Weighted Statistical Convergence of Bögel Continuous Functions by Positive Linear Operator

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Abstract In the present work, we have introduced a weighted statistical approximation theorem for sequences of positive linear operators defined on the space of all real-valued B-continuous functions on a compact subset of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Furthermore, we display an application which shows that our new result is stronger than its classical version.

Keywords Weighted uniform convergence · Double sequences · Statistical convergence · Korovkin-type approximation theorem

Mathematics Subject Classification 40A35 · 41A36

1 Introduction

The classical Korovkin theory is mostly connected with the approximation to continuous functions by means of positive linear operators (see, for instance, [1, 17]). In order to work up the classical Korovkin theory, the space of Bögel-type continuous (or, simply, B-continuous) functions instead of the classical theory has been studied in [2–4]. The concept of statistical convergence for sequences of real numbers was introduced by Fast [14] and Steinhaus [21] independently in the same year 1951. Some Korovkin-type theorems in the setting of a statistical convergence were given by [5, 6, 10–13, 22].

Now we recall some notations and definitions.

A double sequence $x = (x_{mn}),\ \ m, n \in \mathbb{N}$, is convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$ whenever $m, n > N$, then L is called the Pringsheim limit of x and is denoted by $P - \lim x = L$ (see [20]). Also, if there exists a positive number M such that $|x_{mn}| \leq M$ for all $(m, n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, then $x = (x_{mn})$ is said to be bounded. Note that in contrast to

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the case for single sequences, a convergent double sequence need not to be bounded.

Definition 1 ([19]) Let $K \subset \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. Then *density of* K, denoted by $\delta^2(K)$, is given by:

$$\delta^2(K) := P - \lim_{m,n} \frac{|\{j \leq m, k \leq n : (j,k) \in K\}|}{mn}$$

provided that the limit on the right-hand side exists in the Pringsheim sense by $|B|$ we mean the cardinality of the set $B \subset \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. A real double sequence $x = (x_{mn})$ is said to be statistically convergent to L if, for every $\varepsilon > 0$,

$$\delta^2\left(\left\{(m,n) \in \mathbb{N}^2 : |x_{mn} - L| \geq \varepsilon\right\}\right) = 0.$$

In this case, we write $st^2 - \lim x = L$.

The concept of weighted statistical convergence was defined by Karakaya and Chishti [16]. Recently, Mursaleen et al. [18] modified the definition of weighted statistical convergence. In [15], Ghosal showed that both definitions of weighted statistical convergence are not well defined in general. So Ghosal modified the definition of weighted statistical convergence as follows:

Definition 2 Let $\{p_j\}, \{q_k\}$, $j, k \in \mathbb{N}$ be sequences of nonnegative real numbers such that $p_1 > 0, \liminf_{j\to\infty} p_j > 0, q_1 > 0, \liminf_{k\to\infty} q_k > 0$ and $P_m = \sum_{j=1}^{m} p_j$ and $Q_n = \sum_{k=1}^{n} q_k$ where $n, m \in \mathbb{N}$, $P_m \to \infty$ as $m \to \infty$, $Q_n \to \infty$ as $n \to \infty$. The double sequence $x = (x_{jk})$ is said to be weighted statistical convergent (or $S_{\overline{N_2}}$-convergent) to L if for every $\varepsilon > 0$,

$$P - \lim_{m,n} \frac{1}{P_m Q_n} |\{j \leq P_m, k \leq Q_n : p_j q_k |x_{jk} - L| \geq \varepsilon\}| = 0.$$

In this case, we write $st_{\overline{N_2}} - \lim x = L$ and we denote the set of all weighted statistical convergent sequences by $S_{\overline{N_2}}$.

Remark 1 If $p_j = 1$, $q_k = 1$ for all j, k, then weighted statistical convergence is reduced to statistical convergence for double sequences.

Example 1 Let $x = (x_{mn})$ is a sequence defined by

$$x_{mn} := \begin{cases} mn, & m \text{ and } n \text{ are squares}, \\ 0, & \text{otherwise}, \end{cases}.$$

Let $p_j = j, q_k = k$ for all j, k. Then $P_m = \frac{m(m+1)}{2}$ and $Q_n = \frac{n(n+1)}{2}$. Since, for every $\varepsilon > 0$,

$$
\begin{aligned}
&P-\lim_{m,n}\frac{|\{j\le P_m, k\le Q_n : p_j q_k|x_{jk}-0|\ge\varepsilon\}|}{P_m Q_n}\\
&\le P-\lim_{m,n}\frac{|\{j\le P_m, k\le Q_n : p_j q_k|x_{jk}|\ne 0\}|}{P_m Q_n}\\
&\le P-\lim_{m,n}\frac{\sqrt{P_m}\sqrt{Q_n}}{P_m Q_n}=0
\end{aligned}
$$

So $x=(x_{mn})$ is weighted statistical convergent to 0 but not Pringsheim's sense convergent.

In [15], Ghosal showed that both convergences which are weighted statistical convergence and statistical convergence do not imply each other in general.

In the work, using the Definition 2, we prove Korovkin-type approximation theorem for double sequences of B-continuous functions defined on a compact subset of the real two-dimensional space. Finally, we give an application which shows that our new result is stronger than its classical version.

2 A Korovkin-Type Approximation Theorem

Bögel introduced the definition of B-continuity [7–9] as follows:

Let I be a compact subset of $\mathbb{R}^2=\mathbb{R}\times\mathbb{R}$. Then, a function $f: I\to\mathbb{R}$ is called a B-continuous at a point $(x,y)\in I$ if, for every $\varepsilon>0$, there exists a positive number $\delta=\delta(\varepsilon)$ such that

$$\left|\Delta_{xy}\left[f(u,v)\right]\right|<\varepsilon,$$

for any $(u,v)\in I$ with $|u-x|<\delta$ and $|v-y|<\delta$, where the symbol $\Delta_{xy}\left[f(u,v)\right]$ denotes the mixed difference of f defined by

$$\Delta_{xy}\left[f(u,v)\right]=f(u,v)-f(u,y)-f(x,v)+f(x,y).$$

By $C_b(I)$, we denote the space of all B-continuous functions on I. Recall that $C(I)$ and $B(I)$ denote the space of all continuous (in the usual sense) functions on I and the space of all bounded functions on I, respectively. Then, notice that $C(I)\subset C_b(I)$. Moreover, one can find an unbounded B-continuous function, which follows from the fact that, for any function of the type $f(u,v)=g(u)+h(v)$, we have $\Delta_{xy}\left[f(u,v)\right]=0$ for all $(x,y),(u,v)\in I$. $\|f\|$ denotes the supremum norm of f in $B(I)$.

Let L be a linear operator from $C_b(I)$ into $B(I)$. Then, as usual, we say that L is positive linear operator provided that $f\ge 0$ implies $L(f)\ge 0$. Also, we denote the value of $L(f)$ at a point $(x,y)\in I$ by $L(f(u,v);x,y)$ or, briefly, $L(f;x,y)$. Since

$$\Delta_{xy}\left[f(u,y)+f(x,v)-f(u,v)\right]=-\Delta_{xy}\left[f(u,v)\right]$$

holds for all (x, y), $(u, v) \in I$, the B-continuity of f implies the B-continuity of $F_{xy}(u, v) := f(u, y) + f(x, v) - f(u, v)$ for every fixed $(x, y) \in I$. We also use the following test functions

$$f_0(x, y) = 1, \quad f_1(x, y) = x, \quad f_2(x, y) = y \text{ and } f_3(x, y) = x^2 + y^2.$$

We recall that the following lemma for B-continuous functions was proved by Badea et al. [3].

Lemma 1 ([3]) *If $f \in C_b(I)$, then, for every $\varepsilon > 0$, there are two positive numbers $\alpha_1(\varepsilon) = \alpha_1(\varepsilon, f)$ and $\alpha_2(\varepsilon) = \alpha_2(\varepsilon, f)$ such that*

$$\Delta_{xy}\left[f\,(u, v)\right] \leq \frac{\varepsilon}{3} + \alpha_1(\varepsilon)(u - x)^2 + \alpha_2(\varepsilon)(v - y)^2$$

holds for all (x, y), $(u, v) \in I$.

Now we have the following main result.

Theorem 1 *Let (L_{mn}) be a double sequence of positive linear operators acting from $C_b\,(I)$ into $B\,(I)$. Assume that the following conditions hold:*

$$P - \lim_{m,n} \frac{1}{P_m Q_n} \left|\{j \leq P_m, k \leq Q_n : L_{jk}(f_0; x, y) = f_0(x, y) \text{ for all } (x, y) \in I\}\right| = 1 \tag{2.1}$$

and

$$st_{\overline{N_2}} - \lim \left\| L_{mn}\,(f_i; x, y) - f_i(x, y) \right\| = 0, i = 1, 2, 3. \tag{2.2}$$

Then, for all $f \in C_b\,(I)$, we have

$$st_{\overline{N_2}} - \lim \left\| L_{mn}\,(f(u, y) + f(x, v) - f(u, v); x, y) - f(x, y) \right\| = 0. \tag{2.3}$$

Proof Let $(x, y) \in I$ and $f \in C_b\,(I)$ be fixed. Taking

$$A := \left\{ j \leq P_m, k \leq Q_n : L_{jk}(f_0; x, y) = f_0(x, y) = 1 \text{ for all } (x, y) \in I \right\}, \tag{2.4}$$

we obtain from (2.1) that

$$P - \lim_{m,n} \frac{1}{P_m Q_n} \left|\{j \leq P_m, k \leq Q_n : L_{jk}(f_0; x, y) \neq f_0(x, y) \text{ for all } (x, y) \in I\}\right| = 0. \tag{2.5}$$

Using the B-continuity of the function $F_{xy}(u, v) := f(u, y) + f(x, v) - f(u, v)$, Lemma 1 implies that, for every $\varepsilon > 0$, there exist two positive numbers $\alpha_1(\varepsilon)$ *and* $\alpha_2(\varepsilon)$ such that

$$\left|\Delta_{xy}\left[f(u, y) + f(x, v) - f(u, v)\right]\right| \leq \frac{\varepsilon}{3} + \alpha_1(\varepsilon)(u - x)^2 + \alpha_2(\varepsilon)(v - y)^2 \tag{2.6}$$

holds for every $(u, v) \in I$. Also, by (2.12), see that

$$L_{jk}\left(f(u,y)+f(x,v)-f(u,v);x,y\right)-f(x,y)=L_{jk}\left(\Delta_{xy}\left[f(u,y)+f(x,v)-f(u,v)\right];x,y\right) \tag{2.7}$$

holds for all $(j,k)\in A$. We can write for all $(m,n)\in A$ from (2.6) and (2.7),

$$\begin{aligned}\left|L_{jk}\left(f(u,y)+f(x,v)-f(u,v);x,y\right)-f(x,y)\right| &= \left|L_{jk}\left(\Delta_{xy}\left[f(u,y)+f(x,v)-f(u,v)\right];x,y\right)\right|\\ &\le L_{jk}\left(\left|\Delta_{xy}\left[f(u,y)+f(x,v)-f(u,v)\right]\right|;x,y\right)\\ &\le \frac{\varepsilon}{3}+\alpha_1(\varepsilon)L_{jk}\left((u-x)^2;x,y\right)\\ &\quad+\alpha_2(\varepsilon)L_{jk}\left((v-y)^2;x,y\right)\\ &\le \frac{\varepsilon}{3}+\alpha(\varepsilon)\{x^2+y^2+L_{jk}(f_3;x,y)\\ &\quad -2xL_{jk}(f_1;x,y)-2yL_{jk}(f_2;x,y)\},\end{aligned}$$

where $\alpha(\varepsilon)=\max\{\alpha_1(\varepsilon),\alpha_2(\varepsilon)\}$. It follows from the last inequality that

$$\left|L_{jk}\left(f(u,y)+f(x,v)-f(u,v);x,y\right)-f(x,y)\right|\le\frac{\varepsilon}{3}+\alpha(\varepsilon)\sum_{i=1}^{3}\left|L_{jk}\left(f_i;x,y\right)-f_i(x,y)\right| \tag{2.8}$$

holds for all $(j,k)\in A$. Taking supremum over $(x,y)\in I$ on both sides of inequality (2.8), we obtain, for all $(j,k)\in I$, that

$$\left\|L_{jk}\left(f(u,y)+f(x,v)-f(u,v);x,y\right)-f(x,y)\right\|\le\frac{\varepsilon}{3}+\alpha(\varepsilon)\sum_{i=1}^{3}\left\|L_{jk}\left(f_i;x,y\right)-f_i(x,y)\right\|.$$

Because of ε is arbitrary, we obtain

$$\left\|L_{jk}\left(f(u,y)+f(x,v)-f(u,v);x,y\right)-f(x,y)\right\|\le\alpha(\varepsilon)\sum_{i=1}^{3}\left\|L_{jk}\left(f_i;x,y\right)-f_i(x,y)\right\|.$$

Hence,

$$p_jq_k\left\|L_{jk}\left(f(u,y)+f(x,v)-f(u,v);x,y\right)-f(x,y)\right\|\le\alpha(\varepsilon)\sum_{i=1}^{3}p_jq_k\left\|L_{jk}\left(f_i;x,y\right)-f_i(x,y)\right\|. \tag{2.9}$$

Now for a given $r>0$, consider the following sets:

$$\begin{aligned}U &:=\left\{j\le P_m,k\le Q_n: p_jq_k\left\|L_{jk}\left(f(u,y)+f(x,v)-f(u,v);x,y\right)-f(x,y)\right\|\ge r\right\},\\ U_i &:=\left\{j\le P_m,k\le Q_n: p_jq_k\left\|L_{jk}\left(f_i;x,y\right)-f_i(x,y)\right\|\ge\frac{r}{3\alpha(\varepsilon)}\right\},\quad i=1,2,3,\end{aligned}$$

Hence, inequality (2.9) yields that

$$\frac{|U\cap A|}{P_mQ_n}\le\frac{|U_1\cap A|}{P_mQ_n}+\frac{|U_2\cap A|}{P_mQ_n}+\frac{|U_3\cap A|}{P_mQ_n},$$

which gives,

$$P-\lim\frac{|U\cap A|}{P_mQ_n}\leq\sum_{i=1}^{3}\left\{P-\lim\frac{|U_i\cap A|}{P_mQ_n}\right\}\leq\sum_{i=1}^{3}\left\{P-\lim\frac{|U_i|}{P_mQ_n}\right\}\tag{2.10}$$

Letting $m,n\to\infty$ (in any manner) and also using (2.13), we see from (2.10) that

$$P-\lim\frac{|U\cap A|}{P_mQ_n}=0.\tag{2.11}$$

Furthermore, if we use the inequality

$$\begin{aligned}\frac{|U|}{P_mQ_n}&=\frac{|U\cap A|}{P_mQ_n}+\frac{\left|U\cap\left(\mathbb{N}^2\backslash A\right)\right|}{P_mQ_n}\\&\leq\frac{|U\cap A|}{P_mQ_n}+\frac{\left|\mathbb{N}^2\backslash A\right|}{P_mQ_n}\end{aligned}$$

and if we take limit as $m,n\to\infty$, then it follows from (2.5) and (2.11) that

$$P-\lim\frac{|U|}{P_mQ_n}=0,$$

which means

$$st_{\overline{N_2}}-\lim\left\|L_{mn}\left(f(u,y)+f(x,v)-f(u,v);x,y\right)-f(x,y)\right\|=0=0.$$

This completes the proof. □

If $p_j=1$ and $q_k=1$ with $j,k\in\mathbb{N}$, then we obtain the statistical case of the Korovkin-type result for a double sequences on $C_b(I)$ introduced in [13],

Theorem 2 ([13]) *Let (L_{mn}) be a sequence of positive linear operators acting from $C_b(I)$ into $B(I)$. Assume that the following conditions hold:*

$$\delta^2\left\{(m,n)\in\mathbb{N}^2:L_{mn}(f_0;x,y)=1\textit{ for all }(x,y)\in I\right\}=1\tag{2.12}$$

and

$$st^2-\lim_{m,n}\left\|L_{mn}\left(f_i;x,y\right)-f_i(x,y)\right\|=0\textit{ for }i=1,2,3.\tag{2.13}$$

Then, for all $f\in C_b(I)$, we have

$$st^2-\lim_{m,n}\left\|L_{mn}(f(u,y)+f(x,v)-f(u,v);x,y)-f(x,y)\right\|=0.$$

Now we present an example for double sequences of positive linear operators. The first one shows that Theorem 1 does not work but Theorem 2 works. The second one gives that our approximation theorem and Theorem 2 work.

Example 2 Let $I = [0, 1] \times [0, 1]$. Consider the double Bernstein polynomials

$$B_{mn}(f; x, y) = \sum_{s=0}^{m} \sum_{t=0}^{n} f\left(\frac{s}{m}, \frac{t}{n}\right) x^s y^t (1-x)^{m-s} (1-y)^{n-t}$$

on $C_b(I)$.

(a) Using these polynomials, we introduce the following positive linear operators on $C_b(I)$:

$$P_{mn}(f; x, y) = (1 + \alpha_{mn}) B_{mn}(f; x, y), \quad (x, y) \in I \text{ and } f \in C_b(I), \tag{2.14}$$

where $\alpha := (\alpha_{mn})$ is given by $\alpha_{mn} := \begin{cases} 1 & m, n \text{ are squares}, \\ \frac{1}{\sqrt{mn}} & \text{otherwise}, \end{cases}$. Let $p_j = 2j + 1$, $q_k = k$ for all j, k. Then $P_m = m^2$ and $Q_n = \frac{n(n+1)}{2}$. Note that $\alpha = (\alpha_{mn})$ statistical convergent to 0 but it is not convergent and weighted statistical convergent to 0. Then, observe that

$$\begin{aligned} P_{mn}(f_0; x, y) &= (1 + \alpha_{mn}) f_0(x, y), \\ P_{mn}(f_1; x, y) &= (1 + \alpha_{mn}) f_1(x, y), \\ P_{mn}(f_2; x, y) &= (1 + \alpha_{mn}) f_2(x, y), \\ P_{mn}(f_3; x, y) &= (1 + \alpha_{mn}) \left[f_3(x, y) + \frac{x - x^2}{m} + \frac{y - y^2}{n} \right]. \end{aligned}$$

Since $st^2 - \lim \alpha_{mn} = 0$, we conclude that

$$st^2 - \lim \| P_{mn}(f_i; x, y) - f_i(x, y) \| = 0 \text{ for each } i = 0, 1, 2.$$

However, since α is statistically convergent, the sequence $\{P_{mn}(f; x, y)\}$ given by (2.14) does satisfy the Theorem 2 for all $f \in C_b(I)$. But Theorem 1 does not work since $\alpha = (\alpha_{mn})$ is not weighted statistical convergent to 0.

(b) Now we consider the following positive linear operators on $C_b(I)$:

$$T_{mn}(f; x, y) = (1 + \beta_{mn}) B_{mn}(f; x, y), \quad (x, y) \in I \text{ and } f \in C_b(I), \tag{2.15}$$

where $\beta := (\beta_{mn})$ is given by $\beta_{mn} := \begin{cases} mn & m, n \text{ are squares}, \\ 0 & \text{otherwise}, \end{cases}$. Let $p_j = j$, $q_k = k$ for all j, k. Then $P_m = \frac{m(m+1)}{2}$ and $Q_n = \frac{n(n+1)}{2}$. Note that $\alpha = (\alpha_{mn})$ statistical and weighted statistical convergent to 0 but it is not convergent to 0. Then, observe that

$$\begin{aligned} T_{mn}(f_0; x, y) &= (1 + \beta_{mn}) f_0(x, y), \\ T_{mn}(f_1; x, y) &= (1 + \beta_{mn}) f_1(x, y), \\ T_{mn}(f_2; x, y) &= (1 + \beta_{mn}) f_2(x, y), \end{aligned}$$

$$T_{mn}(f_3; x, y) = (1 + \beta_{mn}) \left[f_3(x, y) + \frac{x - x^2}{m} + \frac{y - y^2}{n} \right].$$

Since $st_{\overline{N_2}} - \lim \beta_{mn} = 0$, we conclude that

$$st_{\overline{N_2}} - \lim \| T_{mn} (f_i; x, y) - f_i(x, y) \| = 0 \text{ for each } i = 1, 2, 3.$$

So, by Theorem 1, we have

$$st_{\overline{N_2}} - \lim \| T_{mn} (f(u, y) + f(x, v) - f(u, v); x, y) - f(x, y) \| = 0 \text{ for all } f \in C_b (I).$$

However, since β is weighted statistical convergent to 0, we can say that Theorem 1 works for our operators defined by (2.15).

Therefore, this application clearly shows that our Theorem 1 is a non-trivial generalization of the classical case of the Korovkin result introduced in [3].

References

1. F. Altomare, M. Campiti, *Korovkin-Type Approximation Theory and Its Applications*, vol. 17, De Gruyter Studies in Mathematics (Walter de Gruyter, Berlin, 1994)
2. I. Badea, Modulus of continuity in Bögel sense and some applications for approximation by a Bernstein-type operator. Studia Univ. Babeş-Bolyai Ser. Math. Mech. **18**, 69–78 (1973) (in Romanian)
3. C. Badea, I. Badea, H.H. Gonska, A test function and approximation by pseudopolynomials. Bull. Aust. Math. Soc. **34**, 53–64 (1986)
4. C. Badea, C. Cottin, Korovkin-type theorems for generalized Boolean sum operators, *Approximation Theory (Kecskemét, 1990)*, vol. 58, Colloquia Mathematica Societatis János Bolyai (North-Holland, Amsterdam, 1991), pp. 51–68
5. C. Bardaro, A. Boccuto, K. Demirci, I. Mantellini, S. Orhan, Triangular A-statistical approximation by double sequences of positive linear operators. Results Math. **68**, 271–291 (2015)
6. C. Bardaro, A. Boccuto, K. Demirci, I. Mantellini, S. Orhan, Korovkin-type theorems for modular -A-statistical convergence. J. Funct. Spaces 11 (2015). Article ID 160401
7. K. Bögel, Mehrdimensionale differentiation von funktionen mehrerer veränderlicher. J. Reine Angew. Math. **170**, 197–217 (1934)
8. K. Bögel, Über mehrdimensionale differentiation, integration und beschränkte variation. J. Reine Angew. Math. **173**, 5–29 (1935)
9. K. Bögel, Über die mehrdimensionale differentiation. Jahresber. Deutsch. Mat. Verein. **65**, 45–71 (1962)
10. K. Demirci, S. Orhan, Statistically relatively uniform convergence of positive linear operators. Results Math. **69**(3–4), 359–367 (2016)
11. F. Dirik, K. Demirci, Korovkin type approximation theorem for functions of two variables in statistical sense. Turk. J. Math. **34**, 73–83 (2010)
12. F. Dirik, K. Demirci, Approximation in statistical sense to n-variate B-continuous functions by positive linear operators. Math. Slovaca **60**, 877–886 (2010)
13. F. Dirik, O. Duman, K. Demirci, Approximation in statistical sense to B-continuous functions by positive linear operators. Studia Sci. Math. Hungarica **47**(3), 289–298 (2010)
14. H. Fast, Sur la convergence statistique. Colloq. Math. **2**, 241–244 (1951)

15. S. Ghosal, Weighted statistical convergence of order α and its applications. J. Egypt. Math. Soc. **24**, 60–67 (2016)
16. V. Karakaya, T.A. Chishti, Weighted statistical convergence. Iran. J. Sci. Technol. Trans. A Sci. **33**, 219–223 (2009)
17. P.P. Korovkin, *Linear Operators and Approximation Theory* (Hindustan, Delhi, 1960)
18. M. Mursaleen, V. Karakaya, M. Erturk, F. Gursoy, Weighted statistical convergence and its application to Korovkin type approximation theorem. Appl. Math. Comput. **218**, 9132–9137 (2012)
19. F. Móricz, Statistical convergence of multiple sequences. Arch. Math. **81**(1), 82–89 (2003)
20. A. Pringsheim, Zur theorie der zweifach unendlichen zahlenfolgen. Math. Ann. **53**, 289–321 (1900)
21. H. Steinhaus, Sur la convergence ordinaire et la convergence asymtotique. Colloq. Math. **2**, 73–74 (1951)
22. B. Yılmaz, K. Demirci, S. Orhan, Relative modular convergence of positive linear operators. Positivity **20**(3), 565–577 (2016)