Multidimensional Indicators of Inequality and Poverty

Satya R. Chakravarty and Maria Ana Lugo

Abstract This chapter reviews the main features of multidimensional indices of inequality and poverty. For each of these cases, the discussion is divided into two approaches: a direct approach, where desirable properties are specified and a measure of inequality or poverty obtained; and the inclusive measure of well-being approach, where an index of individual well-being is defined in a first step, and the measure of inequality or poverty obtained in a second step. The emphasis will be on the properties that different measures satisfy and on the main justifications put forward when properties disagree.

Keywords Multidimensional indicators · Inequality · Poverty · Well-being

Prepared for Handbook on Well-Being and Public Policy, edited by Matthew Adler and Marc Fleurbaey, Oxford University Press (OUP), Oxford.

1 Introduction

The traditional focus for the assessment of the well-being or destitution of individuals has been on the income distribution. It is indeed true that a person's income often determines how much of different goods he or she can consume; higher income allows a person to consume more of some of the goods and/or shift consumption to higher quality variants. But income as the only attribute of well-being is often inappropriate. A sub-optimal supply of a public good in a community might not be sufficient for

The kind consent of Oxford University Press to reprint this article from its volume 'Oxford Handbook of Well-Being and Public Policy', 2016 is acknowledged with utmost gratitude.

S. R. Chakravarty
Indian Statistical Institute, Kolkata 700108, India
e-mail: satyarchakravarty@gmail.com

M. A. Lugo (✉)
World Bank, 1850 I Street NW (I4-405), Washington, DC 20433, United States of America
e-mail: mlugo1@worldbank.org

S. R. Chakravarty (ed.), *Poverty, Social Exclusion and Stochastic Dominance*, Themes in Economics, https://doi.org/10.1007/978-981-13-3432-0_14

the residents. For example, many people in developing countries suffer or even die from malaria because the malaria prevention program, a local public good, is not well organized or available at any price. Thus, it may not be possible to trade-off income for a better position in a non-income attribute which is non-tradable in a market. Likewise, a healthy porter who just earns hand-to-mouth daily by loading and unloading of cargos in a dockyard cannot tradeoff his good health for any additional income. These illustrations indicate that non-income dimensions of well-being contribute significantly to quality of life. Examples of such dimensions are literacy, housing, life expectancy, public goods, social cohesion, human security and so on. This supports the view that traditional economic indices of well-being should be supplemented with alternative indicators that capture non-economic or non-material dimensions of human life. In fact, it is now commonly accepted that human well-being should be regarded as a multidimensional phenomenon along the lines advocated by Rawls (1971), Kolm (1977), Townsend (1979), Streeten (1981), Atkinson and Bourguignon (1982), Sen (1985, 1993), Stewart (1985), Doyal and Gough (1991), Ramsay (1992), Cummins (1996), Ravallion (1996), Nussbaum (2000) and Thorbecke (2008).[1]

Consequently, in recent years a very important development in the research on the measurement of well-being of a population is the shift of emphasis from a single monetary dimension to a multidimensional framework that incorporates non-monetary aspects as well. One of the most influential formalization of this is the capability approach—discussed in more detail in Alkire chapter 21, the OUP Handbook. For nearly two decades now, Sen (1985, 1993) has emphasized the need to move away from the space of incomes or resources for assessing individuals' well-being in favor of a focus on the spaces of functionings and capabilities. Functionings are "parts of the state of a person in particular the things that he or she manages to do or be in leading a life" (Sen 1993, 31) (e.g., being healthy, riding a bicycle), whereas the capability set is the set of potential functionings vectors available to the person. The key idea behind the capabilities approach is that individuals differ in their ability to transform resources into well-being or "flourishing". Even for those goods for which markets exist, there is no reason to believe that relative market prices between the particular goods included as proxies for certain functionings is an appropriate approximation for the well-being trade-off between the functionings themselves since the rate of transformation of goods into functionings may differ and also vary across individuals.

The recognition that well-being and deprivation are multi-faceted does not necessarily lead to a multidimensional indicator—a single number summarizing society's overall condition, the degree of inequality, or the degree of poverty as a function of

[1]The World Development Report 2000–2001 stressed the view that traditional view of poverty should be supplemented with low achievements in health and education. The multidimensional nature of well-being is implicitly recognized by the set of dimensions considered by the European Union to judge the performance of its member countries (Atkinson et al. 2002). European Union policy recommends that for measuring failure in material living conditions income-based poverty should be combined with low employment and material deprivation (Bossert et al. 2013). The Commission on the Measurement of Economic Performance and Social Progress has also insisted on looking at well-being of a population from a multidimensional perspective (Stiglitz et al. 2009).

the pattern of individuals' achievements along the multiple well-being dimensions. Some have argued that a portfolio of indicators (Atkinson et al. 2002; Ravallion 2011), whereby each dimension is assessed separately, is to be preferred so that the efforts are focused on the "best possible distinct measures of the various dimensions of poverty […] rather than a single 'multidimensional index'" (Ravallion 2011, 13). This approach also avoids requiring agreement on the relative importance of each dimension. On the other hand, the often called 'dashboard approach', while looking at the distribution of each of the components, will overlook the dependency structure in the joint distribution of these achievements, which may represent an important aspect in the comparison of distributions (Tsui 1999; Pogge 2002; Stiglitz et al. 2009). Others have favoured an intermediate approach which combines a dimension-wise assessment with a description of the dependency structure (Atkinson et al. 2010; Decancq 2014; Ferreira and Lugo 2013). A third and influential approach is through the use of a multivariate version of stochastic dominance (for instance, Atkinson and Bourguignon 1982, 1987; Duclos et al. 2006; Muller and Trannoy 2011, 2012). The multivariate stochastic dominance approach is more readily applied when the number of dimensions is limited; for a discussion, see Duclos and Tiberti, chapter 23, the OUP Handbook.

However, multidimensional indicators of social welfare (overall social condition), inequality and deprivation have been embraced by both among academics and policy makers. Since 1990 the United Nations Development Program has been using the Human Development Index, which combines income with life expectancy at birth and educational achievement, instead of the per capita GDP, to rank countries.[2] Recently, the Multidimensional Poverty Index developed by Alkire and Santos (2010)[3] has been incorporated into the UNDP's core indicators. The OECD launched the Better Life Index website where the user can build her own index assigning weights to eleven dimensions of well-being that have been found to be essential in many countries and cultures (OECD 2011). Countries are also proposing their own measures of multidimensional poverty.[4] The National Council for the Evaluation of Social Policy (CONEVAL) in Mexico adopted a multidimensional index of poverty as the country's official poverty measure (CONEVAL 2010). A similar multidimensional measure is used in Colombia and Bhutan and various other countries (such as El Salvador, Pakistan, and Malaysia) are considering following these examples.

Undoubtedly, multidimensional indices are appealing in that they provide unique rankings, and thus are seen as useful tools for governments and analysts to readily obtain a picture of the distribution of well-being of a society.

---

[2] Alkire and Foster (2010) consider an inequality adjusted HDI, which uses an Atkinson-type aggregation for each dimension.

[3] See Alkire, chapter 21, the OUP Handbook, for a discussion of the Multidimensional Poverty Index.

[4] While the interest in developing a multidimensional poverty measure in Latin America and Europe has gained force in recent years, there is a long tradition of using the counting approach to consider the existence of multiple deprivations at the same time—for instance, the Basic Needs Approach widely used in Latin American countries since the 1980s and still relevant nowadays. See Atkinson (2003).

Several normative issues are involved in the selection of a multidimensional indicator—of overall social condition, poverty or inequality. Of critical importance, one must decide on a functional form to aggregate attributes and on the relative weights to be assigned to each of these attributes.[5] The rest of the chapter will concentrate on alternative functional forms proposed for measuring multidimensional inequality and poverty.[6] But weights also play a crucial role in determining the set of dimensions to be included in the analysis (a dimension with zero weight is excluded) and the trade-offs between the selected dimensions. See Decancq and Lugo (2013).

The literature suggests a variety of approaches for specifying multidimensional indices of inequality and poverty. These include the axiomatic approach, which starts with desirable properties of the indicator and derives a family of indices that satisfies these principles; the fuzzy set approach; information theory; and the statistical approach. Often there may be insufficient information concerning achievements of different attributes. In a situation of this type where indefiniteness arises from ambiguity, the fuzzy set approach is quite sensible (Chakravarty 2006). The statistical approach relies on multivariate statistical techniques such as principal components or latent variable models to aggregate dimensions (Klasen 2000; Krishnakumar and Nadar 2008). In the information theory-based approach aggregation of achievements relies on the Shannon entropy formula (Maasoumi 1986; Maasoumi and Lugo 2008). In this chapter, our focus will be on the axiomatic approach. For ease of exposition, the emphasis will be on the properties that different measures satisfy (rather than on the set of axioms that characterize them) and on the main justifications put forward when properties disagree.

The next section introduces the notation and framework that will be used throughout the chapter. Sections 3 and 4 discuss the properties and provide some examples of indicators of multidimensional inequality and poverty, respectively. Functional forms of indicators when dimensions are measured on different scales, e.g. ratio or ordinal, are discussed. Section 5 concludes.

2 Preliminaries

For simplicity of exposition we refer to the population under consideration as a society and the unit of analysis in the society as a person (see Chiappori, chapter 27, the OUP Handbook, for inferring individual achievements from household data). Since some

[5]Other key decisions include: choosing the (set of) indicators for each dimension and the transformation function where the variables are not measured in the same measurement units and made comparable. On transformation functions see Jacobs et al. (2004) and Nardo et al. (2005).

[6]Weymark (2006) discusses indicators of overall social condition defined directly on multidimensional matrices. Of course, a traditional social welfare function (SWF) is also such an indicator. SWFs are discussed in Weymark, chapter 5, the OUP Handbook, and in this chapter with reference to the inclusive-measure of well-being approach (IMWB) to multidimensional inequality and poverty metrics.

concepts relevant to our exposition, such as inequality, become meaningless for a single-person society, it is assumed that each society contains at least two persons.

We denote the number of persons in the society by n (with $n \in N$), where N is the set of positive integers. Let d be the number of such dimensions, where $d \geq 2$ is an integer. We assume that the number of dimensions d is fixed—and exogenously given—in order to make meaningful comparisons of well-being across populations.

Let $x_{ij} \geq 0$ be the achievement of person i in attribute or dimension j, where achievement indicates the performance of a person in a given dimension such as income or education. Person i's achievements in different dimensions are summarized in a d-dimensional vector $x_{i.} = (x_{i1}, x_{i2}, \ldots, x_{id})$. The row vector $x_{i.}$ is the ith row of an $n \times d$ distribution matrix X. The column vector $x_{.j}$, which summarizes the distribution of achievements in dimension j $(j = 1, 2, \ldots, d)$ among n persons, is the jth column of X and we denote the mean of this vector by $\mu\left(x_{.j}\right)$.

In a four-person society with three dimensions of well-being (say, years of education, a six-point health score, and income), an example of a distribution matrix X is

$$X = \begin{bmatrix} 9 & 5 & 1200 \\ 6 & 3 & 900 \\ 8 & 6 & 1000 \\ 1 & 4 & 400 \end{bmatrix}.$$

The entry in the third row and first column of the above matrix show that persons 3's achievement in dimension 1 (education) is 8. Other entries of the matrix can be similarly explained. If the set all $n \times d$ matrices with non-negative entries is represented by M, then $X \in M$.

Finally, we define a d-dimensional vector $z = (z_1, z_2, \ldots, z_d)$, where each element z_j is the poverty threshold for dimension d. An individual i is considered deprived (or poor) in dimension d if her achievement $x_{id} < z_d$. For instance, a relevant poverty lines vector for the matrix X above could be,

$$z = \begin{bmatrix} 9 & 5 & 500 \end{bmatrix}.$$

In this example, person 1 will be rich in all three dimensions, since her achievements lie always above or at the respective threshold, whereas person 2 is deprived in education and health but his income level (900) is above the minimum required to be considered deprived.

3 Multidimensional Inequality

We divide our discussion into two subsections. Section 3.1 describes the direct approach, whereby axioms and indicators are specified directly in terms of distribution matrices. Section 3.2 describes the derivation of multidimensional inequality metrics from an inclusive measure of well-being (IWMB).

3.1 The Direct Approach

The distribution of well-being has been the concern of social scientists since at least Smith's (1776) *An Inquiry into the Nature and Causes of the Wealth of Nations*.[7] In the last 50 years, as household data became more easily available, economists attempted to define ways of measuring the extent to which the observed distributions differ from some ideal one. In the beginning of the 1970s, almost simultaneously, Atkinson (1970), Kolm (1969), and Sen (1973) proposed a normative view to measuring inequality as the loss in social welfare due to the fact that income (seen here as the measure of each individual's well-being) is not distributed equally among all individuals. This approach is univariate (unidimensional) because $d = 1$; no dimension of individual achievement other than income is included.

At this point several important families of univariate inequality indices have been characterized using Aktinson–Kolm–Sen's normative approach. Among the relative inequality indices, these include the Gini coefficient, Atkinson index, Theil 0 and Theil 1 (belonging to the General Entropy class of measures), and the Dalton Index. Within the absolute measures, the Kolm index, the variance, and the absolute Gini coefficient are the most widely used ones. All of these measures have been characterized axiomatically, from a set of desirable properties that either the underlying social welfare function or the inequality index itself is required to satisfy (Ebert 1988). By setting the desiderata upfront, all values are made explicit. The family of measures derived is the one that satisfies these postulates simultaneously. (Detailed discussions along this line are available in Cowell's chapter of the OUP Handbook.)

In the last 20 years, various authors have presented generalizations of the most salient univariate inequality measures along with their extensions in the multidimensional context. In this chapter, the focus will be on the discussion on the postulates behind multidimensional indicators where the extension is less straightforward.[8] In particular, we will discuss invariance, distributional, and decomposability properties (formal definitions of axioms are relegated to the appendix). We will also present a selection of multidimensional indices to illustrate how these properties are applied.

A multidimensional inequality indicator I is a real-valued continuous function defined on set of well-being matrices M. More precisely, $I : M \to \Re^1$, where $\Re^1$ is the set of real numbers. For any $X \in M$, $I(X)$ determines the extent of inequality that exists in the distribution matrix X.

We divide this subsection in three parts, based on the nature of the properties.

[7] See also Rousseau's (1754) *Discourse on the Origin and Basis of Inequality Among Men.*

[8] We do not discuss here Normalization, Symmetry, Population Replication Invariance, and Continuity which are presented in the appendix with formal notation.

3.1.1 Invariance Properties

Relative inequality indices are those that satisfy a property known as ratio scale invariance. In the unidimensional context, this property ensures that the measurement of inequality does not vary when each person's achievement is multiplied by the same positive constant, such as when incomes are expressed in a different currency unit or when everyone's incomes are increased by the same proportion. The extension to the multidimensional context requires more careful attention, since often the achievements in different dimensions are measured in different units of measurement.

- **Ratio Scale Invariance (RSI)** says that inequality is invariant to proportional changes in the achievements in different dimensions. If, for instance, the duration of education is measured in months instead of years, the evaluation of inequality should not change. The RSI property allows for the rescaling factor to differ across the different dimensions. This is particularly attractive when the variables are expressed in different measurement units, such as income in dollars and schooling in years. Importantly, this property permits the standardization of each vector by an entry-specific rescaling such as division by their respective mean or range. For instance, if distribution X^* expressed each attribute as a proportion of its respective median, a multidimensional inequality index satisfying RSI will consider that,

$$I(X) = I(X^*), \quad \text{where } X = \begin{bmatrix} 9 & 5 & 1200 \\ 6 & 3 & 900 \\ 8 & 6 & 1000 \\ 1 & 4 & 400 \end{bmatrix} \text{ and } X^* = \begin{bmatrix} 1.3 & 1.1 & 1.3 \\ 0.9 & 0.7 & 0.9 \\ 1.1 & 1.3 & 1.1 \\ 0.1 & 0.9 & 0.4 \end{bmatrix}$$

  Note that X^* is obtained by dividing each of the elements in matrix X by the median of each of the attributes (columns).[9] For instance, the median of the first attribute (years of education) equals 7, thus the first element in matrix $X^* = 9/7 = 1.3$. Similar calculation holds for other entries in X^*.
  An inequality indicator satisfying RSI is called *relative.*
  On the other hand, RSI can be disputed because it implies that proportional changes in one dimension (say, doubling of incomes) have no impact on overall inequality, ignoring possible interactions across dimensions. A stronger version of this property (strong RSI) requires instead that the inequality index should remain constant only when all attributes are rescaled *by the same factor*. That is, when all attributes are doubled, then the measurement of multidimensional inequality should not change. This property is particularly appealing when all attributes are measured in the same scale.
- **Unit Consistency (UCO)** is a weaker form of ratio scale invariance which demands that the inequality ordering[10] of two distribution matrices should remain unaltered

[9]The median of an odd number of observations that are non-decreasingly ordered is the middle-most observation. For the first column of X, the non-decreasingly ordered rearrangement is (1, 6, 8, 9) and the median of these numbers is 7, the average of the two middle numbers.

[10]By inequality ordering, we mean the ranking of matrices by the inequality index.

under changes in the scales of dimensions (Zheng 2007a, b; Diez et al. 2008; Chakravarty and D'Ambrosio 2012). To illustrate this, suppose of two countries, I and II, country I has lower multidimensional inequality than country II. Assume that in both the countries incomes are expressed in the currency of country I and the unit of educational attainment is one year. Now, let incomes in the two countries be converted into the currency of country II and educational attainments be measured in months, while the units of measurement of all other dimensions are assumed to remain unaltered. Unit consistency demands that the ranking of the two countries by the multidimensional inequality index should remain unchanged under this alteration of units of measurement of two dimensions. As we will observe, all ratio scale invariant multidimensional inequality indices are unit consistent, but there exist unit consistent indices which do not satisfy ratio scale invariance.

- **Translation Scale Invariance** (**TSI**), suggested by Kolm (1976), requires that the addition of a constant to the quantities of different attributes does not alter the level of inequality. If everyone's health scores move up two points, then overall inequality will not change. The implication is that from a normative perspective, it does not matter where the zero is set. An inequality indicator satisfying this property is called *absolute*.

Ratio scale invariance and translation scale invariance represent two different value judgments concerning inequality invariance. These two axioms cannot be satisfied simultaneously by a multidimensional inequality indicator—except for a trivial indicator that assigns the same number to all distribution matrices.

3.1.2 Distributional Properties

Distributional axioms specify when a redistribution of achievements *between* individuals increases or decreases inequality. In the unidimensional framework, distributional concerns are generally introduced through the Pigou–Dalton transfers principle (Pigou 1912; Dalton 1920). This postulate demands that a progressive transfer, a transfer of income from a person to a poorer one, should decrease inequality, provided that the donor does not become poorer than the recipient as a result of the transfer and all other incomes remain unaffected. There are a number of ways in which this principle has been extended to the multivariate framework. In the present review, we will include three of the most widely used ones.

Formally, a Pigou–Dalton transfer can be expressed in terms of a T-transformation. The formulation can be motivated by an example. Let $y = (3, 6, 7)$ and $x = (4, 5, 7)$ be two income distributions so that x is obtained from y by a Pigou–Dalton transfer of 1 unit of income from the second person to the first person. This transfer can also be expressed in the following way: $(4, 5, 7) = (3, 6, 7)\left(\frac{2}{3}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\right)$.
The first matrix within the first bracketed term on the right-hand side is a 3×3 identity matrix each of whose diagonal elements is one and off-diagonal elements is zero. The

second matrix is a 3×3 permutation matrix, a matrix with entries 0 and 1, and each of whose rows and columns sums to one. This matrix is obtained by exchanging the first two rows of the identity matrix. The remaining row corresponds to the person unaffected by the transfer. A weighted average of these two 3×3 matrices, where the weights are respectively $\frac{2}{3}$ and $\frac{1}{3}$, after matrix-multiplication with (3, 6, 7), gives us the distribution (4, 5, 7). The weighted average $\left(\frac{2}{3}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\frac{1}{3}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\right)$ is known as a T-transformation (for more on this, see Marshall et al. 2011; Weymark 2006; Chakravarty 2009).

The unidimensional Pigou–Dalton transfer principle can be extended straightforwardly to the multidimensional case by applying the same sequence of T-transformations to all the dimensions, as per the following postulate:

- **Uniform Pigou–Dalton Transfers Principle (UPD)** says that for any two distribution matrices X and Y if X is obtained from Y by multiplying by a finite number of T-transformations, then X has less inequality than Y.

However, the justifiability of UPD can be disputed. The complexity of extending the Pigou–Dalton principle to multiple dimensions arises because of, precisely, the existence of the other dimensions. Consider a case in which a Pigou–Dalton transfer is implemented for each dimension between two individuals. If the donor has more achievements than the recipient in some dimension (say, income) but less in others (say, health and education), then it is not clear whether an income transfer from the donor to the recipient reduces multidimensional inequality. Fleurbaey and Trannoy (2003) offer a restricted version of the above, confining the relevant transfers to be among individuals where the giver is at least as well-off as the recipient in every dimension:

- **Pigou–Dalton Bundle Transfers Principle (PBT)** represents the idea that if, between two individuals, one has at least as much achievement in every dimension as the other and strictly more in at last one dimension, then dimension-wise Pigou–Dalton transfers from the former to the latter in one or more dimensions reduces multidimensional inequality, given that achievements of all other individuals remain unaffected.

Unfortunately PBT comes into conflict with efficiency. Fleurbaey and Trannoy (2003) formally demonstrated that under certain very mild conditions a social ranking of distribution matrices cannot simultaneously satisfy PBT and the Weak Pareto Principle, which demands that if each individual prefers her vector of achievements in ne matrix to a second, then the first is socially better than the latter.[11] See also Fleurbaey and Maniquet (2011), Weymark (2013) and Bourguignon and Chakravarty (2003) for a variant of PBT, referred to as Multidimensional Transfer Principle.

[11] It may be worthwhile to mention that, following the literature, our formulation in this chapter uses directly individual achievements. Therefore, our presentation has ignored individual preferences.

A third alternative for extending the unidimensional Pigou–Dalton transfer principle to the multidimensional context is presented by Kolm (1976)—for discussions, see also Marshall et al. (2011), Duclos et al. (2006, 2007), Weymark (2006) and Chakravarty (2009). In this case, the series of transfers are the same (in percentage terms) in all dimensions. Specifically, the following is noteworthy.

- **Uniform Majorization Principle** (**UM**) requires that if there is a similar smoothing of achievements in all the dimensions, multidimensional inequality should decrease. For example, consider a matrix Y which is obtained from X after a sequence of (mean-preserving) equalizing transfers across individuals for each dimension.[12]

$$X = \begin{bmatrix} 9 & 5 & 1200 \\ 6 & 3 & 900 \\ 8 & 6 & 1000 \\ 1 & 4 & 400 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 6.2 & 4.5 & 890 \\ 6.7 & 3.9 & 970 \\ 7.6 & 5.5 & 1000 \\ 3.5 & 4.1 & 640 \end{bmatrix}.$$

We note that the sum of all entries in each column of the two matrices X and Y is the same. Under this operation there is a smoothing of the distribution of achievements in each dimension and all the dimensions are considered simultaneously. UM says that Y should have lower inequality than X.

Lasso de la Vega et al. (2010) made a systematic comparison between PBT and UM. Under UM one distribution matrix is obtained from another by transferring achievements in all dimensions in the same proportions. This notion of transfer is not valid if some of dimensions are ordinally measurable (see Sect. 4.3 for a discussion on ordinal measurability of dimensions). In addition, and crucially, if transfers are made between two persons in all dimensions where one is not unambiguously richer than the other, then there is ambiguity regarding treatment of the new distributions as more equitable. PBT takes care of all these difficulties. By definition, the transfer is performed between two persons, one richer than the other. Also, the transfers in different dimensions need not be made in the same proportions, or even at all in some dimensions. The distinction between these two principles is particularly important since, as Lasso de la Vega et al. (2010) noted, not all inequality indices, including those satisfying UM, will satisfy PBT.

While the different versions of the Pigou–Dalton principle focus on the redistribution of attributes among the persons, there is a second important form of inequality that arises only in the multidimensional context. Atkinson and Bourguignon (1982)

[12] Formally, uniform mean-preserving averaging (smoothing) can be obtained by multiplying the distribution matrix X by a bistochastic matrix, which is a square nonnegative matrix of appropriate order where all the rows and columns add up to 1. UM says that BX should have lower inequality than X. The B matrix in this case is

$$B = \begin{bmatrix} 0.2 & 0.3 & 0.3 & 0.2 \\ 0.4 & 0.5 & 0 & 0.1 \\ 0.3 & 0 & 0.6 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.6 \end{bmatrix}.$$

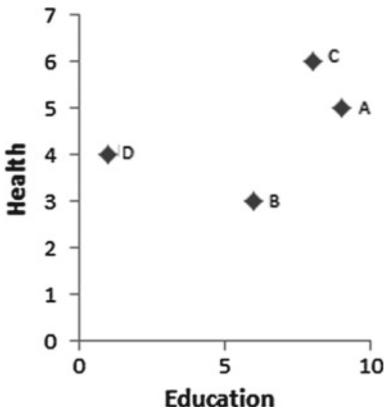


Fig. 1 Example: Distribution matrix X of health and education in a four-person society

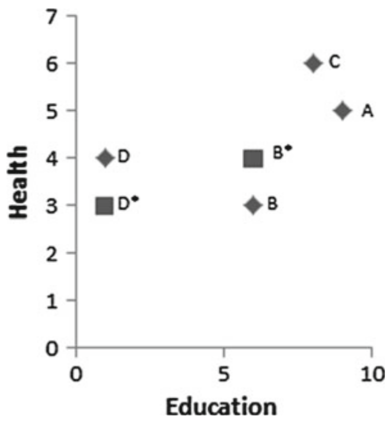
argued that a multidimensional inequality indicator should capture the association (more precisely, rank correlation) between distributions of achievements. Following Epstein and Tanny (1980) and Tchen (1980), the authors introduced the concept of a *correlation increasing switch* between two individuals, whereby one individual receives at least as much of every attribute as the other and more of at least one attribute (see also Boland and Prochan 1988; Decancq 2012). To understand this, suppose that in the original distribution X presented above $x_{11} > x_{21}$ but $x_{22} > x_{12}$. That is, the second person (person B) has six years of education (while person D has only one), and scores three points in health (whereas person D scores 4). This situation is represented in Fig. 1, with diamond-shaped dots—for simplicity of exposition income is ignored in this figure.

If we make a switch of the second attribute, say health, between the two individuals, then their achievements after the switch are given by $y_{11} = x_{11}$, $y_{12} = x_{22}$, $y_{21} = x_{21}$ and $y_{22} = x_{12}$ (positions B* and D* in Fig. 2 for persons B and D, respectively). Person B, who had higher achievement in education, has higher achievement in health as well after the switch. Consequently, the correlation between the attributes has gone up. Note that a correlation increasing switch keeps the mean of each attribute constant, like UPD, PBT and UM.

Tsui (1999) formally introduced this idea to the literature on multidimensional inequality indices via an axiom know as Correlation Increasing Majorization:

- **Correlation Increasing Majorization** (**CIM**) states that if a distribution Y is obtained from another distribution X by a switch in attributes such that the correlation across these attributes is increased, then Y is more unequal than X.

Fig. 2 Example: Distribution matrix X of health and education after a correlation increasing switch



In the example, consider the distributions $X = \begin{bmatrix} 9 & 5 & 1200 \\ 6 & 3 & 900 \\ 8 & 6 & 1000 \\ 1 & 4 & 400 \end{bmatrix}$ and $Y = \begin{bmatrix} 9 & 5 & 1200 \\ 6 & 4 & 900 \\ 8 & 6 & 1000 \\ 1 & 3 & 400 \end{bmatrix}$.

In all of the dimensions, except in dimension 2 (say, health), the achievements of the second person (B) in the distribution X are more than the corresponding achievements of the fourth person (D). The distribution matrix Y obtained from X by a switch in achievements in health between these two individuals is such that the second person has now higher achievements than the fourth one in all three dimensions. This transfer has increased the correlation between dimensions which implies that the situation of the person who was better off in some dimensions is now also better off in the other dimension. CIM will assess this new distribution as being less equal (not preferable) to the original one.

Tsui (1999) showed that UM and CIM are independent axioms. That is, there exist indicators that satisfy both UM and CIM; and also there are indicators that satisfy UM but are violators of CIM and vice versa. Weymark (2006) and Chakravarty (2009) provide further discussion along this line.

A weaker version of CIM has been proposed by Dardanoni (1996) as follows:

- **Unfair Rearrangement Principle** (**UR**) requires that the initial distribution matrix is preferred to one in which the distributional profiles in all dimensions are unaltered but where the dimensions are perfectly rank-correlated.

To understand this property, let us assume a new distribution Z where one person is ranked first in all dimensions, another one is ranked second in all dimensions and so on. For instance, consider

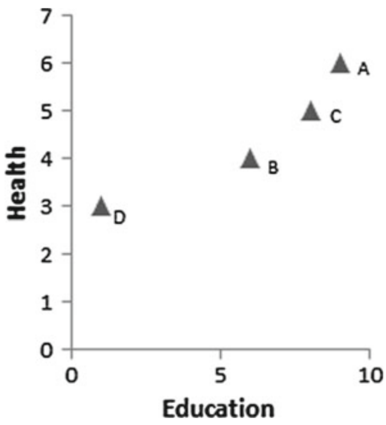


Fig. 3 Example: Distribution matrix $\bar{X}$ of health and education after a switch that makes dimensions perfectly correlated

$Z = \begin{bmatrix} 9 & 6 & 1200 \\ 6 & 4 & 900 \\ 8 & 5 & 1000 \\ 1 & 3 & 400 \end{bmatrix}$, depicted in Fig. 3 [once again, for clarity of exposition the figure only depicts education and health, but income also follows the same rule]. UR implies that distributions X and Y will be preferred to distribution Z, but does not determine the ranking of X versus Y. Thus, UR is indeed a weak property and can be seen as a minimum requirement to be imposed when correlation across dimension is deemed undesirable.

3.1.3 Decomposability Properties

Since the mid-1980s, many multidimensional inequality indicators have been proposed in the literature that can be seen as extensions of the most widely used measures on inequality in the unidimensional framework, including Gini, Atkinson, Generalized Entropy (Theils), and Kolm indices. Table 1 presents a selection of extension of these indices, and the properties that they satisfy. Only for exposition purposes, we consider at least one measure for each family of unidimensional indices, but recognize that the literature contains many more measures that are not discussed here.

Tsui developed a characterization of the multidimensional Atkinson inequality indicator (Tsui 1995), as well as an extension of the Generalized Entropy inequality index (Tsui 1999). These indices have the advantage of being able to satisfy a convenient property related to the decomposability of the measures. One such property is the following:

- **Subgroup Decomposability** (**SDE**): For any partitioning of the population into subgroups such as race, religion, sex, ethnic groups, age etc., overall inequality can be expressed in terms of inequality levels of subgroups, vectors of means

Table 1 Inequality

Authors	Family	Measure	Properties					
			RSI	UCO	TSI	UM	CIM	UR
Tsui (1995)	Atkinson	$I_{AM}(X)=1-\left[\frac{1}{n}\sum_{i=1}^{n}\prod_{j=1}^{d}\left(\frac{x_{ij}}{\mu(x_j)}\right)^{c_j}\right]^{\frac{1}{\sum_{j=1}^{d}c_j}}$ $I_{AM}(X)=1-\left[\frac{1}{n}\prod_{i=1}^{n}\prod_{j=1}^{d}\left(\frac{x_{ij}}{\mu(x_{.j})}\right)^{\frac{c_j}{\sum_{j=1}^{d}c_j}}\right]^{\frac{1}{n}}$	Yes	Yes		ucc	ucc	
Gajdos and Weymark (2005)	Generalized Gini	$I_{GWR}(X)=1-\frac{\left[\sum_{j=1}^{d}a_j\left(\sum_{i=1}^{n}b_{ik}x_{ij}^{0}\right)^{\omega}\right]^{\frac{1}{\omega}}}{\sum_{j=i}^{d}a_j\mu(x_{.j})}$ $I_{GWR}(X)=1-\frac{\prod_{j=1}^{d}\left(\sum_{i=1}^{n}b_{ij}x_{ij}^{0}\right)^{a_j}}{\prod_{j=1}^{d}\mu(x_{.j})^{a_j}}$	Yes	Yes		ucc		
Decancq and Lugo (2012)	Generalized Gini	$I_{DL}(X)=1-\frac{\sum_{i=1}^{n}b_i\left(\sum_{j=1}^{d}a_j(x_{ij})^{\omega}\right)^{\frac{1}{\omega}}}{\left(\sum_{j=1}^{d}a_j(\mu(x_j))^{\omega}\right)^{\frac{1}{\omega}}}$ $where\quad b_i=\left(\frac{r^i}{n}\right)^{\delta}-\left(\frac{r^i-1}{n}\right)^{\delta}$	Yes	Yes		ucc		ucc

(continued)

Table 1 (continued)

Authors	Family	Measure	Properties					
			RSI	UCO	TSI	UM	CIM	UR
Bosmans, Decancq and Ooghe (2013)	CES	$BDO^{(X)} = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\sum_{i=j}^{d}\left(a_j x_{ij}^{1-\beta}\right)}{\sum_{j=1}^{d}\left(a_j \mu(x_{.j})^{1-\beta}\right)}\right)^{\frac{1-\alpha}{1-\beta}}\right]^{\frac{1}{1-\alpha}}$	Yes	Yes		Yes	ucc	
Tsui (1999)	Generalized Entropy	$I_{TME} = \frac{e}{n}\sum_{i=1}^{n}\left(\prod_{j=1}^{d}\left(\frac{x_{ij}}{\mu(x_{.j})}\right)^{c_j} - 1\right)$	Yes	Yes		ucc	ucc	
Maasoumi (1986)	Generalized Entropy	$I_{MM}(X) = \begin{cases} \frac{1}{nc(c-1)}\sum_{i=1}^{n}\left[\left(\frac{x_i}{\mu(x)}\right)^{c} - 1\right], c \neq 0, 1, \\ \frac{1}{n}\sum_{i=1}^{n}\left[\log\left(\frac{\mu(x)}{x_i}\right)\right], c = 0, \\ \frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{x_i}{\mu(x)}\right)\log\left(\frac{x_i}{\mu(x)}\right)\right], c = 1, \end{cases}$	Yes	Yes				
Bourguignon (1999)	Dalton	$I_{BDM}(X) = 1 - \frac{\sum_{i=1}^{n}\left(a_2 x_{i1}^{-\delta} + a_2 x_{x2}^{-\delta}\right)^{\frac{-(1+c)}{\delta}}}{n\left(a_1(\mu(x_{.1}))^{-\delta} + (a_2\mu(x_{.2}))^{-\delta}\right)^{\frac{-(1+c)}{\delta}}}$		Yes		Yes		ucc
Tsui (1995)	Kolm	$I_{KPM}(X) = \frac{1}{\sum_{j=1}^{d}\beta_j}\log\left[\frac{1}{n}\sum_{i=1}^{n}\exp\left(\sum_{j=1}^{d}\beta_j\left(\mu(x_{.j}) - x_{ij}\right)\right)\right]$			Yes	ucc		

Note: "ucc" refers to "under certain conditions" and means that the measures could satisfy the property by setting restrictions to the parameters. RSI: Ratio scale invariance, UCO: Unit consistency, TSI: Translation scale invariance, UM: uniform majorization, CIM: correlation increasing majorization, UR: unfair rearrangement principle of attributes corresponding to different subgroups and population sizes of the subgroups.

Such decompositions become particularly useful for policy makers interested in determining the significance of variations of attributes corresponding to these various characteristics.

A second well-known family of measures is associated with the widely used Gini inequality index. This index has several multivariate extensions (see for example, Koshevoy and Mosler 1996; List 1999; Gajdos and Weymark 2005; Banerjee 2010; Decancq and Lugo 2012). In this chapter, we consider two that are explicitly normative, as characterized by Gajdos and Weymark (2005) and by Decancq and Lugo (2012), respectively. The two indices differ mainly in the order of aggregation of dimensions and individuals; the former aggregates first across individuals and then across dimensions whereas the latter does the reverse. As in the univariate case, these multidimensional extensions of the Gini index are not subgroup decomposable. Yet, the measure proposed by Gajdos and Weymark is separable across dimensions of well-being (that is, overall inequality can be calculated as a function of the inequality in each of the separate dimensions). Formally, this measure satisfies a restricted form of attribute separability proposed by Shorrocks (1982).

- **Factor Decomposability** (**FDE**): Overall inequality is the sum of attribute-wise indicators. FDE becomes helpful for assessment of inequality contribution of different dimensions of well-being.

The Gajdos and Weymark generalized Gini index satisfies FD for $\alpha = \theta = 1$ (see Table 1). Nonetheless, the cost of satisfying FDE is that Gajdos and Weymark's measure (as any two-step measure that first aggregate across dimensions and then across attributes) is insensitive to changes in the correlation across the different attributes. Instead, Decancq and Lugo's Gini measure is able to satisfy UR for specific choices of parameter values at the expense of a weaker separability axiom, that is, the axiom of rank-dependent separability which states that the comparison of two distributions is not affected by the magnitude of the common attributes as long as the initial ranking is maintained (formal definition in the appendix).

3.2 The Inclusive—Measure-of Well-Being Approach

This section considers the inclusive measure of well-being approach (IMWB), which assigns a well-being number to each person i as a function of the person's achievements in all d dimensions. These indices of individual well-being can be then aggregated across persons to arrive at an evaluation of "social welfare" (overall social condition), inequality or poverty. Formally, person i's IMWB is denoted by $U(x_{i.}) = U(x_{i1}, x_{i2}, \ldots, x_{id})$, where $U : Q \to \Re^1$ is the individual well-being function, $Q \subset \Re^d$ being the set of all achievements that individuals can possess in the d dimensions. A social policy evaluation metric W ranks outcomes by incorporating the associated well-being numbers. In other words, the distribution of achievements $(x_{1..}, x_{2.}, \ldots, x_{.n.})'$ is at least as good as the distribution $(y_{1..}, y_{2..}, \ldots, y_{n..})'$ if and only if W ranks $(U(x_{1..}), U(x_{2...}), \ldots, U(x_{n..}))$ at least as good as $(U(y_{1..}), U(y_{2...}), \ldots, U(y_{n..}))$.

There are two distinct ways in which the IMWB approach can be used to derive a multidimensional inequality indicator. In the first variation, W is a social welfare function (SWF), which is then used to construct a multidimensional inequality indicator. In the second variation, W is a unidimensional inequality index, which is applied directly to the vector of individual well-being levels (as in Maasoumi 1986).

In the first alternative, a SWF ranks vectors of individual well-being numbers, or "utilities." See Weymark, chapter 5, the OUP Handbook. In the literature on income inequality, individual income is often seen as a proxy for individual welfare, and thus a SWF is used to rank vectors of individual incomes. See Cowell, chapter 4, the OUP Handbook. The Atkinson-Kolm-Sen (AKS) approach, used to deriving an income inequality metric from an SWF applied to incomes, is as follows. The AKS representative income x_e corresponding to the distribution x is the level of income which, if enjoyed by everybody, would make the distribution x ethically indifferent, that is, $W(x_{e.}, x_e, \ldots, x_e) = W(x)$. The AKS relative inequality index I_{AKS} is thus defined as the proportionate gap between x_e and the mean income $\mu(x)$. When efficiency considerations are absent, that is, when the mean income is fixed, an increase in social welfare is equivalent to a reduction in inequality and vice versa. From a policy perspective, this inequality index determines the fraction of total income that could be saved if the society distributed incomes equally without any loss of social welfare or, in other words, the fractional social welfare loss resulting from the existence of inequality.

We can now describe the first variation of the IMWB approach. Kolm (1977) extends the AKS approach to the multidimensional context—showing how to derive a multidimensional inequality indicator from a social ranking of matrices, such as the ranking defined by an SWF.

Let us define X_λ as the distribution matrix in which each person enjoys the average level of achievements in each dimension $\mu\big(x_{.j}\big)$ so that X_μ represents the perfectly equal situation. Now, define $\Lambda(X)$ implicitly by $W\big(X_\mu \Lambda(X)\big) = W(X)$, that is, as a positive scalar which, when multiplied by the ideal distribution matrix X_μ, is socially or ethically indifferent to the existing distribution matrix X (according to W). $\Lambda(X)$ is the multidimensional counterpart to the Atkinson-Kolm-Sen representative income. Given appropriate assumptions about W, $\Lambda(X)$ is well-defined and $0 < X_\mu < 1$ if $X \neq X_\mu$ and takes on the maximal value 1 when each attribute is equally distributed among the individuals (Weymark 2006).

The multidimensional Kolm (1977) inequality indicator $I_{\text{KM}} : M \to \Re^1$ is defined as $I_{\text{KM}}(X) = 1 - \Lambda(X)$, where $X \in M$ is arbitrary. I_{KM} determines the fraction of welfare loss incurred by moving from the ideal distribution X_μ to the actual distribution X. If there is only one dimension, say income, I_{KM} coincides with the Atkinson–Kolm–Sen inequality index. Assume that the W fulfils the strong Pareto principle, is continuous and increasing under a smoothing of the distribution of achievements. Given these assumptions, the continuous indicator I_{KM} satisfies symmetry (SYM) and UM.[13] For an unequal distribution matrix X, I_{KM} is positive and bounded above by one. I_{KM} takes on the minimum value zero if $X = X_{\mu}$. The behaviour of I_{KM} under a correlation increasing switch depends on the form of the utility function.

The procedure may be illustrated using some examples. Tsui's (1995) characterization of the multidimensional Atkinson inequality index can be accommodated within the IMWB approach. Tsui characterized the symmetric utilitarian social welfare function $W(X) = \sum_{i=1}^{n} U\left(x_{i.}\right)$, where the identical individual multi-attribute utility function is either of the product type or of the logarithmic type.[14] For this form of the utility function, the resulting Kolm (1977) inequality index becomes the multidimensional Atkinson inequality index. If there is only one dimension, the formula coincides with the single dimensional Atkinson (1970) index.

Another example of the first variation of the IMWB-based approach is the double-CES multi-attribute inequality indicator suggested by Bosmans et al. (2013). The individual utility function aggregates individual achievements (assumed to be always positive) using a CES-type aggregator. Next, the social welfare function uses a CES function to aggregate utilities at the social level. The corresponding Kolm (1977) multi-attribute inequality indicator is the Bosmans–Decancq–Ooghe (2013) multidimensional inequality index. This symmetric index satisfies UM for all permissible values of the parameters. It is unambiguously increasing under a correlation increasing switch if the parameter associated with the CES utility function is higher than the corresponding parameter in the welfare function.[15]

These two examples clearly demonstrate that there are multi-attribute inequality indices that relate to social welfare functions applied to some measure of individual well-being. They show that the two-stage approach can be justified by a solid theoretical background within the normative framework. But there also exists inequality indices that cannot be supported by the IMWB structure. The Gajdos–Weymark generalized Gini index is an example of an inequality indicator that cannot be supported by the IMWB structure because in this case aggregation is first done across individuals and then the obtained values are aggregated across dimensions. It is in fact the Kolm index I_{KM} where the underlying multidimensional generalized Gini social evaluation function is defined directly on the set of distribution matrices, rather than being a social welfare function operating on vectors of individual utilities. The social evaluation function is assumed to satisfy continuity, strong Pareto principle and increasingness under a smoothing of the distribution of achievements (see Weymark 2006).

[13]SYM demands that any reordering of the individuals does not change inequality. That is, any characteristic other than the achievement levels, for example, the names of the individuals, is irrelevant to the measurement of inequality.

[14]Formally, the individual utility function is defined as a strictly increasing concave function assuming the forms: $a + b\prod_{j=1}^{d} x_{ij}^{c_j}$ or $a + b\sum_{j=1}^{d} c_j \log x_{ij}$, where a is an arbitrary constant, and the parameters b and c_j should be appropriately restricted to ensure that $U(.)$ is increasing and strictly concave.

[15]For a characterization of a multidimensional social welfare function where the individual well-being function is linear, see Bosmans et al. (2009).

The second variation of the IMWB approach is suggested by Maasoumi (1986). The author developed the first extension of the Generalized Entropy index to the multidimensional set up using a CES-type utility function in the first step to aggregate dimensional achievements of an individual, and a Generalized Entropy-type aggregation of individual utilities in the second step. In other words, Maasoumi employs a uni-dimensional inequality metric, instead of a welfare function, to aggregate individual well-beings. Unfortunately, Maasoumi's index has the weakness that it may not satisfy UM or other multivariate formulations of the Pigou–Dalton principle. The Pigou–Dalton principle is satisfied, however, in the (unidimensional) well-being space.[16]

4 Multidimensional Poverty

Even in the early twenty-first century, poverty alleviation remains one of the major economic policies in many countries of the world. In order to understand the depth and threat of poverty, it is helpful to quantify poverty and measure its change over time. The objective of this section is to briefly outline different poverty measurement methodologies that have been suggested in the literature and that adopt an explicitly multidimensional structure, as adopted, among others, by Tsui (2002), Bourguignon and Chakravarty (2003) and Alkire and Foster (2011). As in Sect. 3, we first discuss the direct approach—the main approach in the literature—whereby axioms and poverty measures are defined directly on multidimensional matrices. See also Duclos and Tiberti's chapter 23 in the OUP Handbook for a similar discussion. We then turn to the IMWB perspective on poverty measurement.

4.1 The Direct Approach

We divide the discussion in this subsection into several parts.

4.1.1 Properties

Since well-being of a population is a multidimensional phenomenon, poverty, which arises because of insufficiency of achievements in one or more dimensions, is as well a multidimensional aspect of human life. As Sen (1976) argued, in income poverty measurement two exercises are involved: (i) the identification of the poor and (ii)

[16]Tsui (1999) also proposes a multidimensional extension of the Generalized Entropy index but he does it in one stage, and thus it is not directly based on individual well-being levels—i.e. is based on a direct approach. The literature contains many more indices that do not use such a two-step aggregation method (for further discussion see Chakravarty 2009, Chap. 5).

aggregation of the characteristics of the poor into an overall indicator of poverty in society. The former problem requires the specification of a poverty line, the income necessary for a subsistence standard of living. A person is regarded as income poor if his income falls below the poverty line. The second problem requires aggregation of income shortfalls of the poor from the poverty line. See Cowell, chapter 4, the OUP Handbook, discussing univariate (income) poverty measures.

Following Sen (1976), various authors have suggested extensions of the standard properties associated with each of these two steps for the multivariate setting and derived multidimensional poverty measures. The introduction of multiple dimensions requires an additional step in the derivation of the poverty measure, which is the aggregation across dimensions. The dimension-wise aggregation is done before the aggregation across individuals (step ii above) but can be done either before or after the specification of the poverty threshold (step i). The decision on the sequence of these steps will have implications in terms of the substitutability assumed across dimensions. In fact, most of the proposals in the literature opt to set the poverty thresholds for each dimension and then aggregate each individual's dimension-specific achievements into a single indicator of each individual's poverty.[17] The argument is that each attribute is considered essential so no substitution across dimension should be permitted above and below the "minimum acceptable levels" (Sen 1992, 139). See, for instance, Tsui (2002), Bourguignon and Chakravarty (2003), Chakravarty and Silber (2008) and Alkire and Foster (2011). These exogenously given "minimally acceptable levels" are the threshold limits for different dimensions for a person to be non-deprived in the dimensions.

Formally, we can define a vector of poverty thresholds $z = (z_1, \ldots, z_d) \in Z \subset \Re^d_{++}$, where $\Re^d_{++}$ is the strictly positive part of the d-dimensional Euclidean space. Person i is said to be deprived or non-deprived in dimension j according as $x_{ij} < z_j$ or $x_{ij} \geq z_j$ and he is called non-deprived if $x_{ij} \geq z_j$ for all j. A multidimensional poverty index P is a non-constant real-valued continuous function defined on $M \times Z$, that is, $P : M \times Z \rightarrow \Re^1$. For any $n \in N$, $X \in M$ and $z \in Z$, $P(X; z)$ gives the level of poverty associated with X and the threshold limit vector z.

When thresholds are imposed before the aggregation across dimensions, the identification of who is to be considered poor presents the additional challenge of defining the number of dimensions in which a person needs to be deprived in order to consider her multidimensionally poor. One extreme is known as the *union* method of identification which says that a person is poor if she is deprived in at least one dimension. On the other hand, the *intersection* criterion identifies a person as poor if she is deprived in all d dimensions (see Tsui 2002; Atkinson 2003; Bourguignon and Chakravarty 2003). The Alkire–Foster (2011)'s counting approach, in turn, propose an *intermediate* option that contains these two extremes as special cases. According to these

[17] The alternative method, of aggregating first across dimensions and then setting a poverty threshold, will be discussed in Sect. 4.1 below.

authors a person is identified as multidimensionally poor if she is deprived in at least k dimensions, where $1 \leq k \leq d$, whenever dimensions are weighted equally.[18]

To illustrate the concepts better, suppose threshold vector is $z = (9, 5, 500)$ and consider the matrix X presented above, which we repeat here for ease of exposition.

$$X = \begin{bmatrix} 9 & 5 & 1200 \\ 6 & 3 & 900 \\ 8 & 6 & 1000 \\ 1 & 4 & 400 \end{bmatrix}$$

According to the union rule the second, third, and fourth persons are multidimensionally poor since each of them is deprived in at least one attribute. Person two is deprived in dimensions 1 and 2, person three is only deprived in dimension 1, whereas person four is deprived in all the dimensions. The intersection approach instead would only identify the fourth person as poor. Finally, if a person is considered multidimensionally poor if she is deprived in at least two dimensions ($k = 2$), then the intermediate approach will identify persons two and four as poor, while person three will be considered non-poor.

As in the case of inequality indices presented in Sect. 3, multidimensional poverty measures can be obtained by defining a set of desirable properties (axioms) that the index should satisfy. Most of the postulates we consider below are immediate generalizations of different axioms proposed for an income poverty index.[19] Unless stated otherwise, all the axioms and indicators presented in this section follow the union rule of identification.[20]

One of the most important postulates in poverty measurement is the requirement of focus on the poor, that is, those whose well-being fall below the poverty threshold. Extending this principle to the multivariate setting has two main variations:

- **Weak Focus** (**WFC**): Poverty does not change under an improvement in the achievement of a non-poor person (Bourguignon and Chakravarty 2003).

In the example presented above, since person 1 in distribution X is non-deprived in all three attributes, an increase in this person's achievement in any dimension should not affect poverty. A stronger version of this axiom has also been put forward.

- **Strong Focus** (**SFC**): If a person is non-deprived in a dimension, then an increase in his/her achievement in the dimension does not change poverty. This holds irrespective of whether the person is deprived or not in any other dimension. Strong Focus rules out the possibility of reducing poverty by subsidizing a poor per-

---

[18]When dimensions are not weighted equally, the condition for a person to be considered multidimensionally poor is when the *minimum dimension weight* $\leq k \leq d$.

[19]For discussion on properties of an income poverty index, see Sen (1976), Foster et al. (1984), Donaldson and Weymark (1986), Chakravarty (1983, 2009), Foster and Shorrocks (1991), and Zheng (1997).

[20]We can as well state these axioms for other rules of identifying the poor.

son in a non-deprived dimension but leaving unaffected her achievements in the dimensions where she is deprived.[21]

For example, if the achievement in dimension 3 (income) of the second person reduces to 750, then the distribution matrix becomes $Y = \begin{bmatrix} 9 & 5 & 1200 \\ 6 & 3 & 750 \\ 8 & 6 & 1000 \\ 1 & 4 & 400 \end{bmatrix}$. SFC demands that poverty remains unchanged because this person, while deprived in dimensions 1 and 2, is not deprived in dimension 3.

If instead of affecting the attainments of dimensions for which the person is not deprived, one modifies the achievement in the deprived dimension, poverty measurement should be impacted. For instance, if we reduce achievement in second dimension of person 4 from 4 to 3, poverty should increase. This is required by **monotonicity**.

- **Monotonicity** (**MON**): A reduction in the achievement of a deprived dimension of a poor person increases poverty.

A second type of monotonicity, relevant for the multidimensional setting, has been introduced by Alkire and Foster (2011).

- **Dimensional Monotonicity** (**DIM**): Poverty should not decrease if a poor person who is non-deprived in a dimension becomes deprived in the dimension.

For instance, if person 3 who is not deprived in dimension 2 in X sees her attainment reduced from 6 to 4, and the distribution becomes $Y = \begin{bmatrix} 9 & 5 & 1200 \\ 6 & 3 & 900 \\ 8 & 4 & 1000 \\ 1 & 4 & 400 \end{bmatrix}$, then DIM requires that $P(X; z) \leq P(Y; z)$. While this property is consistent with both the union and intersection approaches, it is particularly important for the intermediate option proposed by Alkire and Foster, where the number of deprivations suffered by individuals plays a crucial role in the measurement of poverty.

As in the case of inequality indices reviewed in Sect. 3, multidimensional poverty indicators are desired to satisfy three postulates related to decomposition, distribution sensitivity within dimensions and correlation sensitivity across dimensions, in addition to invariance and normalization axioms stated in the Appendix.

Two decomposability postulates are used in the poverty measurement context; the first relates to decomposing the measure across population groups and the second across attributes.

- **Subgroup Decomposability** (**SUD**) says that for any partitioning of the population into subgroups with respect to individuals' exogenous characteristic, like age, sex,

[21] Alkire and Foster (2011) refer to strong focus axiom as poverty focus and weak focus as deprivation focus.

region etc., the overall poverty becomes the population share weighted average of poverty levels of individual subgroups.

SUD shows that the percentage contribution made by subgroup i to the overall poverty is $\frac{n_i P(X^i;z)}{nP(X;z)} * 100$, where n_i is the population size of group i. Such contributions become helpful in isolating subgroups of the population that are more affected by poverty and hence to formulate anti-poverty policy (see Anand 1997; Chakravarty 1983, 2009; Foster et al. 1984; Foster and Shorrocks 1991). Assuming that $P(X;z)$ satisfies SUD, repeated application of the axiom shows that we can write the poverty indicator as

$$P(X;z) = \frac{1}{n}\sum_{i=1}^{n} P(x_{i.};z).$$

Given that $P(x_{i.};z)$ depends only on person i's achievements, we call it 'individual poverty function'. Thus, under SUD, the overall poverty is a simple average of individual poverty levels.

- **Factor Decomposability** (**FAD**) says that the overall poverty is a weighted average of dimensional poverty levels, such that $P(X;z) = \sum_{j=1}^{d} b_j P\left(x_{.j}, z_j\right)$, where $b_j \geq 0$ is the weight assigned to the poverty in dimension j and $\sum_{j=1}^{d} b_j = 1$.

The contribution of dimension j to overall poverty is given by $\frac{b_j P\left(x_{.j};z_j\right)}{P(X;z)}$. FAD was introduced by Chakravarty et al. (1998) and adopted by Alkire and Foster (2011), and is stated under the assumption that only the deprivations of the poor are taken into account and the deprivations of the non-poor are ignored. The coefficient b_j may be interpreted as the importance that a policy maker assigns to eliminating poverty from dimension j. Being able to decompose poverty into the different dimensions is particularly attractive in structuring government policy to reduce poverty—by indicating the attributes where deprivations are the largest. But, as in the case of FDE for inequality measures, the cost of imposing this property is that it makes the measure insensitive to changes in the degree of dependence across attributes.

Regarding distributional properties, we consider the poverty counterpart to UM. Discussions for other variants are similar.

- **Multidimensional Transfers Principle** (**MT**) requires that if a new distribution is obtained by an averaging of achievements among the deprived dimensions of the poor, then poverty should decrease.

In addition, it is often thought to be appropriate that poverty indicators reflect the dependency structure across dimensions. Formally, we consider the following, which is the stronger version of a Bourguignon and Chakravarty (2003) axiom:

- **Increasing Poverty under Correlation Increasing Switch** (**IPC**) requires that poverty should go up after a switch such that the correlation across dimensions is increased. This property is the equivalent of CIM presented above, where attributes are seen as substitutes.

The intuitive reasoning of this property is that among dimensions that fall below their respective poverty thresholds, one can compensate the insufficiency in one attribute (say, education) with additional quantities of another attribute (say, income). If a switch in the quantities of one of the dimensions is performed across two poor individuals such that the person who is more deprived on a second dimension (income) becomes worse off in the first (education) after the switch and poor person who was richer in income has now higher education, poverty should increase. The corresponding property when the attributes are seen as complements requires poverty to decrease under such a switch (DPC). If a poverty indicator remains insensitive to a correlation—increasing switch, then the attributes are regarded as 'independents'. It is evident that a poverty indicator satisfying FAD cannot satisfy at the same time IPC or DPC. In other words, multidimensional poverty measures that are required to be able to be decomposable by dimension, need to assume that deficiencies in one attribute cannot be compensated or complemented with additions of the other attributes.

4.1.2 Indicators

Table 2 presents some examples of multidimensional poverty measures presented in the literature. Chakravarty et al. (1998) (CMR) were among the first to suggest axiomatic multidimensional poverty indicators. One of the most attractive features of this measure is that the function $f(.)$ can be defined such that it becomes generalizations of three well-known one-dimensional poverty indices: the Foster–Greer–Thorbecke (1984), the Chakravarty (1983), and the Watts (1968) unidimensional poverty indices. Tsui (2002) presented a slightly different version of this index that has as special cases both the Charkravarty and the Watts indices.

The CMR indicator satisfies the axioms introduced above, as well as ratio-scale invariance (RSI), except for being sensitive to changes in correlation across attributes. As explained, this is due to the fact that the indicator satisfies FAD which is incompatible with IPC/DPC. In contrast, Tsui's measure is a violator of FAD but satisfies all other axioms including IPC. Tsui also presented a translation invariant poverty index that includes a generalization of the Zheng (2000) single dimensional index and the multidimensional extension of the absolute poverty gap, as special cases.

A highly influential paper in this literature is by Bourguignon and Chakravarty (2003). The measure proposed aggregates a weighted average of individual deprivations across dimensions by taking a power function type transformation over the set of poor persons. The dimension weight a_j may be interpreted as the importance that a policy maker assigns to dimension j. The measure $P_{\alpha,\theta}$ is a single-parameter generalization of the Foster–Greer–Thorbecke (1984) single dimensional index.[22] Since $P_{\alpha,\theta}$ is, in general, not additive across dimensions it does not satisfy FAD; however it fulfills all other axioms for all positive values of parameters and MT for a

[22] Bourguignon and Chakravarty (2003) suggested an alternative generalization of this family using the transformation $f(t) = t^{\alpha_j}$, where $\alpha_j > 1$ is a parameter, in CMR indicator.

Table 2 Poverty

Authors	Measure	Properties						
		SFC	WFC	DIM	SUD	FAD	MTP	IPC/DPC
Chakravarty, Mukherjee and Ranade (1998)	$P_{CMR}(X;z) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{d} b_j\ f\left(\frac{\hat{x}_{ij}}{z_j}\right)$ where *f:[0,1] -> R*^{1} is continuous, decreasing, strictly convex, *f(0)* = 1 and *f(1)*=0	yes	yes	yes	yes	yes	yes	
Tsui (2002)	$P_{TCM}(X;z) = \frac{1}{n}\sum_{j=1}^{n}\left[\prod_{j=1}^{d}\left(\frac{z_j}{\hat{x}_{ij}}\right)^{ej} - 1\right]$	yes	yes	yes		yes	yes	yes
Bourguignon and Chakravarty (2003)	$P_{BC}(X;z) = \frac{1}{n}\sum_{i\in\pi(X;z)}\left[\sum_{j=1}^{d} a_j\left(1-\frac{\hat{x}_{ij}}{z_j}\right)^{\theta}\right]^{\frac{\alpha}{\theta}}$ where α, θ>0 are parameters, and a_i >0 for all $j = 1, 2, \ldots, d$ with	yes	yes	yes	yes		ucc	ucc
Alkire and Foster (2011)	$P_{AFM}(X;z) = \frac{1}{nd}\sum_{i\in\pi(X;z)}\sum_{j=1}^{d} d_{ij}^{\alpha}(k)$	yes	yes	yes	yes	yes	ucc	

(continued)

Table 2 (continued)

Authors	Measure	Properties						
		SFC	WFC	DIM	SUD	FAD	MTP	IPC/DPC
Diez, lasso de la Vega and Urrutia (2008) and Chakravarty and D'Ambrosio (2012)	$P_U(X;z) = \frac{\rho}{n\prod_{j=1}^{d} z_j^{\mu_j-\varepsilon}} \sum_{i\in\pi(X;z)} \left[\prod_{j=1}^{d} z_j^{\mu_j} - \prod_{j=1}^{d} \hat{x}_{ij}^{\mu_j}\right]$	yes		yes	yes		ucc	ucc
Maasoumi and Lugo (2008)	$P_{LMM}(X;z) = \left(\begin{array}{l} \frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{\left(\sum_{j==1}^{d} c_j(\hat{x}_{ij})^{-\delta}\right)^{\frac{-1}{\delta}}}{\left(\sum_{j==1}^{d} c_j(z_j)^{-\delta}\right)^{\frac{-1}{\delta}}}\right)^{\alpha}, \delta \neq 0, \\ \frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{\prod_{j=1}^{d}(\hat{x}_i)^{c_j}}{\prod_{j=1}^{d}(z_j)^{c_j}}\right)^{\alpha}, \delta = 0, \end{array} \right.$		yes	yes	yes		yes	yes
Decancq, Fleurbaey, and Maniquet (2013)	$P\_DFM(X;\Re\_((n))) =$ $\left[1/n\sum 24\_(i=1)^\wedge n \quad \phi(1-\min\{1,\eta(x\_i,\Re\_i)\})\right]$ where $\Re\_((n))$ is the vector of preference of individual n	yes	yes	yes	yes		yes	ucc

Note: "ucc" refers to "under certain conditions", meaning that the measures could satisfy the property by setting restrictions to the parameters subset of these. $P_{\alpha,\theta}$ satisfies IPC or DPC (and even independence) under alternative assumptions about the parameters.

Alkire and Foster (2011) adopted an intermediate identification method, where people are identified as multidimensionally poor if they are deprived in at least k dimensions, where $1 \leq k \leq d$, when dimensions are equally weighted or in at least minimum weight dimensions, where this is $\leq k \leq d$. P_{AFM} is the sum of αth powers of the normalized achievement gaps of the poor divided by the maximal value that this sum can assume. This measure is subgroup decomposable, and meets MTP for values of the parameter $\alpha > 1$. However, the Alkire–Foster measure is non-decreasing under a correlation increasing switch for all $\alpha > 0$, even if transformations of such measure can permit IPC to be satisfied (see Silber and Yalonetzky 2013, for a recent discussion).

Diez et al. (2008) and Chakravarty and D'Ambrosio (2012) axiomatically characterized the family of unit consistent multidimensional poverty indicator. This family of indices satisfies IPC/DPC under certain conditions, allowing for attributes to be considered substitutes or complement, but does not comply with FAD. In addition, if there are only two dimensions MTP holds for a subset of parameter values.

4.2 IMWB-Based Approach to Poverty

In this subsection we briefly analyze the possibility of accommodating multidimensional poverty indices within the IMWB-based approach to poverty measurement. First, we ask whether standard multidimensional poverty indicators, which use a series of dimension-specific poverty thresholds, correspond to a univariate poverty metric applied to a vector of individual well-being numbers.

The issue can be illustrated using some examples. The first example we consider is Tsui's (2002) generalization of the Chakravarty index. From the formulation (in Table 2) it appears that at the first stage for each individual, a product-type well-being function is used to aggregate allocation of the d dimensions into a measure of personal well-being and then at the second stage a simple averaging is applied to aggregate a transformation of these well-being levels. (All achievement quantities are assumed to be positive.) But this well-being function is implausible in the sense that it is not uniformly sensitive to the given person's achievements below and above poverty thresholds for different dimensions. All achievements above any threshold, however small or large they may be, are replaced by the threshold itself. Therefore, the Tsui (2002) index cannot be regarded as an IMWB-based index. The same remark applies to the Chakravarty–Mukherjee–Ranade (1998), Bourguignon–Chakravarty (2003), and Alkire–Foster (2011) indices.

Different from all these previous proposals, Maasoumi and Lugo (2008) (ML) suggested an indicator of multidimensional poverty that inverts the sequence of steps to derive the measure. Relying on an information theory-based approach, the authors in a first stage aggregate attributes of well-being—as done in Maasoumi (1986)—to obtain an individual well-being function. Dimension-specific poverty thresholds are aggregated using the same criterion defining a poverty frontier. Thus, in the second stage, a person's poverty levels are obtained as the shortfall of the ratio between the aggregated achievements and the aggregated poverty thresholds. The third step involves applying a Foster–Greer–Thorbecke (1984) type transformation over the individual poverty functions across persons to arrive at the overall poverty indicator. By construction, the indicator allows for some degree of substitution across attributes even between those that fall above the dimension-specific poverty threshold. This implies, for instance, that if a person does not have the "minimum acceptable level" of one dimension, say education, but she is, say, extremely income rich, she might be considered non poor. Essentiality of attributes is relaxed, at least to a certain degree, depending on the parameter defining the degree of substitution allowed. Therefore, in terms of postulates, the ML measure satisfies the weak version of the focus axiom, but not the strong one—SFC. In addition, the measure meets MT unambiguously and is subgroup decomposable. However, it satisfies only IPC, that is, all the dimensions are implicitly assumed to be substitutes and compensation across dimensions is allowed.

By construction an IMWB-based index is a violator of the strong focus axioms. One way to resolve this issue is to adopt Decancq et al. (2013) suggestion to look to individual preferences in order to identify the poor and aggregate dimensional achievements. Under these authors' approach, the strong Pareto principle is satisfied among the poor. Furthermore, the assessment of complementarity or substitutability between dimensions is left to the individuals themselves. This contrasts with the direct approach where the complementarity-substitutability issue is resolved by imposing parameter restrictions in the form of composite indicator, which may or may not respect individual preferences.

Specifically, Decancq et al. (2013) have characterized a poverty indicator based on the idea that there is a single poverty threshold vector z and a person is treated as poor if and only if he/she prefers z over his/her current consumption bundle. Thus, this contribution offers a two-fold suggestion: endogenizing the poverty thresholds and using individual preferences in the context of identification of the poor.

4.3 Measurement of Multidimensional Poverty for Ordinally Measurable Dimensions

While some of the typical dimensions of well-being and deprivation correspond to ratio scale variables (for instance, income and wealth), others such as health and literacy are generally represented by ordinal variables. (See Alkire's chapter 21 in the OUP Handbook for a similar discussion.) Ordinal variables like gender, ethnicity, and religion have one or more categories or types and their categories have a well-defined ordering rule. For instance, self-reported health is often presented in the following six categories 'very poor', 'poor', 'fair', 'good', 'very good' and 'excellent'. To each of these categories, one can assign positive integral values in an increasing order. This assignment of integral values is arbitrary; the only restriction is that to preserve the ordering a higher number should be assigned to a better category—so that 'very good' should get a higher number than 'good' (see Allison and Foster 2004). A second example can be ordering of educational achievement levels of individuals in a society starting from illiteracy to university education by assigning numbers in an increasing way (see Chakravarty and Zoli 2012). Several indicators of multidimensional poverty have been proposed in the literature to incorporate ordinal characteristic of the dimensions. Ordinal measurability information invariance for a multidimensional poverty indicator requires that the poverty level based on x_{ij} and z_j values should be same as that based on any arbitrary increasing transformation applied to these values, where the transformations need not be the same across dimensions.

The headcount ratio, while a violator of DIM, is an appropriate indicator of multidimensional poverty if some of the dimensions are measurable on ordinal scale and the other dimensions have ratio scale significance. The Alkire–Foster (2011) dimension adjusted headcount ratio also survives this requirement. It is defined as the ratio between the deprivation score of the poor in the Alkire–Foster (intermediate) sense and nd, which is the society deprivation count when all the persons become deprived in all the dimensions.[23] (This is a limiting case of P_{AFM} as $\alpha \rightarrow 0$ and it satisfies DIM (see Table 2)).

Chakravarty and D'Ambrosio (2006) suggested an indicator of multidimensional social exclusion when the dimensions have ordinal significance. This normalized indicator verifies SFC, SUD, DIM but not FAD. It is non-decreasing under a correlation increasing switch, but not increasing.[24] A related indicator is proposed by Bossert et al. (2013) who characterize a multidimensional indicator where the dimensions are discrete in nature and used it for evaluating material deprivation in the European Union. They have defined a person as materially deprived if his deprivation score is at least one. The measure satisfies similar properties as the previous index.

5 Conclusions

The increasing interest among both academics and policy makers in alternative conceptualizations of well-being and deprivation that take into account multiple dimensions has spawned the development of a wide range of measures of multidimensional inequality and poverty. The present chapter attempted to summarize, in a structured way, the main relevant considerations in developing these measures. Within both inequality and poverty, the discussion has been divided into two lines: the direct approach—where a set of desirable properties or postulates in terms of multidimensional matrices are first identified and then measures satisfying these properties

[23]For a recent discussion on the counting approach to multidimensioned deprivation, see Dutta and Yalonetzky (2014).

[24]Jayaraj and Subramanian (2009) employed this indicator to determine multidimensional poverty in India.

are obtained; and the inclusive measure of well-being approach, where a multidimensional indicator of inequality or poverty is derived by applying a social welfare function, univariate inequality measure, or univariate poverty measure to a vector of individual well-being numbers that take account of each individual's multidimensional achievements.

Irrespective of the approach and set of properties chosen, selecting any scalar indicator to summarize the complete distribution of well-being or deprivation attributes across individuals involves imposing important value judgments. There is no escape from that, and thus, there will be always grounds to object to any given multidimensional indicator. However, it is vitally important that policy makers be aware of the full range of normatively plausible options. It may be worthwhile to mention that, following the literature, our formulation in this chapter uses directly individual achievements. Therefore, our presentation has ignored individual preferences. Research on multidimensional poverty and inequality metrics continues to be extremely fertile; alternative new postulates and indicators are proposed on a regular basis. Although there has been tremendous progress in this area, reviewed in this chapter, there is much still to learn.

Acknowledgements We are grateful to the editors for their comments and suggestions. We also thank Koen Decancq and Sabina Alkire who went through an earlier draft of the chapter and offered many helpful suggestions.

Appendix

Let $x_{ij} \geq 0$ be the achievement of person i in attribute or dimension j. An achievement indicates the performance of a person in a dimension, for instance, how much is his or her income. Person i's achievements in different dimensions are summarized by a d-dimensional vector $x_{i.} = (x_{i1}, x_{i2}, \ldots, x_{id})$. The row vector $x_{i.}$ is the ith row of an $n \times d$ distribution matrix X. The column vector $x_{.j}$, which summarizes the distribution of achievements in dimension j $(j = 1, 2, \ldots, d)$ among n persons, is the jth column of X and we denote the mean of this vector by $\mu\big(x_{.j}\big)$. If we denote the set all $n \times d$ matrices whose entries are non-negative real numbers by M_1^n, then $X \in M_1^n$. Similarly, M_2^n stands for the set of all distribution matrices such that $x_{ij} \geq 0$ for all pairs $(i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, d\}$ and $\mu\big(x_{.j}\big) > 0$ for all $1 \leq j \leq d$. Finally, M_3^n denotes the set of all distribution matrices such that $x_{ij} > 0$ for all pairs $(i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, d\}$. Thus, for matrices in the sets M_2^n and M_3^n the mean of each attribute is positive. Since our analysis will often involve different-sized populations, it will be necessary to consider the set $M_1 = \cup_{n \in N} M_1^n$ of all distribution matrices with d columns. Let M_2 and M_3 be the corresponding sets associated with M_2^n and M_3^n and $\boldsymbol{M} = \{M_1, M_2, M_3\}$. We denote an arbitrary element of the set $\boldsymbol{M}$ by M, that is, the set M can be anyone of the three M_i sets.

An $n \times n$ matrix B with non-negative entries is called a bistochastic matrix of order n if each of its columns and rows sums to unity. Any permutation matrix is a bistochastic matrix, but the converse is not true.

An $n \times n$ matrix is called a diagonal matrix of order n if its off-diagonal elements are equal to zero, but diagonal elements may or may not be equal to zero. Throughout this chapter we will consider diagonal matrices with positive diagonal entries. We will denote a diagonal matrix Ω of order n by $\Omega = \text{diag}(\omega_1, \omega_2, \ldots, \omega_d)$, where $\omega_i > 0$ for all i.

For any $n \in N$, $X, Y \in M^n$, X is said to be obtained from Y by a simple increment if $x_{ij} = y_{ij} + \delta$ for some pair $(i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, d\}$, where $\delta > 0$ is a scalar and $x_{lk} = y_{lk}$ for all pairs $(l, k) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, d\}$ such that $(l, k) \neq (i, j)$.

Axioms for multidimensional inequality indices

- **Ratio Scale Invariance** (**RSI**): An inequality indicator $I : M \to \Re^1$ is a ratio scale invariant or relative indicator if for all $n \in N$, $X \in M^n$,

$$I(X\Omega) = I(X), \tag{1}$$

where $\Omega = \text{diag}(\omega_1, \omega_2, \ldots, \omega_d)$, $\omega_i > 0$ for all i.

- **Unit Consistency** (**UCO**): For any $n \in N$, $X^1, X^2 \in M^n$, $I(X^1) < I(X^2)$ implies that $I(X^1\Omega) < I(X^2\Omega)$ for all $\Omega = \text{diag}(\omega_1, \omega_2, \ldots \omega_d)$, $\omega_i > 0$ for all i.
- **Translation Scale Invariance** (**TSI**): An inequality indicator $I : M \to \Re^1$ is a translation scale invariant or an absolute indicator if for all $n \in N$, $X \in M^n$,

$$I(X + A) = I(X), \tag{2}$$

where A is any $n \times d$ matrix with identical rows such that $X + A \in M$.

- **Symmetry (SYM)**: For all $n \in N$, $X \in M^n$, $I(\Pi X) = I(X)$, where Π is any $n \times n$ permutation matrix.
- **Population Replication Invariance** (**PRI**): For all $n \in N$, $X \in M^n$, $I(X) = I(X^{(l)})$, where $X^{(l)}$ is the l-fold replication X, that is, $X^{(l)} = \begin{pmatrix} X^1 \\ X^2 \\ \vdots \\ X^l \end{pmatrix}$ with each $X^i = X$, and $l \geq 2$ is any integer.
- **Normalization** (**NOM**): For all $n \in N$, $X \in M^n$, if X has identical rows, then $I(X) = 0$.
- **Continuity** (**CON**): For all $n \in N$, $I(X)$ is a continuous function.
- **Uniform Pigou–Dalton Transfers Principle** (**UPD**): For all $n \in N$, $X, Y \in M^n$ if X is obtained by pre-multiplying Y by a T-transformation, then $I(X) < I(Y)$,

where a T-transformation is a linear transformation defined by an $n \times n$ matrix T of the form $T = tIM_n + (1-t)\Pi_{ij}$, for some $t \in (0, 1)$, IM_n is the $n \times n$ identity matrix, and Π_{ij} is the $n \times n$ permutation matrix that interchanges the i and j coordinates for some $i, j \in \{1, 2, \ldots, n\}$.

Definition: Let $X, Y \in M^n$. Distribution Y is derived from X by a *PDB transfer* if there exist two individuals p, q such that: (i) $x_q > x_p$; (ii) $y_m = x_m \forall m \neq p, q$; (iii) $y_q = x_q - \delta$ and $y_p = x_p + \delta$ where $\delta = (\delta_1, \ldots, \delta_d) \in \Re^d_+$ with at least one $\delta_j > 0$; (iv) $y_q \geq y_p$.

- **Pigou–Dalton Bundle Transfer Principle** (**PBT**): For all $n \in N$, $X, Y \in M^n$ if Y is obtained from X by a finite sequence of PBD transfers, then $I(Y) \leq I(X)$.
- **Uniform Majorization Principle** (**UM**): For all $n \in N$, $X, Y \in M^n$, if $X = BY$ for some $n \times n$ bistochastic matrix B that is not a permutation matrix, then $I(X) < I(Y)$.

Definition: For $a, b \in R^d$, define $a \vee b = (\max\{a_1, b_1\}, \ldots, \max\{a_d, b_d\})$ and $a \wedge b = (\min\{a_1, b_1\}, \ldots, \min\{a_d, b_d\})$. For $X, Y \in M^n$, we say that X is obtained from Y by a *correlation increasing switch* if $X \neq Y$ and there exist $1 \leq i, l \leq n$ such that (i) $x_{i.} = y_{i.} \wedge y_{l.}$, (ii) $x_{l.} = y_{i.} \vee y_{l.}$, (iii) $x_{i_1.} = y_{i_1.}$ for all $i_1 \notin \{i, l\}$. That is, a correlation increasing switch between two individuals means a rearrangement of their achievements such that one of them (l) receives at least as much of every attribute as the other (i) and more of at least one attribute.

- **Correlation Increasing Majorization** (**CIM**): For all $n \in N$, $X, Y \in M^n$, if Y is obtained from X by a correlation increasing switch, then $I(X) < I(Y)$.
- **Unfair Rearrangement** (**UR**): For all $n \in N$, $X, Y \in M^n$, if Y is obtained from X by a sequence of dimension-wise permutations which make one individual in Y top-ranked in all dimensions, another individual second-ranked in all dimensions as so forth, and $Y \neq X$, then $I(X) < I(Y)$.
- **Subgroup Decomposability** (**SDE**): For all $n_1, n_2 \in N$, $X \in M^{n_1}$, $Y \in M^{n_2}$, $I(X, Y)' = A\Big(I(X), I(Y); \underline{\mu}(X), \underline{\mu}(Y); n_1, n_2\Big)$, where the aggregative function A is continuous and increasing in first two arguments, $\underline{\mu}(X)$ and $\underline{\mu}(Y)$ are the vectors of means of attributes corresponding to the distribution matrices X and Y respectively and / denotes transpose.
- **Factor Decomposability** (**FDE**): For all $n \in N$, $X \in M^n$, $I(X) = \sum_{j=1}^{d} I\big(x_{.j}\big)$.

Axioms for multidimensional poverty indices

- **Normalization** (**NOM**): For any $(X; z) \in M^n \times Z$ if $x_{ij} \geq z_j$ for all i and j, then $P(X; z) = 0$.
- **Symmetry** (**SYM**): For any $(X; z) \in M^n \times Z$, $P(X; z) = P(\Pi X; z)$, where Π is any $n \times n$ permutation matrix.
- **Population Replication Principle** (**PRI**): For any $(X; z) \in M^n \times Z$, $P(X; z) = P\big(X^{(l)}; z\big)$, where $X^{(l)}$ is the l-fold replication of X.
- **Ratio Scale Invariance** (**RSI**): For all $(X; z) \in M^n \times Z$, $P(X; z) = P(X\Omega; z\Omega)$, where $\Omega = \text{diag}(\omega_1, \omega_2, \ldots \omega_d)$, $\omega_i > 0$ for all i.

- **Weak Focus** (**WFC**): For $X, Y \in M^n$ if for some i, $x_{ij} \geq z_j$ for all j and for some $j \in \{1, 2, \ldots, d\}$, $y_{ij} = x_{ij} + \eta$, where $\eta > 0$, and $x_{hk} = y_{hk}$ for all $(h, k) \neq (i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, d\}$, then $P(Y; z) = P(X; z)$.
- **Strong Focus** (**SFC**): Suppose $Y \in M^n$ is obtained from $X \in M^n$ such that for some pair $(i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, d\}$, $y_{ij} = x_{ij} + \eta$, where $x_{ij} \geq z_j$, $\eta > 0$, and $x_{hk} = y_{hk}$ for all $(h, k) \neq (i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, d\}$. Then $P(Y; z) = P(X; z)$.
- **Monotonicity (MON)**: Suppose $Y \in M^n$ is obtained from $X \in M^n$ such that for some pair $(i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, d\}$, $y_{ij} = x_{ij} - c$, where $i \in \pi(X)$, $x_{ij} < z_j$, $c > 0$, and $x_{hk} = y_{hk}$ for all $(h, k) \neq (i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, d\}$. Then, $P(Y; z) > P(X; z)$.
- **Dimensional Monotonicity** (**DIM**): Suppose $Y \in M^n$ is obtained from $X \in M^n$ such that for some pair $(i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, d\}$, $y_{ij} = x_{ij} - c < z_j$, where $i \in \pi(X)$, where $\pi(X)$ is the set of poor persons in X, $x_{ij} \geq z_j$, $c > 0$, and $x_{hk} = y_{hk}$ for all $(h, k) \neq (i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, d\}$. Then, $P(Y; z) > P(X; z)$.
- **Subgroup Decomposability** (**SUD**): For any $X^1, X^2, \ldots, X^l \in M$ and $z \in Z$, $P(X; z) = \sum_{i=1}^{l} \frac{n_i}{n} P\left(X^i; z\right)$, where $X = (X^1, \ldots, X^l)^{/} \in M^n$, / denotes transpose, n_i is the population size associated with X^i and $\sum_{i=1}^{l} n_i = n$.
- **Factor Decomposability** (**FAD**): For any $(X; z) \in M^n \times Z$, $P(X; z) = \sum_{j=1}^{d} b_j P\left(x_{.j}, z_j\right)$, where $b_j \geq 0$ is the weight assigned to the poverty in dimension j and $\sum_{j=1}^{d} b_j = 1$.
- **Multidimensional Transfers Principle** (**MT**): For any $X, Y \in M^n$ if X is obtained from Y by an averaging of achievements among the deprived dimensions of the poor, then $P(X; z) < P(Y; z)$.
- **Increasing Poverty under Correlation Increasing Switch (IPC)**: Under SUD, for any $X \in M^n$, if $Y \in M^n$ is obtained from X by a correlation-increasing switch between two poor persons, then $P(X; z) < P(Y; z)$, given that the two attributes are substitutes.

References

Alkire, S., & Foster, J. E. (2010). *Designing the inequality-adjusted Human Development Index* (*IHDI*). Human Development Research Papers (2009 to Present) HDRP-2010-28, Human Development Report Office (HDRO), United Nations Development Programme.

Alkire, S., & Foster, J. E. (2011). Counting and multidimensional poverty measurement. *Journal of Public Economics, 95,* 476–487.

Alkire, S., & Santos, M. E. (2010). *Acute multidimensional poverty: A new index for developing countries*. OPHI Working Paper Series #38. Oxford, UK: Oxford University, Oxford Poverty and Human Development Initiative.

Allison, R. A., & Foster, J. E. (2004). Measuring health inequality using qualitative data. *Journal of Health Economics, 23*(3), 505–524.

Anand, S. (1997). Aspects of poverty in Malaysia. *Review of Income and Wealth, 23,* 1–16.

Atkinson, A. B. (1970). On the measurement of inequality. *Journal of Economic Theory, 2,* 244–263.

Atkinson, A. B. (2003). Multidimensional deprivation: Contrasting social welfare and counting approaches. *Journal of Economic Inequality, 1,* 51–65.

Atkinson, A. B., & Bourguignon, F. (1982). The comparison of multidimensioned distributions of economic status. *Review of Economic Studies, 49,* 183–201.

Atkinson, A. B., & Bourguignon, F. (1987). Income distribution and differences in needs. In G. F. Feiwel (Ed.), *Arrow and the foundation of the theory of economic policy* (pp. 350–370). London: Macmillan.

Atkinson, A. B., Cantillion, B., Marlier, E., & Nolan, B. (2002). *Social indicators. The EU and social inclusion.* Oxford: Oxford University Press.

Atkinson, A. B., Marlier, E., Monatigne, F., & Reinstadler, A. (2010). Income poverty and income inequality. In A. B. Atkinson & E. Marlier (Eds.), *Income and living conditions in Europe*. Luxemburg: Eurostat.

Banerjee, A. K. (2010). A multidimensional Gini index. *Mathematical Social Sciences, 60*(2), 87–93.

Boland, P. J., & Prochan, F. (1988). Multivariate arrangement increasing functions with applications in probability and statistics. *Journal of Multivariate Analysis, 25,* 286–298.

Bosmans, K., Decancq, K., & Ooghe, E. (2013). What do normative indices of multidimensional inequality really measures? CORE DP 2013/35.

Bosmans, K., Lauwers, L., & Ooghe. E. (2009). A consistent multidimensional Pigou–Dalton transfer principle. *Journal of Economic Theory, 144*, 1358–1371.

Bossert, W., Chakravarty, S. R., & D'Ambrosio, C. (2013). Multidimensional poverty and material deprivation with discrete data. *Review of Income and Wealth, 59,* 29–43.

Bourguignon, F. (1999). Comment on multidimensioned approaches to welfare analysis by E. Maasoumi. In J. Silber (Ed.), *Handbook of income inequality measurement* (pp. 477–484). London: Kluwer Academic.

Bourguignon, F., & Chakravarty, S. R. (2003). The measurement of multidimensional poverty. *Journal of Economic Inequality, 1,* 25–49.

Chakravarty, S. R. (1983). A new index of poverty. *Mathematical Social Sciences, 6,* 307–313.

Chakravarty, S. R. (2006). An axiomatic approach to multidimensional poverty measurement via fuzzy sets. In G. Betti & A. Lemmi (Eds.), *Fuzzy set approach to multidimensional poverty measurement* (pp. 49–92). New York: Springer.

Chakravarty, S. R. (2009). *Inequality, polarization and poverty: Advances in distributional analysis*. New York: Springer.

Chakravarty, S. R., & D'Ambrosio, C. (2006). The measurement of social exclusion. *Review of Income and Wealth, 52,* 377–398.

Chakravarty, S. R., & D'Ambrosio, C. (2012). A family of unit consistent multidimensional poverty indices. In V. Berenger & F. Bresson (Eds.), *Monetary poverty and social exclusion around the mediterranean sea*. New York: Springer.

Chakravarty, S. R., Mukherjee, D., & Ranade, R. (1998). On the family of subgroup and factor decomposable measures of multidimensional poverty. *Research on Economic Inequality, 8,* 175–194.

Chakravarty, S. R., & Silber, J. (2008). Measuring multidimensional poverty: The axiomatic approach. In N. Kakwani & J. Silber (Eds.), *Quantitative approaches to multidimensional poverty measurement*. New York: Palgrave Macmillan.

Chakravarty, S. R., & Zoli, C. (2012). Stochastic dominance relations for integer variables. *Journal of Economic Theory, 147*(4), 1331–1341.

CONEVAL. (2010). Methodology for Multidimensional Poverty Measurement in Mexico. Consejo Nacional de Evaluación de la Política de Desarrollo Social.

Cummins, R. A. (1996). Domains of life satisfaction: An attempt to order chaos. *Social Indicators Research, 38*(3), 303–328.

Dalton, H. (1920). The measurement of the inequality of incomes. *Economic Journal, 30,* 348–461.

Dardanoni, V. (1996). On multidimensional inequality measurement. In C. Dagum & A. Lemmi (Eds.), *Research on economic inequality* (Vol. 6, pp. 201–205). London: JAI Press.

Decancq, K. (2012). Elementary multivariate rearrangements and stochastic dominance on a Fréchet class. *Journal of Economic Theory, 147*(4), 1450–1459.

Decancq, K. (2014). Copula-based measurement of dependence between dimensions of well-being. *Oxford Economic Papers, 66*(3), 681–701.

Decancq, K., Fleurbaey, M., & Maniquet, F. (2013). *Multidimensional poverty measurement: Shouldn't we take preferences into account*? Mimeographed. Brussels: CORE.

Decancq, K., & Lugo, M. A. (2012). Inequality of well-being: A multidimensional approach. *Economica, 79,* 721–746.

Decancq, K., & Lugo, M. A. (2013). Weights in multidimensional indices of well-being: An overview. *Econometric Reviews, 32,* 7–34.

Diez, H., Lasso de la Vega, M. C., & Urrutia, A. M. (2008). Multidimensional unit- and subgroup consistent inequality and poverty measures: Some characterization results. *Research on Economic Inequality, 16,* 189–211.

Donaldson, D., & Weymark, J. A. (1986). Properties of fixed-population poverty indices. *International Economic Review, 27,* 667–688.

Doyal, L., & Gough, I. (1991). *A theory of human need*. Basinkstoke: Macmillan.

Duclos, J. Y., Sahn, D. E., & Younger, S. D. (2006). Robust multidimensional poverty comparisons. *Economic Journal, 116,* 943–968.

Duclos, J. Y., Sahn, D. E., & Younger, S. D. (2007). Robust multidimensional poverty comparisons with discrete indicators of well-being. In S. P. Jenkins & J. Micklewright (Eds.), *Poverty inequality re-examined*. Oxford: Oxford University Press.

Dutta, I., & Yalonetzky, G. (2014). *The 'Counting' approach to multidimensional deprivation*. Mimeo.

Ebert, U. (1988). Measurement of inequality: An attempt at unification and generalization. *Social Choice and Welfare, 5,* 147–169.

Epstein, L. G., & Tanny, S. M. (1980). Increasing generalized correlation: A definition and some economic consequences. *Canadian Journal of Economics, 13*(1), 16–34.

Ferreira, F., & Lugo, M. A. (2013). Multidimensional poverty analysis: Looking for a middle ground. *The World Bank Research Observer, 28*, 220–235.

Fleurbaey, M., & Maniquet, F. (2011). Compensation and responsibility. In K. J. Arrow, A. K. Sen, & K. Suzumura (Eds.), *Handbook of social choice and welfare* (Vol. 2). Amsterdam: North-Holland.

Fleurbaey, M., & Trannoy, A. (2003). The impossibility of a Paretian egalitarian. *Social Choice and Welfare, 21,* 243–264.

Foster, J. E., Greer, J., & Thorbecke, E. (1984). A class of decomposable poverty measures. *Econometrica, 42,* 761–766.

Foster, J. E., & Shorrocks, A. F. (1991). Subgroup consistent poverty indices. *Econometrica, 59,* 687–709.

Gajdos, T., & Weymark, J. A. (2005). Multidimensional generalized Gini indices. *Economic Theory, 26,* 471–496.

Jacobs, R., Smith, P., & Goddard, M. (2004). *Measuring performance: An examination of composite performance indicators*. CHE Technical Paper Series 29, Centre for Health Economics, University of York.

Jayaraj, D., & Subramanian, S. (2009). A Chakravarty-D'Ambrosio view of multidimensional deprivation: Some estimates for India. *Economic and Political Weekly, 45*(6), 53–65.

Klasen, S. (2000). Measuring poverty and deprivation in South Africa. *Review of Income and Wealth, 46*(1), 33–58.

Kolm, S.-C. (1969). The optimal production of social justice. In J. Margolis & H. Guitton (Eds.), *Public economics* (pp. 145–200). London: Macmillan.

Kolm, S. C. (1976). Unequal inequalities I. *Journal of Economic Theory, 12,* 416–442.

Kolm, S. C. (1977). Multidimensional egalitarianism. *Quarterly Journal of Economics, 91*, 1–13.

Koshevoy, G., & Mosler, K. (1996). The Lorenz zonoid of a multivariate distribution. *Journal of the American Statistical Association, 91*(434), 873–882.

Krishnakumar, J., & Nadar, A. (2008). On exact statistical properties of multidimensional indices based on principal components, factor analysis, MIMIC and structural equation models. *Social Indicators Research, 86*(3), 481–496.

Lasso de la Vega, C., Urrutia, A., & Sarachu, A. (2010). Characterizing multidimensional inequality measures which fulfil the Pigou-Dalton bundle principle. *Social Choice and Welfare, 35*(2), 319–329.

List, C. (1999). *Multidimensional inequality measurement: A proposal*. Economics Series Working Papers 1999-W27, University of Oxford, Department of Economics.

Maasoumi, E. (1986). The measurement and decomposition of multi-dimensional inequality. *Econometrica, 54,* 991–997.

Maasoumi, E., & Lugo, M. A. (2008). The information basis of multivariate poverty assessments. In N. Kakwani & J. Silber (Eds.), *Quantitative approaches to multidimensional poverty measurement*. New York: Palgrave Macmillan.

Marshall, A. W., Olkin, I., & Arnold, B. (2011). *Inequalities: Theory of majorization and its applications*. New York: Springer.

Muller, C., & Trannoy, A. (2011). A dominance approach to the appraisal of the distribution of well-being across countries. *Journal of Public Economics, 95*(3–4), 239–246.

Muller, C., & Trannoy, A. (2012). Multidimensional inequality comparisons: A compensation perspective. *Journal of Economic Theory, 147*(4), 1427–1449.

Nardo, M., Saisana, M., Saltelli, A., & Taranto, S. (2005). *Tools for composite indicators building.* EUR 21682 EN, European Communities.

Nussbaum, M. C. (2000). *Women and human development: The capabilities approach*. Cambridge: Cambridge University Press.

OECD. (2011). *How is life? Measuring well-being*. Paris: OECD Publishing. http://dx.doi.org/10.1787/9789264121164-en.

Pigou, A. C. (1912). *Wealth and welfare*. London: Macmillan.

Pogge, T. (2002). *World poverty and human rights: Cosmopolitan responsibilities and reforms*. Cambridge: Polity Press.

Ramsay, M. (1992). *Human needs and the market*. Aldershot: Avebury.

Ravallion, M. (1996). Issues in measuring and modeling poverty. *Economic Journal, 106,* 1328–1343.

Ravallion, M. (2011). On multidimensional indices of poverty. *Journal of Economic Inequality, 9,* 235–248.

Rawls, J. (1971). *A theory of justice*. Cambridge: Harvard University Press.

Rousseau, J. J. (1754). *Discourse on the origin and basis of inequality among men*. Paris.

Sen, A. K. (1973). *On economic inequality*. New York: Norton.

Sen, A. K. (1976). Poverty: On ordinal approach to measurement. *Econometrica, 44,* 219–231.

Sen, A. K. (1985). *Commodities and capabilities*. Amsterdam: North-Holland.

Sen, A. K. (1992). *Inequality re-examined*. Cambridge, MA: Harvard University Press.

Sen, A. K. (1993). In M. Nussbaum (Ed.), *The quality of life*. Oxford: Clarendon Press.

Shorrocks, A. F. (1982). Inequality decomposition by factor components. *Econometrica, 50*(1), 193–211.

Silber, J., & Yalonetzky, G. (2013). Measuring multidimensional deprivation with dichotomized and ordinal variables. In G. Betti & A. Lemmi (Eds.), *Poverty and social exclusion: New methods of analysis*. Abingdon: Routledge.

Smith, A. (1776). *An inquiry into the nature and causes of the wealth of the nations* (1 ed.). London: W. Strahan.

Stewart, F. (1985). *Basic needs in developing countries*. Baltimore, MD: Johns Hopkins University Press.

Stiglitz, J., Sen, A. K., & Fitoussi, J. P. (2009). *Report by the Commission on the Measurement of Economic Performance and Social Progress*. www.stiglitz-sen-fitoussi.fr.

Streeten, P. (1981). *First things first: Meeting basic human needs in developing countries*. New York, Oxford: Oxford University Press.

Tchen, A. (1980). Inequalities for distributions with given margins. *The Annals of Probability, 8*, 814–827.

Thorbecke, E. (2008). Multidimensional poverty: Conceptual and measurement issue. In N. Kakwani & J. Silber (Eds.), *The many dimensions of poverty* (pp. 3–20). New York: Palgrave Macmillan.

Townsend, P. (1979). *Poverty in the United Kingdom: A survey of household resources and standards of living*. Harmondsworth: Penguin Books.

Tsui, K.-Y. (1995). Multidimensional generalizations of the relative and absolute inequality indices: The Atkinson-Kolm-Sen approach. *Journal of Economic Theory, 67,* 251–265.

Tsui, K.-Y. (1999). Multidimensional inequality and multidimensional generalized entropy measures: An axiomatic derivation. *Social Choice and Welfare, 16*.

Tsui, K.-Y. (2002). Multidimensional poverty indices. *Social Choice and Welfare, 19,* 69–93.

Watts, H. (1968). An economic definition of poverty. In D. P. Moynihan (Ed.), *On understanding poverty*. New York: Basic Books.

Weymark, J. A. (2006). The normative approach to the measurement of multidimensional inequality. In F. Farina & E. Savaglio (Eds.), *Inequality and economic integration*. London: Routledge.

Weymark, J. A. (2013). *Conundrums for nonconsequentialists*. Vanderbilt University Department of Economics Working Papers 13-00010, Vanderbilt University Department of Economics.

Zheng, B. (1997). Aggregate poverty measures. *Journal of Economic Surveys, 11,* 123–162.

Zheng, B. (2000). Minimum distribution-sensitivity, poverty aversion and poverty orderings. *Journal of Economic Theory, 95*.

Zheng, B. (2007a). Unit-consistent decomposable inequality measures. *Economica, 74,* 97–111.

Zheng, B. (2007b). Unit-consistent poverty indices. *Economic Theory, 31,* 113–142.