Ranking by Fuzzy Weak Autocatalytic Set

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Abstract. A relation between objects can be presented in a form of a graph. An autocatalytic set (ACS) is a directed graph where every node has incoming link. A fuzzy weak autocatalytic set (FWACS) is introduced to handle uncertainty in a ranking. The FWACS is found to be comparable to eigenvector method (EM) and potential method (PM) for ranking purposes.

Keywords: Ranking · Fuzzy graph · Fuzzy weak autocatalytic set

1 Introduction

The study of decision problems has a long history. Mathematical modeling has been used by economist and mathematicians in decision making problems, in particular multiple criteria decision making (MCDM) (Rao 2006; Lu and Ruan 2007). In early 1950s, Koopmans (1951) worked on MCDM and Saaty (1990) introduced analytic hierarchy process (AHP) which brought advances to MCDM techniques.

In general, there are many situations in which the aggregate performance of a group of alternatives must be evaluated based on a set of criteria. The determination of weights is an important aspect of AHP. The ranks of alternatives are obtained by their associated weights (Saaty 1978; 1979). In AHP, the eigenvector method (EM) is used to calculate the alternative weights. The following section is a review on EM.

2 Eigenvector Method

The AHP is based on comparing n alternatives in pair with respect to their relative weights. Let $C_1, \ldots, C_n$ be n objects and their weights by $W = (w_1, \ldots, w_m)^T$. The pairwise comparisons can be presented in a form of a square matrix $A(a_{ij})$.

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$$A = (a_{ij})_{n\times n} = \begin{array}{c} \\ C_1 \\ C_2 \\ \vdots \\ C_n \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \end{array},$$

where $a_{ij} = 1/a_{ji}$ and $a_{ii} = 1$ for $i, j = 1, 2, \ldots, n$.

Saaty (1977) proposed the EM to find the weight vector from pairwise comparison. He developed the following steps.

Step 1: From the pairwise comparison matrix A, the weight vector W can be determined by solving the following equation.

$$AW = \lambda_{\max} W$$

where $\lambda_{\max}$ is the largest eigenvalue of A.

Step 2: Calculate the consistency ratio (CR). This is the actual measure of consistency. It is defined as follows.

$$CR = \frac{(\lambda_{\max} - n)/(n-1)}{RI}$$

where RI is the consistency index. Table 1 shows the RI values for the pairwise comparison matrices. The pairwise comparison matrix is consistent if $CR \leq 0.1$, otherwise it need to be revised.

Table 1. Random Index for matrices of various size (Saaty 1979)

n	1	2	3	4	5	6	7	8	9	10	11
RI	0.0	0.0	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49	1.51

Step 3: The overall weight of each alternative is calculated using the following formula.

$$w_{A_i} = \sum_{j=1}^{m} w_{ij} w_j, \quad i = 1, \ldots, n$$

where $w_j (j = 1, \ldots, m)$ are the weights of criteria, $w_{ij} (j = 1, \ldots, n)$ are the weights of alternatives with respect to criterion j, and $w_{A_i} (j = 1, \ldots, n)$ are the overall weights of alternatives.

Further, a ranking function using preference graph, namely Potential Method (PM) was introduced by Lavoslav Čaklović in 2002. The following section is a brief review on PM.

3 Potential Method

The Potential Method is a tool in a decision making process which utilizes graph, namely preference graph. A preference graph is a structure generated by comparing on a set of alternatives (Čaklović 2002). Čaklović (2002; 2004) used preference graph to model pairwise comparisons of alternatives. Suppose V be a set of alternatives in which some preferences are being considered. If an alternative u is preferred over alternative v (denoted as $u \succ v$), it can be presented as a directed edge from vertex v to vertex u. The edge is denoted as (u, v) (Fig. 1).

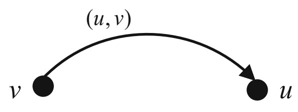

Fig. 1. An alternative u is preferred than alternative v

The preference is described with an intensity from a certain scale (e.g. equal, weak, moderate, strong, or absolute preference) which is expressed by a nonnegative real number, $\mathbb{R}$. The directed edge from v to u has a weight, i.e., it has a preference flow denoted by $F_{(u,v)}$. The formal definition of a preference graph is stated as below.

Definition 1 Čaklović and Kurdija (2017).
A preference graph is a triple $G = (V, E, F)$ where V is a set of $n \in \mathbb{N}$ vertices (representing alternatives), $E \subseteq V \times V$ is a set of directed edges, and $F : E \to \mathbb{R}$ is a preference flow which maps each edge (u, v) to the corresponding intensity $F_{(u,v)}$.
The following are the steps to determine weights and ranks by PM.

Step 1: Build a preference graph $G = (V, E, F)$ for a given problem.
Step 2: Construct incidence, A and flow difference, F matrices.
An $m \times n$ incidence matrix is given by

$$A_{\alpha,v} = \begin{cases} -1, & \text{if the edge } \alpha \text{ leaves } v \\ 1, & \text{if the edge } \alpha \text{ enters } v \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

Step 3: Build the Laplacian matrix, L
The Laplacian matrix is $L = A^T A$ with entries define as

$$L_{i,j} = \begin{cases} -1, & \text{if the edge } (i,j) \text{ or } (j,i), \\ \deg(i), & \text{if } i = j, \\ 0, & \text{else.} \end{cases} \quad (2)$$

such that $\deg(i)$ is the degree of vertex i.

Step 4: Generate the flow difference, ∇.

Let the flow difference be $\nabla := A^T F$. The component of ∇ is determined as below.

$$\begin{aligned}\nabla_v &= \sum_{\alpha=1}^{m} A^T_{v.\alpha} F_\alpha \\ &= \sum_{\alpha \text{ enters } v} F_\alpha - \sum_{\alpha \text{ leaves } v} F_\alpha\end{aligned} \tag{3}$$

whereby ∇_v is the difference between the total flow which enters v and the total flow which leaves v.

Step 5: Determine potential, X

Potential, X is a solution of the Laplacian system

$$LX = \nabla \tag{4}$$

such that $\sum X_v = 0$ on its connected components.

Step 6: Check the consistency degree, $\beta < 12^0$

The measure of inconsistency is defined as

$$\text{Inc}(F) = \frac{\|F - AX\|_2}{\|AX\|_2} \tag{5}$$

where $\|\,.\,\|_2$ denotes 2-norm and $\beta = \arctan(\text{Inc}(F))$ is the angle of inconsistency. The ranking is considered acceptable whenever $\beta < 12^0$.

Step 7: Determine the weight, w. The following equation is used to obtain the weight.

$$w = \frac{a^X}{\|a^x\|_1} \tag{6}$$

where $\|\,.\,\|_1$ represents l_1-norm and parameter a is chosen to be 2 suggested by Čaklović (2002).

Step 8: Rank the objects by their associated weights.

The PM is meant for crisp edges (Čaklović 2004). It is not equipped for fuzzy edges. The following section introduces a special kind of graph, namely weak autocatalytic set (WACS) as a tool for ranking purposes.

4 Weak Autocatalytic Set

Jain and Krishna introduced the concept of autocatalytic set (ACS) set in form of a graph in 1998. An ACS is described by a directed graph with vertices represent species and the directed edges represent catalytic interactions among them (Jain and Krishna 1998; 1999). An edge from vertex j to vertex i indicates that species j catalyses i. The formal definition of an ACS is given as follows.

Definition 2 (Jain and Krishna 1998).

An ACS is a subgraph, each of whose nodes has at least one incoming link from vertices belonging to the same subgraph (Fig. 2).

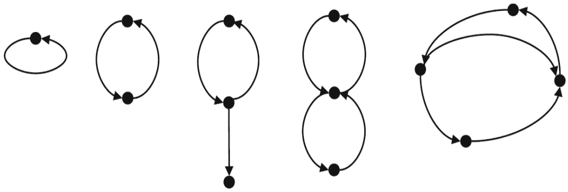

Fig. 2. Some examples of ACS

A weak form of an ACS i.e. WACS was proposed by Mamat et al. (2018). A WACS allows some freedom in connectivity of its vertices in a system. The WACS is defined as follows.

Definition 3 (Mamat *et al.* 2018).

A WACS is a non-loop subgraph which contains a vertex with no incoming link (Fig. 3).

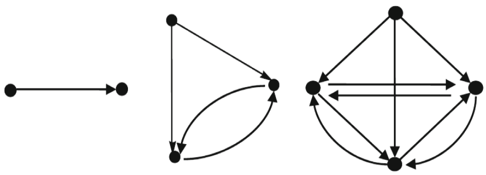

Fig. 3. Several WACS

Some uncertainties may happen in a WACS. The fuzzification of WACS has led to a new structure namely Fuzzy Weak Autocatalytic Set (FWACS). The definition of a FWACS is formalized in Definition 4 as follows.

Definition 4 (Mamat *et al.* 2018)

A FWACS is a WACS such that each edge e_i has a membership value $\mu(e_i) \in [0, 1]$ for $e_i \in E$ (Fig. 4).

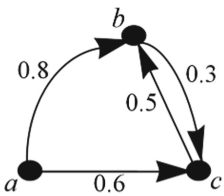

Fig. 4. A FWACS

A FWACS is used for ranking. The following section describes the propose method.

5 Ranking by FWACS

This section presents an algorithm for ranking by FWACS. The input are the membership values of edges obtained in pairwise comparison of objects. The orientation of edges can be represented by an incidence matrix, A. The membership values of edges denoted by F are represented by a $m \times 1$ matrix. The procedure of ranking with FWACS is given as follows.

1. Build a FWACS, $G = (V, E_\mu)$ for a given problem and determine the membership value for edges. The V is a set of vertices and E_μ is the corresponding fuzzy edges.
2. Construct incidence matrix, A and fuzzy flow matrix, F_μ. A $m \times n$ incidence matrix is given by Eq. 1.
3. Define Laplacian matrix, L using Eq. 2.
4. Generate flow difference, D_μ using Eq. 3.
5. By using Eq. 4, the potential, X is calculated.
6. Check the consistency $(\beta < 12^0)$ by solving Eq. 5.
7. Determine the weight, w using Eq. 6.
8. Rank the objects with respect to their associated weights.

The ranking procedure is illustrated in the following flowchart in Fig. 5 which is followed by its algorithm in Fig. 6.

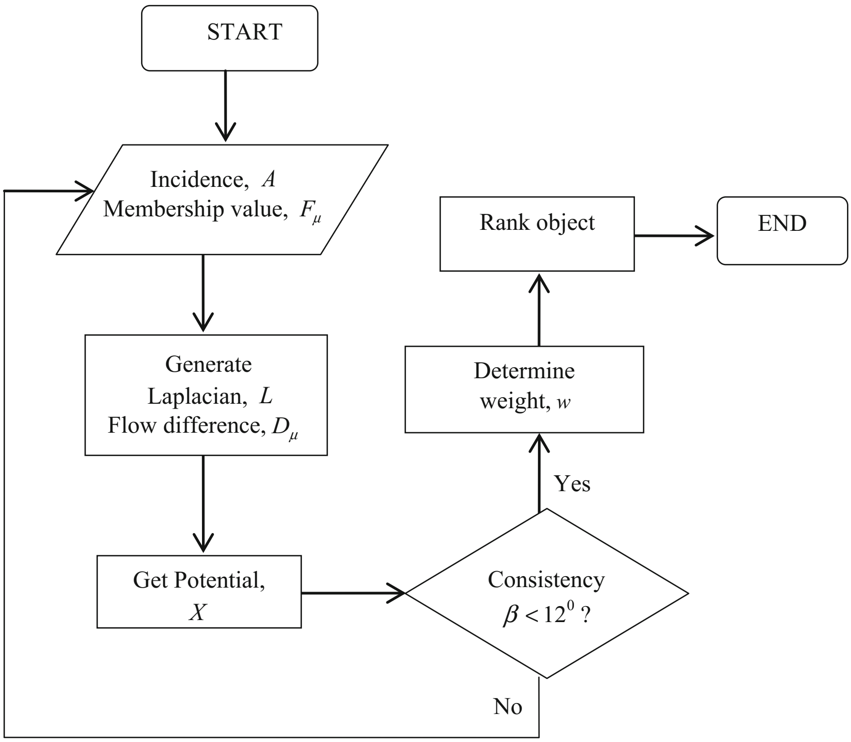

Fig. 5. Ranking flowchart

Algorithm 1 Ranking with FWACS

Begin
Input: $A = (a_{ij})_{m \times n}$ Incidence matrix
$F = (f_1, f_2, f_3, \ldots, f_m)$ Flow matrix
Output: $w = (w_1, w_2, w_3, \ldots, w_n)$ Criteria weights
Procedure 1: [Define laplacian, L]
$L = (l_{ij})_{n \times n}$
return L
Procedure 2: [Generate flow difference, D]
$D = (D_1, D_2, D_3, \ldots, D_n)$
return D
Procedure 3: [Get potential, X]
$X = (x_1, x_2, x_3, \ldots, x_n)$
return X
Procedure 4: [Consistency degree, β]
β
return β
Procedure 5: [Determine weight, w]
$w = (w_1, w_2, w_3, \ldots, w_n)$
return w
End

Fig. 6. Ranking algorithm

An implementation of ranking using FWACS on a problem described in Morano *et al.* (2016) is presented in the following section.

6 Implementation on Culture Heritage Valorization

The Rocca Estate is located in the municipality of Finale Emilia which was erected in 1213 as a defense tower to the city. The building is characterized over the centuries by different interventions, which ended the recovery activities in 2009. However, in 2012 an earthquake struck which caused serious damage to the fortress. An urgent action was needed to restore the building.

The main task is to identify the "total quality" of the building with the support of evaluator (see Fig. 7). The "total quality" takes into account the compatibility of the alternative respect to multiple instances described through the criteria at level 2. The criteria are derived from expertise in different aspects namely technical, economic, legal, social and others. The alternatives are given in level 3.

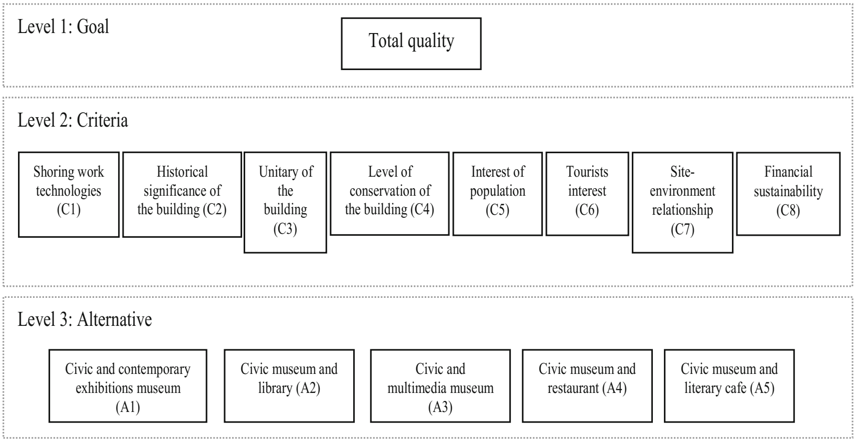

Fig. 7. The hierarchy of decision levels

The evaluation matrix for the goal is given in Table 2 and Fig. 8 illustrates the FWACS for the identified goal.

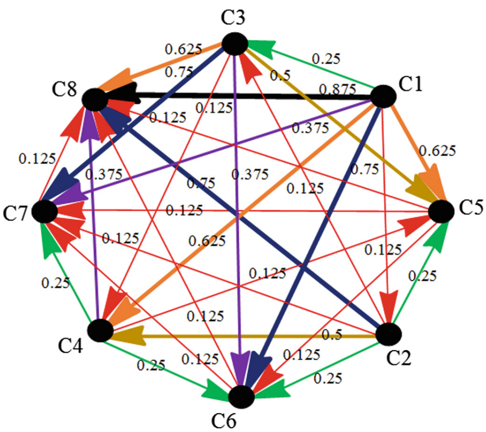

Fig. 8. The FWACS for culture heritage valorization goal

There are 8 criteria need to be considered in order to achieve the goal. Hence, a pairwise comparison within each criterion is made. There exist 28 comparisons in this level. The comparison is represented by an incidence matrix. An arrow pointing from C1 to C2 in Fig. 8 signifies that C2 is more preferred than C1. The incidence matrix and its corresponding membership values are given as follow.

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \text{ and } F = \begin{bmatrix} 0.125 \\ 0.25 \\ 0.625 \\ 0.625 \\ 0.75 \\ 0.375 \\ 0.875 \\ 0.125 \\ 0.5 \\ 0.25 \\ 0.25 \\ 0.125 \\ 0.75 \\ 0.125 \\ 0.5 \\ 0.375 \\ 0.75 \\ 0.625 \\ 0.125 \\ 0.25 \\ 0.25 \\ 0.375 \\ 0.125 \\ 0.125 \\ 0.125 \\ 0.125 \\ 0.125 \\ 0.125 \end{bmatrix}$$

The potential, X of $[-0.453 \quad -0.234 \quad -0.25 \quad -0.031 \quad 0.141 \quad 0.188 \quad 0.203 \quad 0.375]^T$ is determined by solving Eq. 4. In this paper, we made a comparison result using EM taken from Morano et al. (2016) with the result using PM and FWACS. The EM weights are listed alongside our calculated PM and FWACS weights in Table 2.

Table 2. Pairwise comparisons for goal

Criteria	C1	C2	C3	C4	C5	C6	C7	C8	Priority vector		
									EM	PM	FWACS
C1	1	1/2	1/3	1/6	1/6	1/7	1/4	1/8	0.024	0.005	0.090
C2	2	1	1/2	1/5	1/3	1/3	1/2	1/7	0.044	0.015	0.105
C3	3	2	1	1/2	1/5	1/4	1/7	1/6	0.046	0.014	0.103
C4	6	5	2	1	1/2	1/3	1/3	1/4	0.095	0.066	0.126
C5	6	3	5	2	1	1/2	1/2	1/2	0.135	0.122	0.136
C6	7	3	4	3	2	1	1/2	1/2	0.169	0.158	0.140
C7	4	2	7	3	2	2	1	1/2	0.203	0.172	0.142
C8	8	7	6	4	2	2	2	1	0.285	0.447	0.159

Table 2 presented the weights for each criterion for goal. The weights obtained using EM, PM and FWACS signified that the criterion C8 has the highest weight whereas the lowest weight is assigned to criterion C1.

Then, the comparison for alternatives with respect to each criterion is made. The pairwise comparisons of criteria are presented in Table 3.

Table 3. Pairwise comparisons of each criterion (Morano *et al.* 2016)

C1 Shoring work technologies

	A1	A2	A3	A4	A5
A1	1	1/3	1/8	1/6	1/9
A2	3	1	1/5	1/3	1/7
A3	8	5	1	5	1/2
A4	6	3	1/5	1	1/6
A5	9	7	2	6	1

C2 Historical significance of building

	A1	A2	A3	A4	A5
A1	1	1/2	1/6	1/2	1/4
A2	2	1	1/4	3	2
A3	6	4	1	5	3
A4	2	1/3	1/5	1	2
A5	4	1/2	1/3	1/2	1

C3 Unitary of building

	A1	A2	A3	A4	A5
A1	1	1	3	5	6
A2	1	1	3	5	6
A3	1/3	1/3	1	2	4
A4	1/5	1/5	1/2	1	2
A5	1/6	1/6	1/4	1/2	1

C4 Level of conservation of the building

	A1	A2	A3	A4	A5
A1	1	2	7	8	9
A2	1/2	1	6	7	8
A3	1/7	1/6	1	2	3
A4	1/8	1/7	1/2	1	4
A5	1/9	1/8	1/3	1/4	1

C5 Interest of population

	A1	A2	A3	A4	A5
A1	1	1/3	1/2	1/4	1/7
A2	3	1	2	2	1/3
A3	2	1/2	1	1/2	1/6
A4	4	1/2	2	1	1/3
A5	7	3	6	3	1

C6 Touristic interest

	A1	A2	A3	A4	A5
A1	1	3	1/5	1/4	1/8
A2	1/3	1	1/7	1/7	1/9
A3	5	7	1	1/3	1/3
A4	4	7	3	1	1/3
A5	8	9	3	3	1

C7 site-environment relationship

	A1	A2	A3	A4	A5
A1	1	3	1/6	1/4	1/5
A2	1/3	1	1/7	1/5	1/7
A3	6	7	1	2	1/3
A4	4	5	1/2	1	1/3
A5	5	7	3	3	1

C8 Financial stability

	A1	A2	A3	A4	A5
A1	1	3	1/3	1/5	1/6
A2	1/3	1	1/6	1/8	1/9
A3	3	6	1	1/2	1/4
A4	5	8	2	1	1/5
A5	6	9	4	5	1

The comparison between EM from Morano et al. (2016), PM and FWACS weights of alternatives with respect to each criterion are given in Table 4.

Table 4. Comparison between EM, PM and FWACS weights

Priorities									
		C1	C2	C3	C4	C5	C6	C7	C8
EM	A1	0.03	0.06	0.36	0.48	0.05	0.06	0.07	0.06
	A2	0.06	0.20	0.36	0.34	0.20	0.03	0.04	0.03
	A3	0.32	0.49	0.15	0.08	0.09	0.17	0.27	0.14
	A4	0.12	0.12	0.08	0.07	0.16	0.26	0.17	0.22
	A5	0.47	0.13	0.05	0.03	0.49	0.47	0.45	0.54
PM	A1	0.002	0.025	0.431	0.654	0.017	0.009	0.019	0.014
	A2	0.010	0.117	0.431	0.327	0.119	0.002	0.006	0.002
	A3	0.290	0.710	0.094	0.010	0.039	0.115	0.307	0.073
	A4	0.032	0.059	0.031	0.007	0.104	0.175	0.134	0.145
	A5	0.666	0.089	0.013	0.002	0.721	0.689	0.534	0.766
FWACS	A1	0.132	0.167	0.238	0.281	0.160	0.157	0.165	0.166
	A2	0.162	0.202	0.238	0.258	0.204	0.132	0.143	0.132
	A3	0.246	0.252	0.197	0.167	0.178	0.215	0.233	0.204
	A4	0.187	0.185	0.172	0.159	0.201	0.226	0.210	0.223
	A5	0.273	0.195	0.155	0.136	0.256	0.269	0.249	0.274

The priorities listed in Table 4 for PM and FWACS are aggregated with the weights identified in Table 2 using Eq. 6. Table 5 lists the overall priority vectors.

Table 5. Priority vector for goal

Alternatives	Overall Priority vector					
	FWACS	RANK	PM	RANK	EM (Morano *et al.* 2016)	RANK
A1	0.18283	4	0.02919	4	0.115	4
A2	0.17940	5	0.01081	5	0.108	5
A3	0.20186	3	0.13020	3	0.181	3
A4	0.20731	2	0.16876	2	0.182	2
A5	0.22859	1	0.66104	1	0.414	1

The results are summarized in Table 5, whereby A5 is the dominant. The result is in order $A5 \succ A4 \succ A3 \succ A1 \succ A2$. Furthermore, the outcome is in agreement with Morano et al. (2016).

The weights differences between A4 and A3 is 0.001 using EM. However, the weights are different by 0.03856 and 0.00545 by PM and FWACS, respectively. The differences between A1 and A2 is 0.007, 0.01838 and 0.00343 by EM, PM and FWACS respectively.

7 Conclusion

The aim of this paper is to introduce a method for ranking of uncertainty environments. A problem posted in Morano et al. (2016) is considered. The result obtained from FWACS is found to be comparable to EM and PM. Furthermore, FWACS can accommodate the uncertainty environment.

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