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P. V. Subrahmanyam

Elementary Fixed Point Theorems

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Elementary Fixed Point Theorems

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Preface

Fixed point theorems constitute an important and interesting aspect of applicable mathematics and provide solutions to several linear and nonlinear problems arising in biological, engineering and physical sciences. This book, as the title suggests, deals with some fixed point theorems and applications. The choice of the topics is largely guided by personal preferences, and the book will serve as a subjective sampler of topics in fixed point theorems.

This volume neither exhausts all the important fixed point theorems nor elaborates the most general formulations of the theorems presented. No attempt is made to provide a historical/chronological background for the topics covered. Nevertheless, it hopefully supplements the extant publications and promotes interest for further study among the readers.

A quick run-through of the highlights of the individual chapters is not out of place. While Chap. 1 collects the analytic and topological preliminaries needed in the sequel, Chap. 2 describes the basic properties of iterates of real and complex functions. It details the theorems of Thron on the rates of convergence of certain classes of real functions. Cohen's common fixed point theorem for commuting continuous full surjections on a compact interval, Shield's theorem on the existence of a common fixed point for a commuting family of analytic functions on the closed unit disc, an elementary proof of Sharkovsky's theorem on periodic points of real functions and Bergweiler's theorem on the existence of fixed points of meromorphic functions are other noteworthy features of Chap. 2. Chapter 3 explores the existence of fixed points in the setting of partially ordered sets. Knaster–Tarski principle is formulated leading to Tarski's theorem, and its applications to set theory, general topology, nonlinear complementarity problem, parabolic differential equations and formal languages are described. Also discussed are some generalizations due to Merrifield and Stein.

Most of Chap. 4 deals with Ward's theory of partially ordered topological spaces culminating in Schweigert's fixed point theorem. Manka's fixed point theorem on inductively and acyclically ordered posets is proved and used to deduce the fixed point property of continuous functions on dendroids. Also highlighted is Klee's counterexample in fixed point theory. Contraction principle and some of its variants are taken up in Chap. 5, including Kupka's topological generalization and Nadler's extension to multifunctions. Jachymski's proof of the converse of the contraction principle is also elaborated. Applications to differential equations, functional equations, algebraic Weierstrass preparation theorem, Cauchy–Kowalevsky theorem and the central limit theorem are detailed in Chap. 6. Chapter 7 is devoted to Caristi's fixed point theorem, a generalization of the contraction principle. Separate proofs of Caristi's theorem due to Siegel, Penot and Kirk are provided. The equivalence of Caristi's theorem, Ekeland's variational principle and Takahashi's minimization theorem is proved. That these three equivalent theorems characterize metrical completeness is also established.

Chapter 8 is on fixed points of contractive and non-expansive maps. Goebel's proof of Browder–Gohde–Kirk fixed point theorem for nonexpansive mappings in the setting of uniformly convex Banach spaces is provided. Pasicki's theory of bead and discus spaces and his theorem on the fixed points of nonexpansive maps with an application of Matkowski to certain functional equations are other highlights. Ishikawa's theorems on iterates are detailed as also a fixed point theorem due to Merrifield et al. on generalized contractions using combinatorial ideas.

Chapter 9 using the concepts of the geometry of Banach spaces establishes that convex weakly compact subsets of nearly uniformly smooth (NUS) and non-square Banach spaces have fixed point property for nonexpansive mappings. Brouwer's theorem is treated in Chap. 10. Besides an analytic proof due to Rogers, proof based on Sperner's lemma is provided. The existence of Walrasian and Nash equilibria for economies is deduced. Whyburn's proof of Stallings's generalization of Brouwer's fixed point theorem for connectivity maps has also been elaborated. Schauder's theorem and its extensions and applications constitute the major part of Chap. 11. Besides Tychonoff's fixed point theorem and Ky Fan–Browder–Glicksberg theorem for multifunctions, Kakutani–Markov and Ryll–Nardzewski theorems are described. The existence of Banach limits, Haar integral on compact groups and Nash equilibria is provided besides an introduction to measures of noncompactness. Chapter 12 describes the finite-dimensional degree theory as presented by Heinz. Appendices A and B summarize the classical counterexamples due to Huneke and Kinoshita, respectively. Appendix C deals with fractals and fixed points.

The text closely follows the notation and organization of the papers quoted extensively, within reasonable limits, to aid the readers to peruse these papers with ease. A judicious choice of sections from Chaps. 2–12 can constitute a basic course on fixed point theorems.

No expression of gratitude is adequate for the award of a sabbatical leave granted by the competent authority of the Indian Institute of Technology Madras to me for the preparation of an earlier concise version of this tract. I am thankful to my colleagues for their support during the preparation of this tract and to my wife

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Suggestions for improvement and corrections of errors and misprints are earnestly solicited.

Chennai, India P. V. Subrahmanyam

Contents

About the Author

P. V. Subrahmanyam is Professor Emeritus at the Indian Institute of Technology Madras (IIT Madras), India. He received his Ph.D. in mathematics from IIT Madras for his dissertation on 'Topics in Fixed-Point Theory' under the supervision of (late) Dr. V. Subba Rao. He received his M.Sc. in mathematics from IIT Madras and B.Sc. in mathematics from the University of Madras, India. He has held several important administrative positions, such as Senior Professor and Head of the Department of Mathematics at IIT Madras; Founder and Head of the Department of Mathematics at the Indian Institute of Technology Hyderabad (IIT Hyderabad); Executive Chairman of the Association of Mathematics Teachers of India (AMTI) and President of the Forum for Interdisciplinary Mathematics (FIM). Before joining IIT Madras, he served as Faculty Member at Loyola College, University of Madras, and Hyderabad Central University. His areas of interest include classical analysis, nonlinear analysis and fixed point theory, fuzzy set theory, functional equations and mathematics education. He has published over 70 papers and served on the editorial board of several journals including *Differential Equations and Dynamical Systems* and *The Journal of Analysis*. He received an award for his outstanding contributions to mathematical sciences in 2004 and the Lifetime Achievement Award from FIM in 2016. He has given various invited talks at international conferences and completed brief visiting assignments in many countries such as Canada, the Czech Republic, Germany, Greece, Japan, Slovak Republic and the USA. He is also Life Member of the Association of Mathematics Teachers of India, FIM, Indian Mathematical Society and the Society for Industrial and Applied Mathematics (SIAM).

Chapter 1
Prerequisites

This chapter is a precis of the basic definitions and theorems used in the sequel. It is presumed that the reader is familiar with naive set theory (see Halmos [4]) and the properties of real numbers and real functions (see Bartle [1]). Other mathematical concepts and theorems relevant to specific sections of a chapter will be recalled therein.

1.1 Topological Spaces

This section collects important concepts and results from topology. For proofs and other details, Dugundji [3], Kelley [7], Munkres [9] and Simmons [13] may be consulted.

Definition 1.1.1 Let X be a non-empty set. A collection of J of subsets of X is called a topology on X, if

(i) $\phi, X \in \mathscr{T}$,
(ii) $G_1 \cap G_2 \in \mathscr{T}$ for $G_1, G_2 \in \mathscr{T}$ and
(iii) $\bigcup_{G \in \mathcal{F}} G \in \mathscr{T}$ for any $\mathcal{F} \subseteq \mathscr{T}$.

Any subset of X belonging to $\mathscr{T}$ is called an open set or more precisely $\mathscr{T}$-open set. The pair $(X, \mathscr{T})$ is called a topological space. Given a topological space $(X, \mathscr{T})$, the interior of $A \subseteq X$, denoted by A^0 is the largest open subset of A.

For a subset S of X, where $(X, \mathscr{T})$ is a topological space, $\mathscr{T}_S = \{G \cap S : G \in \mathscr{T}\}$ is a topology on S, called the relative topology (or subspace topology) on S.

Example 1.1.2 For a non-empty set X, the family $\{\phi, X\}$ is a topology on X called the indiscrete topology on X, 2^X, the power set of X or the set of all subsets of X is a topology on X called the discrete topology on X. The family of all subsets of X

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whose complements are finite sets together with the empty set is also a topology on X called the co-finite topology on X.

Since the intersection of any collection of topologies on X is a topology on X, for any family $\mathcal{F}$ of subsets of X, there is the smallest topology on X containing $\mathcal{F}$, called the topology generated by $\mathcal{F}$.

Example 1.1.3 A subset G of real numbers is called open if for each $x \in G$, an open interval containing x lies in G. (Evidently the empty set is open.) This collection of all open subsets of $\mathbb{R}$, the real number system is a topology on $\mathbb{R}$, called the usual topology on $\mathbb{R}$.

Definition 1.1.4 Let $(X, \mathscr{T})$ be a topological space. A neighbourhood of a point $x \in X$ is any subset of X containing an open subset $G \in \mathscr{T}$, containing x. A neighbourhood base or local base at x is a family $\mathcal{N}_x$ of neighbourhoods of x such that for any neighbourhood N of x, there is a neighbourhood $N_x \in \mathcal{N}_x$ such that $x \in N_x \subseteq N$. A topological space is called first countable if for each point there is a countable local base. An interior point of A is a point $a \in A$ such that A contains a neighbourhood of a.

Definition 1.1.5 A subset F of a topological space $(X, \mathscr{T})$ is called a closed subset of X if $X - F$ is $\mathscr{T}$-open. The closure of a subset A of X denoted by $\overline{A}$ is the smallest closed set containing A. A subset S of X is said to be dense in X if $\overline{S} = X$. A topological space $(X, \mathscr{T})$ is called separable if it has a countable dense subset.

Remark 1.1.6 Let $(X, \mathscr{T})$ be a topological space and $A, B \subseteq X$. Then

(i) $\phi^0 = \phi$, $\overline{\phi} = \phi$, $X^0 = X$ and $\overline{X} = X$;
(ii) $\overline{A} \supseteq A$ and $A^0 \subseteq A$;
(iii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$, $(A \cap B)^\circ = A^\circ \cap B^\circ$;
(iv) $\overline{(\overline{A})} = \overline{A}$ and $(A^\circ)^\circ = A^\circ$. Further $\overline{A} = \{x \in X :$ every neighbourhood of x has a non-void intersection with $A\}$. $A^0 = \{a \in A : a$ is an interior point of $A\}$.

Definition 1.1.7 For a topological space $(X, \mathscr{T})$ $\mathcal{B} \subseteq \mathscr{T}$ is called a base (or basis) for $\mathscr{T}$ is for $A_1, A_2 \in \mathcal{B}$ and $x \in A_1 \cap A_2$, there exists $A_3 \in \mathcal{B}$ such that $x \in A_3 \subseteq A_1 \cap A_2$. A subfamily $\mathscr{S}$ of $\mathscr{T}$ is called a subbase for $\mathscr{T}$ of $\mathcal{B}$, if the family of intersections of all finite subfamilies of $\mathscr{S}$ is a base for $\mathscr{T}$. If the topology $\mathscr{T}$ has a countable base, then the topological space is called second countable.

Remark 1.1.8 If $\mathscr{S}$ is a family of subsets of X with $\cup\{S : S \in \mathscr{S}\} = X$, then $\mathscr{S}$ is a subbase for a topology on X, for which $\mathcal{B}$ the family of subsets of X which are the intersections of finite subfamilies of $\mathscr{S}$ is a base for this topology.

Remark 1.1.9 The family of all subintervals of the form $[a, b)$, $a < b$, $a, b \in \mathbb{R}$ is a base for a topology on $\mathbb{R}$, called the lower limit topology on $\mathbb{R}$. Similarly, the family $\{(a, b] : a < b, a, b \in \mathbb{R}\}$ is a base for a topology on $\mathbb{R}$ called the upper limit topology on $\mathbb{R}$. The usual (standard) topology on $\mathbb{R}$ has the family of all open intervals (a, b), $a < b$, $a, b \in \mathbb{R}$ as a base. $\mathbb{R}$ with the usual topology is separable and second countable. However, $\mathbb{R}$ with the lower limit topology is separable and first countable but is not second countable.

Definition 1.1.10 A binary relation $\leq$ on a non-empty subset X is called a quasi-order if the following conditions are satisfied:

(i) $x \leq x$ for all $x \in X$ (reflexivity);
(ii) if $x \leq y$ and $y \leq z$, for $x, y, z \in X$, then $x \leq z$ (transitivity).
If, in addition a quasi-order $\leq$ satisfies
(iii) if $x \leq y$ and $y \leq x$, then $x = y$ (anti-symmetry),

then the quasi-order $\leq$ is called a partial order. Accordingly if $\leq$ is a quasi-order on X, then $(X, \leq)$ is called a quasi-ordered space. If $\leq$ is a partial order on X, then $(X, \leq)$ is called a partially ordered set or poset.

Definition 1.1.11 A partial order $\leq$ on a set X is called a linear order or total order if for any pair of elements $x, y \in X$ either $x \leq y$ or $y \leq x$. A linearly ordered set is also called a chain.

Definition 1.1.12 A partially ordered set $(D, \leq)$ is called a directed set if for any pair $x, y \in X$, there exists $z \in D$ such that $x \leq z$ and $y \leq z$.

Definition 1.1.13 A net in a topological space X is a pair $(S, \geq)$ where S is a function from a directed set $(D, \geq)$ into X. A net $(S, \geq)$ in a topological space is said to converge to an element $x \in X$ if for each open set G containing x, there is an element m of D such that for $n \geq m$, $n \in D$, $S(n) \in G$. Clearly, a sequence in a topological space is a net directed by the set of natural numbers with the usual ordering.

Proposition 1.1.14 *A subset S of a topological space $(X, \mathscr{T})$ is closed if and only if no net in S converges to an element of $X - S$. An element $s \in \overline{S}$ for $S \subseteq X$ if and only if there is a net in S converging to s.*

Definition 1.1.15 Let $(X_i, \mathscr{T}_i), i = 1, 2$ be topological spaces. A map $f : X_1 \to X_2$ is said to be continuous if for each $\mathscr{T}_2$-open subset G of X_2, $f^{-1}(G)$ is $\mathscr{T}_1$-open in X_1. If f is one-one and onto X_2 and if both f and f^{-1} are continuous maps, then f (as also f^{-1}) is called a homeomorphism from X_1 onto X_2 (from X_2 onto X_1).

A map $f : X_1 \to X_2$ is said to be continuous at $x \in X_1$, if for each neighbourhood $N_{f(x)}$ of $f(x)$ in X_2, there is a neighbourhood N_x of x such that $f(N_x) \subseteq N_{f(x)}$. A map $g : X_1 \to X_2$ is called open if it maps open subsets of X_1 onto open subsets of X_2.

The following theorem is well-known.

Theorem 1.1.16 *Let $(X_i, \mathscr{T}_i)$, $i = 1, 2$ be topological spaces and $f : X_1 \to X_2$ be a map. The following statements are equivalent:*

(i) f is continuous on X_1;
(ii) f is continuous at each point of X_1;
(iii) $f^{-1}(F)$ is closed in $(X_1, \mathscr{T}_1)$ for each closed subset F of X_2;
(iv) if $G \in \mathscr{S}$, a subbase for $\mathscr{T}_2$, then $f^{-1}(G) \in \mathscr{T}_1$;

(v) *for each net* $(S, \geq)$ *converging to* x *in* X_1, $(f(S), \geq)$ *converges to* $f(x)$ *in* X_2;
(vi) *for each subset* A *of* X_1, $f(\overline{A}) \subseteq \overline{f(A)}$;
(vii) *for each subset* B *of* Y, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

Theorem 1.1.17 *A topological space* $(X, \mathscr{T})$ *is said to be disconnected if* $X = A \cup B$ *where* A *and* B *are non-empty disjoint proper open subsets of* X. *A pair of sets* A *and* B *is said to be separated if* $\overline{A} \cap B = \overline{B} \cap A = \phi$, *where* A *and* B *are non-empty. A topological space is called connected if it is not disconnected (A connected space is not the union of two non-void separated sets). A subset* Y *of* X *is called connected if* Y *is connected in the subspace topology. A maximal connected subset of* X *is called a component.*

Definition 1.1.18 A topological space is called totally disconnected if the only connected subsets are singletons.

Definition 1.1.19 A topological space is said to be locally connected if the family of open connected subsets is a base for the topology.

Remark 1.1.20 A discrete topological space with more than one element is locally connected, though totally disconnected. The set $(0, 1) \cup (2, 3)$ with the subspace topology inherited from $\mathbb{R}$ with the usual topology is locally connected and disconnected though not totally disconnected.

Theorem 1.1.21 *Let* $(X, \mathscr{T})$ *be a topological space. Then*

(i) *if* A *is a connected subset of* X *and* $A \subseteq B \subseteq \overline{A}$, *then* B *is a connected subset;*
(ii) *the union of a family of connected subsets of* X, *no two of which are separated is connected;*
(iii) *components of* X *are closed and any two components are either identical or disjoint;*
(iv) *any component of an open subset of a locally connected space is open.*

Definition 1.1.22 A family of open sets $\{G_\lambda : \lambda \in \Lambda\}$ of a topological space $(X, \mathscr{T})$ is called an open cover for X, if $X = \bigcup_{\lambda \in \Lambda} G_\lambda$. If every open cover of X has a countable subcover, the topological space is said to be Lindelof. If each open cover of X has a finite subcover, then the topological space is called compact.

Definition 1.1.23 A topological space is called locally compact, if each element has a compact neighbourhood.

Definition 1.1.24 Let $(X_\lambda, \mathscr{T}_\lambda)$, $\lambda \in \Lambda$, $\Lambda \neq \phi$ be a family of topological spaces. The Cartesian product of all these sets X_λ denoted by $X = \prod_{\lambda \in \Lambda} X_\lambda$ is the set of all functions $f : \Lambda \to \bigcup_{\lambda \in \Lambda} X_\lambda$ such that $f(\lambda) \in X_\lambda$ for each $\lambda \in \Lambda$. The map $P_\lambda : X \to X_\lambda$ such that $P_\lambda(f) = f(\lambda)$ for each $f \in X$ is called the projection of the set X into the λth coordinate set X_λ. The topology of X having $\{P_\lambda^{-1}(U) : U \in \mathscr{T}_\lambda, \lambda \in \Lambda\}$ as a subbase is called the product topology on X and X with this topology is referred as the product (topological) space.

Theorem 1.1.25 *Let* $\{(X_\lambda, \mathscr{T}_\lambda) : \lambda \in \Lambda, \Lambda \neq \phi\}$ *be a family of topological spaces and* X *be the product space with the product topology* $\mathscr{T}$*. Then*

(i) P_λ*, the projection of* X *into* X_λ *is continuous for each* $\lambda \in \Lambda$*;*
(ii) a map $f : Y \to X$*, where* Y *is a topological space is continuous if and only if* $P_\lambda \circ f : Y \to X_\lambda$ *is continuous for each* $\lambda \in \Lambda$*;*
(iii) a net S *in* X *converges to an element* s *if and only if its projection in each coordinate space converges to the projection of* s*.*
(iv) X *is connected if and only if each* X_λ *is connected;*
(v) (Tychonoff's theorem) X *is compact if and only if each* X_λ *is compact.*

Definition 1.1.26 A topological space X is said to be

(i) T_1 if for each pair of distinct elements x and y, there exist neighbourhoods N_x of x not containing y and N_y of y not containing x;
(ii) T_2 (Hausdorff) if each pair of distinct elements has disjoint neighbourhoods;
(iii) regular, if for each $x \in X$ and any closed subset F of X not containing x, there exist disjoint open sets G_1 and G_2 with $x \in G_1$ and $F \subseteq G_2$ (X is called T_3 if it is T_1 and regular);
(iv) normal, if for each pair of disjoint closed subsets $F_i, i = 1, 2$ of X, there exist disjoint open sets $G_i, i = 1, 2$ with $F_i \subseteq G_i, i = 1, 2$ (X is called T_4 if it is T_1 and normal).
(Every T_4 space is T_3 and each T_3 space is T_2, while a T_2 space is necessarily T_1).

Theorem 1.1.27 (Urysohn's Lemma) *If* A *and* B *are disjoint closed subsets of a normal space* X*, then there is a continuous function* $f : X \to [0, 1]$ *such that* $f \equiv 0$ *on* A *and* $f \equiv 1$ *on* B*.*

Theorem 1.1.28 *Let* X *and* Y *be topological spaces and* $f : X \to Y$ *be a continuous map. If* X *is compact, then* $f(X)$ *is a compact subset of* Y*. If* X *is connected, then* $f(X)$ *is a connected subset of* Y*.*

Corollary 1.1.29 *If* X *is a compact topological space and* $f : X \to \mathbb{R}$ *is a continuous map, then* f *attains its maximum and minimum on* X*. If* X *is a connected space and* $f : X \to \mathbb{R}$ *is continuous, then* $f(X)$ *is an interval.*

1.2 Metric Spaces

In this section, basic concepts and theorems from the theory of metric spaces are recalled. For details, in addition to the references cited in Sect. 1.1, Kaplansky [6] may be consulted.

Definition 1.2.1 Let X be a non-void set. A map $d : X \times X \to [0, \infty)$ $(=\mathbb{R}^+)$ is called a metric if

(i) $d(x, x) = 0$ for all $x \in X$;
(ii) $d(x, y) = 0$ implies $x = y$;
(iii) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(iv) $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y, z (triangle inequality).

The pair (X, d) is called a metric space. A map d satisfying (i), (iii) and (iv) is called a pseudometric and the corresponding (X, d) is called a pseudometric space.

Definition 1.2.2 If (X, d) is a metric space, the set $B(x_0; r) = \{x \in X : d(x_0, x) < r\}$ for $r > 0$ is called an open sphere of radius r centred at x_0, while the set $\{x \in X : d(x_0, x) \leq r\}$ is referred as the closed sphere of radius r with centre x_0.

Remark 1.2.3 The family of all open spheres $\{B(x; r) : x \in X, r > 0\}$ is a base for a topology on X called the metric topology on X induced by d.

Example 1.2.4 (i) If X is a non-empty set the map $d : X \times X \to \mathbb{R}^+$ defined by $d(x, y) = 1$ for $x \neq y$ and $d(x, x) = 0$ is a metric on X called the discrete metric. The corresponding metric topology on X is the discrete topology.

(ii) On $\mathbb{R}$, $d(x, y) = |x - y|$, the absolute value of $x - y$ defines a metric called the usual (or standard) metric on $\mathbb{R}$ and the topology induced is the usual topology on $\mathbb{R}$ (with the base comprising all open intervals).

(iii) On $\mathbb{R}^n$, the set of all n-tuples of real numbers, $d(\overline{x}, \overline{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}}$, where $\overline{x} = (x_1, x_2, \ldots, x_n)$ and $\overline{y} = (y_1, y_2, \ldots, y_n)$ defines a metric, called the Euclidean metric on $\mathbb{R}^n$.

(iv) $\mathcal{C}[a, b]$, the set of all continuous real-valued function on the closed interval $[a, b]$, where $a < b$, $a, b \in \mathbb{R}$ is a metric space under the metric

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [a, b]\}$$

where $f, g \in \mathcal{C}[a, b]$. This metric is called Tschebyshev or uniform metric.

(v) More generally $\mathcal{C}(X)$, the set of all continuous real-valued functions on a compact topological space becomes a metric space with the metric d defined by $d(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$ where $f, g \in \mathcal{C}(X)$.

(vi) $d(f, g) = \int_a^b |f(t) - g(t)|dt$ also defines a metric on $\mathcal{C}[a, b]$ the set of all continuous real-valued functions on $[a, b]$.

(vii) If (X, d) is a metric space and $S \subseteq X$, then the restriction of d to $S \times S$ is a metric and this metric topology is precisely the topology of S relative to the metric topology on X.

Theorem 1.2.5 *A metric space is second countable if and only if it is separable.*

Theorem 1.2.6 *Every metric space is a Hausdorff normal space.*

Definition 1.2.7 A sequence (x_n) in a metric space (X, d) is called Cauchy (fundamental) if $d(x_m, x_n) \to 0$ as $m, n \to \infty$. A metric space is said to be complete if every Cauchy sequence in X converges to an element of X.

Theorem 1.2.8 (Baire) *No complete metric space can be written as a countable union of closed sets having empty interior.*

Definition 1.2.9 Let (X_i, d_i), $i = 1, 2$ be metric spaces. A map $T : X_1 \to X_2$ is called an isometry if $d_2(Tx_1, Tx_2) = d_1(x_1, x_2)$ for all $x_1, x_2 \in X_1$.

Theorem 1.2.10 *Each metric space (X, d) can be isometrically embedded in a complete metric space $(\overline{X}, \overline{d})$ as a dense subset. Further such a complete metric space $\overline{X}$, called the completion of X is unique up to isometry.*

Remark 1.2.11 In example 1.2.4, except the space described in (vi), the metric spaces in examples (i)–(v) are complete.

Theorem 1.2.12 *If (X, d) is a metric space, then $d_1(x, y) = \min\{1, d(x, y)\}$, $x, y \in X$ defines a metric on X and the topologies induced on X by these metrices are the same.*

Theorem 1.2.13 *If (X_n, d_n), $x \in \mathbb{N}$ is a sequence of metric spaces, then $X = \prod_{n\in\mathbb{N}} X_n$ is a metric space under the metric d defined by*

$$d(\overline{x}, \overline{y}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} \right),$$

where $\overline{x} = (x_n)$ and $\overline{y} = (y_n)$ are in X. Further, if each (X_n, d_n) is complete, then (X, d) is complete.

The following metrization theorem is classical.

Theorem 1.2.14 (Urysohn) *A regular T_1 second countable topological space is metrizable (in the sense that there is a metric on this space whose metric topology is the given topology).*

A concept basic to the study of the metrization problem is defined below.

Definition 1.2.15 A family $\mathcal{F}$ of subsets of a topological space $(X, \mathscr{T})$ is called

(i) locally finite, if each point of the space has a neighbourhood that intersects only finitely many sets in $\mathcal{F}$;
(ii) discrete if each point of the space has a neighbourhood that intersects at most one member of $\mathcal{F}$;
(iii) σ-locally finite (σ-locally discrete) if it is the union of a countable collection of locally finite (finite) subfamilies.

Theorem 1.2.16 (Metrization theorems) *A topological space is metrizable if and only if it is T_1 and regular with*

a σ-locally finite base (Nagata–Smirnov);
or
a σ-discrete base (Bing).

Another important notion is that of paracompactness formulated below.

Definition 1.2.17 A topological space X is called paracompact if each open cover $\mathcal{U}$ of X has an open locally finite refinement $\mathcal{U}^*$ (viz. $\mathcal{U}^*$ is locally finite and each member of $\mathcal{U}^*$ is open and is a subset of some set in $\mathcal{U}$).

Theorem 1.2.18 *Every pseudometric space is paracompact and a paracompact T_2 space is normal.*

Definition 1.2.19 Let X be a topological space. A family $\{f_\lambda : \lambda \in \Lambda \neq \phi\}$ of continuous functions mapping X into [0, 1] is called a partition of unity if for each $x \in X$, $\sum\limits_{\lambda \in \Lambda} f_\lambda(x) = 1$ and all but a finite number of f_λ's vanish on some neighbourhood of x. A partition of unity $\{f_\lambda : \lambda \in \Lambda \neq \phi\}$ in subordinate to a cover $\mathcal{U}$ if each f_λ vanishes outside some member of $\mathcal{U}$.

Theorem 1.2.20 *A regular T_1 space is paracompact if and only if for each open covering of X, there is a partition of unity subordinate to this covering. For every compact T_2 space, every open cover has a partition of unity subordinate to it.*

Definition 1.2.21 A subset S of a metric space (X, d) is said to be totally bounded, if for each $\epsilon > 0$, there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ (depending on ϵ) such that $S \subseteq \bigcup\limits_{i=1}^{n} B(x_i; \epsilon)$. A subset S is said to be bounded if $S \subseteq B(x; r)$ for some $x \in X$ and some $r > 0$.

Theorem 1.2.22 *For a metric space (X, d), the following are equivalent:*

(i) *X is compact;*
(ii) *X is complete and totally bounded;*
(iii) *every sequence in X has a convergent subsequence;*
(iv) *X has the Bolzano–Weierstrass property, viz. for every infinite subset A of X has a limit point $x_0 \in X$, i.e. a point x_0 such that every neighbourhood of x_0 meets A.*

Theorem 1.2.23 *Let (X_i, d_i), $i = 1, 2$ be metric spaces and $f : X_1 \to X_2$ be a continuous map. If X_1 is compact, then f is uniformly continuous in the sense that for each $\epsilon > 0$ there exists $\delta > 0$ depending only on ϵ so that $d_2(f(x), f(y)) < \epsilon$ whenever $x, y \in X_1$ and $d_1(x, y) < \delta$.*

Definition 1.2.24 For a non-void subset A of a metric space (X, d), the distance of a point x from A is defined as $D(x, A) = inf\{d(x, a) : a \in A\}$.

Theorem 1.2.25 *Let A be a non-void subset of a metric space (X, d). Then $\overline{A} = \{x \in X : D(x, A) = 0\}$. Further, $|D(x, A) - D(y, A)| \leq d(x, y)$ for any $x, y \in X$ and the map $x \to D(x, A)$ is a continuous map of X into $\mathbb{R}^+$.*

1.3 Normed Linear Spaces

Normed linear spaces, constituting the base of Functional Analysis are metric spaces with a richer (algebraic) structure. They provide a natural setting for mathematical modelling of many natural phenomena. Bollabos [2], Kantorovitch and Akhilov [5], Lyusternik and Sobolev [8], Rudin [11, 12], Simmons [13] and Taylor [14] may be consulted for a detailed exposition of the following concepts and theorems. It is assumed that the reader is familiar with the concepts of groups, rings and fields.

Definition 1.3.1 A linear space or vector space over a field F is a triple $(V, +, \cdot)$, where $+$ is a binary operation (called vector addition or simply addition) and $\cdot$ is a mapping from $F \times V$ into V (called scalar multiplication) satisfying the following conditions;

(i) $(V, +)$ is a commutative group with θ (called zero vector) as its identity element;
(ii) for all $\lambda \in F, x, y \in V$ $\lambda.(x + y) = \lambda x + \lambda y$;
(iii) for all $\lambda, \mu \in F$ and $x \in V$ $(\lambda + \mu).x = \lambda.x + \mu.x$ and $\lambda \cdot (\mu \cdot x) = (\lambda \cdot \mu) \cdot x$ (where $\lambda\mu$ is the product of λ and μ under the multiplication in the field F);
(iv) $0 \cdot x = \theta$, $1 \cdot x = x$ for all $x \in V$, where 0 is the additive identity and 1 the multiplicative identity of the field F.

Often 0 is also used to represent the zero vector and the context will clarify this without much difficulty. If F is the field of real (complex) numbers then V is called a real (complex) vector space. In what follows, we will be concerned only with real or complex vector spaces. Also a linear subspace V_1 of V is a subset of V which is a linear space over F with vector addition and scalar multiplication of V restricted to V_1.

Definition 1.3.2 A subset S of a linear space V over a field F is said to be linearly independent if for every finite subset $\{s_1, \ldots, s_n\}$ of S, $\sum_{i=1}^{n} \alpha_i s_i = \theta$ implies $\alpha_i = 0$ for $i = 1, 2, \ldots, n$ where $\alpha_i \in F$. An element of the form $\sum_{i=1}^{n} \alpha_i s_i$ where $\alpha_i \in F$ and $s_i \in S$ is called a finite linear combination of members of S.

Definition 1.3.3 A subset S of a linear space V over a field F is said to span V if every element of V can be written as a finite linear combination of elements from S. A maximal linearly independent subset of a linear space V over F is called a basis for V.

Any two bases of a vector space have the same cardinality.

Definition 1.3.4 The cardinality of a basis of a linear space V is called the dimension of the linear space. If a linear space has a finite dimension, then it is called a finite-dimensional vector space. Otherwise the linear space is infinite-dimensional.

Definition 1.3.5 Let $(X, +, \cdot)$ be a linear space over $F = \mathbb{R}$ or $\mathbb{C}$. A map $\| \cdot \| : X \to \mathbb{R}^+$ is called a norm if the following conditions are satisfied:

1. $\|x\| = 0$ if and only if $x = 0$;
2. $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in X$ (triangle inequality);
3. $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in F$ and all $x \in X$, $|\alpha|$ being the modulus of α. The pair $(X, \|\cdot\|)$ is called a normed linear space.

Remark 1.3.6 If $(X, \|\cdot\|)$ is a normed linear space, then $d(x, y) = \|x - y\|, x, y \in X$ is a metric on X.

Definition 1.3.7 A normed linear space $(X, \|\cdot\|)$ is called a Banach space if it is complete in the metric induced by the norm.

Remark 1.3.8 The linear spaces in (ii)–(v) of Example 1.2.4 are Banach spaces with the norms defined by $\|x\| = d(x, \theta)$ where d is the metric described in the corresponding case, while (vi) of Example 1.2.4 is a normed linear space under the norm $\int_a^b |f(t)|dt$. However, this is not a Banach space.

Definition 1.3.9 Let $(X_i, \|\cdot\|_i), i = 1, 2$ be normed linear spaces over $F = \mathbb{R}$ or $\mathbb{C}$. A linear operator is a map $T : X_1 \to X_2$ such that $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $\alpha, \beta \in F$ and $x, y \in X_1$. If X_2 is the base field F (=$\mathbb{R}$ or $\mathbb{C}$ with the modulus or absolute value as a norm) which is also a normed linear space, the linear operator is called a linear functional. If the linear operator T is continuous as a map between the metric spaces X_1 and X_2 with metrics induced by the norms, then it is called a continuous linear operator.

Theorem 1.3.10 *Let $(X_i, \|\cdot\|_i)$, $i = 1, 2$ be normed linear spaces over $F = \mathbb{R}$ or $\mathbb{C}$ and $T : X_1 \to X_2$ a linear operator. Then the following are equivalent:*

(i) T is continuous on X_1;
(ii) T is continuous at some $x_0 \in X_1$;
(iii) there exists $K > 0$ such that $\|Tx\|_2 \le K\|x\|_1$ for all $x \in X_1$.

Remark 1.3.11 A linear operator satisfying (iii) of Theorem 1.3.10 is called bounded. In view of the theorem above, bound linear operators are precisely continuous linear operators.

If $T : X_1 \to X_2$ is a continuous linear operator where $(X_i, \|\cdot\|_i)$, $i = 1, 2$ are normed linear spaces, then

$$\inf\{K \ge 0 : x \in X_1 \text{ and } \|Tx\|_2 \le K\|x\|_1\} = \sup\{\|Tx\|_2 : \|x\|_1 = 1, x \in X_1\}$$

is finite and is called the norm of the linear operator and is denoted by $\|T\|$.

Theorem 1.3.12 *For $i = 1, 2$, let $(X_i, \|\cdot\|_i)$ be normed linear spaces. $B(X_1, X_2)$ the set of all bounded (continuous) linear operators is a normed linear space under the norm described in Remark 1.3.11. If $(X_2, \|\cdot\|_2)$ is complete so is $B(X_1, X_2)$ under this norm.*

Theorem 1.3.13 (Hahn–Banach) *If f is a bounded linear functional from a linear subspace N of a normed linear space $(X, \|\cdot\|)$, then there is a bounded linear functional f^* on X such that $f^* \equiv f$ on $\mathbb{N}$ and $\|f\| = \|f^*\|$.*

The Hahn–Banach theorem insures the abundance of continuous linear functionals in any nontrivial normed linear space.

Definition 1.3.14 Given a normed linear space $(X, \|\cdot\|)$, the space of all continuous linear functionals on X is called the dual or conjugate of X and is denoted by X^*. The dual of X^* denoted by X^{**} is called the second dual or second conjugate of X.

Even, if X is incomplete, X^* and X^{**} are complete.

Theorem 1.3.15 *Let $(X, \|\cdot\|)$ be a normed linear space. The map f_x defined by $f \to f(x)$ for each $x \in X$ is a bounded linear functional on X^* and $\|f_x\| = \|x\|$. The map $\varphi : X \to X^{**}$ defined by $\phi(x) = f_x$ is one-one, isometric linear map of X into X^{**} and is called the duality mapping. The duality mapping is the natural embedding (of X into X^{**}).*

Definition 1.3.16 If the duality mapping φ maps X onto X^{**}, the second dual of X, then X is said to be reflexive.

Definition 1.3.17 If $(X, \|\cdot\|)$ is a normed linear space, then the weak topology on X is the smallest topology on X with respect to which all the functionals of X^* are continuous. The weak * topology on X^* is the smallest topology on X^* such that $\varphi(x)$ $(= f_x)$, φ being the natural embedding of X into X^{**} is continuous.

Theorem 1.3.18 (Alaoglu) *The unit sphere S^* in X^* is compact in the weak * topology on X^*.*

Theorem 1.3.19 *A Banach space is reflexive if and only if the closed unit sphere $S = \{x \in X : \|x\| \leq 1\}$ is compact in the weak topology.*

The following three theorems are basic to Functional Analysis.

Theorem 1.3.20 (Open Mapping Theorem) *For $i = 1, 2$, let $(X_i, \|\cdot\|_i)$ be Banach spaces. If $T : X_1 \to X_2$ is a continuous linear operator mapping X_1 onto X_2, then T is an open mapping (i.e. a function for which the image of any open set is open). Consequently a continuous linear bijection of X_1 onto X_2 is a linear homeomorphism.*

Theorem 1.3.21 (Closed Graph Theorem) *For $i = 1, 2$, let $(X_i, \|\cdot\|_i)$ be Banach spaces and $T : X_1 \to X_2$ a linear operator. T is continuous if and only if the graph of $T = \{(x, Tx) : x \in X_1\}$ is a closed subset of the product topological space $X_1 \times X_2$.*

Theorem 1.3.22 (Banach–Steinhaus theorem) *Let $T_\lambda : X_1 \to X_2$, $\lambda \in \Lambda \neq \phi$ be continuous linear operators mapping a Banach space X_1 into a normed linear space X_2 such that for each $x \in X_1$, $\{\|T_\lambda(x)\| : \lambda \in \Lambda\}$ is a bounded set of real numbers. Then $\{\|T_\lambda\| : \lambda \in \Lambda\}$ is bounded.*

Theorem 1.3.23 *A normed linear space is finite-dimensional if and only if the unit sphere is compact.*

Theorem 1.3.24 *If* $\|\cdot\|_i$, $i = 1, 2$ *are two norms on a finite-dimensional normed linear space then* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *are equivalent in the sense that there exist two positive numbers* K_1 *and* K_2 *such that*

$$K_1\|x\|_1 \leq \|x_2\| \leq K_2\|x\|_1 \text{ for all } x \in X.$$

Consequently a finite-dimensional normed linear space over $\mathbb{R}$ *or* $\mathbb{C}$ *is equivalent to* $\mathbb{R}^n$ *or* $\mathbb{C}^n$ *with the norm given by* $\|(x_1, \ldots, x_n)\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$ *and is a Banach space.*

Among normed linear spaces, inner product spaces have rich geometric properties. Many features of the Euclidean spaces carry over to inner product spaces. Parseval identity for orthogonal functions has a crisp functional analytic formulation.

Definition 1.3.25 A linear space $(V, +, \cdot)$ over $F = \mathbb{R}$ or $\mathbb{C}$ is called an inner product space, if there is a map $< >: V \times V \to F$ called an inner product satisfying the following conditions:

(i) $< \alpha x + \beta y, z >= \alpha < x, z > +\beta < y, z >$ for all $x, y, z \in V$ and $\alpha, \beta \in F$;
(ii) $< x, y > = < \overline{y, x} >$ for all $x, y \in V$, $\overline{z}$ being the complex conjugate of $z \in \mathbb{C}$;
(iii) $< x, x > \geq 0$ for all $x \in V$ and $< x, x > = 0$ if and only if $x = \theta$.

Proposition 1.3.26 *An inner product space is a normed linear space with the norm* $\|\cdot\|$ *defined by* $\|x\| = \sqrt{< x, x >}$ *(the positive square root of* $< x, x >$*), as* $| < x, y > | \leq \|x\|\|y\|$ *for all* x, y *in* V *(Schwarz inequality).*

Definition 1.3.27 A Hilbert space is an inner product space which is complete in the norm induced by the inner product.

Example 1.3.28 (i) $\mathbb{R}^n$ or $\mathbb{C}^n$ is a Hilbert space in the norm induced by the inner product defined by $< x, y >= \sum_{i=1}^{n} x_i y_i \left(\sum_{i=1}^{n} x_i \overline{y}_i\right)$ for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ $(\mathbb{C}^n)$.

(ii) ℓ_2, the space of complex sequences (z_n) with $\sum_{n=1}^{\infty} |z_n|^2 < +\infty$ is a Hilbert space with the inner product defined by $< x, y >= \sum_{n=1}^{\infty} x_n \overline{y}_n$ for $x = (x_n)$, $y = (y_n) \in \ell_2$.

(iii) $C_{\mathbb{C}}[a, b]$, the linear space of all continuous complex-valued functions on $[a, b]$ is an inner product space under the inner product $< f, g >= \int_a^b f(x)\overline{g(x)}dx$. This is not a Hilbert space as the space is not complete in the induced norm $\|f\| = \left(\int_a^b |f(x)|^2 dx\right)^{\frac{1}{2}}$. However, its completion is $L_2[a, b]$, the Hilbert space of Lebesgue measurable complex functions which are square-integrable with respect to the Lebesgue measure.

Theorem 1.3.29 *Every incomplete inner product space can be isometrically embedded as a dense subspace of a Hilbert space.*

Theorem 1.3.30 *A normed linear space $(X, \| \ \|)$ is an inner product space if and only if the following parallelogram law is valid:*

$$\text{for } x, y \in X, \|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Definition 1.3.31 A subset C of a linear space V over $F = \mathbb{R}$ or $\mathbb{C}$ is called convex if $tx + (1-t)y \in C$ for all $t \in [0, 1]$, whenever $x, y \in C$.

Theorem 1.3.32 *A non-empty closed convex subset of a Hilbert space contains a unique element with least norm.*

Definition 1.3.33 Let $(V, <, >)$ be an inner product space over $\mathbb{R}$ or $\mathbb{C}$. $x \in V$ is said to be orthogonal to $y \in V$ if $< x, y > = 0$ and we write $x \perp y$ (or $y \perp x$). For $S \subseteq V, S^{\perp} = \{v \in V : v \perp s, \text{for all } s \in \mathcal{S}\}$. $S^{\perp}$ is called the orthogonal complement of S.

Theorem 1.3.34 *If M is a proper closed linear subspace of a Hilbert space H, then*

(i) *$M^{\perp}$ is a closed linear subspace of H;*
(ii) *$M \cap M^{\perp} = \{\theta\}$;*
(iii) *each $h \in H$ can be written uniquely as $h = m_1 + m_2$, where $m_1 \in M, m_2 \in M^{\perp}$ and $\|h\|^2 = \|m_1\|^2 + \|m_2\|^2$.*
(In this case, we write $H = M \oplus M^{\perp}$ and call H the direct sum of M and its orthogonal complement $M^{\perp}$).

Definition 1.3.35 A non-empty set S of a Hilbert space H is called orthogonal if $< x, y > = 0$ whenever $x, y \in S$ and $x \neq y$. S is called orthonormal if each element of S has unit norm and S is orthogonal.

Theorem 1.3.36 *If $S = \{e_\lambda : \lambda \in \Lambda \neq \phi\}$ is an orthonormal set in a Hilbert space H and if $x \in H$, then $\{e_\lambda :< x, e_\lambda > \neq 0\}$ is either empty or countable. Also $\sum_{\lambda \in \Lambda} | < x, e_\lambda > |^2 \leq \|x\|^2$. Further, a nonzero Hilbert space has a maximal orthonormal set of vectors, called an orthonormal basis. If $\{e_\lambda : \lambda \in \Lambda \neq \phi\}$ is an orthonormal basis for H and $x \in H$, then $x = \sum_{\lambda \in \Delta} a_\lambda e_\lambda$, where $a_\lambda =< x, e_\lambda >$.*

Theorem 1.3.37 (Parseval identity) *If $\{e_\lambda : \lambda \in \Lambda \neq \phi\}$ is an orthonormal basis of H then $\|x\|^2 = \sum_{\lambda \in \Lambda} | < x, e_\lambda > |^2$.*

Example 1.3.38 (i) $L_2[0, 2\pi]$, the space of complex-valued Lebesgue measurable functions f on $[0, 2\pi]$ which are square-integrable in the sense that $\int_0^{2\pi} |f(x)|^2 dx < +\infty$ is a Hilbert space under the inner product $< f, g >= \int_0^{2\pi} f(x)\overline{g}(x)dx$. The set $\left\{\frac{e^{inx}}{\sqrt{2\pi}} : n = 0, \pm 1, \pm 2, \ldots\right\}$ is an orthonormal basis.

(ii) $L_2(\mathbb{R})$, the space of all Lebegue-measurable functions for which $\int_{-\infty}^{\infty} |f^2(x)|dx$ is finite, is also a Hilbert space under the inner product $< f, g >= \int_{-\infty}^{\infty} f(x) \overline{g}(x)dx$. $\{x^n e^{-\frac{x^2}{2}}, n = 0, 1, 2, \ldots\}$ gives rise to an orthonormal basis for $L_2(\mathbb{R})$ via the Gram–Schmidt orthogonalization process (see Simmons [13]).

Among normed linear spaces, strictly convex spaces and the more specialized uniformly convex spaces resemble the Euclidean spaces geometrically.

Definition 1.3.39 A normed linear space $(N, \|\cdot\|)$ over $F = \mathbb{R}$ or $\mathbb{C}$ is said to be strictly convex if for $x, y \in N$ with $\|x\| = \|y\| = 1$ and $x \neq y$, $\left\|\frac{x+y}{2}\right\| < 1$.

Definition 1.3.40 A normed linear space $(N, \|\cdot\|)$ is called uniformly convex if there exists an increasing positive function $\delta : (0, 2] \to (0, 1]$ such that for $x, y \in N$, $\|x\|, \|y\| \leq r$ and $\|x - y\| \geq \epsilon r$ imply that $\left\|\frac{x+y}{2}\right\| < (1 - \delta(\epsilon))r$.

Remark 1.3.41 The above definition is equivalent to the requirement that for $\|x_n\|, \|y_n\| \leq 1$ and $\|x_n + y_n\| \to 2$, $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Clearly, every Hilbert space is uniformly convex. Also $L_p[0, 1]$ for $p \geq 2$ is uniformly convex. While every uniformly convex space is strictly convex, $C[0, 1]$ is not even strictly convex.

Hilbert spaces are isometric to their duals, in view of the following.

Theorem 1.3.42 (Riesz Representation Theorem) *Let H be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$ and $f \in H^*$, the dual of H. Then there exists a unique element $y_f \in H$ such that $f(x) =< x, y_f >$ for each $x \in H$ and $\|f\| = \|y_f\|$.*

For $f_y \in H^*$ defined by $f_y(x) = < x, y >$ the correspondence $T_y = f_y$ maps H onto H^* so that $\|T(y)\| = \|y\|$, $T(y_1 + y_2) = Ty_1 + Ty_2$ and $T(\alpha y) = \overline{\alpha}T(y)$ for all $y \in H$.

Theorem 1.3.43 *Every Hilbert space is reflexive.*

In view of the above theorems for a bounded linear operator $T : H \to H$, H being a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$, there is a unique bounded linear operator $T^* : H \to H$ such that $< Tx, y > = < x, T^*y >$ for all $x, y \in H$.

Definition 1.3.44 Let H be a Hilbert space and $T : H \to H$ a bounded linear operator. A linear operator $T^* : H \to H$ satisfying, $< Tx, y > = < x, T^*y >$ for all $x, y \in H$ is called an adjoint operator of T.

Theorem 1.3.45 *If $T \in B(H)$, the space of all bounded linear operators mapping H into itself, then T^* the adjoint of T is uniquely defined. Further,*

(i) $(T_1 + T_2)^* = T_1^* + T_2^*$,
(ii) $(\alpha T)^* = \overline{\alpha}T^*$,
(iii) $(T_1T_2)^* = T_2^*T_1^*$,
(iv) $(T^*)^* = T^*$,
(v) $\|T^*\|^2 = \|T\|^2 = \|T^*T\|$.

Definition 1.3.46 A linear operator $T \in B(H)$, the space of all bounded linear operators on a Hilbert space H is said to be self-adjoint if $T = T^*$.

Definition 1.3.47 For $T \in B(H)$, the spectrum of T is the set $\{\lambda \in \mathbb{C} : T - \lambda I$ is not invertible$\}$, I being the identity operator. An eigenvalue of T is a number $\lambda \in \mathbb{C}$ such that there exists a nonzero vector $x_0 \in H$ with $Tx_0 = \lambda x_0$ and in this case x_0 is called an eigenvector (corresponding to the eigenvalue λ).

Theorem 1.3.48 *For $T \in B(H)$, the space of all bounded linear operators on a Hilbert space H is self-adjoint if and only if $< Tx, x >$ is real for all $x \in H$. So the eigenvalues of a self-adjoint operator are real. Further $\sigma(T)$, the spectrum of T lies in $[m, M]$, where $m = \inf\{< Tx, x >: x \in H$ and $\|x\| = 1\}$ and $M = \sup\{< Tx, x >: x \in H$ and $\|x\| = 1\}$. Also, $m, M \in \sigma(T)$.*

Definition 1.3.49 A linear operator P in $B(H)$ is called a projection if P is self-adjoint and $P^2 = P$.

Remark 1.3.50 If P is a projection on a Hilbert space H, then $P = M \oplus M^\perp$ where $M = \{Px : x \in H\}$, the range of P and $M^\perp$, the range of $I – P$. Further, every representation of H as the orthogonal sum $M + M^\perp$ defines a unique projection of H onto M.

Theorem 1.3.51 *For any self-adjoint operators T in $B(H)$, there is a family $\{P_\lambda : \lambda \in \mathbb{R}\}$ of projections on H satisfying the following conditions:*

(i) *if $TC = CT$ for $C \in B(H)$, then $P_\lambda C = CP_\lambda$ for all $\lambda \in \mathbb{R}$;*
(ii) *$P_\lambda P_\mu = P_\lambda$, if $\lambda < \mu$;*
(iii) *$P_{\lambda-0} = \lim_{\mu \to \lambda - 0} P_\mu = P_\lambda$ (i.e. P_λ is continuous from the left with respect to λ);*
(iv) *$P_\lambda = 0$ if $\lambda \le m$ and $P_\lambda = I$ for $\lambda > M$.*
(Such a family of projections P_λ is called a resolution of identity generated by T).

Theorem 1.3.52 (Spectral theorem) *For every self-adjoint operator $T \in B(H)$ and any $\epsilon > 0$,*

$$T = \int_m^{M+\epsilon} \lambda dP_\lambda$$

where the Stieltjes integral is the limit of (appropriate) integral sums in the operator-norm topology.

Definition 1.3.53 A linear operator $T : N_1 \to N_2$ where N_1 and N_2 are normed linear spaces is said to be a compact operator if $\overline{T(U)}$ is compact in N_2 for each bounded subset U of N_1.

Theorem 1.3.54 *Let $T : B \to B$ be a compact linear operator on a Banach space B. $\sigma(T)$, the spectrum of T is finite or countably infinite and is contained in $[-\|T\|, \|T\|]$. Every nonzero number in $\sigma(T)$ is an eigenvalue of T. If $\sigma(T)$ is countably infinite, then 0 is the only limit point of $\sigma(T)$.*

1.4 Topological Vector Spaces

It is convenient to recall the definition of a topological group and list some of its properties (see Kelley [7], Rudin [11] and Royden [10]).

Definition 1.4.1 Let $(G, \cdot)$ be a group with the identity element e and for each $x \in G$, x^{-1} denote the inverse of x (with respect to the binary operation $\cdot$). The triple $(G, \cdot, \mathscr{T})$ is called a topological group where $\mathscr{T}$ is a topology on the group G with the binary operation $\cdot$ such that the map $(x, y) \to xy^{-1}$ mapping $G \times G$ into G is continuous. (Here $G \times G$ carries the product topology.)

If $(G, \cdot)$ is a group and $A, B \subseteq G$, we write $A \cdot B = \{a \cdot b : a \in A, b \in B\}$.

Theorem 1.4.2 *Let $(G, \cdot, \mathscr{T})$ be a topological group with the identity e. Then*

(i) *the map $x \to x^{-1}$ mapping G into G and the map $(x, y) \to xy$ mapping $G \times G$ into G are continuous. Conversely if $\mathscr{T}_1$ is a topology on a group $(G, \cdot)$ such that $x \to x^{-1}$ and $(x, y) \to xy$ are continuous on G with the topology $\mathscr{T}_1$, then $(G, \cdot, \mathscr{T}_1)$ is a topological group.*
(ii) *the inversion map i, defined by $i(x) = x^{-1}$ is a homeomorphism of G onto G; for each $a \in G$, $L_a(R_a)$ called the left (right) translation by a, defined by $L_a(x) = ax$ $(R_a(x) = xa)$ are homeomorphisms;*
(iii) *a subset S of G is open if and only if for each $x \in S$, $x^{-1}S$ (or equivalently Sx^{-1}) is a neighbourhood of e;*
(iv) *the family $\mathcal{N}$ of all neighbourhoods of e has the following properties:*

(iv-a) *for $U, V \in \mathcal{N}$, $U \cap V \in \mathcal{N}$;*
(iv-b) *for $U \in \mathcal{N}$, $V.V^{-1} \subseteq U$ for some $V \in \mathcal{N}$;*
(iv-c) *for $U \in \mathcal{N}$ and $x \in G$, $x.U.x^{-1} \in \mathcal{N}$;*

(v) *the closure of a (normal) subgroup of G is a (normal) subgroup of G;*
(vi) *every subgroup G_1 of G with an interior point is both open and closed and G_1 is closed or $\overline{G_1} - G_1$ is dense in G_1;*
(vii) *G is Hausdorff if it is a T_0 space in the sense that for every pair of distinct points, there is a point for which some neighbourhood does not contain the other point.*

A topological vector space can be defined in analogy with a topological group.

Definition 1.4.3 The quadruple $(X, +, \cdot, \mathscr{T})$ where $(X, +, \cdot)$ is a vector space over $F = \mathbb{R}$ or $\mathbb{C}$ and $\mathscr{T}$ is a topology on X is called a topological vector space (linear topological space) if the following assumptions are satisfied:

(i) $(X, \mathscr{T})$ is a T_1-space;
(ii) the function $(x, y) \to x + y$ mapping $X \times X$ into X is continuous and
(iii) the function $(\alpha, x) \to \alpha.x$ mapping $F \times X$ into X is continuous.

Often, we simply say that X is a topological vector space (or t.v.s for short) when the topology $\mathscr{T}$ on X and the vector space operations are clear from the context.

Definition 1.4.4 A subset S of a topological vector space X is said to be bounded for every neighbourhood V of θ in X, there is a real number s such that $\mathcal{S} \subseteq t.V$ for every $t > s$. $S \subseteq X$ is called balanced if $\alpha.S \subseteq S$ for all $\alpha \in F$ with $|\alpha| \leq 1$. $\mathcal{S}$ is called absorbing if $X = \bigcup_{t>0} t.S$.

Theorem 1.4.5 *Let X be a t.v.s. For each $a \in X$ and $\lambda \neq 0 \in F$ define the translation operator T_a and the multiplication operator M_λ by the rules $T_a(x) = x + a$ and $M_\lambda(x) = \lambda.x$ respectively for each $x \in X$. Then, T_a and M_λ are homeomorphism of X onto X.*

Further $G \subseteq X$ is open if and only if $T_a(G)$ is open for each $a \in X$. So the local base at 0 completely determines the local base at any $x \in X$ and hence the topology on X.

Remark 1.4.6 Every normed linear space is a t.v.s.

Definition 1.4.7 A function p mapping a vector space X over $F(=\mathbb{R}$ or $\mathbb{C})$ into F is called a seminorm if

(i) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$ and
(ii) $p(\alpha x) = |\alpha| p(x)$ for all $x \in X$ and all $\alpha \in F$.

A seminorm is a norm if $p(x) \neq 0$ for $x \neq \theta$. A family $\mathcal{P}$ of seminorms is separating if for each $x \neq y$, there is a seminorm $p \in \mathcal{P}$ with $p(x - y) \neq 0$.

Theorem 1.4.8 *If $\mathcal{P}$ is a separating family of seminorms on a vector space V, then $V(p, n) = \{x \in X : p(x) < \frac{1}{n}\}$, $p \in \mathcal{P}$ is a local base of convex sets for a topology $\mathscr{T}$ on X. Thus, $(X, \mathscr{T})$ is locally convex and each p is continuous. Also, E is bounded if and only if $p(E)$ is bounded for each $p \in \mathcal{P}$.*

Definition 1.4.9 For an absorbing subset A of a t.v.s. X, the map $\mu_A : X \to \mathbb{R}$ defined by $\mu_A(x) = \inf\{t > 0 : t^{-1}x \in A\}$ is called the Minkowski functional of A.

Listed below are some of the basic properties and features of a topological vector space.

Theorem 1.4.10 *Let X be a topological vector space*

(i) if $S \subseteq X$, $\overline{S} = \cap\{S + V : V$ is a neighbourhood of $0\}$;
(ii) if $S_1, S_2 \subseteq X$, $\overline{S_1} + \overline{S_2} \subseteq \overline{S_1 + S_2}$;
(iii) if $C \subseteq X$ is convex, so are C^0 and $\overline{C}$;
(iv) if $B \subseteq X$ is balanced, so is $\overline{B}$ and if in addition $0 \in B^0$, B^0 is balanced;
(v) the closure of a bounded set is also bounded;
(vi) every neighbourhood of 0 also contains a balanced neighbourhood of 0 and so X has a balanced local base;
(vii) every convex neighbourhood of 0 contains a balanced convex neighbourhood of 0;
(viii) if V is a neighbourhood of 0 and $r_n \uparrow +\infty$ where $r_1 > 0$, $X = \bigcup_{n=1}^{\infty} r_n V$;

(ix) if V is a bounded neighbourhood of 0 and $\delta_n \downarrow 0$, $\delta_1 > 0$, $\{\delta_n V : n \in \mathbb{N}\}$ is a local base at 0;
(x) if X is first countable, then it is metrizable and the metric is translation invariant;
(xi) if X is locally compact, then X is finite dimensional.

Theorem 1.4.11 *If A is a convex absorbing subset of a vector space X, then*

(i) $\mu_A(x + y) \leq \mu_A(x) + \mu_A(y)$ for all $x, y \in X$;
(ii) $\mu_A(tx) \leq t\mu_A(x)$ for $t \geq 0$;
(iii) μ_A is a seminorm, when A is balanced;
(iv) $B = \{x : \mu_A(x) < 1\} \subseteq A \subseteq C = \{x : \mu_A(x) \leq 1\}$ and $\mu_A = \mu_B = \mu_C$.

Theorem 1.4.12 *If $\mathcal{B}$ is a local base for a t.v.s. (X, J) comprising convex balanced neighbourhood, then $\{\mu_V : V \in \mathcal{B}\}$ is a family of continuous seminorms that are separating (i.e. for $x, y \in X$, then there is a μ_V such that $\mu_V(x) \neq \mu_V(y)$). Further, the topology having a local base generated by these seminorms of the form $\{x : \mu_V(x) < \frac{1}{n}\}$, $V \in \mathcal{B}$, $n \in \mathbb{N}$ coincides with the topology on X.*

Definition 1.4.13 A t.v.s is said to be locally convex if it has a local base of convex sets. It is called an F-space if the topology is generated by complete translation-invariant metric. A locally convex F-space is called a Frechet space.

Theorem 1.4.14 *If $\mathcal{P} = \{p_i : i \in \mathbb{N}\}$ is a countable separating family of seminorms on a vector space X, then the topology on X induced by $\mathcal{P}$ is metrizable and this metric d is given by*

$$d(x, y) = \sum_{i=1}^{\infty} \frac{p_i(x, y)}{2^i(1 + p_i(x, y))}$$

is translation invariant.

Theorem 1.4.15 (Kolmogorov) *A topological vector space is normable if and only if the origin has a convex balanced neighbourhood.*

Example 1.4.16 Let Ω be the union of a sequence of compact sets $K_n \subseteq R^m$ for $n = 1, 2, \ldots$ with $K_n \subseteq K^o_{n+1}$, $n = 1, 2, \ldots$. Define for each $f \in C(\Omega)$, the set of all complex-valued functions on Ω, $p_n(f) = \sup\{|f(x)| : x \in K_n\}$. Then, $\{p_n, n = 1, 2, \ldots\}$ is a separating family of continuous seminorms defining a complete translation-invariant metric on $C(\Omega)$. As the origin has no bounded neighbourhood, $C(\Omega)$ is non-normable. Since $C(\Omega)$ is locally convex, it is a Frechet space.

If Ω is any non-empty open subset $\mathbb{C}$, then $H(\Omega)$, the set of all complex functions analytic on Ω is a closed subspace of $\mathcal{C}(\Omega)$. $H(\Omega)$ too is not normable.

Example 1.4.17 Let Ω be a non-void open set in $\mathbb{R}^n$. A multi-index α is an ordered n-tuple of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$ where α_i are non-negative integers. For each multi-index, the differential operator D^α associated is defined by $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$ whose order is $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and for $|\alpha| = 0$,

$D^\alpha f = f$. A complex-valued function defined on Ω is said to belong to $C^\infty(\Omega)$ if $D^\alpha f \in C(\Omega)$ for every multi-index α. Let $\Omega = \bigcup_{m=1}^{\infty} K_m$ where each K_m is compact and $K_m \subseteq K^0_{m+1}, m = 1, 2, \ldots$. Define the seminorms ϕ_m on $C^\infty(\Omega), m = 1, 2, \ldots,$ by $\phi_m(f) = \sup\{|D^\alpha f(a)| : x \in K_m, |\alpha| \le m\}$. Then, $C^\infty(\Omega)$ is a Frechet space under the topology generated by the seminorms ϕ_m. Although every closed bounded subset of $C^\infty(\Omega)$ is sequentially compact (and hence compact in this case), $C^\infty(\Omega)$ is not locally bounded and hence not normable.

Example 1.4.18 For $0 < p < 1$, let $L_p[0, 1]$ be the linear space of all Lebegue-measurable functions f on $[0, 1]$ for which $\delta(f) = \int_0^1 |f(a)|^p dx < +\infty$. Then d, defined by $d(f, g) = \delta(f - g)$ defines a translation-invariant metric on $L_p[0, 1]$ and this metric is complete. Thus $L_p[0, 1]$ is an F-space. However, it is not locally convex. Indeed $L_p[0, 1]$ is the only open convex set. So, 0 is the only continuous linear functional on $L_p[0, 1]$ for $0 < p < 1$ (See Rudin [12]).

Definition 1.4.19 Let X be a topological vector space. The dual of X, denoted by X^* is the set of all continuous linear functionals on X.

Theorem 1.4.20 *If X is a locally convex t.v.s, then X^* separates points in X.*

Definition 1.4.21 Let K be a non-empty subset of a vector space X. A point $s \in K$ is called an extreme point of K if $s = tx + (1 - t)y$ for $t \in (0, 1)$ for some $x, y \in K$ implies $x = y = s$. The convex hull of a set $E \subseteq X$ is the smallest convex set in X containing E. The closed convex hull of E is the closure of its convex hull.

Theorem 1.4.22 (Krein-Milman [11]) *If X is a topological vector space on which X^* separates points. Every compact convex set in X is the closed convex hull of the set of its extreme points. So in a locally convex t.v.s X every compact convex set in X is the closed convex hull of the set of its extreme points.*

In this context, it is pertinent to recall Riesz Representation theorem (see Rudin [12]).

Theorem 1.4.23 (Riesz-Representation) *Let X be a locally compact T_2 space and L be a positive linear functional on $C_C(X)$ the linear space of all continuous complex-valued functions with compact support and the supremum norm. Then, there exists a σ-algebra $\mathscr{S}$ on X containing all the Borel subsets of X and a unique positive measure μ on $\mathscr{S}$ representing L according to the formula*

$$Lf = \int_X f d\mu \ \ for \ \ f \in C_C(X)$$

with the following properties:

(i) $\mu(K) < +\infty$ *for each compact subset of X;*
(ii) *for each $E \in \mathscr{S}$, $\mu(E) = \inf\{\mu(G) : G \supseteq E$ and G is open in $X\}$;*

(iii) $\mu(E) = \sup\{\mu(K) : K \subseteq E$ *and* K *compact*$\}$ *is true for each open set* E *and for any* $E \in \mathscr{S}$ *with* $\mu(E) < +\infty$*;*
(iv) *for* $E \in \mathscr{S}$ *with* $\mu(E) = 0$, $A \in \mathscr{S}$ *for any* $A \subseteq E$ *and* $\mu(A) = 0$.

When X *is compact,* μ *can be chosen so that* $\mu(X) = 1$*, i.e. a Borel probability measure.*

Remark 1.4.24 In a Frechet space, for the convex hull H of a compact set, $\overline{H}$ is compact and in a finite-dimensional space $\mathbb{R}^n$, H itself is compact. Also if an element x lies in the convex hull and a set $E \subseteq \mathbb{R}^n$, then it lies in the convex hull of a subset of E that contains at most $n + 1$ points.

We now proceed to define vector-valued integrals. Rudin [11] may be consulted for further details.

Definition 1.4.25 Let (Q, J, μ) be a measure space, X a t.v.s for which X^* separates points and $f : Q \to X$ be a function such that Λf is integrable with respect to μ for each $\Lambda \in X^*$ (here $(\Lambda f)(q) = \Lambda(f(q))$ for $q \in Q$). If there exist $y \in X$ such that

$$\Lambda y = \int_Q \Lambda f \, d\mu$$

for each $\Lambda \in X^*$, then we define

$$\int_Q f \, d\mu = y.$$

Theorem 1.4.26 *Let* X *be a t.v.s such that* X^* *separates points and* μ *be a Borel probability measure on a compact Hausdorff space* Q*. If* $f : Q \to X$ *is continuous and if the convex hull* H *of* $f(Q)$ *has compact closure* $\overline{H}$ *in* X*, then the integral*

$$y = \int_Q f \, d\mu$$

exists (as per Definition 1.4.25).

Theorem 1.4.27 *Let* X *be a t.v.s such that* X^* *separates points and* Q*, a compact subset of* X *and* $\overline{H}$*, the closed convex hull of* Q *be compact.*

$y \in \overline{H}$ *if and only if there is a regular Borel probability measure* μ *on* Q *such that*

$$y = \int_Q x \, d\mu.$$

When X is a Banach space we also have

Theorem 1.4.28 *Let* Q *be a compact* T_2 *space,* X *a Banach space,* $f : Q \to X$ *a continuous map and* μ *a positive Borel probability measure on* Q*. Then*

$$\| \int_Q f d\mu \| \leq \int_Q \| f \| d\mu.$$

Indeed vector-valued integrals can also be defined more directly as limits of (integral) sums.

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Chapter 2
Fixed Points of Some Real and Complex Functions

This chapter highlights some fixed point theorems for certain real and complex functions.

2.1 Fixed Points of Continuous Maps on Compact Intervals of $\mathbb{R}$

The following definitions are well-known.

Definition 2.1.1 Let $f, g : X \to Y$ be maps, X and Y being non-empty sets. An element $x_0 \in X$ is called a coincidence point of f and g if $f(x_0) = g(x_0)$. If $f : X \to X$ is a map and if for some $x_0 \in X$, $f(x_0) = x_0$, then x_0 is called a fixed point (fix point) of f. If $f, g : X \to X$ are maps such that for some $x_0 \in X, x = f(x_0) = g(x_0)$, then x_0 is called a common fixed point of f and g.

Definition 2.1.2 Let $f : X \to X$ be a map on a non-void set X. The sequence $\{f^n(x)\}$ called the sequence of f iterates is defined recursively by : $f^0(x) = x$, $f^1(x) = f(x)$, $f^{n+1}(x) = f(f^n(x))$, $n = 0, 1, 2, \ldots,$. This sequence is called a sequence of (f) iterates generated at x. We also call the set $\{f^k(x) : k = 0, 1, 2, \ldots\}$ the orbit of x under f and denote it by $O_f(x)$. $f^m(x)$ is called the mth iterate of f at x.

Definition 2.1.3 For a map $f : X \to X$, $x_0 \in X$ is called a periodic point of period m if $f^m(x_0) = x_0$ and $f^n(x_0) \neq x_0$ for $n < m$.

The classical intermediate value theorem for real functions due to Bolzano is equivalent to Brouwer's fixed point theorem for real functions on intervals of real numbers. In a sense, Bolzano's theorem can be viewed as the harbinger of fixed point theory.

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Theorem 2.1.4 (Bolzano's Intermediate Value Theorem) *If $g : [a, b] \to \mathbb{R}$ is a continuous function then for every real number r between $g(a)$ and $g(b)$, there is an element $c = c(r)$ between a and b such that $g(c) = r$.*

Proof Without loss of generality, we can assume that $g(a) \neq g(b)$. Since g is continuous, $g[a, b]$ is a connected subset of $\mathbb{R}$ containing $g(a)$ and $g(b)$. Since connected subsets of $\mathbb{R}$ are intervals, the interval with $g(a)$ and $g(b)$ as endpoints is in the range of g. Hence if r lies between $g(a)$ and $g(b)$, there is an element $c = c(r)$ between a and b such that $g(c) = r$. □

As an immediate consequence, we have

Theorem 2.1.5 (Brouwer's fixed point theorem in $\mathbb{R}$) *If $f : [a, b] \to [a, b]$ is a continuous function, then f has a fixed point.*

Proof If $f(a) = a$ or $f(b) = b$, then the theorem is true. So without loss of generality we assume that $f(a) \neq a$ and $f(b) \neq b$. Since function $g : [a, b] \to \mathbb{R}$ defined by $g(x) = f(x) - x$ is continuous on $[a, b]$ and $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$ (as $f(a), f(b) \in (a, b)$) by Theorem 2.1.4, there is a point $c \in [a, b]$ such that $g(c) = 0 \in [g(b), g(a)]$. Thus c is a fixed point of f. □

Remark 2.1.6 The above fixed point theorem, a consequence of the intermediate value theorem, is indeed equivalent to this theorem.

Let $g : [a, b] \to \mathbb{R}$ be continuous. Without loss of generality let $g(a) < r < g(b)$. Define the map $f : [-1, 1] \to [-1, 1]$ by

$$f(t) = \rho\left(t - \frac{\{r - g\left(\frac{(1-t)a}{2} + \frac{(1+t)b}{2}\right)}{g(b) - g(a)}\right)$$

where $\rho(x) = -1$ for $x < -1$ and $\rho(x) = 1$ for $x > 1$ and $\rho(x) = x$ for other real numbers. Since g is continuous and ρ is continuous on $\mathbb{R}$, clearly f is continuous and maps $[-1, 1]$ into itself. So by Theorem 2.1.5, f has a fixed point $t_0 \in [-1, 1]$. Further t_0 is neither -1 nor 1 and $-1 < t_0 < 1$. So $t_0 = f(t_0) = t_0 - \left\{\frac{r - g\left\{\frac{(1-t_0)a}{2} + \frac{(1+t_0)b}{2}\right\}}{g(b)-g(a)}\right\}$. Hence $r = g\left(\frac{(1-t_0)a}{2} + \frac{(1+t_0)b}{2}\right)$. In short, g has the intermediate value property.

The following is another useful fixed point theorem.

Theorem 2.1.7 *Let $f : [a, b] \to \mathbb{R}$ be a continuous map such that $f[a, b] \supseteq [a, b]$. Then f has a fixed point.*

Proof Since $f[a, b] \supseteq [a, b]$, $[a, b] = [f(c), f(d)]$ for some interval with end points c and d lying $[a, b]$. If $c \leq d$, then $f(c) \leq a \leq c \leq d \leq b \leq f(d)$. Thus $f(x) - x$ changes sign in $[c, d]$ and hence by Theorem 2.1.4 has a zero, which is a fixed point of f. If $c \geq d$, then $f(d) \leq d \leq c \leq f(c)$. Thus again $f(x) - x$ changes sign in $[d, c]$ and so has a fixed point. □

Remark 2.1.8 Theorem 2.1.4 is not true if the interval is not compact. the map $x \to x+1$ is continuous but has no fixed point in $(-\infty, \infty)$ or $[0, \infty)$. The continuous map $x \to \frac{1+x}{2}$ on $[0, 1)$ has no fixed point in $[0, 1)$. Theorem 2.1.4 fails even if f is continuous everywhere on $[a, b]$ except at a single point. For instance $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} \frac{x}{2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

has no fixed point, and $x = 0$ is the only point of discontinuity of f.

Remark 2.1.9 F_f, the set of fixed points of a continuous map on $[a, b]$ is closed. Indeed $F_f = \{x \in [a, b] : f(x) = x\} = g^{-1}(0)$ where $g : [a, b] \to \mathbb{R}$ is defined by $g(x) = f(x) - x$. Since $\{0\}$ is a closed set and g is continuous $g^{-1}\{0\}$ is a closed subset. So F_f is a closed subset of $[a, b]$ ($[a, b]$ being compact, F_f is also compact).

Remark 2.1.10 Indeed we can prove that for each closed subset F of $[0, 1]$ there is a continuous map $f : [0, 1] \to [0, 1]$ for which F is the set of fixed points of f. For proving this we can, without loss of generality, assume that $0, 1 \in F$. So $[0, 1] - F = G$ is open and is a countable union of disjoint open intervals (a_i, b_i), $i \in \mathbb{N}$. Now we consider the case when this collection is countably infinite, leaving the case of finite collection as an exercise.

For $n \in \mathbb{N}$ define $f_n : [0, 1] \to [0, 1]$ by

$$f_n(x) = \begin{cases} x, & x \in F \cup \bigcup_{i=n}^{\infty} (a_i, b_i), \\ a_i, & \text{if } x \in \left[a_i, \frac{a_i+b_i}{2}\right] \text{ for } i < n, \\ 2x - b_i, & \text{if } x \in \left[\frac{a_i+b_i}{2}, b_i\right] \text{ for } i < n. \end{cases}$$

It can be seen that the sequence of continuous functions (f_n) converges uniformly to a continuous function f for which $f(x) = x$ when $x \in F$ and $f(x) \neq x$ if $x \notin F$. In fact, the result is true for any non-empty closed subset of $\mathbb{R}$.

2.2 Iterates of Real Functions

In this section, some theorems on the behaviour of iterates of real functions are discussed. First, Krasnoselskii's theorem on the convergence of special iterates of non-expansive maps of $[a, b]$, following Bailey's [2] proof using elementary properties of subsequential limits is discussed in detail. Theorems 2.2.6–2.2.8 detail the rates of convergence of iterates of special class of functions and are due to Thron [30].

Theorem 2.2.1 (Krasnoselskii [20], Bailey [2]) *Let $f : I (= [a, b]) \to I$ be a map such that $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in I$. For any $x \in I$, the sequence (x_n) defined recursively by $x_{n+1} = \frac{1}{2}(x_n + f(x_n))$, $n = 1, 2, \ldots,$ converges to some fixed point of f.*

Proof Suppose that (x_n) does not converge to a fixed point. We show that this leads to a contradiction. To this end, the proof is divided into several steps.

Step I. If (x_n) converges to $z \in I$, then (x_{n+1}) also converges to z. As $x_{n+1} = \frac{1}{2}(x_n + f(x_n))$, and f is continuous, x_{n+1} converges to $\frac{f(z)+z}{2}$. So $z = f(z)$.

Step II. No subsequence of (x_n) converges to a fixed point of f. For, if (x_{n_i}) converges to z and $f(z) = z$, then $|z - x_{n_i+1}| \le |z - \frac{1}{2}(x_{n_i} + f(x_{n_i})| \le \frac{1}{2}|z - x_{n_i}| + \frac{1}{2}|f(z) - f(x_{n_i})|$ (as $z = \frac{1}{2}(z + f(z))$) $\le |z - x_{n_i}|$ (since $|f(x) - f(y)| \le |x - y|$). This shows that (x_n) itself converges to z, a fixed point of f, contradicting our assumption that (x_n) does not converge to a fixed point of f.

Step III. Since (x_n) lies in the compact interval $I = [a, b]$, it has a subsequential limit p for which $f(p) > p$. Otherwise for all subsequential limits p of (x_n), $f(p) \le p$. Let z be the infimum of all subsequential limits. Then z itself is a subsequential limit of (x_n). So $f(z) \le z$. If $f(z) < z$, then $f(z) < \frac{1}{2}(f(z) + z) < z$ and $\frac{1}{2}(f(z) + z)$ is a subsequential limit of (x_n) smaller than z, the smallest subsequential limit of (x_n), we get a contradiction, unless $f(z) = z$. But by Step II above, $f(z)$ cannot be z. Thus, there is a subsequential limit p of (x_n) for which $f(p) > p$.

Step IV. By Step II, there exists $\epsilon > 0$ such that $|f(x) - x| \ge \epsilon$ for all subsequential limits x of (x_n). Otherwise, there is a sequence (w_n) of subsequential limits of (x_n) with $|w_n - f(w_n)| < \frac{1}{n}$ for all n. This in turn implies that any subsequential limit of (w_n), which is also a subsequential limit of (x_n) is a fixed point of f, contrary to Step II.

Step V. Let w be the largest subsequential limit of (x_n) such that $f(w) > w$ so $f(w) > Q = \frac{1}{2}(f(w) + w) > w$. Since Q is a subsequential limit exceeding w, $f(Q) < Q$.

By Step IV, there is the least subsequential limit R of (x_n) such that $f(R) < R$ and $w < R < f(w)$ (at least Q satisfies these conditions). Now $f(R) < w$.

Otherwise for $A = \frac{1}{2}[R + f(R)]$, $w < A < R$. If $f(R) \ge w$, then $A = \frac{1}{2}(R + f(R)) \ge \frac{1}{2}(R + w) > \frac{1}{2}(w + w) = w$ and $A = \frac{1}{2}(R + f(R)) < \frac{1}{2}(R + R) = R$. Since A is a subsequential limit greater than w, the largest subsequential limit less than $f(w)$, $f(A) \le A$. As $A < R$ and R is the least subsequential limit with $f(R) < R$, $A \le f(A)$. Hence $A = f(A)$ and this contradicts our assumption that no subsequential limit can be a fixed point of f. Hence $f(R) < w$. Consequently $f(R) < w < R < f(w)$ and $|w - R| = R - w < |f(R) - f(w)| = f(w) - f(R)$. This is a contradiction to the assumption on the map f that $|f(x) - f(y)| \le |x - y|$ for all $x, y \in I$. Hence (x_n) converges to a fixed point of f. □

Remark 2.2.2 However, for any continuous map of I into itself, the sequence of iterates defined in Theorem 2.2.1 may not converge. Let $f : [0, 1] \to [0, 1]$ be defined by

$$f(x) = \begin{cases} \frac{3}{4} & \text{for } 0 \le x \le \frac{1}{4} \\ 3\left(\frac{1}{2} - x\right) & \text{for } \frac{1}{4} < x \le \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} \le x \le 1. \end{cases}$$

Clearly $x = \frac{3}{8}$ is a fixed point of f. For $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}(x_1 + f(x_1) = \frac{1}{2}$, $x_3 = \frac{1}{2}(x_2 + f(x_2)) = \frac{1}{4}$ and so on. This shows that x_n does not converge.

In this context, the following result due to Cohen and Hachigian [10] is pertinent.

Theorem 2.2.3 *Let* $f : [-1, 1] \to [-1, 1]$ *be a continuous map such that* $f(-1) = -1$ *and* $f(1) = 1$. *Then for each* $m = 0, 1, 2, \ldots$, $\|f^{m+1} - I\| \ge \|f^m - I\|$. *Here* I *denotes the identity map and* $\|g\| = sup\{|g(x)| : x \in [-1, 1])$ *for any* $g \in C[-1, 1]$.

Proof If $f \equiv I$, the conclusion is obvious. So suppose that $f \ne I$. Let $F = \{x \in [-1, 1] : f(x) = x\}$. Since F is closed, the complement of F is open and so can be written as a disjoint union of open subintervals S_α of $[-1, 1]$. For $x \in S_\alpha$, $f(x) < x$ or $f(x) > x$. Clearly the conclusion is true for $m = 0$. Suppose the inequality $\|f^{k+1} - I\| \ge \|f^k - I\|$ is true for $k = 1, 2, \ldots, m$. As $[-1, 1]$ is compact and f^m is continuous, there exists p in $[-1, 1]$ such that $|f^m(p) - p| = \|f^m - I\|$.

Suppose without loss of generality $f^m(p) > p$. We claim that $f(p) > p$. Clearly $f(p) \ne p$. If $f(p) < p$, then for $q = f(p)$,

$$\begin{aligned} \|f^{m-1} - I\| &\ge |f^{m-1}(q) - q| = |f^m(p) - q| \\ &= f^m(p) - q \quad (\text{as } q < p < f^m(p)) \\ &> f^m(p) - p = \|f^m - I\|. \end{aligned}$$

As this is a contradiction $f(p) > p$. Let $p \in S_\alpha = (a, b)$. So for $x \in S_\alpha$, $f(x) > x$. As $a, b \notin S_\alpha$, $a = f(a) < p < b = f(b)$. So by the intermediate value property of the continuous function f, there exists $r \in S_\alpha$ with $f(r) = p$. Since $f(x) > x$ in S_α and $r \in S_\alpha$, $f(r) = p > r$. Now

$$\begin{aligned} \|f^{m+1} - I\| &> |f^{m+1}(r) - r| = f^m(p) - r \\ &> f^m(p) - p = \|f^m - I\|. \end{aligned}$$

Thus for f different from I, the identity map

$$\|f^{m+1} - I\| \ge \|f^m - I\|, m = 0, 1, 2, \ldots. \qquad \square$$

Cohen and Hachigian [10] have constructed an example of a continuous self-map on the closed unit disc for which every point on the unit circle is a fixed point, with the property that $\|I - f\| > \|I - f^k\|$ for some iterate f^k of f.

For special real functions Thron [30] had obtained some interesting results on the rates of convergence of iterates. Some of these are relevant to the solution of Schroder's functional equation. They provide useful estimates in approximating fixed points by iterates.

Definition 2.2.4 A map $g : \mathbb{R} \to \mathbb{R}$ is said to belong to the class $H(a_1, k)$ if for some $x_0 > 0$, $0 < g(x) < x$ for $x \in (0, x_0]$ and $g(x) = a_1 x + x^{k+1} h(x)$ for $x \in [0, x_0]$ where $0 \leq a_1 \leq 1$, k is a positive number and k is a continuous function on $[0, x_0]$ with $|h(x)| < M$ in $[0, x_0]$.

Remark 2.2.5 Clearly for $g \in H(a, k)$, 0 is the unique fixed point of g and every sequence (x_n) of g-iterates defined by $x_{n+1} = g(x_n)$, $n \in \mathbb{N}$ and $x_1 \in (0, x_0]$ converges to 0.

Theorem 2.2.6 *Let $g \in H(a_1, k)$ where $0 < a_1 < 1$. Then for the sequence (x_n) of g-iterates, there exists a constant $K_1(g, x)$ such that*

$$\lim_{n\to\infty} \frac{x_n}{a_1^n} = K_1$$

Proof From the definition of g and x_{n+1}

$$\frac{x_{n+1}}{x_n} = \frac{a_1 x_n + x_n^{k+1} h(x_n)}{x_n} = a_1 + x_n^k h(x_n)$$

As (x_n) decreases to zero, there exists $x_0 \in N$ such that for $x \geq n_0$

$$0 < x_n^k M < \frac{1 - a_1}{2}$$

Sor $\frac{x_{n+1}}{x_n} < \frac{1+a_1}{2} < 1$. Hence $\sum x_n$ and $\sum x_n^k h(x_n)$ converge. So, the infinite product $\prod_{n=1}^{\infty} \left(1 + \frac{x_n^k h(x_n)}{a_1}\right)$ converges to a number L (say). Writing $u_n = \frac{x_n}{a_1^n}$ it follows that $\frac{u_{n+1}}{u_n} = \frac{x_{n+1}}{a_1 x_n} = \left(1 + \frac{x_n^k h(x_n)}{a}\right)$.

Since $u_{n+1} = u_1 \prod_{m=1}^{n} \left(1 + \frac{x_m^k h(x_m)}{a_1}\right)$, u_{n+1} converges to $u_1 L$. Hence $u_n = \frac{x_n}{a_1^n}$ converges to $u_1 L$ $(= K_1(g, x_1))$. □

Theorem 2.2.7 *If $g \in H(a_1, k)$ for $a_1 = 0$ and (x_n) is the sequence of iterates generated at $x_1 \in (0, x_0]$, then there is a constant $K_2(g, x_1)$ with $0 < K_2 < 1$ such that $0 < x_n < K_2^{(k+1)^n}$ for all n after some stage. If additionally $\liminf_{x\to 0} h(x) > 0$, then for some $K_3(g, x_1)$ with $0 < K_3 < 1$, $\lim_{x\to\infty} x_n^{(k+1)^{-n}} = K_3$.*

Proof Since $a_1 = 0$ and $x_{n+1} = x_n^{k+1} h(x_n)$, $\log x_{n+1} = (k + 1) \log x_n + \log h(x_n)$. Define $v_n = (k + 1)^{-n} \log x_n$. We obtain for $n \geq n_0$

$$\begin{aligned} v_{n+1} &= v_n + (k + 1)^{-(n+1)} \log h(x_n) \\ &= v_{n_0} + \sum_{m=n_0}^{n} (k + 1)^{-(m+1)} \log h(x_m). \end{aligned} \tag{2.2.1}$$

If $\liminf_{x \to 0} h(x) > 0$, then $\sum_{m=n_0}^{\infty} (k+1)^{-(m+1)} \log h(x_m)$ converges to a number $K_3(g, x_1) - v_{n_0}$, say. So (v_n) converges to $\log K_3$ as $n \to \infty$ or $\lim_{n \to \infty} (x_n)^{(k+1)^{-n}} = K_3$.

Suppose $0 < h(x) < M$ and that $\log h(x_n)$ could approach $-\infty$ so that the series (2.2.1) might not converge. Nevertheless, we have from (2.2.1)

$$\begin{aligned} v_{n+1} &< \sum_{m=n_0}^{n} (k+1)^{-(m+1)} \log M + (k+1)^{-n_0} \log x_{n_0} \\ &= \log M (k+1)^{-(n_0+1)} \left[\frac{1 - (k+1)^{-n+n_0+1}}{1 - (k+1)^{-1}} \right] + (k+1)^{-(n_0+1)} \log x_{n_0}^{k+1} \end{aligned} \tag{2.2.2}$$

If $\log M < 0$, choosing x_0 such that $x_{n_0} < 1$, we get from (2.2.2)

$$v_n < (k+1)^{-(n_0+1)} \log M < 0. \tag{2.2.3}$$

If $\log M \geq 0$, (2.2.2) gives

$$v_n < (k+1)^{-(n_0+1)} \log \left(M^{\frac{1+k}{k}} x_{n_0}^{k+1} \right). \tag{2.2.4}$$

For large n_0, the right-hand side of (2.2.3) or (2.2.4) as the case may be is negative and is set as $\log K_2(g, x_1)$.

Now $v_n < \log K_2$ for $n \geq n_0$. So $0 < x_n < K_2^{(k+1)^n}$. □

Theorem 2.2.8 *Let $g \in H(a_1, k)$ for $a_1 = 1$. Then $B_1 = \liminf_{x \to 0^+} -h(x) \geq 0$, $B_2 = \limsup_{x \to 0^+} -h(x) \leq M$. Given $\epsilon > 0$ for the sequence (x_n) of iterates in $(0, x_0]$ there exists $N(\epsilon, g, x_1)$ so that*

$$x_n > [(B_2 + \epsilon)kn]^{-\frac{1}{k}} \text{ for } n > N.$$

If $B_1 > 0$ and $0 < \epsilon < B_1$, then for some $N'(\epsilon, g, x_1)$

$$x_n < [(B_1 - \epsilon)kn]^{-\frac{1}{k}} \text{ for } n > N'$$

Proof Since $g(x) = x + x^{k+1}h(x)$, $g(x) < x$ and $|h(x)| < M$, $0 \leq -h(x) < M$ for $x \in [0, x_0]$. Hence $B_1 \geq 0$ and $B_2 \leq M$. Writing $-h(x_n) = d_n$, $x_{n+1} = x_n + x_n^{k+1}h(x_n)$ becomes, for $k = 1$

$$x_{n+1} = x_n(1 - x_n d_n)$$

and so

$$\frac{1}{x_{n+1}} = \frac{1}{x_n}\frac{1}{(1 - x_n d_n)}.$$

Choose $n_1(g, x_1, \epsilon)$ so that $x_n d_n < 1, \sum_{m=2}^{\infty} d_n^m x_n^{m-1} < \frac{\epsilon}{3}$ and $B_1 - \frac{\epsilon}{3} < d_n < B_2 + \frac{\epsilon}{3}$.

For $n \geq n_1$

$$\begin{aligned} \frac{1}{x_{n+1}} &= \frac{1}{x_n} + d_n + \sum_{m=2}^{\infty} d_n^m x_n^{m-1} \quad \text{(by Binomial theorem)} \\ &< \frac{1}{x_n} + B_2 + \frac{2\epsilon}{3}. \end{aligned} \tag{2.2.5}$$

So $x_{n_1+m} > \dfrac{1}{m\left(B_2 + \frac{2\epsilon}{3}\right) + \frac{1}{x_{n_1}}}$.

So for $n \geq n_1$

$$\begin{aligned} x_n &> \frac{1}{n\left[\left(1 - \frac{n_1}{n}\right)\left(B_2 + \frac{2\epsilon}{3}\right) + \frac{1}{n x_{n_1}}\right]} \\ &> \frac{1}{n\left[B_2 + \frac{2\epsilon}{3} + \frac{1}{n x_{n_1}}\right]} \end{aligned}$$

Choose $n_1' \geq n_1$ so that $\frac{1}{n x_{n_1}} < \frac{\epsilon}{3}$ for $n \geq n_1'$. So we have for $n \geq n_1'$,

$$x_n > \frac{1}{n(B_2 + \epsilon)}.$$

From (2.2.5) for $n \geq n_1$, we get

$$\frac{1}{x_{n+1}} > \frac{1}{x_n} + B_1 - \frac{\epsilon}{3}.$$

So when $B_1 - \epsilon > 0$, for $n > n_1$

$$\begin{aligned} \frac{1}{x_n} &> \frac{1}{x_{n_1}} + (n - n_1)(B_1 - \epsilon) \text{ or} \\ x_n &< \frac{1}{n\left[\left(1 - \frac{n_1}{n}\right)(B_1 - \epsilon) + (n x_{n_1})^{-1}\right]}. \end{aligned} \tag{2.2.6}$$

Choose $N' > n_1' \geq n_1$, such that for $n > N'$,

$$\left(1-\frac{n_1}{n}\right)\left(B_1-\frac{\epsilon}{3}\right) > B_1-\epsilon.$$

So for $n \geq N'$, we get from (2.2.6)

$$x_n < \frac{1}{n(B_1-\epsilon)}.$$

For the case $k \neq 1$, define $w_n = x_n^k$ then $x_{n+1} = g(x_n) = x_n(1 + x_n^k h(x_n))$. So

$$\begin{aligned} w_{n+1} &= \left[g\left(w_n^{\frac{1}{k}}\right)\right]^k = w_n\left[1 + w_n h\left(w_n^{\frac{1}{k}}\right)\right]^k \\ &= w_n[1 + w_n h_1(w_n)]. \end{aligned}$$

Since $[g(w_n^{\frac{1}{k}})]^k$ is a function of w_n, say g_1, it follows that $g_1(w) \in h_1(1,1)$ for $0 \leq w \leq w_0 = x_0^k$. Also $\liminf_{w\to 0^+} h_1(w) = kB_1$, $\limsup_{w\to 0^+} -h_1(w) = kB_2$. The discussion now reduces the case $k \neq 1$ to the case $k = 1$ for $g_1 \in H(1,1)$. It follows from the previous discussion that for $B_1 > 0$ and $0 < \epsilon < B_1$, there exists $N' \in \mathbb{N}$ such that for $n > N'$

$$x_n < [(B_1-\epsilon)kn]^{-\frac{1}{k}}$$

and for $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for $n > N_0$,

$$x_n > [(B_2+\epsilon)kn]^{-\frac{1}{k}}.$$

□

Remark 2.2.9 Since $g(x) = sin(\frac{x}{2}) \in H(\frac{1}{2}, 2)$ in $(0,1)$, $\lim_{n\to\infty}(2^n sin^n(\dfrac{x}{2}))$ converges for each $x \in (0,1)$ by Theorem 2.2.6.

Remark 2.2.10 Theorem 2.2.7 can be applied to $g(x) = sin(x^{1+\epsilon})$ for any $\epsilon > 0$ in $(0,1)$ to conclude that for any sequence (x_n) of iterates of $sin(x^{1+\epsilon})$, $\lim_{n\to\infty}(x_n^{(1+\epsilon)^{-n}})$ converges.

2.3 Periodic Points of Continuous Real Functions

This section treats Sharkovsky's theorem on the existence of periodic points of continuous self-maps on a compact interval $I \subseteq \mathbb{R}$. Sharkovsky published a fundamental paper [27] on the existence of periodic points of continuous self-maps on compact intervals in 1964, when he was about 27 years old. He introduced a new (total) order on the set of natural numbers, often called Sharkovsky order. Interestingly, if a continuous map has a periodic point of period m, in the compact interval I (which it maps into itself) it has periodic points of all periods 'bigger than' m (with respect to this order). The smallest natural number in this order is 3 and so it turns out that if a continuous function mapping $[a, b]$ into itself has periodic point of period 3, then it has periodic points of all periods. Another implication of Sharkovsky's theorem is that if such a map has an odd periodic point then it has periodic points of all even periods.

The more remarkable feature of Sharkovsky's theorem is that its proof is essentially based on the ingenious applications of the intermediate value theorem. The paper by Li and Yorke [21] in 1975 proving a special case of Sharkovsky's theorem as well as May's paper [22] highlighted the complicated behaviour of iterates of simple functions and brought to limelight Sharkovsky's work. The 'simple proof' of Sharkovsky's theorem presented below is due to Bau-Sen Du [14].

In the following, we assume that $f : I \to I$ is a continuous map, where I is a compact interval in $\mathbb{R}$. The following total ordering in $\mathbb{N}$, the set of natural numbers is called Sharkovsky's ordering $\prec$. $m \prec n$ in the following ordering:
$3 \prec 5 \prec 7 \prec \cdots \prec 2.3 \prec 2.5 \cdots$
$\prec 2^2.3 \prec 2^2.5 \prec 2^2.7 \prec \cdots \prec 2^3.3 \prec 2^3.5 \prec \cdots$
$\prec \cdots \prec 2^n.3 \prec 2^n.5 \prec \cdots$
$\prec \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1$

Sharkovsky's theorem states that if $f : I \to I$ has an m-periodic point then f has an n-periodic point precisely when $m \prec n$.

Lemma 2.3.1 *Let a and b be points of I such that either $f(b) < a < b \leq f(a)$ or $f(b) \leq a < b < f(a)$. Then there exists z, a fixed point of $f < b$, a 2-periodic point y of f with $y < z$ and a point v in (y, z) with $f(v) = b$ and*

$$\max\{f^2(v), y\} < v < z < \min\{f(y), f(v)\}.$$

Further, $f(x) > z$ and $f^2(x) < x$ for $y < x \leq v$.

Proof Whether $f(b) < a < b \leq f(a)$ or $f(b) \leq a < b < f(a)$, $f(x) - x$ changes sign in (a, b) and hence has a zero in (a, b). In other words, f has a fixed point z in (a, b). As $b \leq f(a)$, $a < z < b$, and $f(z) = z$, there exists $v \in [a, z)$ with $f(v) = b$. If $f(x) > z$ when $\min I \leq x \leq v$, let $u = \min I$; otherwise let $u = \max\{x : \min I \leq x \leq v, f(x) = z\}$. Then $f^2(u) \geq u$ and $f(x) > z$ for $u < x \leq v$. Since $f^2(v)(= f(b)) \leq a < v$, f^2 has a fixed point in $[u, v)$ or f has a 2 periodic point in $[u, v)$. If y is the largest 2-periodic point, then $u \leq y < v < z < f(y)$. Since $f^2(v) < v$, $f^2(x) < x$ for each x in $(y, v]$. □

Remark 2.3.2 Let P be a period-m orbit of f with $m \geq 3$. Let p, b $(p < b)$ be points in P such that $f(p) \geq b$ and $f(b) \leq p$. So f has a fixed point in $[p, b]$. Let $a \in [p, b)$ be such that $f(a) = b$. Since $f(b) < a$ $(< b = f(a))$, the hypotheses of Lemma 2.3.1 are satisfied. Also b, as a point in P, has least period m.

Theorem 2.3.3 *If f has a periodic point of least period m with $m \geq 3$ and odd then f has periodic points with least period n for each odd integer $n \geq m$.*

Proof Let P be a periodic orbit of f with period m. By Lemma 2.3.1 and Remark 2.3.2. f has a fixed point z, a 2-periodic point y and a point v with $y < v < z < f(y)$ such that $f(v)$ lies in P and $f(x) > z$ and $f^2(x) < x$ when $y < x \leq v$. Define $p_m = v$. As m is odd and y is a 2-periodic point of f, $f^{m+2}(y) = f(y) > y$ and because $f^2(p_m)(= f^2(v))$ is a period-m point of f, $f^{m+2}(p_m) = f^2(p_m) < p_m$. So $p_{m+2} = \min\{x : y \leq x \leq p_m,\ f^{m+2}(x) = x\}$ is well-defined and is an $(m+2)$ periodic point of f. Since $f^{m+4}(y) = f(y) > y$ and $f^{m+4}(p_{m+2}) = f^2(p_{m+2}) < p_{m+2}$ (and it be noted that $f^2(p_{m+2})$ cannot be p_{m+2}). So $p_{m+4} = \min\{x : y \leq x \leq p_{m+2},\ f^{m+4}(x) = x\}$ exists and is a periodic point of f with period $(m+4)$. Thus proceeding, we obtain a decreasing sequence of points $p_m, p_{m+2}, \ldots, p_{m+2k}, \ldots$ with

$$y < \cdots < p_{m+2k+2} < p_{m+2k} < \cdots < p_{m+2} < p_m = v$$

such that p_{m+2k} is a periodic point of f with period $m + 2k$ $(k = 1, 2, \ldots)$. □

Theorem 2.3.4 *If f has a periodic point of least period m with $m \geq 3$ and odd, then f has periodic points of all even periods. Further, there exist disjoint closed subintervals I_0 and I_1 of I such that $f^2(I_0) \cap f^2(I_1) \supseteq I_0 \cup I_1$.*

Proof Let P be an m-orbit of P. By Lemma 2.3.1 and Remark 2.3.2, there is a fixed point z of f, a 2-periodic point y of f and a point v such that $f(v) = b \in P$,

$$\max\{f^2(v), y\} < v < z < b = f(v) = f^{m+1}(v)$$

and $f^2(x) < x$ and $f(x) > z$ for x in $(y, v]$. Write $g = f^2$ and let $z_0 = \min\{t : v \leq t \leq z, g(t) = t\}$. Then y and z_0 are fixed points of g such that $y < v < z_0 \leq z < b = g^{\frac{m+1}{2}}(v)$. Also $g(x) < x$ and $f(x) > z$ for $y < x < z_0$. If $g(x) < z_0$ for $\min I \leq x \leq z_0$, then $g([\min I, z_0]) \subseteq [\min I, z_0]$ and this contradicts that $g^{\frac{m+1}{2}}(v) = b > z_0$. Hence $d = \max\{x : \min I \leq x \leq y, g(x) = z_0\}$ is well defined and $f(x) > z > z_0 > g(x)$ for all x in (d, z_0). Define $s = \min\{g(x) : d \leq x \leq z_0\}$. If $s \geq d$, then $g([d, z_0]) \subseteq [d, z_0]$. But this contradicts that $g^{\frac{m+1}{2}}(v) = b > z_0$. So $s < d$, $[s, d] \cup [d, z_0]$ are non-overlapping closed subintervals and $f^2[s, d] \cap f^2[d, z_0] \supseteq [s, d] \cup [d, z_0]$. Let $\widehat{g} : [d, z_0] \to [d, z_0]$ be the map defined by $\widehat{g}(x) = \max\{g(x), d\}$. Clearly, $\widehat{g}$ is continuous and onto and let $t = \min\{x : d \leq x \leq z_0, g(x) = d\}$. For each $n \in \mathbb{N}$, define $c_n = \min\{x : d \leq x \leq t, \widehat{g}(x) = x\}$. It is not difficult to note that $d < \cdots < c_4 < c_3 < c_2 < c_1 \leq y$ and that c_n generates an n-period orbit $Q_n \subseteq (d, z_0)$ of $\widehat{g}$. Clearly Q_n is also an n-period orbit of $g = f^2$. Since $x < z_0 \leq z < f(x)$ for x in Q_n, $Q_n \cup f(Q_n)$ is $2n$-period orbit of f. Thus f has periods of all even orders. □

Theorem 2.3.5 (Sharkovsky) *Let $f : I \to I$ be a continuous map, where I is a compact interval of real numbers. Then*

(1) if f has a periodic point of period m and if $m \prec n$ (in the Sharkovsky order), then f has also a periodic point of period n;

(2) for each positive integer n, there exists a continuous map $g : I \to I$ that has a periodic point of period n but no point of period $m \prec n$;

(3) *there is a continuous map* $h : I \to I$ *having a* 2^i*-periodic point for* $0, 1, 2, \ldots,$ *and has no other periodic point.*

Proof If f has j-periodic point with $j \geq 3$ and odd, then by Theorem 2.3.3 f has $(j+2)$ periodic point and by Theorem 2.3.4, f has a periodic point of period (2.3). If f has $(2.j)$ periodic point with $j \geq 3$, and odd, f^2 has j-periodic point. So by Theorem 2.3.3, f^2 has $(j+2)$ periodic point and so f has either $(j+2)$ periodic point or period $2(j+2)$ points. If f has $(j+2)$ periodic point, then by Theorem 2.3.4, f has $2(j+2)$ periodic point. In any case f has $2(j+2)$ periodic point. If f^2 has j-periodic point, by Theorem 2.3.4, f^2 has 2.3 periodic point. So f has $(2^2.3)$ periodic point. So if f has $2^k.j$ periodic point, $j \geq 3$ and odd and if $k \geq 2$, then $f^{2^{k-1}}$ has period $2.j$ points. So from what we have proved, we see that $f^{2^{k-1}}$ has period $2(j+2)$ points and period $2^2.3$ points. It follows that f has period $(2^k.(j+2))$ points and period $(2^{k+1}.3)$ points, with $j \geq 3$. If f has $(2^i.j)$ periodic points, $j \geq 3$ and odd and if $i \geq 0$, then f^{2^i} has j-periodic point. For $\ell \geq i$ $f^{2^\ell} = (f^{2^i})^{2^{\ell-i}}$ has period j points. So by Lemma 2.3.1, f^{2^ℓ} has period 2 points. So f has period $2^{\ell+1}$ points for $\ell \geq i$. Finally when f has 2^k-periodic points for some $k \geq 2$, then $f^{2^{k-2}}$ has 4 periodic point. Again by Lemma 2.3.1 $f^{2^{k-2}}$ has 2 periodic points implying that f has 2^{k-1} periodic points. Hence (1) is true.

For proving (2) and (3), without loss of generality, we can assume that $I = [0, 1]$ and $T(x) = 1 - |2x - 1|$, a map with a triangular graph having vertices at $(0, 0)$, $(\frac{1}{2}, 1)$ and $(1, 0)$. Then for each $n \in \mathbb{N}$, $T^n(x) = x$ has exactly 2^n distinct solutions in I. So T has finitely many n-periodic orbits. Among these let P_n be an orbit of the least diameter ($= \max P_n - \min P_n$). Define T_n on I by $T_n(x) = \max P_n$, if $T(x) \geq \max P_n$, $T_n(x) = \min P_n$, if $T(x) \leq \min P_n$ and $T_n(x) = T(x)$ for $\min P_n \leq T(x) \leq \max P_n$. Clearly T_x is continuous on I and T_x has exactly one-period n orbit, i.e. P_n but has no m-periodic orbit for any $m \prec n$.

Let Q_3 be any 3-periodic orbit of T of minimal diameter. Then $[\min Q_3, \max Q_3]$ contains finitely many 6-periodic orbits of T. If Q_6 is one with smallest diameter, then $[\min Q_6, \max Q_6]$ contains finitely many 12-periodic orbits of T. We choose one, say Q_{12} of minimal diameter and continue this process inductively. Define $q_0 = \sup\{\min Q_{2^i.3} : i \geq 0\}$ and $q_1 = \inf\{\max Q_{2^i.3} : i \geq 0\}$. Define $T' : I \to I$ by
$$T'(x) = \begin{cases} q_0 & \text{if } T(x) \leq q_0 \\ q_1 & \text{if } T(x) \geq q_1 \\ T(x) & \text{if } q_0 \leq T(x) \leq q_1 \end{cases}$$
. Clearly T' is continuous and has 2^i-periodic point for $i = 0, 1, 2, \ldots$ but has no other periodic point. Thus (2) and (3) are true. □

Remark 2.3.6 Lemma 2.3.1 has interesting consequences. Let $x_0 \in I$ and $n \geq 2$ be a natural number such that $f^n(x_0) < x_0 < f(x_0)$. Let $X = \{f^k(x_0) : 0 \leq k \leq n-1\}$ (a finite set), $a = \max\{x \in X : q_0 \leq x < f(x)\}$, and $b \in \{x \in X : a < x \leq f(a)\}$ with $f(b) < a$. From these conditions on $a, b, x_0, f(x_0)$ and X it is clear that $f(b) < a < b \leq f(a)$. If $f^n(x_0) \leq x_0 < f(x_0)$ and n is odd (> 1) then f has n-periodic points.

If in addition $\overline{O_f(c)}$ contains both a fixed point z and a point different from z, then f has periodic points with all even periods. Arguments similar to those in Theorems 2.3.3 and 2.3.4 can be used.

Remark 2.3.7 Sharkovsky's theorem cannot be generalized to continua (compact connected subsets) of the plane. On the unit disc, the map $z \to ze^{\frac{2\pi i}{3}}$ has 0 as the only fixed point and all the other points are 3-periodic points. For each $n \in \mathbb{N}$, the map $z \to ze^{\frac{2\pi i}{n}}$ has only one fixed point and the rest of the points are n-periodic points. No point of fundamental period greater than n exists.

Sharkovsky's result is definitely and unalterably one-dimensional (See Ciesielski and Pogoda [8].) Nevertheless, there has been appropriate generalization of Sharkovsky's theorem to general topological spaces and more general maps than continuous functions. See Schirmer [25].

2.4 Common Fixed Points, Commutativity and Iterates

It is natural to find out if two continuous real functions $f, g : I(= [a, b]) \to I$ have a common fixed point. The maps $x \to \frac{x}{2}$ and $x \to 1 - x$ on $[0, 1]$ have the only fixed points 0 and $\frac{1}{2}$ respectively. Since their compositions are $\frac{1-x}{2}$ and $1 - \frac{x}{2}$, they do not commute. If $f, g : I \to I$ have a common fixed point x_0, then $x = f(x_0) = g(x_0) = gf(x_0) = fg(x_0)$ and thus f and g commute at least on $\{x_0\}$. Ritt [24] showed that if f and g are polynomials that commute, then they are within certain homeomorphisms iterates of the same function, both power of x or both must be Chebyshev polynomials and in both these cases, the commuting polynomials have a common fixed point. So Dyer conjectured that if $f, g : I(= [a, b]) \to I$ are continuous real functions that commute, then f and g have a common fixed point. However, Boyce [5] and Huneke [17] had disproved the conjecture independently by constructing counter-examples to point out that commuting continuous self-maps on a compact real interval may not have a common fixed point. Isbell [18] first recorded this problem in a more general form.

This section discusses some results that ensure the existence of common fixed points of two commuting continuous functions $f, g : I \to I$ under suitable additional assumptions. We recall the following definitions.

Definition 2.4.1 Let $\mathscr{F}$ be a family of maps from a topological space X into a metric space (X, d). It is said to be equicontinuous at $x_0 \in X$, if for each $\epsilon > 0$, there exists an open set O in X containing x_0 such that for each $x \in O$ and $f \in \mathscr{F}$, $d(f(x_0), f(x)) < \epsilon$. $\mathscr{F}$ is said to be equicontinuous on X, if it is equicontinuous at each $x \in X$.

Definition 2.4.2 If $f : X \to X$ is a map, a subset $A \subseteq X$ is said to be f-invariant or invariant (under f) if $f(A) \subseteq A$.

An elementary proposition on invariant subsets of continuous maps on compact intervals is given below.

Proposition 2.4.3 *If $f : I = [a, b] \to I$ is a continuous map on the compact interval I of real numbers, then every non-empty closed invariant subset C of I contains a minimal closed invariant non-empty subset C'.*

Proof Let C be a non-empty closed invariant subset of I and $\mathscr{C}$ be the family of all closed invariant subsets of C. Clearly $C \subset \mathscr{C}$. Let $\mathscr{F}$ be a chain of sets in $\mathscr{C}$. Since $\mathscr{F}$ is a subfamily of non-empty closed subsets of C which are indeed compact subsets of I, $F_0 = \cap\{F : F \in \mathscr{F}\}$ is non-empty and compact. Further $f(F_0) \subseteq f(F) \subseteq F$ for all $F \in \mathscr{F}$ and hence $f(F_0) \subseteq \cap\{F : F \in \mathscr{F}\} = F_0$. Thus, F_0 is an invariant closed subset which is contained in each $F \in \mathscr{F}$. Thus F_0 is the least element of $\mathscr{F}$ in C. So by Zorn's Lemma, $\mathscr{C}$ has a minimal element C_0, which is a non-empty minimal closed invariant subset of C. □

Remark 2.4.4 Indeed if $f : X \to X$ is a continuous map of a compact connected T_2 space, then every non-empty closed invariant subset A of X contains a minimal closed invariant subset of A.

Proposition 2.4.5 *If Y is a minimal non-empty closed invariant subset of I a compact interval of $\mathbb{R}$, then for $y \in Y$, $Y = \overline{O_f(y)}$ where $O_f(y) = \{f^n(y) : n = 0, 1, 2, \dots\}$ is the orbit of y, under f.*

Proof If $y \in Y$, then $O_f(y) \subseteq Y$ as $f(Y) \subseteq Y$. Since Y is closed, $\overline{O_f(y)} \subseteq Y$. Now by the continuity of f, $\overline{O_f(y)} \subseteq Y$. By the minimality of Y, $Y \subseteq \overline{O_f(y)}$. So $Y = \overline{O_f(y)}$. □

Theorem 2.4.6 (Schwartz [26]) *Every non-void closed invariant minimal subset of the continuous function $f : I \to I$ is contained in the closure of P_f, where $P_f = \{x \in I : f^k(x) = x$ for some $k \in \mathbb{N}\}$, the set of periodic points of f.*

Proof Let Y be a non-empty minimal closed invariant subset of I. If Y is the orbit of a periodic point, obviously it is finite and closed and the conclusion is true.

Suppose Y is not a periodic orbit. Let $c = \inf Y$. As Y is closed, $c \in Y$. As Y is minimal closed invariant subset, by Proposition 2.4.5, $Y = \overline{O_f(c)}$. So given $\epsilon > 0$, we can find $k \in \mathbb{N}$ with $|y - f^k(c)| < \frac{\epsilon}{2}$. Also we can find $M, N \in \mathbb{N}$ such that $c < f^{N+M}(c) < f^N(c) < c + \epsilon'$, as $c = \inf Y = \overline{O_f(c)}$. As Y is minimal and is not a periodic orbit, $f^M(c) > c$. Thus, the continuous map f^M maps $[c, f^N(c)]$ into itself and so has a fixed point d. Since $c < f^M(c) < f^{M+N}(c)$, $d \in (c, f^N(c))$. Thus $f^M(d) = d$ is a periodic point and $|c - d| < f^N(c) - c < \epsilon'$.

As f^k is continuous at c, for $\epsilon > 0$ we can find $\delta > 0$ with $\epsilon > \delta$ such that $|f^k(x) - f^k(c)| < \frac{\epsilon}{2}$ for $|x - c| < \delta$. Since $|y - f^k(d)| \leq |y - f^k(c)| + |f^k(d) - f^k(c)|$, choosing $\epsilon' = \delta$, we see that $|y - f^k(d)| < \epsilon$. As $f^M(d) = d$, it is clear that $z = f^k(d)$ is a periodic point of f which is within ϵ (> 0) distance from y. So $Y \subseteq \overline{P(f)}$. □

Corollary 2.4.7 *If Y is a non-empty minimal closed invariant subset of f then Y is nowhere dense.*

Proof Let x_0 be an interior point of Y. Then for some $\epsilon > 0$, $[x_0 - \epsilon, x_0 + \epsilon] \subseteq Y$. If $[x_0 - \epsilon, x_0 + \epsilon]$ contains a periodic point y of Y, then $O_f(y)$ is finite and is closed. Since $y \in Y$, $Y = \overline{O_f(y)} = O_f(y)$ and this contradicts that Y is uncountable (since it has an interior point). So $[x_0 - \epsilon, x_0 + \epsilon]$ has no periodic point. As $x_0 \in Y$, by Theorem 2.4.6, $[x_0 - \epsilon, x_0 + \epsilon]$ must contain a periodic point, contradicting the preceding assertion. Hence Y is nowhere dense. □

Theorem 2.4.8 (Cano [6]) *Let $\mathscr{F} = \mathscr{F}_1 \cup \mathscr{F}_2$ be a collection of continuous functions mapping a compact interval $I = [a, b] \subseteq \mathbb{R}$ into itself, satisfying the following assumptions:*

(i) for $f \in \mathscr{F}_1$, F_f the set of fixed points of f in I is a compact interval $[a_f, b_f]$;
(ii) for $f \in \mathscr{F}_2$, every periodic point of f is a fixed point of f;
(iii) for $f, g \in \mathscr{F}$, $f(g(x)) = g(f(x))$ for all $x \in I$ (f and g commute).

If $h : I \to I$ is a continuous function that commutes with each $f \in \mathscr{F}$, then $\mathscr{F} \cup \{h\}$ has a common fixed point in I.

Proof Let $C_1 \cup \{h\}$ be any finite subset of $\mathscr{F} \cup \{h\}$ of the form $\{f_1, \ldots, f_n\} \cup \{h\} \cup \{g_1, \ldots, g_m\}$ where f_i, $i = 1, 2, \ldots, n \in \mathscr{F}_1$ and $\{g_1, \ldots, g_m\} \subseteq \mathscr{F}_2$. Since F_{f_i} is a compact interval and f_i's commute $\bigcap_{i=1}^{n} F_{f_i}$ is a non-empty compact interval, say $[c, d]$. As h commutes with each $f_i \in C_1$, h maps $[c, d]$ into itself and so has a fixed point $z \in [c, d]$. Now $g_1^n(z)$ has a limit point z_1 in $\overline{P_{g_1}}$ by Theorem 2.4.6. As $P_{g_1} = F_{g_1}$ (by hypothesis (ii), and F_{g_1} is closed, $\overline{P_{g_1}} = P_{g_1}$. Similarly $g_2^n(z_1)$ has a limit point z_2 in $\overline{P_{g_2}} = F_{g_2} = P_{g_2}$ and as F_{g_2} is closed $z_2 \in F_{g_2}$. Thus $z_1, z_2 \in [c, a]$. Thus proceeding, we see that $\{g_j^n(z_{j-1})\}$ has a limit point z_j in P_{g_j} for $j = 2, \ldots, m$ which is fixed for $f_1, \ldots, f_n, h, g_1, \ldots, g_m$. So $\cap F_f \neq \phi$ for all $f \in C_1 \cup \{h\}$. It is also easily seen that for any finite subset C_2 of $\mathscr{F}_1$, $\bigcap_{f \in C_2} F_f \neq \phi$ as also $\bigcap_{f \in C_3} F_f \neq \phi$ for any finite subset C_3 of $\mathscr{F}_2$. Thus, the family of closed subsets $\{F_f : f \in \mathscr{F} \cup \{h\}\}$ of $[a, b]$ has finite intersection property and hence $\cap\{F_r : f \in \mathscr{F} \cup \{h\}\}$ is non-empty, in view of the compactness of $[a, b]$. □

Theorem 2.4.9 (Cano [6]) *Let $f : I(= [a, b]) \to I$ be a continuous function such that $\{f^n : n \in \mathbb{N}\}$ is an equicontinuous family at each $x \in I$. Then*

(1) F_p, the fixed point set of f is a compact subinterval of I;
(2) if F_f is a non-degenerate interval, then $F_f = P_f$ (P_f being the set of periodic points of f).

Proof As $f : I \to I$ is continuous, $F_f \neq \phi$. If F_f is a singleton, the theorem is true. Suppose $a_0, b_0 \in F_f$ and $a_0 < b_0$. Assume that for no $x \in (a_0, b_0)$, $x_0 = f(x_0)$. Then for all $x \in (a_0, b_0)$, $f(x) > x$ or $f(x) < x$. Assume that $f(x) > x$ for all $x \in (a_0, b_0)$.

Case (i) If $f(x) < b_)$ for all $x \in (a_0, b_0)$ then $f^n(x) \in (a_0, b_0)$ for all $n \in \mathbb{N}$ and $f^n(x) < f^{n+1}(x) < b_0$ and so it converges to a fixed point of f, which cannot be in (a_0, b_0) and hence has to be b_0. So given $\epsilon > 0$, by the equicontinuity of $\{f^n\}$ at a_0, there exists $\delta > 0$ such that $|a_0 - x_0| < \delta$ such that for $|a_0 - x_0| < \delta$, $|f^n(a_0) - f^n(x_0)| < \epsilon$. Since $f^n(a_0) = a_0$, for all n, this contradicts that $f^n(x_0)$ converges to b_0.

Case (ii) Suppose for some $x_0 \in (a_0, b_0)$, $f(x_0) \geq b_0$. Then there is a least number z in (a_0, b_0) with $f(z) \geq b_0$. In fact $f(z) = b_0$. Otherwise, there exists $z' < z$ with $f(z') \geq b_0$ by the continuity of f and this contradicts the definition of z. Thus proceeding, we can find a non-increasing sequence (x_n) in $(a_0, z]$ such that (x_n) converges to a_0, $x_1 = z$ and $f(x_n) = x_{n-1}, n = 2, 3, \ldots$. Since $f^n(x_n) = f^{n-1}(x_{n-1}) = \cdots f(x_1) = f(z) = b_0$ for all n, f^n cannot be equicontinuous at a_0. (Note that as (x_n) is non-increasing in (a_0, z) it converges to a number $z' \geq a_0$. $z' > a_0$ is a contradiction as $z' = f(z')$ and by assumption f has no fixed point in (a_0, b_0).)

Suppose $f(x) < x$ for all $x \in (a_0, b_0)$. We consider

Case (i)′ Suppose $f(x) > a_0$ for all $x \in (a_0, b_0)$. Then for all $x \in (a_0, b_0)$, $f^n(x) > f^{n+1}(x)$, $n \in \mathbb{N}$ and $(f_n(x))$ as in Case (i) converges to a_0. However the family of f iterates cannot be equicontinuous at b_0.

Case (ii)′ If for some $x \in (a_0, b_0)$, $f(x) \leq a_0$. Then there is a greatest element z' in (a_0, b_0) with $f(z') \leq a_0$. In fact $f(z') = a_0$. By this process, a non-decreasing sequence (y_n) can be chosen in $(z', b_0]$ with $y_1 = z'$, $f(y_n) = y_{n-1}$, $n = 2, 3, \ldots$. So $f^n(y_n) = f(z') = a_0$. If (y_n) converges to w, then $f(y_n)$ $(= y_{n-1})$ converges to $f(w)$ and so $w = f(w)$. As $w \notin (a_0, b_0)$, (y_n) converges to b_0. Since $f^n(y_n) = f^{n-1}(y_{n-1}) \cdots = f(z') = a_0$. As y_n converges to b_0, there is a contradiction to the equicontinuity of f^n at b_0.

Thus we have shown that F_f is a non-void compact interval. If F_f is non-degenerate let $F_f = [a_0, b_0]$ where $a_0 < b_0$. Let $f^n(x) = x$ for some n and $x \in [a, a_0)$. (If $x \in (b_0, b]$, then a similar argument can be provided). Since f^n has a fixed point and its iterates are equicontinuous at each point, $f^n(y) = y$ for all $y \in [x, a_0]$ by what has been proved in (i) so far. Since $f(y) > y$ for all $y \in [a, a_0)$ and $f(a_0) = a_0$, we can choose y from (x, a_0) close to a_0, such that $a_0 - \frac{1}{k} < y < f(y) \cdots < f^{n-1}(y) < a$ and this implies $f^n(y) > y$, a contradiction. So $a_0 + \frac{1}{k} > f(y) > a_0 > y > a_0 - \frac{1}{k}$. Then $f(y)$ is a fixed point for f. So $f(y) = f^2(y)$ and $f^n(y) = f^{n-2}(f^2(y)) = f^{n-1}(y)$. Thus proceeding, $y = f^n(y) = f^{n-1}(b) \cdots = f(y)$ contradicting $f(y) > a > y$. Thus if $F_f = [a_0, b_0]$, $[a_0, a)$ has no periodic point. Similarly $(b_0, b]$ has no periodic point. □

This leads to the following.

Theorem 2.4.10 (Jachymski [19]) *Let $g : I \to I$ be a continuous map and I, a compact interval $[a, b]$ of real numbers. Then the following are equivalent:*

(i) F_g the set of fixed points of g is a compact subinterval of I;

(ii) either F_g is a singleton or the family $\{g^n : n \in \mathbb{N}\}$ of iterates is equicontinuous on F_g;

(iii) g has a common fixed point with each continuous map $f : I \to I$ that commutes with g on F_g.

Proof (i) $\implies$ (ii). Suppose F_g is not a singleton and is $[a_1, b_1]$ where $a_1 < b_1$. Since for $a_1 < x < b_1$, $g^n(x) = x$ for all $n \in \mathbb{N}$, the continuity of g at x implies that given $\epsilon > 0$ with $b - a > \epsilon$, there is a $\delta(\epsilon) > 0$ such that $(x - \delta, x + \delta) \subseteq (a_1, b_1)$ and $|g(x) - g(x')| < \epsilon$ for $x' \in (x - \delta, x + \delta)$. So $|g^n(x) - g^n(x')| = |g(x) - g(x')| < \epsilon$ for $x' \in (x - \delta, x + \delta)$, proving the equicontinuity of $\{g^n\}$ on (a_1, b_1). We now show that $\{g^n\}$ is equicontinuous at a_1. Since g is continuous at a_1, there exists $\delta(\epsilon) > 0$ with $\epsilon > \delta(\epsilon)$ for a given $\epsilon > 0$ such that for $a_1 - \delta < x < a_1 + \delta$, $|g(x) - g(a_1)| = |g(x) - a_1| < \epsilon$. We now show by the principle of finite induction that $a_1 - \epsilon < g^n(x) < a_1 + \epsilon$ for all $x \in (a_1 - \delta, a_1 + \delta)$ for all $n \in \mathbb{N}$. Clearly, the inequality is true for $n = 1$. Suppose it is true for $n = 1, 2, \dots, k$. Let $x \in (a - \delta, a)$. If $a_1 \le g^k(x) < a_1 + \epsilon$, then $g^k(x) \in F_g$ and so $|g^{k+1}(x) - a_1| = |g^{k+1}(x) - g^{k+1}(a_1)| = |g^k(x) - g^k(a_1)| = |g^k(x) - a_1| < \epsilon$. If $g^k(x) < a_1$, then $g^i(x) < a_1$ for $i = 1, 2, \dots, k$. Otherwise by induction hypothesis for some i, $1 \le i \le k$ and $a_1 \le g^i(a) < a_1 + \epsilon$ or $g^i(x) \in F_g$ and so $g^k(x) \in F_g$ or $g^k(x) \ge a_1$, a contradiction. Since $F_g = [a_1, b_1]$, $g(x) > x$ for $x \in [a, a_1)$. So $g^i(x) > g^{i-1}(x)$ for $i = 1, 2, \dots, k$, implying that $g^k(x) > g^{k-1}(x) > \cdots > x$. As $a_1 - \delta < x$ and $g^k(x) < a_1$, it follows that $g^k(x) \in (a_1 - \delta, a_1)$. So $|g(g^k(x)) - g(a_1)| = |g^{k+1}(x) - a_1| < \epsilon$. For $x \in (a_1, a_1 + \delta) \subseteq [a_1, b_1]$, $|g^n(x) - g^n(a_1)| = |x - a_1| < \epsilon$. Thus g^n is equicontinuous at a_1. By a similar reasoning, (g^n) is equicontinuous at b_1.

(ii) $\implies$ (i). This follows from the proof of Theorem 2.4.9 (i). In fact to prove (i) of Theorem 2.4.9, it suffices to assume that $\{f^n\}$ is equicontinuous on F_f.

(i) $\implies$ (iii). If f commutes with g on F_g then F_g is invariant under f. Since F_g is a compact interval by (i), f has a fixed point in F_g which is a common fixed point of f and g.

(iii) $\implies$ (i). If F_g is not an interval, then there exists $a_1, b_1 \in F_g$ such that $(a_1, b_1) \cap F_g = \phi$. Define $f : [a, b] \to [a_1, b_1]$ by

$$f(x) = \begin{cases} b_1 & \text{for } x \in [a, a_1] \\ b_1 + a_1 - x & \text{for } x \in (a_1, b_1] \\ a_1 & \text{for } x \in (b_1, b] \end{cases}$$

f is continuous on I. Let $x \in F_g$. Then $x \in [a, a_1]$ or $[b_1, b]$. If $x \in [a, a_1]$, then $fg(x) = f(x) = b_1 = gf(x) = g(b_1)$. If $x \in [b_1, b]$, then $fg(x) = f(x) = a_1 = g(a_1) = gf(x)$. Thus, f and g commute on F_g but $F_f \cap F_g = \phi$. Hence the theorem. □

Example 2.4.11 The continuous map $g : [0, 1] \to [0, 1]$ defined by $g(x) = 1$ on $[0, \frac{1}{4}]$, $\frac{3}{2} - 2x$ for $x \in (\frac{1}{4}, \frac{3}{4}]$ and 0 on $(\frac{3}{4}, 1]$ has the only fixed point $x = \frac{1}{2}$. But $g^n(\frac{1}{2} + \delta) = (-2)^n\delta + \frac{1}{2}$ for $0 < \delta < \frac{1}{4}$, as long as $2^n\delta < \frac{1}{4}$ or $\delta < \frac{1}{2^{n+2}}$. Suppose g is equicontinuous at $x = \frac{1}{2}$. Then for $\epsilon = \frac{1}{4}$, there exists $\delta > 0$ such that $|g^n(1 + \delta) - g^n(\frac{1}{2})| < \epsilon$ for all n. Since $g(\frac{1}{2}) = \frac{1}{2}$ and choosing least n_0 such that $2^{n_0}\delta > \frac{1}{4}$, it follows that $g^n(\frac{1}{2} + \delta) = 0$ for all $n \geq n_0$ and $|g^n(\frac{1}{2} + \delta) - g^n(\frac{1}{2})| = |0 - \frac{1}{2}| = |\frac{1}{2}| \not< \frac{1}{4}$, a contradiction. So (g^n) is not equicontinuous.

If $f : [0, 1] \to [0, 1]$ commutes with g at $\frac{1}{2}$, then $fg(\frac{1}{2}) = g(f(\frac{1}{2})) = f(\frac{1}{2})$ (as $g(\frac{1}{2}) = \frac{1}{2}$). Since $f(\frac{1}{2})$ is a fixed point of g and g has the unique fixed point $\frac{1}{2}$, $f(\frac{1}{2}) = \frac{1}{2}$. Thus, f and g have a common fixed point, even though $\{g^n\}$ is not equicontinuous.

This example points out that the hypothesis F_g is a singleton cannot be dropped in Theorem 2.4.10.

The next theorem on the convergence of iterates, due to Coven and Hedlund [12], was also obtained independently by Chu and Moyer [7].

Theorem 2.4.12 *If $f : I = [a, b] \to I$ is continuous and $P_f = F_f$, then for each $x \in I$, there exists $p \in F_f$ such that $\{f^n(x)\}$ converges to p.*

Proof If $\{f^n(x)\}$ converges to p, it follows from the continuity of f, that $p \in F_f$. Thus it suffices to prove the convergence of $\{f^n(x)\}$ for each $x \in I$. If $f^n(x) \in P_f$ for some $n \geq 0$, the conclusion is obvious. Suppose that $f^n(x)$ is not a periodic point of f for any $n \geq 0$. Let C_n be the component of NP_f, the set of non-periodic points of f in I containing $f_n(x)$. Let $\xi_n = +1$ if f is completely positive on C_n (i.e.) $(f(x) > x$ on $C_n)$ and $\xi_n = -1$ if f is totally negative on C (i.e. $f(x) < x$ on C_n). Since f is continuous and C_n is connected, $f(x) - x$ cannot take both positive and negative values on C_n as C_n has no fixed point.

If for some $N \geq 0$, $\xi_n = +1$ for $n \geq N$, then $f^N(x) < f^{N+1}(x)$ and so $f^n(a)$ converges. Similarly if $\xi_n = -1$ for all $n \geq N_1$, then $\{f^n(x)\}$ converges.

Suppose $+1$ and -1 appear infinitely many times in the sequence (ξ_n), $n \geq 0$. Let $A = \{n \geq 0 : \xi_n = +1\} = \{p_1 < p_2 < \cdots\}$ and $B = \{n \geq 0 : \xi_n = -1\} = \{m_1 < m_2 < \cdots\}$. $\{f^{p_i}(x)\}$ is increasing while $\{f^{m_i}(x)\}$ is decreasing in I and hence these subsequences of $\{f^n(x)\}$ converge to p and q respectively in I. Now we can find a subsequence $k_i \in A$ such that $k_i + 1 \in B$. Since $\{f^{k_i}(x)\}$ converges to p $\{f^{k_i+1}(x)\}$ converges to q and f is continuous $f(p) = q$. By a similar reasoning we find that $f(q) = p$. Thus $f^2(p) = f(q) = p$ and $f^2(q) = f(p) = q$. Thus $p \in P_f = F_f$. So $p = f(p) = q$. Hence the theorem. □

Corollary 2.4.13 *If $f : I = [a, b] \to I$ is continuous and the set of least periods or periodic points is finite, then for each $x \in [a, b]$, there exists $p \in P_f$ such that $|f^n(x) - p|$ converges to zero as $n \to \infty$.*

Proof Let N be the least common period of the periodic points. Apply Theorem 2.4.12 to f^N and that $P_{f^N} = F_{f^N}$. (It is to be observed that N must be a power of 2, as can be seen from Sharkovsky's theorem.) □

Our next theorem characterizes functions $f : I \to I$ that are continuous and for which $P_f = F_f$.

Theorem 2.4.14 (Jachymski [19]) *Let $g : I = [a, b] \to I$ be a continuous function. Then the following are equivalent:*

(i) $F_g = P_g$;
(ii) $\{g^n : n \in \mathbb{N}\}$ *is pointwise convergent on* I;
(iii) g *has a common fixed point with every continuous map* $f : I \to I$ *that commutes with* g *on* F_f.

Proof (i) $\implies$ (ii) is precisely Theorem 2.4.12.

(ii) $\implies$ (iii). Let $x \in F_f$. By the commutativity of f and g on F_f, F_f is g-invariant. So $g^n(x) \in F_f$ for all $n \geq 1$. Since $\{g^n(x)\}$ converges to $z \in I$ by (ii) and F_f is closed $z \in F_f$ and as g is continuous $z = g(z)$. Thus $z = f(z) = g(z)$.

(iii) $\implies$ (i). Let C be a non-empty g-invariant closed subset of I. We show that $C \cap F_g \neq \phi$. For such a set, there is a continuous map $f : I \to I$ such that $F_f = C$. If $x \in F_f$, then $g(f(x)) = g(x)$ and $f(g(x)) = g(x)$, since C is g-invariant. So f and g commute on $F(g)$. By assumption (iii) $F_f \cap F_g = C \cap F_g \neq \phi$. Let p be a periodic point of least period M for g. Then $C = \{p, g(p), \ldots, g^{M-1}(p)\}$ is closed and invariant under g. So from what we have shown, C has a fixed point of g. If for $1 \leq i \leq M-1$, $g(g^i(p)) = g^i(p)$, $g^i(p) = g^{(M)}(p) = p$, contradicting p is a periodic point of g with least period M. So $i = 0$ gives $g(p) = p$ or p is a fixed point of g. Thus $P_g = F_g$. □

2.5 Common Fixed Points and Full Functions

In this section, an existence theorem on the common fixed points for two commuting continuous self-maps on a compact real interval, due to Cohen [9] is proved. This supplements the theorems in Sect. 2.4. Without loss of generality we take $I = [0, 1]$. We need the following lemmata and definitions.

Lemma 2.5.1 *Let $f, g : I \to I$ be continuous maps and $h : I \to J = [c, d]$ be a homeomorphism onto J. f and g commute on I and have a common fixed point if and only if hfh^{-1} and hgh^{-1} commute on J and have a common fixed point.*

Proof Let $h : I \to J$ be a homeomorphism onto J and $f, g : I \to I$ be continuous functions. Let $hfh^{-1} : J \to J$ and $hgh^{-1} : J \to J$ be commutative and y_0 be a common fixed point. Then $y_0 = hfh^{-1}(y_0) = hgh^{-1}(y_0$. Since h is a homeomorphism from I onto J, so h^{-1} is a homeomorphism of J onto I. So $h^{-1}y_0 = h^{-1}(hfh^{-1}(y_0)) = h^{-1}hgh^{-1}(y_0)$. Thus $h^{-1}(y_0) = f(h^{-1}(y_0)) = g(h^{-1}(y_0))$ or $x_0 = h^{-1}(y_0)$ belongs to I and is a common fixed point for f and g in I. Also by the commutativity of hfh^{-1} and hgh^{-1} we get $hfgh^{-1} = (hfh^{-1}) \circ hgh^{-1} = (hgh^{-1}) \circ (hfh^{-1}) = hgfh^{-1}$ whence $fg = gh$ on I.

If $f(g(x)) = g(f(x))$ for all $x \in I$ and $h^{-1} : J \to I$ is a homeomorphism, for each $y \in J$, $fgh^{-1}(y) = gfh^{-1}(y)$ and so $fh^{-1}hgh^{-1}y = gh^{-1}hfh^{-1}y$ for $y \in J$. Premultiplying by h we get for $y \in J$

$$(hfh^{-1})(hgh^{-1})y = (hgh^{-1})(hfh^{-1})y.$$

Thus hfh^{-1} and hgh^{-1} commute. If for $x_0 \in I$ $x_0 = f(x_0) = g(x_0)$, then $h(x_0) = hf(x_0) = hg(x_0)$. But $x_0 = h^{-1}(y_0)$ for some $y_0 \in J$. So $y_0 = hfh^{-1}(y_0) = hgf^{-1}(y_0)$. Thus hfh^{-1} and hgh^{-1} have a common fixed point. □

Lemma 2.5.2 *If $f, g : I \to I$ are commuting continuous functions without a common fixed point, then there are commuting functions mapping I onto I without a common fixed point.*

Proof Let $a_1 = \max\{\inf_I f, \inf_I g\}$ and $b_1 = \min\{\sup_I f, \sup_I g\}$. Since f and g commute, $f[0,1] \cap g[0,1] \neq \phi$ both f and g map $[a_1, b_1]$ into itself. Otherwise for some $x \in [a_1, b_1]$, $f(x) > b_1$ would imply that for some $y \in [0,1]$, $g(y) = x$ and $g(f(y)) = fg(y) = f(x) > b_1$. This implies that $b_1 < \min\{\sup_I f, \sup_I g\}$. Similarly $f(x) < a_1$ for some $x \in [a_1, b_1]$ would imply that there exists $y \in [0,1]$ with $g(y) = x$ and $g(f(y)) = fg(y) = f(x) < a_1$. This means that $a_1 > \max\{\inf_I f, \inf_I g\}$, a contradiction. Writing f_1 and g_1 as the restrictions of f and g on $J_1 = [a_1, b_1]$ respectively, we can inductively define a_i, b_i and f_i by $a_i = \max\{\inf_{J_{i-1}} f, \inf_{J_{i-1}} g\}$ and $b_i = \min\{\sup_{J_{i-1}} f, \sup_{J_{i-1}} g\}$ where $J_{i-1} = [a_{i-1}, b_{i-1}]$, $i = 2, 3, \ldots,$ and f_i is the restriction of f_{i-1} to J_{i-1}. Since $[a_i, b_i]$, $i = 1, 2, \ldots,$ form a nested sequence of compact subsets of $[0,1]$, they have a non-void intersection. If this intersection is a singleton, then f and g have a common fixed point contrary to the assumption. Hence, $\bigcap_{i=1}^{\infty}[a_i, b_i]$ is a non-degenerate compact interval $[a, b]$ and the restriction $\overline{f}$ and $\overline{g}$ of f and g respectively map $[a, b]$ onto itself. If h is a homeomorphism of $[a, b]$ onto $I = [0,1]$. Then, the continuous maps $h\overline{f}h^{-1}$ and $h\overline{g}h^{-1}$ map $[0,1]$ onto itself but have no common fixed points by Lemma 2.5.1. □

Lemma 2.5.3 *If $f, g : I \to I$ are commuting continuous functions, so are f and gf. f and g have a common fixed point if and only if f and gf have a common fixed point.*

Proof $f(gf) = gf \circ f$ as $fg = gf$. If $x_0 = f(x_0) = g(x_0)$, then $x_0 = f(x_0) = g(x_0) = g(f(x_0))$. If $x_1 = f(x_1) = g(f(x_1))$, then $x_1 = f(x_1) = g(x_1)$. □

Definition 2.5.4 A continuous function $f : I \to I$ is said to be full if there is a partition $P_f = \{x_0 = 0 < x_1 < x_2 \cdots < x_n = 1\}$ of I such that f on $[x_i, x_{i+1}]$ is a homeomorphism on $[0,1]$ for each $i = 0, 1, \ldots, n-1$.

Definition 2.5.5 A partition $P_f = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ is regular if the length of the subintervals $x_{i+1} - x_i$ is the same for all $i = 0, 1, \ldots, n-1$. A partition P_g refines a partition P_f uniformly if each subintervals in P_f formed by consecutive partition points of P_f is the union of partitioning subintervals of g.

Lemma 2.5.6 *If f_1, g_1 are commuting full functions on [0, 1] without a common fixed point, there are functions f and g with the same properties and additionally $f(0) = g(1) = 0$ and $f(1) = g(0) = 1$, P_f, P_g and P_{fg} are regular and P_g refines P_f uniformly.*

Proof If $f_1(0) = g_1(0) = 0$, then f_1 and g_1 have a common fixed point contrary to the assumption. So essentially two cases arise: (i) $f_1(0) = 0$, $g_1(0) = 1$ and (ii) $f_1(0) = 1 = g_1(0)$. In case (i) $f_1(1) = f_1 g_1(0) = g_1 f_1(0) = 1$ and so $g_1(1) = 0$, as otherwise $g_1(1) = 1$ would imply that f_1 and g_1 have 1 as a common fixed point. In this case let $f_2 = f_1$ and $g_2 = g_1$.

For case (ii), $f_1(1) = f_1 g_1(0) = g_1 f_1(0) = g_1(1)$. So $f_1(1) = g_1(1) = 0$ as otherwise 1 would be a fixed point. In this case let $f_2 = f_1 g_1$ and $g_2 = g_1$, $g_2(0) = g(0) = 1$, $f_2(1) = f_1 g_1(1) = f_1(0) = 1$ and $g_2(1) = g_1(1) = 0$. In either case let $f_3 = f_2$ and $g_3 = g_2 f_2$. Clearly P_{g_3} refines P_{f_2} uniformly. Let h be any order preserving homeomorphism on [0, 1] taking $P_{f_3 g_3}$ into the corresponding regular partition of [0, 1]. Define $f = h f_3 h^{-1}$ and $g = h g_3 h^{-1}$. As f_3 and g_3 have no common fixed point, by Lemma 2.5.1 f and g do not have a common fixed point. Also P_f, P_g, P_{fg} are regular and as P_{g_3} refines P_{f_3} uniformly. P_g refines P_f uniformly. □

Theorem 2.5.7 (Cohen) *Commuting continuous full functions mapping [0, 1] onto [0, 1] have a common fixed point.*

Proof Let $f_1, g_1 : I \to I$ be two commuting full functions without a common fixed point. So using Lemma 2.5.6, we can find commuting full functions f_1, g_1 mapping [0, 1] onto itself such that $f(0) = g(1) = 0$, $f(1) = g(0) = 1$, P_f, P_g and P_{fg} regular partitions with P_g refining P_f uniformly. Let $P_f = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ and $P_g = \{0, \frac{1}{m}, \frac{2}{m}, \ldots, 1\}$ and $P_{fg} = \{0, \frac{1}{mn}, \frac{2}{mn}, \ldots, 1\}$ where m and n are odd. Let f_i and g_i be restrictions of f to $[\frac{i-1}{n}, \frac{i}{n}]$ and g to $[\frac{i-1}{m}, \frac{i}{m}]$, respectively. Let $r = \frac{n+1}{2}$ and $s = \frac{m+1}{2}$. Suppose r is odd and s is even. If $D(f_i, g_j)$ is the domain of $f_i g_j$ for each i and j then it is a subinterval of P_{fg}. In particular

$$\begin{aligned} D(g_1 f_r) &= \left[\frac{r-1}{n}, \frac{r-1}{n} + \frac{1}{mn}\right] \\ D(g_2 f_r) &= \left[\frac{r-1}{n} + \frac{1}{mn}, \frac{r-1}{n} + \frac{2}{mn}\right] \ldots, \\ D(g_s f_r) &= \left[\frac{r-1}{n} + \frac{s-1}{mn}, \frac{r-1}{n} + \frac{s}{mn}\right] \\ &= \left[\frac{mn-1}{2mn}, \frac{mn+1}{2mn}\right] \end{aligned}$$

Similarly

$$D(f_1 g_s) = \left[\frac{s-1}{m}, \frac{s-1}{m} + \frac{1}{mn}\right]$$
$$D(f_2 g_s) = \left[\frac{s-1}{m} + \frac{1}{mn}, \frac{s-1}{m} + \frac{2}{mn}\right] \dots,$$
$$D(f_r g_s) = \left[\frac{s-1}{m} + \frac{r-1}{mn}, \frac{s-1}{m} + \frac{r}{mn}\right]$$
$$= \left[\frac{mn-1}{2mn}, \frac{mn+1}{2mn}\right]$$

Thus $D(f_r g_s) = D(g_s f_r)$. Since g_s is continuous and onto [0, 1], its graph must intersect the diagonal of $I \times I$ and g_s has a fixed point z_1. As $D(g_s) \subseteq D(f_0)$, $z_1 \in D(f_r)$ and thus $z_1 \in D(f_r g_s) = D(g_s f_r)$. So $g_s f_r(z_1) = f_r g_s(z_1) = f_r(z_1)$ and $z_2 = f_r(z_1)$ is a fixed point of g_s. Thus proceeding, we get a sequence z_p of fixed points of g_s with $z_{p+1} = f_r(z_p)$. Since f_r is monotone the sequence z_p converges to z_1 a fixed point of both f and g. The case when r is even and s is odd can be handled similarly. □

Remark 2.5.8 One can show that f is full if and only if f maps [0, 1] onto [0, 1] and is an open map. For related work, Baxter and Joichi [3] may be referred.

2.6 Common Fixed Points of Commuting Analytic Functions

We prove a theorem of Shields [28] on the common fixed points of analytic functions in this section. We denote by G, a non-void bounded open connected set in the complex plane. Let F_G be the family of all analytic functions mapping G into itself. Clearly F_G is a semigroup under composition of mappings. We can consider $H(G)$ the linear space of all functions analytic on G and continuous on $\overline{G}$, with the topology of uniform convergence on compact subsets of G. This topology is a metric topology and indeed it arises from a complete metric and so F_G will inherit this metric topology. The following lemma implies that F_G is a topological semigroup (i.e. the composition map is a continuous function from $G \times G$ into G).

Lemma 2.6.1 *Let $f_n, g_n \in F_G$ and $f_n \to f$, $g_n \to g$ in the topology of uniform convergence on compact subsets of G. Then $f_n(g_n) \to f(g)$ and so F_G is a topological semigroup.*

Proof Let K be a compact subset of G and let U be an open set containing $g(K)$ with $\overline{U}$ compact and lying in G. Since $g_n \to g$ uniformly on K, $g_n(K) \subset U$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Now for all n

$$|f(g(z)) - f_n(g_n(z))| \leq |f(g(z)) - f(g_n(z))| + |f(g_n(z)) - f_n g_n(z)|$$

Since $g(z)$, $g_n(z) \in \overline{U}$ for $z \in K$ for all $n \geq n_0$ and f is uniformly continuous on the compact set $\overline{U}$ and $f_n \to f$ uniformly on $\overline{U}$, the above inequality implies that $f(g_n(z)) \to fg(z)$ and $|f_n(g_n(z)) - f(g_n(z))| \leq \sup_{w \in \overline{U}} |f_n(w) - f(w)| \to 0$ as $n \to \infty$. Hence $(f_n g_n)$ converges uniformly on K to fg. Thus F_G is a topological semigroup. □

A few facts from the theory of topological semigroups will be needed in the sequel. For proofs and other details Numakura [23], Wallace [31] and Ellis [15] may be consulted.

Definition 2.6.2 Let $(S, \cdot)$ be a semigroup. An element e of S is called an idempotent if $e.e = e^2 = e$. An element 0 is termed zero if $0.x = 0$ for all $x \in S$. 1 is called an identity of S if $1.x = x = x.1$ for all $x \in S$. In a semigroup S if $ax = ay$ $(xa = ya)$ implies $x = y$ for all a, x, y in S then S is called a semigroup satisfying the left (right) cancellation law. If S satisfies both the left and right cancellation laws, it is called a semigroup satisfying cancellation law.

The following is a basic result in the theory of topological semigroups and the proof is essentially from Ellis [15].

Lemma 2.6.3 *Let S be a compact Hausdorff topological semigroup. Then S has an idempotent element.*

Proof Let $\mathscr{F}$ be the family of all compact subsets K of S such that $K^2 \subseteq K$. $\mathscr{F} \neq \phi$, as $S \in \mathscr{F}$. $\mathscr{F}$ is partially ordered by set inclusion. As every chain in $\mathscr{F}$ has a lower bound $\mathscr{F}$ has a minimal element A in $\mathscr{F}$. If $r \in A$, then rA is a non-void compact subset of S as rA is the image of the compact set A under the continuous map $x \to r.x$. So $rA \in \mathscr{F}$ and $rA \subseteq A$. Since A is minimal $rA = A$. So there exists $p \in A$ such that $rp = r$. Define $L = \{a \in A : ra = r\}$. Clearly $p \in L$ and L is a compact subset of A. Let $\ell_1, \ell_2 \in L$. Then $r\ell_1\ell_2 = r\ell_2 = r$ and hence $\ell_1 \circ \ell_2 \in L$. So $L^2 \subseteq L$. Hence $L \in \mathscr{F}$. As $L \subseteq A$ and A is minimal $L = A$. Since $r \in A = L$, $r^2 = r$ from the definition of L. Thus S has an idempotent element. □

We skip the proof of the following.

Lemma 2.6.4 *Let S be a compact T_2 topological semigroup which is commutative. For $x \in S$ and $\Gamma(x) = cl\{x, x^2, \ldots,\}$, we have*

(i) *$\Gamma(x)$ contains exactly one idempotent;*
(ii) *if e is an identity for $\Gamma(x)$, then $\Gamma(x)$ is a group and x has an inverse in $\Gamma(x)$;*
(iii) *if e is a zero for $\Gamma(x)$, then $x_n \to e$.*

The following lemma makes use of the basic properties of analytic functions.

Lemma 2.6.5 *If the analytic function $e \in F_G$ is idempotent, then $e(z) \equiv z$ on $e(z)$ is constant for all $z \in G$.*

Proof If $e(z)$ is constant for all $z \in G$, clearly it is an idempotent. Suppose e is a non-constant analytic function on G, then f is an open mapping. So $G_1 = e(G)$ is an open set. Since $e^2(z) = e(z)$, $e(z) = z$ on G_1. As G_1 is uncountable, and the analytic functions, viz. identity function and e coincide on G_1, $e(z)$ must be z at each z in G. □

We also recall some classical results from complex analysis (see Conway [11] and Ahlfohrs [1].

Theorem 2.6.6 (Montel) *Let $H(G)$ be the linear space of analytic functions on the open region G. A family $\mathscr{F}$ in $H(G)$ is normal in the sense that every sequence in $\mathscr{F}$ has a convergent subsequence if and only if $\mathscr{F}$ is locally bounded in $H(G)$ (i.e. for each compact subset K of G, there is a positive constant M_k with $|f(z)| \leq M_k$ for all $f \in \mathscr{F}$ and $z \in K$).*

Theorem 2.6.7 (Hurwitz) *Let $A(G)$ be the linear space of all analytic functions with the topology of uniform convergence on compact subsets of G. If (f_n) converges to f in $H(G)$ and f_n never vanishes on G for each n, then $f \equiv 0$ or f is non-zero throughout G.*

Lemma 2.6.8 *Let D be the open unit disc in the complex plane $\mathbb{C}$ and $f : D \to D$ be a bilinear (Mobius) transformation of D onto D. Then there arise three possibilities:*

(i) $f(z) = z$ *on* D;
(ii) f *has exactly one fixed point in the closed unit disc;*
(iii) f *has two distinct fixed points in the unit circle and the iterates of* f *converge to one of these fixed points.*

Proof The general form of such a bilinear transformation is $f(z) = \alpha \frac{(z-a)}{(1-\overline{a})z}$ where $|\alpha| = 1$, $|a| < 1$.

If f is not the identity function the fixed points $z = f(z)$ are given by

$$\overline{a}z^2 - (1-\alpha)z - \alpha z = 0$$

As this equation is invariant under $z \to \frac{1}{\overline{z}}$, the fixed points of $f(z)$ are inverses of each other with respect to the unit circle. So there is a fixed point inside and another outside the circle or there is a 'double fixed point' or two distinct fixed points on the unit circle. □

Lemma 2.6.9 *Let $f \in F_G$, be the subset of $H(G)$ containing all analytic functions mapping G into itself. Suppose f is not a homeomorphism of G onto itself. Then there is a point z_0 in $\overline{G}$ and a subsequence $\{f_{n_i}\}$ of f-iterates such that $f_{n_i}(z) \to z_0$ uniformly on compact subsets of G.*

Proof Write $\Gamma(f) = cl\{f^n\}$ in $H(G)$. If $\Gamma(f) \subseteq F_G$, then $\Gamma(f)$ is a compact semi-group under composition of functions and contains an idempotent element $e(z)$ by Lemma 2.6.3.

By Lemma 2.6.5 $e(z) \equiv z$ for all $z \in G$ or is a constant z_0 for all $z \in G$. If the identity map belongs to $\Gamma(f)$, then by Lemma 2.6.4, $\Gamma(f)$ is a group and $f \in \Gamma(f) \subseteq F_G$ would be invertible in $F(G)$ contradicting that f is not a homeomorphism. Hence $e(z) \equiv z_0$, for all $z \in G$ and is thus a zero for $\Gamma(f)$. So again by Lemma 2.6.4 $f^n(z)$ converges to z_0 in the topology of F_G.

Suppose $g \in \Gamma(f)$ does not belong to F_G. Since $f_n(G) \subseteq G$, $g(G) \subseteq \overline{G}$. As $g \notin F_G$, there is a point $z' \in G$ with $g(z') = z_0 \notin G$. We claim that $g(z) \equiv z_0$.

As $g \in \Gamma(f)$, we can find f_{n_k}, a subsequence of f iterates converging to g in $H(G)$. Now $f_{n_k}(z) - z_0$ never vanishes in G as $z_0 \in G$ and converges to $g(z) - z_0$. So by Lemma 2.6.7 (Hurwitz Theorem), $g(z) - z_0$ is identically zero in G or never vanishes in G. But already for $z = z' \in G$, $g(z') - z_0 = 0$. So $g(z) \equiv z_0$ for all $z \in G$. □

Lemma 2.6.10 *Let $f \in F_G$ and suppose f is not a homeomorphism of G onto itself. Let z_0 be the element of $\overline{G}$ such that f_{n_i} converges to z_0 in $H(G)$. Then z_0 is a common fixed point for all continuous g on $\overline{G}$ that map G into itself and commute with f.*

Proof By Lemma 2.6.9, there exists $z \in \overline{G}$ with $\lim f_{n_i}(z) = z_0$ in F_G. For $g \in C(\overline{G})$, $g(z_0) = g(\lim f_{n_i}(z)) = \lim f_{n_i}(g(z)) = z_0$.

The following remarks are relevant.

Remark 2.6.11 If f is a bilinear map of the open unit disc D onto itself with two distinct fixed points on the boundary, consider p a bilinear map, mapping D onto the upper half-plane and taking these fixed points into 0 and ∞. For $g = pfp^{-1}$, 0 and ∞ are fixed points of g and g maps the upper half-plane onto itself. Hence g is a dilatation and is of the form $g(z) = az$, $a > 0$ and $a \neq 1$ as $f(z) \not\equiv z$. So $g^n(z) = a^n z$ tends to zero or to ∞. Thus the iterates of f converge to one of the fixed points of f.

Remark 2.6.12 Wolff [32] and Denjoy [13] have shown independently in 1926 that if f is analytic in D and $f(D) \subseteq D$, then either f is a bilinear map of D onto itself with exactly one fixed point or f^n converges to a constant $C \in \overline{D}$.

We are now in a position to prove a theorem of Shields [28] on the fixed points of commuting family of analytic functions on $\overline{D}$.

Theorem 2.6.13 (Shields [28]) *Let F be a commuting family of continuous functions on $\overline{D}$ which are analytic in D. Then there is a common fixed point z_0 for all functions in F.*

Proof If F contains a constant function then that constant is the common fixed point. Suppose it contains only non-constant continuous functions on $\overline{D}$ which are analytic in D. So by the Maximum Modulus Theorem $f(D) \subseteq D$ for each $f \in F$. Suppose not all functions of F are bilinear maps of D onto D. So there exists f, different from the identity map in F. Then Lemma 2.6.10 can be invoked to conclude that there is a common fixed point for each $f \in F$. On the other hand if all the members of F are bilinear, then if one of them has just one fixed point, then it is a common fixed point for all. In case these have two fixed points then by Remark 2.6.11, the iterates converge to one of the two fixed points and so invoking Lemma 2.6.10, we conclude that for each f in F there is a common fixed point. □

Remark 2.6.14 Theorem 2.6.13 due to Shields has been generalized to Hilbert spaces by Suffridge [29].

2.7 Fixed Points of Meromorphic Functions

In this section, an interesting theorem on the fixed points of meromorphic functions, due to Bergweiler [4] is detailed. Bergweiler's short proof is elementary, though it invokes Picard's theorem. We recall

Theorem 2.7.1 (Picard (see Conway [11])) *Suppose an analytic function f has an essential singularity at a. Then in each neighbourhood of a, f assumes each complex number, with one possible exception, infinitely many times.*

Corollary 2.7.2 *An entire function which is not a polynomial assumes every complex number, with one exception infinitely many times.*

In response to a question raised by Gross [16], Bergweiler [4] proved the following.

Theorem 2.7.3 (Bergweiler [4]) *Let f be a meromorphic function that has at least two different poles and let g be a transcendental entire function. Then the composite function $f \circ g$ has infinitely many fixed points.*

The theorem above makes use of the following lemmas.

Lemma 2.7.4 *Let f be a meromorphic function and z_0 be a pole of order p. Then there is a function h, defined and analytic in a neighbourhood of 0 such that $h(0) = 0$ and $f(h(z) + z_0) = z^{-p}$ for $z \neq 0$.*

Proof The function k defined as $k(z)^{-p} = f(z + z_0)$ is analytic in a neighbourhood of 0 and $k'(0) \neq 0$. So $k(z)$ is invertible in a neighbourhood of 0 and this inverse $h(z)$ is analytic in a neighbourhood of 0. Now $k(0) = 0$. So $h(0) = 0$ and $f(h(z) + z_0) = z^{-p}$ for $z \neq 0$. □

Lemma 2.7.5 *Let f and g be meromorphic functions. Then $f \circ g$ has infinitely many fixed points if and only if $g \circ f$ does.*

Proof If $x_0 = fg(x_0)$, then $gx_0 = gf(g(x_0))$ so that $g(x_0)$ is a fixed point of gf. If $x_0 = fg(x_0)$ and $x_1 = fg(x_1)$, then $g(x_0) = g(x_1)$ would imply that $fg(x_0) = fg(x_1)$ so that $x_0 = x_1$. Thus g maps the set of fixed points of $f \circ g$ injectively into the set of fixed points of $g \circ f$. Indeed if x^* is a fixed point of $g \circ f$, then $f(x^*)$ is a fixed point of $f \circ g$. Similarly f maps the set of fixed points of $g \circ f$ injectively into the set of fixed points of $f \circ g$. Thus the sets of fixed points of $f \circ g$ and $g \circ f$ have the same cardinality. (Indeed g maps the set of fixed points of $f \circ g$ bijectively onto the set of fixed points of $g \circ f$). □

Now we provide the proof of Theorem 2.7.3.

Proof Let z_1 and z_2 be poles of f of order p_1 and p_2. Using Lemma 2.7.4 choose the functions h_j for $j \in \{1, 2\}$. Let $k_1(z) = h_1(z^{p_2}) + z_1$ and $k_2(z) = h_2(z^{p_1}) + z_2$. Now $f(k_1(z)) = f(k_2(z)) = z^{-p_1 p_2}$ for $z \neq 0$ in a neighbourhood of 0. Define $u(z) =$

$g(z^{-p_1 p_2})$. Then 0 is an essential singularity of u and in a punctured neighbourhood of 0, $u(z) = g(fk_1(z)) = gf(k_2(z))$.

If $f \circ g$ has only finitely many fixed points, then so has $g \circ f$ only finitely many fixed points by Lemma 2.7.5. So $u(z) \neq k_j(z)$ for $j = 1, 2$ in a punctured neighbourhood of 0, since $k_1(0) = z_1 \neq z_2 = k_2(0)$. Define

$$v(z) = \frac{u(z) - k_1(z)}{k_2(z) - k_1(z)}.$$

0 is an essential singularity for u and v does not take the values 0, 1 and ∞ in a punctured neighbourhood of 0. This contradicts Picard's Theorem 2.7.1. Hence the theorem. □

Remark 2.7.6 It can be similarly shown that if f and g are transcendental meromorphic functions and if either f or g has at least three poles, then $f \circ g$ has infinitely many fixed points.

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Chapter 3
Fixed Points and Order

This chapter deals with fixed points of mappings on partially ordered sets (vide Definition 1.1.10) under diverse hypotheses.

3.1 Fixed Points in Linear Continua

In this section, some elementary fixed point theorems on linear continua due to Andres et al. [3] are discussed.

Definition 3.1.1 A partially ordered set (or a poset, for short) $(X, \leq)$ is said to be linearly ordered or totally ordered or a chain, if for any pair of elements $x, y \in X$, $x \leq y$ or $y \leq x$.

Definition 3.1.2 Let S be a nonempty subset of a poset $(X, \leq)$. $x_0 \in X$ is called an upper bound (lower bound) for S if $s \leq x_0$ $(x_0 \leq s)$ for all s in S. An element $x_0 \in X$ is called least upper bound, (l.u.b. for short) or supremum (greatest lower bound (g.l.b. for short) or infimum) of S if x_0 is an upper bound for S and $x_0 \leq x$ for every upper bound x for S (if x_0 is a lower bound for S and $x \leq x_0$ for every lower bound x of S). A maximal element (minimal element) m of X is an element m of X for which $m \leq x (\in X)$ implies $x = m$ (an element m of X for which $x \leq m$ of $x \in X$ implies $x = m$).

Definition 3.1.3 A linearly ordered set $(X, \leq)$ with more than one element is called a linear continuum, if

(i) it is densely ordered or without gaps if for $x, y \in X$ with $x < y$, there exists $z \in X$ such that $x < z < y$ ($a < b$ if $a \leq b$ and $a \neq b$) and
(ii) for each nonempty subset S of X bounded above there is a least upper bound (l.u.b) in X (called l.u.b property).

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Remark 3.1.4 If $(X, \leq)$ is a linearly ordered set, then the sets of the form $\{x \in X : x < a\}$, $\{x \in X : a < x\}$, $a \in X$ is a subbase for a topology on X, called the order topology on X.

X is connected and Hausdorff in the order topology if and only if X is a linear continuum in the sense of Definition 3.1.3. Moreover, the order topology is the smallest topology on X under which $<$ is continuous (see Kelley [10], pp. 57–58). Also in a linearly ordered set with the order topology every closed and (order) bounded subset of X is compact if and only if it has the l.u.b property (see Kelley [10], p. 162).

Definition 3.1.5 Let X be a linear continuum and 2^X denote the set of all subsets (power set) of X. A map $\varphi : X \to 2^X - \{\phi\}$ is called a multimap. An element $x_0 \in X$ such that $x_0 \in \varphi(x_0)$ is called a fixed point of φ.

Theorem 3.1.6 (Andres et al. [3]) *Let $(X, \leq)$ be a linear continuum and $I = [a, b] = \{x \in X : a \leq x \leq b\}$, where $a, b \in X$. Let $\varphi : I \to X$ be a multimap on I with a connected graph. If either $I \subseteq \varphi(I)$ or $\varphi(I) \subseteq I$, then φ has a fixed point.*

Proof Let $G_\varphi = \{(x, y), y \in \varphi(x), x \in I\}$ be the graph of φ, a subset of $I \times X \subseteq X^2$. Let $D = \{(x, x) : x \in X\}$, $P_1 = \{(x, y) \in X^2 : x < y\}$ and $P_2 = \{(x, y) \in X^2 : y < x\}$. Suppose φ has no fixed point. Then $F_\varphi = \{x \in I : x \in \varphi(x)\}$ is empty. Suppose $I \subseteq \varphi(I)$. Then there exists $c, d \in I$ such that $a \in \varphi(c)$ and $b \in \varphi(d)$. Since $F_\varphi = \phi$, $a < c$ and $d < b$. As $a(\in \phi(c)) < c$, $(c, a) \in P_2 \cap G_\varphi$ and as $b \in \varphi(d)$ and $d < b$, $(d, b) \in P_1 \cap G_\varphi$. Since $G_\varphi \subseteq P_1 \cup P_2$ and P_1 and P_2 are separated open sets and G_φ is connected in X^2, $G_\varphi \subseteq P_1$ or $G_\varphi \subseteq P_2$. This is a contradiction to the fact that $(c, a) \in P_1 \cap G_\varphi$ and $(d, b) \in P_2 \cap G_\varphi$. Hence φ has a fixed point in I.

If $\varphi(I) \subseteq I$, then $a < p$ for all $p \in \varphi(a)$ as we have assumed that $F_\varphi = \phi$. For similar reasons, $q < b$ for all $q \in \varphi(b)$. So $(a, p) \in P_1 \cap G_\varphi$ for all $p \in \varphi(a)$ and $(b, q) \in P_2 \cap G_\varphi$ for all $q \in \varphi(b)$. As $G_\varphi \subseteq P_1 \cup P_2$ and P_1 and P_2 are separated, $G_\varphi \subseteq P_1$ or $G_\varphi \subseteq P_2$. In any case, this is a contradiction as G_φ contains points of P_1 as well as P_2. So, in this case also φ has a fixed point in I. □

Theorem 3.1.6 has a simple corollary generalizing Theorems 2.15 and 2.1.7.

Corollary 3.1.7 *Let $f : I \to X$ be a map with a connected graph where $I = [a, b] \subseteq X$, a linear continuum. If $f(I) \subseteq I$ or $I \subseteq f(I)$, then f has a fixed point in I.*

Example 3.1.8 Define $f : [-1, 1] \to [-1, 1]$ by $f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$. f is not continuous (at $x = 0$). But it has a connected graph and has infinitely many fixed points as it maps continuously $I_n = \left[\frac{1}{(2n+3)\frac{\pi}{2}}, \frac{1}{(2n+1)\frac{\pi}{2}}\right]$ into $[-1, 1]$ (containing I_n) for each n (and so has a fixed point in I_n).

Proposition 3.1.9 *Let $f : L \to 2^L - \{\phi\}$ be a multimap with a connected graph G_f. If f has an n-orbit for some x then f has a fixed point in L.*

Proof Let $\{x_1, x_2, \ldots, x_n\}$ be an n-orbit of f. Thus $x_{i+1} \in F(x_i)$, $i = 1, \ldots, n-1$ and $x_n \in F(x_1)$. Let $F_f = \{x \in L : x \in f(x)\}$, the set of fixed points of f be empty. Let $a = \min\{x_1, \ldots, x_n\}$ and $b = \max\{x_1, x_2, \ldots, x_n\}$. So there exist x_k, x_ℓ in the orbit of f such that $x_k \in f(a)$ and $x_\ell \in f(b)$. Then $a < x_k$ and $x_\ell < b$ since $a, b \notin F_f$. As before $(a, x_k) \in P_1 = \{(x, y) \in G_f$ with $x < y\}$ and $(b, x_\ell) \in P_2 = \{(x, y) \in G_f$ with $y < x\}$. Since $G_f = P_1 \cup P_2$ and P_1 and P_2 are disjoint open subsets of G_f and G_f the graph of f is connected, $G_f \subseteq P_1$ or $G_f \subseteq P_2$. This is a contradiction as $G_f \cap P_1, G_f \cap P_2 \neq \phi$. Hence f has a fixed point. □

Example 3.1.10 Define $f : [0, 1] \to 2^{[0,1]} - \{\phi\}$ as follows:

$$f(x) = \begin{cases} \{\frac{1}{6}, \frac{1}{3}\}, & x \in [0, 1) \\ [\frac{1}{6}, \frac{1}{3}], & x = 1 \end{cases}$$

f has a connected graph, though $f^2(x) = \{\frac{1}{6}, \frac{1}{3}\}$ does not have a connected graph. However f has a fixed point in view of Theorem 3.1.6 as $f(I) \subseteq I$ and so f^2 also has a fixed point.

Corollary 3.1.11 *Let $f : L \to 2^L - \{\phi\}$ be a map on a linear continuum L with a connected graph. If for some $n \in \mathbb{N}$, for the nth iterate f^n of f, there exists a closed interval $I \subseteq L$ such that $I \subseteq f^n(I)$ or $f^n(I) \subseteq I$, then f has a fixed point.*

Proof As f^n has a connected graph, by Theorem 3.1.6, f^n has a fixed point. If this is not a fixed point of f, then there is a nontrivial $k \in N$ such that k factors n. So by Proposition 3.1.9 f has a fixed point. □

Example 3.1.12 Define the multimap f on $[0, 1]$ by

$$f(x) = \begin{cases} [\frac{1}{2}, 1], & x = 0 \\ 1 - x, & x \in (0, 1) - \{\frac{1}{2}\} \\ 0, & x = \frac{1}{2} \\ [0, \frac{1}{2}], & x = 1 \end{cases} \tag{3.1}$$

Now

$$f^2(x) = \begin{cases} [0, \frac{1}{2}], & x = 0 \\ x, & x \in (0, 1) - \{\frac{1}{2}\} \\ [\frac{1}{2}, 1], & x \in \{\frac{1}{2}, 1\} \end{cases}$$

and every point of $[0, 1]$ is a fixed point of f^2 but f has no fixed point. Further the graph of f is not connected. Hence in Corollary 3.1.11 the hypothesis that the graph of f is connected cannot be dropped.

3.2 Knaster–Tarski Principle

A set-theoretical fixed point theorem for maps in the power set of a set proved in 1927 by Knaster [11] and improved by Tarski germinated into the following theorem referred as the Knaster–Tarski principle in the literature.

Definition 3.2.1 A map $f : X \to X$, where $(X, \leq)$ is a poset is said to be isotone if $f(x) \leq f(y)$ for all $x, y \in X$ with $x \leq y$.

Theorem 3.2.2 *Let $(X, \leq)$ be a poset and $f : X \to X$ be an isotone map such that*

(i) $b \leq f(b)$ for some $b \in X$;
(ii) every chain in $X_1 = \{x \in X; b \leq x\}$ has a supremum.

Then F_f, the set of fixed points of f is nonempty and contains a maximal fixed point.

Proof Consider the subset $P = \{x \in X : b \leq x \leq f(x)\}$. P is nonempty as $b \in P$. Every chain C in P, being a subset of X_1 has a supremum u in X_1 and for $c \in C$, $c \leq u$ and by the isotonicity of f and the definition of $C \subseteq P$, $c \leq f(c) \leq f(u)$. So $f(u)$ is an upper bound for C and clearly $b \leq u \leq f(u)$. So $u \in P$. Now by Zorn's Lemma [20] since every chain in P has an upperbound (indeed a supremum), P has a maximal element x_0 in X. Since $x_0 \leq f(x_0)$, $f(x_0) \leq f(f(x_0))$ and so $f(x_0) \in P$. If $x_0 \neq f(x_0)$, there is a contradiction to the maximality of x_0. So $x_0 = f(x_0)$. x_0 being a maximal element of P is a maximal fixed point and $b \leq x_0$. □

In this context the following theorem due to Bourbaki [5], also called Zermelo's theorem [19] is stated and the proof is left as an exercise.

Theorem 3.2.3 (Bourbaki–Zermelo) *Let $(X, \leq)$ be a poset in which every chain has an upper bound. Let $f : X \to X$ be a map such that $x \leq f(x)$ for all $x \in X$. Then f has a fixed point.*

In fact Abian [1] and Moroianu [13] have pointed out that it suffices to assume the completeness of each well-ordered subset of X instead of the completeness of a chain in Bourbaki's theorem. First, we recall

Definition 3.2.4 A subset S of a poset $(X, \leq)$ is called well-ordered if every nonempty subset S_0 of S has a least element in S_0.

Remark 3.2.5 The set of natural numbers with the 'natural order' is well-ordered, though the real number system is not well-ordered. While every nonempty subset of [0, 1] has a least and a greatest element in [0, 1], these need not belong to the subset concerned. So [0, 1] is not well-ordered. The Axiom of Choice is equivalent to the hypothesis that every nonempty set can be well-ordered (see Kelley [10]).

We now prove the following version of Bourbaki's theorem, due to Moroianu [13] which is equivalent to the Axiom of Choice.

Theorem 3.2.6 *If every well-ordered subset A of a poset $(X, \leq)$ has an upper bound in X and $f : X \to X$ is a map such that $x \leq f(x)$ for all $x \in X$, then f has a fixed point in X.*

At this stage, it is convenient to introduce the following remark and a definition.

Remark 3.2.7 A map $f : X \to X$, where $(X, \leq)$ is a poset is called progressive or expansive if $x \leq f(x)$ for all $x \in X$.

Definition 3.2.8 Let $(X, \leq)$ be a poset and W be the set of well-ordered subsets of X. If $A \in W$ and $x \in A$ the initial segment defined by x in A is the set $A_x = \{y \in A : y < x\}$.

Definition 3.2.9 Let $F : W \to X$ be a map, where $(X, \leq)$ is a poset and W, the set of all well-ordered subsets of X. A well-ordered subset A of X is called an F-chain if for each $x \in A$ which is not the least element of A, $f(A_x) = x$.

We prove Theorem 3.2.6, making use of the following lemmata.

Lemma 3.2.10 *For two distinct well-ordered subsets A and B of a poset $(X, \leq)$, the following statements are equivalent:*

(i) one of A, B is an initial segment of the other;
(ii) if $x \in A$ and $y \in B$ are such that $A_x = B_y$ then $x = y$.

Proof Let A be an initial segment of B. Let y_1 = infimum $B - A$. Then $A = B_{y_1}$. Let $x \in A$ and $y \in B$ such that $A_x = \{a \in A : a < x\} = \{b \in B : b < y\} = B_y$. Since $A = B_{y_1}$, $A_x = \{b \in B : b < x \text{ and } b < y_1\} = \{b \in B : b < x^*\} = B_{x^*} = B_y$. So $x^* = y$. Hence $x^* = \min(x, y_1) = y$. Since each $a \in A$ is less than $y_1 \in B - A$ and $x \in A$, $x^* = x$. Thus $x = y$.

Suppose for two distinct well-ordered sets A and B there exist x in A and y in B such that $A_x = B_y$. Clearly A and B have the same least element. Define $S = \{x \in A \cap B : A_x = B_y\}$. S is nonempty as it contains the least element of A and B. Further, for any $x \in S$, S contains A_x as also B_x. So $S = A$ or A_x for some x in S. Since A and B are distinct, one can suppose that $S \neq A$. So $S = A_x$ for some x in A. For similar reasons $S = B$ or B_y for some y in B. Thus $S = A_x = B_y$. Now by hypothesis (ii) $x = y$ and this means that $A_x = B_x$ and by the definition of S, $x \in S = A_x$ implying that $x < x$. So $A_x = S = B$. Thus (i) is true. □

From the proof that (ii) implies (i) in Lemma 3.2.10 we have

Corollary 3.2.11 *If A and B are two F-chains with the same least element then one of them is an initial segment of the other.*

Lemma 3.2.12 *If $\tau(a)$ is the family of F-chains, having $a \in X$ as the least element, then $C = \cup\{A : A \in \tau(a)\}$ is an element of $\tau(a)$.*

Proof By Corollary 3.2.11, C is a well-ordered subset of X with a as the least element. If $x \in C$, then $a \in A$ for some $A \in C(a)$ and $A_x = C_x$. So $f(C_x) = x$ from the definition of A as an F chain. This implies that τ itself is an f chain. □

Corollary 3.2.13 *If $F : W \to X$ is a mapping such that for each $A \in W$, $F(A)$ is an upper bound of A, then $F(C) \in C$.*

Proof If $F(C) \notin C$, let $C' = C \cup \{F(C)\}$. For $x \in C'$, $C'_x = C_x$ if $x \neq F(C)$ and $C'_x = C'$ if $x = F(C)$. This is always $F(C'_x)$ and $C' \in \tau(a)$. Since C' contains C as a proper subset this contradicts the definition of τ. □

Theorem 3.2.14 *Let X be an ordered set such that every well-ordered subset A of X has an upper bound in X. Let $f : X \to X$ be a (progressive) map such that $x \leq f(x)$ for all $x \in X$. Then f has a fixed point.*

Proof As A is well-ordered subset of X, so is $f(A)$. Define a map $F : W \to X$ such that (i) $F(A)$ is an upper bound for each $A \in W$ when A does not contain the greatest element and (ii) $F(A) = f(x)$, when A contains the greatest element x. Clearly F satisfies all the hypotheses of Corollary 3.2.13 and so there is a well-ordered subset C of X such that $F(C)$ is the greatest element of C. In view of (ii) $F(C) = f(F(C))$. Thus $F(C) \in X$ is a fixed point of f. □

Remark 3.2.15 Since each $A \in W$ has an upper bound, U the set of upper bounds of A is nonempty. For each $A \in W$, we can choose one from U_A using the axiom of choice. In case each $A \in W$ has a supremum in X, then $F(A) = f(\sup A)$ is a well-defined map of W into X (and there is no need to invoke the axiom of choice). However, by invoking the Axiom of Choice to X it is clear that X has a maximal element which will, of course, be a fixed point of f, the last element of C. Theorem 3.2.14 is equivalent to the Axiom of Choice, as observed by Abian [2].

Theorem 3.2.16 (Abian [2]) *The Axiom of Choice is equivalent to the statement that every (progressive) map $f : X \to X$, where $(X, \leq)$ is a poset wherein every well-ordered subset has an upper bound and $x \leq f(x)$ for all $x \in X$, has a fixed point.*

Proof The use of the Axiom of Choice in the proof of the fixed point theorem, i.e. Theorem 3.2.14 was already noted. (Zorn's Lemma, equivalent to the Axiom of Choice can be invoked to obtain a maximal element z_0 and as $z_0 \leq f(z_0)$, z_0 is a fixed point of f under the hypotheses of Theorem 3.2.14).

We now show that this fixed point theorem implies the Axiom of Choice. Let S be any set and $(W, \leq)$ be the set of all well-ordered subsets of S which are partially ordered initial segment wise (i.e. $W_1 \leq W_2$ if W_1 is an initial segment of W_2, where $W_1, W_2 \in W$). Let $(\mathbb{N}, \leq)$ be the set of all natural numbers with the usual order. Let $(W \times \mathbb{N}, \leq)$ be the cartesian product of W with $\mathbb{N}$ partially ordered lexicographically (i.e. $(W_1, n_1) \leq (W_2, n_2)$ if $W_1 \leq W_2$ and $W_1 \neq W_2$ and if $W_1 = W_2, n_1 \leq n_2$. Define $f : W \times \mathbb{N} \to W \times \mathbb{N}$ by $f(w, n) = (w, n + 1)$. Clearly $x \leq f(x)$ for all $x \in W \times \mathbb{N}$ and f has no fixed point. In view of Theorem 3.2.14 $(W \times \mathbb{N}, \leq)$ has a well-ordered subset without an upper bound (I).

However, every well-ordered subset of W has an upper bound. On the other hand, not every (well-ordered) subset of $\mathbb{N}$ has an upper bound. (I) is impossible unless $(W, \leq)$ has a maximal element. This implies that S itself is a well-ordered set. □

We conclude this section by proving two theorems on the existence of a common fixed point for a family of mappings.

Theorem 3.2.17 *Let $\mathscr{F}$ be a commuting family of mappings of a poset $(X, \leq)$ into itself. Suppose $x \leq f(x)$ for all $f \in \mathscr{F}$ and all $x \in X$. If each well-ordered set in X has an upper bound then there is a common fixed point for each $f \in \mathscr{F}$.*

Proof For $f \in \mathscr{F}$, in view of Remark 3.2.15 f has a fixed point $x_0 \in X$, which is a maximal element. If $g \in \mathscr{F}$, then $g(x_0) = g(f(x_0)) = f(g(x_0)) \geq g(x_0) \geq x_0 = f(x_0)$ using the commutativity of f and g and the progressive nature of f and g. As x_0 is a maximal element, $g(x_0) = x_0 = f(x_0)$. Thus, x_0 is a common fixed point for all members of $\mathscr{F}$. □

Our next theorem supplements the above theorem.

Theorem 3.2.18 *Let $\mathscr{F}$ be a commuting family of isotone maps of a poset $(X, \leq)$ into itself. Suppose for some fixed element $a \in X$, $a \leq f(a)$ for all $f \in \mathscr{F}$. If each chain in X containing a has a supremum in X, then $\mathscr{F}$ has a common fixed point.*

Proof The set $A = \{x \in X : a \leq x \leq f(x) \text{ for all } f \in \mathscr{F}\}$ is a poset with the partial order inherited from $(X, \leq)$ and every chain in A has a supremum in X. Since $a \leq x$ for $x \in A$ and f is isotone, $f(a) \leq f(x)$. As $a \leq f(a)$, we have $a \leq f(a) \leq f(x)$ for all $x \in A$ and $f \in \mathscr{F}$. For $g \in \mathscr{F}$, $a \leq x \leq f(x)$ implies $a \leq g(a) \leq g(x) \leq g(f(x)) = fg(x)$ for all $f \in F$ and $x \in A$. So each $g \in \mathscr{F}$ maps A into itself. Further every chain C in A has a supremum s in X and as $a \leq x \leq s$ for $x \in C$, $a \leq f(a) \leq f(x) \leq f(s)$ and $x \leq f(x) \leq f(s)$ for each $f \in \mathscr{F}$ and $x \in C$. So $\sup C = s \leq f(s)$. Thus C has an upper bound in A. So there is a maximal element x_0 in A (by Zorn's Lemma). Thus $a \leq x_0 \leq f(x_0)$ for all $f \in \mathscr{F}$. By the maximality of x_0, it follows that $x_0 = f(x_0)$ for all $f \in \mathscr{F}$. Hence, there is a common fixed point for all the functions in $\mathscr{F}$. □

3.3 Tarski's Lattice Theoretical Fixed Point Theorem and Related Theorems

The following well-known definitions figure in the original version of Tarski's fixed point theorem.

Definition 3.3.1 A poset $(X, \leq)$ is called a lattice if it contains the minimum and maximum of every pair of elements.

A lattice is said to be complete if every nonempty subset S of X has a supremum and an infimum in X.

Example 3.3.2 (a) $\mathbb{R}$, the set of all real numbers is a lattice in the usual order. Though it is totally ordered, $\mathbb{R}$ is not a complete lattice. While $\mathbb{N}$, the set of natural numbers is well-ordered, $\mathbb{R}$ is not well-ordered.

(b) $C[0, 1]$, the set of all continuous real-valued functions on $[0, 1]$ with the partial order $\preceq$ defined by $f \prec g$ if and only if $f(x) \le g(x)$ for all $x \in [0, 1]$ is a lattice. For $f, g \in C[0, 1]$, $\min\{f, g\}$ or $\max\{f, g\}$ need not coincide with either f or g. For instance for $f(x) \equiv x$, $g(x) \equiv \frac{1}{2}$ for $x \in [0, 1]$,

$$\min\{f, g\}(x) = \begin{cases} x, & x \in [0, \frac{1}{2}] \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

$$\max\{f, g\}(x) = \begin{cases} \frac{1}{2}, & x \in [0, \frac{1}{2}] \\ x, & \text{otherwise} \end{cases}.$$

However, these are different from both f and g; That $C[0, 1]$ is not a complete lattice is left as an exercise.

(c) Let I be the closed unit interval in $\mathbb{R}$ and for $n \in \mathbb{N}$, $n > 1$, let I^n be the n-fold cartesian product of I with itself. Define the partial order $\prec$ on I^n be $\underline{x} = (x_1, x_2, \ldots, x_n) \preceq \underline{y} = (y_1, y_2, \ldots, y_n)$ for $\underline{x}, \underline{y} \in I^n$ if and only if $x_i \le y_i$ for all $i = 1, 2, \ldots, n$. Though I^n is not totally ordered it is a complete lattice for $n > 1$.

(d) For a nonempty set X, 2^X, the power set of X with set-inclusion as a partial order is a complete lattice.

Theorem 3.3.3 (Tarski [18]) *Let $(X, \le)$ be a complete lattice and $f : X \to X$ an isotone map. Then F_f, the set of all fixed points of f is a nonempty complete lattice under the partial order induced by $\le$. In particular* $\sup F_f = \sup\{x \in X : f(x) \ge x\}$ *and* $\inf F_f = \inf\{x \in X : f(x) \le x\}$ *belong to F_f.*

Proof Let $0 = \inf X$ and $1 = \sup X$ and these exist as X is a complete lattice. So $v = \sup\{x \in X : f(x) \ge x\}$ is well-defined as $f(0) \ge 0$. For any $x \in X$ with $x \le f(x)$, $x \le u$ and by the isotonicity of f, $f(x) \le f(u)$ and so $x \le f(x) \le f(u)$. As $x \le u$, for $x \le f(x)$, $u \le f(u)$. Again by the isotonicity of f, $f(u) \le f(f(u))$. So $f(u) \in \{x \in X : x \le f(x)\}$. Since $u = \sup\{x \in X : x \le f(x)\}$, $f(u) \le u$. Thus $u = f(u)$ and $u \in F_f$. Since for every fixed point x of f, $f(x) \ge x$, $u = \sup F_f$.

The set X with the partial order $\preceq$, defined by $x \preceq y$ if $y \le x$, called the dual-order of $\le$ is a complete lattice on which f is again an isotone map (in the dual order). Further for any subset S of X, $\inf_{(\le)} S = \sup_{(\preceq)} S$ and $\sup_{(\le)} S = \inf_{(\preceq)} S$ where $\preceq$ is the dual-order of the partial order $\le$. So, by the first part of the theorem proved so far, $\inf\{x \in X : f(a) \le x\} = \sup\{x \in X : x \preceq f(x)\}$ is in F_f. Clearly $\inf F_f = \inf\{x \in X : f(x) \le x\}$.

Let Y be any subset of F_f. Then $([\sup Y, 1], \le)$ is a complete lattice, where $[a, b] = \{x \in X : a \le x \le b\}$. If $x \in Y$, then $x \le \sup Y$ and so $x = f(x) \le f(\sup Y)$ as f is isotone and $Y \subseteq F_f$. So $\sup Y \le f(\sup Y)$. If $\sup Y \le z$, then $\sup Y \le f(\sup Y) \le f(z)$. So f', the restriction of f to $[\sup Y, 1]$ maps the complete lattice $[\sup Y, 1]$ into itself and is isotone. So $v = \inf$ of all fixed points of f' (and indeed f) in $[\sup Y, 1]$ is itself a fixed point of f' by the preceding part of the theorem. This is, indeed, the least fixed point of f, which is an upper bound of all elements of Y. Similarly, consider the complete lattice $[0, \inf Y]$. If $x \in Y$, then $\inf Y \le x$ and $f(\inf Y) \le f(x) \le x$, as f is isotone and so $f(\inf Y) \le \inf Y$. Thus f^*, the restriction of f to $[0, \inf Y]$ maps this complete lattice into itself. So $u = \sup$ of all the fixed points of f^* in $[0, \inf Y]$ is itself a fixed point and indeed the greatest lower bound of all fixed points of f in Y. Thus, $\inf Y$ and $\sup Y$ are also fixed points of f. Thus, $(F_f, \le)$ is a complete lattice. □

Remark 3.3.4 The existence of a fixed point in Tarski's theorem can be proved along the lines of Theorem 3.2.2 as well.

The next theorem is also due to Tarski [18] and its proof is left as an exercise.

Theorem 3.3.5 *Let $(X, \le)$ be a complete lattice and G a commuting family of isotone maps of X into itself. Then F_G the set of common fixed points of all the functions f in G is nonempty and $(F_G, \le)$ is a complete lattice. Further* $\sup F_G = \sup\{x \in X : x \le f(x)$ *for all* $f \in G\}$ *and* $\inf F_G = \inf\{x \in X : f(x) \le x$ *for all* $f \in G\}$.

Remark 3.3.6 A decreasing map on a complete lattice may not have a fixed point. For example, the map $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

is decreasing (i.e. $f(x) \ge f(y)$ for $x \le y$) and has no fixed point.

Remark 3.3.7 Theorem 3.3.5 is not true for a non-commutative family of isotone maps. For instance, define $f : [0, 1] \to [0, 1]$ by $f(x) = \frac{1+x}{2}$ and $g : [0, 1] \to [0, 1]$ by $g(x) = \frac{1}{2}$ for all $x \in [0, 1]$. While both f and g are isotone and non-commuting, f and g do not have a common fixed point.

Remark 3.3.8 Any non-decreasing map of $[0, 1]$ into itself (being isotone) will always have a fixed point, even if it is not continuous. Contrast this with Theorem 2.1.5 and Corollary 3.1.7.

Remark 3.3.9 The fixed point guaranteed by Tarski's theorem is not unique. For the map $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} x, & x \in [0, \frac{1}{2}) \\ 1, & x \ge \frac{1}{2}. \end{cases}$$

$F_f = [0, \frac{1}{2}) \cup \{1\}$ is the set of fixed points. Although $\frac{1}{2} = \sup[0, \frac{1}{2})$ in $[0, 1]$, as a subset $Y = [0, \frac{1}{2})$ of the set of fixed points F_f of f the supremum is $1!$. F_f is a complete lattice.

Davis [7] had proved that the converse of Tarski's theorem that a lattice in which every isotone map has a fixed point is complete.

Based on Tarski's observation that $\inf\{x \in X : f(x) \leq x\}$ is a fixed point for f, an isotone self-map on a complete lattice X, Merrifield and Stein and Stein [12, 17] proved some fixed point theorems supplementing/generalizing Tarski's fixed point theorem in complete lattices. A few of these results are highlighted in the sequel, as these embellish the crux of the proof of Tarski's theorem.

Theorem 3.3.10 (Stein [16]) *Let $(X, \leq)$ be a complete lattice and $T : X \to X$, a map. Suppose there exist positive integers k and n such that for $x, y \in X$*

(i) $T^n(x) \leq y$ implies $T^k(T^n(x)) \leq T^k(y)$ and
(ii) $x \leq T^n(y)$ implies $T^k(x) \leq T^k(T^n(y))$.

Then T^k has a fixed point.

Proof Define $H = \{x \in X : x \geq T^{nk}(x)\}$ H is non-void as $1 = \sup X \in H$. For $x \in H$, $T^{nk}(x) \leq x$ and so $T^n(T^{(k-1)n}(x)) \leq x$. So by (i) $T^k(T^{nk}(x)) \leq T^k(x)$. Thus $T^{nk}(T^k(x)) \leq T^k(x)$. So $T^k(x) \in H$, for $x \in H$. Consequently $T^{jk}(x) \in H$, for $j = 1, 2, \ldots$, whenever $x \in H$. Also for $x \in H, x \geq T^k(x) \geq T^{2k}(x) \cdots \geq T^{jk}(x)$, $j \geq 2$.

Since n and k are fixed natural numbers, there exists a natural number of sufficiently large such that $(qn-1)k \geq n$. For $x \in H$, $T^{(qn-1)k}(x) \in H$ in view of the preceding paragraph, as $qn > 1$. Let $h = \inf H$. Then $h \leq T^{(qn-1)k}(x) = T^n(T^{(qn-1)k-n}(x))$. Using (ii), we get that $T^k(h) \leq T^k T^{(qn-1)k}(x) = T^{qnk}(x)$. As $x \in H$, $T^{qnk}(x) \in H$ and $T^{qnk}(x) \leq x$, $T^k(h) \leq T^{qnk}(x) \leq x$. Since x is an arbitrary element of H, $T^k(h) \leq h$. So by definition of H, $h \in H$ and so $T^k(h)$ also is in H. As $h = \inf H$ and $T^k(h)(\in H) \leq h$, $h = T^k(h)$. Thus T^k has a fixed point in X. □

Remark 3.3.11 The transposition $T(0) = 1$ and $T(1) = 0$ on the lattice $\{0, 1\}$ has no fixed point. But T satisfies all the conditions of Theorem 3.3.10 for $k = 2, n = 1$ and hence T^2 has a fixed point.

Next is a theorem on common fixed points.

Theorem 3.3.12 (Stein [12]) *Let $\mathscr{F}$ be a commuting family of maps on a complete lattice X into itself satisfying the conditions for $S, T \in \mathscr{F}$ (i) $x \leq S(y)$ implies $Tx \leq TS(y)$ and (ii) $S(x) \leq y$ implies $TS(x) \leq T(y)$. Then the family $\mathscr{F}$ has a common fixed point.*

Proof Define $H = \{x \in X : x \geq T(x), \text{for all } T \in \mathscr{F}\}$. Clearly $1 = \sup X \in H$. Let $h = \inf H$.

Let $x \in H$ and $T \in \mathscr{F}$. Then it follows from (ii) that for any $S \in \mathscr{F}$ and $y = x$, $TS(x) = ST(x) \leq T(x)$ and hence $T(x) \in H$. Since $T(x) \in H$ for $x \in H$, $h \leq T(x)$ from the definition of h. Invoking (i) for $h \leq Tx$, we get $Th \leq T(T(x)) \leq T(x) \leq x$ for all $x \in H$. So $T(h) \leq h$. From the definition of H, it follows that

$h \in H$. Since $T(x) \in H$ for $x \in H$, $T(h) \in H$. Since $h = \inf H$, $h \leq T(h)$. Hence $h = T(h)$. Thus for each $T \in \mathscr{F}$, h is a fixed point. In other words $\mathscr{F}$ has a common fixed point. □

The following definition is needed for the next theorem.

Definition 3.3.13 Let $(X, \leq)$ be a complete lattice and (x_n) a sequence in X. Then

$$\lim_{n\to\infty} \inf(x_n) = \sup_{n=1,2,\ldots} \{\inf(x_m : m \geq n)\}$$
$$\lim_{n\to\infty} \sup(x_n) = \inf_{n=1,2,\ldots} \{\sup(x_m : m \geq n)\}.$$

If $\lim\limits_{n\to\infty} \inf(x_n) = \lim\limits_{n\to\infty} \sup(x_n) = x$, then (x_n) is said to converge to x (with respect to the partial order $\leq$).

Theorem 3.3.14 *Let $(X, \leq)$ be a complete chain and $T : X \to X$ be a map such that $x \leq y$, $x, y \in X$ implies that there exists a natural number $N = N(x, y)$ such that for all $n \geq N$, $T^n(x) \leq T^n(y)$. Then T has a fixed point.*

Proof Suppose T has no fixed point. Then the sets $A = \{x \in X : x \geq T(x)\}$ and $B = \{x \in X : x \leq T(x)\}$ are disjoint and nonempty as $1 \in A$ and $0 \in B$.

Let $x \in A$. If for some natural number p, $T^p(x) \leq T^{p+1}(x)$, then by hypothesis there exists N_1 such that for $n \geq N_1$, $T^{p+n}(x) \leq T^{p+n+1}(x)$. As $Tx \leq x$, there exists N_2 such that for $n \geq N_2$, $T^{n+1}(x) \leq T^n(x)$. So for $n \geq \max\{N_1 + p, N_2\}$, $T^{n+1}(a) \leq T^n(x) \leq T^{n+1}(x)$ or $T^{n+1}(x) = T^n(x)$. Thus T has a fixed point, viz., $T^n(a)$. Since we have assumed that T has no fixed point, for $x \in A$, $T^p(x) \geq T^{p+1}(x)$ for all p so that $\lim\limits_{n\to\infty} \inf T^n(x) \leq x$. By a similar reasoning $\lim\limits_{n\to\infty} \inf T^n(x) \geq x$ for $x \in B$. Let $a = \inf A$. So for $x \in A$, $a \leq x$. Hence by hypothesis, there exists N such that for all $n \geq N$, $T^n(a) \leq T^n(x)$. So $\lim\limits_{n\to\infty} \inf T^n(a) \leq \lim\limits_{n\to\infty} \inf T^n(x) \leq x$ for each $x \in A$. So $\lim\inf T^n(a) \leq a = \inf A$.

If $a \in B$, then $\lim\limits_{n\to\infty} \inf T^n(a) = a$ implies $a = T(a) = T^2(a)\ldots$ and T has a fixed point.

If $T(a) < a$, then $T(a) \in A$. Since $a > T(a) \geq T^2(a)\ldots$, contradicting that a is a lower bound for A. So $T(a) = a$. Thus T has a fixed point. □

Remark 3.3.15 In a complete chain an operator T such that $\lim\limits_{n\to\infty} \inf T^n(x) \leq \lim\limits_{n\to\infty} \inf T^n(y)$ whenever $x \leq y$ may not have a fixed point. The map $T : [0, 1] \to [0, 1]$ defined by

$$T(x) = \begin{cases} \frac{x}{2}, & \text{for } x \neq 0 \\ 1, & \text{for } x = 0 \end{cases}$$

has no fixed point though $T^n(x) \to 0$ as $n \to \infty$.

The next theorem, also due to Merrifield and Stein [12] insures a common fixed point for a pair of isotone maps S and T for which, at each point of X, an iterate of S is majorized by an iterate of T and vice versa.

Theorem 3.3.16 (Merrifield and Stein [12]) *Let $(X, \leq)$ be a complete lattice and $S, T : X \to X$ be isotone maps. Suppose for each $x \in X$, there exist positive integers $p = p(x)$, $q = q(x)$, $i = i(x)$ and $j = j(x)$ (depending on x) such that $S^p(x) \leq T^q(x)$ and $T^i(x) \leq S^j(x)$. Then S and T have a common fixed point.*

Proof Define $H = \{x \in X : S(x), T(x) \leq x\}$. H is nonempty as $1 \in H$. Let $u = \inf\{S^k(x), T^k(x) : k = 0, 1, 2, \ldots\}$. As $u \leq T^n(h)$ for $h \in H$ for all non-negative integers n, since T is isotone, it follows from the definition of H that $T(u) \leq T^{n+1}(h) \leq T^n(h)$ for all $n = 0, 1, 2, \ldots$, for all $h \in H$.

By hypothesis, there exist non-negative integers p_1, q_1 such that $T^{p_1}(h) \leq S^{q_1}(h)$ for $h \in H$. Suppose we have chosen $p_1 < p_2 \cdots < p_k$, $q_1 < q_2 \cdots < q_k$ integers such that $T^{p_j}(h) \leq S^{q_j}(h)$ for $1 \leq j \leq k$. By hypothesis there exist positive integers a and b such that $T^a(T^{p_k}(h)) \leq S^b(S^{q_k}(h))$ (by the isotonicity of S). Let $p_{k+1} = p_k + a$ and $q_{k+1} = q_k + b$. Then $p_k < p_{k+1}$ and $q_k < q_{k+1}$ and $T^{p_{k+1}}(h) \leq S^{q_{k+1}}(h)$. For $n \geq 0$ choose k such that $q_k \geq n$. Then $u \leq T^{p_k - 1}(h)$ and so $T(u) \leq T^{p_k}(h) \leq S^{q_k}(h) \leq S^n(h)$ and the fact that for $h \in H$, $S^m(h) \leq S^{m-1}(h) \leq \cdots \leq S(h) \leq h$ for all $m \geq 1$. Thus $T(u) \leq T^n(h), S^n(h)$ for all $n = 0, 1, \ldots$ and all $h \in H$. So $T(u) \leq v$. By a similar argument $S(u) \leq u$. It follows from this that $u \in H$. Since $u \in H$, and $u \leq T(h), S(h)$ for all $h \in H$, $u \leq T(u), S(u)$. Hence $u = T(u) = S(u)$. □

Corollary 3.3.17 *Let $(X, \leq)$ be a complete lattice and $S, T : X \to X$ be isotone maps. If $(ST)^k = T^n$ or $(ST)^k S = T^n$ for some natural numbers k and n, then S and T have a common fixed point.*

Proof If $(ST)^k S = T^n$, then $(ST)^k ST = (ST)^{k+1} = T^{n+1}$. So it suffices to prove the first part of the corollary.

If $(ST)^j = T^m$ for natural numbers j and m, then (ST) and T satisfy the hypotheses of Theorem 3.3.16 and hence they have a common fixed point u. Thus $u = T(u) = S(T(u))$. Since $T(u) = u$, $S(T(u)) = S(u)$. Thus $u = T(u) = S(u)$. □

The following is a result akin to the corollary above, also due to Merrifield and Stein [12].

Theorem 3.3.18 *Let S and T be isotone maps on the complete lattice $(X, \leq)$ into itself. If $STS^n = T$ for some natural number n, then S and T have a common fixed point.*

Proof As before let $H = \{x \in X : S(x), T(x) \leq x\}$ H is non-void as $1 \in H$. Since S is isotone, for $h \in H$, $S(S(h)) \leq S(h)$ and also $T(S(h)) = STS^n(Sh) \leq ST(h) \leq S(h)$, as $T(h) \leq h$. Hence $S(h) \in H$. Let $u = \inf H$. Then $u \leq h$ for $h \in H$, $S(u) \leq S(h) \leq h$ and $T(u) \leq T(h) \leq h$ by the isotonicity of S and T. Hence $S(u), T(u) \leq u$ and so $u \in H$. Since $S(h) \in H$ for $h \in H$, $S(u) \in H$. Since $u = \inf H$, $u \leq S(u)$. Hence $u = S(u)$ $STS^n(u) = T(u)$ and so $ST(u) = T(u) \leq u$. So $T(u) \in H$. As $u = \inf H$, $u \leq T(u)$. Hence $u = T(u)$. Thus $u = T(u) = S(u)$. □

Corollary 3.3.19 *Let $(X, \leq)$ be a complete lattice and $S, T : X \to X$ be isotone maps with $STS^n = TS^p$ where $n > p$ and n, p are non-negative integers. Then S and T have a common fixed point.*

Proof In Theorem 3.3.18, replace T by TS^p and n by $n - p$. We now conclude that TS^p and S have a common fixed point which is a common fixed point of S and T as well. □

Merrifield and Stein [12] have provided a number of sufficient conditions to ensure that $u = \inf\{x \in X : S(x), T(x) \leq x\}$ is a common fixed point for a pair of isotone maps S and T in a complete lattice. The following theorem embodying such conditions is a sample and the proof is left as an exercise.

Theorem 3.3.20 *Let $(X, \leq)$ be a complete lattice and $S, T : X \to X$ be isotone maps. S and T have a common fixed point, if for each $x \in X$, $\sup\{S^n(x) : n = 1, 2, \ldots\}$, is equal to any one of the following:*

(a) $\sup\{T^n(x) : n = 1, 2, \ldots\}$;
(b) $\inf\{T^n(x) : n = 1, 2, \ldots\}$;
(c) $\lim_{n\to\infty} \inf\{T^n(x)\}$;
(d) $\lim_{n\to\infty} \sup\{T^n(x)\}$.

While a decreasing or antitone self-map on a complete lattice may not have a fixed point (vide Remark 3.3.6) Roth [14] noted that such a map has a fixed point under additional assumptions. We conclude this section by providing two such results for such maps, using the following.

Definition 3.3.21 Let $(X, \leq)$ be a complete lattice map $f : X \to X$ is called antitone (or decreasing) if $f(y) \leq f(x)$ whenever $x \leq y$, $x, y \in X$. A function $f : X \to X$ is called join antimorphism if $f(\sup A) = \inf f(A)$ for each nonempty subset A of X. $f : X \to X$ is called a meet antimorphism if $f(\inf A) = \sup f(A)$ for each nonempty subset A of X.

Remark 3.3.22 Clearly a join or meet antimorphism is antitone.

Theorem 3.3.23 (Blair and Roth [4]) *Let $h, g : X \to X$ be maps on a complete lattice $(X, \leq)$. Suppose h is isotone and g maps F_h, the set of fixed points of h in X into F_h. Then there exists $x \in X$ such that $x = h(x)$ and $x \leq g(x)$.*

Proof Since h is an isotone map on the complete lattice X, F_h the set of fixed points of h in X is non-void and $x_0 = \inf\{x \in X : h(x) \leq x\}$ is in F_h, as this is the smallest fixed point of h, in view of Tarski's Theorem 3.3.3. Since $x_0 = \inf F_h$ and g maps F_h into itself $h(x_0) = x_0 = \inf F_h \leq g(x_0)$ as $x_0 \in F_h$ and $x_0 \leq g(x)$ for all $x \in F_h$. Thus, there is an element x_0 with $x_0 = h(x_0)$ and $x_0 \leq g(x_0)$. □

Corollary 3.3.24 (Roth [14]) *If $f : X \to X$ is a join antimorphism on a complete lattice $(X, \leq)$, then there exists $x \in X$ such that $x = f^2(x)$ and $x \leq f(x)$.*

Proof As f is join antimorphism, f is antitone and f^2 is isotone. Setting $h = f^2$ and $g = f$ in Theorem 3.3.23 and noting that f maps F_{f^2} into itself, it follows that f^2 has a least fixed point x_0 and $x_0 \leq f(x_0)$. □

Theorem 3.3.25 (Blair and Roth [4]) *Let $(X, \leq)$ be a complete lattice and $u_1, u_2 : X \to X$ be maps such that $f = u_1 \circ u_2$ and $g = u_2 \circ u_1$ are isotone. Then there exist $x, y \in X$ such that $x = f(x) \leq u_1(y)$ and $y = g(y) \leq u_2(x)$.*

Proof Since f and g are isotone and X is a complete lattice F_f and F_g the sets of fixed points of the maps f and g, respectively, are nonempty and $x = \inf F_f \in F_f$ and $y = \inf F_g \in F_g$, by Tarski's fixed point Theorem 3.3.3. So $x = u_1u_2(x)$. So $u_2(x) = u_2u_1(u_2(x)) = g(u_2(x))$. So $u_2(x) \in F_g$, as $g = u_2 \circ u_1$. By definition of y and g, $y = g(y) = u_2u_1(y) \leq u_2(x)$. Similarly, as $y = u_2u_1(y)$, $u_1(y) = u_1(u_2u_1(y)) = (u_1u_2)(u_1(y)) = f(u_1(y))$. Hence, $u_1(y) \in F_f$, f being $u_1 \circ u_2$. From the definition of x, $x = f(x) \leq u_1(y)$. □

Remark 3.3.26 Theorem 3.3.25 and Corollary 3.3.24 have found applications in Game Theory.

3.4 Some Applications

Tarski's fixed point theorem central to lattice theoretical fixed point theorems is of wide applicability. In this section, we use it to prove Schroder–Bernstein theorem a basic result in set-theory, Cantor–Bendixon theorem, a representation theorem for closed sets in general topology and existence theorems for a Cauchy problem for a parabolic partial differential equation, a nonlinear complementarity problem and certain formal languages.

Theorem 3.4.1 (Schroder–Bernstein) *If f is a one-to-one mapping of a set A into a set B and g is a one-to-one mapping from B into A, then there is a bijection (i.e. a one-to-one mapping which is onto) of A onto B.*

Proof Let 2^A and 2^B be the power sets (sets of all subsets) of A and B, respectively. 2^A (and for that matter 2^B as well) is a complete lattice under set-inclusion as partial order (see Example 3.3.2(d)). Define $T : 2^A \to 2^B$ by $T(S) = A - g(B - f(S))$ for each subset S of A. For $A_1, A_2 \subseteq A$ and $A_1 \subseteq A_2$, $g(B - f(A_1)) \supseteq g(B - f(A_2))$ and so $A - g(B - f(A_1)) \subseteq A - g(B - f(A_2))$. In short $T(A_1) \subseteq T(A_2)$ or T is isotone on 2^A. So by Tarski's fixed point Theorem 3.3.3, T has a fixed point $S^* \subseteq A$. Thus $S^* = T(S^*) = A - g(B - f(S^*))$.

Define $h : A \to B$ by

$$h(x) = \begin{cases} f(x), & \text{if } x \in S^*, \\ g^{-1}(x), & \text{if } x \in A - S^* \end{cases}$$

Clearly h is well-defined on S^* (as f). If $x \in A - S^*$, then $x \in A - T(S^*) = g(B - f(S^*))$ and so $g^{-1}(x)$ is well-defined as g^{-1} is 1-1 on $g(B)$ and hence on $g(B - f(S^*))$. Further h is one-to-one and

$$\begin{aligned} h(A) &= h(S^* \cup A - S^*) \\ &= h(S^*) \cup h(A - S^*) \\ &= f(S^*) \cup g^{-1}(g(B - f(S^*)) \\ &= f(S^*) \cup B - f(S^*) \\ &= B. \end{aligned}$$

Hence h is a bijection of A onto B. □

Remark 3.4.2 Schroder–Berstein theorem is quite useful in proving that two sets have the same cardinality, for example (0, 1) ∪ (2, 3) and [−1, 0].

For the statement of Cantor–Bendixon theorem, we recall the definitions of a derived set, perfect set and a scattered set.

Definition 3.4.3 For a subset A of X where $(X, \mathscr{T})$ is a topological space, $x_0 \in X$ is called an accumulation point or a cluster point of A if every neighbourhood of x_0 contains a point of A other than x_0. The set of all accumulation points of A is called the derived set of A and is denoted by A'.

Definition 3.4.4 Let $(X, \mathscr{T})$ be a topological space. A subset A of X is said to be perfect if $A = A'$ (the derived set of A). (In other words, every point of A is an accumulation point of A and every accumulation point of A is in A).

A subset A of X is called scattered if the only subset B of A for which $B' \supseteq B$ is the empty set.

Example 3.4.5 For $\mathbb{R}$ with the usual topology, [0, 1] is a perfect set. So is the Cantor-ternary subset of [0, 1]. $\mathbb{N}$, the set of natural numbers is scattered. If X is a T_1-space, then the derived set (of a set) is closed.

Theorem 3.4.6 (Cantor–Bendixon) *Every closed subset of a topological space $(X, \mathscr{T})$ is a disjoint union of a perfect set and a scattered set.*

Proof Let A be closed subset of X. Define $B = \cup\{S \in 2^X : A \cap S' \supseteq S\}$ and $C = A - B$. Clearly, the map defined by $T(S) = A \cap S'$, $S \in 2^X$ is an isotone map on the complete lattice 2^X. So from Tarski's Theorem 3.3.3 and its proof $B = \sup\{S \in 2^X : T(S) \geq S\}$ is a fixed point for T. So $B = T(B) = A \cap B'$ or $B' \supseteq B$. Thus every point of B is an accumulation point of itself. Since A is closed and contains B and $\overline{B}$, $B' \subseteq \overline{B} \subseteq A$. Hence $A \cap B' = B'$. Thus $B = B'$ and B is perfect.

Clearly B is disjoint from $C = A - B$. For $S \subseteq C$ and $S' \supseteq C$, $A \cap S' \supseteq S$ since $C \subseteq A$. By the definition of B, $B \supseteq S$. So $S = S \cap C \subseteq B \cap C = \phi$. Hence S is empty. Thus C is scattered. The proof is complete. □

Finally, an application of Tarski's fixed theorem for the solution of a Cauchy problem for a parabolic differential equation due to Schäfer [15] is described.

Theorem 3.4.7 (Schäfer [15]) *Let E be a real Banach space and $g : \mathbb{R} \times [0, T] \to E$ be a bounded continuous function such that*

$$\|g(x, t) - g(y, t)\| \leq L|x - y|$$

for $x, y \in \mathbb{R}$ for each $t \in [0, T]$, an interval of real numbers, for some positive constant L.

Then the function u defined by

$$u(x, t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} g(\xi, t) d\xi dt$$

for $x \in \mathbb{R}$, $t \in [0, T]$ is a solution of the Cauchy problem

$$u_t(x, t) - u_{xx}(x, t) = g(x, t), \ x \in \mathbb{R}, t \in [0, T]$$
$$u(x, 0) = 0, \ x \in \mathbb{R}.$$

Remark 3.4.8 Schäfer [15] observed that the above theorem can be proved along the lines of proof of an existence theorem in Friedman [8].

We will use the above theorem to prove the following application of Tarski's theorem to the Cauchy problem for a parabolic equation.

Theorem 3.4.9 *Let $f : \mathbb{R} \times [0, T] \times \ell^\infty(A) \to \ell^\infty(A)$ be a function with the following properties, A being a nonempty set, and $\ell^\infty(A)$ the Banach space of bounded real functions on A.*

(i) f is continuous;
(ii) for a constant $L_1 > 0$, for all $(x, t, z), (y, t, z) \in \mathbb{R} \times [0, T] \times \ell^\infty(A)$

$$\|f(x, t, z) - f(y, t, z)\| \leq L_1|x - y|;$$

(iii) for some $L_2 > 0$, for all $(x, t, z_1), (x, t, z_2)$ in $\mathbb{R} \times [0, T] \times \ell^\infty(A)$

$$\|f(x, t, z_1) - f(x, t, z_2)\| \leq L_2\|z_1 - z_2\|;$$

(iv) for $z_1, z_2 \in \ell^\infty(A)$ for $(x, t) \in \mathbb{R} \times [0, T]$ $z_1(a) \leq z_2(a)$ for all $a \in A$ implies

$$f(x, t, z_1) \leq f(x, t, z_2);$$

(v) for a constant $M > 0$, for all (x, t, z) in $\mathbb{R} \times [0, T] \times \ell^\infty(A)$, $\|f(x, t, z)\| \leq M$.

Then there exists a continuous function $u : \mathbb{R} \times [0, T] \to \ell^\infty(A)$ *with*

$$u_t(x,t) - u_{xx}(x,t) = f(x,t,u(x,t)) \text{ for } (x,t) \in \mathbb{R} \times [0,T] \text{ and}$$
$$u(x,0) = 0 \text{ for } x \in \mathbb{R}$$

Proof Define Ω as the set of functions $\{w : \mathbb{R} \times [0,T] \to \ell^\infty(A)$ with $\|w(x,t) - w(y,s)\| \le L_3|x-y| + L_4|t-s|$, for $x, y \in \mathbb{R}$ and $t, s \in [0,T]$, with $-\Gamma(a) \le w(a) \le \Gamma(a)$ for all $a \in A$, where $\Gamma(a) \equiv TM$, $L_3 = 2M\sqrt{T}$ and $L_4 = M + (L_1 + L_2L_3)2\sqrt{\frac{T}{\pi}}\}$.

As $\Gamma \in \Omega$, Ω is non-void. For each $w \in \Omega$ define

$$\phi(w)(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} f(\xi,\tau,w(\xi,\tau))d\xi dt$$

for $(x,t) \in \mathbb{R} \times [0,T]$.

Since the integrand is continuous $\phi(w)$ is continuous on $\mathbb{R} \times [0,T]$. Further, from the definition of Ω and the hypotheses (ii) and (iii), it follows that for $(x,t), (y,t) \in \mathbb{R} \times [0,T]$.

$$\begin{aligned}\|f(x,t,w(x,t)) - f(y,t,w(y,t))\| &\le \|f(x,t,w(x,t)) - f(y,t,w(x,t))\| \\ &\quad + \|f(y,t,w(x,t)) - f(y,t,w(y,t))\| \\ &\le L_1|x-y| + L_2\|w(x,t) - w(y,t)\| \\ &\le (L_1 + L_2L_3)|x-y|. \qquad (vi)\end{aligned}$$

So by Theorem 3.4.7, for $w \in \Omega$, $(x,t) \in \mathbb{R} \times [0,T]$

$$\phi_t w(x,t) - \phi_{xx} w(x,t) = f(x,t,w(x,t)) \text{ and } \phi(w)(x,0) = 0 \qquad (vii)$$

are satisfied.

We now show that Ω is a complete lattice and ϕ maps Ω into itself and ϕ is isotone. Since $\Gamma \in \Omega$, Ω is nonempty. Let S be a subset of Ω which is non-void. For each $s \in S$, $-TM \le s(x,t)(a) \le TM$ for all $a \in A$ and $(x,t) \in \mathbb{R} \times [0,T]$. $\sup_{a \in A} s(x,t)(a)$ for $(x,t) \in \mathbb{R} \times [0,T]$ lies between $-TM$ and TM. Further, as $\|s(x,t) - s(y,t')\| \le L_3|x-y| + L_4\|t-t'|$, $\|\sup_S s(x,t) - \sup_S s(y,t')\| \le L_3|x-y| + L_4\|t-t'|$. Thus $\sup S \in \Omega$. By a similar reasoning $\inf S \in \Omega$. Hence Ω is a complete lattice.

For $w \in \Omega$, since $-\Gamma \le w(x,t) \le \Gamma$ it follows that $-\Gamma \le \phi(w)(x,t) \le \Gamma$. Further for $s < t$, $s, t \in [0,T]$ $\|\phi(w(x,t)) - \phi(w(x,s))\| \le J_1 + J_2$, where $J_1 = \left\| \int_0^s \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} f(\xi,\tau,w(\xi,\tau)\left[\frac{e^{-\frac{(x-\xi)^2}{4(t-\tau)}}}{\sqrt{4\pi(t-\tau)}} - \frac{e^{-\frac{(x-\xi)^2}{4(s-\tau)}}}{\sqrt{4\pi(s-\tau)}}\right] d\xi d\tau \right\|$ and $J_2 = \left\| \int_s^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} f(\xi,\tau,w(\xi,\tau))d\xi d\tau \right\|$. Clearly $J_2 \le M|t-s|$ as

$\|f(\xi, \tau, w)\| \leq M$. Now

$$\begin{aligned} J_1 &\leq \int_0^s \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-\alpha^2} \| f(x - 2\alpha\sqrt{t-\tau}, \tau, w(x - 2\alpha\sqrt{t-\tau}, \tau) \\ &\qquad - f(x - 2\alpha\sqrt{s-0}, 0, w(x - 2\alpha\sqrt{s-\tau}, \tau) \| d\tau d\alpha \\ &\leq \frac{1}{\sqrt{\pi}}(L_1 + L_2L_3) \int_0^s (\sqrt{t-\tau} - \sqrt{s-\tau}) \int_{-\infty}^{\infty} e^{-\alpha^2} 2|\alpha| d\alpha d\tau \ \text{ using } (vi) \\ &\leq \frac{1}{\sqrt{\pi}}(L_1 + L_2L_3) \frac{2}{3}(t^{\frac{3}{2}} - s^{\frac{3}{2}} - (t-s)^{\frac{3}{2}})2 \\ &\leq \frac{1}{\sqrt{\pi}}(L_1 + L_2L_3)\sqrt{T}|t-s|. \end{aligned}$$

So $J_1 + J_2 \leq \{M + (L_1 + L_2L_3)2\sqrt{\frac{T}{\pi}}\}|t-s|$. Thus $\|\phi(w)(r,t) - \phi(w)(x,s)\| \leq L_4|t-s|$. Also for each $a \in A$, $\phi(w(a))(x,s) - \phi(w(a))(y,s) = \frac{\partial}{\partial x}\phi(w)(a)(\eta, s)(x-y)$ by the mean value theorem, where η lies between x and y. So $|\phi(w)(a)(x,s) - \phi(w(a))(y,s)| \leq 2M\sqrt{T}|y-s| = L_3|y-s|$. Thus $\|\phi(w)(x,t) - \phi(w)(y,s)\| \leq L_3|x-y| + L_4|t-s|$. Clearly $-\Gamma \leq \phi(w)(a) \leq \Gamma$ for $w \in \Omega$. So ϕ maps Ω into itself and is isotone. As Ω is a complete lattice by Tarski's fixed point Theorem 3.3.3 ϕ has a fixed point u. Clearly u satisfies (vii) and is thus a solution to the Cauchy problem. □

Chitra and Subrahmanyam [6] obtained an interesting application of Tarski's fixed point theorem to a nonlinear complementarity problem supplementing the solution of a class of nonlinear complementary problems in the space of real-valued continuous functions on a compact Hausdorff space due to Fujimoto [9]. A brief description of their main theorem is described below.

Let $L_p[a,b]$ be the Banach space of all real-valued Lebesgue measurable functions $f : [a,b] \to \mathbb{R}$ with the norm $\|f\| = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}$, $1 \leq p < \infty$. Let K be the cone of functions in this space which are non-negative almost everywhere. $L_p[a,b]$ is partially ordered by the relation $x \geq y$ $(x, y \in L_p[a,b])$ if $x(t) \geq y(t)$ $a.e.$ Of course, functions which are almost everywhere equal are considered equal.

The nonlinear complementarity problem (NLCP) is to find $x \in K$ $(\subseteq L_p[a,b])$ such that

$$x - Tx - b \geq 0 \text{ and } \min(x, x - Tx - b) = 0$$

where b is a given element in $L_p[a,b]$ and T maps K (the cone in $L_p[a,b]$) into $L_p[a,b]$. We have the following:

Theorem 3.4.10 ([6]) *Suppose*

(i) $T : K \to L_p[a,b]$ *is monotone with respect to* K *(i.e.* $Tx \geq Ty$, *if* $x \geq y$, $x, y \in K$*);*

(ii) there exists an x_0 *in* K *with* $x_0 - Tx_0 - b \geq 0$.

Then the above NLCP has a solution.

Proof Consider the set $D = \{x \in K : x \leq x_0\}$ where x_0 satisfies (ii). The mapping F defined on D by $F(x) = \max(Tx + b, 0)$ is non-negative and $Fx \leq x_0$ for all $x \in D$, by the monotonicity of T and (ii). Thus F is a monotone map of D into itself. From the separability of $L_p[a, b]$ and the fact that every Cauchy sequence in $L_p[a, b]$ has a subsequence which converges pointwise a.e in $L_p[a, b]$, it follows that D is a complete lattice. So by Tarski's theorem, the monotone operator F has a fixed point x^* in the complete lattice D. Clearly for $x^* \in D$, $x^* - Tx^* - b \geq 0$ and $\min(x^*, x^* - Tx^* - b) = 0$. Further $x^* \leq x_0$. □

Remark 3.4.11 The above fixed point x^* can be obtained as the limit of the monotonic sequence of iterates defined by $x_1 = \max(Tx_0 + b, 0)$ $x_{n+1} = \max(Tx_n + b, 0)$ $n \in \mathbb{N}$. Define $E = \{x \in K : x - Tx - b \geq 0\}$. As $\inf E = \inf\{x \in K, x \leq x^0, x - Tx - b \geq 0\} = \inf\{x \in D : Fx \leq x\}$, from Tarski's theorem $\inf E$ is a solution to the NLCP.

Corollary 3.4.12 *Let* $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ *be a function such that*

(i) $f(s, u)$ *is continuous with respect to* u *for almost all* $s \in [0, 1]$ *and measurable with respect to a for all* $u \in \mathbb{R}$*;*
(ii) $|f(s, u)| \leq a_0(s) + b_0|u|$ *where* $b_0 > 0$ *as* $a_0(s) \in L_p[0, 1]$*; and*
(iii) *for almost all* s*,* $f(s, u)$ *is increasing with respect to* u *and there exists* $x_0(s), b(s) \in L_p[0, 1]$ *such that* $x_0(t) - f(t, x_0(t)) - b(t) \geq 0$ *for almost all* $t \in [0, 1]$.

Then the problem of finding $x \in K \subseteq L_p[0, 1]$ *such that* $x(t) - f(t, x(t)) - b(t) \geq 0$ *and* $\min(x(t), x(t) - f(t, x(t)) - b(t)) = 0$ *has a solution.*

Proof Define $T : K \to Lp[0, 1]$ by $Tx(t) = f(t, x(t))$. From (i) and (ii), it follows that $Tx \in L_p[0, 1]$ for $x \in L_p[0, 1]$. Using (iii), it can be seen that conditions (i) and (ii) of Theorem 3.4.10 are fulfilled by T. So by Theorem 3.4.10 F has a fixed point, where $F = \max\{Tx + b, 0\}$ has a fixed point. In other words the NLCP has a solution. □

Remark 3.4.13 For the choice $f(s, u) = \cos\left(\frac{\sqrt{s}}{1+u}\right)$, $x_0 \equiv 1$ and $b \equiv 0$, the nonlinear complementarity problem $x(s) \geq 0$, $x(s) - \cos\left[\frac{\sqrt{s}}{1+x(s)}\right] \geq 0$ and $\min\left(x(s), x(s) - \cos\left(\frac{\sqrt{s}}{1+x(s)}\right)\right) = 0$ has a solution. It can also be proved that this solution is unique. It may be noted that this operator is not completely continuous.

Finally follows an application of Tarski's theorem for proving the existence of certain formal languages. Our alphabet consists of just two letters α and β. Words are formed by concatenation of these letters. For example, $\alpha\beta$, $\alpha\alpha\beta$ are words and words have finite length. The empty word denoted by ϕ is also allowed. There is also a grammar to this formal language. More specifically, it has three grammatical terms:

(i) the term A - consisting of all words containing α and ϕ;
(ii) the term B - comprising all words containing β and ϕ;
(iii) the term C - made of all words belonging to A or B.

The rules of the grammar are clear:

(i) for $\eta \in A$, $\eta\alpha \in A$;
(ii) for $\eta \in B$, $\eta\beta \in B$;
(iii) for $\eta \in C$, if and only if $\eta \in A$ or $\eta \in B$;
(vi) $\phi \in A, B, C$.

Theorem 3.4.14 *There exists a formal language fulfilling the above conditions.*

Proof Let W be the set of all words and X the set of all triples $x = (A, B, C)$ where A, B and C are subsets of words as described above. Define a partial order $\leq$ on X by $(A_1, B_1, C_1) \leq (A_2, B_2, C_2)$ if $A_1 \subseteq A_2$, $B_1 \subseteq B_2$ and $C_1 \subseteq C_2$. It can be seen that X is a complete lattice. Define $f : X \to X$ by $f(A, B, C) = (A\alpha, B\beta, A \cup B)$. Clearly f is isotone and by Tarski's theorem f has a least fixed point. Indeed for $A = \bigcup_{n=0}^{\infty} A_n$, $B = \bigcup_{n=0}^{\infty} B_n$ and $C = \bigcup_{n=0}^{\infty} C_n$, where $A_0 = B_0 = \phi$ and $A_{n+1} = A_n\alpha$, $B_{n+1} = B_n\beta$ and $C_{n+1} = A_n \cup B_n$, $n = 0, 1, 2, \ldots$ (A, B, C) corresponds to this least fixed point where $A = \bigcup_{n=0}^{\infty} A_n$, $B = \bigcup_{n=0}^{\infty} B_n$ and $C = \bigcup_{n=0}^{\infty} C_n$. □

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Chapter 4
Partially Ordered Topological Spaces and Fixed Points

A partial order on a set induces a natural topology on this set, and special properties of the partial order influence this topology significantly. These aspects lead to new and interesting fixed point theorems. The interconnections among partial order, topology and fixed point property were systematically investigated by Wallace [11], Ward [12] and Manka [6]. This chapter highlights these contributions to fixed point theory and supplements the theorems detailed in the preceding chapter.

4.1 A Precis of Partially Ordered Topological Spaces

This section is a precis of the fundamental contributions of Ward Jr. [12], although Wallace [11] had already pointed out the importance of partial order and the induced topology for fixed point theorems. Using the definition of a quasi-ordered set (see Definition 1.1.10), we define certain sets as in the following.

Definition 4.1.1 Let $(X, \leq)$ be a quasi-ordered set (viz. $\leq$ is a reflexive, transitive binary relation on X). For $A \subseteq X$, we define

$$\begin{aligned} L(A) &= \{y \in X : y \leq x \text{ for some } x \in A\}, \\ M(A) &= \{y \in X : x \leq y \text{ for some } x \in A\}, \\ E(A) &= L(A) \cap M(A). \end{aligned}$$

$L(A)$ is called the set of predecessors of A and $M(A)$, the set of successors of A.

Remark 4.1.2 Clearly $A \subseteq E(A)$, as $A \subseteq L(A), M(A)$.

Definition 4.1.3 Let $(X, \leq)$ be a quasi-ordered set and $A \subseteq X$. If $A = L(A)(M(A))$, A is said to be monotone decreasing (monotone increasing) or simply decreasing (increasing).

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Remark 4.1.4 If $(X, \leq)$ is a quasi-ordered set and $A \subseteq X$, and A is decreasing (increasing), then $X - A$ is increasing (decreasing). Also, the union (intersection) of a family of increasing (decreasing) sets is increasing (decreasing). Further, for each subset A of a quasi-ordered set X, $M(A)$ $(L(A))$ is increasing (decreasing).

Definition 4.1.5 Let $(X, \mathscr{T})$ be a topological space with a quasi-order $\leq$. This quasi-order is said to be lower semicontinuous (upper semicontinuous) provided for $a \leq b$ $(b \leq a)$ in X, there exists an open set U with $a \in U$ such that for $x \in U$, $x \leq b$ $(b \nleq x)$. The quasi-order is said to be semicontinuous if it is both lower and upper semicontinuous. The quasi-order is called continuous, if whenever $a \nleq b$, there are open sets U and V in X with $a \in U$, $b \in V$ and for $x \in U$ and $y \in V$, $x \nleq y$.

A quasi-ordered topological space QOTS for short is a topological space with a semicontinuous quasi-order. If the quasi-order is a partial order, then the corresponding quasi-ordered topological space is called a partially ordered topological space or POTS, for short.

Remark 4.1.6 Birkhoff [1] may be referred for a detailed discussion of quasi-order. The definitions and remarks given above are from Ward [12] and Wallace [11].

For (x_1, y_1), (x_2, y_2) belonging to the Euclidean space $\mathbb{R}^2$ a partial order $\preceq$ can be defined by $(x_1, y_1) \preceq (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$. $(\mathbb{R}^2, \preceq)$ with the Euclidean topology is a partially ordered topological space (POTS).

The proof of the following proposition is straightforward and is left as an exercise.

Proposition 4.1.7 *Let $(X, \leq)$ be a quasi-ordered set. It is QOTS if and only if for each $x \in X$. $L(x)$ and $M(x)$ are closed subsets of X. Hence in a QOTS for each x, $E(x)$ is a closed set.*

The following lemmata are also needed in the sequel.

Lemma 4.1.8 *Let X be a topological space with a quasi-order. Then the following are equivalent:*

(i) the quasi-order is continuous;
(ii) the graph of the quasi-order is closed in $X \times X$;
(iii) for $a \nleq b$ in X, there exist disjoint neighbourhoods N and N' of a and b, respectively, such that N is increasing and N' is decreasing.

Proof (i) $\implies$ (ii). Let G be the set $\{(a, b) \in X^2 : a \leq b\}$. Suppose $(a, b) \notin G$. Since $a \nleq b$ and the quasi-order is continuous by Definition 4.1.5, there are open sets U and V with $a \in U$, $b \in V$ such that for $(x, y) \in U \times V$, $x \nleq y$. So $U \times V \subseteq X^2 - G$ and $(a, b) \in U \times V$, an open set containing an arbitrary element (a, b) of $X^2 - G$. So $X^2 - G$ is open and G is closed in X^2.

(ii) $\implies$ (i). Let G, the graph of the quasi-order be closed in X^2. Let $a \nleq b$, where $a, b \in X$. Since $(a, b) \in X^2 - G$ and $X^2 - G$ is open, there exist open neighbourhoods U and V of a and b, respectively, in X such that $U \times V \subseteq X^2 - G$. For $x \in U$ and $y \in V$, $x \leq y$ would imply that $(x, y) \in G$, contradicting that $(x, y) \in U \times V \subseteq X^2 - G$. Hence $x \nleq y$. So the quasi-order is continuous.

(i) $\Longrightarrow$ (iii). Let $a, b \in X$ and $a \nleq b$. Then $N = X - L(b)$ and $N' = X - M(a)$ are open sets containing a and b, respectively. While N is increasing, N' is decreasing. If $x_0 \subseteq N \cap N'$, then $x_0 \nleq x_0$, a contradiction. So $N \cap N' = \phi$.

(iii) $\Longrightarrow$ (i). Let $a \nleq b$. Then there exist disjoint neighbourhoods N and N' of a and b such that N is increasing and N' is decreasing. Let $x \in N$ and $y \in N'$. If $y \in N \cap N'$ contradicting that N and N' are disjoint. So $x \nleq y$. □

Lemma 4.1.9 *A continuous quasi-order is semicontinuous. A partially ordered topological space is a T_1 space and a partially ordered topological space with continuous partial order is a Hausdorff space.*

We merely prove the last part of this lemma, leaving the other parts as exercises.

Proof Let $(X, \leq)$ be a POTS with a continuous partial order. Let a and b be two distinct points. If $a \nleq b$, by Lemma 4.1.8, there are disjoint neighbourhoods of a and b. If $a \leq b$ then $b \nleq a$ and again by Lemma 4.1.8, there are disjoint neighbourhoods of b and a. □

Lemma 4.1.10 *If X is a QOTS with a linear quasi-order, then the quasi-order is continuous.*

Proof Since X is a QOTS, the quasi-order is semicontinuous. To prove the continuity of the quasi-order, let $a, b \in X$ and $a \notin b$. So $b \leq a$. So $b \notin E(a)$. If there exists c such that $b \leq c \leq a$, then $a \in U = X - L(c)$ and $b \in V = X - M(c)$ and U and V are open sets by Proposition 4.1.7. If $x \in U$ and $y \in V$, then $x \nleq c$ or $c \leq x$ and $c \nleq y$ or $y \leq c$. So $y \leq x$ or $x \nleq y$.

If there is no such c with $b \leq c \leq a$, then $U = X - L(b)$ and $V = X - M(a)$ are open sets containing a and b, respectively, and for $x \in U$ and $y \in V$, $x \nleq b$ or $b \leq x$ and $a \nleq y$ or $y \leq a$. If $x \leq y$, then $b \leq a$ and this contradicts the assumption that there is not c with $b \leq c \leq a$. Hence $x \nleq y$. □

The existence of maximal chains (maximal linearly ordered subsets) in a quasi-ordered set was proved by Wallace [11], using Zorn's lemma. Wallace [11] also proved

Theorem 4.1.11 (Wallace [11]) *Every maximal chain in a QOTS is a closed set.*

Proof Let C be a maximal chain in a QOTS. Let $x \in C$. Then $x \in L(x) \cup M(x)$. For $c \in C$, $x \leq c$ or $c \leq x$. If $x \leq c$, then $c \in M(x)$ and if $c \leq x$, then $c \in L(x)$. So $C \subseteq L(x) \cup M(x)$ for each $x \in C$. Hence $C \subseteq D = \bigcap_{x \in C} \{L(x) \cup M(x)\}$. If $x' \in D$, then $x' \leq x$ or $x \leq x'$ for each $x \in C$. If $x' \notin C$, then $x' \cup C$ is a chain containing C properly, contradicting the maximality of C. Hence $x' \in C$ or $D \subseteq C$. Thus $C = D$. By Proposition 4.1.7 $L(x)$ and $M(x)$ and hence $L(x) \cup M(x)$ are closed sets. Hence D $(= C)$ is closed. □

Definition 4.1.12 Let $(X, \leq)$ be a quasi-ordered set. $y \in X$ is called minimal (maximal) if $x \leq y$ $(y \leq x)$ in X implies $y \leq x$ $(x \leq y)$.

For the following basic theorem due to Wallace [11], a proof due to Ward is sketched.

Theorem 4.1.13 *Let X be a non-empty compact space with a lower (upper) semicontinuous quasi-order $\leq$. Then X has a minimal element.*

Proof Let $\mathcal{L} = \{L(x) : x \in X\}$. $\mathcal{L}$ can be partially ordered with respect to set inclusion. Clearly, if $\{L(x_\lambda) : x_\lambda \in X, \lambda \in \Lambda\}$ is a chain in $\mathcal{L}$, $\bigcap\limits_{\lambda\in\Lambda} L(x_\lambda)$ is the intersection of a family of closed sets with finite intersection property in the compact space X. So $\bigcap\limits_{\lambda\in\Lambda} L(x_\lambda)$ is non-empty. Let $x_0 \in \bigcap_{\lambda\in\Lambda} L(x_\lambda)$. So $L(x_0)$ is a lower bound for each $L(x_\lambda)$, $\lambda \in \Lambda$. So by Zorn's Lemma $\mathcal{L}$ has a maximal element $L(x_0)$. If $x' \leq x_0$ and $x' \neq x_0$, then $L(x') \geq L(x_0)$ contradicting the maximality of $L(x_0)$. So $x' \in L(x_0)$ or $x_0 \leq x'$. Thus x_0 is a minimal element. The proof for the upper semicontinuous case is similar and left as an exercise, with the hint that one has to consider the closed set $M(x)$ instead. □

The above theorem leads to an interesting proof of a theorem, due to Moore, on the existence of non-cutpoints of a continuum. A few requisite definitions are recalled.

Definition 4.1.14 A continuum is a compact connected Hausdorff space. A non-degenerate continuum (viz. containing more than a point) is called indecomposable if it is not a union of two of its proper subcontinua. A property is said to be hereditary for a continuum if each of its non-degenerate subcontinua has this property. A continuum is called unicoherent if for each representation of it as a union of two subcontinua, the common part of these subcontinua is connected.

Definition 4.1.15 Let X be a connected topological space. A point $c \in X$ is called a cutpoint if $X - \{c\}$ is disconnected. A non-cutpoint of X is an element which is not a cutpoint.

Remark 4.1.16 Generally, metrizable continua are studied in detail. Apart from those continua defined in Definition 4.1.14, there are various classes of continua such as tree-like continua, Peano continua and so on. Fixed point property for continuous functions on such continua is a topic of active research.

For $[a, b]$ in $\mathbb{R}$, every interior point is a cutpoint while no point of the rectangle $[a, b] \times [c, d]$ or a circle is a cut point.

The machinery of POTS developed so far can be used to prove the following theorem due to Moore. The proof is due to Ward [12].

Theorem 4.1.17 (Moore, See Wilder [16]) *A non-degenerate continuum has at least two non-cutpoints.*

Proof For the non-degenerate continuum X let N be the set of non-cut points. Suppose N has at most one point. So there exists $x_0 \in X - N$. As x_0 is a cutpoint, $X - \{x_0\} = A \cup B$ where A and B are non-empty separated sets. Without loss of generality we can assume that $N \subseteq B$. So each point of A is a cut point. For each $x \in A$, there is a decomposition $X - \{x\} = A(x) \cup B(x)$ where $A(x)$ and $B(x)$ are non-void separated sets with $x_0 \in B(x)$ and so $A(x) \subseteq A$. We can define a partial order on $\overline{A}$ by $x \leq y$ if and only if $A(x) \subseteq A(y)$. Since $\overline{A} \cap B(x) = \phi$, $x \in \overline{A} - A$ cannot be a non-cutpoint different from x_0, possibly the only non-cutpoint lying in B. So $L(x) = \overline{A}(x)$. Hence this partial order is lower semicontinuous. Since X is compact, $\overline{A}$ is compact. So by Theorem 4.1.13, there is a minimal element $p \in \overline{A}$. Since p is minimal, $A(p)$ is empty. Otherwise suppose $q \in A(p)$. Then $q \in \overline{A}(p) = L(p)$. So $q \leq p$. Since p is a minimal element of $\overline{A}$, $q = p$. So $p \in A(p)$ and this contradicts the fact that $X - \{p\} = A(p) \cup B(p)$. So $A(p)$ is empty. This again contradicts the construction that $A(x)$ is non-empty for all $x \in \overline{A}$. Hence our assumption, that there is only one non-cutpoint, viz. x_0 is wrong. So a non-degenerate continuum has at least two non-cut points. □

The concept of convexity given below is useful for the study of fixed point theorems in QOTS.

Definition 4.1.18 A subset A of a QOTS is called convex if $A = E(A)$. X is said to be quasi-locally convex if for $x \in X$ and $E(x) \subseteq U$, an open set, there is a convex open set V such that $E(x) \subseteq V \subseteq U$. X is called locally convex, provided, for $x \in X$ and U an open set, there is a convex open set V with $x \in V \subseteq U$.

The next theorem due to Ward [12], an extension of a theorem of Nachbin on compact POTS, is stated without proof.

Theorem 4.1.19 *Let X be a compact QOTS with a continuous quasi-order and $b \nleq a$, where $a, b \in X$. Then we can find a continuous order-preserving map $f : X \to [0, 1]$ such that $f(a) = 0$ and $f(b) = 1$.*

Using the above theorem, Ward proved

Theorem 4.1.20 *A compact Hausdorff QOTS with a continuous quasi-order is quasi-locally convex.*

Proof Let X be a compact Hausdorff QOTS, $x \in E(x) \subseteq U$, where U is an open subset of X. For $t \in X - U$, $t \nleq x$ or $x \nleq t$. If $t \nleq x$, then by Theorem 4.1.19 above there is a continuous order-preserving map $f_t : X \to [0, 1]$ with $f_t(x) = 0$ and $f_t(t) = 1$. Let $U_t = \{y \in X : f_t(y) < \frac{1}{2}\}$. Then U_t is a decreasing open set containing x and $t \notin \overline{U}_t$. If $x \nleq t$, then $U_t = \{y \in X : \frac{1}{2} < g_t(y)\}$ is an increasing open set containing x and $t \notin \overline{U}_t$, where $g_t : X \to [0, 1]$ is a continuous order-preserving map with $g_t(t) = 0$ and $g_t(x) = 1$, as guaranteed by Theorem 4.1.19. Thus $X - U \leq U\{X - \overline{U}_t : t \in X - U\}$ with $X - U \leq \bigcup_{i=1}^{n} X - \overline{U}_{t_i}$.

$U = \bigcap_{i=1}^{n} U_{t_i}$ is an open set containing x and disjoint from $X - U$. Since $E(x) \subseteq U_{t_i}$ as U_{t_i} is either increasing or decreasing, $x \in E(x) \subseteq V \subseteq U$. So X is quasi-locally convex. □

Noting that a POTS with a continuous partial order is Hausdorff, we have

Corollary 4.1.21 *A compact POTS with a continuous partial order is locally convex.*

We proceed to study the convergence and clustering of nets in QOTS.

Definition 4.1.22 Let $(X, \leq)$ be a quasi-ordered set and $\{x_\lambda, \lambda \in D\}$ be a set in X where $(D, \leq_1)$ is a directed set (x_λ) is said to be monotone increasing (decreasing) if $x_\lambda \leq x_\mu$ for $\lambda \leq_1 \mu$ $(\mu \leq_1 \lambda)$ in D.

Definition 4.1.23 Let X be a topological space. A net $(x_\lambda, \lambda \in D)$, where $(D, \leq_1)$ is a directed set, is said to cluster at $x_0 \in X$ if for any open set U containing x_0 and $\lambda \in D$, there exists $\mu \in D$ with $\lambda \subseteq \mu$ such that $x_\mu \in U$.

Lemma 4.1.24 *Let X be a compact Hausdorff QOTS with a continuous order $\leq$. Then every monotone net in X clusters and the set of cluster points is contained in $E(x_0)$ for some $x_0 \in X$.*

Proof Without loss of generality, we assume that the net is monotone increasing. Since X is compact, every net in X has a subset converging to some $x_0 \in X$, and hence clusters at x_0. This is also true of a monotone increasing net (x_λ). By Theorem 4.1.20 X is quasi-locally convex. So given an open set U containing $E(x_0)$, there is an open convex set V such that $E(x_0) \subseteq V \subseteq U$. Since (x_λ) clusters at x_0, $x_{\lambda_0} \in V$ for some λ_0. For $\lambda_0 \leq_1 \lambda$, there exists λ' with $\lambda \leq_1 \lambda'$ such that $x_{\lambda'} \in V$ (Here the net is indexed over the directed set $(D, \leq_1)$). Since V is convex and $x_{\lambda'} \in V$, $x_\lambda \in V$ for all $\lambda \geq \lambda'$. Suppose $x \notin E(x_0)$. Then there are disjoint open sets U_1 and U_2 such that $x \in U_1$ and $x_0 \in U_2$, as X is Hausdorff. So U_1 and $V \cap U_2$ are disjoint and hence (x_λ) cannot cluster at x as otherwise U_1 and $V \cap U_2$ would intersect. Hence the cluster points of (X_λ) are contained in $E(x_0)$. □

Corollary 4.1.25 *If X is a compact POTS with continuous order, then every monotone net in X is convergent.*

The following lemmata can be proved easily.

Lemma 4.1.26 *Let X be a topological space and $f : X \to X$ be a continuous map. If for some $x \in X$, $\{f^n(x) : n \in \mathbb{N}\}$ clusters at some $x_0 \in X$, then it clusters at $f(x_0)$.*

Lemma 4.1.27 *Let $f : X \to X$ be a continuous map on a topological space X and $\{x_n : n \in \mathbb{N}\}$, a sequence in X such that $x_n = f(x_{n+1})$. If $\{x_n\}$ clusters at x_0, then $\{x_n\}$ clusters at $f(x_0)$.*

The following theorem gives a necessary and sufficient condition for a continuous order-preserving map f on a Hausdorff QOTS so that there exists x in X that can compare with $f(x)$. This theorem is quite useful in obtaining fixed point theorems in the setting of QOTS.

Theorem 4.1.28 (Ward [12]) *Let X be a Hausdorff QOTS with compact maximal chains and $f : X \to X$, a continuous order-preserving map. A necessary and sufficient condition that there exists a non-empty compact set $K \subseteq E(x_0)$ for some $x_0 \in X$ such that $f(K) = K$ is that there exists x in X such that x and $f(x)$ are comparable.*

Proof The necessity is obvious. Suppose x and $f(x)$ are comparable. Then $\{f^n(x) : n \in \mathbb{N}\}$ is a chain, as f is order-preserving and is therefore contained in a maximal chain, which by hypothesis is compact. So by Lemma 4.1.24 it clusters at some x_0 and all its cluster points are contained in $E(x_0)$. By Lemma 4.1.26 $f(E(x_0)) \subseteq E(x_0)$. Define $K = \cap\{f^n(E(x_0)) : n \in \mathbb{N}\}$. Then K is a non-empty compact subset of $E(x_0)$ and $f(K) = K$. □

Corollary 4.1.29 *If X is a POTS with compact maximal chains and $f : X \to X$ is a continuous order-preserving map, then a necessary and sufficient condition that f has a fixed point is that there is an $x \in X$ for which x and $f(x)$ are comparable.*

Proof If $x = f(x)$, then x and $f(x)$ are comparable. Conversely if $x \leq f(x)$ or $f(x) \leq x$, then $\{f^n(x) : n \in \mathbb{N}\}$ is a chain which is contained in a maximal chain which also clusters at some $x_0 \in X$. Since $f(E(x_0)) \subseteq E(x_0)$ and $E(x_0) = \{x_0\}$, X being partially ordered $x_0 = f(x_0)$. □

It is possible to define the concept of boundedness in quasi-ordered spaces, as in the following.

Definition 4.1.30 Let $(X, \leq)$ be a quasi-ordered set with an element $e \in X$ such that $e \leq x$ for all $x \in X$. A subset A of X is said to be bounded away from e if there is $y \in X - E(e)$ with $A \subseteq M(y)$.

Using this concept, some results on fixed points can be obtained.

Theorem 4.1.31 (Ward [12]) *Let X be a Hausdorff QOTS with compact maximal chains and suppose there exists $e \in X$ such that $e \leq x$ for all $x \in X$. Let $f : X \to X$ be a continuous order-preserving map satisfying*

(i) for some $x \in X - E(e)$, x and $f(x)$ are comparable;
(ii) for x satisfying (i), either $\{f^n(x) : n \in \mathbb{N}\}$ is bounded away from e or there exists $y \in X$ with $x \in E(f(y))$ and $f(y) \leq y$.

Then there exists $x_0 \in X - E(e)$ and a non-void compact set $K \subseteq E(x_0)$ such that $f(K) = K$.

Proof Suppose (i) is satisfied for some $x \in X - E(e)$. If $\{f^n(x) : n \in \mathbb{N}\}$ is bounded away from e, define $K = \cap\{f^n(E(x_0)) : n \in \mathbb{N}\}$ where $\{f^n(x) : n \in \mathbb{N}\}$ clusters at x_0 and all its cluster points are in $E(x_0)$ (by Lemma 4.1.24) as in Theorem 4.1.28. For this choice of K, $f(K) = K$ and $K \subseteq E(x_0)$. Clearly $x_0 \notin E(e)$.

If for no x satisfying (i), $\{f^n(x) : n \in \mathbb{N}\}$ is bounded away from e, by (ii) there is a y_1 such that for $x \in E(f(y_1))$. $f(y_1) \leq y_1$. Inductively, (y_n) can be chosen for $n \geq 2$ by

$$x \leq f(y_1) \leq y_1 \leq f(y_2) \leq y_2 \leq \cdots$$

where each $y_n \in E(f(y_{n+1}))$, $n \geq 1$. Since $\{y_n : n \in \mathbb{N}\}$ is a chain, it is contained in a maximal chain which by hypothesis is compact. So by Lemma 4.1.24, $\{y_n\}$ clusters at x_0 and all its cluster points are contained in $E(x_0)$ for some $x_0 \in X - E(e)$. By Lemma 4.1.27, $f(E(x_0)) \subseteq E(x_0)$. For K, as defined in the proof of Theorem 4.1.28, by $K = \cap\{f^n(E(x_0)) : n \in \mathbb{N}\}$, $f(K) = K$, $K \subseteq E(x_0)$ with $x_0 \in X - E(e)$. □

Corollary 4.1.32 *Let X be a POTS with compact maximal chains such that for some $e \in X$, $e \leq x$ for all $x \in X$ and $f : X \to X$ be a continuous order-preserving map satisfying the following:*

(i) for some $x \in X - E(e)$, x and $f(x)$ are comparable and
(ii) if x satisfies (i), then $\{f^n(x) : n \in N\}$ is bounded away from e or there exists $y \in X$ such that $x \in E(f(y))$ and $f(y) \leq y$.

Then f has a fixed point different from e.

Proof By Theorem 4.1.31, for some $x_0 \neq e$, f maps a compact subset K of $E(x_0)$ into K. Since f is partially ordered $E(x_0) = \{x_0\}$ and thus f has a fixed point different from x_0. □

The proof of the following corollaries is easy and left as exercises.

Corollary 4.1.33 *Let $(X, \leq)$ be a POTS which is Hausdorff and having compact maximal chains and $f : X \to X$ an order-preserving continuous map. Suppose (i) for some $u \in X$, $L(u) = X$ and (ii) for $x, y \in X$, there exists z with $x \leq z$, $y \leq z$. Then f has a fixed point.*

Corollary 4.1.34 *Let X be a compact Hausdorff QOTS such that*

(i) $L(u) = X$ for some $u \in X$;
(ii) for $x, y \in X$, there exists $z \in X$ with $x \leq z$, $y \leq z$;
(iii) for some $e \in X$, $e \leq x$ for all $x \in X$ and $X \neq E(e)$.

If $f : X \to X$ is an order-preserving continuous surjection, then there exists a non-void compact set $K \subseteq E(x_0)$ for some $x_0 \in X - E(e)$ for which $f(K) = K$.

Corollary 4.1.35 *Let X, f be as in Corollary 4.1.34. If X is partially ordered, then f has a fixed point different from e.*

Remark 4.1.36 Ward Jr. [12] has given an example to show that in a compact POTS an order-preserving continuous surjection may not have a fixed point different from e, without an additional hypothesis such as (ii) in Theorem 4.1.31.

4.2 Schweigert–Wallace Fixed Point Theorem

This section continues further the theory of partially ordered topological spaces developed by Ward Jr. In particular, the concepts of end point and end element and non-alternating maps and theorems relating to these are described. The section culminates in the proof of Schweigert–Wallace fixed point theorem [9, 11] for homeomorphisms on certain locally connected continua.

Definition 4.2.1 An element e of a topological space X is called an end point if, whenever $e \in U$, an open set, there exists an open set V such that $e \in V \subseteq \overline{V} \subseteq U$ and $\overline{V} - U$ is a singleton.

Two subsets P and Q of a connected topological space X are said to be separated by a set $K \subseteq X$ if $X - K = A \cup B$ and $P \subseteq A$, $Q \subseteq B$ and A and B are separated. If two points p and q of a topological space are not separated by any point, we write $p \sim q$.

Definition 4.2.2 A prime chain is a continuum which is either an endpoint, a cutpoint or a non-degenerate set E containing a distinct pair of elements a and b with $a \sim b$ with the condition that $E = \{x \in X : a \sim x \text{ and } x \sim b\}$. An end element is a prime chain E with the property that if $E \subseteq U$, an open set, then there is an open set V such that $E \subseteq \overline{V} \subseteq U$ and $\overline{V} - V$ is a singleton.

Remark 4.2.3 $\mathbb{R}$, with the usual topology has no endpoint, while $[a, b]$ of real numbers $(a < b)$ has a and b as endpoints. On the other hand, the unit circle has no endpoint.

Lemma 4.2.4 *Let X be a connected locally connected Hausdorff space. If E is an end element of X, then E contains at most one cutpoint of X.*

Proof Suppose E has two distinct cutpoints x_1 and x_2 of X. Then for $i = 1, 2$, $X - \{x_i\} = A_i \cup B_i$, where A_i and B_i are separated and $E - \{x_i\} \subseteq A_i$. Since X is locally connected, we can take A_i to be connected with $x_2 \in A_1$ and $x_1 \in A_2$. Also $B_i - B_j \neq \phi$ if $i \neq j$. Let $y_i \in B_i - B_j$ $(i = 1, 2, i \neq j)$ and let C_i be the component of $X - \{x_i\}$ such that $y_i \in C_i$. If $C_1 \cap C_2 \neq \phi$, $C_1 \cup C_2$ is connected. As $x_2 \nsubseteq C_1$, $C_1 \subseteq B_2$ contradicting that $y_1 \in B_1 - B_2$. So $C_1 \cap C_2 = \phi$.

Choose an open set U such that $E \subseteq U$ and U intersects both C_1 and C_2 but contains neither of these sets. Since E is an end element, there is an open set V such that $E \subseteq \overline{V} \subseteq U$ and $\overline{V} - V$ is a singleton. So $x_1 \in V$ or $x_2 \in V$. Otherwise both $x_1, x_2 \in \overline{V} - V$, a contradiction. It readily follows that $\overline{V} - V$ meets both C_1 and C_2, a contradiction since $C_1 \cap C_2 = \phi$. □

Lemma 4.2.5 *If X is a connected, locally connected Hausdorff space and E is an end element of X with a cutpoint x, then $E - \{x\}$ and $X - E$ are separated.*

Proof Since E is a continuum, there is a component C_0 in $X - \{x\}$ containing $E - \{x\}$. Since X is locally connected, it suffices to prove that $C_0 = E - \{x\}$ otherwise there exists $y \in C_0 - E$. Let C be any component of $X - \{x\}$ distinct from C_0 and P, a connected open set such that $x \in P$, $y \notin P$ and $C - P \neq \phi$. Then $C_0 \cup P - \{y\}$ is an open set containing E. Since E is an end element, there is a connected open set V (in view of local connectedness of X) such that $E \subseteq \overline{V} \subseteq C_0 \cup P - \{y\}$ where $\overline{V} - V$ is a singleton $\{z\}$. Since $y \notin V$ and $y \in C_0 - E$, $z \in C_0$ and so $x \in V$. Since $x \in P, C - P \neq \phi$ and C is a component of $X - \{x\}$ distinct from C_0, the component of $X - \{x\}$ containing $E - \{x\}$, $\overline{V} - V \cap C \neq \phi$. So $z \in C$. This contradicts that $z \in C_0$ and C and C_0 are distinct (and hence disjoint) components. So $C_0 = E - \{x\}$. Further $E - \{x\}$ and $X - E$ are separated sets as $\overline{E - \{x\}} \subseteq E$ and so $\overline{E - \{x\}} \cap X - E = \phi$. If $x_0 \in \overline{X - E} \cap E - \{x\}$, then every neighbourhood of x_0 meets $X - E$. Since X is locally connected, there is an open connected neighbourhood N of x_0 $(\neq x)$ in E containing points of $X - E$ other than x. So $N \cup E - \{x\}$ would be a connected subset of $X - \{x\}$ containing $E - \{x\}$ as a proper set contradicting that $E - \{x\}$ is a component in $X - \{x\}$. So $E - \{x\}$ and $X - E$ are separated. □

Another useful concept in this context is that of a non-alternating map, defined below.

Definition 4.2.6 A continuous map $f : X \to Y$ where X and Y are Hausdorff topological spaces is called non-alternating if for every decomposition, $X - f^{-1}(y_0) = M \cup N$ where M and N are separated in X, there is no y in Y such that $f^{-1}(y)$ intersects both M and N. Further $f(X) = Y$.

A continuous surjection $f : X \to Y$ where X and Y are Hausdorff spaces is called monotone if $f^{-1}(y)$ is connected for each y in Y.

Remark 4.2.7 Clearly every monotone map is non-alternating.

We state below without proof, two results due to Wallace [10] for subsequent use.

Lemma 4.2.8 *Let X and Y be Hausdorff topological spaces and $f : X \to Y$, a continuous surjection. The following are necessary and sufficient conditions for f to be non-alternating for each decomposition $X - f^{-1}(y_0) = M \cup N$ where M and N are separated*

(1) $M = f^{-1}(f(M))$ and $N = f^{-1}(f(N))$;
(2) $f(M) \cap f(N) = \phi$;
(3) $f^{-1}(f(M)) \cap f^{-1}(f(N)) = \phi$.

Lemma 4.2.9 *If $f : X \to Y$ is a closed and non-alternating map of X onto Y and $f^{-1}(y_0)$ separates P and Q in X, then y_0 separates $f(P)$ and $f(Q)$.*

In a locally connected space with an end element, a quasi-order can be defined in a natural way. This is captured in the following lemma and part of its proof can be found in Whyburn [15].

Lemma 4.2.10 *Let X be a locally connected continuum with an end element. Then the relation $\leq$ defined on X by $x \leq y$ for $x \in E$, $x = y$ or x separates E and y in X is a semicontinuous quasi-order. If E is a singleton, then $\leq$ is a partial order.*

Proof The proof that $\leq$ is a quasi-order and that $L(x)$ is closed for each $x \in X$ can be found in Whyburn [15].

If $x \in E$, $M(x) = X$ and is closed. For $x \in X - E$, $M(x) = \{x\} \cup \{y : x$ separates E and y in $X\}$. Let C be the component of $X - \{x\}$ containing E, then $M(x) = X - C$. As X is locally connected and $X - \{x\}$ is open, C is open. So $M(x)$ is closed. □

Lemma 4.2.11 *Let X be a locally connected T_2 continuum with end element E. If $f : X \to X$ is non-alternating and $f(E) \subseteq E$, then f is order-preserving.*

Proof Let $x \leq y$, $x, y \in X$. For $x = y$, $x \in E$. Clearly $f(x) \leq f(y)$, since $f(x) \in E$. If $x \notin E$ and $x \leq y$, then x separates E and y. Since $f(E) \subseteq E$, $f(x) \in X - f(E)$ and $f(x) \neq f(y)$. Then $f^{-1}f(x)$ separates E and y. So by Lemma 4.2.9 $f(x)$ separates E and $f(y)$ or $f(x) \leq f(y)$. □

The next lemma is an important step towards proving Schweigert–Wallace Fixed point theorem.

Lemma 4.2.12 *Let X be a locally connected T_2 continuum with an end element E. If $f : X \to X$ is non-alternating and $f(E) = E$, then there is a cutpoint x of X such that x and $f(x)$ are comparable. Further x can be so chosen that for some $y \in X$ with $x < y$ and $x < f(y)$.*

Proof If $x \in E$ is a cutpoint of X, then as $f(x) \in E$, x and $f(x)$ are comparable. Since $f(X) = X$ and $f(E) \subseteq E$, there is an element $y \in X - E$ such that $f(y) \in X - E$. So by Lemma 4.2.5, $x < y$ and $x < f(y)$.

If E has no cutpoint of X, choose $y \in X - E$ with $f(y) \in X - E$. Since E is an end element, there is an open set A with $E \subseteq \overline{A} \subseteq X - \{y, f(y)\}$ and $\overline{A} - A = \{x\}$, x a cutpoint, since $X - \{x\} = X - \overline{A} \cup A$. Clearly x separates E and $\{y\}$ as also E and $f(y)$. So $x < y, x < f(y)$. Also $X - \{x\} = A \cup B$ where A and B are separated with $E \subseteq A$ and $\{y, f(y)\} \subseteq B$. Also $f(y) \in B \cap f(B)$. If $f(x) \in \overline{B}$, then $x \leq f(x)$ and if $f(x) \in E$, then $f(x) \leq x$. In both these cases the lemma is valid.

Suppose $f(x) \notin B \cup E$. So $f(x) \in A - E$. In this case there are two possibilities.
Case (i): $f(x)$ is not a cutpoint. So by Lemma 4.2.9, $f^{-1}f(x)$ does not separate X. So $f^{-1}f(x)$ contains A or B. If $B \subseteq f^{-1}f(x)$, then $f(y) \in A$, a contradiction and when $f^{-1}f(x)$ contains A, then $f(x) \in E$, a contradiction.
Case (ii): Suppose f is a cutpoint. In this case we prove that $X - \{f(x)\} = f(A) - \{f(x)\} \cup f(B) - \{f(x)\}$, where $f(A) - \{f(x)\}$ and $f(B) - \{f(x)\}$ are non-void and separated. Since $E \subseteq A$ and $f(x) \in X - E$, $f(E) \subseteq f(A) - \{f(x)\} \neq \phi$. As $f(x) \neq f(y)$ $(\in f(B))$, $f(B) - \{f(x)\} \neq \phi$. If $t \in \overline{f(A) - \{f(x)\}} \cap f(B) - \{f(x)\}$, then $t \in f(\overline{A}) \cap f(B)$. Since f is non-alternating, $t = f(x)$, a contradiction. For similar reasons, $f(A) - \{f(x)\} \cap \overline{f(B) - \{f(x)\}}$ is empty. If $x \in f(B)$, then $f(x) \leq x$. If $x \in f(A) - \{f(a)\}$, then $f(A) - \{f(x)\}$ intersects both A and B and as $f(x) \in A$ and $\overline{B}$ is connected, it follows that $\overline{B} \subseteq f(A) - \{f(x)\}$. But this implies that $f(y) \in f(A) - \{f(x)\}$, a contradiction. □

Before proving another interesting theorem, similar to Theorem 4.1.31, we define a concept of convergence for a sequence of subsets of a given set, treated for instance in Whyburn [15].

Definition 4.2.13 ([15]) A sequence $\{A_n\}$ of subsets of a topological space X is said to cluster at a point x in X, if given any open neighbourhood of x infinitely many members of the sequence $\{A_n\}$ meet the given open neighbourhood. The set of cluster points is denoted $\limsup A_n$. The sequence (A_n) is said to converge to x if all but a finite number of the A_n meet any given open neighbourhood of x. The set of convergence points of A_n is denoted by $\liminf A_n$. (Clearly $\liminf A_n \subseteq \limsup A_n$.) If $\limsup A_n = \liminf A_n$, then this set is denoted $\lim A_n$.

Theorem 4.2.14 *Let X be a locally connected continuum with an end element E and $f : X \to X$ be a monotone continuous map with $f(X) = X$ and $f(E) = E$. Then X contains a non-empty subcontinuum K such that $f(K) = K$ and either K is a cutpoint or $K \subseteq X - E$. Further, no point separates any pair of points of K in X.*

Proof By Lemma 4.2.10, X is a QOTS and by Lemma 4.2.11, f is order-preserving. Lemma 4.2.12 insures that there is a cutpoint $x \in X$ such that x and $f(x)$ are comparable. If $x \in E$, then by Lemma 4.2.5, $X - \{x\} = E - \{x\} \cup X - E$ and $X - E$ and $E - \{x\}$ are separated. If $f(x) = x$, the theorem is proved; otherwise $f(x) \in E - \{x\}$ and so by Lemma 4.2.4 $f(x)$ is not a cutpoint. So either $E \subseteq f^{-1}f(x)$ or $X - E \subseteq f^{-1}f(x)$ as f is monotone. But $E \subseteq f^{-1}f(x)$ contradicts $f(E) = E$ and $X - E \subseteq f^{-1}f(x)$ contradicts $f(X) = X$.

In the other case for consideration, $x \in X - E$. If $x \leq f(x)$, then by Theorem 4.1.31, there is a non-empty set $K \subseteq E(x_0)$, $x_0 \in X - E$ such that $f(K) = K$. Since $E(x_0) = \{x_0\}$, the theorem is proved. If $f(x) < x$, let $X - \{x\} = A \cup B$, where A and B are separated, A being the component of $X - \{x\}$ containing E. Then $f(x) \in A$. By Lemma 4.2.12, x can be so chosen that there exists $b \in X$ with $b \in B$, $f(b) \in B$. Note that for any positive integer n, $X - f^{-n}(x) = f^{-n}(A) \cup f^{-n}(B)$ where $f^{-n}(A)$ and $f^{-n}(B)$ are separated. Since f is monotone $f^{-1}(\overline{B})$ is connected and because $x \in f^{-1}(A)$, $b \in B$, $f(b) \in B$, $f^{-1}(\overline{B}) \subseteq B$. So $\overline{A} \subseteq f^{-1}(A)$ and so for $n < m$, $f^{-n}(\overline{A}) \subseteq f^{-m}(A)$. Since X is compact, $\limsup f^{-n}(x) \neq \phi$.
(i) $\limsup f^{-n}(x) = \liminf f^{-n}(x)$. Otherwise there is $x_0 \in \limsup f^{-n}(x)$ and an open connected set U containing x_0 such that if N is a positive integer with $f^{-N}(x) \cap U \neq \phi$, then there is $m > N$ with $f^{-m}(x) \cap U = \phi$. So $U \subset f^{-m}(A)$ or $U \subseteq f^{-m}(B)$; since $f^{-N}(x) \cap U \neq \phi$ and $f^{-N}(x) \subseteq f^{-m}(A)$, it follows that $U \subseteq f^{-m}(A)$. So for $p > m$ $U \subseteq f^{-m}(A) \subseteq f^{-p}(A)$ so that $U \cap f^{-p}(x) = \phi$, contradicting the assumption that $x_0 \in \limsup f^{-n}(x)$.
(ii) $\lim f^{-n}(x)$ is a continuum. Clearly $f^{-n}(x)$ is closed in X and hence compact. Suppose $\lim f^{-n}(x) = P \cup Q$ where P and Q are separated. Since X is normal there exist open sets U and V which are disjoint and for which $P \subseteq U$ and $Q \subseteq V$. Suppose $P \neq \phi \neq Q$, we can choose a positive integer N such that for all $m \geq N$.

$$f^{-m}(x) \cap U \neq \phi \neq f^{-m}(x) \cap V.$$

As each $f^{-m}(x)$ is connected, we can choose a sequence y_n, $n \geq N$ with $y_n \in f^{-n}(x) - U \cup V$. Clearly y_n clusters at some $y_0 \in X - U \cup V$, contradicting that $\lim f^{-n}(x) \subseteq U \cap V$. So $\lim f^{-n}(x)$ is connected and hence is a continuum.
(iii) No point of X separates any pair of points of $\lim f^{-n}(x)$ in X. Otherwise there exists $a \in X$ such that the pair $\{p, q\} \subseteq \lim f^{-n}(x)$ is separated by a. Thus $X - \{a\} = P \cup Q$ where P and Q are separated with $p \in P$ and $q \in Q$. Since $f^{-n}(x)$ converges to both p and q, there is a positive integer M such that $f^{-m}(x)$ meets both P and Q for $m \geq M$. In view of (ii), $a \in \cap\{f^{-m}(x) : m \geq M\}$. So $f^{N+1}(a) = f(x) = x$. This contradicts our assumption that $f(x) < x$. Consequently, it follows that $f(\lim f^{-n}(x)) \subseteq \lim f^{-n}(x)$, so that

$$K = \bigcap_{i \in \mathbb{N}} f^i(L), \text{ where } L = \lim f^{-n}(x)$$

is a non-empty continuum and $f(K) = K$. Further no point of X separates any pair of points of K. As $\overline{A} \subseteq f^{-n}(A)$, for each $n = 1, 2, \ldots$, we conclude that $K \subseteq X - \overline{A} \subseteq X - E$. □

Corollary 4.2.15 *In addition to the hypotheses of Theorem 4.2.14 above if E is a singleton, then $K \subseteq X - E$.*

The following fixed point theorem is due to Schweigert and Wallace.

Theorem 4.2.16 (Schweigert [9] and Wallace [11]) *Let X be a locally connected continuum with an end element E. If $f(X) = X$ and $f : X \to X$ is a homeomorphism with $f(E) = E$, then f has a fixed point not in E.*

Proof By Lemma 4.2.11, f and f^{-1} are order-preserving and if X has a cutpoint p, then p and $f(p)$ are comparable by Lemma 4.2.12. If $p \in E$, then $f(p)$ is a cutpoint. So, by Lemma 4.2.4, $p = f(p)$. If $p \in X - E$, then $p \leq f(p)$ or $f(p) \leq p$. If $p \leq f(p)$, then by Theorem 4.1.31 there exists a non-empty compact set K with $K \subseteq E(x_0)$, $x_0 \in X - E$ such that $f(K) = K$. Since $E(x_0) = x_0$, the theorem is proved. If $f(p) \leq p$, by the same reasoning applied to f^{-1}, f^{-1} (and hence f) has a fixed point in $X - E$. □

Remark 4.2.17 Ward Jr. has noted that for non-alternating maps, the above theorem may not be true as shown by the following example. Let X be the locally connected continuum of points (x, y, t) in $\mathbb{R}^3$, satisfying $x = 0 = y$ for $1 \leq t \leq 2$ and $x^2 + y^2 = 1 - t^2$ for $0 \leq t \leq 1$. $e = (0, 0, 2)$ is an endpoint. The map $f : X \to X$ defined by

$$f(x, y, t) = \begin{cases} \left(-\frac{(1-2t)x}{(x^2+y^2)^{\frac{1}{2}}}, -\frac{(1-2t)y}{(x^2+y^2)^{\frac{1}{2}}}, 2t\right), & 0 \leq t \leq \frac{1}{2}, \\ (0, 0, t), & \frac{1}{2} \leq t \leq 1, \\ (0, 0, 2), & 1 \leq t \leq 2 \end{cases}$$

is continuous and monotone with e as the only fixed point. $K = \{(x, y, t) \in X : t = 0\}$ is fixed under f and is a subcontinuum. Of course f is not one-to-one.

Remark 4.2.18 Schweigert [9] originally proved the theorem under the assumption that X was separable and semi-locally connected. Wallace [11] noted that these hypotheses could be dropped in preference to local connectedness and also pointed to the use of quasi-order in the proof. Wallace, in fact, proved a fixed point theorem for homeomorphisms T such that T and T^{-1} preserve a transitive, reflexive binary relation on the space X.

Remark 4.2.19 Wallace [11] has shown that if X has a cutpoint, then it has an end element. He also noted that prime chains in Peano space are precisely cyclic elements, considered by Whyburn [15]. Wallace [11] has further proved that if the end element E of a continuum X has no cutpoint of X, then $X - E$ is connected and if P is the union of all the end elements of X not containing a cutpoint of X, then $X - P$ is connected and each component of P is an end element.

Remark 4.2.20 It was already stated that fixed points theorems generally prove that for certain topological spaces every continuous self-map has a fixed point. Bing (see [8]) has noted that for the proof of numerous fixed point theorems depend on the 'dead-end method' or the 'dog-chase rabbit argument'. Roughly speaking for a given mapping f, as x moves in X, $f(x)$ moves 'ahead' of x 'relative to some hidden order structure' till a special feature of the underlying space is exploited to locate a point x below $f(x)$ in the order and corner $f(x)$ in a dead end. Ward Jr. [14] has captured this 'dead-end method' in the following theorem, which is related to Theorem 4.1.31 and its corollaries.

Theorem 4.2.21 *Let X be a compact Hausdorff space with a lower semicontinuous partial order Γ, such that every maximal chain is compact. Suppose that if C is a maximal chain and $a \in C$ then $a\Gamma(= \{x : (a, x) \in \Gamma\}) \cap C$ is a closed set. If f is a continuous, order-preserving self-map on X, then a necessary and sufficient condition that for some x_0, X contains a fixed point of f is that there exists $x \in x_0\Gamma$ such that $x \leq f(x)$.*

Proof The necessity is obvious. Suppose $x_0 \leq x \leq f(x)$. Since f is order-preserving, clearly $x \leq f(x) \leq f^2(x) \leq \cdots \leq f^n(x) \leq \cdots$, and so $\{f^n(x)\}_{n\in\mathbb{N}}$ lies is a compact maximal chain C. So $\{f^n(x)\}$ has a cluster point $y \in C$. As $f^n(x)\Gamma \cap C$ is closed for each $n \in \mathbb{N}$, $f^n(x) \leq y$ for $n \in \mathbb{N}$. If for some z, $f^n(x) \leq z < y$ for all $n \in \mathbb{N}$, then $X - z\Gamma$ is an open neighbourhood of y, which fails to have any $f^n(x)$. Hence y is the supremum of $\{f^n(x)\}_{n\in\mathbb{N}}$ which converges to y. Since f is continuous and the space is Hausdorff, $y = f(y)$. □

4.3 Set Theory, Fixed Point Theory and Order

In an insightful paper, Manka [6] described a connection between set theory and fixed point theory via partial order. In this section, Manka's approach is described, based on the following definitions.

Definition 4.3.1 A partial ordered set X is called inductive if for every totally ordered subset of X there exists a least upper bound of this subset in X. Clearly an inductive (partially ordered) set is non-empty. A partially ordered set X is called acyclically ordered if for every $p, r \in X$ with $p \leq r$, the segment $[p, r] = \{x \in X : p \leq x \leq r\}$ is totally ordered.

Manka obtained a fixed point theorem for a class of mappings on an acyclically inductively ordered sets.

Theorem 4.3.2 (Manka [6]) *Let* $(X, \leq)$ *be an inductively and acyclically ordered poset and* $f : X \to X$ *be a map satisfying the following two conditions:*

(i) $p < f(p)$ *implies that for some* $q \in (p, f(p)]$ *with* $q \leq f(q)$*;*
(ii) $q \leq f(q)$ *for all* $q \in Y \subseteq X$ *implies that* $\sup Y \leq f(\sup Y)$.

Then f *has a fixed point.*

Proof Since $(X, \leq)$ is inductively ordered, $P_f = \{q \in X : q \leq f(q)\}$ is also inductively ordered. Since X is acyclically ordered, for each $p \leq f(p)$ in X, there exists in P_f supremum of the set $[p, f(p)] \cap P_f$. Thus the map $\varphi(p) = \sup([p, f(A)] \cap P_f)$ sends P_f into itself. Since $p \leq \varphi(p)$ for all $p \in P_f$, it follows that there exists a fixed point of φ.

We note that for each $p \in P_f$, $p = \varphi(p)$ implies $p = f(p)$. If $p \neq f(p)$, then $p < f(p)$ since $p \in P_f$. By (i) and the definition of P_f, there exists $q \in (p, f(p)] \cap P_f$. So $p < q$ and so $q \leq \varphi(p)$. So $p < \varphi(p)$ for $p \in P_f$ and φ cannot have a fixed point, in P_f a contradiction. □

Manka used the above theorem to prove that certain compact connected topological spaces have fixed point property for continuous functions. We need the following concepts.

Definition 4.3.3 A non-empty connected compact Hausdorff space is called a continuum.

(Generally, metrizable space is considered.)

Definition 4.3.4 A continuum having exactly two points which do not disconnect it is called an arc, including continua with only one point.

Definition 4.3.5 A continuum X is said to be arcwise connected if for any pair of points $p, q \in X$, there exists an arc joining these points in X. X is called one-arcwise connected, if this arc is unique in X and will be denoted by pq.

Definition 4.3.6 Let X be an arbitrary one-arcwise connected continuum such that for every monotone family of arcs $ap_\tau \subseteq X$, $\tau \in T_1$, there exists $b \in X$ with

$$\overline{\cup ap_\tau} = ab.$$

(Such a continuum X is some times called one-arcwise connected nested continuum.)

For the sake of completeness, we provide a list of definitions of various kinds of continua.

Definition 4.3.7 A continuum X is said to be irreducible between the points a and b if X contains a and b and no other subcontinuum of X contains both these points; a (and b as well) is called a point of irreducibility.

Remark 4.3.8 a is a point of irreducibility for a continuum X if and only if X is not the union of two proper subcontinua both of which contain a.

Definition 4.3.9 A continuum X is called unicoherent if the intersection of every pair of continua whose union is X, is a continuum.

Definition 4.3.10 A continuum X is said to be decomposable, if it is a union of two continua not contained in one another.

Definition 4.3.11 A property of a continuum X is called hereditary if every non-trivial subcontinuum has that property (thus we have hereditarily unicoherent and hereditarily decomposable continua).

Definition 4.3.12 A hereditarily unicoherent and arcwise connected metric continuum is called a dendroid. By a λ-dendroid is meant a hereditarily unicoherent and hereditarily decomposable metric continuum.

Example 4.3.13 A cone over an arbitrary hereditarily indecomposable plane continuum is a one-arcwise connected nested continuum which is not hereditarily unicoherent.

In what follows, we will show that arcwise connected hereditarily unicoherent continua have fixed point property for continuous functions (see Manka [6], using Theorem 4.3.2).

Let X be an arcwise connected nested continuum and $a \in X$. The binary relation $\leq_a$ defined by $p \leq_a q$ for $p, q \in X$ whenever the arc $ap \subseteq aq$, is indeed a partial order. With respect to this partial order a is the smallest element of X and a subset $\{p_t \in X : t \in T\}$ is totally ordered if and only if the family of arcs $\{ap_t : t \in T\}$ is monotone.

Lemma 4.3.14 *Let X be an arcwise connected nested continuum. For $a \in X$, $(X, \leq_a)$ is inductively ordered.*

Proof Let $\{ap_t : t \in T\}$ be a monotone family of arcs in X. So $\overline{\bigcup_{t\in T} ap_t} = ab$. Clearly $p_t \leq_a b$. Suppose $p_t \leq_a c$ for all $t \in T$. Then $ap_t \subseteq ac$ for all $t \in T$. So $\bigcup_{t\in T} ap_t \subseteq ac$. As ac is closed, $\overline{\bigcup_{t\in T} ap_t} = ab \subseteq ac$. So by definition of $\leq_a$, $b \leq c$. Thus $\{p_t : t \in T\}$ has c as an upper bound. Thus b is the supremum of $\{p_t : t \in T\}$ under this order. □

Remark 4.3.15 In Lemma 4.3.14, $b = \sup\{p_t : t \in T\}$ if and only if $\overline{\bigcup_{t\in T} ap_t} = ab$.
Clearly $ap_t \subseteq ab$ for all $t \in T$ so that $\overline{\bigcup_{t\in T} ap_t} \subseteq ab$. But the closure of $\bigcup_{t\in T} ap_t$ is a subarc ac of ab and $ap_t \subseteq ac$. So $ac \subseteq ab$. Hence c is an upper bound for $\{p_t \in X : t \in T\}$. Since b is the last upper bound, $c = b$.

In an arcwise connected continuum X, for every pair of arcs pq, pr with the same initial point P, a binary relation $\prec$ can be defined by $pq \prec pr$ if $pq \cap pr \neq \{p\}$. In other words $pq \prec pr$ if $pq \cap pr$ is an arc non-degenerate to the point p. The following lemma is left as an exercise.

Lemma 4.3.16 *The binary relation $\prec$ is an equivalence relation in the family of all non-degenerate arcs with the same initial point in X. If K is an arcwise connected continuum of X and $q, r \in K$, then for $p \notin K$, $pq \prec pr$ as $qr \subseteq K$. Further $ap \subseteq aq$ and $pq \prec pr$ imply $ap \subseteq ar$.*

(For further details and related ideas Manka [5] may be consulted.)

Remark 4.3.17 $ap \subseteq aq$ implies that $p \in aq$ and $ap \cup pq = aq$ so that $ap \subseteq aq$ implies $pq \subseteq aq$.

The above ideas lead to the following.

Theorem 4.3.18 (Manka [6]) *Every one arcwise connected nested continuum X with the partial order $\leq_a$ is an inductively and acyclically ordered set. If $f : X \to X$ is a map such that (a) $f(pq)$ is an arcwise connected continuum for each pq and (b) if for each $p \neq f(p)$ there is $\{p\} \neq pq \subseteq pf(p)$ with $pq \cap f(pq) \neq \phi$, satisfying conditions (i) and (ii) of Theorem 4.3.2 in the order $\leq_a$, then f has a fixed point.*

Proof By Lemma 4.3.14, $(X, \leq_a)$ is inductively ordered. If $p \leq_a q$, then the interval $[p, q]$ is the arc pq by the one-arcwise connectedness of X and the order $\leq_a$ in pq is the natural order of the arc pq (defined in pq as $\leq_p$). So $(X, \leq_a)$ is acyclically ordered.

We will verify that (i) and (ii) of Theorem 4.3.2 are satisfied. If $ap \nsubseteq af(p)$ so that $p \neq f(p)$, by (b) there exists $q \in X$ such that $\{p\} \neq pq \subseteq pf(p)$ and $pq \cap f(pq) = \phi$. So $ap \nsubseteq aq \subseteq af(p)$. Now we show that $aq \subseteq af(q)$. Since $p, q \notin f(pq)$ and $f(pq)$ is an arcwise continuum by (a), $qf(p) \prec qf(q)$ and $pf(p) \prec pf(q)$ (by Lemma 4.3.16). Since $pq \prec pf(p)$ (by (b)), $pq \prec pf(q)$ by transitivity of $\prec$. So $ap \subseteq af(q)$ by transitivity of $\prec$. So $ap \subseteq af(q)$ and consequently $pf(q) \subseteq af(q)$. Clearly, these lead to $pq \subseteq af(q)$. Therefore $pq \subseteq af(q)$. Thus $q \in af(q)$ and $aq \subseteq af(q)$. Thus (i) of Theorem 4.3.2 is true.

Let $\{ap_t : t \in T\}$ be a monotone family of arcs in X with $ap_t \subseteq af(p_t), t \in T$. Let $b \in X$ be the supremum in X with the order $\leq_a$ of the set $\{p_t : t \in T\}$. So by Remark 4.3.15, let $ab = \overline{\bigcup_{t\in T} ap_t}$. Suppose $ab \subseteq af(b)$ is not true. Clearly $p_t \neq b$ for all $t \in T$ as $ap_t \subseteq af(p_t)$. Then $b \notin af(b)$ implying that $ba \prec bf(b)$ by Lemma 4.3.16. As

$b \neq f(b)$ and $\bigcap_{t \in T} bp_t = \{b\}$, $bp_t \subseteq bf(b)$ for bp_t with sufficiently small diameter, there exists $t \in T$ with $bp_t \cap f(bp_t) = \phi$. So we can conclude that $ab \cap f(p_t b) = \phi$.

In the other case, since $p_t \notin f(p_t b)$, $ap_t \cap f(p_t b) \neq \phi$. Take the first point p in the arc ap_t belonging to $f(p_t b)$, the image being a continuum by (a). We would have $p_t \notin ap$. So $p_t \notin ap \cup f(p_t b)$. But $af(p_t) \subseteq ap \cup f(p_t b)$. Hence $p_t \notin af(p_t)$ contrary to the assumption that $ap_t \subseteq af(p_t)$. So $ap_t \cap f(p_t b) = \phi$. Since the continua ab and $f(p_t b)$ are disjoint, by the one-arcwise connectedness of X, there exists in X a unique arc joining them so that $ab \cap pq = \{p\}$ and $pq \cap f(p_t b) = \{q\}$ and any arc joining an arbitrary point of ab with an arbitrary point of $f(p_t b)$ contains pq. In particular $p \in af(b)$. As $b \notin af(b)$, $p \neq b$. However, $ap \cup pb = ab$. Taking an arc $p_{t_1} b \subseteq ab$ of smaller diameter than $p_t b$ and a suitable $q_1 \in f(p_{t_1} b)$ instead of q we can suppose that $ap \cap p_t b = \phi$ without change of notation. But $pq \cap p_t b = \phi$ from the choice of pq. Since $qf(p_t) \subseteq f(p_t b)$, $qf(p_t) \cap p_t b = \phi$. Thus $(ap \cup pq \cup qf(p_t)) \cap p_t b = \phi$. As $af(p_t) \subseteq ap \cup pq \cup qf(p_t)$, we infer that $p_t b \cap af(p_t) = \phi$, implying that $p_t \notin af(p_t)$, contradicting the definition of ap_t. Thus (ii) of Theorem 4.3.2 is also satisfied. So f has a fixed point. □

Corollary 4.3.19 *An arcwise connected hereditarily unicoherent continuum has the fixed point property for continuous maps.*

Proof The corollary follows from the fact that a continuous self-map on such a continuum satisfies (a) and (b) of Theorem 4.3.18. □

The following example due to Manka [6] shows that Theorem 4.3.18 holds even for discontinuous maps.

Example 4.3.20 Let X be the plane continuum defined as $\bigcup_{n=0}^{\infty} I_n$, where I_0 is the line segment joining (0, 0) to (1, 0) and I_n is the line segment joining (0, 0) to $(1, \frac{1}{n})$, $n = 1, 2, \ldots$. Define the map $f : X \to X$ by $f(p) = p$ for $p \in I_n$, $n \in \mathbb{N}$ and $f(p) = \begin{cases} p & \text{for } p \in \{0\} \times [0, \frac{1}{2}] \\ (n, 0) & \text{for } p \in \{0\} \times (\frac{1}{2}, 1] \end{cases}$. Clearly, f satisfies all the conditions of Theorem 4.3.18, though f is not continuous.

4.4 Multifunctions and Dendroids

In this section, we prove a fixed point theorem for dendroids following the techniques developed by Manka, as described in Sect. 4.3. In fact, Manka's fixed point theorem subsumes that of Ward [13] and is proved in [7]. To this end, we need the following.

Definition 4.4.1 Let X be a topological space. A multifunction $F : X \to 2^X - \{\phi\}$ is called upper semicontinuous if $F(x)$ is a closed set for each $x \in X$ and $F^{-1}(A) = \{x \in X : F(x) \cap A \neq \phi\}$ is a closed subset of X for each closed subset A of X. F is called lower semicontinuous, if $F^{-1}(A)$ is open for each open subset A of X.

Remark 4.4.2 For a compact metric space X, upper semicontinuity of F means that $\overline{\lim} F(x_n) \subseteq F(\lim x_n)$ for each convergent sequence (x_n) in X. Similarly, lower semicontinuity of $F : X \to 2^X - \{\phi\}$ means that $F(\lim_n x_n) \subseteq \lim_{n\to\infty} \inf F(x_n)$ for each sequence (x_n) converging in the compact metric space X. F is called continuous if it is both lower and upper semicontinuous. Using the following version of Brouwer reduction theorem, Manka [5] obtained an alternative proof of Ward's fixed point theorem for upper semicontinuous closed valued multifunctions on a dendroid.

Theorem 4.4.3 (Manka [5]) *Every non-empty family $\mathcal{P}$ of closed subsets of a compact metric space X which is closed with respect to the operation of closure of a union of increasing sequences contains a maximal element.*

Proof As X is a compact metric space, it has a countable base $\{B_n\}$. Take $P_0 \in \mathcal{P}$ and define a sequence $P_n \in \mathcal{P}$ inductively as follows: Take $P_1 \in \mathcal{P}$ containing P_0 and intersecting B_1 if such a P_1 does not exist take $P_1 = P_0$. Suppose $P_1 \subset P_2 \cdots \subseteq P_{k-1}$ have been defined take P_k as an element of $\mathcal{P}$ containing P_{k-1} and intersecting B_k. If such a P_k does not exist, set $P_k = P_{k-1}$.

The sequence $P_k \in \mathcal{P}$ defined thus has the property that for each $Q \in \mathcal{P}$ $B_k \cap Q \neq \phi$ and $P_{k-1} \subseteq Q$ imply that $B_k \cap P_k \neq \phi$. As $\{P_k\}$ is increasing $P = \overline{\cup P_k} \in \mathcal{P}$ by definition of $\mathcal{P}$. We claim that P is a maximal set in $\mathcal{P}$. If for some $Q \in \mathcal{P}$, Q contains P properly, then there exists B_k, a basic set such that $P \cap B_k = \phi$ and $B_k \cap Q \neq \phi$. Since $P_{k-1} \subseteq Q$, $B_k \cap P_k \neq \phi$. This contradicts that $P \cap B_k = \phi$. So P is maximal in $\mathcal{P}$. □

We also recall the following results and their proofs are left as exercises.

Remark 4.4.4 Let X be an arcwise connected hereditarily unicoherent metric continuum (dendroid) and $F : X \to 2^X - \{\phi\}$ be an upper semicontinuous multifunction for which $F(x)$ is a continuum for each $x \in X$. Then $F(K) = \bigcup_{x\in K} F(x)$ is a continuum, whenever K is a continuum. If F is continuous and $F(x)$ is a closed subset of X for each $x \in X$, then $F(x)$ intersects every component of $F(K)$ whenever K is a continuum, in X.

We use the following lemmata and assume that X is a hereditarily unicoherent arcwise connected metric continuum (dendroid) and $F : X \to 2^X - \{\phi\}$ is a continuum-valued upper semicontinuous multifunction. We define $\mathcal{P}_a$ as the set of all arcs $a, b \subseteq X$ such that for $p \in ab - \{b\}$ and each $q \in F(p)$, the relation $pq \prec pb$ holds (see the paragraph preceding Lemma 4.3.16 for definition of $\prec$).

Lemma 4.4.5 *If $a \notin F(a)$, then for every $d \in F(a)$, there exists $ab \in \mathcal{P}_a$ such that $ab \subset ad$.*

Proof Suppose $d \in F(a)$ and that an arc $ab \subset ad$ satisfies, by the upper semicontinuity of F, $ab \cap F(ab) = \phi$. Now for each $p \in ab - \{b\}$ and each $q \in F(p)$, we have $p \notin F(ab)$ and $d, q \in F(ab)$. Since $F(ab)$ is a continuum (see Remark 4.4.4, $pq \prec pd$. □

Lemma 4.4.6 *If* $ab \in \mathcal{P}_a$, $b \notin F(b)$ *and* $d \in F(b)$, *then* $ab \subset ad$.

Proof As $b \notin F(b)$, it follows from the upper semicontinuity of F that for some $p' \in ab - \{b\}$, $p'b \cap F(p'b) = \phi$. For each $p \in p'b - \{b\}$, $p \in ab - \{b\}$. Hence $ap \subset ab$ and $pb \prec pq$ for each $q \in F(p)$ by the definition of $\mathcal{P}_a$, By Lemma 4.3.16, $ap \subset aq$. Now $p \notin F(p'b)$ and $q, d \in F(p'b)$. Hence $pq \prec pd$. So by Lemma 4.3.16, $ap \subset ad$ for every $p \in p'b - \{b\}$. Since the union of all such arcs ap is $ab - \{b\}$, we have $ab - \{b\} \subset ad$. So $ab \subset ad$. □

Lemma 4.4.7 *If* $ab \cup ac = ac$, $ab \in \mathcal{P}_a$ *and* $bc \in \mathcal{P}_b$, *then* $ac \in \mathcal{P}_a$.

Proof For $p \in ab - \{b\}$, $pc \prec pb$, since $ab \subset ac$. Since $ab \in \mathcal{P}_a$, $pb \prec pq$ for each $q \in F(p)$. So $pc \prec pq$, by the transitivity of $\prec$. If $p \in bc - \{c\}$, then $pc \prec pq$, since $bc \in \mathcal{P}_b$. □

Lemma 4.4.8 *If* $ab = \overline{\bigcup_{n\in\mathbb{N}} ab_n}$ *and* $ab_n \in \mathcal{P}_a$, $n \in \mathbb{N}$, *then* $ab \in \mathcal{P}_a$.

Proof For $p \in ab - \{b\}$, there exists $n \in \mathbb{N}$ such that $p \in ab_n - \{b_n\}$. So $pb_n \prec pb$ since $ab_n \subset ab$ and $pq \prec pb_n$ for every $q \in F(p)$, in view of $ab_n \in \mathcal{P}_a$. From the reflexivity of $\prec$, we have $pq \prec pb$. □

Theorem 4.4.9 (Ward Jr. [13]) *If* $F : X \to 2^X - \{\phi\}$ *is a continuum-valued upper semicontinuous multifunction on a dendroid* X, *then* F *has a fixed point.*

Proof (As in Manka [7]) If $ab_1 \subset ab_2 \subset \cdots$ is an increasing sequence of arcs in the dendroid X, then $\overline{\bigcup_{n\in\mathbb{N}} ab_n}$ is an arc ab for some b. Since $\overline{\bigcup_{n\in\mathbb{N}} ab_n}$ is a continuum, for a proper subcontinuum this continuum containing a some b_j will not be a member. So $\overline{\bigcup_{n\in\mathbb{N}} ab_n}$ is a continuum which is not the union of two proper subcontinua both containing a. So a is a point of irreducibility of $\overline{\bigcup_{n\in\mathbb{N}} ab_n}$. Thus $\overline{\bigcup_{n\in\mathbb{N}} ab_n} = ab$ for some b. So by Lemmata 4.4.5, 4.4.8 and Theorem 4.4.3, for $a \notin F(a)$, there exists an arc ab maximal in $\mathcal{P}_a$. Now from Lemmata 4.4.5–4.4.7 it follows that $b \in F(b)$. Otherwise, if $b \notin F(b)$ then $ab \subset ad$ for each $d \in F(b)$ and by Lemma 4.4.5 there exists $bc \subset bd$ such that $bc \in \mathcal{P}_b$. So $ab \cup bc = ac \in \mathcal{P}_a$ by Lemma 4.4.7 contradicting the maximality of ab in $\mathcal{P}_a$. □

Corollary 4.4.10 (Borsuk [2]) *Every dendroid has fixed point property for continuous functions.*

4.5 Some Spaces with Fixed Point Property

In this section, elementary methods of constructing spaces with fixed point property are described. Relevant concepts are also presented.

Definition 4.5.1 Let X be a topological space and $Y \subseteq X$. Y is said to be a retract of X if there is a continuous map T from X onto Y such that $r(y) = y$ for all $y \in Y$. r is called a retraction of X onto Y.

Proposition 4.5.2 *If X has fixed point property (for continuous functions), then any retract of X also has fixed point property.*

Proof Let $g : Y \to Y$ be continuous and $r : X \to Y$ be a retraction of X onto Y. Then $g \circ r$ maps X into $Y \subseteq X$. Since $g \circ r$ is continuous and X has the fixed point property for continuous functions, there exists $x_0 \in X$ such that $g(r(x_0)) = x_0$ and $x_0 \in Y$. Since $x_0 \in Y$ and r is a retraction of X onto Y $r(x_0) = x_0$. So $g(x_0) = x_0 \in Y$. So Y has fixed point property for continuous functions. □

Proposition 4.5.3 *If X is a disconnected topological space, then X does not have the fixed point property.*

Proof Since X is disconnected $X = A \cup B$ where A and B are non-empty disjoint proper closed subsets of X. Let $a \in A$ and $b \in B$. The map $f : X \to X$ defined by $f(x) = b$ for $x \in A$ and $f(x) = a$ for $x \in B$ is a continuous map without a fixed point. □

Remark 4.5.4 The unit circle S^1 in $\mathbb{R}^2$ with the usual topology is a connected, locally connected compact metric space without fixed point property. For example, $(x, y) \to (-x, -y)$ on S^1 has no fixed point.

Theorem 4.5.5 *Let (X, d) be a compact metric space. Suppose for each $\epsilon > 0$, there is a continuous map $f_\epsilon : X \to X_\epsilon$, where X_ϵ is a subset of X with fixed point property. If $d(f_\epsilon(x), x) < \epsilon$ for each $x \in X$, then X has the fixed point property.*

Proof Let $f : X \to X$ be a continuous map. For each $\epsilon > 0$, $f_\epsilon \circ f$ maps X_ϵ into itself and is continuous. Since X_ϵ has fixed point property, $f_\epsilon \circ f(x_\epsilon) = x_\epsilon$ for some $x_\epsilon \in X$. Now $d(f(x_\epsilon), f_\epsilon(f(x_\epsilon)) < \epsilon$ by hypothesis. Setting $\epsilon = \frac{1}{n}, n \in \mathbb{N}$, it follows from the compactness of X, that (x_{n_k}) converges to an element x^* of X for some subsequence (x_{n_k}) of (x_n). From the continuity of f, and $d(f(x_{n_k}), x_{n_k}) < \frac{1}{n_k}$. We conclude that $f(x^*) = x^*$. Thus X has the fixed point property. □

The proof of the following proposition is left as an exercise.

Proposition 4.5.6 *Let X and Y be topological spaces such that Y is homeomorphic to X. If X has the fixed point property, so has Y.*

Remark 4.5.7 The subset $K = \bigcup_{n=1}^{\infty} I_k \cup I_0$, where I_0 is the line segment joining $(0, 0)$ and $(0, 1)$ and I_n is the line segment joining $(0, 1)$ to $(\frac{1}{n}, 0)$, $n \in \mathbb{N}$ in $\mathbb{R}^2$ has the fixed point property. Similarly the sine circle $\{(x, \sin \frac{1}{x}) \in \mathbb{R}^2\ 0 < x \leq 1\} \cup \{(0, y) \in \mathbb{R}^2 : -1 \leq y \leq 1\}$ has the fixed point property. Theorem 4.5.5 may be used to prove these conclusions.

Yet another useful idea is that of wedge of two spaces, defined below

Definition 4.5.8 Let X and Y be two disjoint spaces and let $p \in X$ and $q \in Y$. The wedge of X and Y at p and q, denoted by $X \vee_{p,q} Y$ (or simply $X \vee Y$) is the quotient space of $X \cup Y$ obtained by identifying p with q. (Clearly $X \vee Y$ has a natural copy of X in $X \vee Y$.)

Theorem 4.5.9 *Let X and Y be T_1 spaces with fixed point property. Then $X \vee Y$ has the fixed point property.*

Proof Let w denote the point at which X and Y intersect in $X \vee Y$. Let $f : X \vee Y \to X \vee Y$ be a continuous map. Define $r : X \vee V \to X$ by

$$r(p) = \begin{cases} p, & p \in X \\ w, & p \in Y \end{cases}$$

X and Y are closed in $X \vee Y$ and r is continuous. Since X has the fixed point property, $r \circ f$ has a fixed point, p say. Now $rf(p) = p$. If $w = f(w)$, then there is nothing to prove. So let $w \neq f(w)$. So $p \neq w$. So $rf(p) \neq w$. So by definition of r, $f(p) \in X$. So $rf(p) = f(p) = p$. (Note the continuity of r is based on the T_1-hypothesis). □

For these and similar results and examples, Nadler [8] may be consulted.

4.6 An Example in Fixed Point Theory

Connell [3] had given examples of noncompact plane sets U, V and W each having fixed point property such that $cl\ W$, U^2 lack fixed point property and V is locally contractible. Klee [4] had given an example of a space combining all these features. For the sake of completeness, we give the following definitions.

Definition 4.6.1 Let X and Y be topological spaces and I, the closed unit interval [0, 1]. A homotopy in a continuous map $h : X \times I \to Y$. We write h_t to denote the map from X into Y defined by $h_t(x) = h(x, t)$ for all $x \in X$ for any fixed $t \in I$. A continuous map $f : X \to Y$ is said to be homotopic to a continuous map $g : X \to Y$, if there exists a homotopy $h : X \times I \to Y$ such that $h_0 = f$ and $h_1 = g$.

Definition 4.6.2 A continuous map $f : X \to Y$ is null-homotopic or inessential if f is homotopic to a constant map. A continuous map $f : X \to Y$ is essential if it is not null-homotopic.

Definition 4.6.3 A topological space X is called contractible if the identity map on X is null-homotopic.

We now briefly present Klee's [4] example.

Example 4.6.4 Let Y be the set of real sequences (y_n) in the Hilbert space ℓ_2 such that y_i is non-zero for at most one i and $0 \leq y_i \leq 1$. If θ is the zero sequence in ℓ^2 and δ_n the sequence in ℓ^2 which is one in the nth place and zero elsewhere then $Y = \bigcup_{n=1}^{\infty} \sigma_n$, where σ_n is the line segment joining θ to δ_n (and so $\sigma_n = [\theta, \delta_n]$). Clearly, Y is both contractible and locally contractible.

For each n, let r_n be the retraction of Y onto σ_n which is identity in σ_n and maps $Y - \sigma_n$ onto θ. Let $f : Y \to Y$ be a continuous map. Suppose $f(\theta) \neq \theta$. So for some n, $f(\theta) \in \sigma_n - \{\theta\}$. As $r_n f$ maps σ_n onto itself and $r_n f(\theta) = f(\theta) \neq \theta$ and σ_n being essentially a compact real interval has the fixed point property, $r_n f(p) = p$ for some $p \in \sigma_n - \{\theta\}$. Since $r_n f(p) \neq \theta$, $f(p) \in \sigma_n$ and so $r_n f(p) = f(p)$. So $f(p) = p$. Thus Y has the fixed point property.

In the space $\ell^2 \times \ell^2$, let P be the infinite polygon with vertices in the order $(\theta, \delta_1), (\delta_1, \theta), (\theta, \delta_2), (\delta_2, \theta), \ldots, (\theta, \delta_n), (\delta_n, \theta), \ldots$. Clearly, P is closed in $Y \times Y$ and P is homeomorphic with $[0, \infty)$. As $Y \times Y$ would admit a retraction onto P, and P lacks the fixed point property, $Y \times Y$ cannot have the fixed point property, in view of Proposition 4.5.2.

For each $t \in [0, \pi]$ and $n \in \mathbb{N}$ consider τ_n the arc consisting of all points $(x_n(t), y_n(t))$ where $x_n(t) = (-1)^n(1 + \frac{t}{n}) \cos t$ and $y_n(t) = (1 + \frac{t}{n}) \sin t$. Each arc τ_n has $(1, 0)$ as an end point and X, the union of all the arcs τ_n is a homeomorphic of y. But clX contains the unit circle C and has a retraction onto C. But C does not have the fixed point property. So clX does not enjoy the fixed point property.

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Chapter 5
Contraction Principle

The contraction mapping principle proved independently by Banach [1] and Cacciopoli [7] is a fundamental fixed point theorem, with an elementary proof. This theorem has a wide spectrum of applications and is a natural choice in approximating solutions to nonlinear problems. According to Rall [18], the applications of the contraction principle would fill volumes and Bollabos [4] calls it a doyen of fixed point theorems. Charmed by both the simplicity and utility of this theorem, many authors have generalized it in diverse directions. This chapter samples a few of these.

5.1 A Simple Proof of the Contraction Principle

A simple proof of the contraction principle due to Palais [16] is given below. This is preceded by a few definitions and remarks.

Definition 5.1.1 Let (X, d) be a metric space. A map $T : X \to X$ is said to be a Lipschitz map with Lipschitz constant M, if for some $M \in \mathbb{R}^+$, $d(Tx, Ty) \leq Md(x, y)$ for all $x, y \in X$. In this case M is called a Lipschitz constant for the map T. If $M < 1$, then T is called a contraction (mapping) with contraction constant M. If $M = 1$, T is called a non-expansive map. If $d(Tx, Ty) = d(x, y)$ for all $x, y \in X$. T is a distance-preserving map and is called an isometry.

Remark 5.1.2 Every Lipschitz map $T : (X, d) \to (X, d)$ is uniformly continuous. Also an isometry is non-expansive.

Definition 5.1.3 Let (X, d) be a metric space. A map $T : X \to X$ is called contractive if $d(Tx, Ty) < d(x, y)$, whenever $x, y \in X$ and $x \neq y$.

Palais [16] proved the contraction principle, using the following contraction inequality.

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Lemma 5.1.4 (Contraction-Inequality) *Let $T : X \to X$ be a contraction mapping on the metric space (X, d) with contraction constant k. Then for any $x, y \in X$*

$$d(x, y) \le \frac{1}{1-k}[d(x, Tx) + d(y, Ty)]$$

Proof For $x, y \in X$, by triangle inequality

$$\begin{aligned} d(x, y) &\le d(x, Tx) + d(Tx, Ty) + d(Ty, y) \\ &\le d(x, Tx) + kd(x, y) + d(Ty, y) \\ &\qquad\qquad \text{(as } T \text{ is a contraction)} \end{aligned}$$

So $(1-k)d(x, y) \le d(x, Tx) + d(y, Ty)$ for all $x, y \in X$. Since $0 \le k < 1$, $d(x, y) \le \frac{1}{1-k}[d(x, Tx) + d(y, Ty)]$ for all $x, y \in X$. □

Corollary 5.1.5 *A contraction mapping can have at most on fixed point.*

This corollary follows at once from Lemma 5.1.4 by choosing x and y as fixed points of T.

Lemma 5.1.6 (Estimate for iterates) *Let (X, d) be a metric space and $T : X \to X$, a contraction mapping with contraction constant k. Then for any $x \in X$,*

$$d(T^n x, T^m x) \le \frac{k^n + k^m}{(1-k)} d(x, Tx)$$

Proof Replacing x and y by $T^n x$ and $T^m x$, respectively, in the contraction inequality (vide Lemma 5.1.4), we get

$$d(T^n x, T^m x) \le \frac{d(T^n x, T^{n+1}x) + d(T^m x, T^{m+1}x)}{(1-k)}$$

For $j \in \mathbb{N}$,

$$\begin{aligned} d(T^j x, T^{j+1}x) &\le kd(T^{j-1}x, T^j x) \\ &\le k^j d(x, Tx) \end{aligned}$$

for all $x \in X$. It follows that

$$d(T^n x, T^m x) \le \frac{k^n + k^m}{(1-k)} d(x, Tx)$$

for all $x \in X$. As $0 < k < 1$, it follows that $\{T^n x\}$ is a Cauchy sequence. □

Theorem 5.1.7 (Contraction Principle) *Let (X, d) be a complete metric space and $T : X \to X$, a contraction mapping with contraction constant k. Then, T has a* *unique fixed point x^* in X and every sequence $T^n x$ of T-iterates generated at any element $x \in X$, converges to x^*. Further, for $x \in X$*

$$d(x^*, T^n x) \leq \frac{k^n}{1-k} d(x, Tx).$$

Proof For any $x \in X$, $\{T^n x\}$, the sequence of T-iterates generated at x is a Cauchy sequence, in view of Lemma 5.1.6. As (X, d) is a complete metric space $(T^n x)$ converges to an element x^*. Since T is continuous, $\{T(T^n(x))\} = \{T^{n+1}x\}$ converges to Tx^*, $\{T^{n+1}x\}$ being a subsequence of $\{T^n x\}$ must converge to x^*. Since the limit of a convergent sequence is unique in a metric space $x^* = Tx^*$. Thus every sequence of iterates converges to a fixed point of T. By Corollary 5.1.5 this fixed point is unique.

For $x \in X$, according to Lemma 5.1.4,

$$d(T^m x, T^n x) \leq \frac{k^m + k^n}{(1-k)} d(x, Tx).$$

Proceeding to the limit in the above inequality as m tends to ∞, and noting that $\{T^m x\}$ converges to $x^* = Tx^*$, we get

$$d(x^*, T^n x) \leq \frac{k^n}{1-k} d(x, Tx).$$

□

Remark 5.1.8 In order that $T^n x$ is at a distance less than ϵ (> 0) from x^* the fixed point of T, it suffices to choose n such that $\frac{k^n}{1-k} d(x, Tx) < \epsilon$. In other words for $N > \frac{\log \epsilon + \log(1-k) - \log d(x, Tx)}{\log k}$, $d(x^*, T^N x) < \epsilon$. Thus in a specific situation, the fixed point of a contraction T can be approximated by $T^N x$ within an error of a pre-assigned positive number ϵ by choosing N appropriately.

Remark 5.1.9 Since the closed sphere $\overline{B}(a : r) = \{x \in X : d(a, x) \leq r\}$ of a complete metric space (X, d) is complete with respect to the restricted metric, a contraction $T : X \to X$ with contraction constant k maps $\overline{B}(a : r)$ into itself and hence has a unique fixed point in $\overline{B}(a : r)$, provided $d(a, Ta) \leq (1-k)r$.

We also have the following useful:

Corollary 5.1.10 *Let (X, d) be a complete metric space and $T : X \to X$ be a map such that T^n is a contraction for some $n \in N$ with $n > 1$. Then T has a unique fixed point.*

Proof Since T^n is a contraction on the complete metric space (X, d), it has a unique fixed point x^*, say. Now $T^{n+1}(x^*) = T(T^n x^*) = T(x^*) = T(T^n x^*) = T^n(Tx^*)$. Thus Tx^* is a fixed point of T^n. Since T^n has the unique fixed point x^*, $x^* = Tx^*$. If y^* is another fixed point of T, then $T^n(y^*) = y^*$ will be a fixed point of T^n and hence $y^* = x^*$. Thus T has a unique fixed point. □

Remark 5.1.11 A contractive map T which is not a strict contraction may not have a fixed point in a complete metric space. For example, the map $x \to x + \frac{1}{x}$ maps $[2, \infty)$ into itself has no fixed point in $[2, \infty)$ which is complete with respect to the usual metric. For $2 \leq x < y$, $0 < (y + \frac{1}{y}) - (x + \frac{1}{x}) = (y - x)(1 - \frac{1}{xy}) < y - x$ and consequently this map is contractive.

5.2 Metrical Generalizations of the Contraction Principle

It is natural to explore if maps satisfying inequalities similar to the Lipschitz condition have fixed points in a complete metric space. Kannan [13] proved a fixed point theorem in this direction for a class of mappings which are not necessarily continuous, though they satisfy a metrical inequality similar to the contraction condition. Interestingly such (Kannan) maps have unique fixed points to which sequences of iterates always converge. Hardy and Rogers [9] later extended Kannan's theorem to obtain a common generalization of the contraction principle and Kannan's fixed point theorem.

Theorem 5.2.1 (Hardy and Roger [9]) *Let (X, d) be a complete metric space and $T : X \to X$ a map satisfying the condition*

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx)$$

for all $x, y \in X$, where $a_i \geq 0$ for $i = 1, 2, \ldots, 5$ and $\sum_{i=1}^{5} a_i < 1$.

Then T has a unique fixed point and every sequence of T-iterates converges to the unique fixed point.

Proof Interchanging the roles of x and y in the inequality satisfied by T, we get

$$d(Tx, Ty) \leq a_1 d(y, x) + a_2 d(y, Ty) + a_3 d(x, Tx) + a_4 d(y, Tx) + a_5 d(x, Ty).$$

So we get

$$\begin{aligned} d(Tx, Ty) \leq a_1 d(x, y) &+ \frac{a_2 + a_3}{2}[d(x, Tx) + d(y, Ty)] \\ &+ \frac{a_4 + a_5}{2}[d(x, Ty) + d(y, Tx)] \end{aligned} \tag{5.2.1}$$

Letting $y = Tx$ in (5.2.1), we have

$$\begin{aligned} d(Tx, T^2x) &\le a_1 d(x, Tx) + \frac{a_2 + a_3}{2}[d(x, Tx) + d(Tx, T^2x)] \\ &\quad + \frac{a_4 + a_5}{2}[d(x, T^2x)] \\ &\le a_1 d(x, Tx) + \frac{a_2 + a_3 + a_4 + a_5}{2} d(x, Tx) \\ &\quad + \frac{a_2 + a_3 + a_4 + a_5}{2} d(Tx, T^2x) \end{aligned} \tag{5.2.2}$$

upon using $d(x, T^2x) \le d(x, Tx) + d(Tx, T^2x)$.

Thus $d(Tx, T^2x) \le \frac{a_1 + \frac{(a_2+a_3+a_4+a_5)}{2} d(x,Tx)}{1 - \frac{(a_2+a_3+a_4+a_5)}{2}}$.

Writing $k = \frac{a_1 + \frac{(a_2+a_3+a_4+a_5)}{2}}{1 - \frac{(a_2+a_3+a_4+a_5)}{2}}$ we get

$$d(Tx, T^2x) \le k d(x, Tx) \text{ for all } x \in X \tag{5.2.3}$$

Since $0 \le \sum_{i=1}^{5} a_i < 1$, $0 \le k < 1$. Further, $d(T^n x, T^{n+1} x) \le k^n d(x, Tx)$ for $n \in \mathbb{N}$ and $x \in X$. For $n, j \in \mathbb{N}$ and $x \in X$

$$\begin{aligned} d(T^n x, T^{n+j} x) &\le \sum_{i=1}^{j} d(T^{n+i-1} x, T^{n+i} x) \\ &\le \sum_{i=1}^{j} k^{n+i-1} d(x, Tx) \\ &\le \frac{k^n}{1-k} d(x, Tx). \end{aligned}$$

So $\{T^n x\}$ is a Cauchy sequence in the complete metric space X and so it converges to an element $x^* \in X$.

Now

$$\begin{aligned} d(x^*, Tx^*) &\le d(x^*, T^{n+1}x) + d(T^{n+1}x, Tx^*) \\ &\le d(x^*, T^{n+1}x) + a_1 d(x^*, Tx^*) + \frac{a_2 + a_3}{2}[d(x^*, Tx^*) + d(T^n x, T^{n+1} x)] \\ &\quad + \frac{a_4 + a_5}{2}[d(x^*, T^{n+1}x) + d(Tx^*, T^n x)] \\ &\le d(x^*, T^{n+1}x) + a_1 d(x^*, Tx^*) + \frac{a_2 + a_3}{2}[d(x^*, Tx^*) + d(T^n x, T^{n+1} x)] \\ &\quad + \frac{a_4 + a_5}{2}[d(x^*, T^{n+1}x) + d(x^*, Tx^*) + d(Tx^*, T^n x)] \end{aligned} \tag{5.2.4}$$

Allowing n to tend to ∞ in (5.2.4) and noting that $x^* = \lim T^n x = \lim T^{n+1} x$, we get

$$d(x^*, Tx^*) \le \frac{(a_2 + a_3 + a_4 + a_5 + 2a_1)}{2} d(x^*, Tx^*)$$

Since $0 \le \sum_{i=1}^{5} a_i < 1$, it follows that $x^* = Tx^*$.

If y^* is also a fixed point of T, then

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) && (5.2.5)\\ &\le a_1 d(x^*, y^*) + a_2 d(x^*, Tx^*) + a_3 d(y^*, Ty^*)\\ &\quad + a_4 d(x^*, Ty^*) + a_5 d(y^*, Tx^*) && (5.2.6)\\ &\le (a_1 + a_4 + a_5) d(x^*, y^*). && (5.2.7) \end{aligned}$$

Since $0 \le a_1 + a_4 + a_5 \le \sum_{i=1}^{5} a_i < 1, x^* = y^*$. Thus T has a unique fixed point and every sequence of T-iterates converges to the unique fixed point. □

Corollary 5.2.2 (Kannan [13]) *If $T : X \to X$ is a map on a complete metric space (X, d) such that*

$$d(Tx, Ty) \le k_1 d(x, Tx) + k_2 d(y, Ty)$$

for all $x, y \in x$ with $k_1, k_2 \ge 0$ and $k_1 + k_2 < 1$, then T has a unique fixed point and every sequence of T-iterates converges to the unique fixed point.

Proof Set $a_1 = a_4 = a_5 = 0$, $a_2 = k_1$ and $a_3 = k_2$ in Theorem 5.2.1. □

Remark 5.2.3 A mapping satisfying the conditions of Corollary 5.2.2 (or Theorem 5.2.1) need not be continuous, as seen from the following example.

The map $T : [0, 1] \to [0, 1]$ defined by $Tx = \begin{cases} \frac{x}{4}, & 0 \le x \le \frac{1}{2} \\ \frac{x}{5}, & \frac{1}{2} < x \le 1 \end{cases}$ is a discontinuous map with 0 as the unique fixed point. For $x, y \in [0, 1]$, it can be shown that

$$|Tx - Ty| \le \frac{3}{8}[|x - Tx| + |y - Ty|].$$

Corollary 5.2.4 *Theorem 5.1.7 (Contraction Principle).*

Proof Set $a_1 = k$, $a_2 = a_3 = a_4 = a_5 = 0$ in Theorem 5.2.1. □

In another direction, Boyd and Wong [6] generalized the contraction principle by majorizing $d(Tx, Ty)$ by $\psi(d(x, y))$ instead of $kd(x, y)$, imposing suitable assumptions on the real-valued function ψ of the real variable. In this context we recall the following.

Definition 5.2.5 Let $\psi : [a, \infty) \to \mathbb{R}$ be a function, where $a \in \mathbb{R}$. ψ is said to be upper semicontinuous from the right at $c \in [a, \infty)$ if $\lim_{t \to c^+} \sup \psi(t) \le \psi(c)$.

Theorem 5.2.6 *Let* (X, d) *be a complete metric space and* $T : X \to X$*, a map such that for all* $x, y \in X$

$$d(Tx, Ty) \leq \psi(d(x, y))$$

where $\psi : \overline{P} \to [0, \infty)$ *is upper semicontinuous from the right on* $\overline{P}$ *and* $\psi(t) < t$ *for all* $t \in \overline{P}$ *and* $t \neq 0$*,* P *being the range of* d *and* $\overline{P}$ *its closure in* $\mathbb{R}^+$*. Then,* T *has a unique fixed point and every sequence of* T*-iterates converges to this unique fixed point.*

Proof For $x \in X$, define $c_n = d(T^n x, T^{n-1}x)$, $n \in \mathbb{N}$ with $T^0 x = x$. Clearly c_n is non-increasing and non-negative and hence converges to $c \geq 0$. Since $c_{n+1} \leq \psi(c_n)$ for all $n \in \mathbb{N}$, for $c > 0$,

$$c = \lim c_n = \limsup c_n \leq \lim_{t \to c^+} \sup \psi(t) \leq \psi(c),$$

a contradiction. Thus $d(T^{n-1}x, T^n x)$ converges to zero as $n \to \infty$ for each $x \in X$.

We now show that $\{T^n x\}$ is a Cauchy sequence. Suppose $\{T^n x\}$ is not a Cauchy sequence. Then, for some $\epsilon_0 > 0$ and each $k \in \mathbb{N}$, we can find natural numbers $m(k)$ and $n(k)$ with $m(k) > n(k) \geq k$ such that for all $k \in \mathbb{N}$

$$d_k = d(T^{m(k)}x, T^{n(k)}x) \geq \epsilon_0$$

and

$$d(T^{m(k)-1}x, T^{n(k)}x) < \epsilon_0.$$

This can be done by choosing $m(k)$ as the least natural number exceeding $n(k)$ for which $d_k \geq \epsilon_0$. Now

$$\begin{aligned} d_k = d(T^{m(k)}x, T^{n(k)}x) &\leq d(T^{m(k)}x, T^{m(k)-1}x) + d(T^{m(k)-1}x, T^{n(k)}x) \\ &\leq c_{m(k)} + \epsilon_0. \end{aligned}$$

Thus $\epsilon_0 \leq d_k \leq \epsilon_0 + c_{m(k)}$. Consequently $\epsilon_0 \leq \lim_{k \to \infty} \inf d_k \leq \lim_{k \to \infty} \sup d_k \leq \lim_{k \to \infty} \sup \epsilon_0 + c_{m(k)} = \epsilon_0$. Thus $\lim_{k \to \infty} d_k = \epsilon_0$. Indeed $d_k \to \epsilon_0^+$ as $k \to \infty$.

Further,

$$\begin{aligned} d_k = d(T^{m(k)}x, T^{n(k)}x) &\leq d(T^{m(k)}x, T^{m(k)+1}x) + d(T^{m(k)+1}x, T^{n(k)+1}x) \\ &\quad + d(T^{n(k)+1}x, T^{n(k)}x) \\ &\leq c_{m(k)} + \psi(d(T^{m(k)}x, T^{n(k)}x)) + c_{n(k)} \\ &\leq 2c_k + \psi(d_k) \quad \text{(as } (c_j) \text{ is non-increasing).} \end{aligned}$$

Allowing k to tend to $+\infty$ in the above inequality, it follows that

$$\lim_{k \to \infty} \sup d_k = \lim_{k \to \infty} d_k = \epsilon_0^+ \leq \lim_{k \to \infty} \sup \psi(d_k) \leq \psi(\epsilon_0).$$

Since $\epsilon_0 > 0$, this contradicts that $\epsilon_0 > \psi(\epsilon_0)$. Hence $\{T^n x\}$ is a Cauchy sequence in X. As X is complete, it converges to an element x^* in X. Since for all $x, y \in X$, $d(Tx, Ty) \leq \phi(d(x, y)) \leq d(x, y)$, T is continuous. Since $\{T^{n+1}x\}$ converges to Tx^* and is also a subsequence of $\{T^n x\}$, it follows that $x^* = Tx^*$. Since $\phi(t) < t$ for all $t > 0$, it follows that the fixed point of T is unique. □

Remark 5.2.7 Let X be $(-\infty, -1] \cup [1, \infty)$ with the usual metric. Define $T : X \to X$ by

$$Tx = \begin{cases} -\frac{(x+1)}{2}, & \text{if } x \geq 1 \\ \frac{(1-x)}{2}, & \text{if } x \leq -1 \end{cases}$$

and $\psi : (0, \infty) \to (0, \infty)$ by

$$\psi(t) = \begin{cases} \frac{t}{2}, & \text{if } t < 2 \\ 1 + \frac{t}{2}, & \text{if } t \geq 2 \end{cases}$$

Clearly $|Tx - Ty| \leq \psi(|x - y|)$. T has no fixed point, as $\psi(2) = 2 \not< 2$. Thus, $\psi(t) < t$ is not true for all $t > 0$ even though ψ is upper semicontinuous from the right on $(0, \infty)$. On the other hand, $-T$ has both -1 and 1 as fixed points with $|-Tx - (-Ty)| \leq \psi(|x - y|)$ for the same ψ. (In this case uniqueness of the fixed point is lost).

Corollary 5.2.8 (Rakotch [17]) *Let (X, d) be a complete metric space and $T : X \to X$ an operator such that for all $x, y \in X$ $d(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$, where $\alpha : (0, \infty) \to [0, 1)$ is a montonic decreasing function such that $0 \leq \alpha(t) < \alpha(s)$ for $0 < t < s$. Then, T has a unique fixed point and every sequence of T-iterates converges to the unique fixed point.*

Proof Set $\psi(t) = \alpha(t)t$ for all $t > 0$ in Theorem 5.2.6. Since all the assumptions of Theorem 5.2.6 are satisfied, the corollary follows. □

In another direction, Jungck [12] obtained an extension of the contraction principle as a common fixed point theorem.

Theorem 5.2.9 (Jungck [12]) *Let $f : X \to X$ be a continuous mapping, (X, d) being a complete metric space. Let $g : X \to X$ be a map such that f commutes with g, $g(X) \subseteq f(X)$ and for all $x, y \in X$ there exists $\alpha \in (0, 1)$ such that*

$$d(gx, gy) \leq \alpha d(x, y).$$

Then f and g have a unique common fixed point.

Proof As f is continuous and $d(gx, gy) \leq \alpha d(fx, fy)$ for all $x, y \in X$, it follows that g is continuous on X. As $g(X) \subseteq f(X)$, given $x_0 \in X$, we can find $x_1 \in X$ such that $fx_1 = gx_0$. Inductively, we can define a sequence $x_n \in X$ such that $gx_{n-1} = fx_n$ for all $n \in \mathbb{N}$. So $d(fx_{n+1}, fx_n) = d(gx_n, gx_{n-1}) \leq \alpha d(fx_n, fx_{n-1}) \leq \alpha^n d(fx_1,$

fx_0). As $0 < \alpha < 1$, $\{fx_n\} = \{gx_{n-1}\}$ is a Cauchy sequence in X. Since X is complete $\{fx_n\}$ converges to some $x^* \in X$. So $\{gx_n\}$ converges to x^* as $gx_n = fx_{n+1}$. As f and g are both continuous, $\{gfx_n\}$ converges to gx^*, while $\{fgx_n\}$ converges to fx^*. Since $gfx_n = fgx_n$, for all n, $fx^* = gx^*$. Now

$$\begin{aligned} d(gx^*, ggx^*) &\leq \alpha d(fx^*, fgx^*) \\ &= \alpha d(gx^*, ggx^*) \end{aligned}$$

Since $0 < \alpha < 1$, $gx^* = ggx^* = gfx^* = fgx^*$. Clearly gx^* is a common fixed point of f and g. If a and b are two common fixed points of f and g, then

$$d(a, b) = d(ga, gb) \leq \alpha d(fa, fb) = \alpha d(a, b).$$

Since $0 < \alpha < 1$, $a = b$. Thus f and g have a unique common fixed point. □

Corollary 5.2.10 *Let f and g be commuting mappings on a complete metric space (X, d) into itself. Suppose f is continuous and $g(X) \subseteq f(X)$. If for some $\alpha \in (0, 1)$ and a positive integer k, $d(g^k x, g^k y) \leq \alpha d(x, y)$ for all $x, y \in X$, then f and g have a unique common fixed point.*

Proof By Theorem 5.2.9, g^k and f have a unique common fixed point a, say. Then $a = g^k(a) = fa$. So $ga = g^k ga = gfa = fga$, showing that ga is also a fixed point for g^k and f. By the uniqueness of the common fixed point for f and g^k, it follows that $a = ga = fa$. □

Remark 5.2.11 In fact if f has a fixed point, then we can find a commuting map g with a unique fixed point common with f, $g(X) = f(X)$ and $d(gx, gy) \leq \alpha d(fx, fy)$ for all $x, y \in X$ for some $\alpha \in (0, 1)$. This is readily seen by setting $gx \equiv a$, a fixed point of f, and choosing any $\alpha \in (0, 1)$.

Example 5.2.12 $f(x) = x^2$ and $g(x) = x^4$ on $[0, \frac{1}{2}]$ satisfy all the assumptions of Theorem 5.2.9 and 0 is the unique common fixed point.

5.3 Fixed Points of Multivalued Contractions

Nadler [15] generalized the contraction principle for multivalued functions, involving the Hausdorff metric. We need the following

Definition 5.3.1 Let (X, d) be a metric space and $CB(X)$ be the set of all non-empty closed bounded subsets of X. For $C \in CB(X)$ define

$$\begin{aligned} N(C, \epsilon) &= \{x \in X : d(x, c) < \epsilon \text{ for some } c \in C\} \\ &= \bigcup_{c \in C} B(c; \epsilon). \end{aligned}$$

For $A, B \in CB(X)$, define

$$H(A, B) = \inf\{\epsilon \in \mathbb{R} : \epsilon > 0, A \subseteq N(B, \epsilon) \text{ and } B \subseteq N(A, \epsilon)\}.$$

Remark 5.3.2 H defines a metric on $CB(X)$, called the Hausdorff distance on the space $CB(X)$. Further for $x, y \in X$, $H(\{x\}, \{y\}) = d(x, y)$.

Nadler [15] proved a generalization of the contraction principle for mappings of X into $CB(X)$, using the following definition and a lemma.

Definition 5.3.3 Let (X, d) be a metric space. A map $F : X \to CB(X)$ is called a multivalued contraction if there exists $\alpha \in (0, 1)$ such that for all $x, y \in X$, $H(Fx, Fy) \leq \alpha d(x, y)$.

Lemma 5.3.4 *Let (X, d) be a metric space and $A, B \in CB(X)$. Given $\epsilon > 0$ and $a \in A$, we can find $b \in B$ such that $d(a, b) \leq H(A, B) + \epsilon$.*

Proof Let $r = H(A, B)$. For $r = 0$, the lemma is clear. For $r > 0$, by definition of $H(A, B)$, $A \subseteq N(B, r + \epsilon)$. So for $a \in A$, there exists $b \in B$ such that $a \in B(b, r + \epsilon)$ or $d(a, b) < r + \epsilon = H(A, B) + \epsilon$. □

Theorem 5.3.5 (Nadler) *Let (X, d) be a complete metric space and $F : X \to CB(X)$ be a multivalued contraction with contraction constant $\alpha \in (0, 1)$. Then F has a fixed point in X (i.e. an element $x_0 \in X$ with $x_0 \in Fx_0$).*

Proof For $p_0 \in X$, since $F(p_0) \in CB(X)$ for any $p_1 \in F(p_0)$, for $\epsilon = \alpha$, it follows from Lemma 5.3.4 above that there exists $p_2 \in F(p_1)$ such that

$$d(p_1, p_2) \leq H(F(p_0), F(p_1)) + \alpha.$$

Since $F(p_1), F(p_2) \in CB(X)$ and $p_2 \in F(p_1)$ there is a point $p_3 \in F(p_2)$ insured by Lemma 5.3.4 for $\epsilon = \alpha^2$ with

$$d(p_2, p_3) \leq H(F(p_1), F(p_2)) + \alpha^2$$

Thus inductively we can define a sequence of points $\{p_i : i \in \mathbb{N}\}$ such that

$$d(p_i, p_{i+1}) \leq H(F(p_{i-1}), F(p_i)) + \alpha^i$$

for $i \geq 1$. Since F is a multivalued contraction, for all $i \geq 1$,

$$\begin{aligned} d(p_i, p_{i+1}) &\leq H(F(p_{i-1}), Fp_i) + \alpha^i \\ &\leq \alpha d(p_{i-1}, p_i) + \alpha^i \\ &\leq \alpha[H(Fp_{i-2}, Fp_{i-1}) + \alpha^{i-1}] + \alpha^i \\ &\leq \alpha^2 d(p_{i-2}, p_{i-1}) + 2\alpha^i \\ &\leq \alpha^i d(p_0, p_1) + i\alpha^i \end{aligned}$$

So

$$\begin{aligned} d(p_i, p_{i+j}) &\le \sum_{k=1}^{j} d(p_{i+k-1}, p_{i+k}) \\ &\le \sum_{k=1}^{j} (\alpha^{i+k-1} d(p_0, p_1) + [i+k-1]\alpha^{i+k-1}). \end{aligned}$$

As $\sum_1^\infty \alpha^n$ and $\sum_1^\infty n\alpha^n$ converge, it follows that $\{p_n\}$ is a Cauchy sequence in the complete metric space (X, d) converging to an element p^* in X. Writing $d(a, B) = \inf\{d(a, b) : b \in B\}$, it follows that

$$\begin{aligned} d(p^*, Fp^*) &\le d(p^*, Fp_n) + H(Fp^*, Fp_n) \\ &\le d(p^*, p_{n+1}) + \alpha d(p^*, p_n) \\ &\quad \text{(as } p_{n+1} \in Fp_n \text{ and } F \text{ is a} \\ &\quad \text{multivalued contraction).} \end{aligned}$$

This implies that $d(p^*, Fp^*) = 0$ as $\{p_n\}$ converges to p^*. Since Fp^* is closed, $p^* \in Fp^*$. Thus F has a fixed point. □

Corollary 5.3.6 *The contraction principle (Theorem 5.1.7).*

Proof The map $x \to \{Tx\}$ maps the complete metric space (X, d) into $CB(X)$ and is a multivalued contraction, whenever $T : X \to X$ is a contraction, in view of Remark 5.3.2. Hence by Nadler's Theorem 5.3.5, this map has a fixed point, which is clearly a fixed point T. The uniqueness can be proved independently. □

Remark 5.3.7 A fixed point of the multivalued contraction, insured by Nadler's theorem need not be unique. For example, for the map $x \to [0, 1]$ of $\mathbb{R}$ into $CB(\mathbb{R})$, every point of [0, 1] is a fixed point. This map, being a constant map, is clearly, a contraction.

5.4 Contraction Principle in Gauge Spaces

We recall the following definition of a pseudometric on a non-empty set.

Definition 5.4.1 A map $d : X \times X \to \mathbb{R}^+$ is called a pseudometric if it satisfies the following conditions:

(i) $d(x, x) = 0$ for all $x \in X$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Remark 5.4.2 The map $T : X \to X$ on a metric space (X, d) induces a pseudometric d_T defined by $d_T(x, y) = d(Tx, Ty)$ on X in a natural way. d_T is a metric on X if and only if T is one-to-one mapping.

A family of pseudometrics on X defines a topology on X in a natural way, leading to the concept of a gauge space.

Definition 5.4.3 ([8]) Let X be a non-empty set and $G = \{d_\lambda : \lambda \in \Lambda\}$ be a family of pseudometrics on X, where $\Lambda \neq \phi$. G is called a gauge on X. The family $\{B_{d_\lambda}(x, \epsilon) : d_\lambda \in G, x \in X, \epsilon > 0\}$ is a sub-base for a topology on X, where $B_{d_\lambda}(x, \epsilon) = \{y \in X : d_\lambda(x, y) < \epsilon\}$. X with this topology is called a gauge space. (Every neighbourhood of x in this gauge space contains a set of the form $\bigcap_{\lambda \in F} B_{d_\lambda}(x, \epsilon_\lambda)$, where F is a non-void finite subset of Λ and $\epsilon_\lambda > 0$.)

Tan [20] generalized the contraction principle to gauge spaces, using the following concepts.

Definition 5.4.4 A gauge $\{d_\lambda : \lambda \in \Lambda, \Lambda \neq 0\}$ is said to be separating on X if for each pair $x, y \in X$ with $x \neq y$, there exists $\mu \in \Lambda$ with $d_\mu(x, y) > 0$.

Remark 5.4.5 Clearly a gauge space is Hausdorff if and only if the gauge is separating.

Definition 5.4.6 Let $D = \{d_\lambda : \lambda \in \Lambda, \Lambda \neq 0\}$ be a gauge on X. A sequence (x_n) is called a Cauchy sequence in (X, D) if $\lim_{m,n \to \infty} d_\lambda(x_m, x_n) = 0$ for each $\lambda \in \Lambda$. A gauge space X is said to be sequentially complete if every Cauchy sequence in X converges to an element of X.

The following is Tan's generalization [20] of the contraction principle to sequentially complete Hausdorff gauge spaces.

Theorem 5.4.7 (Tan [20]) *Let $D = \{d_\lambda : \lambda \in \Lambda, \Lambda \neq \phi\}$ be a separating gauge on X. Let T be a self-map on X such that for each $\lambda \in \Lambda$, there exists c_λ with $0 \leq c_\lambda < 1$ such that $d_\lambda(Tx, Ty) \leq c_\lambda d_\lambda(x, y)$ for all $x, y \in X$. If (X, D) is sequentially complete, then T has a unique fixed point and every sequence of T-iterates converges to the fixed point.*

Proof Define a sequence $\{x_n\}$ iteratively in X by setting $x_1 = Tx_0$, for an arbitrary x_0 in X and defining $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. For each $\lambda \in \Lambda$, $n, p \in \mathbb{N}$,

$$\begin{aligned} d_\lambda(x_n, x_{n+p}) &\le \sum_{k=1}^{p} d_\lambda(x_{n+k-1}, x_{n+k}) \\ &\le \sum_{k=1}^{p} d_\lambda(T^{n+k-1}x_0, T^{n+k}x_0) \\ &\le \sum_{k=1}^{p} c_\lambda^{n+k-1} d_\lambda(x_0, x_1) \\ &\qquad (\text{as } d_\lambda(x_{j-1}, x_j) \le c_\lambda^{j-1} d(x_0, x_1)) \\ &\le \frac{c_\lambda^n}{1-c_\lambda} d_\lambda(x_0, x_1). \end{aligned}$$

Thus $\lim_{m,n\to\infty} d_\lambda(x_m, x_n) = 0$ for each $\lambda \in \Lambda$ and so (x_n) is a Cauchy sequence in X. As (X, d) is sequentially complete, (x_n) converges to an element $x^* \in X$. Clearly $\lim_{n\to\infty} T(a_n) = T(a)$, whenever (a_n) converges to a in (X, D). So (Tx_n) converges to Tx^*. (Tx_n) being (x_{n+1}), (Tx_n) must also converge to x^*. As D is a separating family, X is Hausdorff in the topology induced by the gauge D. Since the limit of a convergent sequence in a Hausdorff space is unique, $x^* = Tx^*$. Thus T has a fixed point. If $y^* = Ty^*$, then $d_\lambda(x^*, y^*) = d_\lambda(Tx^*, Ty^*) \le c_\lambda d_\lambda(x^*, y^*)$. Since $0 \le c_\lambda < 1$ for all $\lambda \in \Lambda$, it follows that $x^* = y^*$. Thus, the fixed point is unique and every sequence of T-iterates converges to the unique fixed point. □

Remark 5.4.8 The topology of a gauge space need not be metrizable. For example, let X be the space of all mappings of $\mathbb{R}$ into itself. Then for $f, g \in X$ $(= \mathbb{R}^{\mathbb{R}})$, d_x defined by $d_x(f, g) = |fx - gx|$ is a pseudometric on X for each $x \in \mathbb{R}$. $D = \{d_x : x \in \mathbb{R}\}$ is a separating family of pseudometrics on X and (X, D) is sequentially complete. It can be seen that X is not first countable. Hence it cannot be metrized.

Corollary 5.4.9 *The contraction principle (Theorem 5.1.7).*

Proof If (X, d) is a complete metric space and $T : X \to X$ a contraction, then T has a unique fixed point, as $\{d\}$ is a separating sequentially complete gauge on X. □

5.5 A Converse to the Contraction Principle

In this section, a converse to the contraction principle due to Bessaga [3] is proved following a simplified approach due to Jachymski [11].

Theorem 5.5.1 (Bessaga) *Let X be a non-empty set and $T : X \to X$ be a map and $k \in (0, 1)$. Then*

(a) there exists a metric d on X such that $d(Tx, Ty) \le kd(x, y)$ for all $x, y \in X$, whenever T^n has at most one fixed point for each $n \in N$;

(b) if in addition some T^n has a fixed point, X has a complete metric d such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$.

Jachymski's proof [11] is based on the following

Lemma 5.5.2 *Let $T : X \to X$ be a map and $k \in (0, 1)$. The following statements are equivalent:*

(i) there exists a complete metric d on X such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$;
(ii) there exists a function $\varphi : X \to \mathbb{R}^+$ such that $\varphi(Tx) \leq k\varphi(x)$ for all $x \in X$ and $\varphi^{-1}\{0\}$ is a singleton.

Proof (i) $\Rightarrow$ (ii). Since by the contraction principle T has a fixed point x^*, the map φ defined by $\varphi(x) = d(x, x^*)$ is such that $\varphi(Tx) = d(Tx, Tx^*) \leq kd(x, x^*) = k\varphi(x)$ for all $x \in X$ and $\varphi^{-1}\{0\} = \{x^*\}$, a singleton. Thus (ii) is true.

(ii) $\Rightarrow$ (i). Define $d : X \times X \to \mathbb{R}^+$ by

$$d(x, y) = \begin{cases} \varphi(x) + \varphi(y) & \text{for } x \neq y \\ 0, \text{ for } x = y. \end{cases}$$

Clearly d is a metric on X. $d(Tx, Ty) = \varphi(Tx) + \varphi(Ty) \leq k\varphi(x) + k\varphi(y) = kd(x, y)$. So T is a contraction on X with Lipschitz constant k. Consider a Cauchy sequence (x_n) in X. Without loss of generality, suppose that $\{x_n : n \in \mathbb{N}\}$ is an infinite set. Otherwise, (x_n) would have a constant subsequence and being Cauchy would converge in X. So there is an infinite subsequence (x_{n_k}) of distinct elements such that

$$d(x_{n_k}, x_{n_m}) = \varphi(x_{n_k}) + \varphi(x_{n_m}) \text{ for } k \neq m.$$

So $\varphi(x_{n_k}) \to 0$ as $k \to \infty$. Since $\varphi^{-1}(\{0\}$ is a singleton $\{z\}$ by hypothesis (ii), $\varphi(z) = 0$. So $d(x_{n_k}, z) = \varphi(x_{n_k}) + \varphi(z)$ tends to zero as $k \to \infty$. Thus $\lim x_{n_k} = z$. This implies that the entire Cauchy sequence (x_n) converges to z. In other words (X, d) is complete. □

We first prove part (b) of Theorem 5.5.1.

Proof (of part (b) of Theorem 5.5.1).
By assumption T^n has a unique fixed point x^*. By uniqueness of the fixed point of T^n, $x^* = Tx^*$. So by part (a) of Theorem 5.5.1 T^m has the unique fixed point x^* for each $m \in \mathbb{N}$. Define

$$\Phi = \{\varphi : \varphi \text{ is a map with domain } D_\varphi \subseteq X \text{ into } \mathbb{R}^+ \text{ with } \varphi^{-1}\{0\} = \{x^*\} \text{ and } T(D_\varphi) \subseteq D_\varphi \text{ and } \varphi(Tx) \leq \varphi(x) \text{ for all } x \in D_\varphi\}$$

Clearly Φ is non-empty, for $\varphi_1 : \{x^*\} \to \mathbb{R}^+$ defined by $\varphi_1(x^*) = 0$ and $D_{\varphi_1} = \{x^*\}$, belongs to Φ. We can partially order Φ by $\varphi_1 \preceq \varphi_2 \Leftrightarrow D_{\varphi_1} \subseteq D_{\varphi_2}$ and $\varphi_2|_{D_{\varphi_1}} = \varphi_1$.

If $\Phi_0 \subseteq \Phi$ is a chain under $\preceq$, then the set $D = \bigcup_{\varphi \in \Phi_0} D_\varphi$ is T-invariant and $\psi : D \to \mathbb{R}^+$ defined by $\psi(x) = \varphi(x)$, $x \in D_\varphi$ is an upper bound for Φ_0. Thus by Zorn's Lemma there exists a maximal element $\varphi_0 : D_0 \to \mathbb{R}^+$ in $(\Phi, \preceq)$. We now show that $D_0 = X$. Otherwise we can find $x_0 \in X - D_0$. Set $O(x_0) = \{T^{n-1}x_0 : n \in \mathbb{N}\}$.

Suppose $O(x_0) \cap D_0 = \phi$. Then the elements $T^{n-1}(x_0)$ for $n \in \mathbb{N}$ are all distinct. Otherwise, T has a periodic point which necessarily is the unique fixed point x^*, implying that $x^* \in O(x_0)$, a contradiction as $x^* \notin D_0$. Define $D_\varphi = O(x_0) \cup D_0$ and $\varphi : D_\varphi \to \mathbb{R}^+$ by

$$D_\varphi(x) = \begin{cases} \varphi(x), & x \in D_0 \\ k^{n-1}, & x = T^{n-1}(x_0) \in O(x_0), \text{ for } n \in \mathbb{N} \end{cases}$$

Clearly $\varphi \in \Phi$, $\varphi_0 \leq \varphi$ and $\varphi_0 \neq \varphi$, contradicting the maximality of φ_0. So $O(x_0) \cap D$ is non-empty. Now, define $m = \min\{n \in \mathbb{N} : T^n(x_0) \in D\}$. Clearly $T^{m-1}(x_0) \notin D_0$. Define $D_\varphi = \{T^{m-1}(x_0)\} \cup D_0$. Then $T(D_\varphi) = \{T^m x_0\} \cup T(D_0) \subseteq D_0$. Thus D_φ is T-invariant. Consider $\varphi : D_\varphi \to \mathbb{R}^+$ defined by $\varphi|D_0 = \varphi_0$ and $\varphi(T^{m-1}(x_0)) = 1$, if $T^{m-1}(x_0) = x^*$ and $\varphi(T^{m-1}(x_0)) = \frac{1}{k}\varphi_0(T^m(x_0))$ for $T^{m-1}(x_0) \neq x^*$. In both these cases $\varphi \in \Phi$, $\varphi_0 \preceq \varphi$ and $\varphi \neq \varphi_0$, contradicting the maximality of φ_0. So by Lemma 5.5.2, the proof of part (b) of Theorem 5.5.1 is complete. □

The next lemma, not only helps to prove part (a) of theorem but even extends it.

Lemma 5.5.3 *Let X be a non-void set and $T : X \to X$, a map, with $\alpha \in (0, 1)$. The following statements are equivalent.*

(i) T has no periodic point;
(ii) the Schroder functional equation $\varphi(Tx) = \alpha\varphi(x)$ has a solution $\varphi : X \to (0, \infty)$.

Proof We prove that (i) implies (ii). Define $\Phi = \{\varphi : D_\varphi \to (0, \infty) | D_\varphi \neq \phi, D_\varphi \subseteq X, T(D_\varphi) \subseteq D_\varphi$ and $\varphi(Tx) = \alpha\varphi(x)$ for $x \in D_\varphi\}$. Let x_0 be a fixed element of X and $D_{\varphi_1} = O(x_0)$, φ_1 being the map $\varphi_1 : O(x_0) \to (0, \infty)$ defined by $\varphi_1(T^{n-1}(x_0)) = \alpha^{n-1}$, $n \in \mathbb{N}$. Clearly T is 1-1 on D_{φ_1} as T has no periodic point in X by hypothesis. Further $\varphi_1(Tx) = \alpha\varphi_1(x)$ on D_{φ_1}. So $\varphi_1 \in \Phi$ and Φ is non-empty. We can repeat the argument used to prove part (b) of Theorem 5.5.1 involving Zorn's Lemma to ensure that there is a maximal element φ_0 in $(\Phi, \preceq)$. Here, $\preceq$ is the partial order in Φ defined by $\varphi_1 \preceq \varphi_2$ if $D_{\varphi_1} \subseteq D_{\varphi_2}$ and $\varphi_2|_{D_{\varphi_1}} = \varphi_1$. We claim that $D_{\varphi_0} = X$. Otherwise there exists $x_0 \in X - D_0$. If $O(x_0) \cap D_{\varphi_0}$ is empty, then $T^{n-1}(x_0)$ are all distinct, as otherwise T would have a periodic point x_0. Define $D_\varphi = O(x_0) \cup D_{\varphi_0}$, $\varphi|_{D_{\varphi_0}} = \varphi_0$ and $\varphi(T^{n-1}(x_0)) = \alpha^{n-1}$ for $n \in \mathbb{N}$. Clearly $\varphi \in \Phi$, $\varphi \neq \varphi_0$ and $\varphi_0 \preceq \varphi$, a contradiction. Hence $O(x_0) \cap D_{\varphi_0}$ is non-empty.

Define $m = \min\{n \in \mathbb{N} : T^n(x_0) \in D_{\varphi_0}\}$. So $T^{m-1}(x_0) \notin D_{\varphi_0}$. Define $D_\varphi = \{T^{m-1}x_0\} \cup D_{\varphi_0}$. Clearly $T(D_\varphi) \subseteq D_\varphi$. Define $\varphi : D_\varphi \to (0, \infty)$ by $\varphi|_{D_0} = \varphi_0$ and $\varphi(T^{m-1}(x_0)) = \frac{\varphi_0(T^m(x_0))}{\alpha}$. It may be noted that $T^{m-1}(x_0) \neq T^m(x_0)$. Further $\varphi \in \Phi$,

$\varphi \neq \varphi_0$ and $\varphi_0 \preceq \varphi$, contradicting the maximality of φ_0 once again. So $D_{\varphi_0} = X$ and $\varphi_0(T(x)) = \alpha\varphi_0(x)$ for all $x \in X$.

To prove that (ii) implies (i), suppose that $x_0 = T^k(x_0)$ for some $x_0 \in X$, $k \in \mathbb{N}$. By hypothesis (ii) $\varphi(x_0) = \varphi(T^k(x_0)) = \alpha\varphi(x_0)$ and this implies $\varphi(x_0) = 0$, a contradiction. □

We now indicate the proof of (a) of Theorem 5.5.1, below.

Proof of Theorem 5.5.1 (a) If T^m has a fixed point for some $m \in \mathbb{N}$, either T has a fixed point or a periodic point. So part (b) applies and there exists a complete metric d on X under which T is α-contraction.

If T^m has no fixed point for all $m \in \mathbb{N}$, T has no periodic point. By Lemma 5.5.3, there exists $\varphi : X \to (0, \infty)$ such that $\varphi(T(x)) = \alpha\varphi(x)$. Now $d(x, y) = \varphi(x) + \varphi(y)$ for $x \neq y$ and $d(x, y) = 0$ for $x = y$ is a metric on X with $d(Tx, Ty) = \alpha d(x, y)$. Further, the open sphere centred at x and radius $\varphi(x)$ contains only x. So this metric topology is indeed discrete.

It is natural to enquire if there exist incomplete metric spaces in which every contraction has a fixed point. The example due to Borwein [5] describes such a space.

Example 5.5.4 Let L_k be the line segment in $\mathbb{R}^2$ joining $A = (0, 0)$ to $B_k = (1, \frac{1}{2^k})$ for each $k \in \mathbb{N}$. Consider $C = \bigcup_{k\in\mathbb{N}} L_k$ with the usual euclidean metric in $\mathbb{R}^2$. Clearly each L_k is a connected subset of C for $k \in \mathbb{N}$ and C itself is connected. As $\overline{C}$ contains the line segment joining $A = (0, 0)$ with $(1, 0)$, C is not closed in $\mathbb{R}^2$. So C with the euclidean metric is incomplete. Let T be a contraction mapping C into itself. If each $T^n(C)$ contains $(0, 0)$, then $T(0, 0) = (0, 0)$ $(= A)$ as diameter $T^n(C)$ tends to zero. Clearly $A = (0, 0)$ is the fixed point of T. If $T^n(C)$ does not contain $A = (0, 0)$ for some $n \in \mathbb{N}$, then $T^n(C)$ being a connected subset not containing A must lie properly in some L_k. So T^n maps L_k $(\subseteq C)$ into L_k. Being a contraction on the complete space L_k into itself, T^n and hence T has a fixed point in this L_k. Thus every contraction of C into itself has a fixed point, even though C is not complete.

On the other hand, the following theorem ensures the completeness of a metric space (X, d) under the assumption that a class of self-maps satisfying certain metrical inequalities have fixed points (see Subrahmanyam [19]).

Theorem 5.5.5 *Let (X, d) be a metric space in which every map $T : X \to X$ satisfying the following conditions has a fixed point.*

(i) for some $\lambda > 0$ and $x, y \in X$
$d(Tx, Ty) \leq \lambda \max\{d(x, Tx), d(y, Ty)\}$*;*
(ii) $T(X)$ is countable.

Then (X, d) is complete.

Proof Let, if possible, $A = \{x_n\}$ be a non-convergent Cauchy sequence of distinct elements of X. For any $x \in X$, $d(x, A) = \inf\{d(x, a) : a \in A\} > 0$. Since $\{x_n\}$ is Cauchy, there exists a least positive integer $N(x)$ such that for $m, n \geq N(x)$

$$d(x_m, x_n) < \lambda d(x, A) \leq \lambda d(x, x_\ell), \quad \ell \in \mathbb{N}.$$

In particular

$$d(x_m, x_{N(x)}) < \lambda d(x, x_\ell), \quad \ell \in \mathbb{N}, m \geq N(x).$$

By a similar reasoning for $n \in \mathbb{N}$, there exists a least positive integer $n' = n'(n) > n$ such that

$$d(x_m, x_{n'}) < \lambda d(x_n, x_{n'}), \quad m \geq n'.$$

Define $T : X \to X$ by

$$T(x) = \begin{cases} x_{N(x)}, & \text{if } x \notin A \\ x_{n'}, & \text{if } x = x_n \in A. \end{cases}$$

Clearly T has no fixed point, though it satisfies conditions (i) and (ii) above. Indeed for $T(x) = x_n$, $T(y) = x_m$,

$$d(x_m, x_n) < \begin{cases} \lambda d(y, A - \{y\}), & n \geq m \\ \lambda d(x, A - \{x\}), & n < m \end{cases}$$

and consequently (i) is true. This contradiction shows that (X, d) must be complete. □

Corollary 5.5.6 (Converse to Kannan's fixed point Theorem (Corollary 5.2.2)) *Let (X, d) be a metric space. If for each $\lambda \in (0, \frac{1}{2})$ every map $T : X \to X$ satisfying the condition*

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)], \quad x, y \in X$$

has a fixed point, then (X, d) is complete.

Corollary 5.5.7 (Converse to Theorem 5.2.1) *Let (X, d) be a metric space. If for each $\{a_1, a_2, a_3, a_4, a_5\} \subseteq [0, 1]$ with $\sum_{i=1}^{5} a_i < 1$, each map $T : X \to X$ satisfying the inequality*

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_2 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx), \quad x, y \in X$$

has a fixed point, then (X, d) is complete.

Corollary 5.5.7 follows from Corollary 5.5.6.

Indeed, the following theorem can be proved using the argument for Theorem 5.5.5.

Theorem 5.5.8 ([19]) *Let (X, d) be a metric space in which every mapping T satisfying either*

(a) $d(Tx, Ty) \leq \lambda \max\{\inf_{k\in\mathbb{N}} d(x, T^k x), \inf_{k\in\mathbb{N}} d(y, T^k y)\}\ x, y \in X$
or

(b) $d(Tx, Ty) \leq \lambda \max\{\inf_{k\in\mathbb{N}} d(x, T^k y), \inf_{k\in\mathbb{N}} d(y, T^k x)\}\ \ x, y \in X$ *for fixed* $\lambda > 0$
and

(c) $T(X)$ *is countable*
has a periodic point.

Then (X, d) *is complete.*

Earlier Hu [10] proved the following converse of the contraction principle using this argument.

Theorem 5.5.9 ([10]) *Let* (X, d) *be a metric space. If for every closed non-empty subset C of X, any contraction* $T : C \to C$ *has a fixed point, then* (X, d) *is complete.*

5.6 A Topological Contraction Principle

In this section, a topological version of the contraction principle due to Kupka [14] is presented. This fixed point theorem is proved for multifunctions which are feebly topologically contractive, without involving any concept of completeness.

Definition 5.6.1 Let $(X, \mathcal{T})$ be a topological space. By a multifunction on X, we mean a mapping of X into the set of all non-empty subsets of X. The graph of a multifunction F on X is the set $G \circ F = \{(x, y) : y \in Fx, x \in X\}$ and is denoted by GrF.

Definition 5.6.2 Let $(X, \mathcal{T})$ be a topological space. A multifunction $F : X \to 2^X - \{\phi\}$ is said to be feebly topologically contractive if for each open cover $\mathcal{G}$ of X and for any pair of points $a, b \in X$ there exists $k \in \mathbb{N}$ such that for some open set $G \in \mathcal{G}$, $F^k(a) \subseteq G$ and $F^k(b) \cap G \neq \phi$. If $f : X \to X$ is (a single-valued) map, then f is feebly topologically contractive if for any cover $\mathcal{G}$ of X and any pair $a, b \in X$, there exists $k \in \mathbb{N}$ such that for some $G \in \mathcal{G}$, $f^k(a), f^k(b) \in G$.

Kupka obtained the following generalization of the contraction principle.

Theorem 5.6.3 (Kupka [14]) *Let X be an arbitrary topological space and* $F : X \to 2^X - \{\phi\}$*, a feebly contractive multifunction, with a closed graph. Then F has a fixed point. If, X is in addition a* T_1 *topological space, then the fixed point is unique and* $F(z) = \{z\}$*, where z is the fixed point of F.*

Proof Suppose F has no fixed point. So $x \notin Fx$. Since GrF is a closed subset of $X \times X$ with the product topology, $O = X \times X - GrF$ is an open set containing the diagonal $\{(x, x) : x \in X\}$ of X^2. Let $\mathcal{G} = \{v \in \mathcal{T} : V \times V \subseteq O\}$. Clearly $\mathcal{G}$ is an open cover for X as $(x, x) \in O$ for each $x \in X$. For $a \in X$ and $b \in F(a)$, by the feeble-topological contractivity of F, there exists $k \in \mathbb{N}$ and an open set $G \in \mathcal{G}$ such that $F^k(a) \subseteq G$ and $F^k(b) \cap G \neq \phi$. Since $b \in F(a)$, $F^k(b) \subseteq F^{k+1}(a)$. So $F^{k+1}(a) \cap G \supseteq F^k(b) \cap G$ is non-void. Since $F^k(a) \subseteq G$, $F(G) \cap G \neq \phi$. This implies that $Gr\,F \cap (G \times G) \neq \phi$. So $Gr\,F \cap O \neq \phi$, a contradiction to the choice of O. So F has a fixed point in X.

Suppose further that X is a T_1-space. Let x^* be a fixed point of F. Suppose $b \neq x^*$ also belongs to Fx^*. Clearly $\mathcal{G} = \{X - \{x^*\}, X - \{b\}\}$ is an open cover for X. As F is feebly topologically contractive, there is an open set $G \in \mathcal{G}$ such that for some $k \in \mathbb{N}$, $F^k(x^*) \subseteq G$. As $x^* \in Fx^*$, $x^* \in F^{k-1}(x^*)$ and hence $\{z, b\} \subseteq F(x^*) \subseteq F^k(x^*) \subseteq G$. But this is impossible as G either lacks x^* or b. Thus $F(x^*) = \{x^*\}$.

If F has two distinct fixed points a and b, then from the above argument $F(a) = \{a\}$ and $F(b) = \{b\}$. Now $\mathcal{G} = \{X - \{a\}, X - \{b\}\}$ is an open cover for X. Since F is feebly topologically contractive, there exists $k \in \mathbb{N}$ such that for some $G \in G$, $F^k(a) \subseteq G$ and $F^k(b) \cap G \neq \phi$. Since $\{a\} = F(a)$ and $\{b\} = F(b)$, it implies that G must contain both a and b. This is a contradiction. Hence F has a unique fixed point. □

Corollary 5.6.4 *Let $T : X \to X$ be a feebly contractive map with a closed graph. Then T has a fixed point. Further, if X is T_1, then T has a unique fixed point.*

Corollary 5.6.5 *Theorem 5.1.7 (Contraction Principle).*

Proof Every contraction has a closed graph and is feebly topologically contractive (as T has a unique fixed point x^* in a complete metric space X for a given pair $a, b \in X$, G can be chosen as an open set containing x^*). □

Example 5.6.6 Define $G : \mathbb{R} \to 2^{\mathbb{R}} - \{\phi\}$ by $G(x) = \{0, n\}$, if $x = \frac{1}{n}, n \in \mathbb{N} \neq n > 1$ and $G(x) = \{0\}$, otherwise. G has closed graph and is feebly topologically contractive, the topology on $\mathbb{R}$ being the usual topology. Clearly 0 is the fixed point of G.

Example 5.6.7 Let X be $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with the usual metric. Define $f : X \to X$ by $f(0) = 1$ and $f(\frac{1}{k}) = \frac{1}{k+1}$, $k \in \mathbb{N}$. While X is a complete metric space and f is feebly topologically contractive, f has no fixed point. This is because the graph of f is not closed in X^2. Thus the assumption that F is closed cannot be dropped in Theorem 5.6.3.

Kupka [14] had also introduced a stronger notion of contractivity, as in.

Definition 5.6.8 Let $(X, \mathcal{T})$ be a topological space and $T : X \to X$, a map. T is called topologically contractive if for each open cover $\mathcal{G}$ of X, given a pair of points a and b in X, there exists $n \in \mathbb{N}$ such that for all $k \geq n$, there exists $G \in \mathcal{G}$ such that $f^k(a), f^k(b) \in G$.

Clearly every topological contraction is topologically feebly contractive. In fact in Example 5.6.7, the map f is topologically contractive.

Example 5.6.9 Let X be $\{0, 1, 3\}$ and $\mathcal{T}$ be $\{\phi, \{0, 1\}, X\}$. Define $f : X \to X$ by $f(x) = x$ for $x \neq 3$ and $f(3) = 1$. f is feebly contractive with 0 and 1 as fixed points. Evidently $(X, \mathcal{J})$ is not T_1.

5.7 Another Proof of the Contraction Principle

In this section, Baranga's proof [2] of the contraction principle using Kleene's fixed point theorem is presented. To this end, we need the following definitions and theorems.

Definition 5.7.1 Let $(P, \leq)$ be a partially ordered set. For an increasing sequence $(x_n : n \in \mathbb{N})$, we denote the supremum of this sequence by $\vee\{x_n : n \in \mathbb{N}\}$. $(P, \leq)$ is said to be ω-complete if every increasing sequence (x_n) in P has a supremum in P.

Definition 5.7.2 Let $(P, \leq)$ and $(Q, \leq)$ be two partially ordered sets. A map $f : P \to Q$ is said to be ω-continuous if for every increasing sequence (x_n) in P, such that $\vee\{x_n : n \in \mathbb{N}\}$ exists in P, also $\vee\{fx_n : n \in \mathbb{N}\}$ exists in Q and $f(\vee\{x_n : n \in \mathbb{N}\}) = \vee\{f(x_n) : n \in \mathbb{N}\}$. (Clearly any ω-continuous function is increasing).

We now state Kleene's fixed point theorem.

Theorem 5.7.3 (Kleene's fixed point theorem) *Let $(P, \leq)$ be an ω-complete partially ordered set and $f : P \to P$ an ω-continuous function. If $x \in P$ is such that $x \leq f(x)$, then $x^* = \vee\{f^n(x) : n \in \mathbb{N}\}$ has the following properties:*
(i) $f(x^*) = x^*$; (ii) $x^* \geq x$ and for each y in P with $y \geq x$ and $f(y) \leq y$, $x^* \leq y$ (x^* is the least fixed point of f in $\{y \in P : x \leq y\}$.

The proof is left as an exercise.

Given a metric space (X, d), we can define a partially ordered space in a natural way. Consider $X \times \mathbb{R}^+ = X^+$, $\mathbb{R}^+$ being $[0, \infty)$, the set of non-negative real numbers with the usual order. Define the binary relation $\leq$ on X^+ by $(x, a) \leq (y, b)$ by $d(x, y) \leq a - b$. Clearly this is a partial order on X^+.

The following propositions relating convergence in X and convergence in X^+, lead to the contraction principle as a consequence of Kleene's fixed point theorem.

Proposition 5.7.4 *Let (x_n, k_n) be an increasing sequence in $(X^+, \leq)$ where $X^+ = X \times \mathbb{R}^+$ and (X, d) is a metric space. Then:*

(i) *the sequence (k_n) is a decreasing convergent sequence in $\mathbb{R}^+$;*
(ii) $\sum_{n=1}^{\infty} d(x_n, x_{n+1})$ *converges;*
(iii) (x_n) *is a Cauchy sequence in X.*

Proof Since $(x_n, k_n) \leq (x_{n+1}, k_{n+1})$ for $n \in \mathbb{N}$, $d(x_n, x_{n+1}) \leq k_n - k_{n+1}$ for all $n \in \mathbb{N}$. So (k_n) decreases in $\mathbb{R}^+$ and so converges to a non-negative number in $\mathbb{R}^+$, say

k. Clearly $\sum_{j=1}^{n} d(x_j, x_{j+1}) \leq k_1 - k_{n+1}$. Allowing n to tend to $+\infty$, it follows that $\sum_{j=1}^{\infty} d(x_j, x_{j+1}) \leq k_1 - k$. So $\sum_{1}^{\infty} d(x_n, x_{n+1})$ converges. Clearly (x_n) is a Cauchy sequence. □

Proposition 5.7.5 *Let (x_n, k_n) be an increasing sequence in $(X^+, \leq)$, $X^+ = X \times \mathbb{R}^+$, (X, d) being a metric space. The least upper bound of (x_n, k_n) exists in X^+ if and only if (x_n) converges in (X, d). Further, the least upper bound of $\{(x_n, k_n) : n \in \mathbb{N}\}$ is (x, k) where $x = \lim x_n$ and $k = \lim k_n$.*

Proof (Necessity) Let $x = \lim x_n$ and $k = \lim k_n$. For $m, n \in \mathbb{N}$ with $m \geq n$, it follows from the increasing nature of (x_n, k_n), $d(x_n, x_m) \leq k_n - k_m$. Allowing m to tend to $+\infty$, it follows that (x, k) is an upper bound for $\{(x_n, k_n) : n \in \mathbb{N}\}$. If (x', k') is another upper bound, then allowing n to tend to $+\infty$ in $d(x_n, x') \leq k_n - k'$, we get $d(x, x') \leq k - k'$ implying that $(x, k) \leq (x', k')$. So (x, k) is the least upper bound of $\{(x_n, k_n) : n \in \mathbb{N}\}$.

(Sufficiency) Suppose (x, k') is the least upper bound of $\{(x_n, k_n) : n \in \mathbb{N}\}$. By Proposition 5.7.4, (k_n) decreases to some $k \in \mathbb{R}^+$. Since $k_n \geq k'$ for each $n \in \mathbb{N}$, $k = \lim k_n \geq k'$. If $k = k'$, then $k_n - k \geq d(x_n, x)$ would imply that $\lim x_n = x$.

Suppose $k \neq k'$. We claim that for each $\epsilon > 0$, there exists $x_\epsilon \in X$ such that $d(x_n, x_\epsilon) \leq \epsilon + k_n - k$ for each $n \in \mathbb{N}$. Otherwise for some $\epsilon_0 > 0$, for each $y \in X$ we can find $n_y \in \mathbb{N}$ such that

$$d(x_{n_y}, y) > \epsilon_0 + k_{n_y} - k.$$

For $n > n_y$, we have

$$\begin{aligned}\epsilon_0 + k_{n_y} - k < d(x_{n_y}, y) &\leq \sum_{i=n_y}^{n-1} d(x_i, x_{i+1}) + d(x_n, y)\\ &\leq k_{n_y} - k_n + d(x_n, y)\\ &\leq k_{n_y} - k + d(x_n, y).\end{aligned}$$

So $d(x_n, y) > \epsilon_0$ for each $n > n_y$.

As (x_n, k_n) is increasing, it follows from Proposition 5.7.4 that (x_n) is Cauchy. So (x_n) converges to some x^* in X^*, the completion of X. Since $d(x_n, y) > \epsilon_0$ for each $n \geq n_y$. Allowing n $(\geq n_y)$ to tend to $+\infty$, it follows that $d(x^*, y) \geq \epsilon_0$, for each $y \in X$, contradicting that $d(x^*, x_m)$ does not tend to zero in X^*. Hence for each $\epsilon > 0$, there exists $x_\epsilon \in X$ such that $d(x_n, x_\epsilon) \leq \epsilon + k_n - k$ for each $n \in \mathbb{N}$. Choose $0 < \epsilon < k - k'$. It follows that for this choice of ϵ, $(x_\epsilon, k - \epsilon)$ is an upper bound for (x_n, k_n). As (x, k') is the least upper bound of $\{(x_n, k_n) : n \in \mathbb{N}\}$, we have $(x, k') \leq (x_\epsilon, k - \epsilon)$ on $d(x, x_\epsilon) \leq k' - (k - \epsilon) = \epsilon + k' - k < 0$, a contradiction. Hence (x_n) converges to x, and k_n converges to k. □

Corollary 5.7.6 *If the metric space is* (X, d) *is complete, then* $(X^+, \leq)$ *is* ω-*complete.*

A Lipschitz map on a metric space defines a ω-continuous map in a natural way, as described in the following.

Proposition 5.7.7 *Let* (X, d) *be a metric space and* $f : X \to X$ *be a map such that for some* $c > 0$, $d(fx, fy) \leq cd(x, y)$ *for all* $x, y \in X$. *Then the map* $f^+ : X^+ = (= X \times \mathbb{R}^+) \to X^+$ *defined by* $f^+(x, a) = (f(x), ca)$ *is* ω-*continuous.*

Proof Let $(x_n, k_n), n \in \mathbb{N}$ be an increasing sequence in X^+ with $(x, k) = \vee\{(x_n, k_n) : n \in \mathbb{N}\}$. For $(y_1, a_1), (y_2, a_2) \in X^+$ with $(y_1, a_1) \leq (y_2, a_2)$, $d(y_1, y_2) \leq a_1 - a_2$. Now $d(fy_1, fy_2) \leq cd(y_1, y_2) \leq ca_1 - ca_2$. So $(fy_1, ca_1) \leq (fy_2, ca_2)$. So $f^+(y_1, a_1) \leq f^+(y_2, a)$ and f^+ is increasing. Also $f^+(x, k) = (f(x), ck) = \vee\{(f(x_n), ck_n) : n \in \mathbb{N}\}$ by Proposition 5.7.4. Thus $f^+(x, k) = \vee\{f^+(x_n, k_n) : n \in \mathbb{N}\}$ and f^+ is ω-continuous. □

We are now in a position to provide Baranga's proof [2] of the contraction principle 5.1.7 via Kleene's fixed point Theorem 5.7.3.

Proof of Theorem 5.1.7 (*Baranga* [2]).

Let $(X^+, \leq)$ and $f^+ : X^+ \to X^+$ be defined as in Proposition 5.7.7. For $x_0 \in X$, we can find $a > 0$ such that $(1 - c)a > d(x_0, f(x_0))$ so that $(x_0, a) \leq f^+(x_0, a)$. By Kleene's Theorem 5.7.3, $(\bar{x}, 0) = \vee\{(f^+)^n(x_0, a) : n \in \mathbb{N}\}$ is a fixed point of f^+. So $\bar{x} = \lim f^n(x_0)$ is a fixed point of f. Clearly $\bar{x}$ is independent of the choice of a. Also $f^+(y, b) \leq (y, b)$ if and only if $b = 0$ and $f(y) = y$. If y is a fixed point of f, one can choose $a > 0$ so that $(x_0, a) \leq (y, 0)$ and $(x_0, a) \leq f^+(x_0, a)$. By the least fixed point property it follows that $(\bar{x}, 0) \leq (y, 0)$. This implies that $d(\bar{x}, y) = 0$ or f has a unique fixed point $\bar{x}$.

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Chapter 6
Applications of the Contraction Principle

This short chapter offers a few samples of applications of the contraction principle. It was already pointed out that the evergrowing list of applications of this fixed point theorem would fill volumes.

6.1 Linear Operator Equations

We have the following simple result.

Theorem 6.1.1 *Let* $(X, \|\cdot\|)$ *be a Banach space and* $T : X \to X$*, a continuous linear operator. If* $\|T\| < 1$*, then the equation* $x = Tx + a$ *has a unique solution in* X *for each fixed* $a \in X$*. Further* $\{T_1^n(x_0)\}$ *converges to the solution of this equation for each* $x_0 \in X$*,* $T_1(x_0)$ *being* $T(x_0) + a$ *for* $x_0 \in X$*.*

Proof Define $T_1(x) = T(x) + a$ for each $x \in X$. Clearly T_1 maps X into itself and $\|T_1(x) - T_1(y)\| = \|Tx - Ty\| \leq k\|x - y\|$, where $k = \|T\| < 1$. Thus T_1 is a contraction mapping X into itself. Since X is complete, T_1 has a unique fixed point x^*. Thus $Tx + a = x$ has the solution $x = x^*$. Again, by the contraction principle $\{T_1^n(x_0)\}$ converges to the unique solution of the equation $x = Tx + a$. □

Corollary 6.1.2 *Let* $A = (a_{ij})$ *be an* $n \times n$ *real matrix and* $b = (b_1, \ldots, b_n)$ *be a vector in* $\mathbb{R}^n$*. Then the system of linear equations*

$$x_i = \sum_{j=1}^{n} a_{ij} x_j + b_i, \quad i = 1, 2, \ldots, n$$

has a unique solution (i) provided $\sup_i \sum_{j=1}^{n} |a_{ij}| < 1$ *or (ii)* $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 < 1$.

P. V. Subrahmanyam, *Elementary Fixed Point Theorems*,
Forum for Interdisciplinary Mathematics,
https://doi.org/10.1007/978-981-13-3158-9_6

Proof Choose $X = \mathbb{R}^n$ and $T : X \to X$ as the operator $x \to Ax$ where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $Ax = (y_1, \ldots, y_n)$ where $y_i = \sum_{j=1}^{n} a_{ij} x_j$, and $a = (b_1, \ldots, b_n)$.

(i) Let $\|x\| = \sup_{i=1,\ldots,n} |x_i|$ where $x = (x_1, \ldots, x_n)$. Clearly $\|Ax\| = \sup_{i=1,\ldots,n} |y_i| \le \sup_i \sum_{j=1}^{n} |a_{ij}| \|x\|$. Since $\sup_i \sum_{j=1}^{n} |a_{ij}| < 1$, $x_i = \sum_{j=1}^{n} a_{ij} x_j + b_i$ has a unique solution in view of the Theorem 6.1.1.

(ii) Let $\|x\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}$ of $x = (x_1, \ldots, x_n)$. Then $\mathbb{R}^n$ with this Euclidean norm is complete. Now

$$
\begin{aligned}
Ax &= \left(\sum_{i=1}^{n} y_i^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right)^2 \right)^{\frac{1}{2}} \\
&\le \left\{ \sum_{i=1}^{n} \left[\sum_{j=1}^{n} a_{ij}^2 \sum_{j=1}^{n} x_j^2 \right] \right)^{\frac{1}{2}} \\
&\qquad \text{(by Cauchy-Schwarz inequality)} \\
&\le \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}} \|x\|
\end{aligned}
$$

So A is a contraction on $\mathbb{R}^n$, since $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 < 1$. Thus $x_i = \sum_{j=1}^{n} a_{ij} x_j + b_i$, $i = 1, 2, \ldots, n$ has a unique solution, by Theorem 6.1.1. □

Corollary 6.1.3 *Let $a, b \in \mathbb{R}$ and $a < b$. Suppose $K : [a, b] \times [a, b] \to \mathbb{R}$ is continuous and is not zero everywhere and $g : [a, b] \to \mathbb{R}$ is also continuous. Then the Fredholm integral equation*

$$
x(t) = \lambda \int_a^b K(s, t) x(s) dt + g(t), \quad t \in [a, b]
$$

has a unique solution $x \in C[a, b]$ for $|\lambda| < \frac{1}{M(b-a)}$ where $M = Sup\{|K(s, t)| : s, t \in [a, b]\}$.

Proof Define the linear operator $T : X \to X$ by $T(x(t)) = \lambda \int_a^b K(s,t)x(s)ds$, $t \in [a,b]$ for $x \in X = C[a,b]$, with this norm is a Banach space. That, $Tx(t)$ is a continuous function on $[a,b]$, can be proved, using the uniform continuity of K on $[a,b] \times [a,b]$. Let $M = Sup\{|K(s,t)| : s,t \in [a,b]\}$. Since K is continuous on the compact set $[a,b] \times [a,b]$, $0 < M < +\infty$. Now for $x_1, x_2 \in C[a,b]$ and $t \in [a,b]$

$$\begin{aligned} |Tx_1(t) - Tx_2(t)| &\leq |\lambda| \int_a^b |K(s,t)||x_1(s) - x_2(s)|ds \\ &\leq M|\lambda|\|x_1 - x_2\|(b-a) \end{aligned}$$

So $\|Tx_1 - Tx_2\| \leq M(b-a)|\lambda|\|x_1 - x_2\|$ and is a contraction as $|\lambda| < \frac{1}{M(b-a)}$. So by Theorem 6.1.1, T_1 has a unique fixed point, where $T_1 x = Tx + g$, $x \in X$. Thus the Fredholm integral equation has a unique solution. □

Corollary 6.1.4 *For $a, b \in \mathbb{R}$ with $a < b$, let $K : [a,b] \times [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be continuous. Then the Volterra integral equation*

$$x(t) = \lambda \int_a^t K(s,t)x(s)ds + g(t), \quad t \in [a,b]$$

has a unique continuous solution on $[a,b]$ for all $\lambda \in \mathbb{R}$.

Proof Let X be the Banach space $C[a,b]$ with the supremum norm. Then for each $x \in C[a,b]$, $\int_a^t K(s,t)x(s)ds$ defines a continuous function on $[a,b]$. Further $x(t) \to T(x(t)) = \lambda \int_a^t K(s,t)x(s)ds$ is a linear operator on X.

For $x_1, x_2 \in X$, $\lambda \in \mathbb{R}$ and $t > 1$

$$Tx_1(t) - Tx_2(t) = \lambda \int_a^t K(s,t)(x_1(s) - x_2(s))ds$$

So

$$\begin{aligned} |Tx_1(t) - Tx_2(t)| &\leq |\lambda| \int_a^t M\|x_1 - x_2\|ds \\ &\leq |\lambda|M\|x_1 - x_2\|(t-a) \end{aligned}$$

where $M = Sup\{|K(s,t)| : s,t \in [a,b]\}$ and $\|x_1 - x_2\| = Sup\{|x_1(t) - x_2(t)| : t \in [a,b]\}$. Now for $n \in \mathbb{N}$ and $t > a$.

$$|T^{n+1}x_1(t) - T^{n+1}x_2(t)| \leq |\lambda|M\|T^n x_1(t) - T^n x_2(t)\|(t-a)$$

Inductively it can be shown that

$$|T^k x_1(t) - T^k x_2(t)| \leq \frac{|\lambda|^k M^k}{k!}(t-a)^k\|x_1 - x_2\|$$

for $k \in \mathbb{N}$ and $t \in (a, b]$. So

$$\|T^{n+1}x_1 - T^{n+1}x_2\| \leq \frac{|\lambda|^{n+1}M^{n+1}}{(n+1)!}(b-a)^{n+1}\|x_1 - x_2\|$$

Since $\lim_{n\to\infty} \dfrac{|\lambda M(b-a)|^{n+1}}{(n+1)!} = 0$, for some $n_0 \in \mathbb{N}$, $\frac{|\lambda M(b-a)|^n}{n!} < 1$ for $n \geq n_0$. Thus T^{n_0} is a contraction on X and for $T_1 = T + g$, $T_1^{n_0}$ is also a contraction on X. So T_1 has a unique fixed point in $X = C[a, b]$ which is the solution of the given integral equation. Thus the Volterra integral equation has a unique solution in $C[a, b]$ for all $\lambda \in \mathbb{R}$. □

Remark 6.1.5 We can supplement Corollary 6.1.3 on the eigen-value problem for Fredholm integral equations. Suppose $K : [a, b] \times [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are Lebesgue measurable functions such that $g \in L^2[a, b]$ and $0 < \int_a^b \int_a^b K^2(s, t)\, dsdt < +\infty$. Then we can show that the equation

$$x(t) = \lambda \int_a^b K(s, t)ds + g(t)$$

has a unique solution in $L^2[a, b]$ for $|\lambda| \int_a^b \int_a^b K^2(s, t)dsdt < 1$. For the proof we use the Hilbert space $L^2[a, b]$ instead of $C[a, b]$.

The next result insures that certain mappings on Banach spaces are surjections and indeed homeomorphisms.

Theorem 6.1.6 *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a bounded linear transformation of X onto X with a bounded inverse. Let $G : X \to X$ be map such that $\|Gx - Gy\| \leq \alpha\|x - y\|$ for all $x, y \in X$. Suppose $\beta = \|T^{-1}\|$ and $\alpha\beta <$ 1. Then $x \to Tx - Gx$ defines a homeomorphism of X onto itself.*

Proof Let $y \in X$. Consider the map F defined by $Fx = Tx - Gx$, $x \in X$ and the map H defined by $Hx = T^{-1}(Gx + y)$ for each $x \in X$. Then $Hx = T^{-1}(Tx - Fx + y)$ or $Hx = x - T^{-1}Fx + T^{-1}y$. The equation $Fx = y$ has a unique solution in X if and only if H has a unique fixed point. Now $\|Hx_1 - Hx_2\| = \|T^{-1}(Gx_1 - Gx_2)\| \leq \alpha\beta\|x_1 - x_2\|$ for $x_1, x_2 \in X$. Since $0 \leq \alpha\beta < 1$, H is a contraction on the complete space X and hence has a unique fixed point. Thus F is a continuous map of X onto X. Also for $x_1, x_2 \in X$

$$\begin{aligned}\|FT^{-1}x_1 - FT^{-1}x_2\| &= \|x_1 - GT^{-1}x_1 - x_2 + G^{-1}x_2\| \\ &\geq \|x_1 - x_2\| - \|GT^{-1}x_1 - GT^{-1}x_2\| \\ &\geq (1 - \alpha\beta)\|x_1 - x_2\|\end{aligned}$$

So $\|Fx_1 - Fx_2\| \geq (1 - \alpha\beta)\|Tx_1 - Tx_2\|$
or $\|x_1 - x_2\| \geq (1 - \alpha\beta)\|TF^{-1}x_1 - TF^{-1}x_2\|$

This shows that TF^{-1} is continuous and so F^{-1} is continuous. Hence F is a homeomorphism. □

Corollary 6.1.7 *Let $(X, \|\cdot\|)$ be a Banach space and $G : X \to X$ a (strict) contraction. Then $(I - G) : X \to X$ is a homeomorphism.*

Proof In Theorem 6.1.6, set $T = I$, the identity operator. □

Example 6.1.8 For each $(y_1, y_2) \in \mathbb{R}^2$, we can find a unique $(x_1, x_2) \in \mathbb{R}^2$ such that

$$(2x_1 - 3x_2, x_1 - 2x_2) - \left(\cos\left(\frac{x_1 + x_2}{50}\right), \frac{|x_1| + |x_2|}{50(1 + |x_1| + |x_2|)}\right) = (y_1, y_2).$$

6.2 Differential Equations

We prove below an existence theorem for the solution of an initial value problem for a system of first-order ordinary differential equations.

Theorem 6.2.1 *Let G be an open set in $\mathbb{R}^{n+1}$ containing the point $(t_0, x_1^0, x_2^0, \ldots, x_n^0)$ and $f_i : G \to \mathbb{R}$ be continuous functions satisfying the Lipschitz condition*

$$|f_i(t, y_1, \ldots, y_n) - f_i(t, z_1, \ldots, z_n)| \leq M \sum_{j=1}^{n} |y_j - z_j|$$

for each $i = 1, 2, \ldots, n$ for all $(t, y_1, y_2, \ldots, y_n)$ and $(t, z_1, z_2, \ldots, z_n) \in G$. Then we can find $h > 0$ such that there exist continuously differentiable functions $x_1, \ldots, x_n$ mapping $[t_0 - h, t_0 + h]$ into $\mathbb{R}$ satisfying the conditions

$$\frac{dx_i}{dt} = f_i(t, x_1(t), \ldots, x_n(t)) \text{ for } i = 1, 2, \ldots, n$$

and $x_i(t_0) = x_i^0$ for $i = 1, 2, \ldots, n$. Further, these are unique solutions and $(t, x_1(t), x_2(t), \ldots, x_n(t)) \in G$ for all $t \in [t_0 - h, t_0 + h]$.

Proof Since G is open and $(x_1^0, \ldots, x_n^0, t_0) \in G$, we can find a closed rectangle $R = [x_1^0 - a, x_1^0 + a] \times \ldots \times [x_n^0 - a, x_n^0 + a] \times [t_0 - a, t_0 + a], a > 0$ in G. As each f_i is continuous on R and R is compact, we can find $K > 0$ such that $|f_i(p)| \leq K$ for all $i = 1, 2, \ldots, n$ for all $p \in R$. In view of the continuity of each $f_i, i = 1, 2, \ldots, n$, it is clear that $x_i(t)$ is a solution of

$$\frac{dx_i}{dt} = f_i(t, x_1(t), \ldots, x_n(t))$$
$$x_i(t_0) = x_i^{(0)}, \quad i = 1, 2, \ldots, n$$

if and only if

$$x_i(t) = x_i(0) + \int_{t_0}^{t} f_i(\tau, x_1(\tau), \dots, x_n(\tau))d\tau.$$

Choose $h > 0$ such that $Kh < a$. Let (X, ρ) be the metric space of vector-valued continuous real functions $(x_1(t), \dots, x_n(t))$ defined on $I = [t_0 - h, t_0 + h]$ with the metric $\rho(x, y) = \sup_{t \in I} \sum_{i=1}^{n} |x_i(t) - y_i(t)|$, where $x = (x_1(t), \dots, x_n(t))$ and $y = (y_1(t), \dots, y_n(t))$.

Clearly (X, ρ) is complete. For $x = (x_1, \dots, x_n) \in X$, define $Tx = (y_1, \dots, y_n)$ where

$$y_i(t) = x_i^0 + \int_{t_0}^{t} f_i(\tau, x_1(\tau), \dots, x_n(\tau))d\tau.$$

$T(x) \in X$ whenever $x \in X$. Further, $\rho(Tx, Tx') = \sum_{i=1}^{n} |y_i - y_i'| \le \sum_{i=1}^{n} M\rho(x, x')$ $h = nM\rho(x, x')h$. If we further choose h such that $nMh < 1$ then T is a contraction on X and hence has a unique fixed point $x = (x_1, \dots, x_n)$ which is the solution of the initial value problem in $[t_0 - h, t_0 + h]$ and $|x_i^0 - x_i(t)| \le Kh \le a$ for all $t \in [t_0 - h, t_0 + h]$ and so lies in $R \subseteq G$. Thus for $h < \min\{\frac{a}{K}, \frac{1}{nM}\}$, a unique local solution exists in $[t_0 - h, t_0 + h]$ for the initial value problem. □

Corollary 6.2.2 *Let G be an open set in $\mathbb{R}^2$ containing the point (t_0, x_0) and $f : G \to \mathbb{R}$ be a continuous function satisfying the condition that $|f(t, x_1) - f(t, x_2)| \le M|x_1 - x_2|$ for all $(t, x_1), (t, x_2) \in G$. Then there exists $h > 0$ such that the initial value problem*

$$\frac{dx}{dt} = f(t, x(t)), \quad x(t_0) = x_0$$

has a unique solution $x(t)$ on $[t_0 - h, t_0 + h]$ such that $(t, x(t)) \in G$ for $t \in [t_0 - h, t_0 + h]$.

(This pertains to the case $n = 1$ in Theorem 6.2.1.

Next, we prove a theorem on the existence of analytic solutions for a differential equation. To this end, we recall a few definitions.

Definition 6.2.3 A power series of the form $\sum_{i_1, i_2, \dots \ge 0} a_{i_1 \dots i_m} x_1^{i_1} \dots x_m^{i_m}$ where $a_{i_1 i_2 \dots i_m}$ are real numbers is called real-analytic in m variables $x_1, \dots, x_m$ inside the sphere $S(a; \rho) = \{(x_1, \dots, x_m) \in \mathbb{R}^m : \sum_{i=1}^{m} (x_i - a_i)^2 < \rho^2 \ (\rho > 0)\}$ if it converges therein, a being $(a_1, \dots, a_m)$.

Remark 6.2.4 If the power series in m-real variables converges in $S(a; \rho)$, then it converges in $S(a; \rho')$ for $0 < \rho' < \rho$.

Remark 6.2.5 The uniform limit of a convergent sequence of real-analytic functions is real-analytic in any domain interior to the domain of uniform convergence of the sequence. (This result is due to Weierstrass).

We have the following theorem due to Cauchy.

Theorem 6.2.6 (Cauchy) *Consider the initial value problem*

$$\frac{du}{dx} = f(x, y), \quad y(x_0) = y_0,$$

where f *is real-analytic in* $\{(x, y) \in \mathbb{R}^2 : |x - x_0| < h, |y - y_0| < h\}$. *Then there exists a unique solution* $y = \phi(x)$ *for this initial value problem which is a power series in* $[x_0 - \epsilon, x_0 + \epsilon]$ *for some* $\epsilon > 0$ *with* $\epsilon < h$.

Proof As before the initial value problem is equivalent to solving

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt.$$

Since f is a power series in x and y in $\{(x, y) : |x - x_0|, |y - y_0| < h\}$, $\frac{\partial f}{\partial y}$ exists and is a power series in x and y and is continuous in $K = \{(x, y) : |x - x_0| \leq h', |y - y_0| \leq h'\}$ where $0 < h' < h$. Let $M = Sup\{\left|\frac{\partial f}{\partial y}(x, y)\right| : (x, y) \in K\}$. Without loss of generality we may assume that $M > 0$.

Let X be the set of real-analytic function, real-analytic in $\{(x, y) : (x - x_0)^2 + (y - y_0)^2 \leq h'^2\}$ with the supremum metric, d. Define $T : X \to X$ by $T(\phi(x)) = y_0 + \int_{x_0}^{x} f(t, \phi(t))dt$ for all x with $|x - x_0| \leq h'$. Clearly $T\phi$ is well-defined and is real-analytic in $|x - x_0| \leq h'$. Further for $\phi_1, \phi_2 \in X$

$$\begin{aligned} |T\phi_1(x) - T\phi_2(x)| &\leq \int_{x_0}^{x} |f(t_1\phi_1(t) - f(t, \phi_2(t))|dt \\ &\leq h' M d(\phi_1, \phi_2) \end{aligned}$$

If we choose $0 < \epsilon < h'$ such that $\epsilon < \frac{1}{M}$, clearly $d(T\phi_1, T\phi_2) \leq \alpha d(\phi_1, \phi_2)$, where $\alpha = \epsilon M < 1$. Thus T is a contraction on the complete metric space (X, d) and hence has a unique fixed point, which is the unique solution of the initial value problem in $[x_0 - \epsilon, x_0 + \epsilon]$. Further, this solution is a power series in x in this interval. □

Remark 6.2.7 Clearly this theorem can be extended to a system of differential equations involving real-analytic functions of several real variables.

6.3 A Functional Differential Equation

Utz [12] raised the problem of determining conditions for the existence of a real function $y(x)$, not identically zero for which $y'(x) = ay(g(x)))$ where a is a given constant and $g(x)$ a given real function. Ryder [7] gave a solution to this problem under suitable assumptions.

Theorem 6.3.1 (Ryder [7]) *Let $g : D_g \to D_g$ be a continuous function on a interval in $\mathbb{R}$ containing the origin with $|g(x)| \leq k$ for all x in D_g. Then there exists a unique solution $\overline{f}(x) : D_g \to \mathbb{R}^n$ to the initial value problem*

$$\begin{aligned} \overline{y}'(x) &= A\overline{y}(g(x)), \quad x \in D_g \\ \overline{y}(0) &= \overline{f}_0 \end{aligned}$$

with A is a given $n \times n$ real matrix such that $\|A\|k < 1$, $\overline{f}_0 \in \mathbb{R}^n$.

Proof Let S be the set of all functions $\overline{f} : D_g \to \mathbb{R}^n$ such that $\overline{f}(0) = \overline{f}_0$ and $\|\overline{f}(x) - \overline{f}_0\| < L|x|$ for all $x \in D_g$ for some $L > 0$. Define $\rho : S \times S \to \mathbb{R}^+$ by

$$\rho(\overline{f}_1, \overline{f}_2) = \inf\{L : \|\overline{f}_1(x) - \overline{f}_2(x)\| \leq L|x|, \quad x \in D_g\}$$

If can be shown that ρ is a metric on S and in fact (S, ρ) is a complete metric space.

Define the operator $T : S \to S$ by

$$T(\overline{f})(x) = \overline{f}_0 + A\int_0^x \overline{f}(g(s))ds.$$

If $\overline{f} \in S$, then $\overline{f}(g(x))$ is well-defined and continuous on D_G and $T\overline{f}(0) = \overline{f}_0$. Further,

$$\begin{aligned} \|T(\overline{f}(x)) - \overline{f}_0\| &\leq \|A\|\left|\int_0^x [\|\overline{f}_0\| + L|g(s)|]ds\right| \\ &\leq \|A\|(\|\overline{f}_0\| + Lk)|x|, \quad x \in D_G. \end{aligned}$$

So $T(\overline{f}) \in S$ for $\overline{f} \in S$.

If $\overline{f}_1, \overline{f}_2 \in S$, then $\|\overline{f}_1(x) - \overline{f}_2(x)\| \leq L|x|$ for $x \in D_G$. Now for $x \in D_g$,

$$\begin{aligned} \|T(\overline{f}_1)(x) - T(\overline{f}_2)(x)\| &\leq \|A\|\left|\int_0^x L|g(s)|ds\right| \\ &\leq \|A\|kL|x| \end{aligned}$$

Consequently

$$\rho(T\overline{f}_1, T\overline{f}_2) \leq \|A\|k\rho(\overline{f}_1, \overline{f}_2)$$

Since $\|A\|k < 1$, it follows that T is a contraction on the complete metric space and hence has a unique fixed point $\overline{f}(x)$. Since $\overline{f}(x) = \overline{f}_0(x) + A\int_0^x \overline{f}(g(s))ds$, and $\overline{f}$ and g are continuous, if follows that $\overline{f}'(x) = A\overline{f}(g(x))$ and $\overline{f}(0) = f_0(0)$. □

Example 6.3.2 Let $g : [-\frac{1}{2}, 1] \to [-\frac{1}{2}, 1]$ be the continuous function defined by $g(x) = \frac{-x}{x+1}$ and $|g(x)| \le 1$ for all $x \in D_g = [-\frac{1}{2}, 1]$. The functional differential equation $f'(x) = \frac{1}{2}f\left(\frac{-x}{x+1}\right)$, $f(0) = 1$ has a unique solution in $[-\frac{1}{2}, 1]$ in view of Theorem 6.3.1, A being $\frac{1}{2}$. In fact the unique solution is $f(x) = \sqrt{x+1}$.

6.4 A Classical Solution for a Boundary Value Problem for a Second Order Ordinary Differential Equation

We consider the following boundary value problem

$$\frac{d^2x}{dt^2} = \alpha f(t, x(t)), \quad t \in (0, 1)$$

$x(0) = x(1) = 0$.

Theorem 6.4.1 *Let $f : [0, 1] \times [-a, a] \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition $|f(t, x_1) - f(t, x_2)| \le M|x_1 - x_2|$ for all $(t, x_i) \in [0, 1] \times [-a, a]$, $i = 1, 2$. Then the above boundary value problem has a unique solution for sufficiently small values of α.*

Proof The Green's function for the problem $G(t, s)$ is defined by

$$G(t, s) = \begin{cases} (t-1)s, & 0 \le s \le t \\ t(s-1), & t \le s \le 1 \end{cases}$$

Let $M = \{x \in C[0, 1] : \|x\| \le a\}$ where $\|x\| = Sup\{|x(t)| : t \in [0, 1]\}$. As f is continuous, $|f(s, 0)| \le K$ for some $K > 0$ for all $s \in [0, 1]$. Also for $x \in M$, $s \in [0, 1]$

$$\begin{aligned} |f(s, x(s))| &\le |f(s, 0)| + |f(s, 0) - f(s, x(s))| \\ &\le K + M\|x\| \le k + Ma. \end{aligned}$$

The operator $T : M \to C[0, 1]$ defined by

$$T(x)(t) = \alpha \int_0^1 G(t, s) f(s, (x))ds$$

maps M into itself provided

$$|T(x)(t)| \leq \frac{\alpha}{8}\|f(s,x(s))\| \leq \frac{\alpha}{8}(K+Ma) < a$$

That is for $\alpha < \frac{8a}{K+Ma}$. Further, for $x_1, x_2 \in M$, we have for all $t \in [0,1]$

$$\begin{aligned}|T(x_1)(t) - T(x_2)(t)| &= |\alpha \int_0 G(t,s)(f(s,x(s)) - f(s,x(s))ds| \\ &\leq \frac{|\alpha|M}{8}\|x_1 - x_2\|\end{aligned}$$

So if we choose $|\alpha| < \min\left\{\frac{8}{M}, \frac{8a}{K+Ma}\right\}$, then T being a contraction, has a unique fixed point, which indeed is the unique solution to the boundary value problem. □

6.5 An Elementary Proof of the Cauchy–Kowalevsky Theorem

Walter [13] gave an elementary proof of the Cauchy–Kowalevsky theorem using the contraction principle. First the linear version of this theorem is proved using Nagumo's lemma and then the proof for the quasi-linear case is given, as the nonlinear case can be reduced to the quasi-linear case. So we merely discuss the proof of the Cauchy–Kowalevsky theorem for the linear case following Walter [13].

Let G be a non-empty open subset of $\mathbb{C}^{n+1}$ or $\mathbb{R} \times \mathbb{C}^n$, Ω an open set in $\mathbb{C}^n$ with non-empty boundary $\Gamma = \partial\Omega$. Define $d(z) = dist(z, \Gamma)$ measured in the maximum norm $|z| = \max\{|z_i| : i = 1, 2, \ldots, n\}$. We further restrict G by requiring that it consists of all points (t, z), with $z \in \Omega$ and $|t| < \eta d(z)$ where $\eta > 0$ is to be specified.

The linear (Cauchy) problem is to solve the initial value problem

$$u_t = A(t,z)u + \sum_{j=1}^{n} B_j(t,z)u_{z_j} + c(t,z) \ \text{ for } (t,z) \in G \tag{6.5.1}$$

with $u(0,z) = \phi(z)$ in Ω, where $u = (u_1, \ldots, u_m)$ has values in $\mathbb{C}^m$. A and B, are complex $m \times m$ matrices, c and ϕ being complex-valued column vectors. Further, $u_t = \frac{\partial u}{\partial t}$, $u_{z_j} = \frac{\partial u}{\partial z_j}$. We seek a solution of the above problem which is continuous in G and analytic in z for fixed t (real case) and analytic in t and z (complex case). We also write $\|A\| = \max_j \sum_{i=1}^{m} |a_{ij}|$ for $A = (a_{ij})$.

For proving the existence of analytic solution of (6.5.1). We need the following lemma due to Nagumo [6].

Lemma 6.5.1 *Let $f : \Omega \to \mathbb{C}^m$ be analytic and $p \geq 0$. Then*

$$|f(z)| \leq \frac{C}{d^p(z)} \ \textit{ implies } |f_{z_j}(z)| \leq C_p \frac{C}{d^{p+1}(z)}$$

where $C_p = (1+p)(1+\frac{1}{p})^p < e(p+1)$, $C_0 = 1$.

Proof If g is analytic function of single complex variable in $|z-z'| \le r$, then

$$|g'(z)| \le \frac{1}{r} \max_{|z-z'|=r} |g(z')|.$$

This follows from Cauchy's integral formula

$$g'(z) = \int_{|z-z'|=r} \frac{g(z')}{2\pi i (z'-z)^2} dz'.$$

Applying this for $z = z_j$ we get the inequalities

$$|f_{z_j}(z)| \le \frac{1}{r} \max_{|z-z'|=r} |f(z')| \le \frac{C}{r} \max \frac{1}{d^p(z')} \le \frac{C}{r(d-r)^p}$$

where $0 < r < d = d(z)$ and $d(z') \ge d - r$. The choice $r = \frac{d}{p+1}$ leads to the estimate stated in the lemma. □

Remark 6.5.2 The Cauchy problem (6.5.1) can be written as the equivalent integral equation.

$$u(t,z) = g(t,z) + \int_0^t [A(s,z)u(s,z) + \sum_{j=1}^n B_j(s,z) u_{z_j}(s,z)] ds \tag{6.5.2}$$

$$\text{where } g(t,z) = \phi(z) + \int_0^t c(s,z) ds \tag{6.5.3}$$

Theorem 6.5.3 *Suppose*

(i) the functions $A(t,z)$, $B_j(t,z)$, $C(t,z)$ *are continuous in* G *and analytic in* z *for fixed* t, *the function* $\phi(z)$ *being analytic in* z;

(ii) there exist positive constants α, β_j, η, δ *and* p *such that*

$$|A(t,z)| \le \frac{\alpha}{d(t,z)}, \qquad |B_j(t,z)| \le \beta_j,$$

$$|C(t,z)| \le \frac{\eta}{d^{p+1}(t,z)}, \qquad |\phi(z)| \le \frac{\delta}{d^p(z)} \text{ in } G;$$

(iii) $\frac{\alpha}{p} + (1+\frac{1}{p})^{p+1} \sum \beta_j < \frac{1}{\eta}$.

Then the Eq. (6.5.2) has a unique solution u *in* G *satisfying* $|u(t,z)| \le \frac{C}{d^p(t,z)}$ *for* $(t,z) \in G$.

Proof Let X be the Banach space of all functions $u = (t,z)$ which are continuous on G and taking values in $\mathbb{C}^m$ and analytic in z with the finite norm

$$\|u\| = \underset{G}{Sup}|u(t,z)|d^p(t,z)$$

Clearly convergence in the norm is uniform convergence on compact subsets of G. So the limit of a convergent sequence in X is analytic and X is complete in this norm. Equation (6.5.2) can be written as

$$u = g + T(u)$$

where T is the linear operator defined by

$$Tu(t,z) = \int_0^t [A(s,z)u(s,z) + \sum_j B_j(s,z)u_{s_j}(s,z)]ds]$$

Since

$$\left|\int_0^t \frac{ds}{d^{p+1}(s,z)}\right| = \int_0^{|t|} \frac{ds}{\left(d(s) - \frac{s}{\eta}\right)^{p+1}} < \frac{\eta}{pd^p(t,z)}, \quad g(t,z) \in X.$$

From the definition of the norm

$$|u(t,z)| \le \frac{\|u\|}{d^p(t,z)}.$$

Applying Nagumo's Lemma 6.5.1 to Ω_t with distance $d(t,z)$ instead of Ω and $d(z)$ we get,

$$|u_{z_j}(t,z)| \le C_p \frac{\|u\|}{d^{p+1}(t,z)}$$

Assumption (ii) and these estimates give

$$|Au| \le \frac{\alpha\|u\|}{d^{p+1}(t,z)}, \quad |B_j u_{z_j}| \le \frac{\|u\|}{d^{p+1}(t,z)}\beta_j C_p$$

So for $\beta = \sum \beta_j$

$$\begin{aligned} |Tu(t,z)| &\le \|u\|(\alpha + \beta C_p)\left|\int_0^t \frac{ds}{d^{p+1}(s,z)}\right| \\ &\le \frac{1}{p}(\alpha + \beta C_p)\frac{\eta\|u\|}{d^p(t,z)}. \end{aligned}$$

So $\|Tu\| \le q\|u\|$, where $q = \frac{1}{p}[\alpha + \beta(1 + \frac{1}{p})^{p+1}]\eta < 1$. Since $0 < q < 1$, $u \to g + Tu$ is a contraction on the Banach space X and has a unique fixed point. Thus

(6.5.2) has a unique solution. Thus the Cauchy problem (6.5.1) with analytic data has a unique solution. □

Note 6.5.4 In Theorem 6.5.3, $\eta > 0$ can be chosen sufficiently small so that (iii) of Theorem 6.5.3 holds.

6.6 An Application to a Discrete Boundary Value Problem

In this section a solution to a discrete boundary value problem obtained by Tisdell [10], using the contraction principle is described.

Let $f : [0, N] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. The discrete boundary value problem is to solve

$$\frac{\Delta x_i}{h} = f(t_i, x_i), \quad i = 0, 1, \ldots, n, \quad ux_0 + vx_n = w, \quad u + v \neq 0 \tag{6.6.1}$$

where $0 < h < \frac{N}{n} < N$ and $t_i = ih$ $(i = 0, 1, \ldots, n)$ are the grid points with $\Delta x_i = x_{i+1} - x_i, i = 0, \ldots, n-1$ and u, v, w are constants. The problem is to find a vetor $\tilde{x} = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ satisfying (6.6.1).

Theorem 6.6.1 (Tisdell [10]) *Let $f : [0, N] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and $u + v \neq 0$. Suppose there exist constants $L > 0$, $\delta \in (0, N)$, $p, q > 1$ such that*

(i) $|f(t, y) - f(t, z)| \leq L|y - z|$ *for* $t \in [0, N], y, z \in \mathbb{R}$*;*

(ii) $\frac{L\delta}{|u+v|}\left(\sum_{i=0}^{n}[|u|^q i + |v|^q (n-i)]^{\frac{p}{q}}\right)^{\frac{1}{p}} < 1$*, (with* $\frac{1}{p} + \frac{1}{q} = 1$*);*

Then the boundary value problem has a unique solution for $0 < h \leq \delta$*.*

Proof Consider $X = \mathbb{R}^{n+1}$ with the norm $\|x\| = \left(\sum_{i=0}^{n}|x_i|^p\right)^{\frac{1}{p}}$ for $p > 1$. Clearly X is a Banach space. The problem (6.6.1) is equivalent to the system of equations

$$x_i = h\sum_{j=0}^{n-1} G(i, j) f(t_j, x_j) + \frac{w}{u+v}, \quad i = 0, \ldots, n$$

where

$$G(i, j) = \begin{cases} \frac{u}{u+v}, & 0 \leq j \leq i-1 \\ \frac{-u}{u+v}, & i \leq j \leq n-1 \end{cases}$$

Define $T : X \to X$ by

$$(T\tilde{x})_i = h\sum_{j=0}^{n-1} G(i, j) f(t_j, x_j) + \frac{w}{u+v}, \quad i = 0, \ldots, n$$

for $\tilde{x} = (x_0, \ldots, x_n) \in X$.

For $\tilde{x}, \tilde{y} \in X = \mathbb{R}^{n+1}$

$$\begin{aligned}
|T(\tilde{x})_i - T(\tilde{y})_i| &\leq h \sum_{j=0}^{n-1} |G(i,j)||f(t_j, x_j) - f(t_j, y_j)| \\
&\leq Lh \sum_{j=0}^{n-1} |G(i,j)||x_j - y_j| \\
&\leq Lh \left(\sum_{j=0}^{n-1} |G(i,j)|^q\right)^{\frac{1}{q}} \left(\sum_{j=0}^{n-1} |x_j - y_j|^p\right)^{\frac{1}{p}} \\
&\qquad \text{(by Holder's inequality)}
\end{aligned}$$

From the definition of $G(i, j)$ we have

$$\sum_{j=0}^{n-1} |G(i,j)|^q = \frac{i|u|^q + (n-i)|v|^q}{|u+v|^q}, \quad i = 0, \ldots, n.$$

So

$$|T(\tilde{x})_i - T(\tilde{y})_i| \leq \frac{Lh\|\tilde{x} - \tilde{y}\|}{|u+v|}[i|u|^q + (n-i)|v|^q]^{\frac{1}{q}}, \quad i = 0, \ldots, n.$$

So

$$|(T\tilde{x})_i - (T\tilde{y})_i|^p \leq \left(\frac{Lh}{|u+v|}\right)^p \|\tilde{x} - \tilde{y}\|^p [i|u|^q + (n-i)|v|^q]^{\frac{p}{q}}, \quad i = 0, \ldots, n.$$

So

$$\begin{aligned}
\|T\tilde{x} - T\tilde{y}\| &= \left(\sum_{i=0}^{n} |(T\tilde{x})_i - (T\tilde{y})_i|^p\right)^{\frac{1}{p}} \\
&\leq \|\tilde{x} - \tilde{y}\| \frac{Lh}{|u+v|} \left(\sum_{i=0}^{n} [i|u|^q + (n-i)|v|^q]^{\frac{p}{q}}\right)^{\frac{1}{p}} \\
&\leq \alpha \|\tilde{x} - \tilde{y}\|
\end{aligned}$$

where $\alpha = \frac{L\delta}{|u+v|} \left(\sum_{i=0}^{n} [i|u|^q + (n-i)|v|^q]^{\frac{p}{q}}\right)^{\frac{1}{p}} < 1$. So T is a contraction on X and so has a unique fixed point, which is a solution to the discrete boundary value problem. □

Corollary 6.6.2 ([10]) *Let $f : [0, N] \to \mathbb{R}$ be continuous and for $0 \le t \le N$ and $x, y \in \mathbb{R}$, $|f(t, x) - f(t, y)| \le L|x - y|$ for some $L > 0$ and $u + v \ne 0$. Then for $0 < h \le \delta$, the boundary value problem has a unique solution provided $L\sqrt{N^2 + \delta N} < \frac{\sqrt{2}|u+v|}{\sqrt{u^2+v^2}}$.*

The above inequality insures that the conditions of Theorem 6.6.1 are fulfilled.

Example 6.6.3 Set $u = 1$, $v = 2$, $w = 0$, $N = 1$, $f(t, y) = \frac{4}{5}(y + t + \cos y)$. The corresponding boundary value problem satisfies the conditions of Corollary 6.6.2 for $L = \frac{8}{5}$ and has a unique solution for $0 < h \le \frac{1}{4}$.

6.7 Applications to Functional Equations

Contraction principle has been a handy tool in the solution of a variety of functional equations. As a matter of fact the implicit function theorem is a consequence of the contraction principle.

Theorem 6.7.1 (Implicit Function Theorem) *Let U be an open neighbourhood of (x_0, y_0) in $\mathbb{R}^2$. Let $f : U \to \mathbb{R}$ be a continuous function such that $\frac{\partial f}{\partial y}$ exists in U and is continuous at (x_0, y_0). Suppose (i) $\frac{\partial f}{\partial y}(x_0, y_0) \ne 0$ and (ii) $f(x_0, y_0) = 0$. Then there exists a unique continuous function g defined in a neighbourhood $N(x_0)$ of x_0 such that $f(x, g(x)) = 0$ for all $x \in N(x_0)$.*

Proof Define $D = \frac{\partial f}{\partial y}(x_0, y_0)$. Since f is continuous on U and $\frac{\partial f}{\partial y}$ is continuous at (x_0, y_0), we can find a closed rectangle $I = [x_0 - \epsilon, x_0 + \epsilon] \times [y_0 - \epsilon, y_0 + \epsilon]$ inside U such that $\left|\frac{1}{D}\frac{\partial f}{\partial y}(x, y) - 1\right| < \frac{1}{2}$ for all $(x, y) \in I$ and $\left|\frac{1}{D} f(x, b)\right| < \frac{\delta}{2}$ for $x_0 - \epsilon \le x \le x_0 + \epsilon$.

Define $X = \{y : I \to \mathbb{R}$ is continuous and $y(x_0) = y_0$ with $|y(x) - y_0| \le \delta$ for all $x \in I\}$. Clearly X is a closed subset of $C(I)$, the Banach space of all continuous real functions on I with the supremum norm. Hence X is complete. Define the map $T : X \to C(I)$ by

$$(Ty)(x) = y(x) - \frac{1}{D} f(x, y(x)), \quad x \in I.$$

Clearly Ty is continuous for each $y \in X$. If $y(x_0) = y_0$, then $Ty(x_0) = y(x_0) = y_0$ as $f(x_0, y(x_0)) = f(x_0, y_0) = 0$. Now

$$\begin{aligned}\|Ty_0 - y_0\| &= \|\frac{1}{D} f(x, y_0)\| \\ &< \frac{1}{2}\delta \ \text{ for } |x - x_0| < \epsilon \\ &\qquad \text{(by construction of } I)\end{aligned}$$

Also for $y_1, y_2 \in M$, for x with $|x - x_0| \leq \epsilon$

$$\begin{aligned}|Ty_1(x) - Ty_2(x)| &= \left| y_1(x) - \frac{1}{D}\frac{\partial}{\partial y} f(x, y_1(x)) - y_2(x) + \frac{1}{D}\frac{\partial}{\partial y} f(x, y_2(x)) \right| \\ &< \frac{1}{2}|y_1(x) - y_2(x)|, \quad \text{since}\end{aligned}$$

$$\left|\frac{\partial}{\partial y}(y - \frac{1}{D} f(x, y))\right| = \left|(1 - \frac{1}{D}\frac{\partial}{\partial y} f(x, y)\right| < \frac{1}{2}$$

by choice of I.

So $\|Ty_1 - Ty_2\| \leq \frac{1}{2}\|y_1 - y_2\|$. Also for $y \in X$,

$$\begin{aligned}\|Ty - y_0\| &\leq \|Ty - Ty_0\| + \|Ty_0 - y_0\| \\ &\leq \frac{1}{2}\|y - y_0\| + \|Ty_0 - y_0\| \\ &< \frac{1}{2}\delta + \frac{1}{2} = \delta\end{aligned}$$

Thus T maps X into itself and is a contraction (with constant $\frac{1}{2}$). Since X is complete T has a unique fixed point $y_0(x)$ which is a solution of $f(x, y_0(x)) = 0$ for $x \in [x_0 - \epsilon, x_0 + \epsilon]$. □

Remark 6.7.2 The above theorem can be extended to functions taking values in a Banach space after suitable modifications.

The problem of finding a curve which is invariant under a continuous transformation can also be reduced to the solution of a functional equation. Here, we consider a relatively simple planar situation wherein the transformation F is differentiable with a fixed point at (0, 0) such that $F'(0, 0)$ is invertible. Indeed we can transform the axes to coincide with the eigen vectors of $F'(0, 0)$ so that $F'(0, 0) = A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. Under suitable assumptions, we can prove the existence of Lipschitzian solutions to the problem of finding curves invariant under F in a neighbourhood of (0, 0). This problem has been considered by Hadamard, Lattes and Montel (see [4]).

Theorem 6.7.3 *Let $F = (f, g)$ be a continuously differentiable mapping of a neighbourhood U of* (0, 0) *in* $\mathbb{R}^2$ *with* $F' = A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ *where* $0 < |\lambda| < |\mu|$ *and* $|\lambda| < 1$ *or* $0 < |\mu| < |\lambda|$, $|\lambda| > 1$. *Then for each* $L > 0$, *we can find a* $c > 0$ *and a unique function* $\varphi : [-c, c] \to \mathbb{R}$ *satisfying*

(i) $|\varphi(s) - \varphi(t)| \leq L|s - t|$, $s, t \in [-c, c]$ *and*
(ii) $\varphi(0) = 0$.

In fact φ *is differentiable at zero and* $\varphi'(0) = 0$.

Proof Without loss of generality we shall assume that $0 < |\lambda| < |\mu|$ and $|\lambda| < 1$. (In the second case F is replace by F^{-1} which exists in some neighbourhood of (0, 0)).

By the inverse function theorem there exists a continuously differentiable mapping h defined in a neighbour V of $(0, 0)$ satisfying $h(x, g(x, y)) = y$ for $(x, y) \in V$. Also $h(x, y) = (0, \frac{1}{\mu}).(x, y) + o(\sqrt{x^2 + y^2})$, $(x, y) \to (0, 0)$.

Let L be any non-negative real number and $c > 0$ be such that $D = \{(x, y) \in \mathbb{R}^2 : |x| \leq c, |y| \leq L|x|\} \subseteq U \cap V$. Let X be the set of all functions $\varphi : I = [-c, c] \to \mathbb{R}$ such that $\varphi(0) = 0$ and $|\varphi(s) - \varphi(t)| \leq L|s - t|$ for all $s, t \in I$. It can be seen that $\rho : X \times X \to \mathbb{R}^+$ defined by

$$\rho(\varphi, \psi) = \underset{x \in I - \{0\}}{Sup} \left| \frac{\varphi(x) - \psi(x)}{x} \right|$$

is a metric on X and that (X, ρ) is a complete metric space. Define $T : X \to X$ by $(T\varphi)(x) = h(x, \varphi(f(x, \varphi(x))), x \in I$. By taking c small, we can insure that $f(x, \varphi(x)) \in I$ for $x \in I$ and $\varphi \in X$.

For $\varphi \in X$, $(T\varphi)(0) = 0$. For fixed $\epsilon > 0$ from the properties of f and h we have

$$|f(x, y) - f(u, v)| \leq (|\lambda| + \epsilon)|x - u| + \epsilon|y - v|$$

and

$$|h(x, y) - h(u, v)| \leq \epsilon|x - u| + \left(\frac{1}{|\mu|} + \epsilon \right) |y - v|$$

for all $(x, y), (u, v) \in D$.

So for $\varphi \in X$ and $(s, t) \in I \times I$, we have

$$|T(\varphi)(s) - T(\varphi)(t)| \leq \left(\epsilon + \left(\frac{1}{|\mu|} + \epsilon \right) \right) L((|\lambda| + \epsilon) + L\epsilon)|s - t|$$

As $\left| \frac{\lambda}{\mu} \right| < 1$, this shows that we can choose $\epsilon > 0$ small such that

$$|T(\varphi)(s) - T(\varphi)(t)| \leq L|s - t|, \quad s, t \in I.$$

So T maps X into itself. Again we can show that for $\varphi, \psi \in X$.

$$\rho(T\varphi, T\psi) \leq \left(\frac{1}{|\mu|} + \epsilon \right) (|\lambda| + \epsilon + 2L\epsilon)\rho(\varphi, \psi)$$

Decreasing ϵ and hence c we can choose the Lipschitz constant in the above inequality to be less than 1. So T is a contraction and hence has a unique fixed point. $\varphi(x) \in X$. Thus

$$\varphi(x) = h(x, \varphi(fx, \varphi(x)) = T\varphi(x).$$

Since $h(x, g(x, \phi(x)) = \phi(x)$ and is inverse to g with respect to the second variable, $\varphi(fx, \varphi(x)) = g(x, \varphi(x))$. Hence the existence of a Lipschitzian solution follows. □

Finally we prove the existence of a unique solution to an operator equation using Tan's extension [9] of the contraction principle (see Theorem 5.4.7) as treated in Subrahmanyam [8].

Let (Y, d) be a complete metric space, X a set and $s : X \to X$, a function. For an operator P on a sequentially closed subset S_0 of Y^X the operator equation $f(x) = Pf(s(x))$, $x \in X$ is studied in the following. The solutions of this operator equation are fixed points of the map P' on S_0 defined by $P' = P(f \circ s)$. Theorem 6.7.4 below can therefore provide a set of sufficient conditions for the existence of a unique solution for this operator equation.

Theorem 6.7.4 *Let (Y, d) be a complete metric space, X a set and $s : X \to X$ a function. For any function $f_0 \in Y^X$ the set*

$$S = \{f \in Y^X : D_x(f_0, f) < +\infty \text{ for all } x \in X\}$$

is a sequentially complete Hausdorff gauge space under the family F of all pseudometrics D_x defined by

$$D_x(f, g) = \sup\{d(f(s^k(x)), g(s^k(x))) : k = 0, 1, 2, \ldots\}$$

($s^0 : X \to X$ being the identity map).

The proof of Theorem 6.7.4 is omitted. The next result is a fixed point theorem obtained from Tan's Theorem (see [9]) and Theorem 6.7.4.

Theorem 6.7.5 (see [8]) *Let (Y, d) be a complete metric space and $s : X \to X$ a function. Let $P : S_0 \to S_0$ be an operator on a sequentially closed subset S_0 of Y^X with the topology of pointwise convergence such that*

(i) $D_X(f_0, P(f_0)) < +\infty$ for some $f_0 \in S_0$ and all $x \in X$ where D_x is as defined in Theorem 6.7.4;
(ii) for each $x \in X$ there is an $a(x)$ such that given $f, g \in S_0$ there exists a non-negative integer $n = n(x)$ with $d(Pf(x), Pg(x)) \leq a(x)d(f(s^n(x)), g(s^n(x)))$. Further, for each $x \in X$,

$$0 < \sup\{a(s^k(x)) : k = 0, 1, 2, \ldots\} < 1.$$

Then P has a unique fixed point in S_0.

Proof Define $S = \{f \in S_0 : D_x(f_0, f) < +\infty$ for each $x \in X\}$. $S \neq \phi$, since $f_0 \in S$. Also S is a sequentially complete Hausdorff gauge space under the family $F = \{D_x : x \in X\}$ of pseudometrics, in view of Theorem 6.7.4 and the assumption that S_0 is sequentially closed in Y^X.

For the operator P mapping S_0 into itself, (i) implies that $P(f_0) \in S$. Further, for $f \in S$ and $x \in X$,

$$\begin{aligned} D_x(f_0, P(f)) &\le D_x(f_0, P(f_0)) + D_x(P(f_0), P(f)) \\ &\le D_x(f_0, P(f_0)) + A(x)D_x(f_0, f) \ \text{(by (ii))} \\ &< +\infty \end{aligned}$$

where $A(x) = \sup\{a(s^k(x)) : k = 0, 1, 2, \ldots\}$. So P maps S into itself. By (ii) again

$$D_x(P(f), P(g)) \le A(x)D_x(f, g)$$

for each $x \in X$ and $f, g \in S$. As $0 < A(x) < 1$, P is strictly contractive and it maps the Hausdorff sequentially complete gauge space (S, F) into itself. Therefore P has a unique fixed point $g_0 \in S$, by Theorem 5.4.7. Also, as $g_0 \in S$, for each $x \in X$

$$\sup\{d(f_0(s^k(x)), g_0(s^k(x))) : k = 0, 1, 2, \ldots\} < +\infty.$$

The proof is complete. □

An application of Theorem 6.7.5 is now given.

Corollary 6.7.6 (see [8]) *Let B^X be the space of all functions mapping X into a Banach space $(B, \|\cdot\|)$ over the real or complex field K. Let $s : X \to X$, $a : X \to K$, $b : X \to B$ and $h : X \times B \to B$ be given functions. Suppose that for each $x \in X$*

(i) *given $y_1, y_2 \in B$, $\|h(x, y_1) - h(x, y_2)\| \le \|y_1 - y_2\|$;*
(ii) $A(x) = \sup\{|a(s^k(x))| : k = 0, 1, 2, \ldots\} < 1$;
(iii) $\sup\{\|b(s^k(x))\| : k = 0, 1, 2, \ldots\} < +\infty$ *and* $\sup\{\|h(s^k(x), 0)\| : k = 0, 1, 2, \ldots\} < +\infty$.

Then the functional equation

$$f(x) = a(x)h(x, f(s(x)) + b(x) \tag{6.7.1}$$

has a unique solution f in B^X.

Proof Set $Y = (B, \|\cdot\|)$, $S_0 = B^X$ and $f_0 = 0$ in Theorem 6.7.5. Then the class S is simply the set

$$\{f \in B^X : \sup \|f(s^k(x))\| < +\infty, k = 0, 1, 2, \ldots, \text{ for all } x \in X\}.$$

The operator P of Theorem 6.7.5 is defined by

$$P(f(x)) = a(x)h(x, f(s(x))) + b(x), \quad x \in X.$$

For this choice of the operator P and the space S, all the condition of Theorem 6.7.5 are satisfied. So the operator P has a unique fixed point in S. As any other solution of the functional equation (6.7.1) is in S in view of (i), (ii) and (iii), (6.7.1) has a unique solution in B^X. □

Example 6.7.7 The functional equation

$$f(x) = g(x)\cos(x + f(x^2)) \tag{6.7.2}$$

has a unique solution in R^R, the space of all real valued functions of a real variable, where the function g is the one defined below:

$$g(x) = \begin{cases} 1/x, & |x| > 1 \\ x, & |x| \le 1, x \text{ irrational} \\ x/2, & \text{otherwise.} \end{cases}$$

Choosing $B = R$, $a(x) = g(x)$, $b = 0$, $h(x, y) = \cos(x + y)$ and $s(x) = x^2$, the conditions of Corollary 6.7.6 are readily satisfied.

6.8 An Application to Commutative Algebra

In this section a short proof of the algebraic Weierstrass Preparation theorem due to Gersten [2] is highlighted. To this end a few basic definitions are described. For these Lang [5] and Zariski and Samuel [14] may be consulted.

Definition 6.8.1 A ring R is called a local ring if it is commutative and has a unique maximal ideal.

Remark 6.8.2 If R is a local ring with the unique maximal ideal $\mathfrak{M}$, then $x \in A - \mathfrak{M}$ is a unit.

Definition 6.8.3 For a ring R and an ideal I, suppose that $\bigcap_{n=1}^{\infty} I^n = \{0\}$. We can define I^n as a neighbourhood of 0 for each n. I^n is the ideal generated by elements of the form $a_1 a_2 \ldots a_n, a_i \in I$ for $i = 1, 2, \ldots, n$. In the same way we can say that a sequence $\{x_n\}$ in R is Cauchy if given some power I^k of I, there exists an integer M such that for all $m, n \ge M$, $x_m - x_n \in I^k$. A sequence (x_n) in R is said to converge to x in R if for each I^k, there exists an integer M such that $(x_n) \in x + I^k$ for all $n \ge M$. R is said to be complete in the I-adic topology if every Cauchy sequence in R converges.

Definition 6.8.4 Let R be a local ring and $I = \mathfrak{M}$ the maximal ideal. R is called a complete local ring if R is complete in the $\mathfrak{M}$-adic topology and we assume that the $\mathfrak{M}$-adic topology is Hausdorff.

Remark 6.8.5 Let k be a field and $R = k[[X_1, \ldots, X_n]]$ the power series ring in n-variables. Then R is a complete local ring. If $\mathfrak{M}$ is the ideal generated by $X_1, \ldots, X_n$ then $R/\mathfrak{M}$ is isomorphic with k and so $\mathfrak{M}$ is a maximal ideal. Any power series of the form $f(X) = c_0 - f_1(X)$ where $c_0 \neq 0 \in k$ and $f_1(X) \in \mathfrak{M}$ is invertible as

$$(c_0 - f_1(X))^{-1} = c_0^{-1}\left(1 + \frac{f_1(X)}{c_0} + \frac{(f_1(X))^2}{c_0^2} + \cdots .\right)$$

So $\mathfrak{M}$ is the unique maximal ideal and R is local. It can be readily verified that R is complete.

For a ring A, the power series ring in n variables for $n > 1$ can be viewed as the ring of power series in variable X_n over the ring of power series in $(n-1)$ variables $X_1, \ldots, X_{n-1}$. Thus we have the identification $A[[X_1, \ldots, X_n]] = A[[X_1, \ldots, X_{n-1}]][[X_n]]$. When A is a field $A[[X_1, \ldots, X_n]]$ is a complete local ring. Moreover if R is a complete local ring, the power series ring $R[[X]]$ is a complete local ring with the maximal ideal $[\mathfrak{M}, X]$, $\mathfrak{M}$ being the maximal ideal of R. If the power series $\sum a_n X^n$ has unit constant $a_0 \in R - \mathfrak{M}$, then the power series is a unit in $R[[X]]$ as $(a_0 + h)^{-1}$ is $a_0^{-1}\left(1 - \frac{h}{a_0} + \frac{h^2}{a_0^2} - \cdots\right)$.

Proposition 6.8.6 *If A is a complete local ring with maximal ideal $\mathfrak{M}$, then it is a metric space under the metric defined by*

$$d(a, a') = \begin{cases} 0 & \text{if } a = a' \\ 2^{-s} & \text{if } a \neq a' \text{ with } a - a' \in \mathfrak{M}^s - \mathfrak{M}^{s+1}, \mathfrak{M}^0 \text{ being } A \end{cases}$$

The routine proof is omitted.

Proposition 6.8.7 ([2]) *Let A be a complete local ring with maximal ideal $\mathfrak{M}$ which is Hausdorff in the $\mathfrak{M}$-adic topology. Let $B = A[[t]]$ be the power series ring in one variable t over A. Define $d_1(f, f') = \sup_{k \in Z^+} d(a_k, a'_k)$ for $f = \sum_{k=0}^{\infty} a_k t^k$ and $f' = \sum_{k=0}^{\infty} a'_k t^k$ where $a_k, a'_k \in A$ and d is as in Proposition 6.8.6. Then (B, d_1) is a complete metric space.*

Proof Since $f \in B$ is uniquely identified by the sequence $(a_0, a_1, \ldots, \ldots)$ for $f = \sum_0^{\infty} a_k t^k$, $a_i \in A$, given a Cauchy sequence $f_n = \sum_0^{\infty} a_{nk} t^k$ in B, (a_{nk}) is uniformly Cauchy in n. As (A, d) is complete, $a_{nk} \to a'_k \in A$ for each $k > 0$. Further $d_1(f_n, f') \to 0$ as $n \to \infty$ where $f' = \sum_0^{\infty} a'_k t^k \in B$. Thus (B, d_1) is complete. □

Definition 6.8.8 Let A be a complete local ring with maximal ideal $\mathfrak{M}$ and $B = A[[t]]$. A distinguished polynomial in B is of the form $p_0 + p_1 t + \cdots + p_{n-1}t^{n-1} + t^n$ where $p_i \in \mathfrak{M}$.

The following is

Theorem 6.8.9 (Algebraic Weierstrass Preparation Theorem) *If* $f = \sum_{k=0}^{\infty} a_k t^k \in B$, *where* $a_k \in A$ *and if there exists* $n \in \mathbb{N}$ *such that* $a_k \in \mathfrak{M}$ *for* $k < n$ *and* $a_n \notin \mathfrak{M}$, *then* $f = up$ *where* u *is a unit of* B *and* p *is a distinguished polynomial of degree* n. *Also* u *and* p *are uniquely determined.*

The proof of the algebraic preparation theorem is based on the following division theorem for which Gersten [2] has given a proof using the contraction principle.

Theorem 6.8.10 ([2]) *If* $f, b \in B$ *and* $b \in \mathfrak{M}[t]$ *and if* $n \in \mathbb{N}$, *then* $f = q(t^n + b) + r$ *where* $q, b \in B$ *and* r *is a polynomial in* t *of degree* $< n$. *Further* q *and* r *are uniquely determined.*

Proof Define the operator $E : B \to B$ by $x = p + E(x)t^n$, where p is a polynomial in t of *degree* $< n$. Define $T : B \to B$ by $Tx = E(f - xb)$ for $x \in B$. So $f - xb = p + E(f - xb)t^n$. If $x' \in B$, then $f - x'b = p' + E(f - x'b)t^n$. Noting that $Ty = E(f - yb)$, and subtracting the latter equation from the former, we get $-b(x - x') = p'' + (Tx - Tx')t^n$ where p'' is a polynomial of degree $< n$. Note that the coefficients of $Tx - Tx'$ involve only the coefficients of the left-hand side of degree $\geq n$. So

$$d_1(Tx, Tx') \leq d_1(bx, bx') \leq \frac{1}{2} d_1(x, x')$$

since $b \in \mathfrak{M}[[t]]$. Thus T is a contraction on B and hence has a unique fixed point $q \in B$ such that $Tq = q$ or $f - qb = r + qt^n$ where r is a polynomial of degree $< n$. Thus $f = q(t^n + b) + r$. □

We can now deduce the algebraic Weierstrass preparation Theorem 6.8.9 from the Division Theorem 6.8.10

Proof Let $f \in B$ with $f = \sum_{k=0}^{\infty} a_k t^k$, where $a_i \in A$ and for some $n \in \mathbb{N}$,$a_i \in \mathfrak{M}$ for $i < n$ with $a_n \notin \mathfrak{M}$. Let $b = \sum_{k=0}^{n-1} a_k t^k$. As $a_k \in \mathfrak{M}$ for $k < n$, $b \in \mathfrak{M}[t]$. Now by Division Theorem 6.8.10, $f = q(t^n + b) + r$ where $q, r \in B$ and r is a polynomial in t of degree $< n$. Since $b \in \mathfrak{M}[t]$, the distinguished polynomial $p = t^n + b$ is a unit in B. So $(f - r)p^{-1} = q$ or $fp^{-1} = q + rp^{-1} = q' \in b$. So $f = q'p = pq'$ with p being a distinguished polynomial. If $q' = \sum_{0}^{\infty} b_n t^n$, then $a_0 = b_0 a_0$ so q' is a unit. The proof of uniqueness is left as an exercise. □

6.9 A Proof of the Central Limit Theorem

Central limit theorem, a classical theorem of probability theory can be proved using the contraction principle. Trotter [11] gave a proof based on Lindberg condition and invoking the properties of certain linear non-expansive operators leading to a contraction. While Trotter's proof completely avoids the use of characteristic functions, Hamedani and Walter [3] gave a proof of the central limit theorem using the properties of characteristic functions, for sub-independent identically distributed random variables, again using the contraction principle. Before describing the proof of Hamedani and Walter [3], a few basic concepts of probability theory are stated below. See Kai Lai Chung [1].

Definition 6.9.1 A measure space $(\Omega, \mathscr{S}, P)$ where $\mathscr{S}$ is a σ-algebra on the non-empty Ω and P is a measure on $\mathscr{S}$ is called a probability measure space if $P(\Omega) = 1$. A map $X : \Omega \to \mathbb{R}$ is called a random variable if $\{x \in \Omega : X(x) \leq r\} \in \mathscr{S}$ for each $r \in \mathbb{R}$. (Thus a random variable on Ω is simply a measurable function).

Definition 6.9.2 A function $F : \mathbb{R} \to \mathbb{R}$ that is increasing and right continuous with $F(-\infty) = \lim_{t \downarrow -\infty} F(t) = 0$ and $F(+\infty) = \lim_{t \uparrow +\infty} F(t) = 1$ is called a distribution function.

Definition 6.9.3 If X is a random variable on $(\Omega, \mathscr{S}, P)$, then μ defined by $\mu(-\infty, x] = P\{p \in \Omega : X(p) \leq x\} = F(x)$ induces a measure on the Borel subsets of $\mathbb{R}$ called the probability distribution measure of X and F is called the distribution function of the random variable X. A family of random variables having the same distribution is said to be identically distributed.

Definition 6.9.4 For $r > 0$ and $a \in \mathbb{R}$ given a random variable X with distribution function F, the moment of X of order r about a is defined as

$$E(X-a)^r = \int_{\mathbb{R}} (x-a)^r \mu(dx) = \int_{-\infty}^{\infty} (x-a)^r dF(x)$$

μ being the probability distribution measure of X, provided the integral exists for $a = 0, r = 1$ $E(X)$ is called the mean of X. The moments about the mean are called central moments. The central moment of order 2 is called the variance of X and is denoted by $Var(X)$. The positive square root of $Var(X)$ is called the standard deviation of X, denoted by $\sigma(X)$.

Definition 6.9.5 The random variables $\{X_i : 1 \leq i \leq n\}$ are said to be independent if for any Borel sets $B_i, i = 1, 2, \ldots, n$ in $\mathbb{R}$ $P\left(\bigcap_{i=1}^{n} (x : X_i(x) \in B_i)\right) = \prod_{i=1}^{n} P\{x : X_i(x) \in B_i\}$.

Remark 6.9.6 The above definition of (stochastic) independence can be extended to any family of random variables in a natural way. If X_i, $i = 1, 2, \ldots, n$ are independent random variables with finite expectations, then $E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i)$.

Definition 6.9.7 For a random variable X with induced probability measure μ and distribution function F, the characteristic function $f(t)$ is defined as

$$f(t) = E(e^{itX}) = \int_{\Omega} e^{itX(w)} P(dw)$$
$$= \int_{\mathbb{R}} e^{itx} \mu d(x) = \int_{-\infty}^{\infty} e^{itx} dF(x).$$

Remark 6.9.8 For all $t \in \mathbb{R}$, $|f(t)| \le 1 = f(0)$ and $f(-t) = \overline{f(t)}$ ($\overline{z}$ being the complex conjugate of z). f is uniformly continuous on $\mathbb{R}$ and for $a, b \in \mathbb{R}$ $f_{aX+b}(t) = f_X(at)e^{itb}$ where f_X denotes the characteristic function of X. For independent random variables $X_i, i = 1, 2, \ldots, n$, $E(e^{itS_n}) = \prod_{i=1}^{m} E(e^{itX_i})$ or $f_{S_n} = \prod_{i=1}^{n} f_{X_i}$, S_n being $\sum_{i=1}^{n} X_i$.

Definition 6.9.9 The convolution of two distribution functions F_1 and F_2 is defined to be the distribution function F such that

$$F(x) = \int_{-\infty}^{\infty} F_1(x-y)dF_2(y) \text{ for } x \in \mathbb{R}$$

and is written $F = F_1 * F_2$.

Remark 6.9.10 If X_1 and X_2 are independent random variables with distribution function F_1 and F_2, then $X_1 + X_2$ has the distribution function $F_1 * F_2$.

For these and the following theorem, Kai Lai Chung [1] may be consulted.

Theorem 6.9.11 *If the distribution function F has a finite absolute moment of positive integral order $k \ge 1$, then its characteristic function f has a bounded continuous derivative of order k given by*

$$f^{(k)}(t) = \int_{-\infty}^{\infty} (ix)^k e^{itx} dF(x)$$

Further

$$f(t) = \sum_{j=0}^{k} \frac{(i)^j}{j!} m^{(j)} t^j + \frac{\theta_k}{k!} \mu^{(k)} |t|^k$$

where $m^{(j)}$ is the moment of order j, $\mu^{(k)}$ the absolute moment of order k and $|\theta_k| \leq 1$.

Definition 6.9.12 Two random variables X and Y on $(\Omega, \mathscr{S}, P)$ are said to be sub-independent if the distribution of their sum is given by

$$F_{X+Y}(t) = (F_X * F_Y)(t).$$

In terms of characteristic functions it is described by

$$\phi_{X,Y}(t, t) = \phi_X(t)\phi_Y(t) \quad \text{for } t \in \mathbb{R}$$

where $\phi_{X,Y}(t, s)$, $\phi_X(t)$, $\phi_Y(t)$ are characteristic functions corresponding to (X, Y), X and Y respectively. The random variables $X_1, X_2, \ldots, X_n$ are said to be sub-independent if for each subset $\{X_{i_1}, \ldots, X_{i_k}\}$ of $\{X_1, \ldots, X_n\}$

$$\phi_{X_{i_1},\ldots X_{i_k}}(t, t, \ldots, t) = \prod_{i=1}^{k} \phi_{X_i}(t) \quad \text{for } t \in \mathbb{R}$$

Remark 6.9.13 The concept of sub-independence is more general than that of independence. Further the random variable with distribution function $F(x) = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt$ is called the standard normal distribution

For the subsequent discussion culminating in the proof of central limit theorem we introduce the following class of random variables

Definition 6.9.14 Let R_λ, $\lambda \geq 0$ be the set of all random variables X on $(\Omega, \mathscr{S}, P)$ such that

(i) $E(|X|^\lambda) < +\infty$;
(ii) $E(X^\lambda) = m_k$ for $k = 1, 2, \ldots, [\lambda]$ where m_k is the k-th moment of Z, the standard normal variable.

Let M_λ denote the set of distributions of $X \in R_\lambda$.

Definition 6.9.15 Define $d_\lambda : M_\lambda \times M_\lambda \to \mathbb{R}^+$ by $d_\lambda(F, G) = \sup_{t \in R} \left| E\left(\frac{e^{iXt} - e^{iYt}}{|t|^\lambda}\right)\right|$ where F and G are the distribution functions respectively of the random variables X and Y.

In order to deduce the central limit theorem from the contraction principle we need to set up appropriate complete metric spaces.

Proposition 6.9.16 (M_λ, d_λ) *is a metric space.*

Proof Clearly the main issue in the proof is to show that $d_\lambda(F, G)$ is finite-valued for $F, G \in M_\lambda$. If ϕ_1 and ψ_1 are the real parts of the characteristic functions of F and G in M_λ, then for $n = [\lambda]$.

$$\frac{|\phi_1(t) - \psi_1(t)|}{|t^n||t|^{\lambda-n}} = \frac{|\phi_1^{(n)}(\xi) - \psi_1^{(n)}(\xi)|}{n!} \cdot \frac{1}{|t|^{\lambda-n}}$$
$$\leq \frac{|\phi_1^{(n)}(\xi) - m_n|}{n!|t|^{\lambda-n}} \frac{|m_n - \psi_1^{(n)}(\xi)|}{n!|t|^{\lambda-n}}$$

by Theorem 6.9.11, in view of $\phi_1^{(k)}(0) = \psi_1^{(k)}(0)$ for $k \leq n$.

Now

$$\frac{|\phi_1^{(n)}(\xi) - \psi_1^{(n)}(0)|}{n!|t|^{\lambda-n}} \leq \int \frac{|e^{i\xi x} - 1|}{|t|^{\lambda-n}} |x|^n dFx$$
$$\leq \int \frac{|e^{i\xi x} - 1|}{|\xi x|^{\lambda-n}} |x|^\lambda dF(x)$$
$$\leq KE(|X|^\lambda) \text{ for some } K \in \mathbb{R}^+$$

F being the distribution function if X. A similar estimate holds for the imaginary parts of F and G. Thus $d_\lambda(F, G) < +\infty$. □

Proposition 6.9.17 *(M_n, d_n) is a complete metric space when n is an integer. For $n < \lambda$, $\overline{M}_\lambda \subseteq M_n$ where $\overline{M}_\lambda$ is the completion of M_λ.*

Proof Let (F_k) be a Cauchy sequence of distributions in M_n with the corresponding characteristic function ϕ_k. For each $\epsilon > 0$ there is a positive integer K such that for $k, m \geq K$

$$\frac{|\phi_k(t) - \phi_m(t)|}{|t^n|} < \epsilon$$

So $\phi(t) = \lim_{k\to\infty} \phi_k(t)$ exists for each $t \in \mathbb{R}$. Now h_m defined by $h_m(t) = \left\{ \phi_m(t) - \sum_{j=0}^{n} m_j \frac{(it)^j}{j!} \right\} t^{-n}$ is a uniformly Cauchy sequence of continuous functions converging uniformly to a continuous function $h(t)$. Further $h(0) = 0$ as $h_m(0) = 0$ by definition of M_n. Thus

$$\phi(t) - \sum_{j=0}^{m} m_j \frac{(it)^j}{j!} = t^m h(t)$$

Since F_k has the same jth moment for $U = 0, 1, \ldots, n$, and ϕ is continuous at zero F_k converges weakly to F in the sense that $E(X_k Y) \to E(XY)$ for each bounded random variable Y, X_k being the random variable with distribution function F_k and X the random variable with distribution function X. Using Fatou's lemma the existence of jth moment of F follows and $\phi(t)$ is j times differentiable at 0. So its jth derivative for $j = 0, 1, 2, \ldots n$ must be the same as that of ϕ_k. Then $F \in M_n$. That a Cauchy sequence in M_λ for $\lambda > n$ is a Cauchy sequence in M_n is left as an exercise. □

Definition 6.9.18 Let $X, Y \in R_\lambda$ be sub-independent and have identical distributions. For $\alpha > 0$ define $T_\alpha : M_\lambda \to M_0$ as the map taking the distribution function of X into the distribution function of $\frac{X+Y}{\alpha}$. (T_α takes the characteristic function $\phi(t)$ into $\phi^2\left(\frac{t}{\alpha}\right)$).

Proposition 6.9.19 *Let $\alpha^\lambda > 0$ be such that $T_\alpha : M_\lambda \to M_\lambda$. Then*

(i) T_α is non-expansive on (M_λ, d_λ) for $\alpha^\lambda \geq 2$;
(ii) T_α is strictly contractive for $\alpha^\lambda > 2$.

Proof

$$\begin{aligned} d_\lambda(T_\alpha F, T_\alpha G) &= \sup_{t\in R} \frac{\left|\phi^2\left(\frac{t}{\alpha}\right) - \psi^2\left(\frac{t}{\alpha}\right)\right|}{|t|^\lambda} \\ &\leq \sup_{t\in R} \frac{\left|\phi\left(\frac{t}{\alpha}\right) - \psi\left(\frac{t}{\alpha}\right)\right|}{|\alpha|^\lambda} \frac{\sup_{t\in R}\left|\phi\left(\frac{t}{\alpha}\right) + \psi\left(\frac{t}{\alpha}\right)\right|}{|\frac{t}{\alpha}|^\lambda} \\ &\leq 2\alpha^{-\lambda} d_\lambda(F, G). \end{aligned}$$

For $\alpha^\lambda \geq 2$, $d_\lambda(T_\alpha F, T_\alpha G) \leq d_\lambda(F, G)$, while
for $\alpha^\lambda > 2$, $d_\lambda(T_\alpha F, T_\alpha G) \leq \mu d_\lambda(F, G)$
where $\mu = 2\alpha^{-\lambda} < 1$. □

Pursuing a suggestion of J. Blum that central limit theorem 'could be interpreted and proved as a fixed point theorem', Hamedani and Walter [3] proved the following version of the central limit theorem.

Theorem 6.9.20 ([3]) *Let (X_m) be a sequence of sub-independent identically distributed (s.i.i.d) random variables with mean 0 and variance 1 such that $E(|X|^\lambda) < \infty$ for some $\lambda > 2$. Then*

$$\frac{1}{2^{n/2}} \sum_{i=1}^{2^n} X_i \to Z$$

in distribution as $n \to \infty$ and their distribution functions converge in the metric of M_λ. Moreover the rate of convergence is governed by

$$d_\lambda(T^n_{\sqrt{2}} F, \phi) < 2^{n(1-\frac{\lambda}{2})}(E(|X|^\lambda) + E(|Z|^\lambda))$$

Proof By Proposition 6.9.19, $T = T_{\sqrt{2}}$ is a contraction on (M_λ, d_λ). So the iterates of T^n must converge to the unique fixed point in the completion of M_λ. Since T_α takes the distribution function $\phi(t)$ of X into $\phi^2(\frac{t}{\alpha})$, and $\phi^2(\frac{t}{\sqrt{2}}) = \phi(t)$ for $\phi(t) = \int_{-\infty}^t e^{-\frac{s^2}{2}} ds$, the distribution function of standard normal variate. So $d_\lambda(\phi, T^n F) \to 0$ as $n \to \infty$. Thus $\lim_{n\to\infty} \sup_{t\in R} \frac{|\psi_n(t) - e^{-t^2/2}|}{|t|^\lambda} = 0$, ψ_n being the characteristic function of $T^n F$ Thus $(\psi_n(t))$ converges to $e^{-t^2/2}$ for all t in $\mathbb{R}$. So by the well-known Levy-Cramer theorem of probability theory the corresponding distribution functions converge. Thus $2^{-n/2}\sum_{i=1}^{2^n} X_i \to Z$ in distribution. In view of the error estimate from the contraction principle.

$$\begin{aligned} d_\lambda(\phi, T^{n+1}F) &\le 2^{(1-\frac{\lambda}{2})(n+1)} d_\lambda(\phi, F) \\ &\le 2^{(1-\frac{\lambda}{2})(n+1)} d_\lambda(\phi, F) \\ &\le 2^{(1-\frac{\lambda}{2})(n+1)} [E(|x|^\lambda) + E(|z|^\lambda)] \\ \text{as } d_\lambda(\phi, F) &\le [E(|x|^\lambda) + E(|z|^\lambda)] \end{aligned}$$

(from the proof of Proposition 6.9.16). □

The next proposition leads to the central limit theorem for s.i.i.d random variables.

Proposition 6.9.21 ([3]) *Let the random variables X_i, Y_i, $i = 1, 2$ in R_λ have the distribution functions F_i, G_i, $i = 1, 2$ correspondingly and $\alpha > 0$. If X_1 and X_2 are sub-independent as also Y_1 and Y_2, then*

$$d_\lambda(F, G) \le \alpha^{-\lambda}[d_\lambda(F_1, G_2) + d_\lambda(F_2, G_2)$$

where F and G are distribution functions of $\frac{F_1+F_2}{\alpha}$ and $\frac{G_1+G_2}{\alpha}$ respectively.

Proof If ϕ_i is the characteristic function of X_i and ψ_i is the characteristic function of Y_i, $i = 1, 2$, then

$$\begin{aligned} \frac{\left|\phi_1(\frac{t}{\alpha})\phi_2(\frac{t}{\alpha}) - \psi_1(\frac{t}{\alpha})\psi_2(\frac{t}{\alpha})\right|}{|t|^\lambda} &\le \frac{\left|\phi_1(\frac{t}{\alpha})\right|\left|\phi_2(\frac{t}{\alpha}) - \psi_2(\frac{t}{\alpha})\right|}{t^\lambda} \\ &\quad + \frac{\left|\psi_2(\frac{t}{\alpha})\right|\left|\phi_1(\frac{t}{\alpha}) - \psi_1(\frac{t}{\alpha})\right|}{t^\lambda} \end{aligned}$$

So

$$\begin{aligned} d_\lambda(F, G) &= \sup_{t\in R} \frac{\left|\phi_1(\frac{t}{\alpha})\phi_2(\frac{t}{\alpha}) - \psi_1(\frac{t}{\alpha})\psi_2(\frac{t}{\alpha})\right|}{|t|^\lambda} \\ &\le \sup_{t\in R}\left|\phi_1(\frac{t}{\alpha})\right| e^{-\lambda} d_\lambda(F_2, G_2) + \sup_{t\in R}\left|\psi_2(\frac{t}{\alpha})\right| e^{-\lambda} d_\lambda(F_1, G_1) \\ &\le \alpha^{-\lambda}[d_\lambda(F_1, G_1) + d_\lambda(F_2, G_2)] \end{aligned}$$

□

Theorem 6.9.22 (Central limit theorem [3]) *Let $\{X_n\}$ be a sequence of sub-independent random variables in R_λ for some $\lambda > 2$ whose distribution functions belong to a bounded set in M_λ. Then $\dfrac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \to Z$ in distribution as $n \to \infty$.*

Proof Let (Y_n) be an s.i.i.d sequence in R_λ. Repeated application of Proposition 6.9.21 for $\sum_{i=1}^{n+1} \frac{X_i}{\sqrt{n+1}}$ and $\sum_{i=1}^{n+1} \frac{Y_i}{\sqrt{n+1}}$ gives

$$\begin{aligned}
\overline{d}_\lambda\left(\sum_{i=1}^{n+1} \frac{X_i}{\sqrt{n+1}}, \sum_{i=1}^{n+1} \frac{Y_i}{\sqrt{n+1}}\right) \\
&\le \left(\frac{n}{n+1}\right)^{\frac{\lambda}{2}} \overline{d}_\lambda\left(\sum_{i=1}^{n} \frac{X_i}{\sqrt{n+1}}, \sum_{i=1}^{n} \frac{Y_i}{\sqrt{n+1}}\right) + (n+1)^{-\frac{\lambda}{2}} \overline{d}_\lambda(X_{n+1}, Y_{n+1}) \\
&\le \left(\frac{2^m}{n+1}\right)^{\frac{\lambda}{2}} \overline{d}_\lambda\left(\sum_{i=1}^{2^m} \frac{X_i}{2^{m/2}}, \sum_{i=1}^{2^m} \frac{Y_i}{2^{m/2}}\right) + (n+1)^{-\frac{\lambda}{2}} \sum_{i=2^m+1}^{n+1} \overline{d}_\lambda(X_i, Y_i) \\
&\le (n+1)^{-\frac{\lambda}{2}} \sum_{i=1}^{n+1} \overline{d}_\lambda(X_i, Y_i)
\end{aligned}$$

$\overline{d}_\lambda$ being the metric in M_λ. As $\{X_n\}$ and $\{Y_n\}$ are both sequences whose distribution functions are bounded in M_λ, the right-hand side of the last inequality converges to zero as n tends to infinity.

So $\overline{d}_\lambda\left(\sum_{i=1}^{n+1} \frac{X_i}{\sqrt{n+1}}, \sum_{i=1}^{n+1} \frac{Y_i}{\sqrt{n+1}}\right) \to 0$ as $n \to \infty$. By Theorem 6.9.20 $\overline{d}_\lambda\left(\sum_{i=1}^{2^m} \frac{Z_i}{2^{m/2}}, \sum_{i=1}^{2^m} \frac{Y_i}{2^{m/2}}\right) \to 0$ as $m \to \infty$ provided each Z_i is a standard normal variate. Choosing m to be the largest integer with $2^m < n+1$, we get

$$\begin{aligned}
(n+1)^{-\frac{\lambda}{2}} \sum_{i=2^m+1}^{n+1} \overline{d}_\lambda(Z_i, Y_i) &\le (\lambda+1)^{-\frac{\lambda}{2}}(n-2^m)C \\
&\le 2^{-m\left(\frac{\lambda}{2}\right)}(2^{m+1}-2^m)C \\
&\qquad (C,\ \text{a constant})
\end{aligned}$$

As $\lambda > 2$, the right-hand side (and hence the left-hand side) of the last inequality tends to 0 as $n \to \infty$. So $\overline{d}_\lambda\left(Z, \sum_{i=1}^{n+1} \frac{Y_i}{\sqrt{n+1}}\right) \to 0$ as $n \to \infty$. Now

$$\begin{aligned}
\overline{d}_\lambda\left(\sum_{i=1}^{n+1} \frac{X_i}{\sqrt{n+1}}, Z\right) &\le \overline{d}_\lambda\left(\sum_{i=1}^{n+1} \frac{X_i}{\sqrt{n+1}}, \sum_{i=1}^{n+1} \frac{Y_i}{\sqrt{n+1}}\right) \\
&\quad + \overline{d}_\lambda\left(\sum_{i=1}^{n+1} \frac{Y_i}{\sqrt{n+1}}, Z\right)
\end{aligned}$$

So $\sum_{i=1}^{n+1} \frac{X_i}{\sqrt{n+1}}$ converges to Z in distribution. □

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Chapter 7
Caristi's Fixed Point Theorem

7.1 Introduction

In 1976, Caristi [4] published a novel generalization of the contraction principle. Using transfinite arguments which was also later simplified by Wong [15]. Brondstedt [3] provided an alternative proof by introducing an interesting partial order. On the other hand, Ekeland [6] established a variational principle whence deducing Caristi's theorem. Brezis and Browder [2] proved an ordering principle also leading to this fixed point theorem. Subsequently Altman [1], Turinici [14, 15] and others have extended this principle. In this chapter, we discuss some of these as well as proofs of Caristi's theorem by Kirk [8], Penot [10] and Seigel [11]. That both Ekcland's principle and Caristi's theorem characterize completeness is also brought out.

7.2 Siegel's Proof of Caristi's Fixed Point Theorem

Caristi's fixed point theorem is the following.

Theorem 7.2.1 *Let (X, d) be a complete metric space and $\phi : X \to \mathbb{R}^+$ be a lower semicontinuous function (i.e. $\phi(x) \leq \liminf_{n\to\infty} \phi(x_n)$ whenever $\lim_{n\to\infty} x_n = x$). If $T : X \to X$ is a map such that $d(x, Tx) \leq \phi(x) - \phi(Tx)$ for all $x \in X$, then T has a fixed point.*

Siegel's proof [11] is detailed below, based on a few definitions and lemmata, ϕ, T and X being as in Theorem 7.2.1 above.

Definition 7.2.2 Let $\Phi \doteq \{f : X \to X \text{ with } d(x, fx)) \leq \phi(x) - \phi(f(x)) \text{ for } x \in X\}$. Define $\Phi_T = \{f \in \Phi : \phi(f) \leq \phi(T)\}$.

Lemma 7.2.3 *Both Φ and Φ_T are closed under compositions. If ϕ is lower semicontinuous, then these classes are closed under countable compositions.*

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Proof For $f_1, f_2 \in \Phi$ and $x \in X$, $d(x, f_2(f_1(x))) \leq d(x, f_1(x)) + d(f_1(x), f_2(f_1(x))) \leq \phi(x) - \phi(f_1(x)) + \phi(f_1(x)) - \phi(f_2(f_1(x)) = \phi(x) - \phi(f_2(f_1(x)))$. So $f_1 \circ f_2 \in \Phi$. If $f_1, f_2 \in \Phi_T$, then

$$\begin{aligned}\phi(f_2(f_1(x)) &\leq \phi(f_1(x)) \text{ (since } f_2 \in \Phi_T) \\ &\leq \phi(x) \text{ (since } f_1 \in \Phi_T).\end{aligned}$$

Thus $f_1 \circ f_2 \in \Phi_T$.

Let (f_n) be a sequence in Φ. For $x \in X$, define $x_n = f_n(f_{n-1} \ldots (f_1 x)) \ldots) = f_1 \circ f_2 \cdots \circ f_n(x)$ for $n \in \mathbb{N}$. Since $d(x_i, x_{i+1}) \leq \phi(x_i) - \phi(x_{i+1})$ for all $i \in \mathbb{N}$ and ϕ is non-negative, $\{\phi(x_i)\}$ is a non-increasing sequence of non-negative numbers which is convergent. Consequently (x_n) is a Cauchy sequence in the complete metric space (X, d) and so converges to some element $\overline{x}$ in X. Now $d(x_n, \overline{x}) = \lim_{k\to\infty} d(x_n, x_k) \leq \phi(x_n) - \liminf_{k\to\infty} \phi(x_k) \leq \phi(x_n) - \phi(\overline{x})$ (in view of the lower semicontinuity of ϕ). Thus for $f_n \in \phi$, $n \in \mathbb{N}$, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} f_1 \circ \ldots f_n(x) = \overline{x}$ exists for each $x \in X$. Let $\overline{x} = \overline{f}(x) = \lim_{n\to\infty} \prod_{i=1}^{n} f_i(x)$. Now

$$\begin{aligned}d(x, \overline{f}(x)) &\leq d(x, x_n) + d(x_n, \overline{f}(x)) \\ &\leq \phi(x) - \phi(x_n) + d(x_n, \overline{x}) \\ &\leq \phi(x) - \phi(x_n) + \phi(x_n) - \phi(\overline{x}) \\ &\qquad \text{(by the above argument)} \\ &\leq \phi(x) - \phi(\overline{f}(x)) \text{ (as } \overline{x} = \overline{f}(x))\end{aligned}$$

Thus $\overline{f} \in \Phi$ and Φ is closed under countable compositions.

For $f_n \in \Phi_T$, $n \in \mathbb{N}$, let $\overline{x} = \overline{f}(x)$ $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \prod_{i=1}^{n} f_i(x)$ as before. Then $\phi(\overline{f}(x)) = \phi(\overline{x}) = \phi(\lim_{n\to\infty} x_n) \leq \liminf_{n\to\infty} \phi(x_n)$ (by the lower semicontinuity of ϕ) $\leq \phi(T(x))$ as $\phi(x_n) = \phi(\prod_{i=1}^{n} f_i(x)) \leq \phi(T(x))$ for each $n \in \mathbb{N}$ and $x \in X$, since $\prod_{i=1}^{n} f_i \in \Phi_T$. Thus Φ_T is also closed under countable compositions. □

Definition 7.2.4 For $A \subseteq X$, define

(i) $D(A) =$ Diameter of $A = Sup\{d(x, y) : x, y \in A\}$;
(ii) $r(A) = \inf\{\phi(x) : x \in A\}$;
(iii) $S_x = \{f(x) : f \in \Phi'\}$ for $\Phi' \subseteq \Phi$.
Clearly for $B \subseteq A$, $d(B) \leq d(A)$ while $r(B) \geq r(A)$.

Lemma 7.2.5 *For $x \in X$, $D(S_x) \leq 2(\phi(x) - r(S_x))$.*

Proof For $f_1(x), f_2(x) \in S_x$,

$$\begin{aligned} d(f_1(x), f_2(x)) &\le d(x, f_1(x)) + d(x, f_2(x)) \\ &\le \phi(x) - \phi(f_1(x)) + \phi(x) - \phi(f_2(x)) \\ &= 2\phi(x) - (\phi(f_1(x)) + \phi(f_2(x)) \\ &\le 2(\phi(x) - r(S_x)). \end{aligned}$$

□

Theorem 7.2.6 *Let $\Phi' \subseteq \Phi$ be closed under compositions. Suppose $x_0 \in X$.*

(a) If Φ' is also closed under countable compositions, then there exists $\overline{f} \in \Phi'$ such that $\overline{x} = \overline{f}(x_0)$ and $g(\overline{x}) = \overline{x}$ for all $g \in \Phi'$.
(b) If the members of Φ' are continuous functions, then there exists a sequence of functions $(f_n) \in \Phi'$ and $\overline{x} = \lim_{n\to\infty} f_n f_{n-1} \ldots f_1(x_0)$ such that $g(\overline{x}) = \overline{x}$ for all $g \in \Phi'$.

Proof From the definition of $r(S_{x_0})$, it follows that we can find $f_1 \in \Phi'$ such that $0 \le \phi(f_1(x_0)) - r(S_{x_0}) < \frac{1}{2}$. Write $x_1 = f_1(x_0)$. As Φ' is closed under compositions, $S_{x_1} \subseteq S_{x_0}$ and $D(S_{x_1}) \le 2(\phi(x_1) - r(S_{x_1})) \le 2(\phi(f_1(x_0)) - r(S_{x_0})) < 1$. Thus proceeding inductively we can obtain a sequence of functions f_n such that $x_{n+1} = f_n(x_n)$, $S_{x_{n+1}} \subseteq S_{x_n}$ and $D(S_{x_n}) < \frac{1}{n}$.

If (a) is true, define $\overline{f} = \prod_{n=1}^{\infty} f_n$ and $\overline{x} = \overline{f}(x_0)$. Since $\overline{x} = \lim_{k\to\infty} \prod_{j=i+1}^{k} f_j(x_j)$, $\overline{x} \in S_{x_i}$ for each i. Since $\lim_{n\to\infty} D(S_{x_n}) = 0$, $\overline{x} \in \bigcap_{n=1}^{\infty} S_{x_n}$. We claim that for $g \in \Phi'$, $g(\overline{x}) = \overline{x}$. Since $g(\overline{x}) = g\left(\prod_{j=i+1}^{\infty} f_i(x_n)\right)$, $g(\overline{x}) \in S_{x_i}$ for each $i \in \mathbb{N}$. $g(\overline{x}) = \overline{x}$ as $\lim_{n\to\infty} D(S_{x_n}) = 0$.

Suppose (b) is true and $\overline{x} = \lim_{n\to\infty} f_n f_{n-1} \ldots f_1(x_0) = \lim_{n\to\infty} x_n$. Since $x_k \in S_n$ for $k \ge x, \overline{x} \in \overline{S}_n$ for all n. As $D(S_{x_n}) = D(\overline{S}_{x_n})$, $\{\overline{x}\} = \bigcap_{n=1}^{\infty} \overline{S}_{x_n}$. For $g \in \Phi'$, $g(x_n) \in S_{x_n}$ for each n and by the continuity of g, $\lim_{n\to\infty} g(x_n) = g(\overline{x})$. So for any given $\epsilon > 0$ we can find n_0 such that $B(g(\overline{x}), \epsilon) \cap S_{x_n} \ne \phi$ for $n > n_0$. So for $n > n_0$, $d(g(\overline{x}), \overline{x}) < \epsilon + \frac{1}{n}$. Thus $d(g(\overline{x}), \overline{x}) \le \epsilon$. As $\epsilon > 0$ is arbitrary $g(\overline{x}) = \overline{x}$. □

Remark 7.2.7 Caristi's theorem 7.2.1 follows upon setting $\Phi' = \{T^n : n \in \mathbb{N}\}$ the set of finite iterates of T and T itself.

Corollary 7.2.8 *Theorem 5.1.7 (Contraction Principle).*

Proof Let $T : X \to X$ be a contraction with $d(T(x), T(y)) \le kd(x, y)$ for $x, y \in X$ where $0 < k < 1$. Define $\phi : X \to \mathbb{R}^+$ by $\phi(x) = \frac{d(x,Tx)}{1-k}$. Clearly ϕ is continuous. Since $d(x, Tx) + \phi(Tx) = d(x, Tx) + \frac{d(Tx,T^2x)}{1-k} \le d(x, Tx) + \frac{k}{1-k}d(x, Tx) = \frac{d(x,Tx)}{1-k} = \phi(x)$. Thus $d(x, Tx) \le \phi(x) - \phi(Tx)$ for all $x \in X$. Hence by Caristi's theorem T has a fixed point. □

Brondstedt's proof [3] of Caristi's theorem involved a partial order and an application of Zorn's lemma. Penot's constructive proof [10] of Caristi's theorem exploiting this order is given below.

Remark 7.2.9 Penot's proof of Caristi's Theorem [10].

Define the binary relation $\le$ on X by $x \le y$ if $d(x, y) \le \phi(y) - \phi(x)$. Clearly this defines a partial order on X. Define $M(x) = \{y \in X : y \ge x\}$. Define an increasing sequence $\{x_n\}$ inductively as in the following. Choose x arbitrarily and when $x_1, \ldots x_n$ are given choose $x_{n+1} \in M(x_n)$ with $\phi(x_{n+1}) \le \inf\{\phi(x) : x \in M(x_n)\} + \frac{1}{n}$. So $x_{n+1} \ge x_n$ and for each $x \in M(x_{n+1}) \subseteq M(x_n)$ we have $\phi(x) \ge \inf \phi\{M(x_n)\} > \phi(x_{n+1}) - \frac{1}{n}$.

$$d(x, x_{n+1}) \le \phi(x_{n+1} - \phi(x)$$

So the diameter of $M(x_n) \le \frac{2}{n}$. As ϕ is lower semicontinuous, each $M(x)$ is closed. Since $\{M(x_n)\}$ is a decreasing sequence of closed sets with diameter of $M(x_n)$ tending to zero, by the completeness of X, $\bigcap_{n=1}^{\infty} M(x_n) = \{\overline{x}\}$, (invoking the Cantor intersection theorem). Clearly $\overline{x}$ is a maximal element of X under this order. If $y\ (\neq \overline{x}) \ge \overline{x}$ then $y \ge x_n$ for all n and $y \in M(x_n)$ for all n and this would contradict $\bigcap_{n=1}^{\infty} M(x_n) = \{\overline{x}\}$. Since $T(\overline{x}) \ge \overline{x}$, by hypothesis, $\overline{x} = T(\overline{x})$.

Making use of Brondstedt order Kirk [8] gave a proof using Zorn's lemma.

Remark 7.2.10 Kirk's proof of Caristi's Theorem [8].

For $x, y \in X$, define the partial order $\le$ by $x \le y$ if $d(x, y) \le \phi(x) - \phi(y)$ as before. Consider all totally ordered subsets of X (with respect to this order), with respect to set-inclusion. Since every chain in this collection has an upper bound, by Zorn's lemma, there is a maximal totally ordered subset $E = \{x_\alpha : \alpha \in I\}$ of X where I is totally ordered by $x_\alpha \le x_\beta$ if and only if $\alpha \le \beta$ $(\alpha, \beta \in I)$. Since $\{\phi(x_\alpha) : \alpha \in I\}$ is a decreasing net of non-negative real numbers converging to r as α increasing. So for $\epsilon > 0$, there exists $\alpha_0 \in I$ such that $r \le \phi(x_\alpha) < r + \epsilon$ for $\alpha \ge \alpha_0$. So for $\beta \ge \alpha \ge \alpha_0$, $d(x_\alpha, x_\beta) \le \phi(x_\alpha) - \phi(x_\beta) < \epsilon$. Thus $\{x_\alpha\}_{\alpha\in I}$ is a Cauchy net in the complete metric space X and hence converges to some $\overline{x} \in X$. ϕ being lower semicontinuous $\phi(\overline{x}) \le r$. Since $d(x_\alpha, x_\beta) \le \phi(x_\alpha) - \phi(x_\beta)$ letting $\beta \uparrow$, $d(x_\alpha, \overline{x}) \le \phi(x_\alpha) - r \le \phi(x_\alpha) - \phi(\overline{x})$. Since E is maximal, $\overline{x} \in E$. Also by hypothesis $\phi(x) - \phi(T(\overline{x})) \ge d(\overline{x}, T(\overline{x}))$. Thus $x_\alpha \le \overline{x} \le T(\overline{x})$ for all $\alpha \in I$. By the maximality of $\overline{x}$, $\overline{x} = T(\overline{x})$.

As has already been pointed out, the direct converse of the contraction principle is not true. However Kirk [8] has pointed out that if in a metric space every map satisfying the inequality of Caristi's theorem has a fixed point, then the space is complete.

Theorem 7.2.11 (Kirk [8]) *Let* (X, d) *be a metric space and* $\phi : X \to \mathbb{R}^+$, *any lower semicontinuous map. If every map* $T : X \to X$ *satisfying* $d(x, Tx) \le \phi(x) - \phi(T(x))$ *has a fixed point, then* (X, d) *is complete.*

Proof Let (x_n) be a non-convergent Cauchy sequence in (X, d). For each $x \in X$, define $\phi(x) = \lim_{n\to\infty} d(x, x_n)$. (Note that $\phi(x)$ is well-defined as $d(x, x_n)$ is a Cauchy sequence of real numbers and hence is convergent.) Given $x \in X$, define the least natural number $n(x)$ such that $0 < \frac{1}{2}d(x, x_n) < \phi(x) - \phi(x_{n(x)})$. Define $T : X \to X$ by $T(x) = x_{n(x)}$ and $\psi : X \to \mathbb{R}^+$ by $\psi(x) = 2\phi(x)$, for $x \in X$. Since $|\phi(x) - \phi(y)| \le d(x, y)$, both ϕ and ψ are continuous. Clearly from the definition of ψ, $0 < d(x, Tx) \le \psi(x) - \psi(Tx)$ for all $x \in X$. Clearly T has no fixed point. Thus if X is not complete we can find a fixed point free map $T : X \to X$ satisfying $d(x, Tx) \le \psi(x) - \psi(T(x))$ for some lower semicontinuous map $\psi : X \to \mathbb{R}^+$.

Thus Caristi's theorem is characteristic of completeness. The following simple example shows that Caristi's theorem applies even when T is not continuous. □

Example 7.2.12 Let X be $[0, 1]$ with the usual metric and $T : X \to X$ be defined as $Tx = \begin{cases} \frac{x}{2} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$. Then $|x - Tx| \le \phi(x) - \phi(T(x))$ for $\phi(x) = x$. Clearly T is discontinuous and has 0 and 1 as fixed points.

Remark 7.2.13 It may be clarified that while proofs of Caristi's theorem by Caristi [4], Wong [16], Kirk [8] and Brondsted [3] invoke some form of the Axiom of Choice such as Zorn's Lemma, the constructive proofs such as those by Penot [10] and Siegel [11] are indeed based on the axiom of choice for countable families. On the other hand, Manka showed that Caristi's theorem can be proved without choice using Zermelo's fixed point theorem for special posets. (See Kirk [9] and Jachymski [7] for detailed comments and references.)

Jachymski [7] noted that Nadler's fixed point Theorem 5.3.5 can be deduced from Caristi's Theorem 7.2.1.

Theorem 7.2.14 (Jachymski [7]) *Let* (X, d) *be a complete metric space and* $T : X \to cl\,BX$ *(the set of all bounded non-empty closed subsets of* X*) such that for some* $\alpha \in (0, 1)$. $H(Tx, Ty) \le \alpha d(x, y), x, y \in X$. *Then* T *admits a selection* $g : X \to X$ *which satisfies the conditions of Caristi's Theorem 7.2.1 and hence has a fixed point. So does* T.

Proof Let $\beta \in (0, 1)$ be such that $\alpha < \beta$. For each $x \in X$, the set $\{y \in Tx : \beta(x, y) \le d(x, Tx)\}$ is non-empty. By the Axiom of Choice there exists a map $g : X \to X$ such that $gx \in Tx$ and $d(x, gx) \le d(x, Tx)$. So $d(gx, Tx) \le H(Tx, T(g(x))) \le \alpha d(x, g(x))$. So

$$\begin{aligned} d(x, g(x)) &\le \frac{1}{(\beta - a)}(\beta d(x, g(x)) - \alpha d(x, g(x)) \\ &\le \frac{1}{(\beta - a)}[d(x, Tx) - d(g(x), Tg(x))] \\ &\le \phi(x) - \phi(g(x)) \\ &\quad \text{where } \phi(x) = \frac{d(x, Tx)}{(\beta - a)} \end{aligned}$$

Thus g is a Caristi map, a selection from T and hence has a fixed point in X. So does T. Further $|\phi(x) - \phi(y)| \le \frac{d(x,y)+H(Tx,Ty)}{\beta-\alpha} \le \left(\frac{1+\alpha}{\beta-a}\right) d(x, y)$. □

Remark 7.2.15 Nadler noted that the multivalued map of his theorem may not admit a selection which is a contraction. Let X be the unit circle in the complex plane. For each $z = e^{i\alpha}$, $\alpha \in [0, 2\pi]$, let $T(z) = \{e^{i\frac{\alpha}{2}}, e^{i(\pi+\frac{\alpha}{2})}\}$, the set of square roots of z. For $z_i = e^{i\alpha_i}$, $i = 1, 2$,

$$H(Tz_1, Tz_2) = \begin{cases} 2\frac{\sin(\alpha_2-\alpha_1}{4} & \text{if } \alpha_2 - \alpha_1 \le \pi \\ 2\frac{\cos(\alpha_2-\alpha_1}{4} & \text{if } \alpha_2 - \alpha_1 > \pi \end{cases}$$

and $H(Tz_1, Tz_2) \le \frac{\sqrt{2}}{2}d(z_1, z_2) = \frac{\sqrt{2}}{2}|2\frac{\sin(\alpha_2-\alpha_1}{4}|$. Following Jachymski [7] if T has a selection g which is a contraction and if $g(1) = 1$, then by the continuity of g at 1, for some $\alpha_0 > 0$, $g(e^{i\alpha}) = e^{i\frac{\alpha}{2}}$ for all $\alpha \in [0, \alpha_0)$. By the continuity of g at $\alpha^* = \sup\{\alpha_0 \in (0, 2\pi) : g(e^{i\alpha}) = e^{i\frac{\alpha}{2}}$ for all $\alpha \in [0, \alpha_0)\}$. For $\alpha^* < 2\pi$, we get $e^{i\frac{\alpha^*}{2}} = e^{i(\frac{\alpha^*}{2}+\pi)}$ so that $\alpha^* = 2\pi$. While $g(e^{i(2\pi-\frac{1}{n})}) \to e^{i\pi}$ as $n \to \infty$, by the continuity of g at 1, $e^{i(2\pi-\frac{1}{n})} \to -1$ and $g(e^{i\frac{\alpha}{2}}) \to 1$. This is a contradiction. If $g(1) = -1$, a similar contradiction results. Thus T has no contractive selection.

7.3 Ekeland's Variational Principle

Ekeland [6] obtained a theorem in the setting of metric spaces that finds wide use in solving optimization problems and partial differential equations. His proof uses a partial order employed earlier by Brondsted and Rockafeller and Bishop and Phelps.

Theorem 7.3.1 (Ekeland [6]) *Let (X, d) be a complete metric space and $F : X \to \mathbb{R} \cup \{+\infty\}$, a lower semicontinuous function $\not\equiv +\infty$ bounded from below. For each $\epsilon > 0$ and $u \in X$ satisfying*

$$\inf_{x\in X} F(x) \le F(u) \le \inf_{x\in X} F(x) + \epsilon$$

and every $\lambda > 0$ there exists $v \in X$ such that $F(v) \le F(u)$, $d(u, v) \le \lambda$ and for $w \ne v$, $F(w) > F(v) - \frac{\epsilon}{\lambda}d(v, w)$.

The proof is based on the following lemma.

Lemma 7.3.2 *Let (X, d) be a complete metric space. Define the binary relation $\preceq$ on $X \times \mathbb{R}$ by $(x_1, a_1) \preceq (x_2, a_2)$ iff $(a_2 - a_1) + \alpha d(x_1, x_2) \leq 0$ where α is any given positive number. Then $\preceq$ is a partial order on $X \times \mathbb{R}$ such that $\{(x, a) : (x_1, a_1) \leq (x, a)\}$ is closed in $X \times R$ for each (x_1, a_1) in $X \times \mathbb{R}$. Further if S is a closed subset of $X \times \mathbb{R}$ such that for all $(x, a) \in S$, there exists $m \in \mathbb{R}$ such that $a \geq m$. Then for every $(x_1, a_1) \in S$, there exists for the ordering $\preceq$ an element $(\overline{x}, \overline{a})$ which is maximal and greater than (x_1, a_1).*

Proof That $\preceq$ is a continuous partial order is easily verified. Define inductively (x_n, a_n) in S, $n \in \mathbb{N}$ beginning (x_1, a_1). If (x_n, a_n) is known, define

$$\begin{aligned} S_n &= \{(x, a) \in S : (x_n, a_n) \preceq (x, a)\} \\ m_n &= \inf\{a \in \mathbb{R} : (x, a) \in S_n\} \end{aligned}$$

Indeed $m_n \geq m$. Choose $(x_{n+1}, a_{n+1}) \in S_n$ such that $a_n - a_{n+1} \geq \frac{1}{2}(a_n - m_n)$. Now S_n are closed and non-empty and $S_{n+1} \subseteq S_n$ for all n. Also

$$|a_{n+1} - m_{n+1}| \leq \frac{1}{2}|a_n - m_n| \leq \frac{1}{2^n}|a_1 - m|.$$

So for $(x, a) \in S_{n+1}$ we get

$$\begin{aligned} |a_{n+1} - a| &\leq \frac{1}{2^n}|a_1 - m| \\ d(x_{n+1}, x) &\leq \frac{1}{2^n}\frac{1}{\alpha}|a_1 - m| \end{aligned}$$

Thus $Diam(S_n) \to 0$ as $n \to \infty$.

As $X \times \mathbb{R}$ is complete, $\bigcap_{n=1}^{\infty} S_n = \{(\overline{x}, \overline{a})\}$. From the definition of $(\overline{x}, \overline{a})$, $(x_n, a_n) \preceq (\overline{x}, \overline{a})$ for each n and in particular for $n = 1$. If $(x_0, a_0) \in S$ is greater than $(\overline{x}, \overline{a})$, then by transitivity $(x_n, a_n) \preceq (x_0, a_0)$ for each n. So $(x_0, a_0) \in \bigcap_{n=1}^{\infty} S_n$ and hence $(x_0, a_0) = (\overline{x}, \overline{a})$. Thus $(\overline{x}, \overline{a})$ is maximal. □

Proof of Theorem 7.3.1. Let $S = \{(x, a) : x \in X, a \geq F(x)\}$. $\mathcal{S}$, the epigraph of F is a closed subset of $X \times \mathbb{R}$ as F is lower semicontinuous. Set $\alpha = \frac{\epsilon}{\lambda}$ and $(x_1, a_1) = (u, F(u))$ and apply Lemma 7.3.2 to get a maximal element (v, a) in $\mathcal{S}$ satisfying $(u, F(u)) \preceq (v, a)$. The maximality relation can be rephrased as $(v, a) \prec (w, b) \in S \Rightarrow \alpha d(w, v) < a - b$. As $(v, a) \in S$ is maximal, $a = F(v)$. If $w \neq v$, $a = F(v) > b \geq F(w)$ implying that (v, a) is not maximal. So for $w \neq v$, and $w \neq v$, $(w, F(w)) \not\preceq (v_1, F(v))$. Or $F(w) - F(v) + \alpha d(v, w) \geq 0$ or $-\frac{\epsilon}{\lambda} d(v, w) < F(w) - F(v))$ If for this $\epsilon > 0$, $u \in X$ is such that $\inf\{F(x) :$

$x \in X\} \leq F(u) \leq \inf\{F(x) : x \in X\} + \epsilon$, then $(u, F(u)) \not\preceq (v, F(v))$. This implies $\frac{\epsilon}{\lambda}d(u, v) < F(u) - F(v) \leq F(u) - \inf\{f(x) : x \in X\} \leq \epsilon$ or $d(u, v) \leq \lambda$ and for $u \neq v$, $F(u) > F(v) - \frac{\epsilon}{\lambda}d(u, v)$. □

Corollary 7.3.3 *Let (X, d) be a complete metric space and $F : X \to R \cup \{+\infty\}$ a lower semicontinous map which is bounded below and not identically $+\infty$. Let $\epsilon > 0$ be given and u be a point in X such that*

$$F(u) \leq \inf\{F(x) : x \in X\} + \epsilon$$

Then there exists $v \in X$ such that (i) $F(v) \leq F(u)$; (ii) $d(u, v) \leq 1$ and for all $w \neq v$, $F(w) + \epsilon d(v, w) > F(v)$.

Proof In Theorem 7.3.1 set $\lambda = 1$. □

Corollary 7.3.4 *Under the hypotheses of Corollary 7.3.3 given $\epsilon > 0$, there exists $x \in X$ with $f(\overline{x}) < \inf f + \epsilon$.*

Corollary 7.3.5 (Caristi's fixed point theorem)

Proof Let $T : X \to X$ be an operator such that $d(x, Tx) \leq \phi(x) - \phi(T(x))$ for all $x \in X$, where ϕ is a non-negative lower semicontinuous function. □

Proof By Corollary 7.3.3 for $\epsilon = 1$, there exists $v \in X$ such that for all $u \neq v$

$$\phi(v) < \phi(u) + d(v, u)$$

If $v \neq T(v)$, then $\phi(T(v)) - \phi(u) < d(u, Tv)$ a contradiction. So $v = T(v)$. Thus T has a fixed point. □

Indeed we can even prove a multivalued version of Caristi's theorem.

Theorem 7.3.6 (Multivalued version of Caristi's Theorem) *Let (X, d) be a complete metric space and $\phi : X \to R \cup \{+\infty\}$ be a lower semicontinuous function which is bounded below. Let $T : X \to 2^X$ be a multivalued map such that $T(x) \neq \phi$ for each $x \in X$ and $d(x, y) \leq \phi(x) - \phi(y)$ for all $y \in Tx$. Then there exists $x_0 \in X$ with $x_0 \in Tx_0$.*

Proof As before set $\epsilon = 1$ in Corollary 7.3.3. So there exists $v \in X$ such that for all $y \neq v$, $\phi(v) < \phi(y) + d(v, y)$. We claim that $v \in T(v)$. Otherwise for all $y \in T(v)$, $d(v, y) > 0$ and $\phi(v) - \phi(y) < d(v, y)$. But by hypothesis $d(v, y) \leq \phi(v) - \phi(y)$. Thus $\phi(v) - \phi(y) < \phi(v) - \phi(y)$, a contradiction. So for some $y \in T(v)$, $d(v, y) = 0$ or $v \in T(v)$. □

7.4 A Minimization Theorem

Takahashi [13] proved a minimization theorem which can be deduced from Ekeland's variational principle.

Theorem 7.4.1 (Takahashi [13]) *Let (X, d) be a complete metric space and $F : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous map bounded below. Suppose F satisfies the (Takahashi) condition: there exists $\alpha_0 > 0$ such that for $x \in X - Z$, there exists $y \neq x$ with $\alpha_0 d(y, x) \leq F(x) - F(y)$ Z being the set $\{z \in X : F(z) = \inf\limits_X F\}$ (the set of possible minima of $F\}$. Then there exists $x_0 \in X$ with $F(x_0) = \inf\limits_X F$ (or Z is non-void).*

Proof For $0 < \alpha < \alpha_0$, by Ekeland's principle (Corollary 7.3.4) there exists $x \in X$ such that $F(x) < F(y) + \alpha d(y, x), \forall \ y \neq x$. By Takahashi condition for $z \in X - Z$, there exists $y \neq z$ such that $F(y) + \alpha d(y, z) \leq F(z)$. If $x \notin Z$, the last inequality for the choice $z = x$ contradicts its preceding inequality. So $x \in Z$ is a minimizer of F over X. □

We now show that Takahashi's minimization theorem implies Ekeland's principle (Theorem 7.3.1).

Theorem 7.4.2 *Theorem 7.4.1 implies Theorem 7.3.1.*

Proof Let x_0 be the minimum guaranteed by Theorem 7.4.1. Define $X_0 = \{x \in X : F(x) \leq F(x_0) - \frac{\epsilon}{\lambda} d(x_0, x)\}$. Clearly $x_0 \in X_0$ and $X_0 \neq \phi$. Since F is lower semicontinuous and $\frac{d}{\lambda}$ is continuous, X_0 is closed. Further for $x \in X_0$.

$$\frac{\epsilon}{\lambda} d(x_0, x) \leq F(x_0) - F(x) = F(x_0) - \inf_X f \leq \epsilon$$

So $\frac{d(x_0, x)}{\lambda} \leq 1$ or $d(x_0, x) \leq \lambda$. We also have $F(x_0) \leq F(x)$. We claim that for $x \in X_0$ and $y \neq x$, $F(y) > F(x) - \frac{\epsilon}{\lambda} d(x, y)$. Otherwise for some $y \in X$, $y \neq x$, $F(y) \leq F(x) - \frac{\epsilon}{\lambda} d(x, y)$. Then

$$\begin{aligned}\frac{\epsilon}{\lambda} d(y, x_0) &\leq \frac{\epsilon}{\lambda} d(x_0, x) + \frac{\epsilon}{\lambda} d(x, y) \\ &\leq F(x_0) - F(x) + F(x) - F(y) \\ &= F(x_0) - F(y).\end{aligned}$$

So $y \in X_0$. So by Theorem 7.4.1 there exists $\overline{x} \in X$ such that $F(\overline{x}) = \inf_{x \in X} F(x)$. This contradicts that $F(y) < F(x_0)$, a contradiction. □

Theorem 7.4.3 *Caristi's Theorem 7.2.1 implies Ekeland's Variational principle, viz. Corollary 7.3.4.*

Proof Let the hypotheses of Corollary 7.3.4 hold. Clearly $d_1(x, y) = \epsilon d(x, y)$ defines an equivalent metric on X. For $x \in x$, define $T(x) = \{y \in X : F(x) \geq F(y) + d_1(x, y), y \neq x\}$. If the conclusion of Corollary 7.3.4 is false, then Clearly $T(x) \neq \phi$ for all $x \in X$. T is a multivalued map of X into $2^X - \{\phi\}$ satisfying $F(y) \leq F(x) - d_1(x, y)$ for $y \in Tx$. Since (X, d_1) is complete, by Caristi's theorem, T has a fixed point x_0. However for $x_0 \in Tx_0$ the definition of $T(x)$ is violated. This contradiction shows that Corollary 7.3.4 is true. □

Remark 7.4.4 Thus both Caristi's fixed point theorem and Takahashi's minimization theorem are equivalent to Ekeland's variational principle. In other words Caristi's theorem, Ekeland's principle and Takahashi's theorem are equivalent.

Since Carisit's theorem is characteristic of completeness, both Takahashi's theorem and Ekeland's principle are also characteristic of completeness. Sullivan [12] proved that Ekeland's principle implies the completeness of the metric space.

Theorem 7.4.5 (Sullivan [12]) *Let (X, d) be a metric space and $F : X \to R \cup \{+\infty\}$, $F \not\equiv +\infty$ be any continuous map bounded below such that for each $\epsilon > 0$, there exists $x_0 \in M$ satisfying $F(x_0) \leq \inf F + \epsilon$ and for all $x \neq x_0$, $F(x) > F(x_0) - \epsilon d(x, x_0)$.*

Proof Suppose that (x_n) is a non-convergent Cauchy sequence in X. For each $x \in F$, the $x \to F(x) = \lim_{n\to\infty} d(x, x_n)$ is well-defined, continuous on X and bounded below by 0. Since (x_n) is Cauchy $\inf F = 0$ and $F \not\equiv +\infty$. Let $0 < \epsilon < 1$ then by assumptions on F, we can find $x_0 \in X$ such that $F(x_0) \leq \epsilon$ and $F(x) > F(x_0) - \epsilon d(x, x_0)$ for all $x \neq x_0$. As (x_n) is non-convergent $F(x_0) > 0$.

Let η be any positive number less than $\frac{(1-\epsilon)F(x_0)}{4}$. Since (x_n) is Cauchy we can find x_{N_0} such that $d(x_{N_0}, x_n) \leq \eta$ for all $n \geq N_0$ so that $F(x_{N_0}) = \lim_{n\to\infty}(x_{N_0}, x_n) \leq \eta$. Setting $x = x_{N_0}$ in the inequality

$$0 < F(x_0) < F(x) + \epsilon d(x_0, x)$$

we get

$$\begin{aligned}
0 < F(x_0) &< F(x_{N_0}) + \epsilon d(x_0, x_{N_0}) \\
&< F(x_{N_0}) + \epsilon[d(x_0, x_n) + d(x_n, x_{N_0})] \text{ for } n \geq N_0 \\
&\leq F(N_0) + \epsilon \lim_{n\to\infty}[d(x_0, x_n) + d(x_n, x_{N_0})] \\
&\leq F(x_{N_0}) + \epsilon[F(x_0) + \eta] \\
0 < F(x_0) &\leq \eta + \epsilon\eta + \epsilon F(x_0)
\end{aligned}$$

or

$$0 < (1 - \epsilon)F(x_0) < 2\eta < \frac{(1 - \epsilon)F(x_0)}{2}$$

This contradiction shows that $F(x)$ cannot be positive for all $x \in X$. In other words (x_n) must converge in X or (X, d) is complete. □

The next theorem shows that Takahashi's minimization theorem also characterizes completeness.

Theorem 7.4.6 *A metric space (X, d) is complete if for every uniformly continuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, $f \not\equiv +\infty$ and every $x' \in X$ with $\inf_X f < f(x')$ there exists $z \in X$ with $z \neq x'$ and $f(z) + d(x', z) \leq f(x')$, there exists x_0 with $f(x_0) = \inf_X f$.*

Proof Let (x_n) be a Cauchy sequence in X. Define $f : X \to \mathbb{R} \cup \{+\infty\}$ by $f(x) = \lim_{n\to\infty} d(x, x_n)$ for all $x \in X$. Clearly f is well-defined and uniformly continuous. Also $\inf_{x\in X} f(x) = 0$. Let $f(x_0) > 0$. Then there exists $x_m \in X$ with $x_m \neq x_0$, $f(x_m) < \frac{f(x_0)}{3}$ and $d(x_m, x_0) - f(x_0) < f(x_0)$. Thus we have $3f(x_m) + d(x_m, x_0) < f(x_0) + 2f(x_0) = 3f(x_0)$. So there exists $\overline{x} \in X$ with $f(\overline{x}) = \inf f(x) = 0$. Thus $\lim d(x_m, \overline{x}) = 0$ or (x_n) converges in X. Thus (X, d) is complete. □

7.5 An Application of Ekeland's Principle

We begin with some basic ideas of calculus.

Definition 7.5.1 For locally convex linear topological spaces X and Y and $U \subseteq X$ open, the map $F : X \to Y$ is said to be Gateaux differentiable at $u \in U$ if for each $h \in X$, $\lim_{t\to 0} \dfrac{F(u + th) - F(u)}{t}$ exists. This limit, denoted by $dF(u, h)$ is called the Gateaux differential at u and is homogeneous.

Remark 7.5.2 If the Gateaux differential is linear and continuous, it is called Gateaux derivative. Since the Gateaux differential of a discontinuous linear function is itself, it follows that a Gateaux differential can be linear without being continuous.

Definition 7.5.3 A map $F : X \to Y$ where X and Y are normed linear spaces is said to be Frechet differentiable at u if there exists a bounded linear function L such that $F(u + h) = F(u) + L(h) + o(\|h\|)$.

Remark 7.5.4 If F is Frechet differentiable then it is Gateaux differentiable and both the Gateaux and Frechet derivatives coincide. While a Frechet differentiable function is continuous, a Gateaux differentiable function need not be continuous. Indeed even if the Gateaux differential is linear and continuous, the Frechet derivative may not exist.

We proceed to describe an application of Ekeland's principle.

Theorem 7.5.5 *Let X be a Banach space and $\phi : X \to \mathbb{R}$, a lower semicontinuous function bounded below. Suppose ϕ has Gateaux derivative on X. Then for each $\epsilon > 0$ there exists $x_\epsilon \in X$ such that $\phi(x_\epsilon) \leq \inf_X \phi + \epsilon$ and $\|D\phi(x_\epsilon)\| \leq \epsilon$.*

Proof From Ekeland's Principle (Theorem 7.3.1), there exists $x_\epsilon \in X$ such that $\phi(x_\epsilon) \leq \phi(x) + \epsilon\|x - x_\epsilon\|$ for all $x \in X$. Let $h \in X$ and $t > 0$. Setting $x = x_\epsilon + th$ in the above inequality we get

$$\frac{1}{t}[\phi(x_\epsilon) - \phi(x_\epsilon + th)] \leq \epsilon\|h\|$$

As $t \to 0$ we get $-D\phi(x_\epsilon)(h) \leq \epsilon\|h\|$. Writing $-h$ for h we get $D\phi(x_\epsilon)h \leq \epsilon\|h\|$. Thus $|D\phi(x_\epsilon)h| \leq \epsilon\|h\|$. So $D\phi(x_\epsilon)$ is bounded and $\|D\phi(x_\epsilon)\| \leq \epsilon$. □

We can even ensure the existence of a critical point for ϕ which is a minimum under an additional condition.

Definition 7.5.6 Let X be a Banach space and $\phi : X \to \mathbb{R}$ be a function which is continuously differentiable. ϕ is said to satisfy the Palais–Smale condition if each sequence (x_n) in X for which $\phi(x_n)$ is bounded and $\phi'(x_n) \to 0$ in X has a convergent subsequence.

Theorem 7.5.7 *Let X be a Banach space and $\phi : X \to \mathbb{R}$ be a C^1 function bounded below satisfying the Palais–Smale condition. Then there exists $x_0 \in X$ such that $\phi(x_0) = \inf_X \phi$ and $\phi'(x_0) = 0$.*

Proof Setting $\epsilon = \frac{1}{n}$ in Theorem 7.5.5 we get a sequence $(x_n) \in X$ such that

$$\phi(x_n) \leq \inf_X \phi + \frac{1}{n} \quad \text{and} \quad \|\phi'(x_n)\| \leq \frac{1}{n}$$

As ϕ satisfies the Palais–Smale condition (x_n) has a subsequence (x_{n_k}) converging to $x_0 \in X$. Clearly $\phi(x_0) = \inf_X \phi$ and ϕ' being continuous, $\phi'(x_{n_k}) = \phi'(x_0)$ and $\phi'(x_0) = 0$. □

For other applications De Figueiredo [5] may be consulted.

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Chapter 8
Contractive and Non-expansive Mappings

In this chapter, fixed points of contractive and non-expansive mappings are studied, as also the convergence of their iterates.

8.1 Contractive Mappings

The following definition is recalled.

Definition 8.1.1 A mapping $T : X \to X$ where (X, d) is a metric space is called contractive if $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$.

A contractive mapping need not have a fixed point in a complete metric space as seen from the following example.

Remark 8.1.2 The map $x \to e^{-x}$ mapping $\mathbb{R}^+$ into itself has no fixed point. For $0 \leq x < y$, $0 < e^{-x} - e^{-y} = (= |e^{-x} - e^{-y}|) = e^{-\xi}(y - x) < y - x = |x - y|$ by the Mean-value Theorem.

In this connection the theorem below leads to the existence of a fixed point for contractive mappings under a set of suitable conditions.

Theorem 8.1.3 *Let (X, d) be a metric space and $T : X \to X$ a continuous map. Suppose*

(i) $d(Tx, T^2x) \leq d(x, Tx)$ for all $x \in X$;
(ii) for $x \neq Tx$, there exists a natural number $n = n(x)$ such that

$$d(T^n x, T^{n+1}x) < d(x, Tx);$$

(iii) for some $x' \in X$, $x'_{n_k} = T^{n_k}(x')$, a subsequence of T-iterates at x' converges to $u \in X$.

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Then u is a fixed point of T.

Proof As T is continuous, the map $\phi(x) = d(x, Tx)$ is a continuous map of X into $\mathbb{R}^+$. As x'_{n_k} converges to u, $\phi(x'_{n_k}) \to \phi(u)$. Since $\phi(Tx) \leq \phi(x)$, the sequence $\{\phi(x'_n)\}$ is a non-increasing sequence of non-negative real numbers, x'_n being $T^n(x')$. So $\{\phi(x'_n)\}$ converges to a non-negative real number r. $\phi(x'_{n_k})$ being a subsequence converging to $\phi(u)$, $\phi(u) = r$. If $r = \phi(u) > 0$, then by (iii), there exists $m = m(u)$, a natural number such that $\phi(T^m(u)) < \phi(u) = r$. As $\{x'_{n_k}\}$ converges to u, $\{T^m(x'_{n_k})\}$ converges to $T^m(u)$ by the continuity of T^m. So $\{\phi(T^m(x'_{n_k}))\}$ converges to $\phi(T^m(u))$. Since this is a subsequence of $\phi(x'_n)$ which converges to $\phi(u)$, $\phi(T^m(u)) = \phi(u) = r$ contradicting that $\phi(T^m(u)) < \phi(u)$. So $r = 0$. Or $u = T(u)$. □

Corollary 8.1.4 (Edelstein [6]) *Let (X, d) be a metric space and $T : X \to X$ a contractive map. If for some $x' \in X$, $\{x'_{n_k} = T^{n_k}x'\}$ is a subsequence of the T-iterates at x' converging to $u \in X$, then u is a unique fixed point of T. Further $\{T^n x'\}$ converges to u.*

Proof While the existence follows from Theorem 8.1.3 the proof of the uniqueness is left as an exercise. Since (x'_{n_k}) converges to u, given $\epsilon > 0$, there exists $N(\epsilon)$, a positive integer such that $d(u, x'_{n_k}) < \epsilon$. For all $k \geq N(\epsilon)$. Now for $n > n_{N(\epsilon)}$ then for $p = n - n_{N(\epsilon)}$, $d(u, x_n) = d(T^p u, T^p x_{n_N}) \leq d(T^{p-1}u, T^{p-1}x_{n_N}) < d(u, x_{n_N}) < \epsilon$. So (x_n) converges to u. □

Corollary 8.1.5 *Let (X, d) be a compact metric space. Then every contractive self map on X has a unique fixed point to which every sequence of iterates converges.*

Remark 8.1.6 Reduction of the problem of finding the fixed point of the operator T to that of finding the zero of the map $x \to d(x, Tx)$ in the metric setting has been suggested for instance in Dieudonne [3].

Even as Kannan's Corollary 5.2.2 or the more general fixed point Theorem 5.2.1 apply to mappings which are not necessarily continuous, Theorem 8.1.3 has an analogue for operators which need not be continuous.

Theorem 8.1.7 *Let (X, d) be a compact metric space and $T : X \to X$ a mapping satisfying the following conditions:*

(i) for some non-negative real numbers a_1, a_2, a_3, a_4 and a_5 for $x, y \in X$ with $x \neq y$

$$d(Tx, Ty) < a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, Ty) + a_4 d(y, Tx) + a_5 d(x, y);$$

(ii) $a_2 + a_3 < 1$ and $\frac{a_1+a_3+a_5}{1-a_2-a_3} = 1$.

Then T has a fixed point. If further $\frac{a_4}{1-a_3-a_5} < 1$, then the fixed point is unique.

Proof Since $d(x, Tx) \geq 0$ for $x \in X$, let $r = \inf\{d(x, Tx) : x \in X\}$. Let $x_n \in X$ be such that $\lim\limits_{n\to\infty} d(x_n, Tx_n) = r$. As X is compact, $T(x_n)$ has a subsequence $y_k = T(x_{n_k})$ converging to y.

Now $d(y, Ty) \leq d(y, T(x_{n_k})) + d(Tx_{n_k}, Ty)$. But

$$d(Tx_{n_k}, Ty) \leq a_1 d(x_{n_k}, Tx_{n_k}) + a_2 d(y, Ty) + a_3 d(x_{n_k}, Ty) + a_4 d(y, Tx_{n_k}) + a_5 d(x_{n_k}, y)$$

$d(x_{n_k}, Ty) \leq d(x_{n_k}, Tx_{n_k}) + d(Tx_{n_k}, y) + d(y, Ty)$ and $d(x_{n_k}, y) \leq d(x_{n_k}, Tx_{n_k}) + d(Tx_{n_k}, y)$
Using these inequalities we get

$$\begin{aligned} d(y, Ty) \leq\ & (a_1 + a_3 + a_5) d(x_{n_k}, Tx_{n_k}) + (a_2 + a_3) d(y, Ty) + (1 + a_4) d(y, Tx_{n_k}) \\ & + (a_3 + a_5) d(Tx_{n_k}, y) \end{aligned}$$

So

$$\begin{aligned} (1 - a_2 - a_3) d(y, Ty) \leq\ & (a_1 + a_3 + a_5) d(x_{n_k}, Tx_{n_k}) + (1 + a_4) d(y, Tx_{n_k}) \\ & + (a_3 + a_5) d(Tx_{n_k}, y) \end{aligned}$$

Proceeding to the limit as $k \to \infty$ in the above, we get

$$(1 - a_2 - a_3) d(y, Ty) \leq (a_1 + a_3 + a_5) r$$

As $\frac{a_1+a_3+a_5}{1-a_2-a_3} = 1$, it follows that $d(y, Ty) = r$. If $y \neq Ty$, then (ii) gives $(1 - a_2 - a_3) d(Ty, T^2 y) < (a_1 + a_3 + a_5) d(y, Ty)$. So $d(Ty, T^2 y) < d(y, Ty) = r$, contradicting that $r = \inf\{d(x, Tx) : x \in X\}$. Thus $y = Ty$.

If x and y are two fixed points of T, then

$$d(x, y) = d(Tx, Ty) < (a_3 + a_5) d(x, y) + a_4 d(y, x)$$

Thus $(1 - a_3 - a_5) d(x, y) < a_4 d(y, x)$. Since $\frac{a_4}{1-a_3-a_5} < 1$, $x = y$. □

Corollary 8.1.8 (Reich [21]) *Let (X, d) be a compact metric space and $T : X \to X$ be a map such that for $x \neq y$, $x, y \in X$*

$$d(Tx, Ty) < \frac{1}{2}[d(x, Tx) + d(y, Ty)]$$

Then T has a unique fixed point.

Edelstein's result (Corollary 8.1.4) may not be true for contractive maps on closed bounded sets which are not compact, as shown by the following.

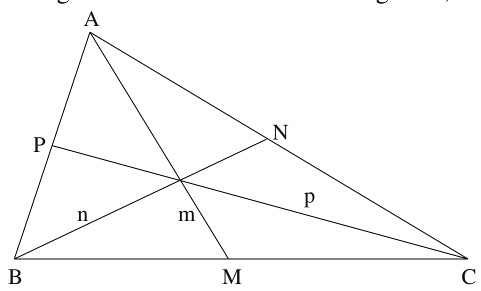
Example 8.1.9 (*Ira Rosenholtz* [22]) Let c_0 be the Banach space of all null real sequences with the supremum norm and S, the closed unit ball in c_0. Define $T : S \to S$ by

$$T(x_1, x_2, \dots, x_n, \dots) = (1, a_1x_1, a_2x_2, a_3x_3, \dots, a_nx_n, \dots)$$

where a_n is a sequence of positive real numbers such that each a_k is less than 1 and $P_n = \prod_{j=1}^{n} a_j$ is bounded away from zero. For example $a_n = \frac{2^n+1}{2^n+2}$. Clearly T maps S into the boundary of S as $\|Tx\| = 1$ for $x \in S$. Also if $x = (x_1, x_2, \dots, x_n, \dots)$ is a fixed point of T, then $Tx = (y_1, \dots, y_n, \dots)$ where $y_1 = 1$ and $y_n = a_{n-1}x_{n-1}$ for $n > 1$, so that $y_n = P_{n-1} = \prod_{k=1}^{n-1} \frac{(1+\frac{1}{2^k})}{(1+\frac{1}{2^{k-1}})} \geq \frac{1+\frac{1}{2^{n-1}}}{2} \geq \frac{1}{2}$. As $(y_n) \notin c_0$, T cannot have a fixed point in S.

Edelstein's theorem can be used to solve a classical problem of geometry. Given a triangle ABC with sides a (= BC), b (= CA) and c (= AB), the problem is to construct the triangle with prescribed lengths of angle bisectors. Originally the problem required the solution to be constructed using ruler and compass. This problem has been independently posed by several persons including Brocard and Terquem. See [4, 16] for a detailed history of this problem. Indeed the internal bisectors of the angles of a triangle are concurrent while the external bisectors form a triangle. Van den Berg [24] showed that given three positive numbers m, n, p the necessary and sufficient conditions for the existence of a triangle with lengths of exterior bisectors of angle m, n, p is that the maximum of $\{mn, np, pm\}$ is larger than the sum of the remaining two of this triplet. He also noted that in this case the problem admits of two solutions. While Korselt showed that it is impossible to construct the triangle by ruler and compass, given the lengths of external bisectors of angles, Neiss proved the impossibility of construction by ruler and compass the triangle for which the lengths of the internal bisectors of the angles of the triangle are given.

In what follows we consider the problem of constructing the triangle, given the lengths of internal bisectors of angles. Let ABC be the triangle with sides a, b, c and let m, n and p be the lengths of internal bisectors of angles A, B and C respectively.



Area of △ ABC = Area of △ ABM + Area of △ AMC. So

$$\frac{1}{2}bc \sin A = \frac{1}{2}cm \sin \frac{A}{2} + \frac{1}{2}bm \sin \frac{A}{2}$$

or

$$m = \frac{2bc}{b+c} \cos\left(\frac{A}{2}\right).$$

Since

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ &= b^2 + c^2 - 2bc \left[2\cos^2\left(\frac{A}{2}\right) - 1 \right] \end{aligned}$$

we get

$$\left.\begin{array}{l} m = \frac{1}{(b+c)}\sqrt{bc[(b+c)^2 - a^2} \\ n = \frac{1}{(c+a)}\sqrt{ca[(c+a)^2 - b^2]} \\ p = \frac{1}{(a+b)}\sqrt{ab[(a+b)^2 - c^2]} \end{array}\right\} \qquad (8.1.1)$$

This can be equivalently written as

$$\begin{aligned} bc[(b+c)^2 - a^2] - (b+c)^2 m^2 = 0 \\ ca[(c+a)^2 - b^2] - (c+a)^2 n^2 = 0 \\ ab[(a+b)^2 - c^2] - (a+b)^2 p^2 = 0 \end{aligned}$$

Thus the problem is reduced to solving for a, b, c in terms of m, n and p using if necessary elimination theory.

Mironescu and Panaitopol [16] viewed it as a problem in fixed point theory. This is described in the following theorem.

Theorem 8.1.10 (Mironescu and Panaitopol [16]) *The problem of finding the triangle, given the lengths of internal angle bisectors is equivalent to a problem of finding the fixed point of a suitable mapping.*

Proof The set of Eq. (8.1.1) can be rewritten suitably. For instance the first equation in (8.1.1) is

$$bc[(b+c)^2 - a^2] - (b+c)^2 m^2 = 0$$

It can be written as

$$4m^2 = \frac{(b+c)^2 - (b-c)^2}{(b+c)^2}[(b+c)^2 - a^2]$$

or

$$4m^2 = (b+c)^2 + \frac{(b-c)^2 a^2}{(b+c)^2} - [a^2 - (b-c)^2]$$

$$= \left[(b+c) \pm \frac{(b-c)a}{(b+c)} \right]^2 - [a \pm (b-c)]^2$$

Eliminating $\frac{(b-c)a}{(b+c)}$ between those two equations one gets

$$2(b+c) = \sqrt{4m^2 + (c+a-b)^2} + \sqrt{4m^2 + (a+b-c)^2}$$

Define x, y, z by

$$a = y + z, \quad b = z + x, \quad c = x + y$$

whence

$$x = \frac{b+c-a}{2}, \quad y = \frac{c+a-b}{2}, \quad z = \frac{a+b-c}{2}$$

Using these in the above expression for m we get

$$x = \frac{1}{2}[\sqrt{m^2 + y^2} - y] + \frac{1}{2}[\sqrt{m^2 + z^2} - z]$$
$$y = \frac{1}{2}[\sqrt{n^2 + z^2} - z] + \frac{1}{2}[\sqrt{n^2 + x^2} - x]$$
$$z = \frac{1}{2}[\sqrt{p^2 + x^2} - x] + \frac{1}{2}[\sqrt{p^2 + y^2} - y]$$

Since $m, n, p > 0$, $x, y, z > 0$. Further $x < m$, $y < n$ and $z < p$. Thus the map $F : K \to K$ where $K = [0, m] \times [0, n] \times [0, p]$ defined by $F(x, y, z) = (f_m(y) + f_m(z), f_n(z) + f_n(z), f_p(x) + f_p(y))$ where $f_\alpha(t) = \frac{1}{2}[\sqrt{\alpha^2 + t^2} - t]$ on $[0, \alpha]$ for $\alpha > 0$ has a fixed point if and only if the system of Eq. (8.1.1) has a solution. a, b, c in terms of m, n and p. □

Remark 8.1.11 Mironescu and Panaitopol invoked Brouwer's fixed point theorem 9.1 - to conclude that F has a fixed point which is the solution to the three internal angle bisectors problem.

On the other hand Dinca and Mawhin [4] deduced it from the contraction principle though it can also be obtained from Edelstein's theorem (Corollary 8.1.4).

Theorem 8.1.12 (Dinca and Mawhin [4]) *The map $F : K \to K$ defined by $F(x, y, z) = (f_m(y) + f_m(z), f_n(z) + f_n(x), f_p(x) + f_p(y))$ where $K = [0, m] \times [0, n] \times [0, p]$ and $f_\alpha(t) = \frac{1}{2}[\sqrt{\alpha^2 + t^2} - t]$ is contractive and has a unique fixed point and every sequence of F-iterates converges to the unique solution of the three internal angle bisectors problem.*

Proof $f_\alpha(t) = \frac{1}{2}[\sqrt{\alpha^2 + t^2} - t]$ is continuous and $f'_\alpha(t) = \frac{1}{2}\left[\frac{t}{\sqrt{\alpha^2+t^2}} - 1\right]$ is negative. Further, $|f'_\alpha(t)| < \frac{1}{2}$. Now K is compact in $\mathbb{R}^3$ with the norm $\|(x, y, z)\| = \max\{|x|, |y|, |z|\}$. Further, $|f_\alpha(t_1) - f_\alpha(t_2)| < \frac{1}{2}|t_1 - t_2|$ for $t_1 \neq t_2$. So for $(x, y, z) \neq (x_1, y_1, z_1)$ $\quad \|F(x, y, z) - F(x_1, y_1, z_2)\| < \max\{|x - x_1|, |y - y_1|, |z - z_1|\}$.

Further as $f_\alpha(t) < \frac{\alpha}{2}$ and F maps K into itself. Thus F is a contractive self-map on the compact space K. So by Edelstein's theorem (Corollary 8.1.4) F has a unique fixed point to which every sequence of F iterates converges. In other words there is a unique triangle up to isometry having with prescribed lengths of internal angle bisectors. □

Remark 8.1.13 Dinca and Mawhin [4] proved Theorem 8.1.12 applying the contraction principle to a sequence of contractions converging pointwise to F.

8.2 Non-expansive Maps

In this section, some elementary results on the fixed points of non-expansive maps are discussed.

Definition 8.2.1 A map $T : X \to X$, where (X, d) is a metric space is called non-expansive if $d(Tx, Ty) \le d(x, y)$ for all $x, y \in X$. T is called an isometry if $d(Tx, Ty) = d(x, y)$ for all $x, y \in X$.

Remark 8.2.2 A non-expansive mapping on a complete metric space may not have a fixed point as is evident by considering the map $x \to x + a$, $a \neq 0$ on $\mathbb{R}$. Indeed themap $e^{it} \to e^{i(t+\alpha)}$ $t \in [0, 2\pi)$ where $\alpha \in (0, 2\pi)$ in the unit circle in the complex plane is an isometry without a fixed point on the compact connected locally connected space S^1.

For $x = (x_n)$ in the closed unit sphere of c_0, the Banach space of all null sequences $x \to Tx = (1, x_1, x_2, \dots)$, $x = (x_n)$ is a fixed point free isometry.

Dotson Jr [5] proved a fixed point theorem for non-expansive maps in the setting of star-shaped subsets of normed linear spaces.

Definition 8.2.3 A non-empty subset S of a linear space X is called star-shaped if there exists an element $a \in S$ such that $ta + (1-t)s \in S$ for all $s \in \mathcal{S}$ for all $t \in [0, 1]$. In this case a is called a star-centre of S.

Remark 8.2.4 While all convex sets are star-shaped about every one of its points a star-shaped subset may not be convex as in evident by considering $S = \{0\} \times [0, 1] \cup [0, 1] \times \{0\}$ in $\mathbb{R}^2$.

Theorem 8.2.5 (Dotson [5]) *Let S be a compact star-shaped subset of a normed linear space and $T : S \to S$, a non-expansive map. Then T has a fixed point in S.*

Proof Without loss of generality we can assume that S is star-shaped about 0 (i.e. 0 is a star-centre of S). Then the map T_n defined by $T_n(x) = (1 - \frac{1}{n})Tx$ $(= \frac{1}{n}.0 + (1 - \frac{1}{n})Tx)$ maps S into itself for each $n \in \mathbb{N}$. As T_n is a contraction on S and S being compact is complete, T_n has a unique fixed point x_n, say. So $T_n(x_n) = (1 - \frac{1}{n})Tx_n = x_n$ for each $n \in \mathbb{N}$. As S is compact and $x_n \in S$ for each $n \in \mathbb{N}$, there is a subsequence

$\{x_{n_k}\}$ of $\{x_n\}$ converging to $x^* \in S$. Since T is continuous, $x_{n_k}(= (1 - \frac{1}{n_k})Tx_{n_k})$ converges to x^*, $Tx_{n_k} = (1 - \frac{1}{n_k})^{-1}x_{n_k}$ converges to $Tx^* = x^*$. Thus x^* is a fixed point of T. □

Corollary 8.2.6 *Every non-expansive self-map on a compact convex subset of a normed linear space has a fixed point.*

In this context an application to approximation theory is described, based on a few concepts and results.

Definition 8.2.7 Let (X, d) be a metric and S, a non-empty subset and $x_0 \in X - S$. If there exists an element $s_0 \in S$ such that $d(x_0, s_0) = \inf\limits_{s \in S} d(x_0, s)$, then s_0 is called a best approximation to x_0 in S.

While every element of $X - S$ of a metric space X need not have a best approximation in $S \subseteq X$, the following proposition gives sufficient conditions for the existence of a best approximation.

Theorem 8.2.8 *Let V be a finite-dimensional subspace of a normed linear space $(X, \|\cdot\|)$ and $f \in X - V$. Then there exists $v_0 \in V$ such that $\|f - v_0\| = \inf\{\|f - v\| : v \in V\}$.*

Proof Since V is a finite-dimensional subspace of X, V is complete and hence closed in X. So $\inf\{\|f - v\| : v \in V\} = d(f, V) > 0$. If $v \in V$ and $\|v\| > \|f\|$ then $d(f, V) \le \|f - 0\| = \|f\| < \|v\|$. So if f has a best approximation, in V, then the best approximation lies in $B(0; \|f\|) \cap V$, the closed sphere centred in 0 of radius $\|f\|$ lying in V. Now $B = B(0; \|f\|) \cap V = \{v \in V : \|v\| \le \|f\|\}$ is the closed sphere centred at 0 and of radius $\|f\|$ in the finite-dimensional space V. So B is compact and so $\inf\{\|f - v\| : v \in B\}$ is attained at some v_0. Thus $\|f - v_0\| = \inf_B\{\|f - v\|\} = \inf_{v \in V} \|f - v\|$. So f has a best approximation in V. □

Theorem 8.2.9 (Invariants of best approximations) *Let X be a normed linear space, V a finite-dimensional subspace of X and $T : X \to X$ a map with a fixed point f such that for all $x, y \in V$, $\|x - y\| \le d(f, V)$ implies $\|Tx - Ty\| \le \|x - y\|$, $d(f, V)$ being* $\inf\{d(f, v) : v \in V\}$. *If T maps V into itself then f has a best approximation in V which is a fixed point of T.*

Proof Since V is finite-dimensional S_f the set of best approximations of f in V is non-empty by Theorem 8.2.8. As $T(V) \subset V$ and $S_f \subseteq V$, $T(S_f) \subseteq V$. Since $Tf = f$, $\|Tg - f\| = \|Tg - Tf\| \le \|g - f\|$ for all $g \in S_f$. So T maps S_f into itself. For $g_1, g_2 \in S_f$, $\|f - tg_1 + (1-t)g_2\| = \|tf - tg_1 + (1-t)f - (1-t)g_2\| \le t\|f - g_1\| + (1-t)\|f - g_2\| \le d(f, V)$. Since S_f is a closed subset of $B(0, \|f\|) \cap V$ and V is finite dimensional, S_f is a compact convex subset of V. As T is a non-expansive self-map on S_f, T has a fixed point f_0 in S_f. Thus f_0 is a best approximation of f which is a fixed point of T. □

Corollary 8.2.10 *If $T : X \to X$ be a non-expansive map with a fixed point f and leaving a finite dimensional subspace V of E invariant, then f has a best approximation in V.*

Corollary 8.2.11 (Meinardus [14]) *Let $T : B \to B$ be a continuous map where B is a compact metric space and $C[B]$ the space of all continuous real (or complex) functions on B with the supremum norm. Let $A : C[B] \to C[B]$ be a non-expansive map and suppose that*

(i) $A(f(T(x))) = f(x)$;
(ii) $A(h(T(x))) \in V$, *whenever* $h(x) \in V$ *where* V *is a finite dimensional subspace of* $C[B]$. *Then there is a best approximation* g *of* f *with respect to* V *such that* $A(g(T(x))) = g(x)$.

This corollary follows from Corollary 8.2.10 upon setting $T_1 : C[B] \to C[B]$ by $T_1(g(x)) = A(g(T(x)))$.

Corollary 8.2.12 *Let f be an even (odd) function in $C[-1, 1]$ with the supremum norm. If V is a finite dimensional subspace of $C[-1, 1]$ such that whenever $h(x) \in V$, $h(-x)$ also is in V, then f has an even (odd) function as best approximation in V.*

8.3 Browder–Gohde–Kirk Fixed Point Theorem

While non-expansive self-maps on bounded closed convex sets in infinite-dimensional Banach spaces may not have fixed points, fixed points for non-expansive mappings can be ensured in Banach spaces with nice geometrical features. Browder [2], Gohde [8] and Kirk [11] independently proved that non-expansive self-maps on non-empty bounded closed convex subsets of a uniformly convex Banach space have fixed points. Indeed Kirk [11] using the concept of normal structure proved a more general fixed point theorem. The study of fixed points of non-expansive mappings on subsets of Banach spaces is an active area of research employing sophisticated analytic and geometric concepts. Goebel [7] gave a simpler proof of the Browder-Gohde-Kirk fixed point theorem in the setting of uniformly convex Banach spaces and this proof is presented below. The definition of uniform convexity (Definition 1.3.40) is repeated below for the sake of both completeness and convenience.

Definition 8.3.1 A Banach space is uniformly convex if there exists an increasing positive function $\delta : I_2 = (0, 2] \to I_1 = (0, 1]$ such that $\|x\|, \|y\| \leq r$ and $\|x - y\| \geq \epsilon\, r$ imply that $\frac{\|x+y\|}{2} \leq (1 - \delta(\epsilon))r$.

Remark 8.3.2 If η is the inverse function of δ, then $\lim_{y \to 0} \eta(y) = 0$. □

Theorem 8.3.3 (Browder-Gohde-Kirk) *Let K be a non-empty bounded closed convex subset of a uniformly convex Banach space X and $T : K \to K$ a non-expansive map. Then T has a fixed point in K.*

Goebel's proof makes use of the following

Lemma 8.3.4 *If u, v, w are elements of a uniformly Banach space B such that $\|u - w\| \le R$, $\|v - w\| \le R$ and $\left\|w - \frac{u+v}{2}\right\| \ge r > 0$, then $\|u - v\| \le R\eta\left(\frac{R-r}{R}\right)$.*

Proof $\left\|w - \frac{u+v}{2}\right\| \ge r$ means that $\left|\frac{(w-u)+(w-v)}{2}\right| \ge r = \left(1 - \frac{R-r}{R}\right) R$ and $R \ge r$ as $R \ge \left|\frac{w-u}{2} + \frac{w-v}{2}\right| \ge r$. So in the contrapositive version of the definition of uniform convexity setting $\epsilon = \eta\left(\frac{R-r}{R}\right)$ we get $\|u - v\| \le R\eta\left(\frac{R-r}{R}\right)$. □

Proof of Theorem 8.3.3 [7]

Without loss of generality we can assume that $0 \in K$. Define $F_\epsilon = (1-\epsilon)F$ for $\epsilon \in I_1 = (0, 1]$. Since F_ϵ is a contraction mapping K into itself for each $\epsilon \in I_1$, F_ϵ has a unique fixed point x_ϵ in K for each $\epsilon \in I_1$. Writing $d(K)$ = diameter of K, we note that $\|x_\epsilon - Fx_\epsilon\| = \|F_\epsilon x_\epsilon - Fx_\epsilon\| = \epsilon\|Fx_\epsilon\| \le \epsilon d(K)$. As $\epsilon \to 0$, $\|x_\epsilon - Fx_\epsilon\| \to 0$. So $\inf_K \|x - Fx\| = 0$. Define $C_\epsilon = \{x \in K : \|x - Fx\| \le \epsilon\}$ and $D_\epsilon = \{x \in C_\epsilon : \|x\| \le a + \epsilon\}$ where $a = \lim_{\epsilon\to 0} a(C_\epsilon)$ and $a(B) = \inf_{x\in B} \|x\|$.

We proceed to show that $\cap C_\epsilon$ is non-empty. Otherwise $a > 0$, since each C_ϵ is closed. Let $u_1, u_1 \in C_\epsilon$. For $i = 1, 2$,

$$\left\|u_i - F\left(\frac{u_1+u_2}{2}\right)\right\| \le \|u_i - F(u_i)\| + \left\|F(u_i) - F\left(\frac{u_1+u_2}{2}\right)\right\| \\ \le \epsilon + \frac{1}{2}\|u_1 - u_2\| \tag{8.3.1}$$

and

$$\left\|u_i - \frac{u_1+u_2}{2}\right\| = \left\|\frac{u_i - u_2}{2}\right\| < \epsilon + \frac{1}{2}\|u_1 - u_2\| \tag{8.3.2}$$

As

$$\|u_1 - u_2\| \le \left\|u_1 - \frac{1}{2}\left(\frac{u_1+u_2}{2} + F\left(\frac{u_1+u_2}{2}\right)\right)\right\| + \left\|u_2 - \frac{1}{2}\left(\frac{u_1+u_2}{2} + F\left(\frac{u_1+u_2}{2}\right)\right)\right\|,$$

the inequality

$$\left\|u_i - \frac{1}{2}\left(\frac{u_1+u_2}{2} + F\left(\frac{u_1+u_2}{2}\right)\right)\right\| \ge \frac{1}{2}\|u_1 - u_2\| \tag{8.3.3}$$

is true for at least one of $i = 1, 2$. From Lemma 8.3.4 and the inequalities (8.3.1), (8.3.2) and (8.3.3)

$$\begin{aligned}\left\|\frac{u_1+u_2}{2}-F\left(\frac{u_1+u_2}{2}\right)\right\| &\leq \left(\epsilon+\frac{1}{2}\|u_1-u_2\|\right)\eta\left(\frac{\epsilon}{\epsilon+\frac{\|u_1-u_2\|}{2}}\right)\\ &\leq \sup_{0<\xi<\frac{d(K)}{2}}(\xi+\epsilon)\eta\left(\frac{\epsilon}{\epsilon+\xi}\right)\\ &\leq \max\left[\sup_{0<\xi\leq\sqrt{\epsilon}-\epsilon}(\epsilon+\xi)\eta\left(\frac{\epsilon}{\epsilon+\xi}\right),\ \sup_{\sqrt{\epsilon}-\epsilon<\xi\leq\frac{d(K)}{2}}(\epsilon+\xi)\eta\left(\frac{\epsilon}{\epsilon+\xi}\right)\right]\\ &\leq \max\left[2\sqrt{\epsilon},\epsilon+\frac{d(K)}{2}\eta(\sqrt{\epsilon})\right]=\phi(\epsilon)\text{ (say).}\end{aligned}$$

So $u_1, u_2 \in C_\epsilon$ implies $\frac{u_1+u_2}{2} \in C_{\phi(\epsilon)}$. Clearly $\phi(\epsilon) \to 0$ as $\epsilon \to 0$.

For $u_1, u_2 \in D_\epsilon$, $\|v_i\| \leq a+\epsilon$ for $i = 1, 2$ and as $\frac{u_1+u_2}{2} \in C_{\phi(\epsilon)}$, $\left\|\frac{u_1+u_2}{2}\right\| \geq a(C_{\phi(\epsilon)})$

Now by Lemma 8.3.4

$$d(D_\epsilon) = \sup_{u_1,u_2\in D_\epsilon}\|u_1-u_2\| \leq (a+\epsilon)\eta\left(\frac{a+\epsilon-a(C_{\phi(\epsilon)})}{a+\epsilon}\right)$$

and $\lim_{\epsilon\to 0} d(D_\epsilon) = 0$. So by Cantor intersection theorem $\cap D_\epsilon \neq \phi$. So $\cap C_\epsilon \neq \phi$. Hence F has a fixed point in K.

As uniformly convex Banach spaces are reflexive and bounded closed convex subsets of a uniformly convex Banach space are weakly compact, it is natural to enquire if a weakly compact convex subset of a non-reflexive Banach space has fixed point property for non-expansive mappings. Alspach [1] has given a counter-example to this conjecture and it is summarized below.

Example 8.3.5 (*Alspach* [1]) Let X be the Banach space $L_1[0,1]$ and $K = \{f \in X : 0 \leq f \leq 2,$ are and $\int_0^1 f\,d\mu = 1\}$. Clearly K is convex and weakly closed in X. As order intervals in X are weakly compact in view of the uniform integrability of its elements, K is weakly compact. Define $T : K \to K$ by

$$Tf(t) = \begin{cases} 2f(2t) \vee 2, & 0 \leq t \leq \frac{1}{2} \\ [2f(2t-1)-2] \vee 0, & \frac{1}{2} < t \leq 1 \end{cases}$$

It can be seen that T is an isometry. If T has a fixed point g, then $g = 2\chi_A$ for some subset A of $[0, 1]$ with measure $\frac{1}{2}$. Now $\{t : g(t) = 2\} = \{t : Tg(t) = 2\} = \{\frac{t}{2} : g(t) = 2\} \cup \{\frac{1+t}{2} : g(t) = 2\} \cup \{\frac{t}{2} : 1 \leq g(t) < 2\}$. Here the union of these sets is a disjoint union. Since $\mu\{t : g(t) = 2\} + \mu\{\frac{1+t}{2} : g(t) = 2\} = \mu\{t : g(t) = 2\}$, $\mu\{t : 1 \leq g(t) < 2\} = 0$. Repeated use of this argument shows that

$$\{t : 0 < g(t) < 2\} = \bigcup_{n=0}^{\infty}\left\{t : \frac{1}{2^n} \leq g(t) < \frac{1}{2^{n-1}}\right\}$$

is of measure zero, as well.

For $g = 2\chi_A$

$$\{t : T^n g(t) = 2\} = \bigcup_{\epsilon_i \in \{0,1\}} \left\{ \sum_{i=1}^{n} \frac{\epsilon_i}{2^i} + \frac{t}{2^n} : t \in A \right\}$$

The above representation can be proved using induction. Since $Tg = g$, $A = \{t : T^n g(t) = 2\}$ for all natural numbers n and the intersection of A with any interval with dyadic end points has measure precisely half the measure of the interval. Since such a measurable set does not exist, T cannot have a fixed point in K.

For other examples Sims [23] may be consulted.

8.4 A Generalization to Metric Spaces

Pasicki [18] obtained an interesting generalization of Browder-Gohde-Kirk Theorem 8.3.3 to special metric spaces. In this section Pasicki's contributions are highlighted.

Definition 8.4.1 ([18]) Let (X, d) be a metric space and A a non-empty bounded subset. $x \in X$ is called central point for A if

$$\begin{aligned} r(a) &:= \inf\{t \in (0, \infty) : A \subseteq B(z, t) \text{ for some } z \in X\} \\ &= \inf\{t \in (0, \infty) : A \subseteq B(x, t)\} \end{aligned}$$

The centre $c(A)$ for A is the set of all central points for A and $r(A)$ is the radius of A.

Proposition 8.4.2 ([18]) *For a metric space (X, d) satisfying for each $r > 0$ and $x, y \in X$ with $x \neq y$ there exists $\delta > 0$, $z \in X$ such that $B(x, r) \cap B(y, r) \subseteq B(z, r - \delta)$ given a bounded non-void subset A of X, $c(A)$ contains at most one point.*

Proof If $x, y \in c(A)$ and $x \neq y$, then for $r = r(A)$, $A \subseteq \overline{B}(x, r) \cap \overline{B}(y, r) \subseteq \overline{B}(z, r - \delta)$. This implies $r(A) = r \leq r - \delta$, a contradiction. □

A centre of a set corresponds to Chebyshev centre of a set.

For developing fixed point theory Pasicki [18] introduced the following definition.

Definition 8.4.3 ([18]) Let Y be a non-void bounded subset of a metric space (X, d) and $F : Y \to 2^Y$, a mapping with $F(y) \neq \phi$ for each $y \in Y$. $x \in X$ is called a central point for F if

$$\begin{aligned} r(F) &:= \inf\{t \in (0, \infty) : F^n(y) \subseteq B(z, t) \text{ for a} \\ &\qquad z \in X \text{ and an } n \in N\} \\ &= \inf\{t \in (0, \infty) : F^n(Y) \subseteq B(x, t) \text{ for an } n \in N\} \end{aligned}$$

The centre $c(F)$ for F is the set of central points for F and $r(F)$ is the radius of F.

It may be noted that if $F^n(y) \subseteq B(z,t)$ then $F^p(Y) \subseteq F^n(Y) \subseteq B(z,t)$ for all $p \geq n$.

Theorem 8.4.4 ([18]) *Let Y be a non-void bounded subset of a metric space (X,d) and $f : X \to X$ a map such that $f(Y) \subseteq Y$ and $c(f|_Y) = \{x\}$. If $d(f(x), f(y)) \leq d(x,y)$ for all $x, y \in Y$, then x is a fixed point for f.*

Proof For $f^{n-1}(Y) \subseteq B(x,t)$, $f^n(Y \cap B(x,t)) \subseteq B(f(x),t)$. As f is non-expansive $f(Y, B(x,t)) \subseteq B(f(x),t)$ and so $f^n(Y) \subseteq B(f(x),t)$ implying that $f(x) \in c(f|_Y)$. As $c(f|_Y)$ is a singleton, $f(x) = x$. □

The above theorem leads to the formulation of a bead space and a discus space.

Definition 8.4.5 ([18]) A metric space is called a bead space if the following condition holds:

for every $r, \beta > 0$, there exists a $\delta > 0$ such that for each pair $x, y \in X$ with $d(x,y) \geq \beta$, there exists a $z \in X$ such that $B(x, r+\delta) \cap B(y, r+\delta) \subseteq B(z, r-\delta)$.

Definition 8.4.6 ([17, 18]) A metric space is called a discus space if there exists a map $\rho : [0,\infty) \times (0,\infty) \to [0,\infty)$ such that

$$\rho(\beta, r) < \rho(0,r) = r, \;\; \beta, r > 0, \tag{8.4.1}$$

$$\rho(\cdot, r) \text{ is non-increasing}, \;\; r > 0, \tag{8.4.2}$$

$$\rho(\delta, \cdot) \text{ is upper semicontinuous for } \delta \geq 0, \tag{8.4.3}$$

$$\begin{aligned}&\text{for each pair } x, y \in X, r, \epsilon > 0 \text{ there exists}\\ &z \in X \text{ such that } \; B(x,r) \cap B(y,r) \subseteq B(z, \rho(d(x,y), r) + \epsilon).\end{aligned} \tag{8.4.4}$$

Proposition 8.4.7 ([18]) *Each discus space is a bead space.*

Proof Let $r > 0$, $x, y \in X$ and $d(x,y) = \beta > 0$ be arbitrary. In view of 8.4.6 (iv) for each $\epsilon, k > 0$, there exists $z \in X$ such that $B(x, r+k) \cap B(y, r+k) \subseteq B(z, \rho(\beta, r+k) + \epsilon)$. Writing $2\eta = r - \rho(\beta, r) = \rho(0,r) - \rho(\beta, r) > 0$ (by (i)). For sufficiently small k, ϵ, we have $\rho(\beta, r+k) + \epsilon \leq \rho(\beta, r) + \eta$ (by (iii)). So one gets $\rho(\beta, r+k) + \epsilon \leq \rho(\beta, r) + \eta = r - 2\eta + \eta = r - \eta$ and $B(x, r+k) \cap B(y, r+k) \subseteq B(z, r-\eta)$. Then for $\delta = \min\{k, \eta\}$, $B(x, r+\delta) \cap B(\eta, r+\delta) \subseteq B(z, r-\delta)$. If $d(x,y) \geq \beta$, then by (ii) $\rho(d(x,y), r+k) \leq \rho(\beta, r+k)$ and the inclusion follows. □

In fact a bead space is also a discus space as noted by Pasicki [19].

Proposition 8.4.8 ([19]) *Each bead space is a discus space.*

Proof Let (X, d) be a bead space. For $\beta, r > 0$ let $\eta(0, r) = 0$ and $\eta(\beta, r) = \sup\{\delta \in (0, r) :$ for each $x, y \in X$ with $d(x, y) \geq \beta$ there exists $z \in X$ with $B(x, r + \delta) \cap B(y, r + \delta) \subseteq B(z, r - \delta)\}$.

Consider $\rho(\beta, r) = r - \eta(\beta, r)$. Clearly ρ maps $[0, \infty) \times (0, \infty)$ into $[0, \infty)$. For $\beta, r > 0$, $\eta(\beta, r) > 0$ and $\rho(\beta, r) < \rho(0, r) = r$. We now show that $\eta(\cdot, r)$ is non-decreasing for $r > 0$. If for $r, \beta > 0$, $\delta > 0$ is such that for $d(x, y) \geq \beta$ $B(x, r + \delta) \cap B(y, r + \delta) \subseteq B(z, r - \delta)$, then it works for $\beta_1 > \beta$ so that $\eta(\beta, r) \leq \eta(\beta_1, r)$. Thus $\eta(\cdot, r)$ is non-decreasing and $\rho(\cdot, r)$ is non-increasing. We now show that $\eta(\beta, \cdot)$ is lower semicontinuous, $\beta \geq 0$. Now $\eta(0, \cdot) = 0$ and consider $\beta > 0$. Suppose $\eta(\beta, x_0) > \alpha > 0$. We show that $\eta(\beta, (r_0 - \epsilon, r_0 + \epsilon)) \subseteq (\alpha, \infty)$ for any $\epsilon > 0$. Since $\eta(\beta, r_0) > r$, there exists $\delta > \alpha$ such that $B(x, r_0 + \delta) \cap B(y, r_0 + \delta) \subset B(z, r_0 - \delta)$ for a $z \in X$. Let $\epsilon > 0$ be such that $r_0 + \delta - 2\epsilon > 0$ and $\delta > \epsilon$. Then $B(x, r_0 - \epsilon + (\delta - \epsilon)) \cap B(y, r_0 - \epsilon + (\delta - \epsilon)) \subset B(x, r + \delta) \cap B(y, r_0 + \delta) \subset B(z, r_0 - \delta) = B(z, r_0 - \epsilon - (\delta - \epsilon))$. So, for $\delta > 2\epsilon$ and $\delta - \epsilon > \alpha$, $\eta(\beta, (r - \epsilon, r_0]) \subseteq (\alpha, \infty)$. However for $\delta > \epsilon + \alpha$, we have

$$\begin{aligned} B(x, r_0 + \epsilon + (\delta - \epsilon)) \cap B(y, x + \epsilon + (\delta - \epsilon)) \\ \subseteq B(x, r_0 + \delta) \cap B(y, r_0 + \delta) \subseteq B(z, r_0 - \delta) \\ = B(z, r_0 + \epsilon - (\delta + \epsilon)) \subseteq B(z, r_0 + \epsilon - (\delta - \epsilon)) \end{aligned}$$

So $\eta(\beta, [r_0, r_0 + \epsilon)) \subseteq (\alpha, \infty)$. Finally one has $\eta(\beta, (r_0 - \epsilon, r_0 + \epsilon)) \subseteq (\alpha, \infty)$ and $\eta(\beta, \cdot)$ is lower semicontinuous. Also $B(x, r) \cap B(y, r) \subseteq B(x, r + \delta) \cap B(y, r + \delta) \subset B(z, r - \delta) \subseteq B(z, r - \delta + \epsilon)$ and so $B(x, r) \cap B(y, r) \subseteq B(z, \rho(d(x, y), r) + \epsilon)$. Thus (X, d) is a discus space. □

Remark 8.4.9 If (X, d) is a metric space and $r > 2\epsilon > 0$, then $B(x, r - 2\epsilon) \subseteq \overline{B}(x, r - \epsilon) \subset B(x, r)$

So the definitions of bead and discus spaces can be formulated using closed balls.

We need the following lemma to decode the geometry of normed bead spaces.

Lemma 8.4.10 ([19]) *In a normed linear space $(X, \|\|)$ the following conditions are equivalent:*

(i) for every $r, \beta > 0$, there exists $\delta > 0$ such that for $x, y \in X$ with $\|x - y\| \geq \beta$, there exists $z \in X$ with $B(x, r + \delta) \cap B(y, r + \delta) \subseteq B(z, r - \delta)$;
(ii) for every $r, \beta > 0$, there exists $\delta > 0$ such that for $x, y \in X$ with $\|x - y\| \geq \beta$, there exists $z \in X$ such that $B(x, r + \delta) \cap B(y, r + \delta) \subseteq B(z, r)$;
(iii) for every $r, \beta > 0$, there exists $\delta > 0$ such that for $x \in X$ and $2\|x\| \geq \beta$ imply $B(-x, r) \cap B(x, r) \subseteq B(0, r - \delta)$.

Proof It is enough to prove the equivalence for $y = -x$. The set $C = B(-x, r) \cap B(x, r)$ is symmetric and therefore $C \subseteq B(z, t)$ implies $C \subseteq B(-z, t) \cap B(z, t) \subseteq B(0, t)$. Thus we may let $z = \frac{x+y}{2} = 0$ in (i), (ii) and (iii). Clearly (i) implies (ii) and (iii). Choosing $r = 1$ in (iii) we get for $\beta > 0$ there exists $\delta > 0$ such that for $u, x \in X$ with $\|u + x\|, \|u - x\| < 1$ and $2\|x\| \geq \beta > 0$

(iv) $\|u\| < 1 - \delta$

Suppose $\|u + x\|, \|u - x\| < r + \epsilon$ and $2\|x\| \geq (r + \epsilon)\beta$ hold. Then $\frac{\|u+x\|}{r+\epsilon}$, $\frac{\|u-x\|}{r+\epsilon} < 1$ and $\frac{2\|x\|}{r+\epsilon} \geq \beta$ and in view of (iv) $\frac{\|u\|}{r+\epsilon} < 1 - \delta$ or $\|u\| < (r + \epsilon)(1 - \delta) < r\left(1 - \frac{\delta}{2}\right)$ for small values of ϵ. For $y = -x$, $\delta_1 = \min\left\{\epsilon, \frac{r\delta}{2}\right\}$, and $2\|x\| \geq \beta_1 = \frac{3r\beta}{2} > (r + \epsilon)\beta$, we get $B(x, r + \delta_1) \cap B(y, r + \delta_1) \subseteq B(0, r - \delta_1)$ which is (i). Thus (i), (iii) and (iv) are equivalent. Now consider (ii). Then $\frac{\|u+x\|}{(r-\epsilon)}, \frac{\|u-x\|}{(r-\epsilon)} < 1 + \delta$ and $\frac{2\|x\|}{(r-\epsilon)} \geq \beta$ imply $\frac{\|u\|}{r-\epsilon} < 1$. (see (ii)). This relation can be written as $\|u + x\|, \|u - x\| < (r - \epsilon)(1 + \delta) = r + r\delta - \epsilon(1 + \delta)$ and $2\|x\| \geq r\beta > (r - \epsilon)\beta$ implying $\|u\| < r - \epsilon$. Thus for $\epsilon < r\delta - \epsilon(1 + \delta)$ or $\epsilon < \frac{r\delta}{2+\delta}$, $\|u + x\|, \|u - x\| < r + \epsilon$ and $2\|x\| \geq r\beta$ implying $\|u\| < r - \epsilon$, verifying (i). □

Theorem 8.4.11 ([19]) *A normed linear space is uniformly convex if and only if for each $\beta > 0$ there exists $\delta > 0$ such that for $u, x \in X$, $\|u + x\|, \|u - x\| < 1$ and $2\|x\| > \beta > 0$ imply*

(v) $\|u\| < 1 - \delta$.

Proof From (iv) of Lemma 8.4.10 setting $u = \frac{y+z}{2}$, $x = \frac{y-z}{2}$ one gets

for every $\beta > 0$, there exists $\delta > 0$ such that $y, z \in X$, $\|y\|, \|z\| < 1$ and $\|y - z\| > \beta > 0$ imply $\left\|\frac{y+z}{2}\right\| < 1 - \delta$. So X is uniformly convex.

If (v) is satisfied then writing $y = u + x$, $z = u - x$, then (iv) of Definition 8.3.1 is satisfied leading to uniformly convexity of X. □

We now have

Theorem 8.4.12 ([19]) *A normed linear $(X, \|\cdot\|)$ is uniformly convex if and only if any one of the conditions (i), (ii), (iii), (iv) of Lemma 8.4.10 and (v) of Theorem 8.4.11 are satisfied.*

From Propositions 8.4.7, 8.4.8 and Theorem 8.4.11 we have

Theorem 8.4.13 ([19]) *For any normed linear space the following conditions are equivalent:*

(i) X is a bead space;
(ii) X is a discus space;
(iii) X is a uniformly convex space.

Remark 8.4.14 ([19]) Each convex subset X of a uniformly convex normed linear space satisfies the definition of a bead space with $z = \frac{x+y}{2}$ (see Definition 8.4.5).

We now proceed to provide Pasicki's generalization of Browder-Gohde-Kirk Theorem 8.3.3 based on the following lemmata.

Lemma 8.4.15 *If (X, d) is a complete discus space, then (iv) of Definition 8.4.6 (of Discus space) can be replaced by*

for each $x, y \in X$ and $r > 0$ there is a $z \in X$ such that $B(x, r) \cap B(y, r) \subseteq B(z, \rho(d(x, y), r))$.

Proof Let $\alpha = \rho(d(x, y), r)$ and $(\alpha_n) \downarrow \alpha$ be such that there exist $x_n \in B(x, r) \cap B(y, r) \subseteq B(x_n, \alpha_n)$. Suppose (x_n) is not Cauchy. Then there is $\beta > 0$ with $d(x_n, x_k) \subset \beta$ for infinitely many k, n with $k < n$. Set $2\eta = \alpha - \rho(\beta, \alpha) = \rho(0, \alpha) - \rho(\beta, \alpha) > 0$ (by (i) of Definition 8.4.6). So $B(x, r) \cap B(y, r) \subseteq B(x_n, \alpha_n) \cap B(x_k, \alpha_k) \subset B(x_n, \alpha_k) \cap B(x_k, \alpha_k) \subset B(z_{n,k}, \rho(d(x_n, x_k), \alpha_k) + \eta)$ for some $z_{n,k} \in X$ (by (iv) of Definition 8.4.6). On the other hand, $\rho(d(x_n, x_k), \alpha_k) \leq \rho(\beta, \alpha_k)$ (by ii) and $\rho(\beta, \alpha_k) \leq \rho(\beta, \alpha) + \eta$ for sufficiently large k (by (iii) of Definition 8.4.6). Now one gets $B(x, r) \cap B(y, r) \subset B(z_{n,k}, \rho(\beta, \alpha) + \eta) = B(z_{n,k}, \alpha - 2\eta + \eta) = B(z_{n,k}, \alpha - \eta) \subset B(z_{n,k}, \alpha)$. Thus the lemma is true. If (x_n) is a Cauchy sequence convergent to $z \in X$, then $B(x_n, \alpha_n) \subset B(z, \alpha + \beta)$ for any $\beta > 0$ and all large n. So $B(x, r) \cap B(y, r) \subseteq B(z, \alpha + \beta)$ for all $\beta > 0$ and $B(x, r) \cap B(y, r) \subseteq \overline{B}(z, \alpha)$. Since $B(x, r) \cap B(y, r)$ is open, the lemma follows. □

Lemma 8.4.16 *Let (X, d) be a complete discus space and let $A \subseteq X$ be non-void and bounded then $c(A)$ is a singleton.*

Proof Let $(r_n) \downarrow r = r(A)$ while $A \subseteq B(x_n, r_n)$. If (x_n) is not Cauchy, then $d(x_n, x_k) \geq \beta > 0$ for infinitely many k, n with $k < n$. We have

$$\begin{aligned} A &\subseteq B(x_n, r_n) \cap B(x_k, r_k) \subset B(x_n, r_k) \cap B(x_k, r_k) \\ &\subseteq B(z_{n,k}, \rho(d(x_n, x_k), r_k)) \subseteq B(z_{n,k}, \rho(\beta_k r_k)) \\ &\qquad\qquad\qquad\qquad \text{(by definition of } \rho). \end{aligned}$$

So $A \subseteq B(z_{n,k}, r(A) - \eta$ (η being $\frac{1}{2}(\alpha - \rho(\beta, \alpha))$) as proceed in the course of the preceding Lemma 8.4.15. This is a contradiction. Let (x_n) converge to x. Then for any $\beta > 0$ $B(x_n, r_n) \subseteq B(x, r + \beta)$ for sufficiently large n implying $A \subseteq B(x, r + \beta)$ for all $\beta > 0$ and so $x \in c(A)$. If $x, y \in c(A)$ and $d(x, y) \geq \beta > 0$, then by Lemma 8.4.15 $A \subseteq \overline{B}(x, r) \cap \overline{B}(y, r) \subseteq \overline{B}(z, \rho(\beta, r)) \subseteq \overline{B}(z, r - \eta)$ for a $\eta > 0$, a contradiction. So $c(A)$ is a singleton. □

Lemma 8.4.17 ([19]) *Let (X, d) be a complete discus space. If $Y \neq \phi \subseteq X$ is bounded and $F : Y \to 2^Y$ is a mapping, then $c(F)$ is a singleton.*

Proof Let $r = r(F)$ (refer Definition 8.4.3 for $r(F)$ and $c(F)$). $F^{n+1}(Y) \subseteq F^n(Y)$ and so there exists $r_n \downarrow r$ and a sequence (x_n) such that $F^n(y) \subseteq B(x_n, r_n)$ for all n. If (x_n) is not a Cauchy sequence, for infinitely many n, k with $k < n$ $d(x_n, x_k) \geq \beta > 0$. We have $F^n(Y) \subseteq F^n(Y) \cap F^k(Y) \subset B(x_n, r_n) \cap B(x_k, r_k) \subseteq B(z_{n,k}, \rho(\beta, r_k))$ and so $F^n(Y) \subseteq B(z_{n,k}, r - \eta)$ for a $\eta > 0$ (as in the preceding proof), a contradiction. Let (x_n) converge to x. We get $F^n(Y) \subseteq B(x, r + \beta)$ for any $\beta > 0$ and sufficiently large n. So $x \in c(F)$. That $c(F)$ is a singleton follows from Lemma 8.4.16. □

Theorem 8.4.4 can be modified as

Theorem 8.4.18 *Let (X, d) be a complete bead (discus) metric space and $f : X \to X$ a non-expansive map. If Y is a bounded non-empty subset of X with $f(Y) \subseteq Y$, then $c(f|_Y)$ is a singleton which is a fixed point of f.*

8.5 An Application to a Functional Equation

In this section, Matkowski's deduction [13] of a fixed point theorem from Pasicki's Theorem 8.4.4 is discussed as well as its application to a functional equation. It is in the setting of a paranormed space.

Definition 8.5.1 Let X be a real linear space over $\mathbb{R}$. A function $p : X \to \mathbb{R}$ is called a paranorm if the following conditions are satisfied:

(i) $p(x) = 0$ if and only if $x = 0$, $p(x) = p(-x)$ for all x;
(ii) $p(x + y) \le p(x) + p(y)$ for all $x, y \in X$;
(iii) for $t_n, t \in \mathbb{R}$ and $x_n, x \in X$ with $t_n \to t$ and $p(x_n - x) \to 0$ as $n \to \infty$, $p(t_n x_n - tx)$ converges to zero as $n \to \infty$.

Remark 8.5.2 If (X, p) is a paranormed linear space, then $d(x, y) = p(x - y)$ defines a metric on X.

Definition 8.5.3 A paranormed space (X, p) is called uniformly convex if for each $r > 0$ and $\epsilon \in (0, 2r)$ there exists $\delta(r, \epsilon) \in (0, r)$ such that for all $x, y \in X$ with $p(x), p(y) \le r$ and $p(x, y) \ge \epsilon$, $p(\frac{x+y}{2}) \le r - \delta(r, \epsilon)$ (the function $\delta : \Delta \to (0, \infty)$ where $\Delta = \{(r, \epsilon) : r > 0, 0 < \epsilon < 2r\}$ is called the modulus of convexity of (X, p).

The following theorem due to Matkowski [13] provides a class of examples of paranormed spaces.

Theorem 8.5.4 *Let $(\Omega, \mathscr{S}, \mu)$ be a measure space and $S = S(\Omega, \mathscr{S}, \mu)$ the real linear space of all μ-integrable simple functions $x : \Omega \to R$. Let $\varphi : [0, \infty) \to [0, \infty)$ be an increasing bijection with $\varphi(0) = 0$. Then the functional $p_\varphi(x) = \varphi^{-1} \int_\Omega \varphi(|x|)d\mu$ is well-defined for each $x \in S$. If $\mu(\Omega) = 1$ and if there exists a set $A \in \mathscr{S}$ with $0 < \mu(A) < 1$, then p_φ is a paranorm if and only if $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ defined by $F(r, s) = \varphi(\varphi^{-1}(r) + \varphi^{-1}(s))$ $r, z \in \mathbb{R}^+$ is concave.*

If $\mu(\Omega) \le 1$ and F is concave, then p_φ is a paranorm on S.

Remark 8.5.5 Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be twice differentiable with $\varphi(0) = 0, \varphi'(r), \varphi''(r) > 0$ for $r > 0$. If $\frac{\varphi'}{\varphi''}$ is super additive in $(0, \infty)$ in the sense that $\frac{\varphi'(r+s)}{\varphi''(r+s)} \ge \frac{\varphi'(r)}{\varphi''(r)} + \frac{\varphi'(s)}{\varphi''(s)}$, $r, s > 0$ then $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ defined by $F(r, s) = \varphi(\varphi^{-1}(r) + \varphi^{-1}(s))$ is concave.

For the proofs of the following lemmata Matkowski [13] may be consulted.

Lemma 8.5.6 ([13]) *Let $(\Omega, \mathscr{S}, \mu)$ be a measure space with $\mu(\Omega) \le 1$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing bijection with $\varphi(0) = 0$ and $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $F(r, s) = \varphi(\varphi^{-1}(r) + \varphi^{-1}(s))$, $r, s \ge 0$. If F is concave, then p_φ defined (in Theorem 8.5.4) is a paranorm on $S = S(\Omega, \mathscr{S}, \mu)$. If further $\varphi(r + s) + \varphi(|r - s|) \ge 2[\varphi(r) + \varphi(s)]$ for all $r, s \ge 0$ (i.e. φ is super quadratic), then for $x, y \in S$*

$$\varphi(p_\varphi(x + y)) + \varphi(p_\varphi(x - y)) \ge 2[\varphi(p_\varphi(x)) + \varphi(p_\varphi(y))].$$

Lemma 8.5.7 ([13]) *Let $(\Omega, \mathscr{S}, \mu)$ be a measure space and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ an increasing bijection such that p_φ is a paranorm on S. If φ is super-quadratic (i.e. $\varphi(r+s) + \varphi(|r-s|) \geq 2[\varphi(r) + \varphi(s)]$ for $r, s \geq 0$), $S = S(\Omega, \mathscr{S}, \mu)$ with the paranorm is uniformly convex, the modulus of convexity being $\delta(r, \epsilon) = r - \varphi^{-1}(\varphi(r) - \varphi(\frac{\epsilon}{2}))$, $r > 0$ and $\epsilon \in (0, 2r)$.*

These lead immediately to

Theorem 8.5.8 ([13]) *Let $(\Omega, \mathscr{S}, \mu)$ be a measure space and $\mu(\Omega) \leq 1$. If $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing bijection and the map $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ defined by $F(r, s) = \varphi(\varphi^{-1}(r) + \varphi^{-1}(s))$, $r, s \geq 0$ is concave then p_φ is a paranorm on $S = S(\Omega, \mathscr{S}, \mu)$. If φ is also superquadratic then S is uniformly convex with the modulus of continuity being $\delta(r, \epsilon) = r - \varphi^{1-}(\varphi(r) - \varphi(\frac{\epsilon}{2})$, $r > 0$ and $0 < \epsilon < 2r$.*

Lemma 8.5.9 ([20]) *Let X be a set in a linear space such that $\frac{X+X}{2} \subseteq X$ and $p : X - X \to R$ be a map such that $d(x, y) = p(x - y)$, $x, y \in X$ defines a metric on X. If for r, $\beta > 0$ there exists $\delta > 0$ such that $s, t, s - t \in X - X$ the inequalities $p(s)$, $p(t) < r + \delta$ and $p(s - t) > \beta$ imply $p(\frac{s+t}{2}) < r - \delta$, then (X, d) is a bead space.*

Proof Set $s = u - x$, $t = v - y$ where $u, v, x, y \in X$. Then $\frac{s+t}{2} = \frac{u+v}{2} - \frac{(x+y)}{2} \in X - X$ as $X + X \subseteq 2X$. So $p(\frac{s+t}{2})$ is well-defined. Let $u, x, y \in X$ be such that $d(u, x), d(u, y) < r + \delta$ and $d(x, y) > \beta$. Then for $s = u - x$, $t = u - y$, $p(s) < r + \delta$, $p(t) < r + \delta$ and for $s - t = y - x \in X - X \subseteq X$, $p(s - t) = p((u - x) - (u - y)) = p(y - x) > \beta$. So by assumption of Lemma 8.5.9, $p(u - \frac{x+y}{2}) = p(\frac{u-x+u-y}{2}) = p(\frac{s+t}{2}) < r - \delta$. Thus $u \in B(z, r - \delta)$ where $z = \frac{x+y}{2} \in X$. Thus (X, d) is a bead space. □

Remark 8.5.10 The proof of the above lemma is due to Pasicki [20]. In the above lemma X can be the whole space.

Remark 8.5.11 From Pasicki's Theorem 8.4.4 it follows that a non-expansive mapping of a non-void bounded closed convex subset C of a complete uniformly convex paranormed space X into itself has a fixed point, the modulus of convexity of X being continuous.

Remark 8.5.12 ([13]) The completion of a uniformly convex paranormed linear space is uniformly convex.

Matkowski deduced an existence theorem for the solution of a functional equation using Pasicki's theorem.

Theorem 8.5.13 (Matkowski [13]) *Let $\Omega = [0, 1]$, $\mathscr{S}$ the σ-algebra of Lebesgue measurable sets and μ the Lebesgue measure. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing, convex, superquadratic function such that $F(r, s) = \varphi(\varphi^{-1}(r) + \varphi^{-1}(s))$ is concave, $r, s > 0$. Let $f : I \to I$ be an increasing differentiable function, $h : I \times \mathbb{R} \to \mathbb{R}$ satisfy the Caratheodory conditions:*

(a) $h(t, .)$ *is continuous for each* x*;*
(b) $h(., x)$ *is measurable for almost all* x*;*
(c) $|h(t, x)| \leq m(t)$ *for all* (t, x) *where* m *is Lebesgue-integrable on* $[0, 1]$.
(d) *there exist* $\alpha, \beta \in S_\phi(\Omega, \mathscr{S}, \mu)$ *with* $\alpha(t) \leq h(t, \alpha(t)) \leq h(t, \beta(t)) \leq \beta(t)$ *for all* $t \in [0, 1]$ *and for all* $t \in [0, 1]$ *and* $x \in [\alpha(t), \beta(t)]$

$$h(t, \alpha(t)) \leq h(t, x) \leq h(t, \beta(t))$$

(e) *there exists a Lebesgue-measurable function* $g : I \to \mathbb{R}^+$ *such that*

$$|h(t, x) - h(t, y)| \leq g(t)|x - y|, t \in I, x, y \in [\alpha(t), \beta(t)].$$

If $\frac{g}{f'} \leq 1$*, then the functional equation* $x(t) = h(t, x(f(t)))$ *has a solution in* $S^\varphi(\Omega, \mathscr{S}, \mu)$*, the completion of the paranormed space* $S = S_\varphi(\Omega, \mathscr{S}, \mu)$*.*

Proof Define $C = \{x \in \mathbb{R}^I : \alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in I\} \cap S^\varphi(\Omega, \mathscr{S}, \mu)$. Clearly C is a bounded closed convex subset of $S^\varphi(\Omega, \mathscr{S}, \mu)$. Define $T : C \to \mathbb{R}^I$ by

$$T(x)(t) = h(t, f(x(t))), t \in I$$

For $x \in S^\varphi(\Omega, \mathscr{S}, \mu)$, $x \circ f$ is measurable and by the Caratheodary conditions (a), (b) and (c), $T(x)$ is also measurable. From (d), $\alpha(t) \leq T(x(t)) \leq \beta(t)$, $t \in I$. So $T(x) \in C$. For $x, y \in C$, from the definition of p_ϕ, and conditions (d) and (e) and the increasing nature of φ,

$$\begin{aligned}
p_\phi(Tx - Ty) &= \varphi^{-1}\left(\int_0^1 \varphi(|T(x) - T(y)|)(t)dt\right) \\
&= \varphi^{-1}\int_0^1 \varphi(|h(t, x(f(t))) - h(t, y(f(t)))|)dt \\
&\leq \varphi^{-1}\left(\int_0^1 \varphi(g(t))|x(f(t)) - y(f(t))|dt\right) \\
&= \varphi\left(\int_0^1 \varphi\left(\frac{g(t)}{f'(t)}|x(f(t)) - y(f(t))|\right) f'(t)dt\right) \\
&\leq \varphi^{-1}\left(\int_0^1 \varphi(|x(f(t)) - y(f(t))|)f'(t)dt\right) \\
&\leq \varphi^{-1}\left(\int_{f(0)}^{f(1)} \varphi(|x(t) - y(t)|)dt\right) \\
&\leq \varphi^{-1}\left(\int_0^1 \varphi(|x(t) - y(t)|)dt\right) \\
&= p_\varphi(x - y).
\end{aligned}$$

Thus T is non-expansive. As S^φ is a complete uniformly convex bead space. T has a fixed point by Pasicki's theorem. Thus the functional equation $x(t) = h(t, x(f(t)))$ has a solution in S^φ. □

It may be mentioned that for the choice $\varphi(t) = t^p$, $p \geq 1$, the p_φ norms coincide with L_p norms.

8.6 Convergence of Iterates in Normed Spaces

While Banach's contraction principle and its variants are concerned with the convergence of the iterates of the operator under consideration it is also relevant to discuss the summability of iterates. For instance, Krasnoselski's theorem investigates the behaviour of the sequence $x_{n+1} = \frac{1}{2}(x_n + Tx_n)$ (vide section 2.2). Mann [12] generalized Krasnoselski's theorem to more general sequences generated by regular matrices. For discussing Mann's theorem we need the following.

Definition 8.6.1 Let $s = (s_n)$ be a sequence of elements in a Banach space $(X, \|\cdot\|)$ and (c_{mn}) be a real or complex matrix with $m = 1, 2, \ldots$ and $n = 1, 2, \ldots$. Let T be the linear transformation represented by (c_{mn}). The transform of s by T denoted by $t = T(s)$ is the sequence (t_m) defined by $t_m = \sum_{n=1}^{\infty} c_{mn}s_n$ for each $m \in N$. The matrix (c_{mn}) (or T) is said to be regular if $t_m \to s$ as $m \to \infty$ whenever $s_m \to s$ as $m \to \infty$.

A classical theorem in summability called the Silverman–Toeplitz theorem is the following.

Theorem 8.6.2 (Silverman–Toeplitz, see Hardy [9]) *An infinite matrix (c_{mn}) is regular if and only if the following are true:*

(i) $\lim_{m\to\infty} c_{mn} = 0$ *for each* $n \in N$;

(ii) $\lim_{m\to\infty} \sum_{n=1}^{\infty} c_{mn} = 1$ *and*

(iii) $\sup_m \left\{ \sum_{n=1}^{\infty} |c_{mn}| \right\} \leq K < +\infty$ *for some* $K > 0$

Corollary 8.6.3 *Let A be the infinite triangular matrix satisfying the following conditions:*

(a) $a_{ij} \geq 0$ *for* $i, j \in \mathbb{N}$;
(b) $a_{ij} = 0$ *for all* $j > i$

(c) $\sum_{j=1}^{i} a_{ij} = 1$ *for all* $i \in \mathbb{N}$.

Then A *is regular.*

Proof Clearly (b) implies (i) and (c) and (a) imply both (ii) and (iii). □

Theorem 8.6.4 (Mann [12]) *Let* $T : E \to E$ *be a continuous map on a compact convex subset* E *if a Banach space* X. *Let* (x_n) *be the sequence of* T*-iterates generated by* $x_1 \in E$. *Define the sequence* (v_n) *inductively by* $v_n = \sum_{k=1}^{n} a_{nk} x_k$ *and* $v_{n+1} = T(v_n)$, *where* $A = (a_{nk})$ *and* A *is the triangular matrix satisfying (a), (b) and (c) of Corollary 8.6.3.*

If either of the sequences (x_n) *and* (v_n) *converges, then the other also converges to the same point and their common limit is a fixed point of* T.

Proof Let (x_n) converge to p. Since $v_n = \sum_{k=1}^{n} a_{nk} x_k$, $p = \sum_{k=1}^{\infty} a_{nk} x_k$ as A is regular, v_n converges to p. Now $x_{n+1} = T(v_n)$ converges to both p and $T(p)$ (T being continuous) so that $p = T(p)$. Thus (x_n) and (v_n) both converge to the same fixed point of T.

If $\lim_{n\to\infty} v_n = q$, then as A is regular and $(v_n) = A(x_n)$, $\lim x_n = q$. Hence $\lim_{n\to\infty} x_{n+1} = q = T(q)$. Thus both (v_n) and (x_n) converge to the same fixed point of T. □

Corollary 8.6.5 *Let* $A = (a_{ik})$ *where*

$$a_{ik} = \begin{cases} \frac{1}{i} & \text{for} k = 1, 2, \ldots, i, i \in \mathbb{N} \\ 0 & \text{if } k > i \end{cases}$$

Then A *is a regular matrix and if* $T : E \to E$ *is a continuous map on the compact convex subset* E *of a Banach space* X, *then* $x_{n+1} = T(v_n)$ *where* $v_n = \sum_{k=1}^{n} \frac{x_k}{n}$ *converges if* (v_n) *converges and vice-versa. In both the cases the limits are the same fixed point of* T.

Theorem 8.6.6 (Mann [12]) *Suppose neither* $\{x_n\}$ *nor* $\{v_n\}$ *(defined in Theorem 8.6.4) is convergent. Let* X *be the set of all limit points of* $\{x_n\}$ *and* V *the set of all limit points of* $\{v_n\}$. *If* A *satisfies additionally* $\lim_{n\to\infty} a_{nn} = 0$ *and* $\lim_{n\to\infty} \sum_{h=1}^{n} |a_{n+1k} - a_{nk}| = 0$, *then* X *and* V *are closed connected sets.*

Proof For a separation of V by two non-void closed sets A_1 and A_2, we can find $v_1 \in A_1$ and $v_2 \in A_2$ such that $d(v_1, v_2) = \inf\{d(x, y) : x \in A \text{ and } y \in B\} = r > 0$. For

v_i, we can find $\sum_{k=1}^{n_i} a_{m_i k} v_{ik} \in B(v_i, \frac{r}{3}) \cap A_i, i = 1, 2,$ leading to a contradiction to the choice of v_1 and v_2 with $d(v_1, v_2) = r$, in view of the Silverman–Toeplitz conditions and the additional hypotheses $\lim_{n\to\infty} a_{nn} = 0$ and $\lim_{n\to\infty} \sum_{k=1}^{n} |a_{n+1k} - a_{nk}| = 0$. So V is connected. Since T is continuous and $X = T(V)$, X is connected. V being compact, so is X.

When T has a unique fixed point and $E = [a, b]$ in $\mathbb{R}$, the sequence (v_n) converges to the unique fixed point. □

Theorem 8.6.7 (Mann [12]) *Let $T : [a, b] \to [a, b]$ be continuous with a unique fixed point p. Define x_n by $x_{n+1} = T(v_n)$ where $v_n = \frac{1}{n}\sum_{k=1}^{n} x_k$ where $x_1 \in [a, b]$. Then x_n converges to p.*

Proof Define $A = (a_{nk})$ by

$$a_{nk} = \begin{cases} \frac{1}{n} & \text{for } k \le n \\ 0 & \text{for } k > n \end{cases}$$

Clearly A is a regular matrix and $v_{n+1} - v_n = \frac{T(v_n)-v_n}{n+1}$. So $\lim_{n\to\infty}(v_{n+1} - v_n) = 0$. As T is continuous with a unique fixed point p, $Tx - x > 0$ for $x < p$ and $Tx - x < 0$ for $x > p$. So for each $\delta > 0$ there exists $\epsilon > 0$ with $|Tx - x| \ge \epsilon$ for $|x - p| \ge \delta$. Since $v_{n+1} = v_1 + \sum_{k=1}^{n} \frac{T(v_k) - v_k}{k+1}$, these constraints imply that $\lim v_n = p$. Now by Theorem 8.6.4, $\lim x_n = p$ as well. □

Remark 8.6.8 Bailey's proof of Krasnoselski's theorem in $\mathbb{R}$ (vide chapter 2) is a special case of the above result.

8.7 Iterations of a Non-expansive Mapping

In this section, two basic results of Ishikawa [10] on the iterates of a non-expansive mapping are detailed. These are in the setting of a Banach space. We make use of the following definitions and lemmata.

Definition 8.7.1 Let D be a subset of a Banach space X, $T : D \to X$ a map and $x_1 \in D$. By $M(x_1, t_n, T)$ we denote the sequence (x_n) defined by
$x_{n+1} = (1 - t_n)x_n + t_n T x_n$, where (t_n) is a real sequence.
$x_1 \in D$ and the real sequence (t_n) are said to satisfy condition (A)
if $0 \le t_n \le b < 1$ for all $n \in N$, $\sum_{n=1}^{\infty} t_n = +\infty$ and $x_n \in D$ for all $n \in N$.

Remark 8.7.2 If $t_n \in [a, b]$ for all n where $0 < a \leq b < 1$, then $\sum_{n=1}^{\infty} t_n = +\infty$ and $0 \leq t_n \leq b < 1$.

Lemma 8.7.3 *Let (s_n) be a real sequence and (u_n) a sequence in a Banach space X. Then for any natural number M*

$$\left(\prod_{i=1}^{M-1} s_i\right)\left(\sum_{i=1}^{M}(1-s_i)u_i\right)$$
$$= \left(1 - \prod_{i=1}^{M} s_i\right) u_M - \sum_{i=1}^{M-1}\left\{\left(\prod_{j=i+1}^{M-1} s_j\right)\left(1 - \prod_{j=1}^{i} s_j\right)(u_{i+1} - s_i u_i)\right\} \tag{8.7.1}$$

When X is the real line and $u_i = 1$ for all i, we have the special case

$$\left(\prod_{i=1}^{M-1} s_i\right)\left(\sum_{i=1}^{M}(1-s_i)\right)$$
$$= 1 - \prod_{i=1}^{M} s_i - \sum_{i=1}^{M-1}\left\{\left(\prod_{j=i+1}^{M-1} s_j\right)\left(1 - \prod_{j=1}^{i} s_j\right)(1 - s_i)\right\}$$

In this and what follows $\sum_{i=m}^{n}$ and $\prod_{i=m}^{n}$ are defined as 0 and 1 for $n < m$.

Proof The proof is by induction on M. When $M = 1$, the result is obvious. Suppose it is true for some $M > 1$, then

$$\sum_{i=1}^{M}\left\{\left(\prod_{j=i+1}^{M} s_j\right)\left(1 - \prod_{j=1}^{i} s_j\right)(u_{i+1} - s_i u_i)\right\}$$
$$= s_M \sum_{i=1}^{M-1}\left\{\left(\prod_{j=i+1}^{M-1} s_j\right)\left(1 - \prod_{j=1}^{i} s_j\right)(u_{i+1} - s_i u_i)\right.$$
$$\left. - s_M\left(1 - \prod_{i=1}^{M} s_i\right) u_M + \left(1 - \prod_{i=1}^{M} s_i\right) u_{M-1}\right.$$
$$= s_M\left\{\left(1 - \prod_{i=1}^{M} s_i\right) u_M - \prod_{i=1}^{M-1}\left(\sum_{i=1}^{M}(1-s_i)u_i\right)\right\}$$
$$- s_M\left(1 - \prod_{i=1}^{M} s_i\right) u_M + \left(1 - \prod_{i=1}^{M} s_i\right) u_{M-1}$$

$$= -\left(\prod_{i=1}^{M} s_i\right)\left(\sum_{i=1}^{M}(1-s_i)u_i\right) + \left(1-\prod_{i=1}^{M} s_i\right)u_{M-1}$$

Whence we have (the right-hand side of (8.7.1) with$M+1$for$M$)

$$= \left(1-\prod_{i=1}^{M+1} s_i\right)u_{M+1} - \sum_{i=1}^{M}\left\{\left(\prod_{j=i+1}^{M} s_j\right)\left(1-\prod_{j=1}^{i} s_j\right)(u_{i+1}-s_iu_i)\right\}$$

$$= \left(1-s_{M+1}\prod_{i=1}^{M} s_i\right)u_{M+1} + \left(\prod_{i=1}^{M} s_i\right)\left(\sum_{i=1}^{M}(1-s_i)u_i - \left(1-\prod_{i=1}^{M} s_i\right)u_{M+1}\right.$$

$$= \left(\prod_{i=1}^{M} s_i\right)\left(\sum_{i=1}^{M+1}(1-s_i)u_i\right)$$

So by induction this lemma follows. □

Lemma 8.7.4 *Let $D \subseteq X$, a Banach space and $T : D \to X$ a non-expansive map. If $x_1 \in D$ and (t_n) satisfy condition A and $M(x_1, t_n, T)$ is bounded, then $x_n - Tx_n \to 0$ as $n \to \infty$.*

Proof

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &= \|(1-t_n)x_n + t_nTx_n - Tx_{n+1}\| \\
&= \|(1-t_n)(x_n - Tx_n) + Tx_n - Tx_{n+1}\| \\
&\leq (1-t_n)\|x_n - Tx_n\| + \|x_n - x_{n+1}\| \\
&= (1-t_n)\|x_n - Tx_n\| + \|x_n - (1-t_n)x_n + t_nTx_n)\| \\
&= \|x_n - Tx_n\|
\end{aligned}$$

Since $(\|x_n - Tx_n\|)$ is a non-increasing sequence of non-negative numbers, $\lim_{n\to\infty} \|x_n - Tx_n\|$ exists. Let $\lim_{n\to\infty} \|x_n - Tx_n\| = r > 0$. So for $\epsilon > 0$ there exists a natural number m such that $r \leq \|x_{m+i} - Tx_{m+i}\| \leq (1+\epsilon)r$ for all $i \in \mathbb{N}$. As T is non-expansive

$$\begin{aligned}
&\|(Tx_{m+i+1} - x_{m+i+1}) - (1-t_{m+i})(Tx_{m+i} - x_{m+i})\| \\
&= \|T((1-t_{m+i})x_{m+i} + t_{m+i}Tx_{m+i}) \\
&\quad - ((1-t_{m+i})x_{m+i} + t_{m+i}Tx_{m+i}) - (1-t_{m+i})(Tx_{m+i} - x_{m+i})\| \\
&= \|T((1-t_{m+i})x_{m+i} + t_{m+i}Tx_{m+i}) - Tx_{m+i}\| \\
&\leq t_{m+i}\|x_{m+i} - Tx_{m+i}\| \leq t_{m+i}(1+\epsilon)r
\end{aligned} \tag{8.7.2}$$

Since (x_n) is bounded and $\sum_{n=1}^{\infty} t_n = +\infty$, we can find a natural number N such that

$$r\sum_{i=1}^{N-1} t_{m+i} \le \delta(M)+1 \le r\sum_{i=1}^{N} t_{m+i}$$

where $\delta(M) = \sup\{|x_i - x_j| : i, j \in \mathbb{N}\}$

Set $s_i = 1 - t_{m+i}$, $v_i = Tx_{m+i} - x_{m+i}$ for $i \in N$ in (8.7.2) to get

$$\begin{aligned}\|u_{i+1} - s_i u_i\| &= \|Tx_{m+i+1} - x_{m+i+1} - (1-t_{m+i})(Tx_{m+i} - x_{m+i})\|\\ &\le t_{m+i}(1+\epsilon)r = (1-s_i)(1+\epsilon)r\end{aligned}$$

and

$$\begin{aligned}x_{m+N+1} - x_{m+1} &= \sum_{i=1}^{N}\{((1-t_{m+i})x_{m+i} + t_{m+i}Tx_{m+i}) - x_{m+i}\}\\ &= \sum_{i=1}^{N} t_{m+i}(Tx_{m+i} - x_{m+i}) = \sum_{i=1}^{N}(1-s_i)u_i\end{aligned}$$

Using Lemma 8.7.3 and the above inequalities we get

$$\begin{aligned}&\left(\prod_{i=1}^{N-1} s_i\right)\|x_{m+N+1} - x_{m+1}\| = \left\|\left(\prod_{i=1}^{N-1} s_i\right)\left(\sum_{i=1}^{N}(1-s_i)u_i\right)\right\|\\ &\ge \left(1-\prod_{i=1}^{N} s_i\right)\|u_N\| - \sum_{i=1}^{N-1}\left\{\left(\prod_{j=i+1}^{N-1} s_j\right)\left(1-\prod_{j=1}^{i} s_j\right)\|u_{i+1} - s_i u_i\|\right\}\\ &\ge \left(1-\prod_{i=1}^{N} s_i\right)r - \sum_{i=1}^{N-1}\left\{\left(\prod_{j=i+1}^{N-1} s_j\right)\left(1-\prod_{j=1}^{i} s_j\right)(1-s_i)(1-\epsilon)r\right\}\\ &= \left[1-\prod_{i=1}^{N} s_i - \sum_{i=1}^{N-1}\left\{\prod_{j=i+1}^{N-1} s_j\left(1-\prod_{j=1}^{i} s_j\right)(1-s_i)\right\}\right]r\\ &\quad - \epsilon r\sum_{i=1}^{N-1}\left\{\left(\prod_{j=i+1}^{N-1} s_j\right)\left(1-\prod_{j=1}^{i} s_j\right)(1-s_i)\right\}\end{aligned}$$

since $s_i = 1 - t_{m+i} \ge 1 - b > 0$, (8.7.1), the choice of m and Lemma 8.7.3 imply

$$\begin{aligned}
\|x_{m+N+1} - x_{m+1}\| &\geq r \sum_{i=1}^{N} (1 - s_i) - \epsilon r \left(\prod_{i=1}^{N-1} s_i \right)^{-1} \\
&\quad \times \left\{ 1 - \prod_{i=1}^{N} s_i - \left(\prod_{i=1}^{N-1} s_i \right) \left(\sum_{i=1}^{N} (1 - s_i) \right) \right\} \\
&\geq r \sum_{i=1}^{N} (1 - s_i) - \epsilon r \left(\prod_{i=1}^{N-1} s_i \right)^{-1} \\
&= r \sum_{i=1}^{N} t_{m+i} - \epsilon r \prod_{i=1}^{N-1} (1 - t_{m+i})^{-1} \\
&\geq \delta M + 1 - \epsilon r \prod_{i=1}^{N-1} (1 - t_{m+i})^{-1} \qquad (8.7.3)
\end{aligned}$$

As $\log(1 + y) \leq y$ for $y > -1$, we have

$$\begin{aligned}
\prod_{i=1}^{N-1} (1 - t_{m+i})^{-1} &= \prod_{i=1}^{N-1} (1 + t_{m+i} (1 - t_{m+i})^{-1}) \\
&= \exp \left\{ \sum_{i=1}^{N-1} \log(1 + t_{m+i} (1 - t_{m+i})^{-1}) \right\} \\
&\leq \exp \left\{ \sum_{i=1}^{N-1} t_{m+i} (1 - t_{m+i})^{-1} \right\} \\
&\leq \exp \left\{ (1 - b)^{-1} \sum_{i=1}^{N-1} t_{m+i} \right\} \\
&\leq \exp\{(1 - b^{-1})(\delta(M) + 1) r^{-1}\} \qquad (8.7.4)
\end{aligned}$$

Now (8.7.3) and (8.7.4) give

$$\begin{aligned}
&\delta(M) + 1 - \epsilon r \exp\{(1 - b)^{-1} (\delta(M) + 1) r^{-1}\} \\
&\quad \leq \|x_{m+N+1} - x_{m+1}\| \leq \delta(M).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, proceeding to the limit in the last inequality as $\epsilon \to 0$, we get

$\delta(M) + 1 \leq \delta(M)$, a contradiction.

So $r = 0$. Thus $\lim_{n\to\infty} \|x_n - T x_n\| = 0$. □

Remark 8.7.5 Let $T : D \to D$ be a non-expansive map of a convex subset D of a Banach space with $T(D)$, a bounded subset of D. For $0 < t < 1$, let $(1 - t)I + tT$ be denoted by T_t where I is the identity map. Then $M(x_1, t, T)$ is bounded since it is a sequence in the convex hull of $T(D) \cup \{x_1\}$. Since $T_t^n x_1 - T_t^{n-1} x_1 = t(Tx_n - x_n)$, by Lemma 8.7.4 T_t is asymptotically regular in the sense that $\lim_{n\to\infty} \|T_t^{n+1}x - T_t^n x\| = 0$ for $x \in D$.

The iteration procedure $M(x_1, t_n, T)$ can be used to approximate a fixed point of a non-expansive map under special conditions.

Theorem 8.7.6 (Ishikawa [10]) *Let $T : D \to X$ be a non-expansive map, where D is a closed subset of a Banach space X such that $T(D)$ is a subset of a compact subset of X. Let $x_1 \in D$ and $\{t_n\}$ a non-negative sequence such that the condition (A) of Definition 8.7.1 is satisfied. Then T has a fixed point in D and $M(x_1, t_n, T)$ converges to a fixed point of T.*

Proof Let D_0 be the closure of the convex hull of $T(D) \cup \{x_1\}$. By Mazur's theorem D_0 is compact. Now $M(x_1, t_n, T)$ lies in D_0. Condition (A) of Definition 8.7.1 along with this implies that this sequence which indeed lies in D being in the compact set D_0 has a subsequence (x_{n_i}) converging to $u \in D_0$. Since (x_{n_i}) is in D and D is closed, $u \in D$. From the boundedness of D_0 and Lemma 8.7.4, $\lim_{i\to\infty} \|x_{n_i} - Tx_{n_i}\| = 0$. As T is non-expansive

$$\begin{aligned}\|u - Tu\| &= \|Tu - Tx_{n_i} + Tx_{n_i} - x_{n_i} + x_{n_i} - u\| \\ &\le 2\|u - x_{n_i}\| + \|x_{n_i} - Tx_{n_i}\|\end{aligned}$$

Hence $u = Tu$, in view of $\lim_{i\to\infty} \|x_{n_i} - Tx_{n_i}\| = \lim_{i\to\infty} \|u - x_{n_i}\| = 0$.
Also

$$\begin{aligned}\|x_{n+1} - u\| &= \|(1 - t_n)x_n + t_n Tx_n - u\| \\ &= \|(1 - t_n)(x_n - u) + t_n(Tx_n - Tu)\| \\ &\le \|x_n - u\| \quad \text{for all } n \in \mathbb{N}.\end{aligned}$$

As $\lim_{i\to\infty} x_{n_i} = u$ and $\|x_n - u\| \le \|x_{n_i} - u\|$ for all $n \ge n_i$, $\lim_{n\to\infty} x_n = u$. □

Corollary 8.7.7 *Let D be a closed subset of a Banach space X and $T : D \to X$ a non-expansive map such that $T(D)$ is contained in a compact subset of X. If for some $t \in (0, 1)$, $(1 - t)x + tTx \in D$ for all $x \in D$ then T has a fixed point in D and for any $x_1 \in D$, $M(x_1, t, T)$ converges to a fixed point of T.*

Corollary 8.7.8 *Let D be a closed convex subset of a Banach space X and T, a non-expansive map from D into a compact subset of D. Then T has a fixed point in D and $M(x_1, \frac{1}{2}, T)$ converges to a fixed point of T, where x_1 is any element of D.*

Remark 8.7.9 Corollary 8.7.8 was proved for uniformly convex Banach spaces by Krasnoselski.

The assumption on the compactness of T can be dispensed with as in the following.

Theorem 8.7.10 (Ishikawa [10]) *Let $T : D \to X$ be a non-expansive mapping where D is a closed subset of a Banach space X with a non-empty set of fixed points F in D. Suppose there exists $f : \mathbb{R}^+ \to \mathbb{R}^+$ $(= [0, \infty))$ such that f is non-decreasing, $f(0) = 0$, $f(r) > 0$ for $r > 0$ with*

$$\|x - Tx\| \geq f(d(x, F)) \text{ for all } x \in D, d(x, F) = \inf\{d(x, y) : y \in F\}. \quad (8.7.5)$$

If for some $x_1 \in D$ and (t_n) condition (A) is satisfied, then $M(x_1, t_n, T)$ converges to some fixed point of T in F.

Proof If $x_1 \in F$, then theorem is obvious. Let $x_1 \notin F$. So for $u \in F$, $\|x_n - u\| \geq \|Tx_n - u\|$. We therefore have

$$\|x_{n+1} - u\| = \|(1 - t_n)x_n + t_n T x_n - u\| \leq \|x_n - u\|$$

So $d(x_{n+1}, F) \leq d(x_n, F)$ for all $n \in \mathbb{N}$. Let $r = \lim_{n\to\infty} d(x_n, F)$. By definition of f,

$$\|x_n - Tx_n\| \geq f(d(x_n, F)) \geq f(r).$$

Clearly $M(x_1, t_n, T)$ is a bounded sequence in D. So by Lemma 8.7.4. $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ and $f(r) = 0$. So $r = 0$ and $\lim_{n\to\infty} d(x_n, F) = 0$. So for each $i \in \mathbb{N}$ we can find $N_i \in \mathbb{N}$ and $u_i \in F$ such that $\|x_{N_i} - u_i\| < 2^{-i}$ with $N_{i+1} > N_i$. So for $n \geq N_i$, $\|x_n - u_i\| < 2^{-i}$, and for $i < j$

$$\begin{aligned} \|u_i - u_j\| &\leq \|u_i - x_{N_{i+1}}\| + \|x_{N_{i+1}} - u_{i+1}\| + \cdots \\ &\quad \|u_{j-1} - x_{N_j}\| + \|x_{N_j} - u_j\| \\ &< 2^{-i} + 2^{-i-1} \cdots + 2^{-j}. \end{aligned}$$

Thus (u_i) is a Cauchy sequence in the closed set F of the Banach space X and hence it converges to some $u \in F$. Given $\epsilon > 0$ we can find $i_0 > 0$ such that $2^{-i_0} < \frac{\epsilon}{2}$ and $\|u_{i_0} - u\| < \frac{\epsilon}{2}$. So for $n > N_{i_0}$,

$$\|x_n - u\| \leq \|x_n - u_{i_0} + \|u_{i_0} - u\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $M(x_1, t_n, T)$ converges to a fixed point of T in D. □

Condition (8.7.5) in the above theorem was originally considered by Senter and Dotson.

8.8 A Generalization of the Contraction Principle Based on Combinatorics

Merrifield, Rothschild and Stein [15] obtained a generalization of the contraction principle based on Ramsey's theorem in combinatorics. Since it involves the study of certain subsequences of iterates, it is discussed in this section. We need the following.

Theorem 8.8.1 (Ramsey) *Let S be an infinite set, n a natural number. Suppose that every subset S of cardinality n is assigned one of a finite number of colours. Then there exists an infinite subset T of S such that T is monochromatic (i.e. every subset T of cardinality n has the same colour).*

The following lemma can be derived from the above theorem.

Lemma 8.8.2 *Let m and n be natural numbers. Suppose G is a graph whose vertex set is a disjoint union of countably many blocks, each of size m. Further suppose that each edge has its two endpoints in distinct blocks and for any n blocks assume that there is at least one edge having its endpoints in two of those blocks. Then there is an infinite path in G, visiting no block more than once.*

Proof Name the blocks as $B_0, B_1, \ldots$, and number the vertices in each block B_i as $v(i, 1), v(i, 2), \ldots, v(i, m)$. Colour the pairs of natural numbers with $m^2 + 1$ colours by assigning as the colour of the pair (i, j) with $i < j$, either some pair (p, q) so that $v(i, p)$ is adjacent to $v(j, q)$ on G or if there is no edge between B_i and B_j, the special colour 'none' is assigned. By Ramsey's Theorem 8.8.1 there is an infinite set H of natural numbers, every pair of which has the same colour. By hypothesis, this colour cannot be 'none'. So let it be (p, q). That is if $i < j$ are both in H, then $v(i, p)$ is adjacent to $v(j, q)$. Let $h(i)$ be the ith element of H. The desired infinite path is $v(h(1), p), v(h(3), q), v(h(2), p), v(h(5), q), v(h(4), p), v(h(7), q), \ldots$. □

The following is an elementary consequence of triangle inequality.

Lemma 8.8.3 *Let (X, d) be a metric space and $0 < \alpha < 1$, J a natural number satisfying*

$$\min\{d(T^i x, T^i y) : i = 1, 2, \ldots, J\} \leq \alpha d(x, y)$$

for all $x, y \in X$. Then for each x, there is a bounded subsequence $\{T^{a(n)}x : n = 1, 2, \ldots\}$ of $\{T^n(x) : n = 1, 2, \ldots\}$ such that $a(n+1) - a(n) \leq J$.

Proof Let $C = \max\{d(x, T^k x) : k = 1, 2, \ldots, J\}$. We show that we can construct a sequence $a(n)$ of integers such that $d(x, T^{a(n)}x) < \frac{C}{1-\alpha}$. Let $a(1) = 1$. Since $0 < \alpha < 1, 0 < 1 - \alpha < 1$, clearly $d(x, Tx) < \frac{C}{1-\alpha}$. Proceeding inductively if $d(x, T^{a(n)} x) < \frac{C}{1-\alpha}$, we have for some $k < J$, $d(T^k x, T^{a(n)+k} x) < \frac{C}{1-\alpha}$. So by triangular inequality we have $d(x, T^{a(n)+k} x) \leq d(x, T^k x) + d(T^k x, T^{a(n)+k} x) \leq C + \frac{\alpha C}{1-\alpha} = \frac{C}{1-\alpha}$. Letting $a(n+1) = a(n) + k$ completes the proof. □

Merrifield, Rothschild and Stein [15] proved the following generalization of the contraction principle.

Theorem 8.8.4 (Merrifield, Rothschild and Stein [15]) *Let $T : X \to X$ be a continuous map. Suppose there is a positive integer J and $0 < \alpha < 1$ such that for $x, y \in X$*

$\min\{d(T^k x T^k y) : 1 \leq k \leq J\} \leq \alpha d(x, y)$. *Then T has a fixed point.*

Proof Let $x \in X$ and $\langle i, k\rangle = d(T^i x, T^k x)$, $T^i x = x$ and for a real number r, $[r]$ denote the greatest integer less than or equal to r.

By Lemma 8.8.3, there is a sequence $\{n_i : i = 1, 2, \ldots\}$ of natural numbers with $\langle 0, n_i\rangle \leq C$ and $n_{i+1} - n_i \leq J$. Applying the generalized contraction hypothesis to each pair of points x and $T^{n_i} x$ we get sequences $\{q_{ij} : j = 1, 2, \ldots\}$ one sequence for each i such that $\langle q_{ij}, n_i + q_{ij}\rangle \leq C\alpha^j$ and $q_{ij+1} - q_i \leq J$. If $q = q_{ij}$, then $j \geq \left[\frac{q}{J}\right] \geq q/J - 1$, in which case $\langle q, n_i + q\rangle \leq C\alpha^{q/J-1} = C_0 Q^q$ where $C_0 = \frac{C}{\alpha}$ and $Q = \alpha^{\frac{1}{J}} < 1$. Note that if we have $\langle q, n_i + q\rangle \leq C_0 Q^q$ and $\langle q, n_j + q\rangle \leq C_0 Q^q$ for two different integers i and j by the triangle inequality we have

$$\begin{aligned}\langle n_i + q, n_j + q\rangle &\leq \langle n_i + q, q\rangle + \langle q, n_j + q\rangle \\ &\leq 2C_0 Q^q\end{aligned}$$

We say that an integer q is represented if there are infinitely many integers i for which $\langle q, n_i + q\rangle \leq C_0 Q^q$. If q is represented and $\langle q, n_i + q\rangle \leq C_0 Q^q$, we say that i is a representative of q. If A is a set of integers, let $r(A) = \{q : q \in a, q \text{ is represented}\}$ and $R(A) = \{i : \text{ for some } q \text{ in } r(A) \text{ is a representative of } q\}$

Let A be a set J consecutive integers. The condition $q_{ij+1} - q_{ij} \leq J$ together with the pigeonhole principle shows that at least one member of A has to be represented. Also for all but finitely many i, there exists $q \in r(A)$ such that i is a representative of q : thus for some integer I_0, $i \geq I_0$ implies $i \in R(A)$.

Again for A, a set of J consecutive integers, there is an integer $\lambda(A)$ such that for $m \geq \lambda(A)$ some integer of the form $n_i + q$ lies in the set $\{m, m+1, \ldots, m + 2J - 1\}$, where $q \in r(A)$ and i is a representative of q. Since there exists I_0 such that $i \geq I_0$ implies $i \in R(A)$.

Let $\lambda(A) = \max\{j : j \in A\} + n_{I_0}$ where A is a set of J consecutive integers. Since each i with $i \geq I_0$ is a representative of some $q \in r(A)$ and if i is a representative of $q \in r(A)$, then $n_{i+1} + q' - (n_i + q) = (n_{i+1} - n_i) + (q' - q) \leq 2J$.

Finally, if A is a set of J consecutive integers, let $NQ(A) = \{n_i + q : q \in r(A) \text{ and } i \text{ is a representative of } q\}$. It may be noted that if $A_1, \ldots, A_{2J+1}$ are disjoint sets of J consecutive integers each, then for $a = \max\{\lambda(A_k) : k = 1, 2, 2J + 1\}$ and $m \geq a$ any set $\{m, m+1, \ldots, m + 2J - 1\}$ must contain an integer common to the sets $NQ(A_j)$ and $NQ(A_k)$ for which $j \neq k$. By partitioning the sequence $\{a, a+1, a+2, \ldots\}$ into blocks of length $2J$ and applying the pigeonhole principle we see that there are two sets $NQ(A_j)$ and $NQ(A_k)$ with $j \neq k$ having infinitely many elements in common. Invoking the pigeonhole principle, there exist integers $q \in A_j$ and $q' \in A_k$ so that there are infinitely many integers common to $NQ(A_j)$

and $NQ(A_k)$ that can be expressed both in the form $n_i + q$ (an integer in $NQ(A_j)$) and $n_p + q'$ (an integer $NQ(A_k)$).

Regard each integer as a vertex in a graph and partition the integers into a disjoint union of blocks $B_k = \{(k-1)J + 1, (k-1)J + 2, \ldots kJ\}$ for $k = 1, 2, \ldots$. We say that two vertices q and q' in distinct blocks B_j and B_k respectively are connected by an edge if there are infinitely many integers that can be expressed in both the forms $n_i + q$ and $n_p + q'$. From the above it is clear that for any collection of $2J + 1$ blocks, at least one edge has endpoints in two distinct blocks.

By Lemma 8.8.2 that there is an infinite path through the graph passing through each block no more than once. Denote the vertices traversed in this path by $\{r_j : j = 1, 2, \ldots\}$. Choose the sequences of integers $\{s_j : j \in \mathbb{N}\}$, and $\{t_j : j \in \mathbb{N}\}$ from $\{n_j : j \in \mathbb{N}\}$ with the following three properties:

(i) if $r_j \in B_k$, both $r_j + s_j$ and $r_j + t_j \in NQ(B_k)$;
(ii) $r_j + t_j = r_{j+1} + s_{j+1}$;
(iii) $r_j + s_j < r_{j+1} + s_{j+1}$

Consider the sequence of iterates with exponents $r_j + s_j$. Observe that

$$\sum_{j=1}^{\infty} \langle r_j + s_j, r_{j+1} + s_{j+1}\rangle = \sum_{j=1}^{\infty} \langle r_j + s_j, r_j + t_j\rangle \leq \sum_{j=1}^{\infty} 2K_0 Q^{r_j}$$

Since the $\{r_j : j \in \mathbb{N}\}$ are all distinct, the above series converges. So the sequence of iterates is a Cauchy sequence. As X is complete, the resulting limit of this sequence will be shown to be a fixed point of T. Let $(T^{m_i} x)$ converge to z. We can even choose it so that $m_{i+1} > m_i + J$ for all i. Since T is continuous, $T^{m_i+k} \to T^k z$ for $1 \leq k \leq J$. Define $L_k = T^k z$ for $0 \leq k \leq J$. We claim that $L_{j+1} = L_j$ for some $j < J$. Since $T(L_j) = L_{j+1}$ it follows that L_j is a fixed point of T.

By the generalized contraction hypothesis applied to x and $y = Tx$, we conclude that for each $i \in \mathbb{N}$, there exists an integer j_i with $0 \leq j_i \leq J - 1$ and $d(T^{m_j + j_i} x, T^{m_i + j_i + 1} x) \leq \alpha^{r_i} d(x, Tx)$ where $r_i \to \infty$ (this can be seen by using an argument in the second paragraph of the proof of this theorem). By the pigeonhole principle, there is an integer k with $0 \leq k \leq J - 1$ and $j_i = k$ for infinitely many i. For those i with $j_i = k$ for infinitely many i we have

$$\begin{aligned} d(L_k, L_{k+1}) &\leq d(L_k, T^{m_j+k} x) + d(T^{m_i+k} x, T^{m_i+k+1} x) \\ &\quad + d(T^{m_i+k+1} x, L_{k+1}) \\ &\leq d(L_k, T^{m_j+k} x) + \alpha^{r_i} d(x, Tx) \\ &\quad + d(T^{m_i+k+1} x, L_{k+1}) \end{aligned}$$

As $i \to \infty$, each of the three terms on the right-hand side of the above inequality tends to zero. Thus $L_k = L_{k+1}$ or $L_k = T^k z$ is a fixed point of T. □

Remark 8.8.5 It is not known if the theorem is true even when T is not continuous for all J.

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Chapter 9
Geometric Aspects of Banach Spaces and Non-expansive Mappings

9.1 Introduction

In this chapter, we outline the proof that a reflexive non-square Banach space has fixed point property for non-expansive mappings on bounded closed convex sets. To this end, some definitions are in order.

Definition 9.1.1 For a Banach space $(X, \|\cdot\|)$, the closed unit ball and the unit sphere in X are denoted by B_X and S_X, respectively. The Clarkson modulus of convexity of X is a function $\delta_X : [0, 2] \to [0, 1]$ defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in S_X, \|x - y\| \geq \epsilon\right\}.$$

X is called uniformly convex if $\delta_X(\epsilon) > 0$ for $0 < \epsilon < 2$.

Remark 9.1.2 δ_X is continuous on [0, 2) increasing on [0, 2], strictly increasing on $[\epsilon_0, 2]$ where $\epsilon_0 = \epsilon_0(X) = \sup\{\epsilon \in (0, 2] : \delta_X(\epsilon) = 0\}$ is called the coefficient of convexity of X. Also $\delta_X(\epsilon) \leq \frac{\epsilon}{2}$, $\lim\limits_{\epsilon \to 2^-} \delta_X(\epsilon) = 1 - \dfrac{\epsilon_0(X)}{2}$ and $\delta_X(\epsilon) \leq 1 - \sqrt{1 - \frac{\epsilon^2}{4}} = \delta_{\ell_2}(\epsilon)$.

For a real Banach space X with $dim\ X \geq 2$ or a complex Banach space X of $dim\ X \geq 1$, it can be shown that

$$\begin{aligned}\delta_X(\epsilon) &= \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in B_X, \|x - y\| \geq \epsilon\right\}\\ &= \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in S_X, \|x - y\| = \epsilon\right\}\\ &= \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in B_X, \|x - y\| = \epsilon\right\}\end{aligned}$$

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See Day [1].

Definition 9.1.3 The modulus of smoothness of a Banach space X is a function $\rho_X : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\rho_X(\epsilon) = \sup\left\{\frac{\|x+ty\| + \|x-ty\|}{2} - 1 : x, y \in S_X\right\}$$

X is called uniformly smooth if $\rho_0'(X) = \lim_{t\to 0^+} \frac{\rho(t)}{t} = 0$.

Remark 9.1.4 ρ_x is increasing, continuous and convex on $\mathbb{R}^+$ and $\rho_X(0) = 0$ with $\rho_X(t) \le t$. Also for a real Banach space X with $dim\ X \ge 2$ (or a complex Banach space X with $dim\ X \ge 1$), $\rho_X(t) \ge \sqrt{1+t^2} - 1 = \rho_{\ell_2}(t)$. Lindenstrauss [13] proved the important relations

$$\rho_{X^*}(t) = \sup\left\{\frac{t\epsilon}{2} - \delta_X(t) : \epsilon \in [0, 2]\right\}$$
$$\rho_X(t) = \sup\left\{\frac{t\epsilon}{2} - \delta_{X^*}(t) : \epsilon \in [0, 2]\right\}.$$

So $\rho_X(t) = \rho_{X^{**}}(t)$ and X is uniformly convex (uniformly smooth) if and only if its dual X^* is uniformly smooth (uniformly convex). Also X is reflexive whenever it is uniformly convex or uniformly smooth.

Definition 9.1.5 (*James* [7]) A Banach space X is called uniformly non-square if there exists $\delta \in (0, 1)$ such that for any $x, y \in S_X$ either $\frac{\|x+y\|}{2} \le 1-\delta$ or $\frac{\|x-y\|}{2} \le 1-\delta$. The constant $J(X)$ defined by

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in S_X\}$$

is called the non-square or James constant of X.

Proposition 9.1.6 (Kato [10]) *Let X be a real Banach space with $dim(X) \ge 2$ (or a complex Banach space wth $dim(X) \ge 1$). Then the following are equivalent.*

(i) *X is uniformly non-square;*
(ii) *$\delta_X(\epsilon) > 0$ for some $\epsilon \in (0, 2)$;*
(iii) *$\epsilon_0(X) < 2$;*
(iv) *$J(X) < 2$;*
(v) *$\rho_X(t_0) < t_0$ for some $t_0 > 0$;*
(vi) *$\rho_X(t) < t$ for all $t > 0$;*
(vii) *$\rho_X'(0) = \lim_{t\to 0} \frac{\rho_X(t)}{t} < 1$;*
(viii) *$\epsilon_0(X^*) < 2$;*
(ix) *$\rho_{X^*}'(0) = \lim_{t\to 0} \frac{\rho_{X^*}(t)}{t} < 1$.*

We provide only the gist of the main arguments in the

Proof We note that X is uniformly non-square
$\Leftrightarrow$ for some $\delta \in (0, 1)$, $x, y \in S_X$ and

$$\frac{\|x - y\|}{2} > 1 - \delta \text{ implies } \frac{\|x + y\|}{2} > 1 - \delta$$

$\Leftrightarrow$ for some $\delta \in (0, 1)$, $x, y \in S_X$ and

$$\frac{\|x - y\|}{2} > 1 - \delta \text{ implies } 1 - \frac{\|x + y\|}{2} \geq \delta$$

$\Leftrightarrow$ for some $\delta \in (0, 1)$, $\delta_X(2 - 2\delta) \geq \delta$
(i) $\Rightarrow$ (ii) If X is uniformly non-square set $\epsilon = 2 - 2\delta \in (0, 2)$ in the above equivalences to get $\delta_X(\epsilon) > 1 - \frac{\epsilon}{2}(= \delta)$.
(ii) $\Rightarrow$ (i) If for some $\epsilon_0 \in (0, 2)$, $\delta_X(\epsilon_0) \geq \eta_0 > 0$, where $\eta_0 \in (0, 1)$, then for $2 - 2\delta = \epsilon \in [\epsilon_0, 2)$, $\delta \in \left(0, 1 - \frac{\epsilon_0}{2}\right)$ and $\delta_X(2 - 2\delta) = \delta_X(\epsilon) = \delta_X(\epsilon_0) \geq \eta_0 > 0$. This means that for any $x, y \in S_X$, with $\|x - y\| \geq 2 - 2\delta$ necessarily $1 - \frac{\|x+y\|}{2} \geq \eta_0$. If $\delta' = \min\{\delta, \eta_0\}$ then $\delta' \in (0, 1)$. If $\frac{\|x-y\|}{2} \leq 1 - \delta'$, we are done. If $\frac{\|x+y\|}{2} > 1 - \delta$, then $1 - \frac{\|x+y\|}{2} \geq \eta_0$ or $\frac{\|x+y\|}{2} \leq 1 - \eta_0 \leq 1 - \delta'$. So X is uniformly non-square. (ii) $\Leftrightarrow$ (iii) and (i) $\Leftrightarrow$ (iv) follow from the definition. (v) $\Leftrightarrow$ (vii) follows from the fact that $\frac{\rho_X(t)}{t}$ is increasing. (vii) $\Leftrightarrow$ (viii) and (ix) $\Leftrightarrow$ (iii) follow from

$$\epsilon_0(X^*) = 2\rho_X'(0) = 2\lim_{t \to 0} \frac{\rho_x(t)}{t} \quad \text{and}$$
$$\epsilon_0(X) = 2\rho_{X^*}'(0) = 2\lim_{t \to 0} \frac{\rho_{x^*}(t)}{t}.$$

We now show that (v) $\Leftrightarrow$ (vi). If $\rho_X(t_0) = t_0$ for some $t_0 > 0$ then $\rho_X(t) = t$ for all $t > 0$. Since $\frac{\rho_X(t)}{t}$ is increasing and $\rho_X(t) \leq t$, it follows that for $t \geq t_0$, $1 = \frac{\rho_X(t_0)}{t_0} \leq \frac{\rho_X(t)}{t} \leq 1$ or $\rho_X(t) = t$ for $t \geq t_0$. Let $0 < t < t_0$ and $\rho_X(t) < t$. Since ρ_X is convex for $t_1 > t_0$

$$\begin{aligned} t_0 = \rho_X(t_0) &= \rho_X\left(\frac{t_0 - t}{t_1 - t}t_1 + \frac{t_1 - t_0}{t_1 - t}t\right) \\ &\leq \frac{t_0 - t}{t_1 - t}\rho_X(t_1) + \frac{t_1 - t_0}{t_1 - t}\rho_X(t) \\ &< \frac{t_0 - t}{t_1 - t}t_1 + \frac{t_1 - t_0}{t_1 - t}t = t_0, \end{aligned}$$

a contradiction. So $\rho_X(t) = t$. □

Remark 9.1.7 A uniformly non-square Banach space is super-reflexive (James [8]) if it has an equivalent norm $|||\cdots|||$ in which it is uniformly convex. The converse is not true in the sense that $\delta_{(X,|\ |)}(\epsilon) = 0$ for $0 < \epsilon < 2$.

Example 9.1.8 For $1 < p < \infty$, consider new norms in ℓ_p as follows:

$$|||x||| = \max\left\{|x_1| + |x_2|, \left(\sum_{k=3}^{\infty} |x_k|^p\right)^{\frac{1}{p}}\right\}$$
$$|||x|||' = \max\left\{|x_1|, |x_2|, \left(\sum_{k=3}^{\infty} |x_k|^p\right)^{\frac{1}{p}}\right\}$$

$||x||_p \leq 2^{\frac{1}{p}} |||x||| \leq 2^{\frac{1}{p+1}}, |||x|||' \leq 2^{\frac{1}{p}+1} |||x||| \leq 4||x||_p$. Since ℓ_p is uniformly convex $(\ell_p, ||| \cdot |||)$ and $(\ell_p, ||| \cdot |||')$ are super-reflexive but not uniformly non- square sine $\delta_X(\epsilon) = 0$ for all $0 < \epsilon < 2$. To see this take $x = (1, 0, 0, \ldots)$ and $y = (0, 1, 0, \ldots)$.

From the above example, one can see that for any real Banach space X with $dim(X) \geq 2$, there is an equivalent norm in which X is not uniformly non-square.

Garcia-Falset et al. [5] proved that in uniformly non- square Banach spaces, every bounded closed convex subset has the fixed point property for non-expansive mappings. This is achieved by studying the properties of certain coefficients associated with the geometry of Banach spaces.

9.2 Coefficients of Banach Spaces and Fixed Points

Dominguez-Benavides [3] defined in a Banach space X, the following parameters, in terms of asymptotic diameter and radius of a sequence.

Definition 9.2.1 For a sequence (x_n) in X

$$diam_a(x_n) = \lim_k \sup\{||x_n - x_m\| : n, m \geq k\}$$
$$r_a(x_n) = \inf\{\limsup_n ||x_n - y|| : y \in (x_n)\}$$
$$WCS(X) = \inf\left\{\frac{diam_a(x_n)}{r_a(x_n)} : (x_n) \text{ is a weakly convergent sequence which is not norm convergent}\right\}$$

($WCS(X)$ is called Bynum's weakly convergent sequence coefficient of X. If $WCS(X) > 1$, then X has weak normal structure in the sense that every weakly compact convex subset of X with more than one element is not diametral. X has normal structure if each bounded convex subset with more than one element has a non-diametral point.

Definition 9.2.2 (*Dominguez Benavides* [3]) For a Banach space X and $a \geq 0$, the parameter

$$R(a, X) := \sup\{\liminf_{n\to\infty} ||x + x_n|| : ||x|| \leq a \text{ and } (x_n) \text{ weakly null sequence with } D[(x_n)] \leq 1\}$$

Here $D[(x_n)] := \limsup_{n\to\infty} \left(\limsup_{m\to\infty} ||x_n - x_m|| \right) \leq 1$.
The coefficient $M(X) := \sup\left\{ \frac{1+a}{R(a,X)} : a \geq 0 \right\}$.
$R(X) := \sup\{\underline{\lim}||x + x_n||(x_n)$ weakly null in B_X and $x \in B_x\}$.

Kirk [11] proved that every closed bounded subset having normal structure in a reflexive Banach space has the fixed point property for non-expansive self-maps. This can be deduced from the following lemma due independently to Goebel [6] and Karlovitz [9], as shown in the proof of Theorem 9.2.5.

Lemma 9.2.3 (Goebel [6], Karlovitz [9]) *Lex X be a Banach space, C_0, a weakly compact convex subset of X and $T : C_0 \to C_0$ a non-expansive map. Let C_0 be a minimal closed convex set that is invariant under T. (i.e. no proper closed convex subset of C_0 is invariant under T). For each sequence of x_n in C_0 with $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$ and for each $x \in C_0$ $\lim_{n\to\infty} ||x_n - Tx_n|| = diam C_0$ (such a sequence x_n with $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$ is called a sequence of approximate fixed points).*

Proof For $y \in C_0$, let $s = \limsup_{n\to\infty} ||y - x_n||$. Let $D = \{x \in C_0 : \limsup_{n\to\infty} ||x - x_n|| \leq s\}$. Clearly D is non-void closed and convex. D is invariant under T, since

$$\begin{aligned} ||Tx - x_n|| &\leq ||Tx - Tx_n|| + ||Tx_n - x_n|| \\ &\leq ||x - x_n|| + ||Tx_n - x_n|| \end{aligned}$$

and $||x_n - Tx_n|| \to 0$. So by the minimality of C_0, $D = C_0$. We can find a subsequence $(x_{n'})$ of (x_n) with $\lim ||y - x_{n'}|| = s'$. Let $z \in C_0$ be such that $||z - x_{n''}|| \to t$ for some subsequence $\{x_{n''}\}$ of $\{x_{n'}\}$. Define $E = \{x \in C_0 : \limsup \|x - x_{n''}|| \leq \min\{t, s'\}\}$. By repeating this argument, it follows that $E = C_0$. So $y, z \in E$ and $t = s'$. Hence, for each $x \in C_0$ $\lim ||x - x_{n'}||$ exists and equals s'.

We claim that $s' = r = diam\ C_0$. Whence it follows that $||x_{n'} - y|| \to r$ whenever $\{||y - x_{n'}||\}$ converges and consequently by the boundedness $||y - x_n|| \to r$ for the entire sequence. Repetition of this argument by replacing $\{x_{n'}\}$ by $\{x_n\}$ shows that $\lim ||x - x_n|| = diam\ C_0$.

Consider $F = \{u \in C_0 : ||u - x|| \leq s'$ for each $x \in C_0\}$. F is non-void as there is a weakly convergent subsequence say $\{x_{n'}\}$ with limit z. Because $||x - x_{n'}|| \to s'$ for each $x \in C_0$, $||x - z|| \leq s'$ for each $x \in C_0$. So $z \in F$. If $s' < r$ then F is a proper subset of C_0, contradicting the minimality of C_0, since F is invariant under T, as well. As C_0 is minimal closed convex subset invariant under T, closed convex hull of $T(C_0) = C_0$. For $u \in C_0$, given $\epsilon > 0$, we can find $v = \sum_{k=1}^{m} \lambda_k T x_k$ with $x_k \in C_0$, $\lambda_k > 0$ and $\sum \lambda_k = 1$ with $||u - v|| \leq \epsilon$. For $w \in F$, $||Tw - u|| \leq ||Tw - v|| + ||u - v|| \leq \sum \lambda_k ||Tw - Tx_k|| + ||v - u|| \leq \sum \lambda_k ||w - x_k|| + ||v - u|| \leq s' + \epsilon$. So $Tw \in F$. Thus F is T-invariant. Hence the lemma. □

At this stage we recall the definition of normal structure.

Definition 9.2.4 A convex subset K of of a Banach space X is said to have normal structure if for each bounded convex subset K_1 of K containing more than one point, there is a non-diametral point in the sense that for some $x_0 \in K_1$, $\sup\{||x_0 - k|| : k \in K_1\} < diam\ K_1$.

With this we can prove Kirk's theorem [11].

Theorem 9.2.5 *Let K be a non-empty closed convex bounded subset of a reflexive Banach space X such that K has normal structure. If $T : K \to K$ is non-expansive then T has a fixed point in K.*

Proof Since K is a bounded closed convex subset of the reflexive Banach space X, K is weakly compact. By a standard application of Zorn's lemma, K has a minimal non-empty closed convex subset C_0 invariant under T. Let $a \in C_0$. Then $T_n(x) = \frac{a}{n} + \left(1 - \frac{1}{n}\right) T x_n$, $n \geq 2$ is a contraction mapping C_0 into itself and hence has a unique fixed point x_n. Thus $x_n = \frac{a}{n} + \left(1 - \frac{1}{n}\right) T x_n$ for each $n \geq 2$. Further $\lim ||x_n - Tx_n|| \to 0$ as $n \to \infty$. So by Goebel-Karlovitz Lemma 9.2.3. $\lim_{n\to\infty} ||x - x_n|| = diam\ C_0$ for all $x \in C_0$. If C_0 contains more than a singleton, it contains a non-diametral point a' such that $\sup ||a' - x|| < diam\ C_0$ as $C_0 \subseteq K$ is a closed convex subset of K and K has normal structure. This implies that $\lim_{n\to\infty} ||a' - x_n|| \leq \sup_n ||a' - x_n|| < diam\ C_0$, contradicting Lemma 9.2.3. So C_0 is a singleton which necessarily is a fixed point of T as $TC_0 \subseteq C_0$. □

Remark 9.2.6 Lin [12] has generalized Goebel-Karlovitz lemma using certain non-standard analytic considerations. Let X be a Banach space with norm $||\cdot||$, $\ell_\infty(X)$, the space of sequences (x_n) in X with norm $\sup\{||x_n|| : n \in \mathbb{N}\}$ and $C_0(X)$ the subspace of $\ell_\infty(X)$ with null sequence (x_n) of X and $[X]$ the quotient space $\ell_\infty(X)/C_0(X)$ with the norm $|||[z_n]||| = \limsup ||z_n||$ where $[z_n]$ is the equivalent class of $\{z_n\} \in \ell_\infty$. $x \in X$ is identified with the class $[x, x, \ldots]$ and consequently X can be considered a subset of $[X]$. For a subset K of X the set $[K] = \{[z_n] \in [X] : z_n \in K$ for every $n \in \mathbb{N}\}$. If $T : K \to K$ is a map then $[T] : [K] \to [K]$ is a map defined in a natural way by $[T]([x_n]) = [Tx_n]$.

Lemma 9.2.7 (Lin [12]) *Let X be a Banach space and K a minimal weakly compact convex subset of X, invariant under T. If $[W]$ is a non-empty closed convex subset of $[K]$ which is invariant under $[T]$ then*

$$\sup\{|||[w_n] - x||| : [w_n] \in [W]\} = diam(K)$$

for each $x \in K$.

We leave the details of the proof to the reader. The following theorem due to Dominguez Benavides makes use of Lin's lemma.

Theorem 9.2.8 (Dominguez Benavides [3]) *If* X *is a Banach space for which* $R(a, X) < 1 + a$ *for some* $a \geq 0$*, then every non-empty bounded convex weakly compact subset of* X *has fixed point property for non-expansive mappings.*

Proof Suppose the theorem is false. Then, there is a non-empty weakly compact convex subset K of X with $diam(K) = 1$ and K is minimal invariant for a non-expansive map T without a fixed point. Further there is a weakly null sequence of approximate fixed points $\{x_n\}$ of T in X. Define

$$[W] := \{[z_n] \in [K] : ||[z_n] - [x_n]|| \leq 1 - t \text{ and } \overline{\lim_n}\, \overline{\lim_m} ||z_n - z_m|| \leq t\}$$

where $t = \frac{1}{1+a}$.

Clearly $[W]$ is a a closed convex $[T]$ invariant set. $[W]$ is non-empty as it contains $[tx_n]$. So by Lin's Lemma 9.2.7 it follows that

$$\sup\{||[w_n] - x|| : [w_n] \in [W]\} = 1$$

for each $x \in [K]$. For $[z_n] \in [W]$ choose a weakly convergent subsequence $\{y_n\}$ of $\{z_n\}$ such that $\overline{\lim}_n ||z_n|| = \lim ||y_n||$ and $\lim_{n,m;n \neq m} ||v_m||$ exists. Now $\lim_{n,m,n \neq m} ||y_n - y_m|| = \overline{\lim_n}\, \overline{\lim_m} ||y_n - y_m|| \leq \overline{\lim_n}\, \overline{\lim_m} ||z_n - z_m|| \leq t$. Let y be the weak limit of $\{y_n\}$. For each $n \in \mathbb{N}$, $||y_n - y|| \leq \liminf_m ||y_n - y_m|| \leq \limsup_m ||y_n - y_m||$. So

$$\overline{\lim_n} ||y_n - y|| \leq \overline{\lim_n}\, \overline{\lim_m} ||y_n - y_m|| \leq t.$$

We can choose $\eta > 0$ such that $\eta R(a, X) < 1 - \frac{R(a,X)}{1+a}$ (i.e. $R(a, X)\left(\eta + \frac{1}{1+a}\right) < 1$. For a large n, we have $||y_n - y|| \leq t + \eta$. Also $||y|| \leq \underline{\lim} ||y_n - x_n|| \leq 1 - t$. So $\left\| \frac{y_n}{t+\eta} \right\| = \left\| \frac{y_n - y}{t+\eta} + \frac{y}{t+\eta} \right\| \leq R\left(\frac{1-t}{t}, X\right) = R(a, X)$. So $\overline{\lim} ||z_n|| = \lim ||y_n|| \leq R(a, X)(t + \eta) < 1$, a contradiction to Lemma 9.2.7. □

At this stage, we can introduce the concept of Banach–Mazur distance between isomorphic Banach spaces, a useful concept in fixed point theory of non-expansive maps in Banach spaces.

Definition 9.2.9 For isomorphic normed spaces X and Y, the Banach–Mazur distance between X and Y denoted by $d(X, Y)$ is defined as

$$d(X, Y) = \inf\{||T||||T^{-1}|| : T : X \to Y \text{ is an isomorphism}\}$$

Remark 9.2.10 For isomorphic normed spaces E, F, G, we have

(i) $d(E, F) = d(F, E)$;
(ii) $d(E, F) \geq 1$;
(iii) $d(E, F) \leq d(E, G)d(G, F)$
(iv) Let $c > 1$ and B_E and B_F be the closed unit balls in E and F. Then, $d(E, F) < c$ if and only if there exist $c_1, c_2 > 0$ with $c_1c_2 < c$ and an invertible $T \in L(E, F)$ such that $\frac{1}{c_1}B(F) \subseteq TB(E) \subseteq c_2B(E)$ holds.

We skip the proof of

Theorem 9.2.11 (Dominguez Benavides [3]) *For isomorphic Banach spaces X and Y*

$$R(a, Y) \leq d(X, Y)R(a, X)$$

where $a \geq 0$.

For a Banach space X, the coefficient $M(X)$ can be defined as in the following

Definition 9.2.12 If X is a Banach space, $M(X)$ is defined as $\sup\left\{\frac{1+a}{R(a,X)} : a \geq 0\right\}$

Theorems 9.2.8 and 9.2.11 lead to

Theorem 9.2.13 (Dominguez Benavides [3]) *If X is a Banach space with $M(X) > 1$ then every non-empty convex weakly compact subset of X has fixed point property for non-expansive mappings. If Y is another Banach space isomorphic to X and $d(X, Y) < M(X)$, then every non-empty convex weakly compact subset of Y has fixed point property for non-expansive mappings.*

Proof Since $M(X) > 1$, for some $a > 0$ $\frac{1+a}{R(a,X)} > 1$. So by Theorem 9.2.8 the first part of theorem follows. Since $M(X) > 1$, for some $a > 0$, $\frac{1+a}{R(a,X)} > 1$. Since by Theorem 9.2.11, $R(a, Y) \leq d(X, Y)R(a, X) < M(X)R(a, X) < \frac{1+a}{R(a,X)}R(a, X) = 1 + a$. So by Theorem 9.2.8, the fixed point property for non-expansive maps of a non-void convex weakly compact subset of Y follows. □

9.3 Nearly Uniformly Smooth Spaces

Definition 9.3.1 Let X be a Banach space with a Schauder basis (e_n) (that is each $x \in X$ has a unique representation $x = \sum_{n=1}^{\infty} x_n e_n$, x_n being scalars). X is called nearly uniformly smooth (NUS) if for each $\epsilon > 0$ there is $\delta > 0$ such that if $0 < t < \delta$ and (x_n) is a basic sequence in B_X there exists $k > 1$ so that $||x_1 + tx_k|| < 1 + t\epsilon$.

X is called weakly near uniformly smooth (WNUS) if the above definition holds for some $\epsilon > 0$.

Garcia Falset [4] proved the following

Theorem 9.3.2 *Let X be a Banach space. The following are equivalent:*

(a) there exists $\epsilon > 0$ and $\delta > 0$ such for all $t \in [0, \delta)$ and every weakly null sequence (x_n) in B_X there is $k > 1$ with $||x_1 + tx_k|| < 1 + t\epsilon$;
(b) there exists $c \in (0, 1)$ such that for each weakly null sequence (x_n) in B_X there is $k > 1$ with $||x_1 + x_k|| \leq 2 - c$;
(c) $R(X) < 2$.

Proof (b) ⇒ (c). Let (x_n) be a weakly null sequence in B_X and $x \in B_X$. Define (y_n) by $y_1 = x$, $y_{n+1} = x_n$ for $n \geq 1$. Then (y_n) is a weakly null sequence in B_X. By (b), there exists $c \in (0, 1)$ and $k_1 > 1$ with $||x + x_{k_1}|| \leq 2 - c$. Define another weakly null sequence $z_n \in B$ defined by $z_1 = x$ and $z_n = x_{k_1+n}$, $n \in \mathbb{N}$. By (b) there exists $k_2 > k_1$ such that $||x + x_{k_2}|| \leq 2 - c$. Thus, proceeding recursively, we can get a subsequence x_{k_n} of weakly null sequence in B_X such that $||x + x_{k_n}|| \leq 2 - c$ for all k_n and $k_n > 1$ for all n. So $\underline{\lim}||x + x_n|| \leq 2 - c$, for any weakly null sequence (x_n) in B_X. So $R(X) < 2$.

(c) ⇒ (b). Let (x_n) be a weakly null sequence in B_X. As $R(X) < 2$, $R(X) < 2 - c$ for some $c \in (0, 1)$. So $\underline{\lim}||x_1 + x_n|| \leq R(X) < 2 - c$. So for some $k > 1$, $||x_1 + x_k|| \leq 2 - c$.

(a) ⇒ (b). By assumption, there exist $\epsilon, \delta > 0$ such that for all $t \in (0, \delta)$ and any weakly null sequence (x_n) in B_X there is $k > 1$ with $||x_1 + x_k|| < 1 + t\epsilon$. Let $\mu = \min\{1, \delta\}$. Then for $t < \delta$, $||x_1 + x_k|| \leq ||x_1 + tx_k|| + (1 - t)||x_k|| \leq 1 + \epsilon t + 1 - t = 2 - t(1 - \epsilon) = 2 - c$, where $c = t(1 + \epsilon) \in (0, 1)$.

(b) ⇒ (a). By hypothesis, there exists $c \in (0, 1)$ such that for each weakly null sequence (x_n) in B_X, there is $k > 1$ with $||x_1 + x_k|| \leq 2 - c$. So for all $t \in (0, 1)$,

$$\begin{aligned} ||x_1 + tx_k|| &\leq t||x_1 + x_k|| + (1 - t)||x_1|| \\ &\leq t(2 - c) + 1 - t + 1 + t(1 - c). \end{aligned}$$

□

Corollary 9.3.3 ([4]) *For a Banach space X, the following are equivalent:*

(a) X is WNUS;
(b) X is reflexive and $R(X) < 2$.

Theorem 9.3.4 (Garcia Falset [4]) *Let X be a Banach space such that $R(X) < 2$. Then, every non-empty weakly compact convex subset of X has fixed point property for non-expansive mappings.*

Proof We prove by the method of contradiction. Suppose the theorem is false. Then, there is a weakly compact convex subset K of X with $diam\ K = 1$ which is minimal for a non-expansive map $T : K \to K$ in the sense of Goebel-Karlovitz Lemma 9.2.3. Let (x_n) be a weakly null sequence of approximate fixed points of T in K.

With the usual notation in Remark 9.2.6, define the subset $[W]$ of $[X]$ by $[W] := \{[z_n] \in [K] : ||[z_n] - [x_n]|| \leq \frac{1}{2}, D([z_n]) \leq \frac{1}{2}\}$. $[W]$ is seen to be a T-invariant closed convex subset $[X]$. By Lemma 9.2.3 $\left[\frac{x_n}{2}\right] \in [W]$. So by Lin's Lemma 9.2.7

$\sup\{||[w_n] - x|| : [w_n] \in [W]\} = 1$ for all $x \in K$. Let $[z_n] \in [W]$. $||[z_n]|| = \overline{\lim_{n\to\infty}} ||z_n|| = \lim_{k\to\infty} ||z_{n_k}||$ for some subsequence (z_{n_k}) of (z_n), in K. As K is weakly compact let (z_{n_k}) converge weakly to $y \in K$ without loss of generality. Then, passing to subsequences and using a diagonal argument, one can assume that for $m \in \mathbb{N}$

$$\left\|1 - \frac{1}{m}\right\| ||z_{n_m} - y|| \leq \underline{\lim_{m\to\infty}} ||z_{n_m} - y||$$

Let $y_m = \left(1 - \frac{1}{m}\right) \frac{z_{n_k} - y}{\max\left\{||y||, \underline{\lim_{m\to\infty}} ||z_{n_m} - y||\right\}}$. As (y_m) is a weakly null sequence in B_X, it follows from the definition of $R(X)$ that

$$R(X) \geq \underline{\lim_{m\to\infty}} \left\| y_m + \frac{y}{\max\left\{\underline{\lim_{m\to\infty}} ||z_{n_m} - y||, ||y||\right\}} \right\|$$

So $\underline{\lim_{m\to\infty}} ||z_{n_m}|| \leq R(X) \max\left\{||y||, \underline{\lim_{m\to\infty}} ||z_{n_m} - y||\right\}$. Since $(z_{n_m} - x_{n_m})$ converges weakly to y, $||y|| \leq \underline{\lim_{m\to\infty}} ||z_{n_m} - x_{n_m}|| \leq \overline{\lim_{m\to\infty}} ||z_n - x_n|| = ||[z_n] - [x_n]|| \leq \frac{1}{2}$. On the other hand, the weak limit of $\{z_{n_m} - y - (z_{n_s} - y)\} = z_{n_m} - y$ as $s \to \infty$. So $||z_{n_m} - y|| \leq \overline{\lim_{m\to\infty}} ||z_{n_m} - y - (z_{n_s} - y)||$ or $\underline{\lim_{m\to\infty}} ||z_{n_m} - y|| \leq \overline{\lim_{m\to\infty}} \left(\overline{\lim_{s\to\infty}} ||z_{n_m} - y - (z_{n_s} - y)||\right.$. As $D(z_{n_m} - y) = D(z_{n_m}) \leq D(z_n)$ and $D([z_n]) \leq \frac{1}{2}$, we have $\overline{\lim_{m\to\infty}} ||z_{n_m} - y|| \leq D([z_n]) \leq \frac{1}{2}$. So $||[z_n] - 0|| = \underline{\lim_{m\to\infty}} ||z_{n_m}|| \leq \frac{R(X)}{2} < 1$, as $\max\left\{\underline{\lim_{m\to\infty}} ||z_{n_m} - y||, ||y||\right\} \leq \frac{1}{2}$. This contradicts Lemma 9.2.7. □

The following are proved using similar arguments.

Theorem 9.3.5 (Garcia Falset [4]) *Let X and Y be isomorphic Banach spaces such that $d(X, Y)R(Y) < 2$. Then, every non-empty convex weakly compact subset of X has the fixed point property for non-expansive maps, provided it satisfies weak Opial's condition. That is,* $\liminf ||x_n|| \leq \liminf ||x_n + x||$ *for each $x \in X$ and each weakly null sequence (x_n).*

Proof Suppose false. Then, there is a convex weakly compact subset K of X with $diam(K) = 1$, which is minimal for a non-expansive map $T : K \to K$ by Goebel-Karlovitz lemma. Thus, there is a weakly null sequence of almost fixed points (x_n) for T in K. As in Remark 9.2.6 consider the subset $[W]$ of $[X]$ defined by

$$[W] = \{[z_n] \in [K] : ||[z_n] - [x_n]|| \leq \frac{1}{2} \text{ and for some } x \in K, ||[z_n] - x|| \leq \frac{1}{2}\}$$

It can be verified that $[W]$ is T-invariant and a closed convex subset of $[K]$. Also $[W]$ is non-empty as $[\frac{x_n}{2}] \in [W]$ as $||[x_n] - 0|| \leq 1$. Thus, by Lin's Lemma 9.2.7 $\sup\{||[w_n] - x|| : [w_n] \in [W]\} = 1$ for all $x \in K$.

Let $[z_n] \in W$ with $||[z_n]|| = \overline{\lim_n} ||z_n|| = \lim_{k\to\infty} ||z_{n_k}||$ where $||z_{n_k}||$ is some subsequence of (z_n). Since K is weakly compact in X, we can assume that (z_{n_k}) weakly converges to $y \in K$. Let T be an isomorphism of X onto Y with $||T||||T^{-1}||. R(Y) < 2$. Clearly $T(z_{n_k})$ converges weakly to $T(y)$. Passing to subsequences we can suppose that for each $k \in N$

$$\left(1 - \frac{1}{k}\right) ||T(z_{n_k}) - T(y)|| \leq \underline{\lim}_{m\to\infty} ||Tz_{n_k} - Ty||.$$

So for all $k \geq 1$, $\left(1 - \frac{1}{k}\right) \max\left\{\underline{\lim}_k ||T(z_{n_k}) - T(y)||, ||Ty||\right\}^{-1} (Tz_{n_k} - Ty)$ and $\max\left\{\underline{\lim}_k ||T(z_{n_k}) - T(y)||, ||Ty||\right\}^{-1} (Ty) \in B_y$. So, from the definition of $R(Y)$ we get $\underline{\lim}_k ||Tz_{n_k}|| = R(Y) \max\left\{\underline{\lim}_k ||T(z_{n_k}) - T(y)||, ||Ty||\right\}$.

On the other hand since $[z_n] \in [W]$, there exists $x_0 \in K$ such that $||[z_n] - x_0|| \leq \frac{1}{2}$ and $||[z_n] - [x_n]|| \leq \frac{1}{2}$. So $||y|| \leq \underline{\lim}_k ||z_{n_k} - x_{n_k}|| \leq ||[z_n] - [x_n]|| \leq \frac{1}{2}$. As Y satisfies weak Opial's condition $\underline{\lim}_k ||Tz_{n_k} - Ty|| \leq \underline{\lim}_k ||Tz_{n_k} - Tx_0||$.

So

$$\begin{aligned}
\underline{\lim}_k ||Tz_{n_k}|| &\leq R(Y)||T|| \max\left\{\underline{\lim}_k ||z_{n_k} - x_0||, ||y||\right\} \\
&\leq R(Y)\frac{||T||}{2}.
\end{aligned}$$

The above conditions imply that

$$\begin{aligned}
||[z_n] - 0|| = \lim_k ||z_{n_k}|| &= \lim_k ||T^{-1}Tz_{n_k}|| \\
&\leq ||T^{-1}|| \lim_k ||Tz_{n_k}|| \\
&\leq ||T^{-1}||||T||\frac{R(Y)}{2} < 1.
\end{aligned}$$

This contradicts Lin's Lemma 9.2.7. □

Corollary 9.3.6 (Garcia Falset [4]) *If X is a weakly nearly uniformly smooth (WNUS) Banach space, then every non-void convex weakly compact subset of X has fixed point property for non-expansive mappings. In particular, a nearly uniformly smooth (NUS) Banach space X also enjoys this property.*

Proof As X is WNUS, X satisfies weak Opial condition. As $d(X, X) = 1$ and $R(X) < 2$ by Theorem 9.3.5 every non-void convex weakly compact subset of X has fixed point property for non-expansive maps. Since an NUS Banach space X has WNUS, every convex weakly compact subset of X has fixed point property for non-expansive mappings. □

9.4 Non-square Banach Spaces

Dominguez Benavides [2] introduced the modulus of nearly uniform smoothness as follows:

Definition 9.4.1 ([2]) The modulus of nearly uniform smoothness of a Banach space X is the function $\Gamma_X : [0, \infty) \to \mathbb{R}$ defined by

$$\Gamma_X(t) = \sup\left\{ \inf\left\{ \frac{||x_1 + tx_n|| + ||x_1 - tx_n||}{2} - 1 : n > 1 \right\} : (x_n) \right.$$
$$\left. \text{is a basic sequence in } B_X. \text{ i.e., } (x_n) \text{ is a Schauder basis for } X. \right\}$$

Remark 9.4.2 $\rho_X(t) = \sup\left\{ \frac{||x+ty||+||x-ty||}{2} : x, y \in B_X \right\}$ is called the modulus of uniform smoothness. Clearly $\rho_X(t) \geq \Gamma_X(t) \geq 0$ for all $t \geq 0$. When X is uniformly smooth, $\rho'(0) = \lim_{t\to 0} \frac{\rho_X(t)}{t} = 0$ by definition. In this case $\lim_{t\to 0} \frac{\Gamma_X(t)}{t} = \Gamma'_X(0) = 0$.

There is an equivalent characterization of near uniform smoothness for reflexive Banach spaces, whose proof is available in [2].

Proposition 9.4.3 ([2]) *Let X be a reflexive Banach space. Then*

$$\Gamma_X(t) = \sup\left\{ \inf\left\{ \frac{||x_1 + tx_n|| + ||x_1 - tx_n||}{2} - 1 : n > 1 \right\} : (x_n) \right.$$
$$\left. \textit{is weakly null in } B_X \right\}$$

As a result we have

Proposition 9.4.4 ([2]) *A Banach space X is nearly uniformly smooth if and only if X is reflexive and* $\lim_{t\to 0} \frac{\Gamma_X(t)}{t} = 0$.

Proof If $\lim_{t\to 0} \frac{\Gamma_X(t)}{t} = 0$, then for $\epsilon > 0$ there exists $\delta > 0$ such that $\Gamma_X(t) \leq t\epsilon$ for $t \in [0, \delta)$. Since X is reflexive, we can find a basic sequence $\{x_n\}$ in B_X as suggested by Prus in §2 of [14] which is not norm convergent for which

$$||x_1 + tx_{n_k}|| \leq \frac{1}{2}\{(1+c)||x_1 - 2tx_{n_k}|| + ||x_1 + 2tx_{n_k}||\}$$

where $c > 1$, $1 + c < \frac{(1+3t\epsilon)}{(1+2t\epsilon)}$ and (x_{n_k}) is a subsequence of $\{x_n\}$. Then for some k $||x_1 + tx_{n_k}|| \leq (1+\epsilon)(1+2t\epsilon) < 1 + 3t\epsilon$.

Conversely if X is nearly uniformly smooth then X is reflexive. Consider a weakly null sequence (x_n). For each $\epsilon > 0$, there is a $\delta' > 0$ with $||x_1 + tz_n|| \leq 1 + \epsilon t$ for all $n > 1$, where z_n is a subsequence of (x_n) with $z_1 = x_1$. Since $\{x_1, -z_2, -z_3, \ldots, \}$ is also weakly null $||x_1 - tz_n|| \leq 1 + \epsilon t$ for some $0 < \delta < \delta'$ for all $t \in [0, \delta)$. So for $0 < t < \delta$

$$\frac{||x_1 + tz_n|| + ||x_1 - tz_n||}{2} - 1 \leq \epsilon t$$

So $\lim_{t\to 0} \frac{\Gamma_X(t)}{t} = 0$. □

For the proof that a non-square Banach space has the fixed point property for non-expansive mappings on non-empty bounded closed convex subsets, Garcia-Falset et al. [5] introduced the coefficient $RW(X)$ and $MW(X)$ relating them to the coefficients $R(X)$ and $W(X)$.

Definition 9.4.5 ([5]) Let X be a Banach space and a, a positive real number. Then

$$RW(a, X) := \sup\left\{\min\left(\underline{\lim}_{n\to\infty} ||x_n + x||, \underline{\lim}_{n\to\infty} ||x_n - x||\right) : x_n \in B_X, x_n \xrightarrow{w} 0, ||x|| \leq a\right\}$$

$$MW(X) := \sup\left\{\frac{1+a}{RW(a, X)} : a > 0\right\}.$$

Remark 9.4.6 For any Banach space X and $a > 0$, $\max\{a, 1\} \leq RW(a, X) \leq 1 + a$. So $1 \leq MW(X) \leq 2$.

Lemma 9.4.7 ([5]) *Let X be a Banach space. Given $x \in X$ and a bounded sequence (x_n) in X, there is a subsequence (x_{n_k}) of (x_n) such that*

$$\underline{\lim}_{n\to\infty} ||(x_{n_k} - x_{n_{k+1}}) + x|| \geq \underline{\lim}_{n\to\infty} \underline{\lim}_{m\to\infty} ||(x_n - x_m) + x||$$

$$\underline{\lim}_{n\to\infty} ||(x_{n_k} - x_{n_{k+1}}) - x|| \geq \underline{\lim}_{n\to\infty} \underline{\lim}_{m\to\infty} ||(x_n - x_m) - x||$$

and

$$\overline{\lim}_{k\to\infty} ||(x_{n_k} - x_{n_{k+1}})|| \leq D[(x_n)]$$

Proof Let

$$a := \underline{\lim}_{n\to\infty} \underline{\lim}_{m\to\infty} ||(x_n - x_m) + x||$$

$$b := \underline{\lim}_{n\to\infty} \underline{\lim}_{m\to\infty} ||(x_n - x_m) - x||$$

By definitions of a, b and $D[(x_n)]$, we can find $n_1 \geq 1$ such that

$$\underline{\lim}_{m\to\infty} ||(x_{n_1} - x_m) + x|| > a - \frac{1}{2}$$
$$\underline{\lim}_{m\to\infty} ||(x_{n_1} - x_m) - x|| > b - \frac{1}{2}$$

and

$$\overline{\lim}_{m\to\infty} ||(x_{n_1} - x_m)|| < D[(x_n)] + \frac{1}{2}.$$

Suppose $n_1 < n_2 < \cdots < n_j$ have been defined such that for each $k = 1, 2, \ldots, j$,

$$\underline{\lim}_{m\to\infty} ||(x_{n_k} - x_m) + x|| > a - \frac{1}{k+1}$$
$$\underline{\lim}_{m\to\infty} ||(x_{n_k} - x_m) - x|| > b - \frac{1}{k+1}$$
$$\overline{\lim}_{m\to\infty} ||x_{n_k} - x_m|| < D[(x_n)] + \frac{1}{k+1}$$

and for each $k = 1, 2, \ldots, j-1$

$$||(x_{n_k} - x_{n_{k+1}}) + x|| > a - \frac{1}{k+1}$$
$$||(x_{n_k} - x_{n_{k+1}}) - x|| > b - \frac{1}{k+1}$$
$$||x_{n_k} - x_{n_{k+1}}|| < D[(x_n)] + \frac{1}{k+1}.$$

From the above inequalities for $k = j$ and the definition of a, b and $D[(x_n)]$, we can find $n_{j+1} > n_j$ such that

$$||(x_{n_j} - x_{n_{j+1}}) + x|| > a - \frac{1}{j+1}$$
$$||(x_{n_j} - x_{n_{j+1}}) - x|| > b - \frac{1}{j+1}$$
$$||x_{n_j} - x_{n_{j+1}}|| < D[(x_n)] + \frac{1}{j+1}$$

$$\underline{\lim}_{m\to\infty} ||(x_{n_{j+1}} - x_m) + x_m|| > a - \frac{1}{j+2}$$
$$\underline{\lim}_{m\to\infty} ||(x_{n_{j+1}} - x_m) - x_m|| > b - \frac{1}{j+2}$$
$$\overline{\lim}_{m\to\infty} ||x_{n_{j+1}} - x_m|| < D[(x_n)] + \frac{1}{j+2}.$$

So by induction, there exists an increasing sequence of natural numbers (n_k) $k \geq 1$ such that $n_k < n_{k+1}$ for all k with

$$\underline{\lim}_{k\to\infty} ||(x_{n_k} - x_{n_{k+1}}) + x|| \geq a,$$
$$\underline{\lim}_{k\to\infty} ||(x_{n_k} - x_{n_{k+1}}) - x|| \geq b$$
$$\overline{\lim}_{k\to\infty} ||x_{n_k} - x_{n_{k+1}}|| \leq D[(x_n)]$$

□

The next theorem makes use of the above lemma.

Theorem 9.4.8 ([5]) *Let X be a Banach space for which B_{X^*} is w^*-sequentially compact. For each $a > 0$, $R(a, X) \leq RW(a, X)$. Hence $M(X) \geq MW(X)$.*

Proof Let $a, \eta > 0$. From the definition of $R(a, X)$, we can find $x \in X$ with $||x|| \leq a$ and a weakly null sequence (x_n) in B_X with $D[(x_n)] \leq 1$ such that

$$\underline{\lim}_{n\to\infty} ||x_n + x|| \geq R(a, X) - \eta$$

For each $n \geq 1$, choose $f_n \in S_{X^*}$ such that

$$f_n(x_n + x) = ||x_n + x||$$

As B_{X^*} is sequentially compact in w^*-topology we can without loss of generality assume that (f_n) converges in w^* topology to some $f \in B_{X^*}$. As (x_m) converges weakly to zero, from the weak lower semicontinuity of the norm, $\underline{\lim}_{n\to\infty} ||(x_n + x_m) + x|| \geq ||x_n + x||$ for all $n \geq 1$. So

$$\underline{\lim}_{n\to\infty} \underline{\lim}_{m\to\infty} ||(x_n - x_m)|| \geq \underline{\lim}_{n\to\infty} ||x_n + x|| \geq R(a, X) - \eta$$

Again for $n \geq 1$, as $f_m \overset{w^*}{\to} f$, $\lim_{m\to\infty} f_m(x_n) = f(x_n)$, we have

$$\begin{aligned}
\varliminf_{m\to\infty} ||(x_n - x_m) - x|| &\geq \varliminf_{m\to\infty} - f_m((x_n - x_m) - x) \\
&= \varliminf_{m\to\infty} f_m(x_m + x) - f(x_n) \\
&= \varliminf_{m\to\infty} ||x_m + x|| - f(x_n) \\
&\geq R(a, X) - \eta - f(x_n).
\end{aligned}$$

So

$$\varliminf_{n\to\infty} \varliminf_{m\to\infty} ||(x_n - x_m) - x|| \geq R(a, X) - \eta,$$

as $x_n \xrightarrow{W} 0$.

Thus we have shown that

$$\min\{\varliminf_{n\to\infty} \varliminf_{m\to\infty} ||(x_n - x_m) + x||, \varliminf_{n\to\infty} \varliminf_{m\to\infty} ||(x_n - x_m) - x|| \geq R(a, X) - \eta$$

Further, $D[(x_n)] \leq 1$. So by Lemma 9.4.7, we can find a subsequence x_{n_k} of x_n with

$$\min\{\varliminf_{k\to\infty} ||(x_{n_k} - x_{n_{k+1}}) + x||, \varliminf_{k\to\infty} ||(x_{n_k} - x_{n_{k+1}}) - x|| \geq R(a, X) - \eta$$

and $\varlimsup_{k\to\infty} ||x_{n_k} - x_{n_{k+1}}|| \leq 1$. So we can find k_0 such that for all $k \geq k_0$, $||x_{n_k} - x_{n_{k+1}}|| \leq 1 + \eta$. Define

$$\begin{aligned}
y_k &:= \frac{x_{n_{k_0+k}} - x_{n_{k_0+k+1}}}{1+\eta}, \ k \geq 1 \\
y &:= \frac{x}{1+\eta}
\end{aligned}$$

(y_k) converges weakly to zero in B_X and $||y|| \leq a$. So by the definition of $RW(a, X)$

$$\begin{aligned}
RW(a, X) &\geq \min\left\{\varliminf_{k\to\infty} ||y_k + y||, \varliminf_{k\to\infty} ||y_k - y||\right\} \\
&= \frac{1}{1+\eta}\left\{\varliminf_{k\to\infty} ||(x_{n_k} - x_{n_{k+1}}) + x||, \varliminf_{k\to\infty} ||(x_{n_k} - x_{n_{k+1}}) - x||\right\} \\
&\geq \frac{R(a, X) - \eta}{1+\eta}
\end{aligned}$$

Allowing η to tend to zero we get $RW(a, X) \geq R(a, X)$. So $M(X) \geq MW(X)$. □

Corollary 9.4.9 *If X is a Banach space with $MW(X) > 1$, then every non-expansive self-map on a convex weakly compact subset of X has a fixed point.*

Proof Clearly by Theorem 9.4.8 $M(X) \geq MW(X) > 1$. We can without loss of generality assume that X is separable and B_{X^*} is w^* sequentially compact. Since

$M(x) > 1$ by Theorems 9.2.13, 9.4.8 implies that on non-void weak compact convex subsets, non-expansive self-maps have fixed points. □

9.5 An Equivalent Definition of $RW(a, X)$ and Fixed Points of Non-expansive Maps in Non-square Banach Spaces

We begin with

Proposition 9.5.1 ([5]) *Let X be a Banach space and $a > 0$. Then*

$$RW(a, X) = \sup\{\inf_{n>1}(||ax_1 + x_n|| \wedge ||ax_1 - x_n||) : (x_n) \text{ a weakly null sequence in } B_X\}$$

Here, $\alpha \wedge \beta$ and $\alpha \vee \beta$ represent, respectively, the minimum and maximum of two numbers α and β.

Proof For $a > 0$, write

$$\widetilde{RW(a, X)} = \sup\{\inf_{n>1}(||ax_1 + x_n|| \wedge ||ax_1 - x_n||) : (x_n) \text{ a weakly null sequence in } B_X\}.$$

Let $\eta > 0$, (x_n) a weakly null sequence in B_X and $x \in X$ with $||x|| \le a$. Define (y_n) by

$$y_n = \begin{cases} x/a, & n = 1 \\ x_n, & n \ge 2 \end{cases}$$

(y_n) is a weakly null sequence in B_X. So

$$\inf_{n>1}(||ay_1 + y_n|| \wedge ||ay_1 - y_n||) \le \widetilde{RW(a, X)}.$$

So there exists $n_1 > 1$ such that

$$||x + x_{n_1}|| \wedge ||x - x_{n_1}|| = ||ay_1 + y_n|| \wedge ||ay_1 - y_n|| < \widetilde{RW(a, X)} + \eta.$$

Suppose $n_1 < n_2 \ldots < n_k$ have been found such that for $j \in \{1, 2, \ldots, k\}$

$$||x + x_{n_j}|| + ||x - x_{n_j}|| < \widetilde{RW(a, X)} + \eta$$

Define (z_n) by $z_1 = \frac{x}{a}$, $z_n = x_{n+n_k}$, $n \ge 2$ (z_n) is a weakly null sequence in B_X with

$$\inf_{n>1}(||az_1 + z_n|| \wedge ||az_1 - z_n||) \le \widetilde{RW(a, X)}$$

so that there is $k > 1$ such that

$$||az_1 + z_k|| \wedge ||az_1 - z_k|| < \widetilde{RW(a, X)} + \eta.$$

Define $n_{k+1} = k + n_k$. We get

$$||x + x_{n_{k+1}}|| \wedge ||x - x_{n_{k+1}}|| = ||az_1 + z_k|| \wedge ||az_1 - z_k|| < \widetilde{RW(a, X)} + \eta.$$

So by induction, there exists a subsequence (x_{n_k}) with

$$||x + x_{n_{k+1}}|| \wedge ||x - x_{n_{k+1}}|| < \widetilde{RW(a, X)} + \eta$$

We can get a subsequence (w_m) of (x_{n_k}) for which $\lim_{m\to\infty} ||x + w_m||$, $\lim_{m\to\infty} ||x - w_m||$ exist.

Now

$$\begin{aligned}\underline{\lim}_{n\to\infty} ||x + x_n|| \wedge \underline{\lim}_{n\to\infty} ||x - x_n|| &\leq \underline{\lim}_{m\to\infty} ||x + w_m|| \wedge \underline{\lim}_{m\to\infty} ||x - w_m|| \\ &= \underline{\lim}_{m\to\infty} (||x + w_m|| \wedge ||x - w_m||) \\ &\leq \widetilde{RW(a, X)} + \eta\end{aligned}$$

As $\eta > 0$ is arbitrary we get

$$RW(a, X) \leq \widetilde{RW(a, X)}.$$

For $\eta > 0$ and (x_n) a weakly null sequence in B_X, suppose

$$\underline{\lim}_{n\to\infty} (||ax_1 + x_n||) \wedge \underline{\lim}_{n\to\infty} (||ax_1 - x_n||) = \underline{\lim}_{n\to\infty} ||ax_1 + x_n||$$

So by definition of $RW(a, X)$,

$$\underline{\lim}_{n\to\infty} ||ax_1 + x_n|| \leq RW(a, X)$$

Thus there exists $k > 1$ such that

$$||ax_1 + x_n|| < RW(a, X) + \eta$$

Hence $\inf_{n>1}(||ax_1 + x_n|| \wedge ||ax_1 - x_n||) < RW(a, X) + \eta$.

So in this case $\widetilde{RW(a, X)} \leq RW(a, X)$, as $\eta > 0$.

If $\left(\underline{\lim}_{n\to\infty} ||ax_1 + x_n||\right) \wedge \left(\underline{\lim}_{n\to\infty} ||ax_1 - x_n||\right) = \underline{\lim}_{n\to\infty} ||ax_1 - x_n||$ by a similar argument $\widetilde{RW(a, X)} \leq RW(a, X)$. □

Theorem 9.5.2 ([5]) *Let X be a reflexive Banach space. Then*

(i) for all $t > 0$, $\Gamma_X(t) \leq \dfrac{t(RW(\frac{1}{t}, X) + 1) - 1}{2}$;

(ii) for all $a > 0$, $RW(a, X) \leq a(1 + \Gamma_X(\dfrac{1}{a}))$ *and in particular* $MW(X) \geq \sup\{\frac{1+t}{1+\Gamma_X(t)} : t > 0\}$

Proof (i) Let $t, \eta > 0$ and (x_n) be a weakly null sequence in B_X. By Proposition 9.5.1

$$\inf_{n>1}(||x_1 + tx_n|| \wedge ||x_1 - tx_n||) = t \inf_{n>1}\{||\frac{1}{t}x_1 + x_n|| \wedge ||\frac{1}{t}x_1 - x_n||\}$$
$$\leq tRW(\frac{1}{t}, X).$$

So for some $k > 1$

$$||x_1 + tx_k|| \wedge ||x_1 - tx_k|| < tRW(\frac{1}{t}, X) + \eta$$

As $||x_1 + tx_k|| \vee ||x_1 - tx_k|| \leq 1 + t$, we have

$$||x_1 + tx_k|| + ||x_1 - tx_k|| = (||x_1 + tx_k|| \wedge ||x_1 - tx_k||) + ||x_1 + tx_k|| \vee ||x_1 - tx_k||$$
$$< tRW(\frac{1}{t}, X) + \eta + 1 + t$$

So

$$\inf_{n>1}\left(\frac{||x_1 + tx_n|| + ||x_1 - tx_n||}{2} - 1\right) \leq \frac{||x_1 + tx_k|| + ||x_1 - tx_k|| - 1}{2}$$
$$< \frac{t(RW(\frac{1}{t}, X) + 1) - 1 + \eta}{2}.$$

So $\Gamma_X(t) < \frac{t(RW(\frac{1}{t},X)+1)-1+\eta}{2}$. As $\eta \to 0$, $\Gamma_X(t) \leq \frac{t(RW(\frac{1}{t},X)+1)-1}{2}$.
(ii) Let $a, \eta > 0$ and (x_n) be a weakly null sequence in B_X. So

$$\inf_{n>1}\left(\frac{||x_1 + \frac{x_n}{a}|| + ||x_1 - \frac{x_n}{a}||}{2} - 1\right) \leq \Gamma_X(\frac{1}{a})$$

So there exists $k > 1$ with

$$||x_1 + \frac{x_k}{a}|| + ||x_1 - \frac{x_k}{a}|| < 2(1 + \Gamma_X(\frac{1}{a}) + \eta)$$

So

$$
\begin{aligned}
\inf_{n>1} ||ax_1 + x_n|| \wedge ||ax_1 - x_n|| &\le ||ax_1 + x_k|| \wedge ||ax_1 - x_k|| \\
&\le \frac{1}{2}(||ax_1 + x_k|| \wedge ||ax_1 - x_k||) \\
&= \frac{a}{2}(||x_1 + \frac{x_k}{a}|| \wedge ||x_1 - \frac{x_k}{a}||) \\
&< a(1 + \Gamma_X(\frac{1}{a}) + \eta)
\end{aligned}
$$

Since $\eta > 0$ is arbitrary, we get

$$RW(a, X) \le a(1 + \Gamma_X(\frac{1}{a})).$$

□

Theorem 9.5.3 ([5]) *If X is a reflexive Banach space for which $\Gamma'_X(0) < 1$, then $M(X) > 1$. So every non-void bounded closed convex subset of X has fixed point property for non-expansive maps.*

Proof Clearly $\Gamma'_X(0) < 1$ if and only if for some $t > 0$ $\sup\{\frac{\Gamma_X(s)}{s} : 0 < s \le t\} < 1$. So for some $t > 0$ $\Gamma_X(t) < t$. So by (ii) $MW(X) \ge \sup\{\frac{1+t}{1+\Gamma_X(t)} : t > 0\}$. Or $MW(X) > 1$. So by Corollary 9.4.9, and the reflexivity of X every non-void closed bounded convex subset of X has fixed point property for non-expansive maps. □

Corollary 9.5.4 ([5]) *If X is a uniformly non-square Banach space, then $M(X) > 1$ and so every non-void closed bounded convex subset of X has fixed point property for non-expansive maps.*

Proof Since X is uniformly non-square, so is X^*. By Proposition 9.1.6, $\epsilon_0(X^*) < 2$. So by Lindenstrauss formulae the modulus of smoothness ρ_X satisfies $\rho'_X(0) < 1$ (again by Proposition 9.1.6). Since $\Gamma_X(t) \le \rho_X(t)$ for all $t > 0$, $\Gamma'_X(0) \le \rho'_X(0) < 1$. As X is reflexive (being non-square) by Theorem 9.5.3 every non-void bounded closed convex subset of X has fixed point property for non-expansive mappings. □

Remark 9.5.5 Garcia Falset et al. [5] have shown that in a reflexive Banach space X, the following are equivalent:

(i) $MW(X) > 1$;
(ii) for some $a > 0$, $RW(a, X) < 1 + a$;
(iii) $\inf\{\frac{1+a}{RW(a,X)} : a > 0\} > 1$;
(iv) $\Gamma'_X(0) < 1$;
(v) for some $t > 0$, $\sup\{\frac{\Gamma_X(s)}{s} : 0 \le s \le t\} < 1$;
(vi) for some $t > 0$, $\Gamma_X(t) < t$.

Remark 9.5.6 If $MW(X) > 1$, X may not be non-square although every non-void closed bounded convex subset of X has fixed point property for non-expansive maps. This is seen by the following example due to Garcia Falset et al. [5].

Example 9.5.7 Let $X = (\mathbb{R}^2, || \, ||_\infty)$ and $Y = (\ell_2, || \cdot ||_2)$ and Z the product space $Z = X \times Y$ with the norm $||(x; y)|| = \max\{||x||_\infty, ||y||_2\}$, where $x \in X = \mathbb{R}^2$ and $y \in Y = \ell_2$. For $z_1 = ((1, 1); 0)$ and $z_2 = (1, -1); 0)$, $||z_1|| = ||z_2|| = 1$, $||z_1 + z_2|| = ||z_1 - z_2|| = 2$. Thus Z is not uniformly non-square. Let $z_n = (x_n; y_n)$ be a weakly null sequence in B_z and $z \in B_Z$, with $z = (x; y)$. If $z_n \overset{W}{\to} 0$, then $x_n \to 0$ and $y_n \to 0$ and $\lim ||x_n + x||_\infty = ||x||_\infty \leq ||z|| \leq 1$ and $\overline{\lim_{n\to\infty}} ||y_n + y||_2^2 = \overline{\lim_{n\to\infty}} ||y_n||_2^2 + ||y||^2 \leq \overline{\lim_{n\to\infty}} ||z_n||^2 + ||z||^2 \leq 2$. Therefore, $\overline{\lim_{n\to\infty}} ||z_n + z|| = \overline{\lim_{n\to\infty}} \max\{||x_n + x||, ||y_n + y||\} \leq \sqrt{2}$. So $RW(1, X) \leq \sqrt{2}$ and $MW(X) > 1$. So in X every non-void closed convex bounded subset has fixed point property for non-expansive maps.

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Chapter 10
Brouwer's Fixed-Point Theorem

10.1 Introduction

It is more than a century since Brouwer [4] proved a fixed- point theorem of great consequence, in the setting of finite-dimensional Euclidean spaces. It was subsequently extended to normed linear spaces by Schauder [25], and later to locally convex linear topological spaces by Tychonoff [31]. Brouwer's theorem was generalized to multifunctions first by Kakutani [12], and later to locally convex linear topological spaces by Glicksberg [8] and Ky Fan [6]. Brouwer's theorem admits of several proofs. Notable among them are those based on Sperner's lemma [28] or concepts of homotopy/homology from algebraic topology (see Dugundji [5] or Munkres [17]) or concepts and results from Real analysis (see Milnor [16], Seki [26], Rogers [23], Kannai [13], Traynor [30]). However, we provide here only the analytic proof of Brouwer's theorem and a proof based on Sperner's lemma. Needless to state that Brouwer's theorem and its generalizations/variants find a wide range of applications in the solution of nonlinear equations, differential and integral equations, mathematical biology and mathematical economics.

10.2 Analytic Preliminaries

We collect in this section the basic theorems of analysis needed in the proof of Brouwer's theorem.

Theorem 10.2.1 (Weierstrass Approximation Theorem) *If f is a continuous real-valued function defined on a closed bounded subset S of $\mathbb{R}^n$, then for any given positive number ϵ, we can find a polynomial P_ϵ of n variables $x_1, \ldots, x_n$ such that $|f(x_1, \ldots, x_n) - P_\epsilon(x_1, \ldots, x_n)| < \epsilon$ for all $(x_1, x_2, \ldots, x_n) \in S$.*

Theorem 10.2.2 (Inverse Function Theorem) *Let G be a non-empty open set in $\mathbb{R}^n$ and $f = (f_1, \ldots, f_n) : G \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping having continuous*

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first-order partial derivatives at all points of G. If for some point $P_0 \in G$, the determinant of the Jacobian matrix $J(x) = \left(\frac{\partial f_i}{\partial x_j}(x)\right)$ is non-zero at P_0, then f is a one-to-one open map in a neighbourhood of P_0.

Theorem 10.2.3 (Change of variables for multiple integrals) *Let $g = (g_1, \ldots, g_n)$ be a function defined on an open connected set $G \subseteq \mathbb{R}^n$ and taking values in $\mathbb{R}^n$ and having continuous first-order partial derivatives at all points of G. Let g be one-to-one on G and $Det(J_g(x)) \neq 0$ for all $x \in G$, where J_g is the Jacobian matrix of g and $f : g(G) \to \mathbb{R}$ be continuous. If X is a Jordan-measurable compact subregion of G, then*

$$\int_X f(x_1, \ldots, x_n) dx_1 dx_2 \ldots dx_n = \int_{g^{-1}(X)} f(g(t_1, \ldots, t_n)) |Det\ J_g(t_1 \ldots, t_n)| dt_1 \ldots dt_n$$

Theorem 10.2.4 (Mean-value Inequality) *If $f : G \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function with continuous first-order partial derivatives which are bounded in the open connected set G, then for some $M > 0$ and all $x, y \in G$,*

$$\|f(x) - f(y)\| \leq M\|x - y\|.$$

For these and other aspects of calculus in finite-dimensional spaces, Apostol [1] may be consulted.

10.3 Brouwer's Fixed-Point Theorem

We state first Brouwer's fixed-point theorem.

Theorem 10.3.1 (Brouwer) *Every continuous function f mapping the closed unit sphere B_n (of $\mathbb{R}^n$) into itself has a fixed point.*

We deduce it from the no-retraction theorem via a lemma, following Rogers [23]. First, we recall the following.

Definition 10.3.2 Let $B \subseteq A \subseteq \mathbb{R}^n$. A continuous map $f : A \to B$ is called a retraction of A onto B if $f(A) = B$ and f is identity on B. B is called a retract of A if there is a retraction of A onto B.

Lemma 10.3.3 *There is no continuously differentiable retraction of B^n (the unit ball in $\mathbb{R}^n$) onto S^{n-1} the unit sphere (in $\mathbb{R}^n$).*

Proof Let if possible $f : B^n \to S^{n-1}$ be such a retraction. Define $g : B^n \to \mathbb{R}^n$ by $g(x) = f(x) - x$, $x \in B^n$. Clearly g is continuous and has continuous first-order partial derivatives. Clearly $\frac{\partial g_i}{\partial x_j}(x)$ is continuous for all $i, j = 1, 2, \ldots, n$, where $g = (g_1, g_2, \ldots, g_n)$ and all $x \in B^n$. Since B^n is compact $\left|\frac{\partial g_i}{\partial x_j}(x)\right|$ is bounded on

B^n for all $i, j = 1, 2, \ldots, n$, in view of the continuity of $\frac{\partial g_i}{\partial x_j}$ on B^n. So by Theorem 10.2.4 for some $k > 1$,

$$\|g(x) - g(y)\| \le k\|x - y\| \text{ for all } x, y \in B^n.$$

Define $f_t : B^n \to \mathbb{R}^n$ by $f_t(x) = x + tg(x) = (1-t)x + tf(x), \forall\, t \in [0, 1]$. Since f is a retraction of B^n onto S^{n-1}, $f_t(x) = x$ for all $x \in S^{n-1}$ for all $t \in [0, 1]$. Now $f_t'(x) = I + tg'(x)$ for each $x \in B^n$, $f_t'(x)$ being the linear operator represented by the Jacobian matrix of f_t at x. So $\|f_t'(x)\| = \|I + tg'(x)\| \ge \|I\| - t\|g'(x)\|$ for $t \in [0, 1]$. So for $t < \frac{1}{k}$ and $x \in B^n$, $\|tg'(x)\| < 1$ so that f_t' is invertible by Corollary 6.1.7. So by the inverse function Theorem 10.2.2, f_t is an open one-to-one map for $t < k$. For $x \in B^0$ = interior B^n, $\|f_t(x)\| \le (1-t)\|x\| + t\|x\| < 1$, when $t < \frac{1}{k}$. Thus, f_t maps B^0 into B^0 when $t < \frac{1}{k}$ and in this case, we claim that $G_t \doteq f_t(B^0) = B^0$. Let if possible $e \in B^n - G_t$. Let $g \in G_t$. Join e to g and choose a point b on the line segment $[e, g]$ meeting the boundary of G_t. Since $f_t(B^n)$ is compact, being the continuous image of a compact set, $b = f_t(x)$ for some $x \in B^n$. Since b, the boundary point of G_t is not in the open set G_t, x cannot be an interior point of B^n. So $x \in S^{n-1}$ and $f_t(x) = x$. Thus $b = x$ and so e as well as b lies on S^{n-1}, the boundary of B^n. Also $f_t(S^{n-1}) = S^{n-1}$, $B^0 = G_t$ and f_t maps B^n bijectively onto B^n.

Let $C(A)$ denote the Jordan content of a subset A of $\mathbb{R}^n$. Clearly B_n has Jordan content and for $t < \frac{1}{k}$

$$\begin{aligned} C(B_n) &= C(f_t(B_n)) \\ &= \int_{B_n} |Det\ f_t'(x)| dx \text{ by Theorem 10.2.3} \\ &= \int_{B_n} Det(f_t'(x)) dx \text{ for } t < \frac{1}{k}, \end{aligned}$$

which is a polynomial in t. However, left-hand side of the above equality has the constant value $C(B^n)$, and so this polynomial is constant for all $t < \frac{1}{k}$. But for all $t \in [0, 1]$. $I_t = \int_{B^n} Det(f_t'(x))dx$ is a polynomial in t, which is $C(B^n)$ for $0 < t < \frac{1}{k}$. So I_t is constant for all $t \in [0, 1]$. Now, $f_1.f_1 = \|f_1(x)\|^2 = \|x\|^2 = 1$ for $x \in B^n$, as $f_1 = f$ is the retraction of B^n onto S^{n-1}. So $\frac{\partial f_1}{\partial x_i}.f_1 = 0$ for $1 \le i \le n$ on B^n. This system of linear equations has non-trivial solutions on B^n only when $Det\left(\frac{\partial f_1}{\partial x_1}\right) = 0$ where $f_1 = (f_{11}, f_{12}, \ldots, f_{1n})$. So $I(1) = 0$ and this contradicts that $C(B^n) > 0$. □

Theorem 10.3.4 (Brouwer's theorem for differentiable maps) *If $f : B^n \to B^n$ is a continuous map having continuous first-order partial derivatives, then f has a fixed point in B^n.*

Proof Suppose $f : B^n \to B^n$ is a continuous function having continuous first-order partial derivatives without a fixed point. So $f(x) \ne x$ for all $x \in B^n$. Define $w : B^n \to \mathbb{R}^n$ by

$$w(x) = x - \frac{(1 - \|x\|^2)}{(1- < x, f(x) >)} f(x)$$

for $x \in B^n$.

Clearly $< x, f(x) > < 1$ for $x \in B_n$. Otherwise $< x, f(x) > \geq 1$ would imply that $1 \leq < x, f(x) > \leq \|x\| \|f(x)\| \leq 1$ by Cauchy–Schwarz inequality leading to $1 = < x, f(x) > = \|x\| \|f(x)\|$. This would mean that $x = cf(x)$ for some $c \neq 0$ and $1 = < x, f(x) > = c < f(x), f(x) > = c\|fx\|^2 = \frac{1}{c}\|x\|^2$ with $c > 0$. Therefore $c = 1$ with $x = f(x)$, contradicting that f has no fixed point. Thus $< x, f(x) > < 1$ for $x \in B^n$.

Suppose $w(x) = 0$. Then $x = \frac{1-\|x\|^2}{1-<x, f(x)>} f(x) = c' f(x)$, where $c' = \frac{1-\|x\|^2}{1-<x, f(x)>}$. Since $x = c' f(x)$, $c' = \frac{1-|c'|^2\|f(x)\|^2}{1-c'\|f(x)\|^2}$, $c' - c'^2\|f(x)\|^2 = 1 - |c'|^2\|f(x)\|^2$. So $c' = 1$, contradicting that f has no fixed point. So $w(x) \neq 0$ for all $x \in B^n$.

Define $g : B^n \to \mathbb{R}^n$ by $g(x) = \frac{w(x)}{\|w(x)\|}$. Clearly $g(B^n) \subseteq S^{n-1}$ and for $x \in S^{n-1}$, $w(x) = x$ and so $g(x) = x$. Thus g is a continuously differentiable retraction of B^n onto S^{n-1}, contradicting Lemma 10.3.3.

Hence every continuously differentiable map of B^n into itself has a fixed point. □

We are now in a position to prove Brouwer's fixed-point theorem 10.3.1.

Proof of Brouwer's Theorem 10.3.1 Let $f : B^n \to B^n$ be a continuous function. By Weierstrass Approximation Theorem 10.2.1, for any $\frac{1}{m} > 0$, $m \in \mathbb{N}$ there exists a polynomial P_m such that $\|f(x) - P_m(x)\| < \frac{1}{m}$ for all $x \in B^n$. So $\|P_m(x)\| < \|f(x)\| + \frac{1}{m} < 1 + \frac{1}{m}$ for all $x \in B^n$. So by Theorem 10.3.4, $\frac{P_m}{(1+\frac{1}{m})}$ mapping B^n into itself has a fixed point x_m in B^n. Now $\|f(x_m) - x_m\| \leq \|f(x_m) - \frac{P_m(x_m)}{1+\frac{1}{m}}\| \leq \|f(x_m) - P_m(x_m)\| + \|P_m(x_m) - \frac{P_m(x_m)}{1+\frac{1}{m}}\| < \frac{2}{m}$, for each $m \in \mathbb{N}$.

Hence $\inf\{\|x - f(x)\| : x \in B^n\} = 0$. As B^n is compact and $x \to \|x - f(x)\|$ is continuous, there exists $x^* \in B^n$ such that $\|x^* - fx^*\| = 0$. Thus, $x^* = f(x^*)$ and f has a fixed point in B^n. □

As a corollary we can deduce the no-retraction theorem.

Corollary 10.3.5 *There is no retraction of B^n onto S^{n-1}.*

Proof If $g : B^n \to S^{n-1}$ is a retraction, then $(-g)$ mapping continuously B^n into B^n has a fixed point x_0 by Brouwer's theorem. So $-g(x_0) = x_0$. Hence $g(x_0) = -x_0$. But $g(x_0) \in S^{n-1}$ and all the points of S^{n-1} are invariant under g. So $x_0 = g(x_0) = -x_0$ contradicts that x_0 is in S^{n-1}. Hence there is no retraction of B^n onto S^{n-1}. □

Remark 10.3.6 From the no-retraction theorem we can deduce Brouwer's theorem.

The next theorem points out that all compact convex subsets of $\mathbb{R}^n$ have the fixed-point property for continuous functions and is a precursor to Schauder's fixed-point theorem.

Theorem 10.3.7 *If K is a non-empty compact convex subset of $\mathbb{R}^n$, then every continuous mapping of K into itself has a fixed point.*

Proof Let f be a continuous map of K into K and B, closed ball containing K. For each $x \in B$, let $N(x)$ be the unique point in K nearest to x. (Note that if y_1 and y_2 are two points in K equidistant from K, then $\frac{x_1+x_2}{2} \in K$ is nearer to x!). Thus $x \to N(x)$, defines a map from B into K. We now note that this map is continuous. Suppose $x \in B$, x_n $(\in B)$ converges to x and that $N(x_n)$ does not converge to $N(x)$. Since $N(x_n) \in K$ and K is compact we can find $(x_{n(k)}) \in K$ a subsequence of (x_n) such that $N(x_{n(k)})$ converges to $y(\neq N(x)) \in K$. Now $\|x_{n(k)} - N(x_{n(k)})\| \leq \|x_{n(k)} - N(x)\|$. Allowing k to tend to ∞ in the above inequality, we get $\|x - y\| \leq \|x - N(x)\|$, contradicting that $y \neq N(x)$. Hence $x \to N(x)$ is continuous on B.

Now, by Brouwer's theorem, $x \to f(N(x))$ maps B into itself continuously and so has a fixed point, x_0 say. Since $f(N(x)) \in K$ for all $x \in B$ and $f(N(x_0)) = x_0$, $x_0 \in K$. If $x_0 \in K$, $N(x_0) = x_0$. Thus $f(x_0) = x_0$ and f has a fixed point in K. □

Remark 10.3.8 It follows that closed bounded intervals such as $[a, b]^n$ in $\mathbb{R}^n$ have the fixed-point property.

10.4 A Proof of Brouwer's Theorem from Sperner's Lemma

Sperner [28] proved a combinatorial theorem wherefrom Brouwer's fixed-point theorem can be deduced. In this section Sperner's lemma is detailed.

Definition 10.4.1 An n-dimensional simplex $\mathcal{S}$ in a linear space is a convex linear combination of $n+1$ points in general position. This is for given vertices $v_1, \ldots, v_{n+1}$, the simplex $S = \left\{ \sum_{i=1}^{n+1} \alpha_i v_i : \alpha_i \geq 0 \text{ and } \sum_{i=1}^{n} \alpha_i = 1 \right\}$ and the set of vertices $\{v_1, \ldots, v_{n+1}\}$ is not contained in any $(n-1)$ dimensional hyperplane of the linear space. It is written $[v_1, \ldots, v_{n+1}]$.

Each v_i is called a vertex of S and each k-simplex $[v_{i_0}, \ldots, v_{i_k}]$ is called a face of S. Thus each vertex is a face as also the whole simplex S.

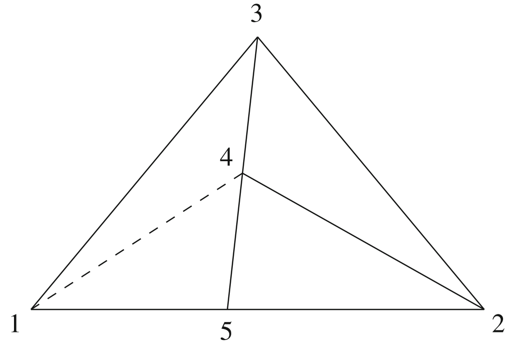
Definition 10.4.2 Let S be the n-dimensional simplex with the $n+1$ vertices $v_1, \ldots, v_{n+1}$ that are in general position. For $y \in S$, define $\psi(y) = \left\{ i \in N : y = \sum_{i=1}^{n+1} \lambda_i v_i \in S \text{ and } \lambda_i \geq 0 \right\}$. If $\psi(y) = \{i_0, \ldots, i_k\}$ then y is in the face $[x_{i_0}, \ldots, x_{i_k}]$. This face is called the carrier of y.

For the n-dimensional simplex S described above, if $y \in S$, then $y = \sum_{i=0}^{n+1} \alpha_i v_i$ and in this representation α_i are uniquely defined. $(\alpha_1, \ldots, \alpha_{n+1})$ are called the barycentric coordinates of y. Thus the carrier of y is well defined.

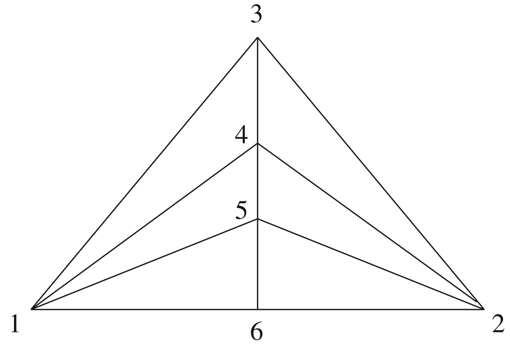
Remark 10.4.3 The standard n-simplex is $\left\{ y = (y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1} : y_i \geq 0, \sum_{i=1}^{n+1} y_i = 1 \right\}$. It is also conveniently written as $\Delta n = [e^1, \ldots, e^{n+1}]$.

Definition 10.4.4 A simplicial subdivision of an n-dimensional simplex S is a partition of S into a finite collection $\{S_i : i = 1, 2, \ldots, k\}$ of simplexes such that $\bigcup_{i=1}^{k} S_i = S$ and $S_i \cap S_j$ is either empty or a common face.

Example 10.4.5



In the above 2-simplex [1, 2, 3], the subdivision {[1,3,5], [2,3,4], [2,4,5], [1,3], [1,5], [2,3], [2,4], [2,5], [3,4], [4,5], 1, 2, 3, [4], [5]} is not simplicial for the intersection of the faces [1,3,5] with [2,3,4] is [3,4] and is not a common face. On the other hand in the following figure:



for the simplex [1, 2, 3] {[1,3,4], [1,4,5], [1,5,6], [2,5,6], [2,4,5], [2,3,4], [1,4], {1}, [1,5], [4,5], [5,6], {5}, [2,4], [3,4], {2}, {4}} is a simplicial subdivision.

Definition 10.4.6 Let $S = [x_1, \ldots, x_{n+1}]$ be an n-dimensional space which is simplicially subdivided, with V being the set of all vertices of all sub-simplexes. A function $\lambda : v \to \{1, \ldots, n+1\}$ satisfying $\lambda(v) \in \psi(v)$ is called a proper labelling where ψ is the carrier function defined in Definition 10.4.2. A sub-simplex is said to be completely labelled if λ takes all the values from 1 to $n+1$ on the set of all its vertices.

Remark 10.4.7 For the simplex $S = [x_1, \ldots, x_{n+1}]$ the centroid $\frac{1}{n+1}\sum_{i=1}^{n+1} x_i$, denoted by $b(\mathcal{S})$ is called the barycentre of S. For simplexes S_1 and S_2, we write $S_1 \leq S_2$ if S_1 is a face of S_2. For simplexes $S_k \leq S_{k-1} \cdots \leq S_1 \leq S_0 \leq S$, the family of all simplexes $b(S_0), \ldots, b(S_k)$ is called a barycentric subdivision of S. Given a simplex S, it is possible to obtain a simplicial subdivision such that the diameter of each sub-simplex is less than any given positive number. The mesh of a simplicial subdivision is the diameter of the largest proper sub-simplex.

We are now in a position to state and prove Sperner's Lemma [28].

Theorem 10.4.8 (Sperner [28]) *Let $S = [x_1, \ldots, x_{n+1}]$ be a simplex which is simplicially subdivided and properly labelled by the function λ. Then there is an odd number of completely labelled sub-simplexes in the subdivision.*

Proof When $n = 0$, the simplex consists of a single point x_1 bearing the label 1 and thus that is one completely labelled sub-simplex x_1 itself.

Suppose the statement is true for $n-1$. Given a simplicial subdivision of S, let A_1 be the number of all completely labelled n-simplexes, A_2 the number of almost completely labelled n-simplexes, i.e. those for which the sense of λ is $\{1, \ldots, n\}$, A_3 the number $(n-1)$ simplexes on the boundary of S that bear all the labels $\{1, \ldots, n\}$ and A_4 the number of all $(n-1)$ simplexes with labels $\{1, \ldots, n\}$ in the interior of S.

An $(n-1)$ simplex lies either on the boundary of S and is the face of a single n-simplex in the subdivision or it is a common face of two n-simplexes. Each simplex of type A_1 exactly one face labelled $\{1, 2, \ldots, n\}$. Each sub-simplex of type A_2 contributes two faces labelled $\{1, 2, \ldots, n\}$. However, inside faces appear in two simplexes while boundary faces appear in one sub-simplex. Thus we get $2A_2 + A_1 = A_3 + 2A_4$.

On the boundary, the only $(n-1)$ dimensional face labelled $\{1, 2, \ldots, n\}$ can be on the face $F \subseteq S$ whose vertices are labelled $\{1, 2, \ldots, n\}$. So we can apply the inductive hypothesis for F which forms a complete labelled $(n-1)$ dimensional sub-simplex. By hypothesis F has an odd number of completely labelled sub-simplexes. Thus by definition of A_3, A_3 is odd. Since $A_1 + 2A_2 = A_3 + 2A_4$, A_1 must be odd. So the theorem is true for all n. □

Brouwer's fixed-point theorem can be deduced from Sperner's lemma. Indeed, it suffices to prove it for n-dimensional simplexes.

Theorem 10.4.9 (Brouwer) *Let $S = [v_1, \ldots, v_{n+1}]$ be the n-dimensional simplex in $\mathbb{R}^{n+1}$ with vertices $v_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots, 0)$ (with 1 in the ith coordinate and zero elsewhere) for $i = 1, 2, \ldots, n+1$. If $f : S \to S$ is continuous then f has a fixed point in S.*

Proof Let $(\mathscr{S}_n)$ be a sequence of simplicial subdivisions of S such that each $\mathscr{S}_k$ is a subdivision of $\mathscr{S}_{k-1}$ with $mesh(\mathscr{S}_k) \to 0$ as $k \to \infty$. Let $\epsilon > 0$ and $\mathscr{S}_k$ be the division such that $mesh(\mathscr{S}_k) < \epsilon$, for $k \geq k(\epsilon)$. Let V be the set of vertices of the subdivision and $\lambda : V \to \{1, \ldots, n+1\}$ be a labelling function. For $v \in [v_{i_1}, \ldots, v_{i_\ell}]$ choose

$$\lambda(v) \in \{i_1, \ldots, i_\ell\} \cap \{i : f_i(v) \leq \lambda_i\}$$

where $v = \sum_{i=1}^{n+1} \lambda_i v_i$ and $f(v) = \sum_{i=1}^{n+1} f_i(v) v_i$.

This intersection is non-empty, for if $f_i(v) > \lambda_i$ for all $i \in \{i_1, \ldots, i_\ell\}$, then we would have

$$1 = \sum_{i=1}^{n+1} f_i(v) > \sum_{j=1}^{\ell} v_{i_\ell} = \sum_{i=1}^{n+1} v_i = 1$$

a contradiction, with the second inequality following from $v \in [v_{i_1}, \ldots, v_{i_\ell}]$. It may be noted that we are using the representation of v, $f(v)$ in barycentric coordinates. Since λ is a labelling function satisfying Sperner's lemma there exists a completely labelled sub-simplex $[p_1^\epsilon, \ldots, p_{m+1}^\epsilon]$ such that $f_i(p_i^\epsilon) \leq (p_i^\epsilon)^i$. As $\epsilon \downarrow 0$ there is a subsequence of simplexes such that $p_i^\epsilon \to q$ as $\epsilon \to 0$ for each $i = 1, 2, \ldots, m+1$. Since f is continuous $f_i(q) \leq q_i$, $i = 1, \ldots, m+1$. If $f(q) \neq q$, $f_i(q) \leq q_i$ for all i and $f_k(q) < q$ for some k would contradict $\sum f_k(q) = \sum q_k = 1$. So $f(q) = q$. □

Using Sperner's lemma, one can deduce a classical result due to Knaster Kuratowski and Mazurkiewicz [14], called Knaster–Kuratowski–Mazurkiewicz Lemma or simply KKM lemma.

Theorem 10.4.10 (KKM Lemma) *Let* Δ *be the simplex* $[e_1, \ldots, e_{m+1}]$ *in* $\mathbb{R}^{m+1}$ *and* $F_1, \ldots, F_{m+1}$ *be a family of non-empty closed subsets of* Δ *such that for each* $A \subseteq \{1, 2, \ldots, m+1\}$ *the convex hull of* $\{e_i : i \in A\} \subseteq \bigcup_{i \in A} F_i$.

Then $\bigcap_{i=1}^{m+1} F_i$ *is non-empty and compact.*

Proof Clearly $\bigcap_{i=1}^{m+1} F_i$ is closed and compact. For $\epsilon > 0$ given subdivide Δ into sub-simplexes with mesh size $\leq \epsilon$. For a vertex v of the subdivision lying on the face $e^{i_1}, \ldots, e^{i_{k+1}}$, by hypothesis there is an index $i \in \{i_1, \ldots, i_{k+1}\}$ with $v \in F_i$. Labelling all the vertices in this way. We observe that it satisfies all the conditions of Sperner's Lemma. So there is a completely labelled sub-simplex $[p_1^\epsilon, \ldots, p_{m+1}^\epsilon]$ with $p_i^\epsilon \in F_i$ for each i. As $\epsilon \downarrow 0$, choosing a subsequence p_i^ϵ converging to q and noting $p_i^\epsilon \in F_i$ for each i, it follows that $z \in F_i$ for each i as F_i is closed for each i. Thus $z \in \bigcap_{i=1}^{m+1} F_i$. □

Now KKM lemma implies Brouwer fixed-point theorem.

Theorem 10.4.11 *Theorem 10.4.10 (KKM Lemma) implies Theorem 10.3.1 (Brouwer fixed-point theorem).*

Proof Let $\Delta_m = [e_1, \ldots, e_{m+1}]$ be the m-dimensional simplex in $\mathbb{R}^{m+1}$ and $f : \Delta_m \to \Delta_m$ a continuous map. Define $F_i = \{x \in \Delta : f_i(x) \le x_i\}$. The collections $\{e_1, \ldots, e_{m+1}\}$ and $\{F_i, \ldots, F_{m+1}\}$ satisfy the hypotheses of KKM Lemma. For $x \in [e_{i_1}, \ldots, e_{i_{k+1}}]$ $\sum_{i=1}^{m+1} f_i(x) = \sum_{i=1}^{k+1} x_{i_j}$. So at least for one x_{i_j}, $f_{i_j}(x) \le x_{i_j}$ and F_i are all closed. So $\bigcap_{i=1}^{m+1} F_i$ is compact and non-empty. But this set is precisely $\{x \in \Delta : f(x) \le x\}$ and is simply the set of fixed points of f. □

We can also prove that Brouwer's theorem implies the KKM Lemma.

Theorem 10.4.12 *Brouwer's Theorem 10.3.1 implies KKM Lemma 10.4.10.*

Proof Let K = convex hull of $\{a_i : i = 1, \ldots, m+1\}$. K is evidently compact and convex. Suppose $\bigcap_{i=1}^{m+1} F_i = \phi$. Then $\{F_i^c : i = 1, \ldots, m+1\}$ is an open cover for K. So there exists a partition of unity $f_1, \ldots, f_{m+1}$ subordinate to this covering by Theorem 1.2.20. Let $g : K \to K$ be defined by $g(x) = \sum_{i=1}^{m+1} f_i(x) a_i$. Clearly g is continuous and maps K into itself. By Brouwer's theorem it has a fixed point p. Let $A = \{i : f_i(p) > 0\}$. Then $p \in co\{a_i : i \in A\}$ and $p \in F_i$ for no i, a contradiction. □

Aliter(Peleg [22]) Let $F_1, \ldots, F_{m+1}$ satisfy the hypotheses of KKM Lemma. Define $g_i(x) = d(x, F_i)$ and $f : \Delta \to \Delta$ by $f_i(x) = \frac{x_i + g_i(x)}{1 + \sum_{j=1}^{m+1} g_j(x)}$. Then f is continuous and so by Brouwer's theorem f has a fixed point $\overline{x}$. So $g_j(\overline{x}) = 0$ for all $j \in \{1, \ldots, m+1\}$. So $\bigcap_{j=1}^{m+1} F_j \ne \phi$.

10.5 Scarf's Algorithm

Scarf [24] provided an algorithm for approximating a fixed point guaranteed by Brouwer's fixed-point Theorem 10.3.1. In this section Scarf's algorithm is outlined. It may be added that many other algorithms improving on Scarf's contribution, due to Eaves, Kuhn and others, have emerged subsequently.

A constructive proof of Sperner's Lemma 10.4.8 is described below in line with Scarf.

Let S_n be the standard simplex in $\mathbb{R}^n$. Let $\{v^1, v^2, \ldots, v^n, v^{n+1}, v^{n+k}\}$ be a restricted simplicial subdivision where $v^1, \ldots, v^n$ are the unit vectors of the initial subdivision with no subdivision along the boundaries. We label each vertex v with $\ell(v) \in \{1, 2, \ldots, n\}$ such that $\ell(v^i) = i$ for $i = 1, 2, \ldots, n$. The remaining vertices in the interior can be labelled arbitrarily from $\{1, 2, \ldots, n\}$. We construct a completely labelled simplex in this restricted subdivision as follows:

We construct the unique initial sub-simplex with vertices $v_2, \ldots, v_n$ by adding an additional vertex v^{n+j} such that the algorithm for completely labelled restricted subdivision halts in $\ell(v^{n+j}) = 1$. If $\ell(v^{n+j}) = k \neq 1$, then k appears in the labelling of this sub-simplex twice. We eliminate the vertex whose label coincides with the label of v^{n+j} leading to another sub-simplex with a new vertex $v^{n+k(j)}$. This step can be repeated again. Thus at each stage of the algorithm, one has a sub-simplex with $n-1$ labels $\{2, \ldots, n\}$ terminating only when we reach a restricted sub-simplex having all the n labels $\{1, 2, \ldots, n\}$.

We note that this algorithm leads to a completely labelled sub-simplex in a finite number of steps which has not been reached previously. Let S' be the first simplex revisited, if possible by this algorithm. If it is not the initial sub-simplex, then it can be reached in two ways through either one of the adjacent simplices with $(n-1)$ distinct labels. As per the algorithm, both these adjoint sub-simplexes had already been reached during the algorithmic construction contradicting that S' is not the first simplex revisited. This computational scheme clearly encapsulates a constructive proof of Sperner's lemma.

Scarf's algorithm for approximating a fixed point of a continuous self-map on a standard simplex S_n can now be briefly described.

Definition 10.5.1 Let $f : S_n \to S_n$ be a continuous map with a fixed point x^*. For $\epsilon > 0$ given, $x_\epsilon \in S_n$ is called ϵ-almost fixed point if f if $\|x^* - x_\epsilon\| \leq \epsilon$.

Theorem 10.5.2 (Scarf algorithm) *Let $f : S_n \to S_n$ be a continuous function. Given $\epsilon > 0$ there exists $\delta > 0$ less then ϵ and for G, a simplicial subdivision of S_n such that each vertex of G is labelled by $i = \min\{j : f_j(x) \leq x, > 0$ where $x = (x_1, \ldots, x_n)\}$ there is a completely labelled sub-simplex S' of G such that for $x \in S'$, $\|f(x) - x\| \leq 2n\epsilon$ and $\|f(y) - f(z)\| < \epsilon$ for $\|y - z\| \leq \delta$ with $y, z \in S'$.*

Proof Since f is continuous on S_n, it is uniformly continuous in view of the compactness of S_n. So given $\epsilon > 0$ there exists $\delta > 0$ with $\delta < \epsilon$ such that $\|y - z\| < \delta$, $y, z \in S_n$ implies $\|f(y) - f(z)\| < \epsilon$.

For each vertex v^j one can associate an index i such that $v_i^j > 0$ ($v^j = (v_1^j, v_2^j, \ldots, v_n^j)$) and $f_i(v^j) \leq v_i^j$, $f(v^j)$ being $(f_1(v^j), \ldots, f_n(v^j))$.

From the constructive proof of the Sperner lemma outlined above, there exists a restricted completely labelled simplicial subdivision S' of G with mesh $S' < \delta$ containing a fixed point x^* of f. So for $x \in S'$, $|x_i - f_i(x)| \leq \|x - f(x)\| \leq \|x - x^*\| + \|x^* - f(x)\| < \delta + \epsilon < 2\epsilon$. So $\|x - f(x)\| < 2n\epsilon$. □

10.6 More on Brouwer's Fixed-Point Theorem

Ky Fan [6] proved the following consequence of KKM Lemma.

Theorem 10.6.1 *Let $X \subseteq \mathbb{R}^n$ and for each $x \in X$, let $F(x)$ be a closed subset of $\mathbb{R}^n$. Suppose*

(i) for any subset $\{x_1, \dots, x_k\} \subseteq X$, $c_0\{x_1, \dots, x_k\} \subseteq \bigcup_{i=1}^{k} F(x_i)$.
(ii) $F(x)$ is compact for some $x \in X$. Then $\bigcap_{x \in X} F(x)$ is non-empty and compact.

Proof Since for each finite non-empty subset S of X, $S \cup \{x\}$ is finite and $conv\{S \cup \{x\}\} \subseteq \bigcup_{s \in S} F(s) \cup F(x) \bigcap_{s \in S} F(s) \cap F(x)$ is non-empty and compact by KKM lemma. Thus $\{F(s) \cap F(x) : s \in X\}$ has finite intersection property and is a collection of closed subsets of $F(x)$. So $\bigcap_{s \in X} F(s)$ is non-empty and closed by the compactness of $F(x)$. □

We use the following.

Definition 10.6.2 Let U be a binary relation on a set K, i.e. a subset of $K \times K$, such that $U(x) = \{(x, y) \in K\} \subseteq K$ (this may be viewed as the set of elements y 'bigger than' x). An element $x \in K$ is called U-maximal if $U(x) = \phi$. The U-maximal set of K is simply $\{x \in K : U(x) = \phi\}$. The graph of U is $\{(x, y) : y \in U(x)\}$.

Theorem 10.6.3 (Sonnenschein [27]) *Let $K \subseteq \mathbb{R}^n$ be a non-empty compact convex subset and U be a binary relation on K, satisfying the following:*

(i) $x \notin$ convex hull of $U(x)$ for all $x \in K$;
(ii) for $y \in U^{-1}(x)$, there exists $x' \in K$ such that $y \in int\ U^{-1}(x')$.

Then K has a U-maximal element and the U-maximal set is compact.

Proof Clearly

$$\begin{aligned}\{x : U(x) = \phi\} &= \bigcap_{x \in K} (K - U^{-1}(x)) \\ &= \bigcap_{y \in K} (K - int\ U^{-1}(y)) \ \text{ by } (ii)\end{aligned}$$

For each y, $F(y) = K - int\ U^{-1}(y)$ is a closed subset of K and hence is compact. For $y \in co\{x_1, \dots x_n : x_i \in K\}$, $y \in \bigcup_{i=1}^{n} F(x_i)$ otherwise $y \notin \bigcup_{i=1}^{n} F(x_i)$ implies $y \notin \bigcup_{i=1}^{n} (K - int\ U^{-1}(x_i)) = K - \bigcap_{i=1}^{n} int\ U^{-1}(x_i)$ or $y \in int\ U^{-1}(x_i) \subseteq U^{-1}(x_i)$ for all

$i = 1, 2, \ldots, n$. So $x_i \in U(y)$ for all $i = 1, 2, \ldots n$. Since y is in the convex hull of x_i, $y \in$ convex hull of $U(y)$, contradicting (i). Thus F satisfies the hypotheses of Theorem 10.6.1 and so $\bigcap_{x \in K} F(x) \neq \phi$ and compact. Thus the set of all U-maximal elements is non-empty and compact. □

Definition 10.6.4 Let S, T be non-void subsets of $\mathbb{R}^n$ and $\psi : S \to 2^T$. ψ is called upper semicontinuous at $x_0 \in S$ if for $x_n \in S, n \in \mathbb{N}$ and $(x_n) \to x_0$ $y_n \in \psi(x_n)$ and $y_n \to y_0$ imply that $y_0 \in \psi(x_0)$. ψ is said to be lower semicontinuous at x_0 if $x_n \in S$ for $n \in \mathbb{N}, x_n \to x_0 \in S$ and $y_0 \in \psi(x_0)$ imply that there exists $y_n \in \psi(x_n)$ for $n \in \mathbb{N}$ such that $y_n \to y_0$. ψ is called continuous if it is both upper and lower semicontinuous $\psi : S \to 2^T$ is said to have open lower sections it $\psi^{-1}\{y\} = \{x \in S : y \in \psi(x)\}$ is open in S.

Remark 10.6.5 $\psi : S \to 2^{\mathbb{R}^n}$ is upper semicontinuous on S if for every open subset V of $\mathbb{R}^n$ $\{x \in S : \psi(x) \subseteq V\}$ is open in S.

Following Krasa and Yannelis [15], we can give an alternative proof of Theorem 10.6.3 using Brouwer's theorem.

Aliter for Theorem 10.6.3 For each $x \in K$, define $\varphi(x) = U^{-1}\{x\}$. φ has open lower sections. Suppose $U^{-1}(x) \neq \phi$ for all x. By assumption (ii) of Theorem 10.6.3 $int\ U^{-1}\{x\} \neq \phi$. So $\{int \varphi(x) : x \in K\}$ is an open cover for K and as K is compact it has a finite subcover $\{int\ U^{-1}(y_1), int\ U^{-1}(y_2), \ldots, int\ U^{-1}(y_m)\}$ say. Let $g_i(x) = dist(x, K - int\ \varphi(y_i)\}$ and $\alpha_i(x) = \frac{g_i(x)}{\sum_{j=1}^m g_j(x)}$. Clearly each α_i is continuous, $0 \leq \alpha_i \leq 1$, $\alpha_i = 0$ on $K - int\ \varphi(y_i)$ and $\sum_{i=1}^m \alpha_i(x) = 1$ for $x \in K$. Thus $\{\alpha_i\}$ is a partition of unity subordinate to the covering $\{int\ \varphi(y_i)\}$ for K. The map $f : K \to K$ defined by $f(x) = \sum_{i=1}^m \alpha_i(x) y_i$ is continuous on the compact convex set K in $\mathbb{R}^n$ and so by Brouwer's fixed-point theorem it has a fixed point $x_0 = f(x_0) = \sum_{i=1}^m \alpha_i(x_0) y_i$ in K. For all the $\alpha_i(x_0) \neq 0$, $x_0 \in int\ \varphi(y_i)$ and $x_0 \in U^{-1}(y_i)$ or $y_i \in U(x_0)$. So $x_0 = \sum_{i=1}^m \alpha_i(x_0) y_i \in$ convex hull of $U(x_0)$ contradicting the hypothesis of Theorem 10.6.3. Thus U has a maximal element.

Some consequences of Theorem 10.6.3 can now be deduced.

Theorem 10.6.6 *Let U be a binary relation on $K \subseteq \mathbb{R}^m$ with values in $\mathbb{R}^m$, K being a non-empty compact convex subset. Suppose*

1. *$x \notin U(x)$ for all $x \in K$;*
2. *$U(x)$ is convex for all $x \in K$;*
3. *$\{(x, y) : y \in U(x)\}$ is open in $K \times K$.*

Then U has a maximal element and the set of all such maximal elements of U is compact.

Theorem 10.6.7 *Let $K \subseteq \mathbb{R}^m$ be a non-void compact convex set and $E \subseteq K \times K$ be closed. Suppose*

1. *$(x, x) \in E$ for all $x \in K$,*
2. *for each $y \in K$, $\{x \in K : (x, y) \notin E\}$ is convex.*

Then there exists $k_0 \in K$ such that $K \times \{k_0\} \subset K$. The set of such k_0 is compact.

The above results are due to Ky Fan.

Proof of Theorem 10.6.6 For each $x \in K$, define $U(x) = \{y : (x, y) \notin E\}$. Clearly $U(x)$ is convex, open and $x \notin U(x)$. So by Theorem 10.6.3, U has a maximal element k_0 and the set of al such maximal elements of U is compact. Thus $U(k_0) = \{k \in K : (k, k_0) \notin E\} = \phi$. Thus $K \times \{k_0\} \subseteq E$.

For the proof of Theorem 10.6.7 set $E = \{(x, y) : (x, y) \notin U\}$. Clearly $(x, x) \in E$ as $x \notin U(x)$. Since $\{(x, y) : y \in U(x)\}$ is open, $\{(x, y) \in E\}$ is closed in $K \times K$. Further $U(x)$ is convex. So by Theorem 10.6.6 for $k_0 \in K$, $K \times \{k_0\} \subseteq E$ or $\{y : y \in U(k_0)\} = \phi$. □

One can deduce the following

Theorem 10.6.8 *Let $f : K \to \mathbb{R}^m$ be a continuous map on a non-void compact convex subset K. Then there exists $k_0 \in K$ such that for all $p \in K$, $< k_0, f(k_0) \geq \langle p, f(k_0)\rangle$.*

Proof Define the binary relation U on K by $y \in U(x)$ if and only if

$$\langle y, f(x)\rangle > \langle (x, f(x)\rangle.$$

Since f is continuous, $\{y \in K : (x, y) \in U(x)\}$ is open in $K \times K$, $x \notin U(x)$ and $U(x)$ is convex, by Theorem 10.6.7, there exists $k_0 \in K$ such that $U(k_0) = \phi$. So for $k_0 \in K$, $\langle k, f(k_0)\rangle \leq \langle k_0, f(k_0)\rangle$, for all $k \in K$. □

The above result is referred as Hartman–Stampacchia Lemma.

We can now prove the existence of Walrasian equilibrium in an exchange economy without production.

Let $x = (x_1, \dots, x_n)$ be an allocation of goods to m consumers (each element in the above list could itself be a vector of goods). Let $w = (w_1, \dots, w_n)$ be the initial endowments of the n consumers. Let $p = (p_1, \dots, p_n)$ be a vector of n prices. In the competitive economy, a Walrasian equilibrium or competitive equilibrium is represented by a list (x^*, p^*) such that x_i^* is preferred to x_i for all x_i satisfying the budget constraint, $p^* x_i \leq p^* w_i$. Thus all consumers maximize their utility. We assume that demand does not exceed supply for each good or $\sum_i (x_i^* - w_i) \leq 0$. Aggregate excess demand function $z(p)$ is defined by $z(p) = \sum_i (x_i^*(p) - w_i)$. Further, we assume that excess demand functions are homogeneous of degree 0 whenever $z(p^*) \leq 0$, where (x^*, p^*) represents a competitive equilibrium. Also if each consumer has strictly increasing and strictly convex preferences then $z(p)$ is continuous. According to the Walras law, $pz(p) = 0$. In practical terms it means that if $z_1 = z_{n-1} = 0$ and if $p_n > 0$ then z_n must be zero. Thus in calculating competitive equilibrium, we have to ensure that $n - 1$ markets are clear for clearing the n^{th} market. The existence of a Walrasian equilibrium can be proved as follows, using Brouwer's fixed-point theorem:

As z is homogeneous of degree zero, $z(tp^*) \leq 0$ whenever $z(p^*) \leq 0$. We can normalize the price vector so that the price vector lies in the $(n-1)$ dimensional unit simplex $S = \{p \in \mathbb{R}^n : p = (p_1, \dots, p_n)$ with $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1\}$. The map $g : S \to S$ defined by $g(p) = (g_1(p), \dots g_n(p))$ where $g_i(p) = \frac{(p_i + \max\{0, z_i(p)\})}{1+\sum_{j=1}^{n} \max\{0, z_j(p)\}}$ is continuous and maps S into itself. By Brouwer's theorem g has a fixed point p^* in S. Thus $p^* = g(p^*)$. So for $p^* = (p_1^*, \dots, p_n^*)$ we have $p_i^* = \frac{p_i^* + \max\{0, z_i(p^*)\}}{1+\sum_{j=1}^{n} \max\{0, z_j(p^*)\}}$. Thus $p_i^* \sum_{j=1}^{n} \max\{0, z_j(p^*)\} = \max\{0, z_i(p^*)\}$. So $\sum_{i=1}^{n} p_i^* z(p_i^*) \sum_{j=1}^{n} \max\{0, z_j(p^*)\}$ $= \sum_{i=1}^{n} z_i(p^*) \max\{0, z_i(p^*)\}$. By the Walras law the right-hand side of the above equation is zero. Since each of the n-terms in this sum is non-negative, $z_i(p^*) \leq 0$ for each i. Since the Walrus law holds at p^*, p^* is a competitive equilibrium.

Uzawa [32] has proved that Walras equilibrium theorem implies Brouwer's theorem and Uzawa's theorem is stated and proved below.

Theorem 10.6.9 (Uzawa [32]) *Walras' theorem implies Brouwer fixed-point theorem.*

Proof Let S be the standard unit simplex $\{(\pi_1, \dots, \pi_n) : \pi_i \geq 0, \sum_{i=1}^{n} \pi_i = 1\}$ in $\mathbb{R}^n$ and $f : S \to S$ be a continuous map. Construct an excess demand function by $x(p) = (x_1(p), \dots, x_2(p))$ defined by

$$x_i(p) = f_i\left(\frac{p}{\lambda(p)}\right) - p_i \mu(p), i = 1, 2, \dots$$

where $p = (p_1, \dots, p_n)$, $\lambda(p) = \sum_{i=1}^{n} p_i$ and $\mu(p) = \dfrac{\sum_{i=1}^{n} p_i f_i\left(\frac{p}{\lambda(p)}\right)}{\sum_{i=1}^{n} p_i^2}$.

Clearly $f_i\left(\frac{p}{\lambda(p)}\right)$ and $p_i\mu(p)$ are positive homogeneous of order zero. Thus the excess demand function $x(p)$ satisfies

1. continuity;
2. homogeneity of order 0; i.e. $x(tp) = x(p)$ for $t > 0$;
3. the Walras law; i.e. $\sum_{i=1}^{n} p_i x(p_i) = 0$.

So by Walras equilibrium theorem it has an equilibrium p^*. So by definition of $x(p)$, $f_i\left(\frac{p^*}{\lambda(p^*)}\right) \leq p_i^* \mu(p^*)$ for $i = 1, \dots, n$ with equality unless $p_i = 0$. Writing $\pi^* =$

$\frac{p^*}{\lambda(p^*)}$ and $\beta = \lambda(p^*)\mu(p^*)$ we get $f_i(\pi^*) \leq \beta\pi_i^*$ with equality unless $\pi_i^* = 0$. Since S is the standard simplex and π^*, $f(\pi^*) \in S = \{(\pi_1, \dots \pi_n) : \sum_{i=1}^n \pi_i = 1, \pi_i \geq 0\}$ and so $\beta = 1$. So $f_i(\pi^*) \leq \pi_i^*$ with equality unless $\pi_i^* = 0$. This again implies that $f_i(\pi^*) = \pi_i^*$, $i = 1, 2, \dots, n$. Thus π^* is a fixed point for f. □

Complementing von Neumann's concept of cooperative games, Nash [18] defined the concept of a non-cooperative game and its equilibrium point. Using Brouwer's theorem, he proved the existence of an equilibrium point (or Nash equilibrium) for an n-person non-cooperative game. In what follows the basic ideas for the existence of a Nash equilibrium of non-cooperative games are described, following closely Nash [19].

An n-person games is a set of n-players (or positions), each associated with a finite set of pure strategies. Corresponding to each player i, a pay-off function p_i mapping the set of all n-tuples of pure strategies into the set of real numbers. By n-tuple is meant a set of n elements, each element being associated with a different player. A mixed strategy of player i is a set of non-negative numbers with unit sum and are in one-one correspondence with his pure strategies. We write $s_i = \sum_\alpha C_{i\alpha}\pi_{i\alpha}$ with $C_{i\alpha} \geq 0$ $\sum_\alpha C_\alpha = 1$ where $\pi_{i\alpha}$ are pure strategies of the player i (s_i) can be regarded as points in a simplex with vertices $\pi_{i\alpha}$. Viewing it as a convex set enables one to get a natural way of linear combination for mixed strategies. Players can be denoted by suffixes i, j, k while pure strategies are represented by α, β, γ etc. While s_i, t_i denote mixed strategies, $\pi_{i\alpha}$ would mean αth pure strategy of the ith player. The pay-off function p_i has a unique extension to the n-tuples of mixed strategies which is linear in the mixed strategy of each player. This linear extension denoted by p_i, is also written $p_i(s_1, \dots, s_n)$. $\underline{s}, \underline{t}$ denote the n-tuple of mixed strategies. For $\underline{s} = (s_1, \dots, s_n)$, $p_i(\underline{s}) = p_i(s_1, \dots, s_n)$ is the corresponding pay-off function. Such an n-tuple $\underline{s}$ is a point in the product space of the vector spaces containing the mixed strategies.

The notation $(\underline{s}, t_i)$ denotes $(s_1, s_2, \dots, s_{i-1}, t_i, s_{i+1}, \dots, s_n)$ where $\underline{s} = (s_1, \dots, s_n)$. The resultant of successive substitutions $((\underline{s}, t_i); r_j)$ is denoted by $(\underline{s}; t_i; r_j)$. An n-tuple $\underline{s}$ is an equilibrium point if and only if for each i

$$p_i(\underline{s}) = \max_{r_i}[p(\underline{s}; r_i)]$$

In other words, the equilibrium point is an n-tuple such that each player's mixed strategy maximizes his pay-off if the strategies of others are held fixed. So each player's strategy is optimal against those of the remaining players. A mixed strategy s_i is said to use a pure strategy $\pi_{i\alpha}$ if $s_i = \sum_\beta C_{i\beta}\pi_{i\beta}$ and $C_{i\alpha} > 0$. If $\underline{s} = (s_1, \dots, s_n)$ and s_i uses $\pi_{i\alpha}$, we say that $\underline{s}$ uses $\pi_{i\alpha}$. Since $p_i(\underline{s})$ is linear in s_i

$$\max_{r_i}[p_i(\underline{s}; r_i)] = \max_\alpha[p_i(\underline{s}; \pi_{i\alpha})]$$

Define $p_{i\alpha}(\underline{s}) = p_i(\underline{s}; \pi_{i\alpha})$. Then a trivial necessary and sufficient condition for $\underline{s}$ to be an equilibrium point is $p_i(\underline{s}) = \max_\alpha p_{i\alpha}(\underline{s})$. For $\underline{s} = (s_1, \dots, s_n)$, $s_i = \sum_\alpha c_{i\alpha}\pi_{i\alpha}$, then $p_i(\underline{s}) = \sum_\alpha c_{i\alpha} p_{i\alpha}(\underline{s})$. So for the validity of $p_i(\underline{s}) = \max_\alpha p_{i\alpha}(\underline{s})$, we must have $c_{i\alpha} = 0$ whenever $p_{i\alpha}(\underline{s}) < \max_\beta p_{i\beta}(\underline{s})$. This simply means that $\underline{s}$ does not use $\pi_{i\alpha}$ unless it is an optimal pure strategy. With these preliminaries Nash [19] proved the following:

Theorem 10.6.10 *Every finite game has an equilibrium point.*

Proof Let $\underline{s}$ be an n-tuple of mixed strategies, $p_i(\underline{s})$ the corresponding pay-off to player i and $p_{i\alpha}(\underline{s})$ the pay-off to player i if he changes his αth pure strategy $\pi_{i\alpha}$ while the others continue to use their respective mixed strategies from $\underline{s}$. Define for each $\underline{s} = (s_1, \dots, s_n)$ the map $\phi_{i\alpha}(\underline{s}) = \max(0, p_{i\alpha}(\underline{s}) - p_i(s))$ and for each component s_i of $\underline{s}$ define $s_i' = \frac{s_i + \sum_\alpha \varphi_{i\alpha}(\underline{s})\pi_{i\alpha}}{1 + \sum_\alpha \varphi_{i\alpha}(s)}$ and write $\underline{s}' = (s_1', \dots, s_n')$. Consider the map $T(\underline{s}) = (s_1', \dots, s_n') = \underline{s}'$. Since the pay-off functions are all continuous $\underline{s} \to T(\underline{s})$ is a continuous map on the cell formed by all the strategies. So by Brouwer's fixed point theorem T has a fixed point $\underline{\varsigma} = (\varsigma_1, \dots, \varsigma_n)$. That part of $\pi_{i\alpha}$ used in ς_i must not be decreased by T. So for all β, $\varphi_{i\beta}(\underline{\varsigma}) = 0$ so that ς_i does not exceed 1. Thus if $\underline{\varsigma}$ is fixed by T, $\varphi_{i\beta}(\underline{\varsigma}) = 0$ for all β. So no player can improve his pay-off by moving to a pure strategy $\pi_{i\beta}$. This is precisely the definition that $\underline{\varsigma}$ is an equilibrium point. Conversely if $\underline{\varsigma}$ is an equilibrium point, then all φ's vanish so that $\underline{\varsigma}$ is a fixed point of T.

By a symmetry (or automorphism) of a game is meant a permutation of its strategies such that if two strategies belong to a single player they move to two strategies belonging to a single player. Thus, a permutation ϕ of the pure strategies induces a permutation of the players. Each vector of pure strategies permutes into another vector of pure strategies. Let ψ be the induced permutation of these vectors. If $\underline{\xi}$ is a vector of pure strategies and $p_i(\underline{\xi})$ the pay-off to player i when the vector $\underline{\xi}$ is applied, then symmetry requires if $j = i^\psi$ then $p_j(\xi^\psi) = p_i(\xi)$. This permutation ϕ has a unique linear extension to the mixed strategies. Thus for $s_i = \sum_\alpha C_{i\alpha}\pi_{i\alpha}$ $(s_i)^\phi = \sum_\alpha C_{i\alpha}(\pi_{i\alpha})^\phi$. From this extension to mixed strategies, one gets an obvious extension of ψ to n-tuples of mixed strategies. A symmetric n-tuple $\underline{s}$ of a game is defined by $\underline{s}^\psi = \underline{s}$ for all the extensions ψ. □

Theorem 10.6.11 (Nash [19]) *A finite cooperative game has a symmetric equilibrium point.*

Proof $s_i' = \sum_\alpha \pi_{i\alpha} / \sum_\alpha 1$ has the property $(s_i')^\phi = s_j'$ where $j = i^\psi$ so that the n-tuple $\underline{s}' = (s_1', \dots, s_n')$ is fixed under any ψ. So any game has at least one symmetric n-tuple. Clearly if $\underline{s}$ and $\underline{t}$ are symmetric, so is their convex combination. Thus the set of symmetric n-tuples is convex. It is also closed. Consider the map T taking $\underline{s}$ to $\underline{s}'$

defined in the proof of Theorem 10.6.10 and ψ is a symmetry on the game, then for $\underline{s}' = T\underline{s}$, $(\underline{s}')^{\psi} = T(\underline{s}^{\psi})$. So T maps the closed convex subset of symmetric n-tuples into itself and by the continuity of T and Brouwer's theorem T has a fixed point in the set of symmetric n-tuples. Thus there is a symmetric equilibrium point. $\square$

In an earlier paper, Nash [18] outlined the proof of the existence of the equilibrium point using Kakutani's fixed-point theorem. This seminal work culminated in the award of a Nobel prize (jointly in economics) for Nash. For a perspective on the impact of Nash equilibrium on social sciences Holt and Roth [10] may be referred.

10.7 A Proof of the Fundamental Theorem of Algebra

Arnold [2] and later Niven [21] attempted a proof of the fundamental theorem of algebra based on Brouwer's fixed-point theorem. Subsequently, it was noted in [3] that both the proofs contained errors. Later, Fort [7] salvaged it and we present his proof first of Brouwer's fixed-point theorem in the plane and then that of the fundamental theorem of algebra.

By S we denote the set of all complex numbers z with $|z| = 1$. For $z \in S$, $A(z)$ is the set of all real numbers θ for which $z = e^{i\theta}$. Thus $A(z)$ is the set of all arguments of z. A continuous function defined on a subset X of the plane of complex numbers and taking values in S is said to have a continuous logarithm on X if there exists a real-valued continuous function ϕ of X such that $f(z) = e^{i\phi(z)}$ for all $z \in X$. Two basic properties of complex numbers, used in the sequel, are the following:

(a) for $z_1, z_2 \in S$, $|z_1 - z_2| < 2$ and $\theta_1 \in A(z_1)$, then for a unique $\theta_2 \in A(z_2)$ with $|\theta_1 - \theta_2| < \pi$.
(b) if $\theta_i \in A(w_i)$, $i = 1, 2$, and $|\theta_1 - \theta_2| < \pi$, then $|\theta_1 - \theta_2| \le \pi|z_1 - z_2|$.

Theorem 10.7.1 *If $f : D \to S$ is a continuous mapping, then f has a continuous logarithm, D being a closed disc in the plane.*

Proof Let D be the disc $\{z : |z - q| \le r\}$. Thus q is the centre of D and r its radius. From the uniform continuity of f, it follows that there exists $\delta > 0$ such that for $z_1, z_2 \in D$ with $|z_1 - z_2| < \delta$, $|f(z_1) - f(z_2)| < \frac{1}{3}$. Choose $n \in \mathbb{N}$ such that $\frac{r}{n} < \delta$. Define $D_k = \{z : |z - q| \le \frac{rk}{n}\}$ for $k \in \mathbb{N}$ with $0 \le k \le n$. Define ϕ on D by defining it successively on $D_0, \ldots, D_n$. $\phi(D_0)$ is defined as $\phi(q) = \theta \in A(f(q))$ such that $0 \le \theta < 2\pi$. If ϕ is defined on D_k and $z \in D_{k+1}$, let z' be the nearest point of D_k to z. Since $|z - z'| < \delta$, $|f(z) - f(z')| < \frac{1}{3}$. So by property (a) stated above, we may define $\phi(z)$ to be the unique number $A(f(z))$ that differs from $\phi(z')$ by less than π.

Let S_k be the statement: if $z_1, z_2 \in D_k$ and $|z_1 - z_2| < \delta$, then $|\phi(z_1) - \phi(z_2)| < \pi$. Clearly S_0 is true as D_0 contains only one point. Suppose S_k is true. For $z_1, z_2 \in D_{k+1}$ with $|z_1 - z_2| < \delta$. Consider z_1' and z_2' the nearest points of z_1 and z_2 in D_k respectively. From the definition of ϕ, it follows that $|\phi(z_i) - \phi(z_i')| < \pi$, for $i = 1, 2$. It is readily seen that $|z_1' - z_2'| < \delta$ and by inductive hypothesis $|\phi(z_1) - \phi(z_2)| < \pi$. So we have

$$\begin{aligned}|\phi(z_1) - \phi(z_2)| &\leq |\phi(z_1) - \phi(z_1')| + |\phi(z_1') - \phi(z_2')| + |\phi(z_2') - \phi(z_2)| \\ &\leq \pi\{|f(z_1) - f(z_1')| + |f(z_1')| - f(z_2')| + |f(z_2')| - f(z_2)|\} \\ &< \frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{3} = \pi\end{aligned}$$

So S_{k+1} is true. Thus S_j is true for all j with $0 \leq j \leq n$.

Using S_n and (b), we get that for $z_1, z_2 \in D$ with $|z_1 - z_2| < \delta$, $|\phi(z_1) - \phi(z_2)| \leq \pi|f(z_1) - f(z_2)|$. This implies that ϕ is continuous in view of f being continuous. As $\phi(z) \in A(f(z))$ for $z \in D$, f has a continuous logarithm on D. □

Proposition 10.7.2 *The identity map on S does not have a continuous logarithm.*

Proof If the identity map on S has a continuous logarithm, then for $z \in S$, $z = e^{i\phi(z)}$, where ϕ is continuous and real-valued. The map $g : [0, 2\pi] \to \mathbb{R}$ defined by $g(\theta) = \phi(e^{i\theta}) - \theta$ is continuous and an integral multiple of 2π. So g must be a constant function. Now $g(0) = g(1)$ and $g(2\pi) = g(1) - 2\pi$, a contradiction. □

The following no-retraction theorem can be deduced.

Theorem 10.7.3 *There does not exist a retraction of the closed unit disc onto its boundary S.*

Proof If f is such a mapping, then by Theorem 10.7.1, there is a continuous function ϕ on D such that $f(z) = e^{i\phi(z)}$ for all $z \in D$. Since $f(z) = z$ on S, this contradicts Proposition 10.7.2. □

Theorem 10.7.4 (Brouwer's fixed-point theorem for plane) *If g is a continuous function mapping the closed unit disc into itself, then g has a fixed point in D.*

Proof Suppose g has no fixed point in D. For each $z \in D$, let $f(z)$ be the unique point of S such that z lies on the segment joining $f(z)$ to $g(z)$. It can be shown that $z \to f(z)$ is continuous and maps D onto S with $f(z) = z$ on S. This contradicts Theorem 10.7.3. Hence g must have a fixed point. □

Remark 10.7.5 Since any two closed discs in the plane are homeomorphic it follows that any closed disc in the plane has the fixed-point property for continuous functions.

For proving the fundamental theorem of algebra using Brouwer's theorem, we need the following.

Proposition 10.7.6 *If f is a continuous map on a closed disc D into the set of non-zero complex numbers, then there exist n distinct continuous mappings $h_1, \ldots, h_n$ such that $[h_k(z)]^n = f(z)$ for $z \in D$.*

Proof $f(z) = |f(z)|\frac{f(z)}{|f(z)|}$. By Theorem 10.7.1, as $z \to \frac{f(z)}{|f(z)|}$ maps D into S continuously there exists a continuous map ϕ such that $\frac{f(z)}{|f(z)|} = e^{i\phi(z)}$. Define $h_k(z) = \sqrt[n]{|f(z)|}.e^{\frac{i(\phi(z)+2k\pi)}{n}}$ for $k = 1, \ldots, n$. Then these are the required continuous nth roots of $f(z)$. □

Theorem 10.7.7 (Fundamental Theorem of Algebra) *If p is a polynomial of degree $n > 0$, then for some $z \in \mathbb{C}$, $p(z) = 0$.*

Proof Without loss of generality we may assume that $p(z) \neq 0$ for all $z \in \mathbb{C}$ and the coefficient of z^n in $p(z)$ is $\frac{1}{2}$. Now $\lim_{z\to\infty} \dfrac{p(z)}{z^n} = \dfrac{1}{2}$. So for some $r > 0$, $p(z)$ and z^n have arguments which differ by less than $\frac{\pi}{2}$ for $|z| \geq r$ and $|p(z)| < |z|^n$. Now select $R > r$ such that $\sqrt[n]{|p(z)|} + |z| < R$ for $|z| \leq r$. Consider D the disc centred at 0 with radius R. By Proposition 10.7.6, as p is non-zero and maps D into non-zero complex numbers p has continuous nth root f on D. We can select f such that $f(r)$ and r have arguments differing by less than $\frac{\pi}{3n}$ for all z with $|z| \geq r$. (If for some z' with $|z'| \geq r$, $f(z')$ and z' have arguments that differ exactly by $\frac{\pi}{3n}$, then the arguments of $p(z)$ and z^n differ by $\frac{\pi}{3}$, an impossibility.)

For $|z| \geq r$, $|p(z)| < |z^n|$ and so $|f(z)| < |z|$ and z and $f(z)$ both lie in the circular sector of radius $|z|$ and angle $\frac{\pi}{3n}$. Thus $|z - f(z)| \leq |z|$ for $|z| \geq r$. But when $|z| \leq r$, $|z - f(z)| \leq |z| + |f(z)| < R$. So the function $z \to z - f(z)$ maps D into itself continuously and hence has a fixed point, say z_0. So $f(z_0) = 0$ or $p(z_0) = 0$, a contradiction. □

10.8 A Generalization of Brouwer's Theorem

Hamilton [9] extended the Brouwer's fixed-point theorem for peripherally continuous maps, while Stallings [29] generalized Brouwer's theorem for connectivity functions. Whyburn [35] extended an intersection theorem due to Hurewicz and Wallman [11], whence he deduced both the generalizations of Hamilton and Stallings. In this section, Whyburn's approach to these fixed-point theorems is described. See also [34].

Definition 10.8.1 A subset E of a topological space X is said to be quasi-closed or of external dimension zero, if for each $p \in X/E$ every neighbourhood of p contains an open set having p, whose boundary does not intersect E. A subset G of X is quasi-open if its complement is quasi-closed.

Definition 10.8.2 A function $f : X \to Y$ where X and Y are topological spaces is said to be peripherallay continuous if for $x \in X$ and open sets U in X containing x and V in Y containing $f(x)$ there exists an open set W containing x such that $W \subseteq U$ and boundary $Fr(W)$ is mapped in V by f.

A map f is called a connectivity function if the associated graph function $x \to (x, f(x))$ maps connected subsets of X onto connected subsets of the graph of f.

Lemma 10.8.3 (Whyburn [36]) *A function $f : X \to Y$ is peripherally continuous if and only if the inverse of every closed (open) set in Y is quasi-closed (quasi-open) in X.*

Proof Suppose f is peripherally continuous. If C is closed in Y then for any $p \in X - f^{-1}(C)$ and an open set U containing p for the open set $V = Y - C$, $f(p) \in V$.

So by peripheral continuity of f, there exists an open set W in X with $p \in W \subseteq U$ and $f[Fr(W)] \subseteq V$. So $Fr(W) \cap f^{-1}(C) = \phi$. Thus for $p \in V$ there exists an open set $W \subseteq V$ containing p for which $fr(W) \subseteq V$. Thus $f^{-1}(C)$ is quasi-closed.

Suppose for each closed subset C of Y $f^{-1}(C)$ is quasi-closed. Let $p \in X$ and $P = f(p)$ and let U and V be open sets containing p and P respectively. Let $C = Y - V$. As $p \in X - f^{-1}(C)$ and C is closed there is an open set W with $p \in W \subseteq U$ and $Fr(W) \cap f^{-1}(C) = \phi$. So $f[Fr(W)] \subseteq Y - C = V$. Thus f is peripherally continuous. □

Hereinafter, we assume that the topological spaces are regular T_1 spaces. The following definitions are needed in the sequel.

Definition 10.8.4 (*Whyburn* [36]) A topological space X is called locally cohesive if it is connected and each open set containing a point p of X contains the closure of a canonical region about p, i.e. a connected open set R with connected boundary $Fr(R)$ such that $\overline{R}$ is unicoherent between x and $Fr(R)$ or equivalently between x and $X - R$. Recall that a connected space or set M is unicoherent or cohesive between disjoint connected subsets (or points) A and B of M if $H_a \cap H_b$ is connected for every representation $M = H_a \cup H_b$, H_a, H_b being closed connected subsets containing A and B, respectively, in their interiors relative to M.

Remark 10.8.5 A locally cohesive space is locally connected and has no local cut point. If W is a canonical region in X about $a \in X$, any set K separating a and $Fr(W)$ in $\overline{W}$ contains the boundary of a canonical region R lying in W. Thus in a locally cohesive space X, given a closed subset E of X and an open set U containing $a \in X - E$ contains a canonical region R about a with $R \subset U$ and $E \cap Fr(R) = \phi$. Thus for G quasi-open, any open set U containing $a \in G$ has a canonical region R about a so that $R \subseteq U$ and $Fr(R) \subseteq G$.

Definition 10.8.6 Two subsets A and B of a connected space X are weakly separated in X by a set E provided no component of $X - E$ meets both A and B.

Theorem 10.8.7 *Let A and B be disjoint nondegenerate closed connected sets in a locally cohesive space X. Any quasi-closed set L that separates A and B weakly in X contains a non-void closed set K that separates $A - K$ and $B - K$ in X.*

Proof If A (or B) $\subseteq L$, then $K = A$ (or B). Suppose neither A nor B is contained in L. Let H be the union of all components in $X - L$ intersecting A and V be the union of all components of $X - (\overline{H} \cup A)$ intersecting B. Define $K = Fr(V) \cup \overline{H} \cap B$ and define $U = X - (V \cup K)$. U and V are open and disjoint, K is closed and $K \subseteq \overline{H} \cup (A \cap L)$. From these it follows that $X - K = U \cup V$ and $B - K \subset B - \overline{H} \subset V$. Also since $A \cap \overline{V} \subseteq K$, we have $A - K \subseteq X - (V \cup K) = U$. We claim that $K = Fr(V) \cup \overline{H} \cap B \subseteq L$. Otherwise for some $x \in K, x \notin L$. Let R be a canonical region about x so that $\overline{R}$ contains neither A nor B intersects A or B only in case x is in A or B respectively. So $Fr(C)$ of R lies in H. In any case C intersects $X - L$ and lies in some component Q of $X - L$ in H. If $x \in A$, this is true as C intersects A; for $x \notin A$, it follows since some component of $X - L$ in H intersects both R and A.

For $x \in B$, this, however is not possible as $C \cap B \neq \phi$; for $x \notin B$ this is not possible as C intersects some component of V. Thus $K = Fr(V) \cup \overline{H} \cap B \subseteq L$. □

Theorem 10.8.8 ([35]) *If X is locally cohesive, any connected set in X lying in the union of two disjoint quasi-open sets lies entirely in one of them.*

Proof Let E be connected and lie in $U \cup V$ where U and V are two disjoint quasi-open sets. Let $a \in E \cap U$ and $b \in E \cap V$. For each $x \in E$, let Q_x be a canonical region containing x whose boundary C_x lies in U or V according as $x \in U$ or V. E being connected there exists a simple chain of such regions $a \in Q_1, Q_2, \ldots, Q_n \ni b$ from a to b only the first in the chain containing a and only the last containing b. Let $C_i = Fr(Q_i)$ for each i. Then for $1 < k < n$, both Q_{k-1} and Q_{k+1} must intersect C_k as each meets Q_k but is not contained in Q_k. So C_k intersects both C_{k-1} and C_{k+1} because C_k is connected and $Q_{k-1} \cap Q_{k+1} = \phi$. However this implies that $C_i \subseteq U$ for all $i = 1, \ldots, n$ contradicting $C_n \subseteq V$. □

Corollary 10.8.9 ([35]) *Any peripherally continuous function $f : X \to Y$ of a locally cohesive space X into a completely normal T_1 space Y preserves connectedness. It is a connectivity function whenever $X \times Y$ is completely normal.*

Proof If E is connected in X, and if $f(E)$ were not connected, there would exist disjoint open sets U_1 and V_1 in Y intersecting $f(E)$ such that $f(E) \subseteq U_1 \cup V_1$. But $U = f^{-1}(U_1)$ and $V = f^{-1}(V_1$ would then be quasi-open sets intersecting E, contradicting Theorem 10.8.8. So $f(E)$ is connected. The proof that f is a connectivity map is left as an exercise. □

The following extension theorem due to Whyburn is an intersection theorem improving on Hurewicz–Wallman intersection theorem [11].

Theorem 10.8.10 (Whyburn [35]) *Given quasi-closed sets $C_1, C_2, \ldots, C_n$ in $I^n = [0, 1]^n$ such that for each i, $1 \leq i \leq n$, C_i weakly separates A_i and B_i in I^n. Then $\bigcap_{i=1}^{n} C_i \neq \phi$, A_i and B_i being the faces of I^n on which $x_i = 0$ and $x_i = 1$, respectively.*

Proof For each i, by Theorem 10.8.7, C_i contains a closed set K_i separating $A_i - K_i$ and $B_i - K_i$ in I^n with $I^n - K_i = U_i \cup V_i$ where U_i and V_i are disjoint and open and contain $A_i - K_i$ and $B_i - K_i$ respectively. Define $f(x)$ for $x \in I^n$, by letting $f(x)$ as the terminal end of the position vector $x + d(x)$ in $\mathbb{R}^n$ where the ith component d_i of the vector $d(x)$ is $\pm\rho(x, K_i)$, the sign being $+$ for $x \in U_i$ and $-$ for $x \in V_i$. Then for $x \in U_i$ and each i, $d_i = \rho(x, K_i) \leq 1 - x_i$ so that $0 \leq x_i + d_i \leq 1$ while for $x \in V_i, d_i = -\rho(x, K_i) \geq -x_i$ and again $0 \leq x_i + d_i \leq x_i \leq 1$. Thus ρ maps I^n into itself. Further f is continuous and so by Brouwer's fixed-point theorem f has a fixed point $x_0 \in I^n$. This implies that $d(x_0) = 0$ and $x_0 \in \bigcap_{i=1}^{n} K_i \subset \bigcap_{i=1}^{n} C_i$. □

Theorem 10.8.11 (Hamilton-Stallings) *Any peripherally continuous function of I^n into itself, $n \geq 2$ has at least one fixed point. The same is true of any connectivity function of itself for $n \geq 1$.*

Proof Let $f : I^n \to I^n$, $n \geq 2$ be a peripherally continuous function. For $x = (x_1, \ldots, x_n) \in I^n$, let $f(x) = (x'_1, x'_2, \ldots, x'_n) \in (I')^n = I^{n'}$. Let $g : I^n \to I^n \times I'^n$ be the graph function of f. Thus $g(x) = (x, f(x))$. For $1 \leq i \leq n$, let $p_i = \pi_i g : I^n \to I_i \times I_i$ be the projection of the graph of f into the planar cell $I_i \times I'_i$ defined by $p_i(x) = \pi_i(x, x') = (x_i, x'_i)$ where $f(x) = (x'_1, x'_2, \ldots, x'_n)$. Write $\Delta_i = \{(x_i, x'_i) : x_i = x'_i, x_i \in I_i = I'_i\}$.

We now show that $p_i, i = 1, \ldots, n$ is peripherally continuous. If $E \subseteq I_i \times I'_i$ is closed, then $\pi_i^{-1}(E)$ is closed as π_i is closed. So $g^{-1}\pi_i^{-1}(E) = p_i^{-1}(E)$ is closed as f is peripherally continuous.

Since Δ_i is closed, $C_i = p_i^{-1}(\Delta_i)$ is quasi-closed for $1 \leq i \leq n$ as p_i is peripherally continuous. We claim that C_i weakly separates in I^n the faces A_i and B_i of I^n for which $x_i = 0$ and $x_i = 1$ respectively. Otherwise some component Q of $I^n - C_i$ intersects both A_i and B_i and $p_i(Q)$ would be connected by Corollary 10.8.9. Then $p_i(Q)$ would intersect Δ_i as it contains $p_i(a)$ for some $a \in A_i$ for which $x'_i \geq x_i$ and also $b \in B_i$ where $x'_i \leq x_i$. So C_i weakly separates in I^n the faces A_i and B_i for $1 \leq i \leq n$. So by Theorem 10.8.10 $\bigcap_{i=1}^{n} C_i \neq \phi$. Since $p_i^{-1}(\Delta_i) = C_i$, $x \in \bigcap_{i=1}^{n} C_i$ implies $x = x'$ or $x = f(x)$.

For $n = 1$, that every connectivity map of [0, 1] into itself has a fixed point has already been proved (see Corollary 3.1.7). For $n \geq 2$ we note that such a map is a peripherally continuous map in view of the following Lemma 10.8.19 and so by the first part of the theorem has a fixed point. □

The proof of this lemma is better understood on the basis of the following concepts and propositions (see Whyburn [33]).

Definition 10.8.12 A collection of sets $\mathcal{G}$ in a topological space is called upper semicontinuous if for $G \in \mathcal{G}$ and U an open set containing G, there is an open set V containing G such that any $H \in \mathcal{G}$ and intersecting V, $H \subseteq U$.

Definition 10.8.13 Let $(X, \mathscr{T})$ be a topological space and $\mathcal{G}$ a collection of non-empty subsets of X. The limit superior of $\mathcal{G}$, written $\limsup \mathcal{G}$ is the set of all points p in X such that every neighbourhood of p contains points from infinitely many sets in $\mathcal{G}$. The set of all points q in X such that every neighbourhood of q has points from all but a finite number of sets from $\mathcal{G}$ is called the limit inferior of $\mathcal{G}$ and is written $\liminf \mathcal{G}$.

For the proofs of the following propositions Whyburn [33] may be consulted.

Proposition 10.8.14 *In a compact metric space, a necessary and sufficient condition for a collection $\mathcal{G}$ of closed sets to be upper semicontinuous is that for each sequence (G_n) in $\mathcal{G}$ with* $\liminf G_n \cap G$ *for some $G \in \mathcal{G}$ implies* $\limsup G_n \subseteq G$.

Proposition 10.8.15 *If X is a compact space, the collection $\mathcal{G}$ of disjoint closed sets in the union of three subcollections $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ where $\mathcal{G}_1$ is the collection of components of a closed set, $\mathcal{G}_2$ is a null sequence (i.e. a sequence of sets such that* *for each $\epsilon > 0$, it contains at most a finite number of sets with diameter exceeding ϵ) and $\mathcal{G}_3$ is a collection of singletons, then $\mathcal{G}$ is upper semicontinuous.*

Proposition 10.8.16 *Proposition 10.8.14 is true when X is locally compact and all the sets in $\mathcal{G}$ are continua.*

Definition 10.8.17 A collection $\mathcal{G}$ of subsets of a metric space X is called an upper semicontinous decomposition of X if $\bigcup_{G \in \mathcal{G}} G = X$, each set in $\mathcal{G}$ is compact and $\mathcal{G}$ is an upper semicontinuous collection.

Remark 10.8.18 If $\mathcal{G}$ is an upper semicontinuous decomposition of a metric space, then a topology on $\mathcal{G}$ can be defined by declaring a neighbourhood of $G \in \mathcal{G}$ as a subcollection $\mathcal{U}$ of $\mathcal{G}$ such that $\cup\{\mathcal{O} : \mathcal{O} \in \mathcal{U}\}$ is open in X and contains G. This topology on $\mathcal{G}$ is called the hyperspace topology on $\mathcal{G}$.

Lemma 10.8.19 ([35]) *If X is a locally compact, locally cohesive metric space and Y a regular T_1 space, then any connectivity map $f : X \to Y$ is peripherally continuous.*

Proof For $x \in X$, let U and V be open sets containing x and $f(x)$, respectively. Without loss of generality let U be a canonical region with compact closure since X is T_1, regular and locally compact. So the boundary B of U is connected and $\overline{U}$ is unicoherent between x and B. Let U_1 and V_1 be open sets such that $x \in U_1 \subseteq \overline{U}_1 \subseteq U$ and $f(x) \in V_1 \subseteq \overline{V}_1 \subseteq V$. Let $D = \overline{U}_1 \cap f^{-1}(\overline{V}_1)$. If x is an interior point of A comprising the component A_0 of D containing x together with the union of all components of $\overline{U} - A_0$ except the one containing B or x is separated in $\overline{U}$ from B by a component H of D, we get an open set $W \subseteq U$ with $x \in W$ and $Fr(W) \subseteq A_0$ or $Fr(W) \subseteq H$. In the former case, let $W = int\ A$ and in the latter case, choose W = component of $\overline{U} - H$. In both the cases $f(Fr(W)) \subseteq V$.

Suppose x is neither in the interior of A nor is separated in $\overline{U}$ from B by any single component D. So the decomposition of $\overline{U}$ into the sets A, B components of D not contained in A and single points of $U - A - D$ is upper semicontinuous. If $\phi(\overline{U}) = M$ is the natural mapping of this decomposition, then ϕ is closed and monotone (i.e. $\phi^{-1}(y)$ is a continuum for each y in the range of ϕ). So M is a locally connected continuum. If $a = \phi(x)$ and $b = \phi(B)$ and $N = C(a, b)$ the cyclic element taken in M, then no point of $\phi(D) \cap N$ is a cutpoint of N as no such point can separate a and b in M or N. Since $\overline{U}$ is unicoherent between x and $Fr(U)$, N is unicoherent. As $\phi(D)$ is totally disconnected $R = N - \phi(D)$ is connected. So $a \subseteq \overline{R}$. So $\phi^{-1}(R) = Q$ is connected as ϕ is monotone and closed. Also $\overline{Q} \supseteq \{x\}$, as any region S in U_1 must intersect Q. If S is not in A and a is not a cutpoint of M. $\phi(S) \cap N$ is non-degenerate and connected and so is not connected and is not contained in $\phi(D)$. But then $Q \cup \{x\}$ is connected while $(x, f(x))$ is an isolated point of the graph of $f|Q \cup x$, because $(q, f(q))$ is not in $U_1 \times V_1$ for $q \in Q$, since $f(q) \in Y - V_1$ for all $q \in Q$. This contradiction implies that x is either in the interior of A or is separated in $\overline{U}$ from B by some single component of D. So f is peripherally continuous. □

Remark 10.8.20 Since $f : I_n \to I_n$ is a connectivity map it is a peripherally continuous map. So by first part of Theorem 10.8.11. f has a fixed point for $n \geq 2$. For $n = 1$, a connectivity map, $f : I \to I$ has a fixed point. For $n = 1$, let the map $f : X = [0, \xi] \to [0, \xi]$ be defined by $f(x) = \begin{cases} \xi, & x \in X \text{ is rational} \\ 0, & x \in X \text{ is irrational} \end{cases}$ where ξ is a positive irrational number less than 1. Clearly f is peripherally continuous and has no fixed point.

It may be added that Nash [20] raised the question of whether a connectivity map on I^n has a fixed point.

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Chapter 11
Schauder's Fixed Point Theorem and Allied Theorems

11.1 Introduction

Attempts to extend Brouwer's fixed point theorem to infinite-dimensional spaces culminated in Schauder's fixed point theorem [20]. The need for such an extension arose because existence of solutions to nonlinear equations, especially nonlinear integral and differential equations can be formulated as fixed point problems in function-spaces. This chapter discusses Schauder's and allied fixed point theorems with their applications including the existence of Haar integral, invariant mean and Banach limit. In this context, the following simple example shows that Brouwer's theorem is not true for closed balls in infinite-dimensional spaces.

Example 11.1.1 B, be the closed unit ball in C_0 the space of all null real sequences $x = (x_n)$ with the norm $\|x\| = \sup\limits_n |x_n|$ does not have the fixed point property. For example, the map $x \to Tx$ where $T(x) = (1, x_1, x_2, \ldots)$, x being $(x_1, x_2, \ldots, x_n, \ldots)$ maps B into itself. If $T(x) = x$, then $x = (x_n)$ with $x_n \equiv 1$ for all n contradicting that x_n is a null sequence.

Theorem 11.1.2 (Schauder [20]) *If K is a compact convex subset of a normed linear space, then every continuous function f mapping K into itself has a fixed point.*

Proof Given any $\epsilon > 0$, by the compactness of K, we can find a finite number of points $x_1, \ldots, x_N$ in K such that each $x \in K$ lies in an open ball centred at x_i for some $i = 1, 2, \ldots, N$ and of radius ϵ. Define $g_j : K \to \mathbb{R}^+$ by

$$g_j(x) = \begin{cases} \epsilon - \|x - x_j\|, & \text{if } \|x - x_j\| < \epsilon \\ 0, & \text{if } \|x - x_j\| \geq \epsilon,\ j = 1, 2, \ldots, N. \end{cases}$$

Clearly, each g_j is continuous and so is h_j defined by

$$h_j(x) = \frac{g_j(x)}{\sum_{i=1}^{N} g_i(x)}.$$

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Further, $\sum_{j=1}^{N} h_j(x) = 1\ \forall\, x \in K$ and $h_j(x) = 0$ if $\|x - x_j\| \geq \epsilon$. The map $x \to V(x)$ defined by

$$V(x) = \sum_{j=1}^{N} h_j(x) x_j$$

maps K into the convex hull of $\{x_1, x_2, \dots, x_N\}$. Also, for $x \in K$

$$x - V(x) = \sum_{j=1}^{N} h_j(x)(x - x_j)$$

where the sum is only over those j for which $\|x - x_j\| < \epsilon$ has positive contribution. Thus for $x \in K$

$$\|x - V(x)\| \leq \sum h_j(x) \|x - x_j\| < \epsilon.$$

If we denote by K_ϵ, the convex hull of $x_1, \dots, x_N$, then the map $x \to V(f(x))$ maps continuously K_ϵ, a compact convex set into itself. So, by Theorem 10.3.7, $V(f x_\epsilon) = x_\epsilon$, for some x_ϵ.

Set $\epsilon = \frac{1}{n}$ for each $n \in \mathbb{N}$. Thus we have $x_n \in K$ such that

$$Vf(x_n) = x_n.$$

Now, by compactness of K, there is a subsequence $x_{n(k)}$ converging to some x^*. So, by triangle inequality and continuity of Vf, we have

$$\|f(x^*) - x^*\| \leq \|f(x^*) - f(x_{n(k)})\| + \|f(x_{n(k)}) - Vf(x_{n(k)})\| + \|x_{n(k)} - x^*\|$$

Allowing k to tend to ∞ in the above, it follows by continuity of f that $f(x^*) = x^*$. □

Schauder's theorem as mentioned earlier has far-reaching applications. We merely sketch the well-known Peano's existence theorem for initial value problem of first-order ordinary differential equations. For this purpose, we recall the concept of an equicontinuous family of functions and Arzela–Ascoli theorem.

Definition 11.1.3 A family F of real-valued functions on a subset S of $\mathbb{R}$ is said to be equicontinuous if given any $\epsilon > 0$ there exists $\delta > 0$ (independent of $f \in F$) such that $|f(x) - f(y)| < \epsilon$ for all $f \in F$ whenever $|x - y| < \delta$, $x, y \in S$.

Theorem 11.1.4 (Arzela–Ascoli) *A subset K of the space $C[0, 1]$ with supremum norm is totally bounded if and only if it is uniformly bounded and equicontinuous.*

We make use of the above criterion for compactness in the proof of Peano's Theorem.

Theorem 11.1.5 (Peano's existence theorem) *Let* $f : J = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b] \to \mathbb{R}$ *be continuous* ($x_0, t_0 \in \mathbb{R}$ *and* $a, b > 0$)*. Then the initial value problem*

$$\frac{dx}{dt} = f(t, x) \ \text{ and } \ x(t_0) = x_0$$

has a solution in $[t_0 - h, t_0 + h]$ *for some positive* h.

Sketch of Proof. This initial value problem is equivalent to solving the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

Since f is continuous on the rectangle J, f is bounded on J. Let $K = \max_{(t,x)\in J} |f(t, x)|$. Set $h = \min[a, \frac{b}{K}]$. Clearly, $h > 0$. Consider the space $C[t_0 - h, t_0 + h]$ and define the operator A on this space by

$$Ax = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

It is easy to see that A is continuous.

Let S be the closed sphere $\{f \in C[t_0 - h, t_0 + h] : |f(t) - x_0| \leq b\}$. We can show using the definition of h, that A maps S into itself. Further, $A(S)$ is a uniformly bounded and equicontinuous family (Prove it!). Then, S^*, the smallest closed convex set containing $A(S)$ (viz., the closed convex hull of $A(S)$ is also uniformly bounded and equicontinuous. Thus, S^* is a convex, compact set (in view of Arzela–Ascoli Theorem 11.1.4) in $C[t_0 - h, t_0 + h]$. Clearly, as $A(S) \subseteq S$, $A(S^*) \subseteq A(S) \subseteq S^*$ (by definition of S^*) $\subseteq S$.

So, by Schauder's fixed point Theorem 11.1.2, A has a fixed point in S and this is a solution of the initial-value problem.

Along similar lines one can prove the following existence theorem.

Theorem 11.1.6 *Let* $F : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function and* $g : [0, 1] \to \mathbb{R}$ *be continuous. Then the following integral equation has a continuous solution* f *in* $[0, 1]$.

$$g(s) = f(s) + \int_{0}^{1} F(s, t, f(t))dt.$$

Even when a continuous function does not map a compact convex set into itself it may have a fixed point under some additional assumptions. Rothe [16] obtained the following fixed point theorem belonging to this category. For this purpose, we need the following.

Definition 11.1.7 A map $T : S(\subseteq X) \to X$ is called compact if $T(S)$ is contained in a compact subset of X. Here S is a subset of a topological space X.

Theorem 11.1.8 (Rothe [16]) *Let X be a normed linear space, B its closed unit ball and S the unit sphere. Let $T : B \to X$ be a compact continuous map such that $T(\partial B) \subseteq B$. Then T has a fixed point in B.*

Proof Define $r : X \to B$ by

$$r(x) = \begin{cases} x, & \text{if } x \in B \\ \frac{x}{\|x\|}, & \text{if } x \notin B. \end{cases}$$

Then r, called the radial retraction maps X continuously onto B. For $x \in B^0, rx = x$, while for $x \notin B, r(x) \in \partial B$.

Since $T : B \to X$ is continuous $r \circ T$ maps B continuously into B and is compact. So by Schauder's theorem rT has a fixed point x_0. If $x_0 \in \partial B$ then $Tx_0 \in B$ so that $x_0 = rTx_0 = Tx_0$. If $x_0 \in B^0$, then $Tx_0 \in B_0$ so that $x_0 = rTx_0 = Tx_0$. If $Tx_0 \in \partial B$, then $rTx_0 = Tx_0 \in \partial B \subseteq B$. So in this case too $x_0 = Tx_0$. Thus T has a fixed point. □

11.2 An Application of Schauder's Theorem to an Iterative Functional Equation

11.2.1 Introduction

In what follows an application of Schauder's fixed point theorem to the solution of an iterative functional equation with variable coefficients is described. Solution of iterative functional equations has been studied by Abel, Babbage, Schroder and other well-known mathematicians. For the history of this problem, Kuczma et al. [13] may be consulted. A special case called the iterative root problem arises in the theory of invariant curves and also in the problem of embedding a function into a flow in dynamical systems. Murugan and Subrahmanyam [15] used Schauder's fixed point theorem to solve a special class of iterative functional equations of finding a continuous function $f : I = [a, b] \to \mathbb{R}$ such that

$$\sum_{i=1}^{\infty} \lambda_i(x) H_i(f^i(x)) = F(x), \ x \in I$$

under suitable assumption on the functions $\lambda_i, H_i, i = 1, 2, \ldots$. This section details the solution described in [15].

11.2.2 A Subset of $C(I, R)$

For $I = [a, b]$, $a < b$, $a, b \in \mathbb{R}$ let $C(I, R)$ be the Banach space of continuous real-valued functions on I with supremum norm $\|\cdot\|$. For $M \geq 0$, we define

$$Q(M) = \{f \in C(I, R) : |f(x) - f(y)| \leq M|x - y|, \text{ for } x, y \in I\}$$

and for $\delta \geq 0$,

$$F_\delta(M) = \{f \in C(I, R) : f(a) = a, f(b) = b, \delta(x - y) \leq f(x) - f(y) \leq M(x - y), \\ x, y \in I \text{ and } x \geq y\}$$

we have

Proposition 11.2.1 ([15]) If $M < 1$ or $\delta > 1$ $F_\delta(M) = \emptyset$. If $M = 1$ or $\delta = 1$, then $F_\delta(M)$ is the singleton containing the identity map.

Proof Let $f \in F_\delta(M)$. If $M < 1$, then $f(x) - f(y) < x - y$ for $x > y$. Setting $x > y = a$, it follows that $f(x) < x$ as $f(a) = a$. This implies that $f(b) = b < b$, a contradiction. So $F_\delta(M) = \emptyset$. A similar argument shows that $F_\delta(M) = \emptyset$ for $\delta > 1$. For $M = 1$ and $f \in F_\delta(M)$ and $x > y$, $f(x) - f(y) \leq x - y$. For $y = a$ we get $f(x) \leq f(a)$ for all x. For $x = b$, $y \leq f(y)$ for all y, so that $f(x) = x$ on I. A similar argument for $\delta = 1$ implies that $F_\delta(M)$ is a singleton containing only the identity function. □

Proposition 11.2.2 ([15]) $F_\delta(M)$ is a compact convex subset of $C(I, \mathbb{R})$.

Proof Clearly, $F_\delta(M)$ is a closed and convex subset of $C[I, R]$. Also $|f(x)| \leq max\{|a|, |b|\}$ for $x \in I$. As $\delta(x - y) \leq f(x) - f(y) \leq M(x - y)$ for $x > y, x, y \in I$, for $f \in F_\delta(M)$, $F_\delta(M)$ is uniformly bounded and equicontinuous. So by Arzela–Ascoli Theorem $F_\delta(M)$ is compact. □

Lemma 11.2.3 *If $f \in F_\delta(M)$ where $0 \leq \delta \leq 1 \leq M$, then f is a self-homeomorphism on I and $f^{-1} \in F_{\frac{1}{M}}\left(\frac{1}{\delta}\right)$.*

Proof $f \in F_\delta(M)$ is strictly increasing self-map onto $I = [a, b]$. So f^{-1} exists on I and $f^{-1}(a) = a$ and $f^{-1}(b) = b$. So for $x \geq y$ in I, $\frac{1}{M}(x - y) \leq f^{-1}(x) - f^{-1}(y) \leq \frac{1}{\delta}(x - y)$. Thus $f^{-1} \in F_{M^{-1}}(\delta^{-1})$.

□

Lemma 11.2.4 *Let f_1, f_2 be self-homeomorphisms on I and for $x \geq y$, $x, y \in I$ $\delta(x - y) \leq f_i(x) - f_i(y) \leq M(x - y)$ for some $\delta > 0$ and $M > 0$ where $i = 1, 2$. Then $\delta\|f_1^{-1} - f_2^{-1}\| \leq \|f_1 - f_2\| \leq M\|f_1^{-1} - f_2^{-1}\|$.*

For a proof, Zhang and Baker [24] may be consulted. The following proposition implies that it suffices to solve the functional equation in [0, 1].

Proposition 11.2.5 *f is a solution of the functional equation*

$$\sum_{i=1}^{\infty} \lambda_i(x) H_i(f^i(x)) = F(x) \text{ for } x \in I = [a, b]$$

if and only if $g(x)h^{-1}(fh(x))$ is a solution for

$$\sum_{i=1}^{\infty} \mu_i(x) R_i(g^i(x)) = G(x), \ x \in [0, 1]$$

where $h(x) = a + x(b - a)$, $\mu_i(x) = \lambda_i(h(x))$, $R_i(x) = h^{-1}(H_i(h(x)))$, $G(x) = h^{-1}(Fh(x))$ and $\lambda_i(x) \geq 0$ with $\sum_{i=1}^{\infty} \lambda_i(x) = 1$.

Proof Let f be a solution for

$$\sum_{i=1}^{\infty} \lambda_i(x) H_i(\rho^i(x)) = F(x), \ x \in I = [a, b].$$

Noting that h and h^{-1} are affine continuous maps and $\sum_{i=1}^{\infty} \mu_i(x) = 1$ on [0, 1], for $x \in [0, 1]$

$$\begin{aligned}
\sum_{i=1}^{\infty} \mu_i(x) R_i(g^i(x)) &= \lim_{n\to\infty} \frac{1}{\sum_{i=1}^{n} \mu_i(x)} \sum_{i=1}^{n} \mu_i(x) R_i(g^i(x)) \\
&= \lim_{n\to\infty} \frac{1}{\sum_{i=1}^{n} \mu_i(x)} \sum_{i=1}^{n} \mu_i(x) h^{-1}(H_i h h^{-1} f^i(h(x))) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\mu_i(x)}{\sum_{j=1}^{n} \mu_j(x)} h^{-1}(H_i f^i(h(x)) \\
&= h^{-1}\left(\lim_{n\to\infty} \sum_{i=1}^{n} \mu_i(x) H_i f^i(h(x))\right) \\
&= h^{-1} F(h(x)) = G(x).
\end{aligned}$$

The converse is straightforward. □

Lemma 11.2.6 *If $f, g \in Q(M)$, $M > 1$ and map I onto itself, then for $i \in \mathbb{N}$*

$$\|f^i - g^i\| \leq \frac{M^i - 1}{M - 1}\|f - g\|.$$

This can be proved using induction.

We state and prove an existence theorem making use of subsequent lemmata.

Theorem 11.2.7 ([15]) Let $\lambda_i(x)$ be a sequence of nonnegative continuous functions on $I = [0, 1]$ such that $\lambda_i(x) \in Q(\alpha_i)$ and $1 \leq \gamma_i \leq \lambda_i(x) \leq \Lambda_i$ for $i = 1, 2, \ldots$ and $\sum_{i=1}^{\infty} \lambda_i(x) = 1$ for $x \in I$. Let $l_i, L_i \geq 0$ and $H_i \in F_{l_i}(L_i)$, for $i = 1, 2, \ldots$. Suppose $0 < \delta < 1$, $M > 1$ and

(i) $\sum_{i=1}^{\infty} \Lambda_i L_i M^i < \infty$,

(ii) $K_0 = \sum_{i=1}^{\infty} \gamma_i l_i \delta^{i-1} - \frac{1}{\delta}\sum_{i=1}^{\infty} \alpha_i > 0.$

Then for any function F in $F_{K_1\delta}(K_0 M)$, the functional equation

$$\sum_{i=1}^{\infty} \lambda_i(x) H_i(f^i(x)) = F(x),\ x \in I$$

has a solution f in $F_\delta(M)$ where

$$K_1 = \sum_{i=1}^{\infty} \left\{\frac{\alpha_i}{\delta} + \Lambda_i L_i M^{i-1}\right\}$$

For $f \in F_\delta(M)$, define $L_f : I \to I$ by

$$L_f(x) = \sum_{i=1}^{\infty} \lambda_i(f^{-1}(x)) H_i(f^{i-1}(x)),\ x \in I.$$

Lemma 11.2.8 *Suppose that in addition to the hypotheses of the above theorem $f \in F(M)$. Then $L_f \in F_{K_0}(K_1)$ where K_0 and K_1 are as defined in that theorem.*

Proof It is easy to see that $L_f(0) = 0$ and $L_f(1) = 1$. For $x \geq y$, $x, y \in I$

$$\begin{aligned}
L_f(x) - L_f(y) &= \sum_{i=1}^{\infty} \lambda_i(f^{-1}(x)) H_i(f^{i-1}(x)) - \sum_{i=1}^{\infty} \lambda_i(f^{-1}(y)) H_i(f^{i-1}(y)) \\
&= \sum_{i=1}^{\infty} \{[\lambda_i(f^{-1}(x)) - \lambda_i(f^{-1}(y))] H_i(f^{i-1}(x))
\end{aligned}$$

$$+\lambda_i(f^{-1}(y))[H_i(f^{i-1}(x)) - H_i(f^{i-1}(x))]$$

From the definitions of λ_i, H_i and f and by Lemma 11.2.6 one gets

$$\begin{aligned} L_f(x) - L_f(y) &\le \sum_{i=1}^{\infty}\{\alpha_i(f^{-1}(x) - f^{-1}(y))\Lambda_i L_i M^{i-1}(x-y)\} \\ &\le \sum_{i=1}^{\infty}\left\{\frac{\alpha_i}{\delta}(x-y) + \Lambda_i L_i M^{i-1}(x-y)\right\} = K_1(x-y) \end{aligned}$$

Similarly

$$L_f(x) - L_f(y) \ge \sum_{i=1}^{\infty}\left\{\gamma_i l_i \delta^{i-1} - \frac{\alpha_i}{\delta}\right\}(x-y) = K_0(x-y)$$

So $L_f \in F_{K_0}(K_1)$ and by Lemma 11.2.3 $L_f^{-1} \in F_{1/K_1}(1/K_0)$.

□

Lemma 11.2.9 *Besides the hypotheses of Theorem 11.2.7, suppose that* $f \in F_\delta(M)$. *Then* $\|L_f - L_g\| \le K_2\|f-g\|$ *and* $\|L_f^{-1} - L_g^{-1}\| \le \frac{K_2}{K_0}\|f-g\|$, K_2 *being* $\sum_{i=1}^{\infty}\left\{\frac{\alpha_i}{\delta} + \frac{\Lambda_i L_i(M^{i-1})}{M-1}\right\}$.

Proof If $f, g \in F_\delta(M)$ and $x \in I$

$$\begin{aligned} |L_f(x) - L_g(x)| &\le \left|\sum_{i=1}^{\infty}[\lambda_i(f^{-1}(x))H_i(f^{i-1}(x)) - \lambda_i(g^{-1}(x))H_i(g^{i-1}(x))]\right| \\ &\le \sum_{i=1}^{\infty}\{|\lambda_i(f^{-1}(x)) - \lambda_i(g^{-1}(x))||H_i(f^{i-1}(x))| \\ &\quad + |\lambda_i(g^{-1}(x))||H_i(f^{i-1}(x)) - H_i(g^{i-1}(x))|\} \end{aligned}$$

From Lemmata 11.2.3 and 11.2.6 and the definitions of $\lambda_i(x)$ and $H_i(x)$, one obtains

$$\begin{aligned} |L_f(x) - L_g(x)| &\le \sum_{i=1}^{\infty}\alpha_i|f^{-1}(x) - g^{-1}(x)| + \Lambda_i L_i\|f^{i-1} - g^{i-1}\| \\ &\le \sum_{i=1}^{\infty}\left(\frac{\alpha_i}{\delta} + \Lambda_i L_i\frac{M^i - 1}{M-1}\right)\|f-g\| \end{aligned}$$

Thus $\|L_f - L_g\| \le K_2\|f-g\|$. As $f \in F_\delta(M)$, for $x \ge y$, $L_f^{-1}(x) - L_f^{-1}(y) \le \frac{1}{K_0}(x-y)$. So by Lemma 11.2.4, $\|L_f^{-1} - L_g^{-1}\| \le \frac{K_2}{K_0}\|f-g\|$. □

The above lemmata lead to the proof of Theorem 11.2.7.

Proof of Theorem 11.2.7. Define $T : F_\delta(M) \to C(I, R)$ by $Tf(x) = L_f^{-1}(F(x))$ for $x \in I$. Clearly, $Tf(0) = 0$ and $Tf(1) = 1$. If $x \geq y$, $x, y \in I$

$$\begin{aligned} Tf(x) - Tf(y) &= L_f^{-1}(F(x)) - L_f^{-1}(F(y)) \leq \frac{1}{K_0}[F(x) - F(y)] \\ &\leq M(x - y). \end{aligned}$$

Further,

$$Tf(x) - Tf(y) \geq \frac{1}{K_1}|F(x) - F(y)| \geq \frac{1}{K_1}K_1\delta(x - y) = \delta(x - y).$$

Thus $Tf \in F_\delta(M)$ and maps $F_\delta(M)$ into itself. For $f, g \in F_\delta(M)$ and $x \in I$

$$\begin{aligned} |Tf(x) - Tg(x)| &= |L_f^{-1}(F(x)) - L_g^{-1}(F(x))| \\ &\leq \|L_f^{-1} - L_g^{-1}(x)\| \end{aligned}$$

Thus $\|Tf - Tg\| \leq \frac{K_2}{K_0}\|f - g\|$ in view of Lemma 11.2.8. By Proposition 11.2.2, $F_\delta(M)$ is a compact convex subset of $C[I, R]$. So by Schauder fixed point theorem T has a fixed point in $F_\delta(M)$ which is a solution of the functional equation. □

11.3 Measures of Noncompactness and Fixed Point Theorems

Given a noncompact space, a natural question is to find out if noncompactness can be measured. In a complete metric space a measure of noncompactness can be formulated in a natural way since compact subsets of such a space are precisely totally bounded closed subsets. Such a measure, called Kuratowski's measure of noncompactness is the following.

Definition 11.3.1 Let (M, d) be a complete metric space and X a bounded subset of M. Kuratowski measure of noncompactness of X denoted by $\alpha(X)$ is defined by

$$\begin{aligned} \alpha(X) = \inf\{\epsilon > 0 : X &\text{ can be covered by a finite} \\ &\text{number of sets of diameter less than } \epsilon\}. \end{aligned}$$

Using the above definition, the following proposition can be proved easily.

Proposition 11.3.2 *Let (M, d) be a complete metric space, X, Y bounded subsets of M and $\alpha(\cdot)$ the Kuratowski measure of noncompactness function. Then*

(i) $\alpha(X) = 0$ *if and only if* $\overline{X}$ *is compact;*
(ii) $\alpha(X) = \alpha(\overline{X})$*;*
(iii) $X \subseteq Y \Rightarrow \alpha(X) \leq \alpha(Y)$*;*
(iv) $\alpha(X \cup Y) = \max\{\alpha(X), \alpha(Y)\}$*;*
(v) $\alpha(X \cap Y) \leq \min\{\alpha(x), \alpha(Y)\}$.

Proposition 11.3.3 *Let* (X_n) *be a non-increasing sequence of nonempty bounded closed subsets of a complete metric space* (M, d) *such that* $\lim_{n\to\infty} \alpha(X_n) = 0$. *Then* $\bigcap_{n=1}^{\infty} X_n$ *is nonempty and compact.*

Proof Define a sequence $x_n \in X_n$ for each $n \in N$ and a set $A_k = \{x_n : n \geq k\}$ for $k \in \mathbb{N}$. Clearly, $A_n \subseteq X_n\ \forall\ n$ and $\alpha(A_1) = \alpha(A_n) = \alpha(\overline{A_n})$ in view of (i), (ii), (iii) and (iv). So $\alpha(A_1) = 0$. So $\alpha(\overline{A_1}) = 0$. Thus $\overline{A_1}$ is compact. As every cluster point of A_1 is a cluster point of A_n and $\alpha(A_n) = \alpha(\overline{A_n}) = 0$, $\overline{A_n}$ are all compact. Since $\overline{A_n}$ and hence $\{X_n \supseteq \overline{A_n} : n \in \mathbb{N}\}$ has finite intersection property $\bigcap_{n=1}^{\infty} \overline{A_n}$ and hence $\bigcap_{n=1}^{\infty} X_n \neq \phi$. □

Remark 11.3.4 Proposition 11.3.3 is a generalization of the Cantor intersection theorem.

If the metric space (M, d) is a Banach space with the metric d arising from the norm $\| \ \|$, then the (Kuratowski) measure of noncompactness satisfies additional properties. These are collected in

Proposition 11.3.5 *If* $(E, \| \cdot \|)$ *is a Banach space, the Kuratwoski measure of noncompactness satisfies the additional properties stated below:*

(vi) $\alpha(X + Y) \leq \alpha(X) + \alpha(Y)$*;*
(vii) $\alpha(cX) = |c|\alpha(X)$*; for* $c \in \mathbb{R}$*;*
(viii) $\alpha(conv\ X) = \alpha(X)$*;*

where X, Y *are bounded subsets of* E.

Proof We prove only (viii). For $x = \sum_{i=1}^{n} t_i x_i$ and $y = \sum_{j=1}^{m} s_j y_j$ for $x_i, y_j \in X$ and $t_i, s_j \geq 0$ with $\sum_{i=1}^{n} t_i = \sum_{j=1}^{m} s_j = 1$, $\|x - y\| = \left\| \sum_{i=1}^{n} t_i x_i - \sum_{j=1}^{m} s_j y_j \right\| = \left\| \sum_{j=1}^{m} s_j \sum_{i=1}^{n} t_i x_i - \sum_{i=1}^{n} t_i \sum_{j=1}^{m} y_j \right\| \leq \sum_{i,j} s_j t_i \|x_i - y_j\| \leq diam\ X$. Consequently $\alpha(conv\ X) \leq \alpha(X)$. We get $\alpha(conv\ X) \leq \alpha(X)$. As $X \subseteq conv\ X$, $\alpha(X) \leq \alpha(conv\ X)$. Thus $\alpha(X) = \alpha(conv\ X)$. □

Remark 11.3.6 When X is compact, the above result leads to Mazur's theorem that the closed convex hull of a compact set is compact. The calculation of the measure of noncompactness is not always easy. In fact, the proof that $\alpha(B(0, 1)) = 2$ for the unit open ball $B(0, 1)$ in a Banach space is not obvious.

Remark 11.3.7 $\alpha\left(\bigcup_{0\le t\le t_0} tX\right) = t_0\alpha(X)$. This may be deduced from $\bigcup_{0\le t\le t_0} tX \subseteq conv[t_0X \cup \{0\}]$.

Remark 11.3.8 One may alternatively cover a bounded subset X of a metric space (X, d) by a finite number of open balls of radius smaller than $\epsilon > 0$. This leads to the Hausdorff (or ball) measure of noncompactness of X. This has been used by Gohberg, Goebel, Nussbaum and others. This measure is closely related to the concept of Hausdorff metric and is denoted by χ. It can be proved that $\psi(X) = \inf\{H(X, F) : F$ a nonempty compact subset of M and H the Hausdorff distance on the space of all closed nonempty bounded subsets of $M\}$.

It is, therefore, possible to propose an abstract concept of a measure of noncompactness based on an axiomatic approach.

Definition 11.3.9 A nonempty subfamily $\mathcal{P}$ of the family $\mathscr{N}$ of nonempty relatively compact subsets of E is said to be the kernel (of a measure of noncompactness) if the following conditions are satisfied:

(i) $X \in \mathcal{P} \Rightarrow \overline{X} \in \mathcal{P}$;
(ii) $X \in \mathcal{P}, \phi \neq Y \subseteq X \Rightarrow Y \in \mathcal{P}$;
(iii) $X, Y \in \mathcal{P} \Rightarrow \lambda X + (1-\lambda)Y \in \mathcal{P}$ for $\lambda \in [0, 1]$;
(iv) the family of compact sets in $\mathcal{P}$ is closed in the family of nonempty compact sets with Hausdorff metric.

Definition 11.3.10 Let $\mathfrak{M}$ be the family of non-void bounded sets of E. A function $\mu : \mathfrak{M} \to [0, \infty)$ is said to be a measure of noncompactness with the kernel $\mathcal{P}$ if it satisfies the following:

(i) $\mu(X) = 0 \Leftrightarrow X \in \mathcal{P}$;
(ii) $\mu(\overline{X}) = \mu(X)$;
(iii) $X \subseteq Y \Rightarrow \mu(X) \le \mu(Y)$;
(iv) $\mu(conv\ X) = \mu(X)$;
(v) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$;
(vi) for $X_n \in \mathfrak{M}$ and $X_n = \overline{X}_n$ and $X_{n+1} \subseteq X_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mu(X_n) = 0$ imply $\bigcap_{n=1}^{\infty} X_n \neq \phi$.

We skip the proof of the following.

Theorem 11.3.11 *For any kernel $\mathcal{P}$ the function $\mu(X) = \inf\{H(X, P) : P \in \mathcal{P}\}$ is a measure of noncompactness with kernel P.*

Remark 11.3.12 If $\mathcal{P}$ is $\mathfrak{N}$, the family of nonempty relatively compact subsets of E then the kernel is full and the measure is said to be complete. The Kuratowski and Hausdorff measures are full. Given a nontrivial closed subset V of the Banach space E, for a nonempty bounded subset X of E, $\mu(X) = \chi(X) + d(X, F)$, where

$$d(X, F) = \inf\{\epsilon > 0 : X \subseteq N(F; \epsilon)\}$$

is a measure of noncompactness on the family of non-void compact sets X. $\|X\|$ and $diam(X)$ are also measures. The kernels are respectively $\{\theta\}$ and the family of singletons.

Definition 11.3.13 A measure μ is said to have maximum property if $\mu(X \cup Y) = \max(\mu(X), \mu(Y))$ for all $x, y \in \mathcal{P}$. If $\mu(\lambda X) = |\lambda|\mu(X)$ for all scalars λ, μ is called homogeneous and if $\mu(X + Y) \leq \mu(X) + \mu(Y)$, μ is called subadditive. (A homogeneous subadditive measure is also called sublinear measure.)

A measure of noncompactness is called regular if it is full, sublinear and has maximum property. For further details and applications Banas and Goebel [1] may be referred.

We now state and prove Darbo's fixed point theorem [4].

Theorem 11.3.14 (Darbo [4]) *Let C be a non-void closed convex set which is bounded and μ be any measure of noncompactness with the kernel of the family of bounded nonempty convex subsets of E. Let $T : C \to C$ be a μ-contraction in the sense that $\mu(T(X)) \leq k\mu(X)$ for any nonempty convex bounded subset X, where $0 \leq k < 1$. If T is continuous, then T has a fixed point and the set of fixed points of T in C belongs to the kernel of μ.*

Proof Define $C_1 = C$, $C_{n+1} = \overline{conv}(TC_n)$ for $n \geq 1$. Now $\mu(C_{n+1}) = \mu(\overline{con}(TC_n)) = \mu(TC_n) \leq k\mu(C_n) \leq k^{n-1}\mu(C)$ for all $n \geq 1$. As $0 < k < 1$, proceeding to the limit as $n \to \infty$, it follows that $C^* = \bigcap_{n=1}^{\infty} C_n \neq \phi$ is in the kernel of μ and $\mu(C^*) = 0$. Further, C^* is convex and closed. As C^* is precompact, C^* is a non-void compact convex subset of the Banach space E. Also $TC^* \subseteq TC_n \subseteq (\overline{con}(TC_n)) = C_{n+1}$ for all $n \geq 1$. So $TC^* \subseteq \bigcap_{n=1}^{\infty} C_{n+1} \cap C_1 = C^*$. Thus T maps C^* into itself and is continuous. So by Schauder's fixed point theorem T has a fixed point in C. If F is the set of fixed points of T in C, then $\mu(F) = \mu(T(F)) \leq k\mu(F)$. So $\mu(F) = 0$. Since F is closed, F is compact and so belongs to the kernel of μ. □

Corollary 11.3.15 *Let C be a non-void bounded closed convex subset of a Banach space and T be a contraction on C with Lipschitz constant $k < 1$ and S a continuous map mapping bounded subsets of C into precompact (totally bounded) subsets of C such that $F = T + S$ maps C into itself. Then F has a fixed point.*

Proof Let α be the Kuratowski measure of noncompactness and B a bounded non-void subset of C. Then $\alpha(F(B)) = \alpha(T+S)(B)) \leq \alpha T(B) + \alpha S(B) = k\alpha(B)$ as $\alpha(S(B)) = 0$. Thus F is a α-set contraction and so by Darbo's Theorem 11.3.14 has a fixed point. □

Example 11.3.16 The functional equation $x(t) = \frac{3}{4}x(\sqrt{t}) + \int_0^t \frac{\cos(x(\frac{s+1}{2}))}{4} ds$ for $t \in [0, 1]$ has a solution in $C[0, 1]$. Let B be the closed unit ball in $C[0, 1]$ and T and S the maps on B defined by $Tx(t) = \frac{3}{4}x(\sqrt{t})$ and $S(x(t)) = \frac{1}{4}\int_0^t \cos(x(\frac{s+1}{2}))ds$. Clearly, T is a contraction on B with Lipschitz constant $\frac{3}{4}$ while S is a compact operator since $|Sx(t_1) - Sx(t_2)| \leq \frac{1}{4}\left|\int_{t_1}^{t_2} \cos(x(\frac{s+1}{2}))ds\right| \leq \frac{1}{4}|t_1 - t_2|$ for $t_1, t_2 \in [0, 1]$. Further, $(T+S)x \in B$ for $x \in B$. So by Corollary 11.3.15. $F = T + S$ has a fixed point in $B \subseteq C[0, 1]$. Thus the above functional equation has a solution in $C[0, 1]$.

Sadovskii [19] obtained a generalization of Darbo's theorem using the concept of a condensing operator.

Definition 11.3.17 Let λ be a measure of noncompactness with a kernel $\mathcal{P}$. Let C be a bounded non-void closed convex subset of a Banach space E. A map $T : C \to E$ is called condensing if $\lambda(T(A)) < \lambda(A)$ for all $A \subseteq C$ with $\lambda(A) > 0$.

Theorem 11.3.18 (Sadovskii [19]) *Let C be a nonempty bounded closed convex subset of a Banach space and $T : C \to C$ be a continuous condensing operator. Then T has a fixed point, provided λ has maximum property.*

Proof Let $c \in C$. Denote by $\mathscr{F}$ the set of all closed convex subsets K of C such that $c \in K$ and $T(K) \subseteq K$. Define $B = \bigcap_{K \in \mathscr{F}} K$ and $C^* = \overline{con}(T(B) \cup \{c\})$. Clearly, $\mathscr{F} \neq \phi$ as $C \in \mathscr{F}$ and $B \neq \phi$ as $c \in B$ by definition of $\mathscr{F}$. Since $T(B) \subseteq T(K) \subseteq K$ for each $K \in \mathscr{F}$, $T(B) \subseteq \bigcap_{K \in \mathscr{F}} K = B$. Thus T maps B into itself. We claim that $B = C^*$. As $c \subset B$ and $T(B) \subseteq B$, $B \in \mathscr{F}$, $C^* \subseteq B$. So $T(C^*) \subseteq T(B) \subseteq C^*$. So $C^* \in \mathscr{F}$ and by definition of B, $C^* \supseteq B$. Thus $B = C^*$. By the properties of λ, the measure of noncompactness, $\lambda(B) = \lambda(C) = \lambda\{T(B) \cup \{c\}\} = \lambda(T(B))$. This implies $\lambda(B) = 0$ as T is condensing. So B is compact. Since T is continuous, B is nonempty compact and convex and $T(B) \subseteq B$, T has a fixed point by Schauder's fixed point theorem. □

Remark 11.3.19 Let B be the closed unit ball in the Hilbert space of real sequences ℓ_2. The operator $F : B \to B$ defined by

$$\begin{aligned} F(x) &= F(x_1, \ldots, x_n, \ldots) = (\sqrt{1 - \|x\|^2}, x_1, x_2, \ldots) \\ &= T(x) + S(x) = (\sqrt{1 - \|x\|^2}, 0, 0, \ldots) + (0, x_1, x_2, \ldots) \\ &\qquad \text{for } x = (x_1, x_2, \ldots, x_n, \ldots) \end{aligned}$$

Clearly, S, the shift operator is an isometry and T is a compact operator. Further, $\alpha(F(B)) = \alpha(T(B) + S(B)) \leq \alpha(SB) = \alpha(B)$. So F is 1-set contraction with respect to α has no fixed point. Thus Darbo's theorem or Sadovskii's theorem is not true for 1-set contractions.

A fixed point theorem for mappings satisfying the so-called Leray-boundary condition involving measures of noncompactness can also be proved.

Theorem 11.3.20 *Let μ be a measure with kernel $\mathcal{P}$ having maximum property such that $\{\theta\} \in \mathcal{P}$. Suppose C is an open and bounded neighbourhood of θ of a Banach space E and $T : \overline{C} \to E$ is a continuous k-set contraction with contractive constant k (< 1) such that for any $x \in \partial C$, $Tx \neq \lambda x$ for $\lambda > 1$. Then T has a fixed point in $\overline{C}$ and the set of fixed points of T belongs to $\mathcal{P}$.*

Proof Let $K = \{x \in C : x = cTx \text{ for some } c \in [0, 1]\}$. K is nonempty as $\theta \in K$ and is obviously closed. As $K \subseteq conv(TK \cup \{\theta\})$ we have $\mu(K) \leq \mu(T(K) \cup \{0\}) = \mu(T(K)) \leq k\mu(K)$ and $K \in \mathcal{P}$. Clearly, $K \cap \partial C = \phi$. Since K is compact and $E - C$ is non-void closed set and disjoint from K, by Urysohn's Lemma there is a continuous function $g : E \to [0, 1]$ such that $g(x) = 1$ for $x \in K$ and $g(x) = 0$ for $x \notin C$ and $0 < g(x) < 1$ for $x \in C - K$. Define the map $F : E \to E$ by

$$F(x) = \begin{cases} g(x)T(x), & \text{for } x \in C \\ 0, & \text{for } x \notin C \end{cases}$$

Clearly, F maps each ball $B(\theta; r)$ ($r > 0$) containing C into itself. For any set X, $F(X) \subseteq conv(T(X \cap C) \cup \{\theta\})$ and so

$$\begin{aligned} \mu(F(X)) &\leq \mu conv(T(X \cap C) \cup \{\theta\}) = \mu(T(X \cap C)) \\ &\leq \mu(T(X)) \leq k\mu(X). \end{aligned}$$

As T is continuous on $\overline{C}$. F is continuous on the closed ball $B(\theta; r)$ containing C and is a k-set contraction. So by Darbo's fixed point Theorem 11.3.14. F has a fixed point x_0 in $B(\theta; r)$. Clearly, $x_0 \neq \theta$ and is not in the complement of C. So $x_0 \in C$. If $x_0 \in C - K$, then $0 < g(x_0) < 1$ and $x_0 = g(x_0)T(x_0)$ contradicts that $x_0 \notin K$ (by the definition of K). So $x_0 \in K$. So $g(x_0) = 1$ and $F(x_0) = g(x_0)T(x_0) = x_0$ $T(x_0) = x_0$. Thus x_0 is a fixed point of T. That the set of fixed points of T in $\overline{C}$ is compact is left as an exercise. □

11.4 Kakutani-Ky Fan–Glicksberg Fixed Point Theorem

Kakutani [10] generalized Brouwer's fixed point theorem to multivalued functions on Euclidean spaces. Inspired by this theorem. Glicksberg [8] and Ky Fan [7] extended it to topological linear spaces. We need the following.

Definition 11.4.1 Let X and Y be topological spaces and 2^Y denote the set of all subsets of Y. Let T be a multivalued function mapping X into $2^Y - \{\phi\}$. T is said to be upper semicontinuous if for each $x_0 \in X$ and any neighbourhood W of $T(x_0)$ in Y, there exists a neighbourhood V of x_0 such that $x \in V$ implies $T(x) \subseteq W$.

Theorem 11.4.2 (Ky Fan–Glicksberg [7, 8]) *Let K be a non-void compact convex subset of a Hausdorff locally convex linear topological space X. Let $T : K \to 2^K$ be an upper semicontinuous multivalued function such that $T(x)$ is a nonempty closed convex subset of K for each x in K. Then there exists $x_0 \in K$ such that $x_0 \in Tx_0$.*

The following proof due to Terkelsen [21] employs a technique of Browder [2].

Proof Let $\{U_i : i \in I\}$ be a neighbourhood base at 0 in X comprising open convex circled sets. For each $i \in I$, there exists a finite set $\{x_{ij} : j \in J(i)\} \subseteq K$ such that $K \subseteq \bigcup_{j \in J(i)} (x_{ij} + U_i)$. Now there is a continuous partition of unity subordinate to this covering, i.e. for $j \in J(i)$, continuous functions $\alpha_{i,j} : K \to \mathbb{R}$ exist with $\alpha_{ij}(x) \geq 0$ for $x \in K$, $\alpha_{ij}(x) = 0$ for $x \notin x_{ij} + U_i$ and $\sum_{j \in J(i)} \alpha_{ij}(x) = 1$ for $x \in K$. Let $y_{ij} \in T(x_{ij})$ and define $f_i : K \to X$ by $f_i(x) = \sum_{j \in J(i)} \alpha_{ij}(x) y_{ij}$. Let C_i be the convex hull of $\{y_{ij} : j \in J(i)\}$. C_i is homeomorphic to a closed convex set in a finite-dimensional euclidean space, where $C_i \subseteq K$. So by Brouwer's fixed point theorem f_i has a fixed point x_i in C_i.

Let $x_0 \in K$ be the cluster point of the net $\{x_i : i \in I\}$ directed by $\{U_i, \supset : i \in I\}$. Suppose $x_0 \notin Tx_0$. By separation theorem there exists a closed convex neighbourhood W of Tx_0 with $x_0 \notin W$. As T is upper semicontinuous we can find a neighbourhood V of x_0 with $V \cap W = \phi$ such that $x \in V \cap K$ implies $T(x) \subseteq W$. Let $m \in I$ be for an open set $U_m \ni \theta$ such that $U_m + U_m \subset V - x_0$. We can find $i \in I$ such that $U_m \supset U_i$, $x_i \in x_0 + U_m$ and $x_i + U_i \subset V$. Now for $j \in J(i)$ with $\alpha_{ij}(x_i) \neq 0$, $x_i \in x_{ij} + U_i$. So $x_{ij} \in V$. This implies that $y_{ij} \in W$. Consequently $x_i = f_i(x_i) = \sum_{j \in J(i)} a_{ij}(x_i) y_{ij} \in W$ contradicting that $x_i \in V$ and V is disjoint from W. Hence $x_0 \in T(x_0)$ for some $x_0 \in K$. □

Corollary 11.4.3 (Kakutani [10]) *Let K be a compact convex nonempty subset of $\mathbb{R}^n$ and $T : K \to 2^K$ be an upper semicontinuous multivalued function such that $T(x)$ is a nonempty compact convex subset of K for each $x \in K$. Then T has a fixed point.*

Corollary 11.4.4 (Tychonoff [22]) *Let X be a linear topological space which is both locally convex and Hausdorff and K a nonempty compact convex subset of X. If $T : K \to K$ is continuous then T has a fixed point in K.*

Proof The map $F : K \to 2^K$ defined by $F(x) = \{T(x)\}$ satisfies all the hypothesis of Theorem 11.4.2. So T has a fixed point in K. □

Cauty [3] has shown that Tychonoff (Schauder) Theorem is true in arbitrary linear topological spaces. Tychonoff's theorem may be used to deduce a common fixed point theorem for a commuting family of affine continuous mappings, due to Markov [14] and Kakutani [11].

Definition 11.4.5 A mapping f on a convex set C into a linear space is called affine if $f(\alpha x + (1-\alpha)y) = \alpha f(x) + (1-\alpha) f(y)$ for all $x, y \in C$ and $\alpha \in [0, 1]$.

Theorem 11.4.6 *Let K be a nonempty compact convex subset of a Hausdorff locally convex topological vector space and $\mathscr{F}$ be a commuting family of affine continuous functions mapping K into itself. Then there is a common fixed point for mappings of $\mathscr{F}$ in K.*

Proof By Tychonoff's theorem (Corollary 11.4.4), $F(T)$, the set of fixed points, is nonempty for each $T \in \mathscr{F}$. As T is affine, $F(T)$ is closed and convex as well. If $S \in \mathscr{F}$, then S maps $F(T)$ into itself by the commutativity of S and T and again by Tychonoff's Theorem has a fixed point in $F(T)$. So $F(T) \cap F(S) \neq \phi$. In fact, the family $\{F(T) : T \in \mathscr{F}\}$ is a family of nonempty compact convex subsets of K with finite intersection property. Since K is compact, $\cap\{F(T) : T \in \mathscr{F}\} \neq \phi$. Any element in this intersection is a common fixed point for mappings in $\mathscr{F}$. □

Kakutani [11] has also given a more elementary proof of this theorem and it is described below.

Proof (*(Aliter) (Kakutani* [11])) Given a commuting family A of continuous self-maps on a non-void compact convex subset of a locally convex linear topological space, it is clear that for $T \in A$, T_n defined by

$$T_{(n)}(x) = \frac{1}{n}(x + tx + \cdots + T^{n-1}(x))$$

for $x \in M$ and $n > 1$. T^k being the kth iterate of T is affine, maps M into itself and commutes with each $U \in A$. Further, a fixed point of T is also a fixed point of $T_{(n)}$ for all $n > 1$. So we may without loss of generality assume that A is a semigroup containing such convex combinations of the iterates of T. If S and T are in A then $S(T(M)) \subseteq S(M) \subseteq M$ and $S(T(M)) = T(S(M)) \subseteq T(M) \subseteq M$. So $ST(M) \subseteq S(M) \cap T(M) \subseteq M$. Since $ST \in A$ $ST(M) \neq \phi$. Thus $S(M) \cap T(M) \neq \phi$. If therefore follows that if A_1 is a finite subset of A, then $\bigcap_{U \in A_1} U(M) \neq \phi$. Since $U(M)$ is compact for each $U \in A$, $\bigcap_{T \in A} T(M) \neq \phi$. Let $x^* \in \bigcap_{T \in A} T(M)$. Then $x^* = T_{(n)} y$ for some $y \in M$. So $Tx^* - x^* = \frac{1}{n}(T^n y - y) \in \frac{1}{n} M_1$ where $M_1 = \{x - y : x, y \in M\}$. As M is compact, M_1 is compact and bounded. Since $n \in N$ is arbitrary, $Tx^* = x^*$. T being an arbitrary element of A, it follows that x^* is a common fixed point for all $T \in A$.

As an application Hahn–Banach theorem on the extension of linear functionals on a linear subspace of a locally convex linear topological space can be deduced.

Theorem 11.4.7 *Let E be a linear space and f a linear functional defined on a linear subspace E_0 of E. Let $p : E \to R$ be a function such that $p(tx) = tp(x)$ for $t \geq 0$ and $p(x + y) \leq p(x) + p(y)$ where $x, y \in E$. Suppose $f(x) \leq p(x)$ for any $x \in E_0$. Then there exists a linear functional F on E such that $p(F(x)) \leq p(x)$ for $x \in E$ and $F(x) = f(x)$ for $x \in E_0$.*

Proof Let $B = \{F \in R^E : -p(-x) \leq F(x + y) - F(y) \leq p(x)$ for all $x, y \in E$ and $F(x) = f(x)$ for $x \in E_0\}$. Since B is a closed subset of $\prod_{x\in E}[-p(x), p(x)]$ in the product topology of $\prod_{x\in E}[-p(x), p(x)]$, B is compact. Note that R^E with the product topology is a Hausdorff locally convex linear topological space. The convexity of B is also clear. Let Γ be the group of all linear transformations on B generated by S_t and T_y, $t \in R$, $y \in E$ defined by $S_t(F(x)) = \frac{F(tx)}{t}$, $T_y F(x) = F(x + y) - F(y)$. Clearly it is abelian. So by Kakutani's common fixed point theorem it has a fixed point F which coincides with f on E_0, $F(x) \leq p(x)$ for $x \in E$ and $F(tx) = tFx$ as $S_t F = F$ for all t and $F(x + y) - F(y) = F(x)$ as $T_y F = F$ for all y. Thus F is a linear extension of f to E. □

Werner [23] has proved that Hahn–Banach theorem implies Markov–Kakutani theorem. To this end, we use the Hahn–Banach theorem in the form of separation theorem, to prove the following lemma leading to Markov–Kakutani theorem.

Lemma 11.4.8 ([23]) *Let $T : K \to K$ be a continuous affine map on K, a compact convex subset of a locally convex Hausdorff linear topological space E. Then T has a fixed point in K.*

Proof Suppose the lemma is false. Then the diagonal $\Delta = \{(x, x) : x \in K\}$ and the graph of T, viz $\Gamma = \{(x, Tx) : x \in K\}$ are disjoint compact convex subsets of $E \times E$. So by the Hahn–Banach theorem there exist continuous linear functional ℓ_1 and ℓ_2 on E and real numbers α and β such that

$$\ell_1(x) + \ell_2(x) \leq \alpha < \beta < \ell_1(y) + \ell_2(T(y))$$

for all $x, y \in K$. So $\ell_2(T(x)) - \ell_2(x) \geq \beta - \alpha$ for all $x \in K$. Since $\ell_2(T^n(x)) - \ell_2(T^{n-1}(x)) \geq \beta - \alpha$ for all $n > 1$, it follows $\ell_2(T^n(x)) - \ell_2(x) \geq n(\beta - \alpha)$ for all $x \in K$. This implies that $\ell_2(T^n(x)) \to +\infty$ contradicting that $\ell_2(K)$ is compact. So $\Gamma \cap \Delta \neq \phi$ or T has a fixed point in K. □

We can now deduce Markov–Kakutani theorem from the preceding lemma.

Proof Let $T_\lambda : K \to K$ be an affine continuous map on a compact convex subset of a Hausdorff locally convex linear topological space for each $\lambda \in \Lambda$. Further, $T_\lambda T_\mu = T_\mu T_\lambda$ for any $\lambda, \mu \in \Lambda$. By Lemma 11.4.8 K_λ, the set of fixed points of T_λ is non-void for each $\lambda \in \Lambda$. K_λ is compact being a closed subset of K and is convex as T_λ is affine for each $\lambda \in \Lambda$. Let $k \in K_\mu$. Then $T_\mu(k) = k$. Also $T_\lambda k = T_\lambda T_\mu k = T_\mu(T_\lambda k)$. So $T_\lambda(k)$ is a fixed point of T_μ. So $T_\lambda(K_\mu) \subseteq K_\mu$ and so T_λ has a fixed point K_μ. Thus

$K_\lambda \cap K_\mu \neq \phi$. From the principle of finite induction it follows that every non-void finite subfamily of $\{K_\lambda : \lambda \in \Lambda\}$ has non-void intersection. So $K^* = \bigcap_{\lambda \in \Lambda} K_\lambda \neq \phi$. Clearly, every point if K^* is a fixed point of T_λ, $\lambda \in \Lambda$. □

Markov–Kakutani theorem also provides other applications, for example the existence of an invariant mean and the existence of Banach limits. These are described below.

Definition 11.4.9 Let $P(X)$ be the set of all Borel probability measures on a compact Hausdorff space X. Let $\mu \in P(X)$. A μ-measurable map $f : X \to X$ is called measure preserving with respect to μ if $\mu(B) = \mu(f^{-1}(B)$ for every Borel set $B \subseteq X$. In this case μ is called an invariant measure for f.

Remark 11.4.10 For a compact Hausdorff space the dual of $C(X)$ can be identified with $M(X)$, the space of complex regular Borel measures on X, (in view of the Riesz Representation theorem) with $\|\mu\|$ = total variation of μ.

If $f : X \to X$ is continuous, then $\tilde{f}_\mu$ defined by $\tilde{f}_\mu(B) = \mu(f^{-1}(B))$ for each Borel set B defines a probability measure on X.

We have

Lemma 11.4.11 *If $f : X \to X$ is continuous on a compact Hausdorff space, then the map $\tilde{f} : P(X) \to P(X)$ defined by $\tilde{f}_\mu(B) = \mu(f^{-1}(B))$, B any Borel set is continuous in the Weak* topology.*

Proof For $g \in C(X)$, $\int_X g d\tilde{f}_\mu = \int_X g \circ f d\mu$ is well-defined for g and an application of monotone convergence theorem to an increasing sequence of nonnegative simple functions clarifies that it is well-defined for all $g \in C(X)$. If $\{\mu_\lambda\}_{\lambda\in\Lambda}$ is a net converging to $\mu \in P(X)$, then for every $g \in C(X)$ it can be seen that

$$\begin{aligned}\lim_{\lambda\in\Lambda} \int_X g d(\tilde{f}_\mu)_\lambda &= \lim_{\lambda\in\Lambda} \int_X (g \circ f) d\mu_\lambda \\ &= \int_X (g \circ f) d\mu = \int_X g d(\tilde{f}_\mu)\end{aligned}$$

□

Hence we have

Theorem 11.4.12 *For $f \in C(X)$, there exists $\mu \in P(X)$ for which f is measure preserving.*

Proof The map $\tilde{f} : P(X) \to P(X)$ defined by $\tilde{f}_\mu(B) = \mu(f^{-1}(B))$ for any Borel subset of X is continuous in the weak* topology by Lemma 11.4.11 and $P(X)$ is a convex and closed subset of the closed unit sphere of $M(X)$. $M(X)$ is compact by Alaoglu's theorem. So by Tychonoff's theorem $\tilde{f}$ has a fixed point μ^* in $P(X)$ and $\mu^*(B) = \tilde{f}_{\mu^*}(B) = \mu^*(f^{-1}(B))$ or f is measure preserving with respect to μ^*. □

The next application insures the existence of an invariant mean in a semigroup.

Definition 11.4.13 Let $(S, \circ)$ be a semigroup and $B(S)$ be the real Banach space of all bounded real functions in S with the supremum norm. For $t \in S$, the left-translation operator $L_t : B(S) \to B(S)$ is an operator defined by

$$(L_t f)(s) = f(t \circ s) \ \text{ for } s \in S$$

where $\circ$ is the associative binary operation on S.

Definition 11.4.14 A left-invariant mean on a semigroup $(S, \circ)$ is a positive linear functional μ on $B(S)$ satisfying the following conditions:

(i) $\mu(1) = 1$; (ii) $\mu(L_s f) = \mu(f)\ \forall\ s \in S$ and $f \in B(S)$.

If a semigroup has a mean, then the semigroup is called amenable (one can define a right-invariant mean similarly and in an abelian semigroup both the means coincide).

The next theorem is due to Day [5].

Theorem 11.4.15 *Let S be an abelian semigroup. Then S is amenable.*

Proof Let $K = \{\ell \in (B(S))^* : \|\ell\| = \ell(1) = 1\}$. Clearly, K is convex and closed subset of the closed unit ball in the weak* topology of $B(S)$. Since the closed unit ball in the weak* topology is compact by the Banach–Alaoglu theorem. K is compact and convex. It is nonempty by Hahn–Banach Theorem. Define the family of linear operators $T_s : B(S)^* \to B(S)^*$ by $T_s(\lambda)(f) = \Lambda(L_s(f))$ for $s \in S$. If $V = \{\Lambda \in B(S)^* : |\Lambda f_i| < \epsilon_i, i = 1, \ldots, n\}$ where $f_i \in B(S)$ and $\epsilon_i > 0$, $i = 1, 2, \ldots, n$, then $T_s^{-1}(V) = \{\Lambda \in B(S)^* : |T_s\Lambda(f_i)| < \epsilon_i, i = 1, 2, \ldots, n\} = \{\Lambda \in B(S)^* : |\Lambda(L_s f_i)| < \epsilon_i,\ i = 1, 2, \ldots, n\}$. So $T_s^{-1}(U)$ is a neighbourhood of zero even as V is a neighbourhood of zero in $B(S)$ with the weak* topology.

If $s \in K$, then $T_s\Lambda(1) = \Lambda L_s(1) = \Lambda 1 = 1$ and

$$\begin{aligned}\|T_S\Lambda\| &= \sup_{\|f\|\le 1} |T_S(\Lambda)(f)| = \sup_{\|f\|\le 1} |\Lambda(L_S(f))| \\ &\le \sup_{\|f\|\le 1} |\Lambda f| = \|\Lambda\| = 1\end{aligned}$$

as $\|L_s f\| \le \|f\|$. So $T_S(K) \subseteq K$. Further,

$$\begin{aligned}T_s T_t(\Lambda) &= T_s(\Lambda \circ L_t) = (\Lambda \circ L_t) \circ L_s \\ &= \Lambda \circ L_{st} = \Lambda \circ L_{ts} \\ &= T_t T_s(\Lambda)\ \forall\ \Lambda \in B(S)^*.\end{aligned}$$

Thus $\{T_s\}$ is a commuting family of linear operators mapping the compact convex set into itself and so has a fixed point Λ^* in K. Clearly, Λ^* is left-invariant (and right-invariant as S is commutative) and $\Lambda^* L_s(1) = \Lambda^* 1 = 1$. We now show that each element of K is positive. Suppose not. Then there exists $f \in B(S)$, $f \ge 0$ such that for some $\Lambda \in K$, $\Lambda f = \beta < 0$. So for small $\epsilon > 0$, we get

$$\|1-\epsilon f\| = \sup_{s\in S} |1-\epsilon f(s)| \leq 1 \text{ (as } f > 0)$$

So

$$\begin{aligned} 1 < 1-\epsilon\beta &\leq |1-\epsilon\beta| = |\Lambda(1-\epsilon f)| \\ &\leq \|1-\epsilon f\| \leq 1, \end{aligned}$$

a contradiction. So each element if Λ is positive. Thus Λ^* is a left-invariant functional which is positive with $1 = \Lambda^*(1) = \|\Lambda^*\|$. Thus Λ^* is an invariant mean on S and S is amenable. □

The existence of a generalized limit or Banach limit for bounded sequences can also be deduced from fixed point theorems.

We can also deduce the existence of a generalized or Banach limit of a bounded sequence from Tychonoff's theorem.

Definition 11.4.16 A generalized or Banach limit of a bounded sequence of reals $a = (a_n)$ is a real number $L(a)$ satisfying the following:

(i) L is a linear functional on ℓ_∞, the linear space of bounded sequences;
(ii) $L(1, 1, \ldots) = 1$;
(iii) $L(a) \geq 0$ if $a \geq 0$
(iv) $L(a_1, a_2, \ldots) = L(a_2, a_3, \ldots)$

Remark 11.4.17 If $a = (a_n) \in m$, then $\inf a_n \leq L(a) \leq \sup a_n$. Let $m = \inf a_n$ and $M = \sup a_n$. Clearly, $m \leq a_n \leq M$ for n. So $a_n - m \geq 0$. So $L(a_n - m) \geq 0$, by (iii). Since $L(a_n - m) = L(a_n - m(1)) = L(a_n) - mL(1, 1, \ldots) \geq 0$. So $L(a_n) \geq mL(1) = m$ by (ii). Similarly $L(a_n) \leq L(M) = ML(1, 1, \ldots) = M$. Thus $\inf a_n \leq L(a) \leq \sup a_n$. This in turn implies $|L(a)| \leq \sup |a_n| = \|a\|$. Since $L(1) = 1$, $\|L\| = 1$. From (iv) and $\inf(a_n) \leq L(a) \leq \sup(a_n)$ it follows that $\underline{\lim}(a_n) \leq L(a) \leq \overline{\lim}(a_n)$ for $a = (a_n) \in \ell_\infty$ (v). Also (i) and (v) imply (ii) and (iii).

Theorem 11.4.18 *A generalized (or Banach) limit of a bounded sequence always exists.*

Proof Define $K = \{L \in m^* : L$ satisfies (i), (ii), (iii) and (iv) of Definition 11.4.16$\}$. Clearly, K is non-void as $L(a) = a_1$ for $a = (a_n)$ lies in K. Also K is convex. Since $K = \cap\{L \in m^* : L(a) \leq \sup a_n\} \cap \{L \in m^* : L(a) \geq \inf a_n\}$ and $\{L : L(a) \leq \xi\}$ and $\{L : L(a) \geq \eta\}$ are weak* closed in m^*, K is weak* closed. As $\|L\| = 1$, K is a weak* closed subset of the unit ball in m^*; K is weak* compact by the Banach–Alaoglu theorem. K is thus compact and convex. Define the map $T : K \to K$ by $T(a) = TL(a) = L(a_2, a_3, \ldots)$ for $a = (a_1, a_2, \ldots)$ $T \in m^{**}$. For $a \in m$ and $Q \in m^{**}$, $T^{-1}(N(Q; a)) = \{L : TL \in N(Q, a)\} = \{L : |TL(a) - Q(a)| < 1\} = \{L : |L(a_2, a_3, \ldots) - Q(a_1, \ldots, a_n, \ldots)| < 1\} = N(Q_1; (a_2, a_3, \ldots))$, Q_1 being in m^* for which $Q_1(a_2, a_3, \ldots) = Q(a_1, \ldots)$.

Thus T is continuous on K with the weak* topology. So by Tychonoff's theorem T has a fixed point $L_0 \in K$ such that $TL_0 = L_0$. Thus there is a generalized (Banach) limit on m, satisfying (i)–(iv). □

Kakutani [11] proved another related fixed point theorem and is stated below.

Theorem 11.4.19 *Let K be a compact convex subset of a locally convex space and let $\mathcal{F}$ be a group of affine transformations of K into itself. If $\mathcal{F}$ is equicontinuous on K (in the sense that for each neighbourhood N of zero, there is a neighbourhood V of origin with the property that $Tx - Ty \in N$ for all $T \in \mathcal{F}$ whenever $x - y \in V$), then $\mathcal{F}$ has a common fixed point.*

The above theorem due to Kakutani follows from a more general theorem of Ryll-Nardzewski and its proof discussed in the following is based on the ideas of Hahn [9]. To this end, we need the following.

Definition 11.4.20 A family $\mathcal{F}$ of functions mapping a topological space X into itself is called distal it for every pair x, y of distinct points in X, there is an open cover $\{G_\alpha : \alpha \in \Lambda\}$ of X such that $F(y) \notin \cup\{G_\alpha : F(x) \in G_\alpha\}$ for each $F \in \mathcal{F}$. (In other words for $x \neq y$ the set $\{(Fx, Fy) : F \in \mathcal{F}$ is disjoint from some neighbourhood $\cup\{V_\alpha \times V_\alpha : \alpha \in \Lambda\}$ of the diagonal.)

Theorem 11.4.21 *Let K be a non-void compact convex subset of a locally convex space E and F a semigroup of affine continuous self-maps on K. If $\mathcal{F}$ is distal in each minimal $\mathcal{F}$-invariant closed set in K, then $\mathcal{F}$ has a common fixed point.*

Proof Let $\mathscr{K}$ be the family of all non-void compact convex subsets which are $\mathcal{F}$ invariant. Since $K \in \mathscr{K}$, $\mathscr{K}$ is nonempty. $\mathscr{K}$ is partially ordered by set inclusion and every chain K_α has the lower bound $\cap K_\alpha$; by Zorn's lemma there is a minimal element $K_0 \subseteq K$ in $\mathscr{K}$. Let $\mathscr{K}_0$ be the family of all non-void compact subsets of K_0 that are $\mathcal{F}$ invariant. As before an application of Zorn's lemma insures the existence of a closed minimal $\mathcal{F}$ invariant subset S_0 of K_0. We claim that S_0 is a singleton. If $x, y, x \neq y$ are in S_0, then $\frac{x+y}{2} \in K_0$ and K_0 is F invariant. Moreover $A = \left\{F\left(\frac{x+y}{2}\right) : F \in \mathcal{F}\right\} \subseteq K_0$ and $\overline{A} \subseteq K_0$ is compact. Also $\overline{A}$ is $\mathcal{F}$ invariant. As each F is affine, $\overline{conv A} \subset K_0$ is also $\mathcal{F}$ invariant and so by the minimal property of K_0, $\overline{con\ A} = K_0$. Let z be an extreme point of K_0 which exists by the Krein–Milman theorem in $\overline{A}$. So $z = \lim_\lambda F_\lambda\left(\dfrac{x+y}{2}\right)$ for some net. $F_\lambda(x)$ and $F_\lambda(y)$ belong to S_0 and S_0 is compact. So we may assume without loss of generality $F_\alpha x \to s \in S_0$ and $F_\alpha y \to t \in S_0$ so that $= \lim \dfrac{1}{2}[F_\alpha x + F_\alpha y] = \dfrac{1}{2}(s + t)$. As z is an extreme point of K_0, $z = s = t$. So for each open covering $\{G_\alpha\}$ of S_0 any set G_{α_0} containing s will contain all the $F_\alpha x, F_\alpha y$ eventually. So $\mathcal{F}$ is not distal in the closed minimal $\mathcal{F}$ invariant set S_0. This contradiction to the hypothesis implies that S_0 must be a singleton $\mathcal{F}$ invariant set. Thus $\mathcal{F}$ has a common fixed point. □

Corollary 11.4.22 (Hahn [9]) *Let $\mathcal{K}$ be a compact convex non-void subset of a locally convex topological vector space and $\mathcal{F}$ a semigroup of affine continuous self-maps on K. If $\mathcal{F}$ is distal on K, then $\mathcal{F}$ has a common fixed point.*

Following Dugundji and Granas [6], we state and prove

Theorem 11.4.23 (Ryll-Nardzewski [18]) *Let K be a non-void compact convex subset of a locally convex topological vector space E and $\mathcal{F}$ a semigroup of weakly continuous self-maps on K. If $\mathcal{F}$ is strongly distal on K, then $\mathcal{F}$ has a common fixed point.*

Proof To prove that $\bigcap_{F\in\mathcal{F}} FixF \neq \phi$, where $FixF$ is the set of fixed points of F in K, it suffices to show that $\{FixF, F \in \mathcal{F}\}$ being a family of weakly closed subsets of the weakly compact set K has finite intersection property.

Let $F_1, \ldots, F_n$ be a finite subfamily of $\mathcal{F}$ and $\mathcal{S}$ be the semigroup generated by $\{F_1, \ldots, F_n\}$. $\mathcal{S}$ is countable and it suffices to show that $\mathcal{S}$ has a common fixed point. To this end, consider Q the convex closure of $\{F(k_0) : F \in \mathcal{S}\}$ where $k_0 \in K$. Clearly, Q is strongly separable and as each $F \in \mathcal{S}$ is affine Q is $\mathcal{S}$ invariant and is weakly closed being a closed convex subset of K. K being weakly compact, Q too is weakly compact. In other words, the proof of the theorem follows from the proof for the special case of $K = Q$ and $\mathcal{F} = \mathcal{S}$.

We now show that $\mathcal{S}$ is weakly distal on each weakly closed minimal $\mathcal{S}$ invariant set X in Q. For $x, y \in X$, $x \neq y$. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a strongly open cover satisfying Definition 11.4.20. Since E is locally convex $\{V_\alpha\}$ has an open refinement $\{X \cap B_\beta, \beta \in \mathcal{B}\}$ with each B_β a convex strongly open set in E and for each B_β, $X \cap \overline{B}_\beta \subset V_\alpha$ for some α. Due to strong separability, there is a countable subcover $\{X \cap B_i : i \in Z\}$. Each $\overline{B}_i$ is strongly closed convex and is weakly closed. Now $\{X \cap \overline{B}_i : i \in Z\}$ is a countable weakly closed cover of the weakly compact set X. So by Baire's theorem, at least one of these sets must contain a weakly open set U with $U \subseteq X \cap \overline{B}_i \subset V_\alpha$.

The family $\{F^{-1}(U) : F \in \mathcal{S}\}$ of weakly open sets satisfies the Definition 11.4.20 for the points $x \neq y$. These sets cover X as otherwise $X - \cup\{F^{-1}(U) : F \in \mathcal{S}\}$ would be a weakly compact $\mathcal{S}$-invariant proper subset of X contradicting the minimality of X. Also for no $G \in \mathcal{S}$, Gx and Gy belong to the same $F^{-1}(U)$; otherwise FGx and FGy would be in $U \subseteq X \cap \overline{B}_i \subseteq V_\alpha$ and as $FG \in \mathcal{S}$ this contradicts that $\{U_\alpha\}$ is a strong open cover. Hence the theorem. □

Proof of Theorem 11.4.19. Given a compact convex non-void set K of a locally convex linear topological space E and $\mathcal{F}$ an equicontinuous group of affine maps of K into itself, $\mathcal{F}$ must be distal on K. Otherwise there exist pair of distinct elements on K, with the property for each neighbourhood U of the origin there is an $F_U \in \mathcal{F}$ such that $F_U(x)$ and $F_U(y)$ lie in a common set of the open cover $\{U + c : c \in K\}$ of K. Let W be a neighbourhood of the origin such that $y \notin x + W$; then for each V a neighbourhood of the origin there exists U (a neighbourhood of the origin) with $U = U \subseteq V$ and $F_U(x) - F_U(y) \in V$. However for $F_U^{-1} \in \mathcal{F}$, $F_U^{-1}(F_U x) - F_U^{-1}(F_U y) = x - y \notin W$. This contradicts the equicontinuity of $\mathcal{F}$ on K. Hence $\mathcal{F}$ is distal on K. So by Theorem 11.4.21. $\mathcal{F}$ has a common fixed point. □

We can wind up with some applications.

Definition 11.4.24 Let G be a group and f a bounded function in G. f is called left-uniformly almost periodic if the set

$$M_f = \{f_\lambda : \lambda \in G\}$$

is precompact in the uniform norm, f_λ being the function $f_\lambda(x) = f(\lambda x), \lambda, x \in G$). f is called left weakly almost periodic if the set M_f defined above is precompact in the weak topology in the linear space of bounded functions on G with uniform norm.

Theorem 11.4.25 *Let G be a group and f a bounded function and $M_f = \{f_\lambda : \lambda \in G$, where $f_\lambda(x) = f(\lambda x)$ for all $x \in G\}$. Let $K = \overline{co} M_f$. If*

(i) f is left almost periodic, then K contains constant functions;
(ii) if f is almost periodic and M_f is precompact in the weak topology, then too K contains constant functions.

Proof (i) K is a compact convex subset of $B(G)$ the space of bounded functions on G. The operators T_λ defined by $T_\lambda g = g_\lambda$ is a group of isometries mapping K into itself. So by Kakutani's Theorem 11.4.19. T_λ has a common fixed point f in M. Thus $f(x) = f(\lambda x)\ \forall\ \lambda \in G$. Thus $f(x) = f(e)$ for all $x \in G$ or f is constant.

If M_f is weakly compact, K is weakly compact and convex. Let T_λ be the group of isometries mapping M into itself as before. Then by Ryll-Nardzewski Theorem 11.4.23. T_λ has a common fixed point which is a constant function again. □

Next, following Rudin [17] we deduce the existence of left-invariant Haar measure on a compact group. Let G be a compact topological group and $C(G)$ the Banach space of all continuous complex-valued functions with the supremum norm (Recall that a topological group is a group with a Hausdorff topology such that $(a, b) \to ab^{-1}$ is continuous.

Lemma 11.4.26 *Let G be a compact group, $f \in C(G)$ and $H_L(f)$ the closed convex hull of the left translates $L_s f$ of f. (Thus $H_L(f) = conv\{L_s f : L_s f(x) = f(sx), s \in G\}$. Then (a) f is uniformly continuous and (b) $H_L(f)$ is totally bounded in $\mathcal{C}(G)$.*

Proof $f : G \to \mathbb{C}$ is called uniformly continuous if for each $\epsilon > 0$ there is a neighbourhood N_e of e the identity in G such that $|f(x) - f(y)| < \epsilon$ for $x, y \in G$ with $x^{-1}y \in N_e$.

As f is continuous on G, for each $a \in G$ and $\epsilon > 0$, there is a neighbourhood N_a of e such that $|f(x) - f(a)| < \frac{\epsilon}{2}$ for all $x \in aN_a$. From the definition of the topology on G, we can find neighbourhoods U_a of e such that $U_a U_a^{-1} \subseteq N_a$. As G is compact and $\{aU_a : a \in G\}$ is an open cover for G, $G \subseteq \bigcup_{i=1}^{k} a_i U_{a_i}$ for some $k \in \mathbb{N}$. Let $U = \bigcap_{i=1}^{k} U_{a_i}$. Let $x^{-1}y \in U$. Choose a_i, $i = 1, 2, \ldots, k$ such that $y \in a_i U_{a_i}$. Then $|f(a) - f(a_i)| < \frac{\epsilon}{2}$. Now $|f(a_i) - f(x)| < \frac{\epsilon}{2}$ as $x \in yU^{-1} \subseteq a_i U_{a_i} U^{-1} \subseteq aN_a$. So $|f(x) - f(y)| \leq |f(x) - f(a_i)| + |f(a_i) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Thus f is uniformly continuous on G.

For $x^{-1}y \in U$, $|f(x) - f(y)| < \epsilon$. But for all $s \in G$, $x^{-1}y = (sx)^{-1}(sy)$ and $|L_s f(x) - L_s f(y)| = |f(sx) - f(sy)| < \epsilon$ as $(sx)^{-1}(sy) = x^{-1}y \in U$. Thus $H_L(f)$ is an equicontinuous subfamily of $C(G)$. Hence the lemma. □

Theorem 11.4.27 *Given a compact group G, there is a unique regular Borel probability measure μ which is left-invariant. That is $\int_G f d\mu = \int_G (L_s f) d\mu$ for $s \in G$ and $f \in C(G)$. This μ is also right-invariant and satisfies the relation*

$$\int_G f(x) d\mu = \int_G f(x^{-1}) d\mu$$

for $f \in C(G)$.

Proof The operators L_s on $C(G)$ defined by $L_s f(x) = f(sx)$ for any given $s \in G$ and $f \in C(G)$ for all x form a semigroup and indeed a group of isometries on G. So it is equicontinuous. If $f \in C(G)$, K_f the closure of $H_L(f)$ is compact by Lemma 11.4.26. Clearly, $L_s(K_f) = K_f$. So by Kakutani's Theorem 11.4.19, there is a common fixed point ϕ in K_f for all L_s. Thus $L_s\phi = \phi$ for all $s \in G$. Thus $\phi(x) = \phi(sx)$ for all $s \in G$ so that $\phi(x) = \phi(sx)$ for all $s \in G$ so that $\phi(x) = \phi(e)$ for all x. Hence ϕ is constant. Since $K_f = cl\ H_L(f)$. $\phi(e)$ can be uniformly approximated by functions in $H_L(f)$. So for each $f \in C(G)$, there exists a constant k which can be uniformly approximated by convex combinations of left translates of f on G. Similarly there is a constant k' that can be uniformly approximated on G by convex combinations of right translates of f. We will show that $k' = k$. Let $\epsilon > 0$ be any prescribed number. There exist finite sets $A = \{a_i\}$ and $B = \{b_i\}$ in G with $\alpha_i, \beta_j > 0$ and $\sum_A \alpha_i = \sum_B \beta_j = 1$ and

(I) $\left|k - \sum_A \alpha_i f(a_i x)\right| < \epsilon, x \in G$ and
(II) $\left|k' - \sum_B \beta_j f(b_j x)\right| < \epsilon, x \in G$

Setting $x = b_j$ in (I), multiplying (I) by β, and add over j to get

(III) $\left|k - \sum_{i,j} \alpha_i \beta_j f(a_i b_j)\right| < \epsilon$

Similarly setting $x = a_i$ in (II), multiplying (II) by α_i and add over i to get

(IV) $\left|k' - \sum_{i,j} \alpha_i \beta_j f(a_i b_j)\right| < \epsilon$

From (III) and (IV) it follows that $k = k'$.

Thus for each $f \in C(G)$ there is a unique number written as I_f such that it can be uniformly approximated by convex combinations of left translates of f (as well as convex combinations of right translates of f) with the following properties:

$$\begin{aligned} I_f &\geq 0 \text{ for } f \geq 0, \\ I_1 &= 1 \\ I_{\alpha f} &= \alpha I_f \text{ for any scalar } \alpha \\ I_{(L_s f)} &= I_f = I_{(R_s f)} \text{ for each } s \in G \end{aligned}$$

We now show that $I_{f+g} = I_f + I_g$. Given $\epsilon > 0$ for a finite subset $A = \{a_i\} \subseteq G$ and $\alpha_i > 0$ with $\sum \alpha_i = 1$, $\left| I_f - \sum_A \alpha_i f(a_i x) \right| < \epsilon$ for all $x \in G$. Define $h(x) = \sum_i \alpha_i g(a_i x)$, then $h \in K_g$. So $K_h \subseteq K_g$. Since each of K_f and K_g contains unique constant functions I_h and I_g, it follows that $I_h = I_g$. So there is a finite set $B = \{b_j\} \subseteq G$ with $\beta_j > 0$ and $\sum \beta_j = 1$ such that for all $x \in G$

$$\left| I_g - \sum_j \beta_j h(b_j x) \right| < \epsilon.$$

From the definition of h we get for all $x \in G$

$$\left| I_g - \sum_{i,j} \alpha_i \beta_j g(a_i b_j x) \right| < \epsilon.$$

Since for all $x \in G$

$$\left| I_f - \sum_j \alpha_j f(a_j x) \right| < \epsilon$$

replacing x by $b_j x$ and multiplying by β_j and suming over j we get as $\sum \beta_j = 1$

$$\left| I_f - \sum_{i,j} \alpha_i \beta_j f(a_i b_j x) \right| < \epsilon$$

So $\left| I_f + I_g - \sum_{i,j} \alpha_i \beta_j (f+g)(a_i b_j x) \right| < 2\epsilon$ for all $x \in G$. Since $\sum_{i,j} \alpha_i \beta_j = 1$ and $\alpha_i \beta_j \geq 0$ it follows that $I_f + I_g = I_{(f+g)}$.

Since $I_f \geq 0$, $I_f = 1$, $I_{\alpha f} = \alpha I_f$ for $\alpha \in \mathbb{C}$ and $I_{f+g} = I_f + I_g$, it follows from the Riesz representation theorem that there is a unique regular Borel probability measure μ such that

$$I_f = \int_G f d\mu \text{ for } f \in C(G).$$

From the construction of I_f, the left invariance of μ follows.

$I'_f = \int_G f(x^{-1}) d\mu$ is well-defined on $C(G)$ and $I'_f \geq 0$ for $f \geq 0$, $I_1 = 1$, $I'_f(\alpha f + \beta g) = \alpha I'_f + \beta I'_g$ for $f, g \in C(G)$ and $\alpha, \beta \in \mathbb{C}$. So by the uniqueness $I'_f = I_f$. □

Kitamura and Kusano [12] proved the existence of oscillatory solutions for a first-order nonlinear functional differential equation under suitable assumptions using Tychonoff's theorem. In what follows this existence result is described.

Definition 11.4.28 A solution of a first-order differential or functional differential equation defined on $[\alpha, \infty)$ $\alpha > 0$ is called oscillatory if it has arbitrarily large zeros. Otherwise it is called nonoscillatory.

Theorem 11.4.29 *Consider the differential equation*

$$x'(t) = \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))), t > a > 0 \tag{11.1}$$

Suppose

(a) $q_i, g_i : [a, \infty) \to \mathbb{R}$ *are continuous functions, with* $q_i(t) \geq 0$ *and* $\lim_{t\to\infty} g_i(t) = \infty$ *for* $i = 1, 2, \ldots, N$*;*
(b) $f_i : R \to R$ *is continuous, nondecreasing and* $tf_i(t) > 0$ *for* $t \neq 0$ *for* $i = 1, 2, \ldots, N$*;*
(c) $\sum_{i=1}^{N} \int_a^{\infty} q_i(t)dt < \infty$*;*

Then the differential equation (11.1) has only nonoscillatory solutions.

Proof For an arbitrary positive constant k consider the integral equation

$$x(t) = k + \sum_{i=1}^{N} \int_T^t q_i(s) f_i(x(g_i(s)))ds$$

where $T > a$ is chosen such that

$$\sum_{i=1}^{N} f_i(2k) \int_T^{\infty} q_i(s)ds < k$$

in view of (c) and the continuity of $f_i, i = 1, 2, \ldots, N$. Define $T_0 = \min_{i=1,\ldots,N}\{\inf g_i(t) : t \geq T\}$ so that $T_0 > 0$ (this can be done by rechoosing T in view of the assumption $\lim_{t\to\infty} g_i(t) = \infty$ for $i = 1, 2, \ldots, N$). Let M be the locally convex space of all real-valued continuous functions on $[T_0, \infty)$ with the topology of uniform convergence on compact subintervals of $[T_0, \infty)$. Let $X = \{x \in M : k \leq x(t) \leq 2k \text{ for } t \geq T_0\}$. Define the operator $\phi : X \to M$ by

$$\phi(x(t)) = \begin{cases} k + \sum_{i=1}^{N} \int_T^t q_i(s) f_i(x(g_i(s)))ds & \\ k & \text{for } T_0 \leq t \leq T \end{cases}$$

Clearly, $\phi(x) \in X$ for each $x \in X$ in view of the choice of T. Further, X is a closed convex subset of M and ϕ maps X into a compact subset of X. So by Tychonoff's fixed point theorem ϕ has a fixed point which is a solution for the integral equation leading to the solution of the functional differential equation (11.1). Clearly, the solution is nonoscillatory. □

The next theorem follows similarly and its proof is left as an exercise.

Theorem 11.4.30 *Under (a), (b) and (c) of Theorem 11.4.29, the equation*

$$x'(t) + \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))) = 0 \tag{11.2}$$

has nonoscillatory solutions.

Example 11.4.31 (*Kitamura and Kusano* [12]) For $\alpha > 0$ and $\beta \geq 1$, the equation

$$x'(t) = \frac{|x(t + \sin t)|^{\alpha} sgn\ x(t + \sin t)}{t^{\beta}(\log(t + \sin t))^{\alpha}}$$

has nonoscillatory solutions for $t \geq 2\pi$ as $\displaystyle\int_{2\pi}^{\infty} \frac{dt}{t^{\beta}(\log(t + \sin t))^{\alpha}} < \infty$ and conditions (a) and (b) of Theorem 11.4.29 are satisfied.

Remark 11.4.32 It has been observed in [12] that under the assumptions (a) and (b) of Theorem 11.4.29, along with $\displaystyle\sum_{i=1}^{N} \int_{a}^{\infty} q_i(t)dt = \infty$, for the equation

$$x'(t) = \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))) \tag{11.3}$$

every solution is oscillatory when $g_i(t) > t$ for $i = 1, \ldots, N$; for the equation

$$x'(t) + \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))) = 0 \tag{11.4}$$

every solution is oscillatory when $g_i(t) < t$ for $i = 1, \ldots, N$.

Next we state and prove an inequality due to Ky Fan.

Theorem 11.4.33 *Let K be a nonempty compact convex subset of a linear topological space X and $g : K \times K \to R$ be a map such that*

(i) $g(\cdot, y)$ is lower semicontinuous for each $y \in K$ (i.e. $g^{-1}(\cdot, y)\ (\alpha, \infty]$ is open in K for each $y \in K$);

(ii) *$g(x, \cdot)$ is concave for each $x \in K$ (i.e. $-g(x, \cdot)$ is a convex function for each $x \in K$).*

Then there exists $x_0 \in K$ such that

$$\sup_{y \in K} g(x_0, y) \leq \sup_{y \in K} g(y, y).$$

Proof For each $\epsilon > 0$, and a given $x \in K$ we can find $y_x \in K$ and a neighbourhood N_x of x such that

$$g(z, y_x) > \sup_{y \in K} g(x, y) - \epsilon \text{ for all } z \in N_x$$

Since $K \subseteq \bigcup_{x \in K} N_x$ and K is compact we can find a finite number of elements $x_1, x_2, \ldots, x_n$ in K with $K \subseteq \bigcup_{i=1}^{n} N_{x_i}$. Let $\{\varphi_1, \ldots, \varphi_n\}$ be a partition of unity for K subordinate to the covering $\{N_{x_i} : i = 1, \ldots, n\}$. Then the map f defined by

$$f(x) = \sum_{i=1}^{n} \varphi_i(x) y_{x_i}$$

maps the closed convex hull of $\{y_{x_1}, \ldots, y_{x_n}\}$ into itself and is continuous. Since X is a linear topological space and the subspace topology on the closed convex hull of $\{y_{x_1}, \ldots, y_{x_n}\}$ is euclidean, f has a fixed point x^* by Brouwer's fixed point theorem. So

$$\begin{aligned}
\sup_{y \in K} g(y, y) &\geq g(x^*, x^*) \\
&\geq \sum_{i=1}^{n} \varphi(x^*)(g(x^*, y_{x_i}) - \epsilon) \\
&\geq \sum_{i=1}^{n} \varphi(x^*)(\sup_{y \in K} g(x_i, y) - \epsilon) \\
&\geq \sum_{i=1}^{n} \varphi(x^*) \sup_{y \in K} g(x_i, y) - \epsilon \\
&\geq \inf_{x \in K} \sup_{y \in K} g(x, y) - \epsilon \\
&= \sup g(x_0, y) - \epsilon \text{ for some } x_0
\end{aligned}$$

Allowing ϵ to tend to zero we conclude that

$$\sup_{y \in K} g(x_0, y) \leq \sup_{y \in K} g(y, y)$$

for some x_0. □

Consider a game involving $n(\geq 2)$ players who pursue a strategy depending on the strategies of other players. Let the strategy set of the ith player be denoted by K_I and K be the set $K_1 \times K_2 \times \cdots \times K_n$. An element of K is called a strategy profile. For each player let $f_i : K \to \mathbb{R}$ be the loss function of the ith player. If $\sum_{k=1}^{n} f_k(x) = 0$, then this game is called zero-sum game.

Definition 11.4.34 A Nash equilibrium is a strategy profile $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n) \in K$ such that for each $i = 1, 2, \ldots, n$

$$f_i(\overline{x}) \leq f_i(\overline{x}_1, \ldots, \overline{x}_{i-1}, x, \overline{x}_i, \ldots, \overline{x}_n) \text{ for all } x \in K_i$$

(In other words this strategy profile minimizes the loss for each player.)

The existence of a Nash equilibrium for an n-person game can now be proved in the setting of a locally convex topological vector space.

Theorem 11.4.35 *For $i = 1, 2, \ldots, n$, let K_i be a non-void compact convex subset of a locally convex topological vector space X_i. Suppose that for each $i = 1, 2, \ldots, n$ the loss function $f_i : K \to R$ is continuous and for each fixed $x_j \in K$, with $j \neq i$, the function $f_i(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n) : K_i \to R$ is convex. Here $K = K_1 \times K_2 \times \cdots \times K_n$. Then there is a Nash equilibrium in K.*

Proof Define $g : K \times K \to R$ by

$$g(x, y) = \sum_{i=1}^{n} f_i(x) - f_i(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Since g is continuous and $g(x, \cdot)$ is concave for each fixed $x \in K$, by Theorem 11.4.33 there exists $\overline{x} \in K$ such that $\sup_{y \in K} g(\overline{x}, y) \leq \sup_{y \in K} g(y, y) = 0$. Setting $\overline{y} = (\overline{x}_1, \ldots, \overline{x}_{i-1}, x_i, \overline{x}_{i+1}, \ldots, \overline{x}_n)$ for each $x_i \in K_i$ we get

$$g(\overline{x}, \overline{y}) \leq 0 \text{ for each } x_i \in K_i$$

This means that for each $i = 1, 2, \ldots, n$

$$f_i(\overline{x}) = f_i(\overline{x}_1, \ldots, \overline{x}_{i-1}, x_i, \overline{x}_{i+1}, \ldots, \overline{x}_n)$$

for all $x_i \in K_i$. □

In the case of two person zero-sum game the hypotheses can be further weakened and we get a theorem due to von Neumann.

Clearly, $g(x_1, x_2) = f_1(x_1, x_2) + f_2(x_1, x_2) = 0$ or $f_1(x_1, x_2) = -f_2(x_1, x_2)$.

Theorem 11.4.36 *Let X_1 and X_2 be two locally convex spaces and $K_i \subseteq X_i$ $i = 1, 2$ be non-void compact convex subsets of X_i. Let $\psi : K_1 \times K_2 \to \mathbb{R}$ be such that*

(i) $\psi(\cdot, x_2)$ is lower semicontinuous and convex for each $x_2 \in K_2$;
(ii) $\psi(x_1, \cdot)$ is upper semicontinuous and concave for each $x_2 \in K_2$;

Then there is a Nash equilibrium $(\overline{x}_1, \overline{x}_2) \in K_1 \times K_2$.

For the proof define $g : K \times K \to R$ by $g((x_1, x_2), (y_1, y_2)) = -\psi(y_1, x_2) + \psi(x_1, y_2)$ and apply Theorem 11.4.35 to $K = K_1 \times K_2$. From the theorem, it also follows that $\inf_{x \in K_1} \sup_{y \in K_2} \psi(x, y) = \sup_{y \in K_2} \inf_{x \in K_1} \psi(x, y)$. So this theorem is also called minimax theorem.

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Chapter 12
Basic Analytic Degree Theory of a Mapping

12.1 Introduction

The problem of finding the number of solutions of a given equation has engaged a number of mathematicians. Brouwer, Bohl, Cauchy, Descartes, Gauss, Hadamard, Hermite, Jacobi, Kronecker, Ostrowski, Picard, Sturm and Sylvester had contributed to this topic. The Argument principle propounded by Cauchy on the zeros of a function inside a domain and Sturm's theorem on the number of zeros of a real polynomial in a closed bounded interval have evolved into Degree theory of mappings. Even as the degree of a nonconstant polynomial gives the number of zeros of a polynomial, the degree of a mapping provides the number of zeros of nonlinear mapping in a domain. In this chapter, an elementary degree theory of mappings is described from an analytic point of view proposed by Heinz [3]. For more elaborate treatment, Cronin [1], Deimling [2], Lloyd [4] Outerelo and Ruiz [6] and Rothe [7] may be referred. It should be mentioned that Ortega and Rheinboldt [5] had provided a more accessible version of Heinz's treatment.

12.2 Heinz's Elementary Analytic Theory of Mapping Degree in Finite Dimensional Spaces

Heinz [3] based his approach on some lemmata and relevant definitions. Throughout we assume that Ω is an open set in $\mathbb{R}^n$, $\partial\Omega$ its boundary and $\overline{\Omega}$ its closure with respect to the topology generated by the euclidean norm. We consider a map $y : \Omega \to \mathbb{R}^n$ with $y = (y_1, \ldots, y_n)$ where $y_i = y_i(x_1, \ldots, x_n)$ is the ith component function of y, where $i = 1, 2, \ldots, n$. For $y \in C^1(\Omega)$, the Jacobian $J[y(x)]$ is well-defined for each $x \in \Omega$. Let $A_{ij}(x)$ be the cofactor of $a_{ij}(x)$ in the determinant of $J[y(x)] = det(a_{ij}(x))$.

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Lemma 12.2.1 *Let $y : \Omega \to \mathbb{R}^n$ be a function in $C^2(\Omega)$. Let $A_{ij}(x)$ be the cofactor of $a_{ij}(x)$ in the determinant of $J[y(x)] = det(a_{ij}(x))$. Then $\sum_{j=1}^{n} \frac{\partial}{\partial x_j} A_{ij}(x) = 0$, $i = 1, \ldots, n$, $x \in \Omega$.*

Proof Suppose $(a_{ij}(x))$ is invertible and $b_{ij} = (a_{ij}(x))^{-1}$. Then for $d = det(a_{ij}(x))$. Clearly, $dI = (a_{ij})(A_{ij})^T$ or
$d\delta_{ij} = \sum_{k=1}^{n} a_{ik} A_{jk}$. So $\frac{\partial d}{\partial a_{ij}} = A_{ij}$
$db_{ij} = A_{ji}$.

For $y'(x_0)$ invertible, by the inverse function theorem y maps some open neighbourhood of U of x_0 onto an open neighbourhood of $y(x_0)$ and if $z = y^{-1}$, then $y'(x_0)z'(y(x_0)) = I$. So $Jy(x_0) = (a_{ij})$. $J(z(y(x_0)) = (b_{ij})$, $d = det(Jy(x_0))$ and $e = det\ J[z(y(x_0)]$. Let B_{ij} be the cofactor of the (i, j)th element of $J[z(y(x_0))] = b_{ij}$. Then $\frac{\partial e}{\partial y_k} = \sum_{i,j=1}^{n} \frac{\partial e}{\partial b_{ij}} \frac{\partial b_{ij}}{\partial y_k} = \sum_{i,j=1}^{n} B_{ij} \frac{\partial b_{ij}}{\partial y_k} = e \sum_{i,j=1}^{n} a_{ji} \frac{\partial b_{ij}}{\partial y_k}$.

But

$$\frac{\partial b_{ik}}{\partial x_i} = \frac{\partial^2 z_i}{\partial x_i \partial y_k} = \sum_{j=1}^{n} \frac{\partial}{\partial y_k} \frac{\partial z_i}{\partial y_j} \frac{\partial y_j}{\partial x_i}$$
$$= \sum_{j=1}^{n} a_{ji} \frac{\partial b_{ij}}{\partial y_k}$$

and hence

$$\frac{\partial e}{\partial y_k} = e \sum_{i=1}^{n} \frac{\partial b_{ik}}{\partial y_k} \tag{12.2.1}$$

As $de = 1$, $d\frac{\partial e}{\partial y_k} + e\frac{\partial d}{\partial y_k} = 0$, we get from (12.2.1), $0 = \frac{\partial d}{\partial y_k} + d\sum_{i=1}^{n} \frac{\partial b_{ik}}{\partial x_i} = \sum_{i=1}^{n} \left[\frac{\partial d}{\partial x_i} b_{ik} + e\frac{\partial b_{ik}}{\partial x_i}\right] = \sum_{i=1}^{n} \frac{\partial}{\partial y_k}(db_{ik}) = \sum_{i=1}^{n} \frac{\partial}{\partial y_k} A_{ki}$, $k = 1, \ldots n$.

If $y'(x_0)$ is singular, then the mapping $y_\epsilon = y(x) + \epsilon x$ is invertible. So by the first part of the lemma $\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(A^\epsilon_{ji}) = 0$ where A^ϵ_{ji} is the cofactor of (i, j)th element in $J[y_\epsilon(x)]$. By continuity $\sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_{ji} = \lim_{\epsilon > 0} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} A^\epsilon_{ji} = 0$. Thus in this case also the Lemma is true. □

Lemma 12.2.2 *Suppose*

(i) $y : \Omega \to \Omega$ is a function with $y \in C^1(\Omega)$, Ω being an open bounded subset of $\mathbb{R}^n$, y is continuous on $\overline{\Omega}$ and for some $\epsilon > 0$ $\|y(x)\| > \epsilon$ for all $x \in \partial\Omega$;

(ii) $\varphi : [0, \infty) \to \mathbb{R}$ is continuous, vanishing in $[\epsilon, \infty)$ and also in a neighbourhood of 0 and $\int_0^\infty r^{n-1}\varphi(r)dr = 0$.

Then $\int_\Omega \varphi(\|y(x)\|)J[y(x)]dx = 0$.

Proof Since functions in $C(\overline{\Omega})$ can be uniformly approximated by polynomials in Ω, it suffices to prove the lemma for $y \in C^2(\Omega)$.

Define $\psi : [0, \infty) \to \mathbb{R}$ by

$$\psi(r) = \begin{cases} r^{-n} \int_0^r \rho^{n-1}\varphi(\rho)d\rho, & \text{for } r > 0 \\ 0, & \text{for } r = 0 \end{cases}$$

Clearly, $\psi \in C^1[0, \infty)$ and vanishes in a neighbourhood of 0 and in $[\epsilon, \infty)$ since φ vanishes in a neighbourhood of 0 and in $[\epsilon, \infty)$. Further $r\psi'(r) + n\psi(r) = \varphi(r)$ for $r \geq 0$.

For each $i = 1, \ldots, n$, $f_i(y) = \psi(\|y\|)y_i$ is in $C^1(\mathbb{R}^n)$ and vanishes for $\|y\| \geq \epsilon$. From Lemma 12.2.1 we get

$$\begin{aligned}\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \sum_{j=1}^{n} A_{ij}(x) f_j(y(x)) &= J[y(x)] \sum_{j=1}^{n} \left(\frac{\partial f_j}{\partial y_j}\right)_{y=y(x)} \\ &= J[(y(x)](r\psi'(r) + n\psi(r))_{r=\|y(x)\|} \\ &= \varphi(\|y(x)\|)J[y(x)]. \qquad (12.2.2)\end{aligned}$$

For $F : \mathbb{R}^n \to \mathbb{R}$, a continuously differentiable function with compact support

$$\int_{\mathbb{R}^n} div\ F(x)dx = 0$$

So integrating the above equality (12.2.2) over $\mathbb{R}^n$ we get

$$\int_{\mathbb{R}^n} \varphi(\|y(x)\|)J[y(x)]dx = 0$$

□

Lemma 12.2.3 *Let* $y : \overline{\Omega} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ *be a function in* $C^1(\Omega)$ *where* Ω *is a bounded open set and let* y *be a continuous on* $\overline{\Omega}$. *Let* $z \in \mathbb{R}^n$ *be such that* $z \neq y(x)$ *for* $x \in \partial\Omega$ *and* $\Phi : [0, \infty) \to \mathbb{R}$ *be continuous function such that*

(i) it vanishes in a neighbourhood of 0 and also in $[\epsilon, \infty)$ *where* $0 < \epsilon < \min\{x \in \partial A : |y(x) - z|\}$*;*
(ii) $\int_{\mathbb{R}^n} \Phi(\|x\|)dx = 1$.

Then the number $d[y(x); \Omega; z]$ *is uniquely defined by*

$$d[y(x); \Omega; z] = \int_\Omega \Phi(\|y(x) - z\|)J[y(x)]dx$$

Proof Let D be the linear space of all continuous real-valued functions satisfying (i). Define $L, M, N : D \to \mathbb{R}$ by

$$L\Phi = \int_0^\infty r^{n-1}\Phi(r)dr$$
$$M\Phi = \int_{\mathbb{R}^n} \Phi(\|x\|)dx$$
$$N\Phi = \int_\Omega \Phi(|y(x) - z|)J[y(x)]dx$$

Clearly, L, M and N are linear functionals on D. Applying Lemma 12.2.2 to $y = x$ (for $\|x\| < 2\epsilon$) and $y = y(x) - z$ ($x \in \Omega$) it follows that $L\Phi = 0$ for $\Phi \in D$ implies $M\Phi = N\Phi = 0$. For $\Phi_1, \Phi_2 \in D$ with $M\Phi_1 = M\Phi_2 = 1$, $L(L\Phi_1.\Phi_2 - L\Phi_2.\Phi_1) = 0$ and so $L\Phi_2.M\Phi_1 - L\Phi_1.M\Phi_2 = 0$. So $L(\Phi_1 - \Phi_2) = 0$. So $N(\Phi_1 - \Phi_2) = 0$ and $N\Phi_1 = N\Phi_2$. Thus the definition of $N\Phi$ is independent of Φ in D and $d[y(x); \Omega, z]$ is independent of Φ and is uniquely defined. □

Definition 12.2.4 Let $y : \overline{\Omega} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ (where Ω is a bounded open set in $\mathbb{R}^n$) a continuous function such that $y \in C^1(\Omega)$. The number $d[y; \Omega, z]$ uniquely obtained in Lemma 12.2.3 is called the (Brouwer) degree of y.

Lemma 12.2.5 *Let $y_i : \overline{\Omega} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be mappings such that y_i is in $C^1(\Omega)$ and continuous on $\overline{\Omega}$ for each $i = 1, 2$. Suppose for $z \in \mathbb{R}^n$*

(i) $\|y_i(x) - z\| > 7\epsilon > 0$ for $x \in \partial\Omega$ for $i = 1, 2$ and
(ii) $\|y_1(x) - y_2(x)\| > \epsilon$ for $x \in \overline{\Omega}$.

Then $d[y_1(x); \Omega, z] = d[y_2(x); \Omega, z]$.

Proof Clearly, $d[y_i(x); \Omega, z]$ is well-defined and equals $d[y_i(x) - z; \Omega, 0]$ for each $i = 1, 2$. Thus without loss of generality it may be assumed that $z = 0$. Let $f : [0, \infty) \to \mathbb{R}$ be a continuously differentiable function which vanishes in $[3\epsilon, \infty)$ and is 1 in $[0, 2\epsilon]$ with range in $[0, 1]$. Define $y_3 : \overline{\Omega} \to \mathbb{R}^n$ by $y_3(x) = [1 - f(\|y_1(x)\|)]y_1(x) + f(\|y_1(x)\|)y_2(x)$. Since $0 \notin \overline{\Omega}$, $y_3 \in C^1(\Omega)$ as f, y_1, y_2 are C^1-functions. Further $y_3 \in C(\overline{\Omega})$. Clearly, $\|y_i(x) - y_j(x)| < \epsilon$ for $x \in \overline{\Omega}$, for $i, j \in \{1, 2, 3\}$. From the definition of f, for $\|y_1(x)\| > 3\epsilon$, $y_3(x) = y_1(x)$ and for $||y_1(x)|| < 2\epsilon$, $y_3(x) = y_2(x)$.

Let $\Phi_i : [0, \infty) \to \mathbb{R}$ be be two continuous functions that vanish in a neighbourhood of 0 satisfying

$$\Phi_1(r) = 0 \text{ for } r \in [0, 4\epsilon] \cup [5\epsilon, \infty)$$
$$\Phi_2(r) = 0 \text{ in } [\epsilon, \infty) \text{ and}$$
$$\int_{\mathbb{R}^n} \Phi_i(\|x\|)dx = 1 \text{ for } i = 1, 2.$$

From the definitions of Φ_i and the choice of y_3 it follows that for $x \in \Omega$

$$\Phi_1(\|y_3(x)\|)J[y_3(x)] = \Phi_1(\|y_1(x)\|)J[y_1(x)]$$

and

$$\Phi_2(\|y_3(x)\|)J[y_3(x)] = \Phi_2(\|y_2(x)\|)J[y_2(x)].$$

Upon integrating we get

$$d[y_3(x); \Omega, 0] = d[y_1(x); \Omega, 0]$$

and

$$d[y_3(x); \Omega, 0] = d[y_2(x); \Omega, 0]$$

whence $d[y_1(x); \Omega, 0] = d[y_2(x); \Omega, 0]$ as required. □

Lemma 12.2.6 *Let $y : \overline{\Omega} \to \mathbb{R}^n$ be continuous, Ω being a nonvoid bounded open set in $\mathbb{R}^n$. Suppose $z \in \mathbb{R}^n$ is such that $z \neq y(x)$ for all $x \in \partial\Omega$. If $y_k : \overline{\Omega} \to \mathbb{R}^n$ is continuous for each $k \in \mathbb{N}$ and is in $C^1(\Omega)$ with $y_k(x) \neq z$ for all $x \in \partial\Omega$ and (y_k) converges uniformly in $\overline{\Omega}$ to y then $d[y(x); \Omega, z) = \lim_{k\to\infty} d[y_k(x); \Omega, z]$.*

Proof Be Lemma 12.2.5, $\{d(y_k(x); \Omega, z)\}$ is a Cauchy sequence of real numbers and hence is convergent. (Indeed it is constant after some stage.) Again as $\sup\{\|y(x) - y_k(x)\| : x \in \overline{\Omega}\}$ can be made less than $\frac{1}{10}\sup\{|y(x) - z\| : x \in \partial\Omega\}$. $d(y(x); \Omega, z) = d(y_k(x); \Omega, z)$ after some stage. Thus $d(y(x); \Omega, z) = \lim\limits_{k\to\infty} d(y_k(x); \Omega, z)$ for $z \notin \{y(x); x \in \partial\Omega\}$. □

Remark 12.2.7 For $y \in C(\overline{\Omega})$ where Ω is a nonvoid bounded open set and $y(\Omega) \subseteq \mathbb{R}^n$, for $z \notin \{y(x); x \in \partial\Omega\}$, $d[y(x); \Omega, z]$ is well-defined. Indeed by Weierstrass approximation theorem there is a sequence of polynomials (y_k) on $\overline{\Omega}$ such that (y_k) converges uniformly on $\overline{\Omega}$ to y and $y_k(x) \neq z$ for all $x \in \partial\Omega$ for all k. So $d[y(x); \Omega, z)$ is well-defined for all $y \in C(\overline{\Omega})$ with $z \notin y(\partial\Omega)$ and is called the degree of y.

Remark 12.2.8 For a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$, $\Omega = B(0, 1)$ the unit open ball.

$$d[A, \Omega = B(0,1), 0] = \begin{cases} +1, & \text{if det of } A > 0 \\ -1, & \text{if det of } A < 0 \\ \text{not defined} & \text{if det of } A = 0 \end{cases}$$

12.3 Properties of the Degree

In this section, we describe the elementary properties of the degree of a mapping.

Theorem 12.3.1 *Let Ω_1 and Ω_2 be two disjoint bounded nonempty open sets in $\mathbb{R}^n$. Let $y : \overline{\Omega}_1 \cup \overline{\Omega}_2 \to \mathbb{R}^n$ be a continuous map such that $y(x) \neq z$ for x in $\overline{\Omega}_1 \cup \overline{\Omega}_2$ for some fixed $z \in \mathbb{R}^n$. Then*

$$d[y(x); \Omega_1 \cup \Omega_2, z] = d[y(x); \Omega_1, z] + d[y(x); \Omega_2, z]$$

Proof By definition of the degree

$$\begin{aligned} d[y(x); \Omega_1 \cup \Omega_2, z] &= \lim_{k\to\infty} d[y_k(x); \Omega_1 \cup \Omega_2, z] \\ &= \lim_{k\to\infty} [d[y_k(x); \Omega_1, z] + d(y_k(x); \Omega_2, z)] \\ &= d[y(x); \Omega_1, z] + d[y(x); \Omega_2, z] \end{aligned}$$

where $y_k \in C^1(\overline{\Omega}_1 \cup \overline{\Omega}_2)$ converges uniformly on $\overline{\Omega}_1 \cup \overline{\Omega}_2$ to y. It may be noted that the linearity of the integral and Lemma 12.2.5 lead to the conclusion. □

Corollary 12.3.2 *Let $y : \overline{\Omega} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be continuous, where Ω is a nonempty bounded open set. Suppose $\overline{\Omega} = \bigcup_{i=1}^{m} \overline{\Omega}_i$ where each Ω_i is a nonempty bounded open set in $\mathbb{R}^n$ with $\Omega_i \cap \Omega_j = \phi$ for $i \neq j$. If $z \notin y(\partial\Omega)$ then $d[y(x); \Omega, z] = \sum_{i=1}^{m} d[y(x); \Omega_i, z]$.*

The next theorem generalizes Lemma 12.2.5.

Theorem 12.3.3 *Let Ω be a nonvoid bounded open subset of $\mathbb{R}^n$ and $y : \overline{\Omega} \to \mathbb{R}^n$ a continuous map. If $z \in \mathbb{R}^n$ is such that $\min\{\|y(x) - z\| : x \in \partial\Omega\} > \epsilon > 0$, then $d[y(x); \Omega, z] = d[\hat{y}(x); \Omega, z]$ for any $\hat{y} : \overline{\Omega} \to \mathbb{R}^n$ for which $\sup\{\|y(x) - \hat{y}(x)\| : x \in \Omega\} < \frac{\epsilon}{7}$.*

Proof We can find sequences $y_k, \hat{y}_j : \Omega_0 \to \mathbb{R}^n$ such that $(y_k), (\hat{y}_j) \in C^1(\Omega_0)$, where Ω_0 is a bounded open set containing $\overline{\Omega}$ such that (y_k) and $(\hat{y}_j)$ converge uniformly on $\overline{\Omega}$ to y and $\hat{y}$ respectively. So there exists $m_0 \in \mathbb{N}$ such that for all $k, j \geq m_0$ and all $x \in \overline{\Omega}$

$$\begin{aligned} \|\hat{y}_j(x) - y_k(x)\| &\leq \|\hat{y}_j(x) - \hat{y}(x)\| + \|\hat{y}(x) - y(x)\| + \|y(x) - y_k(x)\| \\ &< \frac{\epsilon}{7}. \end{aligned}$$

(Since $\sup\{\|y(x) - \hat{y}(x)\| : x \in \Omega\} = \eta < \frac{\epsilon}{7}$, we have to choose m_0 so that $\sup\{\|y(x) - y_k(x)\| : x \in \overline{\Omega}\}$ and $\sup\{\|\hat{y}_j(x) - \hat{y}_j(x) : x \in \overline{\Omega}\} < \frac{1}{2}\left(\frac{\epsilon}{7} - \eta\right)$.) Also we can choose (y_k) and $(\hat{y}_j)$ such that for $k, j \geq m_0$, $\inf\{\|y_k(x) - z\| : x \in \partial\Omega\}$ and $\inf\{\|\hat{y}_j(x) - z\| : x \in \partial\Omega\} > \epsilon$. So by Lemma 12.2.5, $d[\hat{y}_j(x); \Omega, z] = d[y_k(x); \Omega, z]$ for all $j, k \geq m_0$. Proceeding to the limit we get $d[\hat{y}(x); \Omega, z] = d[y(x); \Omega, z]$ by Lemma 12.2.6. □

Next theorem establishes the invariance of the degree under homotopy.

Theorem 12.3.4 *Let Ω be a nonvoid bounded open set in $\mathbb{R}^n$ and $H : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n$ be a continuous map. Suppose for $z \in \mathbb{R}^n$ for which $H(x, t) \neq z$ for all $x \in \partial\Omega$ and $t \in [0, 1]$. Then $d[H(x, t); \Omega, z]$ is a constant for all $t \in [0, 1]$.*

Proof As $\partial\Omega \times [0, 1]$ is compact (in the product topology), $\min\{\|H(x, t) - z\| : x \in \partial\Omega, t \in [0, 1]\} > \epsilon > 0$ for some ϵ. As H is uniformly continuous on $\overline{\Omega} \times [0, 1]$, for $\frac{\epsilon}{7}$ we can find $\delta > 0$ such that $\sup\{\|H(x, t) - H(x, s)\| : x \in \overline{\Omega}, s, t \in [0, 1]\} < \frac{\epsilon}{7}$ for $|s - t| < \delta$. So by Theorem 12.3.3 above $d[H(x, t); \Omega, z] = d[H(x, s); \Omega, z]$ for $|s - t| < \delta$, $s, t \in [0, 1]$. Since $[0, 1]$ can be covered by a finite number of subintervals of length δ, it follows that $d[H(x, t); \Omega, z]$ is constant for all $t \in [0, 1]$. □

The next theorem, also called Poincare–Bohl theorem, is a consequence of the above result.

Theorem 12.3.5 *Let $y, \hat{y} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be two continuous maps on a nonvoid bounded open set Ω. If $z \notin \{v \in \mathbb{R}^n : v = ty(x) + (1 - t)\hat{y}(x), x \in \partial\Omega, t \in [0, 1]\}$ then $d[y(x); \Omega, z) = d[\hat{y}(x); \Omega, z]$.*

Proof Define $H : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n$ by $H(x, t) = ty(x) + (1 - t)\hat{y}(x)$. By assumption $z \notin \{v = ty(x) + (1 - t)\hat{y}(x), x \in \partial\Omega, t \in [0, 1]\}$. Further H is a homotopy between $y(x)$ and $\hat{y}(x)$, with $H(x, 0) = \hat{y}(x)$ and $H(x, 1) = y(x)$ for $x \in \overline{\Omega}$. So by Theorem 12.3.4 $d[y(x); \Omega, z] = d[\hat{y}(x); \Omega, z]$. □

Corollary 12.3.6 *Let Ω be a nonvoid bounded open subset of $\mathbb{R}^n$ such that $y, \hat{y} : \overline{\Omega}(\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ are continuous. Suppose $y(x) = \hat{y}(x)$ for $x \in \partial\Omega$. Then for any $z \notin y(\partial\Omega)$. $d(y(x); \Omega, z) = d[\hat{y}(x); \Omega, z]$.*

Proof $H : \overline{\Omega} \times [0, 1] \times \mathbb{R}^n$ defined by $H(x, t) = ty(x) + (1 - t)\hat{y}(x)$ $x \in \overline{\Omega}$, $t \in [0, 1]$ is a homotopy with $H(x, t) = y(x) \neq z$ for all $x \in \partial\Omega$ and $t \in [0, 1]$ since $y(x) = \hat{y}(x)$ on $\partial\Omega$. So by Theorem 12.3.5, $d(y(x); \Omega, z) = d[\hat{y}(x); \Omega, z]$. □

Remark 12.3.7 The above corollary implies that the degree of a mapping depends only on its value on the boundary of Ω.

The following theorem shows that the degree remains the same under certain translates.

Theorem 12.3.8 *Let $y : \overline{\Omega} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map on a nonvoid bounded open set Ω. If $z \notin y(\partial\Omega)$ and $p \in \mathbb{R}^n$. Then $d[y(x) - p; \Omega, z - p] = d[y(x); \Omega, z]$.*

Proof If $y : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable on an open bounded set $D \supseteq \overline{\Omega}$, then for $\hat{y}(x) = y(x) - p$. Let $\Phi : [0, \infty) \to \mathbb{R}$ be a continuous function vanishing in a neighbourhood of z as well as in $[\epsilon, \infty)$ where $0 < \epsilon < \min\{\|y(x) - z\| : x \in \partial\Omega\}$, such that $\int_{\mathbb{R}^n} \Phi(\|x\|)dx = 1$. Then $\Phi(\|y(x) - z\|)det\ \ y'(x) = \Phi(\|\hat{y}(x) - (z - p)\|)det\ \ \hat{y}'(x)$. Consequently $d(\hat{y}(x); \Omega, z - p) = d[y(x); \Omega, z]$. Since there is a sequence (y_k) of functions in $C^1(D)$ converging uniformly on D to y, where D is a bounded open set containing $\overline{\Omega}$ and $d(\hat{y}_k(x); \Omega, z - p) = d[y_k(x); \Omega, z]$ by the proceeding argument, and proceeding to the limit as k tends to infinity, we get $d(\hat{y}(x); \Omega, z - p) = d[y(x); \Omega, z]$. □

The above theorem in conjunction with the property of homotopy invariance of the degree leads to

Theorem 12.3.9 *Let $y : \overline{\Omega} \to \mathbb{R}^n$ be a continuous map on a nonvoid bounded open set Ω in $\mathbb{R}^n$. Let p_1 and p_2 be two points in $\mathbb{R}^n$ such that there is a continuous path $\eta : [0, 1] \to \mathbb{R}^n$ which does not meet $y(\partial\Omega)$ with $\eta(0) = p_1$ and $\eta(1) = p_2$. Then $d(y(x); \Omega, p_1) = d[y(x); \Omega, p_2]$.*

Proof The map $H : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n$ defined by $H(x, t) = y(x) - \eta(t)$ is a homotopy for which $H(x, t) \neq 0\, x \in \partial\Omega$ and $t \in [0, 1]$. So by Theorems 12.3.4 and 12.3.8 $d[y(x); \Omega, p_1) = d[H(x, 0), \Omega, \eta(0)) = d[H(x, 1); \Omega, \eta(1)) = d[y(x); \Omega, p_2)$. □

The above results imply that in the computation of the degree of a mapping on Ω, certain portions of Ω can be excised. This point is captured in

Theorem 12.3.10 (Excision property) *Let $y : \overline{\Omega} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map, where Ω is a nonvoid bounded open subset of $\mathbb{R}^n$. Let $z \notin y(\partial\Omega)$ and F any closed subset of $\overline{\Omega}$ such that $z \notin y(F)$. Then $d[y(x); \Omega, z] = d[y(x); \Omega - F, z]$. In particular if $F = \overline{\Omega}$, then $d[y(x); \Omega, z] = 0$.*

Proof Let $y_k \in C^1(D)$ where D is an open set containing $\overline{\Omega}$ such that $\epsilon_0 = \min\{\|y_k(x) - z\| : x \in \partial\Omega\} > 0$ and $\eta = \min\{\|y_k(x) - z\| : x \in F\} > 0$. Now $\epsilon = \min(\epsilon_0, \eta) > 0$. We can find $\Phi : [0, \infty) \to \mathbb{R}$ such that Φ vanishes in a neighbourhood of zero and on $[\epsilon_0, \infty)$ with $\int_{\mathbb{R}^n} \Phi(\|x\|)dx = 1$ and

$$
\begin{aligned}
d[y_k(x); \Omega, z] &= \int_{\Omega} \Phi(\|y_k(x) - z\|)J(y_k(x))dx \\
&= \int_{\Omega - F} \Phi(\|y_k(x) - z\|)J(y_k(x))dx \\
&\qquad \text{as } \Phi(\|y_k(x) - z\|) = 0 \text{ for } x \in F. \\
&= d[y_k(x); \Omega - F, z].
\end{aligned}
$$

Since any $y \in C(\overline{\Omega})$ can be uniformly approximated by a sequence $y_k \in C^1(\overline{\Omega})$ for which $\lim_{k\to\infty} d[y_k(x); \Omega, z] = d[y(x); \Omega, z]$, $d[y(x); \Omega, z] = \lim\limits_{k\to\infty} d[y_k(x); \Omega, z] = \lim\limits_{k\to\infty} [y_k(x); \Omega - F, z] = d[y(x); \Omega - F, z]$. In particular when $\overline{\Omega} = F$, $d[y_k(x); \Omega - F, z] = 0$ so that $d[y(x); \Omega, z] = d[y(x); \Omega - F, z] = 0$. □

Corollary 12.3.11 *Let $y : \overline{\Omega} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be continuous, Ω being a nonvoid bounded open subset of $\mathbb{R}^n$. If $F = \{x \in \overline{\Omega} : y(x) = z\}$ where $z \notin y(\partial\Omega)$ then $d[y(x); \Omega, z] = d[y(x); \Omega - F, z]$.*

The following theorem is the first step in relating the degree of a mapping to the solution of (non-linear) equations.

Theorem 12.3.12 *Let Ω be a nonvoid bounded open set in $\mathbb{R}^n$ and $y : \Omega \to \mathbb{R}^n$ be a continuously differentiable map. Let D be a nonvoid open set contained together with its closure in Ω. Suppose that for a given $z \notin y(\partial D)$, $y'(x)$ is non-singular for all $x \in \Gamma = \{x \in D : y(x) = z\}$.*

Then Γ contains at most finitely many points such that

$$d[y(x); \Omega, z] = \begin{cases} \displaystyle\sum_{i=1}^{m} sgn\ det\ J[y(x_i)], & for\ \Gamma = \{x_1, \ldots, x_m\} \\ 0 & if\ \Gamma = \phi. \end{cases}$$

Proof Since $z \notin y(x)$ for $x \in \partial\Omega$ by hypothesis the solutions of $y(x) = z$ lie in Ω. In other words each solution of $y(x) = z$ is an interior point of $\overline{\Omega}$. Since y'_x is non-singular at these solutions, y is a local homeomorphism at each of the solutions of $y(x) = z$ by the inverse function theorem. Thus for each x with $y(x) = z$, y maps a neighbourhood $N(x)$ of x onto a neighbourhood of y injectively. Such a neighbourhood of x will therefore not contain another solution of $y(x) = z$. Thus the solutions of $y(x) = z$ have no cluster points in Ω or $\overline{\Omega}$. Since $\overline{\Omega}$ is compact, such isolated solutions of $y(x) = z$ can only be finite. Thus Γ is empty or nonempty and finite.

If $\Gamma = \phi$, then we have already shown in Theorem 12.3.10, that $d[y(x); \Omega, z] = 0$.

Suppose $\Gamma \neq \phi$ and $\Gamma = \{x_1, \ldots, x_m\}$. Let $0 < \epsilon < \min\{\|y(x) - z\| : x \in \partial\Omega\}$. We can find $\Phi : [0, \infty) \to \mathbb{R}$ such that Φ is zero in a small neighbourhood of zero as well as in $[\epsilon, \infty)$ with $\int_{\mathbb{R}^n} \phi(\|x\|)dx = 1$. Further $\int_{\Omega} \Phi(\|y(x) - z\|)J[y(x)]dx = 0$. For each $x_i \in \Gamma$ by Inverse function theorem we can find an open neighbourhood U_i of x_i such that the restriction of y on U_i is an open neighbourhood V_i of z and y is a homeomorphism of U_i onto V_i. Since y' is non-singular at each x_i, we can choose U_i such that the sign of $J[y[x]]$ is that of $y'(x_i)$ for each $i = 1, 2, \ldots, m$. We can find $\epsilon_0 \in (0, \epsilon)$ such that the closed ball $\overline{B}(z, \epsilon_0) = K \subseteq V_i$. Define $W_i = y_i^{-1}(K)$ where y_i is the restriction of y to U_i. Now $\displaystyle\bigcup_{i=1}^{m} W_i \subseteq \Omega$ and if $\Phi : [0, \infty) \to \mathbb{R}$ is a continuous map vanishing in $[0, \alpha)$ and in $[\alpha, \infty)$ for $0 < \alpha < \epsilon_0$ with $\int_{\mathbb{R}^n} \phi(\|x\|)dx = 1$, then by the change of variables theorem for the integral applied to each y_i we get

$$\begin{aligned} d[y(x); \Omega, z] &= \sum_{i=1}^{m} \int_{W_i} \Phi(\|y(x) - z\|) J[y(x)] dx \\ &= \sum_{i=1}^{m} \int_{y_i^{-1}(K)} \phi(\|y_i(x) - z\|) J[y_i(x)] dx \\ &= \sum_{i=1}^{m} sgn\ det\ J[y_i(x)] \int_K \phi(\|x\|) dx \\ &= \sum_{i=1}^{m} sgn\ det\ J[y_i(x)] \end{aligned}$$

as $\int_K \phi(\|x\|)dx = \int_{\mathbb{R}^n} \phi(\|x\|)dx = 1$.

Thus in this case degree of the mapping y at z with respect to Ω is an integer. □

Naturally one would like to know if the degree of the mapping $y(x)$ at a point remains an integer even if the Jacobian of the mapping in singular at some of the roots of $y(x) = z$. Thanks to Sard's theorem detailed below the answer to this question is in the affirmative.

Theorem 12.3.13 (Sard's Theorem [8]) *Let $y : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on the nonempty bounded open set Ω. Let K be a compact subset of Ω and $C = \{x \in K.y'(x)$ is singular$\}$. Then $y(C)$ has zero measure.*

Proof Since $K \subseteq \Omega$ is compact and Ω is open K can be covered by a finite number of cubes of any prescribed length. So it suffices to assume that K is a cube Q, of side length ℓ. Divide Q into m^n subcubes P_j of side $\frac{\ell}{m}$. Suppose some P, a subcube contains u at which y' is singular. Given $\epsilon > 0$, one can choose m so large that $\|y(x) - y(u) - y'_x(x - y)\|_\infty \leq \epsilon\|x - u\|_\infty \leq \epsilon\frac{\ell}{m}$. Let $\sup\{\|y'_x\| : x \in Q\} = \beta$. Clearly, $\beta > 0$. So for $x \in P$,

$$\|y(u) - (y(u) + y'(x - y))\|_\infty \leq \beta\|x - u\|_\infty \leq \beta\frac{\ell}{m}.$$

As $y'(u)$ is singular $T(P)$ lies in a hyperplane of dimension at most $n - 1$ where $T(x) = y(u) + y'(x - u)$. So $y(P)$ is contained in $\frac{\epsilon\ell}{m}$ neighbourhood of $T(P)$. In other words $y(P)$ is contained in a hyper interval of volume at most $\left[2(\beta + \epsilon)\frac{\ell}{m}\right]^{n-1}$ $\left(\frac{2\ell}{m}\epsilon\right) = \left(\frac{2\ell}{m}\right)^n (\beta + \epsilon)^{n-1}\epsilon$. Since $\epsilon > 0$ is arbitrarily small, $y(K)$ is contained in hyper intervals of total volume at most equal to $(2\ell)^n(\beta + \epsilon)^{n-1}\epsilon$. So $y(K)$ is of measure zero. □

In view of Sard's theorem above, Theorem 12.3.12 can be reformulated as

Theorem 12.3.14 *Let $y : \overline{\Omega} \subseteq \mathbb{R}^N \to \mathbb{R}^n$ be continuously differentiable where Ω is a nonvoid bounded open set. Suppose $G \subseteq \Omega$ is open such that $\overline{G} \subseteq \Omega$ and $z \notin y(\partial G) \cup y(\mathscr{S}(\overline{G}))$ where $\mathscr{S}(A) = \{x \in A : y'(x)$ is singular$\}$ for $A \subseteq \Omega$. Then*

$\Gamma = \{x \in G : y(x) = z\}$ *is empty and* $d[y(x); G, z] = 0$ *or has finitely many points* $x_1, \ldots, x_m$ *and*

$$d[y(x); G, z] = \sum_{i=1}^{m} sgn\ det\ J[y(x_i)]$$

Theorem 12.3.15 *Let* $y : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ *be continuously differentiable on the bounded open set* Ω *and* $G \subseteq \Omega$ *an open set with* $\overline{G} \subseteq \Omega$. *If* $z \notin y(\partial G)$, *then there is a sequence* $z_k \notin y(\partial G) \cup y(\mathscr{S}(G))$ $(k = 1, 2, \ldots,)$ *such that* $\lim_{k\to\infty} z_k = z$ *and there exists* k_0 *with* $d[y[x]; G, z] = d[y(x); G, z_k]$ *for all* $k \geq k_0$.

Proof By Sard's Theorem 12.3.13 $y(\mathscr{S}(\overline{G}))$ has measure zero. So there exists a sequence $z_k \notin y(\mathscr{S}(\overline{G})) \cup y(\partial G)$ with $\lim_{k\to\infty} z_k = z$. By assumption there exists $\epsilon > 0$ such that $z_k \in B(z, \epsilon)$ for all $k \geq k_0$ and the line segments joining y to y_k for $k \geq k_0$ lie in $B(z, \epsilon)$. So these paths do not meet $y(\partial G)$. So by Theorem 12.3.9 for $k \geq k_0$. $d[y(x); G, z] = d[y(x); G, z_k]$. □

Corollary 12.3.16 *Let* $y : \overline{\Omega} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous map where* Ω *is a nonvoid bounded open set. Then for any* $z \notin y[\partial\Omega]$, $d[y(x); \Omega, z]$ *is an integer.*

Proof We can find a sequence of functions $y_k \in C^1(\overline{\Omega})$ such that y_k converges uniformly in $\overline{\Omega}$ to y, with $z \notin y_k(\partial\Omega)$. Clearly, $d[y_k(x); \Omega, z]$ is an integer for each k by Theorem 12.3.15. Also we can find k_0 such that $d[y_k(x); \Omega, z] = d[y_{k_0}(x); \Omega, z]$ for all $k \geq k_0$. Proceeding to the limit as k tends to infinity and noting that $d[y_{k_0}(x); \Omega, z)$ is an integer and $\lim_{k\to\infty} d[y_k(x); \Omega, z] = d[y(x); \Omega, z]$ it follows that $d[y(x); \Omega, z]$ is an integer. □

12.4 Some Consequences

The concept of degree of a mapping at a point with respect to a region can be used, along with its properties to prove Brouwer's fixed point theorem as well as Kronecker's theorem on the existence of solution to nonlinear equations.

First, we prove Kronecker's theorem.

Theorem 12.4.1 (Kronecker) *Let* Ω *be a nonempty bounded open set in* $\mathbb{R}^n$ *and* $f : \overline{\Omega} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous map. If* $z \notin f(\partial\Omega)$ *and if* $d[y(x); \Omega, z] \neq 0$, *then the equation* $f(x) = z$ *has a solution in* Ω.

Proof If $f(x) = z$ has no solution in Ω then $z \notin f(\overline{\Omega})$. So by Excision Theorem 12.3.10. $d[f(x); \Omega, z] = 0$, a contradiction. □

Theorem 12.4.2 (Brouwer) *Let* $f : \overline{B}(0, 1) \to \overline{B}(0, 1)$ *be a continuous mapping where* $\overline{B}(0, 1)$ *is the closed unit ball in* $\mathbb{R}^n$, *then* f *has a fixed point.*

Proof Let Ω be the unit open ball $\{x \in \mathbb{R}^n : \|x\| < 1\}$. Suppose $f(x) \neq x$ for all $x \in \overline{\Omega} = \overline{B}(0, 1)$. For each $t \in [0, 1]$ and $x \in \overline{\Omega}$ define $f(x, t) = x - tf(x)$. Now $\|f(x, t)\| = \|x - tf(x)\| \geq \|x\| - t\|f(x)\| = 1 - t\|f(x)\| \geq 1 - t > 0$ for $x \in \partial\Omega$ and $t \in [0, 1)$. By assumption $x \neq f(x)$ for $x \in \partial\Omega$. Thus $f(x, t) \neq 0$ for all $x \in \partial\Omega$ and $t \in [0, 1]$. So by Theorem 12.3.4, $d[f(x, t); \Omega, 0]$ is a constant. Thus $d[f(x, 1); \Omega, 0] = d[f(x_0); \Omega, 0] = 1$ by definition of the degree. So by Theorem 12.4.1, the equation $f(x) - x = 0$ has a solution in Ω, a contradiction. So f has a fixed poin in $\overline{\Omega}$. □

It may be added by way of conclusion that Heinz had also presented Leray's product theorem for the degree of a mapping. This result can be used to deduce important theorems of topology such as Brouwer's domain invariance theorem. The reader is referred to Lloyd [4] for further details.

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Appendix A
A Counterexample on Common Fixed Points

Since every continuous self-map on a bounded closed interval of real numbers has a fixed point, a natural question is whether two commuting continuous self-maps on a closed bounded interval of real numbers have a common fixed point. This has been answered in the negative, independently by Boyce [1] and Huneke [2]. In the sequel, we highlight the solution provided by Huneke in Part II of his paper [2].

Let $h : S \subseteq \mathbb{R} \to \mathbb{R}$ be a map where S is a subset of the set of real numbers $\mathbb{R}$. Define h^* by $h^*(x) = 1 - h(1-x)$ for each x for which $1 - x \in S$. Let $b \in \left[0, \frac{1}{2}\right)$ and let $s = \frac{3-2b+(6-4b)^{\frac{1}{2}}}{1-2b}$. Clearly, s is well-defined and positive. Define three continuous functions h_1, h_2 and h_3 with linear graphs as follows:

$$h_1 : I_1 = \left[b, \frac{(1-b+sb)}{s}\right] \to [b, 1] \text{ by } h_1(x) = s(x-b) + b$$

$$h_2 : I_2 = \left[\frac{(1-b+sb)}{s}, \frac{(2-b+sb)}{s}\right] \to [0, 1] \text{ by } h_2(x) = 2 - sx + sb - b$$

$$h_3 : I_3 = \left[\frac{(2-b+sb)}{s}, \frac{(3-2b+sb)}{s}\right] \to [0, 1-b] \text{ by } h_3(x) = -2 + sx - sb + b$$

Define $h : \left[b, \frac{(3-2b+sb)}{s}\right] \to [0, 1]$ by $h(x) = \{h_j(x), x \in I_j, j = 1, 2, 3$. Clearly, h is continuous and has a linear graph and h_j is invertible for each $j = 1, 2, 3$.

Definition A.1.1 Let C_b be the set of all continuous self-maps on $[0, b]$ with b as a fixed point. For each $g \in C_b$ define $\overline{g} : [0, 1] \to \mathbb{R}$ the unique continuous extension of g defined by

(1) $\overline{g}(x) = g(x)$ on $[0, b]$;
(2) $\overline{g}(x) = h(x)$ on $[b, h_3^{-1}(1-b)]$;
(3) $\overline{g}(x) = (h_1^*)^{-1}\overline{g}(h^*(x))$ on $[h_3^{-1}(1-b), (h_2^*)^{-1}h_2^{-1}(0)]$;
(4) $\overline{g}(x) = (h_2^*)^{-1}\overline{g}(h^*(x))$ on $[(h_2^*)^{-1}(h_2^{-1}(0)), 1-b]$;
(5) $\overline{g}(x) =$ the fixed point of h_2^* on $[1-b, 1]$.

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Remark A.1.2 The verification that $\overline{g}$ defined above is a unique extension of g is left as an exercise.

Remark A.1.3 If $g \in C_b$ satisfies Lipschitz condition with Lipschitz constant s, then $\overline{g}$ too satisfies Lipschitz condition with Lipschitz constant s.

Proof (*attributed to David Boyd in* [2]) Let $g \in C_b$ satisfy Lipschitz condition with Lipschitz constants and L be the set of all s-Lipschitz self-maps on [0, 1] satisfying (1), (2) and (5) for $\overline{g}$ in Definition A.1.1.

Define $T : L \to L$ by

$$Tf(x) = \begin{cases} f(x) & \text{if } x \in [0, h_3^{-1}(1-b)] \cup [1-b, 1] \\ (h_1^*)^{-1} f(h^*(x)), & \text{if } x \in [h_3^{-1}(1-b), (h_2^*)^{-1} h_2^{-1}(0)] \\ (h_2^*)^{-1} f(h^*(x)), & \text{if } x \in [(h_2^*)(h_2^{-1}(0)), 1-b] \end{cases}$$

Since $(h_1^*)^{-1}$ and $(h_2^*)^{-1}$ have linear graphs and $\frac{1}{s}$ Lipschitzian, $(h_1^*)^{-1}(0) = (h_2^*)^{-1}(0)$, h^* is s-Lipschitzian and the fixed points of h^* are $h_3^{-1}(1-b)$, $\frac{(s-1)(1-b)}{s+1}$ and $1-b$. Clearly, L is a complete metric space with the supremum metric on which T is a contraction with Lipschitz constant $\frac{1}{s} < 1$. This follows from the fact that $(h_1^*)^{-1}$ and $(h_2^*)^{-1}$ are contractions with Lipschitz constant $\frac{1}{s}$. So, there is unique function $\overline{g}$ in L with $T(\overline{g}) = \overline{g}$ and it can be seen that $\overline{g}$ satisfies the conditions of Definition A.1.1. □

Lemma A.1.4 *Let $f, g \in C_b$ and $x \in \left[\frac{1}{2}, 1\right]$. Then*

(i) $(\overline{f})^* x = x$ *implies* $\overline{g}(x) \neq x$ *and*
(ii) $(\overline{f})^*(\overline{g}(x)) = \overline{g}((\overline{f})^*(x))$.

Proof (i) The domain of h^* is $[(h_3^*)^{-1}(b), 1-b]$ and $(h_3^*)^{-1}(b) = -\frac{3+2b+s-sb}{s} = 1-b+\frac{(1-2b)(2b-3)}{3-2b+(6-4b)^{1/2}} < \frac{1}{2}$ as $0 \le b < \frac{1}{2}$. So, the fixed points of $(\overline{f})^*$ are in $[1-b, 1]$ or those of h^*. By definition of $\overline{g}$ on $[1-b, 1]$, $\overline{g}(x)$ is the same as as the fixed point of h_2^* and this is $\frac{(1-b)(s-1)}{(1+s)} < 1-b \le x$ for $x \in [1-b, 1]$. So $\overline{g}$ and $(\overline{f})^*$ have no common fixed point in $[1-b, 1]$. The only fixed points of h^* are those of h_j^*, $j = 1, 2, 3$. The fixed point $1-b$ of h_1^* is not a fixed point of $\overline{g}$. If x_2 is the fixed point of h_2^*, then $x_2 = \frac{(1-b)(s-1)}{s+1} < \frac{b-2+s+s^2-s^2b-2s}{s^2} = (h_2^*)^{-1}(h_2^{-1}(0))$. So $\overline{g}(x_2) \in (h_1^*)^{-1}[0, 1] = \left[\frac{s-1+b-bs}{s}, \frac{s+b-sb}{s}\right]$. But $x_2 = \frac{(1-b)(s-1)}{s+1} < \frac{s-1+b-sb}{s} \le \overline{g}(x_2)$. So $x_2 = h_2^*(x_2) \neq \overline{g}(x_2)$. If $x_3 = h_3^*(x_3)$ then $x_3 = \frac{3-s+sb-b}{1-s} = \frac{3-2b+sb}{s} = h_3^{-1}(1-b)$. Thus, $\overline{g}(x_3) = g(h_3^{-1}(1-b)) = 1-b > x_3$. So, $(\overline{f})^*$ and $\overline{g}$ have no common fixed point in $\left[\frac{1}{2}, 1\right]$.

(ii) If $x \in [1-b, 1]$ then $(\overline{f})^*(\overline{g}(x)) = (\overline{f})^*(x_2) = h_2^*(x_2) = x_2$ and $\overline{g}(\overline{f})^*(x) = \overline{g}(1-\overline{f}(1-x)) = \overline{g}(1-f(1-x)) = x_2$ since $f(1-x) \in [0, b]$. Thus, $(\overline{f})^*\overline{g}(x) = \overline{g}(\overline{f})^*(x)$ for $x \in [1-b, 1]$. For $x \in [h_3^{-1}(1-b), (h_2^*)^{-1}(h_2^{-1}(0)))]$, $(\overline{f})^*\overline{g}(x) = \overline{g}(\overline{f}^*(x))$. Now $h_3^{-1}(1-b)$ is the fixed point of h_3^*. So $h_3^*[(h_3^*)^{-1}(b), h_3^{-1}(1-b)] = [b, h_3^{-1}(1-b)]$ and this is the domain of h. Also, $h_3^*(1-b)$ is the fixed point of h_3.

So $h_3[(h_3^*)^{-1}(b), h_3^{-1}(1-b)] = [(h_3^*)^{-1}b, 1-b]$. This equals the domain of definition of h^*. It can be seen that the two piecewise linear functions $h^*(h|_{[(h_3^*)^{-1}b, h_3^{-1}(1-b)]})$ and $h^*(h|_{[(h_3^*)^{-1}b, h_3^{-1}(1-b)]})^*$ are the union of three linear functions mapping $(h_3^*)^{-1}(b)$, $h_3^{-1}(h^*)^{-1}(1))$, $h_3^{-1}((h^*)^{-1}(0))$, $h_3^{-1}(1-b)$ to $b, 1, 0$ and $1-b$, respectively. So these two functions coincide. So for each $x \in \left[\frac{1}{2}, h_3^{-1}(1-b)\right], x \in [(h_3^*)^{-1}b, h_3^{-1}(1-b)]$. Thus, $(\overline{f})^*(\overline{g}(x)) = (\overline{f})^* h_3(x) = h^* h(x) = (h^* h)^*(x) = 1 - h^*(h(1-x)) = h(1-h(1-x)) = hh^*(x) = g(h_3^*(x)) = \overline{g}(\overline{f})^*(x)$. In other words, $(\overline{f})^*$ and $\overline{g}$ commute on $\left[\frac{1}{2}, 1\right]$ and are without a common fixed point in $\left[\frac{1}{2}, 1\right]$. □

Thus, we have

Proposition A.1.5 *For any f and g in C_b, $\overline{f}$ and $(\overline{g})^*$ are commuting functions without a common fixed point.*

Proof For $f, g \in C_b$, by Lemma A.1.4, $\overline{f}$ and $(\overline{g})^*$ commute without a common fixed point in $\left[\frac{1}{2}, 1\right]$. Also, $(\overline{f})^*$ and $\overline{g}$ commute without a common fixed point in $\left[\frac{1}{2}, 1\right]$. So, $((\overline{f})^*)^*$ and $(\overline{g})^*$ commute without a common fixed point in $\left[0, \frac{1}{2}\right]$. But $(\overline{f})^{**} = \overline{f}$. Thus, $\overline{f}$ and $\overline{g}^*$ form a solution to the nonexistence of a common fixed point for commuting maps on [0, 1]. □

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Appendix B
A Compact Contractible Space Without Fixed Point Property

B.1 Introduction

A natural question is whether a compact contractible space has the fixed point property for continuous functions. This question was raised in 1932 by Borsuk [1] and was settled in the negative by Kinoshita [2] in 1953. In the following, Kinoshita's counterexample is described.

Recall that a topological space X is called contractible if the identity map on X is null-homotopic (i.e. homotopic to a constant).

B.2 Kinoshita's Example

We define the set $A = A_1 \cup A_2 \cup A_3$ in $\mathbb{R}^3$ as follows:

$$A_1 = \{(r, \theta, z) : 0 \leq r < 1, \theta \in [0, 2\pi], z = 0\}$$
$$A_2 = \{(r, \theta, z) : r = 1, \theta \in [0, 2\pi], z = [0, 1]\}$$
$$A_3 = \left\{(r, \theta, z) : r = \frac{2}{\pi} \arctan \phi, \theta = \phi \bmod 2\pi, \phi \geq 0, z = [0, 1]\right\}.$$

A inherits the subspace topology of $\mathbb{R}^3$. Clearly, A is closed and bounded in $\mathbb{R}^3$ and so is compact. Define $h_1, h_2 : A \to A$ by $h_1(r, \theta, z) = (r, \theta, 0)$ and $h_2 : A \to A$ by $h_2(r, \theta, z) = (0, \theta, z)$. From the properties of $h_2 \circ h_1(r, \theta, z) = (0, \theta, 0)$, the contractibility of A follows.

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Intuitively the continuous map on A constructed by Kinoshita involves twisting the top of A in counterclockwise direction while rotating the bottom of A in clockwise direction. More precisely, $f : A \to A$ is defined in the following way:

For $(r, \theta, z) \in A_1$,

$$f(r,\theta,z) = \begin{cases} \left(\frac{2}{\pi}\arctan\left(\tan\left(\frac{\pi r}{2}-\pi\right)\right), \theta-\pi, 0\right), & \text{for } r \geq \frac{2}{\pi}\arctan(\pi) \\ \left(0, 0, 1-\frac{1}{\pi}\arctan\left(\frac{\pi r}{2}\right)\right), & \text{for } 0 \leq r \leq \frac{2}{\pi}\arctan(\pi). \end{cases}$$

f maps $(0, 0, 0)$ onto $(0, 0, 1)$, the circle $r = \frac{2}{\pi}\arctan(\pi)$ to $(0, 0, 0)$ and the interior of this circle into the segment $r = 0$, $z \in [0, 1]$. The annulus of the circle is rotated π radians while the inner boundary is contracted to $(0, 0, 0)$. f has no fixed points in A_1.

For $(r, \theta, z) \in A_2$, since $r = 1$

$$f(1,\theta,z) = \begin{cases} (1, (\theta-\pi+2\pi z) \bmod 2\pi, z+\frac{z}{2}, & z \in \left[0, \frac{1}{2}\right] \\ (1, (\theta-\pi+2\pi z) \bmod 2\pi, \frac{1}{2}+\frac{z}{2}, & z \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Geometrically, f rotates the top of the cylinder counterclockwise by π radians, the bottom clockwise by π radians, while the circle $r = \frac{2}{\pi}\arctan(\pi)$, $z = 1$ is pulled upwards to $z = \frac{z}{4}$, stretching the bottom half of the cylinder with it and compressing the top half of the cylinder between $\frac{3}{4} \leq z \leq 1$. No point in A_2 off the circle at $z = \frac{1}{2}$ is fixed by f even as no point of the circle is fixed due to the lifting motion of the map.

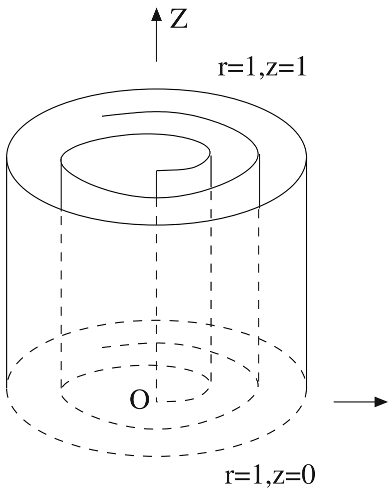
On A_3, for $0 \leq \phi \leq \pi$ f is defined by

$$f(r,\theta,z) = \begin{cases} \left(\frac{2}{\pi}\arctan((\phi+\pi)z, (\phi+\pi)z \bmod 2\pi, 1+\frac{\phi}{\pi}(\frac{3z}{2}-1)\right) & 0 \leq z \leq \frac{1}{2} \\ \left(\frac{2}{\pi}\arctan((\phi+\pi)z, (\phi+\pi)z \bmod 2\pi, 1+\frac{\phi}{\pi}(\frac{z-1}{2})\right) & \frac{1}{2} \leq z \leq 1 \end{cases}$$

On A_3, for $\pi \leq \phi < +\infty$, f is defined by

$$f(r,\theta,z) = \begin{cases} \left(\frac{2}{\pi}\arctan((\phi-\pi+2\pi z), (\phi-\pi+2\pi z)z \bmod 2\pi, \frac{3z}{2})\right) & 0 \leq z \leq \frac{1}{2} \\ \left(\frac{2}{\pi}\arctan((\phi-\pi+2\pi z), (\phi-\pi+2\pi z) \bmod 2\pi, \frac{1+z}{2})\right) & z \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Clearly f is well-defined on $A_1 \cap A_2$, $A_1 \cap A_3$ and continuous not only on each of A_1, A_2 and A_3 but also on $A_1 \cap A_2$, $A_2 \cap A_3$ and $A_1 \cap A_3$. By construction f has no fixed point in A (see sketch below for A).



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Appendix C
Fractals via Fixed Points

C.1 Introduction

Fractals may intuitively be understood as highly irregular non-smooth sets with 'non-integral dimension' arising often from a recursive process of construction displaying self-similarity. Some of these fractals can be realized as fixed points of set-functions. (see Hutchinson [2]).

Mandelbrot has pointed out how fractals can be used to model several physical phenomena. Falconer [1] treats interesting aspects of fractals from a geometric point of view.

C.2 Hausdorff Measure and Hausdorff Dimension

The following concepts and results are used in the sequel.

Definition C.2.1 Let E be any subset of $\mathbb{R}^n$ and $s \geq 0$. The Hausdorff s-dimensional outer measure of E denoted by $H^s(E)$ is defined by $H^s(E) = \lim\limits_{\delta \to 0} H^s_\delta(E)$, where

$$H^s_\delta(E) = \inf \left\{ \sum_{i=1}^{\infty} (diam\ U_i)^s : E \subseteq \bigcup_{i=1}^{\infty} U_i \text{ and } 0 < diam\ U_i < \delta \right\}.$$

Here $diam\ U = \sup\{\|x - y\| : x, y \in U\}$.

Remark C.2.2 One can show that $H^s(E) = \sup\limits_{\delta > 0} H^s_\delta(E)$ for any $E \subseteq \mathbb{R}^v$. Further when H^s in restricted to the σ-algebra of H^s-measurable subsets of $\mathbb{R}^n$, it is a measure called s-dimensional Hausdorff measure.

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Remark C.2.3 If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a similarity map in the sense that $\|Tx - Ty\| = \lambda\|x - y\|$ for some $\lambda > 0$ for all $x, y \in \mathbb{R}^n$, then for a H^s-measurable subset E of $\mathbb{R}^n$. $T(E)$ is H^s-measurable and $H^s(T(E)) = \lambda^s H^s(E)$.

Remark C.2.4 For each $s \geq 0$ H^s is a regular outer measure. Thus all Borel subsets of $\mathbb{R}^n$ are H^s-measurable.

Definition C.2.5 Let $E \subseteq \mathbb{R}^n$. The unique real number, $dim_H E$ is called the Hausdorff dimension of E provided

$$H^s(E) = \begin{cases} \infty & \text{for } 0 \leq s < dim_H(E) \\ 0 & \text{for } dim_H(E) < s < \infty. \end{cases}$$

Using the concept of iterated function systems (IFS), a large class of fractal sets can be constructed with explicit computation of their Hausdorff dimension.

The theorem below stated without proof is used in the existence theorem.

Theorem C.2.6 *Let (X, d) be a complete metric space. $K(X)$ the set of all nonempty compact subsets of X is a complete metric space under the Hausdorff metric (vide Definition 5.3.1).*

Remark C.2.7 If $(A_n) \in K(X)$ is a Cauchy sequence with the Hausdorff metric then it converges to $\limsup(A_n)$ in $K(X)$.

C.3 Construction of Fractal Sets

Definition C.3.1 Let F be a nonvoid closed subset of $\mathbb{R}^n$. An iterated function system (IFS for short) is a finite set of contractions $\{S_1, \ldots, S_m\}$, $m \geq 2$ on F. A subset D of F is called an attractor of the IFS $\{S_1, \ldots, S_n\}$ if $D = \bigcup_{i=1}^{m} S_i(D)$.

Theorem C.3.2 *Let $F \subseteq \mathbb{R}^n$ be a nonempty closed subset of $\mathbb{R}^n$ and $\{S_1, \ldots, S_m\}$ be an IFS on F. Let $K(F)$ be the space of nonvoid compact subsets of F with the Hausdorff metric. For each $E \in K(F)$ define $S(E) = \bigcup_{i=1}^{m} S_i(E)$. Then there is a unique attractor D of S in $K(F)$ and for each $E \in K(F)$ with $S(E) \subseteq E$,*

$$F = \bigcap_{k=0}^{\infty} S^k(E)$$

with $S^0(E) = E$, $S^{k+1}(E) = S(S^k(E))$ for $k = 1, 2, \ldots$.

Proof Let c_i be the contraction constants for S_i, for each $i = 1, 2, \ldots, m$. Now,

$$\begin{aligned} H(SA, SB) &= H\left(\bigcup_{i=1}^{m} S_i(A), \bigcup_{i=1}^{m} S_i(B)\right) \\ &\leq \max_{i=1,\ldots,m} H(S_i(A), S_i(B)) \\ &\leq \max_{i=1,\ldots,m} c_i H(A, B) \end{aligned}$$

for $A, B \in K(F)$. Since for each c_i, $0 \leq c_i < 1$, $0 \leq c = \max\{c_i : i = 1, 2, \ldots, m\} < 1$. Thus $A \to S(A)$ is a contraction on $K(F)$ with contraction constant c. As $(K(F), H)$ is complete, S has a unique fixed point D say. Thus $S(D) = D \in K(F)$. So the IFS $\{S_1, \ldots, S_m\}$ has a unique attractor D. If for $E \in K(F)$, $S(E) \subseteq E$ then $S^k(E) \in K(F)$ is a sequence of S-iterates which is decreasing and Cauchy in $K(F)$ and converges to the fixed point D by the contraction principle. But $\lim S^R(E) = \limsup S^k(E)$. Thus $\bigcap_{k=0}^{\infty} S^k(E) = D$. □

Remark C.3.3 If the contractions are similarity mappings satisfying an additional condition called open set condition, then the Hausdorff dimension of the attractor of the IFS, can be computed. This in turn leads to the construction of a number of self-similar fractals.

Definition C.3.4 An IFS of similarity contractions $\{S_1, \ldots, S_m\}$ is said to satisfy the open set condition (or Moran's condition) if for some nonvoid bounded open set V, $V \supseteq \bigcup_{i=1}^{m} S_i(V)$ with $S_i(V) \cap S_j(V) = \phi$ for $i \neq j$.

The following proposition is left as an exercise.

Proposition C.3.5 *Let $\{V_i\}$ be a family of disjoint nonvoid open sets in $\mathbb{R}^n$, such that each V_i contains a ball of radius αr and is contained in a ball if radius βr. Then any ball of radius r intersects at most $(1 + 2\beta)^n \alpha^{-n}$ of the closures of V_i.*

Proposition C.3.6 *Suppose for some finite positive measurable subset E of $\mathbb{R}^n$ there exist $s, k, \delta > 0$ such that $\mu(E) \leq k(diam\ U)^s$ for all sets U with $diam U \leq \delta$. Then $H^s(E) \geq \frac{\mu(E)}{k}$ and $s \leq dim_H E$.*

Proof For a cover $\{U_i\}$ of E with $diam U_i \leq \delta$. Then from the properties of measure $\mu(E) \subseteq \mu(UU_i) \leq \sum_i \mu(U_i) \leq k \sum_i (diam(U_i))^s$. So $kH^s(E) \geq \mu(E) > 0$. As $\mu(E) > 0$. $dim_H(E) \geq s$. □

Theorem C.3.7 *Let the IFS $\{S_1, \ldots, S_m\}$ of contracting similarities with contraction constant c_i satisfy the open set condition and F be the attractor of the IFS guaranteed by Theorem C.3.2, then $dim_H F = s$ where $\sum_{i=1}^{m} c_i^s = 1$. Further $0 < H^s(F) < \infty$.*

Proof Clearly for the function $t \to \sum_{i=1}^{m} c_i^t$, there is a unique s with $\sum_{i=1}^{m} c_i^s = 1$. For $I_k = \{(i_1, \ldots, i_k) : i_k \in \{1, \ldots, m\}\}$ and any set B and a given element $(i_1, \ldots, i_k)$ of I_k, define $B_{i_1 \ldots i_k} = S_{i_1} \circ \cdots \circ S_{i_k}(B)$. As F is fixed under the IFS, $F = \bigcup_{I_k} F_{i_1 \ldots i_k}$. So $\{F_{i_1 \ldots i_k}\}$ cover F. Since $S_i \circ \cdots \circ S_k$ is a contraction similarity with constant $c_{i_1} \ldots c_{i_k}$,

$$\sum_{I_k} (diam\ F_{i_1 \ldots i_k})^s = \sum (c_{i_1} \ldots c_{i_k})^s (diam\ F)^s$$

$$= \left(\sum c_{i_1}^s\right) \ldots \left(\sum c_{i_k}^s\right) (diam\ F)^s$$

$$= (diam\ F)^s \ \text{ since } \sum_{i=1}^{m} c_i^s = 1.$$

For $\delta > 0$, we can choose k such that $(\max\ c_i)^k\ diam\ F \leq \delta$. Since $diam\ F_{i_1 \ldots i_k} \leq (\max_i c_i)^k\ diam\ F$ the sets $\{F_{i_1 \ldots i_k}\}$ cover F with diameter $\leq \delta$, for sequences $(i_1, \ldots, i_k)$ in I_k. So $H^s(F) \leq \sum_{I_k} (diam(F_{i_1} \ldots F_{i_k}))^s$.

Consider I the set of all sequences (i_n) such that $i_k \in \{1, \ldots, m\}$ for all k and let $I_{i_1 \ldots i_k}$ be the subset of I consisting of only those sequences starting with $i_1, \ldots, i_k$. Define $\mu(I_{i_1, \ldots, i_k}) = (c_{i_1} \ldots c_{i_k})^s$. Define $x_{i_1, i_2, \ldots} = \bigcap_{k=1}^{\infty} F_{i_1 \ldots i_k}$. For a subset A of F define $\overline{\mu}(A) = \mu\{I_{i_1 \ldots i_k} : x_{i_1, \ldots, i_k}, \cdots \in A\}$. Now $\overline{\mu}(F) = \mu(I) = 1$ as F contains every point $x_{i_1, i_2, \ldots}$ for each $(i_1, i_2, \ldots) \in I$.

Since the IFS satisfies an open set condition, there is an open set V with $V \supset \bigcup_{i=1}^{m} S_i(V)$, where $S_i(V) \cap S_j(V) = \phi$ for $i \neq j$. So $S(\overline{V}) = \bigcup_{i=1}^{m} S_i(\overline{V})$ and $\{S^k(\overline{V})\}$ converges to F. So $F \subseteq \overline{V}_{i_1, \ldots, i_k}$ for any sequence $\{i_1, \ldots, i_k\}$. For a ball B of radius $0 < r < 1$, each sequence in I is truncated at the first i_k for which

$$\left(\min_{1 \leq i \leq m} c_i\right) r \leq c_{i_1} \ldots c_{i_k} \leq r$$

where c_i are the contraction constants of the corresponding similarities. As $V_p \cap V_q = \phi$ for $p \neq q$, $V_{i_1, \ldots, i_k, p}$ and $V_{i_1, \ldots, i_k, q}$ are disjoint. Let J be the finite set of all such truncated sequences in I. Clearly,

$$F = \bigcup_{I} F_{i_1, i_2, \ldots} \subseteq \bigcup_{J} F_{i_1, \ldots, i_k} \subseteq \bigcup_{J} \overline{V}_{i_1, \ldots, i_k}$$

Choose α, β such that V contains a ball of radius αr and is contained in a ball of radius βr. So for all $(i_1, \ldots, i_k) \in J$ the set $V_{i_1 \ldots i_k}$ contains a ball of radius $c_{i_1} \ldots c_{i_k} \alpha r$ and is contained in a ball of radius $c_{i_1} \ldots c_{i_k} \beta r$. Now every $V_{i_1 \ldots i_k}$ contains a ball of radius $(\min_i c_i r)\alpha$ and is contained in a ball of radius $r\beta$. Let J' be be the subset of I consisting of only those sequences $(i_1, \ldots, i_k)$ such that $\overline{V}_{i_1 \ldots i_k}$ intersects B. By Proposition C.3.5 there exist at most $\gamma = (1 + 2\beta)^n (\min_i c_i \alpha)^{-n}$ elements in J'. Now

$$\begin{aligned}
\overline{\mu}(B) &= \overline{\mu}(F \cap B) \\
&= \mu(\{I_{i_1,\ldots,i_k} : x_{i_1,\ldots,i_k\ldots} \in F \cap B\}) \\
&= \mu(\{I_{i_1,\ldots,i_k} : x_{i_1,\ldots,i_k\ldots} \in (\bigcup_{J'} \overline{V}_{i_1\ldots i_k}) \cap B\}).
\end{aligned}$$

So

$$\begin{aligned}
\overline{\mu}(B) &\leq \mu(\{I_{i_1,\ldots,i_k} : x_{i_1,\ldots,i_k\ldots} \in \bigcup_{J'} \overline{V}_{i_1\ldots i_k}\}) \\
&= \mu(\bigcup_{J'} I_{i_1,\ldots i_k}) \\
&< \sum_{J'} \mu(I_{i_1} \ldots I_{i_k}) \\
&\leq \sum_{J'} (c_{i_1} \ldots c_{i_k})^s \leq \sum_{J'} r^s = qr^s.
\end{aligned}$$

So for any set U contained in a ball B_U of radius $diam\ U$, $\overline{\mu}(U) \leq \overline{\mu}(B_U) \leq q(diam\ U)^s$. So by Proposition C.3.6, $H^s(F) \geq \frac{\overline{\mu}(F)}{q} = \frac{1}{q} > 0$. Thus $\dim_H F = s$. □

If the dimension of the set F is non-integral, then F is called a fractal. For instance, the Cantor set is the unique attractor of the IFS $\{S_1, S_2\}$ on $[0, 1]$ where $S_1(x) = \frac{x}{3}$ and $S_2(x) = 1 - \frac{x}{3}$. It satisfies the open set condition with $V = (0, 1)$. So by the above theorem the Hausdorff-dimension of the Cantor set is s with $2(\frac{1}{3})^s = 1$ and $s = \frac{\log 2}{\log 3}$. The Cantor set results from throwing off the middle third intervals recursively from $[0, 1]$ and a small section of the Cantor set is a scaled version of the entire set. This points to the self-similar nature of the set. The von Koch curve constructed by throwing off the middle third of a line segment and constructing an equilateral triangle over the removed middle third segment and continuing recursively this process results in a fractal set and its dimension is $\frac{\log 4}{\log 3}$.

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Postscript

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