Chapter 2
Fixed Points of Some Real and Complex Functions

This chapter highlights some fixed point theorems for certain real and complex functions.

2.1 Fixed Points of Continuous Maps on Compact Intervals of $\mathbb{R}$

The following definitions are well-known.

Definition 2.1.1 Let $f, g : X \to Y$ be maps, X and Y being non-empty sets. An element $x_0 \in X$ is called a coincidence point of f and g if $f(x_0) = g(x_0)$. If $f : X \to X$ is a map and if for some $x_0 \in X$, $f(x_0) = x_0$, then x_0 is called a fixed point (fix point) of f. If $f, g : X \to X$ are maps such that for some $x_0 \in X, x = f(x_0) = g(x_0)$, then x_0 is called a common fixed point of f and g.

Definition 2.1.2 Let $f : X \to X$ be a map on a non-void set X. The sequence $\{f^n(x)\}$ called the sequence of f iterates is defined recursively by : $f^0(x) = x$, $f^1(x) = f(x)$, $f^{n+1}(x) = f(f^n(x))$, $n = 0, 1, 2, \ldots,$. This sequence is called a sequence of (f) iterates generated at x. We also call the set $\{f^k(x) : k = 0, 1, 2, \ldots\}$ the orbit of x under f and denote it by $O_f(x)$. $f^m(x)$ is called the mth iterate of f at x.

Definition 2.1.3 For a map $f : X \to X$, $x_0 \in X$ is called a periodic point of period m if $f^m(x_0) = x_0$ and $f^n(x_0) \neq x_0$ for $n < m$.

The classical intermediate value theorem for real functions due to Bolzano is equivalent to Brouwer's fixed point theorem for real functions on intervals of real numbers. In a sense, Bolzano's theorem can be viewed as the harbinger of fixed point theory.

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Theorem 2.1.4 (Bolzano's Intermediate Value Theorem) *If $g : [a, b] \to \mathbb{R}$ is a continuous function then for every real number r between $g(a)$ and $g(b)$, there is an element $c = c(r)$ between a and b such that $g(c) = r$.*

Proof Without loss of generality, we can assume that $g(a) \neq g(b)$. Since g is continuous, $g[a, b]$ is a connected subset of $\mathbb{R}$ containing $g(a)$ and $g(b)$. Since connected subsets of $\mathbb{R}$ are intervals, the interval with $g(a)$ and $g(b)$ as endpoints is in the range of g. Hence if r lies between $g(a)$ and $g(b)$, there is an element $c = c(r)$ between a and b such that $g(c) = r$. □

As an immediate consequence, we have

Theorem 2.1.5 (Brouwer's fixed point theorem in $\mathbb{R}$) *If $f : [a, b] \to [a, b]$ is a continuous function, then f has a fixed point.*

Proof If $f(a) = a$ or $f(b) = b$, then the theorem is true. So without loss of generality we assume that $f(a) \neq a$ and $f(b) \neq b$. Since function $g : [a, b] \to \mathbb{R}$ defined by $g(x) = f(x) - x$ is continuous on $[a, b]$ and $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$ (as $f(a), f(b) \in (a, b)$) by Theorem 2.1.4, there is a point $c \in [a, b]$ such that $g(c) = 0 \in [g(b), g(a)]$. Thus c is a fixed point of f. □

Remark 2.1.6 The above fixed point theorem, a consequence of the intermediate value theorem, is indeed equivalent to this theorem.

Let $g : [a, b] \to \mathbb{R}$ be continuous. Without loss of generality let $g(a) < r < g(b)$. Define the map $f : [-1, 1] \to [-1, 1]$ by

$$f(t) = \rho\left(t - \frac{\{r - g\left(\frac{(1-t)a}{2} + \frac{(1+t)b}{2}\right)}{g(b) - g(a)}\right)$$

where $\rho(x) = -1$ for $x < -1$ and $\rho(x) = 1$ for $x > 1$ and $\rho(x) = x$ for other real numbers. Since g is continuous and ρ is continuous on $\mathbb{R}$, clearly f is continuous and maps $[-1, 1]$ into itself. So by Theorem 2.1.5, f has a fixed point $t_0 \in [-1, 1]$. Further t_0 is neither -1 nor 1 and $-1 < t_0 < 1$. So $t_0 = f(t_0) = t_0 - \left\{\frac{r - g\left\{\frac{(1-t_0)a}{2} + \frac{(1+t_0)b}{2}\right\}}{g(b)-g(a)}\right\}$. Hence $r = g\left(\frac{(1-t_0)a}{2} + \frac{(1+t_0)b}{2}\right)$. In short, g has the intermediate value property.

The following is another useful fixed point theorem.

Theorem 2.1.7 *Let $f : [a, b] \to \mathbb{R}$ be a continuous map such that $f[a, b] \supseteq [a, b]$. Then f has a fixed point.*

Proof Since $f[a, b] \supseteq [a, b]$, $[a, b] = [f(c), f(d)]$ for some interval with end points c and d lying $[a, b]$. If $c \leq d$, then $f(c) \leq a \leq c \leq d \leq b \leq f(d)$. Thus $f(x) - x$ changes sign in $[c, d]$ and hence by Theorem 2.1.4 has a zero, which is a fixed point of f. If $c \geq d$, then $f(d) \leq d \leq c \leq f(c)$. Thus again $f(x) - x$ changes sign in $[d, c]$ and so has a fixed point. □

Remark 2.1.8 Theorem 2.1.4 is not true if the interval is not compact. the map $x \to x+1$ is continuous but has no fixed point in $(-\infty, \infty)$ or $[0, \infty)$. The continuous map $x \to \frac{1+x}{2}$ on $[0, 1)$ has no fixed point in $[0, 1)$. Theorem 2.1.4 fails even if f is continuous everywhere on $[a, b]$ except at a single point. For instance $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} \frac{x}{2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

has no fixed point, and $x = 0$ is the only point of discontinuity of f.

Remark 2.1.9 F_f, the set of fixed points of a continuous map on $[a, b]$ is closed. Indeed $F_f = \{x \in [a, b] : f(x) = x\} = g^{-1}(0)$ where $g : [a, b] \to \mathbb{R}$ is defined by $g(x) = f(x) - x$. Since $\{0\}$ is a closed set and g is continuous $g^{-1}\{0\}$ is a closed subset. So F_f is a closed subset of $[a, b]$ ($[a, b]$ being compact, F_f is also compact).

Remark 2.1.10 Indeed we can prove that for each closed subset F of $[0, 1]$ there is a continuous map $f : [0, 1] \to [0, 1]$ for which F is the set of fixed points of f. For proving this we can, without loss of generality, assume that $0, 1 \in F$. So $[0, 1] - F = G$ is open and is a countable union of disjoint open intervals (a_i, b_i), $i \in \mathbb{N}$. Now we consider the case when this collection is countably infinite, leaving the case of finite collection as an exercise.

For $n \in \mathbb{N}$ define $f_n : [0, 1] \to [0, 1]$ by

$$f_n(x) = \begin{cases} x, & x \in F \cup \bigcup_{i=n}^{\infty} (a_i, b_i), \\ a_i, & \text{if } x \in \left[a_i, \frac{a_i+b_i}{2}\right] \text{ for } i < n, \\ 2x - b_i, & \text{if } x \in \left[\frac{a_i+b_i}{2}, b_i\right] \text{ for } i < n. \end{cases}$$

It can be seen that the sequence of continuous functions (f_n) converges uniformly to a continuous function f for which $f(x) = x$ when $x \in F$ and $f(x) \neq x$ if $x \notin F$. In fact, the result is true for any non-empty closed subset of $\mathbb{R}$.

2.2 Iterates of Real Functions

In this section, some theorems on the behaviour of iterates of real functions are discussed. First, Krasnoselskii's theorem on the convergence of special iterates of non-expansive maps of $[a, b]$, following Bailey's [2] proof using elementary properties of subsequential limits is discussed in detail. Theorems 2.2.6–2.2.8 detail the rates of convergence of iterates of special class of functions and are due to Thron [30].

Theorem 2.2.1 (Krasnoselskii [20], Bailey [2]) *Let* $f : I(= [a, b]) \to I$ *be a map such that* $|f(x) - f(y)| \leq |x - y|$ *for all* $x, y \in I$. *For any* $x \in I$, *the sequence* (x_n) *defined recursively by* $x_{n+1} = \frac{1}{2}(x_n + f(x_n))$, $n = 1, 2, \ldots,$ *converges to some fixed point of* f.

Proof Suppose that (x_n) does not converge to a fixed point. We show that this leads to a contradiction. To this end, the proof is divided into several steps.

Step I. If (x_n) converges to $z \in I$, then (x_{n+1}) also converges to z. As $x_{n+1} = \frac{1}{2}(x_n + f(x_n))$, and f is continuous, x_{n+1} converges to $\frac{f(z)+z}{2}$. So $z = f(z)$.

Step II. No subsequence of (x_n) converges to a fixed point of f. For, if (x_{n_i}) converges to z and $f(z) = z$, then $|z - x_{n_i+1}| \leq |z - \frac{1}{2}(x_{n_i} + f(x_{n_i})| \leq \frac{1}{2}|z - x_{n_i}| + \frac{1}{2}|f(z) - f(x_{n_i})|$ (as $z = \frac{1}{2}(z + f(z)))$ $\leq |z - x_{n_i}|$ (since $|f(x) - f(y)| \leq |x - y|$). This shows that (x_n) itself converges to z, a fixed point of f, contradicting our assumption that (x_n) does not converge to a fixed point of f.

Step III. Since (x_n) lies in the compact interval $I = [a, b]$, it has a subsequential limit p for which $f(p) > p$. Otherwise for all subsequential limits p of (x_n), $f(p) \leq p$. Let z be the infimum of all subsequential limits. Then z itself is a subsequential limit of (x_n). So $f(z) \leq z$. If $f(z) < z$, then $f(z) < \frac{1}{2}(f(z) + z) < z$ and $\frac{1}{2}(f(z) + z)$ is a subsequential limit of (x_n) smaller than z, the smallest subsequential limit of (x_n), we get a contradiction, unless $f(z) = z$. But by Step II above, $f(z)$ cannot be z. Thus, there is a subsequential limit p of (x_n) for which $f(p) > p$.

Step IV. By Step II, there exists $\epsilon > 0$ such that $|f(x) - x| \geq \epsilon$ for all subsequential limits x of (x_n). Otherwise, there is a sequence (w_n) of subsequential limits of (x_n) with $|w_n - f(w_n)| < \frac{1}{n}$ for all n. This in turn implies that any subsequential limit of (w_n), which is also a subsequential limit of (x_n) is a fixed point of f, contrary to Step II.

Step V. Let w be the largest subsequential limit of (x_n) such that $f(w) > w$ so $f(w) > Q = \frac{1}{2}(f(w) + w) > w$. Since Q is a subsequential limit exceeding w, $f(Q) < Q$.

By Step IV, there is the least subsequential limit R of (x_n) such that $f(R) < R$ and $w < R < f(w)$ (at least Q satisfies these conditions). Now $f(R) < w$.

Otherwise for $A = \frac{1}{2}[R + f(R)]$, $w < A < R$. If $f(R) \geq w$, then $A = \frac{1}{2}(R + f(R)) \geq \frac{1}{2}(R + w) > \frac{1}{2}(w + w) = w$ and $A = \frac{1}{2}(R + f(R)) < \frac{1}{2}(R + R) = R$. Since A is a subsequential limit greater than w, the largest subsequential limit less than $f(w)$, $f(A) \leq A$. As $A < R$ and R is the least subsequential limit with $f(R) < R$, $A \leq f(A)$. Hence $A = f(A)$ and this contradicts our assumption that no subsequential limit can be a fixed point of f. Hence $f(R) < w$. Consequently $f(R) < w < R < f(w)$ and $|w - R| = R - w < |f(R) - f(w)| = f(w) - f(R)$. This is a contradiction to the assumption on the map f that $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in I$. Hence (x_n) converges to a fixed point of f. □

Remark 2.2.2 However, for any continuous map of I into itself, the sequence of iterates defined in Theorem 2.2.1 may not converge. Let $f : [0, 1] \to [0, 1]$ be defined by

$$f(x) = \begin{cases} \frac{3}{4} & \text{for } 0 \le x \le \frac{1}{4} \\ 3\left(\frac{1}{2} - x\right) & \text{for } \frac{1}{4} < x \le \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} \le x \le 1. \end{cases}$$

Clearly $x = \frac{3}{8}$ is a fixed point of f. For $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}(x_1 + f(x_1) = \frac{1}{2}$, $x_3 = \frac{1}{2}(x_2 + f(x_2)) = \frac{1}{4}$ and so on. This shows that x_n does not converge.

In this context, the following result due to Cohen and Hachigian [10] is pertinent.

Theorem 2.2.3 *Let* $f : [-1, 1] \to [-1, 1]$ *be a continuous map such that* $f(-1) = -1$ *and* $f(1) = 1$. *Then for each* $m = 0, 1, 2, \ldots$, $\|f^{m+1} - I\| \ge \|f^m - I\|$. *Here* I *denotes the identity map and* $\|g\| = sup\{|g(x)| : x \in [-1, 1])$ *for any* $g \in C[-1, 1]$.

Proof If $f \equiv I$, the conclusion is obvious. So suppose that $f \ne I$. Let $F = \{x \in [-1, 1] : f(x) = x\}$. Since F is closed, the complement of F is open and so can be written as a disjoint union of open subintervals S_α of $[-1, 1]$. For $x \in S_\alpha$, $f(x) < x$ or $f(x) > x$. Clearly the conclusion is true for $m = 0$. Suppose the inequality $\|f^{k+1} - I\| \ge \|f^k - I\|$ is true for $k = 1, 2, \ldots, m$. As $[-1, 1]$ is compact and f^m is continuous, there exists p in $[-1, 1]$ such that $|f^m(p) - p| = \|f^m - I\|$.

Suppose without loss of generality $f^m(p) > p$. We claim that $f(p) > p$. Clearly $f(p) \ne p$. If $f(p) < p$, then for $q = f(p)$,

$$\begin{aligned} \|f^{m-1} - I\| &\ge |f^{m-1}(q) - q| = |f^m(p) - q| \\ &= f^m(p) - q \quad (\text{as } q < p < f^m(p)) \\ &> f^m(p) - p = \|f^m - I\|. \end{aligned}$$

As this is a contradiction $f(p) > p$. Let $p \in S_\alpha = (a, b)$. So for $x \in S_\alpha$, $f(x) > x$. As $a, b \notin S_\alpha$, $a = f(a) < p < b = f(b)$. So by the intermediate value property of the continuous function f, there exists $r \in S_\alpha$ with $f(r) = p$. Since $f(x) > x$ in S_α and $r \in S_\alpha$, $f(r) = p > r$. Now

$$\begin{aligned} \|f^{m+1} - I\| &> |f^{m+1}(r) - r| = f^m(p) - r \\ &> f^m(p) - p = \|f^m - I\|. \end{aligned}$$

Thus for f different from I, the identity map

$$\|f^{m+1} - I\| \ge \|f^m - I\|, m = 0, 1, 2, \ldots. \qquad \square$$

Cohen and Hachigian [10] have constructed an example of a continuous self-map on the closed unit disc for which every point on the unit circle is a fixed point, with the property that $\|I - f\| > \|I - f^k\|$ for some iterate f^k of f.

For special real functions Thron [30] had obtained some interesting results on the rates of convergence of iterates. Some of these are relevant to the solution of Schroder's functional equation. They provide useful estimates in approximating fixed points by iterates.

Definition 2.2.4 A map $g : \mathbb{R} \to \mathbb{R}$ is said to belong to the class $H(a_1, k)$ if for some $x_0 > 0$, $0 < g(x) < x$ for $x \in (0, x_0]$ and $g(x) = a_1 x + x^{k+1} h(x)$ for $x \in [0, x_0]$ where $0 \le a_1 \le 1$, k is a positive number and k is a continuous function on $[0, x_0]$ with $|h(x)| < M$ in $[0, x_0]$.

Remark 2.2.5 Clearly for $g \in H(a, k)$, 0 is the unique fixed point of g and every sequence (x_n) of g-iterates defined by $x_{n+1} = g(x_n)$, $n \in \mathbb{N}$ and $x_1 \in (0, x_0]$ converges to 0.

Theorem 2.2.6 *Let $g \in H(a_1, k)$ where $0 < a_1 < 1$. Then for the sequence (x_n) of g-iterates, there exists a constant $K_1(g, x)$ such that*

$$\lim_{n\to\infty} \frac{x_n}{a_1^n} = K_1$$

Proof From the definition of g and x_{n+1}

$$\frac{x_{n+1}}{x_n} = \frac{a_1 x_n + x_n^{k+1} h(x_n)}{x_n} = a_1 + x_n^k h(x_n)$$

As (x_n) decreases to zero, there exists $x_0 \in N$ such that for $x \ge n_0$

$$0 < x_n^k M < \frac{1 - a_1}{2}$$

Sor $\frac{x_{n+1}}{x_n} < \frac{1+a_1}{2} < 1$. Hence $\sum x_n$ and $\sum x_n^k h(x_n)$ converge. So, the infinite product $\prod_{n=1}^{\infty} \left(1 + \frac{x_n^k h(x_n)}{a_1}\right)$ converges to a number L (say). Writing $u_n = \frac{x_n}{a_1^n}$ it follows that $\frac{u_{n+1}}{u_n} = \frac{x_{n+1}}{a_1 x_n} = \left(1 + \frac{x_n^k h(x_n)}{a}\right)$.

Since $u_{n+1} = u_1 \prod_{m=1}^{n} \left(1 + \frac{x_m^k h(x_m)}{a_1}\right)$, u_{n+1} converges to $u_1 L$. Hence $u_n = \frac{x_n}{a_1^n}$ converges to $u_1 L$ $(= K_1(g, x_1))$. □

Theorem 2.2.7 *If $g \in H(a_1, k)$ for $a_1 = 0$ and (x_n) is the sequence of iterates generated at $x_1 \in (0, x_0]$, then there is a constant $K_2(g, x_1)$ with $0 < K_2 < 1$ such that $0 < x_n < K_2^{(k+1)^n}$ for all n after some stage. If additionally $\liminf_{x\to 0} h(x) > 0$, then for some $K_3(g, x_1)$ with $0 < K_3 < 1$, $\lim_{x\to\infty} x_n^{(k+1)^{-n}} = K_3$.*

Proof Since $a_1 = 0$ and $x_{n+1} = x_n^{k+1} h(x_n)$, $\log x_{n+1} = (k+1)\log x_n + \log h(x_n)$. Define $v_n = (k+1)^{-n} \log x_n$. We obtain for $n \ge n_0$

$$\begin{aligned} v_{n+1} &= v_n + (k+1)^{-(n+1)} \log h(x_n) \\ &= v_{n_0} + \sum_{m=n_0}^{n} (k+1)^{-(m+1)} \log h(x_m). \end{aligned} \tag{2.2.1}$$

If $\liminf_{x\to 0} h(x) > 0$, then $\sum_{m=n_0}^{\infty}(k+1)^{-(m+1)}\log h(x_m)$ converges to a number $K_3(g,x_1) - v_{n_0}$, say. So (v_n) converges to $\log K_3$ as $n\to\infty$ or $\lim_{n\to\infty}(x_n)^{(k+1)^{-n}} = K_3$.

Suppose $0 < h(x) < M$ and that $\log h(x_n)$ could approach $-\infty$ so that the series (2.2.1) might not converge. Nevertheless, we have from (2.2.1)

$$v_{n+1} < \sum_{m=n_0}^{n}(k+1)^{-(m+1)}\log M + (k+1)^{-n_0}\log x_{n_0}$$
$$= \log M(k+1)^{-(n_0+1)}\left[\frac{1-(k+1)^{-n+n_0+1}}{1-(k+1)^{-1}}\right] + (k+1)^{-(n_0+1)}\log x_{n_0}^{k+1} \tag{2.2.2}$$

If $\log M < 0$, choosing x_0 such that $x_{n_0} < 1$, we get from (2.2.2)

$$v_n < (k+1)^{-(n_0+1)}\log M < 0. \tag{2.2.3}$$

If $\log M \geq 0$, (2.2.2) gives

$$v_n < (k+1)^{-(n_0+1)}\log\left(M^{\frac{1+k}{k}}x_{n_0}^{k+1}\right). \tag{2.2.4}$$

For large n_0, the right-hand side of (2.2.3) or (2.2.4) as the case may be is negative and is set as $\log K_2(g,x_1)$.

Now $v_n < \log K_2$ for $n \geq n_0$. So $0 < x_n < K_2^{(k+1)^n}$. □

Theorem 2.2.8 *Let $g \in H(a_1,k)$ for $a_1 = 1$. Then $B_1 = \liminf_{x\to 0^+} -h(x) \geq 0$, $B_2 = \limsup_{x\to 0^+} -h(x) \leq M$. Given $\epsilon > 0$ for the sequence (x_n) of iterates in $(0,x_0]$ there exists $N(\epsilon,g,x_1)$ so that*

$$x_n > [(B_2+\epsilon)kn]^{-\frac{1}{k}} \text{ for } n > N.$$

If $B_1 > 0$ and $0 < \epsilon < B_1$, then for some $N'(\epsilon,g,x_1)$

$$x_n < [(B_1-\epsilon)kn]^{-\frac{1}{k}} \text{ for } n > N'$$

Proof Since $g(x) = x + x^{k+1}h(x)$, $g(x) < x$ and $|h(x)| < M$, $0 \leq -h(x) < M$ for $x \in [0,x_0]$. Hence $B_1 \geq 0$ and $B_2 \leq M$. Writing $-h(x_n) = d_n$, $x_{n+1} = x_n + x_n^{k+1}h(x_n)$ becomes, for $k = 1$

$$x_{n+1} = x_n(1 - x_n d_n)$$

and so

$$\frac{1}{x_{n+1}} = \frac{1}{x_n} \frac{1}{(1 - x_n d_n)}.$$

Choose $n_1(g, x_1, \epsilon)$ so that $x_n d_n < 1, \sum_{m=2}^{\infty} d_n^m x_n^{m-1} < \frac{\epsilon}{3}$ and $B_1 - \frac{\epsilon}{3} < d_n < B_2 + \frac{\epsilon}{3}$.

For $n \geq n_1$

$$\begin{aligned} \frac{1}{x_{n+1}} &= \frac{1}{x_n} + d_n + \sum_{m=2}^{\infty} d_n^m x_n^{m-1} \quad \text{(by Binomial theorem)} \\ &< \frac{1}{x_n} + B_2 + \frac{2\epsilon}{3}. \end{aligned} \tag{2.2.5}$$

So $x_{n_1+m} > \dfrac{1}{m\left(B_2 + \frac{2\epsilon}{3}\right) + \frac{1}{x_{n_1}}}$.

So for $n \geq n_1$

$$\begin{aligned} x_n &> \frac{1}{n\left[\left(1 - \frac{n_1}{n}\right)\left(B_2 + \frac{2\epsilon}{3}\right) + \frac{1}{n x_{n_1}}\right]} \\ &> \frac{1}{n\left[B_2 + \frac{2\epsilon}{3} + \frac{1}{n x_{n_1}}\right]} \end{aligned}$$

Choose $n_1' \geq n_1$ so that $\frac{1}{n x_{n_1}} < \frac{\epsilon}{3}$ for $n \geq n_1'$. So we have for $n \geq n_1'$,

$$x_n > \frac{1}{n(B_2 + \epsilon)}.$$

From (2.2.5) for $n \geq n_1$, we get

$$\frac{1}{x_{n+1}} > \frac{1}{x_n} + B_1 - \frac{\epsilon}{3}.$$

So when $B_1 - \epsilon > 0$, for $n > n_1$

$$\begin{aligned} \frac{1}{x_n} &> \frac{1}{x_{n_1}} + (n - n_1)(B_1 - \epsilon) \text{ or} \\ x_n &< \frac{1}{n\left[\left(1 - \frac{n_1}{n}\right)(B_1 - \epsilon) + (n x_{n_1})^{-1}\right]}. \end{aligned} \tag{2.2.6}$$

Choose $N' > n_1' \geq n_1$, such that for $n > N'$,

$$\left(1-\frac{n_1}{n}\right)\left(B_1-\frac{\epsilon}{3}\right) > B_1-\epsilon.$$

So for $n \geq N'$, we get from (2.2.6)

$$x_n < \frac{1}{n(B_1-\epsilon)}.$$

For the case $k \neq 1$, define $w_n = x_n^k$ then $x_{n+1} = g(x_n) = x_n(1 + x_n^k h(x_n))$. So

$$\begin{aligned} w_{n+1} &= \left[g\left(w_n^{\frac{1}{k}}\right)\right]^k = w_n\left[1 + w_n h\left(w_n^{\frac{1}{k}}\right)\right]^k \\ &= w_n[1 + w_n h_1(w_n)]. \end{aligned}$$

Since $[g(w_n^{\frac{1}{k}})]^k$ is a function of w_n, say g_1, it follows that $g_1(w) \in h_1(1,1)$ for $0 \leq w \leq w_0 = x_0^k$. Also $\liminf\limits_{w\to 0^+} h_1(w) = kB_1$, $\limsup\limits_{w\to 0^+} -h_1(w) = kB_2$. The discussion now reduces the case $k \neq 1$ to the case $k = 1$ for $g_1 \in H(1,1)$. It follows from the previous discussion that for $B_1 > 0$ and $0 < \epsilon < B_1$, there exists $N' \in \mathbb{N}$ such that for $n > N'$

$$x_n < [(B_1-\epsilon)kn]^{-\frac{1}{k}}$$

and for $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for $n > N_0$,

$$x_n > [(B_2+\epsilon)kn]^{-\frac{1}{k}}.$$

□

Remark 2.2.9 Since $g(x) = sin(\frac{x}{2}) \in H(\frac{1}{2}, 2)$ in $(0, 1)$, $\lim\limits_{n\to\infty}(2^n sin^n(\dfrac{x}{2}))$ converges for each $x \in (0, 1)$ by Theorem 2.2.6.

Remark 2.2.10 Theorem 2.2.7 can be applied to $g(x) = sin(x^{1+\epsilon})$ for any $\epsilon > 0$ in $(0, 1)$ to conclude that for any sequence (x_n) of iterates of $sin(x^{1+\epsilon})$, $\lim\limits_{n\to\infty}(x_n^{(1+\epsilon)^{-n}})$ converges.

2.3 Periodic Points of Continuous Real Functions

This section treats Sharkovsky's theorem on the existence of periodic points of continuous self-maps on a compact interval $I \subseteq \mathbb{R}$. Sharkovsky published a fundamental paper [27] on the existence of periodic points of continuous self-maps on compact intervals in 1964, when he was about 27 years old. He introduced a new (total) order on the set of natural numbers, often called Sharkovsky order. Interestingly, if a continuous map has a periodic point of period m, in the compact interval I (which it maps into itself) it has periodic points of all periods 'bigger than' m (with respect to this order). The smallest natural number in this order is 3 and so it turns out that if a continuous function mapping $[a, b]$ into itself has periodic point of period 3, then it has periodic points of all periods. Another implication of Sharkovsky's theorem is that if such a map has an odd periodic point then it has periodic points of all even periods.

The more remarkable feature of Sharkovsky's theorem is that its proof is essentially based on the ingenious applications of the intermediate value theorem. The paper by Li and Yorke [21] in 1975 proving a special case of Sharkovsky's theorem as well as May's paper [22] highlighted the complicated behaviour of iterates of simple functions and brought to limelight Sharkovsky's work. The 'simple proof' of Sharkovsky's theorem presented below is due to Bau-Sen Du [14].

In the following, we assume that $f : I \to I$ is a continuous map, where I is a compact interval in $\mathbb{R}$. The following total ordering in $\mathbb{N}$, the set of natural numbers is called Sharkovsky's ordering $\prec$. $m \prec n$ in the following ordering:
$3 \prec 5 \prec 7 \prec \cdots \prec 2.3 \prec 2.5 \cdots$
$\prec 2^2.3 \prec 2^2.5 \prec 2^2.7 \prec \cdots \prec 2^3.3 \prec 2^3.5 \prec \cdots$
$\prec \cdots \prec 2^n.3 \prec 2^n.5 \prec \cdots$
$\prec \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1$

Sharkovsky's theorem states that if $f : I \to I$ has an m-periodic point then f has an n-periodic point precisely when $m \prec n$.

Lemma 2.3.1 *Let a and b be points of I such that either $f(b) < a < b \leq f(a)$ or $f(b) \leq a < b < f(a)$. Then there exists z, a fixed point of $f < b$, a 2-periodic point y of f with $y < z$ and a point v in (y, z) with $f(v) = b$ and*

$$\max\{f^2(v), y\} < v < z < \min\{f(y), f(v)\}.$$

Further, $f(x) > z$ and $f^2(x) < x$ for $y < x \leq v$.

Proof Whether $f(b) < a < b \leq f(a)$ or $f(b) \leq a < b < f(a)$, $f(x) - x$ changes sign in (a, b) and hence has a zero in (a, b). In other words, f has a fixed point z in (a, b). As $b \leq f(a)$, $a < z < b$, and $f(z) = z$, there exists $v \in [a, z)$ with $f(v) = b$. If $f(x) > z$ when $\min I \leq x \leq v$, let $u = \min I$; otherwise let $u = \max\{x : \min I \leq x \leq v, f(x) = z\}$. Then $f^2(u) \geq u$ and $f(x) > z$ for $u < x \leq v$. Since $f^2(v)(= f(b)) \leq a < v$, f^2 has a fixed point in $[u, v)$ or f has a 2 periodic point in $[u, v)$. If y is the largest 2-periodic point, then $u \leq y < v < z < f(y)$. Since $f^2(v) < v$, $f^2(x) < x$ for each x in $(y, v]$. □

Remark 2.3.2 Let P be a period-m orbit of f with $m \geq 3$. Let p, b $(p < b)$ be points in P such that $f(p) \geq b$ and $f(b) \leq p$. So f has a fixed point in $[p, b]$. Let $a \in [p, b)$ be such that $f(a) = b$. Since $f(b) < a$ $(< b = f(a))$, the hypotheses of Lemma 2.3.1 are satisfied. Also b, as a point in P, has least period m.

Theorem 2.3.3 *If f has a periodic point of least period m with $m \geq 3$ and odd then f has periodic points with least period n for each odd integer $n \geq m$.*

Proof Let P be a periodic orbit of f with period m. By Lemma 2.3.1 and Remark 2.3.2. f has a fixed point z, a 2-periodic point y and a point v with $y < v < z < f(y)$ such that $f(v)$ lies in P and $f(x) > z$ and $f^2(x) < x$ when $y < x \leq v$. Define $p_m = v$. As m is odd and y is a 2-periodic point of f, $f^{m+2}(y) = f(y) > y$ and because $f^2(p_m)(= f^2(v))$ is a period-m point of f, $f^{m+2}(p_m) = f^2(p_m) < p_m$. So $p_{m+2} = \min\{x : y \leq x \leq p_m, f^{m+2}(x) = x\}$ is well-defined and is an $(m+2)$ periodic point of f. Since $f^{m+4}(y) = f(y) > y$ and $f^{m+4}(p_{m+2}) = f^2(p_{m+2}) < p_{m+2}$ (and it be noted that $f^2(p_{m+2})$ cannot be p_{m+2}). So $p_{m+4} = \min\{x : y \leq x \leq p_{m+2}, f^{m+4}(x) = x\}$ exists and is a periodic point of f with period $(m+4)$. Thus proceeding, we obtain a decreasing sequence of points $p_m, p_{m+2}, \ldots, p_{m+2k}, \ldots$ with

$$y < \cdots < p_{m+2k+2} < p_{m+2k} < \cdots < p_{m+2} < p_m = v$$

such that p_{m+2k} is a periodic point of f with period $m + 2k$ $(k = 1, 2, \ldots)$. □

Theorem 2.3.4 *If f has a periodic point of least period m with $m \geq 3$ and odd, then f has periodic points of all even periods. Further, there exist disjoint closed subintervals I_0 and I_1 of I such that $f^2(I_0) \cap f^2(I_1) \supseteq I_0 \cup I_1$.*

Proof Let P be an m-orbit of P. By Lemma 2.3.1 and Remark 2.3.2, there is a fixed point z of f, a 2-periodic point y of f and a point v such that $f(v) = b \in P$,

$$\max\{f^2(v), y\} < v < z < b = f(v) = f^{m+1}(v)$$

and $f^2(x) < x$ and $f(x) > z$ for x in $(y, v]$. Write $g = f^2$ and let $z_0 = \min\{t : v \leq t \leq z, g(t) = t\}$. Then y and z_0 are fixed points of g such that $y < v < z_0 \leq z < b = g^{\frac{m+1}{2}}(v)$. Also $g(x) < x$ and $f(x) > z$ for $y < x < z_0$. If $g(x) < z_0$ for $\min I \leq x \leq z_0$, then $g([\min I, z_0]) \subseteq [\min I, z_0]$ and this contradicts that $g^{\frac{m+1}{2}}(v) = b > z_0$. Hence $d = \max\{x : \min I \leq x \leq y, g(x) = z_0\}$ is well defined and $f(x) > z > z_0 > g(x)$ for all x in (d, z_0). Define $s = \min\{g(x) : d \leq x \leq z_0\}$. If $s \geq d$, then $g([d, z_0]) \subseteq [d, z_0]$. But this contradicts that $g^{\frac{m+1}{2}}(v) = b > z_0$. So $s < d$, $[s, d] \cup [d, z_0]$ are non-overlapping closed subintervals and $f^2[s, d] \cap f^2[d, z_0] \supseteq [s, d] \cup [d, z_0]$. Let $\widehat{g} : [d, z_0] \to [d, z_0]$ be the map defined by $\widehat{g}(x) = \max\{g(x), d\}$. Clearly, $\widehat{g}$ is continuous and onto and let $t = \min\{x : d \leq x \leq z_0, g(x) = d\}$. For each $n \in \mathbb{N}$, define $c_n = \min\{x : d \leq x \leq t, \widehat{g}(x) = x\}$. It is not difficult to note that $d < \cdots < c_4 < c_3 < c_2 < c_1 \leq y$ and that c_n generates an n-period orbit $Q_n \subseteq (d, z_0)$ of $\widehat{g}$. Clearly Q_n is also an n-period orbit of $g = f^2$. Since $x < z_0 \leq z < f(x)$ for x in Q_n, $Q_n \cup f(Q_n)$ is $2n$-period orbit of f. Thus f has periods of all even orders. □

Theorem 2.3.5 (Sharkovsky) *Let $f : I \to I$ be a continuous map, where I is a compact interval of real numbers. Then*

(1) if f has a periodic point of period m and if $m \prec n$ (in the Sharkovsky order), then f has also a periodic point of period n;

(2) for each positive integer n, there exists a continuous map $g : I \to I$ that has a periodic point of period n but no point of period $m \prec n$;

(3) *there is a continuous map* $h : I \to I$ *having a* 2^i*-periodic point for* $0, 1, 2, \ldots,$ *and has no other periodic point.*

Proof If f has j-periodic point with $j \geq 3$ and odd, then by Theorem 2.3.3 f has $(j+2)$ periodic point and by Theorem 2.3.4, f has a periodic point of period (2.3). If f has $(2.j)$ periodic point with $j \geq 3$, and odd, f^2 has j-periodic point. So by Theorem 2.3.3, f^2 has $(j+2)$ periodic point and so f has either $(j+2)$ periodic point or period $2(j+2)$ points. If f has $(j+2)$ periodic point, then by Theorem 2.3.4, f has $2(j+2)$ periodic point. In any case f has $2(j+2)$ periodic point. If f^2 has j-periodic point, by Theorem 2.3.4, f^2 has 2.3 periodic point. So f has $(2^2.3)$ periodic point. So if f has $2^k.j$ periodic point, $j \geq 3$ and odd and if $k \geq 2$, then $f^{2^{k-1}}$ has period $2.j$ points. So from what we have proved, we see that $f^{2^{k-1}}$ has period $2(j+2)$ points and period $2^2.3$ points. It follows that f has period $(2^k.(j+2))$ points and period $(2^{k+1}.3)$ points, with $j \geq 3$. If f has $(2^i.j)$ periodic points, $j \geq 3$ and odd and if $i \geq 0$, then f^{2^i} has j-periodic point. For $\ell \geq i$ $f^{2^\ell} = (f^{2^i})^{2^{\ell-i}}$ has period j points. So by Lemma 2.3.1, f^{2^ℓ} has period 2 points. So f has period $2^{\ell+1}$ points for $\ell \geq i$. Finally when f has 2^k-periodic points for some $k \geq 2$, then $f^{2^{k-2}}$ has 4 periodic point. Again by Lemma 2.3.1 $f^{2^{k-2}}$ has 2 periodic points implying that f has 2^{k-1} periodic points. Hence (1) is true.

For proving (2) and (3), without loss of generality, we can assume that $I = [0, 1]$ and $T(x) = 1 - |2x - 1|$, a map with a triangular graph having vertices at $(0, 0)$, $(\frac{1}{2}, 1)$ and $(1, 0)$. Then for each $n \in \mathbb{N}$, $T^n(x) = x$ has exactly 2^n distinct solutions in I. So T has finitely many n-periodic orbits. Among these let P_n be an orbit of the least diameter ($= \max P_n - \min P_n$). Define T_n on I by $T_n(x) = \max P_n$, if $T(x) \geq \max P_n$, $T_n(x) = \min P_n$, if $T(x) \leq \min P_n$ and $T_n(x) = T(x)$ for $\min P_n \leq T(x) \leq \max P_n$. Clearly T_x is continuous on I and T_x has exactly one-period n orbit, i.e. P_n but has no m-periodic orbit for any $m \prec n$.

Let Q_3 be any 3-periodic orbit of T of minimal diameter. Then $[\min Q_3, \max Q_3]$ contains finitely many 6-periodic orbits of T. If Q_6 is one with smallest diameter, then $[\min Q_6, \max Q_6]$ contains finitely many 12-periodic orbits of T. We choose one, say Q_{12} of minimal diameter and continue this process inductively. Define $q_0 = \sup\{\min Q_{2^i.3} : i \geq 0\}$ and $q_1 = \inf\{\max Q_{2^i.3} : i \geq 0\}$. Define $T' : I \to I$ by
$$T'(x) = \begin{cases} q_0 & \text{if } T(x) \leq q_0 \\ q_1 & \text{if } T(x) \geq q_1 \\ T(x) & \text{if } q_0 \leq T(x) \leq q_1 \end{cases}$$
. Clearly T' is continuous and has 2^i-periodic point for $i = 0, 1, 2, \ldots$ but has no other periodic point. Thus (2) and (3) are true. □

Remark 2.3.6 Lemma 2.3.1 has interesting consequences. Let $x_0 \in I$ and $n \geq 2$ be a natural number such that $f^n(x_0) < x_0 < f(x_0)$. Let $X = \{f^k(x_0) : 0 \leq k \leq n-1\}$ (a finite set), $a = \max\{x \in X : q_0 \leq x < f(x)\}$, and $b \in \{x \in X : a < x \leq f(a)\}$ with $f(b) < a$. From these conditions on a, b, x_0, $f(x_0)$ and X it is clear that $f(b) < a < b \leq f(a)$. If $f^n(x_0) \leq x_0 < f(x_0)$ and n is odd (> 1) then f has n-periodic points.

If in addition $\overline{O_f(c)}$ contains both a fixed point z and a point different from z, then f has periodic points with all even periods. Arguments similar to those in Theorems 2.3.3 and 2.3.4 can be used.

Remark 2.3.7 Sharkovsky's theorem cannot be generalized to continua (compact connected subsets) of the plane. On the unit disc, the map $z \to ze^{\frac{2\pi i}{3}}$ has 0 as the only fixed point and all the other points are 3-periodic points. For each $n \in \mathbb{N}$, the map $z \to ze^{\frac{2\pi i}{n}}$ has only one fixed point and the rest of the points are n-periodic points. No point of fundamental period greater than n exists.

Sharkovsky's result is definitely and unalterably one-dimensional (See Ciesielski and Pogoda [8].) Nevertheless, there has been appropriate generalization of Sharkovsky's theorem to general topological spaces and more general maps than continuous functions. See Schirmer [25].

2.4 Common Fixed Points, Commutativity and Iterates

It is natural to find out if two continuous real functions $f, g : I(= [a, b]) \to I$ have a common fixed point. The maps $x \to \frac{x}{2}$ and $x \to 1 - x$ on $[0, 1]$ have the only fixed points 0 and $\frac{1}{2}$ respectively. Since their compositions are $\frac{1-x}{2}$ and $1 - \frac{x}{2}$, they do not commute. If $f, g : I \to I$ have a common fixed point x_0, then $x = f(x_0) = g(x_0) = gf(x_0) = fg(x_0)$ and thus f and g commute at least on $\{x_0\}$. Ritt [24] showed that if f and g are polynomials that commute, then they are within certain homeomorphisms iterates of the same function, both power of x or both must be Chebyshev polynomials and in both these cases, the commuting polynomials have a common fixed point. So Dyer conjectured that if $f, g : I(= [a, b]) \to I$ are continuous real functions that commute, then f and g have a common fixed point. However, Boyce [5] and Huneke [17] had disproved the conjecture independently by constructing counter-examples to point out that commuting continuous self-maps on a compact real interval may not have a common fixed point. Isbell [18] first recorded this problem in a more general form.

This section discusses some results that ensure the existence of common fixed points of two commuting continuous functions $f, g : I \to I$ under suitable additional assumptions. We recall the following definitions.

Definition 2.4.1 Let $\mathscr{F}$ be a family of maps from a topological space X into a metric space (X, d). It is said to be equicontinuous at $x_0 \in X$, if for each $\epsilon > 0$, there exists an open set O in X containing x_0 such that for each $x \in O$ and $f \in \mathscr{F}$, $d(f(x_0), f(x)) < \epsilon$. $\mathscr{F}$ is said to be equicontinuous on X, if it is equicontinuous at each $x \in X$.

Definition 2.4.2 If $f : X \to X$ is a map, a subset $A \subseteq X$ is said to be f-invariant or invariant (under f) if $f(A) \subseteq A$.

An elementary proposition on invariant subsets of continuous maps on compact intervals is given below.

Proposition 2.4.3 *If $f : I = [a, b] \to I$ is a continuous map on the compact interval I of real numbers, then every non-empty closed invariant subset C of I contains a minimal closed invariant non-empty subset C'.*

Proof Let C be a non-empty closed invariant subset of I and $\mathscr{C}$ be the family of all closed invariant subsets of C. Clearly $C \subset \mathscr{C}$. Let $\mathscr{F}$ be a chain of sets in $\mathscr{C}$. Since $\mathscr{F}$ is a subfamily of non-empty closed subsets of C which are indeed compact subsets of I, $F_0 = \cap\{F : F \in \mathscr{F}\}$ is non-empty and compact. Further $f(F_0) \subseteq f(F) \subseteq F$ for all $F \in \mathscr{F}$ and hence $f(F_0) \subseteq \cap\{F : F \in \mathscr{F}\} = F_0$. Thus, F_0 is an invariant closed subset which is contained in each $F \in \mathscr{F}$. Thus F_0 is the least element of $\mathscr{F}$ in C. So by Zorn's Lemma, $\mathscr{C}$ has a minimal element C_0, which is a non-empty minimal closed invariant subset of C. □

Remark 2.4.4 Indeed if $f : X \to X$ is a continuous map of a compact connected T_2 space, then every non-empty closed invariant subset A of X contains a minimal closed invariant subset of A.

Proposition 2.4.5 *If Y is a minimal non-empty closed invariant subset of I a compact interval of $\mathbb{R}$, then for $y \in Y$, $Y = \overline{O_f(y)}$ where $O_f(y) = \{f^n(y) : n = 0, 1, 2, \dots\}$ is the orbit of y, under f.*

Proof If $y \in Y$, then $O_f(y) \subseteq Y$ as $f(Y) \subseteq Y$. Since Y is closed, $\overline{O_f(y)} \subseteq Y$. Now by the continuity of f, $\overline{O_f(y)} \subseteq Y$. By the minimality of Y, $Y \subseteq \overline{O_f(y)}$. So $Y = \overline{O_f(y)}$. □

Theorem 2.4.6 (Schwartz [26]) *Every non-void closed invariant minimal subset of the continuous function $f : I \to I$ is contained in the closure of P_f, where $P_f = \{x \in I : f^k(x) = x$ for some $k \in \mathbb{N}\}$, the set of periodic points of f.*

Proof Let Y be a non-empty minimal closed invariant subset of I. If Y is the orbit of a periodic point, obviously it is finite and closed and the conclusion is true.

Suppose Y is not a periodic orbit. Let $c = \inf Y$. As Y is closed, $c \in Y$. As Y is minimal closed invariant subset, by Proposition 2.4.5, $Y = \overline{O_f(c)}$. So given $\epsilon > 0$, we can find $k \in \mathbb{N}$ with $|y - f^k(c)| < \frac{\epsilon}{2}$. Also we can find $M, N \in \mathbb{N}$ such that $c < f^{N+M}(c) < f^N(c) < c + \epsilon'$, as $c = \inf Y = \overline{O_f(c)}$. As Y is minimal and is not a periodic orbit, $f^M(c) > c$. Thus, the continuous map f^M maps $[c, f^N(c)]$ into itself and so has a fixed point d. Since $c < f^M(c) < f^{M+N}(c)$, $d \in (c, f^N(c))$. Thus $f^M(d) = d$ is a periodic point and $|c - d| < f^N(c) - c < \epsilon'$.

As f^k is continuous at c, for $\epsilon > 0$ we can find $\delta > 0$ with $\epsilon > \delta$ such that $|f^k(x) - f^k(c)| < \frac{\epsilon}{2}$ for $|x - c| < \delta$. Since $|y - f^k(d)| \le |y - f^k(c)| + |f^k(d) - f^k(c)|$, choosing $\epsilon' = \delta$, we see that $|y - f^k(d)| < \epsilon$. As $f^M(d) = d$, it is clear that $z = f^k(d)$ is a periodic point of f which is within ϵ (> 0) distance from y. So $Y \subseteq \overline{P(f)}$. □

Corollary 2.4.7 *If Y is a non-empty minimal closed invariant subset of f then Y is nowhere dense.*

Proof Let x_0 be an interior point of Y. Then for some $\epsilon > 0$, $[x_0 - \epsilon, x_0 + \epsilon] \subseteq Y$. If $[x_0 - \epsilon, x_0 + \epsilon]$ contains a periodic point y of Y, then $O_f(y)$ is finite and is closed. Since $y \in Y$, $Y = \overline{O_f(y)} = O_f(y)$ and this contradicts that Y is uncountable (since it has an interior point). So $[x_0 - \epsilon, x_0 + \epsilon]$ has no periodic point. As $x_0 \in Y$, by Theorem 2.4.6, $[x_0 - \epsilon, x_0 + \epsilon]$ must contain a periodic point, contradicting the preceding assertion. Hence Y is nowhere dense. □

Theorem 2.4.8 (Cano [6]) *Let $\mathscr{F} = \mathscr{F}_1 \cup \mathscr{F}_2$ be a collection of continuous functions mapping a compact interval $I = [a, b] \subseteq \mathbb{R}$ into itself, satisfying the following assumptions:*

(i) for $f \in \mathscr{F}_1$, F_f the set of fixed points of f in I is a compact interval $[a_f, b_f]$;
(ii) for $f \in \mathscr{F}_2$, every periodic point of f is a fixed point of f;
(iii) for $f, g \in \mathscr{F}$, $f(g(x)) = g(f(x))$ for all $x \in I$ (f and g commute).

If $h : I \to I$ is a continuous function that commutes with each $f \in \mathscr{F}$, then $\mathscr{F} \cup \{h\}$ has a common fixed point in I.

Proof Let $C_1 \cup \{h\}$ be any finite subset of $\mathscr{F} \cup \{h\}$ of the form $\{f_1, \dots, f_n\} \cup \{h\} \cup \{g_1, \dots, g_m\}$ where f_i, $i = 1, 2, \dots, n \in \mathscr{F}_1$ and $\{g_1, \dots, g_m\} \subseteq \mathscr{F}_2$. Since F_{f_i} is a compact interval and f_i's commute $\bigcap_{i=1}^{n} F_{f_i}$ is a non-empty compact interval, say $[c, d]$. As h commutes with each $f_i \in C_1$, h maps $[c, d]$ into itself and so has a fixed point $z \in [c, d]$. Now $g_1^n(z)$ has a limit point z_1 in $\overline{P_{g_1}}$ by Theorem 2.4.6. As $P_{g_1} = F_{g_1}$ (by hypothesis (ii), and F_{g_1} is closed, $\overline{P_{g_1}} = P_{g_1}$. Similarly $g_2^n(z_1)$ has a limit point z_2 in $\overline{P_{g_2}} = F_{g_2} = P_{g_2}$ and as F_{g_2} is closed $z_2 \in F_{g_2}$. Thus $z_1, z_2 \in [c, a]$. Thus proceeding, we see that $\{g_j^n(z_{j-1})\}$ has a limit point z_j in P_{g_j} for $j = 2, \dots, m$ which is fixed for $f_1, \dots, f_n, h, g_1, \dots, g_m$. So $\cap F_f \neq \phi$ for all $f \in C_1 \cup \{h\}$. It is also easily seen that for any finite subset C_2 of $\mathscr{F}_1$, $\bigcap_{f \in C_2} F_f \neq \phi$ as also $\bigcap_{f \in C_3} F_f \neq \phi$ for any finite subset C_3 of $\mathscr{F}_2$. Thus, the family of closed subsets $\{F_f : f \in \mathscr{F} \cup \{h\}\}$ of $[a, b]$ has finite intersection property and hence $\cap\{F_r : f \in \mathscr{F} \cup \{h\}\}$ is non-empty, in view of the compactness of $[a, b]$. □

Theorem 2.4.9 (Cano [6]) *Let $f : I (= [a, b]) \to I$ be a continuous function such that $\{f^n : n \in \mathbb{N}\}$ is an equicontinuous family at each $x \in I$. Then*

(1) F_p, the fixed point set of f is a compact subinterval of I;
(2) if F_f is a non-degenerate interval, then $F_f = P_f$ (P_f being the set of periodic points of f).

Proof As $f : I \to I$ is continuous, $F_f \neq \phi$. If F_f is a singleton, the theorem is true. Suppose $a_0, b_0 \in F_f$ and $a_0 < b_0$. Assume that for no $x \in (a_0, b_0)$, $x_0 = f(x_0)$. Then for all $x \in (a_0, b_0)$, $f(x) > x$ or $f(x) < x$. Assume that $f(x) > x$ for all $x \in (a_0, b_0)$.

Case (i) If $f(x) < b_)$ for all $x \in (a_0, b_0)$ then $f^n(x) \in (a_0, b_0)$ for all $n \in \mathbb{N}$ and $f^n(x) < f^{n+1}(x) < b_0$ and so it converges to a fixed point of f, which cannot be in (a_0, b_0) and hence has to be b_0. So given $\epsilon > 0$, by the equicontinuity of $\{f^n\}$ at a_0, there exists $\delta > 0$ such that $|a_0 - x_0| < \delta$ such that for $|a_0 - x_0| < \delta$, $|f^n(a_0) - f^n(x_0)| < \epsilon$. Since $f^n(a_0) = a_0$, for all n, this contradicts that $f^n(x_0)$ converges to b_0.

Case (ii) Suppose for some $x_0 \in (a_0, b_0)$, $f(x_0) \geq b_0$. Then there is a least number z in (a_0, b_0) with $f(z) \geq b_0$. In fact $f(z) = b_0$. Otherwise, there exists $z' < z$ with $f(z') \geq b_0$ by the continuity of f and this contradicts the definition of z. Thus proceeding, we can find a non-increasing sequence (x_n) in $(a_0, z]$ such that (x_n) converges to $a_0, x_1 = z$ and $f(x_n) = x_{n-1}, n = 2, 3, \ldots$. Since $f^n(x_n) = f^{n-1}(x_{n-1}) = \cdots f(x_1) = f(z) = b_0$ for all n, f^n cannot be equicontinuous at a_0. (Note that as (x_n) is non-increasing in (a_0, z) it converges to a number $z' \geq a_0$. $z' > a_0$ is a contradiction as $z' = f(z')$ and by assumption f has no fixed point in (a_0, b_0).)

Suppose $f(x) < x$ for all $x \in (a_0, b_0)$. We consider

Case (i)′ Suppose $f(x) > a_0$ for all $x \in (a_0, b_0)$. Then for all $x \in (a_0, b_0)$, $f^n(x) > f^{n+1}(x), n \in \mathbb{N}$ and $(f_n(x))$ as in Case (i) converges to a_0. However the family of f iterates cannot be equicontinuous at b_0.

Case (ii)′ If for some $x \in (a_0, b_0)$, $f(x) \leq a_0$. Then there is a greatest element z' in (a_0, b_0) with $f(z') \leq a_0$. In fact $f(z') = a_0$. By this process, a non-decreasing sequence (y_n) can be chosen in $(z', b_0]$ with $y_1 = z'$, $f(y_n) = y_{n-1}$, $n = 2, 3, \ldots$. So $f^n(y_n) = f(z') = a_0$. If (y_n) converges to w, then $f(y_n)$ $(= y_{n-1})$ converges to $f(w)$ and so $w = f(w)$. As $w \notin (a_0, b_0)$, (y_n) converges to b_0. Since $f^n(y_n) = f^{n-1}(y_{n-1}) \cdots = f(z') = a_0$. As y_n converges to b_0, there is a contradiction to the equicontinuity of f^n at b_0.

Thus we have shown that F_f is a non-void compact interval. If F_f is non-degenerate let $F_f = [a_0, b_0]$ where $a_0 < b_0$. Let $f^n(x) = x$ for some n and $x \in [a, a_0)$. (If $x \in (b_0, b]$, then a similar argument can be provided). Since f^n has a fixed point and its iterates are equicontinuous at each point, $f^n(y) = y$ for all $y \in [x, a_0]$ by what has been proved in (i) so far. Since $f(y) > y$ for all $y \in [a, a_0)$ and $f(a_0) = a_0$, we can choose y from (x, a_0) close to a_0, such that $a_0 - \frac{1}{k} < y < f(y) \cdots < f^{n-1}(y) < a$ and this implies $f^n(y) > y$, a contradiction. So $a_0 + \frac{1}{k} > f(y) > a_0 > y > a_0 - \frac{1}{k}$. Then $f(y)$ is a fixed point for f. So $f(y) = f^2(y)$ and $f^n(y) = f^{n-2}(f^2(y)) = f^{n-1}(y)$. Thus proceeding, $y = f^n(y) = f^{n-1}(b) \cdots = f(y)$ contradicting $f(y) > a > y$. Thus if $F_f = [a_0, b_0]$, $[a_0, a)$ has no periodic point. Similarly $(b_0, b]$ has no periodic point. □

This leads to the following.

Theorem 2.4.10 (Jachymski [19]) *Let $g : I \to I$ be a continuous map and I, a compact interval $[a, b]$ of real numbers. Then the following are equivalent:*

(i) F_g the set of fixed points of g is a compact subinterval of I;

(ii) either F_g is a singleton or the family $\{g^n : n \in \mathbb{N}\}$ of iterates is equicontinuous on F_g;

(iii) g has a common fixed point with each continuous map $f : I \to I$ that commutes with g on F_g.

Proof (i) $\implies$ (ii). Suppose F_g is not a singleton and is $[a_1, b_1]$ where $a_1 < b_1$. Since for $a_1 < x < b_1$, $g^n(x) = x$ for all $n \in \mathbb{N}$, the continuity of g at x implies that given $\epsilon > 0$ with $b - a > \epsilon$, there is a $\delta(\epsilon) > 0$ such that $(x - \delta, x + \delta) \subseteq (a_1, b_1)$ and $|g(x) - g(x')| < \epsilon$ for $x' \in (x - \delta, x + \delta)$. So $|g^n(x) - g^n(x')| = |g(x) - g(x')| < \epsilon$ for $x' \in (x - \delta, x + \delta)$, proving the equicontinuity of $\{g^n\}$ on (a_1, b_1). We now show that $\{g^n\}$ is equicontinuous at a_1. Since g is continuous at a_1, there exists $\delta(\epsilon) > 0$ with $\epsilon > \delta(\epsilon)$ for a given $\epsilon > 0$ such that for $a_1 - \delta < x < a_1 + \delta$, $|g(x) - g(a_1)| = |g(x) - a_1| < \epsilon$. We now show by the principle of finite induction that $a_1 - \epsilon < g^n(x) < a_1 + \epsilon$ for all $x \in (a_1 - \delta, a_1 + \delta)$ for all $n \in \mathbb{N}$. Clearly, the inequality is true for $n = 1$. Suppose it is true for $n = 1, 2, \ldots, k$. Let $x \in (a - \delta, a)$. If $a_1 \leq g^k(x) < a_1 + \epsilon$, then $g^k(x) \in F_g$ and so $|g^{k+1}(x) - a_1| = |g^{k+1}(x) - g^{k+1}(a_1)| = |g^k(x) - g^k(a_1)| = |g^k(x) - a_1| < \epsilon$. If $g^k(x) < a_1$, then $g^i(x) < a_1$ for $i = 1, 2, \ldots, k$. Otherwise by induction hypothesis for some i, $1 \leq i \leq k$ and $a_1 \leq g^i(a) < a_1 + \epsilon$ or $g^i(x) \in F_g$ and so $g^k(x) \in F_g$ or $g^k(x) \geq a_1$, a contradiction. Since $F_g = [a_1, b_1]$, $g(x) > x$ for $x \in [a, a_1)$. So $g^i(x) > g^{i-1}(x)$ for $i = 1, 2, \ldots, k$, implying that $g^k(x) > g^{k-1}(x) > \cdots > x$. As $a_1 - \delta < x$ and $g^k(x) < a_1$, it follows that $g^k(x) \in (a_1 - \delta, a_1)$. So $|g(g^k(x)) - g(a_1)| = |g^{k+1}(x) - a_1| < \epsilon$. For $x \in (a_1, a_1 + \delta) \subseteq [a_1, b_1]$, $|g^n(x) - g^n(a_1)| = |x - a_1| < \epsilon$. Thus g^n is equicontinuous at a_1. By a similar reasoning, (g^n) is equicontinuous at b_1.

(ii) $\implies$ (i). This follows from the proof of Theorem 2.4.9 (i). In fact to prove (i) of Theorem 2.4.9, it suffices to assume that $\{f^n\}$ is equicontinuous on F_f.

(i) $\implies$ (iii). If f commutes with g on F_g then F_g is invariant under f. Since F_g is a compact interval by (i), f has a fixed point in F_g which is a common fixed point of f and g.

(iii) $\implies$ (i). If F_g is not an interval, then there exists $a_1, b_1 \in F_g$ such that $(a_1, b_1) \cap F_g = \phi$. Define $f : [a, b] \to [a_1, b_1]$ by

$$f(x) = \begin{cases} b_1 & \text{for } x \in [a, a_1] \\ b_1 + a_1 - x & \text{for } x \in (a_1, b_1] \\ a_1 & \text{for } x \in (b_1, b] \end{cases}$$

f is continuous on I. Let $x \in F_g$. Then $x \in [a, a_1]$ or $[b_1, b]$. If $x \in [a, a_1]$, then $fg(x) = f(x) = b_1 = gf(x) = g(b_1)$. If $x \in [b_1, b]$, then $fg(x) = f(x) = a_1 = g(a_1) = gf(x)$. Thus, f and g commute on F_g but $F_f \cap F_g = \phi$. Hence the theorem. □

Example 2.4.11 The continuous map $g : [0, 1] \to [0, 1]$ defined by $g(x) = 1$ on $[0, \frac{1}{4}]$, $\frac{3}{2} - 2x$ for $x \in (\frac{1}{4}, \frac{3}{4}]$ and 0 on $(\frac{3}{4}, 1]$ has the only fixed point $x = \frac{1}{2}$. But $g^n(\frac{1}{2} + \delta) = (-2)^n\delta + \frac{1}{2}$ for $0 < \delta < \frac{1}{4}$, as long as $2^n\delta < \frac{1}{4}$ or $\delta < \frac{1}{2^{n+2}}$. Suppose g is equicontinuous at $x = \frac{1}{2}$. Then for $\epsilon = \frac{1}{4}$, there exists $\delta > 0$ such that $|g^n(1 + \delta) - g^n(\frac{1}{2})| < \epsilon$ for all n. Since $g(\frac{1}{2}) = \frac{1}{2}$ and choosing least n_0 such that $2^{n_0}\delta > \frac{1}{4}$, it follows that $g^n(\frac{1}{2} + \delta) = 0$ for all $n \geq n_0$ and $|g^n(\frac{1}{2} + \delta) - g^n(\frac{1}{2})| = |0 - \frac{1}{2}| = |\frac{1}{2}| \not< \frac{1}{4}$, a contradiction. So (g^n) is not equicontinuous.

If $f : [0, 1] \to [0, 1]$ commutes with g at $\frac{1}{2}$, then $fg(\frac{1}{2}) = g(f(\frac{1}{2})) = f(\frac{1}{2})$ (as $g(\frac{1}{2}) = \frac{1}{2}$). Since $f(\frac{1}{2})$ is a fixed point of g and g has the unique fixed point $\frac{1}{2}$, $f(\frac{1}{2}) = \frac{1}{2}$. Thus, f and g have a common fixed point, even though $\{g^n\}$ is not equicontinuous.

This example points out that the hypothesis F_g is a singleton cannot be dropped in Theorem 2.4.10.

The next theorem on the convergence of iterates, due to Coven and Hedlund [12], was also obtained independently by Chu and Moyer [7].

Theorem 2.4.12 *If $f : I = [a, b] \to I$ is continuous and $P_f = F_f$, then for each $x \in I$, there exists $p \in F_f$ such that $\{f^n(x)\}$ converges to p.*

Proof If $\{f^n(x)\}$ converges to p, it follows from the continuity of f, that $p \in F_f$. Thus it suffices to prove the convergence of $\{f^n(x)\}$ for each $x \in I$. If $f^n(x) \in P_f$ for some $n \geq 0$, the conclusion is obvious. Suppose that $f^n(x)$ is not a periodic point of f for any $n \geq 0$. Let C_n be the component of NP_f, the set of non-periodic points of f in I containing $f_n(x)$. Let $\xi_n = +1$ if f is completely positive on C_n (i.e.) $(f(x) > x$ on $C_n)$ and $\xi_n = -1$ if f is totally negative on C (i.e. $f(x) < x$ on C_n). Since f is continuous and C_n is connected, $f(x) - x$ cannot take both positive and negative values on C_n as C_n has no fixed point.

If for some $N \geq 0$, $\xi_n = +1$ for $n \geq N$, then $f^N(x) < f^{N+1}(x)$ and so $f^n(a)$ converges. Similarly if $\xi_n = -1$ for all $n \geq N_1$, then $\{f^n(x)\}$ converges.

Suppose $+1$ and -1 appear infinitely many times in the sequence (ξ_n), $n \geq 0$. Let $A = \{n \geq 0 : \xi_n = +1\} = \{p_1 < p_2 < \cdots\}$ and $B = \{n \geq 0 : \xi_n = -1\} = \{m_1 < m_2 < \cdots\}$. $\{f^{p_i}(x)\}$ is increasing while $\{f^{m_i}(x)\}$ is decreasing in I and hence these subsequences of $\{f^n(x)\}$ converge to p and q respectively in I. Now we can find a subsequence $k_i \in A$ such that $k_i + 1 \in B$. Since $\{f^{k_i}(x)\}$ converges to p $\{f^{k_i+1}(x)\}$ converges to q and f is continuous $f(p) = q$. By a similar reasoning we find that $f(q) = p$. Thus $f^2(p) = f(q) = p$ and $f^2(q) = f(p) = q$. Thus $p \in P_f = F_f$. So $p = f(p) = q$. Hence the theorem. □

Corollary 2.4.13 *If $f : I = [a, b] \to I$ is continuous and the set of least periods or periodic points is finite, then for each $x \in [a, b]$, there exists $p \in P_f$ such that $|f^n(x) - p|$ converges to zero as $n \to \infty$.*

Proof Let N be the least common period of the periodic points. Apply Theorem 2.4.12 to f^N and that $P_{f^N} = F_{f^N}$. (It is to be observed that N must be a power of 2, as can be seen from Sharkovsky's theorem.) □

Our next theorem characterizes functions $f : I \to I$ that are continuous and for which $P_f = F_f$.

Theorem 2.4.14 (Jachymski [19]) *Let $g : I = [a, b] \to I$ be a continuous function. Then the following are equivalent:*

(i) $F_g = P_g$;
(ii) $\{g^n : n \in \mathbb{N}\}$ *is pointwise convergent on* I;
(iii) g *has a common fixed point with every continuous map* $f : I \to I$ *that commutes with* g *on* F_f.

Proof (i) $\implies$ (ii) is precisely Theorem 2.4.12.

(ii) $\implies$ (iii). Let $x \in F_f$. By the commutativity of f and g on F_f, F_f is g-invariant. So $g^n(x) \in F_f$ for all $n \geq 1$. Since $\{g^n(x)\}$ converges to $z \in I$ by (ii) and F_f is closed $z \in F_f$ and as g is continuous $z = g(z)$. Thus $z = f(z) = g(z)$.

(iii) $\implies$ (i). Let C be a non-empty g-invariant closed subset of I. We show that $C \cap F_g \neq \phi$. For such a set, there is a continuous map $f : I \to I$ such that $F_f = C$. If $x \in F_f$, then $g(f(x)) = g(x)$ and $f(g(x)) = g(x)$, since C is g-invariant. So f and g commute on $F(g)$. By assumption (iii) $F_f \cap F_g = C \cap F_g \neq \phi$. Let p be a periodic point of least period M for g. Then $C = \{p, g(p), \ldots, g^{M-1}(p)\}$ is closed and invariant under g. So from what we have shown, C has a fixed point of g. If for $1 \leq i \leq M - 1$, $g(g^i(p)) = g^i(p)$, $g^i(p) = g^{(M)}(p) = p$, contradicting p is a periodic point of g with least period M. So $i = 0$ gives $g(p) = p$ or p is a fixed point of g. Thus $P_g = F_g$. □

2.5 Common Fixed Points and Full Functions

In this section, an existence theorem on the common fixed points for two commuting continuous self-maps on a compact real interval, due to Cohen [9] is proved. This supplements the theorems in Sect. 2.4. Without loss of generality we take $I = [0, 1]$. We need the following lemmata and definitions.

Lemma 2.5.1 *Let $f, g : I \to I$ be continuous maps and $h : I \to J = [c, d]$ be a homeomorphism onto J. f and g commute on I and have a common fixed point if and only if hfh^{-1} and hgh^{-1} commute on J and have a common fixed point.*

Proof Let $h : I \to J$ be a homeomorphism onto J and $f, g : I \to I$ be continuous functions. Let $hfh^{-1} : J \to J$ and $hgh^{-1} : J \to J$ be commutative and y_0 be a common fixed point. Then $y_0 = hfh^{-1}(y_0) = hgh^{-1}(y_0$. Since h is a homeomorphism from I onto J, so h^{-1} is a homeomorphism of J onto I. So $h^{-1}y_0 = h^{-1}(hfh^{-1}(y_0)) = h^{-1}hgh^{-1}(y_0)$. Thus $h^{-1}(y_0) = f(h^{-1}(y_0)) = g(h^{-1}(y_0))$ or $x_0 = h^{-1}(y_0)$ belongs to I and is a common fixed point for f and g in I. Also by the commutativity of hfh^{-1} and hgh^{-1} we get $hfgh^{-1} = (hfh^{-1}) \circ hgh^{-1} = (hgh^{-1}) \circ (hfh^{-1}) = hgfh^{-1}$ whence $fg = gh$ on I.

If $f(g(x)) = g(f(x))$ for all $x \in I$ and $h^{-1} : J \to I$ is a homeomorphism, for each $y \in J$, $fgh^{-1}(y) = gfh^{-1}(y)$ and so $fh^{-1}hgh^{-1}y = gh^{-1}hfh^{-1}y$ for $y \in J$. Premultiplying by h we get for $y \in J$

$$(hfh^{-1})(hgh^{-1})y = (hgh^{-1})(hfh^{-1})y.$$

Thus hfh^{-1} and hgh^{-1} commute. If for $x_0 \in I$ $x_0 = f(x_0) = g(x_0)$, then $h(x_0) = hf(x_0) = hg(x_0)$. But $x_0 = h^{-1}(y_0)$ for some $y_0 \in J$. So $y_0 = hfh^{-1}(y_0) = hgf^{-1}(y_0)$. Thus hfh^{-1} and hgh^{-1} have a common fixed point. □

Lemma 2.5.2 *If $f, g : I \to I$ are commuting continuous functions without a common fixed point, then there are commuting functions mapping I onto I without a common fixed point.*

Proof Let $a_1 = \max\{\inf_I f, \inf_I g\}$ and $b_1 = \min\{\sup_I f, \sup_I g\}$. Since f and g commute, $f[0,1] \cap g[0,1] \neq \phi$ both f and g map $[a_1, b_1]$ into itself. Otherwise for some $x \in [a_1, b_1]$, $f(x) > b_1$ would imply that for some $y \in [0,1]$, $g(y) = x$ and $g(f(y)) = fg(y) = f(x) > b_1$. This implies that $b_1 < \min\{\sup_I f, \sup_I g\}$. Similarly $f(x) < a_1$ for some $x \in [a_1, b_1]$ would imply that there exists $y \in [0,1]$ with $g(y) = x$ and $g(f(y)) = fg(y) = f(x) < a_1$. This means that $a_1 > \max\{\inf_I f, \inf_I g\}$, a contradiction. Writing f_1 and g_1 as the restrictions of f and g on $J_1 = [a_1, b_1]$ respectively, we can inductively define a_i, b_i and f_i by $a_i = \max\{\inf_{J_{i-1}} f, \inf_{J_{i-1}} g\}$ and $b_i = \min\{\sup_{J_{i-1}} f, \sup_{J_{i-1}} g\}$ where $J_{i-1} = [a_{i-1}, b_{i-1}]$, $i = 2, 3, \ldots$, and f_i is the restriction of f_{i-1} to J_{i-1}. Since $[a_i, b_i]$, $i = 1, 2, \ldots$, form a nested sequence of compact subsets of $[0,1]$, they have a non-void intersection. If this intersection is a singleton, then f and g have a common fixed point contrary to the assumption. Hence, $\bigcap_{i=1}^{\infty}[a_i, b_i]$ is a non-degenerate compact interval $[a, b]$ and the restriction $\overline{f}$ and $\overline{g}$ of f and g respectively map $[a, b]$ onto itself. If h is a homeomorphism of $[a, b]$ onto $I = [0,1]$. Then, the continuous maps $h\overline{f}h^{-1}$ and $h\overline{g}h^{-1}$ map $[0,1]$ onto itself but have no common fixed points by Lemma 2.5.1. □

Lemma 2.5.3 *If $f, g : I \to I$ are commuting continuous functions, so are f and gf. f and g have a common fixed point if and only if f and gf have a common fixed point.*

Proof $f(gf) = gf \circ f$ as $fg = gf$. If $x_0 = f(x_0) = g(x_0)$, then $x_0 = f(x_0) = g(x_0) = g(f(x_0))$. If $x_1 = f(x_1) = g(f(x_1))$, then $x_1 = f(x_1) = g(x_1)$. □

Definition 2.5.4 A continuous function $f : I \to I$ is said to be full if there is a partition $P_f = \{x_0 = 0 < x_1 < x_2 \cdots < x_n = 1\}$ of I such that f on $[x_i, x_{i+1}]$ is a homeomorphism on $[0,1]$ for each $i = 0, 1, \ldots, n-1$.

Definition 2.5.5 A partition $P_f = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ is regular if the length of the subintervals $x_{i+1} - x_i$ is the same for all $i = 0, 1, \ldots, n-1$. A partition P_g refines a partition P_f uniformly if each subintervals in P_f formed by consecutive partition points of P_f is the union of partitioning subintervals of g.

Lemma 2.5.6 *If f_1, g_1 are commuting full functions on [0, 1] without a common fixed point, there are functions f and g with the same properties and additionally $f(0) = g(1) = 0$ and $f(1) = g(0) = 1$, P_f, P_g and P_{fg} are regular and P_g refines P_f uniformly.*

Proof If $f_1(0) = g_1(0) = 0$, then f_1 and g_1 have a common fixed point contrary to the assumption. So essentially two cases arise: (i) $f_1(0) = 0$, $g_1(0) = 1$ and (ii) $f_1(0) = 1 = g_1(0)$. In case (i) $f_1(1) = f_1g_1(0) = g_1f_1(0) = 1$ and so $g_1(1) = 0$, as otherwise $g_1(1) = 1$ would imply that f_1 and g_1 have 1 as a common fixed point. In this case let $f_2 = f_1$ and $g_2 = g_1$.

For case (ii), $f_1(1) = f_1g_1(0) = g_1f_1(0) = g_1(1)$. So $f_1(1) = g_1(1) = 0$ as otherwise 1 would be a fixed point. In this case let $f_2 = f_1g_1$ and $g_2 = g_1$, $g_2(0) = g(0) = 1$, $f_2(1) = f_1g_1(1) = f_1(0) = 1$ and $g_2(1) = g_1(1) = 0$. In either case let $f_3 = f_2$ and $g_3 = g_2f_2$. Clearly P_{g_3} refines P_{f_2} uniformly. Let h be any order preserving homeomorphism on [0, 1] taking $P_{f_3g_3}$ into the corresponding regular partition of [0, 1]. Define $f = hf_3h^{-1}$ and $g = hg_3h^{-1}$. As f_3 and g_3 have no common fixed point, by Lemma 2.5.1 f and g do not have a common fixed point. Also P_f, P_g, P_{fg} are regular and as P_{g_3} refines P_{f_3} uniformly. P_g refines P_f uniformly. □

Theorem 2.5.7 (Cohen) *Commuting continuous full functions mapping [0, 1] onto [0, 1] have a common fixed point.*

Proof Let $f_1, g_1 : I \to I$ be two commuting full functions without a common fixed point. So using Lemma 2.5.6, we can find commuting full functions f_1, g_1 mapping [0, 1] onto itself such that $f(0) = g(1) = 0$, $f(1) = g(0) = 1$, P_f, P_g and P_{fg} regular partitions with P_g refining P_f uniformly. Let $P_f = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ and $P_g = \{0, \frac{1}{m}, \frac{2}{m}, \ldots, 1\}$ and $P_{fg} = \{0, \frac{1}{mn}, \frac{2}{mn}, \ldots, 1\}$ where m and n are odd. Let f_i and g_i be restrictions of f to $[\frac{i-1}{n}, \frac{i}{n}]$ and g to $[\frac{i-1}{m}, \frac{i}{m}]$, respectively. Let $r = \frac{n+1}{2}$ and $s = \frac{m+1}{2}$. Suppose r is odd and s is even. If $D(f_i, g_j)$ is the domain of f_ig_j for each i and j then it is a subinterval of P_{fg}. In particular

$$\begin{aligned} D(g_1 f_r) &= \left[\frac{r-1}{n}, \frac{r-1}{n} + \frac{1}{mn}\right] \\ D(g_2 f_r) &= \left[\frac{r-1}{n} + \frac{1}{mn}, \frac{r-1}{n} + \frac{2}{mn}\right] \ldots, \\ D(g_s f_r) &= \left[\frac{r-1}{n} + \frac{s-1}{mn}, \frac{r-1}{n} + \frac{s}{mn}\right] \\ &= \left[\frac{mn-1}{2mn}, \frac{mn+1}{2mn}\right] \end{aligned}$$

Similarly

$$D(f_1 g_s) = \left[\frac{s-1}{m}, \frac{s-1}{m} + \frac{1}{mn}\right]$$
$$D(f_2 g_s) = \left[\frac{s-1}{m} + \frac{1}{mn}, \frac{s-1}{m} + \frac{2}{mn}\right] \ldots,$$
$$D(f_r g_s) = \left[\frac{s-1}{m} + \frac{r-1}{mn}, \frac{s-1}{m} + \frac{r}{mn}\right]$$
$$= \left[\frac{mn-1}{2mn}, \frac{mn+1}{2mn}\right]$$

Thus $D(f_r g_s) = D(g_s f_r)$. Since g_s is continuous and onto [0, 1], its graph must intersect the diagonal of $I \times I$ and g_s has a fixed point z_1. As $D(g_s) \subseteq D(f_0)$, $z_1 \in D(f_r)$ and thus $z_1 \in D(f_r g_s) = D(g_s f_r)$. So $g_s f_r(z_1) = f_r g_s(z_1) = f_r(z_1)$ and $z_2 = f_r(z_1)$ is a fixed point of g_s. Thus proceeding, we get a sequence z_p of fixed points of g_s with $z_{p+1} = f_r(z_p)$. Since f_r is monotone the sequence z_p converges to z_1 a fixed point of both f and g. The case when r is even and s is odd can be handled similarly. □

Remark 2.5.8 One can show that f is full if and only if f maps [0, 1] onto [0, 1] and is an open map. For related work, Baxter and Joichi [3] may be referred.

2.6 Common Fixed Points of Commuting Analytic Functions

We prove a theorem of Shields [28] on the common fixed points of analytic functions in this section. We denote by G, a non-void bounded open connected set in the complex plane. Let F_G be the family of all analytic functions mapping G into itself. Clearly F_G is a semigroup under composition of mappings. We can consider $H(G)$ the linear space of all functions analytic on G and continuous on $\overline{G}$, with the topology of uniform convergence on compact subsets of G. This topology is a metric topology and indeed it arises from a complete metric and so F_G will inherit this metric topology. The following lemma implies that F_G is a topological semigroup (i.e. the composition map is a continuous function from $G \times G$ into G).

Lemma 2.6.1 *Let $f_n, g_n \in F_G$ and $f_n \to f$, $g_n \to g$ in the topology of uniform convergence on compact subsets of G. Then $f_n(g_n) \to f(g)$ and so F_G is a topological semigroup.*

Proof Let K be a compact subset of G and let U be an open set containing $g(K)$ with $\overline{U}$ compact and lying in G. Since $g_n \to g$ uniformly on K, $g_n(K) \subset U$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Now for all n

$$|f(g(z)) - f_n(g_n(z))| \leq |f(g(z)) - f(g_n(z))| + |f(g_n(z)) - f_n g_n(z)|$$

Since $g(z),\ g_n(z) \in \overline{U}$ for $z \in K$ for all $n \geq n_0$ and f is uniformly continuous on the compact set $\overline{U}$ and $f_n \to f$ uniformly on $\overline{U}$, the above inequality implies that $f(g_n(z)) \to fg(z)$ and $|f_n(g_n(z)) - f(g_n(z))| \leq \sup_{w \in \overline{U}} |f_n(w) - f(w)| \to 0$ as $n \to \infty$. Hence $(f_n g_n)$ converges uniformly on K to fg. Thus F_G is a topological semigroup. □

A few facts from the theory of topological semigroups will be needed in the sequel. For proofs and other details Numakura [23], Wallace [31] and Ellis [15] may be consulted.

Definition 2.6.2 Let $(S, \cdot)$ be a semigroup. An element e of S is called an idempotent if $e.e = e^2 = e$. An element 0 is termed zero if $0.x = 0$ for all $x \in S$. 1 is called an identity of S if $1.x = x = x.1$ for all $x \in S$. In a semigroup S if $ax = ay$ $(xa = ya)$ implies $x = y$ for all a, x, y in S then S is called a semigroup satisfying the left (right) cancellation law. If S satisfies both the left and right cancellation laws, it is called a semigroup satisfying cancellation law.

The following is a basic result in the theory of topological semigroups and the proof is essentially from Ellis [15].

Lemma 2.6.3 *Let S be a compact Hausdorff topological semigroup. Then S has an idempotent element.*

Proof Let $\mathscr{F}$ be the family of all compact subsets K of S such that $K^2 \subseteq K$. $\mathscr{F} \neq \phi$, as $S \in \mathscr{F}$. $\mathscr{F}$ is partially ordered by set inclusion. As every chain in $\mathscr{F}$ has a lower bound $\mathscr{F}$ has a minimal element A in $\mathscr{F}$. If $r \in A$, then rA is a non-void compact subset of S as rA is the image of the compact set A under the continuous map $x \to r.x$. So $rA \in \mathscr{F}$ and $rA \subseteq A$. Since A is minimal $rA = A$. So there exists $p \in A$ such that $rp = r$. Define $L = \{a \in A : ra = r\}$. Clearly $p \in L$ and L is a compact subset of A. Let $\ell_1, \ell_2 \in L$. Then $r\ell_1\ell_2 = r\ell_2 = r$ and hence $\ell_1 \circ \ell_2 \in L$. So $L^2 \subseteq L$. Hence $L \in \mathscr{F}$. As $L \subseteq A$ and A is minimal $L = A$. Since $r \in A = L$, $r^2 = r$ from the definition of L. Thus S has an idempotent element. □

We skip the proof of the following.

Lemma 2.6.4 *Let S be a compact T_2 topological semigroup which is commutative. For $x \in S$ and $\Gamma(x) = cl\{x, x^2, \ldots, \}$, we have*

(i) *$\Gamma(x)$ contains exactly one idempotent;*
(ii) *if e is an identity for $\Gamma(x)$, then $\Gamma(x)$ is a group and x has an inverse in $\Gamma(x)$;*
(iii) *if e is a zero for $\Gamma(x)$, then $x_n \to e$.*

The following lemma makes use of the basic properties of analytic functions.

Lemma 2.6.5 *If the analytic function $e \in F_G$ is idempotent, then $e(z) \equiv z$ on $e(z)$ is constant for all $z \in G$.*

Proof If $e(z)$ is constant for all $z \in G$, clearly it is an idempotent. Suppose e is a non-constant analytic function on G, then f is an open mapping. So $G_1 = e(G)$ is an open set. Since $e^2(z) = e(z)$, $e(z) = z$ on G_1. As G_1 is uncountable, and the analytic functions, viz. identity function and e coincide on G_1, $e(z)$ must be z at each z in G. □

We also recall some classical results from complex analysis (see Conway [11] and Ahlfohrs [1].

Theorem 2.6.6 (Montel) *Let $H(G)$ be the linear space of analytic functions on the open region G. A family $\mathscr{F}$ in $H(G)$ is normal in the sense that every sequence in $\mathscr{F}$ has a convergent subsequence if and only if $\mathscr{F}$ is locally bounded in $H(G)$ (i.e. for each compact subset K of G, there is a positive constant M_k with $|f(z)| \leq M_k$ for all $f \in \mathscr{F}$ and $z \in K$).*

Theorem 2.6.7 (Hurwitz) *Let $A(G)$ be the linear space of all analytic functions with the topology of uniform convergence on compact subsets of G. If (f_n) converges to f in $H(G)$ and f_n never vanishes on G for each n, then $f \equiv 0$ or f is non-zero throughout G.*

Lemma 2.6.8 *Let D be the open unit disc in the complex plane $\mathbb{C}$ and $f : D \to D$ be a bilinear (Mobius) transformation of D onto D. Then there arise three possibilities:*

(i) *$f(z) = z$ on D;*
(ii) *f has exactly one fixed point in the closed unit disc;*
(iii) *f has two distinct fixed points in the unit circle and the iterates of f converge to one of these fixed points.*

Proof The general form of such a bilinear transformation is $f(z) = \alpha \frac{(z-a)}{(1-\overline{a})z}$ where $|\alpha| = 1$, $|a| < 1$.

If f is not the identity function the fixed points $z = f(z)$ are given by

$$\overline{a}z^2 - (1-\alpha)z - \alpha z = 0$$

As this equation is invariant under $z \to \frac{1}{z}$, the fixed points of $f(z)$ are inverses of each other with respect to the unit circle. So there is a fixed point inside and another outside the circle or there is a 'double fixed point' or two distinct fixed points on the unit circle. □

Lemma 2.6.9 *Let $f \in F_G$, be the subset of $H(G)$ containing all analytic functions mapping G into itself. Suppose f is not a homeomorphism of G onto itself. Then there is a point z_0 in $\overline{G}$ and a subsequence $\{f_{n_i}\}$ of f-iterates such that $f_{n_i}(z) \to z_0$ uniformly on compact subsets of G.*

Proof Write $\Gamma(f) = cl\{f^n\}$ in $H(G)$. If $\Gamma(f) \subseteq F_G$, then $\Gamma(f)$ is a compact semigroup under composition of functions and contains an idempotent element $e(z)$ by Lemma 2.6.3.

By Lemma 2.6.5 $e(z) \equiv z$ for all $z \in G$ or is a constant z_0 for all $z \in G$. If the identity map belongs to $\Gamma(f)$, then by Lemma 2.6.4, $\Gamma(f)$ is a group and $f \in \Gamma(f) \subseteq F_G$ would be invertible in $F(G)$ contradicting that f is not a homeomorphism. Hence $e(z) \equiv z_0$, for all $z \in G$ and is thus a zero for $\Gamma(f)$. So again by Lemma 2.6.4 $f^n(z)$ converges to z_0 in the topology of F_G.

Suppose $g \in \Gamma(f)$ does not belong to F_G. Since $f_n(G) \subseteq G$, $g(G) \subseteq \overline{G}$. As $g \notin F_G$, there is a point $z' \in G$ with $g(z') = z_0 \notin G$. We claim that $g(z) \equiv z_0$.

As $g \in \Gamma(f)$, we can find f_{n_k}, a subsequence of f iterates converging to g in $H(G)$. Now $f_{n_k}(z) - z_0$ never vanishes in G as $z_0 \in G$ and converges to $g(z) - z_0$. So by Lemma 2.6.7 (Hurwitz Theorem), $g(z) - z_0$ is identically zero in G or never vanishes in G. But already for $z = z' \in G$, $g(z') - z_0 = 0$. So $g(z) \equiv z_0$ for all $z \in G$. □

Lemma 2.6.10 *Let $f \in F_G$ and suppose f is not a homeomorphism of G onto itself. Let z_0 be the element of $\overline{G}$ such that f_{n_i} converges to z_0 in $H(G)$. Then z_0 is a common fixed point for all continuous g on $\overline{G}$ that map G into itself and commute with f.*

Proof By Lemma 2.6.9, there exists $z \in \overline{G}$ with $\lim f_{n_i}(z) = z_0$ in F_G. For $g \in C(\overline{G})$, $g(z_0) = g(\lim f_{n_i}(z)) = \lim f_{n_i}(g(z)) = z_0$.

The following remarks are relevant.

Remark 2.6.11 If f is a bilinear map of the open unit disc D onto itself with two distinct fixed points on the boundary, consider p a bilinear map, mapping D onto the upper half-plane and taking these fixed points into 0 and ∞. For $g = pfp^{-1}$, 0 and ∞ are fixed points of g and g maps the upper half-plane onto itself. Hence g is a dilatation and is of the form $g(z) = az$, $a > 0$ and $a \neq 1$ as $f(z) \not\equiv z$. So $g^n(z) = a^n z$ tends to zero or to ∞. Thus the iterates of f converge to one of the fixed points of f.

Remark 2.6.12 Wolff [32] and Denjoy [13] have shown independently in 1926 that if f is analytic in D and $f(D) \subseteq D$, then either f is a bilinear map of D onto itself with exactly one fixed point or f^n converges to a constant $C \in \overline{D}$.

We are now in a position to prove a theorem of Shields [28] on the fixed points of commuting family of analytic functions on $\overline{D}$.

Theorem 2.6.13 (Shields [28]) *Let F be a commuting family of continuous functions on $\overline{D}$ which are analytic in D. Then there is a common fixed point z_0 for all functions in F.*

Proof If F contains a constant function then that constant is the common fixed point. Suppose it contains only non-constant continuous functions on $\overline{D}$ which are analytic in D. So by the Maximum Modulus Theorem $f(D) \subseteq D$ for each $f \in F$. Suppose not all functions of F are bilinear maps of D onto D. So there exists f, different from the identity map in F. Then Lemma 2.6.10 can be invoked to conclude that there is a common fixed point for each $f \in F$. On the other hand if all the members of F are bilinear, then if one of them has just one fixed point, then it is a common fixed point for all. In case these have two fixed points then by Remark 2.6.11, the iterates converge to one of the two fixed points and so invoking Lemma 2.6.10, we conclude that for each f in F there is a common fixed point. □

Remark 2.6.14 Theorem 2.6.13 due to Shields has been generalized to Hilbert spaces by Suffridge [29].

2.7 Fixed Points of Meromorphic Functions

In this section, an interesting theorem on the fixed points of meromorphic functions, due to Bergweiler [4] is detailed. Bergweiler's short proof is elementary, though it invokes Picard's theorem. We recall

Theorem 2.7.1 (Picard (see Conway [11])) *Suppose an analytic function f has an essential singularity at a. Then in each neighbourhood of a, f assumes each complex number, with one possible exception, infinitely many times.*

Corollary 2.7.2 *An entire function which is not a polynomial assumes every complex number, with one exception infinitely many times.*

In response to a question raised by Gross [16], Bergweiler [4] proved the following.

Theorem 2.7.3 (Bergweiler [4]) *Let f be a meromorphic function that has at least two different poles and let g be a transcendental entire function. Then the composite function $f \circ g$ has infinitely many fixed points.*

The theorem above makes use of the following lemmas.

Lemma 2.7.4 *Let f be a meromorphic function and z_0 be a pole of order p. Then there is a function h, defined and analytic in a neighbourhood of 0 such that $h(0) = 0$ and $f(h(z) + z_0) = z^{-p}$ for $z \neq 0$.*

Proof The function k defined as $k(z)^{-p} = f(z + z_0)$ is analytic in a neighbourhood of 0 and $k'(0) \neq 0$. So $k(z)$ is invertible in a neighbourhood of 0 and this inverse $h(z)$ is analytic in a neighbourhood of 0. Now $k(0) = 0$. So $h(0) = 0$ and $f(h(z) + z_0) = z^{-p}$ for $z \neq 0$. □

Lemma 2.7.5 *Let f and g be meromorphic functions. Then $f \circ g$ has infinitely many fixed points if and only if $g \circ f$ does.*

Proof If $x_0 = fg(x_0)$, then $gx_0 = gf(g(x_0))$ so that $g(x_0)$ is a fixed point of gf. If $x_0 = fg(x_0)$ and $x_1 = fg(x_1)$, then $g(x_0) = g(x_1)$ would imply that $fg(x_0) = fg(x_1)$ so that $x_0 = x_1$. Thus g maps the set of fixed points of $f \circ g$ injectively into the set of fixed points of $g \circ f$. Indeed if x^* is a fixed point of $g \circ f$, then $f(x^*)$ is a fixed point of $f \circ g$. Similarly f maps the set of fixed points of $g \circ f$ injectively into the set of fixed points of $f \circ g$. Thus the sets of fixed points of $f \circ g$ and $g \circ f$ have the same cardinality. (Indeed g maps the set of fixed points of $f \circ g$ bijectively onto the set of fixed points of $g \circ f$). □

Now we provide the proof of Theorem 2.7.3.

Proof Let z_1 and z_2 be poles of f of order p_1 and p_2. Using Lemma 2.7.4 choose the functions h_j for $j \in \{1, 2\}$. Let $k_1(z) = h_1(z^{p_2}) + z_1$ and $k_2(z) = h_2(z^{p_1}) + z_2$. Now $f(k_1(z)) = f(k_2(z)) = z^{-p_1 p_2}$ for $z \neq 0$ in a neighbourhood of 0. Define $u(z) =$

$g(z^{-p_1 p_2})$. Then 0 is an essential singularity of u and in a punctured neighbourhood of 0, $u(z) = g(fk_1(z)) = gf(k_2(z))$.

If $f \circ g$ has only finitely many fixed points, then so has $g \circ f$ only finitely many fixed points by Lemma 2.7.5. So $u(z) \neq k_j(z)$ for $j = 1, 2$ in a punctured neighbourhood of 0, since $k_1(0) = z_1 \neq z_2 = k_2(0)$. Define

$$v(z) = \frac{u(z) - k_1(z)}{k_2(z) - k_1(z)}.$$

0 is an essential singularity for u and v does not take the values 0, 1 and ∞ in a punctured neighbourhood of 0. This contradicts Picard's Theorem 2.7.1. Hence the theorem. □

Remark 2.7.6 It can be similarly shown that if f and g are transcendental meromorphic functions and if either f or g has at least three poles, then $f \circ g$ has infinitely many fixed points.

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