Chapter 1
Prerequisites

This chapter is a precis of the basic definitions and theorems used in the sequel. It is presumed that the reader is familiar with naive set theory (see Halmos [4]) and the properties of real numbers and real functions (see Bartle [1]). Other mathematical concepts and theorems relevant to specific sections of a chapter will be recalled therein.

1.1 Topological Spaces

This section collects important concepts and results from topology. For proofs and other details, Dugundji [3], Kelley [7], Munkres [9] and Simmons [13] may be consulted.

Definition 1.1.1 Let X be a non-empty set. A collection of J of subsets of X is called a topology on X, if

(i) $\phi, X \in \mathscr{T}$,
(ii) $G_1 \cap G_2 \in \mathscr{T}$ for $G_1, G_2 \in \mathscr{T}$ and
(iii) $\bigcup_{G \in \mathcal{F}} G \in \mathscr{T}$ for any $\mathcal{F} \subseteq \mathscr{T}$.

Any subset of X belonging to $\mathscr{T}$ is called an open set or more precisely $\mathscr{T}$-open set. The pair $(X, \mathscr{T})$ is called a topological space. Given a topological space $(X, \mathscr{T})$, the interior of $A \subseteq X$, denoted by A^0 is the largest open subset of A.

For a subset S of X, where $(X, \mathscr{T})$ is a topological space, $\mathscr{T}_S = \{G \cap S : G \in \mathscr{T}\}$ is a topology on S, called the relative topology (or subspace topology) on S.

Example 1.1.2 For a non-empty set X, the family $\{\phi, X\}$ is a topology on X called the indiscrete topology on X, 2^X, the power set of X or the set of all subsets of X is a topology on X called the discrete topology on X. The family of all subsets of X

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whose complements are finite sets together with the empty set is also a topology on X called the co-finite topology on X.

Since the intersection of any collection of topologies on X is a topology on X, for any family $\mathcal{F}$ of subsets of X, there is the smallest topology on X containing $\mathcal{F}$, called the topology generated by $\mathcal{F}$.

Example 1.1.3 A subset G of real numbers is called open if for each $x \in G$, an open interval containing x lies in G. (Evidently the empty set is open.) This collection of all open subsets of $\mathbb{R}$, the real number system is a topology on $\mathbb{R}$, called the usual topology on $\mathbb{R}$.

Definition 1.1.4 Let $(X, \mathscr{T})$ be a topological space. A neighbourhood of a point $x \in X$ is any subset of X containing an open subset $G \in \mathscr{T}$, containing x. A neighbourhood base or local base at x is a family $\mathcal{N}_x$ of neighbourhoods of x such that for any neighbourhood N of x, there is a neighbourhood $N_x \in \mathcal{N}_x$ such that $x \in N_x \subseteq N$. A topological space is called first countable if for each point there is a countable local base. An interior point of A is a point $a \in A$ such that A contains a neighbourhood of a.

Definition 1.1.5 A subset F of a topological space $(X, \mathscr{T})$ is called a closed subset of X if $X - F$ is $\mathscr{T}$-open. The closure of a subset A of X denoted by $\overline{A}$ is the smallest closed set containing A. A subset S of X is said to be dense in X if $\overline{S} = X$. A topological space $(X, \mathscr{T})$ is called separable if it has a countable dense subset.

Remark 1.1.6 Let $(X, \mathscr{T})$ be a topological space and $A, B \subseteq X$. Then

(i) $\phi^0 = \phi$, $\overline{\phi} = \phi$, $X^0 = X$ and $\overline{X} = X$;
(ii) $\overline{A} \supseteq A$ and $A^0 \subseteq A$;
(iii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$, $(A \cap B)^\circ = A^\circ \cap B^\circ$;
(iv) $\overline{(\overline{A})} = \overline{A}$ and $(A^\circ)^\circ = A^\circ$. Further $\overline{A} = \{x \in X :$ every neighbourhood of x has a non-void intersection with $A\}$. $A^0 = \{a \in A : a$ is an interior point of $A\}$.

Definition 1.1.7 For a topological space $(X, \mathscr{T})$ $\mathcal{B} \subseteq \mathscr{T}$ is called a base (or basis) for $\mathscr{T}$ is for $A_1, A_2 \in \mathcal{B}$ and $x \in A_1 \cap A_2$, there exists $A_3 \in \mathcal{B}$ such that $x \in A_3 \subseteq A_1 \cap A_2$. A subfamily $\mathscr{S}$ of $\mathscr{T}$ is called a subbase for $\mathscr{T}$ of $\mathcal{B}$, if the family of intersections of all finite subfamilies of $\mathscr{S}$ is a base for $\mathscr{T}$. If the topology $\mathscr{T}$ has a countable base, then the topological space is called second countable.

Remark 1.1.8 If $\mathscr{S}$ is a family of subsets of X with $\cup\{S : S \in \mathscr{S}\} = X$, then $\mathscr{S}$ is a subbase for a topology on X, for which $\mathcal{B}$ the family of subsets of X which are the intersections of finite subfamilies of $\mathscr{S}$ is a base for this topology.

Remark 1.1.9 The family of all subintervals of the form $[a, b)$, $a < b$, $a, b \in \mathbb{R}$ is a base for a topology on $\mathbb{R}$, called the lower limit topology on $\mathbb{R}$. Similarly, the family $\{(a, b] : a < b, a, b \in \mathbb{R}\}$ is a base for a topology on $\mathbb{R}$ called the upper limit topology on $\mathbb{R}$. The usual (standard) topology on $\mathbb{R}$ has the family of all open intervals (a, b), $a < b$, $a, b \in \mathbb{R}$ as a base. $\mathbb{R}$ with the usual topology is separable and second countable. However, $\mathbb{R}$ with the lower limit topology is separable and first countable but is not second countable.

Definition 1.1.10 A binary relation $\leq$ on a non-empty subset X is called a quasi-order if the following conditions are satisfied:

(i) $x \leq x$ for all $x \in X$ (reflexivity);
(ii) if $x \leq y$ and $y \leq z$, for $x, y, z \in X$, then $x \leq z$ (transitivity).
If, in addition a quasi-order $\leq$ satisfies
(iii) if $x \leq y$ and $y \leq x$, then $x = y$ (anti-symmetry),

then the quasi-order $\leq$ is called a partial order. Accordingly if $\leq$ is a quasi-order on X, then $(X, \leq)$ is called a quasi-ordered space. If $\leq$ is a partial order on X, then $(X, \leq)$ is called a partially ordered set or poset.

Definition 1.1.11 A partial order $\leq$ on a set X is called a linear order or total order if for any pair of elements $x, y \in X$ either $x \leq y$ or $y \leq x$. A linearly ordered set is also called a chain.

Definition 1.1.12 A partially ordered set $(D, \leq)$ is called a directed set if for any pair $x, y \in X$, there exists $z \in D$ such that $x \leq z$ and $y \leq z$.

Definition 1.1.13 A net in a topological space X is a pair $(S, \geq)$ where S is a function from a directed set $(D, \geq)$ into X. A net $(S, \geq)$ in a topological space is said to converge to an element $x \in X$ if for each open set G containing x, there is an element m of D such that for $n \geq m$, $n \in D$, $S(n) \in G$. Clearly, a sequence in a topological space is a net directed by the set of natural numbers with the usual ordering.

Proposition 1.1.14 *A subset S of a topological space $(X, \mathscr{T})$ is closed if and only if no net in S converges to an element of $X - S$. An element $s \in \overline{S}$ for $S \subseteq X$ if and only if there is a net in S converging to s.*

Definition 1.1.15 Let $(X_i, \mathscr{T}_i), i = 1, 2$ be topological spaces. A map $f : X_1 \to X_2$ is said to be continuous if for each $\mathscr{T}_2$-open subset G of X_2, $f^{-1}(G)$ is $\mathscr{T}_1$-open in X_1. If f is one-one and onto X_2 and if both f and f^{-1} are continuous maps, then f (as also f^{-1}) is called a homeomorphism from X_1 onto X_2 (from X_2 onto X_1).

A map $f : X_1 \to X_2$ is said to be continuous at $x \in X_1$, if for each neighbourhood $N_{f(x)}$ of $f(x)$ in X_2, there is a neighbourhood N_x of x such that $f(N_x) \subseteq N_{f(x)}$. A map $g : X_1 \to X_2$ is called open if it maps open subsets of X_1 onto open subsets of X_2.

The following theorem is well-known.

Theorem 1.1.16 *Let $(X_i, \mathscr{T}_i)$, $i = 1, 2$ be topological spaces and $f : X_1 \to X_2$ be a map. The following statements are equivalent:*

(i) f is continuous on X_1;
(ii) f is continuous at each point of X_1;
(iii) $f^{-1}(F)$ is closed in $(X_1, \mathscr{T}_1)$ for each closed subset F of X_2;
(iv) if $G \in \mathscr{S}$, a subbase for $\mathscr{T}_2$, then $f^{-1}(G) \in \mathscr{T}_1$;

(v) *for each net* $(S, \geq)$ *converging to* x *in* X_1, $(f(S), \geq)$ *converges to* $f(x)$ *in* X_2;
(vi) *for each subset* A *of* X_1, $f(\overline{A}) \subseteq \overline{f(A)}$;
(vii) *for each subset* B *of* Y, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

Theorem 1.1.17 *A topological space* $(X, \mathscr{T})$ *is said to be disconnected if* $X = A \cup B$ *where* A *and* B *are non-empty disjoint proper open subsets of* X. *A pair of sets* A *and* B *is said to be separated if* $\overline{A} \cap B = \overline{B} \cap A = \phi$, *where* A *and* B *are non-empty. A topological space is called connected if it is not disconnected (A connected space is not the union of two non-void separated sets). A subset* Y *of* X *is called connected if* Y *is connected in the subspace topology. A maximal connected subset of* X *is called a component.*

Definition 1.1.18 A topological space is called totally disconnected if the only connected subsets are singletons.

Definition 1.1.19 A topological space is said to be locally connected if the family of open connected subsets is a base for the topology.

Remark 1.1.20 A discrete topological space with more than one element is locally connected, though totally disconnected. The set $(0, 1) \cup (2, 3)$ with the subspace topology inherited from $\mathbb{R}$ with the usual topology is locally connected and disconnected though not totally disconnected.

Theorem 1.1.21 *Let* $(X, \mathscr{T})$ *be a topological space. Then*

(i) *if* A *is a connected subset of* X *and* $A \subseteq B \subseteq \overline{A}$, *then* B *is a connected subset;*
(ii) *the union of a family of connected subsets of* X, *no two of which are separated is connected;*
(iii) *components of* X *are closed and any two components are either identical or disjoint;*
(iv) *any component of an open subset of a locally connected space is open.*

Definition 1.1.22 A family of open sets $\{G_\lambda : \lambda \in \Lambda\}$ of a topological space $(X, \mathscr{T})$ is called an open cover for X, if $X = \bigcup_{\lambda \in \Lambda} G_\lambda$. If every open cover of X has a countable subcover, the topological space is said to be Lindelof. If each open cover of X has a finite subcover, then the topological space is called compact.

Definition 1.1.23 A topological space is called locally compact, if each element has a compact neighbourhood.

Definition 1.1.24 Let $(X_\lambda, \mathscr{T}_\lambda)$, $\lambda \in \Lambda$, $\Lambda \neq \phi$ be a family of topological spaces. The Cartesian product of all these sets X_λ denoted by $X = \prod_{\lambda \in \Lambda} X_\lambda$ is the set of all functions $f : \Lambda \to \bigcup_{\lambda \in \Lambda} X_\lambda$ such that $f(\lambda) \in X_\lambda$ for each $\lambda \in \Lambda$. The map $P_\lambda : X \to X_\lambda$ such that $P_\lambda(f) = f(\lambda)$ for each $f \in X$ is called the projection of the set X into the λth coordinate set X_λ. The topology of X having $\{P_\lambda^{-1}(U) : U \in \mathscr{T}_\lambda, \lambda \in \Lambda\}$ as a subbase is called the product topology on X and X with this topology is referred as the product (topological) space.

Theorem 1.1.25 *Let $\{(X_\lambda, \mathscr{T}_\lambda) : \lambda \in \Lambda, \Lambda \neq \phi\}$ be a family of topological spaces and X be the product space with the product topology $\mathscr{T}$. Then*

(i) *P_λ, the projection of X into X_λ is continuous for each $\lambda \in \Lambda$;*
(ii) *a map $f : Y \to X$, where Y is a topological space is continuous if and only if $P_\lambda \circ f : Y \to X_\lambda$ is continuous for each $\lambda \in \Lambda$;*
(iii) *a net S in X converges to an element s if and only if its projection in each coordinate space converges to the projection of s.*
(iv) *X is connected if and only if each X_λ is connected;*
(v) *(Tychonoff's theorem) X is compact if and only if each X_λ is compact.*

Definition 1.1.26 A topological space X is said to be

(i) T_1 if for each pair of distinct elements x and y, there exist neighbourhoods N_x of x not containing y and N_y of y not containing x;
(ii) T_2 (Hausdorff) if each pair of distinct elements has disjoint neighbourhoods;
(iii) regular, if for each $x \in X$ and any closed subset F of X not containing x, there exist disjoint open sets G_1 and G_2 with $x \in G_1$ and $F \subseteq G_2$ (X is called T_3 if it is T_1 and regular);
(iv) normal, if for each pair of disjoint closed subsets F_i, $i = 1, 2$ of X, there exist disjoint open sets G_i, $i = 1, 2$ with $F_i \subseteq G_i$, $i = 1, 2$ (X is called T_4 if it is T_1 and normal).
(Every T_4 space is T_3 and each T_3 space is T_2, while a T_2 space is necessarily T_1).

Theorem 1.1.27 (Urysohn's Lemma) *If A and B are disjoint closed subsets of a normal space X, then there is a continuous function $f : X \to [0, 1]$ such that $f \equiv 0$ on A and $f \equiv 1$ on B.*

Theorem 1.1.28 *Let X and Y be topological spaces and $f : X \to Y$ be a continuous map. If X is compact, then $f(X)$ is a compact subset of Y. If X is connected, then $f(X)$ is a connected subset of Y.*

Corollary 1.1.29 *If X is a compact topological space and $f : X \to \mathbb{R}$ is a continuous map, then f attains its maximum and minimum on X. If X is a connected space and $f : X \to \mathbb{R}$ is continuous, then $f(X)$ is an interval.*

1.2 Metric Spaces

In this section, basic concepts and theorems from the theory of metric spaces are recalled. For details, in addition to the references cited in Sect. 1.1, Kaplansky [6] may be consulted.

Definition 1.2.1 Let X be a non-void set. A map $d : X \times X \to [0, \infty)$ $(=\mathbb{R}^+)$ is called a metric if

(i) $d(x, x) = 0$ for all $x \in X$;
(ii) $d(x, y) = 0$ implies $x = y$;
(iii) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(iv) $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y, z (triangle inequality).

The pair (X, d) is called a metric space. A map d satisfying (i), (iii) and (iv) is called a pseudometric and the corresponding (X, d) is called a pseudometric space.

Definition 1.2.2 If (X, d) is a metric space, the set $B(x_0; r) = \{x \in X : d(x_0, x) < r\}$ for $r > 0$ is called an open sphere of radius r centred at x_0, while the set $\{x \in X : d(x_0, x) \leq r\}$ is referred as the closed sphere of radius r with centre x_0.

Remark 1.2.3 The family of all open spheres $\{B(x; r) : x \in X, r > 0\}$ is a base for a topology on X called the metric topology on X induced by d.

Example 1.2.4 (i) If X is a non-empty set the map $d : X \times X \to \mathbb{R}^+$ defined by $d(x, y) = 1$ for $x \neq y$ and $d(x, x) = 0$ is a metric on X called the discrete metric. The corresponding metric topology on X is the discrete topology.

(ii) On $\mathbb{R}$, $d(x, y) = |x - y|$, the absolute value of $x - y$ defines a metric called the usual (or standard) metric on $\mathbb{R}$ and the topology induced is the usual topology on $\mathbb{R}$ (with the base comprising all open intervals).

(iii) On $\mathbb{R}^n$, the set of all n-tuples of real numbers, $d(\overline{x}, \overline{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}}$, where $\overline{x} = (x_1, x_2, \ldots, x_n)$ and $\overline{y} = (y_1, y_2, \ldots, y_n)$ defines a metric, called the Euclidean metric on $\mathbb{R}^n$.

(iv) $\mathcal{C}[a, b]$, the set of all continuous real-valued function on the closed interval $[a, b]$, where $a < b$, $a, b \in \mathbb{R}$ is a metric space under the metric

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [a, b]\}$$

where $f, g \in \mathcal{C}[a, b]$. This metric is called Tschebyshev or uniform metric.

(v) More generally $\mathcal{C}(X)$, the set of all continuous real-valued functions on a compact topological space becomes a metric space with the metric d defined by $d(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$ where $f, g \in \mathcal{C}(X)$.

(vi) $d(f, g) = \int_a^b |f(t) - g(t)| dt$ also defines a metric on $\mathcal{C}[a, b]$ the set of all continuous real-valued functions on $[a, b]$.

(vii) If (X, d) is a metric space and $S \subseteq X$, then the restriction of d to $S \times S$ is a metric and this metric topology is precisely the topology of S relative to the metric topology on X.

Theorem 1.2.5 *A metric space is second countable if and only if it is separable.*

Theorem 1.2.6 *Every metric space is a Hausdorff normal space.*

Definition 1.2.7 A sequence (x_n) in a metric space (X, d) is called Cauchy (fundamental) if $d(x_m, x_n) \to 0$ as $m, n \to \infty$. A metric space is said to be complete if every Cauchy sequence in X converges to an element of X.

Theorem 1.2.8 (Baire) *No complete metric space can be written as a countable union of closed sets having empty interior.*

Definition 1.2.9 Let (X_i, d_i), $i = 1, 2$ be metric spaces. A map $T : X_1 \to X_2$ is called an isometry if $d_2(Tx_1, Tx_2) = d_1(x_1, x_2)$ for all $x_1, x_2 \in X_1$.

Theorem 1.2.10 *Each metric space (X, d) can be isometrically embedded in a complete metric space $(\overline{X}, \overline{d})$ as a dense subset. Further such a complete metric space $\overline{X}$, called the completion of X is unique up to isometry.*

Remark 1.2.11 In example 1.2.4, except the space described in (vi), the metric spaces in examples (i)–(v) are complete.

Theorem 1.2.12 *If (X, d) is a metric space, then $d_1(x, y) = \min\{1, d(x, y)\}$, $x, y \in X$ defines a metric on X and the topologies induced on X by these metrices are the same.*

Theorem 1.2.13 *If (X_n, d_n), $x \in \mathbb{N}$ is a sequence of metric spaces, then $X = \prod_{n\in\mathbb{N}} X_n$ is a metric space under the metric d defined by*

$$d(\overline{x}, \overline{y}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} \right),$$

where $\overline{x} = (x_n)$ and $\overline{y} = (y_n)$ are in X. Further, if each (X_n, d_n) is complete, then (X, d) is complete.

The following metrization theorem is classical.

Theorem 1.2.14 (Urysohn) *A regular T_1 second countable topological space is metrizable (in the sense that there is a metric on this space whose metric topology is the given topology).*

A concept basic to the study of the metrization problem is defined below.

Definition 1.2.15 A family $\mathcal{F}$ of subsets of a topological space $(X, \mathscr{T})$ is called

(i) locally finite, if each point of the space has a neighbourhood that intersects only finitely many sets in $\mathcal{F}$;
(ii) discrete if each point of the space has a neighbourhood that intersects at most one member of $\mathcal{F}$;
(iii) σ-locally finite (σ-locally discrete) if it is the union of a countable collection of locally finite (finite) subfamilies.

Theorem 1.2.16 (Metrization theorems) *A topological space is metrizable if and only if it is T_1 and regular with*

a σ-locally finite base (Nagata–Smirnov);
or
a σ-discrete base (Bing).

Another important notion is that of paracompactness formulated below.

Definition 1.2.17 A topological space X is called paracompact if each open cover $\mathcal{U}$ of X has an open locally finite refinement $\mathcal{U}^*$ (viz. $\mathcal{U}^*$ is locally finite and each member of $\mathcal{U}^*$ is open and is a subset of some set in $\mathcal{U}$).

Theorem 1.2.18 *Every pseudometric space is paracompact and a paracompact T_2 space is normal.*

Definition 1.2.19 Let X be a topological space. A family $\{f_\lambda : \lambda \in \Lambda \neq \phi\}$ of continuous functions mapping X into [0, 1] is called a partition of unity if for each $x \in X$, $\sum_{\lambda \in \Lambda} f_\lambda(x) = 1$ and all but a finite number of f_λ's vanish on some neighbourhood of x. A partition of unity $\{f_\lambda : \lambda \in \Lambda \neq \phi\}$ in subordinate to a cover $\mathcal{U}$ if each f_λ vanishes outside some member of $\mathcal{U}$.

Theorem 1.2.20 *A regular T_1 space is paracompact if and only if for each open covering of X, there is a partition of unity subordinate to this covering. For every compact T_2 space, every open cover has a partition of unity subordinate to it.*

Definition 1.2.21 A subset S of a metric space (X, d) is said to be totally bounded, if for each $\epsilon > 0$, there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ (depending on ϵ) such that $S \subseteq \bigcup_{i=1}^{n} B(x_i; \epsilon)$. A subset S is said to be bounded if $S \subseteq B(x; r)$ for some $x \in X$ and some $r > 0$.

Theorem 1.2.22 *For a metric space (X, d), the following are equivalent:*

(i) *X is compact;*
(ii) *X is complete and totally bounded;*
(iii) *every sequence in X has a convergent subsequence;*
(iv) *X has the Bolzano–Weierstrass property, viz. for every infinite subset A of X has a limit point $x_0 \in X$, i.e. a point x_0 such that every neighbourhood of x_0 meets A.*

Theorem 1.2.23 *Let (X_i, d_i), $i = 1, 2$ be metric spaces and $f : X_1 \to X_2$ be a continuous map. If X_1 is compact, then f is uniformly continuous in the sense that for each $\epsilon > 0$ there exists $\delta > 0$ depending only on ϵ so that $d_2(f(x), f(y)) < \epsilon$ whenever $x, y \in X_1$ and $d_1(x, y) < \delta$.*

Definition 1.2.24 For a non-void subset A of a metric space (X, d), the distance of a point x from A is defined as $D(x, A) = inf\{d(x, a) : a \in A\}$.

Theorem 1.2.25 *Let A be a non-void subset of a metric space (X, d). Then $\overline{A} = \{x \in X : D(x, A) = 0\}$. Further, $|D(x, A) - D(y, A)| \leq d(x, y)$ for any $x, y \in X$ and the map $x \to D(x, A)$ is a continuous map of X into $\mathbb{R}^+$.*

1.3 Normed Linear Spaces

Normed linear spaces, constituting the base of Functional Analysis are metric spaces with a richer (algebraic) structure. They provide a natural setting for mathematical modelling of many natural phenomena. Bollabos [2], Kantorovitch and Akhilov [5], Lyusternik and Sobolev [8], Rudin [11, 12], Simmons [13] and Taylor [14] may be consulted for a detailed exposition of the following concepts and theorems. It is assumed that the reader is familiar with the concepts of groups, rings and fields.

Definition 1.3.1 A linear space or vector space over a field F is a triple $(V, +, \cdot)$, where $+$ is a binary operation (called vector addition or simply addition) and $\cdot$ is a mapping from $F \times V$ into V (called scalar multiplication) satisfying the following conditions;

(i) $(V, +)$ is a commutative group with θ (called zero vector) as its identity element;
(ii) for all $\lambda \in F$, $x, y \in V$ $\lambda.(x + y) = \lambda x + \lambda y$;
(iii) for all $\lambda, \mu \in F$ and $x \in V$ $(\lambda + \mu).x = \lambda.x + \mu.x$ and $\lambda \cdot (\mu \cdot x) = (\lambda \cdot \mu) \cdot x$ (where $\lambda\mu$ is the product of λ and μ under the multiplication in the field F);
(iv) $0 \cdot x = \theta$, $1 \cdot x = x$ for all $x \in V$, where 0 is the additive identity and 1 the multiplicative identity of the field F.

Often 0 is also used to represent the zero vector and the context will clarify this without much difficulty. If F is the field of real (complex) numbers then V is called a real (complex) vector space. In what follows, we will be concerned only with real or complex vector spaces. Also a linear subspace V_1 of V is a subset of V which is a linear space over F with vector addition and scalar multiplication of V restricted to V_1.

Definition 1.3.2 A subset S of a linear space V over a field F is said to be linearly independent if for every finite subset $\{s_1, \ldots, s_n\}$ of S, $\sum_{i=1}^{n} \alpha_i s_i = \theta$ implies $\alpha_i = 0$ for $i = 1, 2, \ldots, n$ where $\alpha_i \in F$. An element of the form $\sum_{i=1}^{n} \alpha_i s_i$ where $\alpha_i \in F$ and $s_i \in S$ is called a finite linear combination of members of S.

Definition 1.3.3 A subset S of a linear space V over a field F is said to span V if every element of V can be written as a finite linear combination of elements from S. A maximal linearly independent subset of a linear space V over F is called a basis for V.

Any two bases of a vector space have the same cardinality.

Definition 1.3.4 The cardinality of a basis of a linear space V is called the dimension of the linear space. If a linear space has a finite dimension, then it is called a finite-dimensional vector space. Otherwise the linear space is infinite-dimensional.

Definition 1.3.5 Let $(X, +, \cdot)$ be a linear space over $F = \mathbb{R}$ or $\mathbb{C}$. A map $\| \cdot \| : X \to \mathbb{R}^+$ is called a norm if the following conditions are satisfied:

1. $\|x\| = 0$ if and only if $x = 0$;
2. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (triangle inequality);
3. $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in F$ and all $x \in X$, $|\alpha|$ being the modulus of α. The pair $(X, \|\cdot\|)$ is called a normed linear space.

Remark 1.3.6 If $(X, \|\cdot\|)$ is a normed linear space, then $d(x, y) = \|x - y\|, x, y \in X$ is a metric on X.

Definition 1.3.7 A normed linear space $(X, \|\cdot\|)$ is called a Banach space if it is complete in the metric induced by the norm.

Remark 1.3.8 The linear spaces in (ii)–(v) of Example 1.2.4 are Banach spaces with the norms defined by $\|x\| = d(x, \theta)$ where d is the metric described in the corresponding case, while (vi) of Example 1.2.4 is a normed linear space under the norm $\int_a^b |f(t)|dt$. However, this is not a Banach space.

Definition 1.3.9 Let $(X_i, \|\cdot\|_i), i = 1, 2$ be normed linear spaces over $F = \mathbb{R}$ or $\mathbb{C}$. A linear operator is a map $T : X_1 \to X_2$ such that $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $\alpha, \beta \in F$ and $x, y \in X_1$. If X_2 is the base field F (=$\mathbb{R}$ or $\mathbb{C}$ with the modulus or absolute value as a norm) which is also a normed linear space, the linear operator is called a linear functional. If the linear operator T is continuous as a map between the metric spaces X_1 and X_2 with metrics induced by the norms, then it is called a continuous linear operator.

Theorem 1.3.10 *Let $(X_i, \|\cdot\|_i)$, $i = 1, 2$ be normed linear spaces over $F = \mathbb{R}$ or $\mathbb{C}$ and $T : X_1 \to X_2$ a linear operator. Then the following are equivalent:*

(i) T is continuous on X_1;
(ii) T is continuous at some $x_0 \in X_1$;
(iii) there exists $K > 0$ such that $\|Tx\|_2 \leq K\|x\|_1$ for all $x \in X_1$.

Remark 1.3.11 A linear operator satisfying (iii) of Theorem 1.3.10 is called bounded. In view of the theorem above, bound linear operators are precisely continuous linear operators.

If $T : X_1 \to X_2$ is a continuous linear operator where $(X_i, \|\cdot\|_i)$, $i = 1, 2$ are normed linear spaces, then

$$\inf\{K \geq 0 : x \in X_1 \text{ and } \|Tx\|_2 \leq K\|x\|_1\} = \sup\{\|Tx\|_2 : \|x\|_1 = 1, x \in X_1\}$$

is finite and is called the norm of the linear operator and is denoted by $\|T\|$.

Theorem 1.3.12 *For $i = 1, 2$, let $(X_i, \|\cdot\|_i)$ be normed linear spaces. $B(X_1, X_2)$ the set of all bounded (continuous) linear operators is a normed linear space under the norm described in Remark 1.3.11. If $(X_2, \|\cdot\|_2)$ is complete so is $B(X_1, X_2)$ under this norm.*

Theorem 1.3.13 (Hahn–Banach) *If f is a bounded linear functional from a linear subspace N of a normed linear space $(X, \|\cdot\|)$, then there is a bounded linear functional f^* on X such that $f^* \equiv f$ on $\mathbb{N}$ and $\|f\| = \|f^*\|$.*

The Hahn–Banach theorem insures the abundance of continuous linear functionals in any nontrivial normed linear space.

Definition 1.3.14 Given a normed linear space $(X, \|\cdot\|)$, the space of all continuous linear functionals on X is called the dual or conjugate of X and is denoted by X^*. The dual of X^* denoted by X^{**} is called the second dual or second conjugate of X.

Even, if X is incomplete, X^* and X^{**} are complete.

Theorem 1.3.15 *Let $(X, \|\cdot\|)$ be a normed linear space. The map f_x defined by $f \to f(x)$ for each $x \in X$ is a bounded linear functional on X^* and $\|f_x\| = \|x\|$. The map $\varphi : X \to X^{**}$ defined by $\phi(x) = f_x$ is one-one, isometric linear map of X into X^{**} and is called the duality mapping. The duality mapping is the natural embedding (of X into X^{**}).*

Definition 1.3.16 If the duality mapping φ maps X onto X^{**}, the second dual of X, then X is said to be reflexive.

Definition 1.3.17 If $(X, \|\cdot\|)$ is a normed linear space, then the weak topology on X is the smallest topology on X with respect to which all the functionals of X^* are continuous. The weak * topology on X^* is the smallest topology on X^* such that $\varphi(x)$ $(= f_x)$, φ being the natural embedding of X into X^{**} is continuous.

Theorem 1.3.18 (Alaoglu) *The unit sphere S^* in X^* is compact in the weak * topology on X^*.*

Theorem 1.3.19 *A Banach space is reflexive if and only if the closed unit sphere $S = \{x \in X : \|x\| \leq 1\}$ is compact in the weak topology.*

The following three theorems are basic to Functional Analysis.

Theorem 1.3.20 (Open Mapping Theorem) *For $i = 1, 2$, let $(X_i, \|\cdot\|_i)$ be Banach spaces. If $T : X_1 \to X_2$ is a continuous linear operator mapping X_1 onto X_2, then T is an open mapping (i.e. a function for which the image of any open set is open). Consequently a continuous linear bijection of X_1 onto X_2 is a linear homeomorphism.*

Theorem 1.3.21 (Closed Graph Theorem) *For $i = 1, 2$, let $(X_i, \|\cdot\|_i)$ be Banach spaces and $T : X_1 \to X_2$ a linear operator. T is continuous if and only if the graph of $T = \{(x, Tx) : x \in X_1\}$ is a closed subset of the product topological space $X_1 \times X_2$.*

Theorem 1.3.22 (Banach–Steinhaus theorem) *Let $T_\lambda : X_1 \to X_2$, $\lambda \in \Lambda \neq \phi$ be continuous linear operators mapping a Banach space X_1 into a normed linear space X_2 such that for each $x \in X_1$, $\{\|T_\lambda(x)\| : \lambda \in \Lambda\}$ is a bounded set of real numbers. Then $\{\|T_\lambda\| : \lambda \in \Lambda\}$ is bounded.*

Theorem 1.3.23 *A normed linear space is finite-dimensional if and only if the unit sphere is compact.*

Theorem 1.3.24 *If $\|\cdot\|_i$, $i = 1, 2$ are two norms on a finite-dimensional normed linear space then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent in the sense that there exist two positive numbers K_1 and K_2 such that*

$$K_1\|x\|_1 \leq \|x_2\| \leq K_2\|x\|_1 \textit{ for all } x \in X.$$

Consequently a finite-dimensional normed linear space over $\mathbb{R}$ or $\mathbb{C}$ is equivalent to $\mathbb{R}^n$ or $\mathbb{C}^n$ with the norm given by $\|(x_1, \ldots, x_n)\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$ and is a Banach space.

Among normed linear spaces, inner product spaces have rich geometric properties. Many features of the Euclidean spaces carry over to inner product spaces. Parseval identity for orthogonal functions has a crisp functional analytic formulation.

Definition 1.3.25 A linear space $(V, +, \cdot)$ over $F = \mathbb{R}$ or $\mathbb{C}$ is called an inner product space, if there is a map $< >: V \times V \to F$ called an inner product satisfying the following conditions:

(i) $< \alpha x + \beta y, z >= \alpha < x, z > + \beta < y, z >$ for all $x, y, z \in V$ and $\alpha, \beta \in F$;
(ii) $< x, y > = < \overline{y, x} >$ for all $x, y \in V$, $\overline{z}$ being the complex conjugate of $z \in \mathbb{C}$;
(iii) $< x, x > \geq 0$ for all $x \in V$ and $< x, x > = 0$ if and only if $x = \theta$.

Proposition 1.3.26 *An inner product space is a normed linear space with the norm $\|\cdot\|$ defined by $\|x\| = \sqrt{< x, x >}$ (the positive square root of $< x, x >$), as $| < x, y > | \leq \|x\|\|y\|$ for all x, y in V (Schwarz inequality).*

Definition 1.3.27 A Hilbert space is an inner product space which is complete in the norm induced by the inner product.

Example 1.3.28 (i) $\mathbb{R}^n$ or $\mathbb{C}^n$ is a Hilbert space in the norm induced by the inner product defined by $< x, y >= \sum_{i=1}^{n} x_i y_i \left(\sum_{i=1}^{n} x_i \overline{y}_i\right)$ for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ $(\mathbb{C}^n)$.

(ii) ℓ_2, the space of complex sequences (z_n) with $\sum_{n=1}^{\infty} |z_n|^2 < +\infty$ is a Hilbert space with the inner product defined by $< x, y >= \sum_{n=1}^{\infty} x_n \overline{y}_n$ for $x = (x_n)$, $y = (y_n) \in \ell_2$.

(iii) $C_{\mathbb{C}}[a, b]$, the linear space of all continuous complex-valued functions on $[a, b]$ is an inner product space under the inner product $< f, g >= \int_a^b f(x)\overline{g(x)}dx$. This is not a Hilbert space as the space is not complete in the induced norm $\|f\| = \left(\int_a^b |f(x)|^2 dx\right)^{\frac{1}{2}}$. However, its completion is $L_2[a, b]$, the Hilbert space of Lebesgue measurable complex functions which are square-integrable with respect to the Lebesgue measure.

Theorem 1.3.29 *Every incomplete inner product space can be isometrically embedded as a dense subspace of a Hilbert space.*

Theorem 1.3.30 *A normed linear space* $(X, \|\ \|)$ *is an inner product space if and only if the following parallelogram law is valid:*

$$\text{for } x, y \in X, \|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Definition 1.3.31 A subset C of a linear space V over $F = \mathbb{R}$ or $\mathbb{C}$ is called convex if $tx + (1 - t)y \in C$ for all $t \in [0, 1]$, whenever $x, y \in C$.

Theorem 1.3.32 *A non-empty closed convex subset of a Hilbert space contains a unique element with least norm.*

Definition 1.3.33 Let $(V, <, >)$ be an inner product space over $\mathbb{R}$ or $\mathbb{C}$. $x \in V$ is said to be orthogonal to $y \in V$ if $< x, y > = 0$ and we write $x \perp y$ (or $y \perp x$). For $S \subseteq V, S^\perp = \{v \in V : v \perp s, \text{for all } s \in \mathcal{S}\}$. $S^\perp$ is called the orthogonal complement of S.

Theorem 1.3.34 *If* M *is a proper closed linear subspace of a Hilbert space* H*, then*

(i) $M^\perp$ *is a closed linear subspace of* H*;*
(ii) $M \cap M^\perp = \{\theta\}$*;*
(iii) each $h \in H$ *can be written uniquely as* $h = m_1 + m_2$*, where* $m_1 \in M, m_2 \in M^\perp$ *and* $\|h\|^2 = \|m_1\|^2 + \|m_2\|^2$.
(In this case, we write $H = M \oplus M^\perp$ *and call* H *the direct sum of* M *and its orthogonal complement* $M^\perp$*).*

Definition 1.3.35 A non-empty set S of a Hilbert space H is called orthogonal if $< x, y > = 0$ whenever $x, y \in S$ and $x \neq y$. S is called orthonormal if each element of S has unit norm and S is orthogonal.

Theorem 1.3.36 *If* $S = \{e_\lambda : \lambda \in \Lambda \neq \phi\}$ *is an orthonormal set in a Hilbert space* H *and if* $x \in H$*, then* $\{e_\lambda :< x, e_\lambda > \neq 0\}$ *is either empty or countable. Also* $\sum_{\lambda \in \Lambda} | < x, e_\lambda > |^2 \leq \|x\|^2$. *Further, a nonzero Hilbert space has a maximal orthonormal set of vectors, called an orthonormal basis. If* $\{e_\lambda : \lambda \in \Lambda \neq \phi\}$ *is an orthonormal basis for* H *and* $x \in H$*, then* $x = \sum_{\lambda \in \Delta} a_\lambda e_\lambda$*, where* $a_\lambda =< x, e_\lambda >$.

Theorem 1.3.37 (Parseval identity) *If* $\{e_\lambda : \lambda \in \Lambda \neq \phi\}$ *is an orthonormal basis of* H *then* $\|x\|^2 = \sum_{\lambda \in \Lambda} | < x, e_\lambda > |^2$.

Example 1.3.38 (i) $L_2[0, 2\pi]$, the space of complex-valued Lebesgue measurable functions f on $[0, 2\pi]$ which are square-integrable in the sense that $\int_0^{2\pi} |f(x)|^2 dx < +\infty$ is a Hilbert space under the inner product $< f, g >= \int_0^{2\pi} f(x)\overline{g}(x)dx$. The set $\left\{\frac{e^{inx}}{\sqrt{2\pi}} : n = 0, \pm 1, \pm 2, \dots\right\}$ is an orthonormal basis.

(ii) $L_2(\mathbb{R})$, the space of all Lebegue-measurable functions for which $\int_{-\infty}^{\infty} |f^2(x)|dx$ is finite, is also a Hilbert space under the inner product $< f, g >= \int_{-\infty}^{\infty} f(x) \overline{g}(x)dx$. $\{x^n e^{-\frac{x^2}{2}}, n = 0, 1, 2, \dots\}$ gives rise to an orthonormal basis for $L_2(\mathbb{R})$ via the Gram–Schmidt orthogonalization process (see Simmons [13]).

Among normed linear spaces, strictly convex spaces and the more specialized uniformly convex spaces resemble the Euclidean spaces geometrically.

Definition 1.3.39 A normed linear space $(N, \|\cdot\|)$ over $F = \mathbb{R}$ or $\mathbb{C}$ is said to be strictly convex if for $x, y \in N$ with $\|x\| = \|y\| = 1$ and $x \neq y$, $\left\|\frac{x+y}{2}\right\| < 1$.

Definition 1.3.40 A normed linear space $(N, \|\cdot\|)$ is called uniformly convex if there exists an increasing positive function $\delta : (0, 2] \to (0, 1]$ such that for $x, y \in N$, $\|x\|, \|y\| \leq r$ and $\|x - y\| \geq \epsilon r$ imply that $\left\|\frac{x+y}{2}\right\| < (1 - \delta(\epsilon))r$.

Remark 1.3.41 The above definition is equivalent to the requirement that for $\|x_n\|, \|y_n\| \leq 1$ and $\|x_n + y_n\| \to 2$, $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Clearly, every Hilbert space is uniformly convex. Also $L_p[0, 1]$ for $p \geq 2$ is uniformly convex. While every uniformly convex space is strictly convex, $C[0, 1]$ is not even strictly convex.

Hilbert spaces are isometric to their duals, in view of the following.

Theorem 1.3.42 (Riesz Representation Theorem) *Let H be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$ and $f \in H^*$, the dual of H. Then there exists a unique element $y_f \in H$ such that $f(x) =< x, y_f >$ for each $x \in H$ and $\|f\| = \|y_f\|$.*

For $f_y \in H^*$ defined by $f_y(x) = < x, y >$ the correspondence $T_y = f_y$ maps H onto H^* so that $\|T(y)\| = \|y\|$, $T(y_1 + y_2) = Ty_1 + Ty_2$ and $T(\alpha y) = \overline{\alpha}T(y)$ for all $y \in H$.

Theorem 1.3.43 *Every Hilbert space is reflexive.*

In view of the above theorems for a bounded linear operator $T : H \to H$, H being a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$, there is a unique bounded linear operator $T^* : H \to H$ such that $< Tx, y > = < x, T^*y >$ for all $x, y \in H$.

Definition 1.3.44 Let H be a Hilbert space and $T : H \to H$ a bounded linear operator. A linear operator $T^* : H \to H$ satisfying, $< Tx, y > = < x, T^*y >$ for all $x, y \in H$ is called an adjoint operator of T.

Theorem 1.3.45 *If $T \in B(H)$, the space of all bounded linear operators mapping H into itself, then T^* the adjoint of T is uniquely defined. Further,*

(i) $(T_1 + T_2)^* = T_1^* + T_2^*$,
(ii) $(\alpha T)^* = \overline{\alpha}T^*$,
(iii) $(T_1T_2)^* = T_2^*T_1^*$,
(iv) $(T^*)^* = T^*$,
(v) $\|T^*\|^2 = \|T\|^2 = \|T^*T\|$.

Definition 1.3.46 A linear operator $T \in B(H)$, the space of all bounded linear operators on a Hilbert space H is said to be self-adjoint if $T = T^*$.

Definition 1.3.47 For $T \in B(H)$, the spectrum of T is the set $\{\lambda \in \mathbb{C} : T - \lambda I$ is not invertible$\}$, I being the identity operator. An eigenvalue of T is a number $\lambda \in \mathbb{C}$ such that there exists a nonzero vector $x_0 \in H$ with $Tx_0 = \lambda x_0$ and in this case x_0 is called an eigenvector (corresponding to the eigenvalue λ).

Theorem 1.3.48 *For $T \in B(H)$, the space of all bounded linear operators on a Hilbert space H is self-adjoint if and only if $< Tx, x >$ is real for all $x \in H$. So the eigenvalues of a self-adjoint operator are real. Further $\sigma(T)$, the spectrum of T lies in $[m, M]$, where $m = \inf\{< Tx, x >: x \in H$ and $\|x\| = 1\}$ and $M = \sup\{< Tx, x >: x \in H$ and $\|x\| = 1\}$. Also, $m, M \in \sigma(T)$.*

Definition 1.3.49 A linear operator P in $B(H)$ is called a projection if P is self-adjoint and $P^2 = P$.

Remark 1.3.50 If P is a projection on a Hilbert space H, then $P = M \oplus M^\perp$ where $M = \{Px : x \in H\}$, the range of P and $M^\perp$, the range of $I - P$. Further, every representation of H as the orthogonal sum $M + M^\perp$ defines a unique projection of H onto M.

Theorem 1.3.51 *For any self-adjoint operators T in $B(H)$, there is a family $\{P_\lambda : \lambda \in \mathbb{R}\}$ of projections on H satisfying the following conditions:*

(i) if $TC = CT$ for $C \in B(H)$, then $P_\lambda C = CP_\lambda$ for all $\lambda \in \mathbb{R}$;
(ii) $P_\lambda P_\mu = P_\lambda$, if $\lambda < \mu$;
(iii) $P_{\lambda-0} = \lim_{\mu \to \lambda - 0} P_\mu = P_\lambda$ (i.e. P_λ is continuous from the left with respect to λ);
(iv) $P_\lambda = 0$ if $\lambda \leq m$ and $P_\lambda = I$ for $\lambda > M$.
(Such a family of projections P_λ is called a resolution of identity generated by T).

Theorem 1.3.52 (Spectral theorem) *For every self-adjoint operator $T \in B(H)$ and any $\epsilon > 0$,*

$$T = \int_m^{M+\epsilon} \lambda dP_\lambda$$

where the Stieltjes integral is the limit of (appropriate) integral sums in the operator-norm topology.

Definition 1.3.53 A linear operator $T : N_1 \to N_2$ where N_1 and N_2 are normed linear spaces is said to be a compact operator if $\overline{T(U)}$ is compact in N_2 for each bounded subset U of N_1.

Theorem 1.3.54 *Let $T : B \to B$ be a compact linear operator on a Banach space B. $\sigma(T)$, the spectrum of T is finite or countably infinite and is contained in $[-\|T\|, \|T\|]$. Every nonzero number in $\sigma(T)$ is an eigenvalue of T. If $\sigma(T)$ is countably infinite, then 0 is the only limit point of $\sigma(T)$.*

1.4 Topological Vector Spaces

It is convenient to recall the definition of a topological group and list some of its properties (see Kelley [7], Rudin [11] and Royden [10]).

Definition 1.4.1 Let $(G, \cdot)$ be a group with the identity element e and for each $x \in G, x^{-1}$ denote the inverse of x (with respect to the binary operation $\cdot$). The triple $(G, \cdot, \mathscr{T})$ is called a topological group where $\mathscr{T}$ is a topology on the group G with the binary operation $\cdot$ such that the map $(x, y) \to xy^{-1}$ mapping $G \times G$ into G is continuous. (Here $G \times G$ carries the product topology.)

If $(G, \cdot)$ is a group and $A, B \subseteq G$, we write $A \cdot B = \{a \cdot b : a \in A, b \in B\}$.

Theorem 1.4.2 *Let $(G, \cdot, \mathscr{T})$ be a topological group with the identity e. Then*

(i) the map $x \to x^{-1}$ mapping G into G and the map $(x, y) \to xy$ mapping $G \times G$ into G are continuous. Conversely if $\mathscr{T}_1$ is a topology on a group $(G, \cdot)$ such that $x \to x^{-1}$ and $(x, y) \to xy$ are continuous on G with the topology $\mathscr{T}_1$, then $(G, \cdot, \mathscr{T}_1)$ is a topological group.
(ii) the inversion map i, defined by $i(x) = x^{-1}$ is a homeomorphism of G onto G; for each $a \in G$, $L_a(R_a)$ called the left (right) translation by a, defined by $L_a(x) = ax$ $(R_a(x) = xa)$ are homeomorphisms;
(iii) a subset S of G is open if and only if for each $x \in S$, $x^{-1}S$ (or equivalently Sx^{-1}) is a neighbourhood of e;
(iv) the family $\mathcal{N}$ of all neighbourhoods of e has the following properties:

(iv-a) for $U, V \in \mathcal{N}$, $U \cap V \in \mathcal{N}$;
(iv-b) for $U \in \mathcal{N}$, $V.V^{-1} \subseteq U$ for some $V \in \mathcal{N}$;
(iv-c) for $U \in \mathcal{N}$ and $x \in G$, $x.U.x^{-1} \in \mathcal{N}$;

(v) the closure of a (normal) subgroup of G is a (normal) subgroup of G;
(vi) every subgroup G_1 of G with an interior point is both open and closed and G_1 is closed or $\overline{G_1} - G_1$ is dense in G_1;
(vii) G is Hausdorff if it is a T_0 space in the sense that for every pair of distinct points, there is a point for which some neighbourhood does not contain the other point.

A topological vector space can be defined in analogy with a topological group.

Definition 1.4.3 The quadruple $(X, +, \cdot, \mathscr{T})$ where $(X, +, \cdot)$ is a vector space over $F = \mathbb{R}$ or $\mathbb{C}$ and $\mathscr{T}$ is a topology on X is called a topological vector space (linear topological space) if the following assumptions are satisfied:

(i) $(X, \mathscr{T})$ is a T_1-space;
(ii) the function $(x, y) \to x + y$ mapping $X \times X$ into X is continuous and
(iii) the function $(\alpha, x) \to \alpha.x$ mapping $F \times X$ into X is continuous.

Often, we simply say that X is a topological vector space (or t.v.s for short) when the topology $\mathscr{T}$ on X and the vector space operations are clear from the context.

Definition 1.4.4 A subset S of a topological vector space X is said to be bounded for every neighbourhood V of θ in X, there is a real number s such that $\mathcal{S} \subseteq t.V$ for every $t > s$. $S \subseteq X$ is called balanced if $\alpha.S \subseteq S$ for all $\alpha \in F$ with $|\alpha| \leq 1$. $\mathcal{S}$ is called absorbing if $X = \bigcup\limits_{t>0} t.S$.

Theorem 1.4.5 *Let X be a t.v.s. For each $a \in X$ and $\lambda \neq 0 \in F$ define the translation operator T_a and the multiplication operator M_λ by the rules $T_a(x) = x + a$ and $M_\lambda(x) = \lambda.x$ respectively for each $x \in X$. Then, T_a and M_λ are homeomorphism of X onto X.*

Further $G \subseteq X$ is open if and only if $T_a(G)$ is open for each $a \in X$. So the local base at 0 completely determines the local base at any $x \in X$ and hence the topology on X.

Remark 1.4.6 Every normed linear space is a t.v.s.

Definition 1.4.7 A function p mapping a vector space X over $F(=\mathbb{R}$ or $\mathbb{C})$ into F is called a seminorm if

(i) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$ and
(ii) $p(\alpha x) = |\alpha| p(x)$ for all $x \in X$ and all $\alpha \in F$.

A seminorm is a norm if $p(x) \neq 0$ for $x \neq \theta$. A family $\mathcal{P}$ of seminorms is separating if for each $x \neq y$, there is a seminorm $p \in \mathcal{P}$ with $p(x - y) \neq 0$.

Theorem 1.4.8 *If $\mathcal{P}$ is a separating family of seminorms on a vector space V, then $V(p, n) = \{x \in X : p(x) < \frac{1}{n}\}$, $p \in \mathcal{P}$ is a local base of convex sets for a topology $\mathscr{T}$ on X. Thus, $(X, \mathscr{T})$ is locally convex and each p is continuous. Also, E is bounded if and only if $p(E)$ is bounded for each $p \in \mathcal{P}$.*

Definition 1.4.9 For an absorbing subset A of a t.v.s. X, the map $\mu_A : X \to \mathbb{R}$ defined by $\mu_A(x) = \inf\{t > 0 : t^{-1}x \in A\}$ is called the Minkowski functional of A.

Listed below are some of the basic properties and features of a topological vector space.

Theorem 1.4.10 *Let X be a topological vector space*

(i) if $S \subseteq X$, $\overline{S} = \cap\{S + V : V$ is a neighbourhood of 0$\}$;
(ii) if $S_1, S_2 \subseteq X$, $\overline{S_1} + \overline{S_2} \subseteq \overline{S_1 + S_2}$;
(iii) if $C \subseteq X$ is convex, so are C^0 and $\overline{C}$;
(iv) if $B \subseteq X$ is balanced, so is $\overline{B}$ and if in addition $0 \in B^0$, B^0 is balanced;
(v) the closure of a bounded set is also bounded;
(vi) every neighbourhood of 0 also contains a balanced neighbourhood of 0 and so X has a balanced local base;
(vii) every convex neighbourhood of 0 contains a balanced convex neighbourhood of 0;
(viii) if V is a neighbourhood of 0 and $r_n \uparrow +\infty$ where $r_1 > 0$, $X = \bigcup\limits_{n=1}^{\infty} r_n V$;

(ix) if V is a bounded neighbourhood of 0 and $\delta_n \downarrow 0$, $\delta_1 > 0$, $\{\delta_n V : n \in \mathbb{N}\}$ is a local base at 0;
(x) if X is first countable, then it is metrizable and the metric is translation invariant;
(xi) if X is locally compact, then X is finite dimensional.

Theorem 1.4.11 *If A is a convex absorbing subset of a vector space X, then*

(i) $\mu_A(x + y) \leq \mu_A(x) + \mu_A(y)$ for all $x, y \in X$;
(ii) $\mu_A(tx) \leq t\mu_A(x)$ for $t \geq 0$;
(iii) μ_A is a seminorm, when A is balanced;
(iv) $B = \{x : \mu_A(x) < 1\} \subseteq A \subseteq C = \{x : \mu_A(x) \leq 1\}$ and $\mu_A = \mu_B = \mu_C$.

Theorem 1.4.12 *If $\mathcal{B}$ is a local base for a t.v.s. (X, J) comprising convex balanced neighbourhood, then $\{\mu_V : V \in \mathcal{B}\}$ is a family of continuous seminorms that are separating (i.e. for $x, y \in X$, then there is a μ_V such that $\mu_V(x) \neq \mu_V(y)$). Further, the topology having a local base generated by these seminorms of the form $\{x : \mu_V(x) < \frac{1}{n}\}$, $V \in \mathcal{B}$, $n \in \mathbb{N}$ coincides with the topology on X.*

Definition 1.4.13 A t.v.s is said to be locally convex if it has a local base of convex sets. It is called an F-space if the topology is generated by complete translation-invariant metric. A locally convex F-space is called a Frechet space.

Theorem 1.4.14 *If $\mathcal{P} = \{p_i : i \in \mathbb{N}\}$ is a countable separating family of seminorms on a vector space X, then the topology on X induced by $\mathcal{P}$ is metrizable and this metric d is given by*

$$d(x, y) = \sum_{i=1}^{\infty} \frac{p_i(x, y)}{2^i(1 + p_i(x, y))}$$

is translation invariant.

Theorem 1.4.15 (Kolmogorov) *A topological vector space is normable if and only if the origin has a convex balanced neighbourhood.*

Example 1.4.16 Let Ω be the union of a sequence of compact sets $K_n \subseteq R^m$ for $n = 1, 2, \ldots$ with $K_n \subseteq K^o_{n+1}$, $n = 1, 2, \ldots$. Define for each $f \in C(\Omega)$, the set of all complex-valued functions on Ω, $p_n(f) = \sup\{|f(x)| : x \in K_n\}$. Then, $\{p_n, n = 1, 2, \ldots\}$ is a separating family of continuous seminorms defining a complete translation-invariant metric on $C(\Omega)$. As the origin has no bounded neighbourhood, $C(\Omega)$ is non-normable. Since $C(\Omega)$ is locally convex, it is a Frechet space.

If Ω is any non-empty open subset $\mathbb{C}$, then $H(\Omega)$, the set of all complex functions analytic on Ω is a closed subspace of $\mathcal{C}(\Omega)$. $H(\Omega)$ too is not normable.

Example 1.4.17 Let Ω be a non-void open set in $\mathbb{R}^n$. A multi-index α is an ordered n-tuple of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$ where α_i are non-negative integers. For each multi-index, the differential operator D^α associated is defined by $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$ whose order is $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and for $|\alpha| = 0$,

$D^\alpha f = f$. A complex-valued function defined on Ω is said to belong to $C^\infty(\Omega)$ if $D^\alpha f \in C(\Omega)$ for every multi-index α. Let $\Omega = \bigcup_{m=1}^{\infty} K_m$ where each K_m is compact and $K_m \subseteq K_{m+1}^0, m = 1, 2, \ldots$. Define the seminorms ϕ_m on $C^\infty(\Omega), m = 1, 2, \ldots,$ by $\phi_m(f) = \sup\{|D^\alpha f(a)| : x \in K_m, |\alpha| \leq m\}$. Then, $C^\infty(\Omega)$ is a Frechet space under the topology generated by the seminorms ϕ_m. Although every closed bounded subset of $C^\infty(\Omega)$ is sequentially compact (and hence compact in this case), $C^\infty(\Omega)$ is not locally bounded and hence not normable.

Example 1.4.18 For $0 < p < 1$, let $L_p[0, 1]$ be the linear space of all Lebegue-measurable functions f on $[0, 1]$ for which $\delta(f) = \int_0^1 |f(a)|^p dx < +\infty$. Then d, defined by $d(f, g) = \delta(f - g)$ defines a translation-invariant metric on $L_p[0, 1]$ and this metric is complete. Thus $L_p[0, 1]$ is an F-space. However, it is not locally convex. Indeed $L_p[0, 1]$ is the only open convex set. So, 0 is the only continuous linear functional on $L_p[0, 1]$ for $0 < p < 1$ (See Rudin [12]).

Definition 1.4.19 Let X be a topological vector space. The dual of X, denoted by X^* is the set of all continuous linear functionals on X.

Theorem 1.4.20 *If X is a locally convex t.v.s, then X^* separates points in X.*

Definition 1.4.21 Let K be a non-empty subset of a vector space X. A point $s \in K$ is called an extreme point of K if $s = tx + (1 - t)y$ for $t \in (0, 1)$ for some $x, y \in K$ implies $x = y = s$. The convex hull of a set $E \subseteq X$ is the smallest convex set in X containing E. The closed convex hull of E is the closure of its convex hull.

Theorem 1.4.22 (Krein-Milman [11]) *If X is a topological vector space on which X^* separates points. Every compact convex set in X is the closed convex hull of the set of its extreme points. So in a locally convex t.v.s X every compact convex set in X is the closed convex hull of the set of its extreme points.*

In this context, it is pertinent to recall Riesz Representation theorem (see Rudin [12]).

Theorem 1.4.23 (Riesz-Representation) *Let X be a locally compact T_2 space and L be a positive linear functional on $C_C(X)$ the linear space of all continuous complex-valued functions with compact support and the supremum norm. Then, there exists a σ-algebra $\mathscr{S}$ on X containing all the Borel subsets of X and a unique positive measure μ on $\mathscr{S}$ representing L according to the formula*

$$Lf = \int_X f d\mu \ \ for \ \ f \in C_C(X)$$

with the following properties:

(i) $\mu(K) < +\infty$ *for each compact subset of X;*
(ii) *for each* $E \in \mathscr{S}$, $\mu(E) = \inf\{\mu(G) : G \supseteq E$ *and G is open in X*$\}$;

(iii) $\mu(E) = \sup\{\mu(K) : K \subseteq E$ *and* K *compact*$\}$ *is true for each open set* E *and for any* $E \in \mathscr{S}$ *with* $\mu(E) < +\infty$*;*
(iv) *for* $E \in \mathscr{S}$ *with* $\mu(E) = 0$*,* $A \in \mathscr{S}$ *for any* $A \subseteq E$ *and* $\mu(A) = 0$*.*

When X *is compact,* μ *can be chosen so that* $\mu(X) = 1$*, i.e. a Borel probability measure.*

Remark 1.4.24 In a Frechet space, for the convex hull H of a compact set, $\overline{H}$ is compact and in a finite-dimensional space $\mathbb{R}^n$, H itself is compact. Also if an element x lies in the convex hull and a set $E \subseteq \mathbb{R}^n$, then it lies in the convex hull of a subset of E that contains at most $n + 1$ points.

We now proceed to define vector-valued integrals. Rudin [11] may be consulted for further details.

Definition 1.4.25 Let (Q, J, μ) be a measure space, X a t.v.s for which X^* separates points and $f : Q \to X$ be a function such that Λf is integrable with respect to μ for each $\Lambda \in X^*$ (here $(\Lambda f)(q) = \Lambda(f(q))$ for $q \in Q$). If there exist $y \in X$ such that

$$\Lambda y = \int_Q \Lambda f \, d\mu$$

for each $\Lambda \in X^*$, then we define

$$\int_Q f \, d\mu = y.$$

Theorem 1.4.26 *Let* X *be a t.v.s such that* X^* *separates points and* μ *be a Borel probability measure on a compact Hausdorff space* Q*. If* $f : Q \to X$ *is continuous and if the convex hull* H *of* $f(Q)$ *has compact closure* $\overline{H}$ *in* X*, then the integral*

$$y = \int_Q f \, d\mu$$

exists (as per Definition 1.4.25).

Theorem 1.4.27 *Let* X *be a t.v.s such that* X^* *separates points and* Q*, a compact subset of* X *and* $\overline{H}$*, the closed convex hull of* Q *be compact.*

$y \in \overline{H}$ if and only if there is a regular Borel probability measure μ on Q such that

$$y = \int_Q x \, d\mu.$$

When X is a Banach space we also have

Theorem 1.4.28 *Let* Q *be a compact* T_2 *space,* X *a Banach space,* $f : Q \to X$ *a continuous map and* μ *a positive Borel probability measure on* Q*. Then*

$$\| \int_Q f \, d\mu \| \leq \int_Q \| f \| d\mu.$$

Indeed vector-valued integrals can also be defined more directly as limits of (integral) sums.

References

1. Bartle, R.G.: The Elements of Real Analysis, 2nd edn. Wiley, New York (1976)
2. Bollobas, B.: Linear Analysis, An Introductory Course, 2nd edn. Cambridge University Press, Cambridge (1999)
3. Dugundji, J.: Topology. Allyn and Bacon Inc., Boston (1966)
4. Halmos, P.R.: Naive Set Theory. D. Van Nostrand Co., Princeton (1960)
5. Kantorovich, L.V., Akilov, G.P.: Functional Analysis, Translated from the Russian by Howard L. Silcock, 2nd edn. Pergamon Press, Oxford-Elmsford (1982)
6. Kaplansky, I.: Set Theory and Metric Spaces, 2nd edn. Chelsea Publishing Co., New York (1977)
7. Kelley, J.L.: General Topology. D. Van Nostrand Company, Inc., Toronto (1955)
8. Lusternik, L.A., Sobolev, V.J.: Elements of Functional Analysis. Hindustan Publishing Corporation, Delhi (1974)
9. Munkres, J.R.: Topology: A First Course. Prentice-Hall, Inc., Englewood Cliffs (1975)
10. Royden, H.L.: Real Analysis, 3rd edn. Macmillan Publishing Company, New York (1988)
11. Rudin, W.: Functional Analysis. Tata McGraw-Hill Publishing Co. Ltd., New Delhi (1974)
12. Rudin, W.: Real and Complex Analysis, 3rd edn. McGraw-Hill Book Co., New York (1987)
13. Simmons, G.F.: Introduction to Topology and Modern Analysis. McGraw-Hill Book Co., Inc., New York (1963)
14. Taylor, A.E.: Introduction to Functional Analysis. Wiley, New York; Chapman Hall, Ltd., London (1958)