On Generalized Picard Integral Operators

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Abstract In the paper, we constructed a class of linear positive operators generalizing Picard integral operators which preserve the functions $e^{\mu x}$ and $e^{2\mu x}$, $\mu > 0$. We show that these operators are approximation processes in a suitable weighted spaces. The uniform weighted approximation order of constructed operators is given via exponential weighted modulus of smoothness. We also obtain their shape preserving properties considering exponential convexity.

Keywords Voronovskaya-type theorems · Weighted modulus of continuity

2000 Mathematics Subject Classification Primary 41A36 · Secondary 41A25

1 Introduction

According to P.P. Korovkin and H. Bohman theorem, the convergence of a sequence $(L_n)_{n\geq 1}$ of the linear positive operators to the identity operator is essentially connected with the set $\{e_0, e_1, e_2\}$ with $e_i(t) = t^i$, $i = 0, 1, 2$. Since many classical linear positive operators fix e_0 and e_1, their theorem is one of the most powerful and spectacular criteria in approximation theory. It is known that for the study of convergence of linear positive operators the set $\left\{e_0, \exp_\mu, \exp_\mu^2\right\}$, with $\exp_\mu(x) = e^{\mu x}$, $\mu > 0$, also play an important role. For this purpose, recently in [1], the authors introduced and investigated generalized Picard $\left(P_n^*\right)_{n\geq 1}$ operators fixing e_0 and $\exp_\mu^2$ given by

$$\left(P_n^* f\right)(x) = P_n\left(f; \alpha_n^*(x)\right),$$

where

$$\alpha_n^*(x) = x - \frac{1}{2a}\ln\left(\frac{n}{n-4a^2}\right),\ n > n_a,$$

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S. A. Mohiuddine and T. Acar (eds.), *Advances in Summability and Approximation Theory*, https://doi.org/10.1007/978-981-13-3077-3_9

$n_a = \left[4a^2\right]$, $[\cdot]$ indicating the integer part function or so-called floor function and $(P_n)_{n\geq 1}$ classical Picard operators defined by

$$(P_n f)(x) = P_n(f;x) = \frac{\sqrt{n}}{2}\int_{\mathbb{R}} f(x+t)\, K_n(t)\, dt, \quad x \in \mathbb{R} \tag{1.1}$$

where

$$K_n(t) = e^{-\sqrt{n}|t|} \quad . \tag{1.2}$$

(See [5].) In here, the function f is selected such that the integrals are finite. Note that similar ideas for different linear positive operators were discussed in [2–4, 7].

In this paper, we want to obtain a new construction of the classical Picard operators fixing not only the function $\exp_\mu$ but also the function $\exp_\mu^2$. We aim to show that the new operators are positive approximation processes in the setting of large classes of weighted spaces. Using a technique developed in [6] by T. Coşkun which is based on a weighted Korovkin type theorem for linear positive operators acting on spaces which have different weights, we obtain weighted uniform convergence of the operators. Note that obtained asymptotic formulae for the new operators are different from those given for the corresponding classical operators on the line group.

The modification of our interest in this paper is defined by

$$\left(P_n^{**} f\right)(x) = \frac{\sqrt{n}}{2}\int_{\mathbb{R}} e^{-\mu\left(\alpha_n^{**}(x)+t\right)} e^{\mu x} f\left(\alpha_n^{**}(x)+t\right) K_n(t)\, dt, \quad n > n_\mu, \quad x \in \mathbb{R} \tag{1.3}$$

where

$$\alpha_n^{**}(x) = x - \frac{1}{\mu}\ln\left(\frac{n}{n-\mu^2}\right) \tag{1.4}$$

$\mu > 0$, $n_\mu = \left[\mu^2\right] + 1$ and K_n defined in (1.2).

Their close connection with the classical Picard operators is now displayed:

$$\left(P_n^{**} f\right)(x) = \exp_\mu(x)\, P_n\left(\frac{f}{\exp\mu}; \alpha_n^{**}(x)\right).$$

It is obvious that $\left(P_n^{**}\right)_{n>n_\mu}$ are positive and linear operators. On the other hand, whereas Picard operators $(P_n)_{n\geq 1}$ fix the functions e_0 and e_1, it can be checked easily that the operators $\left(P_n^{**}\right)_{n>n_\mu}$ reproduce $\exp_\mu$ and $\exp_\mu^2$, i.e.

$$\left(P_n^{**}\exp_\mu\right)(x) = \exp_\mu(x) \tag{1.5}$$

and

$$\left(P_n^{**}\exp_\mu^2\right)(x) = \exp_\mu^2(x). \tag{1.6}$$

2 Auxiliary Results

In this section, we will give some elementary properties of the generalized Picard integral operators defined in (1.3).

By means of elementary calculations, we obtain:

Lemma 1 *For each $n > n_\mu$ and $x \in \mathbb{R}$, the following identities hold:*
$\left(P_n^{**}e_0\right)(x) = \frac{n^2}{\left(n-\mu^2\right)^2}$
$\left(P_n^{**}\exp_\mu^3\right)(x) = e^{3\mu x}\frac{\left(n-\mu^2\right)^2}{n\left(n-4\mu^2\right)}$
$\left(P_n^{**}\exp_\mu^4\right)(x) = e^{4\mu x}\frac{\left(n-\mu^2\right)^3}{n^2\left(n-9\mu^2\right)}$

Lemma 2 *For each $n > n_\mu$ and $x \in \mathbb{R}$, the following identities hold:*
$\left(P_n^{**}e_1\right)(x) = \frac{n^2}{\left(n-\mu^2\right)^3}\left(\left(n-\mu^2\right)\alpha_n^{**}(x) - 2\mu\right),$
$\left(P_n^{**}e_2\right)(x) = \frac{n^2}{\left(n-\mu^2\right)^4}[2\left(n+3\mu^2\right) + \left(\mu^2-n\right)\left(4\mu x - \left(\mu^2-n\right)x^2\right)$
$-\frac{1}{\mu^2}\left(4\mu^2 + \left(\mu^2-n\right)\mu\left(x+\alpha_n^{**}(x)\right)\right)]$

3 Approximation on Weighted Spaces

Now we recall the concept of weighted function and weighted spaces considered in [6]. Let $\mathbb{R}$ denote the set of real numbers. A real-valued function ρ is called weight function if it is continuous on $\mathbb{R}$ and

$$\lim_{|x|\to\infty}\rho(x) = \infty, \quad \rho(x) \geq 1 \text{ for all } x \in \mathbb{R}. \tag{3.1}$$

We consider the weighted spaces $C_\rho(\mathbb{R})$ and $B_\rho(\mathbb{R})$ of the real function defined on real line defined by $B_\rho(\mathbb{R}) := \left\{f : |f(x)| \leq M_f\rho(x), x \in \mathbb{R}\right\}$ and $C_\rho(\mathbb{R}) = \left\{f : f \in B_\rho(\mathbb{R}), f \text{ continuous}\right\}$. The spaces $B_\rho(\mathbb{R})$ and $C_\rho(\mathbb{R})$ are Banach spaces endowed with the ρ-norm

$$\|f\|_\rho = \sup_{x\in\mathbb{R}}\frac{|f(x)|}{\rho(x)}.$$

Now we give some properties of a linear positive operator acting between two spaces with different weights.

(1) A positive linear operator L_n, defined on $C_{\rho_1}(\mathbb{R})$, maps $C_{\rho_1}(\mathbb{R})$ into $B_{\rho_2}(\mathbb{R})$ iff

$$L_n\rho_1 \in B_{\rho_2}(\mathbb{R}).$$

(2) Let $L_n : C_{\rho_1}(\mathbb{R}) \to B_{\rho_2}(\mathbb{R})$ be a positive linear operator. Then

$$\|L_n\|_{C_{\rho_1}\to B_{\rho_2}} = \|L\rho_1\|_{\rho_2}.$$

(3) For $n \in \mathbb{N}$, let $L_n : C_{\rho_1}(\mathbb{R}) \to B_{\rho_2}(\mathbb{R})$ be a positive linear operator. Suppose that there exists $M > 0$ such that for all $x \in \mathbb{R}$, $\rho_1(x) \le M\rho_2(x)$. If

$$\lim_{n\to\infty} \|L_n(\rho_1) - \rho_1\|_{\rho_2} = 0,$$

then the sequence of norms $\|L_n\|_{C_{\rho_1}\to B_{\rho_2}}$ is uniformly bounded.

Let φ_1 and φ_2 be two continuous functions, monotonically increasing on the real axis such that

$$\lim_{|x|\to\infty} \varphi_1(x) = \lim_{|x|\to\infty} \varphi_2(x) = \pm\infty \text{ and } \rho_k(x) = 1 + \varphi_k^2(x), \quad k = 1, 2.$$

Theorem A ([6]) Assume that ρ_1 and ρ_2 are weight functions satisfying the equality $\lim_{|x|\to\infty} \frac{\rho_1(x)}{\rho_2(x)} = 0$. If the sequence of linear positive operators $L_n : C_{\rho_1}(\mathbb{R}) \to B_{\rho_2}(\mathbb{R})$ satisfies the following three conditions

$$\lim_{n\to\infty} \|L_n(\varphi_1^\nu) - \varphi_1^\nu\|_{\rho_2} = 0, \quad \nu = 0, 1, 2, \tag{3.2}$$

then

$$\lim_{n\to\infty} \|L_n(f) - f\|_{\rho_2} = 0,$$

for all $f \in C_{\rho_1}(\mathbb{R})$.

Now we show that Theorem A can be applied to our new operators $\left(P_n^{**}\right)_{n>n_\mu}$ can be applicable to. Let $\rho_1(x) = 1 + x^2$ and $\rho_2(x) = 1 + x^4$ with $\varphi_1(x) = x$ and $\varphi_2(x) = x^2$. In this case the test functions set is $\{1, e_0, e_2\}$. Using Lemma 1 and (1.6) we have

$$\begin{aligned}\left(P_n^{**}\rho_1\right)(x) &= \left(P_n^{**}\left(e_0 + e_1^2\right)\right)(x) \\ &= \frac{n^2}{\left(n-\mu^2\right)^2} + \frac{n^2}{\left(n-\mu^2\right)^4}\left(2\left(n+3\mu^2\right) + \left(\mu^2 - n\right)\left(4\mu x - \left(\mu^2 - n\right)x^2\right)\right. \\ &\quad \left. - \frac{1}{\mu^2}\left(4\mu^2 + \left(\mu^2 - n\right)\mu\left(x + \alpha_n^{**}(x)\right)\right)\right)\end{aligned}$$

and thus there exists $C > 0$ such that the inequality

$$\frac{\left(P_n^{**}\rho_1\right)(x)}{\rho_2(x)} \le C$$

holds for $n > n_\mu$. Thus, $\left(P_n^{**}\right)_{n>n_\mu}$ are linear positive operators acting from $C_{\rho_1}(\mathbb{R})$ into $B_{\rho_2}(\mathbb{R})$. Also $\left(P_n^{**}\right)_{n>n_\mu}$ is a uniformly bounded sequence of positive linear operators from $C_{\rho_1}(\mathbb{R})$ into $B_{\rho_2}(\mathbb{R})$. Now we check the conditions in (3.2). For $\nu = 0$, we see that

$$\lim_{n\to\infty} \|P_n^{**}e_0 - e_0\|_{\rho_2} = \sup_{x\in\mathbb{R}} \frac{1}{1+x^4}\left[\left(\frac{n}{n-\mu^2}\right)^2 - 1\right] = 0$$

For $\nu = 1$, we have

$$\begin{aligned}\lim_{n\to\infty} \|P_n^{**}e_1 - e_1\|_{\rho_2} &\le \lim_{n\to\infty}\sup_{x\in\mathbb{R}} \frac{1}{1+x^4}\left[\frac{n^2}{\left(n-\mu^2\right)^2}\alpha_n^{**}(x) - x\right] \\ &\le \lim_{n\to\infty}\sup_{x\in\mathbb{R}} \frac{x}{1+x^4}\left[\frac{n^2}{\left(n-\mu^2\right)^2} - 1\right] = 0\end{aligned}$$

Similarly for $\nu = 2$, we have

$$\begin{aligned}\|P_n^{**}e_2 - e_2\|_{\rho_2} \le{}& \frac{2\left(n+3\mu^2\right)n^2}{\left(n-\mu^2\right)^4} + \frac{4\mu n^2}{\left(n-\mu^2\right)^3}\sup_{x\in\mathbb{R}}\frac{x}{1+x^4} + \left(\frac{n^2}{\left(n-\mu^2\right)^2} - 1\right)\sup_{x\in\mathbb{R}}\frac{x^2}{1+x^4} \\ &+ \frac{4n^2}{\left(n-\mu^2\right)^4} + \frac{n^2}{\mu\left(n-\mu^2\right)^3}\sup_{x\in\mathbb{R}}\frac{2x}{1+x^4}.\end{aligned}$$

Thus, we get

$$\lim_{n\to\infty} \|P_n^{**}e_2 - e_2\|_{\rho_2} = 0$$

Since all conditions of Theorem A are fulfilled, for all $f \in C_{\rho_1}(\mathbb{R})$, we have the following theorem.

Theorem 1 *Let P_n^{**}, $n > n_\mu$, be the operators defined by (1.3). For each $f \in C_{\rho_1}(\mathbb{R})$, the relation*

$$\lim_{n\to\infty} \|P_n^{**}f - f\|_{\rho_2} = 0$$

holds, where $\rho_1(x) = 1 + x^2$ and $\rho_2(x) = 1 + x^4$.

4 A Quantitative Result

The order of convergence of the operators $\left(P_n^{**}\right)_{n>n_\mu}$ in an exponential weighted space will be studied by using the following modulus of continuity. For function $f \in C_{\rho_3}(\mathbb{R})$, $\rho_3(x) = e^{\mu|x|}$, we consider the modulus of continuity defined in [8]:

$$\widetilde{\omega}(f;\delta) = \sup_{|h|<\delta} e^{-\mu|x|}\left|f(x+h) - f(x)\right|, \tag{4.1}$$

where $\delta > 0$ and $\mu > 1$. The weighted modulus of continuity has the following properties:

$$\widetilde{\omega}(f;\lambda\delta) \le (1+\lambda)\, e^{\lambda\mu\delta}\widetilde{\omega}(f;\delta), \quad \lambda > 0. \tag{4.2}$$

Similar weighted modulus of continuity was also given in [10].

Theorem 2 *For function* $f \in C_{\rho_3}(\mathbb{R})$*, we have*

$$\left\| P_n^{**} f - f \right\|_{\rho_3} \le \| f \|_{\rho_3} \left(\frac{n^2}{\left(n-\mu^2\right)^2} - 1 \right) + \frac{n}{n-\mu^2} \left(\frac{\sqrt{n}}{\sqrt{n}-\mu} + 1 \right) \widetilde{\omega} \left(\frac{f}{\exp_\mu}; \frac{\sqrt{n}}{\left(\sqrt{n}-\mu\right)^2} \right)$$

Proof Since

$$\left(P_n^{**} \rho_3\right)(x) \le e^{\mu|x|} \frac{n^2}{\left(n-\mu^2\right)^2} \left(1 + \frac{\sqrt{n}}{\sqrt{n}-\mu}\right),$$

$P_n^{**} f$ is a sequence of linear positive operators acting $C_{\rho_3}(\mathbb{R})$ into itself. From Lemma 1, we can write

$$\left(P_n^{**} f\right)(x) - f(x) = f(x) \left(\left(P_n^{**} e_0\right)(x) - 1\right) + \frac{\sqrt{n}}{2} \int_{\mathbb{R}} \left(e^{-\mu\left(\alpha_n^{**}(x)+t\right)} e^{\mu x} f\left(\alpha_n^{**}(x) + t\right) - f(x) \right) e^{-\sqrt{n}|t|} dt$$

Using (4.1), it is not difficult to deduce that

$$\begin{aligned} &\left| e^{-\mu\left(\alpha_n^{**}(x)+t\right)} f\left(\alpha_n^{**}(x)+t\right) - e^{-\mu x} f(x) \right| \\ &\le \left| e^{-\mu\left(\alpha_n^{**}(x)+t\right)} f\left(\alpha_n^{**}(x)+t\right) - e^{-\mu \alpha_n^{**}(x)} f\left(\alpha_n^{**}(x)\right) \right| + \left| e^{-\mu \alpha_n^{**}(x)} f\left(\alpha_n^{**}(x)\right) - e^{-\mu x} f(x) \right| \\ &\le e^{\mu\left|\alpha_n^{**}(x)\right|} \widetilde{\omega}\left(\frac{f}{\exp}; |t|\right) + \left| e^{-\mu \alpha_n^{**}(x)} f\left(\alpha_n^{**}(x)\right) - e^{-\mu x} f(x) \right| \end{aligned}$$

and then we conclude that from (4.2),

$$\begin{aligned} \left(P_n^{**} f\right)(x) - f(x) &= f(x)\left(\left(P_n^{**} e_0\right)(x) - 1\right) + e^{\mu\left|\alpha_n^{**}(x)\right|} \frac{\sqrt{n}}{2} \int_{\mathbb{R}} \widetilde{\omega}\left(\frac{f}{\exp}; |t|\right) e^{-\sqrt{n}|t|} dt \\ &\quad + \left| e^{-\mu \alpha_n^{**}(x)} f\left(\alpha_n^{**}(x)\right) - e^{-\mu x} f(x) \right| \left(P_n^{**} e_0\right)(x) \\ &= f(x)\left(\left(P_n^{**} e_0\right)(x) - 1\right) + \widetilde{\omega}\left(\frac{f}{\exp}; \delta_n\right) e^{\mu\left|\alpha_n^{**}(x)\right|} \frac{\sqrt{n}}{2} \int_{\mathbb{R}} \left(1 + \frac{|t|}{\delta_n}\right) e^{\mu|t|} e^{-\sqrt{n}|t|} dt \\ &\quad + \left| e^{-\mu \alpha_n^{**}(x)} f\left(\alpha_n^{**}(x)\right) - e^{-\mu x} f(x) \right| \left(P_n^{**} e_0\right)(x) \\ &= f(x)\left(\left(P_n^{**} e_0\right)(x) - 1\right) + \frac{n}{n-\mu^2} \widetilde{\omega}\left(\frac{f}{\exp}; \delta\right) e^{\mu|x|} \left(\frac{\sqrt{n}}{\sqrt{n}-\mu} + \frac{1}{\delta_n} \frac{\sqrt{n}}{\left(\sqrt{n}-\mu\right)^2} \right). \\ &\quad + \left| e^{-\mu \alpha_n^{**}(x)} f\left(\alpha_n^{**}(x)\right) - e^{-\mu x} f(x) \right| \left(P_n^{**} e_0\right)(x). \end{aligned}$$

Choosing $\delta = \frac{\sqrt{n}}{\left(\sqrt{n}-\mu\right)^2}$, we have desired result. □

5 Voronovskaya-Type Theorem

Using exponential moments, we shall prove the Voronovskaya-type theorem for $\left(P_n^{**}\right)_{n\geq 1}$.

Theorem 3 *If $f \in C_{\rho_3}(\mathbb{R})$ has a second derivative at a point $x \in \mathbb{R}$, then we have*

$$\lim_{n\to\infty} n\left(P_n^{**} f\right)(x) - f(x) = f''(x) - 3\mu f'(x) + 2\mu^2 f(x). \tag{5.1}$$

Proof We can use Taylor formula in the form

$$\begin{aligned} f(x+t) &= \left(f \circ \log_\mu\right)\left(e^{\mu(x+t)}\right) \\ &= \left(f \circ \log_\mu\right)\left(e^{\mu x}\right) + \left(f \circ \log_\mu\right)'\left(e^{\mu x}\right)\left(e^{\mu(x+t)} - e^{\mu x}\right) \\ &\quad + \frac{1}{2}\left(f \circ \log_\mu\right)''\left(e^{\mu x}\right)\left(e^{\mu(x+t)} - e^{\mu x}\right)^2 + h_x(t)\left(e^{\mu(x+t)} - e^{\mu x}\right)^2, \end{aligned}$$

where $h_x(t)$ is a continuous function which vanishes at 0.

Replacing x with $\alpha_n^{**}(x)$ in above equality and applying the operator $\left(P_n^{**}\right)_{n>n_\mu}$, one has

$$\begin{aligned} \left(P_n^{**} f\right)(x) = {} & f\left(\alpha_n^{**}(x)\right)\left(P_n^{**} e_0\right)(x) + \left(f \circ \log_\mu\right)'\left(e^{\mu\alpha_n^{**}(x)}\right)\left(\left(P_n^{**} \exp_\mu\right)(x) - e^{\mu\alpha_n^{**}(x)}\left(P_n^{**} e_0\right)(x)\right) \\ & + \frac{1}{2}\left(f \circ \log_\mu\right)''\left(e^{\mu\alpha_n^{**}(x)}\right) \\ & \times \left(\left(P_n^{**} \exp_\mu^2\right)(x) - 2e^{\mu\alpha_n^{**}(x)}\left(P_n^{**} \exp_\mu\right)(x) + e^{2\mu\alpha_n^{**}(x)}\left(P_n^{**} e_0\right)(x)\right) \\ & + \left(P_n^{**} h_x(t)\left(e^{\mu\left(\alpha_n^{**}(x)+t\right)} - e^{\mu x}\right)^2\right)(x). \end{aligned}$$

This equality can be arranged as

$$\begin{aligned} \left(P_n^{**} f\right)(x) = {} & f(x)\left(P_n^{**} e_0\right)(x) + \left[f\left(\alpha_n^{**}(x)\right) - f(x)\right]\left(P_n^{**} e_0\right)(x) \\ & + \left[\left(f \circ \log_\mu\right)'\left(e^{\mu\alpha_n^{**}(x)}\right) - \left(f \circ \log_\mu\right)'\left(e^{\mu x}\right)\right] \\ & \times \left(\left(P_n^{**} \exp_\mu\right)(x) - e^{\mu\alpha_n^{**}(x)}\left(P_n^{**} e_0\right)(x)\right) \\ & + \left(f \circ \log_\mu\right)'\left(e^{\mu x}\right)\left(\left(P_n^{**} \exp_\mu\right)(x) - e^{\mu\alpha_n^{**}(x)}\left(P_n^{**} e_0\right)(x)\right) \\ & + \frac{1}{2}\left[\left(f \circ \log_\mu\right)''\left(e^{\mu\alpha_n^{**}(x)}\right) - \left(f \circ \log_\mu\right)''\left(e^{\mu x}\right)\right] \\ & \times \left(\left(P_n^{**} \exp_\mu^2\right)(x) - 2e^{\mu\alpha_n^{**}(x)}\left(P_n^{**} \exp_\mu\right)(x) + e^{2\mu\alpha_n^{**}(x)}\left(P_n^{**} e_0\right)(x)\right) \\ & + \frac{1}{2}\left(f \circ \log_\mu\right)''\left(e^{\mu x}\right) \\ & \times \left(\left(P_n^{**} \exp_\mu^2\right)(x) - 2e^{\mu\alpha_n^{**}(x)}\left(P_n^{**} \exp_\mu\right)(x) + e^{2\mu\alpha_n^{**}(x)}\left(P_n^{**} e_0\right)(x)\right) \end{aligned}$$

$$+\left(P_n^{**}h_x\left(e^{\mu\left(\alpha_n^{**}(x)+t\right)}-e^{\mu x}\right)^2\right)(x).$$

Since

$$\lim_{n\to\infty}\alpha_n^{**}(x)=x,\quad \lim_{n\to\infty}n\left(e^{\mu\alpha_n^{**}(x)}-e^{\mu x}\right)=-\mu^2e^{\mu x}$$
$$\lim_{n\to\infty}n\left(e^{2\mu\alpha_n^{**}(x)}-e^{2\mu x}\right)=-2\mu^2e^{2\mu x}$$

and

$$\lim_{n\to\infty}n\left(\left(P_n^{**}e_0\right)(x)-1\right)=2\mu^2,$$

we get

$$\begin{aligned}
&\lim_{n\to\infty}n\left(P_n^{**}f\right)(x)-f(x)\\
&=2\mu^2f(x)+\left(f\circ\log_\mu\right)'\left(e^{\mu x}\right)\lim_{n\to\infty}n\left[\left(P_n^{**}\exp_\mu\right)(x)-e^{\mu x}\left(P_n^{**}e_0\right)(x)\right]\\
&\quad+\frac{1}{2}\left(f\circ\log_\mu\right)''\left(e^{\mu x}\right)\\
&\quad\times\lim_{n\to\infty}n\left[\left(P_n^{**}\exp_\mu^2\right)(x)-2e^{\mu x}\left(P_n^{**}\exp_\mu\right)(x)+e^{2\mu x}\left(P_n^{**}e_0\right)(x)\right]\\
&\quad+\lim_{n\to\infty}n\left(P_n^{**}h_x\left(e^{\mu\left(\alpha_n^{**}(x)+t\right)}-e^{\mu x}\right)^2\right)(x)
\end{aligned}$$

Using (1.5), (1.6) and Lemma 1, one finds that

$$\begin{aligned}
\lim_{n\to\infty}n\left[\left(\left(P_n^{**}\exp_\mu\right)(x)-e^{\mu x}\left(P_n^{**}e_0\right)(x)\right)\right]&=e^{\mu x}\lim_{n\to\infty}n\left[1-\left(P_n^{**}e_0\right)(x)\right]\\
&=-2\mu^2e^{\mu x}
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty}n\left[\left(P_n^{**}\exp_\mu^2\right)(x)-2e^{\mu x}\left(P_n^{**}\exp_\mu\right)(x)+e^{2\mu x}\left(P_n^{**}e_0\right)(x)\right]\\
&=e^{2\mu x}\lim_{n\to\infty}n\left[\left(P_n^{**}e_0\right)(x)-1\right]=2\mu^2e^{2\mu x}.
\end{aligned}$$

Since

$$\left(f\circ\log_\mu\right)'\left(e^{\mu x}\right)=e^{-\mu x}\mu^{-1}f'(x)\ \text{ and }\ \left(f\circ\log_\mu\right)''\left(e^{\mu x}\right)=e^{-2\mu x}\left(\mu^{-2}f''(x)-\mu^{-1}f'(x)\right),$$

we have

$$\lim_{n\to\infty} n\left(P_n^{**}f\right)(x) - f(x) = f^{''}(x) - 3\mu f^{'}(x) + 2\mu^2 f(x) + \lim_{n\to\infty} n\left(P_n^{**}h_x\left(e^{\mu\left(\alpha_n^{**}(x)+t\right)} - e^{\mu x}\right)^2\right)(x).$$

The proof of the theorem will be over if we prove

$$\lim_{n\to\infty} n\left(P_n^{**}h_x\left(e^{\mu\left(\alpha_n^{**}(x)+t\right)} - e^{\mu x}\right)^2\right)(x) = 0.$$

From Cauchy–Schwarz inequality, we can write

$$n\left|\left(P_n^{**}h_x\left(e^{\mu\left(\alpha_n^{**}(x)+t\right)} - e^{\mu x}\right)^2\right)\right| \le \sqrt{\left(P_n^{**}h_x^2\right)(x)}\sqrt{n^2\left(P_n^{**}\exp_{\mu,x}^4\right)(x)}.$$

Since

$$\begin{aligned}\left(P_n^{**}\exp_{\mu,x}^4\right)(x) &= \left(P_n^{**}\exp_\mu^4\right)(x) - 4e^{\mu x}\left(P_n^{**}\exp_\mu^3\right)(x) + 6e^{2\mu x}\left(P_n^{**}\exp_\mu^2\right)(x)\\ &\quad -4e^{3\mu x}\left(P_n^{**}\exp_\mu\right)(x) + e^{4\mu x}\left(P_n^{**}e_0\right)(x)\\ &= e^{4\mu x}\left(\frac{\left(n-\mu^2\right)^3}{n^2\left(n-9\mu^2\right)} - 4\frac{\left(n-\mu^2\right)^2}{n\left(n-4\mu^2\right)} + 2 + \frac{n^2}{\left(n-\mu^2\right)^2}\right)\end{aligned}$$

and

$$\lim_{n\to\infty} n^2\left(P_n^{**}\exp_{\mu,x}^4\right)(x) = 24\mu^4 e^{4\mu x},$$

we have desired result. □

6 Shape Preserving Properties

In this section, we will present some shape preserving properties of the operator (1.3). Also we will give the global smoothness preservation properties of mentioned operators. First, we have the following simple results.

Let $f \in C_{\rho_3}^2(\mathbb{R})$, we consider the operators for $x \in \mathbb{R}$, $n \in \mathbb{N}$,

$$\frac{\left(P_n^{**}f\right)(x)}{e^{\mu x}} = \frac{\sqrt{n}}{2}\int_{\mathbb{R}} e^{-\mu\left(\alpha_n^{**}(x)+t\right)} f\left(\alpha_n^{**}(x)+t\right) K_n(t)\,dt.$$

With simple calculations, we have

$$\Delta_h\left(\frac{P_n^{**}f}{\exp_\mu}\right)(x)=\frac{\sqrt{n}}{2}\int_{\mathbb{R}}\Delta_h\left(\frac{f}{\exp_\mu}\right)\left(\alpha_n^{**}(x)+t\right)K_n(t)\,dt$$

and

$$\Delta_h^2\left(\frac{P_n^{**}f}{\exp_\mu}\right)(x)=\frac{\sqrt{n}}{2}\int_{\mathbb{R}}\Delta_h^2\left(\frac{f}{\exp_\mu}\right)\left(\alpha_n^{**}(x)+t\right)K_n^P(t)\,dt,$$

where $\Delta_h(f)(x)=f(x+h)-f(x)$ and $\Delta_h^2(f)(x)=f(x+2h)-2f(x+h)+f(x)$.

Thus from previous expression, since $K_n(t)$ is positive for all $t\in\mathbb{R}$, if $\frac{f}{\exp_\mu}$ is increasing $\left(\Delta_h\left(\frac{f}{\exp_\mu}\right)(x)\geq 0\right)$ then $\Delta_h\left(\frac{P_n^{**}f}{\exp_\mu}\right)(x)\geq 0$, and so $\frac{P_n^{**}f}{\exp_\mu}$ is also increasing. If $\frac{f}{\exp_\mu}$ is convex $\left(\Delta_h^2\left(\frac{f}{\exp_\mu}\right)(x)\geq 0\right)$, then $\Delta_h^2\left(\frac{P_n^{**}f}{\exp_\mu}\right)(x)\geq 0$ and so $\frac{P_n^{**}f}{\exp_\mu}$ is also convex.

We want to give the connection of the operators $\left(P_n^{**}\right)_{n>n_\mu}$ with generalized convexity. Now we recall the definition of generalized convexities with respect to the functions $\exp_\mu$ and $\exp_\mu^2$.

Definition 1 A function f defined on $\mathbb{R}$ is said to be convex with respect to $\left\{\exp_\mu\right\}$, denoted by $f\in\mathcal{F}\left(\exp_\mu\right)$, if

$$\begin{vmatrix} e^{\mu x_0} & e^{\mu x_1}\\ f(x_0) & f(x_1)\end{vmatrix}\geq 0,\quad x_0<x_1.$$

f is said to be convex with respect to $\left\{\exp_\mu,\exp_\mu^2\right\}$, denoted by $f\in\mathcal{F}\left(\exp_\mu,\exp_\mu^2\right)$, if

$$\begin{vmatrix} e^{\mu x_0} & e^{\mu x_1} & e^{\mu x_2}\\ e^{2\mu x_0} & e^{2\mu x_1} & e^{2\mu x_2}\\ f(x_0) & f(x_1) & f(x_2)\end{vmatrix}\geq 0,\quad x_0<x_1<x_2.$$

Proposition 1 (see [4]) *Let $f\in C_{\rho_3}^2(\mathbb{R})$. Then the following items hold.*

(1) $f\in\mathcal{F}\left(\exp_\mu\right)$ if and only if $f/\exp_\mu$ is increasing for $x\in\mathbb{R}$,
(2) $f\in\mathcal{F}\left(\exp_\mu,\exp_\mu^2\right)$ if and only if $f''(x)-3\mu f'(x)+2\mu^2 f(x)\geq 0$ for $x\in\mathbb{R}$.

Using above proposition, we have

Theorem 4 *Let $f\in C_{\rho_3}(\mathbb{R})$. Then the following items hold.*

(1) If $f\in\mathcal{F}\left(\exp_\mu,\exp_\mu^2\right)$, then $\left(P_n^{**}f\right)(x)\geq f(x)$ for $x\in\mathbb{R}$,
(2) If $f\in\mathcal{F}\left(\exp_\mu\right)$, then $\left(P_n^{**}f\right)\in\mathcal{F}\left(\exp_\mu\right)$ for $x\in\mathbb{R}$.

Theorem 5 *Let $f\in C_{\rho_3}^2(\mathbb{R})$. Suppose that there exists $n_0\in\mathbb{N}$ such that*

$$f(x)\leq\left(P_n^{**}f\right)(x)\leq\left(P_nf\right)(x),\ \text{for all } n\geq n_0,\quad x\in\mathbb{R}. \tag{6.1}$$

Then

$$f^{''}(x) \geq 3\mu f^{'}(x) - 2\mu^2 f(x) \geq 0, \quad x \in \mathbb{R}. \tag{6.2}$$

In particular, $f^{''}(x) \geq 0$.

Conversely, if (6.2) holds with strictly inequalities at a given point $x \in \mathbb{R}$, *then there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$

$$f(x) \leq \left(P_n^{**} f\right)(x) \leq \left(P_n f\right)(x).$$

Proof From (6.1), we have that

$$0 \leq n\left(\left(P_n^{**} f\right)(x) - f(x)\right) \leq n\left(\left(P_n f\right)(x) - f(x)\right).$$

We know from [9] that

$$\lim_{n \to \infty} n\left(\left(P_n f\right)(x) - f(x)\right) = f^{''}(x). \tag{6.3}$$

Using (5.1) and (6.3), we have the desired result.

Conversely, if (6.2) holds with strict inequalities at a given point $x \in \mathbb{R}$, using again (5.1) and (6.3), we have

$$f(x) \leq \left(P_n^{**} f\right)(x) \leq \left(P_n f\right)(x)$$

for all $n \geq n_0$. □

By using the weighted modulus of continuity defined by (4.1), the result regarding global smoothness preservation properties for the operators of $\left(P_n^{**}\right)_{n > n_\mu}$ will be given as follows:

Theorem 6 *Let* $\delta > 0$, *we have*

$$\widetilde{\omega}\left(\frac{P_n^{**}(f)}{\exp_\mu}; \delta\right) \leq \left(\frac{n}{n - \mu^2}\right) \widetilde{\omega}\left(\frac{f}{\exp_\mu}; \delta\right). \tag{6.4}$$

Proof For $x \in \mathbb{R}$, we have

$$\begin{aligned} & e^{-\mu(x+h)} P_n^{**}(f; x+h) - e^{-\mu x} P_n^{**}(f; x) \\ = & \frac{\sqrt{n}}{2} \int_{\mathbb{R}} \left[e^{-\mu(\alpha_n^{**}(x+h)+t)} f\left(\alpha_n^{**}(x+h)+t\right) - e^{-\mu(\alpha_n^{**}(x+h)+t)} f\left(\alpha_n^{**}(x+h)+t\right) \right] K_n(t)\, dt \end{aligned}$$

Thus, we have for $n > n_\mu$

$$\begin{aligned}
& e^{-\mu|x|}\left|e^{-\mu(x+h)}P_n^{**}(f;x+h)-e^{-\mu x}P_n^{**}(f;x)\right| \\
&\leq \left(\frac{n}{n-\mu^2}\right)\frac{\sqrt{n}}{2}\int_{\mathbb{R}} e^{-\mu\left|\alpha_n^{**}(x)\right|}\left|\frac{f\left(\alpha_n^{**}(x+h)+t\right)}{e^{\mu\left(\alpha_n^{**}(x+h)+t\right)}}-\frac{f\left(\alpha_n^{**}(x)+t\right)}{e^{\mu\left(\alpha_n^{**}(x)+t\right)}}\right|K_n(t)\,dt \\
&\leq \left(\frac{n}{n-\mu^2}\right)\widetilde{\omega}\left(\frac{f}{\exp_\mu};\left|\alpha_n^{**}(x+h)-\alpha_n^{**}(x)\right|\right) \\
&\leq \left(\frac{n}{n-\mu^2}\right)\widetilde{\omega}\left(\frac{f}{\exp_\mu};h\right).
\end{aligned}$$

Thus, we get

$$\widetilde{\omega}\left(\frac{P_n^{**}(f)}{\exp_\mu};\delta\right)\leq\left(\frac{n}{n-\mu^2}\right)\widetilde{\omega}\left(\frac{f}{\exp_\mu};\delta\right).$$

□

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