Approximation Properties of Chlodowsky Variant of (p, q) Szász–Mirakyan–Stancu Operators

M. Mursaleen and A. A. H. AL-Abied

Abstract In the present paper, we introduce the Chlodowsky variant of (p, q) Szász–Mirakyan–Stancu operators on the unbounded domain which is a generalization of (p, q) Szász–Mirakyan operators. We have also derived its Korovkin-type approximation properties and rate of convergence.

Keywords (p, q)-integers · (p, q)-Szász–Mirakyan operators · Chlodowsky polynomials · Weighted approximation

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1 Introduction and Preliminaries

The applications of q-calculus emerged as a new area in the field of approximation theory from last two decades. The development of q-calculus has led to the discovery of various modifications of Bernstein polynomials involving q-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design and solutions of differential equations.

In 1987, Lupaş and in 1997, Phillips introduced a sequence of Bernstein polynomials based on q-integers and investigated its approximation properties.

Mursaleen et al. [12, 13, 18] introduced on Chlodowsky variant of Szász operators by Brenke-type polynomials, rate of convergence of Chlodowsky-type Durrmeyer Jakimovski–Leviatan operators and Dunkl generalization of q-parametric Szász–Mirakjan operators and shape preserving properties.

Several authors produced generalizations of well-known positive linear operators based on q-integers and studied them extensively. For instance, the approximation

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properties of A generalization of Szász–Mirakyan operators based on q-integers [5], convergence of the q-analogue of Szász-Beta operators [9], Dunkl generalization of q-parametric Szász–Mirakjan operators [18] and weighted statistical approximation by Kantorovich-type q-Szász–Mirakyan operators [6].

Recently, Mursaleen et al. introduced (p, q)-calculus in approximation theory and constructed the (p, q)-analogue of Bernstein operators [14], (p, q)-analogue of Bernstein–Stancu operators [15]. Further, Acar [1] has studied recently, (p, q)-generalization of Szász–Mirakyan operators.

In the present paper, we introduce the Chlodowsky variant of (p, q) Szász–Mirakyan–Stancu operators on the unbounded domain. Most recently, the (p, q)-analogue of some more operators has been studied in [2, 4, 10, 11, 14, 17, 19, 21].

The (p, q)-integer or in general the (p, q)-calculus was introduced to generalize or unify several forms of q-oscillator algebras well known in the Physics literature related to the representation theory of single-parameter quantum algebras. The (p, q)-integer is defined by

$$[n]_{p,q} = p^{n-1} + qp^{n-2} + \cdots + pq^{n-2} + q^{n-1} = \begin{cases} \dfrac{p^n - q^n}{p - q} & (p \neq q \neq 1) \\ \dfrac{1 - q^n}{1 - q} & (p = 1) \\ n & (p = q = 1). \end{cases} \tag{1}$$

The (p, q)-binomial expansion is

$$(ax + by)^n_{p,q} := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k} b^k x^{n-k} y^k,$$

$$(x + y)^n_{p,q} := (x + y)(px + qy)(p^2 x + q^2 y) \cdots (p^{n-1} x + q^{n-1} y),$$

$$(1 - x)^n_{p,q} := (1 - x)(p - qx)(p^2 - q^2 x) \cdots (p^{n-1} - q^{n-1} x).$$

The (p, q)-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

The definite integrals of a function f is defined by

$$\int_0^a f(t) d_{p,q} t = (q - p) a \sum_{k=0}^{\infty} f\left(\frac{p^k}{q^{k+1}} a\right) \frac{p^k}{q^{k+1}}, \qquad if \left|\frac{p}{q}\right| < 1,$$

$$\int_0^a f(t)d_{p,q}t = (p-q)a\sum_{k=0}^{\infty} f\left(\frac{q^k}{p^{k+1}}a\right)\frac{q^k}{p^{k+1}}, \qquad if\left|\frac{q}{p}\right| < 1.$$

There are two (p, q)-analogues of the classical exponential function defined as follows

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{p^{\frac{n(n-1)}{2}} x^n}{[n]_{p,q}!},$$

and

$$E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{[n]_{p,q}!},$$

which satisfy the equality $e_{p,q}(x)E_{p,q}(-x) = 1$. For $p = 1$, $e_{p,q}(x)$ and $E_{p,q}(x)$ reduce to q-exponential functions.

For $0 < q < 1$, Aral [5] introduced the generalized q-Szász–Mirakyan operators as follows

$$S_{n,q}(f;x) = \sum_{k=0}^{\infty} s^q_{n,k}(x) f\left(\frac{[k]_q b_n}{[n]_q}\right) \tag{2}$$

where

$$s^q_{n,k}(x) = \frac{1}{E_q([n]_q \frac{x}{b_n})} \frac{([n]_q x)^k}{[k]_q!(b_n)^k}.$$

where $0 \le x < \alpha_q(n)$, $\alpha_q(n) := \dfrac{b_n}{(1-q)[n]_q}$, $f \in C(\mathbb{R}_0)$ and (b_n) is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$.

Mursaleen et al. [10] introduced the (p, q)-analogue of the Szász–Mirakyan operators as follows

$$S_{n,p,q}(f;x) = \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} e_{p,q}(-[n]_{p,q}q^{-k}x) f\left(\frac{[k]_{p,q}}{p^{k-1}[n]_{p,q}}\right). \tag{3}$$

Lemma 1.1 *Let* $0 < q < p \le 1$ *and* $n \in \mathbb{N}$. *We have*

(i) $S_{n,p,q}(1;x) = 1$
(ii) $S_{n,p,q}(t;x) = x$
(iii) $S_{n,p,q}(t^2;x) = \frac{x^2}{p} + \frac{x}{[n]_{p,q}}$
(iv) $S_{n,p,q}(t^3;x) = \frac{x^3}{p^3} + \frac{2p+q}{p^2[n]_{p,q}}x^2 + \frac{x}{[n]^2_{p,q}}$
(v) $S_{n,p,q}(t^4;x) = \frac{x^4}{p^6} + \frac{3p^2+2pq+q^2}{p^5[n]_{p,q}}x^3 + \frac{3p^2+3pq+q^2}{p^3[n]^2_{p,q}}x^2 + \frac{x}{[n]^3_{p,q}}$.

2 Construction of the Operators

We construct the Chlodowsky variant of (p, q) Szász–Mirakyan–Stancu operators as

$$S_{n,p,q}^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}(-[n]_{p,q}q^{-k}\frac{x}{b_n}) f\left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta} b_n\right) \tag{4}$$

where $n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{N}_0$ with $0 \le \alpha \le \beta$, $0 \le x \le b_n$, $0 < q < p \le 1$ and b_n is an increasing sequence of positive terms with the properties $b_n \longrightarrow \infty$ and $\frac{b_n}{[n]_{p,q}} \longrightarrow 0$ as $n \longrightarrow \infty$. We observe that $S_{n,p,q}^{(\alpha,\beta)}$ is positive and linear. Furthermore, in the case of $q = p = 1$ and $\alpha = \beta = 0$, the operators (4) are similar to the classical Szász–Mirakyan operators.

Lemma 2.1 *Let $0 < q < p \le 1$ and $n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{N}_0$ with $0 \le \alpha \le \beta$, $0 \le x \le b_n$, and integer $m \ge 0$, we have*

$$S_{n,p,q}^{(\alpha,\beta)}(t^m;x) = \frac{b_n^m}{([n]_{p,q}+\beta)^m} \sum_{j=0}^{m} \binom{m}{j} \alpha^{m-j} S_{n,p,q}\left(t^j; q^{-1}\frac{x}{b_n}\right). \tag{5}$$

Proof Using the identity

$$[k+1]_{p,q} = p^k + q[k]_{p,q},$$

we can write

$$\begin{aligned}
S_{n,p,q}^{(\alpha,\beta)}(t^m;x) &= \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta} b_n\right)^m \\
&= \frac{b_n^m}{([n]_{p,q}+\beta)^m} \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \left(p^{1-k}[k]_{p,q}+\alpha\right)^m \\
&= \frac{b_n^m}{([n]_{p,q}+\beta)^m} \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \\
&\quad \times \sum_{j=0}^{m} \binom{m}{j} \alpha^{m-j} p^{j(1-k)} [k]_{p,q}^j \\
&= \frac{b_n^m}{([n]_{p,q}+\beta)^m} \sum_{j=0}^{m} \binom{m}{j} \alpha^{m-j} \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} \\
&\quad \times e_{p,q}(-[n]_{p,q}q^{-k}\frac{x}{b_n}) p^{j(1-k)} [k]_{p,q}^j
\end{aligned}$$

$$= \frac{b_n^m}{([n]_{p,q}+\beta)^m}\sum_{j=0}^{m}\binom{m}{j}\alpha^{m-j}S_{n,p,q}(t^j;q^{-1}\frac{x}{b_n})$$

□

which is desired.

Lemma 2.2 *Let $S_{n,p,q}^{(\alpha,\beta)}(f;x)$ be given by (4). Then the following properties hold:*

$$(i)\ S_{n,p,q}^{(\alpha,\beta)}(1;x)=1$$

$$(ii)\ S_{n,p,q}^{(\alpha,\beta)}(t;x)=\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha b_n}{[n]_{p,q}+\beta}$$

$$(iii)\ S_{n,p,q}^{(\alpha,\beta)}(t^2;x)=\frac{[n]_{p,q}^2}{p([n]_{p,q}+\beta)^2}x^2+\frac{(1+2\alpha)b_n[n]_{p,q}}{([n]_{p,q}+\beta)^2}x+\frac{\alpha^2b_n^2}{([n]_{p,q}+\beta)^2}$$

$$(iv)\ S_{n,p,q}^{(\alpha,\beta)}(t^3;x)=\frac{[n]_{p,q}^3}{p^3([n]_{p,q}+\beta)^3}x^3+\frac{(3p\alpha+2p+q)b_n[n]_{p,q}^2}{p^2([n]_{p,q}+\beta)^3}x^2$$
$$+\frac{(1+3\alpha++3\alpha^2)b_n^2[n]_{p,q}}{([n]_{p,q}+\beta)^3}x+\frac{\alpha^3b_n^3}{([n]_{p,q}+\beta)^3}$$

$$(v)\ S_{n,p,q}^{(\alpha,\beta)}(t^4;x)=\frac{[n]_{p,q}^4}{p^6([n]_{p,q}+\beta)^4}x^4+\frac{(3p^2+2pq+q^2+4p\alpha)b_n[n]_{p,q}^3}{p^5([n]_{p,q}+\beta)^4}x^3$$
$$+\frac{(3p^2+3pq+q^2+4pq\alpha+8p^2\alpha+6p^2\alpha^2)b_n^2[n]_{p,q}^2}{p^3([n]_{p,q}+\beta)^4}x^2$$
$$+\frac{(1+4\alpha+6\alpha^2+4\alpha^3)b_n^3[n]_{p,q}}{([n]_{p,q}+\beta)^4}x+\frac{\alpha^4b_n^4}{([n]_{p,q}+\beta)^4}.$$

Proof (i)

$$S_{n,p,q}^{(\alpha,\beta)}(1;x)=S_{n,p,q}(1;q^{-1}\frac{x}{b_n})$$
$$=1.$$

(ii)

$$S_{n,p,q}^{(\alpha,\beta)}(t;x)=\frac{b_n}{([n]_{p,q}+\beta)}\sum_{j=0}^{1}\binom{1}{j}\alpha^{1-j}S_{n,p,q}\left(t^j;q^{-1}\frac{x}{b_n}\right)$$
$$=\frac{b_n}{([n]_{p,q}+\beta)}\left\{\alpha+S_{n,p,q}\left(t;q^{-1}\frac{x}{b_n}\right)\right\}$$
$$=\frac{b_n}{([n]_{p,q}+\beta)}\left\{\alpha+\frac{[n]_{p,q}}{b_n}x\right\}$$

$$= \frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha b_n}{[n]_{p,q}+\beta}.$$

(iii)

$$\begin{aligned}
S_{n,p,q}^{(\alpha,\beta)}(t^2;x) &= \frac{b_n^2}{([n]_{p,q}+\beta)^2}\sum_{j=0}^{2}\binom{2}{j}\alpha^{2-j}S_{n,p,q}(t^j;q^{-1}\frac{x}{b_n})\\
&= \frac{b_n^2}{([n]_{p,q}+\beta)^2}\left\{\alpha^2 + 2\alpha S_{n,p,q}(t;q^{-1}\frac{x}{b_n}) + S_{n,p,q}(t^2;q^{-1}\frac{x}{b_n})\right\}\\
&= \frac{b_n^2}{([n]_{p,q}+\beta)^2}\left\{\alpha^2 + 2\alpha\frac{[n]_{p,q}}{b_n}x + \frac{[n]_{p,q}}{b_n}x + \frac{[n]_{p,q}^2}{pb_n^2}x^2\right\}\\
&= \frac{[n]_{p,q}^2}{p([n]_{p,q}+\beta)^2}x^2 + \frac{(1+2\alpha)b_n[n]_{p,q}}{([n]_{p,q}+\beta)^2}x + \frac{\alpha^2 b_n^2}{([n]_{p,q}+\beta)^2}.
\end{aligned}$$

(iv)

$$\begin{aligned}
S_{n,p,q}^{(\alpha,\beta)}(t^3;x) &= \frac{b_n^3}{([n]_{p,q}+\beta)^3}\sum_{j=0}^{3}\binom{3}{j}\alpha^{3-j}S_{n,p,q}(t^j;q^{-1}\frac{x}{b_n})\\
&= \frac{b_n^3}{([n]_{p,q}+\beta)^3}\left\{\alpha^3 + 3\alpha^2 S_{n,p,q}(t;q^{-1}\frac{x}{b_n}) + 3\alpha S_{n,p,q}(t^2;q^{-1}\frac{x}{b_n})\right.\\
&\quad \left. + S_{n,p,q}(t^3;q^{-1}\frac{x}{b_n})\right\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{b_n^3}{([n]_{p,q}+\beta)^3}\left\{\alpha^3 + 3\alpha^2\frac{[n]_{p,q}}{b_n}x + 3\alpha\left(\frac{[n]_{p,q}}{b_n}x + \frac{[n]_{p,q}^2}{pb_n^2}x^2\right)\right.\\
&\quad \left. + \frac{[n]_{p,q}}{b_n}x + \frac{2[n]_{p,q}^2}{pb_n^2}x^2 + \frac{q[n]_{p,q}^2}{p^2b_n^2}x^2 + \frac{[n]_{p,q}^3}{p^3b_n^3}x^3\right\}\\
&= \frac{[n]_{p,q}^3}{p^3([n]_{p,q}+\beta)^3}x^3 + \frac{(3p\alpha+2p+q)b_n[n]_{p,q}^2}{p^2([n]_{p,q}+\beta)^3}x^2\\
&\quad + \frac{(1+3\alpha++3\alpha^2)b_n^2[n]_{p,q}}{([n]_{p,q}+\beta)^3}x + \frac{\alpha^3 b_n^3}{([n]_{p,q}+\beta)^3}.
\end{aligned}$$

(v)

$$\begin{aligned}
S_{n,p,q}^{(\alpha,\beta)}(t^4;x) &= \frac{b_n^4}{([n]_{p,q}+\beta)^4}\sum_{j=0}^{4}\binom{4}{j}\alpha^{4-j}S_{n,p,q}(t^j;q^{-1}\frac{x}{b_n})\\
&= \frac{b_n^4}{([n]_{p,q}+\beta)^4}\left\{\alpha^4 + 4\alpha^3 S_{n,p,q}(t;q^{-1}\frac{x}{b_n}) + 6\alpha^2 S_{n,p,q}(t^2;q^{-1}\frac{x}{b_n})\right.\\
&\quad \left. + 4\alpha S_{n,p,q}(t^3;q^{-1}\frac{x}{b_n}) + S_{n,p,q}(t^4;q^{-1}\frac{x}{b_n})\right\}
\end{aligned}$$

$$= \frac{b_n^4}{([n]_{p,q}+\beta)^4}\Bigg\{\alpha^4 + 4\alpha^3\frac{[n]_{p,q}}{b_n}x + 6\alpha^2\left(\frac{[n]_{p,q}}{b_n}x + \frac{[n]_{p,q}^2}{pb_n^2}x^2\right)$$
$$+4\alpha\left(\frac{[n]_{p,q}}{b_n}x + \frac{2[n]_{p,q}^2}{pb_n^2}x^2 + \frac{q[n]_{p,q}^2}{p^2b_n^2}x^2 + \frac{[n]_{p,q}^3}{p^3b_n^3}x^3\right) + \frac{[n]_{p,q}^4}{p^6b_n^4}x^4$$
$$+\frac{q^2[n]_{p,q}^3}{p^5b_n^3}x^3 + \frac{2q[n]_{p,q}^3}{p^4b_n^3}x^3 + \frac{q^2[n]_{p,q}^2}{p^3b_n^2}x^2 + \frac{3[n]_{p,q}^3}{p^3b_n^3}x^3 + \frac{3q[n]_{p,q}^2}{p^2b_n^2}x^2$$
$$+\frac{3[n]_{p,q}^2}{pb_n^2}x^2 + \frac{[n]_{p,q}}{b_n}x\Bigg\}$$
$$= \frac{[n]_{p,q}^4}{p^6([n]_{p,q}+\beta)^4}x^4 + \frac{(3p^2+2pq+q^2+4p\alpha)b_n[n]_{p,q}^3}{p^5([n]_{p,q}+\beta)^4}x^3$$
$$+\frac{(3p^2+3pq+q^2+4pq\alpha+8p^2\alpha+6p^2\alpha^2)b_n^2[n]_{p,q}^2}{p^3([n]_{p,q}+\beta)^4}x^2$$
$$+\frac{(1+4\alpha+6\alpha^2+4\alpha^3)b_n^3[n]_{p,q}}{([n]_{p,q}+\beta)^4}x + \frac{\alpha^4b_n^4}{([n]_{p,q}+\beta)^4}.$$

□

Lemma 2.3 *Let $p, q \in (0,1)$. Then for, $x \in [0,\infty)$, we have:*

$$(i)\ S_{n,p,q}^{(\alpha,\beta)}(t-x;x) = \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta} - 1\right)x + \frac{\alpha b_n}{[n]_{p,q}+\beta}$$
$$(ii)\ S_{n,p,q}^{(\alpha,\beta)}((t-x)^2;x) = \frac{((1-p)[n]_{p,q}^2 + p\beta^2)x^2 + ([n]_{p,q}+2\alpha\beta)pb_nx + p\alpha^2b_n^2}{p([n]_{p,q}+\beta)^2}.$$

3 Korovkin-Type Approximation Theorem

Suppose C_ρ is the space of all continuous functions f such that $|f(x)| \le M\rho(x)$, $-\infty < x < \infty$. Then C_ρ is a Banach space with the norm $\|f\|_\rho = \sup\limits_{-\infty<x<\infty}\frac{|f(x)|}{\rho(x)}$. The subsequent results are used for proving Korovkin approximation theorem on unbounded sets.

Theorem 3.1 (See [8]) *There exists a sequence of positive linear operators T_n, acting from C_ρ to B_ρ, satisfying the conditions*

(i)

$$\lim_{n\to\infty}\|T_n(1;x) - 1\|_\rho = 0$$

(ii)

$$\lim_{n\to\infty}\|T_n(\varphi;x) - \varphi\|_\rho = 0$$

(iii)

$$\lim_{n\to\infty} \|T_n(\varphi^2; x) - \varphi^2\|_\rho = 0,$$

where $\varphi(x)$ *is a continuous and increasing function on* $(-\infty, \infty)$, *such that* $\lim_{x\to\pm\infty} \varphi(x) = \pm\infty$, $\rho(x) = 1 + \varphi^2$, *and there exists a function* $f^* \in C_\rho$, *for which* $\overline{\lim_{n\to\infty}} \|T_n f^* - f^*\|_\rho > 0$.

Theorem 3.2 (See [8]) *Conditions (i), (ii), (iii) of above theorem implies that*

$$\lim_{n\to\infty} \|T_n f - f\|_\rho = 0,$$

for any function f belonging to the subset

$$C_\rho^0 = \left\{ f \in C_\rho[0, \infty) : \lim_{x\to\infty} \frac{| f(x) |}{\rho(x)} < \infty \right\}.$$

Consider the weight function $\rho(x) = 1 + x^2$ and operators:

$$T_{n,p,q}^{(\alpha,\beta)}(f; x) = \begin{cases} S_{n,p,q}^{(\alpha,\beta)}(f; x), & x \in [0, b_n] \\ \\ f(x), & x \in [0, \infty)/[0, b_n]. \end{cases} \tag{6}$$

Thus for $f \in C_{1+x^2}$, we have

$$\begin{aligned} \|T_{n,p,q}^{(\alpha,\beta)}(f; x)\|_{1+x^2} &\le \sup_{x\in[0,b_n]} \frac{|T_{n,p,q}^{(\alpha,\beta)}(f; x)|}{1 + x^2} + \sup_{b_n<x<\infty} \frac{|f(x)|}{1 + x^2} \\ &\le \|f\|_{1+x^2} \left(\sup_{x\in[0,\infty)} \frac{|T_{n,p,q}^{(\alpha,\beta)}(1 + t^2; x)|}{1 + x^2} + 1 \right). \end{aligned}$$

Now we will obtain,

$$\|T_{n,p,q}^{(\alpha,\beta)}(f; x)\|_{1+x^2} \le M\|f\|_{1+x^2}$$

if $p := (p_n)$ and $q := (q_n)$ satisfy $0 < q_n < p_n \le 1$ and for n sufficiently large $p_n \to 1, q_n \to 1$ and $p_n^n \to N, q_n^n \to N, N < \infty$ and $\lim_{n\to\infty} \frac{b_n}{[n]_{p,q}} = 0$.

Theorem 3.3 *Let* $p := (p_n)$ *and* $q := (q_n)$ *satisfy* $0 < q_n < p_n \le 1$ *and for n sufficiently large* $p_n \to 1, q_n \to 1$ *and* $p_n^n \to N, q_n^n \to N, N < \infty$ *and* $\lim_{n\to\infty} \frac{b_n}{[n]_{p,q}} = 0$. *Then, for any* $f \in C_{1+x^2}^0$, *we have*

$$\|T_{n,p_n,q_n}^{(\alpha,\beta)}(f; \cdot) - f(\cdot)\|_{1+x^2} = 0.$$

Using the results of Theorem 3.1, Lemma 2.2, we will obtain the following assessments, respectively:

$$\sup_{x\in[0,\infty)}\frac{|T^{(\alpha,\beta)}_{n,p_n,q_n}(1;x)-1|}{1+x^2}=\sup_{0\le x\le b_n}\frac{|S^{(\alpha,\beta)}_{n,p_n,q_n}(1;x)-1|}{1+x^2}=0.$$

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{|T^{(\alpha,\beta)}_{n,p_n,q_n}(t;x)-t|}{1+x^2}&=\sup_{0\le x\le b_n}\frac{|S^{(\alpha,\beta)}_{n,p_n,q_n}(t;x)-x|}{1+x^2}\\&\le\sup_{0\le x\le b_n}\frac{\left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}-1\right)x+\frac{\alpha b_n}{[n]_{p,q}+\beta}}{1+x^2}\\&\le\frac{\alpha b_n}{[n]_{p,q}+\beta}+\left|\frac{[n]_{p,q}}{[n]_{p,q}+\beta}-1\right|\longrightarrow 0,\end{aligned}$$

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{|T^{(\alpha,\beta)}_{n,p_n,q_n}(t^2;x)-t^2|}{1+x^2}&=\sup_{0\le x\le b_n}\frac{|S^{(\alpha,\beta)}_{n,p_n,q_n}(t^2;x)-x^2|}{1+x^2}\\&\le\sup_{0\le x\le b_n}\frac{1}{1+x^2}\left(\frac{((1-p)[n]^2_{p,q}+p\beta^2)x^2}{p([n]_{p,q}+\beta)^2}\right.\\&\quad\left.+\frac{([n]_{p,q}+2\alpha\beta)b_nx}{([n]_{p,q}+\beta)^2}+\frac{\alpha^2b_n^2}{([n]_{p,q}+\beta)^2}\right)\\&\le\frac{\alpha^2b_n^2}{([n]_{p,q}+\beta)^2}+\left|\frac{(1-p)[n]^2_{p,q}+p\beta^2}{p([n]_{p,q}+\beta)^2}\right|\\&\quad+\left|\frac{([n]_{p,q}+2\alpha\beta)b_n}{([n]_{p,q}+\beta)^2}\right|\longrightarrow 0,\end{aligned}$$

whenever $n\longrightarrow\infty$, because we have $\lim_{n\to\infty}q_n=\lim_{n\to\infty}p_n=1$, $\lim_{n\to\infty}\frac{b_n}{[n]_{p,q}}=0$, as $n\longrightarrow\infty$.

Theorem 3.4 *Assuming C as a positive and real number independent of n and f as a continuous function which vanishes on $[C,\infty)$. Let $p:=(p_n)$ and $q:=(q_n)$ satisfy $0<q_n<p_n\le 1$ and for n sufficiently large $p_n\to 1$, $q_n\to 1$ and $p_n^n\to N$, $q_n^n\to N$, $N<\infty$ and $\lim_{n\to\infty}\frac{b_n}{[n]_{p,q}}=0$. Then*

$$\lim_{n\to\infty}\sup_{0\le x\le b_n}|S^{(\alpha,\beta)}_{n,p_n,q_n}(f;x)-f(x)|=0.$$

Proof From the hypothesis on f, it is bounded, i.e. $|f(x)| \leq M(M > 0)$. For any $\varepsilon > 0$, we have

$$\left| f\left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}b_n\right) - f(x)\right| < \varepsilon + \frac{2M}{\delta^2}\left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}b_n - x\right)^2$$

where $x \in [0, b_n]$ and $\delta = \delta(\varepsilon)$ are independent of n. Now since we know,

$$S^{(\alpha,\beta)}_{n,p_n,q_n}((t-x)^2;x) = S^{(\alpha,\beta)}_{n,p_n,q_n}(t^2;x) - 2xS^{(\alpha,\beta)}_{n,p_n,q_n}(t;x) + x^2 S^{(\alpha,\beta)}_{n,p_n,q_n}(1;x).$$

We can conclude by Theorem 3.3,

$$\sup_{0\leq x\leq b_n} |S^{(\alpha,\beta)}_{n,p_n,q_n}(f;x) - f(x)| \leq \varepsilon + \frac{2M}{\delta^2}\Bigg(\frac{\alpha^2 b_n^2}{([n]_{p,q}+\beta)^2} + \frac{((1-p)[n]^2_{p,q}+p\beta^2)x^2}{p([n]_{p,q}+\beta)^2} + \frac{([n]_{p,q}+2\alpha\beta)b_n x}{([n]_{p,q}+\beta)^2}\Bigg).$$

Since $\dfrac{b_n}{[n]_{p,q}} = 0$, as $n \to \infty$, we have the desired result. □

4 Rate of Convergence

Now we give the rate of convergence of the operators $S^{(\alpha,\beta)}_{n,p,q}(f;x)$ in terms of the elements of the usual Lipschitz class $Lip_M(\gamma)$.

Let $f \in C_B[0,\infty)$, $M > 0$ and $0 < \gamma \leq 1$. We recall that f belongs to the class $Lip_M(\gamma)$ if the inequality

$$| f(t) - f(x) | \leq M \mid t - x \mid^{\gamma} \quad t, x \in [0, \infty)$$

is satisfied.

Theorem 4.1 *Let* $0 < q < p \leq 1$. *Then for each* $f \in Lip_M(\gamma)$, *we have*

$$| S^{(\alpha,\beta)}_{n,p,q}(f;x) - f(x) | \leq M(\delta_n(x))^{\frac{\gamma}{2}}$$

where

$$\delta_n(x) = S^{(\alpha,\beta)}_{n,p,q}((t-x)^2;x).$$

Proof For $f \in Lip_M(\gamma)$, we obtain

$$| S^{(\alpha,\beta)}_{n,p,q}(f;x) - f(x) |$$

$$= \Big| \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}(-[n]_{p,q}q^{-k}\frac{x}{b_n}) \left(f\left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta} b_n \right) - f(x) \right) \Big|$$
$$\leq \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \left| f\left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta} b_n \right) - f(x) \right|$$
$$\leq M \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \left| \frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta} b_n - x \right|^{\gamma}.$$

Applying Hölder's inequality with the values $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, we get following inequality,

$$| S_{n,p,q}^{(\alpha,\beta)}(f;x) - f(x) |$$
$$\leq M \left(\sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta} b_n - x \right)^2 \right)^{\frac{\gamma}{2}}$$
$$\left(\sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \right)^{\frac{2-\gamma}{2}}.$$

From Lemma 2.2, we get

$$= M \left(S_{n,p,q}^{(\alpha,\beta)}\left((t-x)^2;x\right) \right)^{\frac{\gamma}{2}} \left(S_{n,p,q}^{(\alpha,\beta)}(1;x) \right)^{\frac{2-\gamma}{2}}$$
$$= M \left(S_{n,p,q}^{(\alpha,\beta)}\left((t-x)^2;x\right) \right)^{\frac{\gamma}{2}}.$$

Choosing $\delta : \delta_n(x) = S_{n,p,q}^{(\alpha,\beta)}\left((t-x)^2;x\right)$,
we obtain

$$| S_{n,p,q}^{(\alpha,\beta)}(f;x) - f(x) | \leq M(\delta_n(x))^{\frac{\gamma}{2}}.$$

Hence, the desired result is obtained. □

We will estimate the rate of convergence in terms of modulus of continuity. Let $f \in C_B[0,\infty)$, and the modulus of continuity of f denoted by $\omega(f,\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by the relation

$$\omega(f,\delta) = \max_{|y-x|\leq\delta} | f(y) - f(x) |, \quad x, y \in [0,\infty).$$

It is known that $\lim_{\delta\to 0+} \omega(f,\delta) = 0$ for $f \in C_B[0,\infty)$ and for any $\delta > 0$ one has

$$| f(y) - f(x) | \leq \left(\frac{| y - x |}{\delta} + 1 \right) \omega(f,\delta). \tag{7}$$

Theorem 4.2 *If* $f \in C_B[0,\infty)$, *then*

$$| S^{(\alpha,\beta)}_{n,p,q}(f;x) - f(x) | \leq 2\omega(f;(\sqrt{\delta_n(x)}),$$

where $\omega(f;\cdot)$ *is modulus of continuity of* f *and* $\delta_n(x)$ *be the same as in Theorem 4.1.*

Proof Using triangular inequality, we get

$$\begin{aligned}
| S^{(\alpha,\beta)}_{n,p,q}(f;x) - f(x) | &= \left| \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \right. \\
&\quad \left. \times \left(f\left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta} b_n\right) - f(x)\right) \right| \\
&\leq \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \\
&\quad \times \left| f\left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta} b_n\right) - f(x)\right|.
\end{aligned}$$

Now using inequality (7), Hölder's inequality and Lemma 2.2, we get

$$\begin{aligned}
| S^{(\alpha,\beta)}_{n,p,q}(f;x) - f(x) | &= \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \\
&\quad \times \left(\frac{| \frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta} b_n - x |}{\delta} + 1 \right) \omega(f,\delta) \\
&\leq \omega(f,\delta) \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \\
&\quad + \frac{\omega(f,\delta)}{\delta} \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \\
&\quad \times \left| \frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta} b_n - x \right| \\
&= \omega(f,\delta) + \frac{\omega(f,\delta)}{\delta} \left(\sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} \right. \\
&\quad \left. \times e_{p,q}(-[n]_{p,q}q^{-k}\frac{x}{b_n}) \left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta} b_n - x\right)^2 \right)^{\frac{1}{2}} \\
&= \omega(f,\delta) + \frac{\omega(f,\delta)}{\delta} \left(S^{(\alpha,\beta)}_{n,p,q}((t-x)^2;x)\right)^{\frac{1}{2}}.
\end{aligned}$$

Now choosing $\delta = \delta_n(x)$ as in Theorem 4.1, we have

$$| S_{n,p,q}^{(\alpha,\beta)}(f; x) - f(x) | \leq 2\omega(f; (\sqrt{\delta_n(x)}). \qquad \square$$

Now let us denote by $C_B^2[0,\infty)$, the space of all functions $f \in C_B[0,\infty)$, such that $f^{'}, f^{''} \in C_B[0,\infty)$. Let $\| f \|$ denote the usual supremum norm of f. Classical Peetre's K-functional and the second modulus of smoothness of the function $f \in C_B[0,\infty)$ are defined, respectively, as

$$K_2(f,\delta) = \inf_{g \in C_B^2[0,\infty)} \{\| f - g \| + \delta \| g^{''} \|\},$$

where $\delta > 0$ and $g \in C_B^2[0,\infty)$. By Theorem 2.4 of [7], there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2(f, \sqrt{\delta}) \tag{8}$$

where

$$\omega_2(f, \sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x,x+h \in I} | f(x+2h) - 2f(x+h) + f(x) |$$

is the second-order modulus of smoothness of $f \in C_B^2[0,\infty)$. The usual modulus of continuity of $f \in C_B^2[0,\infty)$ is defined by

$$\omega(f,\delta) = \sup_{0<h\leq\delta} \sup_{x \in [0,\infty)} | f(x+h) - f(x) | .$$

Theorem 4.3 *Let* $x \in [0, b_n]$, $f \in C_B[0,\infty)$ *and* $0 < q < p \leq 1, 0 \leq \alpha \leq \beta$. *Then for all* $n \in \mathbb{N}$, *there exists a positive constant* $C > 0$ *such that*

$$| S_{n,p,q}^{(\alpha,\beta)}(f; x) - f(x) | \leq C\omega_2(f, \delta_n(x)) + \omega(f, \alpha_n(x)),$$

where

$$\delta_n(x) = \sqrt{S_{n,p,q}^{(\alpha,\beta)}((t-x)^2; x) + (\alpha_n(x))^2}, \quad \alpha_n(x) = \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta} - 1\right)x + \frac{\alpha b_n}{[n]_{p,q}+\beta}.$$

Proof For $x \in [0,\infty)$, we consider the auxiliary operators $\bar{S}_n^*$ defined by

$$\bar{S}_n^*(f; x) = S_{n,p,q}^{(\alpha,\beta)}(f; x) + f(x) - f\left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha b_n}{[n]_{p,q}+\beta}\right).$$

From Lemma 2.2 (i) (ii) and Lemma 2.3 (i), we observe that the operators $\bar{S}_n^*(f; x)$ are linear and reproduce the linear functions. Hence,

$$\bar{S}_n^*(1; x) = S_{n,p,q}^{(\alpha,\beta)}(1; x) + 1 - 1 = 1$$
$$\bar{S}_n^*(t; x) = S_{n,p,q}^{(\alpha,\beta)}(t; x) + x - \left(\frac{[n]_{p,q}}{[n]_{p,q} + \beta}x + \frac{\alpha b_n}{[n]_{p,q} + \beta}\right) = x$$
$$\bar{S}_n^*((t - x); x) = \bar{S}_n^*(t; x) - x\bar{S}_n^*(1; x) = 0.$$

Let $x \in [0, \infty)$ and $g \in C_B^2[0, \infty)$. Using Taylor's formula

$$g(t) = g(x) + g'(x)(t - x) + \int_x^t (t - u)g''(u)\mathrm{d}u.$$

Applying $\bar{S}_n^*$ to both sides of the above equation, we have

$$\begin{aligned}\bar{S}_n^*(g; x) - g(x) &= g'(x)\bar{S}_n^*((t - x); x) + \bar{S}_n^*\left(\int_x^t (t - u)g''(u)\mathrm{d}u; x\right)\\ &= S_{n,p,q}^{(\alpha,\beta)}\left(\int_x^t (t - u)g''(u)\mathrm{d}u; x\right)\\ &\quad - \int_x^{\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha b_n}{[n]_{p,q}+\beta}} \left(\frac{[n]_{p,q}}{[n]_{p,q} + \beta}x + \frac{\alpha b_n}{[n]_{p,q} + \beta} - u\right) g''(u)\mathrm{d}u.\end{aligned}$$

On the other hand, since

$$\left|\int_x^t (t - u)g''(u)\mathrm{d}u\right| \le \int_x^t \mid t - u \mid\mid g''(u) \mid \mathrm{d}u \le\parallel g'' \parallel \left|\int_x^t \mid t - u \mid \mathrm{d}u\right| \le (t - x)^2 \parallel g'' \parallel$$

and

$$\left|\int_x^{\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha b_n}{[n]_{p,q}+\beta}} \left(\frac{[n]_{p,q}}{[n]_{p,q} + \beta}x + \frac{\alpha b_n}{[n]_{p,q} + \beta} - u\right) g''(u)\mathrm{d}u\right|$$
$$\le \left(\frac{[n]_{p,q}}{[n]_{p,q} + \beta}x + \frac{\alpha b_n}{[n]_{p,q} + \beta} - x\right)^2 \parallel g'' \parallel .$$

We conclude that

$$\begin{aligned}\left|\bar{S}_n^*(g; x) - g(x)\right| &\le \left|S_{n,p,q}^{(\alpha,\beta)}\left(\int_x^t (t - u)g''(u)\mathrm{d}u; x\right)\right|\\ &\quad + \left|\int_x^{\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha b_n}{[n]_{p,q}+\beta}} \left(\frac{[n]_{p,q}}{[n]_{p,q} + \beta}x + \frac{\alpha b_n}{[n]_{p,q} + \beta} - u\right) g''(u)\mathrm{d}u\right|\\ &\le \parallel g'' \parallel S_{n,p,q}^{(\alpha,\beta)}((t - x)^2; x) + \parallel g'' \parallel \left(\frac{[n]_{p,q}}{[n]_{p,q} + \beta}x + \frac{\alpha b_n}{[n]_{p,q} + \beta} - x\right)^2\\ &= \parallel g'' \parallel \delta_n^2(x).\end{aligned}$$

Now, taking into account Lemma 2.2 (i), we have

$$| \bar{S}_n^*(f;x) | \leq | S_{n,p,q}^{(\alpha,\beta)}(f;x) | + 2 \parallel f \parallel \leq 3 \parallel f \parallel .$$

Therefore,

$$\begin{aligned} | S_{n,p,q}^{(\alpha,\beta)}(f;x) - f(x) | &\leq | \bar{S}_n^*(f-g;x) - (f-g)(x) | \\ &+ \left| f\left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha b_n}{[n]_{p,q}+\beta}\right) - f(x) \right| + | \bar{S}_n^*(g;x) - g(x) | \\ &\leq 4 \parallel f-g \parallel + \omega(f,\alpha_n(x)) + \delta_n^2(x) \parallel g'' \parallel . \end{aligned}$$

Hence, taking the infimum on the right-hand side over all $g \in C_B^2[0,\infty)$, we have the following result

$$| S_{n,p,q}^{(\alpha,\beta)}(f;x) - f(x) | \leq 4K_2(f,\delta_n^2(x)) + \omega(f,\alpha_n(x)).$$

In view of the property of K-functional, we get

$$| S_{n,p,q}^{(\alpha,\beta)}(f;x) - f(x) | \leq C\omega_2(f,\delta_n(x)) + \omega(f,\alpha_n(x)).$$

This completes the proof of the theorem. □

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