Approximation Results for Urysohn-Type Nonlinear Bernstein Operators

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Abstract In the present work, our aim of this study is generalization and extension of the theory of interpolation of functions to functionals or operators by means of Urysohn-type nonlinear operators. In accordance with this purpose, we introduce and study a new type of Urysohn-type nonlinear operators. In particular, we investigate the convergence problem for nonlinear operators that approximate the Urysohn-type operator. The starting point of this study is motivated by the important applications that approximation properties of certain families of nonlinear operators have in signal–image reconstruction and in other related fields. We construct our nonlinear operators by using a nonlinear forms of the kernels together with the Urysohn-type operator values instead of the sampling values of the function. As far as we know, this will be first use of such kind of operators in the theory of interpolation and approximation. Hence, the present study is a generalization and extension of some previous results.

Keywords Urysohn integral operators · Nonlinear Bernstein operators · Urysohn-type nonlinear Bernstein operators · Approximation.

AMS Subject Classification 41A25 · 41A35 · 47G10 · 47H30.

1 Introduction

For a function defined on the interval [0, 1], the Bernstein operators $(B_n f)$, $n \geq 1$, are defined by

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x) \quad , \quad n \geq 1, \tag{1}$$

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where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the well-known Binomial distribution and called Bernstein basis $(0 \leq x \leq 1)$. These polynomials were introduced by Bernstein [1] in 1912 to give the first constructive proof of the Weierstrass approximation theorem.

The first main approximation result related to pointwise convergence of the Bernstein polynomials reads; let f be a bounded function on [0, 1], then

$$\lim_{n\to\infty} (B_n f)(x) = f(x)$$

holds at each point of continuity x of $f(x)$ and that the relation holds uniformly on [0, 1] if $f(x)$ is uniformly continuous on the interval.

Undoubtedly that the most intensively studied discrete operator is the celebrated Bernstein polynomial, which provides an elegant proof and example to the famous Weierstrass first approximation theorem for continuous function defined on [0, 1]. For detailed approach to this operator, see the classical book of Lorentz [2].

It is worthwhile to note that the linear positive operators have been obtained by starting from the following well-known properties of the probability density functions; for discrete case

$$\sum_{k=0}^{n} p_{n,k}(x) = 1$$

and for continuous case

$$\int_a^b f(t)dt = 1$$

from the probability theory.

Now, in view of the theory of singular integrals, we will characterize the positive linear operators in terms of the singular integrals.

In general, a singular integral may be written in the form

$$(T_n f)(x) = \int_a^b f(t)\, K_n(x,t)\, dt \tag{2}$$

where $K_n(x,t)$ is the kernel, defined for $a \leq x \leq b$, $a \leq t \leq b$, which has the property that for functions f of a certain class and in a certain sense, $(T_n f)(x)$ converges to $f(x)$ as $n \to \infty$.

The Bernstein polynomial (1) is a finite sum of a type corresponding to the integral (2). It is easy to see that (1) and (2) are special cases of singular Stieltjes integrals and hence (1) may be written in the form of a Stieltjes integral in the variable t as follows:

$$(B_n f)(x) = \int_0^1 f(t)\, d_t K_n(x,t)$$

with the kernel

$$K_n(x,t) = \sum_{k \le nt} \binom{n}{k} x^k (1-x)^{n-k} \quad , \quad 0 < t \le 1$$
$$K_n(x,0) = 0$$

which is constant in any interval $k/n \le t < (k+1)/n$, $k = 0, 1, \ldots, n-1$.

At the beginning, the theory of approximation is strongly related to the linearity of the operators. But, thanks to the approach of the famous Polish mathematician Julian Musielak, see [3], and afterwards continuous works of C. Bardaro, G. Vinti and their research group, this theory can be extended to the nonlinear-type operators, under some specific assumptions on its kernel functions; see the fundamental book due to Bardaro, Musielak and Vinti [4]. For further reading, please see [5–11] as well as the monographs [12].

Especially, nonlinear integral operators of type

$$(T_\lambda f)(x) = \int_a^b K_\lambda(t-x, f(t))\, dt, \qquad x \in (a,b),$$

and its special cases were studied by Bardaro, Karsli and Vinti [13, 14] and Karsli [15, 16] in some functional spaces.

In view of the approach due to Musielak [3], recently, Karsli–Tiryaki and Altin [17] introduced the following type nonlinear counterpart of the well-known Bernstein operators:

$$(NB_n f)(x) = \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) \quad , \quad 0 \le x \le 1 \quad , \quad n \in \mathbb{N}, \tag{3}$$

acting on bounded functions f on an interval $[0,1]$, where $P_{n,k}$ satisfy some suitable assumptions. They proved some existence and approximation theorems for the nonlinear Bernstein operators. In particular, they obtain some pointwise convergence for the nonlinear sequence of Bernstein operators (3) to some discontinuity point of the first kind of f, as $n \to \infty$.

As a continuation of the very recent paper of the author [18], the author and his PhD student estimated a Voronovskaya-type formula for this class nonlinear Bernstein operators on the interval [0, 1] (see [19]). Please see also very recent papers of the author [20, 21].

The most important and frequently investigated integral equations in nonlinear functional analysis are the Hammerstein equations

$$x(t) = y(t) + \int_a^b k(t,s) f(s, x(s)) ds, \qquad t \in [a,b],$$

and the Urysohn equations

$$x(t) = y(t) + \int_a^b k(t,s,x(s)) ds, \qquad t \in [a,b].$$

Consider the nonlinear operator equation

$$x = y + K(x)$$

where K is a completely continuous operator defined on a Banach space. An example of such an operator K is the Urysohn integral operator with a kernel function

$$Kx(t) = \int_\Omega k(t,s,x(s)) ds, \qquad t \in \Omega,\ x \in D$$

with a closed bounded region Ω in $\mathbb{R}^m$ for some $m \geq 1$, which includes the Fredholm equations of the first and second kind.

In the present work, we will deal with the following Urysohn equation:

$$x(t) = y(t) + \int_a^b k(t,s,x(s)) ds, \qquad t \in [a,b]$$

and corresponding Urysohn operator

$$Ux(t) = \int_a^b k(t,s,x(s)) ds, \qquad t \in [a,b]$$

where k and y are known functions and x is the unknown function to be determined. In the above equation, $k(t,s,x)$ is called kernel function of the type of Green's function, which is defined on $[a,b] \times [a,b] \times \mathbb{R}$ into $\mathbb{R}$.

The goal of this study is generalization and extension of the theory of interpolation of functions to functionals and operators by introducing the Urysohn-type nonlinear counterpart of the Bernstein operators. Afterwards, we investigate the convergence problem for these nonlinear operators that approximate the Urysohn-type operator in some functional spaces. The main difference between the present work and convergence to a function lies in the use of the Urysohn type operator values instead of the sampling values of a function.

Let us consider a sequence $NBF = (NB_nF)$ of operators, which we call it Urysohn-type nonlinear counterpart of the Bernstein operators, having the form

$$(NB_nF)x\,(t) = \int_0^1 \left[\sum_{k=0}^{n} P_{k,n}\left(x(s), f\left(t, s, \frac{k}{n} \right) \right) \right] ds \;\; , \;\; 0 \le x(s) \le 1 \;\; , \;\; n \in \mathbb{N},$$

acting on bounded functions f on an interval $[0, 1]$, where $P_{k,n}$ satisfy some suitable assumptions. In particular, we will put $Dom\ NBF = \bigcap_{n \in \mathbb{N}} Dom\ NB_nF$, where $Dom\ NB_nF$ is the set of all functions $f : [0, 1] \to \mathbb{R}$ for which the operator is well defined.

Since the theory of approximation is quite different from its linear counterpart, in same cases we can obtain only some estimates related to the convergence problems. Actually, in some cases, it is not possible to obtain exact estimates for nonlinear operators, because of the nonlinearity of their kernel functions.

2 Preliminaries and Auxiliary Results

This section is devoted to collecting some definitions and results which will be needed further on.

Here we consider the following type Urysohn integral operator,

$$Fx(t) = \int_0^1 f(t, s, x(s))ds, \qquad t \in [0, 1] \tag{4}$$

with unknown kernel f. If such a representation exists, then the kernel function $f(t, s, x)$ is called the Green's function, which is strongly related to the function x.

For a constant function $x(s) = a$, we set $Fa(t) := F(a)$.

Equation (4) was investigated by Urysohn in 1923–1924 in [22, 23]. This kind of equations appears in many problems. For example, it occurs in solving problems arising in economics, mathematics, engineering and physics (see [12, 24]).

It is well known that the solution of the following differential equation

$$DG(x, y) = \delta(x - y),$$

represents a Green function $G(x, y)$; here D is a differential operator, δ is the Dirac Delta function satisfying a boundary condition. Note that

$$\delta(x) = \frac{dH(x)}{dx},$$

is true, where

$$H(x) = \begin{cases} 1\ , \ x \geq 0 \\ 0\ , \ x < 0 \end{cases}$$

is the Heaviside function.

In view of these relations, we assume that continuous interpolation conditions hold:

$$Fx_i(t) = \int_0^1 f(t, s, x_i(s))ds, \qquad t \in [0, 1] \tag{5}$$

where $x_i(s) = \frac{i}{n} H(s - \xi)$; $\xi \in [0; 1]$; and $i = 0, 1, 2, \ldots$.

The following equalities are well known:

$$\frac{\partial Fx_i(t)}{\partial s} = \frac{\partial F\left(\frac{i}{n} H(t-s)\right)}{\partial s} = f(t, s, 0) - f(t, s, \frac{i}{n}), \tag{6}$$

where $x_i(t) = \frac{i}{n} H(t - s)$; $s \in [0; 1]$, and $i = 0, 1, 2, \ldots$.

Say

$$F_1\left(t, s, \frac{i}{n}\right) := \frac{\partial Fx_i(t)}{\partial s}. \tag{7}$$

According to the above definition together with (6) and (7), it is possible to construct an approximation operator in order to generalization and extension of the theory of interpolation of functions to operators.

In 2000, Demkiv [25] defined and investigated some properties of the following type Bernstein operators, which is linear with respect to F defined by (5):

$$(B_n F)\, x\, (t) = \int_0^1 \sum_{k=0}^{n} f\left(t, s, \frac{k}{n}\right) p_{n,k}\, (x(s))\, ds$$

In 2012, Makarov and Demkiv [26] considered the problem of approximation to the Urysohn operator (4) by Stancu-type operators, which is based on Polya distribution $p^{\alpha}_{n,k}\, (x(s))$, defined as:

$$\left(P^{\alpha}_n F\right) x\, (t) = \int_0^1 \sum_{k=0}^{n} f\left(t, s, \frac{k}{n}\right) p^{\alpha}_{n,k}\, (x(s))\, ds,$$

where $\alpha \geq 0$.

In 2017, the author [21] defined the following Urysohn type Meyer-K önig and Zeller operators:

$$(M_n F)x(t) = \int_0^1 \left[\sum_{k=0}^{\infty} f\left(t, s, \frac{k}{k+n}\right) m_{n,k}\left(x(s)\right) \right] ds,$$
$$(M_n F)1(t) = F1(t) = F(1),$$

where
$$m_{n,k}\left(x(s)\right) = \binom{n+k-1}{k} \left(x(s)\right)^k \left(1 - x(s)\right)^n,$$

n is a non-negative integer and $0 \leq x(s) < 1$, and obtained some positive results about the convergence problem.

In view of (3) and (5), we introduce the following Urysohn-type nonlinear Bernstein operators:

$$(NB_n F)x\,(t) = \int_0^1 \left[\sum_{k=0}^{n} P_{k,n}\left(x(s), f\left(t, s, \frac{k}{n}\right)\right) \right] ds, \tag{8}$$

where n is a non-negative integer, $P_{k,n}$ satisfy some suitable assumptions and $0 \leq x(s) \leq 1$.

Now, we assemble the main definitions and notations which will be used throughout the paper.

Let X be the set of all bounded Lebesgue measurable functions $f : [0, 1] \to \mathbb{R}_0^+ = [0, \infty)$.

Let Ψ be the class of all functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that the function ψ is continuous and concave with $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$.

We now introduce a sequence of functions. Let $\left\{P_{k,n}\right\}_{n\in\mathbb{N}}$ be a sequence functions $P_{k,n} : [0, 1]$ x $\mathbb{R} \to \mathbb{R}$ defined by

$$P_{k,n}\,(t, u) = p_{k,n}(t) H_n(u) \tag{9}$$

for every $t \in [0, 1]$, $u \in \mathbb{R}$, where $H_n : \mathbb{R} \to \mathbb{R}$ is such that $H_n(0) = 0$ and $p_{k,n}(t)$ is the Bernstein basis.

Throughout the paper, we assume that $\mu : \mathbb{N} \to \mathbb{R}^+$ is an increasing and continuous function such that $\lim\limits_{n\to\infty} \mu(n) = \infty$.

First of all, we assume that the following conditions hold:

(a) $H_n : \mathbb{R} \to \mathbb{R}$ *is such that*

$$\left|H_n(u) - H_n(v)\right| \leq |u - v|\,,$$

holds for every $u, v \in \mathbb{R}$, *for every* $n \in \mathbb{N}$. *That is,* H_n *satisfies a strong Lipschitz condition.*

(b) Denoting by $r_n(u) := H_n(u) - u$, $u \in \mathbb{R}$ *and* $n \in \mathbb{N}$, *such that*

$$\lim_{n\to\infty} r_n(u) = 0$$

uniformly with respect to u.

In other words, for n *sufficiently large*

$$\sup_u |r_n(u)| = \sup_u |H_n(u) - u| \leq \frac{1}{\mu(n)},$$

holds.

The symbol $[a]$ will denote the greatest integer not greater than a .

At first we recall the following results.

Lemma 1 *For* $(B_n t^s)(x)$, $s = 0, 1, 2,$ *one has*

$$\begin{aligned}(B_n 1)(x) &= 1\\(B_n t)(x) &= x\\(B_n t^2)(x) &= x^2 + \frac{x(1-x)}{n}.\end{aligned}$$

For proof of this Lemma, see [2].

By direct calculation, we find the following equalities:

$$(B_n\,(t-x)^2)(x) = \frac{x(1-x)}{n} \quad , \quad (B_n\,(t-x))(x) = 0\,.$$

Lemma 2 *For the central moments of order* $m \in N_0$

$$T_{n,m}(x) := \sum_{k=0}^{n} (k - nx)^m \; p_{k,n}\,(x)\,.$$

One has for each $m = 0, 1, \ldots$ *there is a constant* A_m *such that*

$$0 \leq T_{n,2m}(x) \leq A_m n^m.$$

The presented inequality is the well-known bound for the moments of the Bernstein polynomials, and it can be found in Chap. 10 in [27].

Lemma 3 *The first-order absolute moment for Bernstein polynomial is defined as*

$$M_1\left(p_{n,k}, x(s)\right) = \sum_{k=0}^{n} \left|\frac{k}{n} - x(s)\right| p_{n,k}\,(x(s))$$

and satisfies the following inequality

$$M_1\left(p_{n,k},x(s)\right)\leq\left(\frac{2x(s)\left(1-x(s)\right)}{\pi}\right)^{\frac{1}{2}}\frac{1}{\sqrt{n}}+\frac{B}{n}\frac{1}{\sqrt{x(s)\left(1-x(s)\right)}},$$

where

$$B=\left(\frac{\pi}{2}\right)^{\frac{5}{2}}+\frac{4}{\pi}+\frac{\pi^{\frac{9}{2}}}{54\sqrt{2}}o\left(\frac{1}{\sqrt{n}}\right),$$

which can be found in [28]. Note that the above inequality can be written as

$$M_1\left(p_{n,k},x(s)\right)\leq\left(\frac{2x(s)\left(1-x(s)\right)}{\pi}\right)^{\frac{1}{2}}\frac{1}{\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right).$$

Remark 1 By (6), (NB_nF) satisfies the following inequality:

$$\left|(NB_nF)x\,(t)\right|\leq|F(0)|+\left|\int_0^1\sum_{k=0}^{n}p_{k,n}\left(x(s)\right)\frac{\partial F\left(\frac{k}{n}H(t-s)\right)}{\partial s}ds\right|.$$

We can prove the above equality as follows:

$$\begin{aligned}
(NB_nF)x\,(t)&=\int_0^1\left[\sum_{k=0}^{n}P_{k,n}\left(x(s),\,f\left(t,s,\frac{k}{n}\right)\right)\right]ds\\
&=\int_0^1\sum_{k=0}^{n}p_{k,n}\left(x(s)\right)H_n\left(f\left(t,s,\frac{k}{n}\right)\right)ds\\
&\leq\int_0^1\sum_{k=0}^{n}p_{k,n}\left(x(s)\right)\left|f\left(t,s,\frac{k}{n}\right)\right|ds\\
&=\int_0^1\sum_{k=0}^{n}p_{k,n}\left(x(s)\right)\left|f(t,s,0)-\frac{\partial F\left(\frac{k}{n}H(t-s)\right)}{\partial s}\right|ds\\
&\leq|F(0)|+\left|\int_0^1\sum_{k=0}^{n}p_{k,n}\left(x(s)\right)\frac{\partial F\left(\frac{k}{n}H(t-s)\right)}{\partial s}ds\right|.
\end{aligned}$$

3 Convergence Property

We now introduce some notations and structural hypotheses, which will be fundamental in proving our convergence theorems.

Here, as usual, we denote by $C[0, 1]$ the Banach space of continuous functions $u : [0, 1] \to R$ with norm

$$\|u\| = \sup\{|u(x)| : x \in [0, 1]\}$$

Let Ψ be the set of all continuous, concave and non-decreasing functions $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with

$$\varphi(0) = 0,$$
$$\varphi(u) > 0 \text{ for all } u > 0$$

and

$$\lim_{u \to +\infty} \varphi(u) = +\infty$$

in the usual sense. Such function is called a φ-function.

Assume that the following condition holds:

$f : [0, 1] \times [0, 1] \times [0, 1] \to \mathbb{R}$ is such that

$$|f(t, s, u) - f(t, s, v)| \le \psi(|u - v|), \quad \psi \in \Psi,$$

holds for every $u, v \in [0, 1]$. That is, f satisfies a $(L - \Psi)$ Lipschitz condition with respect to the third variable.

Let $f \in C\left([a, b]^3\right)$ and $\delta > 0$ be given. Then the modulus of continuity is given by:

$$\omega(f; \delta) = \omega(\delta) = \sup_{|u-v| \le \delta\ ,\ t,s \in [a,b]} |f(t, s, u) - f(t, s, v)|. \tag{10}$$

Recall that $\omega(f; \delta)$ has the following properties:

(i) Let $\lambda \in \mathbb{R}^+$, then $\omega(f; \lambda\delta) \le (\lambda + 1)\,\omega(f; \delta)$,
(ii) $\lim_{\delta \to 0^+} \omega(f; \delta) = 0$,
(iii) $|f(t) - f(x)| \le \omega(|t - x|)$,
(iv) $|f(t) - f(x)| \le \left(\frac{|t-x|}{\delta} + 1\right)\omega(\delta)$.

We mention that some additional properties and applications of this modulus of continuity given in [2] and some of its generalizations can be found in [4].

Definition 1 We will say that the sequence $(P_n)_{n \in \mathbb{N}}$ is $(\psi - \alpha)$ −singular if the following assumptions are satisfied:

(P.1) For every $x \in I$ and $\delta > 0$, there holds

$$\psi\left(\sum_{\left|\frac{k}{n} - x\right| \ge \delta} \left|\frac{k}{n} - x\right| p_{n,k}(x)\right) = o\left(n^{-\alpha}\right), \quad (n \to \infty).$$

(P.2) For every $u \in \mathbb{R}$ and for every $x \in I$, we have

$$\lim_{n\to\infty} n^{\alpha}\left[\sum_{k=0}^{n} P_{n,k}(x,u) - u\right] = 0.$$

We are now ready to establish one of the main results of this study:

Theorem 1 *Let F be the Urysohn integral operator with $0 \le x(s) \le 1$. Then (NB_nF) converges to F uniformly in $x \in C[0,1]$. That is,*

$$\lim_{n\to\infty} \|(NB_nF)x\,(t) - Fx(t)\| = 0.$$

Proof In view of the definition of the operator (8), by considering (5), (9), (6) and (7), we have

$$|(NB_nF)x\,(t) - Fx(t)| = \left|\int_0^1 \left[\sum_{k=0}^{n} P_{k,n}\left(x(s), f\left(t,s,\frac{k}{n}\right)\right)\right] ds - Fx(t)\right|$$

$$\le \left|\int_0^1 \left[\sum_{k=0}^{n} p_{k,n}(x(s))\, H_n\left(f\left(t,s,\frac{k}{n}\right)\right)\right] ds - \int_0^1 \left[\sum_{k=0}^{n} p_{k,n}(x(s))\, H_n(f(t,s,x(s)))\right] ds\right|$$
$$+ \left|\int_0^1 \left[\sum_{k=0}^{n} p_{k,n}(x(s))\, H_n(f(t,s,x(s)))\right] ds - \int_0^1 f(t,s,x(s))ds\right|$$

$$\le \left|\int_0^1 \sum_{k=0}^{n} p_{k,n}(x(s))\left[H_n\left(f\left(t,s,\frac{k}{n}\right)\right) - H_n(f(t,s,x(s)))\right] ds\right|$$
$$+ \int_0^1 \sum_{k=0}^{n} p_{k,n}(x(s))\,|H_n(f(t,s,x(s))) - f(t,s,x(s))|\,ds$$
$$: = I_1 + I_2.$$

By assumption (b), the second term, namely I_2, tends to zero as $n \to \infty$. In fact,

$$
\begin{aligned}
I_2 &= \int_0^1 \sum_{k=0}^{n} p_{k,n}(x(s)) \left| H_n(f(t,s,x(s))) - f(t,s,x(s)) \right| ds \\
&\le \int_0^1 \sum_{k=0}^{n} p_{k,n}(x(s)) \frac{1}{\mu(n)} ds \\
&= \frac{1}{\mu(n)},
\end{aligned}
$$

which tends to zero as $n \to \infty$. Now, it is sufficient to evaluate the term I_1. Using the definition of the function $F_1(t,s,x(s))$, we have

$$
\begin{aligned}
I_1 &\le \int_0^1 \sum_{k=0}^{n} p_{k,n}(x(s)) \left| f\left(t,s,\frac{k}{n}\right) - f(t,s,x(s)) \right| ds \\
&= \int_0^1 \sum_{k=0}^{n} p_{k,n}(x(s)) \left| f(t,s,0) - f(t,s,x(s)) - \left[f(t,s,0) - f\left(t,s,\frac{k}{n}\right)\right]\right| ds \\
&= \int_0^1 \sum_{k=0}^{n} p_{k,n}(x(s)) \left| F_1(t,s,x(s)) - F_1\left(t,s,\frac{k}{n}\right)\right| ds.
\end{aligned}
$$

Let us divide the last term into two parts as:

$$
I_1 \le I_{1,1} + I_{1,2},
$$

where

$$
I_{1,1} = \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|<\delta} p_{n,k}(x(s)) \left| F_1(t,s,x(s)) - F_1\left(t,s,\frac{k}{n}\right)\right| ds
$$

and

$$
I_{1,2} = \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|\ge\delta} p_{n,k}(x(s)) \left| F_1(t,s,x(s)) - F_1\left(t,s,\frac{k}{n}\right)\right| ds.
$$

Since $x \in C[0,1]$, then clearly

$$
\left| F_1(t,s,x(s)) - F_1\left(t,s,\frac{k}{n}\right)\right| < \epsilon
$$

holds true when $\left|\frac{k}{n} - x(s)\right| < \delta$, and

$$\left|F_1\left(t,s,x(s)\right)-F_1\left(t,s,\frac{k}{n}\right)\right| \leq \left|F_1\left(t,s,x(s)\right)\right| + \left|F_1\left(t,s,\frac{k}{n}\right)\right| \\ \leq 2M$$

holds true for some $M > 0$, when $\left|\frac{k}{n}-x(s)\right| \geq \delta$.

So

$$\begin{aligned} I_{1,1} &= \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|<\delta} p_{n,k}\left(x(s)\right)\left|F_1\left(t,s,x(s)\right)-F_1\left(t,s,\frac{k}{n}\right)\right| ds \\ &\leq \epsilon \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|<\delta} p_{n,k}\left(x(s)\right) ds \\ &\leq \epsilon, \end{aligned}$$

and

$$\begin{aligned} I_{1,2} &= \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|\geq\delta} p_{n,k}\left(x(s)\right)\left|F_1\left(t,s,x(s)\right)-F_1\left(t,s,\frac{k}{n}\right)\right| ds \\ &\leq 2M \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|\geq\delta} p_{n,k}\left(x(s)\right) ds \\ &\leq 2M \int_0^1 \left[\sum_{\left|\frac{k}{n}-x(s)\right|\geq\delta} \left(\frac{\frac{k}{n}-x(s)}{\delta}\right)^2 p_{n,k}\left(x(s)\right)\right] ds \\ &= \frac{2M}{\delta^2} \int_0^1 \left[\sum_{\left|\frac{k}{n}-x(s)\right|\geq\delta} \left(\frac{k}{n}-x(s)\right)^2 p_{n,k}\left(x(s)\right)\right] ds \\ &\leq \frac{2M}{\delta^2} \int_0^1 \left[\sum_{k=0}^{n} \left(\frac{k}{n}-x(s)\right)^2 p_{n,k}\left(x(s)\right)\right] ds. \end{aligned}$$

In view of Lemma 2, we obtain

$$I_{1,2} \leq \frac{2M}{\delta^2}\frac{A_1}{n}.$$

Collecting these estimates, we have

$$|(NB_nF)x\,(t) - Fx(t)| \leq \epsilon + \frac{2MA_1}{n\delta^2} + \frac{1}{\mu\,(n)}.$$

That is,

$$\lim_{n\to\infty} \|(NB_nF)x\,(t) - Fx(t)\|_{C[0,1]} = 0.$$

This completes the proof.

Theorem 2 *Let F be the Urysohn integral operator with $x \in C[0,1]$, and $0 \leq x(s) \leq 1$. Then for every $\varepsilon > 0$*

$$|(NB_nF)x\,(t) - Fx(t)| \leq \psi\,(\varepsilon) + 2\omega\,(f;\delta) + \frac{1}{\mu\,(n)}$$

holds true, where $\delta = \sqrt{x(s)\,(1 - x(s))\,/n}$.

Proof Clearly, one has

$$|(NB_nF)x\,(t) - Fx(t)| \leq \int_0^1 \left[\sum_{k=0}^{n} p_{k,n}\,(x(s)) \left| f\left(t, s, \frac{k}{n}\right) - f\,(t, s, x(s))\right|\right] ds + \frac{1}{\mu\,(n)}$$
$$:= I_{n,1}\,(x) + \frac{1}{\mu\,(n)}, \tag{11}$$

say. Since $x \in C[0,1]$ we can rewrite (11) as follows:

$$I_{n,1}\,(x) \leq \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|<\delta} p_{n,k}\,(x(s))\,\psi\left(\left|x(s) - \frac{k}{n}\right|\right) ds$$
$$+ \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|\geq\delta} p_{n,k}\,(x(s)) \left| f\left(t, s, \frac{k}{n}\right) - f\,(t, s, x(s))\right| ds$$
$$\leq \psi\,(\varepsilon) + I_{n,1,2}\,(x)\,.$$

Taking into account that $\omega\,(f;\delta)$ is the modulus of continuity defined as (10), $I_{n,1,2}\,(x)$ can be written as

$$I_{n,1,2}\,(x) = \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|\geq\delta} p_{n,k}\,(x(s)) \left| f\left(t, s, \frac{k}{n}\right) - f\,(t, s, x(s))\right| ds$$

$$\leq \int_0^1 \omega(f;\delta) \sum_{\left|\frac{k}{n}-x(s)\right|\geq\delta} \left(\frac{\left|\frac{k}{n}-x(s)\right|}{\delta} + 1 \right) ds$$

$$\leq \omega(f;\delta) \left\{ 1 + \delta^{-1} \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|\geq\delta} \left| \frac{k}{n} - x(s) \right| p_{n,k}(x(s))\, ds \right\}$$

$$\leq \omega(f;\delta) \left\{ 1 + \delta^{-2} \int_0^1 \sum_{k=0}^{n} \left(\frac{k}{n} - x(s) \right)^2 p_{n,k}(x(s))\, ds \right\}$$

$$\leq \omega(f;\delta) \left\{ 1 + \frac{A_1}{\delta^2 n} \right\}.$$

If we choose

$$\delta = \sqrt{\frac{A_1}{n}},$$

then one can obtain the desired estimate, namely

$$|(M_n F)x(t) - Fx(t)| \leq \psi(\varepsilon) + 2\omega(f;\delta) + \frac{1}{\mu(n)}.$$

Thus, the proof is now complete.

Theorem 3 *Let F be the Urysohn integral operator with $x \in C[0,1]$, and $0 < x(s) < 1$. Then*

$$|(NB_n F)x(t) - Fx(t)| \leq \psi\left(\frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{2\pi}} + BA \right) \right) + \frac{1}{\mu(n)}$$

holds true for constants A and B, for which

$$\int_0^1 \frac{ds}{\sqrt{x(s)(1-x(s))}} = A < \infty,$$

and

$$B = \left(\frac{\pi}{2}\right)^{\frac{5}{2}} + \frac{4}{\pi} + \frac{\pi^{\frac{9}{2}}}{54\sqrt{2}}.$$

Proof Using the similar lines to the proof of Theorem 2, one has

$$|(NB_nF)x(t) - Fx(t)| \leq \int_0^1 \left[\sum_{k=0}^{n} p_{k,n}(x(s)) \left| F_1(t,s,x(s)) - F_1\left(t,s,\frac{k}{n}\right) \right| \right] ds + \frac{1}{\mu(n)}$$
$$\leq \int_0^1 \left[\sum_{k=0}^{n} p_{n,k}(x(s))\, \psi\left(\left| x(s) - \frac{k}{n} \right| \right) \right] ds + \frac{1}{\mu(n)}.$$

By concavity of the function ψ, and using Jensen's inequality, we obtain

$$|(NB_nF)x(t) - Fx(t)| \leq \psi\left(\int_0^1 \left[\sum_{k=0}^{n} \left| \left(\frac{k}{n} - x(s) \right) \right| p_{n,k}(x(s)) \right] ds \right) + \frac{1}{\mu(n)}$$

Since ψ is non-decreasing, we apply the inequality of the first absolutely moment given in Remark 1; then we can write

$$|(NB_nF)x(t) - Fx(t)| \leq \psi\left(\int_0^1 \left[\left(\frac{2x(s)(1-x(s))}{\pi} \right)^{\frac{1}{2}} \frac{1}{\sqrt{n}} + \frac{B}{n} \frac{1}{\sqrt{x(s)(1-x(s))}} \right] ds \right)$$
$$\leq \psi\left(\frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{2\pi}} + BA \right) \right) + \frac{1}{\mu(n)}.$$

So we get the desired estimate.

4 Practical Examples, Graphical Representations

In this section, we will apply the theory to the theory of interpolation of functions to functionals or operators by means of Urysohn-type nonlinear operators.

We note that in Figs. 1 and 2, the graph with the red line belongs to the original function, the graph with the green line to the operators with $n = 2$, and finally the graph consisting of blue line to the operators with $n = 10$.

Example 1 Let us consider the operator $Fx(t) = \int_0^1 x^3(t)dt$, and we take its corresponding nonlinear Bernstein operator $(NB_nF)x(t)$; then one has for $n = 2$ and for $n = 10$.

The corresponding numerical evaluation on the left-hand side yields numerically, for $n = 10, 20, 30, 40$,

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$(NB_nF)(0.3)$	0.04674	0.03666	0.0333933	0.0317775 ,
$f(0.3)$	0.027	0.027	0.027	0.027

Fig. 1 Approximation of $Fx(t) = \int_0^1 x^3(t)dt$ by Urysohn-type nonlinear Bernstein operator, for $n = 2$ and $n = 10$

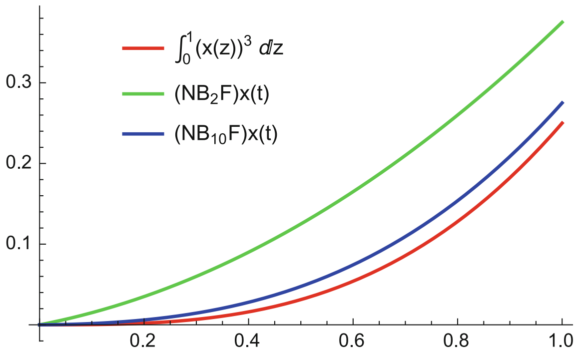
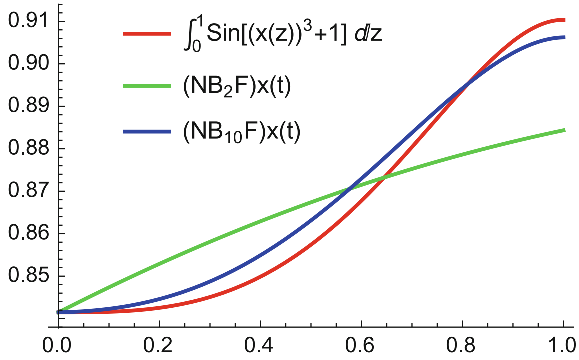


Fig. 2 Approximation of $Fx(t) = \int_0^1 \sin\left[x^3(t) + 1\right] dt$ by Urysohn-type nonlinear Bernstein operator, for $n = 2$ and $n = 10$



	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$(NB_nF)\,(0.5)$	0.1625	0.14375	0.1375	0.134375 ,
$f\,(0.5)$	0.125	0.125	0.125	0.125

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$(NB_nF)\,(0.8)$	0.54944	0.53096	0.524693	0.52154 .
$f\,(0.8)$	0.512	0.512	0.512	0.512

Example 2 Let us consider the operator $Fx(t) = \int_0^1 \sin\left[x^3(t) + 1\right] dt$, and we take its corresponding nonlinear Bernstein operator $(NB_nF)x\,(t)$; then one has for $n = 2$ and for $n = 10$.

Finally, numerically for $n = 10, 20, 30, 40$,

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$(NB_nF)\,(0.3)$	0.864157	0.860156	0.858726	0.857995 ,
$f\,(0.3)$	0.855751	0.855751	0.855751	0.855751

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$(NB_nF)\,(0.5)$	0.908705	0.906317	0.905157	0.904507 ,
$f(0.5)$	0.902268	0.902268	0.902268	0.902268

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$(NB_nF)\,(0.8)$	0.9723	0.984958	0.98933	0.991542 .
$f(0.8)$	0.998272	0.998272	0.998272	0.998272

The situation is similar for other examples studied.

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