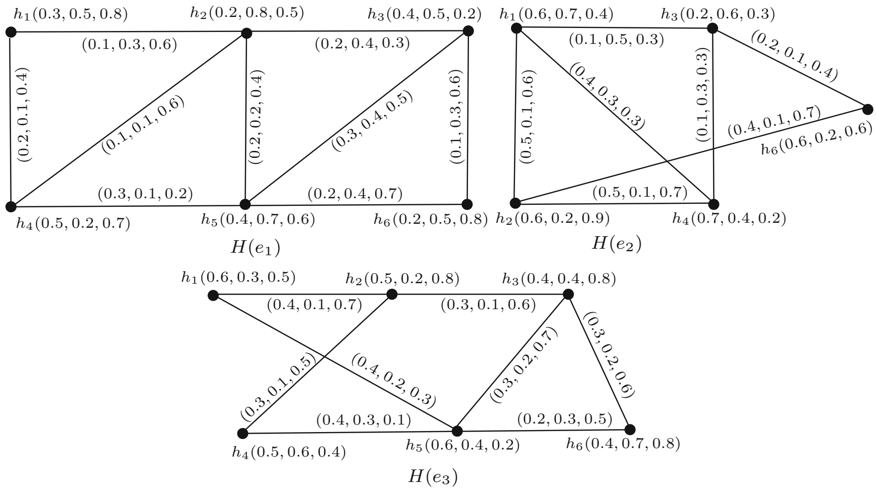


Fig. 7.8 Neutrosophic soft graph $G = \{H(e_1), H(e_2), H(e_3)\}$

$$H(e_2) = \begin{pmatrix} (0,0,0) & (0.5,0.1,0.6) & (0.1,0.5,0.3) & (0.4,0.3,0.3) & (0,0,0) & (0,0,0) \\ (0.5,0.1,0.6) & (0,0,0) & (0,0,0) & (0.5,0.1,0.7) & (0,0,0) & (0.4,0.1,0.7) \\ (0.1,0.5,0.3) & (0,0,0) & (0,0,0) & (0.1,0.3,0.3) & (0,0,0) & (0.2,0.1,0.4) \\ (0.4,0.3,0.3) & (0.5,0.1,0.7) & (0.1,0.3,0.3) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0.4,0.1,0.7) & (0.2,0.1,0.4) & (0,0,0) & (0,0,0) & (0,0,0) \end{pmatrix},$$

and

$$H(e_3) = \begin{pmatrix} (0,0,0) & (0.4,0.1,0.7) & (0,0,0) & (0,0,0) & (0.4,0.2,0.3) & (0,0,0) \\ (0.4,0.1,0.7) & (0,0,0) & (0.3,0.1,0.6) & (0.3,0.1,0.5) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0.3,0.1,0.6) & (0,0,0) & (0,0,0) & (0.3,0.2,0.7) & (0.3,0.2,0.6) \\ (0,0,0) & (0.3,0.1,0.5) & (0,0,0) & (0,0,0) & (0.4,0.3,0.1) & (0,0,0) \\ (0.4,0.2,0.3) & (0,0,0) & (0.3,0.2,0.7) & (0.4,0.3,0.1) & (0,0,0) & (0.2,0.3,0.5) \\ (0,0,0) & (0,0,0) & (0.3,0.2,0.6) & (0,0,0) & (0.2,0.3,0.5) & (0,0,0) \end{pmatrix}.$$

After performing some operations (AND or OR), we obtain the resultant neutrosophic graph $H(e)$, where $e = e_1 \wedge e_2 \wedge e_3$. The incidence matrix of resultant neutrosophic graph is

$$H(e) = \begin{pmatrix} (0,0,0) & (0.1,0.1,0.7) & (0,0,0.3) & (0,0,0.4) & (0,0,0.3) & (0,0,0) \\ (0.1,0.1,0.7) & (0,0,0) & (0,0,0.6) & (0.1,0.1,0.7) & (0,0,0.4) & (0,0,0.7) \\ (0,0,0.3) & (0,0,0.6) & (0,0,0) & (0,0,0.3) & (0,0,0.7) & (0.1,0.1,0.6) \\ (0,0,0.4) & (0.1,0.1,0.7) & (0,0,0.3) & (0,0,0) & (0,0,0.2) & (0,0,0) \\ (0,0,0.3) & (0,0,0.4) & (0,0,0.7) & (0,0,0.2) & (0,0,0) & (0,0,0.7) \\ (0,0,0) & (0,0,0.7) & (0.1,0.2,0.6) & (0,0,0) & (0,0,0.7) & (0,0,0) \end{pmatrix}.$$

Tabular representation of score values of incidence matrix of resultant neutrosophic graph $H(e)$ with average score function $S_k = \frac{T_k + I_k + 1 - F_k}{3}$ and choice value for each house h_k for $k = 1, 2, 3, 4, 5, 6$ are given in Table 7.3.

Table 7.3 Tabular representation of score values with choice values

	h_1	h_2	h_3	h_4	h_5	h_6	$\acute{h}_k$
h_1	0.334	0.167	0.234	0.2	0.234	0.334	1.503
h_2	0.167	0.334	0.133	0.334	0.2	0.334	1.502
h_3	0.234	0.133	0.334	0.234	0.1	0.2	1.235
h_4	0.2	0.167	0.234	0.334	0.267	0.334	1.536
h_5	0.234	0.2	0.1	0.267	0.334	0.1	1.235
h_6	0.334	0.1	0.234	0.334	0.1	0.334	1.436

Clearly, the maximum score value is 1.536, scored by the h_4. Mr. Aslam will buy the house h_4.
We present our method as Algorithm 7.3.1 that is used in our application.

Algorithm 7.3.1

1. Input the set P of choice parameters of Mr. Aslam, M is a subset of P.
2. Input the neutrosophic soft sets (F, M) and (K, M).
3. Construct the neutrosophic soft graph $G = (F, K, M)$.
4. Compute the resultant neutrosophic graph $H(e) = \bigcap_k H(e_k)$ for $e = \bigwedge_k e_k \; \forall \, k$.
5. Consider the neutrosophic graph $H(e)$ and its incidence matrix form.
6. Compute the score S_k of $h_k \; \forall \, k$.
7. The decision is h_k if $\acute{h}_k = \max_i \acute{h}_i$.
8. If k has more than one value, then any one of h_k may be chosen.

7.4 Intuitionistic Neutrosophic Soft Graphs

Bhowmik and Pal [55] introduced intuitionistic neutrosophic set and discussed some of its properties. Broumi and Smarandache [60] proposed intuitionistic neutrosophic soft sets.

Definition 7.21 Let X be an initial universe, and let P be the set of all parameters. $\mathcal{N}(X)$ denotes the set of all intuitionistic single-valued neutrosophic soft sets of X. Let N be a subset of P. A pair (F, N) is called an *intuitionistic single-valued neutrosophic soft set* over X.

Let $\mathcal{N}(X)$ denote the set of all intuitionistic single-valued neutrosophic soft sets of X and $\mathcal{N}(E)$ denote the set of all intuitionistic single-valued neutrosophic soft sets of E.

Definition 7.22 An *intuitionistic single-valued neutrosophic soft graph* on a nonempty X is an three-ordered tuple $G = (F, K, N)$ such that

1. N is a nonempty set of parameters.

2. (F, N) is an intuitionistic single-valued neutrosophic soft set over X.
3. (K, N) is an intuitionistic single-valued neutrosophic soft relation on X, i.e., $K : N \to \mathcal{N}(X \times X)$, where $\mathcal{N}(X \times X)$ is an intuitionistic neutrosophic power set.
4. $(F(e), K(e))$ is an intuitionistic single-valued neutrosophic graph for all $e \in N$.

That is,

$$T_{K(e)}(xy) \leq \min\{T_{F(e)}(x), T_{F(e)}(y)\},$$

$$I_{K(e)}(xy) \leq \min\{I_{F(e)}(x), I_{F(e)}(y)\},$$

$$F_{K(e)}(xy) \leq \max\{F_{F(e)}(x), F_{F(e)}(y)\},$$

such that $0 \leq T_{K(e)}(xy) + I_{K(e)}(xy) + F_{K(e)}(xy) \leq 2 \ \forall\, e \in N,\ x, y \in X$.

The intuitionistic single-valued neutrosophic graph $(F(e), K(e))$ is denoted by $H(e)$. Note that $T_{K(e)}(xy) = I_{K(e)}(xy) = 0$ and $F_{K(e)}(xy) = 1$ for all $xy \in X \times X - E, e \notin N$. (F, N) is called an intuitionistic single-valued neutrosophic soft vertex and (K, N) is called an intuitionistic single-valued neutrosophic soft edge. Thus, $((F, N), (K, N))$ is called an intuitionistic single-valued neutrosophic soft graph if

$$T_{K(e)}(xy) \leq \min\{T_{F(e)}(x), T_{F(e)}(y)\},$$

$$I_{K(e)}(xy) \leq \min\{I_{F(e)}(x), I_{F(e)}(y)\},$$

$$F_{K(e)}(xy) \leq \max\{F_{F(e)}(x), F_{F(e)}(y)\},$$

such that $0 \leq T_{K(e)}(xy) + I_{K(e)}(xy) + F_{K(e)}(xy) \leq 2 \ \forall\, e \in N,\ x, y \in X$. In other words, an intuitionistic single-valued neutrosophic soft graph is a parameterized family of intuitionistic single-valued neutrosophic graphs. The class of all intuitionistic single-valued neutrosophic soft graphs is denoted by $\mathcal{INS}(G^*)$. The *order* of an intuitionistic single-valued neutrosophic soft graph is

$$O(G) = \left(\sum_{e_i \in N} \left(\sum_{w \in X} T_{F(e_i)}(w) \right), \sum_{e_i \in N} \left(\sum_{w \in X} I_{F(e_i)}(w) \right), \sum_{e_i \in N} \left(\sum_{w \in X} F_{F(e_i)}(w) \right) \right).$$

The *size* of an intuitionistic single-valued neutrosophic soft graph is

$$S(G) = \left(\sum_{e_i \in N} \left(\sum_{wv \in E} T_{K(e_i)}(wv) \right), \sum_{e_i \in N} \left(\sum_{wv \in E} I_{K(e_i)}(wv) \right), \sum_{e_i \in N} (\sum_{wv \in E} F_{K(e_i)}(wv)) \right).$$

Example 7.9 Consider a simple graph G^* such that $X = \{w_1, w_2, w_3, w_4, w_5\}$ and $E = \{w_1w_2, w_2w_3, w_1w_3, w_1w_5\}$. Let $N = \{e_1, e_2, e_3\}$ be a set of parameters, and let (F, N) be an intuitionistic neutrosophic soft set over X with intuitionistic neutrosophic approximation function $F : N \to \mathcal{N}(X)$ defined by

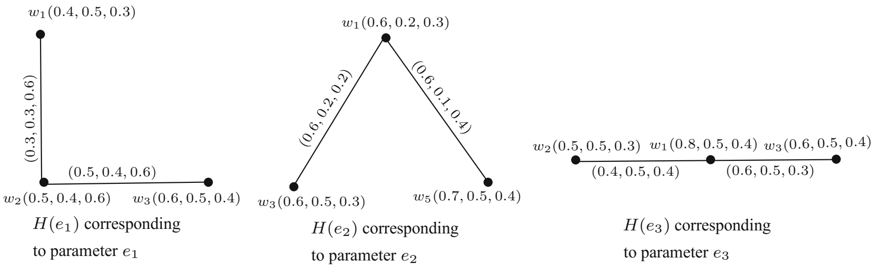


Fig. 7.9 Intuitionistic neutrosophic soft graph $G = \{H(e_1), H(e_2), H(e_3)\}$

$F(e_1) = \{(w_1, 0.4, 0.5, 0.3), (w_2, 0.5, 0.4, 0.6), (w_3, 0.6, 0.5, 0.4)\}$,
$F(e_2) = \{(w_1, 0.6, 0.2, 0.3), (w_3, 0.6, 0.5, 0.3), (w_5, 0.7, 0.5, 0.4)\}$,
$F(e_3) = \{(w_1, 0.8, 0.5, 0.4), (w_2, 0.5, 0.5, 0.3), (w_3, 0.6, 0.5, 0.4)\}$.
Let (K, N) be an intuitionistic neutrosophic soft set over E with intuitionistic neutrosophic approximation function $K : N \to \mathcal{N}(E)$ defined by
$K(e_1) = \{(w_1w_2, 0.3, 0.3, 0.6), (w_2w_3, 0.5, 0.4, 0.6)\}$,
$K(e_2) = \{(w_1w_3, 0.6, 0.2, 0.2), (w_1w_5, 0.6, 0.1, 0.4)\}$,
$K(e_3) = \{(w_1w_2, 0.4, 0.5, 0.4), (w_1w_3, 0.6, 0.5, 0.3)\}$.
Clearly, $H(e_1) = (F(e_1), K(e_1))$, $H(e_2) = (F(e_2), K(e_2))$ and $H(e_3) = (F(e_3), K(e_3))$ are intuitionistic neutrosophic graphs corresponding to the parameters e_1, e_2 and e_3, respectively, as shown in Fig. 7.9.

Hence $G = \{H(e_1), H(e_2), H(e_3)\}$ is an intuitionistic neutrosophic soft graph of G^*. Tabular representation of an intuitionistic neutrosophic soft graph is given in Table 7.4.

Table 7.4 Tabular representation of an intuitionistic neutrosophic soft graph

F	w_1	w_2	w_3	w_4	w_5
e_1	(0.4, 0.5, 0.3)	(0.5, 0.4, 0.6)	(0.6, 0.5, 0.4)	(0.0, 0.0, 0.0)	(0.0, 0.0, 0.0)
e_2	(0.6, 0.2, 0.3)	(0.0, 0.0, 0.0)	(0.6, 0.5, 0.3)	(0.0, 0.0, 0.0)	(0.7, 0.5, 0.4)
e_3	(0.8, 0.5, 0.4)	(0.5, 0.5, 0.3)	(0.6, 0.5, 0.4)	(0.0, 0.0, 0.0)	(0.0, 0.0, 0.0)

K	w_1w_2	w_2w_3	w_1w_3	w_1w_5
e_1	(0.3, 0.3, 0.6)	(0.5, 0.4, 0.6)	(0.0, 0.0, 0.0)	(0.0, 0.0, 0.0)
e_2	(0.0, 0.0, 0.0)	(0.0, 0.0, 0.0)	(0.6, 0.2, 0.2)	(0.6, 0.1, 0.4)
e_3	(0.4, 0.5, 0.4)	(0.0, 0.0, 0.0)	(0.6, 0.5, 0.3)	(0.0, 0.0, 0.0)

The order of intuitionistic neutrosophic soft graph G is $O(G) = \big((0.4 + 0.5 + 0.6) + (0.6 + 0.6 + 0.7) + (0.8 + 0.5 + 0.6), (0.5 + 0.4 + 0.5) + (0.2 + 0.5 + 0.5) + (0.5 + 0.5 + 0.5), (0.3 + 0.6 + 0.4) + (0.3 + 0.3 + 0.4) + (0.4 + 0.3 + 0.4)\big) = (5.3, 4.1, 3.4)$. The size of intuitionistic neutrosophic soft graph G is $S(G) = \big((0.3 +$

0.5) + (0.6 + 0.6) + (0.4 + 0.6), (0.3 + 0.4) + (0.2 + 0.1) + (0.5 + 0.5), (0.6 + 0.6) + (0.2 + 0.4) + (0.4 + 0.3)) = (3.0, 2.0, 2.5).

Definition 7.23 Let $G_1 = (F_1, K_1, N_1)$ and $G_2 = (F_2, K_2, N_2)$ be two intuitionistic neutrosophic soft graphs of G_1^* and G_2^*, respectively. The *Cartesian product* of G_1 and G_2 is an intuitionistic neutrosophic soft graph $G = G_1 \times G_2 = (F, K, N_1 \times N_2)$, where $(F = F_1 \times F_2, N_1 \times N_2)$ is an intuitionistic neutrosophic soft set over $X = X_1 \times X_2$, $(K = K_1 \times K_2, N_1 \times N_2)$ is an intuitionistic neutrosophic soft set over $E = \{((w, v_1), (w, v_2)) : w \in X_1, (v_1, v_2) \in E_2\} \cup \{((w_1, v), (w_2, v)) : v \in X_2, (w_1, w_2) \in E_1\}$ defined as

(i) $T_{F(e_1,e_2)}(w, v) = T_{F_1(e_1)}(w) \wedge T_{F_2(e_2)}(v)$,
$I_{F(e_1,e_2)}(w, v) = I_{F_1(e_1)}(w) \wedge I_{F_2(e_2)}(v)$,
$F_{F(e_1,e_2)}(w, v) = F_{F_1(e_1)}(w) \vee F_{F_2(e_2)}(v) \ \forall\ (w, v) \in X$, $(e_1, e_2) \in N_1 \times N_2$,

(ii) $T_{K(e_1,e_2)}\big((w, v_1), (w, v_2)\big) = T_{F_1(e_1)}(w) \wedge T_{K_2(e_2)}(v_1, v_2)$,
$I_{K(e_1,e_2)}\big((w, v_1), (w, v_2)\big) = I_{F_1(e_1)}(w) \wedge I_{K_2(e_2)}(v_1, v_2)$,
$F_{K(e_1,e_2)}\big((w, v_1), (w, v_2)\big) = F_{F_1(e_1)}(w) \vee F_{K_2(e_2)}(v_1, v_2) \ \forall\ w \in X_1$, $(v_1, v_2) \in E_2$,

(iii) $T_{K(e_1,e_2)}\big((w_1, v), (w_2, v)\big) = T_{F_2(e_2)}(v) \wedge T_{K_1(e_1)}(w_1, w_2)$,
$I_{K(e_1,e_2)}\big((w_1, v), (w_2, v)\big) = I_{F_2(e_2)}(v) \wedge I_{K_1(e_1)}(w_1, w_2)$,
$F_{K(e_1,e_2)}\big((w_1, v), (w_2, v)\big) = F_{F_2(e_2)}(v) \vee F_{K_1(e_1)}(w_1, w_2) \ \forall\ v \in X_2$, $(w_1, w_2) \in E_1$.

$H(e_1, e_2) = H_1(e_1) \times H_2(e_2)$ for all $(e_1, e_2) \in N_1 \times N_2$ are intuitionistic neutrosophic graphs.

Definition 7.24 The *cross product* of G_1 and G_2 is an intuitionistic neutrosophic soft graph $G = G_1 \circledcirc G_2 = (F, K, N_1 \times N_2)$, where $(F, N_1 \times N_2)$ is an intuitionistic neutrosophic soft set over $X = X_1 \times X_2$, $(K, N_1 \times N_2)$ is an intuitionistic neutrosophic soft set over $E = \{((w_1, v_1), (w_2, v_2)) : (w_1, w_2) \in E_1, (v_1, v_2) \in E_2\}$ defined as

(i) $T_{F(e_1,e_2)}(w, v) = T_{F_1(e_1)}(w) \wedge T_{F_2(e_2)}(v)$,
$I_{F(e_1,e_2)}(w, v) = I_{F_1(e_1)}(w) \wedge I_{F_2(e_2)}(v)$,
$F_{F(e_1,e_2)}(w, v) = F_{F_1(e_1)}(w) \vee F_{F_2(e_2)}(v) \ \forall\ (w, v) \in X$, $(e_1, e_2) \in N_1 \times N_2$

(ii) $T_{K(e_1,e_2)}\big((w_1, v_1), (w_2, v_2)\big) = T_{K_1(e_1)}(w_1, w_2) \wedge T_{K_2(e_2)}(v_1, v_2)$,
$I_{K(e_1,e_2)}\big((w_1, v_1), (w_2, v_2)\big) = I_{K_1(e_1)}(w_1, w_2) \wedge I_{K_2(e_2)}(v_1, v_2)$,
$F_{K(e_1,e_2)}\big((w_1, v_1), (w_2, v_2)\big) = F_{K_1(e_1)}(w_1, w_2) \vee F_{K_2(e_2)}(v_1, v_2) \ \forall\ (w_1, w_2) \in E_1$, $(v_1, v_2) \in E_2$.

$H(e_1, e_2) = H_1(e_1) \circledcirc H_2(e_2)$ for all $(e_1, e_2) \in N_1 \times N_2$ are intuitionistic neutrosophic graphs.

Definition 7.25 The *lexicographic product* of G_1 and G_2 is an intuitionistic neutrosophic soft graph $G = G_1 \odot G_2 = (F, K, N_1 \times N_2)$, where $(F, N_1 \times N_2)$ is an intuitionistic neutrosophic soft set over $X = X_1 \times X_2$, $(K, N_1 \times N_2)$ is an intuitionistic neutrosophic soft set over $E = \{((w, v_1), (w, v_2)) : w \in X_1, (v_1, v_2) \in E_2\} \cup \{((w_1, v_1), (w_2, v_2)) : (w_1, w_2) \in E_1, (v_1, v_2) \in E_2\}$ defined as

(i) $T_{F(e_1,e_2)}(w, v) = T_{F_1(e_1)}(w) \wedge T_{F_2(e_2)}(v)$,
$I_{F(e_1,e_2)}(w, v) = I_{F_1(e_1)}(w) \wedge I_{F_2(e_2)}(v)$,
$F_{F(e_1,e_2)}(w, v) = F_{F_1(e_1)}(w) \vee F_{F_2(e_2)}(v) \ \forall\ (w, v) \in X, (e_1, e_2) \in N_1 \times N_2$,

(ii) $T_{K(e_1,e_2)}\big((w, v_1), (w, v_2)\big) = T_{F_1(e_1)}(w) \wedge T_{K_2(e_2)}(v_1, v_2)$,
$I_{K(e_1,e_2)}\big((w, v_1), (w, v_2)\big) = I_{F_1(e_1)}(w) \wedge I_{K_2(e_2)}(v_1, v_2)$,
$F_{K(e_1,e_2)}\big((w, v_1), (w, v_2)\big) = F_{F_1(e_1)}(w) \vee F_{K_2(e_2)}(v_1, v_2) \ \forall\ w \in X_1, (v_1, v_2) \in E_2$,

(iii) $T_{K(e_1,e_2)}\big((w_1, v_1), (w_2, v_2)\big) = T_{K_1(e_1)}(w_1, w_2) \wedge T_{K_2(e_2)}(v_1, v_2)$,
$I_{K(e_1,e_2)}\big((w_1, v_1), (w_2, v_2)\big) = I_{K_1(e_1)}(w_1, w_2) \wedge I_{K_2(e_2)}(v_1, v_2)$,
$F_{K(e_1,e_2)}\big((w_1, v_1), (w_2, v_2)\big) = F_{K_1(e_1)}(w_1, w_2) \vee F_{K_2(e_2)}(v_1, v_2) \quad \forall \quad (w_1, w_2) \in E_1, (v_1, v_2) \in E_2$.

$H(e_1, e_2) = H_1(e_1) \odot H_2(e_2)$ for all $(e_1, e_2) \in N_1 \times N_2$ are intuitionistic neutrosophic graphs.

Definition 7.26 The *strong product* of G_1 and G_2 is an intuitionistic neutrosophic soft graph $G = G_1 \otimes G_2 = (F, K, N_1 \times N_2)$, where $(F, N_1 \times N_2)$ is an intuitionistic neutrosophic soft set over $X = X_1 \times X_2$, $(K, A \times N_2)$ is an intuitionistic neutrosophic soft set over $E = \{((w, v_1), (w, v_2)) : w \in X_1, (v_1, v_2) \in E_2\} \cup \{((w_1, v), (w_2, v)) : v \in X_2, (w_1, w_2) \in E_1\} \cup \{((w_1, v_1), (w_2, v_2)) : (w_1, w_2) \in E_1, (v_1, v_2) \in E_2\}$ such that

(i) $T_{F(e_1,e_2)}(w, v) = T_{F_1(e_1)}(w) \wedge T_{F_2(e_2)}(v)$,
$I_{F(e_1,e_2)}(w, v) = I_{F_1(e_1)}(w) \wedge I_{F_2(e_2)}(v)$,
$F_{F(e_1,e_2)}(w, v) = F_{F_1(e_1)}(w) \vee F_{F_2(e_2)}(v) \ \forall\ (w, v) \in X, (e_1, e_2) \in N_1 \times N_2$,

(ii) $T_{K(e_1,e_2)}\big((w, v_1), (w, v_2)\big) = T_{F_1(e_1)}(w) \wedge T_{K_2(e_2)}(v_1, v_2)$,
$I_{K(e_1,e_2)}\big((w, v_1), (w, v_2)\big) = I_{F_1(e_1)}(w) \wedge I_{K_2(e_2)}(v_1, v_2)$,
$F_{K(e_1,e_2)}\big((w, v_1), (w, v_2)\big) = F_{F_1(e_1)}(w) \vee F_{K_2(e_2)}(v_1, v_2) \ \forall\ w \in X_1, (v_1, v_2) \in E_2$,

(iii) $T_{K(e_1,e_2)}\big((w_1, v), (w_2, v)\big) = T_{F_2(e_2)}(v) \wedge T_{K_1(e_1)}(w_1, w_2)$,
$I_{K(e_1,e_2)}\big((w_1, v), (w_2, v)\big) = I_{F_2(e_2)}(v) \wedge I_{K_1(e_1)}(w_1, w_2)$,
$F_{K(e_1,e_2)}\big((w_1, v), (w_2, v)\big) = F_{F_2(e_2)}(v) \vee F_{K_1(e_1)}(w_1, w_2) \ \forall\ v \in X_2, (w_1, w_2) \in E_1$,

(iv) $T_{K(e_1,e_2)}\big((w_1, v_1), (w_2, v_2)\big) = T_{K_1(e_1)}(w_1, w_2) \wedge T_{K_2(e_2)}(v_1, v_2)$,
$I_{K(e_1,e_2)}\big((w_1, v_1), (w_2, v_2)\big) = I_{K_1(e_1)}(w_1, w_2) \wedge I_{K_2(e_2)}(v_1, v_2)$,
$F_{K(e_1,e_2)}\big((w_1, v_1), (w_2, v_2)\big) = F_{K_1(e_1)}(w_1, w_2) \vee F_{K_2(e_2)}(v_1, v_2) \ \forall\ (w_1, w_2) \in E_1, (v_1, v_2) \in E_2$.

$H(e_1, e_2) = H_1(e_1) \otimes H_2(e_2)$ for all $(e_1, e_2) \in N_1 \times N_2$ are intuitionistic neutrosophic graphs.

Definition 7.27 The *composition* of G_1 and G_2 is an intuitionistic neutrosophic soft graph $G = G_1[G_2] = (F, K, N_1 \times N_2)$, where $(F, N_1 \times N_2)$ is an intuitionistic neutrosophic soft set over $X = X_1 \times X_2$, $(K, N_1 \times N_2)$ is an intuitionistic neutrosophic soft set over $E = \{((w, v_1), (w, v_2)) : w \in X_1, (v_1, v_2) \in E_2\} \cup$

$\{((w_1, v), (w_2, v)) : v \in X_2, (w_1, w_2) \in E_1\} \cup \{((w_1, v_1), (w_2, v_2)) : (w_1, w_2) \in E_1, v_1 \neq v_2\}$ defined as

(i) $T_{F(e_1,e_2)}(w, v) = T_{F_1(e_1)}(w) \wedge T_{F_2(e_2)}(v)$,
$I_{F(e_1,e_2)}(w, v) = I_{F_1(e_1)}(w) \wedge I_{F_2(e_2)}(v)$,
$F_{F(e_1,e_2)}(w, v) = F_{F_1(e_1)}(w) \vee F_{F_2(e_2)}(v) \ \forall\ (w, v) \in X, (e_1, e_2) \in N_1 \times N_2$,

(ii) $T_{K(e_1,e_2)}((w, v_1), (w, v_2)) = T_{F_1(e_1)}(w) \wedge T_{K_2(e_2)}(v_1, v_2)$,
$I_{K(e_1,e_2)}((w, v_1), (w, v_2)) = I_{F_1(e_1)}(w) \wedge I_{K_2(e_2)}(v_1, v_2)$,
$F_{K(e_1,e_2)}((w, v_1), (w, v_2)) = F_{F_1(e_1)}(w) \vee F_{K_2(e_2)}(v_1, v_2) \ \forall\ w \in X_1, (v_1, v_2) \in E_2$,

(iii) $T_{K(e_1,e_2)}\big((w_1, v), (w_2, v)\big) = T_{F_2(e_2)}(v) \wedge T_{K_1(e_1)}(w_1, w_2)$,
$I_{K(e_1,e_2)}\big((w_1, v), (w_2, v)\big) = I_{F_2(e_2)}(v) \wedge I_{K_1(e_1)}(w_1, w_2)$,
$F_{K(e_1,e_2)}\big((w_1, v), (w_2, v)\big) = F_{F_2(e_2)}(v) \vee F_{K_1(e_1)}(w_1, w_2) \ \forall\ v \in X_2, (w_1, w_2) \in E_1$,

(iv) $T_{K(e_1,e_2)}\big((w_1, v_1), (w_2, v_2)\big) = T_{F_1(e_1)}(w_1, w_2) \wedge T_{F_2(e_2)}(v_1) \wedge T_{F_2(e_2)}(v_2)$,
$I_{K(e_1,e_2)}\big((w_1, v_1), (w_2, v_2)\big) = I_{F_1(e_1)}(w_1, w_2) \wedge I_{F_2(e_2)}(v_1) \wedge I_{F_2(e_2)}(v_2)$,
$F_{K(e_1,e_2)}\big((w_1, v_1), (w_2, v_2)\big) = F_{F_1(e_1)}(w_1, w_2) \vee F_{F_2(e_2)}(v_1) \vee F_{F_2(e_2)}(v_2) \ \forall (w_1, w_2) \in E_1$, where $v_1 \neq v_2, v_1, v_2 \in X_2$.

$H(e_1, e_2) = H_1(e_1)[H_2(e_2)]$ for all $(e_1, e_2) \in N_1 \times N_2$ are intuitionistic neutrosophic graphs.

Proposition 7.4 *The Cartesian product, cross product, lexicographic product, strong product and composition of two intuitionistic neutrosophic soft graphs are an intuitionistic neutrosophic soft graph.*

Definition 7.28 Let $G_1 = (F_1, K_1, N_1)$ and $G_2 = (F_2, K_2, N_2)$ be two intuitionistic neutrosophic soft graphs. The *intersection* of G_1 and G_2 is an intuitionistic neutrosophic soft graph denoted by $G = G_1 \cap G_2 = (F, K, N_1 \cup N_2)$, where $(F, N_1 \cup N_2)$ is an intuitionistic neutrosophic soft set over $X = X_1 \cap X_2$, $(K, N_1 \cup N_2)$ is an intuitionistic neutrosophic soft set over $E = E_1 \cap E_2$, and the truth-membership, indeterminacy-membership and falsity-membership functions of G for all $w, v \in X$ are defined by,

(i) $$T_{F(e)}(v) = \begin{cases} T_{F_1(e)}(v) & \text{if } e \in N_1 - N_2; \\ T_{F_2(e)}(v) & \text{if } e \in N_2 - N_1; \\ T_{F_1(e)}(v) \wedge T_{F_2(e)}(v), & \text{if } e \in N_1 \cap N_2. \end{cases}$$

$$I_{F(e)}(v) = \begin{cases} I_{F_1(e)}(v) & \text{if } e \in N_1 - N_2; \\ I_{F_2(e)}(v) & \text{if } e \in N_2 - N_1; \\ I_{F_1(e)}(v) \wedge I_{F_2(e)}(v), & \text{if } e \in N_1 \cap N_2. \end{cases}$$

$$F_{F(e)}(v) = \begin{cases} F_{F_1(e)}(v) & \text{if } e \in N_1 - N_2; \\ F_{F_2(e)}(v) & \text{if } e \in N_2 - N_1; \\ F_{F_1(e)}(v) \vee F_{F_2(e)}(v), & \text{if } e \in N_1 \cap N_2. \end{cases}$$

(ii) $$T_{K(e)}(wv) = \begin{cases} T_{K_1(e)}(wv) & \text{if } e \in N_1 - N_2; \\ T_{K_2(e)}(wv) & \text{if } e \in N_2 - N_1; \\ T_{K_1(e)}(wv) \wedge T_{K_2(e)}(wv), & \text{if } e \in N_1 \cap N_2. \end{cases}$$

$$I_{K(e)}(wv) = \begin{cases} I_{K_1(e)}(wv) & \text{if } e \in N_1 - N_2; \\ I_{K_2(e)}(wv) & \text{if } e \in N_2 - N_1; \\ I_{K_1(e)}(wv) \wedge I_{K_2(e)}(wv), & \text{if } e \in N_1 \cap N_2. \end{cases}$$

$$F_{K(e)}(wv) = \begin{cases} F_{K_1(e)}(wv) & \text{if } e \in N_1 - N_2; \\ F_{K_2(e)}(wv) & \text{if } e \in N_2 - N_1; \\ F_{K_1(e)}(wv) \vee F_{K_2(e)}(wv), & \text{if } e \in N_1 \cap N_2. \end{cases}$$

Definition 7.29 Let $G_1 = (F_1, K_1, N_1)$ and $G_2 = (F_2, K_2, N_2)$ be two intuitionistic neutrosophic soft graphs. The *union* of G_1 and G_2 may or may not be intuitionistic neutrosophic soft graph denoted by $G = G_1 \cup G_2 = (F, K, N_1 \cup N_2)$, where $(F, N_1 \cup N_2)$ is an intuitionistic neutrosophic soft set over $X = X_1 \cup X_2$, $(K, N_1 \cup N_2)$ is an intuitionistic neutrosophic soft set over $E = E_1 \cup E_2$, and the truth-membership, indeterminacy-membership and falsity-membership functions of G for all $w, v \in X$ are defined by,

(i) $$T_{F(e)}(v) = \begin{cases} T_{F_1(e)}(v) & \text{if } e \in N_1 - N_2; \\ T_{F_2(e)}(v) & \text{if } e \in N_2 - N_1; \\ T_{F_1(e)}(v) \vee T_{F_2(e)}(v), & \text{if } e \in N_1 \cap N_2. \end{cases}$$

$$I_{F(e)}(v) = \begin{cases} I_{F_1(e)}(v) & \text{if } e \in N_1 - N_2; \\ I_{F_2(e)}(v) & \text{if } e \in N_2 - N_1; \\ I_{F_1(e)}(v) \wedge I_{F_2(e)}(v), & \text{if } e \in N_1 \cap N_2. \end{cases}$$

$$F_{F(e)}(v) = \begin{cases} F_{F_1(e)}(v) & \text{if } e \in N_1 - N_2; \\ F_{F_2(e)}(v) & \text{if } e \in N_2 - N_1; \\ F_{F_1(e)}(v) \wedge F_{F_2(e)}(v), & \text{if } e \in N_1 \cap N_2. \end{cases}$$

(ii) $$T_{K(e)}(wv) = \begin{cases} T_{K_1(e)}(wv) & \text{if } e \in N_1 - N_2; \\ T_{K_2(e)}(wv) & \text{if } e \in N_2 - N_1; \\ T_{K_1(e)}(wv) \vee T_{K_2(e)}(wv), & \text{if } e \in N_1 \cap N_2. \end{cases}$$

$$I_{K(e)}(wv) = \begin{cases} I_{K_1(e)}(wv) & \text{if } e \in N_1 - N_2; \\ I_{K_2(e)}(wv) & \text{if } e \in N_2 - N_1; \\ I_{K_1(e)}(wv) \wedge I_{K_2(e)}(wv), & \text{if } e \in N_1 \cap N_2. \end{cases}$$

$$F_{K(e)}(wv) = \begin{cases} F_{K_1(e)}(wv) & \text{if } e \in N_1 - N_2; \\ F_{K_2(e)}(wv) & \text{if } e \in N_2 - N_1; \\ F_{K_1(e)}(wv) \wedge F_{K_2(e)}(wv), & \text{if } e \in N_1 \cap N_2. \end{cases}$$

Remark 7.1 Let G_1 and G_2 be two intuitionistic neutrosophic soft graphs over G^* then $G_1 \cup G_2$ may or may not be intuitionistic neutrosophic soft graph.

Definition 7.30 Let G_1 and G_2 be two intuitionistic neutrosophic soft graphs. The *join* of G_1 and $\mathbb{G}_2$ may or may not be intuitionistic neutrosophic soft graph denoted by $G_1 + G_2 = (F_1 + F_2, K_1 + K_2, N_1 \cup N_2)$, where $(F_1 + F_2, N_1 \cup N_2)$ is an intuitionistic neutrosophic soft set over $X_1 \cup X_2$, $(K_1 + K_2, N_1 \cup N_2)$ is an intuitionistic neutrosophic soft set over $E_1 \cup E_2 \cup \acute{E}$ defined by

(i) $(F_1 + F_2, N_1 \cup N_2) = (F_1, N_1) \cup (F_2, N_2)$,
(ii) $(K_1 + K_2, N_1 \cup N_2) = (K_1, N_1) \cup (K_2, N_2)$ if $wv \in E_1 \cup E_2$, where $e \in N_1 \cap N_2$, $wv \in \acute{E}$ and $\acute{E}$ is the set of all edges joining the vertices of X_1 and X_2, and the truth-membership, indeterminacy-membership and falsity-membership functions are defined by

$$T_{K_1+K_2(e)}(wv) = \min\{T_{F_1(e)}(w), T_{F_2(e)}(v)\},$$
$$I_{K_1+K_2(e)}(wv) = \min\{I_{F_1(e)}(w), I_{F_2(e)}(v)\},$$
$$F_{K_1+K_2(e)}(wv) = \max\{F_{F_1(e)}(w), F_{F_2(e)}(v)\}\ \forall wv \in \acute{E}.$$

Proposition 7.5 *If G_1 and G_2 are two intuitionistic neutrosophic soft graphs, then their join $G_1 + G_2$ may or may not be intuitionistic neutrosophic soft graph.*

Definition 7.31 The *complement* of an intuitionistic neutrosophic soft graph $G = (F, K, N)$ denoted by $G^c = (F^c, K^c, N^c)$ is defined as follows:

(i) $N^c = N$,
(ii) $F^c(e) = F(e)$,
(iii) $T_{K^c(e)}(w, v) = T_{F(e)}(w) \wedge T_{F(e)}(v) - T_{K(e)}(w, v)$,
(iv) $I_{K^c(e)}(w, v) = I_{F(e)}(w) \wedge I_{F(e)}(v) - I_{K(e)}(w, v)$ and
(v) $F_{K^c(e)}(w, v) = F_{F(e)}(w) \vee F_{F(e)}(v) - F_{K(e)}(w, v)$, for all $w, v \in X$, $e \in N$.

Example 7.10 Let G^* be a crisp graph with $X = \{v_1, v_2, v_3, v_4\}$ and $E=\{v_1v_2, v_1v_4, v_1v_3, v_2v_3, v_3v_4\}$. Let $N = \{e_1, e_2\}$ be a set of parameters, and let (F, N) be an intuitionistic neutrosophic soft set over X with intuitionistic neutrosophic approximation function $F : N \to \mathcal{N}(X)$ defined by
$F(e_1)=\{(v_1, 0.4, 0.6, 0.1), (v_2, 0.5, 0.4, 0.7), (v_3, 0.5, 0.3, 0.4), (v_4, 0.5, 0.6, 0.2)\}$,
$F(e_2)=\{(v_1, 0.4, 0.2, 0.2), (v_2, 0.5, 0.3, 0.4), (v_3, 0.6, 0.3, 0.5), (v_4, 0.5, 0.4, 0.2)\}$.
Let (K, N) be an intuitionistic neutrosophic soft set over E with intuitionistic neutrosophic approximation function $K : N \to \mathcal{N}(E)$ defined by
$K(e_1) = \{(v_1v_2, 0.3, 0.3, 0.5), (v_1v_4, 0.2, 0.5, 0.2), (v_1v_3, 0.4, 0.3, 0.4), (v_2v_3, 0.5, 0.3, 0.5)\}$,
$K(e_2) = \{(v_1v_3, 0.3, 0.2, 0.5), (v_1v_4, 0.4, 0.1, 0.1), (v_3v_4, 0.5, 0.3, 0.4), (v_3v_2, (0.5, 0.3, 0.5)\}$.
Clearly, $G = \{H(e_1) = (F(e_1), K(e_1)), H(e_2) = (F(e_2), K(e_2))\}$ is intuitionistic neutrosophic soft graph, and $H(e_1)$ and $H(e_2)$ are intuitionistic neutrosophic graphs corresponding to the parameters e_1 and e_2, respectively, as shown in Fig. 7.10.

Now, the complement of intuitionistic neutrosophic soft graph $G = \{H(e_1), H(e_2)\}$ is the complement of intuitionistic neutrosophic graphs $H(e_1)$ and $H(e_2)$ which are shown in Fig. 7.11.

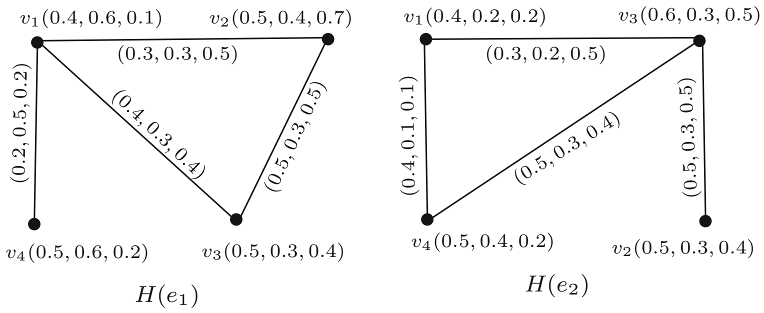


Fig. 7.10 Intuitionistic neutrosophic soft graph $G = \{H(e_1), H(e_2)\}$

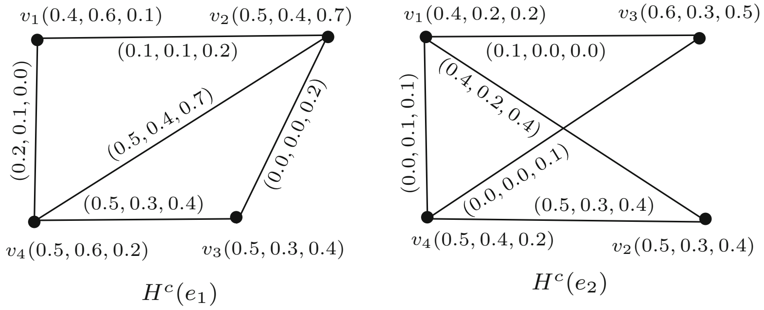


Fig. 7.11 Complement of intuitionistic neutrosophic soft graph $G^c = \{H^c(e_1), H^c(e_2)\}$

Definition 7.32 An intuitionistic neutrosophic soft graph G is a *complete intuitionistic neutrosophic soft graph* if $H(e)$ is a complete intuitionistic neutrosophic graph for all $e \in N$, i.e.,

$$\begin{aligned} T_{K(e)}(wv) &= \min(T_{F(e)}(w), T_{F(e)}(v)), \\ I_{K(e)}(wv) &= \min(I_{F(e)}(w), I_{F(e)}(v)), \\ F_{K(e)}(wv) &= \max(F_{F(e)}(w), F_{F(e)}(v)) \end{aligned}$$

$\forall\, w, v \in X, e \in N.$

Definition 7.33 An intuitionistic neutrosophic soft graph G is a *strong intuitionistic neutrosophic soft graph* if $H(e)$ is a strong intuitionistic neutrosophic graph for all $e \in N$.

Example 7.11 Consider the simple graph G^* where $X = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{v_1v_2, v_2v_5, v_3v_5, v_1v_3, v_1v_4, v_3v_6, v_5v_6\}$. Let $N = \{e_1, e_2\}$. Let (F, N) be an intuitionistic neutrosophic soft set over X with its approximation function $F : N \to \mathcal{N}(X)$ defined by

$F(e_1) = \{(v_1, 0.4, 0.5, 0.7), (v_2, 0.6, 0.5, 0.5), (v_3, 0.6, 0.3, 0.5), (v_4, 0.7, 0.5, 0.4), (v_5, 0.7, 0.4, 0.5), (v_6, 0.3, 0.5, 0.7)\}$,

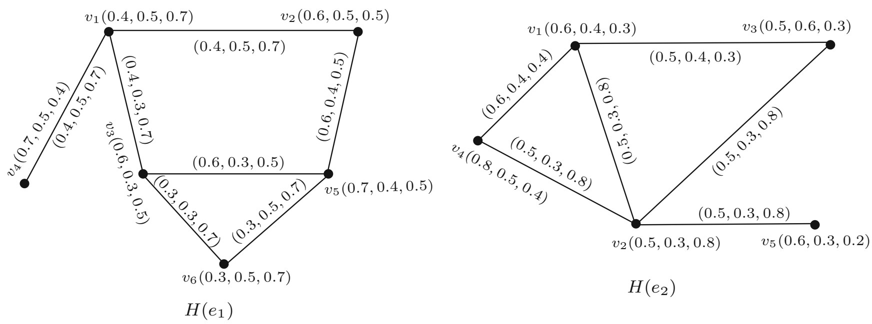


Fig. 7.12 Strong intuitionistic neutrosophic soft graph $G = \{H(e_1), H(e_2)\}$

$F(e_2) = \{(v_1, 0.6, 0.4, 0.3), (v_2, 0.5, 0.3, 0.8), (v_3, 0.5, 0.6, 0.3), (v_4, 0.8, 0.5, 0.4), (v_5, 0.6, 0.3, 0.2)\}$.

Let (K, N) be an intuitionistic neutrosophic soft set over E with its approximation function $K : N \to \mathcal{N}(E)$ defined by

$K(e_1) = \{(v_1v_2, 0.4, 0.5, 0.7), (v_1v_3, 0.4, 0.3, 0.7), (v_1v_4, 0.4, 0.5, 0.7), (v_2v_5, 0.6, 0.4, 0.5), (v_3v_5, 0.6, 0.3, 0.5), (v_3v_6, 0.3, 0.3, 0.7), (v_5v_6, 0.3, 0.5, 0.7)\}$,

$K(e_2) = \{(v_1v_3, 0.5, 0.4, 0.3), (v_1v_4, 0.6, 0.4, 0.4), (v_1v_2, 0.5, 0.3, 0.8), (v_2v_3, 0.5, 0.3, 0.8), (v_2v_4, 0.5, 0.3, 0.8), (v_2v_5, 0.5, 0.3, 0.8)\}$.

$H(e_1) = (F(e_1), K(e_1))$ and $H(e_2) = (F(e_2), K(e_2))$ are strong intuitionistic neutrosophic graphs corresponding to the parameters e_1 and e_2, respectively, as shown in Fig. 7.12. Hence $G = \{H(e_1), H(e_2)\}$ is a strong intuitionistic neutrosophic soft graph of G^*.

Proposition 7.6 *If G_1 and G_2 are strong intuitionistic neutrosophic soft graphs, then $G_1 \times G_2$ and $G_1[G_2]$ are strong intuitionistic neutrosophic soft graphs.*

Remark 7.2 The union of two strong intuitionistic neutrosophic soft graphs is not necessarily strong intuitionistic neutrosophic soft graph.

Example 7.12 Let $N_1 = \{e_1\}$ and $N_2 = \{e_1, e_2\}$ be the parameter sets. Let G_1 and G_2 be the two strong intuitionistic neutrosophic soft graphs defined as follows:

$G_1 = \{H_1(e_1), H_1(e_2)\} = \{(\{(w_1, 0.5, 0.6, 0.4), (w_2, 0.7, 0.4, 0.5), (w_3, 0.5, 0.8, 0.4)\}, \{(w_1w_2, 0.5, 0.4, 0.5), (w_2w_3, 0.5, 0.4, 0.5)\}), (\{(w_1, 0.4, 0.6, 0.5), (w_3, 0.5, 0.7, 0.4)\}, \{(w_1w_3, 0.4, 0.6, 0.5)\})\}$,

$G_2 = \{H_2(e_1)\} = \{(w_1, 0.4, 0.9, 0.3), (w_2, 0.5, 0.6, 0.4), (w_1w_2, 0.4, 0.6, 0.4)\}$.

The union of G_1 and G_2 is $G = G_1 \cup G_2 = (H, N_1 \cup N_2)$, where $N_1 \cup N_2 = \{e_1, e_2\}$, $H(e_1) = H_1(e_1) \cup H_2(e_1)$ and $H(e_2) = H_1(e_2)$ are as shown in Fig. 7.13. Clearly, $G = \{H(e_1), H(e_2)\}$ is not a strong intuitionistic neutrosophic soft graph as shown in Fig. 7.14.

Proposition 7.7 *If $G_1 \times G_2$ is strong intuitionistic neutrosophic soft graph, then at least G_1 or G_2 must be strong intuitionistic neutrosophic soft graph.*

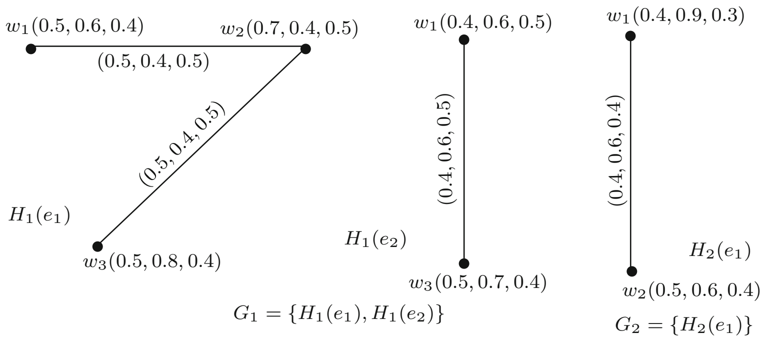
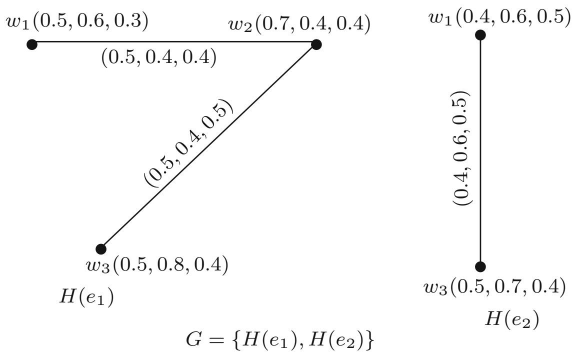


Fig. 7.13 Strong intuitionistic neutrosophic soft graphs G_1 and G_2

Fig. 7.14 Union of two strong intuitionistic neutrosophic soft graphs



Proposition 7.8 *If $G_1[G_2]$ is strong intuitionistic neutrosophic soft graph, then at least G_1 or G_2 must be strong intuitionistic neutrosophic soft graph.*

Definition 7.34 The complement of a strong intuitionistic neutrosophic soft graph $G = (F, K, N)$ is an intuitionistic neutrosophic soft graph $G^c = (F^c, K^c, N^c)$ defined by

(i) $N^c = N$,

(ii) $F^c(e)(w) = F(e)(w)$ for all $e \in N$ and $w \in X$,

(iii) $$T_{K^c(e)}(w, v) = \begin{cases} 0 & \text{if } T_{K(e)}(w, v) > 0, \\ \min\{T_{F(e)}(w), T_{F(e)}(v)\}, & \text{if } T_{K(e)}(w, v) = 0, \end{cases}$$

$$I_{K^c(e)}(w, v) = \begin{cases} 0 & \text{if } I_{K(e)}(w, v) > 0, \\ \min\{I_{F(e)}(w), I_{F(e)}(v)\}, & \text{if } I_{K(e)}(w, v) = 0, \end{cases}$$

$$F_{K^c(e)}(w, v) = \begin{cases} 0 & \text{if } F_{K(e)}(w, v) > 0, \\ \max\{F_{F(e)}(w), F_{F(e)}(v)\}, & \text{if } F_{K(e)}(w, v) = 0, \end{cases}$$

Proposition 7.9 *If G is a strong intuitionistic neutrosophic soft graph over G^*, then G^c is also a strong intuitionistic neutrosophic soft graph.*

Theorem 7.6 *If G and G^c are strong intuitionistic neutrosophic soft graphs of G^*, then $G \cup G^c$ is a complete intuitionistic neutrosophic soft graph.*

7.5 Isomorphism of Intuitionistic Neutrosophic Soft Graphs

Definition 7.35 Let $G_1 = (F_1, K_1, N)$ and $G_2 = (F_2, K_2, N)$ be two intuitionistic neutrosophic soft graphs of $G_1^* = (X_1, E_1)$ and $G_2^* = (X_2, E_2)$, respectively. A *homomorphism* $f_N : G_1 \to G_2$ is a mapping $f_N : X_1 \to X_2$ which satisfies the following conditions:

(i) $T_{F_1(e)}(v_1) \leq T_{F_2(e)}(f_e(v_1))$, $I_{F_1(e)}(v_1) \leq I_{F_2(e)}(f_e(v_1))$, $F_{F_1(e)}(v_1) \geq F_{F_2(e)}(f_e(v_1))$,

(ii) $T_{K_1(e)}(v_1v_2) \leq T_{K_2(e)}(f_e(v_1)f_e(v_2))$, $I_{K_1(e)}(v_1v_2) \leq I_{K_2(e)}(f_e(v_1)f_e(v_2))$, $F_{K_1(e)}(v_1v_2) \geq F_{K_2(e)}(f_e(v_1)f_e(v_2))$, for all $e \in N$, $v_1 \in X_1$, $v_1v_2 \in E_1$.

A bijective homomorphism is called a *weak isomorphism* if
$T_{F_1(e)}(v_1) = T_{F_2(e)}(f_e(v_1))$, $I_{F_1(e)}(v_1) = I_{F_2(e)}(f_e(v_1))$, $F_{F_1(e)}(v_1) = F_{F_2(e)}(f_e(v_1))$, $\forall e \in N$, $v_1 \in X_1$.
A bijective homomorphism $f_N : G_1 \to G_2$ such that
$T_{K_1(e)}(v_1v_2) = T_{K_2(e)}(f_e(v_1)f_e(v_2))$, $I_{K_1(e)}(v_1v_2) = I_{K_2(e)}(f_e(v_1)f_e(v_2))$, $F_{K_1(e)}(v_1v_2) = F_{K_2(e)}(f_e(v_1)f_e(v_2))$, for all $e \in N$, $v_1v_2 \in E_1$ is called a *coweak isomorphism.*

An *endomorphism* of intuitionistic neutrosophic soft graph G with X as the underlying set is a homomorphism of G into itself.

Definition 7.36 Let $G_1 = (F_1, K_1, N)$ and $G_2 = (F_2, K_2, N)$ be two intuitionistic neutrosophic soft graphs of $G_1^* = (X_1, E_1)$ and $G_2^* = (X_2, E_2)$, respectively. An *isomorphism* $f_N : G_1 \to G_2$ is a mapping $f_N : X_1 \to X_2$ which satisfies the following conditions:

(i) $T_{F_1(e)}(v_1) = T_{F_2(e)}(f_e(v_1))$, $I_{F_1(e)}(v_1) = I_{F_2(e)}(f_e(v_1))$, $F_{F_1(e)}(v_1) = F_{F_2(e)}(f_e(v_1))$,

(ii) $T_{K_1(e)}(v_1v_2) = T_{K_2(e)}(f_e(v_1)f_e(v_2))$, $I_{K_1(e)}(v_1v_2) = I_{K_2(e)}(f_e(v_1)f_e(v_2))$, $F_{K_1(e)}(v_1v_2) = F_{K_2(e)}(f_e(v_1)f_e(v_2))$, for all $e \in N$, $v_1 \in X_1$, $v_1v_2 \in E_1$.

Example 7.13 Let $N = \{e_1, e_2\}$ be a parameter set. $G_1 = (F_1, K_1, N)$ and $G_2 = (F_1, K_2, N)$ are two intuitionistic neutrosophic soft graphs defined as follows:
$G_1 = \{H_1(e_1), H_1(e_2)\} = \{(\{(v_1, 0.3, 0.4, 0.7), (v_2, 0.7, 0.4, 0.3)\},$
$\{(v_1v_2, 0.2, 0.3, 0.6)\}),$
$(\{(v_1, 0.3, 0.4, 0.8), (v_2, 0.2, 0.1, 0.6), (v_3, 0.4, 0.5, 0.3)\}, \{(v_1v_2, 0.1, 0.1, 0.7),$
$(v_1v_3, 0.1, 0.3, 0.7)\})\},$

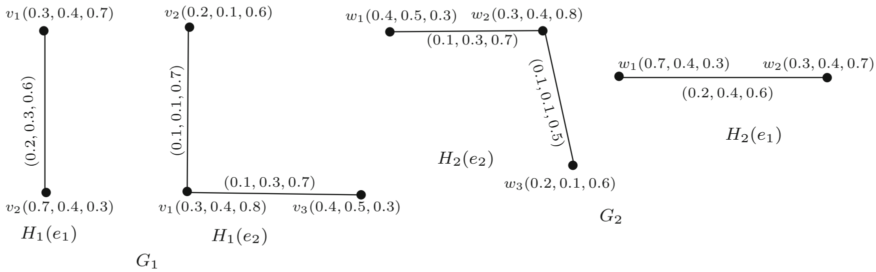


Fig. 7.15 $G_1 = \{H_1(e_1), H_1(e_2)\}$ and $G_2 = \{H_2(e_1), H_2(e_2)\}$

$G_2 = \{H_2(e_1), H_2(e_2)\} = \{(\{(w_1, 0.7, 0.4, 0.3), (w_2, 0.3, 0.4, 0.7)\}, \{(w_1w_2, 0.2, 0.4, 0.6)\}), (\{(w_1, 0.4, 0.5, 0.3), (w_2, 0.3, 0.4, 0.8), (w_3, 0.2, 0.1, 0.6)\}, \{(w_1w_2, 0.1, 0.3, 0.7), (w_2w_3, 0.1, 0.1, 0.5)\})\}$.
A mapping $f_N : X_1 \to X_2$ is defined by $f_{e_1}(v_1) = w_2$, $f_{e_1}(v_2) = w_1$ and $f_{e_2}(v_1) = w_2$, $f_{e_2}(v_2) = w_3$, and $f_{e_2}(v_3) = w_1$, then $T_{F_1(e_1)}(v_1) = T_{F_2(e_1)}(w_2)$, $I_{F_1(e_1)}(v_1) = I_{F_2(e_1)}(w_2)$, $F_{F_1(e_1)}(v_1) = F_{F_2(e_1)}(w_2)$, and $T_{F_1(e_1)}(v_2) = T_{F_2(e_1)}(w_1)$, $I_{F_1(e_1)}(v_2)=I_{F_2(e_1)}(w_1)$, $F_{F_1(e_1)}(v_2)=F_{F_2(e_1)}(w_1)$, but $T_{K_1(e_1)}(v_1v_2) = T_{K_2(e_1)}(w_2w_1)$, $I_{K_1(e_1)}(v_1v_2) \neq I_{K_2(e_1)}(w_2w_1)$, $F_{K_1(e_1)}(v_1v_2) = F_{K_2(e_1)}(w_2w_1)$. Clearly, $H_1(e_1)$ is weak isomorphic to $H_2(e_1)$. By routine computation, we can see that $H_1(e_2)$ is weak isomorphic to $H_2(e_2)$.

Hence G_1 is weak isomorphic to G_2 but not isomorphic as shown in Fig. 7.15.

Example 7.14 Let $N = \{e_1, e_2\}$ be a parameter set. $G_1 = (F_1, K_1, N)$ and $G_2 = (F_1, K_2, N)$ are two intuitionistic neutrosophic soft graphs as shown in Fig. 7.16. A mapping $f_N : X_1 \to X_2$ is defined by $f_{e_1}(w_1) = v_2$, $f_{e_1}(w_2) = v_1$, $f_{e_1}(w_3) = v_4$, $f_{e_1}(w_4) = v_3$ and $f_{e_2}(w_1) = v_1$, $f_{e_2}(w_2) = v_2$ and $f_{e_2}(w_3) = v_3$. By routine computations, we can see that G_1 is coweak isomorphic to G_2 but not isomorphic as $T_{F_1(e_1)}(w_2) = T_{F_2(e_1)}(v_1)$, $I_{F_1(e_1)}(w_2) \neq I_{F_2(e_1)}(v_1)$, $F_{F_1(e_1)}(w_2) \neq F_{F_2(e_1)}(v_1)$ and $T_{F_1(e_2)}(w_3) \neq T_{F_2(e_2)}(v_3)$, $I_{F_1(e_2)}(w_3) \neq I_{F_2(e_2)}(v_3)$, $F_{F_1(e_2)}(w_3) \neq F_{F_2(e_2)}(v_3)$.

Theorem 7.7 *For any two isomorphic intuitionistic neutrosophic soft graphs their orders and sizes are same.*

Definition 7.37 Let G be an intuitionistic neutrosophic soft graph with X as the underlying set. A one-to-one onto map $f_N : X \to X$ is an *automorphism* of G if

(i) $T_{F_1(e)}(v_1) = T_{F_2(e)}(f_e(v_1))$, $I_{F_1(e)}(v_1) = I_{F_2(e)}(f_e(v_1))$, $F_{F_1(e)}(v_1) = F_{F_2(e)}(f_e(v_1))$,
(ii) $T_{K_1(e)}(v_1v_2) = T_{K_2(e)}(f_e(v_1)f_e(v_2))$, $I_{K_1(e)}(v_1v_2) = I_{K_2(e)}(f_e(v_1)f_e(v_2))$, $F_{K_1(e)}(v_1v_2) = F_{K_2(e)}(f_e(v_1)f_e(v_2))$, for all $e \in N$, $v_1, v_2 \in X$.

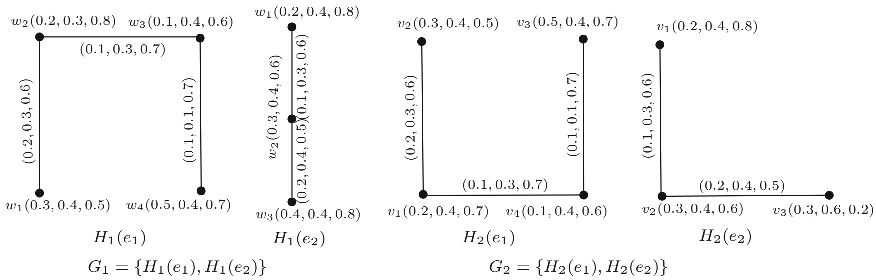


Fig. 7.16 $G_1 = \{H_1(e_1), H_1(e_2)\}$ and $G_2 = \{H_2(e_1), H_2(e_2)\}$

Definition 7.38 An intuitionistic neutrosophic soft graph $G = (F, K, N)$ of G^* is an *ordered intuitionistic neutrosophic soft graph* if it satisfies the following conditions:
$T_{F(e)}(v_1) \leq T_{F(e)}(v_2)$, $I_{F(e)}(v_1) \leq I_{F(e)}(v_2)$, $F_{F(e)}(v_1) \geq F_{F(e)}(v_2)$,
$T_{F(e)}(w_1) \leq T_{F(e)}(w_2)$, $I_{F(e)}(w_1) \leq I_{F(e)}(w_2)$, $F_{F(e)}(w_1) \geq F_{F(e)}(w_2)$,
for $v_1, v_2, w_1, w_2 \in X$, $v_1 \neq w_1$, $v_2 \neq w_2$, for all $e \in N$, imply
$T_{K(e)}(v_1w_1) \leq T_{K(e)}(v_2w_2)$, $I_{K(e)}(v_1w_1) \leq I_{K(e)}(v_2w_2)$, $F_{K(e)}(v_1w_1) \geq F_{K(e)}(v_2w_2)$.

Proposition 7.10 *Let G_1, G_2 and G_3 be intuitionistic neutrosophic soft graphs. Then the isomorphism between these intuitionistic neutrosophic soft graphs is an equivalence relation.*

Proof Let $G_1 = (F_1, K_1, N)$, $G_2 = (F_2, K_2, N)$ and $G_3 = (F_3, K_3, N)$ be three intuitionistic neutrosophic soft graphs with the underlying sets X_1, X_2 and X_3, respectively.

(1) Reflexive: Consider identity mapping $f_N : X_1 \to X_1$, $f_e(v) = v$ for all $v \in X_1$,satisfying
$T_{F_1(e)}(v) = T_{F_2(e)}(f_e(v))$, $I_{F_1(e)}(v) = I_{F_2(e)}(f_e(v))$, $F_{F_1(e)}(v) = F_{F_2(e)}(f_e(v))$,
$T_{K_1(e)}(uv) = T_{K_2(e)}(f_e(u)f_e(v))$, $I_{K_1(e)}(uv) = I_{K_2(e)}(f_e(u)f_e(v))$, $F_{K_1(e)}(uv) = F_{K_2(e)}(f_e(u)f_e(v))$,
for all $u, v \in X_1$, $e \in N$. Hence f_N is an isomorphism of intuitionistic neutrosophic soft graph to itself.
(2) Symmetric: Let $f_N : X_1 \to X_2$ be an isomorphism of G_1 onto G_2, $f_e(v) = v'$ for all $v \in X_1$, such that
$T_{F_1(e)}(v) = T_{F_2(e)}(f_e(v))$, $I_{F_1(e)}(v) = I_{F_2(e)}(f_e(v))$, $F_{F_1(e)}(v) = F_{F_2(e)}(f_e(v))$,
$T_{K_1(e)}(uv) = T_{K_2(e)}(f_e(u)f_e(v))$, $I_{K_1(e)}(uv) = I_{K_2(e)}(f_e(u)f_e(v))$, $F_{K_1(e)}(uv) = F_{K_2(e)}(f_e(u)f_e(v))$,
for all $u, v \in X_1$, $e \in N$.
As f_N is a bijective mapping, $f^{-1}(v') = v$ for all $v' \in X_2$, then
$T_{F_2(e)}(v') = T_{F_1(e)}(f^{-1}(v'))$, $I_{F_2(e)}(v') = I_{F_1(e)}(f^{-1}(v'))$, $F_{F_2(e)}(v') = F_{F_1(e)}(f^{-1}(v'))$,
$T_{K_2(e)}(u'v') = T_{K_1(e)}(f^{-1}(u')f^{-1}(v'))$, $I_{K_2(e)}(u'v') = I_{K_1(e)}(f^{-1}(u')f^{-1}(v'))$,
$F_{K_2(e)}(u'v') = F_{K_1(e)}(f^{-1}(u')f^{-1}(v'))$ for all $u', v' \in X_2$, $e \in N$.

Hence $f^{-1}: X_2 \to X_1$ is an isomorphism from G_2 to G_1; that is, $G_1 \cong G_2$ implies $G_2 \cong G_1$.

(3) Transitive: Let $f_N : X_1 \to X_2$ and $g_N : X_2 \to X_3$ be isomorphisms of the intuitionistic neutrosophic soft graphs G_1 onto G_2 and G_2 onto G_3, respectively. For transitive relation we consider a bijective mapping $g_N \circ f_N : X_1 \to X_3$ such that $(g_N \circ f_N)(u) = g_e(f_e(u))$ for all $u \in X_1$.
As $f_N : X_1 \to X_2$ is an isomorphism from G_1 onto G_2, such that $f_e(v) = v'$ for all $v \in X_1$, then
$T_{F_1(e)}(v) = T_{F_2(e)}(f_e(v)) = T_{F_2(e)}(v')$, $I_{F_1(e)}(v) = I_{F_2(e)}(f_e(v)) = I_{F_2(e)}(v')$,
$F_{F_1(e)}(v) = F_{F_2(e)}(f_e(v)) = F_{F_2(e)}(v')$, and
$T_{K_1(e)}(uv) = T_{K_2(e)}(f_e(u)f_e(v)) = T_{K_2(e)}(u'v')$, $I_{K_1(e)}(uv) = I_{K_2(e)}(f_e(u)f_e(v)) = I_{K_2(e)}(u'v')$,
$F_{K_1(e)}(uv) = F_{K_2(e)}(f_e(u)f_e(v)) = F_{K_2(e)}(u'v')$, for all $u, v \in X_1, e \in N$.

As $g_N : X_2 \to X_3$ is an isomorphism from G_2 onto G_3 such that $g_e(v') = v''$ for all $v' \in X_2$, then

$T_{F_2(e)}(v') = T_{F_3(e)}(g_e(v')) = T_{F_2(e)}(v'')$, $I_{F_2(e)}(v') = I_{F_3(e)}(g_e(v')) = I_{F_3(e)}(v'')$,
$F_{F_2(e)}(v') = F_{F_3(e)}(g_e(v')) = F_{F_3(e)}(v'')$, and
$T_{K_2(e)}(u'v') = T_{K_3(e)}(g_e(u')g_e(v')) = T_{K_3(e)}(u''v'')$, $I_{K_2(e)}(u'v') = I_{K_3(e)}(g_e(u')g_e(v')) = I_{K_2(e)}(u''v'')$,
$F_{K_2(e)}(u'v') = F_{K_3(e)}(g_e(u')g_e(v')) = F_{K_3(e)}(u''v'')$, for all $u', v' \in X_2, e \in N$.

For transitive relation we consider a bijective mapping $g_N \circ f_N : X_1 \to X_3$; then

$T_{F_1(e)}(v) = T_{F_2(e)}(f_e(v)) = T_{F_2(e)}(v') = T_{F_3(e)}(g_e(f_e(v)))$,
$I_{F_1(e)}(v) = I_{F_2(e)}(f_e(v)) = I_{F_2(e)}(v') = I_{F_3(e)}(g_e(f_e(v)))$,
$F_{F_1(e)}(v) = F_{F_2(e)}(f_e(v)) = F_{F_2(e)}(v') = F_{F_3(e)}(g_e(f_e(v)))$, and
$T_{K_1(e)}(uv) = T_{K_2(e)}(f_e(u)f_e(v)) = T_{K_2(e)}(u'v') = T_{K_3(e)}(g_e(f_e(u))g_e(f_e(v)))$,
$I_{K_1(e)}(uv) = I_{K_2(e)}(f_e(u)f_e(v)) = I_{K_2(e)}(u'v') = I_{K_3(e)}(g_e(f_e(u))g_e(f_e(v)))$,
$F_{K_1(e)}(uv) = F_{K_2(e)}(f_e(u)f_e(v)) = F_{K_2(e)}(u'v') = F_{K_3(e)}(g_e(f_e(u))g_e(f_e(v)))$
for all $u, v \in X_1, e \in N$.
Therefore $g_N \circ f_N$ is an isomorphism between G_1 and G_3.

Hence isomorphism between intuitionistic neutrosophic soft graphs by (1), (2) and (3) is an equivalence relation.

Proposition 7.11 *Let G_1, G_2 and G_3 be intuitionistic neutrosophic soft graphs. Then the weak isomorphism between these intuitionistic neutrosophic soft graphs is a partial order relation*

Proof Let $G_1 = (F_1, K_1, N)$, $G_2 = (F_2, K_2, N)$ and $G_3 = (F_3, K_3, N)$ be three intuitionistic neutrosophic soft graphs with the underlying sets X_1, X_2 and X_3, respectively.

(1) Reflexive: Consider identity mapping $f_N : X_1 \to X_1$, $f_e(v) = v$ for all $v \in X_1$, satisfying

$T_{F_1(e)}(v) = T_{F_2(e)}(f_e(v))$, $I_{F_1(e)}(v) = I_{F_2(e)}(f_e(v))$, $F_{F_1(e)}(v) = F_{F_2(e)}(f_e(v))$,
$T_{K_1(e)}(uv) = T_{K_2(e)}(f_e(u)f_e(v))$, $I_{K_1(e)}(uv) = I_{K_2(e)}(f_e(u)f_e(v))$, $F_{K_1(e)}(uv) = F_{K_2(e)}(f_e(u)f_e(v))$,
for all $u, v \in X_1$, $e \in N$. Hence f_N is a weak isomorphism of intuitionistic neutrosophic soft graph to itself. Thus G_1 is a weak isomorphic to itself.

(2) Antisymmetric: Let $f_N : X_1 \to X_2$ be an isomorphism of G_1 onto G_2, $f_e(v) = v'$ for all $v \in X_1$, such that
$T_{F_1(e)}(v) = T_{F_2(e)}(f_e(v))$, $I_{F_1(e)}(v) = I_{F_2(e)}(f_e(v))$, $F_{F_1(e)}(v) = F_{F_2(e)}(f_e(v))$,
$T_{K_1(e)}(uv) \leq T_{K_2(e)}(f_e(u)f_e(v))$, $I_{K_1(e)}(uv) \leq I_{K_2(e)}(f_e(u)f_e(v))$, $F_{K_1(e)}(uv) \geq F_{K_2(e)}(f_e(u)f_e(v))$,
for all $u, v \in X_1$, $e \in N$.
Let $g_N : X_2 \to X_1$ be an isomorphism of G_2 onto G_1, $g_e(v') = v$ for all $v' \in X_2$, such that
$T_{F_2(e)}(v')=T_{F_1(e)}(g_e(v'))$, $I_{F_2(e)}(v')=I_{F_1(e)}(g_e(v'))$, $F_{F_2(e)}(v') = F_{F_2(e)}(g_e(v'))$,
$T_{K_2(e)}(u'v') \leq T_{K_1(e)}(g_e(u')g_e(v'))$, $I_{K_2(e)}(u'v') \leq I_{K_1(e)}(g_e(u')g_e(v'))$, $F_{K_2(e)}(u'v') \geq F_{K_1(e)}(g_e(u')g_e(v'))$,
for all $u', v' \in X_2$, $e \in N$.
Both weak isomorphisms f_N from G_1 onto G_2 and g_N from G_2 onto G_3 are holds when G_1 and G_2 have same number of edges, and the corresponding edges have same truth-membership degree, indeterminacy-membership degree and falsity-membership degree corresponding to the parameter to the set of parameters. Hence G_1 and G_2 are identical.

(3) Transitive: Let $f_N : X_1 \to X_2$ and $g_N : X_2 \to X_3$ be weak isomorphisms of the intuitionistic neutrosophic soft graphs G_1 onto G_2 and G_2 onto G_3, respectively. For transitive relation we consider a bijective mapping $g_N \circ f_N : X_1 \to X_3$ such that $(g_N \circ f_N)(u) = g_e(f_e(u))$ for all $u \in X_1$.
As $f_N : X_1 \to X_2$ is a weak isomorphism from G_1 onto G_2, such that $f_e(v) = v'$ for all $v \in X_1$, then

$T_{F_1(e)}(v) = T_{F_2(e)}(f_e(v)) = T_{F_2(e)}(v')$, $I_{F_1(e)}(v) = I_{F_2(e)}(f_e(v)) = I_{F_2(e)}(v')$,
$F_{F_1(e)}(v) = F_{F_2(e)}(f_e(v)) = F_{F_2(e)}(v')$, and
$T_{K_1(e)}(uv) \leq T_{K_2(e)}(f_e(u)f_e(v)) = T_{K_2(e)}(u'v')$, $I_{K_1(e)}(uv) \leq I_{K_2(e)}(f_e(u)f_e(v)) = I_{K_2(e)}(u'v')$,
$F_{K_1(e)}(uv) \geq F_{K_2(e)}(f_e(u)f_e(v)) = F_{K_2(e)}(u'v')$, for all $u, v \in X_1$, $e \in N$.

As $g_N : X_2 \to X_3$ is an isomorphism from G_2 onto G_3 such that $g_e(v') = v''$ for all $v' \in X_2$, then

$T_{F_2(e)}(v') = T_{F_3(e)}(g_e(v')) = T_{F_3(e)}(v'')$, $I_{F_2(e)}(v') = I_{F_3(e)}(g_e(v')) = I_{F_3(e)}(v'')$,
$F_{F_2(e)}(v') = F_{F_3(e)}(g_e(v')) = F_{F_3(e)}(v'')$, and
$T_{K_2(e)}(u'v') \leq T_{K_3(e)}(g_e(u')g_e(v')) = T_{K_3(e)}(u''v'')$, $I_{K_2(e)}(u'v') \leq I_{K_3(e)}(g_e(u')g_e(v')) = I_{K_3(e)}(u''v'')$,

$F_{K_2(e)}(u'v') \geq F_{K_3(e)}(g_e(u')g_e(v')) = F_{K_3(e)}(u''v'')$, for all $u', v' \in X_2$, $e \in N$.

For transitive relation we consider a bijective mapping $g_N \circ f_N : X_1 \to X_3$; then

$T_{F_1(e)}(v) = T_{F_2(e)}(f_e(v)) = T_{F_2(e)}(v') = T_{F_3(e)}(g_e(f_e(v)))$,
$I_{F_1(e)}(v) = I_{F_2(e)}(f_e(v)) = I_{F_2(e)}(v') = I_{F_3(e)}(g_e(f_e(v)))$,
$F_{F_1(e)}(v) = F_{F_2(e)}(f_e(v)) = F_{F_2(e)}(v') = F_{F_3(e)}(g_e(f_e(v)))$, and
$T_{K_1(e)}(uv) \leq T_{K_2(e)}(f_e(u)f_e(v)) = T_{K_2(e)}(u'v') \leq T_{K_3(e)}(g_e(f_e(u))g_e(f_e(v)))$,
$I_{K_1(e)}(uv) \leq I_{K_2(e)}(f_e(u)f_e(v)) = I_{K_2(e)}(u'v') \leq I_{K_3(e)}(g_e(f_e(u))g_e(f_e(v)))$,
$F_{K_1(e)}(uv) \geq F_{K_2(e)}(f_e(u)f_e(v)) = F_{K_2(e)}(u'v') \geq F_{K_3(e)}(g_e(f_e(u))g_e(f_e(v)))$
for all $u, v \in X_1, e \in N$.
Therefore $g_N \circ f_N$ is a weak isomorphism between G_1 and G_3, i.e., weak isomorphism satisfying transitivity.

Hence isomorphism between intuitionistic neutrosophic soft graphs by (1), (2) and (3) is a partial order relation.

Definition 7.39 An intuitionistic neutrosophic soft graph G is *self-complementary* if $G \approx G^c$.

Proposition 7.12 *Let G_1 and G_2 be intuitionistic neutrosophic soft graphs. Then $G_1 \cong G_2$ if and only if $G_1^c \cong G_2^c$.*

Proof Let G_1 and G_2 be the two intuitionistic neutrosophic soft graphs. Suppose that $G_1 \cong G_2$, then there exist a bijective mapping $f_N : X_1 \to X_2$ such that $f_e(v) = v'$ for all $v \in X_1$, $T_{F_1(e)}(v) = T_{F_2(e)}(f_e(v))$, $I_{F_1(e)}(v) = I_{F_2(e)}(f_e(v))$, $F_{F_1(e)}(v) = F_{F_2(e)}(f_e(v))$ and $T_{K_1(e)}(uv) = T_{K_2(e)}(f_e(u)f_e(v))$,
$I_{K_1(e)}(uv) = I_{K_2(e)}(f_e(u)f_e(v))$, $F_{K_1(e)}(uv) = F_{K_2(e)}(f_e(u)f_e(v))$, for all $u, v \in X_1$, $e \in N$. By the definition of complement of intuitionistic neutrosophic soft graphs

$$
\begin{aligned}
T^c_{K_1(e)}(uv) &= T_{F_1(e)}(u) \wedge T_{F_1(e)}(v) - T_{K_1(e)}(uv),\\
&= T_{F_2(e)}(f_e(u)) \wedge T_{F_2(e)}(f_e(v)) - T_{K_2(e)}(f_e(u)f_e(v))\\
&= T^c_{K_2(e)}(f_e(u)f_e(v)),\\
I^c_{K_1(e)}(uv) &= I_{F_1(e)}(u) \wedge I_{F_1(e)}(v) - I_{K_1(e)}(uv),\\
&= I_{F_2(e)}(f_e(u)) \wedge T_{F_2(e)}(f_e(v)) - I_{K_2(e)}(f_e(u)f_e(v))\\
&= I^c_{K_2(e)}(f_e(u)f_e(v)),\\
F^c_{K_1(e)}(uv) &= F_{F_1(e)}(u) \vee F_{F_1(e)}(v) - F_{K_1(e)}(uv),\\
&= F_{F_2(e)}(f_e(u)) \wedge F_{F_2(e)}(f_e(v)) - F_{K_2(e)}(f_e(u)f_e(v))\\
&= F^c_{K_2(e)}(f_e(u)f_e(v))
\end{aligned}
$$

Hence $G_1^c \cong G_2^c$.

Conversely, assume that $G_1^c \cong G_2^c$, then there exist an isomorphism $g_N : X_1 \to X_2$ such that $g_e(v) = v'$,
$T_{F_1(e)}(v) = T_{F_2(e)}(g_e(v))$, $I_{F_1(e)}(v) = I_{F_2(e)}(g_e(v))$, $F_{F_1(e)}(v) = F_{F_2(e)}(f_e(v))$, for all $v \in X_1, e \in N$,$T^c_{K_1(e)}(uv) = T^c_{K_2(e)}(g_e(u)g_e(v))$,
$I^c_{K_1(e)}(uv) = I^c_{K_2(e)}(g_e(u)g_e(v))$, $F^c_{K_1(e)}(uv) = F^c_{K_2(e)}(g_e(u)g_e(v))$, for all $u, v \in X_1$, $e \in N$.
By using the definition of complement of intuitionistic neutrosophic soft graph

$$T^c_{K_1(e)}(uv) = T^c_{F_1(e)}(u) \wedge T^c_{F_1(e)}(v) - T_{K_1(e)}(uv),$$
$$T^c_{K_2(e)}(g_e(u)g_e(v)) = T^c_{F_2(e)}(g_e(u)) \wedge T^c_{F_2(e)}(g_e(v)) - T_{K_2(e)}(g_e(u)g_e(v)),$$
$$I^c_{K_1(e)}(uv) = I^c_{F_1(e)}(u) \wedge I^c_{F_1(e)}(v) - I_{K_1(e)}(uv),$$
$$I^c_{K_2(e)}(g_e(u)g_e(v)) = I^c_{F_2(e)}(g_e(u)) \wedge I^c_{F_2(e)}(g_e(v)) - I_{K_2(e)}(g_e(u)g_e(v)),$$
$$F^c_{K_1(e)}(uv) = F^c_{F_1(e)}(u) \vee F^c_{F_1(e)}(v) - F_{K_1(e)}(uv),$$
$$F^c_{K_2(e)}(g_e(u)g_e(v)) = F^c_{F_2(e)}(g_e(u)) \vee F^c_{F_2(e)}(g_e(v)) - F_{K_2(e)}(g_e(u)g_e(v)).$$

As $T^c_{K_1(e)}(uv) = T^c_{K_2(e)}(g_e(u)g_e(v))$, $I^c_{K_1(e)}(uv) = I^c_{K_2(e)}(g_e(u)g_e(v))$, $F^c_{K_1(e)}(uv) = F^c_{K_2(e)}(g_e(u)g_e(v))$, for all $u, v \in X_1, e \in N$, $g_N : X_1 \to X_2$ is an isomorphism between G_1 and G_2, that is $G_1 \cong G_2$.

Proposition 7.13 *If G_1 is coweak isomorphic to G_2, then there can be a homomorphism between G_1^c and G_2^c.*

Proposition 7.14 *If G_1 is weak isomorphic to G_2, then G_1^c and G_2^c are weak isomorphic intuitionistic neutrosophic soft graphs.*

7.6 Applications of Intuitionistic Neutrosophic Soft Graphs

Intuitionistic neutrosophic soft graph has several applications in decision-making problems and used to deal with uncertainties from our different daily life problems. In this section, we apply the concept of intuitionistic neutrosophic soft sets in decision-making problems. Many practical problems can be represented by graphs. We present an application of intuitionistic neutrosophic soft graph to a multiple criteria decision-making problem. We present an algorithm for most appropriate selection of an object in a multiple criteria decision-making problem.

Algorithm 7.6.1

1. Input the set of parameters $e_1, e_2, \ldots, e_k$.
2. Input the intuitionistic neutrosophic soft sets (F, N) and (K, N).
3. Input the intuitionistic neutrosophic graphs $H(e_1), H(e_2), \ldots, H(e_k)$.

4. Calculate the score values of intuitionistic neutrosophic graphs $H(e_1)$, $H(e_2)$, $\ldots$, $H(e_k)$ using formula

$$S_{ij} := \sqrt{(T_j)^2 + (I_j)^2 + (1 - F_j)^2} \tag{7.1}$$

   Tabular representation of score values of intuitionistic neutrosophic graphs $H(e_k)$, $\forall\, k$.
5. Compute the choice values of $C_p = \sum_j S_{ij}$ for all $i = 1, 2, \ldots, n$ and $p = 1, 2, \ldots, k$.
6. The decision is S_i if $S_i = \max_{i=1}^{n}\{\min_{p=1}^{k} C_p\}$.
7. If i has more than one value, then any one of S_i may be chosen.

An algorithm for the selection of optimal object based on given set of information.

7.6.1 An Appropriate Selection of a Machine

An appropriate selection of a machine for a specific task is an important decision-making problem for a machine manufacturing corporation. The performance of a manufacturing corporation is badly affected by the wrong selection. The main purpose in machine selection is that machine will achieve the require tasks within possible short time and minimum cost. The main purpose is to select the machine that will complete the required task within the time available for the lowest possible cost. Rate of productivity, automatic system and price are important aspects considered in selection of a machine. The rate of productivity, value of product and charge of manufacturing depend upon the performance of machine. Mr. X should be an expert or at least familiar with the machine properties, to select the best machine among the parameters (alternatives), i.e., "price", "rate of productivity" and "automatic system". Let $X = \{m_1, m_2, m_3, m_4, m_5, m_6\}$ be set of six machines to be considered as the universal set and $N = \{e_1, e_2, e_3\}$ be the set of parameters that characterize the machine, and the parameters e_1, e_2 and e_3 stands for "price", "rate of productivity" and "automatic system", respectively. Consider the intuitionistic neutrosophic soft set (F, N) over X which define the "efficiency of machines" corresponding to the given parameters that Mr. X want to select. (K, N) is an intuitionistic neutrosophic soft set over E=$\{m_1m_2, m_2m_3, m_6m_1, m_1m_3, m_1m_4, m_1m_5, m_2m_4, m_2m_5, m_2m_6, m_3m_4, m_3 m_5, m_3m_6, m_4m_5, m_4m_6, m_5m_6\}$ and defines degree of truth-membership, degree of indeterminacy and degree of falsity-membership of the connection between two machines corresponding to the selected attributes e_1, e_2 and e_3. The intuitionistic neutrosophic graphs $H(e_1)$, $H(e_2)$ and $H(e_3)$ of intuitionistic neutrosophic soft graph $G = \{H(e_1), H(e_2), H(e_3)\}$ corresponding to the parameters "price", "rate of productivity" and "automatic system", respectively, are shown in Fig. 7.17.

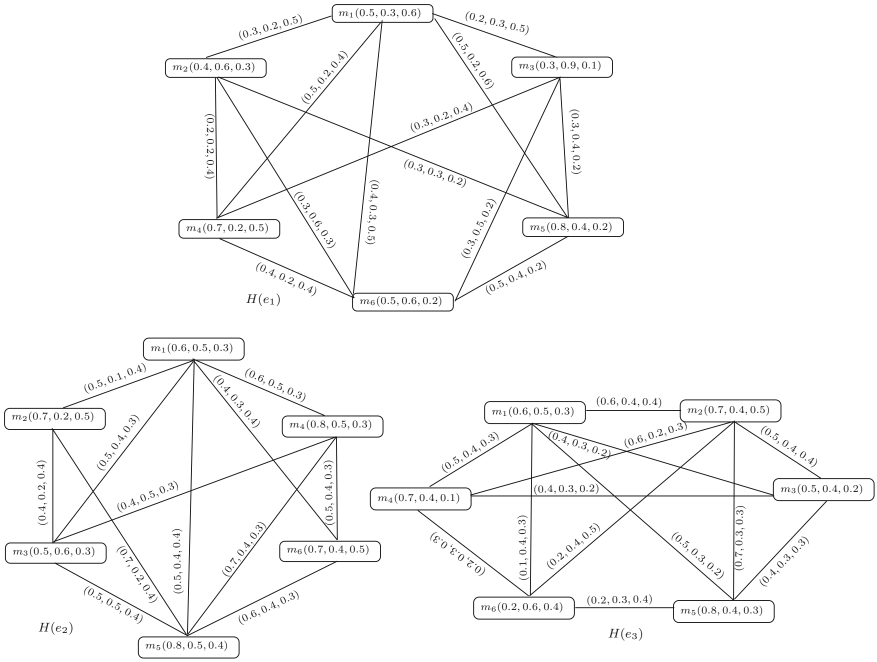


Fig. 7.17 Intuitionistic neutrosophic soft graph $G = \{H(e_1), H(e_2), H(e_3)\}$

Table 7.5 Tabular representation of score values and choice values of $H(e_1)$

	m_1	m_2	m_3	m_4	m_5	m_6	$\acute{m}_k$
m_1	0	0.62	0.62	0.80	0.67	0.71	3.42
m_2	0.62	0	0	0.66	0.91	0.97	3.16
m_3	0.62	0	0	0.70	0.94	0.99	3.25
m_4	0.80	0.66	0.70	0	0	0.75	2.91
m_5	0.67	0.91	0.94	0	0	1.0	3.52
m_6	0.71	0.97	0.94	0.75	1.0	0	4.37

Tabular representation Tables 7.3, 7.6 and 7.7 of score values of intuitionistic neutrosophic graphs $H(e_1)$, $H(e_2)$ and $H(e_3)$ with normalized score function $S_{ij} = \sqrt{(T_j)^2 + (I_j)^2 + (1 - F_j)^2}$ and choice value for each machine m_i for $i = 1, 2, 3, 4, 5, 6$ are given in Table 7.5.

The decision is S_i if $S_i = \max_{i=1}^{6}\{\min_{p=1}^{3} m_p\} = \max_{i=1}^{6}\{3.42, 2.48, 3.25, 2.91, 3.52, 2.73\} = 3.52$. Clearly, the maximum score value is 3.52, scored by the m_5. Mr. X will buy the machine m_5.

Table 7.6 Tabular representation of score values and choice values of $H(e_2)$

	m_1	m_2	m_3	m_4	m_5	m_6	$\acute{m}_k$
m_1	0	0.79	0.94	1.0	0.88	0.78	4.39
m_2	0.79	0	0.75	0	0.94	0	2.48
m_3	0.94	0.75	0	0.95	0.93	0	3.57
m_4	1.0	0	0.95	0	1.0	0.95	3.9
m_5	0.88	0.94	0.93	1.0	0	1.0	4.75
m_6	0.78	0	0	0.95	1.0	0	2.73

Table 7.7 Tabular representation of score values and choice values of $H(e_3)$

	m_1	m_2	m_3	m_4	m_5	m_6	$\acute{m}_k$
m_1	0	0.94	0.94	0.95	0.99	0.81	4.63
m_2	0.94	0	0.94	0.94	1.0	0.67	4.49
m_3	0.94	0.94	0	0.94	0.86	0	3.68
m_4	0.95	0.94	0.94	0	0	0.79	3.62
m_5	0.99	1.0	0.86	0	0	0.70	3.55
m_6	0.81	0.67	0	0.79	0.70	0	2.97

7.6.2 Selection of Brand in Product Marketing

We present a multicriteria decision-making problem for product marketing if there are multiple brands of a product; product marketing has intuitionistic neutrosophic behaviour. Consider Mr. X who is a retail owner wants to maximize his profit by selling some electronic items which meets all the requirements which is set by a retail outlet owner. Let $X = \{S_1, S_2, S_3, S_4, S_5\}$ be a set of five brands of an item to be sold in an international market, and let $N = \{e_1$ = "price", e_2 = "quality"$\}$ be a set of parametric factors in product marketing. Let (F, N) be the intuitionistic neutrosophic soft set over X, which describes the effectiveness of the brands, $T_{F(e_k)}(S_i)$, $T_{F(e_k)}(S_i)$ and $T_{F(e_k)}(S_i)$, for $i = 1, 2, \ldots, 5$, $k = 1, 2$ represent the degree of membership (goodness), degree of indeterminacy and degree of nonmembership (poorness) of the brands corresponding to the parameters e_1 = "price" and e_2 = "quality", respectively, and (K, N) be the intuitionistic neutrosophic soft set on $E = \{S_1S_2, S_1S_4, S_1S_3, S_2S_3, S_3S_4, S_2S_5, S_3S_5, S_1S_5, S_4S_5\}$ which describes the relationship between brands corresponding to the parameters e_1 = "price" and e_2 = "quality". The intuitionistic neutrosophic soft graph is shown in Fig. 7.18. The method for selection of brand in product marketing is presented in Algorithm 7.6.2.

Algorithm 7.6.2

1. Input the set of parameters $e_1, e_2, \ldots, e_k$.
2. Input the intuitionistic neutrosophic soft sets (F, N) and (K, N).
3. Construct intuitionistic neutrosophic graph $H(e_1) \cap H(e_2) \cap \ldots \cap H(e_k)$.

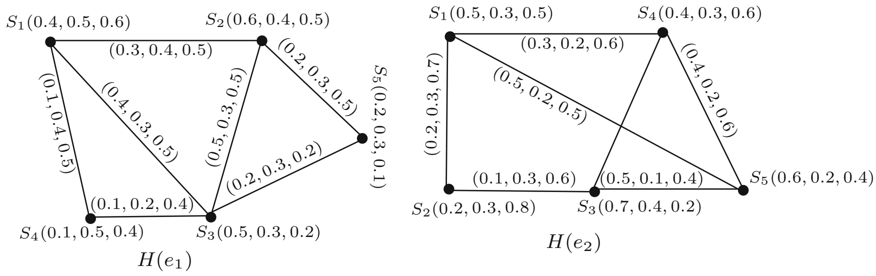


Fig. 7.18 Intuitionistic neutrosophic soft graph

Fig. 7.19 $H(e_1) \cap H(e_2)$

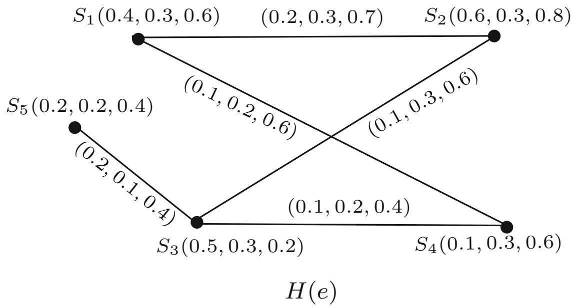


Table 7.8 Tabular representation of score values with choice values

	S_1	S_2	S_3	S_4	S_5	$\acute{C}_i$
S_1	0	0.27	0	0.23	0	0.5
S_2	0.27	0	0.27	04	0	0.54
S_3	0	0.27	0	0.30	0.30	0.87
S_4	0.23	0	0.30	0	0	0.53
S_5	0	0	0.30	0	0	0.30

4. Calculate the average score values of intuitionistic neutrosophic graphs $H(e)$ using formula

$$\zeta_{ij} := \frac{T_{j\,F(e)} + I_{j\,F(e)} + 1 - F_{j\,F(e)}}{3}, \tag{7.2}$$

   Tabular representation of score values of intuitionistic neutrosophic graphs $H(e)$.
5. Compute the choice values of $C_i = \sum_j \zeta_{ij}$ for all $i = 1, 2, \ldots, n$.
6. The decision is S_i if $S_i = \max_{i=1}^{n} C_i$.
7. If i has more than one value, then any one of S_i may be chosen.

The intuitionistic neutrosophic graph $H(e_1) \cap H(e_2)$ is shown in Fig. 7.19, and tabular representation of average score values of intuitionistic neutrosophic graph is shown in Table 7.8.

Clearly, the maximum score value is 0.87, scored by the S_3. Mr. X will choose the brand S_3.

Chapter 8
Neutrosophic Soft Rough Graphs

Neutrosophic soft rough set model is a hybrid model by combining neutrosophic soft sets with rough sets. We apply neutrosophic soft rough sets to graphs. We present the concept of neutrosophic soft rough graphs and describe different methods of their construction. We develop an efficient algorithm of our method to solve decision-making problems. This chapter is due to [17].

8.1 Introduction

Pawlak [142] introduced the concept of rough set. He was a Polish mathematician (citizen of Poland) and computer scientist. Rough means approximate or inexact. Rough set theory expresses vagueness in terms of a boundary region of a set not in terms of membership function as in fuzzy set. The idea of rough set theory is a generalization of classical set theory to study the intelligence systems containing inexact, uncertain or incomplete information. It is an effective drive for bestowal with uncertain or incomplete information. Rough set theory is a novel mathematical approach to imprecise knowledge. Rough set theory expresses vagueness by means of a boundary region of a set. The emptiness of boundary region of a set shows that this is a crisp set, and nonemptiness shows that this is a rough set. Nonemptiness of boundary region also describes the deficiency of our knowledge about a set. A subset of a universe in rough set theory is expressed by two approximations which are known as lower and upper approximations. Equivalence classes are the basic building blocks in rough set theory, for upper and lower approximations. Dubois and Prade [74] investigated rough sets and fuzzy sets and concluded that these two theories are different approaches to handle vagueness. They reported that these are not opposite theories and to obtain beneficial results, both theories can be combined. Following this idea, Broumi et al. [61] introduced the concept of rough neutrosophic sets. Yang et al. [177] proposed single-valued neutrosophic rough sets by combining single-valued neutrosophic sets and rough sets, and established an algorithm for decision-making problem based on single-valued neutrosophic rough sets on two

M. Akram, *Single-Valued Neutrosophic Graphs*, Infosys Science Foundation Series, https://doi.org/10.1007/978-981-13-3522-8_8

Table 8.1 List of notations

Symbols	Stand for
X	Universal set
P	Parameter set
M	Subset of parameter set
$\mathbb{R}$	Neutrosophic soft relation on X
(F, A)	Neutrosophic soft set
A	Neutrosophic set on M
$\mathbb{R}A$	Neutrosophic soft rough set on X
$\underline{\mathbb{R}}(A)$	Lower neutrosophic soft rough approximation on X
$\overline{\mathbb{R}}(A)$	Upper neutrosophic soft rough approximation on X
$\acute{X}$	$X \times X$
E	Subset of $\acute{X}$
$\acute{M}$	$M \times M$
L	Subset of $\acute{M}$
$\mathbb{S}$	Neutrosophic soft relation on E
B	Neutrosophic set on L
$\mathbb{S}B$	Neutrosophic soft rough relation on $\acute{X}$
$\underline{\mathbb{S}}(B)$	Lower neutrosophic soft rough approximation on E
$\overline{\mathbb{S}}(B)$	Upper neutrosophic soft rough approximation on E
α	The sum of upper neutrosophic soft rough set and lower neutrosophic soft rough set
β	The sum of upper neutrosophic soft rough relation and lower neutrosophic soft rough relation
γ	The score function

universes. Zhang et al. [203] presented the notion of intuitionistic fuzzy rough sets. The notions of soft rough neutrosophic sets and neutrosophic soft rough sets as hybrid models are described in [26]. We give a list of notations in Table 8.1.

Definition 8.1 Let X be an initial universal set, P a universal set of parameters and $M \subseteq P$. For an arbitrary neutrosophic soft relation $\mathbb{R}$ over $X \times M$, $(X, M, \mathbb{R})$ is called neutrosophic soft approximation space.

For any neutrosophic set $A \in \mathcal{N}(M)$, we define the upper neutrosophic soft rough approximation and the lower neutrosophic soft rough approximation operators of A with respect to $(X, M, \mathbb{R})$ denoted by $\overline{\mathbb{R}}(A)$ and $\underline{\mathbb{R}}(A)$, respectively, as follows:

$$\overline{\mathbb{R}}(A) =\{(x, T_{\overline{\mathbb{R}}(A)}(x), I_{\overline{\mathbb{R}}(A)}(x), F_{\overline{\mathbb{R}}(A)}(x)) \mid x \in X\},$$
$$\underline{\mathbb{R}}(A) =\{(x, T_{\underline{\mathbb{R}}(A)}(x), I_{\underline{\mathbb{R}}(A)}(x), F_{\underline{\mathbb{R}}(A)}(x)) \mid x \in X\},$$

where

Table 8.2 Neutrosophic soft relation $\mathbb{R}$

$\mathbb{R}$	x_1	x_2	x_3	x_4
m_1	(0.3, 0.4, 0.5)	(0.4, 0.2, 0.3)	(0.1, 0.5, 0.4)	(0.2, 0.3, 0.4)
m_2	(0.1, 0.5, 0.4)	(0.3, 0.4, 0.6)	(0.4, 0.4, 0.3)	(0.5, 0.3, 0.8)
m_3	(0.3, 0.4, 0.4)	(0.4, 0.6, 0.7)	(0.3, 0.5, 0.4)	(0.5, 0.4, 0.6)

$$T_{\overline{\mathbb{R}}(A)}(x) = \bigvee_{m\in M} \big(T_{\mathbb{R}(A)}(x,m) \wedge T_A(m)\big), \quad I_{\overline{\mathbb{R}}(A)}(x) = \bigwedge_{m\in M} \big(I_{\mathbb{R}(A)}(x,m) \vee I_A(m)\big),$$
$$F_{\overline{\mathbb{R}}(A)}(x) = \bigwedge_{m\in M} \big(F_{\mathbb{R}(A)}(x,m) \vee F_A(m)\big); \quad T_{\underline{\mathbb{R}}(A)}(x) = \bigwedge_{m\in M} \big(F_{\mathbb{R}(A)}(x,m) \vee T_A(m)\big),$$
$$I_{\underline{\mathbb{R}}(A)}(x) = \bigvee_{m\in M} \big((1 - I_{\mathbb{R}(A)}(x,m)) \wedge I_A(m)\big), \; F_{\underline{\mathbb{R}}(A)}(x) = \bigvee_{m\in M} \big(T_{\mathbb{R}(A)}(x,m) \wedge F_A(m)\big).$$

The pair $(\underline{\mathbb{R}}(A), \overline{\mathbb{R}}(A))$ is called *neutrosophic soft rough set* of A w.r.t. $(X, M, \mathbb{R})$, *and* $\underline{\mathbb{R}}$ and $\overline{\mathbb{R}}$ are referred to as the lower neutrosophic soft rough approximation and the upper neutrosophic soft rough approximation operators, respectively.

Example 8.1 Suppose that $X = \{x_1, x_2, x_3, x_4\}$ is the set of careers under consideration, Mr. X wants to select best suitable career. $M = \{m_1, m_2, m_3\}$ be a set of decision parameters. Mr. X describe the "most suitable career" by defining a neutrosophic soft set $\mathbb{R} = (F, M)$ on X which is a neutrosophic relation from X to M as shown in Table 8.2.

Now, Mr. X gives the most favourable decision object A which is a neutrosophic set on M defined as follows: A={(m_1,0.5,0.2,0.4),(m_2,0.2,0.3,0.1),(m_3,0.2,0.4,0.6)}. By Definition 8.1, we have

$$T_{\overline{\mathbb{R}}(A)}(x_1) = 0.3, \quad I_{\overline{\mathbb{R}}(A)}(x_1) = 0.4, \quad F_{\overline{\mathbb{R}}(A)}(x_1) = 0.4,$$
$$T_{\overline{\mathbb{R}}(A)}(x_2) = 0.4, \quad I_{\overline{\mathbb{R}}(A)}(x_2) = 0.2, \quad F_{\overline{\mathbb{R}}(A)}(x_2) = 0.4,$$
$$T_{\overline{\mathbb{R}}(A)}(x_3) = 0.2, \quad I_{\overline{\mathbb{R}}(A)}(x_3) = 0.4, \quad F_{\overline{\mathbb{R}}(A)}(x_3) = 0.3,$$
$$T_{\overline{\mathbb{R}}(A)}(x_4) = 0.2, \quad I_{\overline{\mathbb{R}}(A)}(x_4) = 0.3, \quad F_{\overline{\mathbb{R}}(A)}(x_4) = 0.4.$$

Similarly,

$$T_{\underline{\mathbb{R}}(A)}(x_1) = 0.4, \quad I_{\underline{\mathbb{R}}(A)}(x_1) = 0.4, \quad F_{\underline{\mathbb{R}}(A)}(x_1) = 0.3,$$
$$T_{\underline{\mathbb{R}}(A)}(x_2) = 0.5, \quad I_{\underline{\mathbb{R}}(A)}(x_2) = 0.4, \quad F_{\underline{\mathbb{R}}(A)}(x_2) = 0.4,$$
$$T_{\underline{\mathbb{R}}(A)}(x_3) = 0.4, \quad I_{\underline{\mathbb{R}}(A)}(x_3) = 0.4, \quad F_{\underline{\mathbb{R}}(A)}(x_3) = 0.3,$$
$$T_{\underline{\mathbb{R}}(A)}(x_4) = 0.5, \quad I_{\underline{\mathbb{R}}(A)}(x_4) = 0.4, \quad F_{\underline{\mathbb{R}}(A)}(x_4) = 0.5.$$

Thus, we obtain

$$\overline{\mathbb{R}}(A) = \{(x_1, 0.3, 0.4, 0.4), (x_2, 0.4, 0.2, 0.4), (x_3, 0.2, 0.4, 0.3), (x_4, 0.2, 0.3, 0.4)\},$$
$$\underline{\mathbb{R}}(A) = \{(x_1, 0.4, 0.4, 0.3), (x_2, 0.5, 0.4, 0.4), (x_3, 0.4, 0.4, 0.3), (x_4, 0.5, 0.4, 0.5)\}.$$

Hence $(\underline{\mathbb{R}}(A), \overline{\mathbb{R}}(A))$ is a neutrosophic soft rough set of A.

The conventional neutrosophic soft set is a mapping from a parameter to the neutrosophic subset of universe, and let $\mathbb{R}=(F, M)$ be neutrosophic soft set. Now, we present the constructive definition of neutrosophic soft rough relation by using a neutrosophic soft relation $\mathbb{S}$ from $M \times M = \acute{M}$ to $\mathcal{N}(X \times X = \acute{X})$, where X be a universal set and M be a set of parameter.

Definition 8.2 A *neutrosophic soft rough relation* $(\underline{\mathbb{S}}(B), \overline{\mathbb{S}}(B))$ on X is a neutrosophic soft rough set, and $\mathbb{S} : \acute{M} \to \mathcal{N}(\acute{X})$ is a neutrosophic soft relation on X defined by $\mathbb{S}(m_i m_j) = \{x_i x_j \mid \exists x_i \in \mathbb{R}(m_i), x_j \in \mathbb{R}(m_j)\}$, $x_i x_j \in \acute{X}$, such that

$$T_{\mathbb{S}}(x_i x_j, m_i m_j) \le \min\{T_{\mathbb{R}}(x_i, m_i), T_{\mathbb{R}}(x_j, m_j)\}$$
$$I_{\mathbb{S}}(x_i x_j, m_i m_j) \le \max\{I_{\mathbb{R}}(x_i, m_i), I_{\mathbb{R}}(x_j, m_j)\}$$
$$F_{\mathbb{S}}(x_i x_j, m_i m_j) \le \max\{F_{\mathbb{R}}(x_i, m_i), F_{\mathbb{R}}(x_j, m_j)\}.$$

For any $B \in \mathcal{N}(\acute{M})$, $B = \{\big(m_i m_j, T_B(m_i m_j), I_B(m_i m_j), F_B(m_i m_j)\big)\ m_i m_j \in \acute{M}\}$,

$$T_B(m_i m_j) \le \min\{T_A(m_i), T_A(m_j)\},$$
$$I_B(m_i m_j) \le \max\{I_A(m_i), I_A(m_j)\},$$
$$F_B(m_i m_j) \le \max\{F_A(m_i), F_A(m_j)\}.$$

The upper neutrosophic soft approximation and the lower neutrosophic soft approximation of B *w.r.t.* $(\acute{X}, \acute{M}, \mathbb{S})$ are defined as follows:

$$\overline{\mathbb{S}}(B) = \{(x_i x_j, T_{\overline{\mathbb{S}}(B)}(x_i x_j), I_{\overline{\mathbb{S}}(B)}(x_i x_j), F_{\overline{\mathbb{S}}(B)}(x_i x_j)) \mid x_i x_j \in \acute{X}\},$$

$$\underline{\mathbb{S}}(B) = \{(x_i x_j, T_{\underline{\mathbb{S}}(B)}(x_i x_j), I_{\underline{\mathbb{S}}(B)}(x_i x_j), F_{\underline{\mathbb{S}}(B)}(x_i x_j)) \mid x_i x_j \in \acute{X}\},$$

where

$$T_{\overline{\mathbb{S}}(B)}(x_i x_j) = \bigvee_{m_i m_j \in \acute{M}} \big(T_{\mathbb{S}}(x_i x_j, m_i m_j) \wedge T_B(m_i m_j)\big),$$
$$I_{\overline{\mathbb{S}}(B)}(x_i x_j) = \bigwedge_{m_i m_j \in \acute{M}} \big(I_{\mathbb{S}}(x_i x_j, m_i m_j) \vee I_B(m_i m_j)\big),$$
$$F_{\overline{\mathbb{S}}(B)}(x_i x_j) = \bigwedge_{m_i m_j \in \acute{M}} \big(F_{\mathbb{S}}(x_i x_j, m_i m_j) \vee F_B(m_i m_j)\big),$$

$$T_{\underline{\mathbb{S}}(B)}(x_ix_j) = \bigwedge_{m_im_j \in \acute{M}} \big(F_{\mathbb{S}}(x_ix_j, m_im_j) \vee T_B(m_im_j)\big),$$
$$I_{\underline{\mathbb{S}}(B)}(x_ix_j) = \bigvee_{m_im_j \in \acute{M}} \big((1 - I_{\mathbb{S}}(x_ix_j, m_im_j)) \wedge I_B(m_im_j)\big),$$
$$F_{\underline{\mathbb{S}}(B)}(x_ix_j) = \bigvee_{m_im_j \in \acute{M}} \big(T_{\mathbb{S}}(x_ix_j, m_im_j) \wedge F_B(m_im_j)\big).$$

The pair $(\underline{\mathbb{S}}(B), \overline{\mathbb{S}}(B))$ is called *neutrosophic soft rough relation*, and $\underline{\mathbb{S}}, \overline{\mathbb{S}} : \mathcal{N}(\acute{M}) \to \mathcal{N}(\acute{X})$ are called the *lower neutrosophic soft rough approximation* and the *upper neutrosophic soft rough approximation* operators, respectively.

Remark 8.1 Consider a neutrosophic set B on $\acute{M}$ and a neutrosophic set A on M; according to the definition of neutrosophic soft rough relation, we get

$$T_{\overline{\mathbb{S}}(B)}(x_ix_j) \leq \min\{T_{\overline{\mathbb{R}}(A)}(x_i), T_{\overline{\mathbb{R}}(A)}(x_j)\},$$
$$I_{\overline{\mathbb{S}}(B)}(x_ix_j) \leq \max\{I_{\overline{\mathbb{R}}(A)}(x_i), I_{\overline{\mathbb{R}}(A)}(x_j)\},$$
$$F_{\overline{\mathbb{S}}(B)}(x_ix_j) \leq \max\{F_{\overline{\mathbb{R}}(A)}(x_i), F_{\overline{\mathbb{R}}(A)}(x_j)\}.$$

Similarly, for lower neutrosophic soft rough approximation operator $\underline{\mathbb{S}}(B)$,

$$T_{\underline{\mathbb{S}}(B)}(x_ix_j) \leq \min\{T_{\underline{\mathbb{R}}(A)}(x_i), T_{\underline{\mathbb{R}}(A)}(x_j)\},$$
$$I_{\underline{\mathbb{S}}(B)}(x_ix_j) \leq \max\{I_{\underline{\mathbb{R}}(A)}(x_i), I_{\underline{\mathbb{R}}(A)}(x_j)\},$$
$$F_{\underline{\mathbb{S}}(B)}(x_ix_j) \leq \max\{F_{\underline{\mathbb{R}}(A)}(x_i), F_{\underline{\mathbb{R}}(A)}(x_j)\}.$$

Example 8.2 Let $X = \{x_1, x_2, x_3\}$ be a universal set and $M = \{m_1, m_2, m_3\}$ a set of parameters. A neutrosophic soft set $\mathbb{R} = (F, M)$ on X can be defined in Table 8.3 as follows.

Let $E = \{x_1x_2, x_2x_3, x_2x_2, x_3x_2\} \subseteq \acute{X}$ and $L = \{m_1m_3, m_2m_1, m_3m_2\} \subseteq \acute{M}$. Then a soft relation $\mathbb{S}$ on E (from L to E) can be defined in Table 8.4 as follows. Let $A = \{(m_1, 0.2, 0.4, 0.6), (m_2, 0.4, 0.5, 0.2), (m_3, 0.1, 0.2, 0.4)\}$ be a neutrosophic set on M, then
$\overline{\mathbb{R}}(A) = \{(x_1, 0.4, 0.2, 0.4), (x_2, 0.3, 0.4, 0.3), (x_3, 0.4, 0.2, 0.3)\}$
$\underline{\mathbb{R}}(A) = \{(x_1, 0.3, 0.5, 0.4), (x_2, 0.2, 0.5, 0.6), (x_3, 0.4, 0.5, 0.6)\}$.
Let $B = \{(m_1m_3, 0.1, 0.3, 0.5), (m_2m_1, 0.2, 0.4, 0.3), (m_3m_2, 0.1, 0.2, 0.3)\}$ be a neutrosophic set on L, then

Table 8.3 Neutrosophic soft set $\mathbb{R} = (F, M)$

$\mathbb{R}$	x_1	x_2	x_3
m_1	(0.4, 0.5, 0.6)	(0.7, 0.3, 0.2)	(0.6, 0.3, 0.4)
m_2	(0.5, 0.3, 0.6)	(0.3, 0.4, 0.3)	(0.7, 0.2, 0.3)
m_3	(0.7, 0.2, 0.3)	(0.6, 0.5, 0.4)	(0.7, 0.2, 0.4)

Table 8.4 Neutrosophic soft relation $\mathbb{S}$

$\mathbb{S}$	x_1x_2	x_2x_3	x_2x_2	x_3x_2
m_1m_3	(0.4, 0.4, 0.5)	(0.6, 0.3, 0.4)	(0.5, 0.4, 0.2)	(0.5, 0.4, 0.3)
m_2m_1	(0.3, 0.3, 0.4)	(0.3, 0.2, 0.3)	(0.2, 0.3, 0.3)	(0.7, 0.2, 0.2)
m_3m_2	(0.3, 0.3, 0.2)	(0.5, 0.3, 0.2)	(0.2, 0.4, 0.4)	(0.3, 0.4, 0.4)

$$\overline{\mathbb{S}}(B) = \{(x_1x_2, 0.2, 0.3, 0.3), (x_2x_3, 0.2, 0.3, 0.3), (x_2x_2, 0.2, 0.4, 0.3), (x_3x_2, 0.2, 0.4, 0.3)\}$$
$$\underline{\mathbb{S}}(B) = \{(x_1x_2, 0.2, 0.4, 0.4), (x_2x_3, 0.2, 0.4, 0.5), (x_2x_2, 0.3, 0.4, 0.5), (x_3x_2, 0.2, 0.4, 0.5)\}$$

Hence $\mathbb{S}B = (\underline{\mathbb{S}}(B), \overline{\mathbb{S}}(B))$ is neutrosophic soft rough relation.

8.2 Neutrosophic Soft Rough Information

Definition 8.3 A *neutrosophic soft rough graph* on a nonempty X is a four-ordered tuple $(X, M, \mathbb{R}A, \mathbb{S}B)$ such that

(i) M is a set of parameters.
(ii) $\mathbb{R}$ is an arbitrary neutrosophic soft relation over $X \times M$.
(iii) $\mathbb{S}$ is an arbitrary neutrosophic soft relation over $\acute{X} \times \acute{M}$.
(vi) $\mathbb{R}A = (\underline{\mathbb{R}}(A), \overline{\mathbb{R}}(A))$ is a neutrosophic soft rough set of X.
(v) $\mathbb{S}B = (\underline{\mathbb{S}}(B), \overline{\mathbb{S}}(B))$ is a neutrosophic soft rough relation on $\acute{X} \subseteq X \times X$.

$G = (\mathbb{R}A, \mathbb{S}B)$ is a neutrosophic soft rough graph, where $\underline{G} = (\underline{\mathbb{R}}(A), \underline{\mathbb{S}}(B))$ and $\overline{G} = (\overline{\mathbb{R}}(A), \overline{\mathbb{S}}(B))$ are lower neutrosophic approximate graph and upper neutrosophic approximate graph, respectively, of neutrosophic soft rough graph $G = (\mathbb{R}A, \mathbb{S}B)$.

Example 8.3 Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be a vertex set and $M = \{m_1, m_2, m_3\}$ a set of parameters. A neutrosophic soft relation over $X \times M$ can be defined in Table 8.5 as follows.
Let $A = \{(m_1, 0.5, 0.4, 0.6), (m_2, 0.7, 0.4, 0.5), (m_3, 0.6, 0.2, 0.5)\}$ be a neutrosophic set on M, then

Table 8.5 Neutrosophic soft relation $\mathbb{R}$

$\mathbb{R}$	x_1	x_2	x_3	x_4	x_5	x_6
m_1	(0.4, 0.5, 0.6)	(0.7, 0.3, 0.5)	(0.6, 0.2, 0.3)	(0.4, 0.4, 0.2)	(0.5, 0.5, 0.6)	(0.4, 0.5, 0.6)
m_2	(0.5, 0.4, 0.2)	(0.6, 0.4, 0.5)	(0.7, 0.3, 0.4)	(0.5, 0.3, 0.2)	(0.4, 0.5, 0.4)	(0.6, 0.5, 0.4)
m_3	(0.5, 0.4, 0.1)	(0.6, 0.3, 0.2)	(0.5, 0.4, 0.3)	(0.6, 0.2, 0.3)	(0.5, 0.4, 0.4)	(0.7, 0.3, 0.5)

$$\overline{\mathbb{R}}(A) = \{(x_1, 0.5, 0.4, 0.5), (x_2, 0.6, 0.3, 0.5), (x_3, 0.7, 0.4, 0.5), (x_4, 0.6, 0.2, 0.5), (x_5, 0.5, 0.4, 0.5), (x_6, 0.6, 0.3, 0.5)\},$$
$$\underline{\mathbb{R}}(A) = \{(x_1, 0.6, 0.4, 0.5), (x_2, 0.5, 0.4, 0.6), (x_3, 0.5, 0.4, 0.6), (x_4, 0.5, 0.4, 0.5), (x_5, 0.6, 0.4, 0.5), (x_6, 0.6, 0.4, 0.5)\}.$$

Let $E = \{x_1x_1, x_1x_2, x_2x_1, x_2x_3, x_4x_5, x_3x_4, x_5x_2, x_5x_6\} \subseteq \acute{X}$ and $L = \{m_1m_3, m_2m_1, m_3m_2\} \subseteq \acute{M}$. Then a neutrosophic soft relation $\mathbb{S}$ on E (from L to E) can be defined in Tables 8.6 and 8.7 as follows.
Let $B = \{(m_1m_2, 0.4, 0.4, 0.5), (m_2m_3, 0.5, 0.4, 0.5), (m_1m_3, 0.5, 0.2, 0.5)\}$ be a neutrosophic set on L, then

$$\overline{\mathbb{S}}(B) = \{(x_1x_1, 0.5, 0.4, 0.5), (x_1x_2, 0.4, 0.2, 0.5), (x_2x_1, 0.4, 0.2, 0.5), (x_2x_3, 0.5, 0.3, 0.5), (x_3x_4, 0.5, 0.2, 0.5), (x_4x_5, 0.4, 0.3, 0.5), (x_5x_2, 0.5, 0.3, 0.5), (x_5x_6, 0.5, 0.3, 0.5)\},$$
$$\underline{\mathbb{S}}(B) = \{(x_1x_1, 0.4, 0.4, 0.5)(x_1x_2, 0.5, 0.4, 0.4), (x_2x_1, 0.5, 0.4, 0.4), (x_2x_3, 0.4, 0.4, 0.5), (x_3x_4, 0.4, 0.4, 0.5), (x_4x_5, 0.4, 0.4, 0.4), (x_5x_2, 0.4, 0.4, 0.5), (x_5x_6, 0.4, 0.4, 0.5)\}.$$

Hence $\mathbb{S}B = (\underline{\mathbb{S}}(B), \overline{\mathbb{S}}(B))$ is neutrosophic soft rough relation on $\acute{X}$. Thus, $\underline{G} = (\underline{\mathbb{R}}(A), \underline{\mathbb{S}}(B))$ and $\overline{G} = (\overline{\mathbb{R}}(A), \overline{\mathbb{S}}(B))$ are lower neutrosophic approximate graph and upper neutrosophic approximate graph, respectively, as shown in Fig. 8.1. Hence, $G = (\underline{G}, \overline{G})$ is neutrosophic soft rough graph.

Definition 8.4 Let $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ be two neutrosophic soft rough graphs on X. The *union* of G_1 and G_2 is a neutrosophic soft rough graph $G = G_1 \cup G_2 = (\underline{G}_1 \cup \underline{G}_2, \overline{G}_1 \cup \overline{G}_2)$, where $\underline{G}_1 \cup \underline{G}_2 = (\underline{\mathbb{R}}(A_1) \cup \underline{\mathbb{R}}(A_2), \underline{\mathbb{S}}(B_1) \cup \underline{\mathbb{S}}(B_2))$ and $\overline{G}_1 \cup \overline{G}_2 = (\overline{\mathbb{R}}(A_1) \cup \overline{\mathbb{R}}(A_2), \overline{\mathbb{S}}(B_1) \cup \overline{\mathbb{S}}(B_2))$ are neutrosophic graphs, such that

(i) $\forall x \in \mathbb{R}A_1$ but $x \notin \mathbb{R}A_2$.

Table 8.6 Neutrosophic soft relation $\mathbb{S}$

$\mathbb{S}$	x_1x_1	x_1x_2	x_2x_1	x_2x_3
m_1m_2	(0.4, 0.4, 0.2)	(0.4, 0.4, 0.5)	(0.4, 0.4, 0.5)	(0.6, 0.3, 0.4)
m_2m_3	(0.5, 0.4, 0.1)	(0.4, 0.3, 0.2)	(0.4, 0.3, 0.2)	(0.5, 0.3, 0.2)
m_1m_3	(0.4, 0.4, 0.1)	(0.4, 0.2, 0.2)	(0.4, 0.2, 0.2)	(0.5, 0.3, 0.3)

Table 8.7 Neutrosophic soft relation $\mathbb{S}$

$\mathbb{S}$	x_3x_4	x_4x_5	x_5x_2	x_5x_6
m_1m_2	(0.4, 0.2, 0.2)	(0.4, 0.4, 0.2)	(0.4, 0.3, 0.4)	(0.3, 0.2, 0.3)
m_2m_3	(0.6, 0.2, 0.4)	(0.3, 0.2, 0.1)	(0.4, 0.3, 0.2)	(0.4, 0.3, 0.4)
m_1m_3	(0.4, 0.2, 0.3)	(0.4, 0.3, 0.1)	(0.5, 0.3, 0.2)	(0.5, 0.3, 0.5)

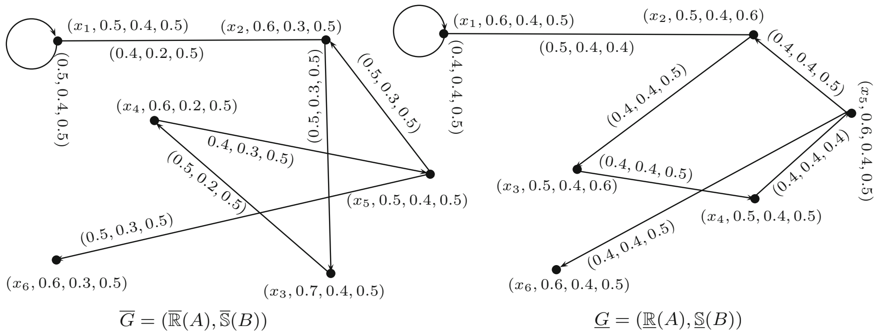


Fig. 8.1 Neutrosophic soft rough graph $G = (\underline{G}, \overline{G})$

$$
\begin{aligned}
T_{\overline{\mathbb{R}}(A_1)\cup\overline{\mathbb{R}}(A_2)}(x) &= T_{\overline{\mathbb{R}}(A_1)}(x),\ T_{\underline{\mathbb{R}}(A_1)\cup\underline{\mathbb{R}}(A_2)}(x) = T_{\underline{\mathbb{R}}(A_1)}(x),\\
I_{\overline{\mathbb{R}}(A_1)\cup\overline{\mathbb{R}}(A_2)}(x) &= I_{\overline{\mathbb{R}}(A_1)}(x),\ \ I_{\underline{\mathbb{R}}(A_1)\cup\underline{\mathbb{R}}(A_2)}(x) = I_{\underline{\mathbb{R}}(A_1)}(x),\\
F_{\overline{\mathbb{R}}(A_1)\cup\overline{\mathbb{R}}(A_2)}(x) &= F_{\overline{\mathbb{R}}(A_1)}(x),\ F_{\underline{\mathbb{R}}(A_1)\cup\underline{\mathbb{R}}(A_2)}(x) = F_{\underline{\mathbb{R}}(A_1)}(x).
\end{aligned}
$$

(ii) $\forall x \notin \mathbb{R}A_1$ but $x \in \mathbb{R}A_2$.

$$
\begin{aligned}
T_{\overline{\mathbb{R}}(A_1)\cup\overline{\mathbb{R}}(A_2)}(x) &= T_{\overline{\mathbb{R}}(A_2)}(x),\ T_{\underline{\mathbb{R}}A_1\cup\underline{\mathbb{R}}(A_2)}(x) = T_{\underline{\mathbb{R}}(A_2)}(x),\\
I_{\overline{\mathbb{R}}(A_1)\cup\overline{\mathbb{R}}(A_2)}(x) &= I_{\overline{\mathbb{R}}(A_2)}(x),\ \ I_{\underline{\mathbb{R}}(A_1)\cup\underline{\mathbb{R}}(A_2)}(x) = I_{\underline{\mathbb{R}}(A_2)}(x),\\
F_{\overline{\mathbb{R}}(A_1)\cup\overline{\mathbb{R}}(A_2)}(x) &= F_{\overline{\mathbb{R}}(A_2)}(x),\ F_{\underline{\mathbb{R}}(A_1)\cup\underline{\mathbb{R}}(A_2)}(x) = F_{\underline{\mathbb{R}}(A_2)}(x).
\end{aligned}
$$

(iii) $\forall x \in \mathbb{R}A_1 \cap \mathbb{R}A_2$

$$
\begin{aligned}
T_{\overline{\mathbb{R}}(A_1)\cup\overline{\mathbb{R}}(A_2)}(x) &= \max\{T_{\overline{\mathbb{R}}(A_1)}(x), T_{\overline{\mathbb{R}}(A_2)}(x)\},\\
T_{\underline{\mathbb{R}}(A_1)\cup\underline{\mathbb{R}}(A_2)}(x) &= \max\{T_{\underline{\mathbb{R}}(A_1)}(x), T_{\underline{\mathbb{R}}(A_2)}(x)\},\\
I_{\overline{\mathbb{R}}(A_1)\cup\overline{\mathbb{R}}(A_2)}(x) &= \min\{I_{\overline{\mathbb{R}}(A_1)}(x), I_{\overline{\mathbb{R}}(A_2)}(x)\},\\
I_{\underline{\mathbb{R}}(A_1)\cup\underline{\mathbb{R}}(A_2)}(x) &= \min\{I_{\underline{\mathbb{R}}(A_1)}(x), I_{\underline{\mathbb{R}}(A_2)}(x)\},\\
F_{\overline{\mathbb{R}}(A_1)\cup\overline{\mathbb{R}}(A_2)}(x) &= \min\{F_{\overline{\mathbb{R}}(A_1)}(x), F_{\overline{\mathbb{R}}(A_2)}(x)\},\\
F_{\underline{\mathbb{R}}(A_1)\cup\underline{\mathbb{R}}(A_2)}(x) &= \min\{F_{\underline{\mathbb{R}}(A_1)}(x), F_{\underline{\mathbb{R}}(A_2)}(x)\}.
\end{aligned}
$$

(iv) $\forall xy \in \mathbb{S}B_1$ but $xy \notin \mathbb{S}B_2$.

$$
\begin{aligned}
T_{\overline{\mathbb{S}}(B_1)\cup\overline{\mathbb{S}}(B_2)}(xy) &= T_{\overline{\mathbb{S}}(B_1)}(xy),\ T_{\underline{\mathbb{S}}(B_1)\cup\underline{\mathbb{S}}(B_2)}(xy) = T_{\underline{\mathbb{S}}(B_1)}(xy),\\
I_{\overline{\mathbb{S}}(B_1)\cup\overline{\mathbb{S}}(B_2)}(xy) &= I_{\overline{\mathbb{S}}(B_1)}(xy),\ \ I_{\underline{\mathbb{S}}(B_1)\cup\underline{\mathbb{S}}(B_2)}(xy) = I_{\underline{\mathbb{S}}(B_1)}(xy),\\
F_{\overline{\mathbb{S}}(B_1)\cup\overline{\mathbb{S}}(B_2)}(xy) &= F_{\overline{\mathbb{S}}(B_1)}(xy),\ F_{\underline{\mathbb{S}}(B_1)\cup\underline{\mathbb{S}}(B_2)}(xy) = F_{\underline{\mathbb{S}}(B_1)}(xy).
\end{aligned}
$$

(v) $\forall xy \notin \mathbb{S}B_1$ but $xy \in \mathbb{S}B_2$

Table 8.8 Neutrosophic soft relation $\mathbb{R}$

$\mathbb{R}$	x_1	x_2	x_3	x_4
m_1	(0.5, 0.4, 0.3)	(0.7, 0.6, 0.5)	(0.7, 0.6, 0.4)	(0.5, 0.7, 0.4)
m_2	(0.3, 0.5, 0.6)	(0.4, 0.5, 0.1)	(0.3, 0.6, 0.5)	(0.4, 0.8, 0.2)
m_3	(0.7, 0.5, 0.8)	(0.2, 0.3, 0.8)	(0.7, 0.3, 0.5)	(0.6, 0.4, 0.3)

$$
\begin{aligned}
T_{\overline{\mathbb{S}}(B_1)\cup\overline{\mathbb{S}}(B_2)}(xy) &= T_{\overline{\mathbb{S}}(B_2)}(xy),\ T_{\underline{\mathbb{S}}(B_1)\cup\underline{\mathbb{S}}(B_2)}(xy) = T_{\underline{\mathbb{S}}(B_2)}(xy),\\
I_{\overline{\mathbb{S}}(B_1)\cup\overline{\mathbb{S}}(B_2)}(xy) &= I_{\overline{\mathbb{S}}(B_2)}(xy),\ \ I_{\underline{\mathbb{S}}(B_1)\cup\underline{\mathbb{S}}(B_2)}(xy) = I_{\underline{\mathbb{S}}(B_2)}(xy),\\
F_{\overline{\mathbb{S}}(B_1)\cup\overline{\mathbb{S}}(B_2)}(xy) &= F_{\overline{\mathbb{S}}(B_2)}(xy),\ F_{\underline{\mathbb{S}}(B_1)\cup\underline{\mathbb{S}}(B_2)}(xy) = F_{\underline{\mathbb{S}}(B_2)}(xy).
\end{aligned}
$$

(vi) $\forall xy \in \mathbb{S}B_1 \cap \mathbb{S}(B_2)$

$$
\begin{aligned}
T_{\overline{\mathbb{S}}(B_1)\cup\overline{\mathbb{S}}(B_2)}(xy) &= \max\{T_{\overline{\mathbb{S}}(B_1)}(xy),\ T_{\overline{\mathbb{S}}(B_2)}(xy)\},\\
T_{\underline{\mathbb{S}}(B_1)\cup\underline{\mathbb{S}}(B_2)}(xy) &= \max\{T_{\underline{\mathbb{S}}(B_1)}(xy),\ T_{\underline{\mathbb{S}}(B_2)}(xy)\},\\
I_{\overline{\mathbb{S}}(B_1)\cup\overline{\mathbb{S}}(B_2)}(xy) &= \min\{I_{\overline{\mathbb{S}}(B_1)}(xy),\ I_{\overline{\mathbb{S}}(B_2)}(xy)\},\\
I_{\underline{\mathbb{S}}(B_1)\cup\underline{\mathbb{S}}(B_2)}(xy) &= \min\{I_{\underline{\mathbb{S}}(B_1)}(xy),\ I_{\underline{\mathbb{S}}(B_2)}(xy)\},\\
F_{\overline{\mathbb{S}}(B_1)\cup\overline{\mathbb{S}}(B_2)}(xy) &= \min\{F_{\overline{\mathbb{S}}(B_1)}(xy),\ F_{\overline{\mathbb{S}}(B_2)}(xy)\},\\
F_{\underline{\mathbb{S}}(B_1)\cup\underline{\mathbb{S}}(B_2)}(xy) &= \min\{F_{\underline{\mathbb{S}}(B_1)}(xy),\ F_{\underline{\mathbb{S}}(B_2)}(xy)\}.
\end{aligned}
$$

Example 8.4 Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of universe and $M = \{m_1, m_2, m_3\}$ a set of parameters. Then a neutrosophic soft relation over $X \times \mathbb{M}$ can be written as in Table 8.8.

Let $A_1 = \{(m_1, 0.5, 0.7, 0.8), (m_2, 0.7, 0.5, 0.3), (m_3, 0.4, 0.5, 0.3)\}$, and $A_2 = \{(m_1, 0.6, 0.3, 0.5), (m_2, 0.5, 0.8, 0.2), (m_3, 0.5, 0.7, 0.2)\}$ be two neutrosophic sets on M, Then $\mathbb{R}A_1 = (\underline{\mathbb{R}}(A_1), \overline{\mathbb{R}}(A_1))$ and $\mathbb{R}A_2 = (\underline{\mathbb{R}}(A_2), \overline{\mathbb{R}}(A_2))$ are neutrosophic soft rough sets, where

$$
\begin{aligned}
\underline{\mathbb{R}}(A_1) &= \{(x_1, 0.5, 0.6, 0.5), (x_2, 0.5, 0.5, 0.7)(x_3, 0.5, 0.5, 0.7), (x_4 0.4, 0.5, 0.5)\},\\
\overline{\mathbb{R}}(A_1) &= \{(x_1, 0.5, 0.5, 0.6), (x_2, 0.5, 0.5, 0.3), (x_3, 0.5, 0.5, 0.5), (x_4 0.5, 0.5, 0.3)\};\\
\underline{\mathbb{R}}(A_2) &= \{(x_1, 0.6, 0.5, 0.5), (x_2, 0.5, 0.7, 0.5), (x_3, 0.5, 0.7, 0.5), (x_4, 0.5, 0.6, 0.5)\},\\
\overline{\mathbb{R}}(A_2) &= \{(x_1, 0.5, 0.4, 0.5), (x_2, 0.6, 0.6, 0.2), (x_3, 0.6, 0.6, 0.5), (x_4, 0.5, 0.7, 0.2)\}.
\end{aligned}
$$

Let $E = \{x_1x_2, x_1x_4, x_2x_2, x_2x_3, x_3x_3, x_3x_4\} \subseteq X \times X$, and $L = \{m_1m_2, m_1m_3, m_2m_3\} \subset \mathbb{M}$. Then a neutrosophic soft relation on E can be written as in Table 8.9

Let $B_1 = \{(m_1m_2, 0.5, 0.4, 0.5), (m_1m_3, 0.3, 0.4, 0.5), (m_2m_3, 0.4, 0.4, 0.3)\}$, and $B_2 = \{(m_1m_2, 0.5, 0.3, 0.2), (m_1m_3, 0.4, 0.3, 0.3), (m_2m_3, 0.4, 0.6, 0.2)\}$ be two neutrosophic sets on L. Then $\mathbb{S}B_1 = (\underline{\mathbb{S}}(B_1), \overline{\mathbb{S}}(B_1))$ and $\mathbb{S}B_2 = (\underline{\mathbb{S}}(B_2), \overline{\mathbb{S}}(B_2))$ are neutrosophic soft rough relations, where

Table 8.9 Neutrosophic soft relation $\mathbb{S}$

$\mathbb{S}$	x_1x_2	x_1x_4	x_2x_2	x_2x_3	x_3x_3	x_3x_4
m_1m_2	(0.3, 0.4, 0.1)	(0.4, 0.4, 0.2)	(0.4, 0.5, 0.1)	(0.3, 0.5, 0.4)	(0.3, 0.4, 0.4)	(0.4, 0.5, 0.2)
m_1m_3	(0.2, 0.3, 0.3)	(0.4, 0.3, 0.2)	(0.2, 0.3, 0.5)	(0.4, 0.3, 0.3)	(0.5, 0.3, 0.3)	(0.5, 0.4, 0.3)
m_2m_3	(0.2, 0.3, 0.5)	(0.3, 0.3, 0.3)	(0.2, 0.3, 0.1)	(0.4, 0.3, 0.1)	(0.3, 0.3, 0.5)	(0.3, 0.4, 0.3)

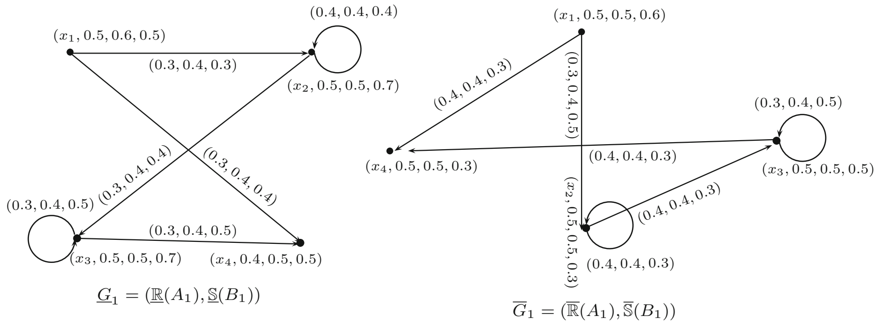


Fig. 8.2 Neutrosophic soft rough graph $G_1 = (\underline{G}_1, \overline{G}_1)$

$$\underline{\mathbb{S}}(B_1) =\{(x_1x_2, 0.3, 0.4, 0.3), (x_1x_4, 0.3, 0.4, 0.4), (x_2x_2, 0.4, 0.4, 0.4), (x_2x_3, 0.3, 0.4, 0.4), (x_3x_3, 0.3, 0.4, 0.5), (x_3x_4, 0.3, 0.4, 0.5)\},$$
$$\overline{\mathbb{S}}(B_1) =\{(x_1x_2, 0.3, 0.4, 0.5), (x_1x_4, 0.4, 0.4, 0.3), (x_2x_2, 0.4, 0.4, 0.3), (x_2x_3, 0.4, 0.4, 0.3), (x_3x_3, 0.3, 0.4, 0.5), (x_3x_4, 0.4, 0.4, 0.3)\};$$
$$\underline{\mathbb{S}}(B_2) =\{(x_1x_2, 0.4, 0.6, 0.2), (x_1x_4, 0.4, 0.6, 0.3), (x_2x_2, 0.4, 0.6, 0.2), (x_2x_3, 0.4, 0.6, 0.3), (x_3x_3, 0.4, 0.6, 0.3), (x_3x_4, 0.4, 0.6, 0.3)\},$$
$$\overline{\mathbb{S}}(B_2) =\{(x_1x_2, 0.3, 0.3, 0.2), (x_1x_4, 0.4, 0.3, 0.2), (x_2x_2, 0.4, 0.3, 0.2), (x_2x_3, 0.4, 0.3, 0.2), (x_3x_3, 0.4, 0.3, 0.3), (x_3x_4, 0.4, 0.4, 0.2)\}.$$

Thus $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ are neutrosophic soft rough graphs, where $\underline{G}_1 = (\underline{\mathbb{R}}(A_1), \underline{\mathbb{S}}(B_1))$, $\overline{G}_1 = (\overline{\mathbb{R}}(A_1), \overline{\mathbb{S}}(B_1))$ as shown in Fig. 8.2

$\underline{G}_2 = (\underline{\mathbb{R}}(A_2), \underline{\mathbb{S}}(B_2))$, $\overline{G}_2 = (\overline{\mathbb{R}}(A_2), \overline{\mathbb{S}}(B_2))$ as shown in Fig. 8.3.

The union of $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ is neutrosophic soft rough graph $G = G_1 \cup G_2 = (\underline{G}_1 \cup \underline{G}_2, \overline{G}_1 \cup \overline{G}_2)$ as shown in Fig. 8.4.

Definition 8.5 Let $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ be two neutrosophic soft rough graphs on X. The *intersection* of G_1 and G_2 is a neutrosophic soft rough graph $G = G_1 \cap G_2 = (\underline{G}_1 \cap \underline{G}_2, \overline{G}_1 \cap \overline{G}_2)$, where $\underline{G}_1 \cap \underline{G}_2 = (\underline{\mathbb{R}}(A_1) \cap \underline{\mathbb{R}}(A_2), \underline{\mathbb{S}}(B_1) \cap \underline{\mathbb{S}}(B_2))$ and $\overline{G}_1 \cap \overline{G}_2 = (\overline{\mathbb{R}}(A_1) \cap \overline{\mathbb{R}}(A_2), \overline{\mathbb{S}}(B_1) \cap \overline{\mathbb{S}}(B_2))$ are neutrosophic graphs, respectively, such that

(i) $\forall x \in \mathbb{R}A_1$ but $x \notin \mathbb{R}A_2$.

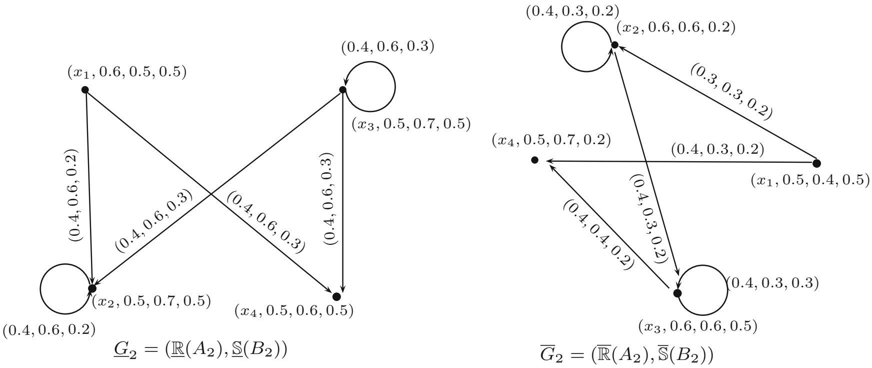


Fig. 8.3 Neutrosophic soft rough graph $G_2 = (\underline{G}_2, \overline{G}_2)$

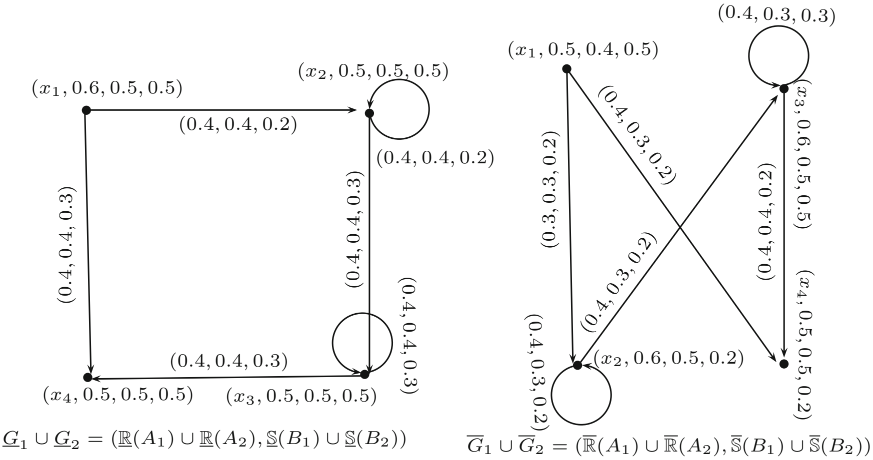


Fig. 8.4 Neutrosophic soft rough graph $G_1 \cup G_2 = (\underline{G}_1 \cup \underline{G}_2, \overline{G}_1 \cup \overline{G}_2)$

$$
\begin{aligned}
T_{\overline{\mathbb{R}}(A_1)\cap\overline{\mathbb{R}}(A_2)}(x) &= T_{\overline{\mathbb{R}}(A_1)}(x),\; T_{\underline{\mathbb{R}}(A_1)\cap\underline{\mathbb{R}}(A_2)}(x) = T_{\underline{\mathbb{R}}(A_1)}(x),\\
I_{\overline{\mathbb{R}}(A_1)\cap\overline{\mathbb{R}}(A_2)}(x) &= I_{\overline{\mathbb{R}}(A_1)}(x),\; I_{\underline{\mathbb{R}}(A_1)\cap\underline{\mathbb{R}}(A_2)}(x) = I_{\underline{\mathbb{R}}(A_1)}(x),\\
F_{\overline{\mathbb{R}}(A_1)\cap\overline{\mathbb{R}}(A_2)}(x) &= F_{\overline{\mathbb{R}}(A_1)}(x),\; F_{\underline{\mathbb{R}}(A_1)\cap\underline{\mathbb{R}}(A_2)}(x) = F_{\underline{\mathbb{R}}(A_1)}(x).
\end{aligned}
$$

(ii) $\forall x \notin \mathbb{R}A_1$ but $x \in \mathbb{R}A_2$.

$$
\begin{aligned}
T_{\overline{\mathbb{R}}(A_1)\cap\overline{\mathbb{R}}(A_2)}(x) &= T_{\overline{\mathbb{R}}(A_2)}(x),\; T_{\underline{\mathbb{R}}(A_1)\cap\underline{\mathbb{R}}(A_2)}(x) = T_{\underline{\mathbb{R}}(A_2)}(x),\\
I_{\overline{\mathbb{R}}(A_1)\cap\overline{\mathbb{R}}(A_2)}(x) &= I_{\overline{\mathbb{R}}(A_2)}(x),\; I_{\underline{\mathbb{R}}(A_1)\cap\underline{\mathbb{R}}(A_2)}(x) = I_{\underline{\mathbb{R}}(A_2)}(x),\\
F_{\overline{\mathbb{R}}(A_1)\cap\overline{\mathbb{R}}(A_2)}(x) &= F_{\overline{\mathbb{R}}(A_2)}(x),\; F_{\underline{\mathbb{R}}(A_1)\cap\underline{\mathbb{R}}(A_2)}(x) = F_{\underline{\mathbb{R}}(A_2)}(x).
\end{aligned}
$$

(iii) $\forall x \in \mathbb{R}A_1 \cap \mathbb{R}A_2$

$$\begin{aligned}
T_{\overline{\mathbb{R}}(A_1)\cap\overline{\mathbb{R}}(A_2)}(x) &= \min\{T_{\overline{\mathbb{R}}(A_1)}(x), T_{\overline{\mathbb{R}}(A_2)}(x)\},\\
T_{\underline{\mathbb{R}}(A_1)\cap\underline{\mathbb{R}}(A_2)}(x) &= \min\{T_{\underline{\mathbb{R}}(A_1)}(x), T_{\underline{\mathbb{R}}(A_2)}(x)\},\\
I_{\overline{\mathbb{R}}(A_1)\cap\overline{\mathbb{R}}(A_2)}(x) &= \max\{I_{\overline{\mathbb{R}}(A_1)}(x), I_{\overline{\mathbb{R}}(A_2)}(x)\},\\
I_{\underline{\mathbb{R}}(A_1)\cap\underline{\mathbb{R}}(A_2)}(v) &= \max\{I_{\underline{\mathbb{R}}(A_1)}(x), I_{\underline{\mathbb{R}}(A_2)}(x)\},\\
F_{\overline{\mathbb{R}}(A_1)\cap\overline{\mathbb{R}}(A_2)}(x) &= \max\{F_{\overline{\mathbb{R}}(A_1)}(x), F_{\overline{\mathbb{R}}(A_2)}(x)\},\\
F_{\underline{\mathbb{R}}(A_1)\cap\underline{\mathbb{R}}(A_2)}(v) &= \max\{F_{\underline{\mathbb{R}}(A_1)}(x), F_{\underline{\mathbb{R}}(A_2)}(x)\}.
\end{aligned}$$

(iv) $\forall xy \in \mathbb{S}B_1$ but $xy \notin \mathbb{S}B_2$.

$$\begin{aligned}
T_{\overline{\mathbb{S}}(B_1)\cap\overline{\mathbb{S}}(B_2)}(xy) &= T_{\overline{\mathbb{S}}(B_1)}(xy),\ T_{\underline{\mathbb{S}}(B_1)\cap\underline{\mathbb{S}}(B_2)}(xy) = T_{\underline{\mathbb{S}}(B_1)}(xy),\\
I_{\overline{\mathbb{S}}(B_1)\cap\overline{\mathbb{S}}(B_2)}(xy) &= I_{\overline{\mathbb{S}}(B_1)}(xy),\ I_{\underline{\mathbb{S}}(B_1)\cap\underline{\mathbb{S}}(B_2)}(xy) = I_{\underline{\mathbb{S}}(B_1)}(xy),\\
F_{\overline{\mathbb{S}}(B_1)\cap\overline{\mathbb{S}}(B_2)}(xy) &= F_{\overline{\mathbb{S}}(B_1)}(xy),\ F_{\underline{\mathbb{S}}(B_1)\cap\underline{\mathbb{S}}(B_2)}(xy) = F_{\underline{\mathbb{S}}(B_1)}(xy).
\end{aligned}$$

(v) $\forall xy \notin \mathbb{S}B_1$ but $xy \in \mathbb{S}B_2$

$$\begin{aligned}
T_{\overline{\mathbb{S}}(B_1)\cap\overline{\mathbb{S}}(B_2)}(xy) &= T_{\overline{\mathbb{S}}(B_2)}(xy),\ T_{\underline{\mathbb{S}}(B_1)\cap\underline{\mathbb{S}}(B_2)}(xy) = T_{\underline{\mathbb{S}}(B_2)}(xy),\\
I_{\overline{\mathbb{S}}(B_1)\cap\overline{\mathbb{S}}(B_2)}(xy) &= I_{\overline{\mathbb{S}}(B_2)}(xy),\ I_{\underline{\mathbb{S}}(B_1)\cap\underline{\mathbb{S}}(B_2)}(xy) = I_{\underline{\mathbb{S}}(B_2)}(xy),\\
F_{\overline{\mathbb{S}}(B_1)\cap\overline{\mathbb{S}}(B_2)}(xy) &= F_{\overline{\mathbb{S}}(B_2)}(xy),\ F_{\underline{\mathbb{S}}(B_1)\cap\underline{\mathbb{S}}(B_2)}(xy) = F_{\underline{\mathbb{S}}(B_2)}(xy).
\end{aligned}$$

(vi) $\forall xy \in \mathbb{S}B_1 \cap \mathbb{S}(B_2)$

$$\begin{aligned}
T_{\overline{\mathbb{S}}(B_1)\cap\overline{\mathbb{S}}(B_2)}(xy) &= \min\{T_{\overline{\mathbb{S}}(B_1)}(xy), T_{\overline{\mathbb{S}}(B_2)}(xy)\},\\
T_{\underline{\mathbb{S}}(B_1)\cap\underline{\mathbb{S}}(B_2)}(xy) &= \min\{T_{\underline{\mathbb{S}}(B_1)}(xy), T_{\underline{\mathbb{S}}(B_2)}(xy)\},\\
I_{\overline{\mathbb{S}}(B_1)\cap\overline{\mathbb{S}}(B_2)}(xy) &= \max\{I_{\overline{\mathbb{S}}(B_1)}(xy), I_{\overline{\mathbb{S}}(B_2)}(xy)\},\\
I_{\underline{\mathbb{S}}(B_1)\cap\underline{\mathbb{S}}(B_2)}(xy) &= \max\{I_{\underline{\mathbb{S}}(B_1)}(xy), I_{\underline{\mathbb{S}}(B_2)}(xy)\},\\
F_{\overline{\mathbb{S}}(B_1)\cap\overline{\mathbb{S}}(B_2)}(xy) &= \max\{F_{\overline{\mathbb{S}}(B_1)}(xy), F_{\overline{\mathbb{S}}(B_2)}(xy)\},\\
F_{\underline{\mathbb{S}}(B_1)\cap\underline{\mathbb{S}}(B_2)}(xy) &= \max\{F_{\underline{\mathbb{S}}(B_1)}(xy), F_{\underline{\mathbb{S}}(B_2)}(xy)\}.
\end{aligned}$$

Definition 8.6 Let $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ be two neutrosophic soft rough graphs on X. The *join* of G_1 and G_2 is a neutrosophic soft rough graph $G = G_1 + G_2 = (\underline{G}_1 + \underline{G}_2, \overline{G}_1 + \overline{G}_2)$, where $\underline{G}_1 + \underline{G}_2 = (\underline{\mathbb{R}}(A_1) + \underline{\mathbb{R}}(A_2), \underline{\mathbb{S}}(B_1) + \underline{\mathbb{S}}(B_2))$ and $\overline{G}_1 + \overline{G}_2 = (\overline{\mathbb{R}}(A_1) + \overline{\mathbb{R}}(A_2), \overline{\mathbb{S}}(B_1) + \overline{\mathbb{S}}(B_2))$ are neutrosophic graphs, respectively, such that

(i) $\forall x \in \mathbb{R}A_1$ but $x \notin \mathbb{R}A_2$.

$$\begin{aligned}
T_{\overline{\mathbb{R}}(A_1)+\overline{\mathbb{R}}(A_2)}(x) &= T_{\overline{\mathbb{R}}(A_1)}(x),\ T_{\underline{\mathbb{R}}(A_1)+\underline{\mathbb{R}}(A_2)}(x) = T_{\underline{\mathbb{R}}(A_1)}(x),\\
I_{\overline{\mathbb{R}}(A_1)+\overline{\mathbb{R}}(A_2)}(x) &= I_{\overline{\mathbb{R}}(A_1)}(x),\ I_{\underline{\mathbb{R}}(A_1)+\underline{\mathbb{R}}(A_2)}(x) = I_{\underline{\mathbb{R}}(A_1)}(x),\\
F_{\overline{\mathbb{R}}(A_1)+\overline{\mathbb{R}}(A_2)}(x) &= F_{\overline{\mathbb{R}}(A_1)}(x),\ F_{\underline{\mathbb{R}}(A_1)+\underline{\mathbb{R}}(A_2)}(x) = F_{\underline{\mathbb{R}}(A_1)}(x).
\end{aligned}$$

(ii) $\forall x \notin \mathbb{R}A_1$ but $x \in \mathbb{R}A_2$.

$$T_{\overline{\mathbb{R}}(A_1)+\overline{\mathbb{R}}(A_2)}(x) = T_{\overline{\mathbb{R}}(A_2)}(x),\ T_{\underline{\mathbb{R}}(A_1)+\underline{\mathbb{R}}(A_2)}(x) = T_{\underline{\mathbb{R}}(A_2)}(x),$$
$$I_{\overline{\mathbb{R}}(A_1)+\overline{\mathbb{R}}(A_2)}(x) = I_{\overline{\mathbb{R}}(A_2)}(x),\ I_{\underline{\mathbb{R}}(A_1)+\underline{\mathbb{R}}(A_2)}(x) = I_{\underline{\mathbb{R}}(A_2)}(x),$$
$$F_{\overline{\mathbb{R}}(A_1)+\overline{\mathbb{R}}(A_2)}(x) = F_{\overline{\mathbb{R}}(A_2)}(x),\ F_{\underline{\mathbb{R}}(A_1)+\underline{\mathbb{R}}(A_2)}(x) = F_{\underline{\mathbb{R}}(A_2)}(x).$$

(iii) $\forall x \in \mathbb{R}A_1 \cap \mathbb{R}A_2$

$$T_{\overline{\mathbb{R}}(A_1)+\overline{\mathbb{R}}(A_2)}(x) = \max\{T_{\overline{\mathbb{R}}(A_1)}(x), T_{\overline{\mathbb{R}}(A_2)}(x)\},$$
$$T_{\underline{\mathbb{R}}(A_1)+\underline{\mathbb{R}}(A_2)}(x) = \max\{T_{\underline{\mathbb{R}}(A_1)}(x), T_{\underline{\mathbb{R}}(A_2)}(x)\},$$
$$I_{\overline{\mathbb{R}}(A_1)+\overline{\mathbb{R}}(A_2)}(x) = \min\{I_{\overline{\mathbb{R}}(A_1)}(x), I_{\overline{\mathbb{R}}(A_2)}(x)\},$$
$$I_{\underline{\mathbb{R}}(A_1)+\underline{\mathbb{R}}(A_2)}(x) = \min\{I_{\underline{\mathbb{R}}(A_1)}(x), I_{\underline{\mathbb{R}}(A_2)}(x)\},$$
$$F_{\overline{\mathbb{R}}(A_1)+\overline{\mathbb{R}}(A_2)}(x) = \min\{F_{\overline{\mathbb{R}}(A_1)}(x), F_{\overline{\mathbb{R}}(A_2)}(x)\},$$
$$F_{\underline{\mathbb{R}}(A_1)+\underline{\mathbb{R}}(A_2)}(x) = \min\{F_{\underline{\mathbb{R}}(A_1)}(x), F_{\underline{\mathbb{R}}(A_2)}(x)\}.$$

(iv) $\forall xy \in \mathbb{S}B_1$ but $xy \notin \mathbb{S}B_2$.

$$T_{\overline{\mathbb{S}}(B_1)+\overline{\mathbb{S}}(B_2)}(xy) = T_{\overline{\mathbb{S}}(B_1)}(xy),\ T_{\underline{\mathbb{S}}(B_1)+\underline{\mathbb{S}}(B_2)}(xy) = T_{\underline{\mathbb{S}}(B_1)}(xy),$$
$$I_{\overline{\mathbb{S}}(B_1)+\overline{\mathbb{S}}(B_2)}(xy) = I_{\overline{\mathbb{S}}(B_1)}(xy),\ I_{\underline{\mathbb{S}}(B_1)+\underline{\mathbb{S}}(B_2)}(xy) = I_{\underline{\mathbb{S}}(B_1)}(xy),$$
$$F_{\overline{\mathbb{S}}(B_1)+\overline{\mathbb{S}}(B_2)}(xy) = F_{\overline{\mathbb{S}}(B_1)}(xy),\ F_{\underline{\mathbb{S}}(B_1)+\underline{\mathbb{S}}(B_2)}(xy) = F_{\underline{\mathbb{S}}(B_1)}(xy).$$

(v) $\forall xy \notin \mathbb{S}B_1$ but $xy \in \mathbb{S}B_2$

$$T_{\overline{\mathbb{S}}(B_1)+\overline{\mathbb{S}}(B_2)}(xy) = T_{\overline{\mathbb{S}}(B_2)}(xy),\ T_{\underline{\mathbb{S}}(B_1)+\underline{\mathbb{S}}(B_2)}(xy) = T_{\underline{\mathbb{S}}(B_2)}(xy),$$
$$I_{\overline{\mathbb{S}}(B_1)+\overline{\mathbb{S}}(B_2)}(xy) = I_{\overline{\mathbb{S}}(B_2)}(xy),\ I_{\underline{\mathbb{S}}(B_1)+\underline{\mathbb{S}}(B_2)}(xy) = I_{\underline{\mathbb{S}}(B_2)}(xy),$$
$$F_{\overline{\mathbb{S}}(B_1)+\overline{\mathbb{S}}(B_2)}(xy) = F_{\overline{\mathbb{S}}(B_2)}(xy),\ F_{\underline{\mathbb{S}}(B_1)+\underline{\mathbb{S}}(B_2)}(xy) = F_{\underline{\mathbb{S}}(B_2)}(xy).$$

(vi) $\forall xy \in \mathbb{S}B_1 \cap \mathbb{S}(B_2)$

$$T_{\overline{\mathbb{S}}(B_1)+\overline{\mathbb{S}}(B_2)}(xy) = \max\{T_{\overline{\mathbb{S}}(B_1)}(xy), T_{\overline{\mathbb{S}}(B_2)}(xy)\},$$
$$T_{\underline{\mathbb{S}}(B_1)+\underline{\mathbb{S}}(B_2)}(xy) = \max\{T_{\underline{\mathbb{S}}(B_1)}(xy), T_{\underline{\mathbb{S}}(B_2)}(xy)\},$$
$$I_{\overline{\mathbb{S}}(B_1)+\overline{\mathbb{S}}(B_2)}(xy) = \min\{I_{\overline{\mathbb{S}}(B_1)}(xy), I_{\overline{\mathbb{S}}(B_2)}(xy)\},$$
$$I_{\underline{\mathbb{S}}(B_1)+\underline{\mathbb{S}}(B_2)}(xy) = \min\{I_{\underline{\mathbb{S}}(B_1)}(xy), I_{\underline{\mathbb{S}}(B_2)}(xy)\},$$
$$F_{\overline{\mathbb{S}}(B_1)+\overline{\mathbb{S}}(B_2)}(xy) = \min\{F_{\overline{\mathbb{S}}(B_1)}(xy), F_{\overline{\mathbb{S}}(B_2)}(xy)\},$$
$$F_{\underline{\mathbb{S}}(B_1)+\underline{\mathbb{S}}(B_2)}(xy) = \min\{F_{\underline{\mathbb{S}}(B_1)}(xy), F_{\underline{\mathbb{S}}(B_2)}(xy)\}.$$

(vii) $\forall xy \in \tilde{E}$, where $\tilde{E}$ is the set of edges joining vertices of $\mathbb{R}A_1$ and $\mathbb{R}A_2$.

$$
\begin{aligned}
T_{\overline{\mathbb{S}}(B_1)+\overline{\mathbb{S}}(B_2)}(xy) &= \min\{T_{\overline{\mathbb{R}}(A_1)}(x), T_{\overline{\mathbb{R}}(A_2)}(y)\},\\
T_{\underline{\mathbb{S}}(B_1)+\underline{\mathbb{S}}(B_2)}(xy) &= \min\{T_{\underline{\mathbb{R}}(A_1)}(x), T_{\underline{\mathbb{R}}(A_2)}(y)\},\\
I_{\overline{\mathbb{S}}(B_1)`+\overline{\mathbb{S}}(B_2)}(xy) &= \max\{I_{\overline{\mathbb{R}}(A_1)}(x), I_{\overline{\mathbb{R}}(A_2)}(y)\},\\
I_{\underline{\mathbb{S}}(B_1)+\underline{\mathbb{S}}(B_2)}(xy) &= \max\{I_{\underline{\mathbb{R}}(A_1)}(x), I_{\underline{\mathbb{R}}(A_2)}(y)\},\\
F_{\overline{\mathbb{S}}(B_1)+\overline{\mathbb{S}}(B_2)}(xy) &= \max\{F_{\overline{\mathbb{R}}(A_1)}(x), F_{\overline{\mathbb{R}}(A_2)}(y)\},\\
F_{\underline{\mathbb{S}}(B_1)+\underline{\mathbb{S}}(B_2)}(xy) &= \max\{F_{\underline{\mathbb{R}}(A_1)}(x), F_{\underline{\mathbb{R}}(A_2)}(y)\}.
\end{aligned}
$$

Definition 8.7 The *Cartesian product* of G_1 and G_2 is a $G = G_1 \ltimes G_2 = (\underline{G}_1 \ltimes \underline{G}_2, \overline{G}_1 \ltimes \overline{G}_2)$, where $\underline{G}_1 \ltimes \underline{G}_2 = (\underline{\mathbb{R}}(A_1) \ltimes \underline{\mathbb{R}}(A_2), \underline{\mathbb{S}}(B_1) \ltimes \underline{\mathbb{S}}(B_2))$ and $\overline{G}_1 \ltimes \overline{G}_2 = (\overline{\mathbb{R}}(A_1) \ltimes \overline{\mathbb{R}}(A_2), \overline{\mathbb{S}}(B_1) \ltimes \overline{\mathbb{S}}(B_2))$ are neutrosophic graphs, such that

(i) $\forall\ (x, y) \in \mathbb{R}A_1 \times \mathbb{R}A_2$.

$$
\begin{aligned}
T_{\overline{\mathbb{R}}(A_1)\ltimes\overline{\mathbb{R}}(A_2)}(x, y) &= \min\{T_{\overline{\mathbb{R}}(A_1)}(x), T_{\overline{\mathbb{R}}(A_2)}(x)\},\\
T_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x, y) &= \min\{T_{\underline{\mathbb{R}}(A_1)}(x), T_{\underline{\mathbb{R}}(A_2)}(x)\},\\
I_{\overline{\mathbb{R}}(A_1)\ltimes\overline{\mathbb{R}}(A_2)}(x, y) &= \max\{I_{\overline{\mathbb{R}}(A_1)}(x), I_{\overline{\mathbb{R}}(A_2)}(x)\},\\
I_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x, y) &= \max\{I_{\underline{\mathbb{R}}(A_1)}(x), I_{\underline{\mathbb{R}}(A_2)}(x)\},\\
F_{\overline{\mathbb{R}}(A_1)\ltimes\overline{\mathbb{R}}(A_2)}(x, y) &= \max\{F_{\overline{\mathbb{R}}(A_1)}(x), F_{\overline{\mathbb{R}}(A_2)}(x)\},\\
F_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x, y) &= \max\{F_{\underline{\mathbb{R}}(A_1)}(x), F_{\underline{\mathbb{R}}(A_2)}(x_1)\}.
\end{aligned}
$$

(ii) $\forall y_1y_2 \in \mathbb{S}B_2, x \in \mathbb{R}A_1$.

$$
\begin{aligned}
T_{\overline{\mathbb{S}}(B_1)\ltimes\overline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \min\{T_{\overline{\mathbb{R}}(A_1)}(x), T_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
T_{\underline{\mathbb{S}}(B_1)\ltimes\underline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \min\{T_{\underline{\mathbb{R}}(A_1)}(x), T_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
I_{\overline{\mathbb{S}}(B_1)\ltimes\overline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \max\{I_{\overline{\mathbb{R}}(A_1)}(x), I_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
I_{\underline{\mathbb{S}}(B_1)\ltimes\underline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \max\{I_{\underline{\mathbb{R}}(A_1)}(x), I_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
F_{\overline{\mathbb{S}}(B_1)\ltimes\overline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \max\{F_{\overline{\mathbb{R}}(A_1)}(x), F_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
F_{\underline{\mathbb{S}}(B_1)\ltimes\underline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \max\{F_{\underline{\mathbb{R}}(A_1)}(x), F_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\}.
\end{aligned}
$$

(iii) $\forall x_1x_2 \in \mathbb{S}B_1, y \in \mathbb{R}A_2$.

$$
\begin{aligned}
T_{\underline{\mathbb{S}}(B_1)\ltimes\underline{\mathbb{S}}(B_2)}\big((x_1, y)(x_2, y)\big) &= \min\{T_{\underline{\mathbb{S}}(B_1)}(x_1x_2), T_{\underline{\mathbb{R}}(A_2)}(y)\},\\
T_{\overline{\mathbb{S}}(B_1)\ltimes\overline{\mathbb{S}}(B_2)}\big((x_1, y)(x_2, y)\big) &= \min\{T_{\overline{\mathbb{S}}(B_1)}(x_1x_2), T_{\overline{\mathbb{R}}(A_2)}(y)\},\\
I_{\overline{\mathbb{S}}(B_1)\ltimes\overline{\mathbb{S}}(B_2)}\big((x_1, y)(x_2, y)\big) &= \max\{I_{\overline{\mathbb{S}}(B_1)}(x_1x_2), I_{\overline{\mathbb{R}}(A_2)}(y)\},\\
I_{\underline{\mathbb{S}}(B_1)\ltimes\underline{\mathbb{S}}(B_2)}\big((x_1, y)(x_2, y)\big) &= \max\{I_{\underline{\mathbb{S}}(B_1)}(x_1x_2), I_{\underline{\mathbb{R}}(A_2)}(y)\},\\
F_{\overline{\mathbb{S}}(B_1)\ltimes\overline{\mathbb{S}}(B_2)}\big((x_1, y)(x_2, y)\big) &= \max\{F_{\overline{\mathbb{S}}(B_1)}(x_1x_2), F_{\overline{\mathbb{R}}(A_2)}(y)\},\\
F_{\underline{\mathbb{S}}(B_1)\ltimes\underline{\mathbb{S}}(B_2)}\big((x_1, y)(x_2, y)\big) &= \max\{F_{\underline{\mathbb{S}}(B_1)}(x_1x_2), F_{\underline{\mathbb{R}}(A_2)}(y)\}.
\end{aligned}
$$

Definition 8.8 The *cross product* of G_1 and G_2 is a neutrosophic soft rough graph $G = G_1 \circledcirc G_2 = (\underline{G}_1 \circledcirc \underline{G}_2, \overline{G}_1 \circledcirc \overline{G}_2)$, where $\underline{G}_1 \circledcirc \underline{G}_2 = (\underline{\mathbb{R}}(A_1) \circledcirc \underline{\mathbb{R}}(A_2), \underline{\mathbb{S}}(B_1) \circledcirc \underline{\mathbb{S}}(B_2))$ and $\overline{G}_1 \circledcirc \overline{G}_2 = (\overline{\mathbb{R}}(A_1) \circledcirc \overline{\mathbb{R}}(A_2), \overline{\mathbb{S}}(B_1) \circledcirc \overline{\mathbb{S}}(B_2))$ are neutrosophic graphs, respectively, such that

(i) $\forall\, (x, y) \in \mathbb{R}A_1 \times \mathbb{R}A_2$.

$$\begin{aligned}
T_{\overline{\mathbb{R}}(A_1)\circledcirc\overline{\mathbb{R}}(A_2)}(x, y) &= \min\{T_{\overline{\mathbb{R}}(A_1)}(x), T_{\overline{\mathbb{R}}(A_2)}(x)\},\\
T_{\underline{\mathbb{R}}(A_1)\circledcirc\underline{\mathbb{R}}(A_2)}(x, y) &= \min\{T_{\underline{\mathbb{R}}(A_1)}(x), T_{\underline{\mathbb{R}}(A_2)}(x)\},\\
I_{\overline{\mathbb{R}}(A_1)\circledcirc\overline{\mathbb{R}}(A_2)}(x, y) &= \max\{I_{\overline{\mathbb{R}}(A_1)}(x), I_{\overline{\mathbb{R}}(A_2)}(x)\},\\
I_{\underline{\mathbb{R}}(A_1)\circledcirc\underline{\mathbb{R}}(A_2)}(x, y) &= \max\{I_{\underline{\mathbb{R}}(A_1)}(x), I_{\underline{\mathbb{R}}(A_2)}(x)\},\\
F_{\overline{\mathbb{R}}(A_1)\circledcirc\overline{\mathbb{R}}(A_2)}(x, y) &= \max\{F_{\overline{\mathbb{R}}(A_1)}(x), F_{\overline{\mathbb{R}}(A_2)}(x)\},\\
F_{\underline{\mathbb{R}}(A_1)\circledcirc\underline{\mathbb{R}}(A_2)}(x, y) &= \max\{F_{\underline{\mathbb{R}}(A_1)}(x), F_{\underline{\mathbb{R}}(A_2)}(x)\}.
\end{aligned}$$

(ii) $\forall x_1x_2 \in \mathbb{S}B_1,\ y_1y_2 \in \mathbb{S}B_2$.

$$\begin{aligned}
T_{\overline{\mathbb{S}}(B_1)\circledcirc\overline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \min\{T_{\overline{\mathbb{S}}(B_1)}(x_1x_2), T_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
T_{\underline{\mathbb{S}}(B_1)\circledcirc\underline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \min\{T_{\underline{\mathbb{S}}(B_1)}(x_1x_2), T_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
I_{\overline{\mathbb{S}}(B_1)\circledcirc\overline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \max\{I_{\overline{\mathbb{S}}(B_1)}(x_1x_2), I_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
I_{\underline{\mathbb{S}}(B_1)\circledcirc\underline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \max\{I_{\underline{\mathbb{S}}(B_1)}(x_1x_2), I_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
F_{\overline{\mathbb{S}}(B_1)\circledcirc\overline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \max\{F_{\overline{\mathbb{S}}(B_1)}(x_1x_2), F_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
F_{\underline{\mathbb{S}}(B_1)\circledcirc\underline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \max\{F_{\underline{\mathbb{S}}(B_1)}(x_1x_2), F_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\}.
\end{aligned}$$

Definition 8.9 The *rejection* of G_1 and G_2 is a neutrosophic soft rough graph $G = G_1|G_2 = (\underline{G}_1|\underline{G}_2, \overline{G}_1|\overline{G}_2)$, where $\underline{G}_1|\underline{G}_2 = (\underline{\mathbb{S}}A_1|\underline{\mathbb{S}}A_2, \underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2))$ and $\overline{G}_1|\overline{G}_2 = (\overline{\mathbb{S}}A_1|\overline{\mathbb{S}}A_2, \overline{\mathbb{S}}(B_1)|\overline{\mathbb{S}}(B_2))$ are neutrosophic graphs such that

(i) $\forall\, (x, y) \in \mathbb{R}A_1 \times \mathbb{R}A_2$.

$$\begin{aligned}
T_{\overline{\mathbb{R}}(A_1)|\overline{\mathbb{R}}(A_2)}(x, y) &= \min\{T_{\overline{\mathbb{R}}(A_1)}(x), T_{\overline{\mathbb{R}}(A_2)}(y)\},\\
T_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x, y) &= \min\{T_{\underline{\mathbb{R}}(A_1)}(x), T_{\underline{\mathbb{R}}(A_2)}(y)\},\\
I_{\overline{\mathbb{R}}(A_1)|\overline{\mathbb{R}}(A_2)}(x, y) &= \max\{I_{\overline{\mathbb{R}}(A_1)}(x), I_{\overline{\mathbb{R}}(A_2)}(y)\},\\
I_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x, y) &= \max\{I_{\underline{\mathbb{R}}(A_1)}(x), I_{\underline{\mathbb{R}}(A_2)}(y)\},\\
F_{\overline{\mathbb{R}}(A_1)|\overline{\mathbb{R}}(A_2)}(x, y) &= \max\{F_{\overline{\mathbb{R}}(A_1)}(x), F_{\overline{\mathbb{R}}(A_2)}(y)\},\\
F_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x, y) &= \max\{F_{\underline{\mathbb{R}}(A_1)}(x), F_{\underline{\mathbb{R}}(A_2)}(y)\}.
\end{aligned}$$

(ii) $\forall y_1y_2 \notin \mathbb{S}B_2,\ x \in \mathbb{R}A_1$.

$$\begin{aligned}
T_{\overline{\mathbb{S}}(B_1)|\overline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \min\{T_{\overline{\mathbb{R}}(A_1)}(x), T_{\overline{\mathbb{R}}(A_2)}(y_1), T_{\overline{\mathbb{R}}(A_2)}(y_2)\},\\
T_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{R}}B_2)}\big((x, y_1)(x, y_2)\big) &= \min\{T_{\underline{\mathbb{R}}(A_1)}(x), T_{\underline{\mathbb{R}}(A_2)}(y_1), T_{\underline{\mathbb{R}}(A_2)}(y_2)\},
\end{aligned}$$

$$(I_{\overline{\mathbb{S}}(B_1)|\overline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) = \max\{I_{\overline{\mathbb{R}}(A_1)}(x), I_{\overline{\mathbb{R}}(A_2)}(y_1), I_{\overline{\mathbb{R}}(A_2)}(y_2)\},$$
$$(I_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) = \max\{I_{\underline{\mathbb{R}}(A_1)}(x), I_{\underline{\mathbb{R}}(A_2)}(y_1), I_{\underline{\mathbb{R}}(A_2)}(y_2)\},$$
$$(F_{\overline{\mathbb{S}}(B_1)|\overline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) = \max\{F_{\overline{\mathbb{R}}(A_1)}(x), F_{\overline{\mathbb{R}}(A_2)}(y_1), F_{\overline{\mathbb{R}}(A_2)}(y_2)\},$$
$$(F_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) = \max\{F_{\underline{\mathbb{R}}(A_1)}(x), F_{\underline{\mathbb{R}}(A_2)}(y_1), F_{\underline{\mathbb{R}}(A_2)}(y_2)\}.$$

(iii) $\forall x_1x_2 \notin \mathbb{S}B_1,\ y \in \mathbb{R}A_2,$

$$T_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}\big((x_1, y)(x_2, y)\big) = \min\{T_{\underline{\mathbb{R}}(A_1)}(x_1), T_{\underline{\mathbb{R}}(A)_1}(x_2), T_{\underline{\mathbb{R}}(A_2)}(y)\},$$
$$I_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}\big((x_1, y)(x_2, y)\big) = \max\{I_{\underline{\mathbb{R}}(A_1)}(x_1), I_{\underline{\mathbb{R}}(A)_1}(x_2), I_{\underline{\mathbb{R}}(A_2)}(y)\},$$
$$F_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}\big((x_1, y)(x_2, y)\big) = \max\{F_{\underline{\mathbb{R}}(A_1)}(x_1), F_{\underline{\mathbb{R}}(A)_1}(x_2), F_{\underline{\mathbb{R}}(A_2)}(y)\},$$
$$T_{\overline{\mathbb{S}}(B_1)|\overline{\mathbb{S}}(B_2)}\big((x_1, y)(x_2, y)\big) = \min\{T_{\overline{\mathbb{R}}(A_1)}(x_1), T_{\overline{\mathbb{R}}(A)_1}(x_2), T_{\overline{\mathbb{R}}(A_2)}(y)\},$$
$$I_{\overline{\mathbb{S}}(B_1)|\overline{\mathbb{S}}(B_2)}\big((x_1, y)(x_2, y)\big) = \max\{I_{\overline{\mathbb{R}}(A_1)}(x_1), I_{\overline{\mathbb{R}}(A)_1}(x_2), I_{\overline{\mathbb{R}}(A_2)}(y)\},$$
$$F_{\overline{\mathbb{S}}(B_1)|\overline{\mathbb{S}}(B_2)}\big((x_1, y)(x_2, y)\big) = \max\{F_{\overline{\mathbb{R}}(A_1)}(x_1), F_{\overline{\mathbb{R}}(A)_1}(x_2), F_{\overline{\mathbb{R}}(A_2)}(y)\}.$$

(iv) $\forall x_1x_2 \notin \mathbb{S}B_1,\ y_1y_2 \notin \mathbb{S}B_2,\ x_1 \neq x_2, y_1 \neq y_2.$

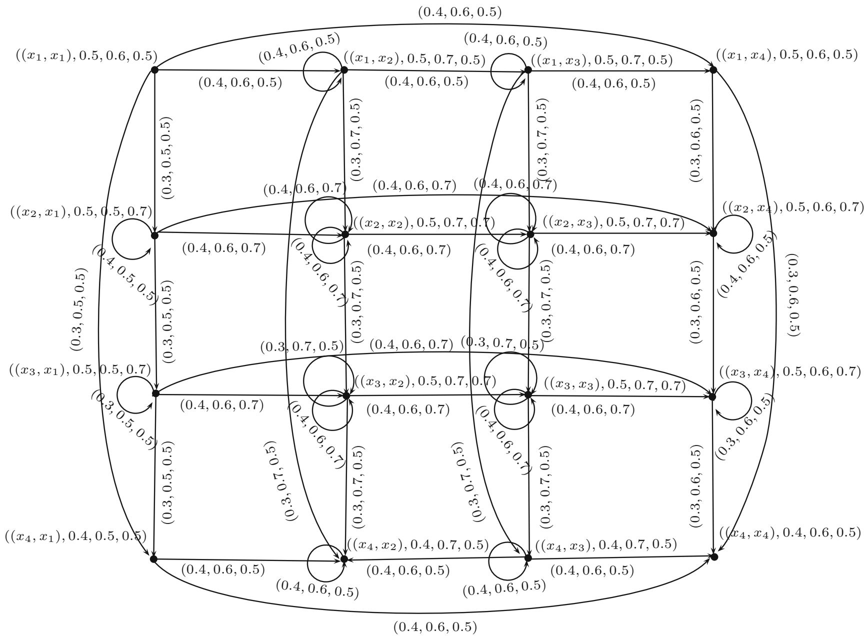
$$T_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) = \min\{T_{\underline{\mathbb{R}}(A_1)}(x_1), T_{\underline{\mathbb{R}}(A)_1}(x_2), T_{\underline{\mathbb{R}}(A_2)}(y_1), T_{\underline{\mathbb{R}}(A_2)}(y_2)\},$$
$$I_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) = \max\{I_{\underline{\mathbb{R}}(A_1)}(x_1), I_{\underline{\mathbb{R}}(A)_1}(x_2), I_{\underline{\mathbb{R}}(A_2)}(y_1), I_{\underline{\mathbb{R}}(A_2)}(y_2)\},$$
$$F_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) = \max\{F_{\underline{\mathbb{R}}(A_1)}(x_1), F_{\underline{\mathbb{R}}(A)_1}(x_2), F_{\underline{\mathbb{R}}(A_2)}(y_1), F_{\underline{\mathbb{R}}(A_2)}(y_2)\},$$
$$T_{\overline{\mathbb{S}}(B_1)|\overline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) = \min\{T_{\overline{\mathbb{R}}(A_1)}(x_1), T_{\overline{\mathbb{R}}(A)_1}(x_2), T_{\overline{\mathbb{R}}(A_2)}(y_1), T_{\overline{\mathbb{R}}(A_2)}(y_2)\},$$
$$I_{\overline{\mathbb{S}}(B_1)|\overline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) = \max\{I_{\overline{\mathbb{R}}(A_1)}(x_1), I_{\overline{\mathbb{R}}(A)_1}(x_2), I_{\overline{\mathbb{R}}(A_2)}(y_1), I_{\overline{\mathbb{R}}(A_2)}(y_2)\},$$
$$F_{\overline{\mathbb{S}}(B_1)|\overline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) = \max\{F_{\overline{\mathbb{R}}(A_1)}(x_1), F_{\overline{\mathbb{R}}(A)_1}(x_2), F_{\overline{\mathbb{R}}(A_2)}(y_1), F_{\overline{\mathbb{R}}(A_2)}(y_2)\},$$

Example 8.5 Let $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ be two neutrosophic soft rough graphs on X, where $\underline{G}_1 = (\underline{\mathbb{R}}(A_1), \underline{\mathbb{S}}(B_1))$ and $\overline{G}_1 = (\overline{\mathbb{R}}(A_1), \overline{\mathbb{S}}(B_1))$ are neutrosophic graphs as shown in Fig. 8.2 and $\underline{G}_2 = (\underline{\mathbb{R}}(A_2), \underline{\mathbb{S}}(B_2))$ and $\overline{G}_2 = (\overline{\mathbb{R}}(A_2), \overline{\mathbb{S}}(B_2))$ are neutrosophic graphs as shown in Fig. 8.3. The Cartesian product of $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ is neutrosophic soft rough graph $G = G_1 \times G_2 = (\underline{G}_1 \times \underline{G}_2, \overline{G}_1 \times \overline{G}_2)$ as shown in Fig. 8.5.

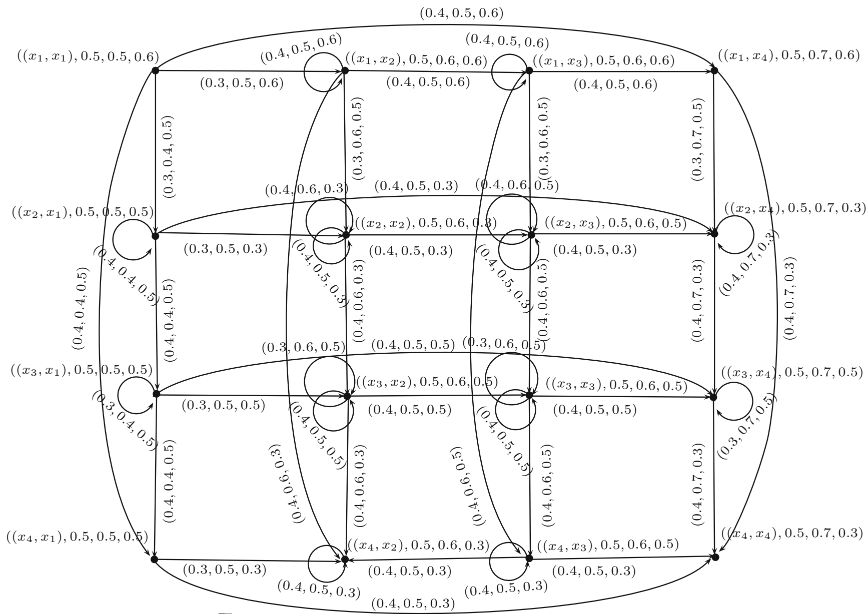
Definition 8.10 The *symmetric difference* of G_1 and G_2 is a neutrosophic soft rough graph $G = G_1 \oplus G_2 = (\underline{G}_1 \oplus \underline{G}_2, \overline{G}_1 \oplus \overline{G}_2)$, where $\underline{G}_1 \oplus \underline{G}_2 = (\underline{\mathbb{R}}(A_1) \oplus \underline{\mathbb{R}}(A_2), \underline{\mathbb{S}}(B_1) \oplus \underline{\mathbb{S}}(B_2))$ and $\overline{G}_1 \oplus \overline{G}_2 = (\overline{\mathbb{R}}(A_1) \oplus \overline{\mathbb{R}}(A_2), \overline{\mathbb{S}}(B_1) \oplus \overline{\mathbb{S}}(B_2))$ are neutrosophic graphs, respectively, such that

(i) $\forall\ (x, y) \in \mathbb{R}A_1 \times \mathbb{R}A_2.$

$$T_{\overline{\mathbb{R}}(A_1)\oplus\overline{\mathbb{R}}(A_2)}(x, y) = \min\{T_{\overline{\mathbb{R}}(A_1)}(x), T_{\overline{\mathbb{R}}(A_2)}(y)\},$$
$$T_{\underline{\mathbb{R}}(A_1)\oplus\underline{\mathbb{R}}(A_2)}(x, y) = \min\{T_{\underline{\mathbb{R}}(A_1)}(x), T_{\underline{\mathbb{R}}(A_2)}(y)\},$$



$\underline{G}_1 \times \underline{G}_2 = (\underline{\mathbb{S}}A_1 \times \underline{\mathbb{S}}A_2, \underline{\mathbb{S}}(B_1) \times \underline{\mathbb{S}}(B_2))$



$\overline{G}_1 \times \overline{G}_2 = (\overline{\mathbb{S}}A_1 \times \overline{\mathbb{S}}A_2, \overline{\mathbb{S}}(B_1) \times \overline{\mathbb{S}}(B_2))$

Fig. 8.5 Cartesian product of two neutrosophic soft rough graphs $G_1 \times G_2$

$$I_{\overline{\mathbb{R}}(A_1)\oplus\overline{\mathbb{R}}(A_2)}(x,y)=\max\{I_{\overline{\mathbb{R}}(A_1)}(x),I_{\overline{\mathbb{R}}(A_2)}(y)\},$$
$$I_{\underline{\mathbb{R}}(A_1)\oplus\underline{\mathbb{R}}(A_2)}(x,y)=\max\{I_{\underline{\mathbb{R}}(A_1)}(x),I_{\underline{\mathbb{R}}(A_2)}(y)\},$$
$$F_{\overline{\mathbb{R}}(A_1)\oplus\overline{\mathbb{R}}(A_2)}(x,y)=\max\{F_{\overline{\mathbb{R}}(A_1)}(x),F_{\overline{\mathbb{R}}(A_2)}(y)\},$$
$$F_{\underline{\mathbb{R}}(A_1)\oplus\underline{\mathbb{R}}(A_2)}(x,y)=\max\{F_{\underline{\mathbb{R}}(A_1)}(x),F_{\underline{\mathbb{R}}(A_2)}(y)\}.$$

(ii) $\forall y_1y_2\in\mathbb{S}B_2,\ x\in\mathbb{R}A_1.$

$$T_{\overline{\mathbb{S}}(B_1)\oplus\overline{\mathbb{S}}(B_2)}\big((x,y_1)(x,y_2)\big)=\min\{T_{\overline{\mathbb{R}}(A_1)}(x),T_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},$$
$$T_{\underline{\mathbb{S}}(B_1)\oplus\underline{\mathbb{S}}(B_2)}\big((x,y_1)(x,y_2)\big)=\min\{T_{\underline{\mathbb{R}}(A_1)}(x),T_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\},$$
$$I_{\overline{\mathbb{S}}(B_1)\oplus\overline{\mathbb{S}}(B_2)}\big((x,y_1)(x,y_2)\big)=\max\{I_{\overline{\mathbb{R}}(A_1)}(x),I_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},$$
$$I_{\underline{\mathbb{S}}(B_1)\oplus\underline{\mathbb{S}}(B_2)}\big((x,y_1)(x,y_2)\big)=\max\{I_{\underline{\mathbb{R}}(A_1)}(x),I_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\},$$
$$F_{\overline{\mathbb{S}}(B_1)\oplus\overline{\mathbb{S}}(B_2)}\big((x,y_1)(x,y_2)\big)=\max\{F_{\overline{\mathbb{R}}(A_1)}(x),F_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},$$
$$F_{\underline{\mathbb{S}}(B_1)\oplus\underline{\mathbb{S}}(B_2)}\big((x,y_1)(x,y_2)\big)=\max\{F_{\underline{\mathbb{R}}(A_1)}(x),F_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\}.$$

(iii) $\forall x_1x_2\in\mathbb{S}B_1,\ y\in\mathbb{R}A_2.$

$$T_{\overline{\mathbb{S}}(B_1)\oplus\overline{\mathbb{S}}(B_2)}\big((x_1,y)(x_2,y)\big)=\min\{T_{\overline{\mathbb{S}}(B_1)}(x_1x_2),T_{\overline{\mathbb{R}}(A_2)}(y)\},$$
$$T_{\underline{\mathbb{S}}(B_1)\oplus\underline{\mathbb{S}}(B_2)}\big((x_1,y)(x_2,y)\big)=\min\{T_{\underline{\mathbb{S}}(B_1)}(x_1x_2),T_{\underline{\mathbb{R}}(A_2)}(y)\},$$
$$I_{\overline{\mathbb{S}}(B_1)\oplus\overline{\mathbb{S}}(B_2)}\big((x_1,y)(x_2,y)\big)=\max\{I_{\overline{\mathbb{S}}(B_1)}(x_1x_2),I_{\overline{\mathbb{R}}(A_2)}(y)\},$$
$$I_{\underline{\mathbb{S}}(B_1)\oplus\underline{\mathbb{S}}(B_2)}\big((x_1,y)(x_2,y)\big)=\max\{I_{\underline{\mathbb{S}}(B_1)}(x_1x_2),I_{\underline{\mathbb{R}}(A_2)}(y)\},$$
$$F_{\overline{\mathbb{S}}(B_1)\oplus\overline{\mathbb{S}}(B_2)}\big((x_1,y)(x_2,y)\big)=\max\{F_{\overline{\mathbb{S}}(B_1)}(x_1x_2),F_{\overline{\mathbb{R}}(A_2)}(y)\},$$
$$F_{\underline{\mathbb{S}}(B_1)\oplus\underline{\mathbb{S}}(B_2)}\big((x_1,y)(x_2,y)\big)=\max\{F_{\underline{\mathbb{S}}(B_1)}(x_1x_2),F_{\underline{\mathbb{R}}(A_2)}(y)\}.$$

(iv) $\forall x_1x_2\in\mathbb{S}B_1,\ y_1y_2\notin\mathbb{S}B_2.$

$$T_{\overline{\mathbb{S}}(B_1)\oplus\overline{\mathbb{S}}(B_2)}\big((x_1,y_1)(x_2,y_2)\big)=\min\{T_{\overline{\mathbb{S}}(B_1)}(x_1x_2),T_{\overline{\mathbb{R}}(A_2)}(y_1),T_{\overline{\mathbb{R}}(A_2)}(y_2)\},$$
$$T_{\underline{\mathbb{S}}(B_1)\oplus\underline{\mathbb{S}}(B_2)}\big((x_1,y_1)(x_2,y_2)\big)=\min\{T_{\underline{\mathbb{S}}(B_1)}(x_1x_2),T_{\underline{\mathbb{R}}(A_2)}(y_1),T_{\underline{\mathbb{R}}(A_2)}(y_2)\},$$
$$I_{\overline{\mathbb{S}}(B_1)\oplus\overline{\mathbb{S}}(B_2)}\big((x_1,y_1)(x_2,y_2)\big)=\max\{I_{\overline{\mathbb{S}}(B_1)}(x_1x_2),I_{\overline{\mathbb{R}}(A_2)}(y_1),I_{\overline{\mathbb{R}}(A_2)}(y_2)\},$$
$$I_{\underline{\mathbb{S}}(B_1)\oplus\underline{\mathbb{S}}(B_2)}\big((x_1,y_1)(x_2,y_2)\big)=\max\{I_{\underline{\mathbb{S}}(B_1)}(x_1x_2),I_{\underline{\mathbb{R}}(A_2)}(y_1),I_{\underline{\mathbb{R}}(A_2)}(y_2)\},$$
$$F_{\overline{\mathbb{S}}(B_1)\oplus\overline{\mathbb{S}}(B_2)}\big((x_1,y_1)(x_2,y_2)\big)=\max\{F_{\overline{\mathbb{S}}(B_1)}(x_1x_2),F_{\overline{\mathbb{R}}(A_2)}(y_1),F_{\overline{\mathbb{R}}(A_2)}(y_2)\},$$
$$F_{\underline{\mathbb{S}}(B_1)\oplus\underline{\mathbb{S}}(B_2)}\big((x_1,y_1)(x_2,y_2)\big)=\max\{F_{\underline{\mathbb{S}}(B_1)}(x_1x_2),F_{\underline{\mathbb{R}}(A_2)}(y_1),F_{\underline{\mathbb{R}}(A_2)}(y_2)\}.$$

(v) $\forall x_1x_2\notin\mathbb{S}B_1,\ y_1y_2\in\mathbb{S}B_2.$

$$T_{\overline{\mathbb{S}}(B_1)\oplus\overline{\mathbb{S}}(B_2)}\big((x_1,y_1)(x_2,y_2)\big)=\min\{T_{\overline{\mathbb{R}}(A_1)}(x_1),T_{\overline{\mathbb{R}}(A_1)}(x_2),T_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},$$
$$T_{\underline{\mathbb{S}}(B_1)\oplus\underline{\mathbb{S}}(B_2)}\big((x_1,y_1)(x_2,y_2)\big)=\min\{T_{\underline{\mathbb{R}}(A_1)}(x_1),T_{\underline{\mathbb{R}}(A_1)}(x_2),T_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\},$$
$$I_{\overline{\mathbb{S}}(B_1)\oplus\overline{\mathbb{S}}(B_2)}\big((x_1,y_1)(x_2,y_2)\big)=\max\{I_{\overline{\mathbb{R}}(A_1)}(x_1),I_{\overline{\mathbb{R}}(A_1)}(x_2),I_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},$$

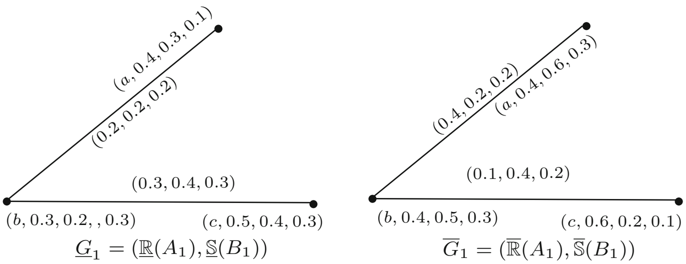


Fig. 8.6 Neutrosophic soft rough graph $G_1 = (\underline{G}_1, \overline{G}_1)$

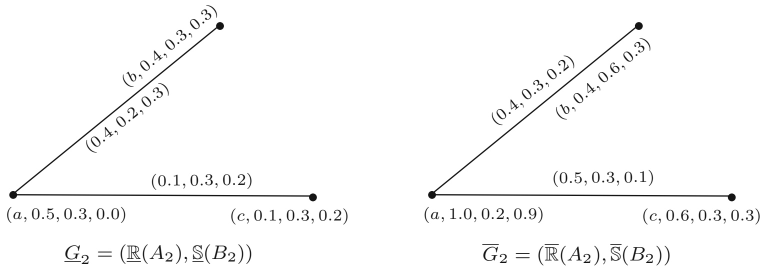


Fig. 8.7 Neutrosophic soft rough graph $G_2 = (\underline{G}_2, \overline{G}_2)$

$$
\begin{aligned}
I_{\underline{\mathbb{S}}(B_1)\oplus\underline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \max\{I_{\underline{\mathbb{R}}(A_1)}(x_1), I_{\underline{\mathbb{R}}(A_1)}(x_2), I_{\underline{\mathbb{S}}(B_2)}(y_1 y_2)\},\\
F_{\overline{\mathbb{S}}(B_1)\oplus\overline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \max\{F_{\overline{\mathbb{R}}(A_1)}(x_1), F_{\overline{\mathbb{R}}(A_1)}(x_2), F_{\overline{\mathbb{S}}(B_2)}(y_1 y_2)\},\\
F_{\underline{\mathbb{S}}(B_1)\oplus\underline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \max\{F_{\underline{\mathbb{R}}(A_1)}(x_1), F_{\underline{\mathbb{R}}(A_1)}(x_2), F_{\underline{\mathbb{S}}(B_2)}(y_1 y_2)\}.
\end{aligned}
$$

Example 8.6 Let $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ be two neutrosophic soft rough graphs on X, where $\underline{G}_1 = (\underline{\mathbb{R}}(A_1), \underline{\mathbb{S}}(B_1))$ and $\overline{G}_1 = (\overline{\mathbb{R}}(A_1), \overline{\mathbb{S}}(B_1))$ are neutrosophic graphs as shown in Fig. 8.6 and $\underline{G}_2 = (\underline{\mathbb{R}}(A_2), \underline{\mathbb{S}}(B_2))$ and $\overline{G}_2 = (\overline{\mathbb{R}}(A_2), \overline{\mathbb{S}}(B_2))$ are neutrosophic graphs as shown in Fig. 8.7

The symmetric difference of G_1 and G_2 is $G = G_1 \oplus G_2 = (\underline{G}_1 \oplus \underline{G}_2, \overline{G}_1 \oplus \overline{G}_2)$, where $\underline{G}_1 \oplus \underline{G}_2 = (\underline{\mathbb{R}}(A_1) \oplus \underline{\mathbb{R}}(A_2), \underline{\mathbb{S}}(B_1) \oplus \underline{\mathbb{S}}(B_2))$ and $\overline{G}_1 \oplus \overline{G}_2 = (\overline{\mathbb{R}}(A_1) \oplus \overline{\mathbb{R}}(A_2), \overline{\mathbb{S}}(B_1) \oplus \overline{\mathbb{S}}(B_2))$ are neutrosophic graphs as shown in Fig. 8.8.

Definition 8.11 The *lexicographic product* of G_1 and G_2 is a neutrosophic soft rough graph $G = G_1 \odot G_2 = (G_{1*} \odot G_{2*}, G_1^* \odot G_2^*)$, where $G_{1*} \odot G_{2*} = (\underline{\mathbb{R}}A_1 \odot \underline{\mathbb{R}}A_2, \underline{\mathbb{S}}B_1 \odot \underline{\mathbb{S}}B_2)$ and $G_1^* \odot G_2^* = (\overline{\mathbb{R}}A_1 \odot \overline{\mathbb{R}}A_2, \overline{\mathbb{S}}B_1 \odot \overline{\mathbb{S}}B_2)$ are neutrosophic graphs, respectively, such that

(i) $\forall\, (x, y) \in \mathbb{R}A_1 \times \mathbb{R}A_2$.

$$
\begin{aligned}
T_{\overline{\mathbb{R}}(A_1)\odot\overline{\mathbb{R}}(A_2)}(x, y) &= \min\{T_{\overline{\mathbb{R}}(A_1)}(x), T_{\overline{\mathbb{R}}(A_2)}(y)\},\\
T_{\underline{\mathbb{R}}(A_1)\odot\underline{\mathbb{R}}(A_2)}(x, y) &= \min\{T_{\underline{\mathbb{R}}(A_1)}(x), T_{\underline{\mathbb{R}}(A_2)}(y)\},
\end{aligned}
$$

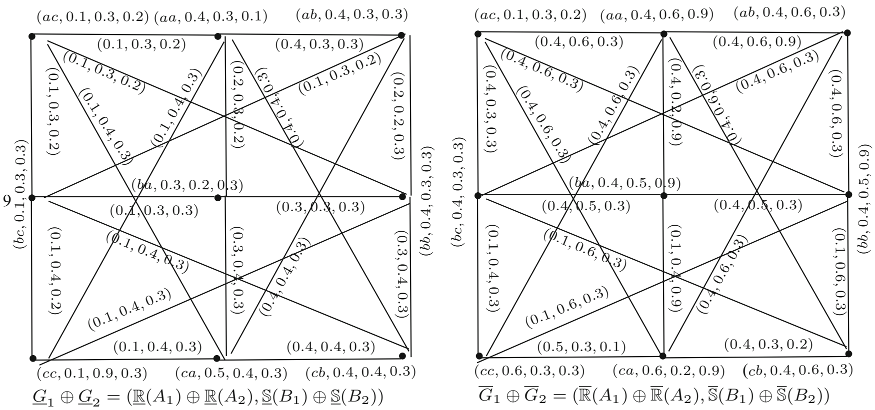


Fig. 8.8 Neutrosophic soft rough graph $G_1 \oplus G_2 = (\underline{G}_1 \oplus G_2, \overline{G}_1 \oplus \overline{G}_2)$

$$\begin{aligned}
I_{\overline{\mathbb{R}}(A_1)\odot\overline{\mathbb{R}}(A_2)}(x, y) &= \max\{I_{\overline{\mathbb{R}}(A_1)}(x), I_{\overline{\mathbb{R}}(A_2)}(y)\},\\
I_{\underline{\mathbb{R}}(A_1)\odot\underline{\mathbb{R}}(A_2)}(x, y) &= \max\{I_{\underline{\mathbb{R}}(A_1)}(x), I_{\underline{\mathbb{R}}(A_2)}(y)\},\\
F_{\overline{\mathbb{R}}(A_1)\odot\overline{\mathbb{R}}(A_2)}(x, y) &= \max\{F_{\overline{\mathbb{R}}(A_1)}(x), F_{\overline{\mathbb{R}}(A_2)}(y)\},\\
F_{\underline{\mathbb{R}}(A_1)\odot\underline{\mathbb{R}}(A_2)}(x, y) &= \max\{F_{\underline{\mathbb{R}}(A_1)}(x), F_{\underline{\mathbb{R}}(A_2)}(y)\}.
\end{aligned}$$

(ii) $\forall y_1y_2 \in \mathbb{S}B_2,\ x \in \mathbb{R}A_1.$

$$\begin{aligned}
T_{\overline{\mathbb{S}}(B_1)\odot\overline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \min\{T_{\overline{\mathbb{R}}(A_1)}(x), T_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
T_{\underline{\mathbb{S}}(B_1)\odot\underline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \min\{T_{\underline{\mathbb{R}}(A_1)}(x), T_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
I_{\overline{\mathbb{S}}(B_1)\odot\overline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \max\{I_{\overline{\mathbb{R}}(A_1)}(x), I_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
I_{\underline{\mathbb{S}}(B_1)\odot\underline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \max\{I_{\underline{\mathbb{R}}(A_1)}(x), I_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
F_{\overline{\mathbb{S}}(B_1)\odot\overline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \max\{F_{\overline{\mathbb{R}}(A_1)}(x), F_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
F_{\underline{\mathbb{S}}(B_1)\odot\underline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \max\{F_{\underline{\mathbb{R}}(A_1)}(x), F_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\}.
\end{aligned}$$

(iii) $\forall x_1x_2 \in \mathbb{S}B_1,\ y_1y_2 \in \mathbb{S}B_2.$

$$\begin{aligned}
T_{\overline{\mathbb{S}}(B_1)\odot\overline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \min\{T_{\overline{\mathbb{S}}(B_1)}(x_1x_2), T_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
T_{\underline{\mathbb{S}}(B_1)\odot\underline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \min\{T_{\underline{\mathbb{S}}(B_1)}(x_1x_2), T_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
I_{\overline{\mathbb{S}}(B_1)\odot\overline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \max\{I_{\overline{\mathbb{S}}(B_1)}(x_1x_2), I_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
I_{\underline{\mathbb{S}}(B_1)\odot\underline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \max\{I_{\underline{\mathbb{S}}(B_1)}(x_1x_2), I_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
F_{\overline{\mathbb{S}}(B_1)\odot\overline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \max\{F_{\overline{\mathbb{S}}(B_1)}(x_1x_2), F_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
F_{\underline{\mathbb{S}}(B_1)\odot\underline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \max\{F_{\underline{\mathbb{S}}(B_1)}(x_1x_2), F_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\}.
\end{aligned}$$

Definition 8.12 The *strong product* of G_1 and G_2 is a neutrosophic soft rough graph $G = G_1 \otimes G_2 = (G_{1*} \otimes G_{2*}, G_1^* \otimes G_2^*)$, where $G_{1*} \otimes G_{2*} = (\underline{\mathbb{R}}A_1 \otimes \underline{\mathbb{R}}A_2, \underline{\mathbb{S}}B_1 \otimes \underline{\mathbb{S}}B_2)$ and $G_1^* \otimes G_2^* = (\overline{\mathbb{R}}A_1 \otimes \overline{\mathbb{R}}A_2, \overline{\mathbb{S}}B_1 \otimes \overline{\mathbb{S}}B_2)$ are neutrosophic graphs, respectively, such that

(i) $\forall\,(x, y) \in \mathbb{R}A_1 \times \mathbb{R}A_2$.

$$\begin{aligned}
T_{\overline{\mathbb{R}}(A_1)\otimes\overline{\mathbb{R}}(A_2)}(x, y) &= \min\{T_{\overline{\mathbb{R}}(A_1)}(x), T_{\overline{\mathbb{R}}(A_2)}(y)\},\\
T_{\underline{\mathbb{R}}(A_1)\otimes\underline{\mathbb{R}}(A_2)}(x, y) &= \min\{T_{\underline{\mathbb{R}}(A_1)}(x), T_{\underline{\mathbb{R}}(A_2)}(y)\},\\
I_{\overline{\mathbb{R}}(A_1)\otimes\overline{\mathbb{R}}(A_2)}(x, y) &= \max\{I_{\overline{\mathbb{R}}(A_1)}(x), I_{\overline{\mathbb{R}}(A_2)}(y)\},\\
I_{\underline{\mathbb{R}}(A_1)\otimes\underline{\mathbb{R}}(A_2)}(x, y) &= \max\{I_{\underline{\mathbb{R}}(A_1)}(x), I_{\underline{\mathbb{R}}(A_2)}(y)\},\\
F_{\overline{\mathbb{R}}(A_1)\otimes\overline{\mathbb{R}}(A_2)}(x, y) &= \max\{F_{\overline{\mathbb{R}}(A_1)}(x), F_{\overline{\mathbb{R}}(A_2)}(y)\},\\
F_{\underline{\mathbb{R}}(A_1)\otimes\underline{\mathbb{R}}(A_2)}(x, y) &= \max\{F_{\underline{\mathbb{R}}(A_1)}(x), F_{\underline{\mathbb{R}}(A_2)}(y)\}.
\end{aligned}$$

(ii) $\forall y_1y_2 \in \mathbb{S}B_2,\ x \in \mathbb{R}A_1$.

$$\begin{aligned}
T_{\overline{\mathbb{S}}(B_1)\otimes\overline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \min\{T_{\overline{\mathbb{R}}(A_1)}(x), T_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
T_{\underline{\mathbb{S}}(B_1)\otimes\underline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \min\{T_{\underline{\mathbb{R}}(A_1)}(x), T_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
I_{\overline{\mathbb{S}}(B_1)\otimes\overline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \max\{I_{\overline{\mathbb{R}}(A_1)}(x), I_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
I_{\underline{\mathbb{S}}(B_1)\otimes\underline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \max\{I_{\underline{\mathbb{R}}(A_1)}(x), I_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
F_{\overline{\mathbb{S}}(B_1)\otimes\overline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \max\{F_{\overline{\mathbb{R}}(A_1)}(x), F_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
F_{\underline{\mathbb{S}}(B_1)\otimes\underline{\mathbb{S}}(B_2)}\big((x, y_1)(x, y_2)\big) &= \max\{F_{\underline{\mathbb{R}}(A_1)}(x), F_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\}.
\end{aligned}$$

(iii) $\forall x_1x_2 \in \mathbb{S}B_1,\ y \in \mathbb{R}A_2$.

$$\begin{aligned}
T_{\overline{\mathbb{S}}(B_1)\otimes\overline{\mathbb{S}}(B_2)}\big((x_1, y)(x_2, y)\big) &= \min\{T_{\overline{\mathbb{S}}(B_1)}(x_1x_2), T_{\overline{\mathbb{R}}(A_2)}(y)\},\\
T_{\underline{\mathbb{S}}(B_1)\otimes\underline{\mathbb{S}}(B_2)}\big((x_1, y)(x_2, y)\big) &= \min\{T_{\underline{\mathbb{S}}(B_1)}(x_1x_2), T_{\underline{\mathbb{R}}(A_2)}(y)\},\\
I_{\overline{\mathbb{S}}(B_1)\otimes\overline{\mathbb{S}}(B_2)}\big((x_1, y)(x_2, y)\big) &= \max\{I_{\overline{\mathbb{S}}(B_1)}(x_1x_2), I_{\overline{\mathbb{R}}(A_2)}(y)\},\\
I_{\underline{\mathbb{S}}(B_1)\otimes\underline{\mathbb{S}}(B_2)}\big((x_1, y)(x_2, y)\big) &= \max\{I_{\underline{\mathbb{S}}(B_1)}(x_1x_2), I_{\underline{\mathbb{R}}(A_2)}(y)\},\\
F_{\overline{\mathbb{S}}(B_1)\otimes\overline{\mathbb{S}}(B_2)}\big((x_1, y)(x_2, y)\big) &= \max\{F_{\overline{\mathbb{S}}(B_1)}(x_1x_2), F_{\overline{\mathbb{R}}(A_2)}(y)\},\\
F_{\underline{\mathbb{S}}(B_1)\otimes\underline{\mathbb{S}}(B_2)}\big((x_1, y)(x_2, y)\big) &= \max\{F_{\underline{\mathbb{S}}(B_1)}(x_1x_2), F_{\underline{\mathbb{R}}(A_2)}(y)\}.
\end{aligned}$$

(iv) $\forall x_1x_2 \in \mathbb{S}B_1, y_1y_2 \in \mathbb{S}B_2$.

$$\begin{aligned}
T_{\overline{\mathbb{S}}(B_1)\otimes\overline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \min\{T_{\overline{\mathbb{S}}(B_1)}(x_1x_2), T_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
T_{\underline{\mathbb{S}}(B_1)\otimes\underline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \min\{T_{\underline{\mathbb{S}}(B_1)}(x_1x_2), T_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
I_{\overline{\mathbb{S}}(B_1)\otimes\overline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \max\{I_{\overline{\mathbb{S}}(B_1)}(x_1x_2), I_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
I_{\underline{\mathbb{S}}(B_1)\otimes\underline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \max\{I_{\underline{\mathbb{S}}(B_1)}(x_1x_2), I_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
F_{\overline{\mathbb{S}}(B_1)\otimes\overline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_1, y_2)\big) &= \max\{F_{\overline{\mathbb{S}}(B_1)}(x_1x_2), F_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},
\end{aligned}$$

$$F_{\underline{\mathbb{S}}(B_1)\otimes\underline{\mathbb{S}}(B_2)}\big((x_1,x_1)(x_1,x_2)\big) = \max\{F_{\underline{\mathbb{S}}(B_1)}(x_1x_2), F_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\}.$$

Definition 8.13 The *composition* of G_1 and G_2 is a neutrosophic soft rough graph $G = G_1[G_2] = (G_{1*}[G_{2*}], G_1^*[G_2^*])$, where $G_{1*}[G_{2*}] = (\underline{\mathbb{R}}A_1[\underline{\mathbb{R}}A_2], \underline{\mathbb{S}}B_1[\underline{\mathbb{S}}B_2]))$ and $G_1^*[G_2^*] = (\overline{\mathbb{R}}A_1[\overline{\mathbb{R}}A_2], \overline{\mathbb{S}}B_1[\overline{\mathbb{S}}B_2])$ are neutrosophic graphs, respectively, such that

(i) $\forall (x,y) \in \mathbb{R}A_1 \times \mathbb{R}A_2$.

$$\begin{aligned}
T_{\overline{\mathbb{R}}(A_1)\times\overline{\mathbb{R}}(A_2)}(x,y) &= \min\{T_{\overline{\mathbb{R}}(A_1)}(x), T_{\overline{\mathbb{R}}(A_2)}(y)\},\\
T_{\underline{\mathbb{R}}(A_1)\times\underline{\mathbb{R}}(A_2)}(x,y) &= \min\{T_{\underline{\mathbb{R}}(A_1)}(x), T_{\underline{\mathbb{R}}(A_2)}(y)\},\\
I_{\overline{\mathbb{R}}(A_1)\times\overline{\mathbb{R}}(A_2)}(x,y) &= \max\{I_{\overline{\mathbb{R}}(A_1)}(x), I_{\overline{\mathbb{R}}(A_2)}(y)\},\\
I_{\underline{\mathbb{R}}(A_1)\times\underline{\mathbb{R}}(A_2)}(x,y) &= \max\{I_{\underline{\mathbb{R}}(A_1)}(x), I_{\underline{\mathbb{R}}(A_2)}(y)\},\\
F_{\overline{\mathbb{R}}(A_1)\times\overline{\mathbb{R}}(A_2)}(x,y) &= \max\{F_{\overline{\mathbb{R}}(A_1)}(x), F_{\overline{\mathbb{R}}(A_2)}(y)\},\\
F_{\underline{\mathbb{R}}(A_1)\times\underline{\mathbb{R}}(A_2)}(x,y) &= \max\{F_{\underline{\mathbb{R}}(A_1)}(x), F_{\underline{\mathbb{R}}(A_2)}(y)\}.
\end{aligned}$$

(ii) $\forall y_1y_2 \in \mathbb{S}B_2,\ x \in \mathbb{R}A_1$.

$$\begin{aligned}
T_{\overline{\mathbb{S}}(B_1)\times\overline{\mathbb{S}}(B_2)}\big((x,y_1)(x,y_2)\big) &= \min\{T_{\overline{\mathbb{R}}(A_1)}(x), T_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
T_{\underline{\mathbb{S}}(B_1)\times\underline{\mathbb{S}}(B_2)}\big((x,y_1)(x,y_2)\big) &= \min\{T_{\underline{\mathbb{R}}(A_1)}(x), T_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
I_{\overline{\mathbb{S}}(B_1)\times\overline{\mathbb{S}}(B_2)}\big((x,y_1)(x,y_2)\big) &= \max\{I_{\overline{\mathbb{R}}(A_1)}(x), I_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
I_{\underline{\mathbb{S}}(B_1)\times\underline{\mathbb{S}}(B_2)}\big((x,y_1)(x,y_2)\big) &= \max\{I_{\underline{\mathbb{R}}(A_1)}(x), I_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
F_{\overline{\mathbb{S}}(B_1)\times\overline{\mathbb{S}}(B_2)}\big((x,y_1)(x,y_2)\big) &= \max\{F_{\overline{\mathbb{R}}(A_1)}(x), F_{\overline{\mathbb{S}}(B_2)}(y_1y_2)\},\\
F_{\underline{\mathbb{S}}(B_1)\times\underline{\mathbb{S}}(B_2)}\big((x,y_1)(x,y_2)\big) &= \max\{F_{\underline{\mathbb{R}}(A_1)}(x), F_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\}.
\end{aligned}$$

(iii) $\forall x_1x_2 \in \mathbb{S}B_1,\ y \in \mathbb{R}A_2$.

$$\begin{aligned}
T_{\overline{\mathbb{S}}(B_1)\times\overline{\mathbb{S}}(B_2)}\big((x_1,y)(x_2,y)\big) &= \min\{T_{\overline{\mathbb{S}}(B_1)}(x_1x_2), T_{\overline{\mathbb{R}}(A_2)}(y)\},\\
T_{\underline{\mathbb{S}}(B_1)\times\underline{\mathbb{S}}(B_2)}\big((x_1,y)(x_2,y)\big) &= \min\{T_{\underline{\mathbb{S}}(B_1)}(x_1x_2), T_{\underline{\mathbb{R}}(A_2)}(y)\},\\
I_{\overline{\mathbb{S}}(B_1)\times\overline{\mathbb{S}}(B_2)}\big((x_1,y)(x_2,y)\big) &= \max\{I_{\overline{\mathbb{S}}(B_1)}(x_1x_2), I_{\overline{\mathbb{R}}(A_2)}(y)\},\\
I_{\underline{\mathbb{S}}(B_1)\times\underline{\mathbb{S}}(B_2)}\big((x_1,y)(x_2,y)\big) &= \max\{I_{\underline{\mathbb{S}}(B_1)}(x_1x_2), I_{\underline{\mathbb{R}}(A_2)}(y)\},\\
F_{\overline{\mathbb{S}}(B_1)\times\overline{\mathbb{S}}(B_2)}\big((x_1,y)(x_2,y)\big) &= \max\{F_{\overline{\mathbb{S}}(B_1)}(x_1x_2), F_{\overline{\mathbb{R}}(A_2)}(y)\},\\
F_{\underline{\mathbb{S}}(B_1)\times\underline{\mathbb{S}}(B_2)}\big((x_1,y)(x_2,y)\big) &= \max\{F_{\underline{\mathbb{S}}(B_1)}(x_1x_2), F_{\underline{\mathbb{R}}(A_2)}(y)\}.
\end{aligned}$$

(iv) $\forall x_1x_2 \in \mathbb{S}B_1,\ y_1 \neq y_2 \in \mathbb{R}A_2$.

$$\begin{aligned}
T_{\overline{\mathbb{S}}(B_1)\times\overline{\mathbb{S}}(B_2)}\big((x_1,y_1)(x_2,y_2)\big) &= \min\{T_{\overline{\mathbb{S}}(B_1)}(x_1x_1), T_{\overline{\mathbb{R}}(A_2)}(y_1), T_{\overline{\mathbb{R}}(A_2)}(y_2)\},\\
T_{\underline{\mathbb{S}}(B_1)\times\underline{\mathbb{S}}(B_2)}\big((x_1,y_1)(x_2,y_2)\big) &= \min\{T_{\underline{\mathbb{S}}(B_1)}(x_1x_1), T_{\underline{\mathbb{R}}(A_2)}(y_1), T_{\underline{\mathbb{R}}(A_2)}(y_2)\},\\
I_{\overline{\mathbb{S}}(B_1)\times\overline{\mathbb{S}}(B_2)}\big((x_1,y_1)(x_2,y_2)\big) &= \max\{I_{\overline{\mathbb{S}}(B_1)}(x_1x_1), I_{\overline{\mathbb{R}}(A_2)}(y_1), I_{\overline{\mathbb{R}}(A_2)}(y_2)\},
\end{aligned}$$

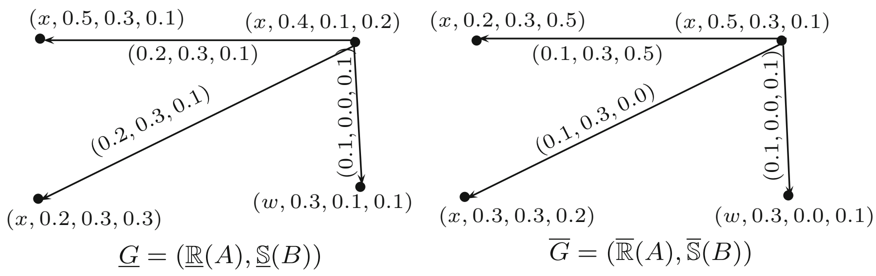


Fig. 8.9 Neutrosophic soft rough graph $G = (\underline{G}, \overline{G})$

$$
\begin{aligned}
I_{\underline{\mathbb{S}}(B_1)\times\underline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \max\{I_{\underline{\mathbb{S}}(B_1)}(x_1x_1), I_{\underline{\mathbb{R}}(A_2)}(y_1), I_{\underline{\mathbb{R}}(A_2)}(y_2)\},\\
F_{\overline{\mathbb{S}}(B_1)\times\overline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \max\{F_{\overline{\mathbb{S}}(B_1)}(x_1x_1), F_{\overline{\mathbb{R}}(A_2)}(y_1), F_{\overline{\mathbb{R}}(A_2)}(y_2)\},\\
F_{\underline{\mathbb{S}}(B_1)\times\underline{\mathbb{S}}(B_2)}\big((x_1, y_1)(x_2, y_2)\big) &= \max\{F_{\underline{\mathbb{S}}(B_1)}(x_1x_1), F_{\underline{\mathbb{R}}(A_2)}(y_1), F_{\underline{\mathbb{R}}(A_2)}(y_2)\}.
\end{aligned}
$$

Definition 8.14 Let $G = (\underline{G}, \overline{G})$ be a neutrosophic soft rough graph. The *complement* of G, denoted by $\acute{\underline{G}} = (\acute{\underline{G}}, \acute{\overline{G}})$, is a neutrosophic soft rough graph, where $\acute{\underline{G}} = (\underline{\mathbb{R}}(\acute{A}), \underline{\mathbb{S}}(\acute{B}))$ and $\acute{\overline{G}} = (\overline{\mathbb{R}}(\acute{A}), \overline{\mathbb{S}}(\acute{B}))$ are neutrosophic graphs such that

(i) $\forall x \in \mathbb{R}A.$

$$
\begin{aligned}
T'_{\overline{\mathbb{R}}(A)}(x) &= T_{\overline{\mathbb{R}}(A)(x)}, \quad I'_{\overline{\mathbb{R}}(A)}(x) = I_{\overline{\mathbb{R}}(A)(x)}, \quad F'_{\overline{\mathbb{R}}(A)}(x) = F_{\overline{\mathbb{R}}(A)(x)},\\
T'_{\underline{\mathbb{R}}(A)}(x) &= T_{\underline{\mathbb{R}}(A)(x)}, \quad I'_{\underline{\mathbb{R}}(A)}(x) = I_{\underline{\mathbb{R}}(A)(x)}, \quad F'_{\underline{\mathbb{R}}(A)}(x) = F_{\underline{\mathbb{R}}(A)(x)}.
\end{aligned}
$$

(ii) $\forall\, v, u \in \mathbb{R}A.$

$$
\begin{aligned}
T'_{\overline{\mathbb{S}}(B)}(xy) &= \min\{T_{\overline{\mathbb{R}}(A)}(x), T_{\overline{\mathbb{R}}(A)}(y)\} - T_{\overline{\mathbb{S}}(B)}(xy),\\
I'_{\overline{\mathbb{S}}(B)}(xy) &= \max\{I_{\overline{\mathbb{R}}(A)}(x), I_{\overline{\mathbb{R}}(A)}(y)\} - I_{\overline{\mathbb{S}}(B)}(xy),\\
F'_{\overline{\mathbb{S}}(B)}(xy) &= \max\{F_{\overline{\mathbb{R}}(A)}(x), F_{\overline{\mathbb{R}}(A)}(y)\} - F_{\overline{\mathbb{S}}(B)}(xy),\\
T'_{\underline{\mathbb{S}}(B)}(xy) &= \min\{T_{\underline{\mathbb{R}}(A)}(x), T_{\underline{\mathbb{R}}(A)}(y)\} - T_{\underline{\mathbb{S}}(B)}(xy),\\
I'_{\underline{\mathbb{S}}(B)}(xy) &= \max\{I_{\underline{\mathbb{R}}(A)}(x), I_{\underline{\mathbb{R}}(A)}(y)\} - I_{\overline{\mathbb{S}}(B)}(xy),\\
F'_{\underline{\mathbb{S}}(B)}(xy) &= \max\{F_{\underline{\mathbb{R}}(A)}(x), F_{\underline{\mathbb{R}}(A)}(y)\} - F_{\underline{\mathbb{S}}(B)}(xy).
\end{aligned}
$$

Example 8.7 Consider a neutrosophic soft rough graphs G as shown in Fig. 8.9. The complement of G is $\acute{G} = (\acute{\underline{G}}, \acute{\overline{G}})$ obtained by using Definition 8.14, where $\acute{\underline{G}} = (\underline{\mathbb{R}}(\acute{A}), \underline{\mathbb{S}}(\acute{B}))$ and $\acute{\overline{G}} = (\overline{\mathbb{R}}(\acute{A}), \overline{\mathbb{S}}(\acute{B}))$ are neutrosophic graphs as shown in Fig. 8.10.

Definition 8.15 A graph G is called self-complement if $G = \acute{G}$, i.e.

(i) $\forall x \in \mathbb{R}A.$

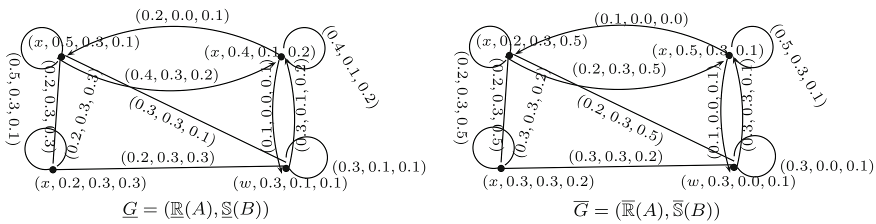


Fig. 8.10 Neutrosophic soft rough graph $\acute{G} = (\acute{\underline{G}}, \acute{\overline{G}})$

Table 8.10 Neutrosophic soft rough set on X

X	$\overline{\mathbb{R}}(A)$	$\underline{\mathbb{R}}(A)$
u	(0.8, 0.5, 0.2)	(0.7, 0.5, 0.2)
v	(0.9, 0.5, 0.1)	(0.7, 0.5, 0.2)
w	(0.7, 0.5, 0.1)	(0.7, 0.5, 0.2)

$$\begin{aligned}
T'_{\overline{\mathbb{R}}(A)}(x) =& T_{\overline{\mathbb{R}}(A)(x)},\ \ I'_{\overline{\mathbb{R}}(A)}(x) = I_{\overline{\mathbb{R}}(A)(x)},\ F'_{\overline{\mathbb{R}}(A)}(x) = F_{\overline{\mathbb{R}}(A)(x)},\\
T'_{\underline{\mathbb{R}}(A)}(x) =& T_{\underline{\mathbb{R}}(A)(x)},\ \ I'_{\underline{\mathbb{R}}(A)}(x) = I_{\underline{\mathbb{R}}(A)(x)},\ \ F'_{\underline{\mathbb{R}}(A)}(x) = F_{\underline{\mathbb{R}}(A)(x)}.
\end{aligned}$$

(ii) $\forall\, x, y \in \mathbb{R}A$.

$$\begin{aligned}
T'_{\overline{\mathbb{S}}(B)}(xy) =& T_{\overline{\mathbb{S}}(B)}(xy),\ \ I'_{\overline{\mathbb{S}}(B)}(xy) = I_{\overline{\mathbb{S}}(B)}(xy),\ \ F'_{\overline{\mathbb{S}}(B)}(xy) = F_{\overline{\mathbb{S}}(B)}(xy),\\
T'_{\underline{\mathbb{S}}(B)}(xy) =& T_{\underline{\mathbb{S}}(B)}(xy),\ \ I'_{\underline{\mathbb{S}}(B)}(xy) = I_{\underline{\mathbb{S}}(B)}(xy),\ \ F'_{\underline{\mathbb{S}}(B)}(xy) = F_{\underline{\mathbb{S}}(B)}(xy).
\end{aligned}$$

Definition 8.16 A neutrosophic soft rough graph G is called *strong neutrosophic soft rough graph* if $\forall xy \in \mathbb{S}B$,

$$\begin{aligned}
T_{\overline{\mathbb{S}}(B)}(xy) &= \min\{T_{\overline{\mathbb{R}}(A)}(x), T_{\overline{\mathbb{R}}(A)}(y)\},\\
I_{\overline{\mathbb{S}}(B)}(xy) &= \max\{I_{\overline{\mathbb{R}}(A)}(x), I_{\overline{\mathbb{R}}(A)}(y)\}),\\
F_{\overline{\mathbb{S}}(B)}(xy) &= \max\{F_{\overline{\mathbb{R}}(A)}(x), F_{\overline{\mathbb{R}}(A)}(y)\},\\
T_{\underline{\mathbb{S}}(B)}(xy) &= \min\{T_{\underline{\mathbb{R}}(A)}(x), T_{\underline{\mathbb{R}}(A)}(y)\},\\
I_{\underline{\mathbb{S}}(B)}(xy) &= \max\{I_{\underline{\mathbb{R}}(A)}(x), I_{\underline{\mathbb{R}}(A)}(y)\},\\
F_{\underline{\mathbb{S}}(B)}(xy) &= \max\{F_{\underline{\mathbb{R}}(A)}(x), F_{\underline{\mathbb{R}}(A)}(y)\}.
\end{aligned}$$

Example 8.8 Consider a graph G such that $X = \{u, v, w\}$ and $E = \{uv, vw, wu\}$. Let $\mathbb{R}A$ be a neutrosophic soft rough set of X, and let $\mathbb{S}B$ be a neutrosophic soft rough set of E defined in Tables 8.10 and 8.11, respectively.

Hence, $G = (\mathbb{R}A, \mathbb{S}B)$ is a strong neutrosophic soft rough graph as shown in Fig. 8.11.

Definition 8.17 A neutrosophic soft rough graph G is called *complete neutrosophic soft rough graph* if $\forall\, x, y \in X$,

Table 8.11 Neutrosophic soft rough set on E

E	$\overline{\mathbb{S}}(B)$	$\underline{\mathbb{S}}(B)$
uv	(0.8, 0.5, 0.2)	(0.7, 0.5, 0.2)
vw	(0.7, 0.5, 0.1)	(0.7, 0.5, 0.2)
wu	(0.7, 0.5, 0.2)	(0.7, 0.5, 0.2)

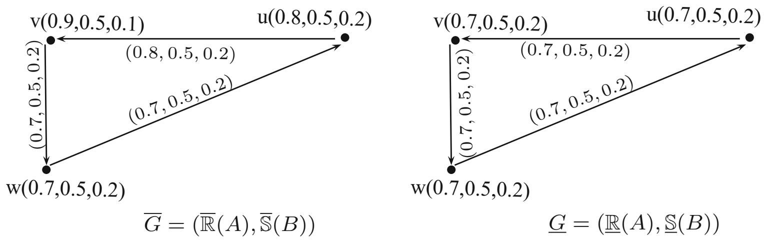


Fig. 8.11 Strong neutrosophic soft rough graph $G = (\mathbb{R}A, \mathbb{S}B)$

$$\begin{aligned}
T_{\overline{\mathbb{S}}(B)}(xy) &= \min\{T_{\overline{\mathbb{R}}(A)}(x), T_{\overline{\mathbb{R}}(A)}(y)\},\\
I_{\overline{\mathbb{S}}(B)}(xy) &= \max\{I_{\overline{\mathbb{R}}(A)}(x), I_{\overline{\mathbb{R}}(A)}(y)\},\\
F_{\overline{\mathbb{S}}(B)}(xy) &= \max\{F_{\overline{\mathbb{R}}(A)}(x), F_{\overline{\mathbb{R}}(A)}(y)\},\\
T_{\underline{\mathbb{S}}(B)}(xy) &= \min\{T_{\underline{\mathbb{R}}(A)}(x), T_{\underline{\mathbb{R}}(A)}(y)\},\\
I_{\underline{\mathbb{S}}(B)}(xy) &= \max\{I_{\underline{\mathbb{R}}(A)}(x), I_{\underline{\mathbb{R}}(A)}(y)\},\\
F_{\underline{\mathbb{S}}(B)}(xy) &= \max\{F_{\underline{\mathbb{R}}(A)}(x), F_{\underline{\mathbb{R}}(A)}(y)\}.
\end{aligned}$$

Remark 8.2 Every complete neutrosophic soft rough graph is a strong neutrosophic soft rough graph. But the converse is not true.

Definition 8.18 A neutrosophic soft rough graph G is *isolated* if $\forall x, y \in X$.

$$T_{\underline{\mathbb{S}}(B)}(xy) = 0,\ I_{\underline{\mathbb{S}}(B)}(xy) = 0,\ F_{\underline{\mathbb{S}}(B)}(xy) = 0,\ T_{\overline{\mathbb{S}}(B)}(xy) = 0,\ I_{\overline{\mathbb{S}}(B)}(xy) = 0,\ F_{\overline{\mathbb{S}}(B)}(xy) = 0,$$

Theorem 8.1 *The rejection of two neutrosophic soft rough graphs is a neutrosophic soft rough graph.*

Proof Let $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ be two neutrosophic soft rough graphs. Let $G = G_1|G_2 = (\underline{G}_1|\underline{G}_2, \overline{G}_1|\overline{G}_2)$ be the rejection of G_1 and G_2, where $\underline{G}_1|\underline{G}_2 = (\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2), \underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2))$ and $\overline{G}_1|\overline{G}_2 = (\overline{\mathbb{R}}(A_1)|\overline{\mathbb{R}}(A_2), \overline{\mathbb{S}}(B_1)|\overline{\mathbb{S}}(B_2))$. We claim that $G = G_1|G_2$ is an neutrosophic soft rough graph. It is enough to show that $\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)$ and $\overline{\mathbb{S}}(B_1)|\overline{\mathbb{S}}(B_2)$ are neutrosophic relations on $\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)$ and $\overline{\mathbb{R}}(A_1)|\overline{\mathbb{R}}(A_2)$, respectively. First, we show that $\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)$ is a neutrosophic relation on $\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)$.
If $x \in \underline{\mathbb{R}}(A_1)$, $y_1y_2 \notin \underline{\mathbb{S}}(B_2)$, then

$$\begin{aligned}
T_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}((x,y_1)(x,y_2)) =& (T_{\underline{\mathbb{R}}(A_1)}(x) \wedge (T_{\underline{\mathbb{R}}(A_2)}(y_2) \wedge T_{\underline{\mathbb{R}}(A_2)}(y_2)))\\
=& (T_{\underline{\mathbb{R}}(A_1)}(x) \wedge T_{\underline{\mathbb{R}}(A_2)}(y_2)) \wedge (T_{\underline{\mathbb{R}}(A_1)}(x) \wedge T_{\underline{\mathbb{R}}(A_2)}(y_2))\\
=& T_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x,y_1) \wedge T_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x,y_2)\\
T_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}((x,y_1)(x,y_2)) =& T_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x,y_1) \wedge T_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x,y_2)\\
\text{Similarly, } I_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}((x,y_1)(x,y_2)) =& I_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x,y_1) \vee I_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x,y_2)\\
F_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}((x,y_1)(x,y_2)) =& F_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x,y_1) \vee F_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x,y_2)
\end{aligned}$$

If $x_1x_2 \notin \underline{\mathbb{S}}(B_1)$, $y \in \underline{\mathbb{R}}(A_2)$, then

$$\begin{aligned}
T_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}((x_1,y)(x_2,y)) =& ((T_{\underline{\mathbb{R}}(A_1)}(x_1) \wedge T_{\underline{\mathbb{R}}(A_1)}(x_2)) \wedge T_{\underline{\mathbb{R}}(A_2)}(y))\\
=& ((T_{\underline{\mathbb{R}}(A_1)}(x_1) \wedge T_{\underline{\mathbb{R}}(A_2)}(y)) \wedge (T_{\underline{\mathbb{R}}(A_1)}(x_2) \wedge T_{\underline{\mathbb{R}}(A_2)}(y)))\\
=& T_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x_1,y) \wedge T_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x_2,y)\\
T_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}((x_1,y)(x_2,y)) =& T_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x_1,y) \wedge T_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x_2,y)\\
\text{Similarly, } I_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}((x_1,y)(x_2,y)) =& I_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x_1,y) \vee I_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x_2,y)\\
F_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}((x_1,y)(x_2,y)) =& F_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x_1,y) \vee F_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x_2,y)
\end{aligned}$$

If $x_1x_2 \notin \underline{\mathbb{S}}(B_1)$, $y_1, y_2 \notin \underline{\mathbb{S}}(B_2)$, then

$$\begin{aligned}
T_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}((x_1,y_1)(x_2,y_2)) =& ((T_{\underline{\mathbb{R}}(A_1)}(x_1) \wedge T_{\underline{\mathbb{R}}(A_1)}(x_2)) \wedge (T_{\underline{\mathbb{R}}(A_2)}(y_1) \wedge T_{\underline{\mathbb{R}}(A_2)}(y_2)))\\
=& (T_{\underline{\mathbb{R}}(A_1)}(x_1) \wedge T_{\underline{\mathbb{R}}(A_2)}(y_1)) \wedge (T_{\underline{\mathbb{R}}(A_1)}(x_2) \wedge T_{\underline{\mathbb{R}}(A_2)}(y_2))\\
=& T_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x_1,y_1) \wedge T_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x_2,y_2)\\
T_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}((x_1,y_1)(x_2,y_2)) =& T_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x_1,y_1) \wedge T_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x_2,y_2)\\
\text{Similarly, } I_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}((x_1,y_1)(x_2,y_2)) =& I_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x_1,y_1) \vee I_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(= y_1,y_2)\\
F_{\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)}((x_1,y_1)(x_2,y_2)) =& F_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x_1,y_1) \vee F_{\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)}(x_2,y_2)
\end{aligned}$$

Thus, $\underline{\mathbb{S}}(B_1)|\underline{\mathbb{S}}(B_2)$ is a neutrosophic relation on $\underline{\mathbb{R}}(A_1)|\underline{\mathbb{R}}(A_2)$. Similarly, we can show that $\overline{\mathbb{S}}(B_1)|\overline{\mathbb{S}}(B_2)$ is a neutrosophic relation on $\overline{\mathbb{R}}(A_1)|\overline{\mathbb{R}}(A_2)$. Hence, G is a neutrosophic soft rough graph.

Theorem 8.2 *The Cartesian product of two neutrosophic soft rough graphs is a neutrosophic soft rough graph.*

Proof Let $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ be two neutrosophic soft rough graphs. Let $G = G_1 \ltimes G_2 = (\underline{G}_1 \ltimes \underline{G}_2, \overline{G}_1 \ltimes \overline{G}_2)$ be the Cartesian product of G_1 and G_2, where $\underline{G}_1 \ltimes \underline{G}_2 = (\underline{\mathbb{R}}(A_1) \ltimes \underline{\mathbb{R}}(A_2), \underline{\mathbb{S}}(B_1) \ltimes \underline{\mathbb{S}}(B_2))$ and $\overline{G}_1 \ltimes \overline{G}_2 = (\overline{\mathbb{R}}(A_1) \ltimes \overline{\mathbb{R}}(A_2), \overline{\mathbb{S}}(B_1) \ltimes \underline{\mathbb{S}}(B_2))$. We claim that $G = G_1 \ltimes G_2$ is a neutrosophic soft rough graph. It is enough to show that $\underline{\mathbb{S}}(B_1) \ltimes \underline{\mathbb{S}}(B_2)$ and $\overline{\mathbb{S}}(B_1) \ltimes \overline{\mathbb{S}}(B_2)$ are neutrosophic relations on $\underline{\mathbb{R}}(A_1) \ltimes \underline{\mathbb{R}}(A_2)$ and $\overline{\mathbb{R}}(A_1) \ltimes \overline{\mathbb{R}}(A_2)$, respectively. We have to show that $\underline{\mathbb{S}}(B_1) \ltimes \underline{\mathbb{S}}(B_2)$ is a neutrosophic relation on $\underline{\mathbb{R}}(A_1) \ltimes \underline{\mathbb{R}}(A_2)$.
If $x \in \underline{\mathbb{R}}(A_1)$, $y_1y_2 \in \underline{\mathbb{S}}(B_2)$, then

$$
\begin{aligned}
T_{\underline{\mathbb{S}}(B_1)\ltimes\underline{\mathbb{S}}(B_2)}((x,y_1)(x,y_2)) &= T_{\underline{\mathbb{R}}(A_1))}(x)\wedge T_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\\
&\leq T_{\underline{\mathbb{R}}(A_1))}(x)\wedge (T_{\underline{\mathbb{R}}(A_2)}(y_1)\wedge T_{\underline{\mathbb{R}}(A_2)}(y_2))\\
&= (T_{\underline{\mathbb{R}}(A_1))}(x)\wedge T_{\underline{\mathbb{R}}(A_2)}(y_1))\wedge (T_{\underline{\mathbb{R}}(A_1))}(x)\wedge T_{\underline{\mathbb{R}}(A_2)}(y_2))\\
&= T_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x,y_1)\wedge T_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x,y_2)\\
T_{\underline{\mathbb{S}}(B_1)\ltimes\underline{\mathbb{S}}(B_2)}((x,y_1)(x,y_2)) &\leq T_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x,y_1)\wedge T_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x,y_2)\\
\text{Similarly, } I_{\underline{\mathbb{S}}(B_1)\ltimes\underline{\mathbb{S}}(B_2)}((x,y_1)(x,y_2)) &\leq I_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x,y_1)\vee I_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x,y_2)\\
F_{\underline{\mathbb{S}}(B_1)\ltimes\underline{\mathbb{S}}(B_2)}((x,y_1)(x,y_2)) &\leq F_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x,y_1)\vee F_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x,y_2)
\end{aligned}
$$

If $x_1x_2 \in \underline{\mathbb{S}}(B_1)$, $z \in \underline{\mathbb{R}}(A_2)$, then

$$
\begin{aligned}
T_{\underline{\mathbb{S}}(B_1)\ltimes\underline{\mathbb{S}}(B_2)}((x_1,z)(x_2,z)) &= T_{\underline{\mathbb{S}}(B_1))}(x_1x_2)\wedge T_{\underline{\mathbb{R}}(A_2)}(z)\\
&\leq (T_{\underline{\mathbb{R}}(A_1))(x_1)\wedge\underline{\mathbb{R}}(A_1))}(x_2))\wedge T_{\underline{\mathbb{R}}(A_2)}(z)\\
&= T_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x_1,z)\wedge T_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x_2,z)\\
T_{\underline{\mathbb{S}}(B_1)\ltimes\underline{\mathbb{S}}(B_2)}((x_1,z)(x_2,z)) &\leq T_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x_1,z)\wedge T_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x_2,z)\\
\text{Similarly, } I_{\underline{\mathbb{S}}(B_1)\ltimes\underline{\mathbb{S}}(B_2)}((x_1,z)(x_2,z)) &\leq I_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x_1,z)\vee I_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x_2,z)\\
F_{\underline{\mathbb{S}}(B_1)\ltimes\underline{\mathbb{S}}(B_2)}((x_1,z)(x_2,z)) &\leq F_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x_1,z)\vee F_{\underline{\mathbb{R}}(A_1)\ltimes\underline{\mathbb{R}}(A_2)}(x_2,z)
\end{aligned}
$$

Therefore, $\underline{\mathbb{S}}(B_1) \ltimes \underline{\mathbb{S}}(B_2)$ is a neutrosophic relation on $\underline{\mathbb{R}}(A_1) \ltimes \underline{\mathbb{R}}(A_2)$. Similarly, $\overline{\mathbb{S}}(B_1) \ltimes \overline{\mathbb{S}}(B_2)$ is a neutrosophic relation on $\overline{\mathbb{R}}(A_1) \ltimes \overline{\mathbb{R}}(A_2)$. Hence, G is a neutrosophic rough graph.

Theorem 8.3 *The cross product of two neutrosophic soft rough graphs is a neutrosophic soft rough graph.*

Proof Let $G_1 = (\underline{G}_1, \overline{G}_1)$ and $G_2 = (\underline{G}_2, \overline{G}_2)$ be two neutrosophic soft rough graphs. Let $G = G_1 \circledcirc G_2 = (\underline{G}_1 \circledcirc \underline{G}_2, \overline{G}_1 \circledcirc \overline{G}_2)$ be the cross product of G_1 and G_2, where $\underline{G}_1 \circledcirc \underline{G}_2 = (\underline{\mathbb{R}}(A_1) \circledcirc \underline{\mathbb{R}}(A_2), \underline{\mathbb{S}}(B_1) \circledcirc \underline{\mathbb{S}}(B_2))$ and $\overline{G}_1 \circledcirc \overline{G}_2 = (\overline{\mathbb{R}}(A_1) \circledcirc \overline{\mathbb{R}}(A_2), \overline{\mathbb{S}}(B_1) \circledcirc \underline{\mathbb{S}}(B_2))$. We claim that $G = G_1 \circledcirc G_2$ is a neutrosophic soft rough graph. It is enough to show that $\underline{\mathbb{S}}(B_1) \circledcirc \underline{\mathbb{S}}(B_2)$ and $\overline{\mathbb{S}}(B_1) \circledcirc \overline{\mathbb{S}}(B_2)$ are neutrosophic relations on $\underline{\mathbb{R}}(A_1) \circledcirc \underline{\mathbb{R}}(A_2)$ and $\overline{\mathbb{R}}(A_1) \circledcirc \overline{\mathbb{R}}(A_2)$, respectively. First, we show that $\underline{\mathbb{S}}(B_1) \circledcirc \underline{\mathbb{S}}(B_2)$ is a neutrosophic relation on $\underline{\mathbb{R}}(A_1) \circledcirc \underline{\mathbb{R}}(A_2)$.
If $x_1x_2 \in \underline{\mathbb{S}}(B_1)$, $y_1y_2 \in \underline{\mathbb{S}}(B_2)$, then

$$
\begin{aligned}
T_{\underline{\mathbb{S}}(B_1)\circledcirc\underline{\mathbb{S}}(B_2)}((x_1,y_1)(x_2,y_2)) &= T_{\underline{\mathbb{S}}(B_1))}(x_1x_2)\wedge T_{\underline{\mathbb{S}}(B_2)}(y_1y_2)\\
&\leq (T_{\underline{\mathbb{R}}(A_1))}(x_1)\wedge T_{\underline{\mathbb{R}}(A_1))}(x_2)\wedge (T_{\underline{\mathbb{R}}(A_2)}(y_1)\wedge T_{\underline{\mathbb{R}}(A_2)}(y_2))\\
&= (T_{\underline{\mathbb{R}}(A_1))}(x_1)\wedge T_{\underline{\mathbb{R}}(A_2)}(x_2))\wedge (T_{\underline{\mathbb{R}}(A_1))}(y_1)\wedge T_{\underline{\mathbb{R}}(A_2)}(y_2))\\
&= T_{\underline{\mathbb{R}}(A_1)\circledcirc\underline{\mathbb{R}}(A_2)}(x_1,x_2)\wedge T_{\underline{\mathbb{R}}(A_1)\circledcirc\underline{\mathbb{R}}(A_2)}(y_1,y_2)\\
T_{\underline{\mathbb{S}}(B_1)\circledcirc\underline{\mathbb{S}}(B_2)}((x_1,x_2)(y_1,y_2)) &\leq T_{\underline{\mathbb{R}}(A_1)\circledcirc\underline{\mathbb{R}}(A_2)}(x_1,y_1)\wedge T_{\underline{\mathbb{R}}(A_1)\circledcirc\underline{\mathbb{R}}(A_2)}(x_2,y_2)\\
\text{Similarly, } I_{\underline{\mathbb{S}}(B_1)\circledcirc\underline{\mathbb{S}}(B_2)}((x_1,y_1)(x_2,y_2)) &\leq I_{\underline{\mathbb{R}}(A_1)\circledcirc\underline{\mathbb{R}}(A_2)}(x_1,y_1)\vee I_{\underline{\mathbb{R}}(A_1)\circledcirc\underline{\mathbb{R}}(A_2)}(x_2,y_2)\\
F_{\underline{\mathbb{S}}(B_1)\circledcirc\underline{\mathbb{S}}(B_2)}((x_1,y_1)(x_2,y_2)) &\leq F_{\underline{\mathbb{R}}(A_1)\circledcirc\underline{\mathbb{R}}(A_2)}(x_1,y_1)\vee F_{\underline{\mathbb{R}}(A_1)\circledcirc\underline{\mathbb{R}}(A_2)}(x_2,y_2)
\end{aligned}
$$

Thus, $\underline{\mathbb{S}}(B_1) \circledcirc \underline{\mathbb{S}}(B_2)$ is a neutrosophic relation on $\underline{\mathbb{R}}(A_1) \circledcirc \underline{\mathbb{R}}(A_2)$. Similarly, we can show that $\overline{\mathbb{S}}(B_1) \circledcirc \overline{\mathbb{S}}(B_2)$ is a neutrosophic relation on $\overline{\mathbb{R}}(A_1) \circledcirc \overline{\mathbb{R}}(A_2)$. Hence, G is a neutrosophic soft rough graph.

8.3 Application of Neutrosophic Soft Rough Graphs

In this section, we apply the concept of neutrosophic soft rough sets to a decision-making problem. In recent times, the object recognition problem has gained considerable importance. The object recognition problem can be considered as a decision-making problem, in which final identification of object is founded on given set of information. A detailed description of the algorithm for the selection of most suitable object based on available set of alternatives is given, and purposed decision-making method can be used to calculate lower and upper approximation operators to progress deep concerns of the problem. The presented algorithms can be applied to avoid lengthy calculations when dealing with a large number of objects. This method can be applied in various domains for multicriteria selection of objects.

Selection of Most Suitable Generic Version of Brand Name Medicine

In pharmaceutical industry, different pharmaceutical companies develop, produce and discover pharmaceutical medicines (drugs) for use as medication. These pharmaceutical companies deals with "brand name medicine" and "generic medicine". Brand name medicine and generic medicine are bioequivalent, generic medicine rate and element of absorption. Brand name medicine and generic medicine have the same active ingredients, and the inactive ingredients may differ. The most important difference is cost. Generic medicine is less expensive as compared to brand name comparators. Usually generic drug manufacturers have competition to produce cost less products. The product may possibly be slightly dissimilar in colour, shape or markings. The major difference is cost. We consider a brand name drug "u = Loratadine" used for seasonal allergies medication. Consider

$$\begin{aligned} X = \{&x_1 = \text{Triamcinolone}, x_2 = \text{Cetirizine/Pseudoephedrine},\\ &x_3 = \text{Pseudoephedrine}, x_4 = \text{loratadine/pseudoephedrine},\\ &x_5 = \text{Fluticasone}\} \end{aligned}$$

is a set of generic versions of "Loratadine". We want to select the most suitable generic version of Loratadine on the basis of parameters e_1 = highly soluble, e_2 = highly permeable, e_3 = rapidly dissolving. $M = \{e_1, e_2, e_3\}$ be a set of paraments. Let $\mathbb{R}$ be a neutrosophic soft relation from X to parameter set M, describes truth-membership, indeterminacy-membership and false-membership degrees of generic version medicines corresponding to the parameters as shown in Table 8.12.

Suppose $A = \{(e_1, 0.2, 0.4, 0.5), (e_2, 0.5, 0.6, 0.4), (e_3, 0.7, 0.5, 0.4)\}$ is most favourable object which is a neutrosophic set on the parameter set M under con-

Table 8.12 Neutrosophic soft set $\mathbb{R} = (F, M)$

$\mathbb{R}$	x_1	x_2	x_3	x_4	x_5
e_1	(0.4, 0.5, 0.6)	(0.5, 0.3, 0.6)	(0.7, 0.2, 0.3)	(0.5, 0.7, 0.5)	(0.6, 0.5, 0.4)
e_2	(0.7, 0.3, 0.2)	(0.3, 0.4, 0.3)	(0.6, 0.5, 0.4)	(0.8, 0.4, 0.6)	(0.7, 0.8, 0.5)
e_3	(0.6, 0.3, 0.4)	(0.7, 0.2, 0.3)	(0.7, 0.2, 0.4)	(0.8, 0.7, 0.6)	(0.7, 0.3, 0.5)

Table 8.13 Neutrosophic soft relation S

S	x_1x_2	x_1x_3	x_4x_1	x_2x_3	x_5x_3	x_2x_4	x_2x_5
e_1e_2	(0.3, 0.4, 0.2)	(0.4, 0.4, 0.5)	(0.4, 0.4, 0.5)	(0.6, 0.3, 0.4)	(0.4, 0.2, 0.2)	(0.4, 0.4, 0.2)	(0.4, 0.3, 0.4)
e_2e_3	(0.5, 0.4, 0.1)	(0.4, 0.3, 0.2)	(0.4, 0.3, 0.2)	(0.3, 0.3, 0.2)	(0.6, 0.2, 0.4)	(0.3, 0.2, 0.1)	(0.3, 0.3, 0.2)
e_1e_3	(0.4, 0.4, 0.1)	(0.4, 0.2, 0.2)	(0.4, 0.2, 0.2)	(0.5, 0.3, 0.3)	(0.4, 0.2, 0.3)	(0.4, 0.3, 0.1)	(0.5, 0.3, 0.2)

sideration. Then $(\underline{\mathbb{R}}(A), \overline{\mathbb{R}}(A))$ is a neutrosophic soft rough set in neutrosophic soft approximation space $(X, M, \mathbb{R})$, where

$$\overline{\mathbb{R}}(A) = \{(x_1, 0.6, 0.5, 0.4), (x_2, 0.7, 0.4, 0.4), (x_3, 0.7, 0.4, 0.4), (x_4, 0.7, 0.6, 0.5), (x_5, 0.7, 0.5, 0.5)\},$$
$$\underline{\mathbb{R}}(A) = \{(x_1, 0.5, 0.6, 0.4), (x_2, 0.5, 0.6, 0.5), (x_3, 0.3, 0.3, 0.5), (x_4, 0.5, 0.6, 0.5), (x_5, 0.4, 0.5, 0.5)\}.$$

Let $E = \{x_1x_2, x_1x_3, x_4x_1, x_2x_3, x_5x_3, x_2x_4, x_2x_5\} \subseteq \acute{X}$ and $L = \{e_1e_3, e_2e_1, e_3e_2\} \subseteq \acute{M}$.
Then a neutrosophic soft relation S on E (from L to E) can be defined in Table 8.13 as follows:

Let $B = \{(e_1e_2, 0.2, 0.4, 0.5), (e_2e_3, 0.5, 0.4, 0.4), (e_1e_3, 0.5, 0.2, 0.5)\}$ be a neutrosophic set on L which describes some relationship between the parameters under consideration, then $SB = (\underline{S}(B), \overline{S}(B))$ is a neutrosophic soft rough relation, where

$$\overline{\mathbb{S}}(B) = \{(x_1x_2, 0.5, 0.4, 0.4), (x_1x_3, 0.4, 0.2, 0.4), (x_4x_1, 0.4, 0.2, 0.4), (x_2x_3, 0.5, 0.3, 0.4), (x_5x_3, 0.5, 0.2, 0.4), (x_2x_4, 0.4, 0.3, 0.4), (x_2x_5, 0.5, 0.3, 0.4)\},$$
$$\underline{\mathbb{S}}(B) = \{(x_1x_2, 0.2, 0.4, 0.4)(x_1x_3, 0.5, 0.4, 0.4), (x_4x_1, 0.5, 0.4, 0.4), (x_2x_3, 0.4, 0.4, 0.5), (x_5x_3, 0.2, 0.4, 0.4), (x_2x_4, 0.2, 0.4, 0.4), (x_2x_5, 0.4, 0.4, 0.5)\}.$$

Thus, $G = (\underline{G}, \overline{G})$ is a neutrosophic soft rough graph as shown in Fig. 8.12.
The sum of two neutrosophic numbers is defined as follows.

Definition 8.19 Let C and D be two single-valued neutrosophic numbers, and the sum of two single-valued neutrosophic numbers is defined as follows:

$$C \oplus D =< T_C + T_D - T_C \times T_D, I_C \times I_D, F_C \times F_D > . \tag{8.1}$$

The sum of upper neutrosophic soft rough set $\overline{\mathbb{R}}A$ and the lower neutrosophic soft rough set $\underline{\mathbb{R}}A$ and sum of lower neutrosophic soft rough relation $\underline{\mathbb{S}}B$ and the upper

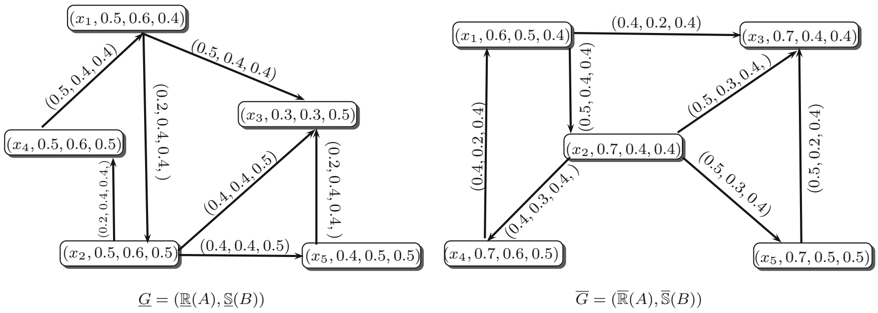


Fig. 8.12 Neutrosophic soft rough graph $G = (\underline{G}, \overline{G})$

neutrosophic soft rough relation $\overline{\mathbb{S}}B$ are neutrosophic sets $\overline{\mathbb{R}}A \oplus \underline{\mathbb{R}}A$ and $\overline{\mathbb{S}}B \oplus \underline{\mathbb{S}}B$, respectively, defined by

$$\alpha = \overline{\mathbb{R}}A \oplus \underline{\mathbb{R}}A = \{(x_1, 0.8, 0.3, 0.16), (x_2, 0.85, 0.24, 0.2), (x_3, 0.79, 0.2, 0.2), (x_4, 0.85, 0.36, 0.25), (x_5, 0.82, 0.25, 0.25)\},$$

$$\beta = \overline{\mathbb{S}}B \oplus \underline{\mathbb{S}}B = \{(x_1x_2, 0.6, 0.16, 0.16), (x_1x_3, 0.7, 0.8, 0.16), (x_4x_1, 0.7, 0.8, 0.16), (x_2x_3, 0.7, 0.12, 0.2), (x_5x_3, 0.6, 0.08, 0.16), (x_2x_4, 0.52, 0.12, 0.16), (x_2x_5, 0.7, 0.12, 0.2), \}.$$

The score function $\gamma(x_k)$ defines for each generic version medicine $x_i \in X$,

$$\gamma(x_i) = \sum_{x_i x_j \in E} \frac{T_\alpha(x_j) + I_\alpha(x_j) - F_\alpha(x_j)}{3 - (T_\beta(x_i x_j) + I_\beta(x_i x_j) - F_\beta(x_i x_j))} \tag{8.2}$$

and x_k with the larger score value $x_k = \max_i \gamma(x_i)$ is the most suitable generic version medicine. By calculations, we have

$$\gamma(x_1) = 0.88, \gamma(x_2) = 0.69, \ \gamma(x_3) = 0.26 \ \gamma(x_4) = 0.57, \ \text{and} \ \gamma(x_5) = 0.33. \tag{8.3}$$

Here, x_1 is the optimal decision, and the most suitable generic version of "Loratadine" is "Triamcinolone". We have used software MATLAB for calculating the required results in the application. The algorithm is given in Algorithm 8.3.1.

Algorithm 8.3.1 *Algorithm for selection of most suitable objects*

1. Input the number of elements in vertex set $X = \{x_1, x_2, \ldots, x_n\}$.
2. Input the number of elements in parameter set $M = \{e_1, e_2, \ldots, e_m\}$.
3. Input a neutrosophic soft relation $\mathbb{R}$ from X to M.

4. Input a neutrosophic set A on M.
5. Compute neutrosophic soft rough vertex set $\mathbb{R}A = (\underline{\mathbb{R}}(A), \overline{\mathbb{R}}(A))$.
6. Input the number of elements in edge set $E = \{x_1x_1, x_1x_2, \ldots, x_kx_1\}$.
7. Input the number of elements in parameter set $\acute{M} = \{e_1e_1, e_1e_2, \ldots, e_le_1\}$.
8. Input a neutrosophic soft relation $\mathbb{S}$ from $\acute{X}$ to $\acute{M}$.
9. Input a neutrosophic set B on $\acute{M}$.
10. Compute neutrosophic soft rough edge set $\mathbb{S}B = (\underline{\mathbb{S}}(B), \overline{\mathbb{S}}(B))$.
11. Compute neutrosophic set $\alpha = (T_\alpha(x_i), I_\alpha(x_i), F_\alpha(x_i))$, where

$$\begin{aligned}
T_\alpha(x_i) &= T_{\overline{\mathbb{R}}(A)}(x_i) + T_{\underline{\mathbb{R}}(A)}(x_i) - T_{\overline{\mathbb{R}}(A)}(x_i) \times T_{\underline{\mathbb{R}}(A)}(x_i),\\
I_\alpha(x_i) &= T_{\overline{\mathbb{R}}(A)}(x_i) \times T_{\underline{\mathbb{R}}(A)}(x_i),\\
F_\alpha(x_i) &= F_{\overline{\mathbb{R}}(A)}(x_i) \times F_{\underline{\mathbb{R}}(A)}(x_i);
\end{aligned}$$

12. Compute neutrosophic set $\beta = (T_\beta(x_ix_i), I_\beta(x_ix_j), F_\beta(x_ix_j))$, where

$$\begin{aligned}
T_\beta(x_ix_j) &= T_{\overline{\mathbb{S}}(B)}(x_ix_j) + T_{\underline{\mathbb{S}}(B)}(x_ix_j) - T_{\overline{\mathbb{S}}(B)}(x_ix_j) \times T_{\underline{\mathbb{S}}(B)}(x_ix_j),\\
I_\beta(x_ix_j) &= T_{\overline{\mathbb{S}}(B)}(x_ix_j) \times T_{\underline{\mathbb{S}}(B)}(x_ix_j),\\
F_\beta(x_ix_j) &= F_{\overline{\mathbb{S}}(B)}(x_ix_j) \times F_{\underline{\mathbb{S}}(B)}(x_ix_j);
\end{aligned}$$

13. Calculate the score values of each object x_i, and score function is defined as follows:

$$\gamma(x_i) = \sum_{x_ix_j \in E} \frac{T_\alpha(x_j) + I_\alpha(x_j) - F_\alpha(x_j)}{3 - (T_\beta(x_ix_j) + I_\beta(x_ix_j) - F_\beta(x_ix_j))};$$

14. The decision is x_i if $\gamma_i = \max_{i=1}^{n} \gamma_i$.
15. If i has more than one value, then any one of x_i may be chosen.

References

1. Abdul-Jabbar, N., Naoom, J. H., & Ouda, E. H. (2009). Fuzzy dual graph. *Journal of Al-Nahrain University*, *12*(4), 168–171.
2. Akram, M. (2011). Bipolar fuzzy graphs. *Information Sciences*, *181*, 5548–5564.
3. Akram, M. (2016). Single-valued neutrosophic planar graphs. *International Journal of Algebra and Statistics*, *5*(2), 157–167.
4. Akram, M., & Davvaz, B. (2012). Strong intuitionistic fuzzy graphs. *FILOMAT*, *26*(1), 177–196.
5. Akram, M., & Dudek, W. A. (2011). Interval-valued fuzzy graphs. *Computers and Mathematics with Applications*, *61*, 289–299.
6. Akram, M., & Dudek, W. A. (2012). Regular bipolar fuzzy graphs. *Neural Computing and Applications*, *21*, 197–205.
7. Akram, M., & Luqman, A. (2017). Certain network models using single-valued neutrosophic directed hypergraphs. *Journal of Intelligent and Fuzzy Systems*, *33*(1), 575–588.
8. Akram, M., & Luqman, A. (2017). Bipolar neutrosophic hypergraphs with applications. *Journal of Intelligent and Fuzzy Systems*, *33*(3), 1699–1713.
9. Akram, M., & Luqman, A. (2017). Intuitionistic single-valued neutrosophic hypergraphs. *OPSEARCH*, *54*(4), 799–815.
10. Akram, M., & Luqman, A. (2017). A new decision-making method based on bipolar neutrosophic directed hypergraphs. *Journal of Applied Mathematics and Computing*. https://doi.org/10.1007/s12190-017-1121-4.
11. Akram, M., & Nasir, M. (2017). Concepts of interval-valued neutrosophic graphs. *International Journal of Algebra and Statistics*, *6*(1–2), 22–41.
12. Akram, M., & Nasir, M. (2017). Interval-valued neutrosophic competition graphs. *Annals of Fuzzy Mathematics and Informatics*, *14*(1), 99–120.
13. Akram, M., & Nasir, M. (2018). Certain bipolar neutrosophic competition graphs. *Journal of the Indonesian Mathematical Society*, *24*(1), 1–25.
14. Akram, M. and Nasir, M. (2018). Novel applications of bipolar neutrosophic competition graphs. *Applied Mathematics-A Journal of Chinese Universities*, In press.
15. Akram, M., & Nasir, M. Certain competition graphs based on intuitionistic neutrosophic environment. *Information*, *8*(4), 2017. https://doi.org/10.3390/info227877.
16. Akram, M., Nabeela, I., Sayed, S., & Smarandache, F. (2018). Decision-making approach based on neutrosophic rough information. *Algorithms*, *11*(5), 59. https://doi.org/10.3390/a11050059.

M. Akram, *Single-Valued Neutrosophic Graphs*, Infosys Science Foundation Series, https://doi.org/10.1007/978-981-13-3522-8

17. Akram, M., Hafsa, M. M., Shahzadi, S., & Smarandache, F. (2018). Neutrosophic soft rough graphs with application. *Axioms*, *7*(1), 14. https://doi.org/10.3390/axioms7010014.
18. Akram, M., Shahzadi, S., & Smarandache, F. (2018). Multi-attribute decision-making method based on neutrosophic soft rough information. *Axioms*, *7*(1), 19. https://doi.org/10.3390/axioms7010019.
19. Akram, M., & Sarwar, M. (2017). Novel multiple criteria decision making methods based on bipolar neutrosophic sets and bipolar neutrosophic graphs. *Italian Journal of Pure and Applied Mathematics*, *38*, 301–322.
20. Akram, M., & Shahzadi, G. (2017). Operations on single-valued neutrosophic graphs. *Journal of Uncertain Systems*, *11*(3), 176–196.
21. Akram, M., Shahzadi, G., & Borumand Saeid, A. (2017). An application of single-valued neutrosophic sets in medical diagnosis. *Neutrosophic Sets and Systems*, *18*, 80–88.
22. Akram, M., & Shahzadi, S. (2017). Neutrosophic soft graphs with application. *Journal of Intelligent and Fuzzy Systems*, *32*(1), 841–858.
23. Akram, M., & Shahzadi, S. (2016). Representation of graphs using intuitionistic neutrosophic soft sets. *Journal of Mathematical Analysis*, *7*(6), 31–53.
24. Akram, M. & Shahzadi, S. (2018). Single-valued neutrosophic hypergraphs. *TWMS Journal of Applied and Engineering Mathematics*. https://doi.org/10.26837/jaem.377615
25. Akram, M., & Shum, K. P. (2017). Bipolar neutrosophic planar graphs. *Journal of Mathematical Research with Applications*, *36*(6), 631–648.
26. Akram, M., & Siddique, S. (2017). Neutrosophic competition graphs with applications. *Journal of Intelligent and Fuzzy Systems*, *33*(2), 921–935.
27. Akram, M., Siddique, S., & Davvaz, B. (2017). New concepts in neutrosophic graphs with application. *Journal of Applied Mathematics and Computing*. https://doi.org/10.1007/s12190-017-1106-3
28. Akram, M., Siddique, S., & Shum, K. P. (2018). Certain properties of bipolar neutrosophic graphs. *Southeast Asian Bulletin of Mathematics*, *42*, 463–490.
29. Akram, M., & Siddique, S. (2018). Certain single-valued neutrosophic graphs. *The Journal of Fuzzy Mathematics*, *26*(3), 593–614.
30. Akram, M., & Sitara, M. (2017). Bipolar neutrosophic graph structures. *Journal of the Indonesian Mathematical Society*, *23*(1), 55–76.
31. Akram, M., & Sitara, M. (2017). Representation of graph structure based on I-V neutrosophic sets. *International Journal of Algebra and Statistics*, *6*(1–2), 56–80.
32. Akram, M., & Sitara, M. (2017). Application of intuitionistic neutrosophic graph structures in decision-making. *Annals of Fuzzy Mathematics and Informatics*, *14*(1), 1–27.
33. Akram, M. & Sitara, M. (2017). Novel applications of single-valued neutrosophic graph structures in decision-making. *Journal of Applied Mathematics and Computing*. https://doi.org/10.1007/s12190-017-1084-5
34. Akram, M. & Sitara, M. (2017). Single-valued neutrosophic graph structures. *Applied Mathematics E -Notes*, *17*, 277–296.
35. Akram, M., & Sitara, M. (2018). Interval-valued neutrosophic graph structures. *Punjab University Journal of Mathematics*, *50*(1), 113–137.
36. Akram, M. & Sitara, M. (2017). Certain concepts in intuitionistic neutrosophic graph structures. *Information*, *8*(4), 154. https://doi.org/10.3390/info8040154
37. Akram, M., Sitara, M. & Smarandache, F. (2017). Graph structures in bipolar neutrosophic environment. *Mathematics*, *5*(4), 60. https://doi.org/10.3390/math5040060
38. Akram, M., Sayed, S., & Smarandache, F. (2018). Neutrosophic incidence graphs with application. *Axioms*, *7*(3), 47. https://doi.org/10.3390/axioms7030047
39. Alaoui, A. (1999). On fuzzification of some concepts of graphs. *Fuzzy Sets and Systems*, *101*, 363–389.
40. Alexanderson, G. (2006). Euler and Königsberg-s bridges: A historical view. *Bulletin of the American Mathematical Society*, *43*(4), 567–573.
41. Alshehri, N. O., & Akram, M. (2015). Bipolar fuzzy competition graphs. *Ars Combinatoria*, *121*, 385–402.

42. Alshehri N. O. & Akram, M. (2013). Cayley bipolar fuzzy graphs. *Scientific World Journal*, **2013**, 8. Article 156786
43. Alshehri, N., & Akram, M. (2014). Intuitionistic fuzzy planar graphs. *Discrete Dynamics in Nature and Society*, **2014**, 9. Article ID 397823
44. Al-Hawary, T. (2011). Complete fuzzy graphs. *International Journal of Mathematical Combinatorics*, *4*, 26–34.
45. Anjali, N., & Mathew, S. (2013). Energy of a fuzzy graph. *Annals Fuzzy Maths and Informatics*, *6*, 455–465.
46. Ashraf, S., Naz, S., Rashmanlou, H., & Malik, M. A. (2017). Regularity of graphs in single valued neutrosophic environment. *Journal of Intelligent and Fuzzy Systems*, *33*(1), 529–542.
47. Atanassov, K. T. (1983). Intuitionistic fuzzy sets. In *Proceedings of the VII ITKR Session, Sofia, Bulgaria, 79 June 1983 (Deposed in Central Sci.- Techn. Library of Bulg. Acad. of Sci., 1697/84) (in Bulgaria)*
48. Atanassov, K. T. (2002). *Intuitionistic fuzzy sets: Theory and applications*., Studies in fuzziness and soft computing New York: Physica-Verl.
49. Banerjee, M., & Pal, S. K. (1996). Roughness of a fuzzy set. *Information Sciences*, *93*, 235–246.
50. Bao, Y. L., & Yang, H. L. (2017). On single valued neutrosophic refined rough set model and its application. *Journal of Intelligent and Fuzzy Systems*,*33*(2), 1235–1248. https://doi.org/10.3233/JIFS-17094
51. Basha, S. S., & Kartheek, E. (2015). Laplacian energy of an intuitionistic fuzzy graph. *Indian Journal of Science and Technology*, *8*, 1–7.
52. Berge, C. (1973). *Graphs and hypergraphs*. Amsterdam: North-Holland Publishing Company.
53. Bhattacharya, P. (1987). Some remarks on fuzzy graphs. *Pattern Recognition Letters*, *6*(5), 297–302.
54. Bhowmik, M., & Pal, M. (2009). Intuitionistic neutrosophic set. *Journal of Information and Computing Science*, *4*(2), 142–152.
55. Bhowmik, M., & Pal, M. (2010). Intuitionistic neutrosophic set relations and some of its properties. *Journal of Information and Computing Science*, *5*(3), 183–192.
56. Bhutani, K. R., & Battou, A. (2003). On M-strong fuzzy graphs. *Information Sciences*, *155*, 103–109.
57. Bhutani, K. R., & Rosenfeld, A. (2003). Fuzzy end vertices in fuzzy graphs. *Information Sciences*, *152*, 323–326.
58. Bhutani, K. R., Mordeson, J. N., & Rosenfeld, A. (2004). On degrees of end vertices in fuzzy graphs. *Iranian Journal of Fuzzy Graphs*, *1*(1), 57–64.
59. Biswas, P., Pramanik, S., & Giri, B. C. (2016). TOPSIS method for multi-attribute group decision-making under single-valued neutrosophic environment. *Neural Computing and Applications*, *27*(3), 727–737.
60. Broumi, S., & Smarandache, F. (2013). Intuitionistic neutrosophic soft set. *Journal of Information and Computing Science*, *8*(2), 130–140.
61. Broumi, S., Smarandache, F., & Dhar, M. (2014). Rough neutrosophic sets. *Neutrosophic Sets and Systems*, *3*, 62–67.
62. Broumi, S., Talea, M., Bakali, A., & Smarandache, F. (2016). Single-valued neutrosophic graphs. *Journal of New Theory*, *10*, 86–101.
63. Chen, S. M. (1997) Interval-valued fuzzy hypergraph and fuzzy partition. *IEEE Transactions on Systems, Man, and Cybernetics, (Cybernetics)*, *27*(4), 725–733
64. Chi, P. P., & Liu, P. D. (2013). An extended TOPSIS method for the multiple attribute decision making problems based on interval neutrosophic set. *Neutrosophic Sets and Systems*, *1*(12), 63–70.
65. Craine, W. L. (1994). Characterization of fuzzy interval graphs. *Fuzzy Sets and Systems*, *68*, 181–193.
66. Deli, I., & Broumi, S. (2015). Neutrosophic soft relations and some properties. *Annals of Fuzzy Mathematics and Informatics*, *9*(1), 169–182.

67. Deli, I., Ali, M. & Smarandache, F. (2015). Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. In *International Conference IEEE on Advanced Mechatronic Systems (ICAMechS)* (pp. 249–254)
68. Delgado, M., Verdegay, J. L., & Vila, M. A. (1985). On fuzzy tree definition. *European Journal of Operational Research*, *22*, 243–249.
69. Deo, N. (1990). *Graph theory with applications to engineering and computer science*. USA: Prentice Hall.
70. Deschrijver, G., & Cornelis, C. (2007). Representability in interval-valued fuzzy set theory. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *15*, 345–361.
71. Deschrijver, G., & Kerre, E. E. (2003). On the relationships between some extensions of fuzzy set theory. *Fuzzy Sets and Systems*, *133*, 227–235.
72. Dinesh, T. (2011). A study on graph structures, incidence algebras and their fuzzy analogues [Ph.D. thesis], Kannur University, Kannur, India
73. Dinesh, T., & Ramakrishnan, T. V. (2011). On generalised fuzzy graph structures. *Applied Mathematical Sciences*, **5**(4), 173–180
74. Dubois, D., & Prade, H. (1990). Rough fuzzy sets and fuzzy rough sets. *International Journal of General System*, *17*, 191–209.
75. Dubios, D., Kaci, S. & Prade, H. (2002). Bipolarity in reasoning and decision, an introduction. In *Inf. Pro. Man. Unc. IPMU-04* (pp. 959–966)
76. Dhavaseelan, R., Vikramaprasad, R., & Krishnaraj, V. (2015). Certain types of neutrosophic graphs. *International Journal of Mathematical Sciences and Applications*, *5*(2), 333–339.
77. Fan, Z. P., & Liu, Y. (2010). An approach to solve group decision-making problems with ordinal interval numbers. *IEEE Transactions on Systems, Man, and Cybernetics Part B (Cybernetics)*, *40*(5), 1413–1423.
78. Fathi, S., Elchawalby, H. and Salama, A. A., A neutrosophic graph similarity measures, *Chapter in book- New Trends in Neutrosophic Theory and Applications- Florentin Smarandache and Surpati Pramanik (Editors)*, (2016), 223-230, ISBN 978-1-59973-498-9.
79. Gallo, G., Longo, G., & Pallottino, S. (1993). Directed hypergraphs and applications. *Discrete Applied Mathematics*, *42*, 177–201.
80. Goetschel, R. H., Craine, W. L., & Voxman, W. (1996). Fuzzy transversals of fuzzy hypergraphs. *Fuzzy Sets and Systems*, *84*(3), 235–254.
81. Gani, A. N., & Latha, S. R. (2012). On irregular fuzzy graphs. *Applied Mathematical Sciences*, *6*, 517–523.
82. Gani, A. N., & Malarvizhi, J. (2010). On antipodal fuzzy graph. *Applied Mathematical Sciences*, *4*(43), 2145–2155.
83. Gani, A. N., & Radha, K. (2008). Regular fuzzy graphs. *Journal of Physical Sciences*, *12*, 33–44.
84. Gutman, I., & Zhou, B. (2006). Laplacian energy of a graph. *Linear Algebra and Its Applications*, *414*, 29–37.
85. Harary, F. (1972). *Graph theory* (3rd ed.). Addison Wesley.
86. Hongmei, J., & Lianhua, W. (2009). Interval-valued fuzzy subsemigroups and subgroups sssociated by interval-valued fuzzy graphs. In *2009 WRI Global Congress on Intelligent Systems* (pp. 484–487).
87. Jurkat, W. B., & Ryser, H. J. (1966). Matrix factorizations of determinants and permanents. *Journal of Algebra*, *3*, 1–27.
88. Kandasamy, W. B. V., & Smarandache, F. (2004). *Basic neutrosophic algebraic structures and their application to fuzzy and neutrosophic models*. USA: HEXIS Church Rock.
89. Karaaslan, F. Neutrosophic soft sets with applications in decision making. *International Journal of Information Science and Intelligent System*, *4*(2), 1–20.
90. Karaaslan, F. Correlation coefficients of single-valued neutrosophic refined soft sets and their applications in clustering analysis. *Neural Computing and Applications*, *28*(9), 2781–2793.
91. Karaaslan, F. (2016). Similarity measure between possibility neutrosophic soft sets and its applications. *University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics*, *78*(3), 155–162.

92. Karaaslan, F., & Davvaz, B. (2018). Properties of single-valued neutrosophic graphs. *Journal of Intelligent and Fuzzy Systems*, *34*(1), 57–79.
93. Karunambigal, M. G., Rangasamy, P., Atanassov, K. T., & Palaniappan, N. (2007). An intuitionistic fuzzy graph method for finding the shortest paths in network. In O. Castill, et al. (Eds.), *Theoretical Advances and Applications of Fuzzy Logic, ASC* (vol. 42, pp. 3–10).
94. Kaufmann, A. (1977). *Introduction a la Thiorie des Sous-Ensemble Flous, 1*. Paris: Masson.
95. Klein, C. M. (1991). Fuzzy shortest paths. *Fuzzy Sets and Systems*, *39*, 27–41.
96. Koczy, L. T. (1992). Fuzzy graphs in the evaluation and optimization of networks. *Fuzzy Sets and Systems*, *46*, 307–319.
97. Krishnaraj, V., Vikramaprasad, R., & Dhavaseelan, R. (2017). Self-centered single valued neutrosophic graphs. *International Journal of Applied Engineering Research*, *12*(24), 15536–15543.
98. Lee, K. M. (2000). Bipolar-valued fuzzy sets and their basic operations. In *Proceedings of the International Conference* (pp. 307–317). Thailand: Bangkok.
99. Lee, K.-M. (2004). Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets and bipolar-valued fuzzy sets. *Journal of Fuzzy Logic and Intelligent Systems*, *14*(2), 125–129.
100. Lee-kwang, H., & Lee, K.-M. (1995). Fuzzy hypergraph and fuzzy partition. *IEEE Transactions on Systems, Man, and Cybernetics*, *25*(1), 196–201.
101. Li, Y., Liu, P. & Chen, Y. (2016). Some single valued neutrosophic number heronian mean operators and their application in multiple attribute group decision making. *Informatica, Lith. Acad. Sci*, *27*(1), 85–110. bibitemZ2 Ma, X., Liu, Q. & Zhan, J. (2017). A survey of decision making methods based on certain hybrid soft set models. *Artificial Intelligence Review*, **47**, 507–530.
102. Maheswari, N. R. S., & Narayanan, S. R. (2015). On (2, (c_1, c_2))-regular bipolar fuzzy graph. *International Journal of Mathematics and Soft computing*, *5*(2), 91–103.
103. Maheswari, M. R. S., & Sekar, C. (2014). On (2, k)-regular fuzzy graphs and totaly (2, k)-regular fuzzy graphs. *International Journal of Mathematics and Soft computing*, *4*(2), 59–69.
104. Maheswari, N. R. S., & Sekar, C. (2014). Some graph product in (r, 2, k)-regular fuzzy graphs. *International Journal of Mathematics and Soft computing*, *4*(2), 173–181.
105. Maheswari, N. R. S., & Sekar, C. (2012). Semi neighbourly irregular graphs. *International Journal of Combinatorial Graph Theory and Applications*, *5*(2), 135–144.
106. Maheshwari, N. R. S., & Sekar, C. (2015). On $m-$ neighbourly irregular fuzzy graphs. *International Journal of Mathematics and Soft Computing*, *5*(2), 145–153.
107. Majumdar, P., & Samanta, S. K. (2014). On similarity and entropy of neutrosophic sets. *Journal of Intelligent and Fuzzy Systems*, *26*(3), 1245–1252.
108. Mathew, S., & Sunitha, M. S. (2011). Cycle connectivity in weighted graphs. *Proyecciones Journal of Mathematics*, *30*(1), 1–7.
109. Mathew, S., & Sunitha, M. S. (2013). Strongest strong cycles and θ-fuzzy graphs. *IEEE Transactions on Fuzzy Systems*, *21*(6), 1096–1104.
110. Mathew, S. & Sunitha, M. S. (2009). Types of arcs in a fuzzy graph. *Information Sciences*, *179*(11), 1760–1768.
111. Maji, P. K. (2013). Neutrosophic soft set. *Annals of Fuzzy Mathematics and Informatics*, *5*(1), 157–168.
112. Maji, P. K., Biswas, R., & Roy, R. (2001). Fuzzy soft sets. *The Journal of fuzzy mathematics*, *9*(3), 589–602.
113. Maji, P. K., Biswas, R., & Roy, R. (2003). Soft set theory. *Computers and Mathematics with Applications*, *45*, 555–562.
114. Masood Malik, H., Akram, M., & Smarandache, F. (2018). Soft rough neutrosophic influence graphs with application. *Mathematics*, *6*(7), 125. https://doi.org/10.3390/math6070125
115. Mehra, S., & Singh, M. (2017). Single valued neutrosophic signedgarphs. *International Journal of computer Applications*, *157*(9), 31–34.
116. Molodtsov, D. (1999). Soft set theory first results. *Computers and Mathematics with Applications*, *37*, 19–31.

117. Mordeson, J. N., & Chang-Shyh, P. (1994). Operations on fuzzy graphs. *Information Sciences*, *79*, 159–170.
118. Mordeson, J. N. (1993). Fuzzy line graph. *Pattern Recognition Letter*, *14*, 381–384.
119. Mordeson, J. N., & Nair, P. S. (1996). Cycles and cocycles of fuzzy graphs. *Information Sciences*, *90*, 39–49.
120. Mordeson, J. N., & Yao, Y. Y. (2002). Fuzzy cycles and fuzzy trees. *The Journal of Fuzzy Mathematics*, *10*, 189–202.
121. Nagoorgani, A., & Radha, K. (2008). Regular properties of fuzzy graphs. *Bulletin of Pure and Applied Sciences*, *27*(2), 411–419.
122. Nandhini, S. P., & Nandhini, E. (2014). Strongly irregular fuzzy graphs. *International Journal of Mathematical Archive*, *5*, 110–114.
123. Nabeela, I, Sayed, S, Akram, M. & Smarandache, F. (2018). Notions of rough neutrosophic digraphs. *Mathematics*, 6(2), 18. https://doi.org/10.3390/math6020018.
124. Naz, S., Akram, M. & Smarandache, F. Certain notions of energy in single-valued neutrosophic graphs. *Axioms*, *7*(3), 50. https://doi.org/10.3390/axioms7030050.
125. Pal, A., Samanta, S., & Pal, M. (2013). Concept of fuzzy planar graphs. *Proceedings of Science and Information Conference* (pp. 557–563). UK: London. October 7–9, 2013
126. Parvathi, R., Akram, M., & Thilagavathi, S. (2013). Intuitionistic fuzzy shortest hyperpath in a network. *Information Processing Letters*, *113*(17), 599–603.
127. Parvathi, R., Karunambigai, M. G., & Atanassov, K. T. (2009). Operations on intuitionistic fuzzy graphs. *IEEE International Conference on Fuzzy Systems*, 1396–1401.
128. Parvathi, R., Thilagavathi, S., & Karunambigai, M. G. (2009). Intuitionistic fuzzy hypergraphs. *Cybernetics and Information Technologies*, *9*(2), 46–53.
129. Pasi, G., Yager, R. & Atanassov, K. T. (2004). Intuitionistic fuzzy graph interpretations of multiperson multi-criteria decision making: generalized net approach. In *Intelligent Systems, Proceedings of the 2004 2nd International IEEE Conference* (vol. 2, pp. 434–439).
130. Pedrycz, W. (1995). *Fuzzy sets engineering*. FL: CRC Press.
131. Praba, B., Chandrasekaran, V. M., & Deepa, G. (2014). Energy of an intutionistic fuzzy graph. *Italian Journal of Pure and Applied Mathematics*, *32*, 431–444.
132. Radha, K., & Kumaravel, N. (2014). Some properties of edge regular fuzzy graphs. *Jamal Academic Research Journal, Special issue*, 121–127.
133. Radha, K., & Kumaravel, N. (2014). On edge regular fuzzy graphs. *International Journal of Mathematical Archive*, *5*(9), 100–112.
134. Radhamani, C., & Radhika, C. (2012). Isomorphism on fuzzy hypergraphs. *IOSR Journal of Mathematics*, *2*(6), 24–31.
135. Radhika, C., Radhamani, C. & Suresh, S. (2013). Isomorphism properties on strong fuzzy hypergraphs. *International Journal of Computer Applications*, *64*(1).
136. Rao, R. V. (2006). A decision-making framework model for evaluating flexible manufacturing systems using digraph and matrix methods. *International Journal of Advanced Manufacturing Technology*, *30*, 1101–1110.
137. Pedrycz, W. (2010). Human centricity in computing with fuzzy sets: an interpretability quest for higher granular constructs. *Journal of Ambient Intelligence and Humanized Computing*, *1*, 65–74.
138. Pedrycz, W., & Bargiela, A. (2010). Fuzzy clustering with semantically distinct families of variables; descriptive and predictive aspects. *Pattern Recognition Letters*, *31*, 1952–1958.
139. Peng, J. J., Wang, J. Q., Zhang, H. Y., & Chen, X. H. (2014). An outranking approach for multi-criteria decision-making problems with simplified neutrosophic sets. *Applied Soft Computing*, *25*, 336–346.
140. Peng, C., Zhang, R., Zhang, Q., & Wang, J. (2010). Dominating sets in directed graphs. *Information Sciences*, *180*, 3647–3652.
141. Perchant, A., & Bloch, I. (2002). Fuzzy morphisms between graphs. *Fuzzy Sets and Systems*, *128*, 149–168.
142. Pawlak, Z. (1982). Rough sets. *International Journal of Computer and Information Sciences*, *11*(5), 341–356.

143. Rivieccio, U. (2008). Neutrosophic logics: prospects and problems. *Fuzzy Sets and System*, *159*, 1860–1868.
144. Roberts, F. (1976). *Discrete mathematical models*. New Jersey: Prentice Hall.
145. Rosenfeld, A. (1975). Fuzzy graphs. In L. A. Zadeh, K. S. Fu., & M. Shimura (Eds.), *Fuzzy sets and their applications* (pp. 77–95). New York: Academic Press.
146. Rosen, K. H. *Discrete mathematics and its applications* (7th ed.). McGraw - Hill. ISBN-13:978-0072899054.
147. Sahin, R., & Kucuk, A. (2014). On similarity and entropy of neutrosophic soft sets. *Journal of Intelligent and Fuzzy Systems*, *27*(5), 2417–2430.
148. Sahin, R. & Kucuk, A. (2014). Subsethood measure for single valued neutrosophic sets. *Journal of Intelligent and Fuzzy Systems*. https://doi.org/10.3233/IFS-141304.
149. Sahin, R., & Liu, P. (2016). Maximizing deviation method for neutrosophic multiple attribute decision making with incomplete weight information. *Neural Computing and Applications*, *27*(7), 2017–2029.
150. Samanta, S., & Pal, M. (2013). Fuzzy k-competition graphs and p-competition fuzzy graphs. *Fuzzy Information and Engineering*, *5*, 191–204.
151. Sampathkumar, E. (2006). Generalized graph structures. *Bulletin of Kerala Mathematics Association*, **3**(2), 65–123.
152. Samanta, S., Akram, M., & Pal, M. (2015). m-step fuzzy competition graphs. *Journal of Applied Mathematics and Computing*, *47*, 461–472.
153. Samanta, S., & Pal, M. (2012). Bipolar fuzzy hypergraphs. *International Journal of Fuzzy Logic Systems*, *2*(1), 17–28.
154. Samanta, S., Pal, M., & Pal, A. (2014). New concepts of fuzzy planar graph. *International Journal of Advanced Research in Artificial Intelligence*, *3*(1), 52–59.
155. Samanta, S., & Pal, M. (2012). Irregular bipolar fuzzy graphs. *Inter. J. Appl. Fuzzy Sets*, *2*, 91–102.
156. Samanta, S., & Pal, M. (2014). Some more results on bipolar fuzzy sets and bipolar fuzzy intersection graphs. *The Journal of Fuzzy Mathematics*, *22*(2), 253–262.
157. Sameena, K., & Sunitha, M. S. (2008). A Characterization of g-selfcentered fuzzy graphs. *The Journal of Fuzzy Mathematics*, *16*(4), 787–791.
158. Santhimaheswari, N. R., & Sekar, C. (2016). On strongly edge irregular fuzzy graphs. *Kragujevac Journal of Mathematics*, *40*, 125–135.
159. Sarwar, M., & Akram, M. (2017). Novel concepts of bipolar fuzzy competition graphs. *Journal of Applied Mathematics and Computing*, *54*, 511–547.
160. Shah, N., & Hussain, A. (2016). Neutrosophic soft graphs. *Neutrosophic Set and Systems*, *11*, 31–44.
161. Sharbaf, S. R., & Fayazi, F. (2014). Laplacian energy of a fuzzy graph. *Iranian Journal of Mathematical Chemistry*, *5*, 1–10.
162. Sidra, S., Ishfaq, N., Akram, M. & Smarandache, F. (2018). Rough neutrosophic digraphs with application. *Axioms 7*(1), 5. https://doi.org/10.3390/axioms7010005.
163. Smarandache, F. (1995). *A unifying field in logics: Neutrosophic logic*. Rehoboth: American Research Press.
164. Smarandache, F. (1998). *Neutrosophy neutrosophic probability, set and logic*. Rehoboth, USA: American Research Press.
165. Smarandache, F. (1999). *A unifying field in logics*., Neutrosophic probability, set and logic Rehoboth: American Research Press.
166. Smarandache, F. Neutrosophic set-a generalization of the intuitionistic fuzzy set. *International Journal of Pure and Applied Mathematics*, *24*(3), 287.
167. Smarandache, F. A geometric interpretation of the neutrosophic set a generalization of the intuitionistic fuzzy set. In *Granular Computing (GrC), 2011 IEEE International Conference*. https://doi.org/10.1109/GRC.2011.6122665.
168. Tripathi, G. (1999). A matrix extension of the Cauchy-Schwarz inequality. *Economics Letters*, *63*(1), 1–3.

169. Wang, H., Smarandache, F., Zhang, Y. Q., & Sunderraman, R. (2010). Single-valued neutrosophic sets. *Multisspace and Multistruct*, *4*, 410–413.
170. Wang, H., Zhang, Y. & Sunderraman, R. (2005). Truth-value based interval neutrosophic sets. In *Granular Computing, 2005 IEEE International Conference* (vol. 1, pp. 274–277). https://doi.org/10.1109/GRC.2005.1547284.
171. Wang, H., Smarandache, F., Zhang, Y. Q. & Sunderram, R. (2005). An interval neutrosophic sets and logic: theory and applications in computing. *Hexis Arizona*.
172. Wang, H., Smarandache, F., Zhang, Y. Q., & Sunderraman, R. (2010). Single valued neutrosophic sets. *Multispace and Multistructure*, *4*, 410–413.
173. Wang, H., Zhang, Y. & Sunderraman, R. (2005). Truth-value based interval neutrosophic sets. In *Granular Computing 2005 IEEE International Conference* (pp. 274–277). https://doi.org/10.1109/GRC.2005.1547284.
174. Wang, H., Smarandache, F., Zhang, Y.Q. & Sunderram, R. (2005). An interval neutrosophic sets and logic: theory and applications in computing. *Hexis Arizona*.
175. Wu, S. Y. (1986). The compositions of fuzzy digraphs. *Journal of Research in Education Science*, *31*, 603–628.
176. Yang, H. L., Guo, Z. L., & Liao, X. (2016). On single-valued neutrosophic relations. *Journal of Intelligent and Fuzzy Systems*, *30*(2), 1045–1056.
177. Yang, H.L., Zhang, C.L., Guo, Z.L., Liu, Y.L., & Liao, X. (2017). A hybrid model of single valued neutrosophic sets and rough sets: single valued neutrosophic rough set model. *Soft computing*. https://doi.org/10.1007/s00500-016-2356-y.
178. Ye, J. (2012). Multicriteria group decision-making method using the distances-based similarity measures between intuitionistic trapezoidal fuzzy numbers. *International Journal of General Systems*, *41*(7), 729–739.
179. Ye, J. (2012). Multicriteria decision-making method using the Dice similarity measure between expected intervals of trapezoidal fuzzy numbers. *Journal of Decision Systems*, *21*(4), 307–317.
180. Ye, J. (2011). Fuzzy cross entropy of interval-valued intuitionistic fuzzy sets and its optimal decision-making method based on the weights of alternatives. *Expert Systems with Applications*, *38*(5), 6179–6183.
181. Ye, J., & Jin, Z. (2013). Non-negative matrix factorisation based on fuzzy K nearest neighbour graph and its applications. *IET Computer Vision*, *7*(5), 346–353.
182. Ye, J. (2014). Some aggregation operators of interval neutrosophic linguistic numbers for multiple attribute decision making. *Journal of Intelligent and Fuzzy Systems*, *27*(5), 2231–2241.
183. Ye, J. (2015). Multiple-attribute decision-making method under a single-valued neutrosophic hesitant fuzzy environment. *Journal of Intelligent Systems*, *24*(1), 23–36.
184. Ye, J. (2017). Multiple attribute decision-making method using correlation coefficients of normal neutrosophic sets. *Symmetry*, *9*(6), 80.
185. Ye, J. (2017). Multiple attribute group decision making based on interval neutrosophic uncertain linguistic variables. *International Journal of Machine Learning and Cybernetics*, *8*(3), 837–848.
186. Ye, J. (2013). Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment. *International Journal of General Systems*, *42*(4), 386–394.
187. Ye, S., & Ye, J. (2014). Dice similarity measure between single valued neutrosophic multisets and its application in medical diagnosis. *Neutrosophic Sets and Systems*, *6*, 48–52.
188. Ye, J. (2014). Single-valued neutrosophic minimum spanning tree and its clustering method. *Journal of Intelligent Systems*, *23*(3), 311–324.
189. Ye, J. (2014). Improved correlation coefficients of single-valued neutrosophic sets and interval neutrosophic sets for multiple attribute decision making. *Journal of Intelligent and Fuzzy Systems*, *27*(5), 2453–2462.
190. Ye, J. (2014). A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. *Journal of Intelligent and Fuzzy Systems*, *26*(5), 2459–2466.

191. Ye, J. (2016). Single-valued neutrosophic clustering algorithms based on similarity measures. *Journal of Classification*. https://doi.org/10.1007/s00357.
192. Ye, J. (2016). A netting method for clustering simplified neutrosophic information. *Soft Computing*. https://doi.org/10.1007/s00500-016-2310-z.
193. Ye, J. (2014). Clustering methods using distance-based similarity measures of single-valued neutrosophic sets. *Journal of Intelligent Systems*, *23*(4), 379–389.
194. Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*(3), 338–353.
195. Zadeh, L. A. (1971). Similarity relations and fuzzy orderings. *Information Sciences*, *3*(2), 177–200.
196. Zadeh, L. A. (2005). From precise to granular probabilities. *Fuzzy Sets and Systems*, *154*, 370–374.
197. Zadeh, L. A. (2008). Is there a need for fuzzy logic? *Information Sciences*, *178*(13), 2751–2779.
198. Zadeh, L. A. (2005). Toward a generalized theory of uncertainty (GTU) an outline. *Information Sciences*, *172*, 1–40.
199. Zadeh, L. A. (1975). The concept of a linguistic and application to approximate reasoning. *Information Sciences*, *8*, 199–249.
200. Zafer, F., & Akram, M. (2018). A novel decision-making method based on rough fuzzy information. *International Journal of Fuzzy Systems*, *20*(3), 1000–1014.
201. Zhang, W. R. (1994). Bipolar fuzzy sets and relations: a computational framework forcognitive modeling and multiagent decision analysis. In *Proceedings of IEEE Conference* (pp. 305–309).
202. Zhang, X., Dai, J., & Yu, Y. (2015). On the union and intersection operations of rough sets based on various approximation spaces. *Information Sciences*, *292*, 214–229.
203. Zhang, X., Zhou, B., & Li, Q. (2012). A general frame for intuitionistic fuzzy rough sets. *Information Sciences*, *216*, 34–49.
204. Zhang, W. R. (1998). Bipolar fuzzy sets. In *Fuzzy Systems Proceedings, IEEE World Congress on Computational Intelligence* (pp. 835–840).

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