Tauberian Conditions Under Which Convergence Follows from Statistical Summability by Weighted Means

Zerrin Önder and İbrahim Çanak

Abstract Let (p_n) be a sequence of nonnegative numbers such that $p_0 > 0$ and

$$P_n := \sum_{k=0}^{n} p_k \to \infty \quad \text{as} \quad n \to \infty.$$

Let (s_n) be a sequence of real and complex numbers. The weighted mean of (s_n) is defined by

$$t_n := \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k \quad \text{for} \quad n = 0, 1, 2, \ldots$$

We obtain some sufficient conditions, under which the existence of the limit $\lim s_n = \mu$ follows from that of st-$\lim t_n = \mu$, where μ is a finite number. If (s_n) is a sequence of real numbers, then these Tauberian conditions are one-sided. If (s_n) is a sequence of complex numbers, these Tauberian conditions are two-sided. These Tauberian conditions are satisfied if (s_n) satisfies the one-sided condition of Landau type relative to (P_n) in the case of real sequences or if (s_n) satisfies the two-sided condition of Hardy type relative to (P_n) in the case of complex numbers.

Keywords Statistical convergence · Slow decreasing · Slow decreasing relative to (P_n) · Slow oscillation · Slow oscillation relative to (P_n) · The one-sided conditions of Landau type · The two-sided conditions of Hardy type · Tauberian theorems · Weighted mean summability method

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Z. Önder · İ. Çanak (✉)
Department of Mathematics, Ege University, 35100 İzmir, Turkey
e-mail: 91150000158@ogrenci.ege.edu.tr

İ. Çanak
e-mail: ibrahim.canak@ege.edu.tr

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1 Introduction

Let (s_n) be a sequence of real or complex numbers and $p = (p_n)$ be a sequence of nonnegative numbers such that

$$p_0 > 0 \quad \text{and} \quad P_n := \sum_{k=0}^{n} p_k \to \infty \quad \text{as} \quad n \to \infty. \tag{1}$$

The weighted mean of (s_n) is defined by

$$t_n := \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k \quad \text{for} \quad n \in \mathbb{N}.$$

A sequence (s_n) is said to be summable by the weighted mean method determined by the sequence p to μ if (t_n) converges to the same number. Weighted mean methods are also called Riesz methods or $(\overline{N}, p)$ methods in the literature. The $(\overline{N}, p)$ summability method is regular if and only if condition (1) is satisfied. In other words, every convergent sequence is also $(\overline{N}, p)$ summable to the same number under condition (1). However, the converse of this statement is not true in general. That the converse of this statement holds true is possible under some suitable condition which is so-called a Tauberian condition on the sequence. Any theorem stating that convergence of a sequence follows from its $(\overline{N}, p)$ summability and some Tauberian condition is said to be a Tauberian theorem for the $(\overline{N}, p)$ summability method. If $p_n = 1$ for all nonnegative integers n, then the $(\overline{N}, p)$ summability method reduces to Cesàro summability method.

We now give the definition of natural density of $K \subset \mathbb{N}$ and present statistically convergent sequences by using this concept. Let $K \subset \mathbb{N}$ be a subset of positive integers and $K_n = \{k \in K : k \leq n\}$. Then the set K has a natural density if the sequence $\left(\frac{|K_n|}{n}\right)$ has a limit. In this case, we write $\delta(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$, where the vertical bar denotes the cardinality of the enclosed set.

A sequence (s_n) is said to be statistically convergent to μ if for every $\epsilon > 0$, the set $K_\epsilon := \{k \in \mathbb{N} : |s_k - \mu| \geq \epsilon,\ k \leq n\}$ has natural density zero, i.e., for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n+1} |\{k \in \mathbb{N} : |s_k - \mu| \geq \epsilon,\ k \leq n\}| = 0. \tag{2}$$

We denote the set of all statistically convergent sequences by st. In this case, we write $st - \lim_{n\to\infty} s_n = \mu$ if the limit (2) exists.

We write down that every convergent sequence is statistically convergent to the same number since all finite subsets of the natural numbers have density zero. Accordingly, the statistical convergence may be considered as a regular summability method.

However, the converse of this statement is not always true. For example, the sequence (s_n) defined by

$$s_n = \begin{cases} n & \text{if } n = k^2;\; k = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

is statistically convergent to 0 since $|\{k \in \mathbb{N} : |s_k - 0| \geq \epsilon,\; k \leq n\}| \leq \sqrt{n}$ for every $\epsilon > 0$, but not convergent in the ordinary sense.

Recall that a sequence (s_n) is called statistically $(\overline{N}, p)$ summable to μ if $st - \lim_{n\to\infty} t_n = \mu$.

We write down that every statistically convergent sequence is also statistically $(\overline{N}, p)$ summable to same number under the boundedness condition of the sequence (cf. [1]).

At present, we define the concepts of slow decrease and slow oscillation relative to (P_n), respectively. In pursuit of defining of these concepts, we mention about how a transition exists between them.

We say that a sequence (s_n) of real numbers is slowly decreasing relative to (P_n) if

$$\liminf_{m\geq n\to\infty} (s_m - s_n) \geq 0 \quad \text{as} \quad 1 \leq \frac{P_m}{P_n} \to 1. \tag{3}$$

Using ϵ's and δ's, (3) is equivalent to the following statement:
To every $\epsilon > 0$, there exist $\delta > 0$ and $n_0 \in \mathbb{N}^0$ such that

$$s_m - s_n \geq -\epsilon \quad \text{whenever} \quad m \geq n \geq n_0 \quad \text{and} \quad 1 \leq \frac{P_m}{P_n} \leq 1 + \delta.$$

We say that a sequence (s_n) of complex numbers is slowly oscillating relative to (P_n) if

$$\limsup_{m\geq n\to\infty} |s_m - s_n| = 0 \quad \text{as} \quad 1 \leq \frac{P_m}{P_n} \to 1. \tag{4}$$

Using ϵ's and δ's, (4) is equivalent to the following statement:
To every $\epsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}^0$ such that

$$|s_m - s_n| \leq \epsilon \quad \text{whenever} \quad m \geq n \geq n_0 \quad \text{and} \quad 1 \leq \frac{P_m}{P_n} \leq 1 + \delta.$$

We emphasize that if $p_n = 1$ for all nonnegative integers n in (3) and (4), then the concepts of slow decrease relative to (P_n) and slow oscillation relative to (P_n) correspond to the concepts of slow decrease and slow oscillation, respectively (cf. [2]). In addition to this, there is a similar relation between the concepts of slow decrease relative to (P_n) and slow oscillation relative to (P_n) like the relation between the concepts of slow decrease and slow oscillation In other words, if the sequence (s_n) is slowly oscillating relative to (P_n), then it is also slowly decreasing relative to (P_n).

We define the concepts of the one-sided condition of Landau type relative to (P_n) and the two-sided condition of Hardy type relative to (P_n), respectively. In pursuit of defining of these concepts, we mention about how a transition exists between them and the concept defined previously.

We say that a sequence (s_n) of real numbers satisfies one-sided condition of Landau type relative to (P_n) if there exist positive constants n_0 and C such that

$$s_n - s_{n-1} \geq -C\frac{p_n}{P_n} \quad \text{whenever} \quad n > n_0. \tag{5}$$

We say that a sequence (s_n) of complex numbers satisfies two-sided condition of Hardy type relative to (P_n) if there exist positive constants n_0 and C such that

$$|s_n - s_{n-1}| \leq C\frac{p_n}{P_n} \quad \text{whenever} \quad n > n_0. \tag{6}$$

We emphasize that if $p_n = 1$ for all nonnegative integers n in (5) and (6), then one-sided condition of Landau type relative to (P_n) and two-sided condition of Hardy type relative to (P_n) correspond to one-sided condition of Landau type and two-sided condition of Hardy type, respectively. We note that if the sequence (s_n) satisfies two-sided condition of Hardy type relative to (P_n), then it also satisfies one-sided condition of Landau type relative to (P_n).

Additionally, it is easy to see that if the sequence (p_n) satisfies condition (1), then one-sided condition of Landau type relative to (P_n) implies condition of slow decrease relative to (P_n) (cf. [3]).

As a matter of fact, we suppose that (p_n) satisfies condition (1). Since one-sided condition of Landau type relative to (P_n) is satisfied, there exist positive constants n_0 and C such that $s_n - s_{n-1} \geq -C\frac{p_n}{P_n}$ whenever $n > n_0$. Let $m \geq n \geq n_0$ and $1 \leq \frac{P_m}{P_n} \leq 1 + \delta$. Then we have for a given $\epsilon > 0$,

$$\begin{aligned} s_m - s_n &= (s_m - s_{m-1}) + (s_{m-1} - s_{m-2}) + \ldots + (s_{n+1} - s_n) \\ &= \sum_{k=n+1}^{m} (s_k - s_{k-1}) \\ &\geq -C \sum_{k=n+1}^{m} \frac{p_k}{P_k} \\ &\geq -C\left(\frac{P_m}{P_n} - 1\right) > -C\delta > -\epsilon \end{aligned}$$

in case we choose $0 < \delta < \frac{\epsilon}{C}$. Therefore, we obtain that the sequence (s_n) of real numbers is slowly decreasing relative to (P_n).

Similarly, if (p_n) satisfies conditions (1) and (11), then two-sided condition of Hardy type relative to (P_n) implies condition of slow oscillation relative to (P_n).

2 Development of Tauberian Theory for Weighted Mean Method of Summability and Its Statistical Convergence

In this section, we begin with some remarks about the late history of the $(\overline{N}, p)$ summability and its Tauberian results that are about the history from the early part of nineteenth century until these days. We shortly mention the emergence of the concept of statistical convergence and the advancement of that in Tauberian theory. In the sequel, bringing together the concepts of the $(\overline{N}, p)$ summability and statistical convergence under the same roof, we refer certain results obtained by several researchers concerning these concepts. After dwelling on the studies that encourage us to do this research, we complete this section summarizing theorems and corollaries attained in this article.

Also called Riesz method since it was investigated for the first time in detail by Riesz, the $(\overline{N}, p)$ summability method has attracted the attention of many researchers, notably Kronecker, Cesàro, and Hardy [4]. One of the researchers who supported the development of this method in Tauberian theory, Tietz [5] revealed many Tauberian conditions that contain some well-known special Tauberian conditions, for the $(\overline{N}, p)$ summability method. In the sequel, Móricz and Rhoades [6] presented two Tauberian theorems which convergence follows from the $(\overline{N}, p)$ summability under necessary and sufficient conditions. Tietz and Zeller [7] established some Tauberian conditions controlling one-sided and two-sided oscillatory behavior of a sequence in certain senses defined in their paper for the $(\overline{N}, p)$ summability method. Móricz and Stadtmüller [8] obtained necessary and sufficient conditions including all classical (one-sided and two-sided) Tauberian conditions given for the $(\overline{N}, p)$ summability method. Móricz and Rhoades [9] arrived more general results than that in [6]. Finally, Sezer and Çanak [10] investigated some conditions needed for the $(\overline{N}, p)$ summable sequences to be convergent by using different approaches.

Contrary to the common belief that the concept of statistical convergence, which is a natural generalization of that of ordinary convergence, was introduced by Fast [11] and Schoenberg [12], this concept firstly came up with by Zygmund [13] who used the term *almost convergence* in place of statistical convergence and proved some theorems related to it. After the definition of statistical convergence was put into final form by Fast [11] and Schoenberg [12], it was associated with Tauberian conditions given by several researchers from past to present. Šalát [14] proved that the statistically convergent sequence needs to be neither bounded nor convergent. Considering statistical convergence as a regular summability method, Fridy [15] indicated that $n\Delta u_n = O(1)$ is a Tauberian condition for statistical convergence. Following the paper [15], Fridy and Khan [16] presented the statistical extension of some classical Tauberian theorems. Móricz [17] found out that necessary conditions for convergence of sequences which are statistically convergent are slow decrease and slow oscillation. In the sequel, Totur and Çanak [18] obtained some results which generalize well-known classical Tauberian theorems given for statistical convergence. There are also some interesting studies related to Tauberian theorems in which statistical convergence is used (see [19, 20]).

After the results obtained related to the concept of statistical convergence were published, it was combined with the $(\overline{N}, p)$ summability method. In relation to that, Móricz and Orhan [21] presented Tauberian theorems which convergence follows from the $(\overline{N}, p)$ summability under necessary and sufficient conditions, statistical slow decrease and statistical slow oscillation conditions. In the sequel, Totur and Çanak [22] arrived some results which improve well-known classical Tauberian theorems given for $(\overline{N}, p)$ summability method and statistical convergence.

Besides the studies mentioned up to now, the studies that encourage us to do this research is in fact those including some results obtained by Móricz [17, 23] for the Cesàro (or (C,1)) and the harmonic (or (H,1)) summability methods. Móricz formulated these results as follows, respectively:

Theorem 2.1 ([17]) *If the real (or complex) sequence (s_n) is statistically $(C, 1)$ summable to μ and slowly decreasing (or slowly oscillating), then (s_n) is convergent to μ.*

Theorem 2.2 ([23]) *If the real (or complex) sequence (s_n) is statistically $(H, 1)$ summable to μ and slowly decreasing (or slowly oscillating) with respect to the (H,1) summability, then (s_n) is convergent to μ.*

In case that $p_n = 1$ and $p_n = \frac{1}{n}$ for all nonnegative integers n, the $(\overline{N}, p)$ summability method reduces to the Cesàro and the harmonic summability methods, respectively. Here, our aim extends the theorems presented by Móricz for the Cesàro and the harmonic summability methods to the $(\overline{N}, p)$ summability method. Therefore, above-mentioned theorems are corollaries of our main results.

In this paper, we indicate that some conditions under which convergence follows from the statistical $(\overline{N}, p)$ summability for real and complex sequence. In Sect. 3, we present some lemmas which will be benefited in the proofs of our main results for real sequences. In the sequel, we prove a Tauberian theorem for real sequences that convergence follows from statistically $(\overline{N}, p)$ summability under the condition of slow decrease relative to (P_n) and additional conditions on (p_n) and we present a corollary related to this theorem. We end this section by giving another Tauberian condition for the $(\overline{N}, p)$ summability method. In Sect. 4, we present some lemmas which will be benefited in the proofs of our main results for complex sequences in parallel with Sect. 3. In the sequel, we prove a Tauberian theorem for complex sequences that convergence follows from statistically $(\overline{N}, p)$ summability under the condition of slow oscillation relative to (P_n) and additional conditions on (p_n) and we present a corollary related to this theorem. We end this section by giving another Tauberian condition for the $(\overline{N}, p)$ summability method.

3 Lemmas and Main Results for Real Sequences

This section essentially consists of two parts. In the first part, we present some lemmas which will be used in the proofs of our main results for real sequences. In the second part, we obtain some Tauberian conditions under which convergence follows from statistically $(\overline{N}, p)$ summability. In the sequel, we end this section by a corollary.

3.1 Lemmas

In this subsection, we express and prove the following assertions which will be benefited in the proofs of our main results for real sequences. The following lemma which were proved by Mikhalin [24] plays a crucial role in the proofs of subsequent two lemmas which are necessary to achieve our main results for real sequences.

Lemma 3.1 ([24, Lemma 2]) *Let* (p_n) *satisfy conditions (1) and* $\dfrac{p_n}{P_n} \to 0$ *as* $n \to \infty$. *If* (s_n) *satisfies condition*

$$\liminf_{m\geq n\to\infty}(s_m - s_n) \geq -r \ \ (0 \leq r < \infty) \ \ \text{as} \ \ 1 \leq \frac{P_m}{P_n} \to 1, \tag{7}$$

then there exist numbers $a > 0$ *and* $b > 0$ *such that* $s_m - s_n \geq -a \log \dfrac{P_m}{P_n} - b$ *for all* $m \geq n \geq 0$.

Due to the fact that condition (7) corresponds to condition of slow decrease relative to (P_n) in the case of $r = 0$ in Lemma 3.1, we prove in the following lemma that the below-mentioned sequence is bounded below under condition of slow decrease relative to (P_n) which is restrictive in comparison with condition (7) and some additional condition on (p_n) by the help of Lemma 3.1.

Lemma 3.2 *Let* (p_n) *satisfy conditions (1) and* $\dfrac{p_n}{P_n} \to 0$ *as* $n \to \infty$. *If* (s_n) *is slowly decreasing relative to* (P_n), *then*

$$\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n(s_m - s_n)\right)$$

is bounded below.

Proof Assume that (p_n) satisfies conditions (1) and $\dfrac{p_n}{P_n} \to 0$ as $n \to \infty$ and (s_n) is slowly decreasing relative to (P_n). Then, by taking these hypotheses into consideration, we conclude by the help of Lemma 3.1 that there exist positive numbers

a and b such that $s_m - s_n \geq -a \log \frac{P_m}{P_n} - b$ for all $m \geq n \geq 0$. In addition to this, since (p_n) satisfies the condition $\frac{p_n}{P_n} \to 0$ as $n \to \infty$, we have

$$\frac{P_n}{P_{n+1}} = 1 - \frac{p_{n+1}}{P_{n+1}} \to 1 \text{ as } n \to \infty. \tag{8}$$

By the fact that $t_n \to \ell$ implies $\frac{1}{t_n} \to \frac{1}{\ell}$ whenever $\ell \neq 0$, we find by (8) that

$$\frac{P_{n+1}}{P_n} \to 1 \quad \text{as} \quad n \to \infty \tag{9}$$

and so,

$$1 \leq \frac{P_m}{P_n} = \frac{P_m}{P_{m-1}} \frac{P_{m-1}}{P_{m-2}} \cdots \frac{P_{n+1}}{P_n} \to 1 \quad \text{as} \quad m \geq n \to \infty. \tag{10}$$

This means that for every $\delta > 0$, there exists $n_0 \in \mathbb{N}^0$ such that $1 \leq \frac{P_m}{P_n} \leq 1 + \delta$ whenever $m \geq n \geq n_0$. Therefore, from the condition of slow decrease relative to (P_n) we declare that for every $\epsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}^0$ such that $s_m - s_n \geq -\epsilon$ whenever $m \geq n \geq n_0$ and $1 \leq \frac{P_m}{P_n} \leq 1 + \delta$. With reference to above inequalities, we obtain that given $\epsilon > 0$

$$\begin{aligned}
\frac{1}{P_m} \sum_{n=0}^{m} p_n (s_m - s_n) &= \frac{1}{P_m} \sum_{n=0}^{n_0} p_n (s_m - s_n) + \frac{1}{P_m} \sum_{n=n_0+1}^{m} p_n (s_m - s_n) \\
&\geq \frac{1}{P_m} \sum_{n=0}^{n_0} p_n \left(-a \log \frac{P_m}{P_n} - b \right) + \frac{1}{P_m} \sum_{n=n_0+1}^{m} p_n (-\epsilon) \\
&\geq \frac{1}{P_m} \sum_{n=0}^{n_0} p_n \left(-a \log \frac{P_m}{P_0} - b \right) + \frac{1}{P_m} \sum_{n=n_0+1}^{m} p_n (-\epsilon) \\
&= \frac{P_{n_0} - P_0}{P_m} \left(-a \log \frac{P_m}{P_0} - b \right) + \frac{P_m - P_{n_0}}{P_m} (-\epsilon) \\
&= \left(P_{n_0} - P_0 \right) \left(-\frac{a}{P_m} \log \frac{P_m}{P_0} \right) + \frac{P_{n_0} - P_0}{P_m} (-b) + \left(1 - \frac{P_{n_0}}{P_m} \right) (-\epsilon) \\
&\geq \left(P_{n_0} - P_0 \right) \left(-\frac{a}{P_0} \right) + \frac{P_{n_0} - P_0}{P_0} (-b) + \left(1 - \frac{P_{n_0}}{P_m} \right) (-\epsilon) \\
&= \left(\frac{P_{n_0}}{P_0} - 1 \right) (-a - b) + \left(1 - \frac{P_{n_0}}{P_m} \right) (-\epsilon)
\end{aligned}$$

for all $m \geq 0$. In conjunction with information obtained up to now if we consider that $\left(\frac{P_{n_0}}{P_m} \right)$ is convergent to 0 by the condition (1) and every convergent sequence is also bounded, then there exists a positive constant H such that

$$\begin{aligned}\frac{1}{P_m}\sum_{n=0}^{m} p_n(s_m - s_n) &\geq \left(\frac{P_{n_0}}{P_0} - 1\right)(-a-b) + \left(1 - \frac{P_{n_0}}{P_m}\right)(-\epsilon)\\ &\geq \left(\frac{P_{n_0}}{P_0} - 1\right)(-a-b) - H\\ &:= -M\end{aligned}$$

for all $m \geq 0$ and some constant $M > 0$. In conclusion, we reach that the sequence $\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n(s_m - s_n)\right)$ is bounded below. □

At present, we offer an alternative proof of Lemma 3.3 which were previously proved by Mikhalin [24].

Lemma 3.3 *Let (p_n) satisfy conditions (1) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$. If (s_n) is slowly decreasing relative to (P_n), then (t_n) is also slowly decreasing relative to (P_n).*

Proof Assume that (p_n) satisfies conditions (1) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$ and (s_n) is slowly decreasing relative to (P_n). Given $\epsilon > 0$. By the definition of slow decrease relative to (P_n), this means that there exist $\delta > 0$ and $n_0 \in \mathbb{N}^0$ such that $s_m - s_n \geq -\epsilon$ whenever $m \geq n \geq n_0$ and $1 \leq \frac{P_m}{P_n} \leq 1 + \delta$.

Let $m \geq n \geq n_0$ and $1 \leq \frac{P_m}{P_n} \leq 1 + \delta'$. By the definition of the weighted means of (s_n) and Lemma 3.2, we obtain that

$$\begin{aligned}t_m - t_n &= \frac{1}{P_m}\sum_{k=0}^{m} p_k s_k - \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k\\ &= \frac{1}{P_m}\left\{\sum_{k=0}^{n} + \sum_{k=n+1}^{m}\right\} p_k s_k - \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k\\ &= \frac{1}{P_m}\sum_{k=0}^{n} p_k s_k + \frac{1}{P_m}\sum_{k=n+1}^{m} p_k s_k + \frac{P_m - P_n}{P_m P_n}\sum_{k=0}^{n} p_k s_n - \frac{P_m - P_n}{P_m P_n}\sum_{k=0}^{n} p_k s_n - \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k\\ &= \frac{1}{P_m}\sum_{k=n+1}^{m} p_k(s_k - s_n) + \left(\frac{1}{P_m} - \frac{1}{P_n}\right)\sum_{k=0}^{n} p_k s_k + \frac{P_m - P_n}{P_m P_n}\sum_{k=0}^{n} p_k s_n\\ &= \frac{P_m - P_n}{P_m}\frac{1}{P_n}\sum_{k=0}^{n} p_k(s_n - s_k) + \frac{1}{P_m}\sum_{k=n+1}^{m} p_k(s_k - s_n)\\ &\geq \left(1 - \frac{P_n}{P_m}\right)(-M) + \frac{1}{P_m}\sum_{k=n+1}^{m} p_k(-\epsilon)\\ &= \left(1 - \frac{P_n}{P_m}\right)(-M - \epsilon)\end{aligned}$$

whenever $m \geq k > n \geq n_0$, $1 < \dfrac{P_k}{P_n} \leq \dfrac{P_m}{P_n} \leq 1 + \delta'$ and for some constant $M > 0$.
Since we have that for $m \geq n \geq n_0$ and $1 \leq \dfrac{P_m}{P_n} \leq 1 + \delta'$

$$0 \leq 1 - \frac{P_n}{P_m} \leq \frac{\delta'}{1+\delta'},$$

if we choose $0 < \delta' \leq \dfrac{\epsilon}{M}$, then we arrive

$$t_m - t_n \geq \left(1 - \frac{P_n}{P_m}\right)(-M - \epsilon) \geq \frac{\delta'}{1+\delta'}(-M - \epsilon) \geq -\epsilon.$$

Therefore, we obtain that (t_n) is also slowly decreasing relative to (P_n). □

Lemma 3.4 ([17, Lemma 6]) *If (s_n) is statistically convergent to μ and slowly decreasing, then (s_n) is convergent to μ.*

Lemma 3.5 ([3, Theorem 4.2.2]) *Let (p_n) satisfy conditions (1) and*

$$\frac{P_n}{P_{n+1}} \to 1 \quad as \quad n \to \infty.$$

If (s_n) is $(\overline{N}, p)$ summable to μ and slowly decreasing relative to (P_n), then (s_n) is convergent to μ.

3.2 Main Results

In this subsection, we prove a Tauberian theorem for real sequences that convergence follows from statistically $(\overline{N}, p)$ summability under the condition of slow decrease relative to (P_n) and additional conditions on (p_n) and we present a corollary related to this theorem. In the sequel, we end this part by giving a Tauberian condition for the $(\overline{N}, p)$ summability method.

Theorem 3.6 *Let (p_n) satisfy conditions (1), $\dfrac{p_n}{P_n} \to 0$ as $n \to \infty$ and*

$$1 \leq \frac{P_m}{P_n} \to 1 \quad whenever \quad 1 < \frac{m}{n} \to 1 \quad as \quad n \to \infty. \tag{11}$$

If (s_n) is statistically $(\overline{N}, p)$ summable to μ and slowly decreasing relative to (P_n), then (s_n) is convergent to μ.

Proof Assume that (p_n) satisfies conditions (1), (11) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$ and that (s_n) is statistically $(\overline{N}, p)$ summable to μ and is slowly decreasing relative to (P_n). In the circumstances, we arrive by the help of Lemma 3.3 that (t_n) is also slowly decreasing relative to (P_n). In other words, we can say by the definition of slow decrease relative to (P_n) that condition

$$\liminf_{m \geq n \to \infty} (t_m - t_n) \geq 0 \quad \text{as} \quad 1 \leq \frac{P_m}{P_n} \to 1$$

holds and so by the condition (11) we obtain that

$$\liminf_{m \geq n \to \infty} (t_m - t_n) \geq 0 \quad \text{as} \quad 1 \leq \frac{m}{n} \to 1.$$

This statement implies the slow decrease of (t_n). Since (t_n) is slowly decreasing and statistically convergent to μ, we reach by the help of Lemma 3.4 that (t_n) is convergent to μ which means that (s_n) is $(\overline{N}, p)$ summable to μ. In addition to this, as (p_n) satisfies condition $\frac{p_n}{P_n} \to 0$ as $n \to \infty$, we attain that

$$\frac{P_n}{P_{n+1}} = \frac{P_n + p_{n+1} - p_{n+1}}{P_{n+1}} = \frac{P_{n+1}}{P_{n+1}} - \frac{p_{n+1}}{P_{n+1}} = 1 - \frac{p_{n+1}}{P_{n+1}} \to 1 \quad \text{as} \quad n \to \infty.$$

If we consider that condition of slowly decreasing relative to (P_n) is Tauberian condition for $(\overline{N}, p)$ summable sequence under additional conditions on (p_n) as a result of Lemma 3.5, then we conclude that (s_n) is convergent to μ. □

Corollary 3.7 *Let (p_n) satisfy conditions (1), (11) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$. If (s_n) is statistically $(\overline{N}, p)$ summable to μ and one-sided condition of Landau type relative to (P_n) is satisfied, then (s_n) is convergent to μ.*

Lemma 3.8 *Let (p_n) satisfy conditions (1) and (11). If the one-sided condition*

$$s_{n+1} - s_n \geq -C\frac{p_n}{P_n} \qquad \textit{for all} \;\; n \in \mathbb{N} \tag{12}$$

is satisfied for some constant $C > 0$, then condition

$$t_n - t_{n-1} \geq -C\frac{p_n}{P_n} \qquad \textit{for all} \;\; n \in \mathbb{N} \tag{13}$$

is also satisfied and (t_n) is slowly decreasing relative to (P_n).

Proof Assume that (p_n) satisfies conditions (1) and (11). By taking these hypotheses and condition (12) into consideration, we obtain that

$$
\begin{aligned}
t_n - t_{n-1} &= \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k - \frac{1}{P_{n-1}}\sum_{k=0}^{n-1} p_k s_k \\
&= \frac{1}{P_n P_{n-1}}\left(P_{n-1}\sum_{k=0}^{n} p_k s_k - P_n \sum_{k=0}^{n-1} p_k s_k\right) \\
&= \frac{1}{P_n P_{n-1}}\left(P_{n-1} p_n s_n - p_n \sum_{k=0}^{n-1} p_k s_k\right) \\
&= \frac{p_n}{P_n P_{n-1}}\sum_{k=0}^{n-1} p_k (s_n - s_k) \\
&= \frac{p_n}{P_n P_{n-1}}\sum_{k=0}^{n-1} p_k \sum_{j=k+1}^{n} (s_j - s_{j-1}) \\
&= \frac{p_n}{P_n P_{n-1}}\sum_{j=0}^{n} P_{j-1}(s_j - s_{j-1}) \\
&\geq -C\frac{p_n}{P_n P_{n-1}}\sum_{j=0}^{n} P_{j-1}\frac{p_{j-1}}{P_{j-1}} \\
&= -C\frac{p_n}{P_n}
\end{aligned}
$$

for all $n \in \mathbb{N}$ and some constant $C > 0$. Therefore, we conclude that condition (13) is satisfied and so (t_n) is slowly decreasing relative to (P_n). □

Remark 3.9 We recall that if (s_n) is a $(\overline{N}, p)$ summable sequence satisfying the two-sided condition

$$|s_{n+1} - s_n| \leq C\frac{p_n}{P_n} \tag{14}$$

for all $n \in \mathbb{N}$ and some constant $C > 0$ and (p_n) holds condition (1), then (s_n) is convergent. However, if we replace the two-sided condition (14) by the one-sided condition (12), then this statement fails in general without additional condition on (p_n) or (s_n) (cf. [3]).
On the other hand, the two-sided condition (14) can be weakened to the one-sided condition (12) by adding the assumption that (p_n) also satisfies the condition $\frac{p_n}{P_n} \to 0$ as $n \to \infty$ (cf. [25]).

In consideration of Lemma 3.8 and Remark 3.9, we can give the following theorem.

Theorem 3.10 *Let (p_n) satisfy conditions (1), (11) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$. If (s_n) is statistically $(\overline{N}, p)$ summable to μ and condition (12) is satisfied, then (s_n) is convergent to μ.*

Proof Assume that (p_n) satisfies conditions (1), (11) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$ and that statistically $(\overline{N}, p)$ summable (s_n) to μ satisfies condition (12). In the circumstances, we arrive by the help of Lemma 3.8 that (t_n) is also slowly decreasing relative to (P_n). In other words, we can say by the definition of slow decrease relative to (P_n) that condition

$$\liminf_{m \geq n \to \infty} (t_m - t_n) \geq 0 \quad \text{as} \quad 1 \leq \frac{P_m}{P_n} \to 1,$$

holds and so by condition (11) we obtain that

$$\liminf_{m \geq n \to \infty} (t_m - t_n) \geq 0 \quad \text{as} \quad 1 \leq \frac{m}{n} \to 1.$$

Since (t_n) is slowly decreasing and statistically convergent to μ, we reach by the help of Lemma 3.4 that (t_n) is convergent to μ which means that (s_n) is $(\overline{N}, p)$ summable to μ. If we consider that condition (12) is a Tauberian condition for $(\overline{N}, p)$ summable sequence under additional conditions on (p_n), then we conclude that (s_n) is convergent to μ. □

4 Lemmas and Main Results for Complex Sequences

This section essentially consists of two parts as lemmas and main results for complex sequences in parallel with Sect. 3. In the first part, we present some lemmas which will be used in the proofs of our main results for complex sequences. In the second part, we prove some Tauberian theorems for complex sequences that convergence follows from statistically $(\overline{N}, p)$ summability under some Tauberian conditions and additional conditions on (p_n). In the sequel, we complete this part by giving a corollary.

4.1 Lemmas

In this subsection, we express and prove the following assertions which will be benefited in the proofs of our main results for complex sequences. The following lemma was proved for real sequences by Mikhalin [24] and it plays a crucial role in the proofs of subsequent two lemmas which are necessary to achieve our main results for complex sequences.

Lemma 4.1 *Let (p_n) satisfy conditions (1) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$. If (s_n) satisfies the condition*

$$\limsup_{m\geq n\to\infty}|s_m-s_n|\leq r\ \ (0\leq r<\infty)\ \ \textit{as}\ \ 1\leq\frac{P_m}{P_n}\to 1, \tag{15}$$

then there exist positive numbers c and d such that $|s_m-s_n|\leq c\log\dfrac{P_m}{P_n}+d$ *for all* $m\geq n\geq 0$.

Proof Assume that (p_n) satisfies conditions (1) and $\dfrac{p_n}{P_n}\to 0$ as $n\to\infty$ and (s_n) satisfies the condition (15). We can say from condition (15) that for every $r+1>0$, there exist $\delta>0$ and $n_0\in\mathbb{N}$ such that $|s_m-s_n|<r+1$ whenever $n>n_0$ and $1\leq\dfrac{P_m}{P_n}\leq 1+\delta$.
Let $n\leq n_0$ and $P_m\leq P_{n_0}(1+\delta)$. In this case, we find that $|s_m-s_n|$ has a maximum depending only on n_0, and so there exist $\delta>0$ and $\ell\geq(r+1)$ such that $|s_m-s_n|<\ell$ for all $m,n\in\mathbb{N}$ related by $0\leq P_m-P_n\leq\delta P_n$. Choose n_1 such that $1\leq\dfrac{P_{n+1}}{P_n}\leq(1+\delta)$ for all $n\geq n_1$. We investigate chosen n_1 in three cases such that $q\geq w\geq n_1$, $0\leq w<n_1\leq q$, and $0\leq w\leq q<n_1$ for arbitrary fixed $q,w\in\mathbb{N}$.

We firstly take into consideration the case $q\geq w\geq n_1$. For this, we define the subsequence (w_{i+1}) where $w_0=w$ and w_{i+1} is the largest natural number for which the inequality $P_n\leq P_{w_i}(1+\delta)$ holds for all $i\in\mathbb{N}$. Therefore, we attain from this defining that the inequalities $P_{w_{i+1}}\leq P_{w_i}(1+\delta)$ and $P_{w_{i+1}+1}>P_{w_i}(1+\delta)$ are valid. In addition to these, let $w_k\leq q-1<w_{k+1}$. Then, we obtain that the inequalities $P_q\leq P_{w_{k+1}}$ and $0\leq P_{w_{i+1}}-P_{w_i}\leq\delta P_{w_i}$ hold for all $k\in\mathbb{N}$, and so we get that

$$\begin{aligned}\left|s_q-s_w\right| &= \left|s_q+s_{w_k}-s_{w_k}-s_{w_0}\right|\\ &= \left|\sum_{j=0}^{k-1}(s_{w_{j+1}}-s_{w_j})+s_q-s_{w_k}\right|\\ &\leq \sum_{j=0}^{k-1}\left|s_{w_{j+1}}-s_{w_j}\right|+\left|s_q-s_{w_k}\right|\\ &\leq \sum_{j=0}^{k-1}\ell+\left|s_q-s_{w_k}\right|\\ &\leq (k+1)\ell.\end{aligned}$$

Due to the fact that we have also the inequalities

$$P_q\geq P_{w_k+1}>P_{w_{k-1}}(1+\delta)\geq P_{w_{k-2}+1}(1+\delta)>P_{w_{k-3}}(1+\delta)^2\geq\cdots\geq P_{w_0}(1+\delta)^{\left[\frac{k}{2}\right]}>P_w(1+\delta)^{\frac{k}{2}-1},$$

we eventually reach the inequality $\log P_q\geq\log P_w+\dfrac{k-2}{2}\log(1+\delta)$. This implies that the inequality $(1+k)\leq\dfrac{\log P_q-\log P_w}{\log(1+\delta)^{1/2}}+3$ holds, and hence, we achieve the inequality

$$\left|s_q - s_w\right| \leq (k+1)\ell < \frac{\ell}{\log(1+\delta)^{1/2}}(\log P_q - \log P_w) + 3\ell \tag{16}$$

for any $q \geq w \geq n_1$.

On the other hand, we take into consideration the case $0 \leq w < n_1 \leq q$. Then, we obtain that

$$\begin{aligned}\left|s_q - s_w\right| = \left|s_q - s_{n_1} + s_{n_1} - s_w\right| &\leq \left|s_q - s_{n_1}\right| + \max_{0\leq w<n_1} |s_{n_1} - s_w| \\ &\leq \frac{\ell}{\log(1+\delta)^{1/2}}(\log P_q - \log P_{n_1}) + 3\ell + \max_{0\leq w<n_1} |s_{n_1} - s_w| \\ &\leq \frac{\ell}{\log(1+\delta)^{1/2}}(\log P_q - \log P_w) + 3\ell + \max_{0\leq w<n_1} |s_{n_1} - s_w|. \end{aligned} \tag{17}$$

Finally, if we consider the case $0 \leq w \leq q < n_1$, then we attain that

$$\left|s_q - s_w\right| \leq \max_{0\leq w\leq q\leq n_1 - 1} \left|s_q - s_w\right| \leq \frac{\ell}{\log(1+\delta)^{1/2}}(\log P_q - \log P_w) + 3\ell + \max_{0\leq w\leq q\leq n_1} \left|s_q - s_w\right|. \tag{18}$$

If we define positive numbers c, d as $c = \frac{\ell}{\log(1+\delta)^{1/2}}$ and $d = \max\{3\ell, 3\ell + \max_{0\leq w\leq q\leq n_1} \left|s_q - s_w\right|\}$, then we conclude by (16)–(18) that

$$\left|s_q - s_w\right| \leq c \log \frac{P_q}{P_w} + d$$

for all $q \geq w \geq 0$. □

Due to the fact that condition (15) corresponds to condition of slow oscillation relative to (P_n) in the case of $r = 0$ in Lemma 4.1, we prove in the following lemma that the below-mentioned sequence is bounded under condition of slow oscillation relative to (P_n) which is restrictive in comparison with condition (15) and some additional condition on (p_n) with the help of Lemma 4.1.

Lemma 4.2 *Let (p_n) satisfy conditions (1) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$. If (s_n) is slowly oscillating relative to (P_n), then*

$$\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n(s_m - s_n)\right)$$

is bounded.

Proof Assume that (p_n) satisfies conditions (1) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$ and (s_n) is slowly oscillating relative to (P_n). Then, by taking these hypotheses into consideration we can say with the help of Lemma 4.1 that there exist positive numbers c and d such that $|s_m - s_n| \le c \log \frac{P_m}{P_n} + d$ for all $m \ge n \ge 0$. In addition to this, as (p_n) satisfies the condition $\frac{p_n}{P_n} \to 0$ as $n \to \infty$, we attain that

$$\frac{P_n}{P_{n+1}} = \frac{P_n + p_{n+1} - p_{n+1}}{P_{n+1}} = \frac{P_{n+1}}{P_{n+1}} - \frac{p_{n+1}}{P_{n+1}} = 1 - \frac{p_{n+1}}{P_{n+1}} \to 1 \text{ as } n \to \infty. \tag{19}$$

Because it is well known that $t_n \to \ell$ implies $\frac{1}{t_n} \to \frac{1}{\ell}$ whenever $\ell \ne 0$ as $n \to \infty$, we find by (19) that

$$\frac{P_{n+1}}{P_n} \to 1 \text{ as } n \to \infty \tag{20}$$

and so,

$$1 \le \frac{P_m}{P_n} = \frac{P_m}{P_{m-1}} \frac{P_{m-1}}{P_{m-2}} \cdots \frac{P_{n+1}}{P_n} \to 1 \text{ as } m \ge n \to \infty. \tag{21}$$

This means that for every $\delta > 0$, there exists $n_0 \in \mathbb{N}^0$ such that $1 \le \frac{P_m}{P_n} \le 1 + \delta$ whenever $m \ge n \ge n_0$. Therefore, from condition of slow oscillation relative to (P_n) we declare that for every $\epsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}^0$ such that $|s_m - s_n| \le \epsilon$ whenever $m \ge n \ge n_0$ and $1 \le \frac{P_m}{P_n} \le 1 + \delta$. With reference to above inequalities, we obtain that for all $m \ge 0$ and given an $\epsilon > 0$

$$\begin{aligned}
\left| \frac{1}{P_m} \sum_{n=0}^{m} p_n (s_m - s_n) \right| &= \left| \frac{1}{P_m} \sum_{n=0}^{n_0} p_n (s_m - s_n) + \frac{1}{P_m} \sum_{n=n_0+1}^{m} p_n (s_m - s_n) \right| \\
&\le \frac{1}{P_m} \sum_{n=0}^{n_0} p_n |s_m - s_n| + \frac{1}{P_m} \sum_{n=n_0+1}^{m} p_n |s_m - s_n| \\
&\le \frac{1}{P_m} \sum_{n=0}^{n_0} p_n \left(c \log \frac{P_m}{P_n} + d \right) + \frac{1}{P_m} \sum_{n=n_0+1}^{m} p_n \epsilon \\
&\le \frac{1}{P_m} \sum_{n=0}^{n_0} p_n \left(c \log \frac{P_m}{P_0} + d \right) + \frac{1}{P_m} \sum_{n=n_0+1}^{m} p_n \epsilon \\
&= \frac{P_{n_0} - P_0}{P_m} \left(c \log \frac{P_m}{P_0} + d \right) + \frac{P_m - P_{n_0}}{P_m} \epsilon \\
&= \left(P_{n_0} - P_0 \right) \left(\frac{c}{P_m} \log \frac{P_m}{P_0} \right) + \frac{P_{n_0} - P_0}{P_m} d + \left(1 - \frac{P_{n_0}}{P_m} \right) \epsilon \\
&\le \left(P_{n_0} - P_0 \right) \left(\frac{c}{P_0} \right) + \frac{P_{n_0} - P_0}{P_0} d + \left(1 - \frac{P_{n_0}}{P_m} \right) \epsilon
\end{aligned}$$

$$= \left(\frac{P_{n_0}}{P_0} - 1\right)(c + d) + \left(1 - \frac{P_{n_0}}{P_m}\right)\epsilon.$$

In conjunction with the information obtained up to now if we consider that $\left(\frac{P_{n_0}}{P_m}\right)$ is convergent to 0 by condition (1) and every convergent sequence is also bounded, then there exists a positive constant H such that

$$\begin{aligned}\left|\frac{1}{P_m}\sum_{n=0}^{m} p_n(s_m - s_n)\right| &\leq \left(\frac{P_{n_0}}{P_0} - 1\right)(c + d) + \left(1 - \frac{P_{n_0}}{P_m}\right)\epsilon \\ &\leq \left(\frac{P_{n_0}}{P_0} - 1\right)(c + d) + H \\ &:= M\end{aligned}$$

for all $m \geq 0$ and some constant $M > 0$. In conclusion, we reach that $\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n(s_m - s_n)\right)$ is bounded. □

An alternative proof of the following lemma can also be done by following the procedure used in [24].

Lemma 4.3 *Let* (p_n) *satisfy conditions (1) and* $\frac{p_n}{P_n} \to 0$ *as* $n \to \infty$. *If* (s_n) *is slowly oscillating relative to* (P_n), *then* (t_n) *is also slowly oscillating relative to* (P_n).

Proof Assume that (p_n) satisfies conditions (1) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$ and (s_n) is slowly oscillating relative to (P_n). Given $\epsilon > 0$. By the definition of slow oscillation relative to (P_n), this means that there exist $\delta > 0$ and $n_0 \in \mathbb{N}^0$ such that $|s_m - s_n| \leq \epsilon$ whenever $m \geq n \geq n_0$ and $1 \leq \frac{P_m}{P_n} \leq 1 + \delta$.

Let $m \geq n \geq n_0$ and $1 \leq \frac{P_m}{P_n} \leq 1 + \delta'$. By the definition of the weighted means of (s_n) and Lemma 4.2, we obtain that

$$\begin{aligned}|t_m - t_n| &= \left|\frac{1}{P_m}\sum_{k=0}^{m} p_k s_k - \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k\right| \\ &= \left|\frac{1}{P_m}\left\{\sum_{k=0}^{n} + \sum_{k=n+1}^{m}\right\} p_k s_k - \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k\right| \\ &= \left|\frac{1}{P_m}\sum_{k=0}^{n} p_k s_k + \frac{1}{P_m}\sum_{k=n+1}^{m} p_k s_k + \frac{P_m - P_n}{P_m P_n}\sum_{k=0}^{n} p_k s_n - \frac{P_m - P_n}{P_m P_n}\sum_{k=0}^{n} p_k s_n - \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k\right| \\ &= \left|\frac{1}{P_m}\sum_{k=n+1}^{m} p_k(s_k - s_n) + \left(\frac{1}{P_m} - \frac{1}{P_n}\right)\sum_{k=0}^{n} p_k s_k + \frac{P_m - P_n}{P_m P_n}\sum_{k=0}^{n} p_k s_n\right| \\ &= \left|\frac{P_m - P_n}{P_m}\frac{1}{P_n}\sum_{k=0}^{n} p_k(s_n - s_k) + \frac{1}{P_m}\sum_{k=n+1}^{m} p_k(s_k - s_n)\right|\end{aligned}$$

$$
\begin{aligned}
&\le \frac{P_m - P_n}{P_m}\left|\frac{1}{P_n}\sum_{k=0}^{n} p_k(s_n - s_k)\right| + \frac{1}{P_m}\sum_{k=n+1}^{m} p_k|s_k - s_n| \\
&\le \left(1 - \frac{P_n}{P_m}\right) M + \frac{1}{P_m}\sum_{k=n+1}^{m} p_k \epsilon \\
&= \left(1 - \frac{P_n}{P_m}\right)(M + \epsilon)
\end{aligned}
$$

whenever $m \ge k > n \ge n_0$, $1 < \dfrac{P_k}{P_n} \le \dfrac{P_m}{P_n} \le 1 + \delta'$ and for some constant $M > 0$. Since we have that for $m \ge n \ge n_0$ and $1 \le \dfrac{P_m}{P_n} \le 1 + \delta'$

$$0 \le \left(1 - \frac{P_n}{P_m}\right) \le \frac{\delta'}{1 + \delta'},$$

if we choose $0 < \delta' \le \dfrac{\epsilon}{M}$, then we arrive

$$|t_m - t_n| \le \left(1 - \frac{P_n}{P_m}\right)(M + \epsilon) \le \frac{\delta'}{1 + \delta'}(M + \epsilon) \le \epsilon.$$

Therefore, we reach that (t_n) is also slowly oscillating relative to (P_n). □

Lemma 4.4 ([17, Lemma 7]) *If (s_n) is statistically convergent to μ and slowly oscillating, then (s_n) is convergent to μ.*

Lemma 4.5 ([3, Corollary of Theorem 4.2.2]) *Let (p_n) satisfy conditions (1) and*

$$\frac{P_n}{P_{n+1}} \to 1 \quad as \quad n \to \infty.$$

If (s_n) is $(\overline{N}, p)$ summable to μ and slowly oscillating relative to (P_n), then (s_n) is convergent to μ.

4.2 Main Results

In this subsection, we prove a Tauberian theorem for complex sequences that convergence follows from statistically $(\overline{N}, p)$ summability under condition of slow oscillation relative to (P_n) and additional conditions on (p_n) and we present a corollary related to this theorem. In the sequel, we complete this part by giving a Tauberian condition for the $(\overline{N}, p)$ summability method.

Theorem 4.6 *Let (p_n) satisfy conditions (1), (11) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$. If (s_n) is statistically $(\overline{N}, p)$ summable to μ and slowly oscillating relative to (P_n), then (s_n) is convergent to μ.*

Proof Assume that (p_n) satisfies conditions (1), (11) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$ and (s_n) which is statistically $(\overline{N}, p)$ summable to μ is slowly oscillating relative to (P_n). In the circumstances, we arrive with the help of Lemma 4.3 that (t_n) is also slowly oscillating relative to (P_n). In other words, we can say by the definition of slow oscillation relative to (P_n) that condition

$$\limsup_{m \geq n \to \infty} |t_m - t_n| = 0 \quad \text{as} \quad 1 \leq \frac{P_m}{P_n} \to 1$$

holds and so by condition (11) we obtain that

$$\limsup_{m \geq n \to \infty} |t_m - t_n| = 0 \quad \text{as} \quad 1 \leq \frac{m}{n} \to 1.$$

The statement implies the slow oscillation of (t_n). Since (t_n) is slowly oscillating and statistically convergent to μ, we reach with the help of Lemma 4.4 that (t_n) is convergent to μ which means that (s_n) is $(\overline{N}, p)$ summable to μ. In addition to this, as (p_n) satisfies condition $\frac{p_n}{P_n} \to 0$ as $n \to \infty$, we attain that

$$\frac{P_n}{P_{n+1}} = \frac{P_n + p_{n+1} - p_{n+1}}{P_{n+1}} = \frac{P_{n+1}}{P_{n+1}} - \frac{p_{n+1}}{P_{n+1}} = 1 - \frac{p_{n+1}}{P_{n+1}} \to 1 \quad \text{as} \quad n \to \infty.$$

If we consider that condition of slowly oscillating relative to (P_n) is a Tauberian condition for $(\overline{N}, p)$ summable sequence under additional conditions on (p_n) as a result of Lemma 4.5, then we conclude that (s_n) is convergent to μ. □

Corollary 4.7 *Let (p_n) satisfy conditions (1), (11) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$. If (s_n) is statistically $(\overline{N}, p)$ summable to μ and two-sided condition of Hardy type relative to (P_n), then (s_n) is convergent to μ.*

Lemma 4.8 *Let (p_n) satisfy conditions (1) and (11). If condition*

$$|s_{n+1} - s_n| \leq C \frac{p_n}{P_n} \quad \textit{for all} \ \ n \in \mathbb{N} \tag{22}$$

is satisfied for some constant $C > 0$, then condition

$$|t_n - t_{n-1}| \leq C \frac{p_n}{P_n} \quad \textit{for all} \ \ n \in \mathbb{N} \tag{23}$$

is also satisfied and (t_n) is slowly oscillating relative to (P_n).

Proof Assume that (p_n) satisfies conditions (1) and (11). By taking these hypotheses and condition (22) into consideration, we obtain that

$$
\begin{aligned}
|t_n - t_{n-1}| &= \left| \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k - \frac{1}{P_{n-1}} \sum_{k=0}^{n-1} p_k s_k \right| \\
&= \frac{1}{P_n P_{n-1}} \left| P_{n-1} \sum_{k=0}^{n} p_k s_k - P_n \sum_{k=0}^{n-1} p_k s_k \right| \\
&= \frac{1}{P_n P_{n-1}} \left| P_{n-1} p_n s_n - p_n \sum_{k=0}^{n-1} p_k s_k \right| \\
&= \frac{1}{P_n P_{n-1}} \left| p_n \sum_{k=0}^{n-1} p_k (s_n - s_k) \right| \\
&= \frac{p_n}{P_n P_{n-1}} \left| \sum_{k=0}^{n-1} p_k \sum_{j=k+1}^{n} (s_j - s_{j-1}) \right| \\
&\le \frac{p_n}{P_n P_{n-1}} \sum_{k=0}^{n-1} p_k \left| \sum_{j=k+1}^{n} (s_j - s_{j-1}) \right| \\
&\le \frac{p_n}{P_n P_{n-1}} \sum_{k=0}^{n-1} p_k \sum_{j=k+1}^{n} |s_j - s_{j-1}| \\
&= \frac{p_n}{P_n P_{n-1}} \sum_{j=0}^{n} P_{j-1} |s_j - s_{j-1}| \\
&\le C \frac{p_n}{P_n P_{n-1}} \sum_{j=0}^{n} P_{j-1} \frac{p_{j-1}}{P_{j-1}} \\
&= C \frac{p_n}{P_n}
\end{aligned}
$$

for all $n \in \mathbb{N}$ and some constant $C > 0$. Therefore, we conclude that condition (23) is satisfied and so (t_n) is slowly oscillating relative to (P_n). □

Remark 4.9 It is known that provided the sequence (p_n) satisfies condition (1), any $(\overline{N}, p)$ summable sequence (s_n) which satisfies two-sided condition (22) is convergent (cf. [3]).

In consideration of Lemma 4.8 and Remark 4.9, we can give the following theorem.

Theorem 4.10 *Let (p_n) satisfy conditions (1) and (11). If (s_n) is statistically $(\overline{N}, p)$ summable to μ and condition (22) is satisfied, then (s_n) is convergent to μ.*

Proof Assume that (p_n) satisfies conditions (1), (11) and statistically $(\overline{N}, p)$ summable (s_n) to μ satisfies condition (22). In the circumstances, we arrive with the help of Lemma 4.8 that (t_n) is also slowly oscillating relative to (P_n). In other words, we can say by the definition of slow oscillation relative to (P_n) that condition

$$\limsup_{m \geq n \to \infty} |t_m - t_n| = 0 \quad \text{as} \quad 1 \leq \frac{P_m}{P_n} \to 1,$$

holds and so by condition (11) we obtain that

$$\limsup_{m \geq n \to \infty} |t_m - t_n| = 0 \quad \text{as} \quad 1 \leq \frac{m}{n} \to 1.$$

The statement implies the slow oscillation of (t_n). Since (t_n) is slowly oscillating and statistically convergent to μ, we reach with the help of Lemma 4.4 that (t_n) is convergent to μ which means that (s_n) is $(\overline{N}, p)$ summable to μ. If we consider that condition (22) is a Tauberian condition for $(\overline{N}, p)$ summable sequence under additional conditions on (p_n), then we conclude that (s_n) is convergent to μ. □

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