Convergence Properties of Genuine Bernstein–Durrmeyer Operators

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Abstract The genuine Bernstein–Durrmeyer operators have notable approximation properties, and many papers have been written on them. In this paper, we introduce a modified genuine Bernstein–Durrmeyer operators. Some approximation results, which include local approximation, error estimation in terms of the modulus of continuity and weighted approximation is obtained. Also, a quantitative Voronovskaya-type approximation will be studied. The convergence of these operators to certain functions is shown by illustrative graphics using MAPLE algorithms.

Keywords Genuine Bernstein–Durrmeyer operators · Rate of convergence · Linear positive operators · Voronovskaja-type theorem

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1 Introduction

In 1912, Bernstein [8] defined the Bernstein polynomials in order to prove Weierstrass's fundamental theorem. These operators are one of the important topics of approximation theory in which it has been studied in great details for a long time. The Bernstein operators are given by

$$B_n : C[0,1] \to C[0,1], \quad B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x), \tag{1}$$

where

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \ x \in [0,1].$$

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Very recently, Cai et al. [9] introduced and considered a new generalization of Bernstein polynomials depending on the parameter λ as follows

$$B_{n,\lambda}(f;x)=\sum_{k=0}^{n}\tilde{b}_{n,k}(\lambda;x)f\left(\frac{k}{n}\right), \tag{2}$$

where $\lambda\in[-1,1]$ and $\tilde{b}_{n,k}$, $k=0,1,\dots$ are defined below

$$\begin{aligned}
\tilde{b}_{n,0}(\lambda;x)&=b_{n,0}(x)-\frac{\lambda}{n+1}b_{n+1,1}(x),\\
\tilde{b}_{n,k}(\lambda;x)&=b_{n,k}(x)+\lambda\left(\frac{n-2k+1}{n^2-1}b_{n+1,k}(x)-\frac{n-2k-1}{n^2-1}b_{n+1,k+1}(x)\right),\\
\tilde{b}_{n,n}(\lambda;x)&=b_{n,n}(x)-\frac{\lambda}{n+1}b_{n+1,n}(x).
\end{aligned}$$

In the particular case, when $\lambda=0$, λ-Bernstein operators reduce to well-known Bernstein operators. The authors of [9] have deeply studied many approximation properties of λ-Bernstein operators such as uniform convergence, rate of convergence in terms of modulus of continuity, Voronovskaya-type pointwise convergence, shape preserving properties.

The genuine Bernstein–Durrmeyer operators were introduced by Chen [10] and Goodman and Sharma [14] and were studied widely by a numbers of authors (see [3, 12, 13, 20, 23]). These operators are given by

$$U_n(f;x)=(n-1)\sum_{k=1}^{n-1}\left(\int_0^1 f(t)b_{n-2,k-1}(t)dt\right)b_{n,k}(x)+(1-x)^n f(0)+x^n f(1),\ \ f\in C[0,1].$$

These operators are limits of the Bernstein–Durrmeyer operators with Jacobi weights (see [6, 7, 21]), namely

$$U_n f=\lim_{\alpha\to-1,\beta\to-1}M_n^{<\alpha,\beta>}f, \text{ where}$$

$$M_n^{<\alpha,\beta>}:C[0,1]\to\Pi_n,\ \ M_n^{<\alpha,\beta>}(f;x)=\sum_{k=0}^{n}b_{n,k}(x)\frac{\int_0^1 w^{(\alpha,\beta)}(t)b_{n,k}(t)f(t)dt}{\int_0^1 w^{(\alpha,\beta)}(t)b_{n,k}(t)dt},$$
$$w^{(\alpha,\beta)}(t)=x^{\beta}(1-x)^{\alpha},\ x\in(0,1),\ \ \alpha,\beta>-1.$$

Also, the genuine Bernstein–Durrmeyer operators can be written as a composition of Bernstein operators and Beta operators, namely $U_n=B_n\circ\overline{\mathbb{B}}_n$. The Beta-type operators $\overline{\mathbb{B}}_n$ were introduced by A. Lupaş [19]. For $n=1,2,3,\dots$ and $f\in C[0,1]$, the explicit form of Beta operators is given by

$$\overline{\mathbb{B}}_n(f;x) := \begin{cases} f(0), \;\; x=0, \\ \dfrac{1}{B(nx, n-nx)} \displaystyle\int_0^1 t^{nx-1}(1-t)^{n-1-nx} f(t)dt, \;\; 0<x<1, \\ f(1), \;\; x=1, \end{cases}$$

where $B(\cdot,\cdot)$ is Euler's Beta function.

Our aim in this paper is to introduce genuine λ-Bernstein–Durrmeyer operators as a composition of λ-Bernstein operators and Beta operators, namely

$$U_{n,\lambda} = B_{n,\lambda} \circ \overline{\mathbb{B}}_n.$$

These operators are given in explicit form by

$$U_{n,\lambda}(f;x) = \tilde{b}_{n,0}(\lambda;x) f(0) + \tilde{b}_{n,n}(\lambda;x) f(1) + (n-1)\sum_{k=1}^{n-1} \tilde{b}_{n,k}(\lambda;x) \int_0^1 b_{n-2,k-1}(t) f(t)dt. \tag{3}$$

2 Preliminary Results

In this section by direct computation, we give the moments and the central moments of genuine λ-Bernstein–Durrmeyer operators.

Lemma 2.1 *The genuine λ-Bernstein–Durrmeyer operators verify*

(i) $U_{n,\lambda}(e_0;x) = 1$;

(ii) $U_{n,\lambda}(e_1;x) = x + \dfrac{\lambda}{n(n-1)}\left(x^{n+1} - (1-x)^{n+1} - 2x + 1\right)$;

(iii) $U_{n,\lambda}(e_2;x) = x^2 + \dfrac{2}{n(n^2-1)}\left\{x(1-x)n^2 + (-2x^2\lambda + \lambda x^{n+1} + x\lambda + x^2 - x)n + \lambda x^{n+1} - x\lambda\right\}$;

(iv) $U_{n,\lambda}(e_3;x) = x^3 + \dfrac{3}{n(n+2)(n^2-1)}\left\{2x^2(1-x)n^3 + (-2\lambda x^3 + x^2\lambda + 2x^3 + \lambda x^{n+1} - 4x^2 + 2x)n^2 + (2\lambda x^3 - 7x^2\lambda + 3\lambda x^{n+1} + 2x\lambda + 2x^2 - 2x)n + 2\lambda x^{n+1} - 2x\lambda\right\}$;

(v) $U_{n,\lambda}(e_4;x) = x^4 + \dfrac{4}{n(n+2)(n+3)(n^2-1)}\left\{3x^3(1-x)n^4 + (-2\lambda x^4 + \lambda x^3 + 3x^4 - 12x^3 + \lambda x^{n+1} + 9x^2)n^3 + (6\lambda x^4 - 18\lambda x^3 - 3x^4 + 6x^2\lambda + 15x^3 + 6\lambda x^{n+1} - 18x^2 + 6x)n^2 + (-4\lambda x^4 + 17\lambda x^3 + 3x^4 - 30x^2\lambda - 6x^3 + 11\lambda x^{n+1} + 6x\lambda + 9x^2 - 6x)n + 6\lambda x^{n+1} - 6x\lambda\right\}$.

Lemma 2.2 *The central moments of genuine λ-Bernstein–Durrmeyer operators are given below:*

(i) $U_{n,\lambda}(t-x;x)=\dfrac{\lambda}{n(n-1)}(x^{n+1}-(1-x)^{n+1}-2x+1)$;

(ii) $U_{n,\lambda}((t-x)^2;x)=2x(1-x)\left\{\lambda\left(\dfrac{x^n+(1-x)^n}{n(n-1)}-\dfrac{2}{n(n^2-1)}\right)+\dfrac{1}{n+1}\right\}$.

Lemma 2.3 *The central moments of genuine λ-Bernstein–Durrmeyer operators verify*

$$|U_{n,\lambda}(t-x;x)|\leq\theta_1(n,\lambda)\ \textit{and}\ U_{n,\lambda}((t-x)^2;x)\leq\theta_2(n,\lambda),\ \textit{for}\ n>2,$$

where $\theta_1(n,\lambda)=\dfrac{|\lambda|}{n(n-1)}$ *and* $\theta_2(n,\lambda)=\dfrac{|\lambda|+n}{2n(n+1)}$.

Lemma 2.4 *The genuine λ-Bernstein–Durrmeyer operators verify:*

(i) $\lim\limits_{n\to\infty} nU_{n,\lambda}(t-x;x)=0$;

(ii) $\lim\limits_{n\to\infty} nU_{n,\lambda}\left((t-x)^2;x\right)=2x(1-x)$;

(iii) $\lim\limits_{n\to\infty} n^2U_{n,\lambda}\left((t-x)^4;x\right)=12x^2(1-x)^2$.

3 Basic Approximation Properties

In this section, we investigate the approximation properties of these operators and we estimate the rate of convergence by using moduli of continuity.

Theorem 3.1 *If $f\in C[0,1]$, then*

$$\lim_{n\to\infty}U_{n,\lambda}(f;x)=f(x)\ \textit{uniformly on}\ [0,1].$$

Proof Using Lemma 2.1 follows that

$$\lim_{n\to\infty}U_{n,\lambda}(e_k;x)=e_k(x)\text{ uniformly on }[0,1],\ \text{ for }k\in\{0,1,2\}.$$

Applying the Bohman–Korovkin theorem, we get the result. □

Theorem 3.2 *If $f\in C[0,1]$, then*

$$\left|U_{n,\lambda}(f;x)-f(x)\right|\leq 2\omega(f;\sqrt{\theta_2(n;\lambda)}),$$

where ω is the usual modulus of continuity.

Proof Using the following property of modulus of continuity

$$|f(t) - f(x)| \leq \omega(f;\delta)\left(\frac{(t-x)^2}{\delta^2} + 1\right),$$

we obtain

$$|U_{n,\lambda}(f;x) - f(x)| \leq U_{n,\lambda}(|f(t) - f(x)|;x) \leq \omega(f;\delta)\left(1 + \frac{1}{\delta^2}U_{n,\lambda}((t-x)^2;x)\right).$$

So, if we choose $\delta = \sqrt{\theta_2(n;\lambda)}$, we have the desired result. □

Theorem 3.3 *If* $f \in C^1[0,1]$, *then*

$$\left|U_{n,\lambda}(f;x) - f(x)\right| \leq \theta_1(n;\lambda)|f'(x)| + 2\sqrt{\theta_2(n;\lambda)}\omega(f', \sqrt{\theta_2(n;\lambda)}).$$

Proof Let $f \in C^1[0,1]$. For any $x, t \in [0,1]$, we have

$$f(t) - f(x) = f'(x)(t-x) + \int_x^t \big(f'(y) - f'(x)\big)\,dy,$$

so, we get

$$U_{n,\lambda}\left(f(t) - f(x);x\right) = f'(x)U_{n,\lambda}(t-x;x) + U_{n,\lambda}\left(\int_x^t (f'(y) - f'(x))dy;x\right).$$

Using the following well-known property of modulus of continuity

$$|f(y) - f(x)| \leq \omega(f;\delta)\left(\frac{|y-x|}{\delta} + 1\right), \quad \delta > 0,$$

we have

$$\left|\int_x^t |f'(y) - f'(x)|dy\right| \leq \omega(f';\delta)\left[\frac{(t-x)^2}{\delta} + |t-x|\right].$$

Therefore,

$$|U_{n,\lambda}(f;x) - f(x)| \leq |f'(x)| \cdot |U_{n,\lambda}(t-x;x)| + \omega(f';\delta)\left\{\frac{1}{\delta}U_{n,\lambda}\left((t-x)^2;x\right) + U_{n,\lambda}(|t-x|;x)\right\}.$$

Using Cauchy–Schwartz inequality, we obtain

$$\begin{aligned}|U_{n,\lambda}(f;x)-f(x)| &\le |f'(x)||U_{n,\lambda}(t-x;x)| \\ &\quad + \omega(f',\delta)\left\{\frac{1}{\delta}\sqrt{U_{n,\lambda}\left((t-x)^2;x\right)}+1\right\}\sqrt{U_{n,\lambda}\left((t-x)^2;x\right)} \\ &\le |f'(x)|\theta_1(n;\lambda)+\omega(f',\delta)\cdot\left\{\frac{1}{\delta}\sqrt{\theta_2(n;\lambda)}+1\right\}\sqrt{\theta_2(n;\lambda)}.\end{aligned}$$

Choosing $\delta=\sqrt{\theta_2(n;\lambda)}$, we find the desired inequality. □

In order to give the next result, we recall the definition of K-functional:

$$K_2(f,\delta):=\inf\left\{\|f-g\|+\delta\left\|g''\right\|:g\in W^2[0,1]\right\},$$

where

$$W^2[0,1]=\left\{g\in C[0,1]:g''\in C[0,1]\right\},$$

$\delta\ge 0$ and $\|\cdot\|$ is the uniform norm on $C[0,1]$. The second-order modulus of continuity is defined as follows

$$\omega_2\left(f,\sqrt{\delta}\right)=\sup_{0<h\le\sqrt{\delta}}\ \sup_{x,x+2h\in[0,1]}\{|f(x+2h)-2f(x+h)+f(x)|\}.$$

It is well known that K-functional and the second-order modulus of continuity $\omega_2\left(f,\sqrt{\delta}\right)$ are equivalent, namely

$$K_2(f,\delta)\le C\omega_2\left(f,\sqrt{\delta}\right), \tag{4}$$

where $\delta\ge 0$ and $C>0$.

Theorem 3.4 *If $f\in C[0,1]$, then*

$$\left|U_{n,\lambda}(f;x)-f(x)\right|\le C\omega_2\left(f,\frac{1}{2}\sqrt{\theta_2(n;\lambda)+\theta_1^2(n,\lambda)}\right)+\omega\left(f,\theta_1(n;\lambda)\right),$$

where C is a positive constant.

Proof Denote $\nu_{n,\lambda}(x)=x+\dfrac{\lambda}{n(n-1)}\left(x^{n+1}-(1-x)^{n+1}-2x+1\right)$ and

$$\tilde{U}_{n,\lambda}(f;x)=U_{n,\lambda}(f;x)+f(x)-f(\nu_{n,\lambda}(x)). \tag{5}$$

It follows immediately

$$\tilde{U}_{n,\lambda}(e_0;x)=U_{n,\lambda}(e_0;x)=1$$

$$\tilde{U}_{n,\lambda}(e_1; x) = U_{n,\lambda}(e_1; x) + x - \nu_{n,\lambda}(x) = x.$$

Applying $\tilde{U}_{n,\lambda}$ to Taylor's formula, we get

$$\tilde{U}_{n,\lambda}(g; x) = g(x) + \tilde{U}_{n,\lambda}\left(\int_x^t (t-y)g''(y)dy; x\right).$$

Therefore,

$$\tilde{U}_{n,\lambda}(g; x) = g(x) + U_{n,\lambda}\left(\int_x^t (t-y)g''(y)dy; x\right) - \int_x^{\nu_{n,\lambda}(x)} \left(\nu_{n,\lambda}(x) - y\right) g''(y)dy.$$

This implies that

$$\begin{aligned}
|\tilde{U}_{n,\lambda}(g; x) - g(x)| &\leq \left|U_{n,\lambda}\left(\int_x^t (t-y)g''(y)dy; x\right)\right| + \left|\int_x^{\nu_{n,\lambda}(x)} \left(\nu_{n,\lambda}(x) - y\right) g''(y)dy\right| \\
&\leq U_{n,\lambda}((t-x)^2; x)\, \|g''\| + \left(\nu_{n,\lambda}(x) - x\right)^2 \|g''\| \\
&\leq \left[\theta_2(n;\lambda) + \theta_1^2(n;\lambda)\right] \|g''\|.
\end{aligned}$$

In view of (5), we obtain

$$|\tilde{U}_{n,\lambda}(f; x)| \leq |U_{n,\lambda}(f; x)| + |f(x)| + |f(\nu_{n,\lambda}(x))| \leq 3\,\|f\|. \tag{6}$$

Now, for $f \in C[0, 1]$ and $g \in W^2[0, 1]$, using (5) and (6) we get

$$\begin{aligned}
&|U_{n,\lambda}(f; x) - f(x)| = \left|\tilde{U}_{n,\lambda}(f; x) - f(x) + f(\nu_{n,\lambda}(x)) - f(x)\right| \\
&\leq \left|\tilde{U}_{n,\lambda}(f-g; x)\right| + \left|\tilde{U}_{n,\lambda}(g; x) - g(x)\right| + |g(x) - f(x)| + \left|f(\nu_{n,\lambda}(x)) - f(x)\right| \\
&\leq 4\,\|f-g\| + \left[\theta_2(n,\lambda) + \theta_1^2(n,\lambda)\right] \|g''\| + \omega\left(f, \theta_1(n,\lambda)\right).
\end{aligned}$$

Taking the infimum on the right side over all $g \in W^2[0, 1]$, we have

$$|U_{n,\lambda}(f; x) - f(x)| \leq 4K_2\left(f, \frac{1}{4}\left(\theta_2(n,\lambda) + \theta_1^2(n,\lambda)\right)\right) + \omega\left(f, \theta_1(n,\lambda)\right).$$

Finally, using the equivalence between K-functional and the second-order modulus of continuity (4), the proof is completed. □

4 Rate of Convergence in Terms of the Ditzian–Totik Modulus of Smoothness

In this section, we study the rate of convergence of genuine λ-Bernstein–Durrmeyer operators in terms of the Ditzian–Totik first-order modulus of smoothness defined as follows:

$$\omega_1^{\phi}(f;t)=\sup_{0<h\leq t}\left\{\left|f\left(x+\frac{h\phi(x)}{2}\right)-f\left(x-\frac{h\phi(x)}{2}\right)\right|, x\pm\frac{h\phi(x)}{2}\in[0,1]\right\}, \tag{7}$$

where $\phi(x)=\sqrt{x(1-x)}$ and $f\in C[0,1]$. The corresponding K-functional of the Ditzian–Totik first-order modulus of smoothness is given by

$$K_{\phi}(f;t)=\inf_{g\in W_{\phi}[0,1]}\{||f-g||+t||\phi g'||\}\ (t>0), \tag{8}$$

where $W_{\phi}[0,1]=\{g: g\in AC_{loc}[0,1], \|\phi g'\|<\infty\}$ and $AC_{loc}[0,1]$ is the class of absolutely continuous functions on every interval $[a,b]\subset[0,1]$. Between K-functional and the Ditzian–Totik first-order modulus of smoothness, there is the following relation

$$K_{\phi}(f;t)\leq C\omega_1^{\phi}(f;t), \tag{9}$$

where $C>0$ is a constant.

Theorem 4.1 *Let $f\in C[0,1]$ and $\phi(x)=\sqrt{x(1-x)}$. Then for every $x\in[0,1]$, we have*

$$|U_{n,\lambda}(f;x)-f(x)|\leq C\omega_1^{\phi}\left(f;\frac{1}{n^{1/2}}\right),$$

where C is a constant independent of n and x.

Proof From the next representation

$$g(t)=g(x)+\int_x^t g'(u)du,$$

we get

$$\left|U_{n,\lambda}(g;x)-g(x)\right|=\left|U_{n,\lambda}\left(\int_x^t g'(u)du;x\right)\right|. \tag{10}$$

For any $x\in(0,1)$ and $t\in[0,1]$, we find that

$$\left|\int_x^t g'(u)du\right|\leq||\phi g'||\left|\int_x^t\frac{1}{\phi(u)}du\right|. \tag{11}$$

But,

$$\left|\int_x^t \frac{1}{\phi(u)}du\right| = \left|\int_x^t \frac{1}{\sqrt{u(1-u)}}du\right| \le \left|\int_x^t \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}}\right)du\right| \tag{12}$$
$$\le 2\Big(\ |\sqrt{t}-\sqrt{x}\ | + |\ \sqrt{1-t}-\sqrt{1-x}\ |\Big)$$
$$= 2|t-x|\left(\frac{1}{\sqrt{t}+\sqrt{x}} + \frac{1}{\sqrt{1-t}+\sqrt{1-x}}\right)$$
$$< 2|t-x|\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}\right) \le \frac{2\sqrt{2}\ |t-x|}{\phi(x)}.$$

Combining (10)–(12) and using Cauchy–Schwarz inequality, we get

$$|U_{n,\lambda}(g;x)-g(x)| < 2\sqrt{2}||\phi g'||\phi^{-1}(x)U_{n,\lambda}(|t-x|;x)$$
$$\le 2\sqrt{2}||\phi g'||\phi^{-1}(x)\Big(U_{n,\lambda}((t-x)^2;x)\Big)^{1/2}.$$

Now using Lemma 2.3, we obtain

$$|U_{n,\lambda}(g;x)-g(x)| \le \frac{2}{n^{1/2}}\|\phi g'\|. \tag{13}$$

Using (3) and (13), we can write

$$|\ U_{n,\lambda}(f;x)-f(x)\ | \le |\ U_{n,\lambda}(f-g;x)\ | + |f(x)-g(x)| + |\ U_{n,\lambda}(g;x)-g(x)\ |$$
$$\le 2\left\{||f-g|| + \frac{1}{n^{1/2}}||\phi g'||\right\}.$$

From the definition of the K-functional (8), we get

$$|U_{n,\lambda}(f;x)-f(x)| \le 2K_\phi\left(f;\frac{1}{n^{1/2}}\right),$$

and considering the relation (9), the proof is completed. □

In the following, we give an estimate by means of Ditzian–Totik modulus of smoothness of second order defined as

$$\omega_2^\phi(f;\delta) = \sup_{0<h\le\delta}\left\{\left|f\left(x+\frac{h\phi(x)}{2}\right) - 2f(x) + f\left(x-\frac{h\phi(x)}{2}\right)\right|; x \pm \frac{h\phi(x)}{2} \in [0,1]\right\}.$$

where $f \in C[0,1]$, $\delta > 0$ and $x \in [0,1]$.

Theorem 4.2 *Let* $f \in C[0,1]$, $\lambda \in [-1,1]$ *and* $h \in \left(0, \frac{1}{2}\right]$. *For all* $x \in (0,1)$ *the relation*

$$|U_{n,\lambda}(f;x) - f(x)| \le \frac{\theta_1(n,\lambda)}{2\,h\phi(x)}\,\omega_1^{\phi}(f,2h) + \left(1 + \frac{3}{2}\frac{\theta_2(n,\lambda)}{h^2\phi^2(x)}\right)\,\omega_2^{\phi}(f,h)$$

holds.

Proof From [22, Theorem 2.5.1], we have

$$|U_{n,\lambda}(f;x) - f(x)| \le \frac{|U_{n,\lambda}(t-x;x)|}{2\,h\phi(x)}\,\omega_1^{\phi}(f,2h) + \left(U_{n,\lambda}(e_0;x) + \frac{3}{2}\frac{U_{n,\lambda}((t-x)^2;x)}{(h\phi(x))^2}\right)\omega_2^{\phi}(f,h)$$

and using Lemma 2.3, we obtain the desired estimation. □

Applying Theorem 4.2 for $h = \sqrt{\theta_2(n,\lambda)}/\phi(x)$, it follows the next result.

Corollary 4.1 *Let* $f \in C[0,1]$, $\lambda \in [-1,1]$ *and* $x \in [0,1]$. *There exist an integer* $n_0 \in \mathbb{N}$, *such that for* $n \ge n_0$, *the following relation*

$$|U_{n,\lambda}(f;x) - f(x)| \le \frac{\theta_1(n,\lambda)}{2\sqrt{\theta_2(n,\lambda)}}\,\omega_1^{\phi}\left(f, \frac{2\sqrt{\theta_2(n,\lambda)}}{\phi(x)}\right) + \frac{5}{2}\,\omega_2^{\phi}\left(f, \frac{\sqrt{\theta_2(n,\lambda)}}{\phi(x)}\right)$$

holds.

5 Voronovskaja-Type Theorems

In the following, we prove a quantitative Voronovskaja-type theorem for the operator $U_{n,\lambda}$ by means of Ditzian–Totik modulus of smoothness. Nowadays such a result has been studied for many operators and for many moduli of continuity in classical and weighted cases (see [1, 4, 15, 18]).

Theorem 5.1 *For any* $f \in C^2[0,1]$ *and* n *sufficiently large the following inequality holds*

$$\left|U_{n,\lambda}(f;x) - f(x) - \Omega_n(x;\lambda)g'(x) - \Psi_n(x;\lambda)f''(x)\right| \le \frac{1}{n}C\phi^2(x)\omega_1^{\phi}\left(f'', n^{-1/2}\right),$$

where

$$\Omega_n(x;\lambda) = \frac{\lambda}{n(n-1)}(x^{n+1} - (1-x)^{n+1} - 2x + 1);$$
$$\Psi_n(x;\lambda) = x(1-x)\left\{\lambda\left(\frac{x^n + (1-x)^n}{n(n-1)} - \frac{2}{n(n^2-1)}\right) + \frac{1}{n+1}\right\}$$

and C *is a positive constant.*

Proof For $f \in C^2[0,1]$, $t, x \in [0,1]$, by Taylor's expansion, we have

$$f(t) - f(x) = (t-x)f'(x) + \int_x^t (t-y)f''(y)dy.$$

Hence,

$$f(t) - f(x) - (t-x)f'(x) - \frac{1}{2}(t-x)^2 g''(x) = \int_x^t (t-y)f''(y)dy - \int_x^t (t-y)f''(x)dy$$
$$= \int_x^t (t-y)[f''(y) - f''(x)]dy.$$

Applying $U_{n,\lambda}(\cdot; x)$ to both sides of the above relation, we get

$$\left|U_{n,\lambda}(f;x) - f(x) - \Omega_n(x;\lambda)f'(x) - \Psi_n(x;\lambda)f''(x)\right| \leq U_{n,\lambda}\left(\left|\int_x^t |t-y||f''(y) - f''(x)|dy\right|; x\right). \tag{14}$$

The quantity $\left|\int_x^t \left|f''(y) - f''(x)\right| |t-y|dy\right|$ was estimated in [11, p. 337] as follows:

$$\left|\int_x^t |f''(y) - f''(x)||t-y|dy\right| \leq 2\|f'' - g\|(t-x)^2 + 2\|\phi g'\|\phi^{-1}(x)|t-x|^3, \tag{15}$$

where $g \in W_\phi[0,1]$.

Using Lemma 2.4 it follows that there exists a constant $C > 0$ such that for n sufficiently large

$$U_{n,\lambda}\left((t-x)^2; x\right) \leq \frac{C}{2n}\phi^2(x) \text{ and } U_{n,\lambda}\left((t-x)^4; x\right) \leq \frac{C}{2n^2}\phi^4(x). \tag{16}$$

From (14)–(16) and applying the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
&\left|U_{n,\lambda}(f;x) - f(x) - \Omega_n(x;\lambda)f'(x) - \Psi_n(x;\lambda)f''(x)\right| \\
&\leq 2\|f'' - g\|U_{n,\lambda}\left((t-x)^2; x\right) + 2\|\phi g'\|\phi^{-1}(x)U_{n,\lambda}\left(|t-x|^3; x\right) \\
&\leq \frac{C}{n}\phi^2(x)\|f'' - g\| + 2\|\phi g'\|\phi^{-1}(x)\left\{U_{n,\lambda}(t-x)^2; x\right\}^{1/2}\left\{U_{n,\lambda}\left((t-x)^4; x\right)\right\}^{1/2} \\
&\leq \frac{C}{n}\phi^2(x)\|f'' - g\| + \phi^2(x)\frac{C}{n\sqrt{n}}\|\phi h'\| \leq \frac{C}{n}\phi^2(x)\left\{\|f'' - g\| + n^{-1/2}\|\phi h'\|\right\}.
\end{aligned}$$

Taking the infimum on the right-hand side of the above relations over $g \in W_\phi[0,1]$, the theorem is proved. □

Corollary 5.1 *If* $f \in C^2[0,1]$*, then*

$$\lim_{n\to\infty} n\left\{U_{n,\lambda}(f;x) - f(x) - \Omega_n(x;\lambda)g'(x) - \Psi_n(x;\lambda)g''(x)\right\} = 0,$$

where $\Omega_n(x;\lambda)$ *and* $\Psi_n(x;\lambda)$ *are defined in Theorem 5.1.*

6 Rate of Convergence for Functions Whose Derivative Are of Bounded Variation

In this section, we study the rate of convergence of genuine λ-Bernstein–Durrmeyer operators for functions whose derivative are of bounded variation on $[0, 1]$. We mention here some of the papers in this direction [2, 5, 16, 17].

An integral representation of the operators $U_{n,\lambda}$ can be given as follows:

$$U_{n,\lambda}(f;x) = \int_0^1 \mathcal{K}_{n,\lambda}(x,t) f(t)dt, \tag{17}$$

where $\mathcal{K}_{n,\lambda}$ is defined as

$$\mathcal{K}_{n,\lambda}(x,t) = (n-1)\sum_{k=1}^{n-1} \tilde{b}_{n,k}(\lambda;x) b_{n-2,k-1}(t) + \tilde{b}_{n,0}(\lambda;x)\delta(t) + \tilde{b}_{n,n}(\lambda;x)\delta(1-t),$$

$\delta(u)$ being the Dirac delta function.

Lemma 6.1 *For a sufficiently large n and a fixed* $x \in (0,1)$*, it follows*

(i) $\eta_{n,\lambda}(x,y) = \displaystyle\int_0^y \mathcal{K}_{n,\lambda}(x,t)dt \le \frac{2(n+|\lambda|)}{n(n+1)} \cdot \frac{x(1-x)}{(x-y)^2},\ 0 \le y < x,$

(ii) $1 - \eta_{n,\lambda}(x,z) = \displaystyle\int_z^1 \mathcal{K}_{n,\lambda}(x,t)dt \le \frac{2(n+|\lambda|)}{n(n+1)} \cdot \frac{x(1-x)}{(z-x)^2},\ x < z < 1.$

Denote $DBV[0,1]$ the class of differentiable functions f defined on $[0, 1]$, whose derivatives f' are of bounded variation on $[0, 1]$. Let $\bigvee_a^b f$ be the total variation of f on $[a, b]$ and f'_x is defined by

$$f'_x(t) = \begin{cases} f'(t) - f'(x-), & 0 \le t < x \\ 0, & t = x \\ f'(t) - f'(x+) & x < t \le 1. \end{cases} \tag{18}$$

Theorem 6.1 *If* $f \in DBV[0,1]$*, then for every* $x \in (0,1)$ *and sufficiently large n, the following inequality*

$$|U_{n,\lambda}(f;x) - f(x)| \leq \sqrt{\frac{(n+|\lambda|)x(1-x)}{2n(n+1)}} \left\{|f'(x+) + f'(x-)| + |f'(x+) - f'(x-)|\right\}$$
$$+ \frac{2}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}} (f'_x)$$

holds.

Proof Since $U_{n,\lambda}(1; x) = 1$, for every $x \in (0, 1)$ we can write

$$\begin{aligned}
U_{n,\lambda}(f;x) - f(x) &= \int_0^1 \mathcal{K}_{n,\lambda}(x,t)(f(t) - f(x))dt \\
&= \int_0^x (f(t) - f(x))\,\mathcal{K}_{n,\lambda}(x,t)dt + \int_x^1 (f(t) - f(x))\,\mathcal{K}_{n,\lambda}(x,t)dt \\
&= -\int_0^x \left[\int_t^x f'(u)du\right] \mathcal{K}_{n,\lambda}(x,t)dt + \int_x^1 \left[\int_x^t f'(u)du\right] \mathcal{K}_{n,\lambda}(x,t)dt \\
&= -\mathcal{A}(x) + \mathcal{B}(x),
\end{aligned}$$

where

$$\mathcal{A}(x) = \int_0^x \left[\int_t^x f'(u)du\right] \mathcal{K}_{n,\lambda}(x,t)dt, \quad \mathcal{B}(x) = \int_x^1 \left[\int_x^t f'(u)du\right] \mathcal{K}_{n,\lambda}(x,t)dt.$$

For any $f \in DBV[0, 1]$, we decompose $f'(t)$ as follows:

$$f'(t) = \frac{1}{2}(f'(x+) + f'(x-)) + f'_x(t) + \frac{1}{2}(f'(x+) - f'(x-))sgn(t-x) \quad (19)$$
$$+ \delta_x(t)\left[f'(x) - \frac{1}{2}(f'(x+) + f'(x-))\right],$$

where

$$\delta_x(t) = \begin{cases} 1\ , & t = x \\ 0\ , & t \neq x \end{cases}.$$

Therefore, we get

$$\begin{aligned}
\mathcal{A}(x) = \int_0^x &\left\{\int_t^x \frac{1}{2}\left(f'(x+) + f'(x-)\right) + f'_x(u) + \frac{f'(x+) - f'(x-)}{2} sgn(u-x)\right. \\
&\left. + \delta_x(u)\left[f'(x) - \frac{1}{2}\left(f'(x+) + f'(x-)\right)\right] du\right\} \mathcal{K}_{n,\lambda}(x,t)dt \\
= &\frac{f'(x+) + f'(x-)}{2} \int_0^x (x-t)\mathcal{K}_{n,\lambda}(x,t)dt + \int_0^x \left[\int_t^x f'_x(u)du\right] \mathcal{K}_{n,\lambda}(x,t)dt \\
&- \frac{f'(x+) - f'(x-)}{2} \int_0^x (x-t)K_{n,\lambda}(x,t)dt
\end{aligned}$$

$$+\left[f'(x)-\frac{f'(x+)+f'(x-)}{2}\right]\int_0^x\left[\int_t^x \delta_x(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt.$$

Since $\displaystyle\int_t^x \delta_x(u)du=0$, we get

$$\begin{aligned}\mathcal{A}(x)=&\frac{f'(x+)+f'(x-)}{2}\int_0^x(x-t)\mathcal{K}_{n,\lambda}(x,t)dt+\int_0^x\left[\int_t^x f'_x(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt\\&-\frac{f'(x+)-f'(x-)}{2}\int_0^x(x-t)\mathcal{K}_{n,\lambda}(x,t)dt.\end{aligned}$$

Using a similar method, we find that

$$\begin{aligned}\mathcal{B}(x)=&\frac{f'(x+)+f'(x-)}{2}\int_x^1(t-x)\mathcal{K}_{n,\lambda}(x,t)dt+\int_x^1\left[\int_x^t f'_x(u)\right]\mathcal{K}_{n,\lambda}(x,t)dt\\&-\frac{f'(x+)-f'(x-)}{2}\int_x^1(t-x)\mathcal{K}_{n,\lambda}(x,t)dt.\end{aligned}$$

Therefore,

$$\begin{aligned}-\mathcal{A}(x)+\mathcal{B}(x)=&\frac{f'(x+)+f'(x-)}{2}\int_0^1(t-x)\mathcal{K}_{n,\lambda}(x,t)dt\\&+\frac{f'(x+)-f'(x-)}{2}\int_0^1|t-x|\mathcal{K}_{n,\lambda}(x,t)dt\\&-\int_0^x\left[\int_t^x f'_x(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt+\int_x^1\left[\int_x^t f'_x(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt.\end{aligned}$$

Then,

$$\begin{aligned}U_{n,\lambda}(f;x)-f(x)=&\frac{f'(x+)+f'(x-)}{2}\int_0^1(t-x)\mathcal{K}_{n,\lambda}(x,t)dt\\&+\frac{f'(x+)-f'(x-)}{2}\int_0^1|t-x|\mathcal{K}_{n,\lambda}(x,t)dt\\&-\int_0^x\left[\int_t^x f'_x(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt+\int_x^1\left[\int_x^t f'_x(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt.\end{aligned}$$

From the above relation, it follows

$$\begin{aligned}\left|U_{n,\lambda}(f;x)-f(x)\right|\le&\left|\frac{f'(x+)+f'(x-)}{2}\right|\left|U_{n,\lambda}(t-x;x)\right|\\&+\left|\frac{f'(x+)-f'(x-)}{2}\right|\left|U_{n,\lambda}(|t-x|;x)\right|\\&+\left|-\int_0^x\left[\int_t^x f'_x(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt\right|\end{aligned}$$

$$+\left|\int_x^1\left[\int_x^t f_x'(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt\right|. \tag{20}$$

According to Lemma 6.1, we write

$$\int_0^x\left[\int_t^x f_x'(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt=\int_0^x\left[\int_t^x f_x'(u)du\right]\frac{\partial}{\partial t}\eta_{n,\lambda}(x,t)dt=\int_0^x f_x'(t)\eta_{n,\lambda}(x,t)dt.$$

Thus,

$$\begin{aligned}\left|-\int_0^x\left[\int_t^x f_x'(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt\right| &\le \int_0^x |f_x'(t)|\eta_{n,\lambda}(x,t)dt\\ &\le \int_0^{x-\frac{x}{\sqrt{n}}}|f_x'(t)|\eta_{n,\lambda}(x,t)dt+\int_{x-\frac{x}{\sqrt{n}}}^x|f_x'(t)|\eta_{n,\lambda}(x,t)dt.\end{aligned}$$

Since $f_x'(x)=0$ and $\eta_{n,\lambda}(x,t)\le 1$, one has

$$\begin{aligned}\int_{x-\frac{x}{\sqrt{n}}}^x|f_x'(t)|\eta_{n,\lambda}(x,t)dt &=\int_{x-\frac{x}{\sqrt{n}}}^x|f_x'(t)-f_x'(x)|\eta_{n,\lambda}(x,t)dt\le\int_{x-\frac{x}{\sqrt{n}}}^x\bigvee_t^x(f_x')dt\\ &\le\bigvee_{x-\frac{x}{\sqrt{n}}}^x(f_x')\int_{x-\frac{x}{\sqrt{n}}}^x dt=\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x(f_x').\end{aligned}$$

From Lemma 6.1, we can write

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}}|f_x'(t)|\eta_{n,\lambda}(x,t)dt &\le\frac{2(n+|\lambda|)}{n(n+1)}x(1-x)\int_0^{x-\frac{x}{\sqrt{n}}}|f_x'(t)|\frac{dt}{(x-t)^2}\\ &=\frac{2(n+|\lambda|)}{n(n+1)}x(1-x)\int_0^{x-\frac{x}{\sqrt{n}}}|f_x'(t)-f_x'(x)|\frac{dt}{(x-t)^2}\\ &\le\frac{2(n+|\lambda|)}{n(n+1)}x(1-x)\int_0^{x-\frac{x}{\sqrt{n}}}\bigvee_t^x(f_x')\frac{dt}{(x-t)^2}.\end{aligned}$$

Using the change of variables $t=x-\frac{x}{u}$, we have

$$\begin{aligned}\frac{2(n+|\lambda|)}{n(n+1)}x(1-x)\int_0^{x-\frac{x}{\sqrt{n}}}\bigvee_t^x(f_x')\frac{dt}{(x-t)^2} &=\frac{2(n+|\lambda|)}{n(n+1)}(1-x)\int_1^{\sqrt{n}}\bigvee_{x-\frac{x}{u}}^x(f_x')du\\ &\le\frac{2(n+|\lambda|)}{n(n+1)}(1-x)\sum_{k=1}^{\sqrt{n}}\bigvee_{x-\frac{x}{k}}^x(f_x')\end{aligned}$$

and hence, we get

$$\left|-\int_0^x \left[\int_t^x f_x'(u)du\right] \mathcal{K}_{n,\lambda}(x,t)dt\right| \le \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} (f_x') + \frac{2(n+|\lambda|)}{n(n+1)}(1-x) \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x} (f_x')$$

$$\le \frac{2x}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x} (f_x') + \frac{2(n+|\lambda|)}{n(n+1)}(1-x) \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x} (f_x') \le \frac{2}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x} (f_x'). \tag{21}$$

Using a similar method, we get

$$\left|\int_x^1 \left[\int_x^t f_x'(u)\right] \mathcal{K}_{n,\lambda}(x,t)dt\right| \le \frac{1-x}{\sqrt{n}} \bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}} (f_x') + \frac{2(n+|\lambda|)x}{n(n+1)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{1-x}{k}} (f_x')$$

$$\le \frac{2(1-x)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{1-x}{k}} (f_x') + \frac{2(n+|\lambda|)x}{n(n+1)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{1-x}{k}} (f_x')$$

$$\le \frac{2}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{1-x}{k}} (f_x'). \tag{22}$$

The relations (20), (21), and (22) complete the proof of the theorem. □

7 Numerical Results

In this section, we will analyze the theoretical results presented in the previous sections by numerical examples.

Example 7.1 Let $f(x) = \sin(2\pi x)$ and $\lambda = 0.5$. Figure 1 is given the graphs of function f and operator $U_{n,\lambda}$ for $n = 10$ and $n = 15$, respectively. This example explains the convergence of the operators $U_{n,\lambda}$ that are going to the function f if the values of n are increasing.

Example 7.2 Let $\lambda = 1$, $f(x) = (x^2 + 3x)e^x$ and $E_{n,\lambda}(f;x) = \left|f(x) - U_{n,\lambda}(f;x)\right|$ be the error function of genuine λ-Bernstein–Durrmeyer operators. Figure 2 is given the graphs of function f and operator $U_{n,\lambda}$, for $n = 5$, $n = 7$ and $n = 10$, respectively. This example explains the convergence of the operators $U_{n,\lambda}$ that are going to the function f if the values of n are increasing. Also, the error of approximation is illustrated in Fig. 3.

Example 7.3 For $\lambda = -1$, the convergence of genuine λ-Bernstein–Durrmeyer operators to $f(x) = \left(x - \frac{1}{4}\right) sin(2\pi x)$ is illustrated in Fig. 4. Also, for $n = 5, 7, 10$ the error functions $E_{n,\lambda}$ are given in Fig. 5.

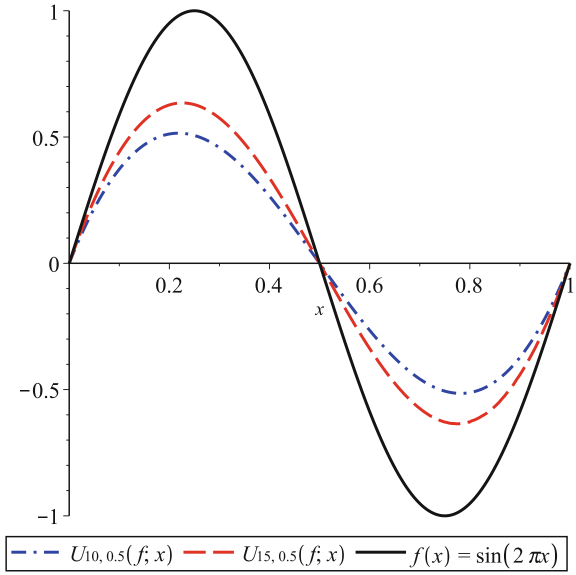


Fig. 1 Convergence of $U_{n,\lambda}(f;x)$ to $f(x)$

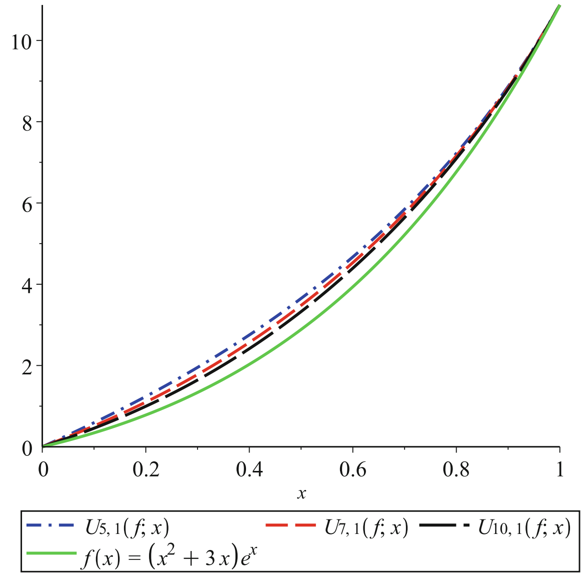


Fig. 2 Approximation process

Example 7.4 Let $f(x) = \left(x - \frac{1}{4}\right)\left(x - \frac{1}{2}\right)\left(x - \frac{3}{8}\right)$ and $n = 7$. In Fig. 6, is illustrated the convergence of genuine λ-Bernstein-Durrmeyer operator for $\lambda = -1, 0, 1$. In Fig. 7, we give the graphs of error functions. We can see that in this special case the error for genuine λ-Bernstein–Durrmeyer operators $U_{7,1}$, is smaller than for $U_{7,0}$, that is classical genuine Bernstein–Durrmeyer operator.

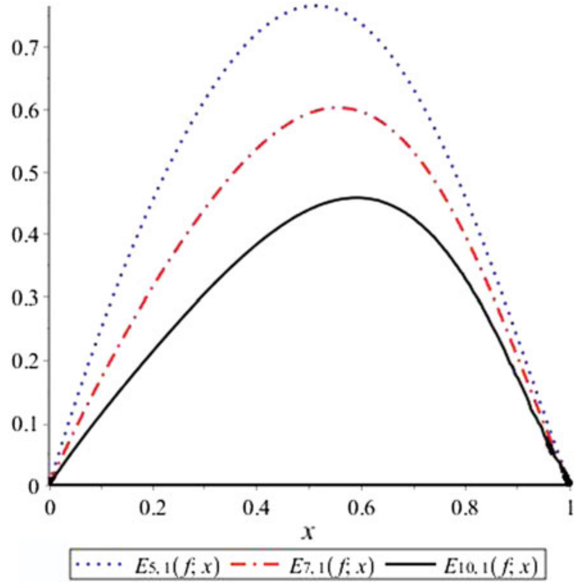


Fig. 3 Error of approximation

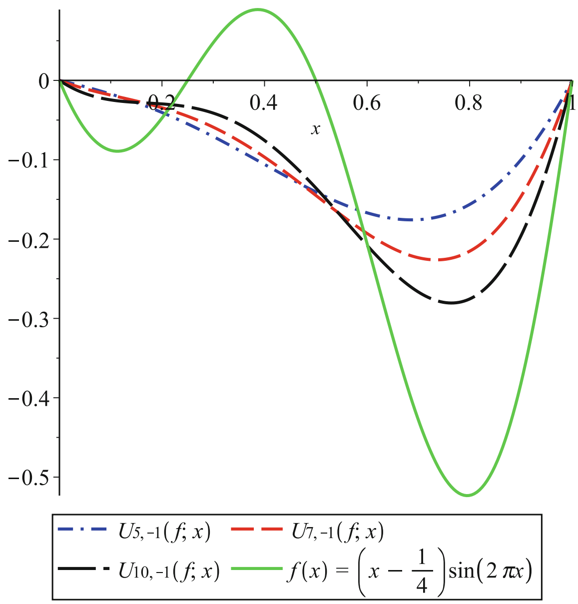


Fig. 4 Approximation process

Fig. 5 Error of approximation

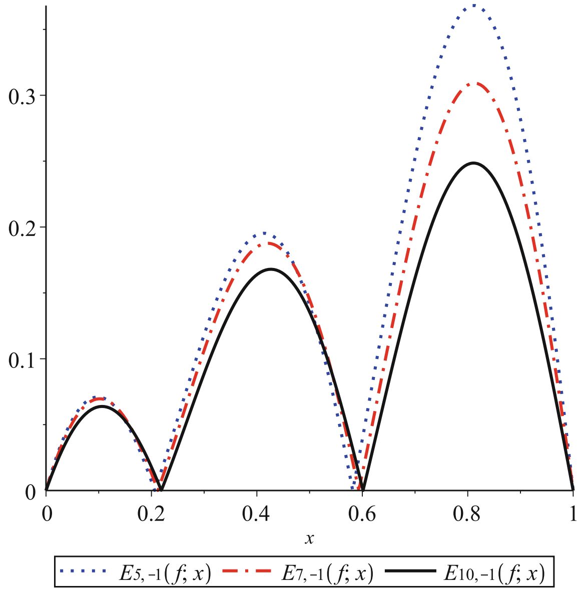
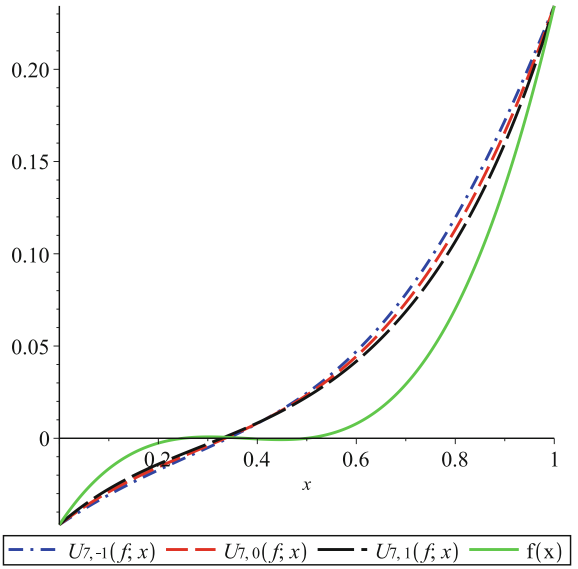


Fig. 6 Approximation process



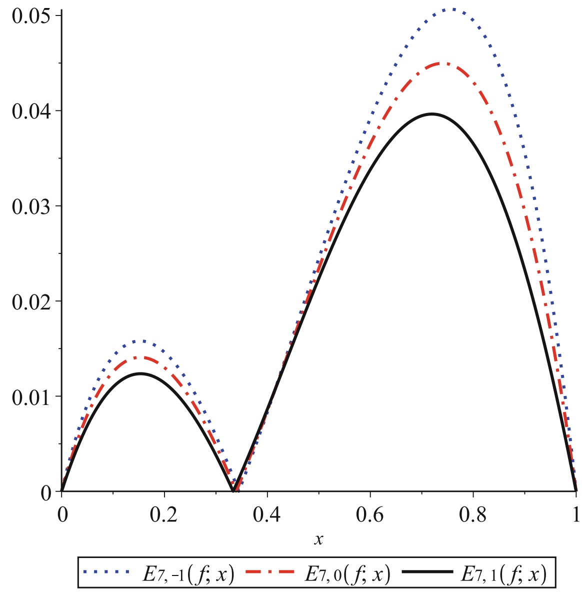


Fig. 7 Error of approximation

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