Statistical Deferred Cesàro Summability Mean Based on (p, q)-Integers with Application to Approximation Theorems

S. K. Paikray, B. B. Jena and U. K. Misra

Abstract This chapter consists of four sections. The first section is introductory in which a concept (presumably new) of statistical deferred Cesàro summability mean based on (p, q)-integers has been introduced and accordingly some basic terminologies are presented. In the second section, we have applied our proposed mean under the difference sequence of order r to prove a Korovkin-type approximation theorem for the set of functions 1, e^{-x} and e^{-2x} defined on a Banach space $C[0, \infty)$ and demonstrated that our theorem is a non-trivial extension of some well-known Korovkin-type approximation theorems. In the third section, we have established a result for the rate of our statistical deferred Cesàro summability mean with the help of the modulus of continuity. Finally, in the last section, we have given some concluding remarks and presented some interesting examples in support of our definitions and results.

Keywords Statistical convergence · Statistical deferred Cesàro summability
Delayed arithmetic mean · Difference sequence of order r · (p, q)-integers
Banach space · Positive linear operators · Korovkin-type approximation theorem
Rate of convergence

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1 Introduction

In the study of sequence spaces, classical convergence has got numerous applications where the convergence of a sequence requires that almost all elements are to satisfy the convergence condition. That is, all the elements of the sequence need to be in an arbitrarily small neighborhood of the limit. However, such restriction is relaxed in statistical convergence, where the validity of convergence condition is achieved only for a majority of elements. The notion of statistical convergence was introduced by Fast [13] and Steinhaus [29]. Recently, statistical convergence has been a dynamic research area due to the fact that it is more general than classical convergence, and such theory is discussed in the study of Fourier analysis, number theory, and approximation theory. For more details, see [7, 9–11, 14, 15, 17, 21, 24, 26–28].

Let ω be the set of all real-valued sequences, and suppose any subspace of ω be the sequence space. Let (x_k) be a sequence with real and complex terms. Suppose ℓ_∞ be the class of all bounded linear spaces, and let c, c_0 be the respective classes for convergent and null sequences with real and complex terms. We have

$$\|x\|_\infty = \sup_k |x_k| \quad (k \in \mathbb{N}),$$

and we recall here that under this norm, the above-mentioned spaces are all Banach spaces.

The notion of difference sequence space was initially studied by Kızmaz [18], and then, it was extended to the difference sequence of natural order $r \quad (r \in \mathbb{N}_0 := \{0\} \cup \mathbb{N})$ by defining

$$\begin{aligned} &\lambda(\Delta^r) = \left\{x = (x_k) : \quad \Delta^r(x) \in \lambda, \quad \lambda \in (\ell_\infty, c_0, c)\right\}; \\ &\Delta^0 x = (x_k); \quad \Delta^r x = (\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1}) \end{aligned}$$

and

$$\Delta^r x_k = \sum_{i=0}^{r} (-1)^i \binom{r}{i} x_{k+i}$$

(see [18]). Also, these are all Banach spaces under the norm defined by

$$\|x\|_{\Delta^r} = \sum_{i=1}^{r} |x_i| + \sup_k |\Delta^r x_k|.$$

For more interest in this direction, see the current works [6, 12, 16].

Let $\mathbb{N}$ be the set of natural numbers, and let $K \subseteq \mathbb{N}$. Also let

$$K_n = \{k : k \leq n \ \text{ and } \ k \in K\}$$

and suppose that $|K_n|$ be the cardinality of K_n. Then, the natural density of K is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{|K_n|}{n} = \lim_{n\to\infty} \frac{1}{n}|\{k : k \leq n \text{ and } k \in K\}|,$$

provided the limit exists.

A given sequence (x_n) is said to be statistically convergent to ℓ if, for each $\epsilon > 0$, the set

$$K_\epsilon = \{k : k \in \mathbb{N} \text{ and } |x_k - \ell| \geq \epsilon\}$$

has zero natural density (see [13, 29]). That is, for each $\epsilon > 0$,

$$\delta(K_\epsilon) = \lim_{n\to\infty} \frac{|K_\epsilon|}{n} = \lim_{n\to\infty} \frac{1}{n}|\{k : k \leq n \text{ and } |x_k - \ell| \geq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat} \lim_{n\to\infty} x_n = \ell.$$

Now, we present an example to show that every convergent sequence is statistically convergent, but the converse is not true in general.

Example 1 Let us consider the sequence $x = (x_n)$ by

$$x_n = \begin{cases} n & \text{when } n = m^2, \text{ for all } m \in \text{N} \\ \frac{1}{n} & \text{otherwise.} \end{cases}$$

Then, it is easy to see that the sequence (x_n) is divergent in the ordinary sense, while 0 is the statistical limit of (x_n) since $\delta(K) = 0$, where $K = \{m^2, \text{ for all } m = 1, 2, 3, \ldots\}$.

In 2002, Móricz [22], introduced the fundamental idea of statistical $(C, 1)$ summability and recently Mohiuddine et al. [20] has established statistical $(C, 1)$ summability as follows.

Let us consider a sequence $x = (x_n)$; the $(C, 1)$ mean of the sequence is given by

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} x_k,$$

and (x_n) is said to be statistical $(C, 1)$ summable to ℓ if, for each $\epsilon > 0$, the set

$$\{k : k \in \mathbb{N} \text{ and } |\sigma_k - \ell| \geq \epsilon\}$$

has zero Cesàro density. That is, for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k : k \leq n \text{ and } |\sigma_k - \ell| \geq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat} \lim_{n\to\infty} \sigma_n = \ell \;\text{ or }\; C_1(\text{stat}) \lim_{n\to\infty} x_n = \ell.$$

Subsequently, with the development of q-calculus, various researchers worked on certain new generalizations of positive linear operators based on q-integers (see [3, 5]). Recently, Mursaleen et al. [23] introduced the (p, q)-analogue of Bernstein operators in connection with (p, q)-integers, and later on, some approximation results for Baskakov operators and Bernstein-Schurer operators are studied for (p, q)-integers by [1].

We now recall some definitions and basic notations on (p, q)-integers for our present study:

For any $(n \in \mathbb{N})$, the (p, q)-integer $[n]_{p,q}$ is defined by,

$$[n]_{p,q} = \begin{cases} \frac{p^n - q^n}{p-q} & (n \geq 1) \\ 0 & (n = 0) \end{cases}$$

where $0 < q < p \leq 1$.

The (p, q)-factorial is defined by

$$[n]!_{p,q} = \begin{cases} [1]_{p,q}[2]_{p,q} \ldots [n]_{p,q} & (n \geq 1) \\ 1 & (n = 0). \end{cases}$$

The (p, q)-binomial coefficient is defined by,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]!_{p,q}}{[k]!_{p,q}\,[n-k]!_{p,q}} \qquad \text{for all } n, k \in \mathbb{N} \text{ and } n \geq k.$$

We also recall that suppose $0 < q < p \leq 1$ and r be a nonnegative integer. Then, the operator

$$\Delta^{[r]}_{p,q} : \omega \to \omega$$

is defined by

$$\Delta^{[r]}_{p,q}(x_n) = \sum_{i=0}^{r} (-1)^i \begin{bmatrix} r \\ i \end{bmatrix}_{p,q} x_{n-i}.$$

That is,

$$\begin{aligned}\Delta_{p,q}^{[r]}(x_n) &= \begin{bmatrix} r \\ 0 \end{bmatrix}_{p,q} x_n - \begin{bmatrix} r \\ 1 \end{bmatrix}_{p,q} x_{n-1} + \begin{bmatrix} r \\ 2 \end{bmatrix}_{p,q} x_{n-2} - \begin{bmatrix} r \\ 3 \end{bmatrix}_{p,q} x_{n-3} + \cdots + (-1)^r \begin{bmatrix} r \\ r \end{bmatrix}_{p,q} x_{n-r} \\ &= x_n - [r]_{p,q} x_{n-1} + \frac{[r]_{p,q}[r-1]_{p,q}}{[2]_{p,q}!} x_{n-2} - \frac{[r]_{p,q}[r-1]_{p,q}[r-2]_{p,q}}{[3]!} x_{n-3} + \cdots + (-1)^r x_{n-r} \\ &= x_n - \left(\frac{p^r - q^r}{p - q}\right) x_{n-1} + \left(\frac{(p^r - q^r)(p^{r-1} - q^{r-1})}{(p-q)^2(p+q)}\right) x_{n-2} \\ &\quad - \left(\frac{(p^r - q^r)(p^{r-1} - q^{r-1})(p^{r-2} - q^{r-2})}{(p-q)^3(p^2 + pq + q^2)(p+q)}\right) x_{n-3} + \cdots + (-1)^m x_{n-r}.\end{aligned}$$

Now, we present an example to see that a sequence is not convergent; however, the associated difference sequence is convergent.

Example 2 Let us consider a sequence $(x_n) = n + 1$ $(n \in \mathbb{N})$. It is clear that the sequence (x_n) is not convergent in the ordinary sense.

Also, we see that

$$\Delta^{[3]}(x_n) = x_n - 3x_{n-1} + 3x_{n-2} - x_{n-3} \quad (x_n = n + 1)$$

converges to 0 $(n \to \infty)$.

For $r = 3$, we obtain that

$$\begin{aligned}\Delta_{p,q}^{[3]}(x_n) &= x_n - [3]_{p,q} x_{n-1} + [3]_{p,q} x_{n-2} - x_{n-3} \quad (x_n = n + 1) \\ &= x_n - (p_n^2 + p_n q_n + q_n^2) x_{n-1} + (p_n^2 + p_n q_n + q_n^2) x_{n-2} - x_{n-3} \\ &= n + 1 - (p_n^2 + p_n q_n + q_n^2) n + (p_n^2 + p_n q_n + q_n^2)(n-1) - (n-2) \quad (x_n = n + 1) \\ &= 3 - (\beta^2 + \alpha\beta + \alpha^2).\end{aligned}$$

Clearly, depending on the choice of the values of p and q, the difference sequence $\Delta_{p,q}^{[3]}(x_n)$ of third order has different limits. This situation is due to the definition of (p, q)-integers. However, in order to obtain a convergence criterion for all values of p and q, belonging to the operator $\Delta_{p,q}^{[r]}$, we must have to overcome this difficulty. This type of difficulties can be avoided in the following two ways. The first one is taking $p = q = 1$, and thus, the operator reduces to the usual difference sequence. Next, the second way is to replace $p = p_n$ and $q = q_n$ under the limits, $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 \leq \alpha, \beta \leq 1)$ where $0 < q_n < p_n \leq 1$, for all $(n \in \mathbb{N})$. Afterward, the difference sequence $\Delta_{p,q}^{[3]}(x_n)$ of third order 3 converges to the value $3 - (\beta^2 + \alpha\beta + \alpha^2)$. Thus, if we take $q_n = \left(\frac{n+1}{n+1+s}\right) < \left(\frac{n+1}{n+1+t}\right) = p_n$ such that $0 < q_n < p_n \leq 1$ $(s > t > 0)$, then $\lim_n q_n = 1 = \lim_n p_n$ and hence $\Delta_{p,q}^{[3]}(x_n) \to 0 \quad (n \to \infty)$.

Remark 1 If $r = 1$, $\lim_n q_n = 1$, and $\lim_n p_n = 1$, then the difference operator $\Delta_{p,q}^{[r]}$ reduces to the $\Delta^{[1]}$. Also, if $r = 0$, $\lim_n q_n = 1$ and $\lim_n p_n = 1$, then the difference operator $\Delta_{p,q}^{[r]}$ reduces to the general sequence (x_n).

Here, we now present the notion of the statistical deferred Cesàro summability under the generalized difference sequence of order r involving (p, q)-integers:

Let (a_n) and (b_n) be sequences of nonnegative integers such that (i) $a_n < b_n$ and (ii) $\lim_{n\to\infty} b_n = \infty$,

then the deferred Cesàro $D(a_n, b_n)$ mean based on (p, q)-integers is defined by,

$$D_{p,q}(a_n, b_n) = D_{p,q}(x_n) = \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} \Delta_{p,q}^{[r]}(x_k). \tag{1}$$

It is well known that $D_{p,q}(a_n, b_n)$ is regular under conditions (i) and (ii) (see Agnew [2]).

Remark 2 If $p = q = 1$, then the deferred Cesàro mean under (p, q)-integers reduces to the deferred Cesàro mean (see [15]).

Let us now introduce the following definitions in support of our proposed work.

Definition 1 Let $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$, and let r is a nonnegative integer. Also, let (a_n) and (b_n) be sequences of integers (nonnegative). A sequence (x_n) is said to be statistical deferred Cesàro summable to ℓ with respect to difference sequence of order r based on (p, q)-integers if, for every $\epsilon > 0$, the set

$$\{k : a_n < k \leq b_n \text{ and } |D_{p,q}(x_n) - \ell| \geq \epsilon\}$$

has natural density zero, that is,

$$\lim_{n\to\infty} \frac{1}{b_n - a_n} |\{k : a_n < k \leq b_n \text{ and } |D_{p,q}(x_n) - \ell| \geq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat} \lim_{n\to\infty} D_{p,q}(x_n) = \ell \text{ or } \lim_{n\to\infty} \text{stat}_{DC}^{p,q} x_n = \ell.$$

Clearly, above definition can be viewed as the generalization of some existing definitions.

Remark 3 If $a_n = n - 1$, $b_n = n$, and $p_n = q_n = 1$, then $D(n - 1, n)$ reduces to the identity transformation, and also, if $a_n = 0$, $b_n = n$, and $p_n = q_n = 1$, then $D(0, n)$ reduces to $(C, 1)$ transformation of x_n, which is often denoted as σ_n. Furthermore, if $a_n = n - 1$, $b_n = n + t - 1$, and $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$ and let r is a nonnegative integer, then

$$D_{p,q}(n - 1, n + t - 1) = \sigma_{n,t}^{p,q} = \left(\frac{t + n}{t}\right) \sigma_{n+t-1}^{p,q} - \left(\frac{n}{t}\right) \sigma_{n-1}^{p,q}, \tag{2}$$

which is called the deferred delayed arithmetic mean. Finally, if $a_n = n - 1$, $b_n = n + t - 1$ and $p_n = q_n = 1$, then

$$D(n-1, n+t-1) = \sigma_{n,t} = \left(\frac{t+n}{t}\right)\sigma_{n+t-1} - \left(\frac{n}{t}\right)\sigma_{n-1},$$

which is called the delayed arithmetic mean (see [31], p. 80).

Definition 2 Let $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$, and let r is a nonnegative integer. Also, let (a_n) and (b_n) be sequences of integers (nonnegative). A sequence (x_n) is said to be statistical deferred delayed arithmetic summable to ℓ if, for every $\epsilon > 0$, the set

$$\{k : n-1 < k \leq n+t-1 \text{ and } |\sigma_{n,t}^{p,q} - \ell| \geq \epsilon\}$$

has zero natural density, that is,

$$\lim_{n\to\infty} \frac{1}{t}|\{k : n-1 < k \leq n+t-1 \text{ and } |\sigma_{n,t}^{p,q} - \ell| \geq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat} \lim_{n\to\infty} \sigma_{n,t}^{p,q} = \ell \text{ or } \text{stat}_{DA}^{p,q} x_n = \ell.$$

Now, we present below an example to show that a sequence is statistically deferred Cesàro summable, whenever it is not statistically Cesàro summable.

Example 3 For $\lim_n q_n = 1$, $\lim_n p_n = 1$, $a_n = 2n$ and $b_n = 4n$ $(\forall\, n \in \mathbb{N})$, consider a sequence $x = (x_n)$,

$$x_n = \begin{cases} \frac{1}{m^2} & (n = m^2 - m, m^2 - m + 1, \ldots, m^2 - 1) \\ -\frac{1}{m^3} & (n = m^2, \quad m > 1) \\ 0 & (\text{otherwise}). \end{cases}$$

We have,

$$\begin{aligned} \Delta_{p,q}^{[r]}(x_k) &= \sum_{i=0}^{r} (-1)^i \begin{bmatrix} r \\ i \end{bmatrix}_{p,q} x_{n-i} \\ &= \left\{ x_n - \begin{bmatrix} r \\ 1 \end{bmatrix}_{p,q} x_{n-1} + \begin{bmatrix} r \\ 2 \end{bmatrix}_{p,q} x_{n-2} - \begin{bmatrix} r \\ 3 \end{bmatrix}_{p,q} x_{n-3} + \cdots + (-1)^r \begin{bmatrix} r \\ r \end{bmatrix}_{p,q} x_{n-r} \right\} \\ &= \left\{ x_n - [r]_{p,q} x_{n-1} + \frac{[r]_{p,q}[r-1]_{p,q}}{[2]_{p,q}!} x_{n-2} - \frac{[r]_{p,q}[r-1]_{p,q}[r-2]_{p,q}}{[3]!} x_{n-3} + \cdots + (-1)^r x_{n-r} \right\}. \end{aligned}$$

Thus,

$$D_{p,q}(a_n, b_n) = D_{p,q}(x_n) = \frac{1}{4n - 2n} \sum_{k=2n+1}^{4n} \Delta_{p,q}^{[r]}(x_k)$$

which implies that

$$D_{p,q}(x_n) \to 0.$$

Hence, (x_n) is not deferred Cesàro summable , even if it is statistical deferred Cesàro summable under the difference operator of order r based on (p, q)-integers.

In the year 2012, Mohiuddine et al. [20] established statistical summability $(C, 1)$ and a Korovkin-type approximation theorem, and then, Jena et al. [15] investigated a Korovkin-type approximation theorem for exponential functions via the statistical deferred Cesàro summability of the real sequence. Very recently, Srivastava et al. [26] has established generalized equi-statistical convergence of the deferred Nörlund summability and its applications to associated approximation theorems, and then, Srivastava et al. [27] established a certain class of weighted statistical convergence and associated Korovkin-type approximation theorems for trigonometric functions. Furthermore, Srivastava et al. [28] has proved some interesting results on deferred weighted $\mathcal{A}$-statistical convergence based on the (p, q)-Lagrange polynomials and its applications to approximation theorems.

The main object of this chapter is to establish some important approximation theorems over the Banach space based on statistical deferred Cesàro summability for (p, q)-integers under difference sequence of order r which will effectively extend and improve most (if not all) of the existing results depending on the choice of sequences of the simple statistical deferred Cesàro means. Furthermore, we intend to estimate the rate of our statistical deferred Cesàro summability and investigate Korovkin-type approximation results.

2 A Korovkin-Type Approximation Theorem

Several researchers have worked on extending or generalizing the Korovkin-type theorems in many ways and to several settings, including Function spaces, Banach Algebras, Banach spaces. This theory is very useful in real analysis, functional analysis, harmonic analysis, measure theory, probability theory, and summability theory. Recently, Jena et al. [15] have proved the Korovkin theorem via statistical deferred Cesàro summability on $\mathcal{C}[0, \infty)$ by using the test functions $1,\ e^{-x}$, and e^{-2x}. In this paper, we generalize the result of Jena, Paikray, and Misra via the notion of statistical deferred Cesàro summability based on difference sequence of order r including (p, q)-integers for the same test functions $1,\ e^{-x}$, and e^{-2x}. We also present an example to justify that our result is stronger than that of Jena, Paikray, and Misra (see [15]).

Let $\mathcal{C}(X)$, be the space of all real-valued continuous functions defined on $[0, \infty)$ under the norm $\|.\|_\infty$. Also, $\mathcal{C}[0, \infty)$ is a Banach space. We have, for $f \in \mathcal{C}[0, \infty)$, the norm of f denoted by $\|f\|$ is given by

$$\|f\|_\infty = \sup_{x\in[0,\infty)} \{|f(x)|\}$$

with

$$\omega(\delta, f) = \sup_{0\leq|h|\leq\delta} \|f(x+h) - f(x)\|_\infty, \;\; f \in \mathcal{C}[0,\infty).$$

The quantities $\omega(\delta, f)$ is called the modulus of continuity of f.

Let $L : \mathcal{C}[0,\infty) \to \mathcal{C}[0,\infty)$ be a linear operator. Then, as usual, we say that L is a positive linear operator provided that

$$f \geq 0 \;\text{ implies }\; L(f) \geq 0.$$

Also, we denote the value of $L(f)$ at a point $x \in [0,\infty)$ by $L(f(u); x)$ or, briefly, $L(f; x)$.

The classical Korovkin theorem states as follows [19]:

Let $L_n : \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ be a sequence of positive linear operators and let $f \in \mathcal{C}[0,\infty)$. Then

$$\lim_{n\to\infty} \|L_n(f; x) - f(x)\|_\infty = 0 \Longleftrightarrow \lim_{n\to\infty} \|L_n(f_i; x) - f_i(x)\|_\infty = 0 \;\; (i = 0, 1, 2),$$

where

$$f_0(x) = 1, \quad f_1(x) = x \;\text{ and }\; f_2(x) = x^2.$$

Now, we prove the following theorem by using the notion of statistical deferred Cesàro summability based on (p, q)-integers.

Theorem 1 *Let $L_m : \mathcal{C}[0,\infty) \to \mathcal{C}[0,\infty)$ be a sequence of positive linear operators. Then, for all $f \in \mathcal{C}[0,\infty)$*

$$stat_{DC}^{p,q} \lim_{m\to\infty} \|L_m(f; x) - f(x)\|_\infty = 0, \tag{3}$$

if and only if

$$stat_{DC}^{p,q} \lim_{m\to\infty} \|L_m(1; x) - 1\|_\infty = 0, \tag{4}$$

$$stat_{DC}^{p,q} \lim_{m\to\infty} \|L_m(e^{-s}; x) - e^{-x}\|_\infty = 0 \tag{5}$$

and

$$stat_{DC}^{p,q} \lim_{m\to\infty} \|L_m(e^{-2s}; x) - e^{-2x}\|_\infty = 0. \tag{6}$$

Proof Since each of $f_i(x) = \{1, e^{-x}, e^{-2x}\} \in \mathcal{C}(X)$ $(i = 0, 1, 2)$ is continuous, the implication (3)$\Longrightarrow$(4)–(6) is obvious. In order to complete the proof of the theorem, we first assume that (4)–(6) hold true. Let $f \in \mathcal{C}[X]$, then there exists a constant

$\mathcal{K} > 0$ such that $|f(x)| \leq \mathcal{K},\ \forall\, x \in X = [0, \infty)$. Thus,

$$|f(s) - f(x)| \leq 2\mathcal{K},\ \ s, x \in X. \tag{7}$$

Clearly, for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(s) - f(x)| < \epsilon \tag{8}$$

whenever $|e^{-s} - e^{-x}| < \delta$, for all $s, x \in X$.

Let us choose $\varphi_1 = \varphi_1(s, x) = (e^{-s} - e^{-x})^2$. If $|e^{-s} - x^{-x}| \geq \delta$, then we obtain:

$$|f(s) - f(x)| < \frac{2\mathcal{K}}{\delta^2}\varphi_1(s, x). \tag{9}$$

From Eqs. (8) and (9), we get

$$|f(s) - f(x)| < \epsilon + \frac{2\mathcal{K}}{\delta^2}\varphi_1(s, x),$$

$$\Rightarrow\ -\epsilon - \frac{2\mathcal{K}}{\delta^2}\varphi_1(s, x) \leq f(s) - f(x) \leq \epsilon + \frac{2\mathcal{K}}{\delta^2}\varphi_1(s, x). \tag{10}$$

Now since $L_m(1; x)$ is monotone and linear, so by applying the operator $L_m(1; x)$ to this inequality, we have

$$L_m(1; x)\left(-\epsilon - \frac{2\mathcal{K}}{\delta^2}\varphi_1(s, x)\right) \leq L_m(1; x)(f(s) - f(x)) \leq L_m(1; x)\left(\epsilon + \frac{2\mathcal{K}}{\delta^2}\varphi_1(s, x)\right). \tag{11}$$

Note that x is fixed and so $f(x)$ is a constant number. Therefore,

$$-\epsilon L_m(1; x) - \frac{2\mathcal{K}}{\delta^2}L_m(\varphi_1; x) \leq L_m(f; x) - f(x)L_m(1; x) \leq \epsilon L_m(1; x) + \frac{2\mathcal{K}}{\delta^2}L_m(\varphi_1; x). \tag{12}$$

But

$$L_m(f; x) - f(x) = [L_m(f; x) - f(x)L_m(1; x)] + f(x)[L_m(1; x) - 1]. \tag{13}$$

Using (12) and (13), we have

$$L_m(f; x) - f(x) < \epsilon L_m(1; x) + \frac{2\mathcal{K}}{\delta^2}L_m(\varphi_1; x) + f(x)[L_m(1; x) - 1]. \tag{14}$$

Now, estimate $L_m(\varphi_1; x)$ as,

$$\begin{aligned} L_m(\varphi_1; x) &= L_m((e^{-s} - e^{-x})^2; x) = L_m(e^{-2s} - 2e^{-x}e^{-s} + e^{-2x}; x) \\ &= L_m(e^{-2s}; x) - 2e^{-x}L_m(e^{-s}; x) + e^{-2s}L_m(1; x) \\ &= [L_m(e^{-2s}; x) - e^{-2x}] - 2e^{-x}[L_m(e^{-s}; x) - e^{-x}] + e^{-2x}[L_m(1; x) - 1]. \end{aligned}$$

Using (14), we obtain

$$\begin{aligned} L_m(f; x) - f(x) &< \epsilon L_m(1; x) + \frac{2\mathcal{K}}{\delta^2}\{[L_m(e^{-2s}; x) - e^{-2s}] - 2e^{-x}[L_m(e^{-s}; x) - e^{-x}] \\ &\quad + e^{-2s}[L_m(1; x) - 1]\} + f(x)[L_m(1; x) - 1]. \\ &= \epsilon[L_m(1; x) - 1] + \epsilon + \frac{2\mathcal{K}}{\delta^2}\{[L_m(e^{-2s}; x) - e^{-2x}] - 2e^{-x}[L_m(e^{-s}; x) - e^{-x}] \\ &\quad + e^{-2x}[L_m(1; x) - 1]\} + f(x)[L_m(1; x) - 1]. \end{aligned}$$

Since ϵ is arbitrary, we can write

$$\begin{aligned} |L_m(f; x) - f(x)| &\leq \epsilon + \left(\epsilon + \frac{2\mathcal{K}}{\delta^2} + \mathcal{K}\right)|L_m(1; x) - 1| \\ &\quad + \frac{4\mathcal{K}}{\delta^2}|L_m(e^{-s}; x) - e^{-x}| + \frac{2\mathcal{K}}{\delta^2}|L_m(e^{-2s}; x) - e^{-2x}| \\ &\leq B\left(|L_m(1; x) - 1| + |L_m(e^{-s}; x) - e^{-x}| + |L_m(e^{-2s}; x) - e^{-2x}|\right), \end{aligned} \tag{15}$$

where

$$B = \max\left(\epsilon + \frac{2\mathcal{K}}{\delta^2} + \mathcal{K}, \frac{4\mathcal{K}}{\delta^2}, \frac{2\mathcal{K}}{\delta^2}\right).$$

Now replace $L_m(f; x)$ by

$$D_{p,q}(x_n) = \frac{1}{b_n - a_n}\sum_{m=a_n+1}^{b_n} \Delta_{p,q}^{[r]}(T_m(f; x))$$

in Eq. (15).

We have for a given $r > 0$, there exists $\epsilon > 0$, such that $\epsilon < r$. Then, by setting

$$\Psi_m(x; r) = \left\{m : a_n < m \leq b_n \text{ and } \left|\frac{1}{b_n - a_n}\sum_{m=a_n+1}^{b_n} \Delta_{p,q}^{[r]}(T_m(f; x)) - f(x)\right| \geq r\right\}$$

and for $i = 0, 1, 2,$

$$\Psi_{i,m}(x; r) = \left\{m : a_n < m \leq b_n \text{ and } \left|\frac{1}{b_n - a_n}\sum_{m=a_n+1}^{b_n} \Delta_{p,q}^{[r]}(T_m(f_i; x)) - f_i(x)\right| \geq \frac{r - \epsilon}{3B}\right\},$$

we obtain

$$\Psi_m(x;r) \leq \sum_{i=0}^{2} \Psi_{i,m}(x;r).$$

Clearly,

$$\frac{\|\Psi_m(x;r)\|_{\mathcal{C}(X)}}{b_n - a_n} \leq \sum_{i=0}^{2} \frac{\|\Psi_{i,m}(x;r)\|_{\mathcal{C}(X)}}{b_n - a_n}. \tag{16}$$

Now, using the above assumption about the implications in (4)–(6) and by Definition 1, the right-hand side of (16) is seen to tend to zero as $n \to \infty$. Consequently, we get

$$\lim_{n\to\infty} \frac{\|\Psi_m(x;r)\|_{\mathcal{C}(X)}}{b_n - a_n} = 0 \;\; (r > 0).$$

Therefore, the implication (3) holds true.

This completes the proof of Theorem 1. □

Corollary 1 *Let $L_m : \mathcal{C}[0,\infty) \to \mathcal{C}[0,\infty)$ be a sequence of positive linear operators, and let $f \in \mathcal{C}[0,\infty)$. Then,*

$$stat_{DA}^{p,q} \lim_{m\to\infty} \|L_m(f;x) - f(x)\|_\infty = 0 \tag{17}$$

if and only if

$$stat_{DA}^{p,q} \lim_{m\to\infty} \|L_m(1;x) - 1\|_\infty = 0, \tag{18}$$

$$stat_{DA}^{p,q} \lim_{m\to\infty} \|L_m(e^{-s};x) - e^{-x}\|_\infty = 0 \tag{19}$$

and

$$stat_{DA}^{p,q} \lim_{m\to\infty} \|L_m(e^{-2s};x) - e^{-2x}\|_\infty = 0. \tag{20}$$

Proof By taking $a_n = n - 1, \; \forall\, n$ and, $b_n = n + k - 1, \; \forall\, n$ and proceeding in the similar line of Theorem 1, the proof of Corollary 1 is established. □

Remark 4 By taking $p_n = q_n = 1 \;\forall\, n$ in Theorem 1, one can obtain the statistical deferred Cesàro summability version of Korovkin-type approximation for the set of functions $1, \; e^{-x}$, and e^{-2x} established by Jena et al. [15].

Now we present below an illustrative example for the sequence of positive linear operators that does not satisfy the conditions of the Korovkin approximation theorems due to Jena et al. [15], Mohiuddine et al. [20], and Boyanov and Veselinov [8] but satisfies the conditions of our Theorem 1. Thus, our theorem is stronger than the results established by Jena et al. [15], Mohiuddine et al. [20] and Boyanov and Veselinov [8].

Example 3 Let $X = [0, 1]$ and consider the (p, q)-analogue of Bernstein operators $\mathcal{B}_{n,p,q}(f; x)$ on $\mathcal{C}[0, 1]$ given by (see [23])

$$\mathcal{B}_{n,p,q}(f; x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right) \ (x \in [0, 1]).$$

Also, observe that

$$\mathcal{B}_{n,p,q}(f_0; x) = 1, \ \ \mathcal{B}_{n,p,q}(f_1; x) = e^{-x} \ \text{ and } \ \mathcal{B}_{n,p,q}(f_2; x) = \frac{p^{n-1}}{[n]_{p,q}} e^{-x} + \frac{q[n-1]_{p,q}}{[n]_{p,q}} e^{-2x}.$$

Let us consider $L_n : \mathcal{C}[0, 1] \to \mathcal{C}[0, 1]$ be sequence of positive linear operators defined as follows:

$$L_n(f; x) = [1 + f_n(x)]x(1 + xD)\mathcal{B}_{n,p,q}(f; x) \ \ (f \in \mathcal{C}[0, 1]), \tag{21}$$

where the operator given by

$$x(1 + xD) \ \ \left(D = \frac{d}{dx}\right)$$

was used earlier by Al-Salam [4] and, more recently, by Viskov and Srivastava [30] (see also the monograph by Srivastava and Manocha [25] for various general families of operators of this kind). If we choose the sequence $f_n(x)$ of functions just as we considered in Example 2, then we have

$$\begin{aligned} L_n(f_0; x) &= [1 + f_n(x)]x(1 + xD) \cdot \mathcal{B}_{n,p,q}(f_0; x) \\ &= [1 + f_n(x)]x(1 + xD) \cdot 1 = [1 + f_n(x)]x, \end{aligned}$$

$$\begin{aligned} L_n(f_1; x) &= [1 + f_n(x)]x(1 + xD) \cdot \mathcal{B}_{n,p,q}(f_1; x) \\ &= [1 + f_n(x)]x(1 + xD) \cdot e^{-x} = [1 + f_n(x)]x(e^{-x} - xe^{-x}), \end{aligned}$$

and

$$\begin{aligned} L_n(f_2; x) &= [1 + f_n(x)]x(1 + xD) \cdot \mathcal{B}_{n,p,q}(f_2; x) \\ &= [1 + f_n(x)]x(1 + xD) \cdot \left\{ \frac{p^{n-1}}{[n]_{p,q}} e^{-x} + \frac{q[n-1]_{p,q}}{[n]_{p,q}} e^{-2x} \right\} \\ &= [1 + f_n(x)]x \left[\frac{p^{n-1}}{[n]_{p,q}} e^{-x} + \frac{q[n-1]_{p,q}}{[n]_{p,q}} e^{-2x} - xe^{-x} \frac{p^{n-1}}{[n]_{p,q}} - 2e^{-2x} \frac{q(n-1)_{p,q}}{[n]_{p,q}} \right]. \end{aligned}$$

So that, we obtain

$$\text{stat}_{DC}^{p,q} \lim_{m\to\infty} \|L_m(1;x) - 1\|_\infty = 0,$$
$$\text{stat}_{DC}^{p,q} \lim_{m\to\infty} \|L_m(e^{-s};x) - e^{-x}\|_\infty = 0$$

and

$$\text{stat}_{DC}^{p,q} \lim_{m\to\infty} \|L_m(e^{-2s};x) - e^{-2x}\|_\infty = 0,$$

that is, the sequence $L_m(f;x)$ satisfies the conditions (4)–(6). Therefore, by Theorem 1, we have

$$\text{stat}_{DC}^{p,q} \lim_{m\to\infty} \|L_m(f;x) - f\|_\infty = 0.$$

Hence, it is statistically deferred Cesàro summable under (p,q)-integers; however, since (x_m) is neither statistically Cesàro summable nor statistically deferred Cesàro summable, so we conclude that earlier works under [15, 20] is not valid for the operators defined by (21), while our Theorem 1 still works.

3 Rate of Statistical Deferred Cesàro Summability

In this section, we study the rates of statistical deferred Cesàro summability based on (p,q)-integers of a sequence of positive linear operators $L(f;x)$ defined on $\mathcal{C}[0,\infty)$ with the help of modulus of continuity.

We now presenting the following definition.

Definition 3 Let $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$, and let r is a nonnegative integer. Also, let (a_n) and (b_n) be sequences of integers (nonnegative). Let (u_n) be a positive non-increasing sequence. A given sequence $x = (x_m)$ is statistically deferred Cesàro summable to a number ℓ with rate $o(u_n)$, if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{u_n(b_n - a_n)} \left|\left\{m : a_n < m \leq b_n \text{ and } |D_{p,q}(x_m) - \ell| \geq \epsilon\right\}\right| = 0.$$

In this case, we may write

$$x_m - \ell = stat_{DC}^{p,q} - o(u_n).$$

We now prove the following basic lemma.

Lemma 1 *Let (u_n) and (v_n) be two positive non-increasing sequences. Let $x = (x_m)$ and $y = (y_m)$ be two sequences such that*

$$x_m - L_1 = stat_{DC}^{p,q} - o(u_n)$$

and

$$y_m - L_2 = stat_{DC}^{p,q} - o(v_n)$$

respectively. Then, the following conditions hold true

(i) $(x_m + y_m) - (\ell_1 + \ell_2) = stat_{DC}^{p,q} - o(w_n)$;
(ii) $(x_m - \ell_1)(y_m - \ell_2) = stat_{DC}^{p,q} - o(u_n v_n)$;
(iii) $\lambda(x_m - \ell_1) = stat_{DC}^{p,q} - o(u_n)$ *(for any scalar λ)*;
(iv) $\sqrt{|x_m - \ell_1|} = stat_{DC}^{p,q} - o(u_n)$,
where

$$w_n = \max\{u_n, v_n\}.$$

Proof In order to prove the condition (i), for $\epsilon > 0$ and $x \in [0, \infty)$, we define the following sets:

$$\mathcal{A}_n(x;\epsilon) = \left|\{m : a_n < m \leq b_n \text{ and } |D_{p,q}(x_m) + D_{p,q}(y_m) - (\ell_1 + \ell_2)| \geq \epsilon\}\right|,$$
$$\mathcal{A}_{0,n}(x;\epsilon) = \left|\left\{m : a_n < m \leq b_n \text{ and } |D_{p,q}(x_m) - \ell_1| \geq \frac{\epsilon}{2}\right\}\right|,$$

and

$$\mathcal{A}_{1,n}(x;\epsilon) = \left|\left\{m : a_n < m \leq b_n \text{ and } |D_{p,q}(y_m) - \ell_2| \geq \frac{\epsilon}{2}\right\}\right|.$$

Clearly, we have

$$\mathcal{A}_n(x;\epsilon) \subseteq \mathcal{A}_{0,n}(x;\epsilon) \cup \mathcal{A}_{1,n}(x;\epsilon).$$

Moreover, since

$$w_n = \max\{u_n, v_n\},$$

by condition (3) of Theorem 1, we obtain

$$\frac{\|\mathcal{A}_m(x;\epsilon)\|_\infty}{w_n(b_n - a_n)} \leq \frac{\|\mathcal{A}_{0,n}(x;\epsilon)\|_\infty}{u_n(b_n - a_n)} + \frac{\|\mathcal{A}_{1,n}(x;\epsilon)\|_\infty}{v_n(b_n - a_n)}. \tag{22}$$

Now, by conditions (4)–(6) of Theorem 1, we obtain

$$\frac{\|\mathcal{A}_n(x;\epsilon)\|_\infty}{w_n(b_n - a_n)} = 0, \tag{23}$$

which establishes (i). Since the proofs of other conditions (ii)–(iv) are similar, we omit them. □

Further, we recall that the modulus of continuity of a function $f \in \mathcal{C}[0,\infty)$ is defined by

$$\omega(f,\delta) = \sup_{|y-x|\leq\delta: x,y\in X} |f(y) - f(x)| \quad (\delta > 0)$$

which implies that

$$|f(y) - f(x)| \leq \omega(f,\delta)\left(\frac{|x-y|}{\delta} + 1\right). \tag{24}$$

Now, we state and prove a result in the form of the following theorem.

Theorem 2 *Let $[0,\infty) \subset \mathbb{R}$, and let $L_m : \mathcal{C}[0,\infty) \to \mathcal{C}[0,\infty)$ be a sequence of positive linear operators. Assume that the following conditions hold true:*

(i) $\|L_m(1;x) - 1\|_\infty = stat_{DC}^{p,q} - o(u_n)$,
(ii) $\omega(f,\lambda_m) = stat_{DC}^{p,q} - o(v_n)$,

where

$$\lambda_m = \sqrt{L_m(\varphi^2;x)} \text{ and } \varphi_1(y,x) = (e^{-y} - x^{-x})^2.$$

Then, for all $f \in \mathcal{C}[0,\infty)$, the following statement holds true:

$$\|L_m(f;x) - f\|_\infty = stat_{DC}^{p,q} - o(w_n), \tag{25}$$

$w_n = \max\{u_n, v_n\}$.

Proof Let $f \in \mathcal{C}[0,\infty)$ and $x \in [0,\infty)$. Using (24), we have

$$\begin{aligned}
|L_m(f;x) - f(x)| &\leq L_m(|f(y) - f(x)|;x) + |f(x)||L_m(1;x) - 1| \\
&\leq L_m\left(\frac{|e^{-x} - e^{-y}|}{\lambda_m} + 1;x\right)\omega(f,\lambda_m) + |f(x)||L_m(1;x) - 1| \\
&\leq L_m\left(1 + \frac{1}{\lambda_m^2}(e^{-x} - e^{-y})^2;x\right)\omega(f,\lambda_m) + |f(x)||L_m(1;x) - 1| \\
&\leq \left(L_m(1;x) + \frac{1}{\lambda_m^2}L_m(\varphi_x;x)\right)\omega(f,\lambda_m) + |f(x)||L_m(1;x) - 1|.
\end{aligned}$$

Putting $\lambda_m = \sqrt{L_m(\varphi^2;x)}$, we get

$$\begin{aligned}
\|L_m(f;x) - f(x)\|_\infty &\leq 2\omega(f,\lambda_m) + \omega(f,\lambda_m)\|L_m(1;x) - 1\|_\infty + \|f(x)\|\|L_m(1;x) - 1\|_\infty \\
&\leq \mathcal{M}\{\omega(f,\lambda_m) + \omega(f,\lambda_m)\|L_m(1;x) - 1\|_\infty + \|L_m(1;x) - 1\|_\infty\},
\end{aligned}$$

where

$$\mathcal{M} = \{\|f\|_\infty, 2\}.$$

Thus,

$$\left\| \frac{1}{b_n - a_n} \sum_{m=a_n+1}^{b_n} L_m(f; x) - f(x) \right\|_\infty$$
$$\leq \mathcal{M} \left\{ \omega(f, \lambda_m) \frac{1}{b_n - a_n} + \omega(f, \lambda_m) \left\| \frac{1}{b_n - a_n} \sum_{m=a_n+1}^{b_n} L_m(f; x) - f(x) \right\|_\infty \right\}$$
$$+\mathcal{M} \left\{ \left\| \frac{1}{b_n - a_n} \sum_{m=a_n+1}^{b_n} L_m(f; x) - f(x) \right\|_\infty \right\}.$$

Now, by using the conditions (i) and (ii) of Theorem 2, in conjunction with Lemma 1, we arrive at the statement (25) of Theorem 2.

This completes the proof of Theorem 2. □

4 Concluding Remarks

In this concluding section of our investigation, we present several further remarks and observations concerning to various results which we have proved here.

Remark 5 Let $(x_m)_{m\in\mathbb{N}}$ be a sequence given in Example 3. Then, since

$$\text{stat}_{DC}^{p,q} - \lim_{m\to\infty} x_m \to 0 \text{ on } [0, \infty),$$

we have

$$\text{stat}_{DC}^{p,q} - \lim_{m\to\infty} \|L_m(f_i; x) - f_i(x)\|_\infty = 0 \quad (i = 0, 1, 2). \tag{26}$$

Thus, we can write (by Theorem 1)

$$\text{stat}_{DC}^{p,q} - \lim_{m\to\infty} \|L_m(f; x) - f(x)\|_\infty = 0, \ (i = 0, 1, 2), \tag{27}$$

where

$$f_0(x) = 1, \quad f_1(x) = e^{-x} \text{ and } f_2(x) = e^{-2x}.$$

However, since (x_m) is not ordinarily convergent, and so also it does not converge uniformly in the ordinary sense. Thus, the classical Korovkin theorem does not work here for the operators defined by (21). Hence, this application clearly indicates that our Theorem 1 is a non-trivial generalization of the classical Korovkin-type theorem (see [19]).

Remark 6 Let $(x_m)_{m\in\mathbb{N}}$ be a sequence as given in Example 3. Then, since

$$\text{stat}_{DC}^{p,q} - \lim_{m\to\infty} x_m \to 0 \text{ on } [0, \infty),$$

so (26) holds true. Now by applying (26) and Theorem 1, condition (27) holds true. However, since (x_m) does not statistical Cesàro summable, so Theorem 2.1 of Jena et al. (see [15]) does not work for our operator defined in (21). Thus, our Theorem 1 is also a non-trivial extension of Theorem 2.1 of Jena et al. [15] (see also [8, 19]). Based on the above results, it is concluded here that our proposed method has successfully worked for the operators defined in (21) and therefore it is stronger than the classical and statistical version of the Korovkin-type approximation (see [8, 19, 20]) established earlier.

Remark 7 Let us suppose that we replace the conditions (i) and (ii) in Theorem 2, by the following condition:

$$|L_m(f_i; x) - f_i| = DC_1(\text{stat}) - o(u_{n_i}) \quad (i = 0, 1, 2). \tag{28}$$

Then, since

$$L_m(\varphi^2; x) = e^{-2x}|L_m(1; x) - 1| - 2e^{-x}|L_m(e^{-x}; x) - e^{-x}| + |L_m(e^{-2x}; x) - e^{-2x}|,$$

we can write

$$L_m(\varphi^2; x) \leq M \sum_{i=0}^{2} |L_m(f_i; x) - f_i(x)|_\infty, \tag{29}$$

where

$$M = \{\|f_2\|_\infty + 2\|f_1\|_\infty + 1\}.$$

Now it follows from (28), (29) and Lemma 1 that

$$\lambda_m = \sqrt{L_m(\varphi^2)} = DC_1(\text{stat}) - o(d_n), \tag{30}$$

where

$$o(d_n) = \max\{u_{n_0}, u_{n_1}, u_{n_2}\}.$$

This implies

$$\omega(f, \delta) = DC_1(\text{stat}) - o(d_n).$$

Now using (30) in Theorem 2, we immediately see that for $f \in \mathcal{C}[0, \infty)$,

$$L_m(f; x) - f(x) = DC_1(\text{stat}) - o(d_n). \tag{31}$$

Therefore, if we use the condition (28) in Theorem 2 instead of (i) and (ii), then we obtain the rates of statistical deferred Cesàro summability of the sequence of positive linear operators in Theorem 1.

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