Applications of Fixed Point Theorems and General Convergence in Orthogonal Metric Spaces

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Abstract In this chapter, we discuss the general convergence methods in orthogonal metric space. Also we study the applications of fixed point theorems to obtain the existence of a solution of differential and integral equations in orthogonal metric spaces.

1 Introduction

The concept of orthogonality in normed linear spaces has been studied by Birkhoff [1–6] among others. The most natural notion of orthogonality arises in the case where there is an inner product $< ., . >$ compatible with the norm $||.||$ on a space X. In this case, $\perp$ is defined by $x \perp y$ if and only if $< x, y >= 0$. Some of the major properties of this relation are as follows:

(1) $x \perp x$ if and only if $x = 0$ for all $x \in X$,
(2) $x \perp y$ implies $\alpha x \perp y$ for all $x, y \in X, \alpha \in \mathbb{R}$ (Homogeneity),
(3) $x \perp y$ implies $y \perp x$ for all $x, y \in X$ (Symmetry),
(4) $x \perp y$ and $x \perp z$ implies $x \perp (y + z)$ for all $x, y, z \in X$ (Additivity),
(5) For every $x, y \in X, x \neq 0$, there exists a real number γ such that $x \perp (\gamma x + y)$.

For general normed linear spaces $(X, ||.||)$, Birkhoff [1] and James [5, 6] formulated definitions of orthogonality which did not require the existence of an inner product as follows:

(B) Birkhoff Orthogonality [1, 5, 6]: $(x \perp y)(B)$ provided $||x|| \leq ||x + \lambda y||$ for all $x, y \in X, \lambda \in \mathbb{R}$,
(P) Pythagorean Orthogonality [4]: $(x \perp y)(P)$ provided $||x - y||^2 = ||x||^2 + ||y||^2$ for all $x, y \in X$,

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(I) Isosceles Orthogonality [4]: $(x \perp y)(I)$ provided $||x-y|| = ||x+y||$ for all $x, y \in X$,

(α) α-Orthogonality [7]: If $\alpha \neq 1$, $(x \perp y)(\alpha)$ provided $(1+\alpha^2)||x-y||^2 = ||x-\alpha y||^2 + ||\alpha x - y||^2$ for all $x, y \in X$,

((α, β)) (α, β)-Orthogonality [8]: If $\alpha, \beta \neq 1$, $(x \perp y)(\alpha, \beta)$ provided $||x-y||^2 + ||\alpha x - \beta y||^2 = ||x - \beta y||^2 + ||y - \alpha x||^2$ for all $x, y \in X$.

It should be noted that Pythagorean and isosceles orthogonalities are particular cases of α-orthogonality, which is in turn a special case of (α, β)-orthogonality. An ordered triples of the form $(X, ||.||, \perp)$, where X is a real linear space, $||.||$ is a norm on X, and $\perp$ is an orthogonality relation on X which has the properties (1)–(5), is an inner product space if there is an inner product $< ., . >$ on X such that $< x, x >= ||x||^2$ for all $x \in X$ and $x \perp y$ if and only if $< x, y >= 0$ for all $x, y \in X$.

Proposition 1.1 ([3, Theorem I]) *If any of* $(x \perp y)(P)$, $(x \perp y)(I)$, $(x \perp y)(\alpha)$ *or* $(x \perp y(\alpha, \beta))$ *implies that* $x \perp y$, *then* $(X, ||.||, \perp)$ *is an inner product space. If* $dimX \geq 3$ *and* $(x \perp y)(B)$ *implies that* $x \perp y$, *then* $(X, ||.||, \perp)$ *is an inner product space.*

Definition 1.2 ([3]) The relation $\perp$ is said to satisfy the *norm invariance property (NIP)* provided the conditions $x \perp y$, $||x|| = ||z||$, $||y|| = ||w||$, and $||x-y|| = ||z-w||$ imply $z \perp w$.

Definition 1.3 ([3]) The relation $\perp$ is said to satisfy the *rotation invariance property (RIP)* provided the following conditions hold

(R1) If $x \perp y$, $||x|| = ||y||$ then $(ax - by) \perp (ax + by)$ for all $a, b \in \mathbb{R}$,

(R2) If $||x|| = ||y||$ then $||\gamma(x, y)x + y|| = ||\gamma(y, x)y + x||$, where $\gamma(x, y)$ and $\gamma(y, x)$ are respective real numbers from (5) such that $x \perp (\gamma(x, y)x + y)$ and $y \perp (\gamma(y, x)y + x)$.

Lemma 1.4 ([3, Theorem III]) $(X, ||.||, \perp)$ *is an inner product space if and only if* $\perp$ *satisfies NIP or RIP.*

Lemma 1.5 ([3, Lemma 1.1]) *If* $x, y \in X, x, y \neq 0$, *and* $x \perp y$, *then* x *and* y *are independent.*

We define a function $\gamma : X \times X \to R$ by

$$\gamma(x, y) = \begin{cases} 0, \ \text{if } x = 0; \\ \text{the unique } \gamma \text{ such that } x \perp (\gamma x + y), \ \text{if } x \neq 0. \end{cases}$$

From definition, we see that $\gamma(x, y) = 0$ if and only if $x \perp y$. Also $\gamma(x, y)$ have the following properties:

Lemma 1.6 ([3, Lemma 1.3])

(i) $\gamma(x, 0) = \gamma(0, x) = 0$ *for all* $x \in X$,
(ii) $\gamma(x, \lambda y) = \lambda\gamma(x, y)$ *for all* $x, y \in X, \lambda \in \mathbb{R}$,
(iii) $\gamma(\lambda x, y) = \frac{1}{\lambda}\gamma(x, y)$ *for all* $x, y \in X, \lambda \in \mathbb{R}, \lambda = neq 0$,
(iv) $\gamma(x, y) = 0$ *if and only if* $\gamma(y, x) = 0$,
(v) $\gamma(x, y + z) = \gamma(x, y) + \gamma(x, z)$ *for all* $x, y, z \in X$.

2 Orthogonal Set

The notion of orthogonal set and orthogonal metric space was introduced by Gordji et al. [9]. They gave an extension of Banach's fixed point theorem in this new structure and applied their results to prove the existence of a solution of an ordinary differential equation. Applications of fixed point theorem in orthogonal metric spaces we refer [10–12].

Definition 2.1 ([9]) Let X be a non-empty set and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ satisfies the following condition

$$\exists x_0 : \ (\forall y, y \perp x_0) \text{ or } (\forall y, x_0 \perp y)$$

then it is called an orthogonal set (written as *O-set*). We denote the orthogonal set by $(X, \perp)$. The element x_0 is called an orthogonal element of X.

Definition 2.2 Let $(X, \perp)$ be an orthogonal set. Any two elements $x, y \in X$ are said to be orthogonally related if $x \perp y$.

Example 2.3 Let $X = \mathbb{Z}$. Define $x \perp y$ if there exists $p \in \mathbb{Z}$ such that $x = py$. We see that $0 \perp x$ for all $x \in \mathbb{Z}$. Hence $(X, \perp)$ is an *O-set*.

Example 2.4 Let X be a non-empty set. Consider $P(X)$ is the power sets of X. We define $\perp$ on $P(X)$ as $A \perp B$ if $A \cap B = \phi$. We have $\phi \cap A = \phi$ for all $A \in P(X)$. Then $(P(X), \perp)$ is an orthogonal set. Similarly we can define $\perp$ on $P(X)$ as $A \perp B$ if $A \cup B = X$. Then $(P(X), \perp)$ is also an orthogonal set.

Example 2.5 Let $X = [3, \infty)$ and define $x \perp y$ if $x \leq y$. Taking $x_0 = 3$, $(X, \perp)$ is an orthogonal set.

Example 2.6 Let $X = [0, \infty)$ and define $x \perp y$ if $xy \in \{x, y\}$. By taking $x_0 = 0$ or $x_0 = 1$ $(X, \perp)$ is an orthogonal set.

Example 2.7 Let (X, d) be a metric space and $T : X \to X$ be a Picard operator; i.e., there exists $z \in X$ such that $\lim_{n\to\infty} T^n(y) = z$ for all $y \in X$. Define $x \leq y$ if $\lim_{n\to\infty} d(z, T^n(y)) = 0$. The $(X, \perp)$ is an orthogonal set.

The following example shows that the orthogonal element x_0 is not unique.

Example 2.8 Suppose M_m is the set of all $m \times m$ matrices and Y is a positive definite matrix. Define the relation $\perp$ on M_m by $A \perp B \Leftrightarrow \exists X \in M_m$ such that $AX = B$. It is easy to see that $I \perp B$, $B \perp O$ and $\sqrt{Y} \perp B$ for all $B \in M_m$, where I and O are the identity and zero matrices in M_m, respectively. Then this orthogonal relation is reflexive and transitive, but it is antisymmetry.

Example 2.9 For any $D \in M_m$, consider the orthogonal relation $\perp_D$ on M_m with respect to D defined by

$$A \perp_D B \Leftrightarrow tr(ABD) = tr(DBA).$$

We have $D \perp_D X$ for all $X \in M_m$. Then this orthogonal relation is reflexive, transitive, and symmetry.

Example 2.10 If $0 < \alpha \leq 1$, let $\Lambda_\alpha([0, 1])$ be the space of Hölder's continuous functions of the exponent α in $[0, 1]$, i.e., $f \in \Lambda_\alpha([0, 1])$ if and only if $||f||_{\Lambda_\alpha} < \infty$, where

$$||f||_{\Lambda_\alpha} = |f(0)| + \sup_{x,y\in[0,1],x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

For all $0 < \alpha \leq 1$, define $\lambda_\alpha([0, 1])$ to be the set of $f \in \Lambda_\alpha([0, 1])$ such that

$$\lim_{x\to y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} = 0 \text{ for all } x, y \in [0, 1].$$

For all $\alpha, \beta \in [0, 1]$, we define $\lambda_\alpha([0, 1]) \perp \lambda_\beta([0, 1])$ if and only if $\lambda_{\frac{\alpha-\beta}{2}}([0, 1])$ is an infinite dimensional closed subspace of $\Lambda_{\frac{\alpha-\beta}{2}}([0, 1])$. Hence $(\lambda_\alpha([0, 1]), \perp)$ is an orthogonal set.

Definition 2.11 ([9]) Let $(X, \perp)$ be an orthogonal set. A sequence (x_n) in X is called an orthogonal sequence (O-sequence) if

$$x_n \perp x_{n+1} \; or \; x_{n+1} \perp x_n, \forall n.$$

Definition 2.12 A mapping $d : X \times X \to [0, \infty)$ is called a metric on the orthogonal set $(X, \perp)$, if the following conditions are satisfied:

(O1) $d(x, y) = d(y, x)$ for any $x, y \in X$ such that $x \perp y$ and $y \perp x$,
(O2) $d(x, y) = 0$ if and only if $x = y$ for any $x, y \in X$ such that $x \perp y$ and $y \perp x$,
(O3) $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in X$ such that $x \perp y$, $y \perp z$ and $x \perp z$.

Then the ordered triple $(X, \perp, d)$ is called an orthogonal metric space.

Example 2.13 Let $X = \mathbb{Q}$. The orthogonal relation on X is defined $x \perp y$ if and only if $x = 0$ or $y = 0$. Then $(X, \perp)$ is an orthogonal set, and with the Euclidean metric, $(X, \perp, d)$ is an orthogonal metric space.

Let X be an orthogonal set and $d : X \times X \to [0, \infty)$ be a mapping. For every $x \in X$ we define the set

$$\mathcal{O}(X, d, x) = \left\{(x_n) \subset X : \lim_{n\to\infty} d(x_n, x) = 0 \text{ and } x_n \perp x, \forall n \in \mathbb{N}\right\}. \qquad (1)$$

Definition 2.14 Let $(X, \perp, d)$ be an orthogonal metric space. A sequence (x_n) in X is said to be

(i) an orthogonal convergent (in short O-convergent) to x if and only if $(x_n) \in \mathcal{O}(X, d, x)$,
(ii) an orthogonal Cauchy (in short O-Cauchy) if and only if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$ and $x_n \perp x_m$ or $x_m \perp x_n$, $\forall n, m \in \mathbb{N}$.

Remark 2.15 In an orthogonal metric space $(X, \perp, d)$, an orthogonal convergent sequence may not be an orthogonal Cauchy.

Definition 2.16 ([9]) An orthogonal metric space $(X, \perp, d)$ is said to be an orthogonal complete (O-complete) if every orthogonal Cauchy sequence converges in X.

Remark 2.17 It is easy to see that every complete metric space is orthogonal complete but the converse is not true. For this remark, see the following examples.

Example 2.18 Let $X = \mathbb{Q}$. Define $x \perp y$ if and only if $x = 0$ or $y = 0$. Then $(X, \perp)$ is an orthogonal set. It is clear that $\mathbb{Q}$ is not a complete metric space with respect to the Euclidean metric, but it is orthogonal complete. If (x_n) is any orthogonal Cauchy sequence in $\mathbb{Q}$, then there exists a subsequence (x_{n_k}) of (x_n) for which $x_{n_k} = 0$ for all $k \geq 1$. Then (x_{n_k}) converges to $0 \in X$. We know that every Cauchy sequence with a convergent subsequence is convergent, so (x_n) is convergent.

Example 2.19 Let $X = [0, 1)$ and define the orthogonal relation on X by

$$x \perp y \Leftrightarrow \begin{cases} x \leq y \leq \frac{1}{4}, \\ \text{or } x = 0. \end{cases}$$

Then $(X, \perp)$ is an orthogonal set. We have X is not a complete metric space with respect to Euclidean metric but it is orthogonal complete. Consider (x_n) is an orthogonal Cauchy sequence in X. Then there exists a subsequence (x_{n_k}) of (x_n) for which $x_{n_k} = 0$ for all $k \geq 1$, or there exists a monotone subsequence (x_{n_k}) if (x_n) for which $x_{n_k} \leq \frac{1}{4}$ for all $k \geq 1$. We see that (x_{n_k}) converges to a point $x \in [0, \frac{1}{4}] \subseteq X$. We know that every Cauchy sequence with a convergent subsequence is convergent, so (x_n) is convergent in X.

Definition 2.20 ([9]) Let $(X, \perp, d)$ be an orthogonal metric space. A function $f : X \to X$ is said to be an orthogonal continuous (O-continuous or $\perp$-continuous) at a point x_0 in X if for each orthogonal sequence (x_n) in X converging to x_0 such that $f(x_n) \to f(x_0)$. Also f is said to be orthogonal continuous on X if f is orthogonal continuous at each point on X.

Remark 2.21 It is easy to see that every continuous mapping is orthogonal continuous. The following examples show that the converse is not true in general.

Example 2.22 Let $X = \mathbb{R}$. Define the orthogonality relation on X by $x \perp y$ if and only if $x = 0$ or $y \neq 0$ in $\mathbb{Q}$. Then $(X, \perp)$ is an orthogonal set. Define a function $f : X \to X$ by

$$f(x) = \begin{cases} 2, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{Q}^c. \end{cases}$$

Then f is an orthogonal continuous but f is not continuous on $\mathbb{Q}$.

Example 2.23 Let $X = \mathbb{R}$. Define $x \perp y$ if $x, y \in \left(q + \frac{1}{7}, q + \frac{2}{7}\right)$ for some $q \in \mathbb{Z}$ or $x = 0$. Then $(X, \perp)$ is an orthogonal set. Define a function $f : X \to X$ by $f(x) = [x]$. Then f is an orthogonal continuous on X. Because for an orthogonal sequence (x_n) in X converging to $x \in X$, then we have

Case-I: If $x_n = 0$ for all n, then $x = 0$ and $f(x_n) = 0 = f(x)$.

Case-II: If $x_{n_0} \neq 0$ for some n_0, then there exists $k \in \mathbb{Z}$ such that $x_n \in \left(k + \frac{1}{7}, k + \frac{2}{7}\right)$ for all $n \geq n_0$. Then $x \in \left[q + \frac{1}{7}, q + \frac{2}{7}\right]$ and $f(x_n) = k = f(x)$. It follows that f is orthogonal continuous on X but it is not continuous on X.

If $X = \mathbb{R}^n$ be a standard inner product space, then the Remark 2.21 is false. It follows from the following theorem.

Lemma 2.24 ([13]) *Let $X = \mathbb{R}^n$ be a standard inner product space and $T : X \to X$ be a mapping, where $T(x) = (T_i(x), T_2(x), \ldots, T_n(x))$ for all $x \in X$, and each T_i is a mapping from $\mathbb{R}^n$ to $\mathbb{R}$ for all $i = 1, 2, \ldots, n$. Then T is continuous at $y = (y_1, y_2, \ldots, y_n)$ if and only if T_i is continuous at y for each $i = 1, 2, \ldots, n$.*

Theorem 2.25 ([14, Theorem 2.1]) *Let $(X, \perp, < ., . >)$ be an orthogonal inner product space, where $X = \mathbb{R}^n$, $< ., . >$ denotes the standard inner product space and $\perp$ is an orthogonal relation on X defined by $x \perp y$ if $< x, y >= 0$ for all $x, y \in X$. Then $f : X \to X$ is orthogonal continuous on X if and only if f is continuous on X.*

Proof Given that $(X, \perp, < ., . >)$ is an orthogonal inner product space, where $X = \mathbb{R}^n$. The orthogonality relation $\perp$ on X is defined by $x \perp y$ if $< x, y >= 0$. Suppose (x_k) be a Cauchy orthogonal sequence converging to x, where $x_k = (x_1^k, x_2^k, \ldots, x_n^k)$ and $x = (x_1, x_2, \ldots, x_n)$. Suppose that $f : X \to X$ is an orthogonal continuous function at x. To show that f is continuous at x.

For any $x, y \in X$, the distance function $d(x, y)$ induced by the inner product is given by

$$d(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{\frac{1}{2}}.$$

Since f is orthogonal continuous at x then for any orthogonal sequence (x_k) converging to x, we have

$$
\begin{aligned}
& \lim_{k\to\infty} d(f(x_k), f(x)) = 0 \\
\Rightarrow & \lim_{k\to\infty} \left[\sum_{i=1}^{n}(f_i(x_k) - f_i(x))^2\right]^{\frac{1}{2}} = 0 \\
\Rightarrow & \lim_{k\to\infty} (f_i(x_k) - f_i(x))^2 = 0, \text{ for each } i \\
\Rightarrow & f_i(x_k) \to f_i(x), \text{ for each } i, \text{ as } k \to \infty \\
\Rightarrow & f_i \text{ is continuous at } x, \text{ for each } i = 1, 2, \dots n \\
\Rightarrow & f \text{ is continuous at } x, \text{(Lemma 2.24)}.
\end{aligned}
$$

Since x is arbitrary, so f is continuous on X.

Conversely, if f is continuous on X, it is easy to show that f is orthogonal continuous on X.

3 Orthogonal Contractions

Definition 3.1 ([9]) Let $(X, \perp, d)$ be an orthogonal metric space and $0 < K < 1$. A mapping $T : X \to X$ is called an orthogonal contraction (O-contraction or $\perp$-contraction) with Lipschitz constant K, if for all $x, y \in X$ with $x \perp y$ then $d(Tx, Ty) \leq Kd(x, y)$

Remark 3.2 It is clear that every contraction is orthogonal contraction but the converse is not true.

Example 3.3 Let $X = [0, 10)$ and d be the Euclidean metric on X. Define $x \perp y$ if $xy \leq x \ or \ y$. Let $F : X \to X$ be a map defined by

$$
F(x) = \begin{cases} \frac{x}{4}, & \text{if } x \leq 4, \\ 0, & \text{if } x > 4. \end{cases}
$$

Let $x \perp y$ and $xy \leq x$ then we have

Case:1 If $x = 0$ and $y \leq 4$ then $F(x) = 0$ and $F(y) = \frac{y}{4}$.

Case:2 If $x = 0$ and $y > 4$ then $F(x) = F(y) = 0$.

Case:3 If $y \leq 3$ and $x \leq 4$ then $F(y) = \frac{y}{4}$ and $F(x) = \frac{x}{4}$.

Case:4 If $y \leq 1$ and $x > 4$ then $x - y > y$, $F(y) = \frac{y}{4}$, ad $F(x) = 0$.

Therefore we have $|F(x) - F(y)| \leq \frac{1}{4}|x - y|$, and hence, F is an orthogonal contraction. But F is not a contraction, because for each $K < 1$ then $|F(5) - F(4)| = 1 > K = K|5 - 4|$.

Example 3.4 Let $X = [0, 1)$ and d be the Euclidean metric on X. Define $x \perp y$ if $xy \in \{x, y\}$ for all $x, y \in X$. Let $F : X \to X$ be a mapping defined by

$$
F(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in \mathbb{Q} \cap X, \\ 0, & \text{if } x \in \mathbb{Q}^c \cap X. \end{cases}
$$

Then F is an orthogonal contraction on X but it is not a contraction.

Definition 3.5 ([9]) Let $(X, \perp, d)$ be an orthogonal metric space. A mapping $T : X \to X$ is said to be an orthogonal preserving (or $\perp$-preserving or O-preserving) if $x \perp y$ implies $Tx \perp Ty$ for all $x, y \in X$.

Definition 3.6 ([9]) Let $(X, \perp, d)$ be an orthogonal metric space. A mapping $T : X \to X$ is said to be a weakly orthogonal preserving (or weakly $\perp$-preserving or weakly O-preserving) if $x \perp y$ implies $Tx \perp Ty$ or $Ty \perp Tx$ for all $x, y \in X$.

Example 3.7 ([9]) Let X be the set of all peoples in the world. We define $x \perp y$ if x can give blood to y. According to the following table, if x_0 is a person such that his/her blood type is O^-, then we have $x_0 \perp y$ for all $y \in X$. Then $(X, \perp)$ is an orthogonal set. In the following, we see that in this orthogonal set x_0 is not unique.

Type	*You can give blood to*	*You can receive blood from*
A^+	A^+, AB^+	A^+, A^-, O^+, O^-
O^+	O^+, A^+, B^+, AB^+	O^+, O^-
B^+	B^+, AB^+	$B^+, B^-, O^+.O^-$
AB^+	AB^+	*Everyone*
A^-	$A^+.A^-, AB^+, AB^-$	A^-, O^-
O^-	*Everyone*	O^-
B^-	B^+, B^-, AB^+, AB^-	B^-, O^-
AB^-	AB^+, AB^-	AB^-, B^-, O^-, A^-

Remark 3.8 We have every orthogonal preserving mapping is weakly preserving, but the converse is not true.

For this let $(X, \perp)$ be an orthogonal set defined in the Example 3.7. Let O_1 in X be a person with blood type O^-; P_1 be a person with blood type A^+. Define a mapping $F : X \to X$ by

$$F(x) = \begin{cases} P_1, & \text{if } x = O_1, \\ O_1, & \text{if } x \in X - \{O_1\}. \end{cases}$$

Let $O_2 \in X - \{O_1\}$ be a person with blood type O^-. Then we get $O_1 \perp O_2$ but we do not have $F(O_1) \perp F(O_2)$. Therefore F is not an orthogonal preserving but it is weakly orthogonal preserving.

Theorem 3.9 ([9, Theorem 3.11]) *Let $(X, \perp, d)$ be an orthogonal complete metric space (not necessarily complete metric space) and $0 < k < 1$. Let $T : X \to X$ be an orthogonal continuous, orthogonal contraction with Lipschitz constant k, and orthogonal preserving. Then T has a unique fixed point $\bar{x} \in X$. Also T is a Picard operator, i.e., $\lim_{n\to\infty} T^n(x) = \bar{x}$ for all $x \in X$.*

Proof Given that $(X, \perp, d)$ is an orthogonal metric space. Therefore by the definition of orthogonality, there exists an element $x_0 \in X$ such that $x_0 \perp y$ or $y \perp x_0$ for all $y \in X$.

It follows that $x_0 \perp Tx_0$ or $Tx_0 \perp x_0$. Let

$$x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \dots, x_{n+1} = Tx_n = T^{n+1}x_0, \forall n \in \mathbb{N}.$$

Since T is orthogonal preserving, (x_n) is an orthogonal sequence in X. Also since T is an orthogonal contraction, so we have

$$d(x_n, x_m) \leq k^n d(x_0, x_1), \forall n \in \mathbb{N}.$$

If $m, n \in \mathbb{N}$ and $m \geq n$ we get

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + \cdots + k^{m-1} d(x_0, x_1) \\ &\leq \frac{k^n}{1-k} d(x_0, x_1). \end{aligned}$$

Since $0 < k < 1$ and $d(x_0, x_1)$ is fixed, $d(x_n, x_m) \to 0$ as $m, n \to \infty$. Therefore (x_n) is an orthogonal Cauchy sequence in X. Since X is orthogonal complete, there exists $\bar{x} \in X$ such that $x_n \to \bar{x}$. Again since T is orthogonal continuous, therefore $T(x_n) \to T(\bar{x})$ and $T(\bar{x}) = \lim_n (Tx_n) = \lim_n x_{n+1} = \bar{x}$. Hence $\bar{x}$ is a fixed point of T.

Next we prove that the uniqueness of $\bar{x}$. Let $\bar{y}$ be another fixed point of T. Then we have $T^n(\bar{x}) = \bar{x}$ and $T^n(\bar{y}) = \bar{y}$ for all $n \in \mathbb{N}$. By the definition of orthogonality, we have

$$x_0 \perp \bar{x} \text{ and } x_0 \perp \bar{y}$$

or

$$\bar{x} \perp x_0 \text{ and } \bar{y} \perp x_0.$$

Since T is orthogonal preserving, we have

$$T^n(x_0) \perp T^n(\bar{x}) \text{ and } T^n(x_0) \perp T^n(\bar{y})$$

or

$$T^n(\bar{x}) \perp T^n(x_0) \text{ and } T^n(\bar{y}) \perp T^n(x_0), \ \forall n \in \mathbb{N}.$$

Now by triangular inequality, we have

$$\begin{aligned} d(\bar{x}, \bar{y}) &= d(T^n(\bar{x}), T^n(\bar{y})) \\ &\leq d(T^n(\bar{x}), T^n(x_0)) + d(T^n(x_0), T^n(\bar{y})) \end{aligned}$$

$$\leq k^n d(\bar{x}, x_0) + k^n d(x_0, \bar{y})$$
$$\to 0 \text{ as } n \to \infty.$$

This shows that $\bar{x} = \bar{y}$.

Finally let $x \in X$ be arbitrary. Similarly we have

$$x_0 \perp \bar{x} \text{ and } x_0 \perp x$$

or

$$\bar{x} \perp x_0 \text{ and } x \perp x_0.$$

Since T is orthogonal preserving, we have

$$T^n(x_0) \perp T^n(\bar{x}) \text{ and } T^n(x_0) \perp T^n(x)$$

or

$$T^n(\bar{x}) \perp T^n(x_0) \text{ and } T^n(x) \perp T^n(x_0), \ \forall n \in \mathbb{N}.$$

Thus for all $n \in \mathbb{N}$ we have

$$\begin{aligned} d(\bar{x}, T^n(x)) &= d(T^n(\bar{x}), T^n(x)) \\ &\leq d(T^n(\bar{x}), T^n(x_0)) + d(T^n(x_0), T^n(x)) \\ &\leq k^n d(\bar{x}, x_0) + k^n d(x_0, x) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

This completes the proof.

Corollary 3.10 (Banach's Contraction Principle) *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that for some $k \in (0, 1)$, $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$. Then T has a unique fixed point in X.*

Proof Suppose that

$$x \perp y \Leftrightarrow d(Tx, Ty) \leq d(x, y).$$

For fix $x_0 \in X$. Since T is a contraction, so for all $y \in X$, $x_0 \perp y$. Hence $(X, \perp)$ is an orthogonal set. It is clear that X is an orthogonal complete and T is an orthogonal contraction, orthogonal continuous, and orthogonal preserving. Then by Theorem 3.9, T has a fixed point in X.

The following example shows that Theorem 3.9 is a real extension of Banach's fixed point theorem.

Example 3.11 Suppose that $(X = [0, 9), \perp, d)$ and $T : X \to X$ is defined by

$$T(x) = \begin{cases} \frac{x}{3}, & \text{if } x \leq 3, \\ 0, & \text{if } x > 3. \end{cases}$$

Then X is orthogonal complete (but not complete), and T is orthogonal continuous (not continuous on X), orthogonal contraction, and orthogonal preserving on X. Therefore by Theorem 3.9 T has a fixed point in X. However T is not a contraction on X, so by Banach's contraction principle, we cannot find any fixed point of T on X.

4 Applications to Ordinary Differential Equations

We apply Theorem 3.9 to prove the existence of a solution for the following differential equation:

$$\begin{cases} u'(t) = f(t, u(t)), \text{ a.e. } t \in I = [0, T] \\ u(0) = a, a \geq 1 \end{cases}, \tag{1}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is an integrable function satisfying the following conditions:

(C1) $f(s, x) \geq 0$ for all $x \geq 0$ and $s \in I$,
(C2) there exists $\alpha \in L^1(I)$ such that

$$|f(s, x) - f(s, y)| \leq \alpha(s)|x - y|$$

for all $s \in I$ and $x, y \geq 0$ with $xy \geq x$ or y.

It is clear that the function $f : I \times \mathbb{R} \to \mathbb{R}$ is not necessarily Lipschitz from the condition (C2). We consider the function

$$f(s, x) = \begin{cases} sx, & \text{if } x \leq \frac{1}{3}, \\ 0, & \text{if } x > \frac{1}{3}. \end{cases}$$

which satifies the conditions (C1) and (C2) but f is not continuous and monotone. For $s \neq 0$ we have

$$\left| f\left(s, \frac{1}{3}\right) - f\left(s, \frac{2}{5}\right) \right| = s\frac{1}{3} > s\frac{1}{15} = s\left|\frac{1}{3} - \frac{2}{5}\right|.$$

Theorem 4.1 ([9, Theorem 4.1]) *Under the conditions (C1) and (C2), for all $T > 0$, the differential equation (1) has a unique positive solution.*

Proof Let $X = \{u \in C(I, \mathbb{R}) : u(t) > 0, \forall t \in I\}$. We consider the orthogonality relation in X as

$$x \perp y \Leftrightarrow x(t)y(t) \geq x(t) \text{ or } y(t), \ \forall t \in I.$$

Let $S(t) = \int_0^t |\alpha(s)| ds$. Then we have $S'(t) = |\alpha(t)|$ for almost every $t \in I$. Define

$$||x||_A = \sup_{x \in I} e^{-S(t)}|x(t)|, \quad d(x, y) = ||x - y||_A, \forall x, y \in X.$$

It is easy to show that (X, d) is a metric space.

We show that X is an orthogonal complete (not necessarily complete) metric space. Consider (x_n) is an orthogonal Cauchy sequence in X. It is easy to show that (x_n) is convergent to a point $x \in C(I)$. It is enough to show that $x \in X$. For $t \in I$ by the definition of $\perp$ we have

$$x_n(t)x_{n+1}(t) \geq x_n(t) \text{ or } x_{n+1}(t) \text{ for each } n \in \mathbb{N}.$$

Since $x_n(t) > 0$ for each $n \in \mathbb{N}$, there exists a subsequence (x_{n_k}) of (x_n) for which $x_{n_k}(t) \geq 1$ for each $k \in \mathbb{N}$. By the convergence of this sequence to a real number, $x(t)$ implies that $x(t) \geq 1$. Since $t \in I$ is arbitrary, so we have $x \in X$.

Define a mapping $F : X \to X$ by

$$F(u(t)) = \int_0^t f(s, u(t))ds + a.$$

The fixed point of F is the solution of the Eq. (1). For this, we need to prove the following steps.

Step-I: F is orthogonal preserving: For all $x, y \in X$ with $x \perp y$ and $t \in I$ we have

$$F(u(t)) = \int_0^t f(s, u(t))ds + a \geq 1$$

which implies that $Fx(t)Fy(t) \geq Fx(t)$ and so $Fx \perp Fy$.

Step-II: F is orthogonal contraction: For all $x, y \in X$ with $x \perp y$ and $t \in I$, the condition (C2) implies that

$$\begin{aligned} e^{-S(t)}|Fx(t) - Fy(t)| &\leq e^{-S(t)} \int_0^t |f(s, x(s)) - f(s, y(s))|ds \\ &\leq e^{-S(t)} \int_0^t |\alpha(s)|e^{S(s)}e^{-S(s)}|x(s) - y(s)|ds \\ &\leq e^{-S(t)} \left(\int_0^t |\alpha(s)|e^{S(s)}ds \right) ||x - y||_A \\ &\leq e^{-S(t)}(e^{S(t)} - 1)||x - y||_A \\ &\leq (1 - e^{-||\alpha||_1})||x - y||_A, \end{aligned}$$

so we have

$$||Fx - Fy||_A \leq (1 - e^{-||\alpha||_1})||x - y||_A.$$

Since $1 - e^{-||\alpha||_1} < 1$, so F is an orthogonal contraction.

Step-III: F is orthogonal continuous: Let (x_n) be an orthogonal sequence in X converging to a point $x \in X$. So we see that $x(t) \geq 1$ for all $t \in I$ and hence $x_n \perp x$ for all $n \in \mathbb{N}$. By condition (C2) we have

$$e^{-S(t)}|Fx_n(t) - Fx(t)| \leq e^{-S(t)} \int_0^t |f(s, x_n(s)) - f(s, x(s))|ds$$
$$\leq 1 - e^{-||\alpha||_1}||x_n - x||_A, \forall n \in \mathbb{N} \text{ and } t \in I.$$

Hence

$$||Fx_n - Fx||_A \leq (1 - e^{-||\alpha||_1})||x_n - x||_A, \forall n \in \mathbb{N}.$$

Therefore $Fx_n \to Fx$.

The uniqueness of the solution follows from Theorem 3.9. This completes the proof.

5 Generalized Metric

A mapping $D : X \times X \to [0, \infty]$ is called a generalized metric on a non-empty set X, if the following conditions are satisfied:

1. $D(x, y) = D(y, x)$ for $x, y \in X$,
2. $D(x, y) = 0 \Leftrightarrow x = y$ for $x, y \in X$,
3. $D(x, z) \leq D(x, y) + D(y, z)$ for $x, y, z \in X$ considering that if $D(x, y) = \infty$ or $D(y, z) = \infty$ then $D(x, y) + D(y, z) = \infty$.

Then the pair (X, D) is called a generalized metric space.

Definition 5.1 [15] A mapping $D : X \times X :\to [0, \infty]$ is called a generalized metric on the orthogonal set $(X, \perp)$, if it satisfy the following conditions:

(GO1) $D(x, y) = D(y, x)$ for any points $x, y \in X$ such that $x \perp y$ and $y \perp x$,
(GO2) $D(x, y) = 0 \Leftrightarrow x = y$ for any points $x, y \in X$, $x \perp y$ and $y \perp x$,
(GO3) $D(x, z) \leq D(x, y) + D(y, z)$ for any points $x, y, z \in X$, $x \perp y$, $y \perp z$, and $x \perp z$, considering that if $D(x, y) = \infty$ or $D(y, z) = \infty$ then $D(x, z) = \infty$.

Then the ordered triple $(X, \perp, D)$ is called generalized orthogonal metric space.

The concept of completeness of a generalized orthogonal metric space is defined in the usual way.

Theorem 5.2 ([15, Theorem 3.2]) *Let $(X, \perp, D)$ be a generalized orthogonal complete metric space. Let $T : X \to X$ be an orthogonal preserving and orthogonal continuous map such that*

(1) $D(Tx, Ty) \leq \lambda D(x, y)$ *for any points* $x, y \in X$ *such that* $x \perp y$ *and* $0 < \lambda < 1$,
(2) *For any* $x \in X$ *there exists* n_0 *such that for* $(T, \perp)$*-orbit* $(T^n x)_{n=0}^{\infty}$ *we have* $D(T^{n_0}x, T^{n_0+1}x) < \infty$,
(3) *If* $x \perp y$, $Tx = x$ *and* $Ty = y$ *then* $D(x, y) < \infty$.

Then there exists a unique fixed point $\bar{x}$ *of the map* T *and* $\lim_{n\to\infty} T^n x = \bar{x}$ *for any* $x \in X$.

Proof Consider the $(T, \perp)$-orbit $(T^n x)_{n=0}^{\infty}$ of an arbitrary point $x \in X$. Suppose that

$$x \perp Tx, Tx \perp T^2x, T^2x \perp T^3x, \ldots, T^n x \perp T^{n+1}x, \ldots.$$

By the given condition (2), we find n_0 such that $D(T^{n_0}x, T^{n_0+1}x) < \infty$. Then for $n \geq n_0$ we have

$$\begin{aligned} D(T^n x, T^{n+1}x) &\leq \lambda D(T^{n-1}x, T^n x) \\ &\leq \lambda^2 D(T^{n-2}x, T^{n-1}x) \\ &\leq \lambda^3 D(T^{n-3}x, T^{n-2}x) \\ &\vdots \\ &\leq \lambda^{n-n_0} D(T^{n_0}x, T^{n_0+1}x) \end{aligned}$$

and

$$\begin{aligned} D(T^n x, T^{n+m}x) &\leq D(T^n x, T^{n+1}x) + D(T^{n+1}x, T^{n+2}x) + \cdots + D(T^{n+m-1}x, T^{n+m}x) \\ &\leq \lambda^{n-n_0} D(T^{n_0}x, T^{n_0+1}x) + \cdots + \lambda^{n+m-1-n_0} D(T^{n_0}x, T^{n_0+1}x) \\ &= \left[\lambda^{n-n_0} + \lambda^{n+1-n_0} + \cdots + \lambda^{n+m-1-n_0} D(T^{n_0}x, T^{n_0+1}x)\right] \\ &\leq \frac{\lambda^{n-n_0}}{1-\lambda} D(T^{n_0}x, T^{n_0+1}x). \end{aligned}$$

Therefore the $(T, \perp)$-orbit $(T^n x)_{n=0}^{\infty}$ is a Cauchy sequence in X, and by the completeness of X, it converges to a point $\bar{x} \in X$. Since T is an orthogonal continuous, so $\bar{x}$ is a fixed point of T. Suppose that $x \perp y$, $Tx = x$ and $Ty = y$, then by the given condition (3) we have $D(x, y) < \infty$, and by condition (1) we get

$$D(x, y) = D(Tx, Ty) \leq \lambda D(x, y)$$

which is a contradiction. So the fixed point is unique, This completes the proof.

Definition 5.3 ([16]) Let $(X, \perp)$ be an orthogonal set. A sequence (x_n) in X is called a strongly orthogonal (SO-orthogonal) if

$$x_n \perp x_{n+m} \text{ or } x_{n+m} \perp x_n, \forall n, m \in \mathbb{N}.$$

Remark 5.4 Every strongly orthogonal sequence is an orthogonal sequence, but the converse is not true.

Example 5.5 Let $X = \mathbb{Z}$. Define the orthogonal relation on X by $x \perp y$ if and only if $xy \in \{x, y\}$. Consider a sequence (x_n) in X as follows

$$x_n = \begin{cases} 3, \text{ if } n = 2k, \text{ for some } k \in \mathbb{Z}, \\ 1, \text{ if } n = 2k+1 \text{ for some } k \in \mathbb{Z}. \end{cases}$$

Then we have $x_n \perp x_{n+1}$ for all $n \in \mathbb{N}$, but x_{2n} is not orthogonal to x_{4n}. So (x_n) is an orthogonal sequence but not a strongly orthogonal sequence.

Definition 5.6 An orthogonal metric space $(X, \perp, d)$ is called strongly orthogonal complete (SO-complete) if every strongly orthogonal Cauchy sequence is convergent.

Remark 5.7 Every complete metric space is strongly orthogonal complete, but the converse is not true.

Example 5.8 Consider $X = \{x \in C([0, 1], \mathbb{R}) : x(t) > 0, \forall t \in [0, 1]\}$. Then X is an incomplete metric space with the supremum norm $||x|| = \sup_{t \in [0,1]} |x(t)|$. Define the orthogonal relation $\perp$ on X by

$$x \perp y \iff x(t)y(t) \geq \max_{t \in [0,1]} \{x(t), y(t)\}.$$

Then X is strongly orthogonal complete. If (x_n) is a strongly orthogonal Cauchy sequence in X, then for all $n \in \mathbb{N}$ and $t \in [0, 1]$, $x_n(t) \geq 1$. Since $C([0, 1], \mathbb{R})$ is a Banach space with the supremum norm, so we can find an element $x \in C([0, 1], \mathbb{R})$ for which $||x_n - x|| \to 0$ as $n \to \infty$. Since uniformly convergent implies the pointwise convergent. Thus $x(t) \geq 1$ for all $t \in [0, 1]$ and hence $x \in X$.

Remark 5.9 Every orthogonal complete metric space is strongly orthogonal complete, but the converse is not true.

Example 5.10 Suppose $X = [1, \infty)$ with the Euclidean metric and the orthogonal relation on X is defined by $x \perp y \iff xy \in \{x, y\}$. Let (x_n) be a strongly orthogonal Cauchy sequence in X, by the definition of $\perp$ we have $x_n = 1$ for all $n \in \mathbb{N}$. Therefore (x_n) converges to 1. Consider a sequence

$$x_n = \begin{cases} 0, \text{ if } n = 2k, \text{ for some } k \in \mathbb{Z}, \\ k+1, \text{ if } n = 2k+1 \text{ for some } k \in \mathbb{Z}. \end{cases}$$

Then (x_n) is an orthogonal sequence but it is not convergent to any element in X.

Definition 5.11 ([16]) Let $(X, \perp, d)$ is an orthogonal metric space. A mapping $T : X \to X$ is called strongly orthogonal continuous (SO-continuous) at $x_0 \in X$, for each strongly orthogonal sequence (x_n) in X if $x_n \to x_0$ then $T(x_n) \to T(x_0)$. Also T is called strongly orthogonal continuous on X if it is strongly orthogonal continuous at each point of X.

Remark 5.12 Every continuous mapping is orthogonal continuous and every orthogonal continuous mapping is strongly orthogonal continuous, but the converse is not true; i.e., every continuous mapping is strongly orthogonal continuous but the converse is not true.

Example 5.13 Let $X = \mathbb{R}$ with the Euclidean metric. Suppose the orthogonal relation $\perp$ as $x \perp y \iff xy \in \{x, y\}$. Define a function $T : X \to X$ by

$$T(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ \frac{1}{x^2}, & \text{if } x \in \mathbb{Q}^c. \end{cases}$$

Then T is not continuous, but T is strongly orthogonal continuous, because we consider $x_n \in \mathbb{Q}$ for enough large n. Then we have $T(x_n) = 1 \to x = 1$. Now we consider a sequence

$$x_n = \begin{cases} 1, & \text{if } n = 2k, \text{ for some } k \in \mathbb{Z}, \\ \frac{\sqrt{2}}{k}, & \text{if } n = 2k+1 \text{ for some } k \in \mathbb{Z}. \end{cases}$$

Then we see that $x_n \to 0$ but the sequence $(T(x_n))$ is not convergent to $T(0)$. So T is not orthogonal continuous.

Definition 5.14 Let $(X, \perp, d)$ be a strongly orthogonal complete metric space. A mapping $T : X \to X$ is called strongly orthogonal Meir–Keeler contraction if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$x \neq y, x \perp y \text{ and } \varepsilon \leq d(x, y) < \varepsilon + \delta(\varepsilon) \implies d(Tx, Ty) < \varepsilon. \qquad (1)$$

Theorem 5.15 *Let $(X, \perp, d)$ be a strongly orthogonal complete metric space (not necessarily complete) with an orthogonal element x_0. Suppose that $T : X \to X$ is orthogonal preserving, strongly orthogonal continuous such that satisfying the strongly orthogonal Meir–Keeler contraction. Then T has a unique fixed point $z \in X$. Also T is a Picard operator, i.e., for all $x \in X$, the sequence $(T^n(x))$ is convergent to z with respect to the metric d.*

Proof By the definition of orthogonality, we have

$$x_0 \perp y \text{ or } y \perp x_0, \forall y \in X.$$

It follows that $x_0 \perp Tx_0$ or $Tx_0 \perp x_0$. Put

$$x_1 = Tx_0, x_2 = T(x_1) = T^2(x_0), \ldots, x_{n+1} = T(x_n) = T^{n+1}(x_0), \forall n \in \mathbb{N}.$$

We have

$$x_0 \perp x_n \text{ or } x_n \perp x_0, \forall n \in \mathbb{N}.$$

Since T is orthogonal preserving, so we get

$$x_m = T^m(x_0) \perp T^m(x_n) = x_{n+m} \text{ or } x_{n+m} = T^m(x_n) \perp T^m(x_0) = x_m, \; \forall n, m \in \mathbb{N}.$$

This gives that (x_n) is a strongly orthogonal sequence.

We divide the proof in the following steps:

Step-I: To show that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

If there exists $m_0 \in \mathbb{N}$, $x_{m_0} = x_{m_0+1}$ then the result is obvious. Let $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then by the Meir–Keeler condition, we have

$$d(x_{n+1}, x_n) < d(x_n, x_{n-1}), \forall n \in \mathbb{N}.$$

This shows that the sequence $(d(x_{n+1}, x_n))$ is strictly decreasing and it converges. Put $\lim_{n\to\infty} d(x_{n+1}, x_n) = t$. We prove that $t = 0$. Suppose that $t > 0$. Using the Meir–Keeler condition for $t > 0$, we can find $\delta(t) > 0$ such that

$$x \neq y, x \perp y \text{ and } t \leq d(x, y) < t + \delta(t) \implies d(Tx, Ty) < t.$$

Since $\lim_{n\to\infty} d(x_{n+1}, x_n) = t$, then there exists $m_0 \in \mathbb{N}$ such that

$$t \leq d(x_{m_0}, x_{m_0-1}) < t + \delta(t) \implies d(Tx_{m_0}, x_{m_0-1}) < t.$$

This implies that $d(x_{m_0+1}, x_{m_0}) < t$, and it contradicts the assumption $\lim_{n\to\infty} d(x_{n+1}, x_n) = t$. Therefore $t = 0$.

Step-II: To prove that (x_n) is a strongly orthogonal Cauchy sequence.

Suppose that (x_n) is not a strongly orthogonal Cauchy sequence. There exist $\varepsilon > 0$ and two sequences (m_k) and (n_k) such that $m_k > n_k \geq m_0$

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon \text{ and } d(x_{m_{k-1}}, x_{n_k}) < \varepsilon \tag{2}$$

To prove the result (2), we suppose that

$$\mathcal{S}_k = \{m \in \mathbb{N} : \exists n_k \geq m_0, d(x_m, x_{n_k}) \geq \varepsilon, m > n_k \geq m_0\}.$$

Clearly $\mathcal{S}_k \neq \phi$ and $\mathcal{S}_k \subseteq \mathbb{N}$, then by the well-ordering principle, the minimum element of $\mathcal{S}_k$ is denoted by m_k, and clearly the result (2) holds. Then there exists $\delta(\varepsilon) > 0$ (which can be chosen as $\delta(\varepsilon) \leq \varepsilon$) satisfy the result (1). Then by Step-I, we show that there exists $m_0 \in \mathbb{N}$ such that $d(x_{m_0}, x_{m_0+1}) < \delta(\varepsilon)$. Then for fix $k \geq m_0$ we have

$$\begin{aligned} d(x_{m_k-1}, x_{n_k-1}) &= d(x_{m_k-1}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}) \\ &< \varepsilon + \delta(\varepsilon). \end{aligned}$$

Now we consider the two cases:

Case-I: Suppose that $d(x_{m_k-1}, x_{n_k-1}) \geq \varepsilon$.

Since x_{n_k-1} and x_{m_k-1} are orthogonal comparable, using the condition (1) we get

$$\varepsilon \leq d(x_{m_k-1}, x_{n_k-1}) < \varepsilon + \delta(\varepsilon) \implies d(x_{m_k}, x_{n_k}) < \varepsilon.$$

Case-II: Suppose that $d(x_{m_k-1}, x_{n_k-1}) < \varepsilon$.

Since x_{m_k-1} and x_{n_k-1} are orthogonal comparable, then by (1) we get

$$d(x_{m_k}, x_{n_k}) < d(x_{m_k-1}, x_{n_k-1}) < \varepsilon.$$

Hence in each case we get $d(x_{m_k}, x_{n_k}) < \varepsilon$ which contradicts the condition (2). Hence (x_n) is a strongly orthogonal Cauchy sequence. Since X is strongly orthogonal complete, then there exists $y_0 \in X$ such that $x_n \to y_0$. So $d(x_n, y_0) \to \infty$ as $n \to \infty$. Also since T is a strongly orthogonal continuous, then for any $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that

$$d(x_{m_0+1}, y_0) < \frac{\varepsilon}{2} \text{ and } d(Tx_{m_0}, Ty_0) < \frac{\varepsilon}{2}.$$

Now

$$\begin{aligned} d(Ty_0, y_0) &\leq d(Ty_0, Tx_{m_0}) + d(Tx_{m_0}, y_0) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

It follows that $Ty_0 = y_0$. Hence T has a fixed point in X.

Now we prove that T is a Picard operator. Let $x \in X$ be arbitrary. By the choice of x_0, we have

$$x_0 \perp y_0 \text{ and } x_0 \perp x$$

or

$$y_0 \perp x_0 \text{ and } x \perp x_0$$

Since T is orthogonal preserving, it implies that

$$x_n \perp y_0 \text{ and } x_n \perp T^n x$$

or

$$y_0 \perp x_n \text{ and } T^n x \perp x_n, \forall n \in \mathbb{N}.$$

Now we show that the sequence $(d(T^n x, x_n))$ converges to zero. For some $m_0 \in \mathbb{N}$, if $T^{m_0} x = x_{m_0}$ then $d(T^n x, x_n) = 0$ for all $n \geq m_0$.

Let $T^n x \neq x_n$ for all $n \in \mathbb{N}$. The Meir–Keeler condition implies that the sequence $(d(T^n x, x_n))$ is strictly decreasing. Using the same argument in Step-I, we get $\lim\limits_{n\to\infty} d(T^n x, x_n) = 0$. For all $n \in \mathbb{N}$ we obtain that

$$d(T^n x, y_0) \le d(T^n x, x_n) + d(x_n, y_0) = 0 \text{ as } n \to \infty.$$

Therefore $T^n x \to y_0$.

Finally we prove that the fixed point y is unique. Let $\bar{y} \in X$ be another fixed point of T, then $T^n \bar{y} = \bar{y}$ for all $n \in \mathbb{N}$. It follows from T is a Picard operator that $\bar{y} = y_0$.

6 Applications to Integral Equations

We study the existence and uniqueness of a solution of the following integral equation

$$u(t) = \int_0^t e^{s-t} \left(\int_0^b e^{-\tau} g(s, \tau, u(\tau)) d\tau \right) ds \tag{1}$$

Let $p > 0$, g be a function from $[0, p] \times [0, p] \times X$ into X and $\Gamma : [0, p] \times [0, p] \times [0, p] \to \mathbb{R}^+$ be an integrable function for which

(P1) (i) $g : (t, ., x) : s \to g(t, s, x)$ is an integrable function for every $x \in X$ and for all $t \in [0, p]$
(ii) $g(t, s, .) : x \to g(t, s, x)$ is d-continuous on X for all $t, s \in [0, p]$.

(P2) (i) $g(t, s, x) \ge 0$ for all $x \ge 0$ and for all $t, s \in [0, p]$
(ii) $g(t, s, x)g(t', r, y) \ge g(t, t', xy)$ for each $x, y \in X$ with $xy \ge 0$ and for all $t, t', r, s \in [0, p]$.

(P3) There exists $\gamma > 0$ such that $d(g(t, s, x), g(t, s, y)) \le \gamma d(x, y)$ for all (t, s, x), $(t, s, y) \in [0, p] \times [0, p] \times X$ with $xy \ge 0$.

(P4) $d(g(t, s, x), g(v, s, x)) \le \Gamma(t, v, s)$ for all $(t, s, x), (v, s, x) \in [0, p] \times [0, p] \times X$ and

$$\lim_{t \to \infty} \int_0^p \Gamma(t, v, s) ds = 0$$

uniformly for all $v \in [0, p]$.

We consider $\mathcal{B} = C([0, p], X)$ the space of all continuous function from $[0, p]$ into X. It is a complete metric space with the metric

$$d_b(u, w) = \sup_{t \in [0, p]} e^{-bt} |u(t) - w(t)|, \text{ where } b \ge 0.$$

We define the operators T and S on $\mathcal{B}$ by

$$Tu(t) = \int_0^p e^{-s} g(t, s, u(s)) ds$$

$$Su(t) = \int_0^t e^{s-t} Tu(s) ds$$

We have the fixed points of S are the solutions of the Eq. (1) and $\mathcal{B}$ is invariant under T and S.

Theorem 6.1 *Under the conditions (P1)–(P4), for all $p \geq 0$ the integral equation (1) has a unique solution in $\mathcal{B}$.*

Proof We consider the orthogonal relation on $\mathcal{B}$ as

$$u \perp w \iff u(t)w(t) \geq 0, \forall t \in [0, p].$$

It is clear that $\mathcal{B}$ is a strongly orthogonal complete metric space. To complete the proof, we need the following steps.

Step-I: S is an orthogonal preserving. For each $u, w \in \mathcal{B}$ with $u \perp w$, by the hypothesis (P1)(i) and (P2)(ii) we have

$$\begin{aligned} Tu(t)Tw(t') &= \int_0^p e^{-s} g(t, s, u(s))ds \int_0^p e^{-r} g(t', r, w(r))dr \\ &= \int_0^p \int_0^p e^{-(s+r)} g(t, s, u(s))g(t', r, w(r))dsdr \\ &\geq \int_0^p \int_0^p e^{-(s+r)} g(t, t', u(s)w(r))dsdr, \quad u(s)w(r) \geq 0 \\ &\geq 0 \text{ for each } t, t' \in [0, p]. \end{aligned}$$

Therefore $Tu \perp Tw$. By the definition of S we have $Su \perp Sw$.

Step-II: To prove that S is d_b-Lipschitz on orthogonal comparable elements. Let $M = \{s_0, s_1, \ldots, s_k\}$ be a subdivision of the interval $[0, p]$. Then we have $\sum_{i=0}^{k-1} (s_{i+1} - s_i)e^{-s_i} x(s_i)$ is norm convergent and consequently d_b-convergent to $\int_0^p e^{-s} x(s)ds$ in $\mathcal{B}$, when $|M| = \sup\{|s_{i+1} - s_i| : i = 0, 1, 2, \ldots, k-1\} \to 0$ as $k \to \infty$. Let $u \perp w$. Then we have

$$\begin{aligned} &\int_0^p e^{-s}(g(t, s, u(s)) - g(t, s, w(s)))ds \\ &= \lim_{k\to\infty} \sum_{i=0}^{k-1} (s_{i+1} - s_i)e^{-s_i}(g(t, s_i, u(s_i)) - g(t, s_i, w(s_i))) \end{aligned}$$

and

$$\sum_{i=0}^{k-1} (s_{i+1} - s_i)e^{-s_i} \leq \int_0^p e^{-s} ds = 1 - e^{-p} < 1$$

by Fatou property and condition (P3). Now we have

$$
\begin{aligned}
d(Tu(t), Tw(t)) &\le \liminf \sum_{i=0}^{k-1} (s_{i+1} - s_i) e^{-s_i} d(g(t, s_i, u(s_i)), g(t, s_i, w(s_i))) \\
&\le \lambda \liminf \sum_{i=0}^{k-1} (s_{i+1} - s_i) e^{-s_i} d(u(s_i), w(s_i)) \\
&\le \lambda \liminf \sum_{i=0}^{k-1} (s_{i+1} - s_i) e^{bs_i} d_b(u, w).
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
e^{-bt} d(Tu(t), Tw(t)) &\le \lambda e^{-bt} \left(\int_0^p e^{bs} ds \right) d_b(u, w) \\
&\le \lambda \frac{e^{bp} - 1}{b} d_b(u, w).
\end{aligned}
$$

Hence

$$
d_b(Tu, Tw) \le \lambda \frac{e^{bp} - 1}{b} d_b(u, w).
$$

By definition of S gives that

$$
d_b(Su, Sw) \le N d_b(u, w), \text{ where } N = \frac{\lambda}{b(b+1)} \left(1 - e^{-(b+1)p}\right) \left(e^{bp} - 1\right).
$$

Step-III: To show that S satisfies the Meir–Keeler condition.

We define

$$
\delta(\varepsilon) = \{d_b(u, w) : d_b(Su, Sw) \ge \varepsilon \text{ and } u \perp w\}.
$$

Let $0 < N < 1$ and $\varepsilon > 0$ be given. If $u \perp w$ and $d_b(Su, Sw) \ge \varepsilon$ then by Step-II, we have

$$
d_b(u, w) \ge N^{-1} \varepsilon.
$$

So $\delta(\varepsilon) \ge N^{-1}\varepsilon > \varepsilon$. By using Theorem 1 of [17] we have S satisfies the Meir–Keeler condition. Therefore by Theorem 5.15, S has a unique fixed point, which is the solution of the integral equation (1).

7 Different Types of Convergence

The asymptotic density or density of a subset U of $\mathbb{N}$, denoted by $\delta(U)$, is given by

$$
\delta(U) = \lim_{n \to \infty} \frac{1}{n} |\{k \le n : k \in U\}|,
$$

if this limit exists, where $|\{k \leq n : k \in U\}|$ denotes the cardinality of the set $\{k \leq n : k \in U\}$. Fast [18] and Steinhaus [19] independently introduced the notion of statistical convergence with the help of asymptotic density, and later on Schoenberg [20] reintroduced it. A sequence $\mathbf{x} = (x_n)$ is said to be statistically convergent to ℓ if for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - \ell| \geq \varepsilon\}$ has density zero. We call ℓ the statistical limit of $\mathbf{x}$. The set of all statistically convergent sequences is denoted by st.

The notion of ideal convergence is the dual (equivalent) to the notion of filter convergence which was introduced by Cartan [21]. The filter convergence is a generalization of the classical notion of convergence of a sequence, and it has been an important tool in general topology and functional analysis. Kostyrko et al. [22] and Nuray and Ruckle [23] independently discussed the ideal convergence which is based on the structure of the admissible ideal $\mathcal{I}$ of subsets of natural numbers $\mathbb{N}$. It was further investigated by many authors, e.g., Šalát et al. [24], and references therein. The statistical convergence and ideal convergence for sequences of real-valued functions were studied by Balcerzak et al. [25].

A non-empty class $\mathcal{I}$ of power sets of a non-empty set X is called an *ideal* on X if and only if (i) $\phi \in \mathcal{I}$ (ii) $\mathcal{I}$ is additive under union (iii) hereditary. An ideal $\mathcal{I}$ is called *non-trivial* if $\mathcal{I} \neq \phi$ and $X \notin \mathcal{I}$. A non-empty class $\mathcal{F}$ of power sets of X is called a *filter* on X if and only if (i) $\phi \notin \mathcal{F}$ (ii) $\mathcal{F}$ is additive under intersection (iii) for each $U \in \mathcal{F}$ and $V \supset U$, implies $V \in \mathcal{F}$. A non-trivial ideal $\mathcal{I}$ is said to be (i) an *admissible ideal* on X if and only if it contains all singletons (ii) *maximal*, if there cannot exist any non-trivial ideal $\mathcal{K} \neq \mathcal{I}$ containing $\mathcal{I}$ as a subset (iii) is said to be a *translation invariant ideal* if $\{n + 1 : n \in U\} \in \mathcal{I}$, for any $U \in \mathcal{I}$.

We recall that a real sequence $\mathbf{x} = (x_n)$ is called ideal convergent (in short $\mathcal{I}$-convergent) to the number l (denoted by $\mathcal{I}\text{-}\lim x_n = l$) if for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - l| \geq \varepsilon\}$ is in $\mathcal{I}$. The set of all ideal convergent sequences denoted by $\mathcal{I}$.

A lacunary sequence $\theta = (k_r)$ is a non-decreasing sequence of positive integers such that $k_0 \neq 0$ and $h_r : k_r - k_{r-1} \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated as q_r. We assume that $\liminf_r q_r > 1$. The notion of lacunary statistical convergence was introduced and studied by Fridy and Orhan [26, 27]. A sequence (x_n) in $\mathbb{R}$ is called lacunary statistically convergent to $x \in \mathbb{R}$ if

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r ; |x_n - x| \geq \varepsilon\}| = 0,$$

for every positive real number ε. The set of all lacunary statistically convergent sequences is denoted by st_θ.

Connor and Grosse-Erdman [28] gave sequential definitions of continuity for real functions calling $\mathcal{G}$-continuity, where a method of sequential convergence, or briefly a method, is a linear function $\mathcal{G}$ defined on a linear subspace of s, denoted by $c_{\mathcal{G}}$, into $\mathbb{R}$. We refer [29] for sequential compactness and [30, 31] for $\mathcal{G}$-sequential continuity. A sequence $\mathbf{x} = (x_n)$ is said to be $\mathcal{G}$-convergent to ℓ if $\mathbf{x} \in c_{\mathcal{G}}$ and $\mathcal{G}(\mathbf{x}) = \ell$. In particular, lim denotes the limit function $\lim \mathbf{x} = \lim_n x_n$ on the linear space c, and $st - \lim$ denotes the statistical limit function $st - \lim \mathbf{x} = st - \lim_n x_n$ on the linear space st and $st_\theta - \lim$ denotes the lacunary statistical limit function $st_\theta - \lim \mathbf{x} = st_\theta - \lim_n x_n$ on the linear space st_θ. Also $\mathcal{I} - \lim$ denotes the $\mathcal{I}$-limit function $\mathcal{I} - \lim \mathbf{x} = \mathcal{I} - \lim_n x_n$ on the linear space $\mathcal{I}(\mathbb{R})$. A method $\mathcal{G}$ is called regular if every convergent sequence $\mathbf{x} = (x_n)$ is $\mathcal{G}$-convergent with $\mathcal{G}(\mathbf{x}) = \lim \mathbf{x}$. A method is called subsequential if whenever $\mathbf{x}$ is $\mathcal{G}$-convergent with $\mathcal{G}(\mathbf{x}) = \ell$, then there is a subsequence (x_{n_k}) of $\mathbf{x}$ with $\lim_k x_{n_k} = \ell$.

8 General Convergence

Let X be an orthogonal set and $d : X \times X \to [0, \infty)$ be a mapping. For every $x \in X$ we define the set

$$\mathcal{GO}(X, d, x) = \left\{ (x_n) \subset X : \mathcal{G} - \lim_{n\to\infty} d(x_n, x) = 0 \text{ and } x_n \perp x, \forall n \in \mathbb{N} \right\}. \quad (1)$$

Definition 8.1 Let $(X, \perp, d)$ be an orthogonal metric space. A sequence (x_n) in X is said to be

(i) $\mathcal{G}$-orthogonal convergent (in short $\mathcal{G}O$-convergent) to x if and only if $(x_n) \in \mathcal{GO}(X, d, x)$,
(ii) $\mathcal{G}$-orthogonal Cauchy (in short $\mathcal{G}O$-Cauchy) if and only if $\mathcal{G} - \lim_{n,m\to\infty} d(x_n, x_m) = 0$ and $x_n \perp x_m$ or $x_m \perp x_n$, $\forall n, m \in \mathbb{N}$.

Theorem 8.2 *Let* $(X, \perp, <.,.>)$ *be an orthogonal inner product space, where* $X = \mathbb{R}^n$, $<.,.>$ *denotes the standard inner product space and* $\perp$ *is an orthogonal relation on* X *defined by* $x \perp y$ *if* $<x, y> = 0$ *for all* $x, y \in X$. *Let* (x_n) *and* (y_n) *be two sequences in* X *with* $(x_n) \in \mathcal{GO}(X, d, x)$ *and* $(y_n) \in \mathcal{GO}(X, d, y)$. *Then*

(a) $(x_n + y_n) \in \mathcal{GO}(X, d, x + y)$,
(b) $<x_n, y_n> \to_{\mathcal{G}} <x, y>$.

Proof The proof is simple. The reader should prove the theorem on its own.

Definition 8.3 Let $(X, \perp, d)$ be an orthogonal metric space. A function $f : X \to X$ is said to be $\mathcal{G}$-orthogonal continuous ($\mathcal{G}O$-continuous) at a point x_0 in X if for each orthogonal sequence (x_n) in X, $\mathcal{G}$-converging to x_0 such that $f(x_n) \to_{\mathcal{G}} f(x_0)$. Also f is said to be $\mathcal{G}$-orthogonal continuous on X if f is $\mathcal{G}$-orthogonal continuous at each point on X.

Definition 8.4 Let $(X, \perp, d)$ be an orthogonal metric space and $E \subset X$. A function $f : E \to X$ is said to be $\mathcal{G}$-orthogonal sequentially continuous ($\mathcal{G}O$-sequentially continuous) at a point x_0 in X if for each orthogonal sequence $(x_n) \in E$, $\mathcal{G}$-converging to x_0 such that $f(x_n) \to_{\mathcal{G}} f(x_0)$.

Theorem 8.5 *Let $\mathcal{G}$ be a regular method and $(X, \perp, d)$ an orthogonal metric space, and $f, g : X \to X$ be functions on X. Then the following are satisfied.*

(a) If f and g are $\mathcal{G}O$-sequentially continuous, then so also is gf,
(b) If f and g are $\mathcal{G}O$-sequentially continuous, then so also is $f + g$.

Proof (a) Let $\mathbf{x}$ be an orthogonal sequence in X such that $\mathcal{G}(\mathbf{x}) = x_0 \in X$. Since f is $\mathcal{G}O$-sequentially continuous at x_0, we get $\mathcal{G}(f(\mathbf{x})) = f(x_0)$ and since g is $\mathcal{G}O$-sequentially continuous at $f(x_0)$ we have $\mathcal{G}(g(f(\mathbf{x}))) = g(f(x_0))$. Therefore the function gf is $\mathcal{G}O$-sequentially continuous.

(b) Let $\mathbf{x}$ be an orthogonal sequence in X such that $\mathcal{G}(\mathbf{x}) = x_0 \in X$. Since the functions f and g are $\mathcal{G}O$-sequentially continuous, so we have $\mathcal{G}(f(\mathbf{x})) = f(x_0)$ and $\mathcal{G}((\mathbf{x})) = g(x_0)$. Therefore by the additivity of $\mathcal{G}$ we get

$$\mathcal{G}((f+g)(\mathbf{x}) = \mathcal{G}(f(\mathbf{x}) + g(\mathbf{x})) = \mathcal{G}(f(\mathbf{x})) + \mathcal{G}(g(\mathbf{x})) = f(x_0) + g(x_0) = (f+g)(x_0)$$

i.e., $f + g$ is $\mathcal{G}O$-sequentially continuous.

Theorem 8.6 *Let $\mathcal{G}$ be a method and $(X, \perp, d)$ be an orthogonal metric space. Then we have the following.*

(i) If $f : X \to X$ is $\mathcal{G}O$-sequentially continuous, then so also is a restriction $f_A : A \to X$ to a subset A,
(ii) The identity map $\mathcal{J} : X \to X$ is $\mathcal{G}O$-sequentially continuous,
(iii) For a subset $A \subseteq X$, the inclusion map $f : A \to X$ is $\mathcal{G}O$-sequentially continuous,
(iv) If $\mathcal{G}$ is regular, then the constant map $\mathcal{C} : X \to X$ is $\mathcal{G}O$-sequentially continuous,
(v) If f is $\mathcal{G}O$-sequentially continuous, then so also is $-f$,
(vi) The inverse function $f : X \to X$; $f(x) = -x$ is $\mathcal{G}O$-sequentially continuous.

Proof (i) Let $\mathbf{x}$ be an orthogonal sequence of the terms in A with $\mathcal{G}(\mathbf{x}) = x_0$. Since f is $\mathcal{G}O$-sequentially continuous, then we have $\mathcal{G}(f(\mathbf{x})) = f(x_0)$.

(ii) Let $\mathcal{G}(\mathbf{x}) = x_0$ for an orthogonal sequence $\mathbf{x}$ in X. Then $\mathcal{G}(\mathcal{J}(\mathbf{x})) = \mathcal{G}(\mathbf{x}) = x_0 = \mathcal{J}(x_0)$ and so $\mathcal{J}$ is $\mathcal{G}O$-sequentially continuous.

(iii) Follows immediately from (i) and (ii).

(iv) Let $\mathcal{C} : X \to X$ be a constant map with $\mathcal{C}(x) = y_0$ and let $\mathbf{x}$ be an orthogonal sequence in X with $\mathcal{G}(\mathbf{x}) = x_0$. Then $\mathcal{C}(\mathbf{x}) = (y_0, y_0, \ldots)$ which $\mathcal{G}$-converges to y_0. Since $\mathcal{G}$ is regular $\mathcal{G}(\mathcal{C}(\mathbf{x})) = y_0 = \mathcal{C}(x_0)$. Therefore $\mathcal{C}$ is $\mathcal{G}O$-sequentially continuous.

(v) Let $\mathbf{x}$ be an orthogonal sequence in X with $\mathcal{G}(\mathbf{x}) = x_0$. Since f is $\mathcal{G}O$-sequentially continuous $\mathcal{G}(f(\mathbf{x})) = f(x_0)$. Therefore $\mathcal{G}(-f(\mathbf{x})) = -\mathcal{G}(f(\mathbf{x})) = -f(x_0)$, and hence, $-f$ is $\mathcal{G}O$-sequentially continuous.

(vi) Follows immediately from (ii) and (v).

Corollary 8.7 *Let $\mathcal{G}$ be a regular method and $C\mathcal{G}O(X)$ the class of $\mathcal{G}O$-sequentially continuous functions. Then $C\mathcal{G}O(X)$ becomes a group with the sum of functions.*

Definition 8.8 Let $(X, \perp, d)$ be an orthogonal metric space. Let $A \subseteq X$ and $x_0 \in X$. Then x_0 is in the $\mathcal{G}O$-sequential closure of A (it is called $\mathcal{G}O$-hull of A) if there is an orthogonal sequence $\mathbf{x} = (x_n)$ of points in A such that $\mathcal{G}(\mathbf{x}) = x_0$. We denote $\mathcal{G}O$-sequential closure of a set A by $\bar{A}^{\mathcal{G}O}$. We say that a subset A is $\mathcal{G}O$-sequentially closed if it contains all the points in its $\mathcal{G}O$-sequential closure, i.e., if $\bar{A}^{\mathcal{G}O} \subseteq A$. It is clear that $\overline{\phi}^{\mathcal{G}O} = \phi$ and $\bar{X}^{\mathcal{G}O} = X$. If $\mathcal{G}$ is a regular method, then $A \subseteq \bar{A} \subseteq \bar{A}^{\mathcal{G}O}$, and hence, A is $\mathcal{G}O$-sequentially closed if and only if $\bar{A}^{\mathcal{G}O} = A$.

Definition 8.9 A subset A of an orthogonal metric space $(X, \perp, d)$ is called $\mathcal{G}O$-sequentially open if its complement is $\mathcal{G}O$-sequentially closed, i.e., $\overline{X\backslash A}^{\mathcal{G}O} \subseteq X\backslash A$.

Definition 8.10 A function f is said to be $\mathcal{G}O$-sequentially open if the image of any $\mathcal{G}O$-sequentially open subset of an orthogonal metric space $(X, \perp, d)$ is $\mathcal{G}O$-sequentially open.

Definition 8.11 Let $(X, \perp, d)$ be an orthogonal metric space. A function f is said to be $\mathcal{G}O$-sequentially closed if the image of any $\mathcal{G}O$-sequentially closed subset of X is $\mathcal{G}O$-sequentially closed.

Theorem 8.12 *Let $(X, \perp, d)$ be an orthogonal metric space and $\mathcal{G}$ be a regular method. A function $f : X \to X$ is $\mathcal{G}O$-sequentially closed if $\overline{f(B)}^{\mathcal{G}O} \subseteq f(\overline{B}^{\mathcal{G}O})$ for every subset B.*

Proof Let $f : X \to X$ be a function such that $\overline{f(B)}^{\mathcal{G}O} \subseteq f(\overline{B}^{\mathcal{G}O})$ for any subset B. Let A be a $\mathcal{G}O$-closed subset. By assumption $\overline{f(A)}^{\mathcal{G}O} \subseteq f(\overline{A}^{\mathcal{G}O})$. Since $\mathcal{G}$ is regular $\overline{B}^{\mathcal{G}O} = B$ and so we have $\overline{f(B)}^{\mathcal{G}O} \subseteq f(B)$ and therefore $f(B)$ is $\mathcal{G}O$-sequentially closed.

Theorem 8.13 *Let $(X, \perp, d)$ be an orthogonal metric space and $\mathcal{G}$ be a regular method. If a function f is $\mathcal{G}O$-sequentially continuous on X, then the inverse image $f^{-1}(A)$ of any $\mathcal{G}O$-sequentially open subset A of X is $\mathcal{G}O$-sequentially open.*

Proof Let $f : X \to X$ be any $\mathcal{G}O$-sequentially continuous function and A be any $\mathcal{G}O$-sequentially open subset of X. Then $X\backslash A$ is $\mathcal{G}O$-sequentially closed. By Lemma 8.12, $f^{-1}(X\backslash A)$ is $\mathcal{G}O$-sequentially closed. On the other hand

$$f^{-1}(X\backslash A) = f^{-1}(X)\backslash f^{-1}(A) = X\backslash f^{-1}(A)$$

and so it follows that $f^{-1}(A)$ is $\mathcal{G}O$-sequentially open. This completes the proof of the theorem.

Definition 8.14 Let $(X, \perp, d)$ be an orthogonal metric space. A point x_0 is called a $\mathcal{G}O$-sequential accumulation point of a subset A of X (or is in the $\mathcal{G}O$-sequential derived set) if there is an orthogonal sequence $\mathbf{x} = (x_n)$ of points in $A\backslash\{x_0\}$ such that $\mathcal{G}(\mathbf{x}) = x_0$.

Definition 8.15 Let $(X, \perp, d)$ be an orthogonal metric space. A subset A of X is called $\mathcal{G}O$-sequentially countably compact if any infinite subset of A has at least one $\mathcal{G}O$-sequential accumulation point in A.

Theorem 8.16 *Let $(X, \perp, <.,.>)$ be an orthogonal inner product space, where $X = \mathbb{R}^n$, $<.,.>$ denotes the standard inner product space and $\perp$ is an orthogonal relation on X defined by $x \perp y$ if $<x, y>= 0$ for all $x, y \in X$. Then any $\mathcal{G}$-orthogonal sequentially continuous function at point x_0 if and only if it is continuous at x_0 in the ordinary sense.*

Proof The proof follows from Theorem 2.25 and the definition.

Theorem 8.17 *Let $(X, \perp, d)$ be an orthogonal metric space. Suppose that G is a regular method. Let $f : X \to X$ be an additive function on X. Then f is $\mathcal{G}O$-sequentially continuous at origin if and only if f is $\mathcal{G}O$-sequentially continuous at any point $b \in X$.*

Proof Let the additive function $f : X \to X$ be $\mathcal{G}O$-sequentially continuous at origin. So for an orthogonal sequence $\mathbf{x} = (x_n) \in X$ such that $\mathcal{G}(f(\mathbf{x})) = 0$, whenever $\mathcal{G}(\mathbf{x}) = 0$. Let $\mathbf{x}$ be a sequence in X with $\mathcal{G} - \lim \mathbf{x} = b$ and $\mathbf{b}$ the constant sequence $\mathbf{b} = (b, b, \ldots)$. Since $\mathcal{G}$ is regular $\mathcal{G}(\mathbf{b}) = b$. Therefore the sequence $\mathbf{x} - \mathbf{b}$ is $\mathcal{G}O$-convergent to 0. So by assumption $\mathcal{G}(f(\mathbf{x} - \mathbf{b})) = 0$. Since f and $\mathcal{G}$ are additive $\mathcal{G}(f(\mathbf{x})) - \mathcal{G}(f(\mathbf{b})) = 0$. Since the constant sequence $f(\mathbf{b})$ tends to $f(b)$ and $\mathcal{G}$ is regular, $\mathcal{G}(f(\mathbf{b})) = f(b)$. Therefore we have that $\mathcal{G}(f(\mathbf{x})) = f(b)$.

Theorem 8.18 *Let $\mathcal{G}$ be a regular subsequential method. Then a subset of X is $\mathcal{G}O$-sequentially compact if and only if it is $\mathcal{G}O$-sequentially countably compact.*

Proof Let A be any $\mathcal{G}O$-sequentially compact subset of X and B be an infinite subset of A. We can choose an orthogonal sequence $\mathbf{x} = (x_n)$ of different points of B. Since A is $\mathcal{G}O$-sequentially compact, so it implies that of B that the orthogonal sequence $\mathbf{x}$ has a convergent subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ with $\mathcal{G}(\mathbf{y}) = x_0$. Since $\mathcal{G}$ is a subsequential method, $\mathbf{y}$ has a convergent subsequence $\mathbf{z} = (z_k)$ of the subsequence $\mathbf{y}$ with $\lim_k z_k = x_0$. By the regularity of $\mathcal{G}$, we obtain that x_0 is a $\mathcal{G}O$-sequential accumulation point of B. Thus A is $\mathcal{G}O$-sequentially countably compact.

Next we suppose that A is any $\mathcal{G}O$-sequentially countably compact subset of X. Let $\mathbf{x} = (x_n)$ be an orthogonal sequence of points in A. We write $P = \{x_n : n \in \mathbb{N}\}$. If P is finite, then there is nothing to prove. If P is infinite, then P has a $\mathcal{G}O$-sequential accumulation point in A. Also each set $P_n = \{x_k : k \geq n\}$, for each positive integer n, has a $\mathcal{G}O$-sequential accumulation point in A. Then the intersection $\bigcap_{n=1}^{\infty} \overline{P_n}^{\mathcal{G}O} \neq \phi$. So there is an element x_0 of A which belongs to the intersection. Since $\mathcal{G}$ is a regular subsequential method, $x_0 \in \bigcap_{n=1}^{\infty} \overline{P_n}$. Then it is not difficult to construct a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence $\mathbf{x}$ with $\mathcal{G}(\mathbf{z}) \in A$. This completes the proof.

9 Orthogonal Sequential Compactness

We consider $(X, \perp, d)$ is an orthogonal metric space.

Definition 9.1 A subset E of X is called an orthogonal sequentially compact if any sequence (x_n) in E has a $\mathcal{G}$-convergent subsequence whose limit is in E.

Definition 9.2 A subset E of X is called $\mathcal{G}O$-sequentially compact if any sequence (x_n) in E, there is subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ of (x_n) such that $\mathcal{G}(\mathbf{y}) = \lim_k y_k$ in E.

Remark 9.3 Any sequentially compact subset E of X is also $\mathcal{G}$-orthogonal sequentially compact and the converse is not always true. For this, see the following example.

Example 9.4 Let $(X = [0, 1), \perp, d)$ be an orthogonal metric space. Define the orthogonal relation $\perp$ on X by

$$x \perp y \Leftrightarrow \begin{cases} x \leq y \leq \frac{1}{2}, \\ \text{or } x = 0. \end{cases}$$

There exists a subsequence $(y_k) = (x_{n_k})$ of (x_n) for which $x_{n_k} = 0$ for all $k \geq 1$ or there exists a monotone subsequence (x_{n_k}) if (x_n) for which $x_{n_k} \leq \frac{1}{2}$ for all $k \geq 1$. We see that (x_{n_k}) is $\mathcal{G}$-convergent to a point $x \in [0, \frac{1}{2}] \subseteq X$. If we consider a subsequence $(y_k) = \left(1 - \frac{1}{k}\right)$ of (x_n), then $\lim_k y_k$ is not in $[0, \frac{1}{2}]$.

Theorem 9.5 *Every $\mathcal{G}O$-sequentially closed subset of a $\mathcal{G}O$-sequentially compact subset of X is $\mathcal{G}O$-sequentially compact.*

Proof Let A be any $\mathcal{G}O$-sequentially compact subset of X and B be a $\mathcal{G}O$-sequentially closed subset of A. Consider an orthogonal sequence $\mathbf{x} = (x_n)$ of points in B. Then $\mathbf{x}$ is a sequence of points in A. Since A is $\mathcal{G}O$-sequentially compact, there is a subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ of sequence $\mathbf{x}$ such that $\mathcal{G}(\mathbf{y}) \in A$. The subsequence $\mathbf{y}$ is also a sequence of points in B. Since B is $\mathcal{G}O$-sequentially closed, so $\mathcal{G}(\mathbf{y}) \in B$. Thus $\mathbf{x}$ has a $\mathcal{G}$-convergent subsequence, with $\mathcal{G}(\mathbf{y}) \in B$. Hence B is $\mathcal{G}O$-sequentially compact.

Theorem 9.6 *Let $\mathcal{G}$ be a regular subsequential method. Every $\mathcal{G}O$-sequentially compact subset of X is $\mathcal{G}O$-sequentially closed.*

Proof Let A be any $\mathcal{G}O$-sequentially compact subset of X. Take any $x_0 \in \overline{A}$. Then there is an orthogonal sequence $\mathbf{x} = (x_n)$ of points in A such that $\mathcal{G}(\mathbf{x}) = x_0$. Since $\mathcal{G}$ is a subsequential method, there is a subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ of the sequence $\mathbf{x}$ such that $\lim_k x_{n_k} = x_0$. Since $\mathcal{G}$ is regular, so we have $\mathcal{G}(\mathbf{y}) = x_0$. By the $\mathcal{G}O$-sequential compactness of A, there is a subsequence $\mathbf{z} = (z_k)$ of the subsequence $\mathbf{y}$ such that $\mathcal{G}(\mathbf{z}) = x_1 \in A$. Since $\lim_k z_k = x_0$ and $\mathcal{G}$ is regular, $\mathcal{G}(\mathbf{z}) = x_0$. Thus $x_0 = x_1$ and hence $x_0 \in A$. Thus A is $\mathcal{G}O$-sequentially closed.

Corollary 9.7 *Let $\mathcal{G}$ be a regular subsequential method. Then a subset of X is $\mathcal{G}O$-sequentially compact if and only if it is sequentially countably compact in the ordinary sense.*

Corollary 9.8 *Let $\mathcal{G}$ be a regular subsequential method. Then a subset of X is $\mathcal{G}O$-sequentially compact if and only if it is countably compact in the ordinary sense.*

Theorem 9.9 *Every $\mathcal{G}O$-sequential continuous image of any $\mathcal{G}O$-sequentially compact subset of X is $\mathcal{G}O$-sequentially compact.*

Proof Let f be any $\mathcal{G}O$-sequentially continuous function on X and A be any $\mathcal{G}O$-sequentially compact subset of X. Consider an orthogonal sequence $\mathbf{y} = (y_n) = (f(x_n))$ of points in $f(A)$. Since A is $\mathcal{G}O$-sequentially compact, there exists a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence $\mathbf{x} = (x_n)$ with $\mathcal{G}(\mathbf{z}) \in A$. Then the sequence $f(\mathbf{z}) = (f(z_k)) = (f(x_{n_k}))$ is a subsequence of the sequence $\mathbf{y}$. Since f is $\mathcal{G}O$-sequentially continuous, so we have $\mathcal{G}(f(\mathbf{z})) = f(z_0) \in f(A)$. Hence $f(A)$ is $\mathcal{G}O$-sequentially compact.

Corollary 9.10 *Let $\mathcal{G}$ be a regular subsequential method. Then every $\mathcal{G}O$-sequentially continuous image of any sequentially compact subset of X is sequentially compact.*

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