S. A. Mohiuddine · Tuncer Acar *Editors*

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Preface

The aim of this book is to provide an up-to-date overview of the problems on summability and approximation theory, obtain some positive linear operators, and prove some approximation results. Of particular interest, the chapters in this book are devoted to the state-of-the-art research and development in summability and approximation theory. This book contains some original research papers by well-established mathematicians from the international mathematical community. This book will be of interest for a wide range of mathematicians whose area of research is summability and approximation theory.

This book consists of 14 chapters. Chapter "Tauberian Conditions Under Which Convergence Follows from Statistical Summability by Weighted Means" is devoted to obtaining some Tauberian theorems for real and complex sequences involving the notion of statistical summability by weighted means. Chapter "Applications of Fixed Point Theorems and General Convergence in Orthogonal Metric Spaces" discusses the general convergence methods in the setting of orthogonal metric space and studies the existence of a solution of differential and integral equations with the help of fixed point theorems in orthogonal metric spaces.

Chapter "Application of Measure of Noncompactness to the Infinite Systems of Second-Order Differential Equations in Banach Sequence Spaces c, ℓ_p and c_0^β" uses a technique to study the existence of the solutions of infinite systems of second-order differential equations in the Banach sequence space based upon the measures of noncompactness in conjunction with Meir–Keeler condensing operators with a view. Some illustrative examples are given in support of the results presented in this chapter. Chapter "Infinite Systems of Differential Equations in Banach Spaces Constructed by Fibonacci Numbers" makes use of the techniques associated with measures of noncompactness to obtain the existence theorems of the Cauchy problem in Banach sequence spaces derived by Fibonacci numbers and provides examples which show that infinite systems of differential equations have solution in these spaces but have no solution in the classical Banach spaces. Chapter "Convergence Properties of Genuine Bernstein–Durrmeyer Operators" presents a new construction of Bernstein–Durrmeyer operators. The local approximation, error estimation in terms of the modulus of continuity, weighted approximation, and a quantitative Voronovskaya-type theorem are investigated for the new operators. The significance of these results is supported by graphical and numerical data.

Chapter "Bivariate Szász-Type Operators Based on Multiple Appell Polynomials" discusses the construction of bivariate Szasz operators by means of multiple Appell polynomials. This chapter also deals with the study of many approximation properties of these operators such as uniform convergence, the degree of approximation via partial moduli of continuity, estimation of error in simultaneous approximation, the rate of convergence for twice continuously differentiable functions by Voronovskaya-type asymptotic theorem, and other related results. Chapter "Approximation Properties of Chlodowsky Variant of (p, q) Szász–Mirakyan–Stancu Operators" is devoted to constructing Chlodowsky variant of (p, q) Szasz–Mirakyan–Stancu operators on the unbounded domain and to obtain its Korovkin-type approximation properties. The rate of convergence of new operators in terms of Lipschitz class and modulus of continuity is discussed. Chapter "Approximation Theorems for Positive Linear Operators Associated with Hermite and Laguerre Polynomials" discusses the approximation behaviors of linear positive operators associated with Hermite and Laguerre polynomials and approximation properties of the Poisson-type integrals in space Lp. The author has given Szasz–Mirakyan-type operators defined by Hermite polynomials. The rate of convergence by means of the modulus of continuity and moduli of smoothness, Voronovskaya-type results, and boundary value problems for the Poisson integrals are discussed.

Chapter "On Generalized Picard Integral Operators" is dedicated to presenting the generation of a famous Picard integral operator which preserves some exponential functions. A few results concerning the weighted approximation properties of new operators, the order of convergence in an exponentially weighted space via exponentially weighted modulus of continuity, and shape-preserving properties are included. Chapter "From Uniform to Statistical Convergence of Binomial-Type Operators" investigates the sequences of binomial operators introduced by using umbral calculus in view of statistical convergence. Bernstein–Sheffer linear positive operators are analyzed, and some particular cases are highlighted. Chapter "Weighted Statistical Convergence of Bögel Continuous Functions by Positive Linear Operator" deals with the study of Korovkin-type approximation theorems for a sequence of linear positive operators for Bögel continuous functions by using the notion of weighted statistical convergence and using bivariate Bernstein polynomials to construct an illustrative example in support of results.

Chapter "Optimal Linear Approximation Under General Statistical Convergence" is devoted to obtaining both qualitative and quantitative results of a sequence of linear positive operators by considering the notion of B-statistical A-summability. Chapter "Statistical Deferred Cesàro Summability Mean Based on (p, q)-Integers with Application to Approximation Theorems" introduces the notion of statistical deferred Cesaro summability mean based on (p, q) integers and uses it under the difference sequence of order to prove a Korovkin-type approximation theorem.

An illustrative example is given to demonstrate that the present theorem is a non-trivial extension of some well-known Korovkin-type approximation theorems. The rate of statistical deferred Cesaro summability means with the help of the modulus of continuity is also discussed. Finally, chapter "Approximation Results for Urysohn-Type Nonlinear Bernstein Operators" introduces the new nonlinear Bernstein operators by using a nonlinear form of the kernels together with the Urysohn-type operator values instead of the sampling values of the function. The results of this chapter deal with the study of convergence problems for these nonlinear operators that approximate the Urysohn-type operator in some functional spaces.

We wish to express our gratitude to the authors who have contributed to this book. We would like to thank our family for moral support during the preparation of this book. Finally, we are also very thankful to Mr. Shamim Ahmad, Editor of Mathematics in Springer, for taking interest in publishing this book.

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About the Editors

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Tuncer Acar is an associate professor in the Department of Mathematics, Faculty of Science, Selçuk University, Turkey. He previously worked in the Department of Mathematics and Statistics, University of Alberta, Canada, as a visiting scholar in 2013 and 2014. He joined the Department of Mathematics, Faculty of Science, Kirikkale University, as a research assistant in 2009. He received his M.Sc. in 2011 and Ph.D. in 2015 from the same university. Born in Amasya, Turkey, in 1985, he graduated from Ataturk University in 2008. He has studied approximation by linear positive operators and published about 40 papers.

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Tauberian Conditions Under Which Convergence Follows from Statistical Summability by Weighted Means

Zerrin Önder and İbrahim Çanak

Abstract Let (p_n) be a sequence of nonnegative numbers such that $p_0 > 0$ and

$$P_n := \sum_{k=0}^{n} p_k \to \infty \quad \text{as} \quad n \to \infty.$$

Let (s_n) be a sequence of real and complex numbers. The weighted mean of (s_n) is defined by

$$t_n := \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k \quad \text{for} \quad n = 0, 1, 2, \ldots$$

We obtain some sufficient conditions, under which the existence of the limit $\lim s_n = \mu$ follows from that of st-$\lim t_n = \mu$, where μ is a finite number. If (s_n) is a sequence of real numbers, then these Tauberian conditions are one-sided. If (s_n) is a sequence of complex numbers, these Tauberian conditions are two-sided. These Tauberian conditions are satisfied if (s_n) satisfies the one-sided condition of Landau type relative to (P_n) in the case of real sequences or if (s_n) satisfies the two-sided condition of Hardy type relative to (P_n) in the case of complex numbers.

Keywords Statistical convergence · Slow decreasing · Slow decreasing relative to (P_n) · Slow oscillation · Slow oscillation relative to (P_n) · The one-sided conditions of Landau type · The two-sided conditions of Hardy type · Tauberian theorems · Weighted mean summability method

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1 Introduction

Let (s_n) be a sequence of real or complex numbers and $p = (p_n)$ be a sequence of nonnegative numbers such that

$$p_0 > 0 \quad \text{and} \quad P_n := \sum_{k=0}^{n} p_k \to \infty \quad \text{as} \quad n \to \infty. \tag{1}$$

The weighted mean of (s_n) is defined by

$$t_n := \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k \quad \text{for} \quad n \in \mathbb{N}.$$

A sequence (s_n) is said to be summable by the weighted mean method determined by the sequence p to μ if (t_n) converges to the same number. Weighted mean methods are also called Riesz methods or $(\overline{N}, p)$ methods in the literature. The $(\overline{N}, p)$ summability method is regular if and only if condition (1) is satisfied. In other words, every convergent sequence is also $(\overline{N}, p)$ summable to the same number under condition (1). However, the converse of this statement is not true in general. That the converse of this statement holds true is possible under some suitable condition which is so-called a Tauberian condition on the sequence. Any theorem stating that convergence of a sequence follows from its $(\overline{N}, p)$ summability and some Tauberian condition is said to be a Tauberian theorem for the $(\overline{N}, p)$ summability method. If $p_n = 1$ for all nonnegative integers n, then the $(\overline{N}, p)$ summability method reduces to Cesàro summability method.

We now give the definition of natural density of $K \subset \mathbb{N}$ and present statistically convergent sequences by using this concept. Let $K \subset \mathbb{N}$ be a subset of positive integers and $K_n = \{k \in K : k \leq n\}$. Then the set K has a natural density if the sequence $\left(\frac{|K_n|}{n}\right)$ has a limit. In this case, we write $\delta(K) = \lim\limits_{n\to\infty} \frac{|K_n|}{n}$, where the vertical bar denotes the cardinality of the enclosed set.

A sequence (s_n) is said to be statistically convergent to μ if for every $\epsilon > 0$, the set $K_\epsilon := \{k \in \mathbb{N} : |s_k - \mu| \geq \epsilon,\ k \leq n\}$ has natural density zero, i.e., for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n+1} \left|\{k \in \mathbb{N} : |s_k - \mu| \geq \epsilon,\ k \leq n\}\right| = 0. \tag{2}$$

We denote the set of all statistically convergent sequences by st. In this case, we write $st - \lim\limits_{n\to\infty} s_n = \mu$ if the limit (2) exists.

We write down that every convergent sequence is statistically convergent to the same number since all finite subsets of the natural numbers have density zero. Accordingly, the statistical convergence may be considered as a regular summability method.

However, the converse of this statement is not always true. For example, the sequence (s_n) defined by

$$s_n = \begin{cases} n & \text{if } \ n = k^2; \ k = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

is statistically convergent to 0 since $|\{k \in \mathbb{N} : |s_k - 0| \geq \epsilon, \ k \leq n\}| \leq \sqrt{n}$ for every $\epsilon > 0$, but not convergent in the ordinary sense.

Recall that a sequence (s_n) is called statistically $(\overline{N}, p)$ summable to μ if $st - \lim_{n\to\infty} t_n = \mu$.

We write down that every statistically convergent sequence is also statistically $(\overline{N}, p)$ summable to same number under the boundedness condition of the sequence (cf. [1]).

At present, we define the concepts of slow decrease and slow oscillation relative to (P_n), respectively. In pursuit of defining of these concepts, we mention about how a transition exists between them.

We say that a sequence (s_n) of real numbers is slowly decreasing relative to (P_n) if

$$\liminf_{m\geq n\to\infty} (s_m - s_n) \geq 0 \quad \text{as} \quad 1 \leq \frac{P_m}{P_n} \to 1. \tag{3}$$

Using ϵ's and δ's, (3) is equivalent to the following statement:
To every $\epsilon > 0$, there exist $\delta > 0$ and $n_0 \in \mathbb{N}^0$ such that

$$s_m - s_n \geq -\epsilon \quad \text{whenever} \quad m \geq n \geq n_0 \quad \text{and} \quad 1 \leq \frac{P_m}{P_n} \leq 1 + \delta.$$

We say that a sequence (s_n) of complex numbers is slowly oscillating relative to (P_n) if

$$\limsup_{m\geq n\to\infty} |s_m - s_n| = 0 \quad \text{as} \quad 1 \leq \frac{P_m}{P_n} \to 1. \tag{4}$$

Using ϵ's and δ's, (4) is equivalent to the following statement:
To every $\epsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}^0$ such that

$$|s_m - s_n| \leq \epsilon \quad \text{whenever} \quad m \geq n \geq n_0 \quad \text{and} \quad 1 \leq \frac{P_m}{P_n} \leq 1 + \delta.$$

We emphasize that if $p_n = 1$ for all nonnegative integers n in (3) and (4), then the concepts of slow decrease relative to (P_n) and slow oscillation relative to (P_n) correspond to the concepts of slow decrease and slow oscillation, respectively (cf. [2]). In addition to this, there is a similar relation between the concepts of slow decrease relative to (P_n) and slow oscillation relative to (P_n) like the relation between the concepts of slow decrease and slow oscillation In other words, if the sequence (s_n) is slowly oscillating relative to (P_n), then it is also slowly decreasing relative to (P_n).

We define the concepts of the one-sided condition of Landau type relative to (P_n) and the two-sided condition of Hardy type relative to (P_n), respectively. In pursuit of defining of these concepts, we mention about how a transition exists between them and the concept defined previously.

We say that a sequence (s_n) of real numbers satisfies one-sided condition of Landau type relative to (P_n) if there exist positive constants n_0 and C such that

$$s_n - s_{n-1} \geq -C\frac{p_n}{P_n} \quad \text{whenever} \quad n > n_0. \tag{5}$$

We say that a sequence (s_n) of complex numbers satisfies two-sided condition of Hardy type relative to (P_n) if there exist positive constants n_0 and C such that

$$|s_n - s_{n-1}| \leq C\frac{p_n}{P_n} \quad \text{whenever} \quad n > n_0. \tag{6}$$

We emphasize that if $p_n = 1$ for all nonnegative integers n in (5) and (6), then one-sided condition of Landau type relative to (P_n) and two-sided condition of Hardy type relative to (P_n) correspond to one-sided condition of Landau type and two-sided condition of Hardy type, respectively. We note that if the sequence (s_n) satisfies two-sided condition of Hardy type relative to (P_n), then it also satisfies one-sided condition of Landau type relative to (P_n).

Additionally, it is easy to see that if the sequence (p_n) satisfies condition (1), then one-sided condition of Landau type relative to (P_n) implies condition of slow decrease relative to (P_n) (cf. [3]).

As a matter of fact, we suppose that (p_n) satisfies condition (1). Since one-sided condition of Landau type relative to (P_n) is satisfied, there exist positive constants n_0 and C such that $s_n - s_{n-1} \geq -C\frac{p_n}{P_n}$ whenever $n > n_0$. Let $m \geq n \geq n_0$ and $1 \leq \frac{P_m}{P_n} \leq 1 + \delta$. Then we have for a given $\epsilon > 0$,

$$\begin{aligned}
s_m - s_n &= (s_m - s_{m-1}) + (s_{m-1} - s_{m-2}) + \ldots + (s_{n+1} - s_n) \\
&= \sum_{k=n+1}^{m} (s_k - s_{k-1}) \\
&\geq -C \sum_{k=n+1}^{m} \frac{p_k}{P_k} \\
&\geq -C\left(\frac{P_m}{P_n} - 1\right) > -C\delta > -\epsilon
\end{aligned}$$

in case we choose $0 < \delta < \frac{\epsilon}{C}$. Therefore, we obtain that the sequence (s_n) of real numbers is slowly decreasing relative to (P_n).

Similarly, if (p_n) satisfies conditions (1) and (11), then two-sided condition of Hardy type relative to (P_n) implies condition of slow oscillation relative to (P_n).

2 Development of Tauberian Theory for Weighted Mean Method of Summability and Its Statistical Convergence

In this section, we begin with some remarks about the late history of the $(\overline{N}, p)$ summability and its Tauberian results that are about the history from the early part of nineteenth century until these days. We shortly mention the emergence of the concept of statistical convergence and the advancement of that in Tauberian theory. In the sequel, bringing together the concepts of the $(\overline{N}, p)$ summability and statistical convergence under the same roof, we refer certain results obtained by several researchers concerning these concepts. After dwelling on the studies that encourage us to do this research, we complete this section summarizing theorems and corollaries attained in this article.

Also called Riesz method since it was investigated for the first time in detail by Riesz, the $(\overline{N}, p)$ summability method has attracted the attention of many researchers, notably Kronecker, Cesàro, and Hardy [4]. One of the researchers who supported the development of this method in Tauberian theory, Tietz [5] revealed many Tauberian conditions that contain some well-known special Tauberian conditions, for the $(\overline{N}, p)$ summability method. In the sequel, Móricz and Rhoades [6] presented two Tauberian theorems which convergence follows from the $(\overline{N}, p)$ summability under necessary and sufficient conditions. Tietz and Zeller [7] established some Tauberian conditions controlling one-sided and two-sided oscillatory behavior of a sequence in certain senses defined in their paper for the $(\overline{N}, p)$ summability method. Móricz and Stadtmüller [8] obtained necessary and sufficient conditions including all classical (one-sided and two-sided) Tauberian conditions given for the $(\overline{N}, p)$ summability method. Móricz and Rhoades [9] arrived more general results than that in [6]. Finally, Sezer and Çanak [10] investigated some conditions needed for the $(\overline{N}, p)$ summable sequences to be convergent by using different approaches.

Contrary to the common belief that the concept of statistical convergence, which is a natural generalization of that of ordinary convergence, was introduced by Fast [11] and Schoenberg [12], this concept firstly came up with by Zygmund [13] who used the term *almost convergence* in place of statistical convergence and proved some theorems related to it. After the definition of statistical convergence was put into final form by Fast [11] and Schoenberg [12], it was associated with Tauberian conditions given by several researchers from past to present. Šalát [14] proved that the statistically convergent sequence needs to be neither bounded nor convergent. Considering statistical convergence as a regular summability method, Fridy [15] indicated that $n\Delta u_n = O(1)$ is a Tauberian condition for statistical convergence. Following the paper [15], Fridy and Khan [16] presented the statistical extension of some classical Tauberian theorems. Móricz [17] found out that necessary conditions for convergence of sequences which are statistically convergent are slow decrease and slow oscillation. In the sequel, Totur and Çanak [18] obtained some results which generalize well-known classical Tauberian theorems given for statistical convergence. There are also some interesting studies related to Tauberian theorems in which statistical convergence is used (see [19, 20]).

After the results obtained related to the concept of statistical convergence were published, it was combined with the $(\overline{N}, p)$ summability method. In relation to that, Móricz and Orhan [21] presented Tauberian theorems which convergence follows from the $(\overline{N}, p)$ summability under necessary and sufficient conditions, statistical slow decrease and statistical slow oscillation conditions. In the sequel, Totur and Çanak [22] arrived some results which improve well-known classical Tauberian theorems given for $(\overline{N}, p)$ summability method and statistical convergence.

Besides the studies mentioned up to now, the studies that encourage us to do this research is in fact those including some results obtained by Móricz [17, 23] for the Cesàro (or (C,1)) and the harmonic (or (H,1)) summability methods. Móricz formulated these results as follows, respectively:

Theorem 2.1 ([17]) *If the real (or complex) sequence (s_n) is statistically $(C, 1)$ summable to μ and slowly decreasing (or slowly oscillating), then (s_n) is convergent to μ.*

Theorem 2.2 ([23]) *If the real (or complex) sequence (s_n) is statistically $(H, 1)$ summable to μ and slowly decreasing (or slowly oscillating) with respect to the (H,1) summability, then (s_n) is convergent to μ.*

In case that $p_n = 1$ and $p_n = \dfrac{1}{n}$ for all nonnegative integers n, the $(\overline{N}, p)$ summability method reduces to the Cesàro and the harmonic summability methods, respectively. Here, our aim extends the theorems presented by Móricz for the Cesàro and the harmonic summability methods to the $(\overline{N}, p)$ summability method. Therefore, above-mentioned theorems are corollaries of our main results.

In this paper, we indicate that some conditions under which convergence follows from the statistical $(\overline{N}, p)$ summability for real and complex sequence. In Sect. 3, we present some lemmas which will be benefited in the proofs of our main results for real sequences. In the sequel, we prove a Tauberian theorem for real sequences that convergence follows from statistically $(\overline{N}, p)$ summability under the condition of slow decrease relative to (P_n) and additional conditions on (p_n) and we present a corollary related to this theorem. We end this section by giving another Tauberian condition for the $(\overline{N}, p)$ summability method. In Sect. 4, we present some lemmas which will be benefited in the proofs of our main results for complex sequences in parallel with Sect. 3. In the sequel, we prove a Tauberian theorem for complex sequences that convergence follows from statistically $(\overline{N}, p)$ summability under the condition of slow oscillation relative to (P_n) and additional conditions on (p_n) and we present a corollary related to this theorem. We end this section by giving another Tauberian condition for the $(\overline{N}, p)$ summability method.

3 Lemmas and Main Results for Real Sequences

This section essentially consists of two parts. In the first part, we present some lemmas which will be used in the proofs of our main results for real sequences. In the second part, we obtain some Tauberian conditions under which convergence follows from statistically $(\overline{N}, p)$ summability. In the sequel, we end this section by a corollary.

3.1 Lemmas

In this subsection, we express and prove the following assertions which will be benefited in the proofs of our main results for real sequences. The following lemma which were proved by Mikhalin [24] plays a crucial role in the proofs of subsequent two lemmas which are necessary to achieve our main results for real sequences.

Lemma 3.1 ([24, Lemma 2]) *Let (p_n) satisfy conditions (1) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$. If (s_n) satisfies condition*

$$\liminf_{m \geq n \to \infty} (s_m - s_n) \geq -r \ \ (0 \leq r < \infty) \quad as \quad 1 \leq \frac{P_m}{P_n} \to 1, \tag{7}$$

then there exist numbers $a > 0$ and $b > 0$ such that $s_m - s_n \geq -a \log \frac{P_m}{P_n} - b$ for all $m \geq n \geq 0$.

Due to the fact that condition (7) corresponds to condition of slow decrease relative to (P_n) in the case of $r = 0$ in Lemma 3.1, we prove in the following lemma that the below-mentioned sequence is bounded below under condition of slow decrease relative to (P_n) which is restrictive in comparison with condition (7) and some additional condition on (p_n) by the help of Lemma 3.1.

Lemma 3.2 *Let (p_n) satisfy conditions (1) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$. If (s_n) is slowly decreasing relative to (P_n), then*

$$\left(\frac{1}{P_m} \sum_{n=0}^{m} p_n (s_m - s_n) \right)$$

is bounded below.

Proof Assume that (p_n) satisfies conditions (1) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$ and (s_n) is slowly decreasing relative to (P_n). Then, by taking these hypotheses into consideration, we conclude by the help of Lemma 3.1 that there exist positive numbers

a and b such that $s_m - s_n \geq -a \log \frac{P_m}{P_n} - b$ for all $m \geq n \geq 0$. In addition to this, since (p_n) satisfies the condition $\frac{p_n}{P_n} \to 0$ as $n \to \infty$, we have

$$\frac{P_n}{P_{n+1}} = 1 - \frac{p_{n+1}}{P_{n+1}} \to 1 \ \text{ as } \ n \to \infty. \tag{8}$$

By the fact that $t_n \to \ell$ implies $\frac{1}{t_n} \to \frac{1}{\ell}$ whenever $\ell \neq 0$, we find by (8) that

$$\frac{P_{n+1}}{P_n} \to 1 \ \text{ as } \ n \to \infty \tag{9}$$

and so,

$$1 \leq \frac{P_m}{P_n} = \frac{P_m}{P_{m-1}} \frac{P_{m-1}}{P_{m-2}} \cdots \frac{P_{n+1}}{P_n} \to 1 \ \text{ as } \ m \geq n \to \infty. \tag{10}$$

This means that for every $\delta > 0$, there exists $n_0 \in \mathbb{N}^0$ such that $1 \leq \frac{P_m}{P_n} \leq 1 + \delta$ whenever $m \geq n \geq n_0$. Therefore, from the condition of slow decrease relative to (P_n) we declare that for every $\epsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}^0$ such that $s_m - s_n \geq -\epsilon$ whenever $m \geq n \geq n_0$ and $1 \leq \frac{P_m}{P_n} \leq 1 + \delta$. With reference to above inequalities, we obtain that given $\epsilon > 0$

$$\begin{aligned}
\frac{1}{P_m} \sum_{n=0}^{m} p_n (s_m - s_n) &= \frac{1}{P_m} \sum_{n=0}^{n_0} p_n (s_m - s_n) + \frac{1}{P_m} \sum_{n=n_0+1}^{m} p_n (s_m - s_n) \\
&\geq \frac{1}{P_m} \sum_{n=0}^{n_0} p_n \left(-a \log \frac{P_m}{P_n} - b \right) + \frac{1}{P_m} \sum_{n=n_0+1}^{m} p_n (-\epsilon) \\
&\geq \frac{1}{P_m} \sum_{n=0}^{n_0} p_n \left(-a \log \frac{P_m}{P_0} - b \right) + \frac{1}{P_m} \sum_{n=n_0+1}^{m} p_n (-\epsilon) \\
&= \frac{P_{n_0} - P_0}{P_m} \left(-a \log \frac{P_m}{P_0} - b \right) + \frac{P_m - P_{n_0}}{P_m} (-\epsilon) \\
&= \left(P_{n_0} - P_0 \right) \left(-\frac{a}{P_m} \log \frac{P_m}{P_0} \right) + \frac{P_{n_0} - P_0}{P_m} (-b) + \left(1 - \frac{P_{n_0}}{P_m} \right) (-\epsilon) \\
&\geq \left(P_{n_0} - P_0 \right) \left(-\frac{a}{P_0} \right) + \frac{P_{n_0} - P_0}{P_0} (-b) + \left(1 - \frac{P_{n_0}}{P_m} \right) (-\epsilon) \\
&= \left(\frac{P_{n_0}}{P_0} - 1 \right) (-a - b) + \left(1 - \frac{P_{n_0}}{P_m} \right) (-\epsilon)
\end{aligned}$$

for all $m \geq 0$. In conjunction with information obtained up to now if we consider that $\left(\frac{P_{n_0}}{P_m} \right)$ is convergent to 0 by the condition (1) and every convergent sequence is also bounded, then there exists a positive constant H such that

$$\begin{aligned}\frac{1}{P_m}\sum_{n=0}^{m} p_n(s_m - s_n) &\geq \left(\frac{P_{n_0}}{P_0} - 1\right)(-a-b) + \left(1 - \frac{P_{n_0}}{P_m}\right)(-\epsilon)\\ &\geq \left(\frac{P_{n_0}}{P_0} - 1\right)(-a-b) - H\\ &:= -M\end{aligned}$$

for all $m \geq 0$ and some constant $M > 0$. In conclusion, we reach that the sequence $\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n(s_m - s_n)\right)$ is bounded below. □

At present, we offer an alternative proof of Lemma 3.3 which were previously proved by Mikhalin [24].

Lemma 3.3 *Let (p_n) satisfy conditions (1) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$. If (s_n) is slowly decreasing relative to (P_n), then (t_n) is also slowly decreasing relative to (P_n).*

Proof Assume that (p_n) satisfies conditions (1) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$ and (s_n) is slowly decreasing relative to (P_n). Given $\epsilon > 0$. By the definition of slow decrease relative to (P_n), this means that there exist $\delta > 0$ and $n_0 \in \mathbb{N}^0$ such that $s_m - s_n \geq -\epsilon$ whenever $m \geq n \geq n_0$ and $1 \leq \frac{P_m}{P_n} \leq 1 + \delta$.

Let $m \geq n \geq n_0$ and $1 \leq \frac{P_m}{P_n} \leq 1 + \delta'$. By the definition of the weighted means of (s_n) and Lemma 3.2, we obtain that

$$\begin{aligned}t_m - t_n &= \frac{1}{P_m}\sum_{k=0}^{m} p_k s_k - \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k\\ &= \frac{1}{P_m}\left\{\sum_{k=0}^{n} + \sum_{k=n+1}^{m}\right\} p_k s_k - \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k\\ &= \frac{1}{P_m}\sum_{k=0}^{n} p_k s_k + \frac{1}{P_m}\sum_{k=n+1}^{m} p_k s_k + \frac{P_m - P_n}{P_m P_n}\sum_{k=0}^{n} p_k s_n - \frac{P_m - P_n}{P_m P_n}\sum_{k=0}^{n} p_k s_n - \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k\\ &= \frac{1}{P_m}\sum_{k=n+1}^{m} p_k(s_k - s_n) + \left(\frac{1}{P_m} - \frac{1}{P_n}\right)\sum_{k=0}^{n} p_k s_k + \frac{P_m - P_n}{P_m P_n}\sum_{k=0}^{n} p_k s_n\\ &= \frac{P_m - P_n}{P_m}\frac{1}{P_n}\sum_{k=0}^{n} p_k(s_n - s_k) + \frac{1}{P_m}\sum_{k=n+1}^{m} p_k(s_k - s_n)\\ &\geq \left(1 - \frac{P_n}{P_m}\right)(-M) + \frac{1}{P_m}\sum_{k=n+1}^{m} p_k(-\epsilon)\\ &= \left(1 - \frac{P_n}{P_m}\right)(-M - \epsilon)\end{aligned}$$

whenever $m \geq k > n \geq n_0$, $1 < \dfrac{P_k}{P_n} \leq \dfrac{P_m}{P_n} \leq 1 + \delta'$ and for some constant $M > 0$.
Since we have that for $m \geq n \geq n_0$ and $1 \leq \dfrac{P_m}{P_n} \leq 1 + \delta'$

$$0 \leq 1 - \frac{P_n}{P_m} \leq \frac{\delta'}{1 + \delta'},$$

if we choose $0 < \delta' \leq \dfrac{\epsilon}{M}$, then we arrive

$$t_m - t_n \geq \left(1 - \frac{P_n}{P_m}\right)(-M - \epsilon) \geq \frac{\delta'}{1 + \delta'}(-M - \epsilon) \geq -\epsilon.$$

Therefore, we obtain that (t_n) is also slowly decreasing relative to (P_n). □

Lemma 3.4 ([17, Lemma 6]) *If (s_n) is statistically convergent to μ and slowly decreasing, then (s_n) is convergent to μ.*

Lemma 3.5 ([3, Theorem 4.2.2]) *Let (p_n) satisfy conditions (1) and*

$$\frac{P_n}{P_{n+1}} \to 1 \quad as \quad n \to \infty.$$

If (s_n) is $(\overline{N}, p)$ summable to μ and slowly decreasing relative to (P_n), then (s_n) is convergent to μ.

3.2 Main Results

In this subsection, we prove a Tauberian theorem for real sequences that convergence follows from statistically $(\overline{N}, p)$ summability under the condition of slow decrease relative to (P_n) and additional conditions on (p_n) and we present a corollary related to this theorem. In the sequel, we end this part by giving a Tauberian condition for the $(\overline{N}, p)$ summability method.

Theorem 3.6 *Let (p_n) satisfy conditions (1), $\dfrac{p_n}{P_n} \to 0$ as $n \to \infty$ and*

$$1 \leq \frac{P_m}{P_n} \to 1 \quad whenever \quad 1 < \frac{m}{n} \to 1 \quad as \quad n \to \infty. \tag{11}$$

If (s_n) is statistically $(\overline{N}, p)$ summable to μ and slowly decreasing relative to (P_n), then (s_n) is convergent to μ.

Proof Assume that (p_n) satisfies conditions (1), (11) and $\dfrac{p_n}{P_n} \to 0$ as $n \to \infty$ and that (s_n) is statistically $(\overline{N}, p)$ summable to μ and is slowly decreasing relative to (P_n). In the circumstances, we arrive by the help of Lemma 3.3 that (t_n) is also slowly decreasing relative to (P_n). In other words, we can say by the definition of slow decrease relative to (P_n) that condition

$$\liminf_{m \geq n \to \infty} (t_m - t_n) \geq 0 \quad \text{as} \quad 1 \leq \frac{P_m}{P_n} \to 1$$

holds and so by the condition (11) we obtain that

$$\liminf_{m \geq n \to \infty} (t_m - t_n) \geq 0 \quad \text{as} \quad 1 \leq \frac{m}{n} \to 1.$$

This statement implies the slow decrease of (t_n). Since (t_n) is slowly decreasing and statistically convergent to μ, we reach by the help of Lemma 3.4 that (t_n) is convergent to μ which means that (s_n) is $(\overline{N}, p)$ summable to μ. In addition to this, as (p_n) satisfies condition $\dfrac{p_n}{P_n} \to 0$ as $n \to \infty$, we attain that

$$\frac{P_n}{P_{n+1}} = \frac{P_n + p_{n+1} - p_{n+1}}{P_{n+1}} = \frac{P_{n+1}}{P_{n+1}} - \frac{p_{n+1}}{P_{n+1}} = 1 - \frac{p_{n+1}}{P_{n+1}} \to 1 \quad \text{as} \quad n \to \infty.$$

If we consider that condition of slowly decreasing relative to (P_n) is Tauberian condition for $(\overline{N}, p)$ summable sequence under additional conditions on (p_n) as a result of Lemma 3.5, then we conclude that (s_n) is convergent to μ. □

Corollary 3.7 *Let (p_n) satisfy conditions (1), (11) and $\dfrac{p_n}{P_n} \to 0$ as $n \to \infty$. If (s_n) is statistically $(\overline{N}, p)$ summable to μ and one-sided condition of Landau type relative to (P_n) is satisfied, then (s_n) is convergent to μ.*

Lemma 3.8 *Let (p_n) satisfy conditions (1) and (11). If the one-sided condition*

$$s_{n+1} - s_n \geq -C\frac{p_n}{P_n} \qquad \text{for all } \ n \in \mathbb{N} \tag{12}$$

is satisfied for some constant $C > 0$, then condition

$$t_n - t_{n-1} \geq -C\frac{p_n}{P_n} \qquad \text{for all } \ n \in \mathbb{N} \tag{13}$$

is also satisfied and (t_n) is slowly decreasing relative to (P_n).

Proof Assume that (p_n) satisfies conditions (1) and (11). By taking these hypotheses and condition (12) into consideration, we obtain that

$$
\begin{aligned}
t_n - t_{n-1} &= \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k - \frac{1}{P_{n-1}}\sum_{k=0}^{n-1} p_k s_k \\
&= \frac{1}{P_n P_{n-1}}\left(P_{n-1}\sum_{k=0}^{n} p_k s_k - P_n \sum_{k=0}^{n-1} p_k s_k\right) \\
&= \frac{1}{P_n P_{n-1}}\left(P_{n-1} p_n s_n - p_n \sum_{k=0}^{n-1} p_k s_k\right) \\
&= \frac{p_n}{P_n P_{n-1}}\sum_{k=0}^{n-1} p_k (s_n - s_k) \\
&= \frac{p_n}{P_n P_{n-1}}\sum_{k=0}^{n-1} p_k \sum_{j=k+1}^{n} (s_j - s_{j-1}) \\
&= \frac{p_n}{P_n P_{n-1}}\sum_{j=0}^{n} P_{j-1}(s_j - s_{j-1}) \\
&\geq -C\frac{p_n}{P_n P_{n-1}}\sum_{j=0}^{n} P_{j-1}\frac{p_{j-1}}{P_{j-1}} \\
&= -C\frac{p_n}{P_n}
\end{aligned}
$$

for all $n \in \mathbb{N}$ and some constant $C > 0$. Therefore, we conclude that condition (13) is satisfied and so (t_n) is slowly decreasing relative to (P_n). □

Remark 3.9 We recall that if (s_n) is a $(\overline{N}, p)$ summable sequence satisfying the two-sided condition

$$
|s_{n+1} - s_n| \leq C\frac{p_n}{P_n} \tag{14}
$$

for all $n \in \mathbb{N}$ and some constant $C > 0$ and (p_n) holds condition (1), then (s_n) is convergent. However, if we replace the two-sided condition (14) by the one-sided condition (12), then this statement fails in general without additional condition on (p_n) or (s_n) (cf. [3]).
On the other hand, the two-sided condition (14) can be weakened to the one-sided condition (12) by adding the assumption that (p_n) also satisfies the condition $\frac{p_n}{P_n} \to 0$ as $n \to \infty$ (cf. [25]).

In consideration of Lemma 3.8 and Remark 3.9, we can give the following theorem.

Theorem 3.10 *Let (p_n) satisfy conditions (1), (11) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$. If (s_n) is statistically $(\overline{N}, p)$ summable to μ and condition (12) is satisfied, then (s_n) is convergent to μ.*

Proof Assume that (p_n) satisfies conditions (1), (11) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$ and that statistically $(\overline{N}, p)$ summable (s_n) to μ satisfies condition (12). In the circumstances, we arrive by the help of Lemma 3.8 that (t_n) is also slowly decreasing relative to (P_n). In other words, we can say by the definition of slow decrease relative to (P_n) that condition

$$\liminf_{m \geq n \to \infty} (t_m - t_n) \geq 0 \quad \text{as} \quad 1 \leq \frac{P_m}{P_n} \to 1,$$

holds and so by condition (11) we obtain that

$$\liminf_{m \geq n \to \infty} (t_m - t_n) \geq 0 \quad \text{as} \quad 1 \leq \frac{m}{n} \to 1.$$

Since (t_n) is slowly decreasing and statistically convergent to μ, we reach by the help of Lemma 3.4 that (t_n) is convergent to μ which means that (s_n) is $(\overline{N}, p)$ summable to μ. If we consider that condition (12) is a Tauberian condition for $(\overline{N}, p)$ summable sequence under additional conditions on (p_n), then we conclude that (s_n) is convergent to μ. □

4 Lemmas and Main Results for Complex Sequences

This section essentially consists of two parts as lemmas and main results for complex sequences in parallel with Sect. 3. In the first part, we present some lemmas which will be used in the proofs of our main results for complex sequences. In the second part, we prove some Tauberian theorems for complex sequences that convergence follows from statistically $(\overline{N}, p)$ summability under some Tauberian conditions and additional conditions on (p_n). In the sequel, we complete this part by giving a corollary.

4.1 Lemmas

In this subsection, we express and prove the following assertions which will be benefited in the proofs of our main results for complex sequences. The following lemma was proved for real sequences by Mikhalin [24] and it plays a crucial role in the proofs of subsequent two lemmas which are necessary to achieve our main results for complex sequences.

Lemma 4.1 *Let (p_n) satisfy conditions (1) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$. If (s_n) satisfies the condition*

$$\limsup_{m\geq n\to\infty}|s_m - s_n| \leq r \ (0\leq r<\infty) \ \ as \ \ 1\leq \frac{P_m}{P_n}\to 1, \tag{15}$$

then there exist positive numbers c and d such that $|s_m - s_n| \leq c\log\dfrac{P_m}{P_n} + d$ *for all* $m \geq n \geq 0$.

Proof Assume that (p_n) satisfies conditions (1) and $\dfrac{p_n}{P_n}\to 0$ as $n\to\infty$ and (s_n) satisfies the condition (15). We can say from condition (15) that for every $r+1>0$, there exist $\delta>0$ and $n_0\in\mathbb{N}$ such that $|s_m-s_n|<r+1$ whenever $n>n_0$ and $1\leq\dfrac{P_m}{P_n}\leq 1+\delta$.
Let $n\leq n_0$ and $P_m\leq P_{n_0}(1+\delta)$. In this case, we find that $|s_m-s_n|$ has a maximum depending only on n_0, and so there exist $\delta>0$ and $\ell\geq(r+1)$ such that $|s_m-s_n|<\ell$ for all $m,n\in\mathbb{N}$ related by $0\leq P_m-P_n\leq\delta P_n$. Choose n_1 such that $1\leq\dfrac{P_{n+1}}{P_n}\leq(1+\delta)$ for all $n\geq n_1$. We investigate chosen n_1 in three cases such that $q\geq w\geq n_1$, $0\leq w<n_1\leq q$, and $0\leq w\leq q<n_1$ for arbitrary fixed $q,w\in\mathbb{N}$.

We firstly take into consideration the case $q\geq w\geq n_1$. For this, we define the subsequence (w_{i+1}) where $w_0=w$ and w_{i+1} is the largest natural number for which the inequality $P_n\leq P_{w_i}(1+\delta)$ holds for all $i\in\mathbb{N}$. Therefore, we attain from this defining that the inequalities $P_{w_{i+1}}\leq P_{w_i}(1+\delta)$ and $P_{w_{i+1}+1}>P_{w_i}(1+\delta)$ are valid. In addition to these, let $w_k\leq q-1<w_{k+1}$. Then, we obtain that the inequalities $P_q\leq P_{w_{k+1}}$ and $0\leq P_{w_{i+1}}-P_{w_i}\leq\delta P_{w_i}$ hold for all $k\in\mathbb{N}$, and so we get that

$$\begin{aligned}
\left|s_q-s_w\right| &= \left|s_q+s_{w_k}-s_{w_k}-s_{w_0}\right| \\
&= \left|\sum_{j=0}^{k-1}(s_{w_{j+1}}-s_{w_j})+s_q-s_{w_k}\right| \\
&\leq \sum_{j=0}^{k-1}\left|s_{w_{j+1}}-s_{w_j}\right|+\left|s_q-s_{w_k}\right| \\
&\leq \sum_{j=0}^{k-1}\ell+\left|s_q-s_{w_k}\right| \\
&\leq (k+1)\ell.
\end{aligned}$$

Due to the fact that we have also the inequalities

$$P_q\geq P_{w_k+1}>P_{w_{k-1}}(1+\delta)\geq P_{w_{k-2}+1}(1+\delta)>P_{w_{k-3}}(1+\delta)^2\geq\cdots\geq P_{w_0}(1+\delta)^{\left[\frac{k}{2}\right]}>P_w(1+\delta)^{\frac{k}{2}-1},$$

we eventually reach the inequality $\log P_q\geq\log P_w+\dfrac{k-2}{2}\log(1+\delta)$. This implies that the inequality $(1+k)\leq\dfrac{\log P_q-\log P_w}{\log(1+\delta)^{1/2}}+3$ holds, and hence, we achieve the inequality

$$\left|s_q - s_w\right| \leq (k+1)\ell < \frac{\ell}{\log(1+\delta)^{1/2}}(\log P_q - \log P_w) + 3\ell \tag{16}$$

for any $q \geq w \geq n_1$.

On the other hand, we take into consideration the case $0 \leq w < n_1 \leq q$. Then, we obtain that

$$\begin{aligned}\left|s_q - s_w\right| = \left|s_q - s_{n_1} + s_{n_1} - s_w\right| &\leq \left|s_q - s_{n_1}\right| + \max_{0\leq w<n_1} |s_{n_1} - s_w| \\ &\leq \frac{\ell}{\log(1+\delta)^{1/2}}(\log P_q - \log P_{n_1}) + 3\ell + \max_{0\leq w<n_1} |s_{n_1} - s_w| \\ &\leq \frac{\ell}{\log(1+\delta)^{1/2}}(\log P_q - \log P_w) + 3\ell + \max_{0\leq w<n_1} |s_{n_1} - s_w|. \end{aligned} \tag{17}$$

Finally, if we consider the case $0 \leq w \leq q < n_1$, then we attain that

$$\left|s_q - s_w\right| \leq \max_{0\leq w\leq q\leq n_1-1} \left|s_q - s_w\right| \leq \frac{\ell}{\log(1+\delta)^{1/2}}(\log P_q - \log P_w) + 3\ell + \max_{0\leq w\leq q\leq n_1} \left|s_q - s_w\right|. \tag{18}$$

If we define positive numbers c, d as $c = \frac{\ell}{\log(1+\delta)^{1/2}}$ and $d = \max\{3\ell, 3\ell + \max_{0\leq w\leq q\leq n_1} \left|s_q - s_w\right|\}$, then we conclude by (16)–(18) that

$$\left|s_q - s_w\right| \leq c \log \frac{P_q}{P_w} + d$$

for all $q \geq w \geq 0$. □

Due to the fact that condition (15) corresponds to condition of slow oscillation relative to (P_n) in the case of $r = 0$ in Lemma 4.1, we prove in the following lemma that the below-mentioned sequence is bounded under condition of slow oscillation relative to (P_n) which is restrictive in comparison with condition (15) and some additional condition on (p_n) with the help of Lemma 4.1.

Lemma 4.2 *Let (p_n) satisfy conditions (1) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$. If (s_n) is slowly oscillating relative to (P_n), then*

$$\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n(s_m - s_n)\right)$$

is bounded.

Proof Assume that (p_n) satisfies conditions (1) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$ and (s_n) is slowly oscillating relative to (P_n). Then, by taking these hypotheses into consideration we can say with the help of Lemma 4.1 that there exist positive numbers c and d such that $|s_m - s_n| \leq c \log \frac{P_m}{P_n} + d$ for all $m \geq n \geq 0$. In addition to this, as (p_n) satisfies the condition $\frac{p_n}{P_n} \to 0$ as $n \to \infty$, we attain that

$$\frac{P_n}{P_{n+1}} = \frac{P_n + p_{n+1} - p_{n+1}}{P_{n+1}} = \frac{P_{n+1}}{P_{n+1}} - \frac{p_{n+1}}{P_{n+1}} = 1 - \frac{p_{n+1}}{P_{n+1}} \to 1 \text{ as } n \to \infty. \tag{19}$$

Because it is well known that $t_n \to \ell$ implies $\frac{1}{t_n} \to \frac{1}{\ell}$ whenever $\ell \neq 0$ as $n \to \infty$, we find by (19) that

$$\frac{P_{n+1}}{P_n} \to 1 \text{ as } n \to \infty \tag{20}$$

and so,

$$1 \leq \frac{P_m}{P_n} = \frac{P_m}{P_{m-1}} \frac{P_{m-1}}{P_{m-2}} \cdots \frac{P_{n+1}}{P_n} \to 1 \text{ as } m \geq n \to \infty. \tag{21}$$

This means that for every $\delta > 0$, there exists $n_0 \in \mathbb{N}^0$ such that $1 \leq \frac{P_m}{P_n} \leq 1 + \delta$ whenever $m \geq n \geq n_0$. Therefore, from condition of slow oscillation relative to (P_n) we declare that for every $\epsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}^0$ such that $|s_m - s_n| \leq \epsilon$ whenever $m \geq n \geq n_0$ and $1 \leq \frac{P_m}{P_n} \leq 1 + \delta$. With reference to above inequalities, we obtain that for all $m \geq 0$ and given an $\epsilon > 0$

$$\begin{aligned}
\left| \frac{1}{P_m} \sum_{n=0}^{m} p_n (s_m - s_n) \right| &= \left| \frac{1}{P_m} \sum_{n=0}^{n_0} p_n (s_m - s_n) + \frac{1}{P_m} \sum_{n=n_0+1}^{m} p_n (s_m - s_n) \right| \\
&\leq \frac{1}{P_m} \sum_{n=0}^{n_0} p_n |s_m - s_n| + \frac{1}{P_m} \sum_{n=n_0+1}^{m} p_n |s_m - s_n| \\
&\leq \frac{1}{P_m} \sum_{n=0}^{n_0} p_n \left(c \log \frac{P_m}{P_n} + d \right) + \frac{1}{P_m} \sum_{n=n_0+1}^{m} p_n \epsilon \\
&\leq \frac{1}{P_m} \sum_{n=0}^{n_0} p_n \left(c \log \frac{P_m}{P_0} + d \right) + \frac{1}{P_m} \sum_{n=n_0+1}^{m} p_n \epsilon \\
&= \frac{P_{n_0} - P_0}{P_m} \left(c \log \frac{P_m}{P_0} + d \right) + \frac{P_m - P_{n_0}}{P_m} \epsilon \\
&= \left(P_{n_0} - P_0 \right) \left(\frac{c}{P_m} \log \frac{P_m}{P_0} \right) + \frac{P_{n_0} - P_0}{P_m} d + \left(1 - \frac{P_{n_0}}{P_m} \right) \epsilon \\
&\leq \left(P_{n_0} - P_0 \right) \left(\frac{c}{P_0} \right) + \frac{P_{n_0} - P_0}{P_0} d + \left(1 - \frac{P_{n_0}}{P_m} \right) \epsilon
\end{aligned}$$

$$= \left(\frac{P_{n_0}}{P_0} - 1\right)(c+d) + \left(1 - \frac{P_{n_0}}{P_m}\right)\epsilon.$$

In conjunction with the information obtained up to now if we consider that $\left(\frac{P_{n_0}}{P_m}\right)$ is convergent to 0 by condition (1) and every convergent sequence is also bounded, then there exists a positive constant H such that

$$\begin{aligned} \left|\frac{1}{P_m}\sum_{n=0}^{m} p_n(s_m - s_n)\right| &\le \left(\frac{P_{n_0}}{P_0} - 1\right)(c+d) + \left(1 - \frac{P_{n_0}}{P_m}\right)\epsilon \\ &\le \left(\frac{P_{n_0}}{P_0} - 1\right)(c+d) + H \\ &:= M \end{aligned}$$

for all $m \ge 0$ and some constant $M > 0$. In conclusion, we reach that $\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n(s_m - s_n)\right)$ is bounded. □

An alternative proof of the following lemma can also be done by following the procedure used in [24].

Lemma 4.3 *Let* (p_n) *satisfy conditions (1) and* $\frac{p_n}{P_n} \to 0$ *as* $n \to \infty$. *If* (s_n) *is slowly oscillating relative to* (P_n), *then* (t_n) *is also slowly oscillating relative to* (P_n).

Proof Assume that (p_n) satisfies conditions (1) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$ and (s_n) is slowly oscillating relative to (P_n). Given $\epsilon > 0$. By the definition of slow oscillation relative to (P_n), this means that there exist $\delta > 0$ and $n_0 \in \mathbb{N}^0$ such that $|s_m - s_n| \le \epsilon$ whenever $m \ge n \ge n_0$ and $1 \le \frac{P_m}{P_n} \le 1 + \delta$.

Let $m \ge n \ge n_0$ and $1 \le \frac{P_m}{P_n} \le 1 + \delta'$. By the definition of the weighted means of (s_n) and Lemma 4.2, we obtain that

$$\begin{aligned} |t_m - t_n| &= \left|\frac{1}{P_m}\sum_{k=0}^{m} p_k s_k - \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k\right| \\ &= \left|\frac{1}{P_m}\left\{\sum_{k=0}^{n} + \sum_{k=n+1}^{m}\right\} p_k s_k - \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k\right| \\ &= \left|\frac{1}{P_m}\sum_{k=0}^{n} p_k s_k + \frac{1}{P_m}\sum_{k=n+1}^{m} p_k s_k + \frac{P_m - P_n}{P_m P_n}\sum_{k=0}^{n} p_k s_n - \frac{P_m - P_n}{P_m P_n}\sum_{k=0}^{n} p_k s_n - \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k\right| \\ &= \left|\frac{1}{P_m}\sum_{k=n+1}^{m} p_k(s_k - s_n) + \left(\frac{1}{P_m} - \frac{1}{P_n}\right)\sum_{k=0}^{n} p_k s_k + \frac{P_m - P_n}{P_m P_n}\sum_{k=0}^{n} p_k s_n\right| \\ &= \left|\frac{P_m - P_n}{P_m}\frac{1}{P_n}\sum_{k=0}^{n} p_k(s_n - s_k) + \frac{1}{P_m}\sum_{k=n+1}^{m} p_k(s_k - s_n)\right| \end{aligned}$$

$$\leq \frac{P_m - P_n}{P_m} \left| \frac{1}{P_n} \sum_{k=0}^{n} p_k (s_n - s_k) \right| + \frac{1}{P_m} \sum_{k=n+1}^{m} p_k |s_k - s_n|$$

$$\leq \left(1 - \frac{P_n}{P_m}\right) M + \frac{1}{P_m} \sum_{k=n+1}^{m} p_k \epsilon$$

$$= \left(1 - \frac{P_n}{P_m}\right) (M + \epsilon)$$

whenever $m \geq k > n \geq n_0$, $1 < \frac{P_k}{P_n} \leq \frac{P_m}{P_n} \leq 1 + \delta'$ and for some constant $M > 0$. Since we have that for $m \geq n \geq n_0$ and $1 \leq \frac{P_m}{P_n} \leq 1 + \delta'$

$$0 \leq \left(1 - \frac{P_n}{P_m}\right) \leq \frac{\delta'}{1 + \delta'},$$

if we choose $0 < \delta' \leq \frac{\epsilon}{M}$, then we arrive

$$|t_m - t_n| \leq \left(1 - \frac{P_n}{P_m}\right) (M + \epsilon) \leq \frac{\delta'}{1 + \delta'} (M + \epsilon) \leq \epsilon.$$

Therefore, we reach that (t_n) is also slowly oscillating relative to (P_n). □

Lemma 4.4 ([17, Lemma 7]) *If* (s_n) *is statistically convergent to* μ *and slowly oscillating, then* (s_n) *is convergent to* μ.

Lemma 4.5 ([3, Corollary of Theorem 4.2.2]) *Let* (p_n) *satisfy conditions (1) and*

$$\frac{P_n}{P_{n+1}} \to 1 \quad as \quad n \to \infty.$$

If (s_n) *is* $(\overline{N}, p)$ *summable to* μ *and slowly oscillating relative to* (P_n)*, then* (s_n) *is convergent to* μ.

4.2 Main Results

In this subsection, we prove a Tauberian theorem for complex sequences that convergence follows from statistically $(\overline{N}, p)$ summability under condition of slow oscillation relative to (P_n) and additional conditions on (p_n) and we present a corollary related to this theorem. In the sequel, we complete this part by giving a Tauberian condition for the $(\overline{N}, p)$ summability method.

Theorem 4.6 *Let (p_n) satisfy conditions (1), (11) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$. If (s_n) is statistically $(\overline{N}, p)$ summable to μ and slowly oscillating relative to (P_n), then (s_n) is convergent to μ.*

Proof Assume that (p_n) satisfies conditions (1), (11) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$ and (s_n) which is statistically $(\overline{N}, p)$ summable to μ is slowly oscillating relative to (P_n). In the circumstances, we arrive with the help of Lemma 4.3 that (t_n) is also slowly oscillating relative to (P_n). In other words, we can say by the definition of slow oscillation relative to (P_n) that condition

$$\limsup_{m \geq n \to \infty} |t_m - t_n| = 0 \quad \text{as} \quad 1 \leq \frac{P_m}{P_n} \to 1$$

holds and so by condition (11) we obtain that

$$\limsup_{m \geq n \to \infty} |t_m - t_n| = 0 \quad \text{as} \quad 1 \leq \frac{m}{n} \to 1.$$

The statement implies the slow oscillation of (t_n). Since (t_n) is slowly oscillating and statistically convergent to μ, we reach with the help of Lemma 4.4 that (t_n) is convergent to μ which means that (s_n) is $(\overline{N}, p)$ summable to μ. In addition to this, as (p_n) satisfies condition $\frac{p_n}{P_n} \to 0$ as $n \to \infty$, we attain that

$$\frac{P_n}{P_{n+1}} = \frac{P_n + p_{n+1} - p_{n+1}}{P_{n+1}} = \frac{P_{n+1}}{P_{n+1}} - \frac{p_{n+1}}{P_{n+1}} = 1 - \frac{p_{n+1}}{P_{n+1}} \to 1 \quad \text{as} \quad n \to \infty.$$

If we consider that condition of slowly oscillating relative to (P_n) is a Tauberian condition for $(\overline{N}, p)$ summable sequence under additional conditions on (p_n) as a result of Lemma 4.5, then we conclude that (s_n) is convergent to μ. □

Corollary 4.7 *Let (p_n) satisfy conditions (1), (11) and $\frac{p_n}{P_n} \to 0$ as $n \to \infty$. If (s_n) is statistically $(\overline{N}, p)$ summable to μ and two-sided condition of Hardy type relative to (P_n), then (s_n) is convergent to μ.*

Lemma 4.8 *Let (p_n) satisfy conditions (1) and (11). If condition*

$$|s_{n+1} - s_n| \leq C \frac{p_n}{P_n} \quad \textit{for all} \ \ n \in \mathbb{N} \tag{22}$$

is satisfied for some constant $C > 0$, then condition

$$|t_n - t_{n-1}| \leq C \frac{p_n}{P_n} \quad \textit{for all} \ \ n \in \mathbb{N} \tag{23}$$

is also satisfied and (t_n) is slowly oscillating relative to (P_n).

Proof Assume that (p_n) satisfies conditions (1) and (11). By taking these hypotheses and condition (22) into consideration, we obtain that

$$
\begin{aligned}
|t_n - t_{n-1}| &= \left| \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k - \frac{1}{P_{n-1}} \sum_{k=0}^{n-1} p_k s_k \right| \\
&= \frac{1}{P_n P_{n-1}} \left| P_{n-1} \sum_{k=0}^{n} p_k s_k - P_n \sum_{k=0}^{n-1} p_k s_k \right| \\
&= \frac{1}{P_n P_{n-1}} \left| P_{n-1} p_n s_n - p_n \sum_{k=0}^{n-1} p_k s_k \right| \\
&= \frac{1}{P_n P_{n-1}} \left| p_n \sum_{k=0}^{n-1} p_k (s_n - s_k) \right| \\
&= \frac{p_n}{P_n P_{n-1}} \left| \sum_{k=0}^{n-1} p_k \sum_{j=k+1}^{n} (s_j - s_{j-1}) \right| \\
&\leq \frac{p_n}{P_n P_{n-1}} \sum_{k=0}^{n-1} p_k \left| \sum_{j=k+1}^{n} (s_j - s_{j-1}) \right| \\
&\leq \frac{p_n}{P_n P_{n-1}} \sum_{k=0}^{n-1} p_k \sum_{j=k+1}^{n} |s_j - s_{j-1}| \\
&= \frac{p_n}{P_n P_{n-1}} \sum_{j=0}^{n} P_{j-1} |s_j - s_{j-1}| \\
&\leq C \frac{p_n}{P_n P_{n-1}} \sum_{j=0}^{n} P_{j-1} \frac{p_{j-1}}{P_{j-1}} \\
&= C \frac{p_n}{P_n}
\end{aligned}
$$

for all $n \in \mathbb{N}$ and some constant $C > 0$. Therefore, we conclude that condition (23) is satisfied and so (t_n) is slowly oscillating relative to (P_n). □

Remark 4.9 It is known that provided the sequence (p_n) satisfies condition (1), any $(\overline{N}, p)$ summable sequence (s_n) which satisfies two-sided condition (22) is convergent (cf. [3]).

In consideration of Lemma 4.8 and Remark 4.9, we can give the following theorem.

Theorem 4.10 *Let (p_n) satisfy conditions (1) and (11). If (s_n) is statistically $(\overline{N}, p)$ summable to μ and condition (22) is satisfied, then (s_n) is convergent to μ.*

Proof Assume that (p_n) satisfies conditions (1), (11) and statistically $(\overline{N}, p)$ summable (s_n) to μ satisfies condition (22). In the circumstances, we arrive with the help of Lemma 4.8 that (t_n) is also slowly oscillating relative to (P_n). In other words, we can say by the definition of slow oscillation relative to (P_n) that condition

$$\limsup_{m \geq n \to \infty} |t_m - t_n| = 0 \quad \text{as} \quad 1 \leq \frac{P_m}{P_n} \to 1,$$

holds and so by condition (11) we obtain that

$$\limsup_{m \geq n \to \infty} |t_m - t_n| = 0 \quad \text{as} \quad 1 \leq \frac{m}{n} \to 1.$$

The statement implies the slow oscillation of (t_n). Since (t_n) is slowly oscillating and statistically convergent to μ, we reach with the help of Lemma 4.4 that (t_n) is convergent to μ which means that (s_n) is $(\overline{N}, p)$ summable to μ. If we consider that condition (22) is a Tauberian condition for $(\overline{N}, p)$ summable sequence under additional conditions on (p_n), then we conclude that (s_n) is convergent to μ. □

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Applications of Fixed Point Theorems and General Convergence in Orthogonal Metric Spaces

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Abstract In this chapter, we discuss the general convergence methods in orthogonal metric space. Also we study the applications of fixed point theorems to obtain the existence of a solution of differential and integral equations in orthogonal metric spaces.

1 Introduction

The concept of orthogonality in normed linear spaces has been studied by Birkhoff [1–6] among others. The most natural notion of orthogonality arises in the case where there is an inner product $< ., . >$ compatible with the norm $||.||$ on a space X. In this case, $\perp$ is defined by $x \perp y$ if and only if $< x, y >= 0$. Some of the major properties of this relation are as follows:

(1) $x \perp x$ if and only if $x = 0$ for all $x \in X$,
(2) $x \perp y$ implies $\alpha x \perp y$ for all $x, y \in X, \alpha \in \mathbb{R}$ (Homogeneity),
(3) $x \perp y$ implies $y \perp x$ for all $x, y \in X$ (Symmetry),
(4) $x \perp y$ and $x \perp z$ implies $x \perp (y + z)$ for all $x, y, z \in X$ (Additivity),
(5) For every $x, y \in X, x \neq 0$, there exists a real number γ such that $x \perp (\gamma x + y)$.

For general normed linear spaces $(X, ||.||)$, Birkhoff [1] and James [5, 6] formulated definitions of orthogonality which did not require the existence of an inner product as follows:

(B) Birkhoff Orthogonality [1, 5, 6]: $(x \perp y)(B)$ provided $||x|| \leq ||x + \lambda y||$ for all $x, y \in X, \lambda \in \mathbb{R}$,
(P) Pythagorean Orthogonality [4]: $(x \perp y)(P)$ provided $||x - y||^2 = ||x||^2 + ||y||^2$ for all $x, y \in X$,

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(I) Isosceles Orthogonality [4]: $(x \perp y)(I)$ provided $||x - y|| = ||x + y||$ for all $x, y \in X$,

(α) α-Orthogonality [7]: If $\alpha \neq 1$, $(x \perp y)(\alpha)$ provided $(1 + \alpha^2)||x - y||^2 = ||x - \alpha y||^2 + ||\alpha x - y||^2$ for all $x, y \in X$,

((α, β)) (α, β)-Orthogonality [8]: If $\alpha, \beta \neq 1$, $(x \perp y)(\alpha, \beta)$ provided $||x - y||^2 + ||\alpha x - \beta y||^2 = ||x - \beta y||^2 + ||y - \alpha x||^2$ for all $x, y \in X$.

It should be noted that Pythagorean and isosceles orthogonalities are particular cases of α-orthogonality, which is in turn a special case of (α, β)-orthogonality. An ordered triples of the form $(X, ||.||, \perp)$, where X is a real linear space, $||.||$ is a norm on X, and $\perp$ is an orthogonality relation on X which has the properties (1)–(5), is an inner product space if there is an inner product $< ., . >$ on X such that $< x, x >= ||x||^2$ for all $x \in X$ and $x \perp y$ if and only if $< x, y >= 0$ for all $x, y \in X$.

Proposition 1.1 ([3, Theorem I]) *If any of* $(x \perp y)(P)$, $(x \perp y)(I)$, $(x \perp y)(\alpha)$ *or* $(x \perp y(\alpha, \beta))$ *implies that* $x \perp y$, *then* $(X, ||.||, \perp)$ *is an inner product space. If* $dimX \geq 3$ *and* $(x \perp y)(B)$ *implies that* $x \perp y$, *then* $(X, ||.||, \perp)$ *is an inner product space.*

Definition 1.2 ([3]) The relation $\perp$ is said to satisfy the *norm invariance property (NIP)* provided the conditions $x \perp y$, $||x|| = ||z||$, $||y|| = ||w||$, and $||x - y|| = ||z - w||$ imply $z \perp w$.

Definition 1.3 ([3]) The relation $\perp$ is said to satisfy the *rotation invariance property (RIP)* provided the following conditions hold

(R1) If $x \perp y$, $||x|| = ||y||$ then $(ax - by) \perp (ax + by)$ for all $a, b \in \mathbb{R}$,

(R2) If $||x|| = ||y||$ then $||\gamma(x, y)x + y|| = ||\gamma(y, x)y + x||$, where $\gamma(x, y)$ and $\gamma(y, x)$ are respective real numbers from (5) such that $x \perp (\gamma(x, y)x + y)$ and $y \perp (\gamma(y, x)y + x)$.

Lemma 1.4 ([3, Theorem III]) $(X, ||.||, \perp)$ *is an inner product space if and only if* $\perp$ *satisfies NIP or RIP.*

Lemma 1.5 ([3, Lemma 1.1]) *If* $x, y \in X$, $x, y \neq 0$, *and* $x \perp y$, *then* x *and* y *are independent.*

We define a function $\gamma : X \times X \to R$ by

$$\gamma(x, y) = \begin{cases} 0, \text{ if } x = 0; \\ \text{the unique } \gamma \text{ such that } x \perp (\gamma x + y), \text{ if } x \neq 0. \end{cases}$$

From definition, we see that $\gamma(x, y) = 0$ if and only if $x \perp y$. Also $\gamma(x, y)$ have the following properties:

Lemma 1.6 ([3, Lemma 1.3])

(i) $\gamma(x, 0) = \gamma(0, x) = 0$ *for all* $x \in X$,
(ii) $\gamma(x, \lambda y) = \lambda\gamma(x, y)$ *for all* $x, y \in X, \lambda \in \mathbb{R}$,
(iii) $\gamma(\lambda x, y) = \frac{1}{\lambda}\gamma(x, y)$ *for all* $x, y \in X, \lambda \in \mathbb{R}, \lambda = neq0$,
(iv) $\gamma(x, y) = 0$ *if and only if* $\gamma(y, x) = 0$,
(v) $\gamma(x, y + z) = \gamma(x, y) + \gamma(x, z)$ *for all* $x, y, z \in X$.

2 Orthogonal Set

The notion of orthogonal set and orthogonal metric space was introduced by Gordji et al. [9]. They gave an extension of Banach's fixed point theorem in this new structure and applied their results to prove the existence of a solution of an ordinary differential equation. Applications of fixed point theorem in orthogonal metric spaces we refer [10–12].

Definition 2.1 ([9]) Let X be a non-empty set and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ satisfies the following condition

$$\exists x_0 : \ (\forall y, y \perp x_0) \text{ or } (\forall y, x_0 \perp y)$$

then it is called an orthogonal set (written as *O-set*). We denote the orthogonal set by $(X, \perp)$. The element x_0 is called an orthogonal element of X.

Definition 2.2 Let $(X, \perp)$ be an orthogonal set. Any two elements $x, y \in X$ are said to be orthogonally related if $x \perp y$.

Example 2.3 Let $X = \mathbb{Z}$. Define $x \perp y$ if there exists $p \in \mathbb{Z}$ such that $x = py$. We see that $0 \perp x$ for all $x \in \mathbb{Z}$. Hence $(X, \perp)$ is an *O-set*.

Example 2.4 Let X be a non-empty set. Consider $P(X)$ is the power sets of X. We define $\perp$ on $P(X)$ as $A \perp B$ if $A \cap B = \phi$. We have $\phi \cap A = \phi$ for all $A \in P(X)$. Then $(P(X), \perp)$ is an orthogonal set. Similarly we can define $\perp$ on $P(X)$ as $A \perp B$ if $A \cup B = X$. Then $(P(X), \perp)$ is also an orthogonal set.

Example 2.5 Let $X = [3, \infty)$ and define $x \perp y$ if $x \leq y$. Taking $x_0 = 3$, $(X, \perp)$ is an orthogonal set.

Example 2.6 Let $X = [0, \infty)$ and define $x \perp y$ if $xy \in \{x, y\}$. By taking $x_0 = 0$ or $x_0 = 1$ $(X, \perp)$ is an orthogonal set.

Example 2.7 Let (X, d) be a metric space and $T : X \to X$ be a Picard operator; i.e., there exists $z \in X$ such that $\lim\limits_{n\to\infty} T^n(y) = z$ for all $y \in X$. Define $x \leq y$ if $\lim\limits_{n\to\infty} d(z, T^n(y)) = 0$. The $(X, \perp)$ is an orthogonal set.

The following example shows that the orthogonal element x_0 is not unique.

Example 2.8 Suppose M_m is the set of all $m \times m$ matrices and Y is a positive definite matrix. Define the relation $\perp$ on M_m by $A \perp B \Leftrightarrow \exists X \in M_m$ such that $AX = B$. It is easy to see that $I \perp B$, $B \perp O$ and $\sqrt{Y} \perp B$ for all $B \in M_m$, where I and O are the identity and zero matrices in M_m, respectively. Then this orthogonal relation is reflexive and transitive, but it is antisymmetry.

Example 2.9 For any $D \in M_m$, consider the orthogonal relation $\perp_D$ on M_m with respect to D defined by

$$A \perp_D B \Leftrightarrow tr(ABD) = tr(DBA).$$

We have $D \perp_D X$ for all $X \in M_m$. Then this orthogonal relation is reflexive, transitive, and symmetry.

Example 2.10 If $0 < \alpha \le 1$, let $\Lambda_\alpha([0, 1])$ be the space of Hölder's continuous functions of the exponent α in $[0, 1]$, i.e., $f \in \Lambda_\alpha([0, 1])$ if and only if $||f||_{\Lambda_\alpha} < \infty$, where

$$||f||_{\Lambda_\alpha} = |f(0)| + \sup_{x,y\in[0,1],x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

For all $0 < \alpha \le 1$, define $\lambda_\alpha([0, 1])$ to be the set of $f \in \Lambda_\alpha([0, 1])$ such that

$$\lim_{x\to y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} = 0 \text{ for all } x, y \in [0, 1].$$

For all $\alpha, \beta \in [0, 1]$, we define $\lambda_\alpha([0, 1]) \perp \lambda_\beta([0, 1])$ if and only if $\lambda_{\frac{\alpha-\beta}{2}}([0, 1])$ is an infinite dimensional closed subspace of $\Lambda_{\frac{\alpha-\beta}{2}}([0, 1])$. Hence $(\lambda_\alpha([0, 1]), \perp)$ is an orthogonal set.

Definition 2.11 ([9]) Let $(X, \perp)$ be an orthogonal set. A sequence (x_n) in X is called an orthogonal sequence (O-sequence) if

$$x_n \perp x_{n+1} \; or \; x_{n+1} \perp x_n, \forall n.$$

Definition 2.12 A mapping $d : X \times X \to [0, \infty)$ is called a metric on the orthogonal set $(X, \perp)$, if the following conditions are satisfied:

(O1) $d(x, y) = d(y, x)$ for any $x, y \in X$ such that $x \perp y$ and $y \perp x$,
(O2) $d(x, y) = 0$ if and only if $x = y$ for any $x, y \in X$ such that $x \perp y$ and $y \perp x$,
(O3) $d(x, z) \le d(x, y) + d(y, z)$ for any $x, y, z \in X$ such that $x \perp y$, $y \perp z$ and $x \perp z$.

Then the ordered triple $(X, \perp, d)$ is called an orthogonal metric space.

Example 2.13 Let $X = \mathbb{Q}$. The orthogonal relation on X is defined $x \perp y$ if and only if $x = 0$ or $y = 0$. Then $(X, \perp)$ is an orthogonal set, and with the Euclidean metric, $(X, \perp, d)$ is an orthogonal metric space.

Let X be an orthogonal set and $d : X \times X \to [0, \infty)$ be a mapping. For every $x \in X$ we define the set

$$\mathcal{O}(X, d, x) = \left\{ (x_n) \subset X : \lim_{n\to\infty} d(x_n, x) = 0 \text{ and } x_n \perp x, \forall n \in \mathbb{N} \right\}. \tag{1}$$

Definition 2.14 Let $(X, \perp, d)$ be an orthogonal metric space. A sequence (x_n) in X is said to be

(i) an orthogonal convergent (in short O-convergent) to x if and only if $(x_n) \in \mathcal{O}(X, d, x)$,
(ii) an orthogonal Cauchy (in short O-Cauchy) if and only if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$ and $x_n \perp x_m$ or $x_m \perp x_n$, $\forall n, m \in \mathbb{N}$.

Remark 2.15 In an orthogonal metric space $(X, \perp, d)$, an orthogonal convergent sequence may not be an orthogonal Cauchy.

Definition 2.16 ([9]) An orthogonal metric space $(X, \perp, d)$ is said to be an orthogonal complete (O-complete) if every orthogonal Cauchy sequence converges in X.

Remark 2.17 It is easy to see that every complete metric space is orthogonal complete but the converse is not true. For this remark, see the following examples.

Example 2.18 Let $X = \mathbb{Q}$. Define $x \perp y$ if and only if $x = 0$ or $y = 0$. Then $(X, \perp)$ is an orthogonal set. It is clear that $\mathbb{Q}$ is not a complete metric space with respect to the Euclidean metric, but it is orthogonal complete. If (x_n) is any orthogonal Cauchy sequence in $\mathbb{Q}$, then there exists a subsequence (x_{n_k}) of (x_n) for which $x_{n_k} = 0$ for all $k \geq 1$. Then (x_{n_k}) converges to $0 \in X$. We know that every Cauchy sequence with a convergent subsequence is convergent, so (x_n) is convergent.

Example 2.19 Let $X = [0, 1)$ and define the orthogonal relation on X by

$$x \perp y \Leftrightarrow \begin{cases} x \leq y \leq \frac{1}{4}, \\ \text{or } x = 0. \end{cases}$$

Then $(X, \perp)$ is an orthogonal set. We have X is not a complete metric space with respect to Euclidean metric but it is orthogonal complete. Consider (x_n) is an orthogonal Cauchy sequence in X. Then there exists a subsequence (x_{n_k}) of (x_n) for which $x_{n_k} = 0$ for all $k \geq 1$, or there exists a monotone subsequence (x_{n_k}) if (x_n) for which $x_{n_k} \leq \frac{1}{4}$ for all $k \geq 1$. We see that (x_{n_k}) converges to a point $x \in [0, \frac{1}{4}] \subseteq X$. We know that every Cauchy sequence with a convergent subsequence is convergent, so (x_n) is convergent in X.

Definition 2.20 ([9]) Let $(X, \perp, d)$ be an orthogonal metric space. A function $f : X \to X$ is said to be an orthogonal continuous (O-continuous or $\perp$-continuous) at a point x_0 in X if for each orthogonal sequence (x_n) in X converging to x_0 such that $f(x_n) \to f(x_0)$. Also f is said to be orthogonal continuous on X if f is orthogonal continuous at each point on X.

Remark 2.21 It is easy to see that every continuous mapping is orthogonal continuous. The following examples show that the converse is not true in general.

Example 2.22 Let $X = \mathbb{R}$. Define the orthogonality relation on X by $x \perp y$ if and only if $x = 0$ or $y \neq 0$ in $\mathbb{Q}$. Then $(X, \perp)$ is an orthogonal set. Define a function $f : X \to X$ by

$$f(x) = \begin{cases} 2, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{Q}^c. \end{cases}$$

Then f is an orthogonal continuous but f is not continuous on $\mathbb{Q}$.

Example 2.23 Let $X = \mathbb{R}$. Define $x \perp y$ if $x, y \in \left(q + \frac{1}{7}, q + \frac{2}{7}\right)$ for some $q \in \mathbb{Z}$ or $x = 0$. Then $(X, \perp)$ is an orthogonal set. Define a function $f : X \to X$ by $f(x) = [x]$. Then f is an orthogonal continuous on X. Because for an orthogonal sequence (x_n) in X converging to $x \in X$, then we have

Case-I: If $x_n = 0$ for all n, then $x = 0$ and $f(x_n) = 0 = f(x)$.

Case-II: If $x_{n_0} \neq 0$ for some n_0, then there exists $k \in \mathbb{Z}$ such that $x_n \in \left(k + \frac{1}{7}, k + \frac{2}{7}\right)$ for all $n \geq n_0$. Then $x \in \left[q + \frac{1}{7}, q + \frac{2}{7}\right]$ and $f(x_n) = k = f(x)$. It follows that f is orthogonal continuous on X but it is not continuous on X.

If $X = \mathbb{R}^n$ be a standard inner product space, then the Remark 2.21 is false. It follows from the following theorem.

Lemma 2.24 ([13]) *Let $X = \mathbb{R}^n$ be a standard inner product space and $T : X \to X$ be a mapping, where $T(x) = (T_i(x), T_2(x), \ldots, T_n(x))$ for all $x \in X$, and each T_i is a mapping from $\mathbb{R}^n$ to $\mathbb{R}$ for all $i = 1, 2, \ldots, n$. Then T is continuous at $y = (y_1, y_2, \ldots, y_n)$ if and only if T_i is continuous at y for each $i = 1, 2, \ldots, n$.*

Theorem 2.25 ([14, Theorem 2.1]) *Let $(X, \perp, <.,.>)$ be an orthogonal inner product space, where $X = \mathbb{R}^n$, $<.,.>$ denotes the standard inner product space and $\perp$ is an orthogonal relation on X defined by $x \perp y$ if $<x, y> = 0$ for all $x, y \in X$. Then $f : X \to X$ is orthogonal continuous on X if and only if f is continuous on X.*

Proof Given that $(X, \perp, <.,.>)$ is an orthogonal inner product space, where $X = \mathbb{R}^n$. The orthogonality relation $\perp$ on X is defined by $x \perp y$ if $<x, y> = 0$. Suppose (x_k) be a Cauchy orthogonal sequence converging to x, where $x_k = (x_1^k, x_2^k, \ldots, x_n^k)$ and $x = (x_1, x_2, \ldots, x_n)$. Suppose that $f : X \to X$ is an orthogonal continuous function at x. To show that f is continuous at x.

For any $x, y \in X$, the distance function $d(x, y)$ induced by the inner product is given by

$$d(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{\frac{1}{2}}.$$

Since f is orthogonal continuous at x then for any orthogonal sequence (x_k) converging to x, we have

$$\lim_{k\to\infty} d(f(x_k), f(x)) = 0$$
$$\Rightarrow \lim_{k\to\infty} \left[\sum_{i=1}^{n} (f_i(x_k) - f_i(x))^2\right]^{\frac{1}{2}} = 0$$
$$\Rightarrow \lim_{k\to\infty} (f_i(x_k) - f_i(x))^2 = 0, \text{ for each } i$$
$$\Rightarrow f_i(x_k) \to f_i(x), \text{ for each } i, \text{ as } k \to \infty$$
$$\Rightarrow f_i \text{ is continuous at } x, \text{ for each } i = 1, 2, \ldots n$$
$$\Rightarrow f \text{ is continuous at } x, \text{ (Lemma 2.24)}.$$

Since x is arbitrary, so f is continuous on X.

Conversely, if f is continuous on X, it is easy to show that f is orthogonal continuous on X.

3 Orthogonal Contractions

Definition 3.1 ([9]) Let $(X, \perp, d)$ be an orthogonal metric space and $0 < K < 1$. A mapping $T : X \to X$ is called an orthogonal contraction (O-contraction or $\perp$-contraction) with Lipschitz constant K, if for all $x, y \in X$ with $x \perp y$ then $d(Tx, Ty) \leq Kd(x, y)$

Remark 3.2 It is clear that every contraction is orthogonal contraction but the converse is not true.

Example 3.3 Let $X = [0, 10)$ and d be the Euclidean metric on X. Define $x \perp y$ if $xy \leq x \text{ } or \text{ } y$. Let $F : X \to X$ be a map defined by

$$F(x) = \begin{cases} \frac{x}{4}, & \text{if } x \leq 4, \\ 0, & \text{if } x > 4. \end{cases}$$

Let $x \perp y$ and $xy \leq x$ then we have

Case:1 If $x = 0$ and $y \leq 4$ then $F(x) = 0$ and $F(y) = \frac{y}{4}$.

Case:2 If $x = 0$ and $y > 4$ then $F(x) = F(y) = 0$.

Case:3 If $y \leq 3$ and $x \leq 4$ then $F(y) = \frac{y}{4}$ and $F(x) = \frac{x}{4}$.

Case:4 If $y \leq 1$ and $x > 4$ then $x - y > y$, $F(y) = \frac{y}{4}$, ad $F(x) = 0$.

Therefore we have $|F(x) - F(y)| \leq \frac{1}{4}|x - y|$, and hence, F is an orthogonal contraction. But F is not a contraction, because for each $K < 1$ then $|F(5) - F(4)| = 1 > K = K|5 - 4|$.

Example 3.4 Let $X = [0, 1)$ and d be the Euclidean metric on X. Define $x \perp y$ if $xy \in \{x, y\}$ for all $x, y \in X$. Let $F : X \to X$ be a mapping defined by

$$F(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in \mathbb{Q} \cap X, \\ 0, & \text{if } x \in \mathbb{Q}^c \cap X. \end{cases}$$

Then F is an orthogonal contraction on X but it is not a contraction.

Definition 3.5 ([9]) Let $(X, \perp, d)$ be an orthogonal metric space. A mapping $T : X \to X$ is said to be an orthogonal preserving (or $\perp$-preserving or O-preserving) if $x \perp y$ implies $Tx \perp Ty$ for all $x, y \in X$.

Definition 3.6 ([9]) Let $(X, \perp, d)$ be an orthogonal metric space. A mapping $T : X \to X$ is said to be a weakly orthogonal preserving (or weakly $\perp$-preserving or weakly O-preserving) if $x \perp y$ implies $Tx \perp Ty$ or $Ty \perp Tx$ for all $x, y \in X$.

Example 3.7 ([9]) Let X be the set of all peoples in the world. We define $x \perp y$ if x can give blood to y. According to the following table, if x_0 is a person such that his/her blood type is O^-, then we have $x_0 \perp y$ for all $y \in X$. Then $(X, \perp)$ is an orthogonal set. In the following, we see that in this orthogonal set x_0 is not unique.

Type	*You can give blood to*	*You can receive blood from*
A^+	A^+, AB^+	A^+, A^-, O^+, O^-
O^+	O^+, A^+, B^+, AB^+	O^+, O^-
B^+	B^+, AB^+	$B^+, B^-, O^+.O^-$
AB^+	AB^+	*Everyone*
A^-	$A^+.A^-, AB^+, AB^-$	A^-, O^-
O^-	*Everyone*	O^-
B^-	B^+, B^-, AB^+, AB^-	B^-, O^-
AB^-	AB^+, AB^-	AB^-, B^-, O^-, A^-

Remark 3.8 We have every orthogonal preserving mapping is weakly preserving, but the converse is not true.

For this let $(X, \perp)$ be an orthogonal set defined in the Example 3.7. Let O_1 in X be a person with blood type O^-; P_1 be a person with blood type A^+. Define a mapping $F : X \to X$ by

$$F(x) = \begin{cases} P_1, \text{ if } x = O_1, \\ O_1, \text{ if } x \in X - \{O_1\}. \end{cases}$$

Let $O_2 \in X - \{O_1\}$ be a person with blood type O^-. Then we get $O_1 \perp O_2$ but we do not have $F(O_1) \perp F(O_2)$. Therefore F is not an orthogonal preserving but it is weakly orthogonal preserving.

Theorem 3.9 ([9, Theorem 3.11]) *Let $(X, \perp, d)$ be an orthogonal complete metric space (not necessarily complete metric space) and $0 < k < 1$. Let $T : X \to X$ be an orthogonal continuous, orthogonal contraction with Lipschitz constant k, and orthogonal preserving. Then T has a unique fixed point $\bar{x} \in X$. Also T is a Picard operator, i.e., $\lim_{n\to\infty} T^n(x) = \bar{x}$ for all $x \in X$.*

Proof Given that $(X, \perp, d)$ is an orthogonal metric space. Therefore by the definition of orthogonality, there exists an element $x_0 \in X$ such that $x_0 \perp y$ or $y \perp x_0$ for all $y \in X$.

It follows that $x_0 \perp Tx_0$ or $Tx_0 \perp x_0$. Let

$$x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \ldots, x_{n+1} = Tx_n = T^{n+1}x_0, \forall n \in \mathbb{N}.$$

Since T is orthogonal preserving, (x_n) is an orthogonal sequence in X. Also since T is an orthogonal contraction, so we have

$$d(x_n, x_m) \leq k^n d(x_0, x_1), \forall n \in \mathbb{N}.$$

If $m, n \in \mathbb{N}$ and $m \geq n$ we get

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + \cdots + k^{m-1} d(x_0, x_1) \\ &\leq \frac{k^n}{1-k} d(x_0, x_1). \end{aligned}$$

Since $0 < k < 1$ and $d(x_0, x_1)$ is fixed, $d(x_n, x_m) \to 0$ as $m, n \to \infty$. Therefore (x_n) is an orthogonal Cauchy sequence in X. Since X is orthogonal complete, there exists $\bar{x} \in X$ such that $x_n \to \bar{x}$. Again since T is orthogonal continuous, therefore $T(x_n) \to T(\bar{x})$ and $T(\bar{x}) = \lim_n (Tx_n) = \lim_n x_{n+1} = \bar{x}$. Hence $\bar{x}$ is a fixed point of T.

Next we prove that the uniqueness of $\bar{x}$. Let $\bar{y}$ be another fixed point of T. Then we have $T^n(\bar{x}) = \bar{x}$ and $T^n(\bar{y}) = \bar{y}$ for all $n \in \mathbb{N}$. By the definition of orthogonality, we have

$$x_0 \perp \bar{x} \text{ and } x_0 \perp \bar{y}$$

or

$$\bar{x} \perp x_0 \text{ and } \bar{y} \perp x_0.$$

Since T is orthogonal preserving, we have

$$T^n(x_0) \perp T^n(\bar{x}) \text{ and } T^n(x_0) \perp T^n(\bar{y})$$

or

$$T^n(\bar{x}) \perp T^n(x_0) \text{ and } T^n(\bar{y}) \perp T^n(x_0), \ \forall n \in \mathbb{N}.$$

Now by triangular inequality, we have

$$\begin{aligned} d(\bar{x}, \bar{y}) &= d(T^n(\bar{x}), T^n(\bar{y})) \\ &\leq d(T^n(\bar{x}), T^n(x_0)) + d(T^n(x_0), T^n(\bar{y})) \end{aligned}$$

$$\leq k^n d(\bar{x}, x_0) + k^n d(x_0, \bar{y})$$
$$\to 0 \text{ as } n \to \infty.$$

This shows that $\bar{x} = \bar{y}$.

Finally let $x \in X$ be arbitrary. Similarly we have

$$x_0 \perp \bar{x} \text{ and } x_0 \perp x$$

or

$$\bar{x} \perp x_0 \text{ and } x \perp x_0.$$

Since T is orthogonal preserving, we have

$$T^n(x_0) \perp T^n(\bar{x}) \text{ and } T^n(x_0) \perp T^n(x)$$

or

$$T^n(\bar{x}) \perp T^n(x_0) \text{ and } T^n(x) \perp T^n(x_0), \ \forall n \in \mathbb{N}.$$

Thus for all $n \in \mathbb{N}$ we have

$$\begin{aligned} d(\bar{x}, T^n(x)) &= d(T^n(\bar{x}), T^n(x)) \\ &\leq d(T^n(\bar{x}), T^n(x_0)) + d(T^n(x_0), T^n(x)) \\ &\leq k^n d(\bar{x}, x_0) + k^n d(x_0, x) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

This completes the proof.

Corollary 3.10 (Banach's Contraction Principle) *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that for some $k \in (0, 1)$, $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$. Then T has a unique fixed point in X.*

Proof Suppose that

$$x \perp y \Leftrightarrow d(Tx, Ty) \leq d(x, y).$$

For fix $x_0 \in X$. Since T is a contraction, so for all $y \in X$, $x_0 \perp y$. Hence $(X, \perp)$ is an orthogonal set. It is clear that X is an orthogonal complete and T is an orthogonal contraction, orthogonal continuous, and orthogonal preserving. Then by Theorem 3.9, T has a fixed point in X.

The following example shows that Theorem 3.9 is a real extension of Banach's fixed point theorem.

Example 3.11 Suppose that $(X = [0, 9), \perp, d)$ and $T : X \to X$ is defined by

$$T(x) = \begin{cases} \frac{x}{3}, & \text{if } x \leq 3, \\ 0, & \text{if } x > 3. \end{cases}$$

Then X is orthogonal complete (but not complete), and T is orthogonal continuous (not continuous on X), orthogonal contraction, and orthogonal preserving on X. Therefore by Theorem 3.9 T has a fixed point in X. However T is not a contraction on X, so by Banach's contraction principle, we cannot find any fixed point of T on X.

4 Applications to Ordinary Differential Equations

We apply Theorem 3.9 to prove the existence of a solution for the following differential equation:

$$\begin{cases} u'(t) = f(t, u(t)), \ \text{a.e. } t \in I = [0, T] \\ u(0) = a, a \geq 1 \end{cases}, \tag{1}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is an integrable function satisfying the following conditions:

(C1) $f(s, x) \geq 0$ for all $x \geq 0$ and $s \in I$,
(C2) there exists $\alpha \in L^1(I)$ such that

$$|f(s, x) - f(s, y)| \leq \alpha(s)|x - y|$$

for all $s \in I$ and $x, y \geq 0$ with $xy \geq x$ or y.

It is clear that the function $f : I \times \mathbb{R} \to \mathbb{R}$ is not necessarily Lipschitz from the condition (C2). We consider the function

$$f(s, x) = \begin{cases} sx, \ \text{if } x \leq \frac{1}{3}, \\ 0, \ \text{if } x > \frac{1}{3}. \end{cases}$$

which satifies the conditions (C1) and (C2) but f is not continuous and monotone. For $s \neq 0$ we have

$$\left| f\left(s, \frac{1}{3}\right) - f\left(s, \frac{2}{5}\right) \right| = s\frac{1}{3} > s\frac{1}{15} = s\left|\frac{1}{3} - \frac{2}{5}\right|.$$

Theorem 4.1 ([9, Theorem 4.1]) *Under the conditions (C1) and (C2), for all $T > 0$, the differential equation (1) has a unique positive solution.*

Proof Let $X = \{u \in C(I, \mathbb{R}) : u(t) > 0, \forall t \in I\}$. We consider the orthogonality relation in X as

$$x \perp y \Leftrightarrow x(t)y(t) \geq x(t) \text{ or } y(t), \ \forall t \in I.$$

Let $S(t) = \int_0^t |\alpha(s)|ds$. Then we have $S'(t) = |\alpha(t)|$ for almost every $t \in I$. Define

$$||x||_A = \sup_{x \in I} e^{-S(t)}|x(t)|, \quad d(x, y) = ||x - y||_A, \forall x, y \in X.$$

It is easy to show that (X, d) is a metric space.

We show that X is an orthogonal complete (not necessarily complete) metric space. Consider (x_n) is an orthogonal Cauchy sequence in X. It is easy to show that (x_n) is convergent to a point $x \in C(I)$. It is enough to show that $x \in X$. For $t \in I$ by the definition of $\perp$ we have

$$x_n(t)x_{n+1}(t) \geq x_n(t) \text{ or } x_{n+1}(t) \text{ for each } n \in \mathbb{N}.$$

Since $x_n(t) > 0$ for each $n \in \mathbb{N}$, there exists a subsequence (x_{n_k}) of (x_n) for which $x_{n_k}(t) \geq 1$ for each $k \in \mathbb{N}$. By the convergence of this sequence to a real number, $x(t)$ implies that $x(t) \geq 1$. Since $t \in I$ is arbitrary, so we have $x \in X$.

Define a mapping $F : X \to X$ by

$$F(u(t)) = \int_0^t f(s, u(t))ds + a.$$

The fixed point of F is the solution of the Eq. (1). For this, we need to prove the following steps.

Step-I: F is orthogonal preserving: For all $x, y \in X$ with $x \perp y$ and $t \in I$ we have

$$F(u(t)) = \int_0^t f(s, u(t))ds + a \geq 1$$

which implies that $Fx(t)Fy(t) \geq Fx(t)$ and so $Fx \perp Fy$.

Step-II: F is orthogonal contraction: For all $x, y \in X$ with $x \perp y$ and $t \in I$, the condition (C2) implies that

$$\begin{aligned} e^{-S(t)}|Fx(t) - Fy(t)| &\leq e^{-S(t)} \int_0^t |f(s, x(s)) - f(s, y(s))|ds \\ &\leq e^{-S(t)} \int_0^t |\alpha(s)|e^{S(s)}e^{-S(s)}|x(s) - y(s)|ds \\ &\leq e^{-S(t)} \left(\int_0^t |\alpha(s)|e^{S(s)}ds \right) ||x - y||_A \\ &\leq e^{-S(t)}(e^{S(t)} - 1)||x - y||_A \\ &\leq (1 - e^{-||\alpha||_1})||x - y||_A, \end{aligned}$$

so we have

$$||Fx - Fy||_A \leq (1 - e^{-||\alpha||_1})||x - y||_A.$$

Since $1 - e^{-||\alpha||_1} < 1$, so F is an orthogonal contraction.

Step-III: F is orthogonal continuous: Let (x_n) be an orthogonal sequence in X converging to a point $x \in X$. So we see that $x(t) \geq 1$ for all $t \in I$ and hence $x_n \perp x$ for all $n \in \mathbb{N}$. By condition (C2) we have

$$\begin{aligned} e^{-S(t)}|Fx_n(t) - Fx(t)| &\leq e^{-S(t)} \int_0^t |f(s, x_n(s)) - f(s, x(s))|ds \\ &\leq 1 - e^{-||\alpha||_1}||x_n - x||_A, \forall n \in \mathbb{N} \text{ and } t \in I. \end{aligned}$$

Hence

$$||Fx_n - Fx||_A \leq (1 - e^{-||\alpha||_1})||x_n - x||_A, \forall n \in \mathbb{N}.$$

Therefore $Fx_n \to Fx$.

The uniqueness of the solution follows from Theorem 3.9. This completes the proof.

5 Generalized Metric

A mapping $D : X \times X \to [0, \infty]$ is called a generalized metric on a non-empty set X, if the following conditions are satisfied:

1. $D(x, y) = D(y, x)$ for $x, y \in X$,
2. $D(x, y) = 0 \Leftrightarrow x = y$ for $x, y \in X$,
3. $D(x, z) \leq D(x, y) + D(y, z)$ for $x, y, z \in X$ considering that if $D(x, y) = \infty$ or $D(y, z) = \infty$ then $D(x, y) + D(y, z) = \infty$.

Then the pair (X, D) is called a generalized metric space.

Definition 5.1 [15] A mapping $D : X \times X :\to [0, \infty]$ is called a generalized metric on the orthogonal set $(X, \perp)$, if it satisfy the following conditions:

(GO1) $D(x, y) = D(y, x)$ for any points $x, y \in X$ such that $x \perp y$ and $y \perp x$,
(GO2) $D(x, y) = 0 \Leftrightarrow x = y$ for any points $x, y \in X$, $x \perp y$ and $y \perp x$,
(GO3) $D(x, z) \leq D(x, y) + D(y, z)$ for any points $x, y, z \in X$, $x \perp y$, $y \perp z$, and $x \perp z$, considering that if $D(x, y) = \infty$ or $D(y, z) = \infty$ then $D(x, z) = \infty$.

Then the ordered triple $(X, \perp, D)$ is called generalized orthogonal metric space.

The concept of completeness of a generalized orthogonal metric space is defined in the usual way.

Theorem 5.2 ([15, Theorem 3.2]) *Let $(X, \perp, D)$ be a generalized orthogonal complete metric space. Let $T : X \to X$ be an orthogonal preserving and orthogonal continuous map such that*

(1) $D(Tx, Ty) \leq \lambda D(x, y)$ *for any points* $x, y \in X$ *such that* $x \perp y$ *and* $0 < \lambda < 1$,
(2) For any $x \in X$ *there exists* n_0 *such that for* $(T, \perp)$*-orbit* $(T^n x)_{n=0}^{\infty}$ *we have* $D(T^{n_0}x, T^{n_0+1}x) < \infty$,
(3) If $x \perp y$, $Tx = x$ *and* $Ty = y$ *then* $D(x, y) < \infty$.

Then there exists a unique fixed point $\bar{x}$ *of the map* T *and* $\lim_{n\to\infty} T^n x = \bar{x}$ *for any* $x \in X$.

Proof Consider the $(T, \perp)$-orbit $(T^n x)_{n=0}^{\infty}$ of an arbitrary point $x \in X$. Suppose that

$$x \perp Tx, Tx \perp T^2x, T^2x \perp T^3x, \ldots, T^nx \perp T^{n+1}x, \ldots.$$

By the given condition (2), we find n_0 such that $D(T^{n_0}x, T^{n_0+1}x) < \infty$. Then for $n \geq n_0$ we have

$$\begin{aligned}
D(T^nx, T^{n+1}x) &\leq \lambda D(T^{n-1}x, T^nx) \\
&\leq \lambda^2 D(T^{n-2}x, T^{n-1}x) \\
&\leq \lambda^3 D(T^{n-3}x, T^{n-2}x) \\
&\vdots \\
&\leq \lambda^{n-n_0} D(T^{n_0}x, T^{n_0+1}x)
\end{aligned}$$

and

$$\begin{aligned}
D(T^nx, T^{n+m}x) &\leq D(T^nx, T^{n+1}x) + D(T^{n+1}x, T^{n+2}x) + \cdots + D(T^{n+m-1}x, T^{n+m}x) \\
&\leq \lambda^{n-n_0} D(T^{n_0}x, T^{n_0+1}x) + \cdots + \lambda^{n+m-1-n_0} D(T^{n_0}x, T^{n_0+1}x) \\
&= \left[\lambda^{n-n_0} + \lambda^{n+1-n_0} + \cdots + \lambda^{n+m-1-n_0} D(T^{n_0}x, T^{n_0+1}x)\right] \\
&\leq \frac{\lambda^{n-n_0}}{1-\lambda} D(T^{n_0}x, T^{n_0+1}x).
\end{aligned}$$

Therefore the $(T, \perp)$-orbit $(T^n x)_{n=0}^{\infty}$ is a Cauchy sequence in X, and by the completeness of X, it converges to a point $\bar{x} \in X$. Since T is an orthogonal continuous, so $\bar{x}$ is a fixed point of T. Suppose that $x \perp y$, $Tx = x$ and $Ty = y$, then by the given condition (3) we have $D(x, y) < \infty$, and by condition (1) we get

$$D(x, y) = D(Tx, Ty) \leq \lambda D(x, y)$$

which is a contradiction. So the fixed point is unique, This completes the proof.

Definition 5.3 ([16]) Let $(X, \perp)$ be an orthogonal set. A sequence (x_n) in X is called a strongly orthogonal (SO-orthogonal) if

$$x_n \perp x_{n+m} \text{ or } x_{n+m} \perp x_n, \forall n, m \in \mathbb{N}.$$

Remark 5.4 Every strongly orthogonal sequence is an orthogonal sequence, but the converse is not true.

Example 5.5 Let $X = \mathbb{Z}$. Define the orthogonal relation on X by $x \perp y$ if and only if $xy \in \{x, y\}$. Consider a sequence (x_n) in X as follows

$$x_n = \begin{cases} 3, \text{ if } n = 2k, \text{ for some } k \in \mathbb{Z}, \\ 1, \text{ if } n = 2k+1 \text{ for some } k \in \mathbb{Z}. \end{cases}$$

Then we have $x_n \perp x_{n+1}$ for all $n \in \mathbb{N}$, but x_{2n} is not orthogonal to x_{4n}. So (x_n) is an orthogonal sequence but not a strongly orthogonal sequence.

Definition 5.6 An orthogonal metric space $(X, \perp, d)$ is called strongly orthogonal complete (SO-complete) if every strongly orthogonal Cauchy sequence is convergent.

Remark 5.7 Every complete metric space is strongly orthogonal complete, but the converse is not true.

Example 5.8 Consider $X = \{x \in C([0, 1], \mathbb{R}) : x(t) > 0, \forall t \in [0, 1]\}$. Then X is an incomplete metric space with the supremum norm $||x|| = \sup_{t\in[0,1]} |x(t)|$. Define the orthogonal relation $\perp$ on X by

$$x \perp y \iff x(t)y(t) \geq \max_{t\in[0,1]} \{x(t), y(t)\}.$$

Then X is strongly orthogonal complete. If (x_n) is a strongly orthogonal Cauchy sequence in X, then for all $n \in \mathbb{N}$ and $t \in [0, 1]$, $x_n(t) \geq 1$. Since $C([0, 1], \mathbb{R})$ is a Banach space with the supremum norm, so we can find an element $x \in C([0, 1], \mathbb{R})$ for which $||x_n - x|| \to 0$ as $n \to \infty$. Since uniformly convergent implies the pointwise convergent. Thus $x(t) \geq 1$ for all $t \in [0, 1]$ and hence $x \in X$.

Remark 5.9 Every orthogonal complete metric space is strongly orthogonal complete, but the converse is not true.

Example 5.10 Suppose $X = [1, \infty)$ with the Euclidean metric and the orthogonal relation on X is defined by $x \perp y \iff xy \in \{x, y\}$. Let (x_n) be a strongly orthogonal Cauchy sequence in X, by the definition of $\perp$ we have $x_n = 1$ for all $n \in \mathbb{N}$. Therefore (x_n) converges to 1. Consider a sequence

$$x_n = \begin{cases} 0, \text{ if } n = 2k, \text{ for some } k \in \mathbb{Z}, \\ k+1, \text{ if } n = 2k+1 \text{ for some } k \in \mathbb{Z}. \end{cases}$$

Then (x_n) is an orthogonal sequence but it is not convergent to any element in X.

Definition 5.11 ([16]) Let $(X, \perp, d)$ is an orthogonal metric space. A mapping $T : X \to X$ is called strongly orthogonal continuous (SO-continuous) at $x_0 \in X$, for each strongly orthogonal sequence (x_n) in X if $x_n \to x_0$ then $T(x_n) \to T(x_0)$. Also T is called strongly orthogonal continuous on X if it is strongly orthogonal continuous at each point of X.

Remark 5.12 Every continuous mapping is orthogonal continuous and every orthogonal continuous mapping is strongly orthogonal continuous, but the converse is not true; i.e., every continuous mapping is strongly orthogonal continuous but the converse is not true.

Example 5.13 Let $X = \mathbb{R}$ with the Euclidean metric. Suppose the orthogonal relation $\perp$ as $x \perp y \iff xy \in \{x, y\}$. Define a function $T : X \to X$ by

$$T(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ \frac{1}{x^2}, & \text{if } x \in \mathbb{Q}^c. \end{cases}$$

Then T is not continuous, but T is strongly orthogonal continuous, because we consider $x_n \in \mathbb{Q}$ for enough large n. Then we have $T(x_n) = 1 \to x = 1$. Now we consider a sequence

$$x_n = \begin{cases} 1, & \text{if } n = 2k, \text{ for some } k \in \mathbb{Z}, \\ \frac{\sqrt{2}}{k}, & \text{if } n = 2k+1 \text{ for some } k \in \mathbb{Z}. \end{cases}$$

Then we see that $x_n \to 0$ but the sequence $(T(x_n))$ is not convergent to $T(0)$. So T is not orthogonal continuous.

Definition 5.14 Let $(X, \perp, d)$ be a strongly orthogonal complete metric space. A mapping $T : X \to X$ is called strongly orthogonal Meir–Keeler contraction if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$x \neq y, x \perp y \text{ and } \varepsilon \leq d(x, y) < \varepsilon + \delta(\varepsilon) \implies d(Tx, Ty) < \varepsilon. \tag{1}$$

Theorem 5.15 *Let $(X, \perp, d)$ be a strongly orthogonal complete metric space (not necessarily complete) with an orthogonal element x_0. Suppose that $T : X \to X$ is orthogonal preserving, strongly orthogonal continuous such that satisfying the strongly orthogonal Meir–Keeler contraction. Then T has a unique fixed point $z \in X$. Also T is a Picard operator, i.e., for all $x \in X$, the sequence $(T^n(x))$ is convergent to z with respect to the metric d.*

Proof By the definition of orthogonality, we have

$$x_0 \perp y \text{ or } y \perp x_0, \forall y \in X.$$

It follows that $x_0 \perp Tx_0$ or $Tx_0 \perp x_0$. Put

$$x_1 = Tx_0, x_2 = T(x_1) = T^2(x_0), \ldots, x_{n+1} = T(x_n) = T^{n+1}(x_0), \forall n \in \mathbb{N}.$$

We have

$$x_0 \perp x_n \text{ or } x_n \perp x_0, \forall n \in \mathbb{N}.$$

Since T is orthogonal preserving, so we get

$$x_m = T^m(x_0) \perp T^m(x_n) = x_{n+m} \text{ or } x_{n+m} = T^m(x_n) \perp T^m(x_0) = x_m, \forall n, m \in \mathbb{N}.$$

This gives that (x_n) is a strongly orthogonal sequence.

We divide the proof in the following steps:

Step-I: To show that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

If there exists $m_0 \in \mathbb{N}$, $x_{m_0} = x_{m_0+1}$ then the result is obvious. Let $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then by the Meir–Keeler condition, we have

$$d(x_{n+1}, x_n) < d(x_n, x_{n-1}), \forall n \in \mathbb{N}.$$

This shows that the sequence $(d(x_{n+1}, x_n))$ is strictly decreasing and it converges. Put $\lim_{n\to\infty} d(x_{n+1}, x_n) = t$. We prove that $t = 0$. Suppose that $t > 0$. Using the Meir–Keeler condition for $t > 0$, we can find $\delta(t) > 0$ such that

$$x \neq y, x \perp y \text{ and } t \leq d(x, y) < t + \delta(t) \implies d(Tx, Ty) < t.$$

Since $\lim_{n\to\infty} d(x_{n+1}, x_n) = t$, then there exists $m_0 \in \mathbb{N}$ such that

$$t \leq d(x_{m_0}, x_{m_0-1}) < t + \delta(t) \implies d(Tx_{m_0}, x_{m_0-1}) < t.$$

This implies that $d(x_{m_0+1}, x_{m_0}) < t$, and it contradicts the assumption $\lim_{n\to\infty} d(x_{n+1}, x_n) = t$. Therefore $t = 0$.

Step-II: To prove that (x_n) is a strongly orthogonal Cauchy sequence.

Suppose that (x_n) is not a strongly orthogonal Cauchy sequence. There exist $\varepsilon > 0$ and two sequences (m_k) and (n_k) such that $m_k > n_k \geq m_0$

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon \text{ and } d(x_{m_{k-1}}, x_{n_k}) < \varepsilon \tag{2}$$

To prove the result (2), we suppose that

$$\mathcal{S}_k = \{m \in \mathbb{N} : \exists n_k \geq m_0, d(x_m, x_{n_k}) \geq \varepsilon, m > n_k \geq m_0\}.$$

Clearly $\mathcal{S}_k \neq \phi$ and $\mathcal{S}_k \subseteq \mathbb{N}$, then by the well-ordering principle, the minimum element of $\mathcal{S}_k$ is denoted by m_k, and clearly the result (2) holds. Then there exists $\delta(\varepsilon) > 0$ (which can be chosen as $\delta(\varepsilon) \leq \varepsilon$) satisfy the result (1). Then by Step-I, we show that there exists $m_0 \in \mathbb{N}$ such that $d(x_{m_0}, x_{m_0+1}) < \delta(\varepsilon)$. Then for fix $k \geq m_0$ we have

$$\begin{aligned} d(x_{m_k-1}, x_{n_k-1}) &= d(x_{m_k-1}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}) \\ &< \varepsilon + \delta(\varepsilon). \end{aligned}$$

Now we consider the two cases:

Case-I: Suppose that $d(x_{m_k-1}, x_{n_k-1}) \geq \varepsilon$.

Since x_{n_k-1} and x_{m_k-1} are orthogonal comparable, using the condition (1) we get

$$\varepsilon \leq d(x_{m_k-1}, x_{n_k-1}) < \varepsilon + \delta(\varepsilon) \implies d(x_{m_k}, x_{n_k}) < \varepsilon.$$

Case-II: Suppose that $d(x_{m_k-1}, x_{n_k-1}) < \varepsilon$.

Since x_{m_k-1} and x_{n_k-1} are orthogonal comparable, then by (1) we get

$$d(x_{m_k}, x_{n_k}) < d(x_{m_k-1}, x_{n_k-1}) < \varepsilon.$$

Hence in each case we get $d(x_{m_k}, x_{n_k}) < \varepsilon$ which contradicts the condition (2). Hence (x_n) is a strongly orthogonal Cauchy sequence. Since X is strongly orthogonal complete, then there exists $y_0 \in X$ such that $x_n \to y_0$. So $d(x_n, y_0) \to \infty$ as $n \to \infty$. Also since T is a strongly orthogonal continuous, then for any $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that

$$d(x_{m_0+1}, y_0) < \frac{\varepsilon}{2} \text{ and } d(Tx_{m_0}, Ty_0) < \frac{\varepsilon}{2}.$$

Now

$$\begin{aligned} d(Ty_0, y_0) &\leq d(Ty_0, Tx_{m_0}) + d(Tx_{m_0}, y_0) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

It follows that $Ty_0 = y_0$. Hence T has a fixed point in X.

Now we prove that T is a Picard operator. Let $x \in X$ be arbitrary. By the choice of x_0, we have

$$x_0 \perp y_0 \text{ and } x_0 \perp x$$

or

$$y_0 \perp x_0 \text{ and } x \perp x_0$$

Since T is orthogonal preserving, it implies that

$$x_n \perp y_0 \text{ and } x_n \perp T^n x$$

or

$$y_0 \perp x_n \text{ and } T^n x \perp x_n, \forall n \in \mathbb{N}.$$

Now we show that the sequence $(d(T^n x, x_n))$ converges to zero. For some $m_0 \in \mathbb{N}$, if $T^{m_0} x = x_{m_0}$ then $d(T^n x, x_n) = 0$ for all $n \geq m_0$.

Let $T^n x \neq x_n$ for all $n \in \mathbb{N}$. The Meir–Keeler condition implies that the sequence $(d(T^n x, x_n))$ is strictly decreasing. Using the same argument in Step-I, we get $\lim_{n\to\infty} d(T^n x, x_n) = 0$. For all $n \in \mathbb{N}$ we obtain that

$$d(T^n x, y_0) \leq d(T^n x, x_n) + d(x_n, y_0) = 0 \text{ as } n \to \infty.$$

Therefore $T^n x \to y_0$.

Finally we prove that the fixed point y is unique. Let $\bar{y} \in X$ be another fixed point of T, then $T^n \bar{y} = \bar{y}$ for all $n \in \mathbb{N}$. It follows from T is a Picard operator that $\bar{y} = y_0$.

6 Applications to Integral Equations

We study the existence and uniqueness of a solution of the following integral equation

$$u(t) = \int_0^t e^{s-t} \left(\int_0^b e^{-\tau} g(s, \tau, u(\tau)) d\tau \right) ds \tag{1}$$

Let $p > 0$, g be a function from $[0, p] \times [0, p] \times X$ into X and $\Gamma : [0, p] \times [0, p] \times [0, p] \to \mathbb{R}^+$ be an integrable function for which

(P1) (i) $g : (t, ., x) : s \to g(t, s, x)$ is an integrable function for every $x \in X$ and for all $t \in [0, p]$
(ii) $g(t, s, .) : x \to g(t, s, x)$ is d-continuous on X for all $t, s \in [0, p]$.

(P2) (i) $g(t, s, x) \geq 0$ for all $x \geq 0$ and for all $t, s \in [0, p]$
(ii) $g(t, s, x)g(t', r, y) \geq g(t, t', xy)$ for each $x, y \in X$ with $xy \geq 0$ and for all $t, t', r, s \in [0, p]$.

(P3) There exists $\gamma > 0$ such that $d(g(t, s, x), g(t, s, y)) \leq \gamma d(x, y)$ for all (t, s, x), $(t, s, y) \in [0, p] \times [0, p] \times X$ with $xy \geq 0$.

(P4) $d(g(t, s, x), g(v, s, x)) \leq \Gamma(t, v, s)$ for all $(t, s, x), (v, s, x) \in [0, p] \times [0, p] \times X$ and

$$\lim_{t \to \infty} \int_0^p \Gamma(t, v, s) ds = 0$$

uniformly for all $v \in [0, p]$.

We consider $\mathcal{B} = C([0, p], X)$ the space of all continuous function from $[0, p]$ into X. It is a complete metric space with the metric

$$d_b(u, w) = \sup_{t \in [0, p]} e^{-bt} |u(t) - w(t)|, \text{ where } b \geq 0.$$

We define the operators T and S on $\mathcal{B}$ by

$$Tu(t) = \int_0^p e^{-s} g(t, s, u(s)) ds$$

$$Su(t) = \int_0^t e^{s-t} Tu(s) ds$$

We have the fixed points of S are the solutions of the Eq. (1) and $\mathcal{B}$ is invariant under T and S.

Theorem 6.1 *Under the conditions (P1)–(P4), for all $p \geq 0$ the integral equation* (1) *has a unique solution in $\mathcal{B}$.*

Proof We consider the orthogonal relation on $\mathcal{B}$ as

$$u \perp w \iff u(t)w(t) \geq 0, \forall t \in [0, p].$$

It is clear that $\mathcal{B}$ is a strongly orthogonal complete metric space. To complete the proof, we need the following steps.

Step-I: S is an orthogonal preserving. For each $u, w \in \mathcal{B}$ with $u \perp w$, by the hypothesis (P1)(i) and (P2)(ii) we have

$$\begin{aligned} Tu(t)Tw(t') &= \int_0^p e^{-s} g(t, s, u(s))ds \int_0^p e^{-r} g(t', r, w(r))dr \\ &= \int_0^p \int_0^p e^{-(s+r)} g(t, s, u(s))g(t', r, w(r))dsdr \\ &\geq \int_0^p \int_0^p e^{-(s+r)} g(t, t', u(s)w(r))dsdr, \quad u(s)w(r) \geq 0 \\ &\geq 0 \text{ for each } t, t' \in [0, p]. \end{aligned}$$

Therefore $Tu \perp Tw$. By the definition of S we have $Su \perp Sw$.

Step-II: To prove that S is d_b-Lipschitz on orthogonal comparable elements. Let $M = \{s_0, s_1, \ldots, s_k\}$ be a subdivision of the interval $[0, p]$. Then we have $\sum_{i=0}^{k-1}(s_{i+1} - s_i)e^{-s_i}x(s_i)$ is norm convergent and consequently d_b-convergent to $\int_0^p e^{-s}x(s)ds$ in $\mathcal{B}$, when $|M| = \sup\{|s_{i+1} - s_i| : i = 0, 1, 2, \ldots, k-1\} \to 0$ as $k \to \infty$. Let $u \perp w$. Then we have

$$\begin{aligned} &\int_0^p e^{-s}(g(t, s, u(s)) - g(t, s, w(s)))ds \\ &= \lim_{k\to\infty} \sum_{i=0}^{k-1}(s_{i+1} - s_i)e^{-s_i}(g(t, s_i, u(s_i)) - g(t, s_i, w(s_i))) \end{aligned}$$

and

$$\sum_{i=0}^{k-1}(s_{i+1} - s_i)e^{-s_i} \leq \int_0^p e^{-s}ds = 1 - e^{-p} < 1$$

by Fatou property and condition (P3). Now we have

$$\begin{aligned} d(Tu(t), Tw(t)) &\leq \liminf \sum_{i=0}^{k-1} (s_{i+1} - s_i) e^{-s_i} d(g(t, s_i, u(s_i)), g(t, s_i, w(s_i))) \\ &\leq \lambda \liminf \sum_{i=0}^{k-1} (s_{i+1} - s_i) e^{-s_i} d(u(s_i), w(s_i)) \\ &\leq \lambda \liminf \sum_{i=0}^{k-1} (s_{i+1} - s_i) e^{bs_i} d_b(u, w). \end{aligned}$$

Therefore we get

$$\begin{aligned} e^{-bt} d(Tu(t), Tw(t)) &\leq \lambda e^{-bt} \left(\int_0^p e^{bs} ds \right) d_b(u, w) \\ &\leq \lambda \frac{e^{bp} - 1}{b} d_b(u, w). \end{aligned}$$

Hence

$$d_b(Tu, Tw) \leq \lambda \frac{e^{bp} - 1}{b} d_b(u, w).$$

By definition of S gives that

$$d_b(Su, Sw) \leq N d_b(u, w), \text{ where } N = \frac{\lambda}{b(b+1)} \left(1 - e^{-(b+1)p}\right) \left(e^{bp} - 1\right).$$

Step-III: To show that S satisfies the Meir–Keeler condition.

We define

$$\delta(\varepsilon) = \{d_b(u, w) : d_b(Su, Sw) \geq \varepsilon \text{ and } u \perp w\}.$$

Let $0 < N < 1$ and $\varepsilon > 0$ be given. If $u \perp w$ and $d_b(Su, Sw) \geq \varepsilon$ then by Step-II, we have

$$d_b(u, w) \geq N^{-1} \varepsilon.$$

So $\delta(\varepsilon) \geq N^{-1}\varepsilon > \varepsilon$. By using Theorem 1 of [17] we have S satisfies the Meir–Keeler condition. Therefore by Theorem 5.15, S has a unique fixed point, which is the solution of the integral equation (1).

7 Different Types of Convergence

The asymptotic density or density of a subset U of $\mathbb{N}$, denoted by $\delta(U)$, is given by

$$\delta(U) = \lim_{n \to \infty} \frac{1}{n} |\{k \leq n : k \in U\}|,$$

if this limit exists, where $|\{k \leq n : k \in U\}|$ denotes the cardinality of the set $\{k \leq n : k \in U\}$. Fast [18] and Steinhaus [19] independently introduced the notion of statistical convergence with the help of asymptotic density, and later on Schoenberg [20] reintroduced it. A sequence $\mathbf{x} = (x_n)$ is said to be statistically convergent to ℓ if for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - \ell| \geq \varepsilon\}$ has density zero. We call ℓ the statistical limit of $\mathbf{x}$. The set of all statistically convergent sequences is denoted by st.

The notion of ideal convergence is the dual (equivalent) to the notion of filter convergence which was introduced by Cartan [21]. The filter convergence is a generalization of the classical notion of convergence of a sequence, and it has been an important tool in general topology and functional analysis. Kostyrko et al. [22] and Nuray and Ruckle [23] independently discussed the ideal convergence which is based on the structure of the admissible ideal $\mathcal{I}$ of subsets of natural numbers $\mathbb{N}$. It was further investigated by many authors, e.g., Šalát et al. [24], and references therein. The statistical convergence and ideal convergence for sequences of real-valued functions were studied by Balcerzak et al. [25].

A non-empty class $\mathcal{I}$ of power sets of a non-empty set X is called an *ideal* on X if and only if (i) $\phi \in \mathcal{I}$ (ii) $\mathcal{I}$ is additive under union (iii) hereditary. An ideal $\mathcal{I}$ is called *non-trivial* if $\mathcal{I} \neq \phi$ and $X \notin \mathcal{I}$. A non-empty class $\mathcal{F}$ of power sets of X is called a *filter* on X if and only if (i) $\phi \notin \mathcal{F}$ (ii) $\mathcal{F}$ is additive under intersection (iii) for each $U \in \mathcal{F}$ and $V \supset U$, implies $V \in \mathcal{F}$. A non-trivial ideal $\mathcal{I}$ is said to be (i) an *admissible ideal* on X if and only if it contains all singletons (ii) *maximal*, if there cannot exist any non-trivial ideal $\mathcal{K} \neq \mathcal{I}$ containing $\mathcal{I}$ as a subset (iii) is said to be a *translation invariant ideal* if $\{n + 1 : n \in U\} \in \mathcal{I}$, for any $U \in \mathcal{I}$.

We recall that a real sequence $\mathbf{x} = (x_n)$ is called ideal convergent (in short $\mathcal{I}$-convergent) to the number l (denoted by $\mathcal{I}\text{-}\lim x_n = l$) if for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - l| \geq \varepsilon\}$ is in $\mathcal{I}$. The set of all ideal convergent sequences denoted by $\mathcal{I}$.

A lacunary sequence $\theta = (k_r)$ is a non-decreasing sequence of positive integers such that $k_0 \neq 0$ and $h_r : k_r - k_{r-1} \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated as q_r. We assume that $\liminf_r q_r > 1$. The notion of lacunary statistical convergence was introduced and studied by Fridy and Orhan [26, 27]. A sequence (x_n) in $\mathbb{R}$ is called lacunary statistically convergent to $x \in \mathbb{R}$ if

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r ; |x_n - x| \geq \varepsilon\}| = 0,$$

for every positive real number ε. The set of all lacunary statistically convergent sequences is denoted by st_θ.

Connor and Grosse-Erdman [28] gave sequential definitions of continuity for real functions calling $\mathcal{G}$-continuity, where a method of sequential convergence, or briefly a method, is a linear function $\mathcal{G}$ defined on a linear subspace of s, denoted by $c_{\mathcal{G}}$, into $\mathbb{R}$. We refer [29] for sequential compactness and [30, 31] for $\mathcal{G}$-sequential continuity. A sequence $\mathbf{x} = (x_n)$ is said to be $\mathcal{G}$-convergent to ℓ if $\mathbf{x} \in c_{\mathcal{G}}$ and $\mathcal{G}(\mathbf{x}) = \ell$. In particular, lim denotes the limit function $\lim \mathbf{x} = \lim_n x_n$ on the linear space c, and $st - \lim$ denotes the statistical limit function $st - \lim \mathbf{x} = st - \lim_n x_n$ on the linear space st and $st_\theta - \lim$ denotes the lacunary statistical limit function $st_\theta - \lim \mathbf{x} = st_\theta - \lim_n x_n$ on the linear space st_θ. Also $\mathcal{I} - \lim$ denotes the $\mathcal{I}$-limit function $\mathcal{I} - \lim \mathbf{x} = \mathcal{I} - \lim_n x_n$ on the linear space $\mathcal{I}(\mathbb{R})$. A method $\mathcal{G}$ is called regular if every convergent sequence $\mathbf{x} = (x_n)$ is $\mathcal{G}$-convergent with $\mathcal{G}(\mathbf{x}) = \lim \mathbf{x}$. A method is called subsequential if whenever $\mathbf{x}$ is $\mathcal{G}$-convergent with $\mathcal{G}(\mathbf{x}) = \ell$, then there is a subsequence (x_{n_k}) of $\mathbf{x}$ with $\lim_k x_{n_k} = \ell$.

8 General Convergence

Let X be an orthogonal set and $d : X \times X \to [0, \infty)$ be a mapping. For every $x \in X$ we define the set

$$\mathcal{GO}(X, d, x) = \left\{ (x_n) \subset X : \mathcal{G} - \lim_{n\to\infty} d(x_n, x) = 0 \text{ and } x_n \perp x, \forall n \in \mathbb{N} \right\}. \quad (1)$$

Definition 8.1 Let $(X, \perp, d)$ be an orthogonal metric space. A sequence (x_n) in X is said to be

(i) $\mathcal{G}$-orthogonal convergent (in short $\mathcal{G}O$-convergent) to x if and only if $(x_n) \in \mathcal{GO}(X, d, x)$,
(ii) $\mathcal{G}$-orthogonal Cauchy (in short $\mathcal{G}O$-Cauchy) if and only if $\mathcal{G} - \lim_{n,m\to\infty} d(x_n, x_m) = 0$ and $x_n \perp x_m$ or $x_m \perp x_n$, $\forall n, m \in \mathbb{N}$.

Theorem 8.2 *Let $(X, \perp, < ., . >)$ be an orthogonal inner product space, where $X = \mathbb{R}^n$, $< ., . >$ denotes the standard inner product space and $\perp$ is an orthogonal relation on X defined by $x \perp y$ if $< x, y >= 0$ for all $x, y \in X$. Let (x_n) and (y_n) be two sequences in X with $(x_n) \in \mathcal{GO}(X, d, x)$ and $(y_n) \in \mathcal{GO}(X, d, y)$. Then*

(a) $(x_n + y_n) \in \mathcal{GO}(X, d, x + y)$,
(b) $< x_n, y_n > \to_{\mathcal{G}} < x, y >$.

Proof The proof is simple. The reader should prove the theorem on its own.

Definition 8.3 Let $(X, \perp, d)$ be an orthogonal metric space. A function $f : X \to X$ is said to be $\mathcal{G}$-orthogonal continuous ($\mathcal{G}O$-continuous) at a point x_0 in X if for each orthogonal sequence (x_n) in X, $\mathcal{G}$-converging to x_0 such that $f(x_n) \to_{\mathcal{G}} f(x_0)$. Also f is said to be $\mathcal{G}$-orthogonal continuous on X if f is $\mathcal{G}$-orthogonal continuous at each point on X.

Definition 8.4 Let $(X, \perp, d)$ be an orthogonal metric space and $E \subset X$. A function $f : E \to X$ is said to be $\mathcal{G}$-orthogonal sequentially continuous ($\mathcal{G}O$-sequentially continuous) at a point x_0 in X if for each orthogonal sequence $(x_n) \in E$, $\mathcal{G}$-converging to x_0 such that $f(x_n) \to_{\mathcal{G}} f(x_0)$.

Theorem 8.5 *Let $\mathcal{G}$ be a regular method and $(X, \perp, d)$ an orthogonal metric space, and $f, g : X \to X$ be functions on X. Then the following are satisfied.*

(a) If f and g are $\mathcal{G}O$-sequentially continuous, then so also is gf,
(b) If f and g are $\mathcal{G}O$-sequentially continuous, then so also is $f + g$.

Proof (a) Let $\mathbf{x}$ be an orthogonal sequence in X such that $\mathcal{G}(\mathbf{x}) = x_0 \in X$. Since f is $\mathcal{G}O$-sequentially continuous at x_0, we get $\mathcal{G}(f(\mathbf{x})) = f(x_0)$ and since g is $\mathcal{G}O$-sequentially continuous at $f(x_0)$ we have $\mathcal{G}(g(f(\mathbf{x}))) = g(f(x_0))$. Therefore the function gf is $\mathcal{G}O$-sequentially continuous.

(b) Let $\mathbf{x}$ be an orthogonal sequence in X such that $\mathcal{G}(\mathbf{x}) = x_0 \in X$. Since the functions f and g are $\mathcal{G}O$-sequentially continuous, so we have $\mathcal{G}(f(\mathbf{x})) = f(x_0)$ and $\mathcal{G}((\mathbf{x})) = g(x_0)$. Therefore by the additivity of $\mathcal{G}$ we get

$$\mathcal{G}((f+g)(\mathbf{x}) = \mathcal{G}(f(\mathbf{x}) + g(\mathbf{x})) = \mathcal{G}(f(\mathbf{x})) + \mathcal{G}(g(\mathbf{x})) = f(x_0) + g(x_0) = (f+g)(x_0)$$

i.e., $f + g$ is $\mathcal{G}O$-sequentially continuous.

Theorem 8.6 *Let $\mathcal{G}$ be a method and $(X, \perp, d)$ be an orthogonal metric space. Then we have the following.*

(i) If $f : X \to X$ is $\mathcal{G}O$-sequentially continuous, then so also is a restriction $f_A : A \to X$ to a subset A,
(ii) The identity map $\mathcal{J} : X \to X$ is $\mathcal{G}O$-sequentially continuous,
(iii) For a subset $A \subseteq X$, the inclusion map $f : A \to X$ is $\mathcal{G}O$-sequentially continuous,
(iv) If $\mathcal{G}$ is regular, then the constant map $\mathcal{C} : X \to X$ is $\mathcal{G}O$-sequentially continuous,
(v) If f is $\mathcal{G}O$-sequentially continuous, then so also is $-f$,
(vi) The inverse function $f : X \to X;\ f(x) = -x$ is $\mathcal{G}O$-sequentially continuous.

Proof (i) Let $\mathbf{x}$ be an orthogonal sequence of the terms in A with $\mathcal{G}(\mathbf{x}) = x_0$. Since f is $\mathcal{G}O$-sequentially continuous, then we have $\mathcal{G}(f(\mathbf{x})) = f(x_0)$.

(ii) Let $\mathcal{G}(\mathbf{x}) = x_0$ for an orthogonal sequence $\mathbf{x}$ in X. Then $\mathcal{G}(\mathcal{J}(\mathbf{x})) = \mathcal{G}(\mathbf{x}) = x_0 = \mathcal{J}(x_0)$ and so $\mathcal{J}$ is $\mathcal{G}O$-sequentially continuous.

(iii) Follows immediately from (i) and (ii).

(iv) Let $\mathcal{C} : X \to X$ be a constant map with $\mathcal{C}(x) = y_0$ and let $\mathbf{x}$ be an orthogonal sequence in X with $\mathcal{G}(\mathbf{x}) = x_0$. Then $\mathcal{C}(\mathbf{x}) = (y_0, y_0, \ldots)$ which $\mathcal{G}$-converges to y_0. Since $\mathcal{G}$ is regular $\mathcal{G}(\mathcal{C}(\mathbf{x})) = y_0 = \mathcal{C}(x_0)$. Therefore $\mathcal{C}$ is $\mathcal{G}O$-sequentially continuous.

(v) Let $\mathbf{x}$ be an orthogonal sequence in X with $\mathcal{G}(\mathbf{x}) = x_0$. Since f is $\mathcal{G}O$-sequentially continuous $\mathcal{G}(f(\mathbf{x})) = f(x_0)$. Therefore $\mathcal{G}(-f(\mathbf{x})) = -\mathcal{G}(f(\mathbf{x})) = -f(x_0)$, and hence, $-f$ is $\mathcal{G}O$-sequentially continuous.

(vi) Follows immediately from (ii) and (v).

Corollary 8.7 *Let $\mathcal{G}$ be a regular method and $C\mathcal{G}O(X)$ the class of $\mathcal{G}O$-sequentially continuous functions. Then $C\mathcal{G}O(X)$ becomes a group with the sum of functions.*

Definition 8.8 Let $(X, \perp, d)$ be an orthogonal metric space. Let $A \subseteq X$ and $x_0 \in X$. Then x_0 is in the $\mathcal{G}O$-sequential closure of A (it is called $\mathcal{G}O$-hull of A) if there is an orthogonal sequence $\mathbf{x} = (x_n)$ of points in A such that $\mathcal{G}(\mathbf{x}) = x_0$. We denote $\mathcal{G}O$-sequential closure of a set A by $\bar{A}^{\mathcal{G}O}$. We say that a subset A is $\mathcal{G}O$-sequentially closed if it contains all the points in its $\mathcal{G}O$-sequential closure, i.e., if $\bar{A}^{\mathcal{G}O} \subseteq A$. It is clear that $\overline{\phi}^{\mathcal{G}O} = \phi$ and $\bar{X}^{\mathcal{G}O} = X$. If $\mathcal{G}$ is a regular method, then $A \subseteq \bar{A} \subseteq \bar{A}^{\mathcal{G}O}$, and hence, A is $\mathcal{G}O$-sequentially closed if and only if $\bar{A}^{\mathcal{G}O} = A$.

Definition 8.9 A subset A of an orthogonal metric space $(X, \perp, d)$ is called $\mathcal{G}O$-sequentially open if its complement is $\mathcal{G}O$-sequentially closed, i.e., $\overline{X \backslash A}^{\mathcal{G}O} \subseteq X \backslash A$.

Definition 8.10 A function f is said to be $\mathcal{G}O$-sequentially open if the image of any $\mathcal{G}O$-sequentially open subset of an orthogonal metric space $(X, \perp, d)$ is $\mathcal{G}O$-sequentially open.

Definition 8.11 Let $(X, \perp, d)$ be an orthogonal metric space. A function f is said to be $\mathcal{G}O$-sequentially closed if the image of any $\mathcal{G}O$-sequentially closed subset of X is $\mathcal{G}O$-sequentially closed.

Theorem 8.12 *Let $(X, \perp, d)$ be an orthogonal metric space and $\mathcal{G}$ be a regular method. A function $f : X \to X$ is $\mathcal{G}O$-sequentially closed if $\overline{f(B)}^{\mathcal{G}O} \subseteq f(\overline{B}^{\mathcal{G}O})$ for every subset B.*

Proof Let $f : X \to X$ be a function such that $\overline{f(B)}^{\mathcal{G}O} \subseteq f(\overline{B}^{\mathcal{G}O})$ for any subset B. Let A be a $\mathcal{G}O$-closed subset. By assumption $\overline{f(A)}^{\mathcal{G}O} \subseteq f(\overline{A}^{\mathcal{G}O})$. Since $\mathcal{G}$ is regular $\overline{B}^{\mathcal{G}O} = B$ and so we have $\overline{f(B)}^{\mathcal{G}O} \subseteq f(B)$ and therefore $f(B)$ is $\mathcal{G}O$-sequentially closed.

Theorem 8.13 *Let $(X, \perp, d)$ be an orthogonal metric space and $\mathcal{G}$ be a regular method. If a function f is $\mathcal{G}O$-sequentially continuous on X, then the inverse image $f^{-1}(A)$ of any $\mathcal{G}O$-sequentially open subset A of X is $\mathcal{G}O$-sequentially open.*

Proof Let $f : X \to X$ be any $\mathcal{G}O$-sequentially continuous function and A be any $\mathcal{G}O$-sequentially open subset of X. Then $X \backslash A$ is $\mathcal{G}O$-sequentially closed. By Lemma 8.12, $f^{-1}(X \backslash A)$ is $\mathcal{G}O$-sequentially closed. On the other hand

$$f^{-1}(X \backslash A) = f^{-1}(X) \backslash f^{-1}(A) = X \backslash f^{-1}(A)$$

and so it follows that $f^{-1}(A)$ is $\mathcal{G}O$-sequentially open. This completes the proof of the theorem.

Definition 8.14 Let $(X, \perp, d)$ be an orthogonal metric space. A point x_0 is called a $\mathcal{G}O$-sequential accumulation point of a subset A of X (or is in the $\mathcal{G}O$-sequential derived set) if there is an orthogonal sequence $\mathbf{x} = (x_n)$ of points in $A \backslash \{x_0\}$ such that $\mathcal{G}(\mathbf{x}) = x_0$.

Definition 8.15 Let $(X, \perp, d)$ be an orthogonal metric space. A subset A of X is called $\mathcal{G}O$-sequentially countably compact if any infinite subset of A has at least one $\mathcal{G}O$-sequential accumulation point in A.

Theorem 8.16 *Let $(X, \perp, <.,.>)$ be an orthogonal inner product space, where $X = \mathbb{R}^n$, $<.,.>$ denotes the standard inner product space and $\perp$ is an orthogonal relation on X defined by $x \perp y$ if $<x, y> = 0$ for all $x, y \in X$. Then any $\mathcal{G}$-orthogonal sequentially continuous function at point x_0 if and only if it is continuous at x_0 in the ordinary sense.*

Proof The proof follows from Theorem 2.25 and the definition.

Theorem 8.17 *Let $(X, \perp, d)$ be an orthogonal metric space. Suppose that G is a regular method. Let $f : X \to X$ be an additive function on X. Then f is $\mathcal{G}O$-sequentially continuous at origin if and only if f is $\mathcal{G}O$-sequentially continuous at any point $b \in X$.*

Proof Let the additive function $f : X \to X$ be $\mathcal{G}O$-sequentially continuous at origin. So for an orthogonal sequence $\mathbf{x} = (x_n) \in X$ such that $\mathcal{G}(f(\mathbf{x})) = 0$, whenever $\mathcal{G}(\mathbf{x}) = 0$. Let $\mathbf{x}$ be a sequence in X with $\mathcal{G} - \lim \mathbf{x} = b$ and $\mathbf{b}$ the constant sequence $\mathbf{b} = (b, b, \ldots)$. Since $\mathcal{G}$ is regular $\mathcal{G}(\mathbf{b}) = b$. Therefore the sequence $\mathbf{x} - \mathbf{b}$ is $\mathcal{G}O$-convergent to 0. So by assumption $\mathcal{G}(f(\mathbf{x} - \mathbf{b})) = 0$. Since f and $\mathcal{G}$ are additive $\mathcal{G}(f(\mathbf{x})) - \mathcal{G}(f(\mathbf{b})) = 0$. Since the constant sequence $f(\mathbf{b})$ tends to $f(b)$ and $\mathcal{G}$ is regular, $\mathcal{G}(f(\mathbf{b})) = f(b)$. Therefore we have that $\mathcal{G}(f(\mathbf{x})) = f(b)$.

Theorem 8.18 *Let $\mathcal{G}$ be a regular subsequential method. Then a subset of X is $\mathcal{G}O$-sequentially compact if and only if it is $\mathcal{G}O$-sequentially countably compact.*

Proof Let A be any $\mathcal{G}O$-sequentially compact subset of X and B be an infinite subset of A. We can choose an orthogonal sequence $\mathbf{x} = (x_n)$ of different points of B. Since A is $\mathcal{G}O$-sequentially compact, so it implies that of B that the orthogonal sequence $\mathbf{x}$ has a convergent subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ with $\mathcal{G}(\mathbf{y}) = x_0$. Since $\mathcal{G}$ is a subsequential method, $\mathbf{y}$ has a convergent subsequence $\mathbf{z} = (z_k)$ of the subsequence $\mathbf{y}$ with $\lim_k z_k = x_0$. By the regularity of $\mathcal{G}$, we obtain that x_0 is a $\mathcal{G}O$-sequential accumulation point of B. Thus A is $\mathcal{G}O$-sequentially countably compact.

Next we suppose that A is any $\mathcal{G}O$-sequentially countably compact subset of X. Let $\mathbf{x} = (x_n)$ be an orthogonal sequence of points in A. We write $P = \{x_n : n \in \mathbb{N}\}$. If P is finite, then there is nothing to prove. If P is infinite, then P has a $\mathcal{G}O$-sequential accumulation point in A. Also each set $P_n = \{x_k : k \geq n\}$, for each positive integer n, has a $\mathcal{G}O$-sequential accumulation point in A. Then the intersection $\bigcap_{n=1}^{\infty} \overline{P_n}^{\mathcal{G}O} \neq \phi$. So there is an element x_0 of A which belongs to the intersection. Since $\mathcal{G}$ is a regular subsequential method, $x_0 \in \bigcap_{n=1}^{\infty} \overline{P_n}$. Then it is not difficult to construct a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence $\mathbf{x}$ with $\mathcal{G}(\mathbf{z}) \in A$. This completes the proof.

9 Orthogonal Sequential Compactness

We consider $(X, \perp, d)$ is an orthogonal metric space.

Definition 9.1 A subset E of X is called an orthogonal sequentially compact if any sequence (x_n) in E has a $\mathcal{G}$-convergent subsequence whose limit is in E.

Definition 9.2 A subset E of X is called $\mathcal{G}O$-sequentially compact if any sequence (x_n) in E, there is subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ of (x_n) such that $\mathcal{G}(\mathbf{y}) = \lim_k y_k$ in E.

Remark 9.3 Any sequentially compact subset E of X is also $\mathcal{G}$-orthogonal sequentially compact and the converse is not always true. For this, see the following example.

Example 9.4 Let $(X = [0, 1), \perp, d)$ be an orthogonal metric space. Define the orthogonal relation $\perp$ on X by

$$x \perp y \Leftrightarrow \begin{cases} x \leq y \leq \frac{1}{2}, \\ \text{or } x = 0. \end{cases}$$

There exists a subsequence $(y_k) = (x_{n_k})$ of (x_n) for which $x_{n_k} = 0$ for all $k \geq 1$ or there exists a monotone subsequence (x_{n_k}) if (x_n) for which $x_{n_k} \leq \frac{1}{2}$ for all $k \geq 1$. We see that (x_{n_k}) is $\mathcal{G}$-convergent to a point $x \in [0, \frac{1}{2}] \subseteq X$. If we consider a subsequence $(y_k) = \left(1 - \frac{1}{k}\right)$ of (x_n), then $\lim_k y_k$ is not in $[0, \frac{1}{2}]$.

Theorem 9.5 *Every $\mathcal{G}O$-sequentially closed subset of a $\mathcal{G}O$-sequentially compact subset of X is $\mathcal{G}O$-sequentially compact.*

Proof Let A be any $\mathcal{G}O$-sequentially compact subset of X and B be a $\mathcal{G}O$-sequentially closed subset of A. Consider an orthogonal sequence $\mathbf{x} = (x_n)$ of points in B. Then $\mathbf{x}$ is a sequence of points in A. Since A is $\mathcal{G}O$-sequentially compact, there is a subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ of sequence $\mathbf{x}$ such that $\mathcal{G}(\mathbf{y}) \in A$. The subsequence $\mathbf{y}$ is also a sequence of points in B. Since B is $\mathcal{G}O$-sequentially closed, so $\mathcal{G}(\mathbf{y}) \in B$. Thus $\mathbf{x}$ has a $\mathcal{G}$-convergent subsequence, with $\mathcal{G}(\mathbf{y}) \in B$. Hence B is $\mathcal{G}O$-sequentially compact.

Theorem 9.6 *Let $\mathcal{G}$ be a regular subsequential method. Every $\mathcal{G}O$-sequentially compact subset of X is $\mathcal{G}O$-sequentially closed.*

Proof Let A be any $\mathcal{G}O$-sequentially compact subset of X. Take any $x_0 \in \overline{A}$. Then there is an orthogonal sequence $\mathbf{x} = (x_n)$ of points in A such that $\mathcal{G}(\mathbf{x}) = x_0$. Since $\mathcal{G}$ is a subsequential method, there is a subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ of the sequence $\mathbf{x}$ such that $\lim_k x_{n_k} = x_0$. Since $\mathcal{G}$ is regular, so we have $\mathcal{G}(\mathbf{y}) = x_0$. By the $\mathcal{G}O$-sequential compactness of A, there is a subsequence $\mathbf{z} = (z_k)$ of the subsequence $\mathbf{y}$ such that $\mathcal{G}(\mathbf{z}) = x_1 \in A$. Since $\lim_k z_k = x_0$ and $\mathcal{G}$ is regular, $\mathcal{G}(\mathbf{z}) = x_0$. Thus $x_0 = x_1$ and hence $x_0 \in A$. Thus A is $\mathcal{G}O$-sequentially closed.

Corollary 9.7 *Let $\mathcal{G}$ be a regular subsequential method. Then a subset of X is $\mathcal{G}O$-sequentially compact if and only if it is sequentially countably compact in the ordinary sense.*

Corollary 9.8 *Let $\mathcal{G}$ be a regular subsequential method. Then a subset of X is $\mathcal{G}O$-sequentially compact if and only if it is countably compact in the ordinary sense.*

Theorem 9.9 *Every $\mathcal{G}O$-sequential continuous image of any $\mathcal{G}O$-sequentially compact subset of X is $\mathcal{G}O$-sequentially compact.*

Proof Let f be any $\mathcal{G}O$-sequentially continuous function on X and A be any $\mathcal{G}O$-sequentially compact subset of X. Consider an orthogonal sequence $\mathbf{y} = (y_n) = (f(x_n))$ of points in $f(A)$. Since A is $\mathcal{G}O$-sequentially compact, there exists a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence $\mathbf{x} = (x_n)$ with $\mathcal{G}(\mathbf{z}) \in A$. Then the sequence $f(\mathbf{z}) = (f(z_k)) = (f(x_{n_k}))$ is a subsequence of the sequence $\mathbf{y}$. Since f is $\mathcal{G}O$-sequentially continuous, so we have $\mathcal{G}(f(\mathbf{z})) = f(z_0) \in f(A)$. Hence $f(A)$ is $\mathcal{G}O$-sequentially compact.

Corollary 9.10 *Let $\mathcal{G}$ be a regular subsequential method. Then every $\mathcal{G}O$-sequentially continuous image of any sequentially compact subset of X is sequentially compact.*

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Application of Measure of Noncompactness to the Infinite Systems of Second-Order Differential Equations in Banach Sequence Spaces c, ℓ_p, and c_0^β

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Abstract In this paper, we establish the existence of solutions of infinite systems of second-order differential equations in Banach sequence spaces by using techniques associated with measures of noncompactness in a combination of Meir–Keeler condensing operators. We illustrate our results with the help of some examples.

1 Intoduction

The theory of infinite system of ordinary differential equations is a very important branch of the theory of differential equations in Banach spaces. Infinite system of ordinary differential equations describes many real-life problems which can found in the theory of neural nets, the theory of branching processes and mechanics, etc (see [3, 4, 6]).

In the functional analysis the study of measure of noncompactness plays a very important role which was introduced by Kuratowski [7]. The idea of measure of noncompactness has been used by many authors in obtaining the existence of solutions of infinite systems of integral equations and differential equations (see [2]). Mursaleen and Mohiuddine [8] and Mohiuddine et al. [9] proved existence theorems for the infinite systems of differential equations in the space ℓ_p. On the other hand, existence theorems for the infinite systems of linear equations in ℓ_1 and ℓ_p were

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given by Alotaibi et al. [10]. Mursaleen and Alotaibi [11] proved existence theorems for the infinite system of differential equations in some BK-spaces.

1.1 Preliminaries

Let M and S are subsets of a metric space (X, d) and $\epsilon > 0$, and then the set S is called $\epsilon - net$ of M if for any $x \in M$ there exists $s \in S$, such that $d(x, s) < \epsilon$. If S is finite, then the $\epsilon - net$ S of M is called finite $\epsilon - net$. The set M is said to be totally bounded if it has a finite $\epsilon - net$ for every $\epsilon > 0$. A subset M of a metric space X is said to be compact if every sequence (x_n) in M has a convergent subsequence and the limit of that subsequence is in M. The set M is called relatively compact if the closure $\bar{M}$ of M is a compact set. If a set M is relatively compact, then M is totally bounded. If the metric space (X, d) is complete, then the set M is relatively compact if and only if it is totally bounded.

If $x \in X$ and $r > 0$, then the open ball with center at x and radius r is denoted by $B(x, r)$, where $B(x, r) = \{y \in X : d(x, y) < r\}$. If X is a normed space, then we denote by B_X the closed unit ball in X and by S_X the unit sphere in X. Let $\mathcal{M}_X$ or simply $\mathcal{M}$ be the family of all nonempty and bounded subsets of a metric space (X, d), and let $\mathcal{M}_X^c$ or simply $\mathcal{M}^c$ be the subfamily of $\mathcal{M}_X$ consisting of all closed sets. Further, let $\mathcal{N}_X$ or simply $\mathcal{N}$ be the family of all nonempty and relatively compact subsets of (X, d). Let $d_H : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be the function defined by

$$d_H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\},$$

where $A, B \in \mathcal{M}_X$. The function d_H is called the Hausdorff distance, and $d_H(A, B)$ is the Hausdorff distance of sets A, B.

Let X and Y be infinite-dimensional complex Banach spaces and denote the set of bounded linear operators from X into Y by $B(X, Y)$. We put $B(X) = B(X, X)$. For T in $B(X, Y)$, $N(T)$ and $R(T)$ denote the null space and the range space of T, respectively. A linear operator L from X to Y is called compact (or completely continuous) if $D(L) = X$ for the domain of L, and for every sequence $(x_n) \in X$ such that $\| x_n \| \leq C$, the sequence $(L(x_n))$ has a subsequence which converges in Y. A compact operator is bounded. An operator L in $B(X, Y)$ is of finite rank if $dim R(L) < \infty$. An operator of finite rank is clearly compact. Let $F(X, Y)$, $C(X, Y)$ denote the set of all finite rank and compact operators from X to Y, respectively. Set $F(X) = F(X, X)$ and $C(X) = C(X, X)$.

If F is a subset of X, then the intersection of all convex sets that contain F is called convex cover or convex hull of F denoted by $co(F)$.

Let Q be a nonempty and bounded subset of a normed space X. Then the convex closure of Q, denoted by $Co(Q)$, is the smallest convex and closed subset of X that contains Q. Note that $Co(Q) = \bar{co}(Q)$.

1.2 Kuratowski Measure of Noncompactness

Definition 1.1 ([1]) Let (X, d) be a metric space and Q a bounded subset of X. Then the Kuratowski measure of noncompactness (α-measure or set measure of noncompactness) of Q, denoted by $\alpha(Q)$, is the infimum of the set of all numbers $\epsilon > 0$ such that Q can be covered by a finite number of sets with diameters $\epsilon > 0$, that is,

$$\alpha(Q) = \inf\left\{\epsilon > 0 : Q \subset \cup_{i=1}^{n} S_i, S_i \subset X, \text{diam}(S_i) < \epsilon\ (i = 1, 2, ..., n)\ , n \in \mathbb{N}\right\}$$

The function α is called Kuratowski measure of noncompactness was introduced by Kuratowski [5]. Clearly,

$$\alpha(Q) \leq \text{diam}(Q) \text{ for each bounded subset } Q \text{ of } X.$$

1.3 Axiomatic Approach to the Concept of a Measure of Noncompactness

Suppose E is a real Banach space with the norm $\|\ .\ \|$. Let $B(x_0, r)$ be a closed ball in E centered at x_0 and with radius r. If X is a nonempty subset of E, then by $\bar{X}$ and Conv(X) we denote the closure and convex closure of X. Moreover, let $\mathcal{M}_E$ denote the family of all nonempty and bounded subsets of E and $\mathcal{N}_E$ its subfamily consisting of all relatively compact sets.

We consider the definition of the concept of a measure of noncompactness defined by Banaś and Lecko [12].

Definition 1.2 A function $\mu : \mathcal{M}_E \to [0, \infty)$ will be called a measure of noncompactness if it satisfies the following conditions:

(i) the family $\ker \mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathcal{N}_E$.
(ii) $X \subset Y \implies \mu(X) \leq \mu(Y)$.
(iii) $\mu\left(\bar{X}\right) = \mu(X)$.
(iv) $\mu(ConvX) = \mu(X)$.
(v) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
(vi) if $X_n \in \mathcal{M}_E,\ X_n = \bar{X}_n,\ X_{n+1} \subset X_n$ for $n = 1, 2, 3, ...$ and $\lim_{n\to\infty} \mu(X_n) = 0$ then $\bigcap_{n=1}^{\infty} X_n \neq \phi$.

The family $ker\mu$ is said to be the *kernel of measure* μ.

A measure μ is said to be the sublinear if it satisfies the following conditions:

(1) $\mu(\lambda X) = |\lambda|\, \mu(X)$ for $\lambda \in \mathbb{R}$.
(2) $\mu(X + Y) \leq \mu(Y) + \mu(Y)$.

A sublinear measure of noncompactness μ satisfying the condition:

$$\mu(X \cup Y) = \max\{\mu(\lambda X), \mu(\lambda Y)\}$$

and such that $ker\mu = \mathcal{N}_E$ is said to be regular.

1.4 Hausdorff Measure of Noncompactness

Definition 1.3 ([2]) Let (X, d) be a metric space, Q be a bounded subset of X and $B(x, r) = \{y \in X : d(x, y) < r\}$. Then the Hausdorff measure of noncompactness $\chi(Q)$ of Q is defined by

$$\chi(Q) := \inf\left\{\epsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i), x_i \in X, r_i < \epsilon \quad (i = 1, 2, \ldots, n), n \in \mathbb{N}\right\}.$$

The definition of the Hausdorff measure of noncompactness of the set Q is not supposed that centers of the balls that cover Q belong to Q. Hence, it can equivalently be stated as follows:

$$\chi(Q) = \inf\{\epsilon > 0 : Q \text{ has a finite } \epsilon\text{−net in } X\}.$$

Then the following results were obtained in [1, 2].

Lemma 1 ([2]) *Let Q, Q_1, and Q_2 be bounded subsets of the complete metric space (X, d). Then*

$$\chi(Q) = 0 \; if \; and \; only \; if \; \bar{Q} \; is \; compact, \tag{1.1}$$

$$\chi(Q) = \chi(\bar{Q}), \tag{1.2}$$

$$Q_1 \subset Q_2, \; implies \; \chi(Q_1) \le \chi(Q_2), \tag{1.3}$$

$$\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\}, \tag{1.4}$$

$$\chi(Q_1 \cap Q_2) \le \min\{\chi(Q_1), \chi(Q_2)\}. \tag{1.5}$$

Consider the following sequence spaces are Banach spaces with their respective norms,

$$c = \left\{x \in \omega : \lim_{k\to\infty} x_k = l, \; l \in \mathbb{C}, \| x \|_\infty = \sup_k |x_k|\right\}$$

the space of all convergent sequences;

$$\ell_p = \left\{ x \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty \; (1 \leq p < \infty), \; \| x \|_p = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{1/p} \right\}$$

the space of all absolutely $p-$ summable series.

In the Banach space $(c, \| \, . \, \|_c)$, the most convenient measure of noncompactness μ can be formulated as follows (see [2]).

$$\mu (B) = \lim_{p \to \infty} \left[\sup_{u \in B} \left\{ \sup_{k \geq n} | \, u_k - \lim_{m \to \infty} u_m \, | \right\} \right]$$

where $u(t) = (u_i(t))_{i=1}^{\infty} \in c$ for each $t \in [0, T]$ and $B \in \mathcal{M}_c$. The measure μ is regular.

Banaś and Mursaleen [2] defined the Hausdorff measure of noncompactness χ for the Banach space $\left(\ell_p, \| \, . \, \|_{\ell_p}\right)$, $(1 \leq p < \infty)$ as follows.

$$\chi (B) = \lim_{n \to \infty} \left[\sup_{u \in B} \left(\sum_{k=n}^{\infty} | \, u_k \, |^p \right)^{1/p} \right]$$

where $u(t) = (u_i(t))_{i=1}^{\infty} \in \ell_p$ for each $t \in [0, T]$ and $B \in \mathcal{M}_{\ell_p}$.

Let us fix a positive nonincreasing real sequence $\beta = (\beta_n)_{n=1}^{\infty}$ such a sequence is called the tempering sequence. Let the set X consisting of all real (or complex) sequences $x = (x_n)_{n=1}^{\infty}$ such that $\beta_n x_n \to 0$ as $n \to \infty$. It is obvious that X forms a linear space over the field of real (or complex) numbers. We will denote the space by the symbol c_0^{β}.It is easy to see that c_0^{β} is a Banach space under the norm

$$\| \, x \, \|_{c_0^{\beta}} = \sup_{n \in \mathbb{N}} \{\beta_n \, |x_n|\} \, .$$

In fact, the Hausdorff measure of noncompactness $\chi(X)$ for $X \in \mathcal{M}_{c_0^{\beta}}$ can be expressed as follows [1]:

$$\chi(X) = \lim_{n \to \infty} \left\{ \sup_{(x_i) \in X} \left[\sup_{i \geq n} (\beta_i \, |x_i|) \right] \right\} .$$

1.5 Condensing Operators, Compact Operators, and Related Results

Definition 1.4 Let μ_1 and μ_2 be measures of noncompactness on Banach spaces E and F, respectively. Let $L : E \to F$ be an operator. Then

(a) L is called (μ_1, μ_2)-contractive operator with constant $k > 0$ (or simply k-(μ_1, μ_2)-contractive) if L is continuous and

$$\mu_2(L(Q)) \leq k\mu_1(Q) \text{ for each } Q \in \mathcal{M}_E.$$

In particular, if $E = F$ and $\mu_1 = \mu_2 = \mu$, then we say that L is a k-μ-contractive operator.

(b) L is called (μ_1, μ_2)-condensing operator with constant $k > 0$ (or simply k-(μ_1, μ_2)-condensing) if L is continuous and

$$\mu_2(L(Q)) < k\mu_1(Q) \text{ for each non} - \text{precompact } Q \in \mathcal{M}_E.$$

In particular, if $E = F$ and $\mu_1 = \mu_2 = \mu$, then we say that L is a k-μ-condensing operator. Moreover, if $k = 1$, we say that L is a μ-condensing operator.

If an operator L is (μ_1, μ_2)-contractive, then the number $\| L \|_{\mu_1,\mu_2}$ defined by

$$\| L \|_{\mu_1,\mu_2} = \inf\{k \geq 0 : \mu_2(L(Q)) \leq k\mu_1(Q) \; for \; each \; Q \in \mathcal{M}_E\} \tag{1.6}$$

is called (μ_1, μ_2)-operator norm of L, or (μ_1, μ_2)-measure of noncompactness of L, or simply measures of noncompactness of L. If $\mu_1 = \mu_2 = \mu$, then we write $\| L \|_\mu$ instead of $\| L \|_{\mu,\mu}$ which we call as the μ-norm of L.

In infinite-dimensional spaces E and F, for any arbitrary measure of noncompactness μ, $\| L \|_\mu$ may be expressed by the equivalent formula

$$\| L \|_\mu = \sup\left\{\frac{\mu(L(Q))}{\mu(Q)} : Q \in \mathcal{M}_E, \; \mu(Q) > 0\right\}. \tag{1.7}$$

Definition 1.5 Let X and Y be Banach spaces and χ_1 and χ_2 be Hausdorff measures on X and Y. Then, the operator $L : X \to Y$ is called $(\chi_1, \chi_2) - bounded$ if $L(Q)$ is bounded subset of Y for every $Q \in \mathcal{M}_X$ and there exists a positive constant K such that $\chi_2(L(Q)) \leq K\chi_1(Q)$ for every $Q \in \mathcal{M}_X$. If an operator L is $(\chi_1, \chi_2)-$bounded, then the number $\| L \|_{(\chi_1,\chi_2)} = \inf\{K \geq 0 : \chi_2(L(Q)) \leq K\chi_1(Q)$ for all bounded $Q \subset X\}$ is called (χ_1, χ_2)-measure of noncompactness of L. In particular, if $\chi_1 = \chi_2 = \chi$, then we write $\| L \|_{(\chi,\chi)} = \| L \|_\chi$.

The idea of compact operators between Banach spaces is closely related to the Hausdorff measure of noncompactness, and it can be given, as follows. Let X and Y be Banach spaces and $L \in B(X, Y)$. Then, the Hausdorff measure of noncompactness $\|L\|_\chi$ of L can be given by $\|L\|_\chi = \chi(L(S_X))$, where $S_X = \{x \in X : \|x\| = 1\}$ and we have L is compact if and only if $\|L\|_\chi = 0$. We also have $\|L\| = \sup_{x \in S_X} \|Lx\|_Y$.

Definition 1.6 ([13]) Let E_1 and E_2 be two Banach spaces and let μ_1 and μ_2 be arbitrary measure of noncompactness on E_1 and E_2, respectively. An operator f

from E_1 to E_2 is called a (μ_1, μ_2) −condensing operator if it is continuous and $\mu_2\,(f(D)) < \mu_1(D)$ for every set $D \subset E_1$ with compact closure.

Remark 1.7 If $E_1 = E_2$ and $\mu_1 = \mu_2 = \mu$, then f is called a $\mu-$ condensing operator.

Theorem 1.8 ([14]) *Let Ω be a nonempty, closed, bounded, and convex subset of a Banach space E and let $f : \Omega \to \Omega$ be a continuous mapping such that there exists a constant $k \in [0, 1)$ with the property $\mu_2\,(f(\Omega)) < k\mu_1(\Omega)$. Then f has a fixed point in Ω.*

Definition 1.9 ([15]) Let (X, d) be a metric space. Then a mapping T on X is said to be a Meir–Keeler contraction if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq d(x, y) < \epsilon + \delta \implies d\,(Tx, Ty) < \epsilon, \forall x, y \in X.$$

Theorem 1.10 ([15]) *Let (X, d) be a complete metric space. If $T : X \to X$ is a Meir–Keeler contraction, then T has a unique fixed point.*

Definition 1.11 ([16]) Let C be a nonempty subset of a Banach space E and let μ be an arbitrary measure of noncompactness on E. We say that an operator $T : C \to C$ is a Meir–Keeler condensing operator if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq \mu\,(X) < \epsilon + \delta \implies \mu\,(T\,(X)) < \epsilon$$

for any bounded subset X of C.

Theorem 1.12 ([16]) *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let μ be an arbitrary measure of noncompactness on E. If $T : C \to C$ is a continuous and Meir–Keeler condensing operator, then T has at least one fixed point and the set of all fixed points of T in C is compact.*

We study an infinite system of second-order differential equation by transforming the system into an infinite system of integral equation with the help of Green's function (see [17]).

Consider the infinite systems of second-order differential equations

$$u_i^{''}(t) = -f_i\,(t, u_1, u_2, u_3, ...) \tag{1.8}$$

where $u_i(0) = u_i(T) = 0,\ t \in [0, T]$ and $i = 1, 2, 3, ...$

Let $C(I, \mathbb{R})$ denote the space of all continuous real functions on the interval $I = [0, T]$ and let $C^2(I, \mathbb{R})$ be the class of functions with second continuous derivatives on I. A function $u \in C^2(I, \mathbb{R})$ is a solution of (1.8) if and only if $u \in C(I, \mathbb{R})$ is a solution of the infinite systems of integral equation

$$u_i(t) = \int_0^T G(t, s) f_i(s, u(s))ds, \tag{1.9}$$

where $f_i(t,u) \in C(I,\mathbb{R}),\ i = 1, 2, 3, ...$ and $t \in I$ and Green's function associated with (1.8) is given by (see [17, 18])

$$G(t,s) = \begin{cases} \frac{t}{T}(T-s), & 0 \le t \le s \le T, \\ \frac{s}{T}(T-t), & 0 \le s \le t \le T. \end{cases} \tag{1.10}$$

The solution of the infinite systems (1.8) in the sequence space ℓ_1 has been studied by Aghajani and Pourhadi [19], and in the sequence spaces, c_0 and ℓ_1 have been studied by Mursaleen and Rizvi [18]. In this article, we establish the existence of solution of the infinite systems (1.8) for the sequence spaces c, $\ell_p\ (1 < p < \infty)$ and c_0^{β}.

2 Solvability of Infinite Systems of Second-Order Differential Equations in *c*

Suppose that

(i) The functions f_i are defined on the set $I \times \mathbb{R}^{\infty}$ and take real values. The operator f defined on the space $I \times c$ into c as

$$(t,u) \to (fu)(t) = (f_1(t,u),\ f_2(t,u),\ f_3(t,u), ...)$$

is such that the class of all functions $((fu)(t))_{t\in I}$is equicontinuous at every point of the space c.

(ii) The following condition hold:

$$f_n(t, u(t)) = p_n(t, u(t)) + q_n(t)u_n(t)$$

where for all $n \in \mathbb{N}$ both $p_n(t,u(t))$ and $q_n(t)$ are real functions and continuous defined on $I \times c$ and I, respectively. Also, there exists a sequence $\{P_n\}$ converges to zero with $|p_n(t,u(t))| \le P_n$ for any $t \in I$, $u(t) \in c$ and the sequence $(q_n(t))$ is uniformly convergent on I.

Let us assume

$$P = \sup_{n\in\mathbb{N}} \{P_n\}$$
$$Q = \sup_{t\in I, n\in\mathbb{N}} \{q_n(t)\}$$

such that $QT^2 < 1$.

Theorem 2.1 *Under the hypothesis (i)–(ii), infinite systems (1.8) has at least one solution* $u(t) = (u_i(t)) \in c$ *for all* $t \in [0,T]$.

Proof We have $\sup_{i\in\mathbb{N}} \mid u_i(t) \mid \leq M$ where M is a finite positive real number for all $u(t) = (u_i(t)) \in c$ and $t \in I$.

By using (1.9) and (ii), we have for all $t \in I$,

$$\begin{aligned}
&\| u(t) \|_c \\
&= \sup_{k\geq 1} \left| \int_0^T G(t,s) f_k(s,u(s)) ds \right| \\
&\leq \sup_{k\geq 1} \int_0^T |G(t,s) f_k(s,u(s))| \, ds \\
&\leq \sup_{k\geq 1} \int_0^T G(t,s) \{|p_k(s,u(s))| + |q_k(s)| \mid u_k(s) \mid\} \, ds \\
&\leq \sup_{k\geq 1} \left\{ \int_0^T T(P+QM) ds \right\} \\
&= (P+QM)T^2 = r(say) \\
&i.e. \ \| u(t) \|_c \leq r.
\end{aligned}$$

Let $u^0(t) = \left(u_i^0(t)\right)$ where $u_i^0(t) = 0 \, \forall \, t \in I, i \in \mathbb{N}$.

Consider $\bar{B} = \bar{B}\left(u^0(t), r\right)$, the closed ball centered at $u_0(t)$ and radius r; thus, $\bar{B}$ is an nonempty, bounded, closed, and convex subset of c. Consider the operator $\mathcal{F} = (\mathcal{F}_i)$ on $C\left(I, \bar{B}\right)$ defined as follows.

For $t \in I$,

$$(\mathcal{F}u)(t) = \{(\mathcal{F}_i u)(t)\} = \left\{ \int_0^T G(t,s) f_i(s,u(s)) ds \right\}$$

where $u(t) = (u_i(t)) \in \bar{B}$ and $u_i(t) \in C(I, \mathbb{R})$.

Since $(f_i\ (t, u(t))) \in c$ for each $t \in I$, we have,

$$\lim_{i\to\infty} (\mathcal{F}_i u)(t) = \lim_{i\to\infty} \int_0^T G(t,s) f_i(s,u(s)) ds = \int_0^T G(t,s) \lim_{i\to\infty} f_i(s,u(s)) ds$$

is finite and unique. Hence, $(\mathcal{F}u)(t) \in c$.

Also, $(\mathcal{F}_i u)(t)$ satisfies boundary conditions, i.e.,

$$\begin{aligned}
(\mathcal{F}_i u)(0) &= \int_0^T G(0,s) f_i(s,u(s)) ds = \int_0^T 0. f_i(s,u(s)) ds = 0 \\
(\mathcal{F}_i u)(T) &= \int_0^T G(T,s) f_i(s,u(s)) ds = \int_0^T 0. f_i(s,u(s)) ds = 0
\end{aligned}$$

Since $\| (\mathcal{F}u)(t) - u^0(t) \|_c \leq r$, thus $\mathcal{F}$ is self-mapping on $\bar{B}$.

The operator $\mathcal{F}$ is continuous on $C\left(I, \bar{B}\right)$ by the assumption (i). Now, we shall show that $\mathcal{F}$ is a Meir–Keeler condensing operator.

For $\epsilon > 0$, we need to find $\delta > 0$ such that $\epsilon \leq \mu\left(\bar{B}\right) < \epsilon + \delta \implies \mu\left(\mathcal{F}\bar{B}\right) < \epsilon$.

We have

$$\begin{aligned}
&\mu\left(\mathcal{F}\bar{B}\right)\\
&= \lim_{p\to\infty}\left[\sup_{u(t)\in\bar{B}}\left\{\sup_{k\geq p}\left|\int_0^T G(t,s)f_k(s,u(s))ds - \lim_{m\to\infty}\int_0^T G(t,s)f_m(s,u(s))ds\right|\right\}\right]\\
&\leq QT\lim_{p\to\infty}\left[\sup_{u(t)\in\bar{B}}\left\{\sup_{k\geq p}\int_0^T\left|u_k(s) - \lim_{m\to\infty}u_m(s)\right|ds\right\}\right].
\end{aligned}$$

i.e., $\mu\left(\mathcal{F}\bar{B}\right) \leq QT^2\mu\left(\bar{B}\right)$

Hence, $\mu\left(\mathcal{F}\bar{B}\right) \leq T^2Q\mu(\bar{B}) < \epsilon \implies \mu\left(\bar{B}\right) < \frac{\epsilon}{T^2Q}$.

Taking $\delta = \frac{\epsilon}{T^2Q}\left(1 - T^2Q\right)$ we get $\epsilon \leq \mu\left(\bar{B}\right) < \epsilon + \delta$. Therefore, $\mathcal{F}$ is a Meir–Keeler condensing operator defined on the set $\bar{B} \subset c$. So $\mathcal{F}$ satisfies all the conditions of Theorem 1.8 which implies $\mathcal{F}$ has a fixed point in $\bar{B}$. Thus, the system (1.8) has a solution in c. □

3 Solvability of Infinite Systems of Second-Order Differential Equations in ℓ_p $(1 < p < \infty)$

Assume that

(i) The functions f_i are defined on the set $I \times \mathbb{R}^\infty$ and take real values. The operator f defined on the space $I \times \ell_p$ into ℓ_p as

$$(t, u) \to (fu)(t) = (f_1(t, u), f_2(t, u), f_3(t, u), ...)$$

is such that the class of all functions $((fu)(t))_{t\in I}$is equicontinuous at every point of the space ℓ_p.

(ii) The following inequality hold:

$$\mid f_n(t, u(t)) \mid^p \leq g_n(t) + h_n(t) \mid u_n(t) \mid^p$$

where $g_n(t)$ and $h_n(t)$ are real functions defined on I, such that $\sum_{k\geq 1} g_k(t)$ converges uniformly on I and the sequence $(h_n(t))$ is equibounded on I.

Let us assume that

$$G = \sup_{t\in I}\left\{\sum_{k\geq 1} g_k(t)\right\}$$

and

$$H = \sup_{n \in \mathbb{N}, t \in I} \{h_n(t)\}$$

such that $T^2 H^{1/p} < 1$.

Theorem 3.1 *Under the hypothesis (i)–(ii), infinite systems (1.8) has at least one solution $u(t) = (u_i(t)) \in \ell_p$ for all $t \in [0, T]$.*

Proof By using (1.9) and (ii), we have for all $t \in I$,

$$\begin{aligned}
&\| u(t) \|_{\ell_p}^p \\
&= \sum_{i=1}^{\infty} \left| \int_0^T G(t,s) f_i(s, u(s)) ds \right|^p \\
&\le T^{p/q} \sum_{i=1}^{\infty} \int_0^T |G(t,s)|^p \left\{ g_i(s) + h_i(s) \, |u_i(s)|^p \right\} ds, \ \ \frac{1}{p} + \frac{1}{q} = 1
\end{aligned}$$

Since $u(t) \in \ell_p$, therefore there exists a finite real number $\bar{M}$ such that $\sum_{i=1}^{\infty} |u_i(t)|^p < \bar{M}$.

Hence, we get

$$\| u(t) \|_{\ell_p}^p \le T^{\frac{p}{q}+p+1} \left(G + H\bar{M} \right) = T^{2p} \left(G + H\bar{M} \right) = r_1^p (say)$$

i.e., $\| u(t) \|_{\ell_p} \le r_1$.

Let $u^0(t) = \left(u_i^0(t) \right)$ where $u_i^0(t) = 0 \, \forall \, t \in I, i \in \mathbb{N}$.

Consider $B_1 = B_1 \left(u^0, r_1 \right)$, the closed ball centered at $u_0(t)$ and radius r_1; thus, B_1 is an nonempty, bounded, closed, and convex subset of ℓ_p. Consider the operator $\mathcal{F} = (\mathcal{F}_i)$ on $C(I, B_1)$ defined as follows.

For $t \in I$,

$$(\mathcal{F}u)(t) = \{(\mathcal{F}_i u)(t)\} = \left\{ \int_0^T G(t,s) f_i(s, u(s)) ds \right\},$$

where $u(t) = (u_i(t)) \in B_1$ and $u_i(t) \in C(I, \mathbb{R})$.

We have that $(\mathcal{F}u)(t) = \{(\mathcal{F}_i u)(t)\} \in \ell_p$ for each $t \in I$. Since $(f_i(t, u(t))) \in \ell_p$ for each $t \in I$, we have,

$$\sum_{i=1}^{\infty} |(\mathcal{F}_i u)(t)|^p = \sum_{i=1}^{\infty} \left| \int_0^T G(t,s) f_i(s, u(s)) ds \right|^p \le r_1^p < \infty.$$

Also, $(\mathcal{F}_i u)(t)$ satisfies boundary conditions, i.e.,

$$(\mathcal{F}_i u)(0) = \int_0^T G(0,s) f_i(s,u(s))ds = \int_0^T 0.f_i(s,u(s))ds = 0$$

and

$$(\mathcal{F}_i u)(T) = \int_0^T G(T,s) f_i(s,u(s))ds = \int_0^T 0.f_i(s,u(s))ds = 0$$

Since $\| (\mathcal{F}u)(t) - u^0(t) \|_{\ell_p} \leq r_1$, thus $\mathcal{F}$ is self-mapping on B_1.

The operator $\mathcal{F}$ is continuous on $C(I, B_1)$ by the assumption (i). Now, we shall show that $\mathcal{F}$ is a Meir–Keeler condensing operator.

For $\epsilon > 0$, we need to find $\delta > 0$ such that $\epsilon \leq \chi(B_1) < \epsilon + \delta \implies \chi(\mathcal{F}B_1) < \epsilon$.

We have

$$\begin{aligned}
&\chi(\mathcal{F}B_1)\\
&= \lim_{n\to\infty}\left[\sup_{u(t)\in B_1}\left\{\sum_{k\geq n}\left|\int_0^T G(t,s) f_k(s,u(s))ds\right|^p\right\}^{\frac{1}{p}}\right]\\
&\leq T^{\frac{1}{q}+1}\lim_{n\to\infty}\left[\sup_{u(t)\in B_1}\left\{\sum_{k\geq n}\int_0^T \left(g_k(s)+h_k(s)\left|u_k(s)\right|^p\right)ds\right\}^{\frac{1}{p}}\right]\\
&\leq T^{\frac{1}{q}+1}\lim_{n\to\infty}\left[\sup_{u(t)\in B_1}\left\{\int_0^T \left(\sum_{k\geq n} g_k(s) + H\sum_{k\geq n}\left|u_k(s)\right|^p\right)ds\right\}^{\frac{1}{p}}\right]
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\chi(\mathcal{F}B_1)\\
&\leq H^{1/p}T^{1+1/p+1/q}\chi(B_1)\\
&= H^{1/p}T^2\chi(B_1).
\end{aligned}$$

Hence, $\chi(\mathcal{F}B_1) \leq H^{1/p}T^2\chi(B_1) < \epsilon \implies \chi(B_1) < \frac{\epsilon}{H^{1/p}T^2}$.

Taking $\delta = \frac{(1-H^{1/p}T^2)}{H^{1/p}.T^2}\epsilon$, we get $\epsilon \leq \chi\left(\bar{B}\right) < \epsilon + \delta$. Therefore, $\mathcal{F}$ is a Meir–Keeler condensing operator defined on the set $B_1 \subset \ell_p$. So $\mathcal{F}$ satisfies all the conditions of Theorem 1.8 which implies $\mathcal{F}$ has a fixed point in B_1. Thus, the system (1.8) has a solution in ℓ_p. □

4 Solvability of Infinite Systems of Second-Order Differential Equations in c_0^β

We assume that

(i) The functions f_i are defined on the set $I \times \mathbb{R}^\infty$ and take real values. The operator f defined on the space $I \times c_0^\beta$ into c_0^β as

$$(t, u) \to (fu)(t) = (f_1(t, u), f_2(t, u), f_3(t, u), ...)$$

is such that the class of all functions $((fu)(t))_{t\in I}$ is equicontinuous at every point of the space c_0^β.

(ii) The following inequality holds for all $n \in \mathbb{N}$, $t \in I$ and $u(t) \in c_0^\beta$,

$$| f_n(t, u(t)) | \leq \bar{p}_n(t) + \bar{q}_n(t) | u_n(t) |$$

where $\bar{p}_n(t)$ and $\bar{q}_n(t)$ are real functions defined and continuous on I, such that $(\beta_n \bar{p}_n(t))$ converges uniformly to zero on I and the sequence $(\bar{q}_n(t))$ is equibounded on I.

Let us consider

$$\bar{P} = \sup_{t\in I, n\in\mathbb{N}} \{\beta_n \bar{p}_n(t)\}$$
$$\bar{Q} = \sup_{n\in\mathbb{N}, t\in I} \{\bar{q}_n(t)\}$$

such that $\bar{Q}T^2 < 1$.

Theorem 4.1 *Under the hypothesis (i)–(ii), infinite systems (1.8) has at least one solution $u(t) = (u_i(t)) \in c_0^\beta$ for all $t \in [0, T]$.*

Proof We have $\sup_{n\in\mathbb{N}} \{\beta_n |u_n(t)|\} \leq M_1$ where M_1 is a finite positive real number for all $u(t) = (u_i(t)) \in c_0^\beta$ and $t \in I$.

By using (1.9) and (ii), we have for all $t \in I$,

$$\begin{aligned}
&\| u(t) \|_{c_0^\beta} \\
&= \sup_{k\geq 1} \left[\beta_k \left| \int_0^T G(t, s) f_k(s, u(s)) ds \right| \right] \\
&\leq \sup_{k\geq 1} \left[\beta_k \int_0^T G(t, s) \{\bar{p}_k(s) + \bar{q}_k(s) | u_k(s) |\} ds \right] \\
&= \sup_{k\geq 1} \left[\int_0^T G(t, s) \{\beta_k \bar{p}_k(s) + \beta_k \bar{q}_k(s) | u_k(s) |\} ds \right]
\end{aligned}$$

$$\leq \sup_{k\geq 1}\left[\int_0^T G(t,s)\left\{\bar{P}+\bar{Q}M_1\right\}ds\right]$$
$$\leq (\bar{P}+\bar{Q}M_1)T^2 = r_2(say)$$

Let $u^0(t) = \left(u_i^0(t)\right)$ where $u_i^0(t) = 0\,\forall\, t \in I, i \in \mathbb{N}$.

Consider $B_2 = B_2\left(u^0(t), r_2\right)$, the closed ball centered at $u_0(t)$ and radius r_2; thus, B_2 is an nonempty, bounded, closed, and convex subset of c_0^β. Consider the operator $\mathcal{F} = (\mathcal{F}_i)$ on $C\,(I, B_2)$ defined as follows.

For $t \in I$,

$$(\mathcal{F}u)\,(t) = \{(\mathcal{F}_i u)\,(t)\} = \left\{\int_0^T G(t,s) f_i(s, u(s))ds\right\}$$

where $u(t) = (u_i(t)) \in B_2$ and $u_i(t) \in C(I, \mathbb{R})$.

We have that $(\mathcal{F}u)\,(t) = \{(\mathcal{F}_i u)\,(t)\} \in c_0^\beta$ for each $t \in I$. Since $(f_i\,(t, u(t))) \in c_0^\beta$ for each $t \in I$, we have

$$\lim_{n\to\infty}[\beta_n\,(\mathcal{F}_n u)\,(t)] = \lim_{n\to\infty}\left[\int_0^T G(t,s) f_n(s,u(s))ds\right] = \lim_{n\to\infty}\left[\int_0^T G(t,s)\beta_n f_n(s,u(s))ds\right] = 0.$$

Also, $(\mathcal{F}_i u)\,(t)$ satisfies boundary conditions, i.e.,

$$(\mathcal{F}_i u)\,(0) = \int_0^T G(0,s) f_i(s,u(s))ds = \int_0^T 0.f_i(s,u(s))ds = 0,$$
$$(\mathcal{F}_i u)\,(T) = \int_0^T G(T,s) f_i(s,u(s))ds = \int_0^T 0.f_i(s,u(s))ds = 0$$

Since $\|\,(\mathcal{F}u)\,(t) - u^0(t)\,\|_{c_0^\beta} \leq r_2$ thus $\mathcal{F}$ is self mapping on B_2.

The operator $\mathcal{F}$ is continuous on $C\,(I, B_2)$ by the assumption (i). Now, we shall show that $\mathcal{F}$ is a Meir–Keeler condensing operator.

For $\epsilon > 0$, we need to find $\delta > 0$ such that $\epsilon \leq \chi\,(B_2) < \epsilon + \delta \implies \chi\,(\mathcal{F}B_2) < \epsilon$.

We have

$$\chi\,(\mathcal{F}B_2)$$
$$= \lim_{n\to\infty}\left[\sup_{u(t)\in B_2}\left\{\sup_{k\geq n}\left(\beta_k\left|\int_0^T G(t,s) f_k(s,u(s))ds\right|\right)\right\}\right]$$
$$\leq \lim_{n\to\infty}\left[\sup_{u(t)\in B_2}\left\{\sup_{k\geq n}\left(\beta_k\int_0^T G(t,s)\left(\bar{p}_k(s) + \bar{q}_k(s)\,|u_k(s)|\right)ds\right)\right\}\right]$$

$$\leq T\bar{Q} \lim_{n\to\infty}\left[\sup_{u(t)\in B_2}\left\{\sup_{k\geq n}\left(\int_0^T (\beta_k |u_k(s)|)\, ds\right)\right\}\right]$$

i.e $\chi(\mathcal{F}B_2) \leq \bar{Q}T^2\chi(B_2)$.

Hence $\chi(\mathcal{F}B_2) \leq \bar{Q}T^2\chi(B_2) < \epsilon \implies \chi(B_2) < \frac{\epsilon}{\bar{Q}T^2}$.

Taking $\delta = \frac{\epsilon}{\bar{Q}T^2}\left(1 - \bar{Q}T^2\right)$, we get $\epsilon \leq \chi(B_2) < \epsilon + \delta$. Therefore, $\mathcal{F}$ is a Meir–Keeler condensing operator defined on the set $B_2 \subset c_0^\beta$. So $\mathcal{F}$ satisfies all the conditions of Theorem 1.8 which implies $\mathcal{F}$ has a fixed point in B_2. Thus, the system (1.8) has a solution in c_0^β. □

5 Illustrative Example

In this section, we illustrate our results with the help of examples.

Example 5.1 Let us consider the following system of second-order differential equations

$$-\frac{d^2u_i(t)}{dt^2} = f_i(t, u(t)) \tag{5.1}$$

with $u_i(0) = u_i(1) = 0$, where $f_i(t, u(t)) = \frac{t}{i^2} + \sum_{m=1}^{i} \frac{u_i(t)}{2m^2}\ \forall i \in \mathbb{N},\ t \in I = [0, 1]$.

If $u(t) \in c$, then

$$\lim_{i\to\infty} f_i(t, u(t)) = \lim_{i\to\infty}\left[\frac{t}{i^2} + \sum_{m=1}^{i}\frac{u_i(t)}{2m^2}\right]$$

is finite and unique. Therefore, $(f_i(t, u(t))) \in c$.

Let $\epsilon > 0$ arbitrary and $z(t) \in c$ with $\| u(t) - z(t) \|_c \leq \delta = \frac{6\epsilon}{\pi^2}$. Then we get

$$\begin{aligned}
&| f_i(t, u(t)) - f_i(t, z(t)) | \\
&= \left|\sum_{m=1}^{i}\frac{u_i(t) - z_i(t)}{2m^2}\right| \\
&\leq \sum_{m=1}^{i}\frac{1}{2m^2}|u_i(t) - z_i(t)| \\
&\leq \delta\sum_{m=1}^{i}\frac{1}{2m^2}
\end{aligned}$$

$$< \delta.\frac{\pi^2}{12} < \epsilon \text{ for any fixed } i.$$

Therefore, $(f_i\ (t, y(t)))_{t\in I}$ is equicontinuous on c. Again, we have

$$\begin{aligned}
&f_i\ (t, u(t)) \\
&= \frac{t}{i^2} + \sum_{m=1}^{i} \frac{u_i(t)}{m^2} \\
&= \frac{t}{i^2} + u_i(t) \sum_{m=1}^{i} \frac{1}{m^2} \\
&= p_i(t, u(t)) + q_i(t)u_i(t),
\end{aligned}$$

where $p_i(t, u(t)) = \frac{t}{i^2}$, $P_i = \frac{1}{i^2}$ and $q_i(t) = \sum\limits_{m=1}^{i} \frac{1}{2m^2}$ defined and $(p_i(t, u(t)))$ and $q_i(t)$ are continuous and P_i converges to zero and $\{q_i(t)\}$ is uniformly convergent on I. Also, we have $P = 1$, $Q = \frac{\pi^2}{12}$ and $T = 1$. Thus, $QT^2 = \frac{\pi^2}{12} < 1$. Hence, by Theorem 2.1, the system (5.1) has unique solution in c.

Example 5.2 Let us consider the following system of second-order differential equations

$$-\frac{d^2u_i(t)}{dt^2} = f_i\ (t, u(t)) \tag{5.2}$$

with $u_i(0) = u_i(1) = 0$, where $f_i\ (t, u(t)) = \frac{e^t cos(t)u_i(t)}{4i^2}\ \forall\ i \in \mathbb{N},\ t \in I = [0, 1]$.

We have $\sum\limits_{k=1}^{\infty} |f_k\ (t, u(t))|^p \le e^p \sum\limits_{k=1}^{\infty} |u_k(t)|^p < \infty$ if $u(t) = (u_i(t)) \in \ell_p$ where $1 < p < \infty$, i.e., $(f_i\ (t, u(t))) \in \ell_p$.

Let us consider a positive arbitrary real number $\epsilon > 0$ and $z(t) \in \ell_p$. Taking $z(t) \in \ell_p$ with $\| u(t) - z(t) \|_{\ell_p} < \delta = \left(\frac{\epsilon}{e^p}\right)^{1/p}$, then

$$\begin{aligned}
&|f_i(t, u(t)) - f_i(t, z(t))|^p \\
&= \left| \frac{e^t cos(t)u_i(t)}{4i^2} - \frac{e^t cos(t)z_i(t)}{4i^2} \right|^p \\
&\le e^p \ \| u(t) - z(t) \|_{\ell_p}^p \\
&i.e. |f_i(t, u(t)) - f_i(t, z(t))| < \epsilon
\end{aligned}$$

which implies the equicontinuity of $((fu)(t))_{t\in I}$ on ℓ_p.

Again, we have for all $i \in \mathbb{N}$ and $t \in I$,

$$|f_i(t, u(t))|^p \le \frac{e^{pt}}{4^p} |u_i(t)|^p = g_i(t) + h_i(t)\, |u_i(t)|^p$$

where $g_i(t) = 0$ and $h_i(t) = \frac{e^{pt}}{4^p}$ are real functions on I and $\sum\limits_{k\geq 1} g_k(t)$ converges uniformly on I and the sequence $\{h_i(t)\}$ is equibounded on I. Also, we have $G = 0$, $H = \frac{e^p}{4^p}$ and $T^2 H^{1/p} = \frac{e}{4} < 1$. Thus, by Theorem 3.1, the system (5.2) has unique solution in ℓ_p.

Example 5.3 Let us consider the following system of second-order differential equations

$$-\frac{d^2u_i(t)}{dt^2} = f_i\,(t, u(t)) \tag{5.3}$$

with $u_i(0) = u_i(1) = 0$, where $f_i\,(t, u(t)) = \frac{t^2}{i^2} + \sum_{m=1}^{i} \frac{u_i(t)}{2m^2}$ $\forall\ i \in \mathbb{N}$, $t \in I = [0, 1]$.

Let $\beta_i = \frac{1}{i^2}$ for all $i \in \mathbb{N}$.

If $u(t) \in c_0^\beta$, then we have for any $t \in I$ that

$$\lim_{i\to\infty} \beta_i f_i\,(t, u(t)) = \lim_{i\to\infty} \left(\frac{t^2}{i^4} + \frac{1}{i^2} \sum_{m=1}^{i} \frac{u_i(t)}{2m^2} \right) = 0.$$

Thus, if $u(t) = (u_i(t)) \in c_0^\beta$, i.e., $(f_i\,(t, u(t))) \in c_0^\beta$.

Let us consider a positive arbitrary real number $\epsilon > 0$ and $z(t) \in c_0^\beta$ where $z(t) = (z_i(t))_{i=1}^\infty$. with $\| u(t) - z(t) \|_{c_0^\beta} < \delta = \frac{12\epsilon}{\pi^2}$, then we have

$$\begin{aligned}
&\beta_i\,|f_i(t, u(t)) - f_i(t, z(t))| \\
&= \frac{1}{i^2} \left| \sum_{m=1}^{i} \left\{ \frac{u_i(t)}{2m^2} - \frac{z_i(t)}{2m^2} \right\} \right| \\
&\leq \frac{1}{i^2}\,|u_i(t) - z_i(t)| \sum_{m=1}^{i} \frac{1}{2m^2} \\
&\leq \frac{\pi^2}{12i^2}\,|u_i(t) - z_i(t)| \\
&\leq \frac{\pi^2}{12}\,\| u(t) - z(t) \|_{c_0^\beta} \\
&i.e.\ \beta_i\,|f_i(t, u(t)) - f_i(t, z(t))| < \epsilon
\end{aligned}$$

which implies the equicontinuity of $((fu)(t))_{t\in I}$ on c_0^β.

Again, we have for all $i \in \mathbb{N}$ and $t \in I$,

$$|f_i(t, u(t))| \leq \frac{t^2}{i^2} + |u_i(t)| \sum_{m=1}^{i} \frac{1}{2m^2} \leq \frac{t^2}{i^2} + \frac{\pi^2}{12}\,|u_i(t)|$$

where $\bar{p}_i(t) = \frac{t^2}{i^2}$ and $\bar{q}_i(t) = \frac{\pi^2}{12}$ are real continuous functions on I and $(\beta_i \bar{p}_i(t)) = \left(\frac{t^2}{i^4}\right)$ converges uniformly to zero on I, and the sequence $\{\bar{q}_i(t)\}$ is equibounded on I. We also have $\bar{P} = 1$, $\bar{Q} = \frac{\pi^2}{12}$ and $\bar{Q}T^2 = \frac{\pi^2}{12} < 1$. Thus, by Theorem 4.1, the system (5.3) has unique solution in c_0^β.

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Infinite Systems of Differential Equations in Banach Spaces Constructed by Fibonacci Numbers

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Abstract In the present paper, we investigate the existence theorem for the Cauchy problem

$$x^{'} = g(t, x), \quad x(0) = x_0$$

in some Banach spaces derived by Fibonacci numbers. For this purpose, we use the Hausdorff measure of noncompactness. Also, we give an example of infinite system of differential equations which has a solution in these spaces but has no solution in the classical Banach sequence spaces c_0 and ℓ_p.

Keywords Differential equations · Fibonacci numbers · Banach spaces Hausdorff measure of noncompactness

2010 Mathematics Subject Classification 34A34 · 11B39 · 34G20

1 Introduction and Preliminaries

The theory of infinite systems of differential equations is an important branch of functional analysis and applied mathematics since it has many applications in the theory of neural nets, branching processes, the theory of dissociation of polymers and so on (cf. [1–4]). Also, several problems in mechanics which lead to infinite systems of differential equations have been investigated by authors in [5–7].

The infinite systems of ordinary differential equations can be considered as a special case of ordinary differential equations in Banach spaces. Recently, Banas

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and Lecko [8], and Mursaleen and Mohuiddine [9] have studied the solvability of infinite systems of differential equations in classical Banach sequence spaces by using the measure of noncompactness. For further results and discussions related to this topic, we refer the papers [10–15] and various references given therein. A considerable number of those results have been formulated in terms of measures of noncompactness. Also, Olszowy has studied the infinite systems of integral equations in Frechet spaces with measure of noncompactness in [16, 17].

Now, we give some basic definitions, notations, and results about sequence spaces and the Hausdorff measure of noncompactness (special case of measure of noncompactness). These can be found in [18, 19].

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{R}$ be the set of all real numbers. By ω we denote the vector space of all real sequences $x = (x_k)_{k\in\mathbb{N}}$. Let c_0 be the set of all null sequences. We write $\ell_p = \{x \in \omega : \sum_k |x_k|^p < \infty\}$ for $1 \le p < \infty$. The spaces c_0 and ℓ_p are Banach spaces with the norms $\|x\|_\infty = \sup_k |x_k|$ and $\|x\|_p = \left(\sum_k |x_k|^p\right)^{1/p}$, respectively.

The Fibonacci numbers are the sequence of numbers $\{f_n\}_{n=0}^{\infty}$ defined by the linear recurrence equations

$$f_0 = f_1 = 1 \text{ and } f_n = f_{n-1} + f_{n-2};\ n \ge 2.$$

There are many interesting properties and applications of Fibonacci numbers in arts, sciences, and architecture. For example, the ratio sequences of Fibonacci numbers converge to the golden ratio α ($\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \alpha$), which is important in sciences and arts. Also, for some basic properties of Fibonacci numbers, one can see [19]. Furtherfor, the inequalities

$$\frac{f_k}{f_{k+1}} \le 1 \text{ and } \frac{f_{k+1}}{f_k} \le 2 \tag{1.1}$$

hold every $k \in \mathbb{N}$.

Let (X, d) $((X, \|.\|))$ be a metric space (a normed space). By $B(x_0, r\ )$ and $\overline{B}(x_0, r)$, we denote, respectively, the open ball and closed ball in X centered x_0 and with radius $r > 0$. Moreover, let M_X be the collection of all nonempty and bounded subsets of X. If $Y \in M_X$, then the *Hausdorff measure of noncompactness* of the set Y, denoted by $\chi\ (Y)$, is defined by

$$\begin{aligned}\chi\,(Y) &:= \inf\left\{\varepsilon > 0 : Y \subset \bigcup_{i=1}^{n} B\,(x_i, r_i)\,,\quad x_i \in X,\quad r_i < \varepsilon \quad (i = 1, 2, \ldots, n)\,,\quad n \in \mathbb{N} - \{0\}\right\} \\ &:= \inf\{\varepsilon > 0 : Y \text{ has a finite } \varepsilon\text{-net in } X\}\,.\end{aligned}$$

The function $\chi : M_X \to [0, \infty)$ is called the Hausdorff measure of noncompactness. Recently the theory of measures of noncompactness has been used in determining the classes of compact operators on some sequence spaces, e.g., [20–24].

The basic properties of the Hausdorff measure of noncompactness can be found in [25].

The following result [26, Theorem 2.8] gives the Hausdorff measure of noncompactness of a bounded set in the spaces c_0 and ℓ_p for $1 \leq p < \infty$.

Lemma 1.1 *Let $X = \ell_p$ or $X = c_0$ and $Y \in M_X$. If $P_m : X \to X$ $(m \in \mathbb{N})$ is the operator defined by $P_m(x) = (x_0, x_1, \ldots, x_m, 0, 0, \ldots)$ for all $x = (x_k) \in X$, then we have*

$$\chi(Y) = \lim_{m\to\infty} \left(\sup_{x\in Y} \|(I - P_m)(x)\| \right),$$

where I is the identity operator on X.

Recently, Kara [27] and Başarır et al. [28] have defined the Banach sequence spaces $\ell_p(F)$ and $c_0(F)$ by using Fibonacci numbers as

$$\ell_p(F) = \left\{ x = (x_n) \in \omega : \sum_{n=0}^{\infty} \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right|^p < \infty \right\}$$

and

$$c_0(F) = \left\{ x = (x_n) \in \omega : \lim_{n\to\infty} \left(\frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right) = 0 \right\},$$

where $1 \leq p < \infty$ and $x_{-1} = 0$. The spaces $\ell_p(F)$ and $c_0(F)$ are Banach spaces with the norms given by

$$\|x\|_{\ell_p(F)} = \left(\sum_{n=0}^{\infty} \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right|^p \right)^{1/p} \text{ and } \|x\|_{c_0(F)} = \sup_{n\in\mathbb{N}} \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right|,$$

respectively. Since the inclusions $\ell_p \subset \ell_p(F)$ and $c_0 \subset c_0(F)$ strictly hold, the spaces $\ell_p(F)$ and $c_0(F)$ are more comprehensive than the spaces ℓ_p and c_0, respectively. Also, Kara et al. [25] and Alotaibi et al. [29] studied the classes of compact operators on the spaces $\ell_p(F)$ and $c_0(F)$ by using the Hausdorff measure of noncompactness. For more details about the Banach spaces derived by Fibonacci numbers, see [30–32].

In this paper, we apply the technique of measure of noncompactness to give solvability of infinite systems of differential equations in Banach spaces $\ell_p(F)$ and $c_0(F)$. Also, we give an example of infinite system of differential equations which has a solution in these spaces but has no solution in the spaces c_0 and ℓ_p.

The following lemma is essential for our results.

Lemma 1.2 *(i) If $Y \in M_{\ell_p(F)}$, then we have that*

$$\chi(Y) = \lim_{n\to\infty} \left(\sup_{(x_k)\in Y} \sum_{k\geq n} \left| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right|^p \right).$$

(ii) If $Y \in M_{c_0(F)}$, then we have that

$$\chi(Y)=\lim_{n\to\infty}\left(\sup_{(x_k)\in Y}\left(\sup_{k\geq n}\left|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\right|^p\right)\right).$$

Proof It is obtained from Lemma 1.1. □

2 Infinite Systems of Differential Equations in $\ell_p(F)$ and $c_0(F)$

In the present section, we study the solvability of the infinite systems of differential equations in the sequence spaces $\ell_p(F)$ and $c_0(F)$. Also, we give some examples.

Let us consider the ordinary differential equation

$$x^{'}=g(t,x) \tag{2.1}$$

with the initial condition

$$x(0)=x_0. \tag{2.2}$$

Now let $(X,\|.\|)$ be a real Banach space and take the interval $J=[0,T]$, $T>0$. Then, we give the following result proved by Banaś and Goebel [33] and modified by Banaś and Lecko [8].

Theorem 2.1 *Assume that $g(t,x)$ is a function defined on $J\times X$ with values in X such that*

$$\|g(t,x)\|\leq Q+R\,\|x\|\,,$$

for any $x\in X$, where Q and R are nonnegative constants. Also, let g be uniformly continuous on $J_1\times\overline{B}(x_0,r)$, where $r=(QT_1+RT_1\,\|x_0\|)/(1-RT_1)$ and $J_1=[0,T_1]\subset J$, $RT_1<1$. Further, assume that for any subset Y of $\overline{B}(x_0,r)$ and for almost $t\in J$ the following inequality holds:

$$\mu(g(t,Y))\leq q(t)\mu(Y) \tag{2.3}$$

with a sublinear measure of noncompactness μ such that $\{x_0\}\in ker\mu$. Then, the problem (2.1)–(2.2) has a solution x such that $\{x(t)\}\in ker\mu$ for $t\in J_1$; where $q(t)$ is an integrable function on J, and $ker\mu=\{E\in M_X:\mu(E)=0\ \}$ is the kernel of the measure μ.

Remark 2.2 If we take the Hausdorff measure of noncompactness χ instead of μ in Theorem 2.1, then the assumption of the uniform continuity of g can be replaced by the continuity of g.

Consider the infinite system of ordinary differential equations

$$x_i^{'}=g_i(t,x_1,x_2,\ldots),\quad t\in J=[0,T], \tag{2.4}$$

with the initial condition

$$x_i(0) = x_i^0 \tag{2.5}$$

for $i = 1, 2, 3, \ldots$.

Let the functions g_i $(i = 1, 2, \ldots)$ be defined on the set $J \times \mathbb{R}^\infty$ and take real values. Further, we assume the following conditions hold:

(i) $x_0 = (x_i^0) \in c_0(F)$,

(ii) the mapping $g = (g_1, g_2, \ldots)$ defined from the set $J \times c_0(F)$ into $c_0(F)$ is continuous,

(iii) there exits an increasing sequence (k_n) of natural numbers ($k_n \to \infty$ as $n \to \infty$) such that for any $t \in J$, $(x_i) \in c_0(F)$ and for $n = 1, 2, \ldots$, the inequality

$$|g_n(t, x_1, x_2, \ldots)| \le p_n(t) + q_n(t) \sup\left\{\left|\frac{f_i}{f_{i+1}}x_i - \frac{f_{i+1}}{f_i}x_{i-1}\right| : i \ge k_n\right\}$$

holds, where p_n and q_n are real-valued continuous functions on J such that $(p_n(t))$ converges uniformly on J to the function vanishing identically and $(q_n(t))$ is equibounded on J.

Also, let us denote

$$q(t) = \sup\{q_n(t) : n = 1, 2, \ldots\}; \quad t \in J,$$

$$Q_1 = \sup\{q(t) : t \in J\},$$

$$P_1 = \sup\{p_n(t) : t \in J, \quad n = 1, 2, \ldots\}.$$

Then we have the following theorem.

Theorem 2.3 *Let $T_1 < T$, $J_1 = [0, T_1]$, $Q = 3Q_1$, and $QT_1 < 1$. Then, under the above assumptions the problem (2.4)–(2.5) has at least one solution $x = x(t) = (x_n(t))$ on J_1. Further, $x(t) \in c_0(F)$ for any $t \in J_1$.*

Proof Let us take an arbitrary $x = (x_n(t)) \in c_0(F)$. By using the above assumptions and (1.1), for any $t \in J$ and for a fixed n we obtain

$$\begin{aligned}
\left|\frac{f_n}{f_{n+1}}g_n(t,x) - \frac{f_{n+1}}{f_n}g_{n-1}(t,x)\right| &= \left|\frac{f_n}{f_{n+1}}g_n(t,x_1,x_2,\ldots) - \frac{f_{n+1}}{f_n}g_{n-1}(t,x_1,x_2,\ldots)\right| \\
&\le \frac{f_n}{f_{n+1}}\left(p_n(t) + q_n(t)\sup\left\{\left|\frac{f_i}{f_{i+1}}x_i - \frac{f_{i+1}}{f_i}x_{i-1}\right| : i \ge k_n\right\}\right) \\
&+ \frac{f_{n+1}}{f_n}\left(p_{n-1}(t) + q_{n-1}(t)\sup\left\{\left|\frac{f_i}{f_{i+1}}x_i - \frac{f_{i+1}}{f_i}x_{i-1}\right| : i \ge k_{n-1}\right\}\right) \\
&\le \left(P_1 + Q_1\sup\left\{\left|\frac{f_i}{f_{i+1}}x_i - \frac{f_{i+1}}{f_i}x_{i-1}\right| : i \ge k_n\right\}\right) \\
&+ 2\left(P_1 + Q_1\sup\left\{\left|\frac{f_i}{f_{i+1}}x_i - \frac{f_{i+1}}{f_i}x_{i-1}\right| : i \ge k_{n-1}\right\}\right) \\
&\le P + Q\,\|x\|_{c_0(F)},
\end{aligned}$$

where $P = 3P_1$.

Thus, we have

$$\|g(t,x)\|_{c_0(F)} \le P + Q\,\|x\|_{c_0(F)}\,. \tag{2.6}$$

Now, let us take the closed ball $\overline{B}(x_0, r)$ in $c_0(F)$, where r is choosen as in Theorem 2.1. Let X be a subset of $\overline{B}(x_0, r)$ and $t \in J_1$. Then, by using Lemma 1.2(ii) and inequality (1.1), we get

$$\begin{aligned}
\chi(g(t,X)) &= \lim_{n\to\infty}\left\{\sup_{x\in X}\left\{\sup\left\{\left|\frac{f_i}{f_{i+1}}g_i(t,x) - \frac{f_{i+1}}{f_i}g_{i-1}(t,x)\right| : i \ge n\right\}\right\}\right\} \\
&= \lim_{n\to\infty}\left\{\sup_{(x_i)\in X}\left\{\sup\left\{\left|\frac{f_i}{f_{i+1}}g_i(t,x_1,x_2,\dots) - \frac{f_{i+1}}{f_i}g_{i-1}(t,x_1,x_2,\dots)\right| : i \ge n\right\}\right\}\right\} \\
&\le \lim_{n\to\infty}\left\{\sup_{(x_i)\in X}\left\{\sup_{i\ge n}\left\{\frac{f_i}{f_{i+1}}p_i(t) + \frac{f_i}{f_{i+1}}q_i(t)\sup\left[\left|\frac{f_p}{f_{p+1}}x_p - \frac{f_{p+1}}{f_p}x_{p-1}\right| : p \ge k_i\right]\right\}\right\}\right\} \\
&\quad + \lim_{n\to\infty}\left\{\sup_{(x_i)\in X}\left\{\sup_{i\ge n}\left\{\frac{f_{i+1}}{f_i}p_{i-1}(t) + \frac{f_{i+1}}{f_i}q_{i-1}(t)\sup\left[\left|\frac{f_p}{f_{p+1}}x_p - \frac{f_{p+1}}{f_p}x_{p-1}\right| : p \ge k_{i-1}\right]\right\}\right\}\right\} \\
&\le \lim_{n\to\infty}\left(\sup_{i\ge n}\{p_i(t)\}\right) + 2\lim_{n\longrightarrow\infty}\left(\sup_{i\ge n}\{p_{i-1}(t)\}\right) \\
&\quad + q(t)\lim_{n\to\infty}\left(\sup_{(x_i)\in X}\left\{\sup_{i\ge n}\left\{\sup\left[\left|\frac{f_p}{f_{p+1}}x_p - \frac{f_{p+1}}{f_p}x_{p-1}\right| : p \ge k_i\right]\right\}\right\}\right) \\
&\quad + 2q(t)\lim_{n\to\infty}\left(\sup_{(x_i)\in X}\left\{\sup_{i\ge n}\left\{\sup\left[\left|\frac{f_p}{f_{p+1}}x_p - \frac{f_{p+1}}{f_p}x_{p-1}\right| : p \ge k_{i-1}\right]\right\}\right\}\right) \\
&\le q_1(t)\chi(X),
\end{aligned} \tag{2.7}$$

where $q_1(t) = 3q(t)$.

It follows from inequalities (2.6) and (2.7), Theorem 2.1, and Remark 2.2 that there exists a solution $x = x(t)$ of the problem (2.4)–(2.5) such that $x(t) \in c_0(F)$ for any $t \in J_1$. □

Now, we consider the solvability of the problem (2.4)–(2.5) in the Banach space $\ell_p(F)$. For this purpose, we assume the following conditions.

(i) $x_0 = (x_n^0) \in \ell_p(F)$,

(ii) the mapping $g = (g_1, g_2, \dots)$ defined from the set $J \times \ell_p(F)$ into $\ell_p(F)$ is continuous, where $g_i : J \times \mathbb{R}^\infty \to \mathbb{R}$ for every $i \in \mathbb{N}$,

(iii) there exist the functions $q_i, r_i : J \to [0,\infty)$ for every $i \in \mathbb{N}$ such that

$$\begin{aligned}
\left|\frac{f_i}{f_{i+1}}g_i(t,x) - \frac{f_{i+1}}{f_i}g_{i-1}(t,x)\right|^p &= \left|\frac{f_i}{f_{i+1}}g_i(t,x_1,x_2,\dots) - \frac{f_{i+1}}{f_i}g_{i-1}(t,x_1,x_2,\dots)\right|^p \\
&\le q_i(t) + r_i(t)\left|\frac{f_i}{f_{i+1}}x_i - \frac{f_{i+1}}{f_i}x_{i-1}\right|^p
\end{aligned}$$

for $t \in J$, $x = (x_n) \in \ell_p(F)$,

(iv) for every $i \in \mathbb{N}$, q_i is continuous on J and the series $\sum_{i=0}^{\infty} q_i(t)$ converges uniformly on J,

(v) the sequence $(q_i(t))$ is equibounded on J and the function $r(t) = \limsup_{i\to\infty} r_i(t)$ is integrable on J.

Now, we give the following result.

Theorem 2.4 *If the conditions (i)–(v) hold then the problem (2.4)–(2.5) has a solution $x(t) = (x_i(t))$ defined on the interval $J = [0, T]$ whenever $R_1 T < 2^{-(p+1)}$, where*

$$R_1 = \sup\{r_i(t) : t \in J, \quad i = 0, 1, 2, \ldots\}.$$

Further, $x(t) \in \ell_p(F)$ for any $t \in J$.

Proof For any $x(t) \in \ell_p(F)$ and $t \in J$, under the above assumptions and using (1.1), we have

$$\begin{aligned}
\|g(t,x)\|^p_{\ell_p(F)} &= \sum_{i=0}^{\infty} \left| \frac{f_i}{f_{i+1}} g_i(t,x) - \frac{f_{i+1}}{f_i} g_{i-1}(t,x) \right|^p \\
&\le \sum_{i=0}^{\infty} 2^p \left(|g_i(t,x)|^p + |2g_{i-1}(t,x)|^p \right) \\
&\le 2^{2p} \sum_{i=0}^{\infty} \left(|g_i(t,x)|^p + |g_{i-1}(t,x)|^p \right) \\
&\le 2^{2p} \sum_{i=0}^{\infty} \left(q_i(t) + r_i(t) \left| \frac{f_i}{f_{i+1}} x_i - \frac{f_{i+1}}{f_i} x_{i-1} \right|^p \right) \\
&\quad + 2^{2p} \sum_{i=0}^{\infty} \left(q_{i-1}(t) + r_{i-1}(t) \left| \frac{f_{i-1}}{f_i} x_{i-1} - \frac{f_i}{f_{i-1}} x_{i-2} \right|^p \right) \\
&\le 2^{2p} \left(\sum_{i=0}^{\infty} q_i(t) + \sup_{i\ge 0}(r_i(t)) \sum_{i=0}^{\infty} \left| \frac{f_i}{f_{i+1}} x_i - \frac{f_{i+1}}{f_i} x_{i-1} \right|^p \right) \\
&\quad + 2^{2p} \left(\sum_{i=0}^{\infty} q_{i-1}(t) + \sup_{i\ge 0}(r_{i-1}(t)) \sum_{i=0}^{\infty} \left| \frac{f_{i-1}}{f_i} x_{i-1} - \frac{f_i}{f_{i-1}} x_{i-2} \right|^p \right) \\
&= 2^{2p+1} \left(\sum_{i=0}^{\infty} q_i(t) + \sup_{i\ge 0}(r_i(t)) \sum_{i=0}^{\infty} \left| \frac{f_i}{f_{i+1}} x_i - \frac{f_{i+1}}{f_i} x_{i-1} \right|^p \right) \\
&\le Q + R \|x\|^p_{\ell_p(F)},
\end{aligned}$$

where $Q = 2^{2p+1} \sup_{t\in J} \sum_{i=0}^{\infty} q_i(t)$ and $R = 2^{2p+1} R_1$.

Now, let us take the closed ball $\overline{B}(x_0, r)$ in $\ell_p(F)$, where $r = (QT + RT \|x_0\|^p_{\ell_p(F)})/(1 - RT)$. Consider the operator $g = (g_i)$ on the set $J \times \overline{B}(x_0, r)$, and let Y be subset of $\overline{B}(x_0, r)$. Then, by using inequality (1.1), we get

$$\begin{aligned}\chi(g(t,Y)) &= \lim_{n\to\infty} \sup_{x\in Y} \left(\sum_{i\geq n} \left| \frac{f_i}{f_{i+1}} g_i(t,x) - \frac{f_{i+1}}{f_i} g_{i-1}(t,x) \right|^p \right) \\ &\leq 2^{2p} \lim_{n\to\infty} \sup_{x\in Y} \left(\sum_{i\geq n} \left(|g_i(t,x)|^p + |g_{i-1}(t,x)|^p \right) \right) \\ &\leq 2^{2p} \lim_{n\to\infty} \left(\sum_{i\geq n} q_i(t) + \sum_{i\geq n} q_{i-1}(t) \right) \\ &+ 2^{2p} \lim_{n\to\infty} \left(\sup_{i\geq n} r_i(t). \sup_{(x_i)\in Y} \left[\sum_{i\geq n} \left| \frac{f_i}{f_{i+1}} x_i - \frac{f_{i+1}}{f_i} x_{i-1} \right|^p \right] \right) \\ &+ 2^{2p} \lim_{n\to\infty} \left(\sup_{i\geq n} r_{i-1}(t). \sup_{(x_i)\in Y} \left[\sum_{i\geq n} \left| \frac{f_{i-1}}{f_i} x_{i-1} - \frac{f_i}{f_{i-1}} x_{i-2} \right|^p \right] \right).\end{aligned}$$

It follows from Assumptions (i)–(v) and Lemma 1.2(i) that

$$\chi(g(t,Y)) \leq r_1(t)\chi(Y),$$

where $r_1(t) = 2^{2p+1} r(t)$. This says that the operator g satisfies (2.3) of Theorem 2.1 and Remark 2.2. Thus, we obtain that there exists a solution $x = x(t)$ of problem (2.4)–(2.5) such that $x(t) \in \ell_p(F)$ for any $t \in J$. Thus, the proof of the theorem is completed. □

Example 2.5 Let us consider the infinite system of differential equations

$$x_i^{'} = x_i; \quad x_i(0) = f_{i+1} \tag{2.8}$$

for $i = 0, 1, 2, \ldots$.

It is easily seen that the solution of (2.8) has the form

$$x(t) = (x(t)) = (f_{i+1}e^t) = (f_1 e^t, f_2 e^t, f_3 e^t, \ldots)$$

on the interval $J = [0, T]$.

Now, for every $t \in J$, we will show that $x(t) \notin c_0(F)$. Let $t \in J$. Then we have

$$\begin{aligned}\left| \frac{f_i}{f_{i+1}} x_i(t) - \frac{f_{i+1}}{f_i} x_{i-1}(t) \right| &= \left| \frac{f_i}{f_{i+1}} f_{i+1} e^t - \frac{f_{i+1}}{f_i} f_i e^t \right| \\ &= e^t \left| f_i - f_{i+1} \right| = e^t f_{i-1}\end{aligned}$$

for $i = 1, 2, 3, \ldots$. Also, it is clear that $\left|\frac{f_i}{f_{i+1}} x_i(t) - \frac{f_{i+1}}{f_i} x_{i-1}(t)\right| = e^t$ for $i = 0$. Thus, we get $\lim_{i\to\infty} \left|\frac{f_i}{f_{i+1}} x_i(t) - \frac{f_{i+1}}{f_i} x_{i-1}(t)\right| = \lim_{i\to\infty} e^t f_{i-1} = \infty$ for every $t \in J$. This shows that $x(t) \notin c_0(F)$ for every $t \in J$. Therefore, the sequence space $c_0(F)$ is not suitable to consider solvability of problem (2.8) in this space. Indeed, this situation appears quite naturally since the initial point $(x_i^0) = (f_{i+1})$ is not in the space $c_0(F)$.

Corollary 2.6 *The problem in Example 2.5 has no solution in the space* $\ell_p(F)$.

Proof The proof is clear since the inclusion $\ell_p(F) \subset c_0(F)$ holds. □

Example 2.7 Let us consider the infinite system of differential equations

$$x_i' = x_i; \quad x_i(0) = f_{i+1}^2 \tag{2.9}$$

for $i = 0, 1, 2, \ldots$ and take the interval $J = [0, T]$. Then, the solution of (2.9) has the form

$$x(t) = (x(t)) = (f_{i+1}^2 e^t) = (f_1^2 e^t, f_2^2 e^t, f_3^2 e^t, \ldots)$$

for every $t \in J$. Also, for every $t \in J$, we have that $\left|\frac{f_i}{f_{i+1}} x_i(t) - \frac{f_{i+1}}{f_i} x_{i-1}(t)\right| = e^t \left|f_i f_{i+1} - f_i f_{i+1}\right| = 0$ for $i \geq 1$ and $\left|\frac{f_i}{f_{i+1}} x_i(t) - \frac{f_{i+1}}{f_i} x_{i-1}(t)\right| = e^t$ for $i = 0$. Thus, the problem (2.9) has a solution in the space $\ell_p(F)$ (or $c_0(F)$). On the other hand, the initial condition $(x_i^0) = \left(f_{i+1}^2\right)$ is not in c_0 (and so ℓ_p). Thus, the problem (2.9) has no solution in the spaces c_0 and ℓ_p according to [8, Theorem 3] and [9, Theorem 3.1].

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Convergence Properties of Genuine Bernstein–Durrmeyer Operators

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Abstract The genuine Bernstein–Durrmeyer operators have notable approximation properties, and many papers have been written on them. In this paper, we introduce a modified genuine Bernstein–Durrmeyer operators. Some approximation results, which include local approximation, error estimation in terms of the modulus of continuity and weighted approximation is obtained. Also, a quantitative Voronovskaya-type approximation will be studied. The convergence of these operators to certain functions is shown by illustrative graphics using MAPLE algorithms.

Keywords Genuine Bernstein–Durrmeyer operators · Rate of convergence · Linear positive operators · Voronovskaja-type theorem

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1 Introduction

In 1912, Bernstein [8] defined the Bernstein polynomials in order to prove Weierstrass's fundamental theorem. These operators are one of the important topics of approximation theory in which it has been studied in great details for a long time. The Bernstein operators are given by

$$B_n : C[0,1] \to C[0,1], \quad B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x), \tag{1}$$

where

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \;\; x \in [0,1].$$

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Very recently, Cai et al. [9] introduced and considered a new generalization of Bernstein polynomials depending on the parameter λ as follows

$$B_{n,\lambda}(f;x)=\sum_{k=0}^{n}\tilde{b}_{n,k}(\lambda;x)f\left(\frac{k}{n}\right), \tag{2}$$

where $\lambda\in[-1,1]$ and $\tilde{b}_{n,k}$, $k=0,1,\dots$ are defined below

$$\begin{aligned}
\tilde{b}_{n,0}(\lambda;x)&=b_{n,0}(x)-\frac{\lambda}{n+1}b_{n+1,1}(x),\\
\tilde{b}_{n,k}(\lambda;x)&=b_{n,k}(x)+\lambda\left(\frac{n-2k+1}{n^2-1}b_{n+1,k}(x)-\frac{n-2k-1}{n^2-1}b_{n+1,k+1}(x)\right),\\
\tilde{b}_{n,n}(\lambda;x)&=b_{n,n}(x)-\frac{\lambda}{n+1}b_{n+1,n}(x).
\end{aligned}$$

In the particular case, when $\lambda=0$, λ-Bernstein operators reduce to well-known Bernstein operators. The authors of [9] have deeply studied many approximation properties of λ-Bernstein operators such as uniform convergence, rate of convergence in terms of modulus of continuity, Voronovskaya-type pointwise convergence, shape preserving properties.

The genuine Bernstein–Durrmeyer operators were introduced by Chen [10] and Goodman and Sharma [14] and were studied widely by a numbers of authors (see [3, 12, 13, 20, 23]). These operators are given by

$$U_n(f;x)=(n-1)\sum_{k=1}^{n-1}\left(\int_0^1 f(t)b_{n-2,k-1}(t)dt\right)b_{n,k}(x)+(1-x)^n f(0)+x^n f(1),\ \ f\in C[0,1].$$

These operators are limits of the Bernstein–Durrmeyer operators with Jacobi weights (see [6, 7, 21]), namely

$$U_n f=\lim_{\alpha\to-1,\beta\to-1}M_n^{<\alpha,\beta>}f, \text{ where}$$

$$M_n^{<\alpha,\beta>}:C[0,1]\to\Pi_n,\ \ M_n^{<\alpha,\beta>}(f;x)=\sum_{k=0}^{n}b_{n,k}(x)\frac{\int_0^1 w^{(\alpha,\beta)}(t)b_{n,k}(t)f(t)dt}{\int_0^1 w^{(\alpha,\beta)}(t)b_{n,k}(t)dt},$$
$$w^{(\alpha,\beta)}(t)=x^{\beta}(1-x)^{\alpha},\ x\in(0,1),\ \ \alpha,\beta>-1.$$

Also, the genuine Bernstein–Durrmeyer operators can be written as a composition of Bernstein operators and Beta operators, namely $U_n=B_n\circ\overline{\mathbb{B}}_n$. The Beta-type operators $\overline{\mathbb{B}}_n$ were introduced by A. Lupaş [19]. For $n=1,2,3,\dots$ and $f\in C[0,1]$, the explicit form of Beta operators is given by

$$\overline{\mathbb{B}}_n(f;x) := \begin{cases} f(0), \; x=0, \\ \dfrac{1}{B(nx,n-nx)} \displaystyle\int_0^1 t^{nx-1}(1-t)^{n-1-nx} f(t)dt, \; 0<x<1, \\ f(1), \; x=1, \end{cases}$$

where $B(\cdot,\cdot)$ is Euler's Beta function.

Our aim in this paper is to introduce genuine λ-Bernstein–Durrmeyer operators as a composition of λ-Bernstein operators and Beta operators, namely

$$U_{n,\lambda} = B_{n,\lambda} \circ \overline{\mathbb{B}}_n.$$

These operators are given in explicit form by

$$U_{n,\lambda}(f;x) = \tilde{b}_{n,0}(\lambda;x)f(0) + \tilde{b}_{n,n}(\lambda;x)f(1) + (n-1)\sum_{k=1}^{n-1}\tilde{b}_{n,k}(\lambda;x)\int_0^1 b_{n-2,k-1}(t)f(t)dt. \tag{3}$$

2 Preliminary Results

In this section by direct computation, we give the moments and the central moments of genuine λ-Bernstein–Durrmeyer operators.

Lemma 2.1 *The genuine λ-Bernstein–Durrmeyer operators verify*

(i) $U_{n,\lambda}(e_0;x) = 1$;

(ii) $U_{n,\lambda}(e_1;x) = x + \dfrac{\lambda}{n(n-1)}\left(x^{n+1} - (1-x)^{n+1} - 2x + 1\right)$;

(iii) $U_{n,\lambda}(e_2;x) = x^2 + \dfrac{2}{n(n^2-1)}\left\{x(1-x)n^2 + (-2x^2\lambda + \lambda x^{n+1} + x\lambda + x^2 - x)n + \lambda x^{n+1} - x\lambda\right\}$;

(iv) $U_{n,\lambda}(e_3;x) = x^3 + \dfrac{3}{n(n+2)(n^2-1)}\left\{2x^2(1-x)n^3 + (-2\lambda x^3 + x^2\lambda + 2x^3 + \lambda x^{n+1} - 4x^2 + 2x)n^2 + (2\lambda x^3 - 7x^2\lambda + 3\lambda x^{n+1} + 2x\lambda + 2x^2 - 2x)n + 2\lambda x^{n+1} - 2x\lambda\right\}$;

(v) $U_{n,\lambda}(e_4;x) = x^4 + \dfrac{4}{n(n+2)(n+3)(n^2-1)}\left\{3x^3(1-x)n^4 + (-2\lambda x^4 + \lambda x^3 + 3x^4 - 12x^3 + \lambda x^{n+1} + 9x^2)n^3 + (6\lambda x^4 - 18\lambda x^3 - 3x^4 + 6x^2\lambda + 15x^3 + 6\lambda x^{n+1} - 18x^2 + 6x)n^2 + (-4\lambda x^4 + 17\lambda x^3 + 3x^4 - 30x^2\lambda - 6x^3 + 11\lambda x^{n+1} + 6x\lambda + 9x^2 - 6x)n + 6\lambda x^{n+1} - 6x\lambda\right\}$.

Lemma 2.2 *The central moments of genuine λ-Bernstein–Durrmeyer operators are given below:*

(i) $U_{n,\lambda}(t-x;x)=\dfrac{\lambda}{n(n-1)}(x^{n+1}-(1-x)^{n+1}-2x+1)$*;*

(ii) $U_{n,\lambda}((t-x)^2;x)=2x(1-x)\left\{\lambda\left(\dfrac{x^n+(1-x)^n}{n(n-1)}-\dfrac{2}{n(n^2-1)}\right)+\dfrac{1}{n+1}\right\}.$

Lemma 2.3 *The central moments of genuine λ-Bernstein–Durrmeyer operators verify*

$$|U_{n,\lambda}(t-x;x)|\le\theta_1(n,\lambda) \text{ and } U_{n,\lambda}((t-x)^2;x)\le\theta_2(n,\lambda), \text{ for } n>2,$$

where $\theta_1(n,\lambda)=\dfrac{|\lambda|}{n(n-1)}$ *and* $\theta_2(n,\lambda)=\dfrac{|\lambda|+n}{2n(n+1)}$.

Lemma 2.4 *The genuine λ-Bernstein–Durrmeyer operators verify:*

(i) $\lim\limits_{n\to\infty} nU_{n,\lambda}\,(t-x;x)=0$*;*

(ii) $\lim\limits_{n\to\infty} nU_{n,\lambda}\left((t-x)^2;x\right)=2x(1-x)$*;*

(iii) $\lim\limits_{n\to\infty} n^2U_{n,\lambda}\left((t-x)^4;x\right)=12x^2(1-x)^2$*.*

3 Basic Approximation Properties

In this section, we investigate the approximation properties of these operators and we estimate the rate of convergence by using moduli of continuity.

Theorem 3.1 *If $f\in C[0,1]$, then*

$$\lim_{n\to\infty} U_{n,\lambda}(f;x)=f(x) \text{ uniformly on } [0,1].$$

Proof Using Lemma 2.1 follows that

$$\lim_{n\to\infty} U_{n,\lambda}(e_k;x)=e_k(x) \text{ uniformly on } [0,1],\ \text{ for } k\in\{0,1,2\}.$$

Applying the Bohman–Korovkin theorem, we get the result. □

Theorem 3.2 *If $f\in C[0,1]$, then*

$$\left|U_{n,\lambda}(f;x)-f(x)\right|\le 2\omega(f;\sqrt{\theta_2(n;\lambda)}),$$

where ω is the usual modulus of continuity.

Proof Using the following property of modulus of continuity

$$|f(t) - f(x)| \leq \omega(f;\delta)\left(\frac{(t-x)^2}{\delta^2} + 1\right),$$

we obtain

$$|U_{n,\lambda}(f;x) - f(x)| \leq U_{n,\lambda}(|f(t) - f(x)|;x) \leq \omega(f;\delta)\left(1 + \frac{1}{\delta^2}U_{n,\lambda}((t-x)^2;x)\right).$$

So, if we choose $\delta = \sqrt{\theta_2(n;\lambda)}$, we have the desired result. □

Theorem 3.3 *If* $f \in C^1[0,1]$, *then*

$$\left|U_{n,\lambda}(f;x) - f(x)\right| \leq \theta_1(n;\lambda)|f'(x)| + 2\sqrt{\theta_2(n;\lambda)}\omega(f', \sqrt{\theta_2(n;\lambda)}).$$

Proof Let $f \in C^1[0,1]$. For any $x, t \in [0,1]$, we have

$$f(t) - f(x) = f'(x)(t-x) + \int_x^t \big(f'(y) - f'(x)\big)\, dy,$$

so, we get

$$U_{n,\lambda}\left(f(t) - f(x);x\right) = f'(x)U_{n,\lambda}(t-x;x) + U_{n,\lambda}\left(\int_x^t (f'(y) - f'(x))dy;x\right).$$

Using the following well-known property of modulus of continuity

$$|f(y) - f(x)| \leq \omega(f;\delta)\left(\frac{|y-x|}{\delta} + 1\right), \quad \delta > 0,$$

we have

$$\left|\int_x^t |f'(y) - f'(x)|dy\right| \leq \omega(f';\delta)\left[\frac{(t-x)^2}{\delta} + |t-x|\right].$$

Therefore,

$$|U_{n,\lambda}(f;x) - f(x)| \leq |f'(x)| \cdot |U_{n,\lambda}(t-x;x)| \\ + \omega(f';\delta)\left\{\frac{1}{\delta}U_{n,\lambda}\left((t-x)^2;x\right) + U_{n,\lambda}(|t-x|;x)\right\}.$$

Using Cauchy–Schwartz inequality, we obtain

$$\begin{aligned}|U_{n,\lambda}(f;x) - f(x)| &\le |f'(x)||U_{n,\lambda}(t-x;x)| \\ &\quad + \omega(f',\delta)\left\{\frac{1}{\delta}\sqrt{U_{n,\lambda}\left((t-x)^2;x\right)}+1\right\}\sqrt{U_{n,\lambda}\left((t-x)^2;x\right)} \\ &\le |f'(x)|\theta_1(n;\lambda) + \omega(f',\delta)\cdot\left\{\frac{1}{\delta}\sqrt{\theta_2(n;\lambda)}+1\right\}\sqrt{\theta_2(n;\lambda)}.\end{aligned}$$

Choosing $\delta = \sqrt{\theta_2(n;\lambda)}$, we find the desired inequality. □

In order to give the next result, we recall the definition of K-functional:

$$K_2(f,\delta) := \inf\left\{\|f-g\| + \delta\left\|g''\right\| : g \in W^2[0,1]\right\},$$

where

$$W^2[0,1] = \left\{g \in C[0,1] : g'' \in C[0,1]\right\},$$

$\delta \ge 0$ and $\|\cdot\|$ is the uniform norm on $C[0,1]$. The second-order modulus of continuity is defined as follows

$$\omega_2\left(f,\sqrt{\delta}\right) = \sup_{0<h\le\sqrt{\delta}}\ \sup_{x,x+2h\in[0,1]}\{|f(x+2h)-2f(x+h)+f(x)|\}.$$

It is well known that K-functional and the second-order modulus of continuity $\omega_2\left(f,\sqrt{\delta}\right)$ are equivalent, namely

$$K_2(f,\delta) \le C\omega_2\left(f,\sqrt{\delta}\right), \tag{4}$$

where $\delta \ge 0$ and $C > 0$.

Theorem 3.4 *If $f \in C[0,1]$, then*

$$\left|U_{n,\lambda}(f;x) - f(x)\right| \le C\omega_2\left(f,\frac{1}{2}\sqrt{\theta_2(n;\lambda)+\theta_1^2(n,\lambda)}\right) + \omega\left(f,\theta_1(n;\lambda)\right),$$

where C is a positive constant.

Proof Denote $\nu_{n,\lambda}(x) = x + \frac{\lambda}{n(n-1)}\left(x^{n+1} - (1-x)^{n+1} - 2x + 1\right)$ and

$$\tilde{U}_{n,\lambda}(f;x) = U_{n,\lambda}(f;x) + f(x) - f(\nu_{n,\lambda}(x)). \tag{5}$$

It follows immediately

$$\tilde{U}_{n,\lambda}(e_0;x) = U_{n,\lambda}(e_0;x) = 1$$

$$\tilde{U}_{n,\lambda}(e_1; x) = U_{n,\lambda}(e_1; x) + x - \nu_{n,\lambda}(x) = x.$$

Applying $\tilde{U}_{n,\lambda}$ to Taylor's formula, we get

$$\tilde{U}_{n,\lambda}(g; x) = g(x) + \tilde{U}_{n,\lambda}\left(\int_x^t (t-y)g''(y)dy; x\right).$$

Therefore,

$$\tilde{U}_{n,\lambda}(g; x) = g(x) + U_{n,\lambda}\left(\int_x^t (t-y)g''(y)dy; x\right) - \int_x^{\nu_{n,\lambda}(x)} \left(\nu_{n,\lambda}(x) - y\right) g''(y)dy.$$

This implies that

$$\begin{aligned}
|\tilde{U}_{n,\lambda}(g; x) - g(x)| &\leq \left|U_{n,\lambda}\left(\int_x^t (t-y)g''(y)dy; x\right)\right| + \left|\int_x^{\nu_{n,\lambda}(x)} \left(\nu_{n,\lambda}(x) - y\right) g''(y)dy\right| \\
&\leq U_{n,\lambda}((t-x)^2; x)\,\|g''\| + \left(\nu_{n,\lambda}(x) - x\right)^2 \|g''\| \\
&\leq \left[\theta_2(n;\lambda) + \theta_1^2(n;\lambda)\right] \|g''\|.
\end{aligned}$$

In view of (5), we obtain

$$|\tilde{U}_{n,\lambda}(f; x)| \leq |U_{n,\lambda}(f; x)| + |f(x)| + |f(\nu_{n,\lambda}(x))| \leq 3\,\|f\|. \tag{6}$$

Now, for $f \in C[0,1]$ and $g \in W^2[0,1]$, using (5) and (6) we get

$$\begin{aligned}
&|U_{n,\lambda}(f; x) - f(x)| = \left|\tilde{U}_{n,\lambda}(f; x) - f(x) + f(\nu_{n,\lambda}(x)) - f(x)\right| \\
&\leq \left|\tilde{U}_{n,\lambda}(f-g; x)\right| + \left|\tilde{U}_{n,\lambda}(g; x) - g(x)\right| + |g(x) - f(x)| + \left|f(\nu_{n,\lambda}(x)) - f(x)\right| \\
&\leq 4\,\|f-g\| + \left[\theta_2(n,\lambda) + \theta_1^2(n,\lambda)\right] \|g''\| + \omega\left(f, \theta_1(n,\lambda)\right).
\end{aligned}$$

Taking the infimum on the right side over all $g \in W^2[0,1]$, we have

$$|U_{n,\lambda}(f; x) - f(x)| \leq 4K_2\left(f, \frac{1}{4}\left(\theta_2(n,\lambda) + \theta_1^2(n,\lambda)\right)\right) + \omega\left(f, \theta_1(n,\lambda)\right).$$

Finally, using the equivalence between K-functional and the second-order modulus of continuity (4), the proof is completed. □

4 Rate of Convergence in Terms of the Ditzian–Totik Modulus of Smoothness

In this section, we study the rate of convergence of genuine λ-Bernstein–Durrmeyer operators in terms of the Ditzian–Totik first-order modulus of smoothness defined as follows:

$$\omega_1^{\phi}(f;t)=\sup_{0<h\le t}\left\{\left|f\left(x+\frac{h\phi(x)}{2}\right)-f\left(x-\frac{h\phi(x)}{2}\right)\right|,x\pm\frac{h\phi(x)}{2}\in[0,1]\right\},\tag{7}$$

where $\phi(x)=\sqrt{x(1-x)}$ and $f\in C[0,1]$. The corresponding K-functional of the Ditzian–Totik first-order modulus of smoothness is given by

$$K_{\phi}(f;t)=\inf_{g\in W_{\phi}[0,1]}\{||f-g||+t||\phi g'||\}\ (t>0),\tag{8}$$

where $W_{\phi}[0,1]=\{g:g\in AC_{loc}[0,1],\|\phi g'\|<\infty\}$ and $AC_{loc}[0,1]$ is the class of absolutely continuous functions on every interval $[a,b]\subset[0,1]$. Between K-functional and the Ditzian–Totik first-order modulus of smoothness, there is the following relation

$$K_{\phi}(f;t)\le C\omega_1^{\phi}(f;t),\tag{9}$$

where $C>0$ is a constant.

Theorem 4.1 *Let $f\in C[0,1]$ and $\phi(x)=\sqrt{x(1-x)}$. Then for every $x\in[0,1]$, we have*

$$|U_{n,\lambda}(f;x)-f(x)|\le C\omega_1^{\phi}\left(f;\frac{1}{n^{1/2}}\right),$$

where C is a constant independent of n and x.

Proof From the next representation

$$g(t)=g(x)+\int_x^t g'(u)du,$$

we get

$$\left|U_{n,\lambda}(g;x)-g(x)\right|=\left|U_{n,\lambda}\left(\int_x^t g'(u)du;x\right)\right|.\tag{10}$$

For any $x\in(0,1)$ and $t\in[0,1]$, we find that

$$\left|\int_x^t g'(u)du\right|\le||\phi g'||\left|\int_x^t\frac{1}{\phi(u)}du\right|.\tag{11}$$

But,

$$\left|\int_x^t \frac{1}{\phi(u)}du\right| = \left|\int_x^t \frac{1}{\sqrt{u(1-u)}}du\right| \leq \left|\int_x^t \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}}\right)du\right| \tag{12}$$
$$\leq 2\left(|\sqrt{t}-\sqrt{x}| + |\sqrt{1-t}-\sqrt{1-x}| \right)$$
$$= 2|t-x|\left(\frac{1}{\sqrt{t}+\sqrt{x}} + \frac{1}{\sqrt{1-t}+\sqrt{1-x}}\right)$$
$$< 2|t-x|\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}\right) \leq \frac{2\sqrt{2}\,|t-x|}{\phi(x)}.$$

Combining (10)–(12) and using Cauchy–Schwarz inequality, we get

$$|U_{n,\lambda}(g;x)-g(x)| < 2\sqrt{2}||\phi g'||\phi^{-1}(x)U_{n,\lambda}(|t-x|;x)$$
$$\leq 2\sqrt{2}||\phi g'||\phi^{-1}(x)\left(U_{n,\lambda}((t-x)^2;x)\right)^{1/2}.$$

Now using Lemma 2.3, we obtain

$$|U_{n,\lambda}(g;x)-g(x)| \leq \frac{2}{n^{1/2}}\|\phi g'\|. \tag{13}$$

Using (3) and (13), we can write

$$|\, U_{n,\lambda}(f;x)-f(x)\,| \leq |\, U_{n,\lambda}(f-g;x)\,| + |f(x)-g(x)| + |\, U_{n,\lambda}(g;x)-g(x)\,|$$
$$\leq 2\left\{||f-g|| + \frac{1}{n^{1/2}}||\phi g'||\right\}.$$

From the definition of the K-functional (8), we get

$$|U_{n,\lambda}(f;x)-f(x)| \leq 2K_\phi\left(f;\frac{1}{n^{1/2}}\right),$$

and considering the relation (9), the proof is completed. □

In the following, we give an estimate by means of Ditzian–Totik modulus of smoothness of second order defined as

$$\omega_2^\phi(f;\delta) = \sup_{0<h\leq\delta}\left\{\left|f\left(x+\frac{h\phi(x)}{2}\right) - 2f(x) + f\left(x-\frac{h\phi(x)}{2}\right)\right|; x \pm \frac{h\phi(x)}{2} \in [0,1]\right\}.$$

where $f \in C[0,1]$, $\delta > 0$ and $x \in [0,1]$.

Theorem 4.2 *Let* $f \in C[0,1]$, $\lambda \in [-1,1]$ *and* $h \in \left(0, \frac{1}{2}\right]$. *For all* $x \in (0,1)$ *the relation*

$$|U_{n,\lambda}(f;x) - f(x)| \leq \frac{\theta_1(n,\lambda)}{2\,h\phi(x)}\,\omega_1^{\phi}(f,2h) + \left(1 + \frac{3}{2}\frac{\theta_2(n,\lambda)}{h^2\phi^2(x)}\right)\,\omega_2^{\phi}(f,h)$$

holds.

Proof From [22, Theorem 2.5.1], we have

$$|U_{n,\lambda}(f;x) - f(x)| \leq \frac{|U_{n,\lambda}(t-x;x)|}{2\,h\phi(x)}\,\omega_1^{\phi}(f,2h) + \left(U_{n,\lambda}(e_0;x) + \frac{3}{2}\frac{U_{n,\lambda}((t-x)^2;x)}{(h\phi(x))^2}\right)\omega_2^{\phi}(f,h)$$

and using Lemma 2.3, we obtain the desired estimation. □

Applying Theorem 4.2 for $h = \sqrt{\theta_2(n,\lambda)}/\phi(x)$, it follows the next result.

Corollary 4.1 *Let* $f \in C[0,1]$, $\lambda \in [-1,1]$ *and* $x \in [0,1]$. *There exist an integer* $n_0 \in \mathbb{N}$, *such that for* $n \geq n_0$, *the following relation*

$$|U_{n,\lambda}(f;x) - f(x)| \leq \frac{\theta_1(n,\lambda)}{2\sqrt{\theta_2(n,\lambda)}}\,\omega_1^{\phi}\left(f, \frac{2\sqrt{\theta_2(n,\lambda)}}{\phi(x)}\right) + \frac{5}{2}\,\omega_2^{\phi}\left(f, \frac{\sqrt{\theta_2(n,\lambda)}}{\phi(x)}\right)$$

holds.

5 Voronovskaja-Type Theorems

In the following, we prove a quantitative Voronovskaja-type theorem for the operator $U_{n,\lambda}$ by means of Ditzian–Totik modulus of smoothness. Nowadays such a result has been studied for many operators and for many moduli of continuity in classical and weighted cases (see [1, 4, 15, 18]).

Theorem 5.1 *For any* $f \in C^2[0,1]$ *and n sufficiently large the following inequality holds*

$$\left|U_{n,\lambda}(f;x) - f(x) - \Omega_n(x;\lambda)g'(x) - \Psi_n(x;\lambda)f''(x)\right| \leq \frac{1}{n}C\phi^2(x)\omega_1^{\phi}\left(f'', n^{-1/2}\right),$$

where

$$\Omega_n(x;\lambda) = \frac{\lambda}{n(n-1)}(x^{n+1} - (1-x)^{n+1} - 2x + 1);$$
$$\Psi_n(x;\lambda) = x(1-x)\left\{\lambda\left(\frac{x^n + (1-x)^n}{n(n-1)} - \frac{2}{n(n^2-1)}\right) + \frac{1}{n+1}\right\}$$

and C is a positive constant.

Proof For $f \in C^2[0, 1]$, $t, x \in [0, 1]$, by Taylor's expansion, we have

$$f(t) - f(x) = (t - x)f'(x) + \int_x^t (t - y)f''(y)dy.$$

Hence,

$$\begin{aligned} f(t) - f(x) - (t - x)f'(x) - \frac{1}{2}(t - x)^2 g''(x) &= \int_x^t (t - y)f''(y)dy - \int_x^t (t - y)f''(x)dy \\ &= \int_x^t (t - y)[f''(y) - f''(x)]dy. \end{aligned}$$

Applying $U_{n,\lambda}(\cdot; x)$ to both sides of the above relation, we get

$$\left|U_{n,\lambda}(f; x) - f(x) - \Omega_n(x; \lambda)f'(x) - \Psi_n(x; \lambda)f''(x)\right| \le U_{n,\lambda}\left(\left|\int_x^t |t - y||f''(y) - f''(x)|dy\right|; x\right). \tag{14}$$

The quantity $\left|\int_x^t \left|f''(y) - f''(x)\right| |t - y|dy\right|$ was estimated in [11, p. 337] as follows:

$$\left|\int_x^t |f''(y) - f''(x)||t - y|dy\right| \le 2\|f'' - g\|(t - x)^2 + 2\|\phi g'\|\phi^{-1}(x)|t - x|^3, \tag{15}$$

where $g \in W_\phi[0, 1]$.

Using Lemma 2.4 it follows that there exists a constant $C > 0$ such that for n sufficiently large

$$U_{n,\lambda}\left((t - x)^2; x\right) \le \frac{C}{2n}\phi^2(x) \text{ and } U_{n,\lambda}\left((t - x)^4; x\right) \le \frac{C}{2n^2}\phi^4(x). \tag{16}$$

From (14)–(16) and applying the Cauchy–Schwarz inequality, we get

$$\begin{aligned} &\left|U_{n,\lambda}(f; x) - f(x) - \Omega_n(x; \lambda)f'(x) - \Psi_n(x; \lambda)f''(x)\right| \\ &\le 2\|f'' - g\|U_{n,\lambda}\left((t - x)^2; x\right) + 2\|\phi g'\|\phi^{-1}(x)U_{n,\lambda}\left(|t - x|^3; x\right) \\ &\le \frac{C}{n}\phi^2(x)\|f'' - g\| + 2\|\phi g'\|\phi^{-1}(x)\left\{U_{n,\lambda}(t - x)^2; x\right\}^{1/2}\left\{U_{n,\lambda}\left((t - x)^4; x\right)\right\}^{1/2} \\ &\le \frac{C}{n}\phi^2(x)\|f'' - g\| + \phi^2(x)\frac{C}{n\sqrt{n}}\|\phi h'\| \le \frac{C}{n}\phi^2(x)\left\{\|f'' - g\| + n^{-1/2}\|\phi h'\|\right\}. \end{aligned}$$

Taking the infimum on the right-hand side of the above relations over $g \in W_\phi[0, 1]$, the theorem is proved. □

Corollary 5.1 *If $f \in C^2[0,1]$, then*

$$\lim_{n\to\infty} n\left\{U_{n,\lambda}(f;x) - f(x) - \Omega_n(x;\lambda)g'(x) - \Psi_n(x;\lambda)g''(x)\right\} = 0,$$

where $\Omega_n(x;\lambda)$ and $\Psi_n(x;\lambda)$ are defined in Theorem 5.1.

6 Rate of Convergence for Functions Whose Derivative Are of Bounded Variation

In this section, we study the rate of convergence of genuine λ-Bernstein–Durrmeyer operators for functions whose derivative are of bounded variation on $[0,1]$. We mention here some of the papers in this direction [2, 5, 16, 17].

An integral representation of the operators $U_{n,\lambda}$ can be given as follows:

$$U_{n,\lambda}(f;x) = \int_0^1 \mathcal{K}_{n,\lambda}(x,t)f(t)dt, \tag{17}$$

where $\mathcal{K}_{n,\lambda}$ is defined as

$$\mathcal{K}_{n,\lambda}(x,t) = (n-1)\sum_{k=1}^{n-1} \tilde{b}_{n,k}(\lambda;x)b_{n-2,k-1}(t) + \tilde{b}_{n,0}(\lambda;x)\delta(t) + \tilde{b}_{n,n}(\lambda;x)\delta(1-t),$$

$\delta(u)$ being the Dirac delta function.

Lemma 6.1 *For a sufficiently large n and a fixed $x \in (0,1)$, it follows*

(i) $\eta_{n,\lambda}(x,y) = \int_0^y \mathcal{K}_{n,\lambda}(x,t)dt \le \dfrac{2(n+|\lambda|)}{n(n+1)} \cdot \dfrac{x(1-x)}{(x-y)^2},\ 0 \le y < x,$

(ii) $1 - \eta_{n,\lambda}(x,z) = \int_z^1 \mathcal{K}_{n,\lambda}(x,t)dt \le \dfrac{2(n+|\lambda|)}{n(n+1)} \cdot \dfrac{x(1-x)}{(z-x)^2},\ x < z < 1.$

Denote $DBV[0,1]$ the class of differentiable functions f defined on $[0,1]$, whose derivatives f' are of bounded variation on $[0,1]$. Let $\bigvee_a^b f$ be the total variation of f on $[a,b]$ and f'_x is defined by

$$f'_x(t) = \begin{cases} f'(t) - f'(x-), & 0 \le t < x \\ 0, & t = x \\ f'(t) - f'(x+) & x < t \le 1. \end{cases} \tag{18}$$

Theorem 6.1 *If $f \in DBV[0,1]$, then for every $x \in (0,1)$ and sufficiently large n, the following inequality*

$$|U_{n,\lambda}(f;x)-f(x)|\leq\sqrt{\frac{(n+|\lambda|)x(1-x)}{2n(n+1)}}\left\{|f'(x+)+f'(x-)|+|f'(x+)-f'(x-)|\right\}$$
$$+\frac{2}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}}(f'_x)$$

holds.

Proof Since $U_{n,\lambda}(1;x)=1$, for every $x\in(0,1)$ we can write

$$\begin{aligned}U_{n,\lambda}(f;x)-f(x)&=\int_0^1\mathcal{K}_{n,\lambda}(x,t)(f(t)-f(x))dt\\&=\int_0^x(f(t)-f(x))\,\mathcal{K}_{n,\lambda}(x,t)dt+\int_x^1(f(t)-f(x))\,\mathcal{K}_{n,\lambda}(x,t)dt\\&=-\int_0^x\left[\int_t^x f'(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt+\int_x^1\left[\int_x^t f'(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt\\&=-\mathcal{A}(x)+\mathcal{B}(x),\end{aligned}$$

where

$$\mathcal{A}(x)=\int_0^x\left[\int_t^x f'(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt,\quad\mathcal{B}(x)=\int_x^1\left[\int_x^t f'(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt.$$

For any $f\in DBV[0,1]$, we decompose $f'(t)$ as follows:

$$f'(t)=\frac{1}{2}(f'(x+)+f'(x-))+f'_x(t)+\frac{1}{2}(f'(x+)-f'(x-))sgn(t-x)\qquad(19)$$
$$+\delta_x(t)\left[f'(x)-\frac{1}{2}(f'(x+)+f'(x-))\right],$$

where

$$\delta_x(t)=\begin{cases}1\ ,\ t=x\\0\ ,\ t\neq x\end{cases}.$$

Therefore, we get

$$\begin{aligned}\mathcal{A}(x)=&\int_0^x\left\{\int_t^x\frac{1}{2}\left(f'(x+)+f'(x-)\right)+f'_x(u)+\frac{f'(x+)-f'(x-)}{2}sgn(u-x)\right.\\&\left.+\,\delta_x(u)\left[f'(x)-\frac{1}{2}\left(f'(x+)+f'(x-)\right)\right]du\right\}\mathcal{K}_{n,\lambda}(x,t)dt\\=&\frac{f'(x+)+f'(x-)}{2}\int_0^x(x-t)\mathcal{K}_{n,\lambda}(x,t)dt+\int_0^x\left[\int_t^x f'_x(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt\\&-\frac{f'(x+)-f'(x-)}{2}\int_0^x(x-t)K_{n,\lambda}(x,t)dt\end{aligned}$$

$$+\left[f'(x)-\frac{f'(x+)+f'(x-)}{2}\right]\int_0^x\left[\int_t^x\delta_x(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt.$$

Since $\displaystyle\int_t^x \delta_x(u)du=0$, we get

$$\begin{aligned}\mathcal{A}(x)=&\frac{f'(x+)+f'(x-)}{2}\int_0^x(x-t)\mathcal{K}_{n,\lambda}(x,t)dt+\int_0^x\left[\int_t^x f_x'(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt\\&-\frac{f'(x+)-f'(x-)}{2}\int_0^x(x-t)\mathcal{K}_{n,\lambda}(x,t)dt.\end{aligned}$$

Using a similar method, we find that

$$\begin{aligned}\mathcal{B}(x)=&\frac{f'(x+)+f'(x-)}{2}\int_x^1(t-x)\mathcal{K}_{n,\lambda}(x,t)dt+\int_x^1\left[\int_x^t f_x'(u)\right]\mathcal{K}_{n,\lambda}(x,t)dt\\&-\frac{f'(x+)-f'(x-)}{2}\int_x^1(t-x)\mathcal{K}_{n,\lambda}(x,t)dt.\end{aligned}$$

Therefore,

$$\begin{aligned}-\mathcal{A}(x)+\mathcal{B}(x)=&\frac{f'(x+)+f'(x-)}{2}\int_0^1(t-x)\mathcal{K}_{n,\lambda}(x,t)dt\\&+\frac{f'(x+)-f'(x-)}{2}\int_0^1|t-x|\mathcal{K}_{n,\lambda}(x,t)dt\\&-\int_0^x\left[\int_t^x f_x'(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt+\int_x^1\left[\int_x^t f_x'(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt.\end{aligned}$$

Then,

$$\begin{aligned}U_{n,\lambda}(f;x)-f(x)=&\frac{f'(x+)+f'(x-)}{2}\int_0^1(t-x)\mathcal{K}_{n,\lambda}(x,t)dt\\&+\frac{f'(x+)-f'(x-)}{2}\int_0^1|t-x|\mathcal{K}_{n,\lambda}(x,t)dt\\&-\int_0^x\left[\int_t^x f_x'(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt+\int_x^1\left[\int_x^t f_x'(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt.\end{aligned}$$

From the above relation, it follows

$$\begin{aligned}\left|U_{n,\lambda}(f;x)-f(x)\right|\le&\left|\frac{f'(x+)+f'(x-)}{2}\right|\left|U_{n,\lambda}(t-x;x)\right|\\&+\left|\frac{f'(x+)-f'(x-)}{2}\right|\left|U_{n,\lambda}(|t-x|;x)\right|\\&+\left|-\int_0^x\left[\int_t^x f_x'(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt\right|\end{aligned}$$

$$+\left|\int_x^1\left[\int_x^t f_x'(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt\right|. \tag{20}$$

According to Lemma 6.1, we write

$$\int_0^x\left[\int_t^x f_x'(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt=\int_0^x\left[\int_t^x f_x'(u)du\right]\frac{\partial}{\partial t}\eta_{n,\lambda}(x,t)dt=\int_0^x f_x'(t)\eta_{n,\lambda}(x,t)dt.$$

Thus,

$$\begin{aligned}\left|-\int_0^x\left[\int_t^x f_x'(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt\right| &\le \int_0^x |f_x'(t)|\eta_{n,\lambda}(x,t)dt\\ \le \int_0^{x-\frac{x}{\sqrt{n}}}|f_x'(t)|\eta_{n,\lambda}(x,t)dt &+\int_{x-\frac{x}{\sqrt{n}}}^x|f_x'(t)|\eta_{n,\lambda}(x,t)dt.\end{aligned}$$

Since $f_x'(x)=0$ and $\eta_{n,\lambda}(x,t)\le 1$, one has

$$\begin{aligned}\int_{x-\frac{x}{\sqrt{n}}}^x|f_x'(t)|\eta_{n,\lambda}(x,t)dt&=\int_{x-\frac{x}{\sqrt{n}}}^x|f_x'(t)-f_x'(x)|\eta_{n,\lambda}(x,t)dt\le\int_{x-\frac{x}{\sqrt{n}}}^x\bigvee_t^x(f_x')dt\\ &\le\bigvee_{x-\frac{x}{\sqrt{n}}}^x(f_x')\int_{x-\frac{x}{\sqrt{n}}}^x dt=\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x(f_x').\end{aligned}$$

From Lemma 6.1, we can write

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}}|f_x'(t)|\eta_{n,\lambda}(x,t)dt&\le\frac{2(n+|\lambda|)}{n(n+1)}x(1-x)\int_0^{x-\frac{x}{\sqrt{n}}}|f_x'(t)|\frac{dt}{(x-t)^2}\\ &=\frac{2(n+|\lambda|)}{n(n+1)}x(1-x)\int_0^{x-\frac{x}{\sqrt{n}}}|f_x'(t)-f_x'(x)|\frac{dt}{(x-t)^2}\\ &\le\frac{2(n+|\lambda|)}{n(n+1)}x(1-x)\int_0^{x-\frac{x}{\sqrt{n}}}\bigvee_t^x(f_x')\frac{dt}{(x-t)^2}.\end{aligned}$$

Using the change of variables $t=x-\frac{x}{u}$, we have

$$\begin{aligned}\frac{2(n+|\lambda|)}{n(n+1)}x(1-x)\int_0^{x-\frac{x}{\sqrt{n}}}\bigvee_t^x(f_x')\frac{dt}{(x-t)^2}&=\frac{2(n+|\lambda|)}{n(n+1)}(1-x)\int_1^{\sqrt{n}}\bigvee_{x-\frac{x}{u}}^x(f_x')du\\ &\le\frac{2(n+|\lambda|)}{n(n+1)}(1-x)\sum_{k=1}^{\sqrt{n}}\bigvee_{x-\frac{x}{k}}^x(f_x')\end{aligned}$$

and hence, we get

$$\left|-\int_0^x \left[\int_t^x f_x'(u)du\right]\mathcal{K}_{n,\lambda}(x,t)dt\right| \leq \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f_x') + \frac{2(n+|\lambda|)}{n(n+1)}(1-x)\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f_x')$$

$$\leq \frac{2x}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f_x') + \frac{2(n+|\lambda|)}{n(n+1)}(1-x)\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f_x') \leq \frac{2}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f_x'). \tag{21}$$

Using a similar method, we get

$$\left|\int_x^1 \left[\int_x^t f_x'(u)\right]\mathcal{K}_{n,\lambda}(x,t)dt\right| \leq \frac{1-x}{\sqrt{n}}\bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}}(f_x') + \frac{2(n+|\lambda|)x}{n(n+1)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{1-x}{k}}(f_x')$$

$$\leq \frac{2(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{1-x}{k}}(f_x') + \frac{2(n+|\lambda|)x}{n(n+1)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{1-x}{k}}(f_x')$$

$$\leq \frac{2}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{1-x}{k}}(f_x'). \tag{22}$$

The relations (20), (21), and (22) complete the proof of the theorem. □

7 Numerical Results

In this section, we will analyze the theoretical results presented in the previous sections by numerical examples.

Example 7.1 Let $f(x) = \sin(2\pi x)$ and $\lambda = 0.5$. Figure 1 is given the graphs of function f and operator $U_{n,\lambda}$ for $n = 10$ and $n = 15$, respectively. This example explains the convergence of the operators $U_{n,\lambda}$ that are going to the function f if the values of n are increasing.

Example 7.2 Let $\lambda = 1$, $f(x) = (x^2+3x)e^x$ and $E_{n,\lambda}(f;x) = \left|f(x) - U_{n,\lambda}(f;x)\right|$ be the error function of genuine λ-Bernstein–Durrmeyer operators. Figure 2 is given the graphs of function f and operator $U_{n,\lambda}$, for $n = 5$, $n = 7$ and $n = 10$, respectively. This example explains the convergence of the operators $U_{n,\lambda}$ that are going to the function f if the values of n are increasing. Also, the error of approximation is illustrated in Fig. 3.

Example 7.3 For $\lambda = -1$, the convergence of genuine λ-Bernstein–Durrmeyer operators to $f(x) = \left(x - \frac{1}{4}\right) sin(2\pi x)$ is illustrated in Fig. 4. Also, for $n = 5, 7, 10$ the error functions $E_{n,\lambda}$ are given in Fig. 5.

Fig. 1 Convergence of $U_{n,\lambda}(f;x)$ to $f(x)$

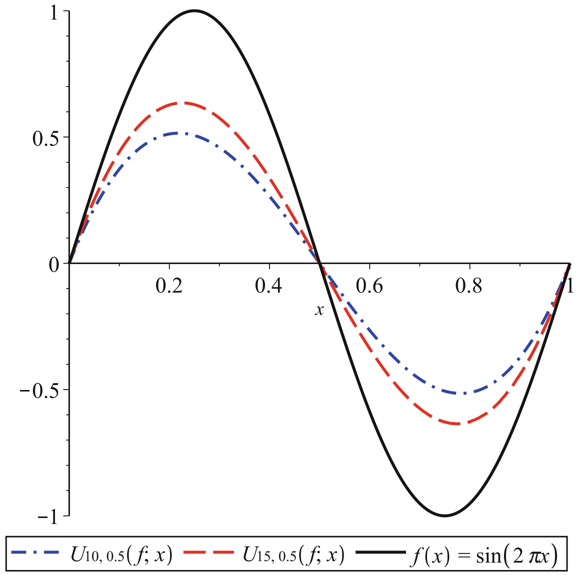
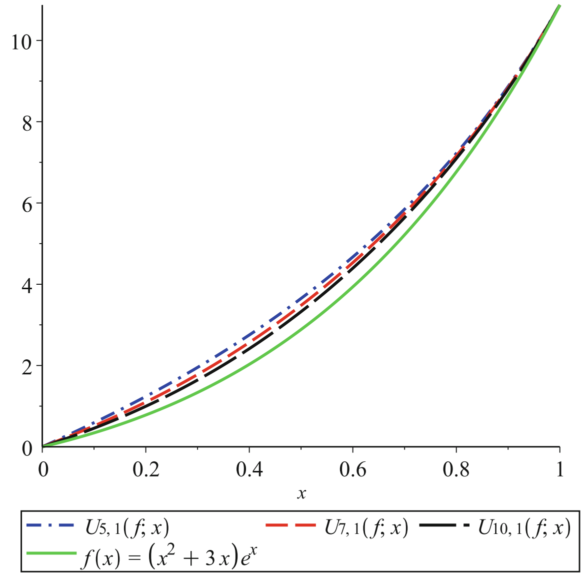


Fig. 2 Approximation process



Example 7.4 Let $f(x) = \left(x - \frac{1}{4}\right)\left(x - \frac{1}{2}\right)\left(x - \frac{3}{8}\right)$ and $n = 7$. In Fig. 6, is illustrated the convergence of genuine λ-Bernstein-Durrmeyer operator for $\lambda = -1, 0, 1$. In Fig. 7, we give the graphs of error functions. We can see that in this special case the error for genuine λ-Bernstein–Durrmeyer operators $U_{7,1}$, is smaller than for $U_{7,0}$, that is classical genuine Bernstein–Durrmeyer operator.

Fig. 3 Error of approximation

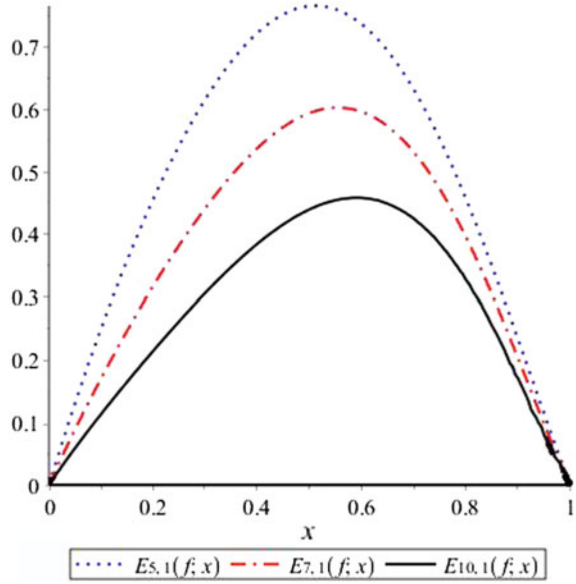


Fig. 4 Approximation process

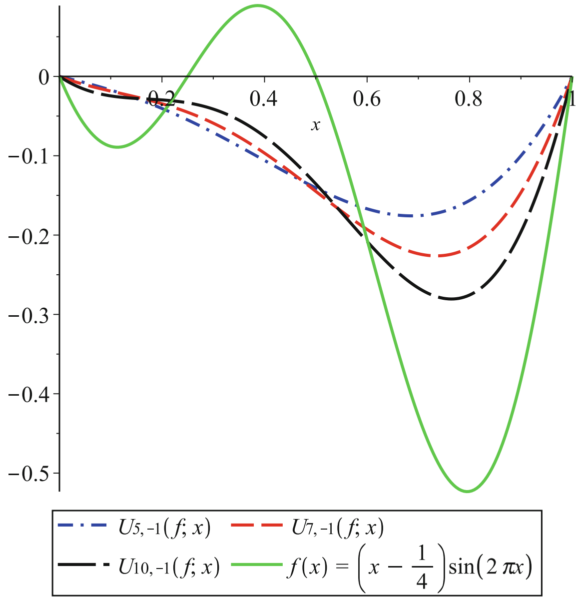


Fig. 5 Error of approximation

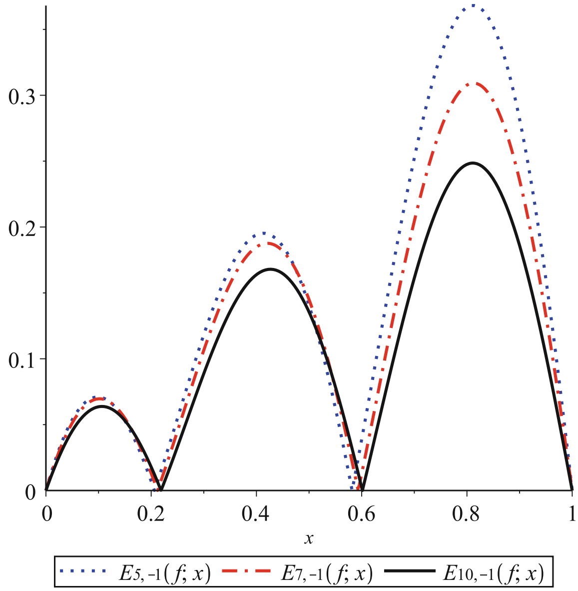
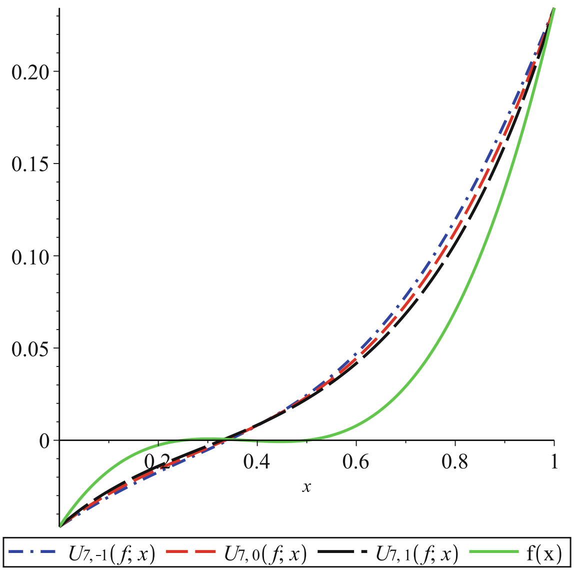


Fig. 6 Approximation process



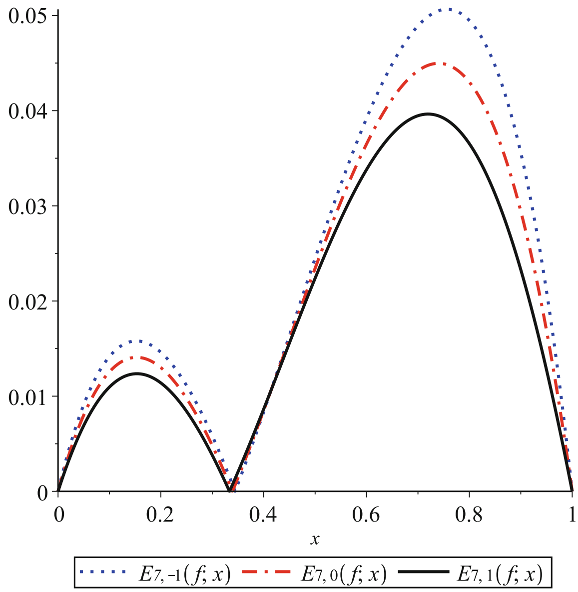


Fig. 7 Error of approximation

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Bivariate Szász-Type Operators Based on Multiple Appell Polynomials

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Abstract We introduce bivariate case of the Szász-type operators based on multiple Appell polynomials introduced by Varma (Stud. Univ. Babeş -Bolyai Math. 58, 361–369 (2013)). We establish a uniform convergence theorem and determine the degree of approximation in terms of the partial moduli of continuity of the approximated function. We estimate the error in simultaneous approximation of the function by the bivariate operators by using finite differences. We investigate the degree of approximation of the bivariate operators by means of the Peetre's K-functional. The rate of convergence of these operators is determined for twice continuously differentiable functions by Voronovskaja-type asymptotic theorem. The weighted approximation properties are derived for unbounded functions with a polynomial growth. Lastly, we introduce the associated generalized boolean sum (GBS) of the bivariate operators to study the approximation of Bögel-continuous and Bögel-differentiable functions and establish the approximation degree with the aid of the Lipschitz class of Bögel-continuous functions and the mixed modulus of smoothness.

Keywords Szász-type operators · Divided differences
Multiple Appell polynomials · Rate of convergence · Modulus of smoothness

Mathematics Subject Classification (2010) 41A10 · 41A25 · 41A36 · 41A63
26A15 · 26A16

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1 Introduction

First, we introduce some facts about Appell polynomials. The well-known Appell polynomials are generated by

$$g(t)e^{xt} = \sum_{k=0}^{\infty} \phi_k(x)\frac{t^n}{n!}.$$

For every power series

$$g(t_1, t_2) = \sum_{k_1,k_2=0}^{\infty} \frac{a_{k_1,k_2} t_1^{k_1} t_2^{k_2}}{k_1!k_2!} \tag{1.1}$$

with $g(0, 0) = a_{0,0} \neq 0$, the multiple Appell polynomials $\phi_{k_1,k_2}(x)$ have the following generating function:

$$g(t_1, t_2)e^{x(t_1+t_2)} = \sum_{k_1,k_2=0}^{\infty} \frac{\phi_{k_1,k_2}(x) t_1^{k_1} t_2^{k_2}}{k_1!k_2!}. \tag{1.2}$$

Remark 1.1 ([12]) Let $\{\phi_{k_1,k_2}(x)\}_{k_1,k_2=0}^{\infty}$ be a multiple Appell polynomial. Then, the following statements are equivalent:

(i) $\{\phi_{k_1,k_2}(x)\}_{k_1,k_2=0}^{\infty}$ is a set of multiple Appell polynomials.
(ii) There exists a sequence $\{\phi_{k_1,k_2}(x)\}_{k_1,k_2=0}^{\infty}$ with $\phi_{0,0}(x) \neq 0$ satisfying the differential recurrence relation

$$\phi_{k_1,k_2}(x) = \sum_{i=0}^{k_1} \sum_{j=0}^{k_2} \binom{k_1}{i} \binom{k_2}{j} \phi_{k_1-i,k_2-j} x^{k_1+k_2}.$$

(iii) For each $k_1 + k_2 \geq 1$, we have

$$\phi'_{k_1,k_2}(x) = k_1\phi_{k_1-1,k_2}(x) + k_2\phi_{k_1,k_2-1}(x).$$

In view of the above remark, let us assume that the multiple Appell polynomials satisfy:

(i) $g(1, 1) \neq 0$, $\frac{\phi_{k_1,k_2}}{g(1,1)} \geq 0$, for $k_1, k_2 \in \mathbb{N}$;
(ii) Equations (1.1) and (1.2) converge for $|t_1| < R_1$, $|t_2| < R_2$ $(R_1, R_2 > 1)$.

Varma [15] gave a generalization of Szász operators involving multiple Appell polynomials:

$$\mathfrak{F}_n(f; x, y) = \sum_{k_1,k_2=0}^{\infty} \frac{e^{-nx}}{g(1, 1)} \frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!} f\left(\frac{k_1 + k_2}{n}\right). \tag{1.3}$$

Also, in the same paper he defined a sequence of Kantorovich-type operators as

$$\mathfrak{L}_n(f;x,y)=\sum_{k_1,k_2=0}^{\infty}\frac{ne^{-nx}}{g(1,1)}\frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\int_{\frac{k_1+k_2}{n}}^{\frac{k_1+k_2+1}{n}}f(t)\,dt. \tag{1.4}$$

The purpose of this paper is to construct and investigate the bivariate Sz ász operators by linking to multiple Appell polynomials.

2 The Construction of the Operators

Let $E=[0,\infty)\times[0,\infty)$. For $\alpha>0$, let $C_\alpha(E)=\{f\in C(E):|f(x,y)|\leq \gamma e^{\alpha(x+y)},$ for some constant $\gamma>0\}$. If $f\in C_\alpha(E)$ then, inspired by [15], we introduce the bivariate case of Szász operators by means of multiple Appell polynomials as follows:

$$\mathfrak{A}_{n,m}(f;x,y)=\frac{e^{-(nx+my)}}{g^2(1,1)}\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!}f\left(\frac{k_1+k_2}{n},\frac{k_3+k_4}{m}\right), \tag{2.1}$$

whenever the right-hand side of (2.1) exists.

An appropriate generalization of the Kantorovich-type operators in the space of continuous functions of two variables is given by

$$\mathfrak{K}_{n,m}(f;x,y)=\frac{nme^{-(nx+my)}}{g^2(1,1)}\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!}\int_{\frac{k_1+k_2}{n}}^{\frac{k_1+k_2+1}{n}}\int_{\frac{k_3+k_4}{m}}^{\frac{k_1+k_2+1}{m}}f(t,s)\,dtds. \tag{2.2}$$

An appropriate generalization of the type Durrmeyer operator is given by

$$\mathfrak{K}_{n,m}(f;x,y)=\frac{nme^{-(nx+my)}}{g^2(1,1)}\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!}\int_0^{\infty}\int_0^{\infty}\Omega_m^n(t,s)f(t,s)\,dtds. \tag{2.3}$$

where $\Omega_m^n(t,s)=\frac{1}{\Gamma(\alpha+k_1+k_2+1)}\frac{1}{\Gamma(\alpha+k_3+k_4+1)}e^{-(nt+ms)}(nt)^{\alpha+k_1+k_2}(ms)^{\beta+k_3+k_4}$, $m,n\in N$ and $\alpha,\beta>-1$.

Throughout this paper, we use the following notation for the partial derivatives;

$$\frac{\partial g}{\partial t_i}=g_{t_i}\ and\ g_{t_it_j}=\frac{\partial^2 g}{\partial t_i\partial t_j}\quad i,j=1,2,3,4.$$

Lemma 2.1 *Let*

$$g(t_1,t_2)g(t_3,t_4)e^{x(t_1+t_2)+y(t_3+t_4)}=\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{\phi_{k_1,k_2}(x)}{k_1!k_2!}\frac{\phi_{k_3,k_4}(y)}{k_3!k_4!}t_1^{k_1}t_2^{k_2}t_3^{k_3}t_4^{k_4}. \tag{2.4}$$

Then we have

$$\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!}=g^2(1,1)e^{nx+my}; \tag{2.5}$$

$$\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{k_1\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!}=\left(g_{t_1}(1,1)g(1,1)+g^2(1,1)\frac{nx}{2}\right)e^{nx+my}; \tag{2.6}$$

$$\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{k_1k_2\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!}=g(1,1)e^{nx+my}\Big\{g_{t_1t_2}(1,1)+g_{t_1}(1,1)\frac{nx}{2} +g_{t_2}(1,1)\frac{nx}{2}+g(1,1)\frac{n^2x^2}{4}\Big\}; \tag{2.7}$$

$$\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\frac{k_3k_4\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!}=g(1,1)e^{nx+my}\Big\{g_{t_3t_4}(1,1)+g_{t_3}(1,1)\frac{my}{2}+ +g_{t_4}(1,1)\frac{my}{2}+g(1,1)\frac{m^2y^2}{4}\Big\}; \tag{2.8}$$

$$\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{k_1^2\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!}=g(1,1)e^{nx+my}\Big\{g(1,1)\frac{n^2x^2}{4}+g_{t_1}(1,1) +nxg_{t_1}(1,1)+g_{t_1t_1}(1,1)+g(1,1)\frac{nx}{2}\Big\}. \tag{2.9}$$

Proof By the generating function (2.4), after simple computations, the proof of the above lemma follows. Hence, we omit the details.

Lemma 2.2 *For the bivariate operators* $\mathfrak{A}_{n,m}$, $e_{ij}(x,y)=x^iy^j$, $0\le i,j\le 2$, *i, j being integers be the two-dimensional test functions, we have*

(i) $\mathfrak{A}_{n,m}(e_{00};x,y)=1$;
(ii) $\mathfrak{A}_{n,m}(e_{10};x,y)=x+\frac{g_{t_1}(1,1)+g_{t_2}(1,1)}{ng(1,1)}$;
(iii) $\mathfrak{A}_{n,m}(e_{01};x,y)=y+\frac{g_{t_3}(1,1)+g_{t_4}(1,1)}{mg(1,1)}$;

(iv)

$$\mathfrak{A}_{n,m}(e_{20}; x, y) = x^2 + \frac{x\big(2g_{t_1}(1,1) + 2g_{t_2}(1,1) + g(1,1)\big)}{ng(1,1)} + \frac{g_{t_1}(1,1) + g_{t_2}(1,1) + g_{t_1t_1}(1,1) + g_{t_2t_2}(1,1) + 2g_{t_1t_2}(1,1)}{n^2 g(1,1)};$$

(iv)

$$\mathfrak{A}_{n,m}(e_{02}; x, y) = y^2 + \frac{y\big(2g_{t_3}(1,1) + 2g_{t_4}(1,1) + g(1,1)\big)}{mg(1,1)} + \frac{g_{t_3}(1,1) + g_{t_4}(1,1) + g_{t_3t_3}(1,1) + g_{t_4t_4}(1,1) + 2g_{t_3t_4}(1,1)}{m^2 g(1,1)}.$$

Proof From (2.4), it is clear that for all $n, m \in \mathbb{N}$:

$$\begin{aligned}\mathfrak{A}_{n,m}(e_{00}; x, y) &= \frac{e^{-(nx+my)}}{g^2(1,1)} \sum_{k_1,k_2=0}^{\infty} \sum_{k_3,k_4=0}^{\infty} \frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!} \frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!} \\ &= \frac{e^{-(nx+my)}}{g^2(1,1)} g^2(1,1)e^{nx+my} = 1.\end{aligned}$$

Also, we obtain

$$\begin{aligned}\mathfrak{A}_{n,m}(e_{10}; x, y) &= \frac{e^{-(nx+my)}}{ng^2(1,1)} \sum_{k_1,k_2=0}^{\infty} \sum_{k_3,k_4=0}^{\infty} \frac{(k_1+k_2)\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!} \frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!} \\ &= \frac{e^{-(nx+my)}e^{nx+my}}{ng^2(1,1)} \Big(g_{t_1}(1,1)g(1,1) + g^2(1,1)\frac{nx}{2} + g_{t_2}(1,1)g(1,1) + g^2(1,1)\frac{nx}{2}\Big) \\ &= x + \frac{g_{t_1}(1,1) + g_{t_2}(1,1)}{ng(1,1)}.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathfrak{A}_{n,m}(e_{01}; x, y) &= \frac{e^{-(nx+my)}}{mg^2(1,1)} \sum_{k_1,k_2=0}^{\infty} \sum_{k_3,k_4=0}^{\infty} \frac{(k_3+k_4)\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!} \frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!} \\ &= \frac{e^{-(nx+my)}e^{nx+my}}{mg^2(1,1)} \Big(g_{t_3}(1,1)g(1,1) + g^2(1,1)\frac{my}{2} + g_{t_4}(1,1)g(1,1) + g^2(1,1)\frac{my}{2}\Big) \\ &= y + \frac{g_{t_3}(1,1) + g_{t_4}(1,1)}{mg(1,1)}.\end{aligned}$$

Finally, from (2.4) and (2.6), we get

$$\begin{aligned}
\mathfrak{A}_{n,m}(e_{20};x,y) &= \frac{e^{-(nx+my)}}{n^2g^2(1,1)}\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{(k_1+k_2)^2\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!}\\
&= \frac{e^{-(nx+my)}}{n^2g^2(1,1)}\Big(g(1,1)e^{nx+my}\Big\{g(1,1)\frac{n^2x^2}{2}+g_{t_1}(1,1)+g_{t_2}(1,1)\\
&+nx\big\{g_{t_1}(1,1)+g_{t_2}(1,1)\big\}+g_{t_1t_1}(1,1)+g_{t_2t_2}(1,1)+g(1,1)nx\\
&+2g_{t_1t_2}(1,1)+2g_{t_1}(1,1)\frac{nx}{2}+2g_{t_2}(1,1)\frac{nx}{2}+2g(1,1)\frac{n^2x^2}{4}\Big\}\Big)\\
&= x^2+\frac{x\big(2g_{t_1}(1,1)+2g_{t_2}(1,1)+g(1,1)\big)}{ng(1,1)}\\
&+\frac{g_{t_1}(1,1)+g_{t_2}(1,1)+g_{t_1t_1}(1,1)+g_{t_2t_2}(1,1)+2g_{t_1t_2}(1,1)}{n^2g(1,1)}.
\end{aligned}$$

Analogously, we can show the formula

$$\begin{aligned}
\mathfrak{A}_{n,m}(e_{02};x,y) &= \frac{e^{-(nx+my)}}{m^2g^2(1,1)}\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{(k_3+k_4)^2\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!}\\
&= \frac{e^{-(nx+my)}}{m^2g^2(1,1)}\Big(g(1,1)e^{nx+my}\Big\{g(1,1)\frac{m^2y^2}{2}+g_{t_3}(1,1)+g_{t_4}(1,1)\\
&+my\big\{g_{t_3}(1,1)+g_{t_4}(1,1)\big\}+g_{t_3t_3}(1,1)+g_{t_4t_4}(1,1)+g(1,1)my\\
&+2g_{t_3t_4}(1,1)+2g_{t_3}(1,1)\frac{my}{2}+2g_{t_4}(1,1)\frac{my}{2}+2g(1,1)\frac{m^2y^2}{4}\Big\}\Big)\\
&= y^2+\frac{y\big(2g_{t_3}(1,1)+2g_{t_4}(1,1)+g(1,1)\big)}{mg(1,1)}\\
&+\frac{g_{t_3}(1,1)+g_{t_4}(1,1)+g_{t_3t_3}(1,1)+g_{t_4t_4}(1,1)+2g_{t_3t_4}(1,1)}{m^2g(1,1)}.
\end{aligned}$$

From (2.1) and by applying relation (2.4) in Lemma 2.1, we obtain the identities (v)–(vii). Hence, we omit the details.

Remark 2.3 For the operator given by (2.1), the first few central moments are given by

(i) $\mathfrak{A}_{n,m}(e_{10}-x;x,y)=\frac{g_{t_1}(1,1)+g_{t_2}(1,1)}{ng(1,1)}$;
(ii) $\mathfrak{A}_{n,m}(e_{01}-y;x,y)=\frac{g_{t_3}(1,1)+g_{t_4}(1,1)}{mg(1,1)}$;
(iii) $\mathfrak{A}_{n,m}((e_{10}-x)^2;x,y)=\frac{x}{n}+\frac{g_{t_1}(1,1)+g_{t_2}(1,1)+g_{t_1t_1}(1,1)+g_{t_2t_2}(1,1)+2g_{t_1t_2}(1,1)}{n^2g(1,1)}$;
(iv) $\mathfrak{A}_{n,m}((e_{01}-y)^2;x,y)=\frac{y}{m}+\frac{g_{t_3}(1,1)+g_{t_4}(1,1)+g_{t_3t_3}(1,1)+g_{t_4t_4}(1,1)+2g_{t_3t_4}(1,1)}{m^2g(1,1)}$.

Definition 2.4 (*See* [2]) For $f\in C(\Pi_{ab})$ where $\Pi_{ab}=[0,a]\times[0,b]$, the complete modulus of continuity for the function $f(x,y)$ is defined by

$$\omega(f;\delta_1,\delta_2)=sup\{|f(u,v)-f(x,y)|:(u,v),(x,y)\in I_{ab},\ |u-x|\le\delta_1,|v-y|\le\delta_2\},$$

and its partial modulus of continuity with respect to x and y is given by

$$\omega^{(1)}(f;\delta) = \sup_{0\le y\le b}\sup_{|x_1-x_2|\le\delta}\{|f(x_1,y)-f(x_2,y)|\},$$

$$\omega^{(2)}(f;\delta) = \sup_{0\le x\le a}\sup_{|y_1-y_2|\le\delta}\{|f(x,y_1)-f(x,y_2)|\}.$$

Now, we investigate the rate of convergence of the approximation by the bivariate operators $\mathfrak{A}_{n,m}$, in terms of the partial of modulus of continuity.

Theorem 2.5 *If $f \in C_\alpha(E)$, then*

$$\lim_{n,m\to\infty}\mathfrak{A}_{n,m}(f;x,y) = f(x,y),$$

the convergence being uniform in each compact Π_{ab}.

Proof The proof is obvious from the Korovkin Theorem [16].

Theorem 2.6 *Let $f \in C_\alpha(E)$. Then, for all $(x,y)\in(E)$, and $t_1,t_2\in[0,\infty)$ we have*

$$\begin{aligned}
&\left|\mathfrak{A}_{n,m}(f;x,y)-f(x,y)\right| \\
&\le \omega^{(1)}(f;\frac{1}{\sqrt{n}})\left\{1+\sqrt{x+\frac{1}{ng(1,1)}\{g_{t_1}(1,1)+g_{t_2}(1,1)+g_{t_{11}}(1,1)+2g_{t_{12}}(1,1)+g_{t_{22}}(1,1)\}}\right\} \\
&+\omega^{(2)}(f;\frac{1}{\sqrt{m}})\left\{1+\sqrt{y+\frac{1}{mg(1,1)}\{g_{t_3}(1,1)+g_{t_4}(1,1)+g_{t_{33}}(1,1)+2g_{t_{34}}(1,1)+g_{t_{44}}(1,1)\}}\right\}.
\end{aligned}$$

Proof Suppose that $f \in C_\alpha(E)$. By Lemma 2.2 case (i), we obtain the following inequality:

$$\begin{aligned}
\left|\mathfrak{A}_{n,m}(f;x,y)-f(x,y)\right| &\le \frac{e^{-(nx+my)}}{g^2(1,1)}\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!} \\
&\times\left|f\left(\frac{k_1+k_2}{n},\frac{k_3+k_4}{m}\right)-f(x,y)\right| \\
&\le \frac{e^{-(nx+my)}}{g^2(1,1)}\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!} \\
&\times\left(\left|f\left(\frac{k_1+k_2}{n},\frac{k_3+k_4}{m}\right)-f(x,\frac{k_3+k_4}{m})\right|\right. \\
&+\left.\left|f\left(x,\frac{k_3+k_4}{m}\right)-f(x,y)\right|\right)
\end{aligned}$$

$$= \frac{e^{-(nx+my)}}{g^2(1,1)} \sum_{k_1,k_2=0}^{\infty} \sum_{k_3,k_4=0}^{\infty} \frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!} \frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!}$$
$$\times \left| f\left(\frac{k_1+k_2}{n}, \frac{k_3+k_4}{m}\right) - f(x, \frac{k_3+k_4}{m}) \right|$$
$$+ \frac{e^{-(nx+my)}}{g^2(1,1)} \sum_{k_1,k_2=0}^{\infty} \sum_{k_3,k_4=0}^{\infty} \frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!} \frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!}$$
$$\times \left| f\left(x, \frac{k_3+k_4}{m}\right) - f(x,y) \right| = I_1 + I_2, \ (say). \tag{2.10}$$

Now, we consider I_1. By using well-known properties of the modulus of continuity, we obtain the formula

$$I_1 = \frac{e^{-(nx+my)}}{g^2(1,1)} \sum_{k_1,k_2=0}^{\infty} \sum_{k_3,k_4=0}^{\infty} \frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!} \frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!} \left| f\left(\frac{k_1+k_2}{n}, \frac{k_3+k_4}{m}\right) - f(x, \frac{k_3+k_4}{m}) \right|$$
$$\leq \omega^{(1)}(f;\delta_n) \left\{ 1 + \frac{1}{\delta_n} \frac{e^{-(nx+my)}}{g^2(1,1)} \sum_{k_1,k_2=0}^{\infty} \sum_{k_3,k_4=0}^{\infty} \frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!} \frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!} \left| \frac{k_1+k_2}{n} - x \right| \right\}.$$

If we apply Cauchy–Schwarz inequality for sums, we obtain

$$I_1 \leq \omega^{(1)}(f;\delta_n) \left\{ 1 + \frac{1}{\delta_n} \left(\frac{e^{-(nx+my)}}{g^2(1,1)} \sum_{k_1,k_2=0}^{\infty} \sum_{k_3,k_4=0}^{\infty} \frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!} \frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!} \right)^{\frac{1}{2}} \right.$$
$$\left. \times \left(\frac{e^{-(nx+my)}}{g^2(1,1)} \sum_{k_1,k_2=0}^{\infty} \sum_{k_3,k_4=0}^{\infty} \frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!} \frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!} \left(\frac{k_1+k_2}{n} - x \right)^2 \right)^{\frac{1}{2}} \right\}$$
$$= \omega^{(1)}(f;\delta_n) \left\{ 1 + \frac{1}{\delta_n} \{\mathfrak{A}_{n,m}(e_{00}; x, y)\}^{\frac{1}{2}} \times \{\mathfrak{A}_{n,m}((e_{10}-x)^2; x, y)\}^{\frac{1}{2}} \right\}. \tag{2.11}$$

In view of Lemma 2.2 and taking $\delta_n = \frac{1}{\sqrt{n}}$, we have:

$$I_1 \leq \omega^{(1)}(f; \frac{1}{\sqrt{n}}) \left\{ 1 + \sqrt{x + \frac{\{g_{t_1}(1,1) + g_{t_2}(1,1) + g_{t_1t_1}(1,1) + 2g_{t_1t_2}(1,1) + g_{t_2t_2}(1,1)\}}{ng(1,1)}} \right\}. \tag{2.12}$$

Similarly, we obtain

$$I_2 = \omega^{(2)}(f; \frac{1}{\sqrt{m}}) \left\{ 1 + \sqrt{y + \frac{\{g_{t_3}(1,1) + g_{t_4}(1,1) + g_{t_{33}}(1,1) + 2g_{t_{34}}(1,1) + g_{t_{44}}(1,1)\}}{mg(1,1)}} \right\}. \tag{2.13}$$

Combining (2.10)–(2.13), we reach the desired result.

Now, we can give the following statement about error estimation in terms of higher-order partial moduli of continuity in simultaneous approximation for the operators (2.1). Let $C_\alpha^r(E) = \{f \in C(E) : f^{(r)} \in C_\alpha(E)\}$

Theorem 2.7 *Let $f \in C_\alpha^r(E)$, then the following estimate holds*

$$(i)\ \left|\frac{\partial^r}{\partial x^r}\mathfrak{A}_{n,m}(f;x,y) - \frac{\partial^r}{\partial x^r}f(x,y)\right|$$
$$\leq r!\omega^{(1)}(f;\frac{1}{\sqrt{n}})\left\{1+\sqrt{x+\frac{1}{ng(1,1)}\{g_{t_1}(1,1)+g_{t_2}(1,1)+g_{t_1t_1}(1,1)+2g_{t_1t_2}(1,1)+g_{t_2t_2}(1,1)\}}\right\}$$
$$+r!\omega^{(2)}(f;\frac{1}{\sqrt{m}})\left\{1+\sqrt{y+\frac{1}{mg(1,1)}\{g_{t_3}(1,1)+g_{t_4}(1,1)+g_{t_3t_3}(1,1)+2g_{t_3t_4}(1,1)+g_{t_4t_4}(1,1)\}}\right\}$$
$$+\omega^{(1)}(\frac{\partial^r f}{\partial x^r};\frac{r}{n}).$$
$$(ii)\ \left|\frac{\partial^r}{\partial y^r}\mathfrak{A}_{n,m}(f;x,y) - \frac{\partial^r}{\partial y^r}f(x,y)\right|$$
$$\leq r!\omega^{(1)}(f;\frac{1}{\sqrt{n}})\left\{1+\sqrt{x+\frac{1}{ng(1,1)}\{g_{t_3}(1,1)+g_{t_3}(1,1)+g_{t_1t_1}(1,1)+2g_{t_1t_2}(1,1)+g_{t_2t_2}(1,1)\}}\right\}$$
$$+r!\omega^{(2)}(f;\frac{1}{\sqrt{m}})\left\{1+\sqrt{y+\frac{1}{mf(1,1)}\{g_{t_1}(1,1)+g_{t_2}(1,1)g_{t_1t_1}(1,1)+2g_{t_2t_1}(1,1)+g_{t_2t_2}(1,1)\}}\right\}$$
$$+\omega^{(2)}(\frac{\partial^r f}{\partial y^r};\frac{r}{m}).$$

Proof (i) The partial derivative of $\mathfrak{A}_{n,m}(f;x,y)$ with respect to x may be written as follows:

$$\frac{\partial}{\partial x}\mathfrak{A}_{n,m}(f;x,y) = \frac{-ne^{-(nx+my)}}{g^2(1,1)}\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!}f\left(\frac{k_1+k_2}{n},\frac{k_3+k_4}{m}\right)$$
$$+\frac{ne^{-(nx+my)}}{2g^2(1,1)}\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{\phi'_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!}f\left(\frac{k_1+k_2}{n},\frac{k_3+k_4}{m}\right).$$

By virtue of the results (iii) from Remark (1.1), we deduce that

$$\frac{\partial}{\partial x}\mathfrak{A}_{n,m}(f;x,y) = \frac{e^{-(nx+my)}}{g^2(1,1)}\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{\phi_{k_1,k_2}(x)}{k_1!k_2!}\frac{\phi_{k_3,k_4}(x)}{k_3!k_4!}\frac{\Delta^1_{\frac{1}{n}}f\left(\frac{k_1+k_2}{n},\frac{k_3+k_4}{m}\right)}{\frac{1}{n}}, \tag{2.14}$$

where $\Delta^1_{\frac{1}{n}}f\left(\frac{k_1+k_2}{n},\frac{k_3+k_4}{m}\right)$ is the difference of order 1 of f with step $\frac{1}{n}$. From (2.14), one computes the rth derivative of $\mathfrak{A}_{n,m}$ as

$$\frac{\partial^r}{\partial x^r}\mathfrak{A}_{n,m}(f;x,y)) = r!\frac{e^{-(nx+my)}}{g^2(1,1)}\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{\phi_{k_1,k_2}(x)}{k_1!k_2!}\frac{\phi_{k_3,k_4}(x)}{k_3!k_4!}\frac{\Delta^r_{\frac{1}{n}} f\left(\frac{k_1+k_2}{n},\frac{k_3+k_4}{m}\right)}{r!\left(\frac{1}{n}\right)^r}$$

$$= r!\frac{e^{-(nx+my)}}{g^2(1,1)}\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{\phi_{k_1,k_2}(x)}{k_1!k_2!}\frac{\phi_{k_3,k_4}(x)}{k_3!k_4!}\left[\frac{k_1+k_2}{n},\frac{k_1+k_2+1}{n},\ldots,\frac{k_1+k_2+r}{n},f;\frac{k_3+k_4}{m}\right]$$

$$= r!\frac{e^{-(nx+my)}}{g^2(1,1)}\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{\phi_{k_1,k_2}(x)}{k_1!k_2!}\frac{\phi_{k_3,k_4}(x)}{k_3!k_4!}h\left(\frac{k_1+k_2}{n},\frac{k_3+k_4}{m}\right), \tag{2.15}$$

where $h(s,t) = (s, s+\frac{1}{n}, s+\frac{2}{n},\ldots,s+\frac{r}{n}; f, t)$ and so we obtain

$$\frac{\partial^r}{\partial x^r}\mathfrak{A}_{n,m}(f;x,y)) = r!\mathfrak{A}_{n,m}(h;x,y)). \tag{2.16}$$

From (2.16), the difference $\left|\frac{\partial^r}{\partial x^r}\mathfrak{A}_{n,m}(f;x,y)) - \frac{\partial^r}{\partial x^r}f(x,y)\right|$ is represented as follows:

$$\left|\frac{\partial^r}{\partial x^r}\mathfrak{A}_{n,m}(f;x,y)) - \frac{\partial^r}{\partial x^r}f(x,y)\right| \le r!\left|\mathfrak{A}_{n,m}(h;x,y)) - h(x,y)\right| + \left|r!h(x,y) - \frac{\partial^r}{\partial x^r}f(x,y)\right|. \tag{2.17}$$

By using Theorem 2.6, we obtain:

$$\left|\frac{\partial^r}{\partial x^r}\mathfrak{A}_{n,m}(f;x,y)) - \frac{\partial^r}{\partial x^r}f(x,y)\right|$$

$$\le r!\omega^{(1)}(h;\frac{1}{\sqrt{n}})\left\{1+\sqrt{x+\frac{1}{nf(1,1)}\{g_{t_1}(1,1)+g_{t_2}(1,1)+g_{t_{11}}(1,1)+2g_{t_{12}}(1,1)+g_{t_{22}}(1,1)\}}\right\}$$

$$+ r!\omega^{(2)}(h;\frac{1}{\sqrt{m}})\left\{1+\sqrt{y+\frac{1}{mg(1,1)}\{g_{t_3}(1,1)+g_{t_4}(1,1)+g_{t_{33}}(1,1)+2g_{t_{34}}(1,1)+g_{t_{44}}(1,1)\}}\right\}$$

$$+\left|r!h(x,y) - \frac{\partial^r}{\partial x^r}f(x,y)\right|. \tag{2.18}$$

On the other hand,

$$\left|h(s+\delta_1,t+\delta_2) - h(s,t)\right| \le |h(s+\delta_1,t+\delta_2) - h(s+\delta_1,t)| + |h(s+\delta_1,t) - h(s,t)| = \Delta_1+\Delta_2,\ say. \tag{2.19}$$

We estimate Δ_1:

$$\begin{aligned}\Delta_1 = \Big|h(s+\delta_1, t+\delta_2) - h(s+\delta_1, t)\Big| \\ &\leq \Big|[\, t+\delta_2, t+\delta_2+\frac{1}{m}, \ldots, t+\delta_2+\frac{r}{m}; f, s+\delta_1] \\ &- [t, t+\frac{1}{m}, \ldots, t+\frac{r}{m}; f, s+\delta_1]\Big| \\ &= \frac{1}{r!}\left|\frac{\partial^r}{\partial t^r} f(s+\delta_1, t+\delta_2+\varphi_1\frac{r}{m}) - \frac{\partial^r}{\partial t^r} f(s+\delta_1, t+\varphi_2\frac{r}{m})\right|\end{aligned}$$

where $\varphi_1, \varphi_2 \in (0, 1)$. Hence, we get

$$\Delta_1 \leq \frac{1}{r!}\omega^{(2)}\left(\frac{\partial^r}{\partial t^r} f; \delta_2 + |\varphi_1 - \varphi_2|\frac{r}{m}\right) \leq \frac{1}{r!}\omega^{(2)}\left(\frac{\partial^r}{\partial t^r} f; \delta_2 + \frac{r}{m}\right). \quad (2.20)$$

Similarly, we have

$$\Delta_2 \leq \frac{1}{r!}\omega^{(1)}\left(\frac{\partial^r}{\partial s^r} f; \delta_1 + \frac{r}{n}\right). \quad (2.21)$$

On other hand, we write

$$\begin{aligned}\left|r!h(x, y) - \frac{\partial^r}{\partial x^r} f(x, y)\right| &= \left|r![x, x+\frac{1}{n}, \ldots, x+\frac{r}{n}; f, y] - \frac{\partial^r}{\partial x^r} f(x, y)\right| \\ &= \left|\frac{\partial^r}{\partial x^r} f(x+\varphi_3\frac{r}{n}, y) - \frac{\partial^r}{\partial x^r} f(x, y)\right| \\ &\leq \omega^{(1)}(\frac{\partial^r}{\partial x^r} f; \varphi_3\frac{r}{n}) \leq \omega^{(1)}(\frac{\partial^r f}{\partial x^r}; \frac{r}{n}), \quad (2.22)\end{aligned}$$

where $\varphi_3 \in [0, 1]$. Using (2.20)–(2.22) in (2.18), the proof of Theorem 2.7 is completed. The proof (ii) can be given in a similar manner.

Let $C^2(\Pi_{ab})$ be the space of all functions f such that $\frac{\partial^i f}{\partial x^i}, \frac{\partial^i f}{\partial y^i} \in C(\Pi_{ab}), i = 1, 2.$ The norm on the space $C^2(\Pi_{ab})$ is defined as

$$\| f \|_{C^2(\Pi_{ab})} = \| f \|_{C(\Pi_{ab})} + \sum_{i=1}^{2}\left(\left\|\frac{\partial^i f}{\partial x^i}\right\|_{C(\Pi_{ab})} + \left\|\frac{\partial^i f}{\partial y^i}\right\|_{C(\Pi_{ab})}\right),$$

where $||.||_{C(\Pi_{ab})}$ is the sup-norm.

In [7], the Peetre's K-functional and the second order modulus of smoothness for function $f \in C(\Pi_{ab})$ are defined, respectively, as

$$K(f; \delta) = \inf_{\eta \in C^2(\Pi_{ab})}\left\{\| f - \eta \|_{C(\Pi_{ab})} + \delta \| \eta \|_{C(\Pi_{ab})}\right\}$$

and

$$\omega_2(f;\delta) = \sup_{\sqrt{u^2+v^2}\le\delta} \| f(x+2u, y+2v) - 2f(x+u, y+v) + f(x,y) \|_{C(\Pi_{ab})} .$$

Theorem 2.8 *If $f \in C^2(\Pi_{ab})$ then for $(x,y) \in \Pi_{ab}$, the following estimate holds:*

$$\left|\mathfrak{A}_{n,m}(f;x,y) - f(x,y)\right| \le 2K(f;\frac{\delta_{n,m}}{2}) \tag{2.23}$$

where $\delta_{n,m} = max\left\{\left|\frac{g_{t_1}(1,1)+g_{t_2}(1,1)}{ng(1,1)}\right|, \frac{1}{2}\left\{\frac{x}{n} + \left|\frac{g_{t_1}(1,1)+g_{t_2}(1,1)+g_{t_1t_1}(1,1)+g_{t_2t_2}(1,1)+2g_{t_1t_2}(1,1)}{n^2g(1,1)}\right|\right\},\right.$ $\left.\left|\frac{g_{t_3}(1,1)+g_{t_4}(1,1)}{mg(1,1)}\right|, \frac{1}{2}\left\{\frac{y}{m} + \left|\frac{g_{t_3}(1,1)+g_{t_4}(1,1)+g_{t_3t_3}(1,1)+g_{t_4t_4}(1,1)+2g_{t_3t_4}(1,1)}{m^2g(1,1)}\right|\right\}\right\}.$

Proof Let $\psi \in C^2(\Pi_{ab})$. Using Taylor's theorem, we can write

$$\begin{aligned}\psi(z_1,z_2)-\psi(x,y) &= \psi(z_1,y) - \psi(x,y) + \psi(z_1,z_2) - \psi(z_1,y)\\ &= \frac{\partial\psi(x,y)}{\partial x}(z_1-x) + \int_x^{z_1}(z_1-\alpha)\frac{\partial^2\psi(x,y)}{\partial\alpha^2}d\alpha\\ &+\frac{\partial\psi(x,y)}{\partial y}(z_2-y) + \int_y^{z_2}(z_2-\beta)\frac{\partial^2\psi(x,y)}{\partial\beta^2}d\beta.\end{aligned} \tag{2.24}$$

Applying $\mathfrak{A}_{n,m}$ to both sides of the relation (2.24), we deduce that

$$\begin{aligned}\left|\mathfrak{A}_{n,m}(\psi;x,y)-\psi(x,y)\right| &\le \left|\frac{\partial\psi}{\partial x}\right|\left|\mathfrak{A}_{n,m}(z_1-x;x,y)\right| + \left|\frac{\partial\psi^2}{\partial x^2}\right|\left|\mathfrak{A}_{n,m}\left(\frac{(z_1-x)^2}{2};x,y\right)\right|\\ &+\left|\frac{\partial\psi}{\partial y}\right|\left|\mathfrak{A}_{n,m}(z_2-y;x,y)\right| + \left|\frac{\partial\psi^2}{\partial y^2}\right|\left|\mathfrak{A}_{n,m}\left(\frac{(z_2-y)^2}{2};x,y\right)\right|.\end{aligned} \tag{2.25}$$

By using Remark 2.3, we obtain

$$\begin{aligned}&\| \mathfrak{A}_{n,m}(\psi;x,y) - \psi(x,y) \|_{C(\Pi_{ab})}\\ &\le \| \frac{\partial\psi}{\partial x} \|_{C(\Pi_{ab})} \left|\frac{g_{t_1}(1,1)+g_{t_2}(1,1)}{ng(1,1)}\right| + \| \frac{\partial\psi}{\partial y} \|_{C(\Pi_{ab})} \left|\frac{g_{t_3}(1,1)+g_{t_4}(1,1)}{mg(1,1)}\right|\\ &+ \| \frac{\partial^2\psi}{\partial x^2} \|_{C(\Pi_{ab})} \frac{1}{2}\left\{\frac{x}{n} + \left|\frac{g_{t_1}(1,1)+g_{t_2}(1,1)+g_{t_1t_1}(1,1)+g_{t_2t_2}(1,1)+2g_{t_1t_2}(1,1)}{n^2g(1,1)}\right|\right\}\\ &+ \| \frac{\partial^2\psi}{\partial y^2} \|_{C(\Pi_{ab})} \frac{1}{2}\left\{\frac{y}{m} + \left|\frac{g_{t_3}(1,1)+g_{t_4}(1,1)+g_{t_3t_3}(1,1)+g_{t_4t_4}(1,1)+2g_{t_3t_4}(1,1)}{m^2g(1,1)}\right|\right\}\\ &\le \delta_{n,m}\left\{\| \frac{\partial\psi}{\partial x} \|_{C(\Pi_{ab})} + \| \frac{\partial^2\psi}{\partial x^2} \|_{C(\Pi_{ab})} + \| \frac{\partial\psi}{\partial y} \|_{C(\Pi_{ab})} + \| \frac{\partial^2\psi}{\partial y^2} \|_{C(\Pi_{ab})}\right\}\\ &\le \delta_{n,m} \| \psi \|_{C(\Pi_{ab})},\end{aligned} \tag{2.26}$$

where $\delta_{n,m} = max\left\{\left|\frac{g_{t_1}(1,1)+g_{t_2}(1,1)}{ng(1,1)}\right|, \frac{1}{2}\left\{\frac{x}{n}+\left|\frac{g_{t_1}(1,1)+g_{t_2}(1,1)+g_{t_1t_1}(1,1)+g_{t_2t_2}(1,1)+2g_{t_1t_2}(1,1)}{n^2g(1,1)}\right|\right\},\right.$
$\left.\left|\frac{g_{t_3}(1,1)+g_{t_4}(1,1)}{mg(1,1)}\right|, \frac{1}{2}\left\{\frac{y}{m}+\left|\frac{g_{t_3}(1,1)+g_{t_4}(1,1)+g_{t_3t_3}(1,1)+g_{t_4t_4}(1,1)+2g_{t_3t_4}(1,1)}{m^2g(1,1)}\right|\right\}\right\}.$
By the linearity property of the operator $\mathfrak{A}_{n,m}$, we have

$$\begin{aligned}\| \mathfrak{A}_{n,m}(f;x,y) - f(x,y) \|_{C(\Pi_{ab})} &\leq \| \mathfrak{A}_{n,m}(f;x,y) - \mathfrak{A}_{n,m}(\psi;x,y) \|_{C(\Pi_{ab})}\\ &+ \| f(x,y) - \psi(x,y) \|_{C(\Pi_{ab})} + \| \mathfrak{A}_{n,m}(\psi;x,y) - \psi(x,y) \|_{C(\Pi_{ab})}\\ &\leq \| f-\psi \|_{C(\Pi_{ab})} \left|\mathfrak{A}_{n,m}(e_{00};x,y)\right| + \| f-\psi \|_{C(\Pi_{ab})} + \delta_{n,m} \| \psi \|_{C^2(\Pi_{ab})}\\ &\leq 2\left\{\| f-\psi \|_{C(\Pi_{ab})} + \frac{1}{2}\delta_{n,m} \| \psi \|_{C^2(\Pi_{ab})}\right\}.\end{aligned}$$

By taking the infimum over $\psi \in C^2(\Pi_{ab})$, the proof is completed.

Theorem 2.9 *If $f \in C_\alpha(E)$ such that $f_x, f_y, f_{xx}, f_{yy}, f_{xy} \in C_\alpha(E)$, we have*

$$\begin{aligned}\lim_{n\to\infty} n\big(\mathfrak{A}_{n,n}(f;x,y) - f(x,y)\big) &= \frac{g_{t_1}(1,1)+g_{t_2}(1,1)}{g(1,1)} f_x(x,y)\\ &+ \frac{g_{t_3}(1,1)+g_{t_4}(1,1)}{g(1,1)} f_y(x,y) + \frac{1}{2}xf_{xx} + \frac{1}{2}yf_{yy},\end{aligned}$$

uniformly in Π_{ab}.

Proof Let $f_x, f_y, f_{xx}, f_{yy}, f_{xy} \in C_\alpha(E)$. For each $(x,y) \in E$, let us define the function $\varkappa$:

$$\varkappa_{(x,y)}(u,v) = \begin{cases} \dfrac{f(u,v)-\sum\limits_{j=0}^{2}\frac{1}{j!}\left[(u-x)\frac{\partial}{\partial x}+(v-y)\frac{\partial}{\partial y}\right]^j f(x,y)}{\sqrt{(u-x)^4+(v-y)^4}}, & (u,v) \neq (x,y),\\ 0, & \text{else,}\end{cases}$$

where $g^{(j)}$ is a derivative of function g for $j = 0, 1, 2$. Then, using our hypothesis we have $\varkappa_{(x,y)}(u,v) = 0$ and $\varkappa_{(x,y)}(\cdot,\cdot) \in C_\alpha(E)$. By the Taylor formula for $f \in C_\alpha(E)$, we may write:

$$\begin{aligned}f(u,v) = f(x,y) + f_x(x,y)(u-x) + f_y(x,y)(v-y) + \frac{1}{2}\Big\{ f_{xx}(x,y)(u-x)^2 + f_{yy}(x,y)(v-y)^2\\ + 2f_{xy}(x,y)(u-x)(v-y)\Big\} + \varkappa_{(x,y)}(u,v)\sqrt{(u-x)^4+(v-y)^4}. \qquad (2.27)\end{aligned}$$

Now,

$$\begin{aligned}n\{\mathfrak{A}_{n,n}(f;x,y) - f(x,y)\} = nf_x(x,y)\mathfrak{A}_{n,n}((u-x);x,y) + nf_y(x,y)\mathfrak{A}_{n,n}((v-y);x,y)\\ + \frac{1}{2}\Big\{ f_{xx}(x,y)n\mathfrak{A}_{n,n}((u-x)^2;x,y) + f_{yy}(x,y)n\mathfrak{A}_{n,m}((v-y)^2;x,y)\\ +2f_{xy}(x,y)n\mathfrak{A}_{n,n}((u-x)(v-y);x,y)\Big\} + n\mathfrak{A}_{n,m}(\varkappa_{(x,y)}(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y).\\ (2.28)\end{aligned}$$

By the Hölder inequality and by the linearity of $\mathfrak{A}_{n,n}$ and Lemma 2.2, we get

$$|\mathfrak{A}_{n,n}(\varkappa_{(x,y)}(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y)| \\ \leq (\mathfrak{A}_{n,n}((u-x)^4+(v-y)^4;x,y))^{\frac{1}{2}} \times (\mathfrak{A}_{n,n}(\varkappa^2_{(x,y)}(u,v);x,y))^{\frac{1}{2}} \tag{2.29}$$

It follows from Theorem 2.5 that $\lim\limits_{n\to\infty} \mathfrak{A}_{n,n}(\varkappa^2_{(x,y)}(u,v);x,y)=0$, uniformly on E. From the foregoing facts and Lemma 2.2, we obtain

$$\lim_{n\to\infty} n\mathfrak{A}_{n,m}(\varkappa^2_{(x,y)}(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y)=0. \tag{2.30}$$

Then, taking limit as $n\to\infty$ in (2.28) and using (2.30), we have

$$\lim_{n\to\infty} n\big(\mathfrak{A}_{n,m}(f;x,y)-f(x,y)\big)=\frac{g_{t_1}(1,1)+g_{t_2}(1,1)}{g(1,1)}f_x(x,y) \\ +\frac{g_{t_3}(1,1)+g_{t_4}(1,1)}{g(1,1)}f_y(x,y)+\frac{1}{2}xf_{xx}+\frac{1}{2}yf_{yy},$$

uniformly in Π_{ab}.

3 Weighted Approximation Properties

Let $\mathbb{R}^2_+=\{(x,y):x\geq 0, y\geq 0\}$ and $\Pi_{ab}=\{(x,y):0\leq x\leq a, 0\leq y\leq b\}$. Let $C(\mathbb{R}^2_+)$ be the space of all continuous function f in $\mathbb{R}^2_+$ such that condition $|f(x,y)|\leq M_f\rho(x,y)$, where $\rho(x,y)=1+x^2+y^2$ and M_f is a constant depending on function f only. It is clear that $C(\mathbb{R}^2_+)$ is a linear normed space with a norm $\| f \|_\rho = \sup\limits_{(x,y)\in\mathbb{R}^2_+} \frac{|f(x,y)|}{\rho(x,y)}$.

Lemma 3.1 ([10, 11]) *For the sequence of positive linear operators $\{A_{n,m}\}_{n,m\geq 1}$ acting from $C_\rho(\mathbb{R}^2_+)$ to $B_\rho(\mathbb{R}^2_+)$, it is necessary and sufficient that inequality*

$$\| A_{n,m}(\rho;x,y) \|_\rho \leq k$$

is fulfilled with some positive constant k.

Theorem 3.2 ([10, 11]) *If a sequence of positive linear operators $A_{n,m}$, acting from $C_\rho(\mathbb{R}^2_+)$ to $B_\rho(\mathbb{R}^2_+)$, satisfies the conditions*

$$\lim_{n,m\to\infty} \| A_{n,m}(1;x,y)-1 \|_\rho = 0, \tag{3.1}$$

$$\lim_{n,m\to\infty} \parallel A_{n,m}(t_1; x, y) - x \parallel_\rho = 0, \tag{3.2}$$

$$\lim_{n,m\to\infty} \parallel A_{n,m}(t_2; x, y) - 1 \parallel_\rho = 0, \tag{3.3}$$

$$\lim_{n,m\to\infty} \parallel A_{n,m}((t_1^2 + t_2^2); x, y) - (x^2 + y^2) \parallel_\rho = 0, \tag{3.4}$$

then, for any function $f \in C_\rho^k(\mathbb{R}_+^2)$,

$$\lim_{n,m\to\infty} \parallel A_{n,m} f - f \parallel_\rho = 0,$$

and there exists a function $f^* \in C_\rho^k(R_+^2)$, *for which*

$$\lim_{n,m\to\infty} \parallel A_{n,m} f^* - f^* \parallel_\rho \geq 1.$$

Theorem 3.3 ([10, 11]) *Let* $A_{n,m}$ *be a sequence of linear operators acting from* $C_\rho(\mathbb{R}_+^2)$ *to* $B_\rho^k(\mathbb{R}_+^2)$, *and let* $\rho_1(x, y) \geq 1$ *be a continuous function for which*

$$\lim_{|v|\to\infty} \frac{\rho(v)}{\rho_1(v)} = 0, \text{ where } v = (x, y). \tag{3.5}$$

If $A_{n,m}$ *satisfies the conditions of Theorem 3.2, then*

$$\lim_{n,m\to\infty} \parallel A_{n,m} f - f \parallel_{\rho_1} = 0,$$

for all $f \in C_\rho(\mathbb{R}_+^2)$.
Now, we consider the following positive linear operators A_{n_1,n_2}, *defined by*

$$A_{n,m}(f; x, y) = \begin{cases} \mathfrak{A}_{n,m}(f; x, y) & \text{when } (x, y) \in \Pi_{ab} \\ \\ f(x, y) & \text{when } (x, y) \in \mathbb{R}_+^2 \backslash \Pi_{ab}. \end{cases} \tag{3.6}$$

Theorem 3.4 *Let* $\rho(x, y) = 1 + x^2 + y^2$, *be a weight function, and* $A_{n,m}(f; x, y)$ *be a sequence of linear positive operators defined by (3.6). Then, for all* $f \in C_\rho(\mathbb{R}_+^2)$, *we have*

$$\lim_{n,m\to\infty} \parallel A_{n,m} f - f \parallel_{\rho_1} = 0,$$

where $\rho_1(x, y)$ *is a continuous function satisfying condition (3.5).*

Proof First, we show that $A_{n,m}$ is acting from $C_\rho(\mathbb{R}_+^2)$ to $B_\rho(\mathbb{R}_+^2)$. Using Lemma 2.2, we can write

$$\begin{aligned}\| A_{n,m}(\rho; x, y) \|_\rho \leq 1 &+ \sup_{(x,y)\in\Pi_{ab}} \frac{x^2}{\rho(x,y)} + \sup_{(x,y)\in\Pi_{ab}} \frac{x}{\rho(x,y)} \frac{\big(2g_{t_1}(1,1) + 2g_{t_2}(1,1) + g(1,1)\big)}{ng(1,1)} \\ &+ \sup_{(x,y)\in\Pi_{ab}} \frac{1}{\rho(x,y)} \frac{g_{t_1}(1,1) + g_{t_2}(1,1) + g_{t_1t_1}(1,1) + g_{t_2t_2}(1,1) + 2g_{t_1t_2}(1,1)}{n^2 g(1,1)} \\ &+ \sup_{(x,y)\in\Pi_{ab}} \frac{y^2}{\rho(x,y)} + \sup_{(x,y)\in\Pi_{ab}} \frac{y}{\rho(x,y)} \frac{\big(2g_{t_3}(1,1) + 2g_{t_4}(1,1) + g(1,1)\big)}{mg(1,1)} \\ &+ \sup_{(x,y)\in\Pi_{ab}} \frac{1}{\rho(x,y)} \frac{g_{t_3}(1,1) + g_{t_4}(1,1) + g_{t_3t_3}(1,1) + g_{t_4t_4}(1,1) + 2g_{t_3t_4}(1,1)}{m^2 g(1,1)} \\ &= 3 + \varphi + \psi\end{aligned}$$

where $\varphi = \frac{\big(2g_{t_1}(1,1)+2g_{t_2}(1,1)+g(1,1)\big)}{ng(1,1)} + \frac{g_{t_1}(1,1)+g_{t_2}(1,1)+g_{t_1t_1}(1,1)+g_{t_2t_2}(1,1)+2g_{t_1t_2}(1,1)}{n^2 g(1,1)}$ and $\psi = \frac{\big(2g_{t_1}(1,1)+2g_{t_2}(1,1)+g(1,1)\big)}{mg(1,1)} + \frac{g_{t_1}(1,1)+g_{t_2}(1,1)+g_{t_1t_1}(1,1)+g_{t_2t_2}(1,1)+2g_{t_1t_2}(1,1)}{m^2 g(1,1)}$. Since $\lim_{n\to\infty} \varphi = 0$ and $\lim_{m\to\infty} \psi = 0$, we have

$$\| A_{n,m}(\rho; x, y) \|_\rho \leq 1 + k.$$

From Lemma 3.1, we have $A_{n,m} : C_\rho(\mathbb{R}^2_+) \to B_\rho(\mathbb{R}^2_+)$. If we can show that conditions of Theorem 3.2 are satisfied, then the proof of Theorem 3.4 is completed. Using Lemma 2.2, we can obtain (3.1)–(3.3). Finally, using Lemma 2.2, we get

$$\| K_{n,m}(e_{20} + e_{02}; x, y) - (x^2 + y^2) \|_\rho \leq \varphi + \psi,$$

and since $\lim_{n\to\infty} \varphi = 0$, $\lim_{m\to\infty} \psi = 0$, we obtain the desired result.

Theorem 3.5 *If $f \in C^0_\rho$, then*

$$\sup_{(x,y)\in R^2_+} \frac{|\mathfrak{A}_{n,m}(f; x, y) - f(x, y)|}{\rho^3(x, y)} \leq K\omega_\rho(f; \delta_n, \delta_m) \tag{3.7}$$

holds true where $\delta_n = \frac{1}{\sqrt{n}}$, $\delta_m = \frac{1}{\sqrt{m}}$ and K is independent of m and n.

Proof Considering the following inequality

$$\begin{aligned}|f(t, s) - f(x, y)| \leq 8(1 + x^2 + y^2)&\omega_\rho(f; \delta_n, \delta_m) \\ &\left(1 + \frac{|t - x|}{\delta_n}\right)\left(1 + \frac{|s - y|}{\delta_m}\right)(1 + (t - x)^2)(1 + (s - y)^2),\end{aligned}$$

We have,

$$\begin{aligned}|\mathfrak{A}_{n,m}(f; x, y) - f(x, y)| \leq 8(1 + x^2 + y^2)&\frac{e^{-(nx+my)}}{g^2(1,1)} \sum_{k_1,k_2=0}^{\infty} \frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\left(1 + \frac{1}{\delta_n}\left|\frac{k_1 + k_2}{n} - x\right|\right) \\ &\left(1 + \left(\frac{k_1 + k_2}{n} - x\right)^2 \sum_{k_3,k_4=0}^{\infty} \frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!}\left(1 + \frac{1}{\delta_m}\left|\frac{k_3 + k_4}{m} - x\right|\right)\right.\end{aligned}$$

$$\left(1+\left(\frac{k_3+k_4}{m}-y\right)^2\right).$$

Now, applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned}|\mathfrak{A}_{n,m}(f;x,y)-f(x,y)| \leq{}& 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\\ &\left[1+\mathfrak{A}_{n,m}((e_{10}-x)^2;x,y)+\frac{1}{\delta_n}\sqrt{\mathfrak{A}_{n,m}((e_{10}-x)^2;x,y)}\right.\\ &\left.+\frac{1}{\delta_n}\sqrt{\mathfrak{A}_{n,m}((e_{10}-x)^2;x,y)\mathfrak{A}_{n,m}((e_{10}-x)^4;x,y)}\right]\\ &\times\left[1+\mathfrak{A}_{n,m}((e_{01}-y)^2;x,y)+\frac{1}{\delta_m}\sqrt{\mathfrak{A}_{n,m}((e_{01}-y)^2;x,y)}\right.\\ &\left.+\frac{1}{\delta_m}\sqrt{\mathfrak{A}_{n,m}((e_{01}-y)^2;x,y)\mathfrak{A}_{n,m}((e_{10}-y)^4;x,y)}\right].\end{aligned}$$

By using Remark 2.3, we get

$$\begin{aligned}|\mathfrak{A}_{n,m}(f;x,y)-f(x,y)| \leq{}& 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\\ &\left[1+O\left(\frac{1}{n}\right)(x+x^2)+\frac{1}{\delta_n}\right.\\ &\left.\sqrt{O\left(\frac{1}{n}\right)(x+x^2)}+\frac{1}{\delta_n}\sqrt{O\left(\frac{1}{n}\right)(x+x^2)(x^4+x^3+x^2+x)}\right]\\ &\times\left[1+O\left(\frac{1}{m}\right)(y+y^2)\right.\\ &+\frac{1}{\delta_m}\sqrt{O\left(\frac{1}{m}\right)(y+y^2)}\\ &\left.+\frac{1}{\delta_m}\sqrt{O\left(\frac{1}{m}\right)(y+y^2)(y^4+y^3+y^2+y)}\right],\end{aligned}$$

from which the desired result is immediate.

4 Approximation in the Space of Bögel-Continuous Functions

In this section, we introduce the generalization of (2.1) to approximate Bögel-continuous functions. For this, we define a GBS operator associated with (2.1) and determine the rate of convergence. Bögel [5, 6] gave a concept to approximate the B-continuous and B-differentiable functions. Badea et al. [3] gave the Korovkin theorem to prove the convergence of these type of functions. For more literature in this direction we refer the reader to (cf. [8, 9, 13, 14]). In the recent years, several authors have made significant contributions in this area of approximation theory ([1, 4] etc.). Let I and J be compact real intervals and $A = I \times J$. For any $f : A \to \mathbb{R}$ and any $(t, s), (x, y) \in A$, let $\triangle_{(t,s)} f(x, y)$ be the bivariate mixed difference operator defined as

$$\triangle_{(t,s)} f(x, y) = f(t, s) - f(t, y) - f(x, s) + f(x, y).$$

A function $f : A \to \mathbb{R}$ is called a B-continuous (Bögel-continuous) function at $(x, y) \in A$ if

$$\lim_{(t,s)\to(x,y)} \triangle_{(t,s)} f(x, y) = 0.$$

If f is B-continuous at every point $(x, y) \in A$, then f is B-continuous on A. We denote by $C_b(A) = \{f | f : A \to \mathbb{R}, f\ is\ B - bounded\ on\ A\}$, the space of all B-continuous functions in A.

A function $f : A \to \mathbb{R}$ is called a B-differentiable on $(x, y) \in A$ if it exists and if the limit is finite:

$$\lim_{(t,s)\to(x,y)} \frac{\triangle_{(t,s)} f(x, y)}{(t - x)(s - y)} = D_B f(x, y) < \infty.$$

We wright by $D_b(A) = \{f | f : A \to \mathbb{R},\ f\ is\ B - differentiable\ on\ A\}$, the space of all B-differentiable functions.

The function $f : A \to R$ is B-bounded on D if there exists $K > 0$ such that $|\triangle_{(t,s)} f(x, y)| \le K$ for any $(t, s), (x, y) \in A$. Here, If A is a compact subset, then each B-continuous function is a B-bounded function on $A \to R$. We denote by $B_b(A)$, the space of all B-bounded functions on A equipped with the norm $\| f \|_B = \sup_{(x,y),(t,s)\in A} \left|\triangle_{(t,s)} f(x, y)\right|$.

In order to evaluate the approximation degree B-continuous function using linear positive operators, an important tool is the mixed modulus of continuity. Let $f \in B_b(\Pi_{ab})$. The mixed modulus of continuity of f is the function $\omega_B : [0, \infty) \times [0, \infty) \to \mathbb{R}$, defined by

$$\omega_B(f; \delta_1, \delta_2) = sup\{\triangle_{(t,s)} f(x, y) : |t - x| \le \delta_1, |s - y| \le \delta_2\},$$

for any $(t, s), (x, y) \in A$.

For $\Pi_{ab} = [0, a] \times [0, b]$, let $C_b(\Pi_{ab})$ denotes the space of all B-continuous functions on Π_{ab} and let $C(\Pi_{ab})$ be the space of all ordinary continuous functions on Π_{ab}. We define the GBS operators of the $\mathfrak{A}_{n,m}$ given by (2.1), for any $f \in C_b(\Pi_{ab})$ and $n, m \in \mathbb{N}$, by

$$\mathfrak{S}_{n,m}(f(t, s); x, y) = \mathfrak{A}_{n,m}(f(t, y) + f(x, s) - f(t, s); x, y),$$

for all $(x, y) \in \Pi_{ab}$. for any $f \in C_b(\Pi_{ab})$, the GBS operator of (2.1) is given by

$$\mathfrak{S}_{n,m}(f;x,y)=\frac{e^{-(nx+my)}}{g^2(1,1)}\sum_{k_1,k_2=0}^{\infty}\sum_{k_3,k_4=0}^{\infty}\frac{\phi_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\frac{\phi_{k_3,k_4}(\frac{my}{2})}{k_3!k_4!}\left(f\left(\frac{k_1+k_2}{n},y\right)\right.$$
$$\left.+f\left(x,\frac{k_3+k_4}{m}\right)-f\left(\frac{k_1+k_2}{n},\frac{k_3+k_4}{m}\right)\right).$$

Theorem 4.1 *If $f\in C_b(\Pi_{ab})$, then for any $(x,y)\in\Pi_{ab}$, and any $m,n\in N$, we have*

$$|\mathfrak{S}^a_{n,m}(f(t,s);x,y)-f(x,y)|\le 4\omega_B(f;\frac{1}{\sqrt{n}},\frac{1}{\sqrt{m}}),$$

Proof By using the properties of ω_B we obtain

$$\left|\triangle_{(t,s)}f(x,y)\right|\le\omega_B(f;|t-x|,|s-y|)\le\left(1+\frac{|t-x|}{\delta_n}\right)\left(1+\frac{|s-y|}{\delta_m}\right)\omega_B(f;\delta_n,\delta_m),$$

for every $(x,y),(t,s)\in\Pi_{ab}$. Hence, from the monotonicity and linearity of the operators $\mathfrak{S}_{n,m}\left(f(t,s);x,y\right)$, and by using Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}|\mathfrak{S}_{n,m}\left(f(t,s);x,y\right)-f(x,y)|&\le\mathfrak{A}_{n,m}\left(|\triangle_{(t,s)}f(x,y)|;x,y\right)\\&\le\left(\mathfrak{A}_{n,m}(e_{00};x,y)+\frac{1}{\delta_n}\left(\mathfrak{A}_{n,m}\left((e_{10}-x)^2;x,y\right)\right)^{1/2}\right.\\&\quad+\frac{1}{\delta_m}\left(\mathfrak{A}_{n,m}\left((e_{01}-y)^2;x,y\right)\right)^{1/2}\\&\quad+\frac{1}{\delta_n}\left(\mathfrak{A}_{n,m}\left((e_{10}-x)^2;x,y\right)\right)^{1/2}\\&\quad\left.\frac{1}{\delta_m}\left(\mathfrak{A}_{n,m}\left((e_{01}-y)^2;x,y\right)\right)^{1/2}\right)\omega_B(f;\delta_n,\delta_m).\\&\le 4\omega_B(f;\frac{1}{\sqrt{n}},\frac{1}{\sqrt{m}}).\end{aligned}$$

Next, we discuss the approximation of Lipschitz class of B-continuous functions.

For $0<\alpha\le 1$, $0<\beta\le 1$ let $Lip_M\left(\alpha,\beta\right)=\left\{f\in C(\Pi_{ab}):|\Delta_{(t,s)}f(x,y)|\le M\left|t-x\right|^{\alpha}\left|s-y\right|^{\beta}\right\}$, where $(t,s),\ (x,y)\in\Pi_{ab}$.

Let $\delta_n(x)=\mathfrak{A}_{n,m}\left(e_{10}-x\right)^2;x,y\big)$ and $\delta_m(y)=\mathfrak{A}_{n,m}\left(e_{01}-y\right)^2;x,y\big)$.

Theorem 4.2 *If $f\in Lip_M\left(\alpha,\beta\right)$, then for any $(x,y)\in\Pi_{ab}$, and any $m,n\in\mathbb{N}$, we have*

$$|\mathfrak{S}_{n,m}\left(f(t,s);x,y\right)-f(x,y)|\le M(\delta_n(x))^{\frac{\alpha}{2}}(\delta_m(y))^{\frac{\beta}{2}},$$

where $M>0$, $\alpha,\beta\in(0,1]$.

Proof By the definition of $\mathfrak{S}_{n,m}(f;x,y)$, we have

$$\begin{aligned}\mathfrak{S}_{n,m}\left(f(t,s);x,y\right) &= \mathfrak{A}_{n,m}\left(f(x,s)+f(t,y)-f(t,s);x,y\right)\\ &= \mathfrak{A}_{n,m}\left(f(x,y)-\Delta_{(t,s)}f(x,y);x,y\right)\\ &= f(x,y)\mathfrak{A}_{n,m}\left(e_{00};x,y\right)-\mathfrak{A}_{n,m}\left(\Delta_{(t,s)}f(x,y);x,y\right).\end{aligned}$$

In view of definition $Lip_M(\alpha,\beta)$, we have

$$\left|\mathfrak{S}_{n,m}\left(f(t,s);x,y\right)-f(x,y)\right|\leq M\mathfrak{A}_{n,m}\left(|t-x|^{\alpha}\,|s-y|^{\beta};x,y\right).$$

Now by using the Hölder's inequality with $p_1=2/\alpha, q_1=2/\left(2-\alpha\right)$ and $p_2=2/\beta, q_2=2/\left(2-\beta\right)$, we get the desired result.

Theorem 4.3 *If* $f\in D_b(\Pi_{ab})$ *and* $D_Bf\in B(\Pi_{ab})$, *then for each* $(x,y)\in\Pi_{ab}$, *we get*

$$|\mathfrak{S}_{n,m}(f;x,y)-f(x,y)|\leq\frac{M}{n^{1/2}m^{1/2}}\left(||D_Bf||_{\infty}+\omega_B(D_Bf;n^{-1/2},m^{-1/2})\right).$$

Proof Since $f\in D_b(\Pi_{ab})$, we have the identity

$$\Delta f[(t,s);(x,y)]=(t-x)(s-y)D_Bf(\xi,\eta),\ \ with\ x<\xi<t;\ \ y<\eta<s.$$

Eventually,

$$D_Bf(\xi,\eta)=\Delta D_Bf(\xi,\eta)+D_Bf(\xi,y)+D_Bf(x,\eta)-D_Bf(x,y).$$

Since $D_Bf\in B(\Pi_{ab})$, by above relations, we can write

$$\begin{aligned}|\mathfrak{A}_{n,m}(\Delta f[(t,s);(x,y)];x,y)| &= |\mathfrak{A}_{n,m}((t-x)(s-y)D_Bf(\xi,\eta);x,y)|\\ &\leq \mathfrak{A}_{n,m}(|t-x||s-y||\Delta D_Bf(\xi,\eta)|;x,y)\\ &\quad+\mathfrak{A}_{n,m}(|t-x||s-y|(|D_Bf(\xi,y)|\\ &\quad+|D_Bf(x,\eta)|+|D_Bf(x,y)|);x,y)\\ &\leq \mathfrak{A}_{n,m}(|t-x||s-y|\omega_B(D_Bf;|\xi-x|,|\eta-y|);x,y)\\ &\quad+3\,||D_Bf||_{\infty}\,\mathfrak{A}_{n,m}(|t-x||s-y|;x,y).\end{aligned}$$

By the above inequality, using the linearity of $\mathfrak{A}_{n,m}$ and applying the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}|\mathfrak{S}_{n,m}(f;x,y)-f(x,y)| &= |\mathfrak{A}_{n,m}\Delta f[(t,s);(x,y)];x,y|\\ &\leq 3||D_Bf||_{\infty}\sqrt{\mathfrak{A}_{n,m}((t-x)^2(s-y)^2;x,y)}\\ &\quad+\Bigg(\mathfrak{A}_{n,m}(|t-x||s-y|;q_{n_1},q_{n_2},x,y)\\ &\quad+\delta_n^{-1}\mathfrak{A}_{n,m}((t-x)^2|s-y|;x,y)\end{aligned}$$

$$
\begin{aligned}
&+\delta_m^{-1}\mathfrak{A}_{n,m}(|t-x|(s-y)^2;x,y)\\
&+\delta_n^{-1}\delta_m^{-1}\mathfrak{A}_{n,m}((t-x)^2(s-y)^2;x,y)\Big)\omega_B(D_Bf;\delta_n,\delta_m)\\
\leq{}& 3||D_Bf||_\infty\sqrt{\mathfrak{A}_{n,m}((t-x)^2(s-y)^2;q_{n_1},q_{n_2},x,y)}\\
&+\Bigg(\sqrt{\mathfrak{A}_{n,m}((t-x)^2(s-y)^2;x,y)}\\
&+\delta_n^{-1}\sqrt{\mathfrak{A}_{n,m}((t-x)^4(s-y)^2;x,y)}\\
&+\delta_m^{-1}\sqrt{\mathfrak{A}_{n,m}((t-x)^2(s-y)^4;x,y)}\\
&+\delta_n^{-1}\delta_m^{-1}\mathfrak{A}_{n,m}((t-x)^2(s-y)^2;x,y)\Bigg)\omega_B(D_Bf;\delta_n,\delta_m).
\end{aligned}
\tag{4.1}
$$

In view of Remark 2.3, for $(t,s)\in\Pi_{ab}$, $(x,y)\in\Pi_{ab}$ and $i,j=1,2$

$$
\begin{aligned}
\mathfrak{A}_{n,m}((t-x)^{2i}(s-y)^{2j};x,y) &= \mathfrak{A}_{n,m}((t-x)^{2i};x,y)\mathfrak{A}_{n,m}((s-y)^{2j};x,y).\\
&\leq \frac{M_1}{n^i}\frac{M_2}{m^j},
\end{aligned}
\tag{4.2}
$$

for some constants $M_1, M_2>0$.

Let $\delta_n=\frac{1}{n^{1/2}}$, and $\delta_m=\frac{1}{m^{1/2}}$.

Thus, combining (4.1)–(4.2), we get the desired result.

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Approximation Properties of Chlodowsky Variant of (p, q) Szász–Mirakyan–Stancu Operators

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Abstract In the present paper, we introduce the Chlodowsky variant of (p, q) Szász–Mirakyan–Stancu operators on the unbounded domain which is a generalization of (p, q) Szász–Mirakyan operators. We have also derived its Korovkin-type approximation properties and rate of convergence.

Keywords (p, q)-integers · (p, q)-Szász–Mirakyan operators · Chlodowsky polynomials · Weighted approximation

AMS Subject Classification (2010) 41A10 · 41A25 · 41A36

1 Introduction and Preliminaries

The applications of q-calculus emerged as a new area in the field of approximation theory from last two decades. The development of q-calculus has led to the discovery of various modifications of Bernstein polynomials involving q-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design and solutions of differential equations.

In 1987, Lupaş and in 1997, Phillips introduced a sequence of Bernstein polynomials based on q-integers and investigated its approximation properties.

Mursaleen et al. [12, 13, 18] introduced on Chlodowsky variant of Szász operators by Brenke-type polynomials, rate of convergence of Chlodowsky-type Durrmeyer Jakimovski–Leviatan operators and Dunkl generalization of q-parametric Szász–Mirakjan operators and shape preserving properties.

Several authors produced generalizations of well-known positive linear operators based on q-integers and studied them extensively. For instance, the approximation

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properties of A generalization of Szász–Mirakyan operators based on q-integers [5], convergence of the q-analogue of Szász-Beta operators [9], Dunkl generalization of q-parametric Szász–Mirakjan operators [18] and weighted statistical approximation by Kantorovich-type q-Szász–Mirakyan operators [6].

Recently, Mursaleen et al. introduced (p, q)-calculus in approximation theory and constructed the (p, q)-analogue of Bernstein operators [14], (p, q)-analogue of Bernstein–Stancu operators [15]. Further, Acar [1] has studied recently, (p, q)-generalization of Szász–Mirakyan operators.

In the present paper, we introduce the Chlodowsky variant of (p, q) Szász–Mirakyan–Stancu operators on the unbounded domain. Most recently, the (p, q)-analogue of some more operators has been studied in [2, 4, 10, 11, 14, 17, 19, 21].

The (p, q)-integer or in general the (p, q)-calculus was introduced to generalize or unify several forms of q-oscillator algebras well known in the Physics literature related to the representation theory of single-parameter quantum algebras. The (p, q)-integer is defined by

$$[n]_{p,q} = p^{n-1} + qp^{n-2} + \cdots + pq^{n-2} + q^{n-1} = \begin{cases} \dfrac{p^n - q^n}{p - q} & (p \neq q \neq 1) \\ \dfrac{1 - q^n}{1 - q} & (p = 1) \\ n & (p = q = 1). \end{cases} \tag{1}$$

The (p, q)-binomial expansion is

$$(ax + by)^n_{p,q} := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k} b^k x^{n-k} y^k,$$

$$(x + y)^n_{p,q} := (x + y)(px + qy)(p^2 x + q^2 y) \cdots (p^{n-1} x + q^{n-1} y),$$

$$(1 - x)^n_{p,q} := (1 - x)(p - qx)(p^2 - q^2 x) \cdots (p^{n-1} - q^{n-1} x).$$

The (p, q)-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

The definite integrals of a function f is defined by

$$\int_0^a f(t) d_{p,q} t = (q - p) a \sum_{k=0}^{\infty} f\left(\frac{p^k}{q^{k+1}} a\right) \frac{p^k}{q^{k+1}}, \qquad if \left|\frac{p}{q}\right| < 1,$$

$$\int_0^a f(t)d_{p,q}t = (p-q)a\sum_{k=0}^{\infty} f\left(\frac{q^k}{p^{k+1}}a\right)\frac{q^k}{p^{k+1}}, \qquad if\left|\frac{q}{p}\right| < 1.$$

There are two (p, q)-analogues of the classical exponential function defined as follows

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{p^{\frac{n(n-1)}{2}} x^n}{[n]_{p,q}!},$$

and

$$E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{[n]_{p,q}!},$$

which satisfy the equality $e_{p,q}(x)E_{p,q}(-x) = 1$. For $p = 1$, $e_{p,q}(x)$ and $E_{p,q}(x)$ reduce to q-exponential functions.

For $0 < q < 1$, Aral [5] introduced the generalized q-Szász–Mirakyan operators as follows

$$S_{n,q}(f; x) = \sum_{k=0}^{\infty} s_{n,k}^q(x) f\left(\frac{[k]_q b_n}{[n]_q}\right) \tag{2}$$

where

$$s_{n,k}^q(x) = \frac{1}{E_q([n]_q \frac{x}{b_n})} \frac{([n]_q x)^k}{[k]_q!(b_n)^k}.$$

where $0 \le x < \alpha_q(n)$, $\alpha_q(n) := \dfrac{b_n}{(1-q)[n]_q}$, $f \in C(\mathbb{R}_0)$ and (b_n) is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$.

Mursaleen et al. [10] introduced the (p, q)-analogue of the Szász–Mirakyan operators as follows

$$S_{n,p,q}(f; x) = \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q} x)^k}{[k]_{p,q}!} e_{p,q}(-[n]_{p,q} q^{-k} x) f\left(\frac{[k]_{p,q}}{p^{k-1}[n]_{p,q}}\right). \tag{3}$$

Lemma 1.1 *Let* $0 < q < p \le 1$ *and* $n \in \mathbb{N}$. *We have*

(i) $S_{n,p,q}(1; x) = 1$
(ii) $S_{n,p,q}(t; x) = x$
(iii) $S_{n,p,q}(t^2; x) = \frac{x^2}{p} + \frac{x}{[n]_{p,q}}$
(iv) $S_{n,p,q}(t^3; x) = \frac{x^3}{p^3} + \frac{2p+q}{p^2[n]_{p,q}} x^2 + \frac{x}{[n]_{p,q}^2}$
(v) $S_{n,p,q}(t^4; x) = \frac{x^4}{p^6} + \frac{3p^2+2pq+q^2}{p^5[n]_{p,q}} x^3 + \frac{3p^2+3pq+q^2}{p^3[n]_{p,q}^2} x^2 + \frac{x}{[n]_{p,q}^3}$.

2 Construction of the Operators

We construct the Chlodowsky variant of (p,q) Szász–Mirakyan–Stancu operators as

$$S_{n,p,q}^{(\alpha,\beta)}(f;x)=\sum_{k=0}^{\infty}\frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}}\frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k}e_{p,q}(-[n]_{p,q}q^{-k}\frac{x}{b_n})f\left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}b_n\right) \tag{4}$$

where $n\in\mathbb{N}$, $\alpha,\beta\in\mathbb{N}_0$ with $0\leq\alpha\leq\beta$, $0\leq x\leq b_n$, $0<q<p\leq 1$ and b_n is an increasing sequence of positive terms with the properties $b_n\longrightarrow\infty$ and $\frac{b_n}{[n]_{p,q}}\longrightarrow 0$ as $n\longrightarrow\infty$. We observe that $S_{n,p,q}^{(\alpha,\beta)}$ is positive and linear. Furthermore, in the case of $q=p=1$ and $\alpha=\beta=0$, the operators (4) are similar to the classical Szász–Mirakyan operators.

Lemma 2.1 *Let $0<q<p\leq 1$ and $n\in\mathbb{N}$, $\alpha,\beta\in\mathbb{N}_0$ with $0\leq\alpha\leq\beta$, $0\leq x\leq b_n$, and integer $m\geq 0$, we have*

$$S_{n,p,q}^{(\alpha,\beta)}(t^m;x)=\frac{b_n^m}{([n]_{p,q}+\beta)^m}\sum_{j=0}^{m}\binom{m}{j}\alpha^{m-j}S_{n,p,q}\left(t^j;q^{-1}\frac{x}{b_n}\right). \tag{5}$$

Proof Using the identity

$$[k+1]_{p,q}=p^k+q[k]_{p,q},$$

we can write

$$\begin{aligned}
S_{n,p,q}^{(\alpha,\beta)}(t^m;x) &= \sum_{k=0}^{\infty}\frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}}\frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k}e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right)\left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}b_n\right)^m\\
&= \frac{b_n^m}{([n]_{p,q}+\beta)^m}\sum_{k=0}^{\infty}\frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}}\frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k}e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right)\left(p^{1-k}[k]_{p,q}+\alpha\right)^m\\
&= \frac{b_n^m}{([n]_{p,q}+\beta)^m}\sum_{k=0}^{\infty}\frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}}\frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k}e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right)\\
&\quad\times\sum_{j=0}^{m}\binom{m}{j}\alpha^{m-j}p^{j(1-k)}[k]_{p,q}^j\\
&= \frac{b_n^m}{([n]_{p,q}+\beta)^m}\sum_{j=0}^{m}\binom{m}{j}\alpha^{m-j}\sum_{k=0}^{\infty}\frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}}\frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k}\\
&\quad\times e_{p,q}(-[n]_{p,q}q^{-k}\frac{x}{b_n})p^{j(1-k)}[k]_{p,q}^j
\end{aligned}$$

$$= \frac{b_n^m}{([n]_{p,q}+\beta)^m}\sum_{j=0}^{m}\binom{m}{j}\alpha^{m-j}S_{n,p,q}(t^j;q^{-1}\frac{x}{b_n})$$

□

which is desired.

Lemma 2.2 *Let $S_{n,p,q}^{(\alpha,\beta)}(f;x)$ be given by (4). Then the following properties hold:*

$$(i)\ S_{n,p,q}^{(\alpha,\beta)}(1;x) = 1$$
$$(ii)\ S_{n,p,q}^{(\alpha,\beta)}(t;x) = \frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha b_n}{[n]_{p,q}+\beta}$$
$$(iii)\ S_{n,p,q}^{(\alpha,\beta)}(t^2;x) = \frac{[n]_{p,q}^2}{p([n]_{p,q}+\beta)^2}x^2 + \frac{(1+2\alpha)b_n[n]_{p,q}}{([n]_{p,q}+\beta)^2}x + \frac{\alpha^2 b_n^2}{([n]_{p,q}+\beta)^2}$$
$$(iv)\ S_{n,p,q}^{(\alpha,\beta)}(t^3;x) = \frac{[n]_{p,q}^3}{p^3([n]_{p,q}+\beta)^3}x^3 + \frac{(3p\alpha+2p+q)b_n[n]_{p,q}^2}{p^2([n]_{p,q}+\beta)^3}x^2 + \frac{(1+3\alpha++3\alpha^2)b_n^2[n]_{p,q}}{([n]_{p,q}+\beta)^3}x + \frac{\alpha^3 b_n^3}{([n]_{p,q}+\beta)^3}$$

$$(v)\ S_{n,p,q}^{(\alpha,\beta)}(t^4;x) = \frac{[n]_{p,q}^4}{p^6([n]_{p,q}+\beta)^4}x^4 + \frac{(3p^2+2pq+q^2+4p\alpha)b_n[n]_{p,q}^3}{p^5([n]_{p,q}+\beta)^4}x^3 + \frac{(3p^2+3pq+q^2+4pq\alpha+8p^2\alpha+6p^2\alpha^2)b_n^2[n]_{p,q}^2}{p^3([n]_{p,q}+\beta)^4}x^2 + \frac{(1+4\alpha+6\alpha^2+4\alpha^3)b_n^3[n]_{p,q}}{([n]_{p,q}+\beta)^4}x + \frac{\alpha^4 b_n^4}{([n]_{p,q}+\beta)^4}.$$

Proof (i)

$$S_{n,p,q}^{(\alpha,\beta)}(1;x) = S_{n,p,q}(1;q^{-1}\frac{x}{b_n}) = 1.$$

(ii)

$$S_{n,p,q}^{(\alpha,\beta)}(t;x) = \frac{b_n}{([n]_{p,q}+\beta)}\sum_{j=0}^{1}\binom{1}{j}\alpha^{1-j}S_{n,p,q}\left(t^j;q^{-1}\frac{x}{b_n}\right)$$
$$= \frac{b_n}{([n]_{p,q}+\beta)}\left\{\alpha + S_{n,p,q}\left(t;q^{-1}\frac{x}{b_n}\right)\right\}$$
$$= \frac{b_n}{([n]_{p,q}+\beta)}\left\{\alpha + \frac{[n]_{p,q}}{b_n}x\right\}$$

$$= \frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha b_n}{[n]_{p,q}+\beta}.$$

(iii)

$$\begin{aligned}
S_{n,p,q}^{(\alpha,\beta)}(t^2;x) &= \frac{b_n^2}{([n]_{p,q}+\beta)^2}\sum_{j=0}^{2}\binom{2}{j}\alpha^{2-j}S_{n,p,q}(t^j;q^{-1}\frac{x}{b_n})\\
&= \frac{b_n^2}{([n]_{p,q}+\beta)^2}\left\{\alpha^2 + 2\alpha S_{n,p,q}(t;q^{-1}\frac{x}{b_n}) + S_{n,p,q}(t^2;q^{-1}\frac{x}{b_n})\right\}\\
&= \frac{b_n^2}{([n]_{p,q}+\beta)^2}\left\{\alpha^2 + 2\alpha\frac{[n]_{p,q}}{b_n}x + \frac{[n]_{p,q}}{b_n}x + \frac{[n]_{p,q}^2}{pb_n^2}x^2\right\}\\
&= \frac{[n]_{p,q}^2}{p([n]_{p,q}+\beta)^2}x^2 + \frac{(1+2\alpha)b_n[n]_{p,q}}{([n]_{p,q}+\beta)^2}x + \frac{\alpha^2 b_n^2}{([n]_{p,q}+\beta)^2}.
\end{aligned}$$

(iv)

$$\begin{aligned}
S_{n,p,q}^{(\alpha,\beta)}(t^3;x) &= \frac{b_n^3}{([n]_{p,q}+\beta)^3}\sum_{j=0}^{3}\binom{3}{j}\alpha^{3-j}S_{n,p,q}(t^j;q^{-1}\frac{x}{b_n})\\
&= \frac{b_n^3}{([n]_{p,q}+\beta)^3}\left\{\alpha^3 + 3\alpha^2 S_{n,p,q}(t;q^{-1}\frac{x}{b_n}) + 3\alpha S_{n,p,q}(t^2;q^{-1}\frac{x}{b_n})\right.\\
&\quad \left. + S_{n,p,q}(t^3;q^{-1}\frac{x}{b_n})\right\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{b_n^3}{([n]_{p,q}+\beta)^3}\left\{\alpha^3 + 3\alpha^2\frac{[n]_{p,q}}{b_n}x + 3\alpha\left(\frac{[n]_{p,q}}{b_n}x + \frac{[n]_{p,q}^2}{pb_n^2}x^2\right)\right.\\
&\quad \left. + \frac{[n]_{p,q}}{b_n}x + \frac{2[n]_{p,q}^2}{pb_n^2}x^2 + \frac{q[n]_{p,q}^2}{p^2b_n^2}x^2 + \frac{[n]_{p,q}^3}{p^3b_n^3}x^3\right\}\\
&= \frac{[n]_{p,q}^3}{p^3([n]_{p,q}+\beta)^3}x^3 + \frac{(3p\alpha+2p+q)b_n[n]_{p,q}^2}{p^2([n]_{p,q}+\beta)^3}x^2\\
&\quad + \frac{(1+3\alpha++3\alpha^2)b_n^2[n]_{p,q}}{([n]_{p,q}+\beta)^3}x + \frac{\alpha^3 b_n^3}{([n]_{p,q}+\beta)^3}.
\end{aligned}$$

(v)

$$\begin{aligned}
S_{n,p,q}^{(\alpha,\beta)}(t^4;x) &= \frac{b_n^4}{([n]_{p,q}+\beta)^4}\sum_{j=0}^{4}\binom{4}{j}\alpha^{4-j}S_{n,p,q}(t^j;q^{-1}\frac{x}{b_n})\\
&= \frac{b_n^4}{([n]_{p,q}+\beta)^4}\left\{\alpha^4 + 4\alpha^3 S_{n,p,q}(t;q^{-1}\frac{x}{b_n}) + 6\alpha^2 S_{n,p,q}(t^2;q^{-1}\frac{x}{b_n})\right.\\
&\quad \left. + 4\alpha S_{n,p,q}(t^3;q^{-1}\frac{x}{b_n}) + S_{n,p,q}(t^4;q^{-1}\frac{x}{b_n})\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{b_n^4}{([n]_{p,q}+\beta)^4}\Bigg\{\alpha^4 + 4\alpha^3\frac{[n]_{p,q}}{b_n}x + 6\alpha^2\left(\frac{[n]_{p,q}}{b_n}x + \frac{[n]_{p,q}^2}{pb_n^2}x^2\right)\\
&\quad +4\alpha\left(\frac{[n]_{p,q}}{b_n}x + \frac{2[n]_{p,q}^2}{pb_n^2}x^2 + \frac{q[n]_{p,q}^2}{p^2b_n^2}x^2 + \frac{[n]_{p,q}^3}{p^3b_n^3}x^3\right) + \frac{[n]_{p,q}^4}{p^6b_n^4}x^4\\
&\quad +\frac{q^2[n]_{p,q}^3}{p^5b_n^3}x^3 + \frac{2q[n]_{p,q}^3}{p^4b_n^3}x^3 + \frac{q^2[n]_{p,q}^2}{p^3b_n^2}x^2 + \frac{3[n]_{p,q}^3}{p^3b_n^3}x^3 + \frac{3q[n]_{p,q}^2}{p^2b_n^2}x^2\\
&\quad +\frac{3[n]_{p,q}^2}{pb_n^2}x^2 + \frac{[n]_{p,q}}{b_n}x\Bigg\}\\
&= \frac{[n]_{p,q}^4}{p^6([n]_{p,q}+\beta)^4}x^4 + \frac{(3p^2+2pq+q^2+4p\alpha)b_n[n]_{p,q}^3}{p^5([n]_{p,q}+\beta)^4}x^3\\
&\quad +\frac{(3p^2+3pq+q^2+4pq\alpha+8p^2\alpha+6p^2\alpha^2)b_n^2[n]_{p,q}^2}{p^3([n]_{p,q}+\beta)^4}x^2\\
&\quad +\frac{(1+4\alpha+6\alpha^2+4\alpha^3)b_n^3[n]_{p,q}}{([n]_{p,q}+\beta)^4}x + \frac{\alpha^4b_n^4}{([n]_{p,q}+\beta)^4}.
\end{aligned}
$$

□

Lemma 2.3 *Let $p, q \in (0,1)$. Then for, $x \in [0,\infty)$, we have:*

$$(i)\ S_{n,p,q}^{(\alpha,\beta)}(t-x;x) = \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta} - 1\right)x + \frac{\alpha b_n}{[n]_{p,q}+\beta}$$

$$(ii)\ S_{n,p,q}^{(\alpha,\beta)}((t-x)^2;x) = \frac{((1-p)[n]_{p,q}^2 + p\beta^2)x^2 + ([n]_{p,q}+2\alpha\beta)pb_nx + p\alpha^2b_n^2}{p([n]_{p,q}+\beta)^2}.$$

3 Korovkin-Type Approximation Theorem

Suppose C_ρ is the space of all continuous functions f such that $|f(x)| \leq M\rho(x)$, $-\infty < x < \infty$. Then C_ρ is a Banach space with the norm $\|f\|_\rho = \sup\limits_{-\infty<x<\infty} \frac{|f(x)|}{\rho(x)}$. The subsequent results are used for proving Korovkin approximation theorem on unbounded sets.

Theorem 3.1 (See [8]) *There exists a sequence of positive linear operators T_n, acting from C_ρ to B_ρ, satisfying the conditions*

(i)

$$\lim_{n\to\infty} \|T_n(1;x) - 1\|_\rho = 0$$

(ii)

$$\lim_{n\to\infty} \|T_n(\varphi;x) - \varphi\|_\rho = 0$$

(iii)

$$\lim_{n\to\infty} \|T_n(\varphi^2; x) - \varphi^2\|_\rho = 0,$$

where $\varphi(x)$ *is a continuous and increasing function on* $(-\infty, \infty)$*, such that* $\lim_{x\to\pm\infty} \varphi(x) = \pm\infty$*,* $\rho(x) = 1 + \varphi^2$*, and there exists a function* $f^* \in C_\rho$*, for which* $\overline{\lim_{n\to\infty}} \|T_n f^* - f^*\|_\rho > 0$.

Theorem 3.2 (See [8]) *Conditions (i), (ii), (iii) of above theorem implies that*

$$\lim_{n\to\infty} \|T_n f - f\|_\rho = 0,$$

for any function f belonging to the subset

$$C_\rho^0 = \left\{ f \in C_\rho[0, \infty) : \lim_{x\to\infty} \frac{| f(x) |}{\rho(x)} < \infty \right\}.$$

Consider the weight function $\rho(x) = 1 + x^2$ and operators:

$$T_{n,p,q}^{(\alpha,\beta)}(f; x) = \begin{cases} S_{n,p,q}^{(\alpha,\beta)}(f; x), & x \in [0, b_n] \\ \\ f(x), & x \in [0, \infty)/[0, b_n]. \end{cases} \tag{6}$$

Thus for $f \in C_{1+x^2}$, we have

$$\begin{aligned} \|T_{n,p,q}^{(\alpha,\beta)}(f; x)\|_{1+x^2} &\le \sup_{x\in[0,b_n]} \frac{|T_{n,p,q}^{(\alpha,\beta)}(f; x)|}{1 + x^2} + \sup_{b_n<x<\infty} \frac{|f(x)|}{1 + x^2} \\ &\le \|f\|_{1+x^2} \left(\sup_{x\in[0,\infty)} \frac{|T_{n,p,q}^{(\alpha,\beta)}(1 + t^2; x)|}{1 + x^2} + 1 \right). \end{aligned}$$

Now we will obtain,

$$\|T_{n,p,q}^{(\alpha,\beta)}(f; x)\|_{1+x^2} \le M\|f\|_{1+x^2}$$

if $p := (p_n)$ and $q := (q_n)$ satisfy $0 < q_n < p_n \le 1$ and for n sufficiently large $p_n \to 1, q_n \to 1$ and $p_n^n \to N, q_n^n \to N, N < \infty$ and $\lim_{n\to\infty} \frac{b_n}{[n]_{p,q}} = 0$.

Theorem 3.3 *Let* $p := (p_n)$ *and* $q := (q_n)$ *satisfy* $0 < q_n < p_n \le 1$ *and for n sufficiently large* $p_n \to 1, q_n \to 1$ *and* $p_n^n \to N, q_n^n \to N, N < \infty$ *and* $\lim_{n\to\infty} \frac{b_n}{[n]_{p,q}} = 0$. *Then, for any* $f \in C_{1+x^2}^0$*, we have*

$$\|T_{n,p_n,q_n}^{(\alpha,\beta)}(f; \cdot) - f(\cdot)\|_{1+x^2} = 0.$$

Using the results of Theorem 3.1, Lemma 2.2, we will obtain the following assessments, respectively:

$$\sup_{x\in[0,\infty)} \frac{|T^{(\alpha,\beta)}_{n,p_n,q_n}(1;x)-1|}{1+x^2} = \sup_{0\leq x\leq b_n} \frac{|S^{(\alpha,\beta)}_{n,p_n,q_n}(1;x)-1|}{1+x^2} = 0.$$

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{|T^{(\alpha,\beta)}_{n,p_n,q_n}(t;x)-t|}{1+x^2} &= \sup_{0\leq x\leq b_n} \frac{|S^{(\alpha,\beta)}_{n,p_n,q_n}(t;x)-x|}{1+x^2}\\ &\leq \sup_{0\leq x\leq b_n} \frac{\left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}-1\right)x + \frac{\alpha b_n}{[n]_{p,q}+\beta}}{1+x^2}\\ &\leq \frac{\alpha b_n}{[n]_{p,q}+\beta} + \left|\frac{[n]_{p,q}}{[n]_{p,q}+\beta}-1\right| \longrightarrow 0,\end{aligned}$$

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{|T^{(\alpha,\beta)}_{n,p_n,q_n}(t^2;x)-t^2|}{1+x^2} &= \sup_{0\leq x\leq b_n} \frac{|S^{(\alpha,\beta)}_{n,p_n,q_n}(t^2;x)-x^2|}{1+x^2}\\ &\leq \sup_{0\leq x\leq b_n} \frac{1}{1+x^2}\left(\frac{((1-p)[n]^2_{p,q}+p\beta^2)x^2}{p([n]_{p,q}+\beta)^2}\right.\\ &\quad \left.+\frac{([n]_{p,q}+2\alpha\beta)b_n x}{([n]_{p,q}+\beta)^2} + \frac{\alpha^2 b_n^2}{([n]_{p,q}+\beta)^2}\right)\\ &\leq \frac{\alpha^2 b_n^2}{([n]_{p,q}+\beta)^2} + \left|\frac{(1-p)[n]^2_{p,q}+p\beta^2}{p([n]_{p,q}+\beta)^2}\right|\\ &\quad +\left|\frac{([n]_{p,q}+2\alpha\beta)b_n}{([n]_{p,q}+\beta)^2}\right| \longrightarrow 0,\end{aligned}$$

whenever $n \longrightarrow \infty$, because we have $\lim_{n\to\infty} q_n = \lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} \frac{b_n}{[n]_{p,q}} = 0$, as $n \longrightarrow \infty$.

Theorem 3.4 *Assuming C as a positive and real number independent of n and f as a continuous function which vanishes on $[C,\infty)$. Let $p := (p_n)$ and $q := (q_n)$ satisfy $0 < q_n < p_n \leq 1$ and for n sufficiently large $p_n \to 1$, $q_n \to 1$ and $p_n^n \to N$, $q_n^n \to N$, $N < \infty$ and $\lim_{n\to\infty} \frac{b_n}{[n]_{p,q}} = 0$. Then*

$$\lim_{n\to\infty} \sup_{0\leq x\leq b_n} |S^{(\alpha,\beta)}_{n,p_n,q_n}(f;x) - f(x)| = 0.$$

Proof From the hypothesis on f, it is bounded, i.e. $|f(x)| \leq M (M > 0)$. For any $\varepsilon > 0$, we have

$$\left| f\left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta} b_n\right) - f(x)\right| < \varepsilon + \frac{2M}{\delta^2}\left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta} b_n - x\right)^2$$

where $x \in [0, b_n]$ and $\delta = \delta(\varepsilon)$ are independent of n. Now since we know,

$$S^{(\alpha,\beta)}_{n,p_n,q_n}((t-x)^2; x) = S^{(\alpha,\beta)}_{n,p_n,q_n}(t^2; x) - 2x S^{(\alpha,\beta)}_{n,p_n,q_n}(t; x) + x^2 S^{(\alpha,\beta)}_{n,p_n,q_n}(1; x).$$

We can conclude by Theorem 3.3,

$$\sup_{0\leq x\leq b_n} |S^{(\alpha,\beta)}_{n,p_n,q_n}(f; x) - f(x)| \leq \varepsilon + \frac{2M}{\delta^2}\left(\frac{\alpha^2 b_n^2}{([n]_{p,q}+\beta)^2} + \frac{((1-p)[n]^2_{p,q} + p\beta^2)x^2}{p([n]_{p,q}+\beta)^2} + \frac{([n]_{p,q}+2\alpha\beta)b_n x}{([n]_{p,q}+\beta)^2}\right).$$

Since $\frac{b_n}{[n]_{p,q}} = 0$, as $n \to \infty$, we have the desired result. □

4 Rate of Convergence

Now we give the rate of convergence of the operators $S^{(\alpha,\beta)}_{n,p,q}(f; x)$ in terms of the elements of the usual Lipschitz class $Lip_M(\gamma)$.

Let $f \in C_B[0, \infty)$, $M > 0$ and $0 < \gamma \leq 1$. We recall that f belongs to the class $Lip_M(\gamma)$ if the inequality

$$| f(t) - f(x) | \leq M | t - x |^{\gamma} \quad t, x \in [0, \infty)$$

is satisfied.

Theorem 4.1 *Let* $0 < q < p \leq 1$. *Then for each* $f \in Lip_M(\gamma)$, *we have*

$$| S^{(\alpha,\beta)}_{n,p,q}(f; x) - f(x) | \leq M(\delta_n(x))^{\frac{\gamma}{2}}$$

where

$$\delta_n(x) = S^{(\alpha,\beta)}_{n,p,q}((t-x)^2; x).$$

Proof For $f \in Lip_M(\gamma)$, we obtain

$$| S^{(\alpha,\beta)}_{n,p,q}(f; x) - f(x) |$$

$$= \Big| \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}(-[n]_{p,q}q^{-k}\frac{x}{b_n}) \Big(f\left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta} b_n\right) - f(x)\Big)\Big|$$
$$\leq \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \left| f\left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta} b_n\right) - f(x)\right|$$
$$\leq M \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \left| \frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta} b_n - x\right|^{\gamma}.$$

Applying Hölder's inequality with the values $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, we get following inequality,

$$| S_{n,p,q}^{(\alpha,\beta)}(f;x) - f(x) |$$
$$\leq M \left(\sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta} b_n - x\right)^2 \right)^{\frac{\gamma}{2}}$$
$$\left(\sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \right)^{\frac{2-\gamma}{2}}.$$

From Lemma 2.2, we get

$$= M\left(S_{n,p,q}^{(\alpha,\beta)}\left((t-x)^2;x\right)\right)^{\frac{\gamma}{2}} \left(S_{n,p,q}^{(\alpha,\beta)}(1;x)\right)^{\frac{2-\gamma}{2}}$$
$$= M\left(S_{n,p,q}^{(\alpha,\beta)}\left((t-x)^2;x\right)\right)^{\frac{\gamma}{2}}.$$

Choosing $\delta : \delta_n(x) = S_{n,p,q}^{(\alpha,\beta)}\left((t-x)^2;x\right)$,
we obtain

$$| S_{n,p,q}^{(\alpha,\beta)}(f;x) - f(x) | \leq M(\delta_n(x))^{\frac{\gamma}{2}}.$$

Hence, the desired result is obtained. □

We will estimate the rate of convergence in terms of modulus of continuity. Let $f \in C_B[0,\infty)$, and the modulus of continuity of f denoted by $\omega(f,\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by the relation

$$\omega(f,\delta) = \max_{|y-x|\leq\delta} | f(y) - f(x) |, \quad x, y \in [0,\infty).$$

It is known that $\lim_{\delta\to 0+} \omega(f,\delta) = 0$ for $f \in C_B[0,\infty)$ and for any $\delta > 0$ one has

$$| f(y) - f(x) | \leq \left(\frac{| y-x |}{\delta} + 1 \right) \omega(f,\delta). \tag{7}$$

Theorem 4.2 *If $f \in C_B[0,\infty)$, then*

$$\mid S_{n,p,q}^{(\alpha,\beta)}(f;x) - f(x) \mid \leq 2\omega(f;(\sqrt{\delta_n(x)}),$$

where $\omega(f;\cdot)$ is modulus of continuity of f and $\delta_n(x)$ be the same as in Theorem 4.1.

Proof Using triangular inequality, we get

$$\begin{aligned}\mid S_{n,p,q}^{(\alpha,\beta)}(f;x) - f(x) \mid &= \Bigg| \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \\ &\quad \times \left(f\left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}b_n\right) - f(x)\right)\Bigg| \\ &\leq \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \\ &\quad \times \left| f\left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}b_n\right) - f(x)\right|.\end{aligned}$$

Now using inequality (7), Hölder's inequality and Lemma 2.2, we get

$$\begin{aligned}\mid S_{n,p,q}^{(\alpha,\beta)}(f;x) - f(x) \mid &= \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \\ &\quad \times \left(\frac{\mid \frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}b_n - x \mid}{\delta} + 1\right)\omega(f,\delta) \\ &\leq \omega(f,\delta) \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \\ &\quad + \frac{\omega(f,\delta)}{\delta} \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} e_{p,q}\left(-[n]_{p,q}q^{-k}\frac{x}{b_n}\right) \\ &\quad \times \left| \frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}b_n - x \right| \\ &= \omega(f,\delta) + \frac{\omega(f,\delta)}{\delta}\Bigg(\sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!(b_n)^k} \\ &\quad \times e_{p,q}(-[n]_{p,q}q^{-k}\frac{x}{b_n})\left(\frac{p^{1-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}b_n - x\right)^2\Bigg)^{\frac{1}{2}} \\ &= \omega(f,\delta) + \frac{\omega(f,\delta)}{\delta}\left(S_{n,p,q}^{(\alpha,\beta)}((t-x)^2;x)\right)^{\frac{1}{2}}.\end{aligned}$$

Now choosing $\delta = \delta_n(x)$ as in Theorem 4.1, we have

$$| S_{n,p,q}^{(\alpha,\beta)}(f; x) - f(x) | \leq 2\omega(f; (\sqrt{\delta_n(x)}). \qquad \square$$

Now let us denote by $C_B^2[0,\infty)$, the space of all functions $f \in C_B[0,\infty)$, such that $f^{'}, f^{''} \in C_B[0,\infty)$. Let $\| f \|$ denote the usual supremum norm of f. Classical Peetre's K-functional and the second modulus of smoothness of the function $f \in C_B[0,\infty)$ are defined, respectively, as

$$K_2(f,\delta) = \inf_{g \in C_B^2[0,\infty)} \{\| f - g \| + \delta \| g^{''} \|\},$$

where $\delta > 0$ and $g \in C_B^2[0,\infty)$. By Theorem 2.4 of [7], there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta}) \tag{8}$$

where

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x,x+h \in I} | f(x+2h) - 2f(x+h) + f(x) |$$

is the second-order modulus of smoothness of $f \in C_B^2[0,\infty)$. The usual modulus of continuity of $f \in C_B^2[0,\infty)$ is defined by

$$\omega(f,\delta) = \sup_{0<h\leq\delta} \sup_{x\in[0,\infty)} | f(x+h) - f(x) | .$$

Theorem 4.3 *Let $x \in [0, b_n]$, $f \in C_B[0,\infty)$ and $0 < q < p \leq 1$, $0 \leq \alpha \leq \beta$. Then for all $n \in \mathbb{N}$, there exists a positive constant $C > 0$ such that*

$$| S_{n,p,q}^{(\alpha,\beta)}(f; x) - f(x) | \leq C\omega_2(f, \delta_n(x)) + \omega(f, \alpha_n(x)),$$

where

$$\delta_n(x) = \sqrt{S_{n,p,q}^{(\alpha,\beta)}((t-x)^2; x) + (\alpha_n(x))^2}, \quad \alpha_n(x) = \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta} - 1\right)x + \frac{\alpha b_n}{[n]_{p,q}+\beta}.$$

Proof For $x \in [0,\infty)$, we consider the auxiliary operators $\bar{S}_n^*$ defined by

$$\bar{S}_n^*(f; x) = S_{n,p,q}^{(\alpha,\beta)}(f; x) + f(x) - f\left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha b_n}{[n]_{p,q}+\beta}\right).$$

From Lemma 2.2 (i) (ii) and Lemma 2.3 (i), we observe that the operators $\bar{S}_n^*(f; x)$ are linear and reproduce the linear functions. Hence,

$$\bar{S}_n^*(1;x) = S_{n,p,q}^{(\alpha,\beta)}(1;x) + 1 - 1 = 1$$
$$\bar{S}_n^*(t;x) = S_{n,p,q}^{(\alpha,\beta)}(t;x) + x - \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha b_n}{[n]_{p,q}+\beta}\right) = x$$
$$\bar{S}_n^*((t-x);x) = \bar{S}_n^*(t;x) - x\bar{S}_n^*(1;x) = 0.$$

Let $x \in [0,\infty)$ and $g \in C_B^2[0,\infty)$. Using Taylor's formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\mathrm{d}u.$$

Applying $\bar{S}_n^*$ to both sides of the above equation, we have

$$\begin{aligned}\bar{S}_n^*(g;x) - g(x) &= g'(x)\bar{S}_n^*((t-x);x) + \bar{S}_n^*\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right)\\ &= S_{n,p,q}^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right)\\ &\quad - \int_x^{\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha b_n}{[n]_{p,q}+\beta}} \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha b_n}{[n]_{p,q}+\beta} - u\right) g''(u)\mathrm{d}u.\end{aligned}$$

On the other hand, since

$$\left|\int_x^t (t-u)g''(u)\mathrm{d}u\right| \le \int_x^t \mid t-u \mid\mid g''(u) \mid \mathrm{d}u \le \parallel g'' \parallel \left|\int_x^t \mid t-u \mid \mathrm{d}u\right| \le (t-x)^2 \parallel g'' \parallel$$

and

$$\left|\int_x^{\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha b_n}{[n]_{p,q}+\beta}} \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha b_n}{[n]_{p,q}+\beta} - u\right) g''(u)\mathrm{d}u\right|$$
$$\le \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha b_n}{[n]_{p,q}+\beta} - x\right)^2 \parallel g'' \parallel .$$

We conclude that

$$\begin{aligned}\left|\bar{S}_n^*(g;x) - g(x)\right| &\le \left|S_{n,p,q}^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right)\right|\\ &\quad + \left|\int_x^{\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha b_n}{[n]_{p,q}+\beta}} \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha b_n}{[n]_{p,q}+\beta} - u\right) g''(u)\mathrm{d}u\right|\\ &\le \parallel g'' \parallel S_{n,p,q}^{(\alpha,\beta)}((t-x)^2;x) + \parallel g'' \parallel \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha b_n}{[n]_{p,q}+\beta} - x\right)^2\\ &= \parallel g'' \parallel \delta_n^2(x).\end{aligned}$$

Now, taking into account Lemma 2.2 (i), we have

$$|\ \bar{S}_n^*(f;x)\ | \leq |\ S_{n,p,q}^{(\alpha,\beta)}(f;x)\ | + 2 \parallel f \parallel \leq 3 \parallel f \parallel .$$

Therefore,

$$\begin{aligned}|\ S_{n,p,q}^{(\alpha,\beta)}(f;x) - f(x)\ | &\leq |\ \bar{S}_n^*(f-g;x) - (f-g)(x)\ | \\ &\quad + \left| f\left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha b_n}{[n]_{p,q}+\beta}\right) - f(x)\right| + |\ \bar{S}_n^*(g;x) - g(x)\ | \\ &\leq 4 \parallel f-g \parallel + \omega(f,\alpha_n(x)) + \delta_n^2(x) \parallel g'' \parallel .\end{aligned}$$

Hence, taking the infimum on the right-hand side over all $g \in C_B^2[0,\infty)$, we have the following result

$$|\ S_{n,p,q}^{(\alpha,\beta)}(f;x) - f(x)\ | \leq 4K_2(f,\delta_n^2(x)) + \omega(f,\alpha_n(x)).$$

In view of the property of K-functional, we get

$$|\ S_{n,p,q}^{(\alpha,\beta)}(f;x) - f(x)\ | \leq C\omega_2(f,\delta_n(x)) + \omega(f,\alpha_n(x)).$$

This completes the proof of the theorem. □

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Approximation Theorems for Positive Linear Operators Associated with Hermite and Laguerre Polynomials

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Abstract We present some results regarding positive linear operators associated with Hermite and Laguerre expansions. We consider Poisson type integrals for orthogonal expansions and discuss their approximation properties in the L^p space. We also investigate operators of Szász–Mirakjan type defined via Hermite polynomials. We give the rates of convergence by means of the modulus of continuity and moduli of smoothness. We present Voronovskaya type theorems for these operators and discuss boundary value problems for Poisson integrals. We also consider some combinations of the operators presented here, study their approximation errors and prove the Voronovskaya type formula.

Keywords Poisson integrals · Linear operators · Hermite and Laguerre expansions · Approximation order · Voronovskaya type theorem

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1 Introduction

In this part, we present some examples of positive linear operators associated with the Hermite and Laguerre expansions. We recall and summarize the approximation results achieved for these operators.

1.1 Modified Szász–Mirakjan Operators

Starting with the identity

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$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!},$$

G.M. Mirakjan (in 1941) and O. Szász (in 1950) considered the linear positive operators (see [6, 9])

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),$$

$x \in \mathbb{R}_0^+ := [0, \infty)$, $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, and proved that if f is bounded in every finite interval, $f \in O(x^k)$ as $x \to \infty$, for some $k > 0$ and f is differentiable at a point $\xi > 0$, then

$$\lim_{n\to\infty} \sqrt{n}|S_n(f; \xi) - f(\xi)| = 0.$$

The so-called Hermite–Kempé de Fériet polynomials (or, in other words, the two variable Hermite polynomials [1]) defined by

$$\widetilde{H}_k(y, z) = k! \sum_{s=0}^{[\frac{k}{2}]} \frac{y^{k-2s} z^s}{(k-2s)!s!}$$

are specified by means of the generating function

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \widetilde{H}_k(y, z) = \exp(yt + zt^2). \tag{1.1}$$

Putting $y = nx$, $z = a$ in the equality (1.1) we can consider the class of operators G_n^a, $n \in \mathbb{N}$, $a \geq 0$, given by the formula (see [5])

$$G_n^a(f; x) = e^{-(nx+ax^2)} \sum_{k=0}^{\infty} \frac{x^k}{k!} \widetilde{H}_k(n, a) f\left(\frac{k}{n}\right), \quad x \in \mathbb{R}_0^+.$$

In paper [5], we study approximation properties of G_n^a for functions $f \in C_B\left(\mathbb{R}_0^+\right)$, where $C_B(\mathbb{R}_0^+)$ is the space of all real-valued functions f continuous and bounded on $\mathbb{R}_0^+$. The norm on $C_B(\mathbb{R}_0^+)$ is defined by

$$||f|| = \sup_{x\in\mathbb{R}_0^+} |f(x)|.$$

The operator G_n^α maps $C_B(\mathbb{R}_0^+)$ into $C_B(\mathbb{R}_0^+)$ and

$$\left|\left|G_n^\alpha(f)\right|\right| \leq ||f||$$

for $f \in C_B(\mathbb{R}_0^+)$.

We recall some estimates of the rate of convergence of the operators G_n^a, $a \geq 0$ for functions $f \in C_B\left(\mathbb{R}_0^+\right)$ using the modulus of continuity and the modulus of smoothness.

Theorem 1.1 ([5]) *For every $f \in C_B(\mathbb{R}_0^+)$, $x \in \mathbb{R}_0^+$ and $n \in \mathbb{N}$, we have*

$$\left|G_n^a(f;x) - f(x)\right| \leq 2\omega_1\left(f, \sqrt{\frac{x}{n} + \frac{4ax^2(ax^2+1)}{n^2}}\right),$$

where

$$\omega_1(f,\delta) = \sup_{\substack{x,y\in\mathbb{R}_0^+\\ |y-x|\leq\delta}} |f(y) - f(x)|, \qquad \delta > 0.$$

Theorem 1.2 ([5]) *If $f \in C_B(\mathbb{R}_0^+)$, then for every $x \in \mathbb{R}_0^+$ we have*

$$\begin{aligned}&\left|G_n^a(f;x) - f(x)\right|\\ &\leq M\omega_2\left(f, \frac{1}{2}\sqrt{\frac{x}{n} + \frac{4ax^2(ax^2+1)}{n^2} + \left(\frac{2ax^2}{n}\right)^2}\right) + \omega_1\left(f, \frac{2ax^2}{n}\right),\end{aligned}$$

where M is some positive constant, $\omega_1(f,\delta)$ is given by (1.1) and

$$\omega_2(f,\delta) = \sup_{\substack{x\in\mathbb{R}_0^+\\ 0<h\leq\delta}} |f(x+2h) - 2f(x+h) + f(x)|, \qquad \delta > 0.$$

The above theorems imply the following result.

Corollary 1.2.1 ([5]) *If f is an uniformly continuous bounded function on $\mathbb{R}_0^+$, then*

$$\lim_{n\to\infty} G_n^\alpha(f;x) = f(x)$$

uniformly on every interval $[a,b] \subset \mathbb{R}_0^+$, $a < b$.

We have also the Voronovskaya type theorem for G_n^a.

Theorem 1.3 ([5]) *Let $x \in \mathbb{R}_0^+$ be a fixed point and let f be an uniformly continuous bounded function on $\mathbb{R}_0^+$. If f is of the class $C^1(\mathbb{R}_0^+)$ in a certain neighbourhood of a point x and $f''(x)$ exists, then*

$$\lim_{n\to\infty} n\left[G_n^a(f;x) - f(x)\right] = 2ax^2 f'(x) + \frac{x}{2} f''(x).$$

Observe that for $a = 0$ we have

$$G_n^0(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{x^k}{k!} n^k f\left(\frac{k}{n}\right), \quad x \in \mathbb{R}_0^+,$$

so G_n^0, $n \in \mathbb{N}$ are the classical Szász–Mirakjan operators S_n.

It is clear that $\widetilde{H}_k(2n, -1) = H_k(n)$, where H_k is the kth classical Hermite polynomial defined by

$$H_k(t) = (-1)^k e^{t^2} \frac{d^k}{dt^k} e^{-t^2}, \quad k \in \mathbb{N}_0 := \{0, 1, \ldots\}. \tag{1.2}$$

In the next paragraph, we recall some approximation theorems for Poisson integrals associated with the classical Hermite polynomials (1.2).

1.2 *The Poisson Integrals for Hermite and Laguerre Expansions*

Let $L^p(\exp(-z^2))$, $p \geq 1$ denote the set of functions f defined on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} |f(t)|^p \exp(-t^2)\, dt < \infty \quad \text{if } 1 \leq p < \infty,$$

and f is bounded almost everywhere on $\mathbb{R}$ if $p = \infty$.

The Hermite polynomials in (1.2) satisfy

$$\left(\frac{d^2}{dt^2} - 2t\frac{d}{dt}\right) H_k(t) = -2k H_k(t)$$

and the system $\{\widetilde{h}_k\}_{k \in \mathbb{N}_0}$, where

$$\widetilde{h}_k(t) = \left(2^k k! \sqrt{\pi}\right)^{-\frac{1}{2}} H_k(t),$$

is a complete orthonormal system on $L^2(\exp(-z^2))$.

Muckenhoupt in [7] studied the Poisson integral of a function $f \in L^p(\exp(-z^2))$ for Hermite expansions defined by

$$\widetilde{A}(f)(r, y) = \widetilde{A}(f; r, y) = \int_{-\infty}^{\infty} \widetilde{K}(r, y, z) f(z) \exp\left(-z^2\right) dz, \quad 0 < r < 1,$$

where

$$\widetilde{K}(r, y, z) = \sum_{n=0}^{\infty} \frac{r^n H_n(y) H_n(z)}{\sqrt{\pi}\, 2^n n!}$$
$$= \frac{1}{\sqrt{\pi(1-r^2)}} \exp\left(\frac{-r^2y^2 + 2ryz - r^2z^2}{1-r^2}\right).$$

Muckenhoupt proved that if $f \in L^p(\exp(-z^2))$, then

(a) $\|\widetilde{A}(f; r, \cdot)\|_p \leq \|f\|_p, \quad 1 \leq p \leq \infty,$

(b) $\|\widetilde{A}(f; r, \cdot) - f(\cdot)\|_p \to 0$ as $r \to 1^-$ for $1 \leq p < \infty$,

(c) $\lim_{r\to 1^-} \widetilde{A}(f; r, y) = f(y)$ almost everywhere, $1 \leq p \leq \infty$,

where $\|f\|_p$ denotes the norm in $L^p\left(\exp(-z^2)\right)$ of a function f defined on $\mathbb{R}$.

In papers [8, 10], the following approximation properties of $\widetilde{A}(f)$ are presented.

Theorem 1.4 ([8]) *Let* $f \in C(\mathbb{R}) \cap L^p(\exp(-z^2))$. *Then*

$$|\widetilde{A}(f; r, y) - f(y)| \leq 2\omega_1\left(f, \sqrt{(1-r)\left(\frac{1}{2}(r+1) + y^2(1-r)\right)}\right)$$

for $0 < r < 1$ *and* $y \in \mathbb{R}$, *where*

$$\omega_1(f, \delta) = \sup_{\substack{x, y \in \mathbb{R} \\ |y-x| \leq \delta}} |f(y) - f(x)|, \qquad \delta > 0.$$

We have also the Voronovskaya type formula for the operator $\widetilde{A}(f)$.

Theorem 1.5 ([10]) *Let* $y \in \mathbb{R}$. *If* $f \in C(\mathbb{R}) \cap L^p(\exp(-z^2))$, f *is of the class* C^1 *in a certain neighbourhood of a point* y *and* $f''(y)$ *exists, then*

$$\lim_{r\to 1^-} \frac{1}{1-r}\left[\widetilde{A}(f; r, y) - f(y)\right] = -yf'(y) + \frac{1}{2}f''(y).$$

The following boundary problem is presented in paper [10].

Theorem 1.6 ([10]) *If* $f \in L^p(\exp(-z)^2)$, *then* $\widetilde{A}(f)$ *is of the class* C^∞ *in the set* $D = \{(r, y) : 0 < r < 1,\ y \in \mathbb{R}\}$ *and* $\widetilde{A}(f)$ *is a solution of the equation*

$$2r\frac{\partial u(r, y)}{\partial r} - 2y\frac{\partial u(r, y)}{\partial y} + \frac{\partial^2 u(r, y)}{\partial y^2} = 0$$

in D.

The above theorem implies

Corollary 1.6.1 ([10]) *If* $f \in C(\mathbb{R}) \cap L^p(\exp(-z)^2)$, *then the function* $\widetilde{A}(f)$ *is a solution of the problem*

$$2r\frac{\partial u(r,y)}{\partial r} - 2y\frac{\partial u(r,y)}{\partial y} + \frac{\partial^2 u(r,y)}{\partial y^2} = 0 \ \ in\ D,$$

$$\lim_{r\to 1} u(r,y) = f(y), \quad y \in \mathbb{R}.$$

Let $L^p(z^\alpha \exp(-z))$, $p \geq 1$, $\alpha > -1$ denote the set of functions f defined on $\mathbb{R}_0^+$ such that

$$\int_0^\infty |f(t)|^p t^\alpha \exp(-t)\,dt < \infty \quad \text{if } 1 \leq p < \infty,$$

and f is bounded almost everywhere on $\mathbb{R}_0^+$ if $p = \infty$.

Note [7] also considered Poisson integrals for Laguerre polynomial expansions. The Poisson integral $\widetilde{B}(f)(r,y)$ of a function $f \in L^p(z^\alpha \exp(-z))$, $\alpha > -1$ is defined by

$$\widetilde{B}(f)(r,y) = \widetilde{B}(f;r,y) = \int_0^\infty \widetilde{P}(r,y,z) f(z) z^\alpha \exp(-z)\,dz, \quad 0 < r < 1$$

with the Poisson kernel

$$\begin{aligned}\widetilde{P}(r,y,z) &= \sum_{n=0}^\infty \frac{r^n n!}{\Gamma(n+\alpha+1)} L_n^\alpha(y) L_n^\alpha(z)\\ &= \frac{(ryz)^{-\frac{\alpha}{2}}}{1-r}\exp\left(\frac{-r(y+z)}{1-r}\right) I_\alpha\left(\frac{2(ryz)^{\frac{1}{2}}}{1-r}\right),\end{aligned}$$

where L_n^α is the nth Laguerre polynomial and I_α is the modified Bessel function ([3])

$$I_\alpha(s) = \sum_{n=0}^\infty \frac{s^{\alpha+2n}}{2^{\alpha+2n} n!\Gamma(\alpha+n+1)}.$$

The Laguerre polynomials

$$L_n^\alpha(x) = e^x \frac{x^{-\alpha}}{n!} \cdot \frac{d^n}{x^n}\left(e^{-x} x^{n+\alpha}\right), \quad n \in \mathbb{N}_0$$

satisfy

$$\left(x\frac{d^2}{dx^2} + (1+\alpha-x)\frac{d}{dx}\right) L_n^\alpha(x) = -n L_n^\alpha(x),$$

and the system $\{l_n^\alpha\}_{n\in\mathbb{N}_0}$, where

$$l_n^\alpha(x) = \left(\frac{n!}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}} L_n^\alpha(x),$$

is a complete orthonormal system on $L^2(z^\alpha \exp(-z))$.

Muckenhoupt proved in [7] that if $f \in L^p\,(z^\alpha \exp(-z))$, then

(a) $\|\widetilde{B}(f; r, \cdot)\|_p \le \|f\|_p, \quad 1 \le p \le \infty$,
(b) $\|\widetilde{B}(f; r, \cdot) - f(\cdot)\|_p \longrightarrow 0$ as $r \to 1^-$ for $1 \le p < \infty$,
(c) $\lim_{r\to 1^-} \widetilde{B}(f; r, y) = f(y)$ almost everywhere in $[0, \infty)$, $1 \le p \le \infty$.

The symbol $\|f\|_p$ is used here to denote the pth norm of a function f defined on $\mathbb{R}_0^+$ with respect to the measure $z^\alpha \exp(-z)\,dz$.

Analogously as for the operator $\widetilde{A}$ we have the results for the operator $\widetilde{B}$.

Theorem 1.7 ([8]) *Let* $f \in C(\mathbb{R}_0^+) \cap L^p(z^\alpha \exp(-z))$. *Then*

$$\begin{aligned}&|\widetilde{B}(f; r, y) - f(y)|\\ &\le 2\omega_1\left(f, \sqrt{(1-r)\left((y^2+\alpha^2+3\alpha+2)(1-r)+2y((\alpha+2)r-\alpha-1)\right)}\right)\end{aligned}$$

for $0 < r < 1$ *and* $y \in \mathbb{R}_0^+$.

Theorem 1.8 ([10]) *Let* $y \in \mathbb{R}_0^+$. *If* $f \in C(\mathbb{R}_0^+) \cap L^p(z^\alpha \exp(-z))$, f *is of the class* C^1 *in a certain neighbourhood of a point* y *and* $f''(y)$ *exists, then*

$$\lim_{r\to 1^-} \frac{1}{1-r}\left[\widetilde{B}(f; r, y) - f(y)\right] = (1+\alpha-y)f'(y) + yf''(y).$$

Theorem 1.9 ([10]) *If* $f \in L^p(z^\alpha \exp(-z))$, *then* $\widetilde{B}(f)$ *is of the class* C^∞ *in the set* $D = \{(r, y) : 0 < r < 1,\ y \in \mathbb{R}_0^+\}$ *and* $\widetilde{B}(f)$ *is a solution of the heat-diffusion equation*

$$r\frac{\partial u(r, y)}{\partial r} + (1+\alpha-y)\frac{\partial u(r, y)}{\partial y} + y\frac{\partial^2 u(r, y)}{\partial y^2} = 0$$

in D.

Corollary 1.9.1 ([10]) *If* $f \in C(\mathbb{R}_0^+) \cap L^p(z^\alpha \exp(-z))$, *then the function* $\widetilde{B}(f)$ *is a solution of the problem*

$$r\frac{\partial u(r, y)}{\partial r} + (1+\alpha-y)\frac{\partial u(r, y)}{\partial y} + y\frac{\partial^2 u(r, y)}{\partial y^2} = 0 \quad \text{in } D,$$

$$\lim_{r\to 1} u(r, y) = f(y), \quad y \in \mathbb{R}_0^+.$$

1.3 The Poisson Integrals Associated with Hermite Functions

Let $L^p(\mathbb{R})$ denote the set of functions f defined on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} |f(t)|^p \, dt < \infty \quad \text{if } 1 \leq p < \infty,$$

and f is bounded almost everywhere on $\mathbb{R}$ if $p = \infty$.

Gosselin and Stempak in [2] studied the heat-diffusion and Poisson integrals for Hermite function expansions. They considered the integral $\widehat{A}(f)$ of a function $f \in L^p(\mathbb{R})$ defined by

$$\widehat{A}(f)(x, y) = \widehat{A}(f; x, y) = \int_{-\infty}^{\infty} \widehat{K}(x, y, z) f(z) \, dz,$$

where

$$\widehat{K}(x, y, z) = \sum_{n=0}^{\infty} h_n(y) \, h_n(z) \exp\left(-(2n+1)x\right), \quad x > 0$$

and

$$h_n(x) = \left(2^n n! \sqrt{\pi}\right)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2}\right) H_n(x).$$

Muckenhoupt in [7] treated the exponential $\exp(-z^2)$ as a weight function. Gosselin and Stempak in [2] considered expansions with respect to the system of Hermite functions $\{h_n\}_{n \in \mathbb{N}_0}$ which is a complete orthonormal system in $L^p(\mathbb{R})$.

The authors obtained the following results.

Theorem 1.10 ([2]) *If* $f \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, *then* $\widehat{A}(f)$ *is of the class* C^∞ *on the set* $(0, \infty) \times \mathbb{R}$ *and* $\widehat{A}(f)$ *is a solution of the differential equation*

$$\frac{\partial u(x, y)}{\partial x} = \frac{\partial^2 u(x, y)}{\partial y^2} - y^2 u(x, y).$$

Theorem 1.11 ([2]) *Let* $f \in L^p(\mathbb{R})$. *Then*

(a) $\|\widehat{A}(f; x, \cdot)\|_p \leq (\cosh 2x)^{-\frac{1}{2}} \|f\|_p, \quad 1 \leq p \leq \infty,$

(b) $\|\widehat{A}(f; x, \cdot) - f(\cdot)\|_p \to 0$ *as* $x \to 0, \quad 1 \leq p < \infty,$

(c) $\lim_{x \to 0} \widehat{A}(f; x, y) = f(y)$ *almost everywhere,* $1 \leq p < \infty,$

where $\|f\|_p$ *denotes the norm in* $L^p(\mathbb{R})$ *of a function* f *defined on* $\mathbb{R}$.

Instead of working with the kernel $\widehat{K}$ we consider the kernel

$$K(r, y, z) = \sum_{n=0}^{\infty} r^n h_n(y)\, h_n(z), \quad 0 < r < 1.$$

Then we have $\widehat{K}(x, y, z) = e^{-x} K(e^{-2x}, y, z)$.

In paper [4], the author studied some approximation properties of the operator $A(f)$, $f \in L^p(\mathbb{R})$, given by

$$A(f)(r, y) = A(f; r, y) = \int_{-\infty}^{\infty} K(r, y, z)\, f(z)\, dz.$$

The following results were obtained.

Lemma 1.1 ([4]) *If $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$, then $A(f; r, \cdot) \in L^p(\mathbb{R})$ and*

$$\|A(f; r, \cdot)\|_p \le \|f\|_p \tag{1.3}$$

for $0 < r < 1$, $y \in \mathbb{R}$.

This means that the operator A is linear, positive, bounded and transform the space $L^p(\mathbb{R})$ into itself.

Theorem 1.12 ([4]) *If $f \in C(\mathbb{R})$ and $f = f_1 + f_2$, where $f_1 \in L^1(\mathbb{R})$, $f_2 \in L^\infty(\mathbb{R})$, then*

$$\lim_{r \to 1^-} A(f; r, y) = f(y)$$

for every $y \in \mathbb{R}$. This convergence is uniform on every closed subset of $\mathbb{R}$.

Moreover, from paper [4] we have the Voronovskaya type theorem.

Theorem 1.13 ([4]) *Let $y \in \mathbb{R}$. If $f \in C(\mathbb{R}) \cap L^p(\mathbb{R})$, f is of the class C^1 in a certain neighbourhood of a point y and $f''(y)$ exists, then*

$$\lim_{r \to 1^-} \frac{1}{1 - r} (A(f; r, y) - f(y)) = -\frac{1}{2} y^2 f(y) + \frac{1}{2} f''(y).$$

The main aim of this survey paper is to present approximation properties for a combination of the operators $A(f)$.

We will consider the sequence of operators of the form

$$A_k(f; r, y) = A(f; 1 - k(1 - r), y), \quad k = 1, 2, \ldots, \quad r \in \left(\frac{k-1}{k}, 1\right).$$

We define the following operator $\overline{A}(f)$

$$\overline{A}(f; n, r, y) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} A_k(f; r, y).$$

2 Auxiliary Results

In paper [4], we presented among other thing some preliminary properties of the operator A. The purpose of this part is to remind the reader of some results which we apply to prove main theorems.

Let $\varphi_{m,y}(z) = (z-y)^m$, $y, z \in \mathbb{R}$, $m \in \mathbb{N}$. If $y = 0$, then we denote $\varphi_{m,y}$ by φ_m. Moreover, we assume that $\varphi_0(z) = 1$.

Lemma 2.1 ([4]) *For every fixed $m \in \mathbb{N}_0$ it follows*

$$A(\varphi_m; r, y) = \left(\frac{2}{1+r^2}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2}\cdot\frac{1-r^2}{1+r^2}\, y^2\right) \times \sum_{p=0}^{[\frac{m}{2}]} \binom{m}{p} \cdot \frac{(m-p)!}{(m-2p)!\, 2^p} \cdot \left(\frac{1-r^2}{1+r^2}\right)^p \left(\frac{2ry}{1+r^2}\right)^{m-2p},$$

$$A(\varphi_{m,y}; r, y) = \left(\frac{2}{1+r^2}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2}\cdot\frac{1-r^2}{1+r^2}\, y^2\right) \times \sum_{p=0}^{[\frac{m}{2}]} \binom{m}{p} \cdot \frac{(m-p)!}{(m-2p)!\, 2^p} \cdot \left(\frac{1-r^2}{1+r^2}\right)^p \left(-\frac{(1-r)^2}{1+r^2}\, y\right)^{m-2p} \tag{2.1}$$

for $0 < r < 1$, $y \in \mathbb{R}$, where $[a]$ denotes the integral part of $a \in \mathbb{R}$.

Observe that if $r \longrightarrow 1^-$, then $1 - k(1-r) \longrightarrow 1^-$. Moreover, the following formula can be written

$$A_k(1; r, y) = \left(\frac{2}{1+r_k^2}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2}\cdot\frac{1-r_k^2}{1+r_k^2}\, y^2\right), \tag{2.2}$$

where $r_k = 1 - k(1-r)$.

From (2.2) and (1.3) we have

$$0 < A_k(1; r, y) \le 1 \quad \text{and} \quad \|A_k(1; r, y)\|_p \le \|f\|_p.$$

By the definition of $\overline{A}$ the following inequalities hold:

$$\left|\overline{A}(1; n, r, y)\right| \le \sum_{k=1}^{n} \binom{n}{k} = 2^n - 1,$$

$$\left\|\overline{A}(f; n, r, \cdot)\right\|_p \le \left(2^n - 1\right) \|f\|_p.$$

Now, we recall results giving some error estimates for the operator A via modulus of continuity

$$\omega_2(f;\delta)_p = \sup_{0<h\le\delta} \|\Delta_h^2(f;\cdot)\|_p\,, \quad \delta>0,$$

where

$$\Delta_h^2(f;y) = f(y+h) - 2f(y) + f(y-h) \quad \text{for } h, y \in \mathbb{R}.$$

Theorem 2.1 ([4]) *If $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$, then there exists a constant $M>0$, independent of f, such that*

$$\|A(f;r,\cdot) - f(\cdot)A(1;r,\cdot)\|_p \le M\left\{\omega_2\left(f;\sqrt{1-r}\right)_p + (1-r)\|f\|_p\right\}$$

for $0<r<1$.

This theorem allows one to write

Corollary 2.1.1 ([4]) *If $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$, then there exists a constant $M>0$, independent of f, such that*

$$\begin{aligned}\|A(f;r,\cdot) - f(\cdot)\|_p \le M&\left\{\omega_2\left(f;\sqrt{1-r}\right)_p + (1-r)\|f\|_p\right\}\\ &+ ||(A(1;r,\cdot)-1)f(\cdot)||_p\end{aligned}$$

for $0<r<1$.

3 Approximation Theorems

Theorem 3.1 *If $f \in C(\mathbb{R})$ and $f = f_1 + f_2$, where $f_1 \in L^1(\mathbb{R})$, $f_2 \in L^\infty(\mathbb{R})$, then*

$$\lim_{r\to1^-} \overline{A}(f;n,r,y) = f(y)$$

for every $y\in\mathbb{R}$. This convergence is uniform on every closed subset of $\mathbb{R}$.

Proof Using Theorem 1.12 we obtain

$$\begin{aligned}\lim_{r\to1^-} \overline{A}(f;n,r,y) &= \lim_{r\to1^-} \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}A_k(f;r,y)\\ &= \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\lim_{r\to1^-} A_k(f;r,y) = \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}f(y)\\ &= f(y)\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}\cdot(-1)+1\right) = f(y).\end{aligned}$$

This ends the proof. □

Theorem 3.2 *If* $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$, $n \in \mathbb{N}$, *then there exists a constant* $M(n, p) > 0$, *independent of* f, *such that*

$$\left\|\overline{A}(f; n, r, \cdot) - f(\cdot)\right\|_p \le M(n, p) \left\{\omega_2 \left(f; \sqrt{1-r}\right)_p + (2-r)\|f\|_p\right\}$$

for $0 < r < 1$.

Proof Observe that

$$\left\|\overline{A}(f; n, r, \cdot) - f(\cdot)\right\|_p \le \sum_{k=1}^{n} \binom{n}{k} \|A_k(f; r, \cdot) - f(\cdot)\|_p .$$

From Corollary 2.1.1 we have

$$\|A(f; r, \cdot) - f(\cdot)\|_p \le M \left\{\omega_2 \left(f; \sqrt{1-r}\right)_p + (1-r)\|f\|_p\right\} \\ + ||(A(1; r, \cdot) - 1) f(\cdot)||_p.$$

Using the above inequality we obtain

$$\begin{aligned} &\left\|\overline{A}(f; n, r, \cdot) - f(\cdot)\right\|_p \\ &\le \sum_{k=1}^{n} \binom{n}{k} \left\{M\left[\omega_2 \left(f; \sqrt{k(1-r)}\right)_p + k(1-r)\|f\|_p\right] + \|(A_k(f; r, \cdot) - 1)\, f(\cdot)\|_p\right\} \\ &\le \sum_{k=1}^{n} \binom{n}{k} \left\{M(k, p)\left[\omega_2 \left(f; \sqrt{1-r}\right)_p + (1-r)\|f\|_p\right] + \|(A_k(f; r, \cdot) - 1)\, f(\cdot)\|_p\right\}. \end{aligned}$$

Consider the term $\|(A_k(f; r, \cdot) - 1)\, f(\cdot)\|_p$. Taking the supremum over $y \in \mathbb{R}$ in $|A_k(1; r, y) - 1|^p$, we can conclude

$$\left\|\overline{A}(f; n, r, \cdot) - f(\cdot)\right\|_p \le M(n, p) \left\{\omega_2 \left(f; \sqrt{1-r}\right)_p + (2-r)\|f\|_p\right\},$$

where $M(n, p)$ is some positive constant, $0 < r < 1$. This completes the proof. □

Let $r_0 = 1$ and $r_k = 1 - k(1-r)$, $k \in \mathbb{N}$ and $A(f; r_0, y) = 0$. Observe that

$$\frac{\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} h(r_k)}{n!(1-r)^n} = [r_0, r_1, \ldots, r_n]h,$$

where $[r_0, r_1, \ldots, r_n]h$ denotes the nth order divided difference of h at the points $r_0, r_1, \ldots, r_n$. Using the properties of $[r_0, r_1, \ldots, r_n]h$ we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}h(r_k) = (1-r)^n h^{(n)}(r_\xi),$$

where $\xi \in (0, 1)$.

Taking into account the definition and properties of the divided difference we can write

$$\begin{aligned}\overline{A}(\varphi_{m,y}; n, r, y) &= -\sum_{k=0}^{n}(-1)^k\binom{n}{k}A_k(\varphi_{m,y}, r, y)\\ &= -\sum_{k=0}^{n}(-1)^k\binom{n}{k}A(\varphi_{m,y}, r_k, y) = -(1-r)^n A^{(n)}(\varphi_{m,y}, r_\xi, y)\end{aligned}$$

and

$$\begin{aligned}\lim_{r\to 1^-}\frac{1}{(1-r)^n}\overline{A}(\varphi_{m,y}; n, r, y) &= -\lim_{r\to 1^-} A^{(n)}(\varphi_{m,y}, r_\xi, y)\\ &= -A^{(n)}(\varphi_{m,y}, 1, y).\end{aligned}$$

Moreover, applying the induction and using the Leibniz rule to the formula (2.1), we can conclude

$$A^{(n)}(\varphi_{4n,y}, r, y) = \mathcal{O}\left((1-r)^n\right) \quad \text{as} \quad r \to 1^-.$$

Hence, by some calculations, it follows the following lemma.

Lemma 3.1 *For every fixed $y \in \mathbb{R}$ and $n \in \mathbb{N}$ we have*

$$\begin{aligned}&\lim_{r\to 1^-}\frac{1}{(1-r)^n}\left(\overline{A}(1; n, r, y) - 1\right) = c_0(y),\\ &\lim_{r\to 1^-}\frac{1}{(1-r)^n}\overline{A}(\varphi_{n,y}; n, r, y) = -A^{(n)}(\varphi_{n,y}; 1, y),\\ &\lim_{r\to 1^-}\frac{1}{(1-r)^{2n}}\overline{A}\left(\varphi_{4n,y}; n, r, y\right) < \infty, \qquad (3.1)\end{aligned}$$

where $c_0(y)$ denotes a constant depending on y.

Now we can prove the Voronovskaya type theorem.

Theorem 3.3 *Let $y \in \mathbb{R}$. If $f \in C(\mathbb{R}) \cap L^p(\mathbb{R})$, f is of the class C^{2n-1} in a certain neighbourhood of a point y and $f^{(2n)}(y)$ exists, then*

$$\lim_{r\to 1^-}\frac{1}{(1-r)^n}\left(\overline{A}(f; n, r, y) - f(y)\right) = \sum_{k=0}^{2n} c_k(y) f^{(k)}(y),$$

where constants $c_k(y)$, $k = 0, 1, \dots, 2n$, *depend on* y.

Proof Let $y \in \mathbb{R}$. By Taylor's formula we get

$$\psi_y(z) = \begin{cases} \dfrac{f(z) - f(y) - \sum_{k=1}^{2n} \frac{1}{k!}(z-y)^k f^{(k)}(y)}{(z-y)^{2n}}, & z \neq y, \\ 0, & z = y, \end{cases}$$

where $\lim_{z \to y} \psi_y(z) = 0$ and ψ_y is continuous on $\mathbb{R}$.

Remark that

$$\begin{aligned} &\overline{A}(f; n, r, y) - f(y) = f(y)\left(\overline{A}(1; n, r, y) - 1\right) + f'(y)\overline{A}(\varphi_{1,y}; n, r, y) \\ &+ \frac{1}{2} f''(y)\overline{A}\left(\varphi_{2,y}; n, r, y\right) + \cdots + \frac{1}{(2n)!} f^{(2n)}(y)\overline{A}\left(\varphi_{2n,y}; n, r, y\right) \\ &+ \overline{A}\left(\psi_y\, \varphi_{2n,y}; n, r, y\right). \end{aligned} \tag{3.2}$$

Using the Hölder inequality we obtain

$$\frac{1}{(1-r)^n}\left|\overline{A}\left(\psi_y\, \varphi_{2n,y}; n, r, y\right)\right| \leq \left|\overline{A}\left(\psi_y^2; n, r, y\right)\right|^{1/2} \left|\frac{1}{(1-r)^{2n}}\overline{A}\left(\varphi_{4n,y}; n, r, y\right)\right|^{1/2}.$$

Moreover, the function ψ_y^2 satisfies the assumption of Theorem 3.1. Hence

$$\lim_{r \to 1^-} \overline{A}\left(\psi_y^2; n, r, y\right) = \psi_y^2(y) = 0.$$

Using (3.1) we obtain

$$\lim_{r \to 1^-} \frac{1}{(1-r)^n}\overline{A}\left(\psi_y\, \varphi_{2n,y}; n, r, y\right) = 0. \tag{3.3}$$

From (3.2), (3.3) and Lemma 3.1 we get the assertion. □

Thus, the operator $\overline{A}(f)$ converges at the rate of $(1-r)^n$.

Corollary 3.3.1 *Let* $y \in \mathbb{R}$. *If* f *is as in Theorem 3.3, then*

$$\left|\overline{A}(f; n, r, y) - f(y)\right| = \mathcal{O}\left((1-r)^n\right)$$

as $r \to 1^-$.

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On Generalized Picard Integral Operators

Ali Aral

Abstract In the paper, we constructed a class of linear positive operators generalizing Picard integral operators which preserve the functions $e^{\mu x}$ and $e^{2\mu x}$, $\mu > 0$. We show that these operators are approximation processes in a suitable weighted spaces. The uniform weighted approximation order of constructed operators is given via exponential weighted modulus of smoothness. We also obtain their shape preserving properties considering exponential convexity.

Keywords Voronovskaya-type theorems · Weighted modulus of continuity

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1 Introduction

According to P.P. Korovkin and H. Bohman theorem, the convergence of a sequence $(L_n)_{n\geq 1}$ of the linear positive operators to the identity operator is essentially connected with the set $\{e_0, e_1, e_2\}$ with $e_i(t) = t^i$, $i = 0, 1, 2$. Since many classical linear positive operators fix e_0 and e_1, their theorem is one of the most powerful and spectacular criteria in approximation theory. It is known that for the study of convergence of linear positive operators the set $\left\{e_0, \exp_\mu, \exp_\mu^2\right\}$, with $\exp_\mu(x) = e^{\mu x}$, $\mu > 0$, also play an important role. For this purpose, recently in [1], the authors introduced and investigated generalized Picard $\left(P_n^*\right)_{n\geq 1}$ operators fixing e_0 and $\exp_\mu^2$ given by

$$\left(P_n^* f\right)(x) = P_n\left(f; \alpha_n^*(x)\right),$$

where

$$\alpha_n^*(x) = x - \frac{1}{2a}\ln\left(\frac{n}{n-4a^2}\right),\ n > n_a,$$

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$n_a = \left[4a^2\right]$, $[\cdot]$ indicating the integer part function or so-called floor function and $(P_n)_{n\geq 1}$ classical Picard operators defined by

$$(P_n f)(x) = P_n(f; x) = \frac{\sqrt{n}}{2} \int_{\mathbb{R}} f(x+t) K_n(t)\, dt, \quad x \in \mathbb{R} \tag{1.1}$$

where

$$K_n(t) = e^{-\sqrt{n}|t|} \quad . \tag{1.2}$$

(See [5].) In here, the function f is selected such that the integrals are finite. Note that similar ideas for different linear positive operators were discussed in [2–4, 7].

In this paper, we want to obtain a new construction of the classical Picard operators fixing not only the function $\exp_\mu$ but also the function $\exp_\mu^2$. We aim to show that the new operators are positive approximation processes in the setting of large classes of weighted spaces. Using a technique developed in [6] by T. Coşkun which is based on a weighted Korovkin type theorem for linear positive operators acting on spaces which have different weights, we obtain weighted uniform convergence of the operators. Note that obtained asymptotic formulae for the new operators are different from those given for the corresponding classical operators on the line group.

The modification of our interest in this paper is defined by

$$\left(P_n^{**} f\right)(x) = \frac{\sqrt{n}}{2} \int_{\mathbb{R}} e^{-\mu\left(\alpha_n^{**}(x)+t\right)} e^{\mu x} f\left(\alpha_n^{**}(x)+t\right) K_n(t)\, dt, \quad n > n_\mu, \quad x \in \mathbb{R} \tag{1.3}$$

where

$$\alpha_n^{**}(x) = x - \frac{1}{\mu} \ln\left(\frac{n}{n-\mu^2}\right) \tag{1.4}$$

$\mu > 0$, $n_\mu = \left[\mu^2\right] + 1$ and K_n defined in (1.2).

Their close connection with the classical Picard operators is now displayed:

$$\left(P_n^{**} f\right)(x) = \exp_\mu(x)\, P_n\left(\frac{f}{\exp \mu}; \alpha_n^{**}(x)\right).$$

It is obvious that $\left(P_n^{**}\right)_{n>n_\mu}$ are positive and linear operators. On the other hand, whereas Picard operators $(P_n)_{n\geq 1}$ fix the functions e_0 and e_1, it can be checked easily that the operators $\left(P_n^{**}\right)_{n>n_\mu}$ reproduce $\exp_\mu$ and $\exp_\mu^2$, i.e.

$$\left(P_n^{**} \exp_\mu\right)(x) = \exp_\mu(x) \tag{1.5}$$

and

$$\left(P_n^{**} \exp_\mu^2\right)(x) = \exp_\mu^2(x). \tag{1.6}$$

2 Auxiliary Results

In this section, we will give some elementary properties of the generalized Picard integral operators defined in (1.3).

By means of elementary calculations, we obtain:

Lemma 1 *For each $n > n_\mu$ and $x \in \mathbb{R}$, the following identities hold:*
$\left(P_n^{**} e_0\right)(x) = \frac{n^2}{\left(n-\mu^2\right)^2}$
$\left(P_n^{**} \exp_\mu^3\right)(x) = e^{3\mu x} \frac{\left(n-\mu^2\right)^2}{n\left(n-4\mu^2\right)}$
$\left(P_n^{**} \exp_\mu^4\right)(x) = e^{4\mu x} \frac{\left(n-\mu^2\right)^3}{n^2\left(n-9\mu^2\right)}$

Lemma 2 *For each $n > n_\mu$ and $x \in \mathbb{R}$, the following identities hold:*
$\left(P_n^{**} e_1\right)(x) = \frac{n^2}{\left(n-\mu^2\right)^3} \left(\left(n-\mu^2\right) \alpha_n^{**}(x) - 2\mu\right),$
$\left(P_n^{**} e_2\right)(x) = \frac{n^2}{\left(n-\mu^2\right)^4} [2\left(n+3\mu^2\right) + \left(\mu^2 - n\right)\left(4\mu x - \left(\mu^2-n\right)x^2\right)$
$-\frac{1}{\mu^2}\left(4\mu^2 + \left(\mu^2-n\right)\mu\left(x+\alpha_n^{**}(x)\right)\right)]$

3 Approximation on Weighted Spaces

Now we recall the concept of weighted function and weighted spaces considered in [6]. Let $\mathbb{R}$ denote the set of real numbers. A real-valued function ρ is called weight function if it is continuous on $\mathbb{R}$ and

$$\lim_{|x|\to\infty} \rho(x) = \infty, \quad \rho(x) \geq 1 \text{ for all } x \in \mathbb{R}. \tag{3.1}$$

We consider the weighted spaces $C_\rho(\mathbb{R})$ and $B_\rho(\mathbb{R})$ of the real function defined on real line defined by $B_\rho(\mathbb{R}) := \left\{f : |f(x)| \leq M_f \rho(x), x \in \mathbb{R}\right\}$ and $C_\rho(\mathbb{R}) = \left\{f : f \in B_\rho(\mathbb{R}),\ f \text{ continuous}\right\}$. The spaces $B_\rho(\mathbb{R})$ and $C_\rho(\mathbb{R})$ are Banach spaces endowed with the ρ-norm

$$\|f\|_\rho = \sup_{x\in\mathbb{R}} \frac{|f(x)|}{\rho(x)}.$$

Now we give some properties of a linear positive operator acting between two spaces with different weights.

(1) A positive linear operator L_n, defined on $C_{\rho_1}(\mathbb{R})$, maps $C_{\rho_1}(\mathbb{R})$ into $B_{\rho_2}(\mathbb{R})$ iff

$$L_n\rho_1 \in B_{\rho_2}(\mathbb{R}).$$

(2) Let $L_n : C_{\rho_1}(\mathbb{R}) \to B_{\rho_2}(\mathbb{R})$ be a positive linear operator. Then

$$\|L_n\|_{C_{\rho_1}\to B_{\rho_2}} = \|L\rho_1\|_{\rho_2}.$$

(3) For $n \in \mathbb{N}$, let $L_n : C_{\rho_1}(\mathbb{R}) \to B_{\rho_2}(\mathbb{R})$ be a positive linear operator. Suppose that there exists $M > 0$ such that for all $x \in \mathbb{R}$, $\rho_1(x) \le M\rho_2(x)$. If

$$\lim_{n\to\infty} \|L_n(\rho_1) - \rho_1\|_{\rho_2} = 0,$$

then the sequence of norms $\|L_n\|_{C_{\rho_1}\to B_{\rho_2}}$ is uniformly bounded.

Let φ_1 and φ_2 be two continuous functions, monotonically increasing on the real axis such that

$$\lim_{|x|\to\infty} \varphi_1(x) = \lim_{|x|\to\infty} \varphi_2(x) = \pm\infty \text{ and } \rho_k(x) = 1 + \varphi_k^2(x), \quad k = 1, 2.$$

Theorem A ([6]) Assume that ρ_1 and ρ_2 are weight functions satisfying the equality $\lim_{|x|\to\infty} \frac{\rho_1(x)}{\rho_2(x)} = 0$. If the sequence of linear positive operators $L_n : C_{\rho_1}(\mathbb{R}) \to B_{\rho_2}(\mathbb{R})$ satisfies the following three conditions

$$\lim_{n\to\infty} \|L_n(\varphi_1^{\nu}) - \varphi_1^{\nu}\|_{\rho_2} = 0, \quad \nu = 0, 1, 2, \tag{3.2}$$

then

$$\lim_{n\to\infty} \|L_n(f) - f\|_{\rho_2} = 0,$$

for all $f \in C_{\rho_1}(\mathbb{R})$.

Now we show that Theorem A can be applied to our new operators $\left(P_n^{**}\right)_{n>n_\mu}$ can be applicable to. Let $\rho_1(x) = 1 + x^2$ and $\rho_2(x) = 1 + x^4$ with $\varphi_1(x) = x$ and $\varphi_2(x) = x^2$. In this case the test functions set is $\{1, e_0, e_2\}$. Using Lemma 1 and (1.6) we have

$$\begin{aligned}\left(P_n^{**}\rho_1\right)(x) &= \left(P_n^{**}\left(e_0 + e_1^2\right)\right)(x) \\ &= \frac{n^2}{\left(n-\mu^2\right)^2} + \frac{n^2}{\left(n-\mu^2\right)^4}\left(2\left(n+3\mu^2\right) + \left(\mu^2 - n\right)\left(4\mu x - \left(\mu^2 - n\right)x^2\right)\right. \\ &\quad \left. - \frac{1}{\mu^2}\left(4\mu^2 + \left(\mu^2 - n\right)\mu\left(x + \alpha_n^{**}(x)\right)\right)\right)\end{aligned}$$

and thus there exists $C > 0$ such that the inequality

$$\frac{\left(P_n^{**}\rho_1\right)(x)}{\rho_2(x)} \le C$$

holds for $n > n_\mu$. Thus, $\left(P_n^{**}\right)_{n>n_\mu}$ are linear positive operators acting from $C_{\rho_1}(\mathbb{R})$ into $B_{\rho_2}(\mathbb{R})$. Also $\left(P_n^{**}\right)_{n>n_\mu}$ is a uniformly bounded sequence of positive linear operators from $C_{\rho_1}(\mathbb{R})$ into $B_{\rho_2}(\mathbb{R})$. Now we check the conditions in (3.2). For $\nu = 0$, we see that

$$\lim_{n\to\infty} \|P_n^{**}e_0 - e_0\|_{\rho_2} = \sup_{x\in\mathbb{R}} \frac{1}{1+x^4}\left[\left(\frac{n}{n-\mu^2}\right)^2 - 1\right] = 0$$

For $\nu = 1$, we have

$$\lim_{n\to\infty} \|P_n^{**}e_1 - e_1\|_{\rho_2} \le \lim_{n\to\infty} \sup_{x\in\mathbb{R}} \frac{1}{1+x^4}\left[\frac{n^2}{\left(n-\mu^2\right)^2}\alpha_n^{**}(x) - x\right]$$
$$\le \lim_{n\to\infty} \sup_{x\in\mathbb{R}} \frac{x}{1+x^4}\left[\frac{n^2}{\left(n-\mu^2\right)^2} - 1\right] = 0$$

Similarly for $\nu = 2$, we have

$$\|P_n^{**}e_2 - e_2\|_{\rho_2} \le \frac{2\left(n+3\mu^2\right)n^2}{\left(n-\mu^2\right)^4} + \frac{4\mu n^2}{\left(n-\mu^2\right)^3}\sup_{x\in\mathbb{R}}\frac{x}{1+x^4} + \left(\frac{n^2}{\left(n-\mu^2\right)^2} - 1\right)\sup_{x\in\mathbb{R}}\frac{x^2}{1+x^4}$$
$$+\frac{4n^2}{\left(n-\mu^2\right)^4} + \frac{n^2}{\mu\left(n-\mu^2\right)^3}\sup_{x\in\mathbb{R}}\frac{2x}{1+x^4}.$$

Thus, we get
$$\lim_{n\to\infty} \|P_n^{**}e_2 - e_2\|_{\rho_2} = 0$$

Since all conditions of Theorem A are fulfilled, for all $f \in C_{\rho_1}(\mathbb{R})$, we have the following theorem.

Theorem 1 *Let P_n^{**}, $n > n_\mu$, be the operators defined by (1.3). For each $f \in C_{\rho_1}(\mathbb{R})$, the relation*
$$\lim_{n\to\infty} \|P_n^{**}f - f\|_{\rho_2} = 0$$

holds, where $\rho_1(x) = 1 + x^2$ and $\rho_2(x) = 1 + x^4$.

4 A Quantitative Result

The order of convergence of the operators $\left(P_n^{**}\right)_{n>n_\mu}$ in an exponential weighted space will be studied by using the following modulus of continuity. For function $f \in C_{\rho_3}(\mathbb{R})$, $\rho_3(x) = e^{\mu|x|}$, we consider the modulus of continuity defined in [8]:

$$\widetilde{\omega}(f;\delta) = \sup_{|h|<\delta} e^{-\mu|x|}\left|f(x+h) - f(x)\right|, \tag{4.1}$$

where $\delta > 0$ and $\mu > 1$. The weighted modulus of continuity has the following properties:
$$\widetilde{\omega}(f;\lambda\delta) \le (1+\lambda)\, e^{\lambda\mu\delta}\widetilde{\omega}(f;\delta), \quad \lambda > 0. \tag{4.2}$$

Similar weighted modulus of continuity was also given in [10].

Theorem 2 *For function $f \in C_{\rho_3}(\mathbb{R})$, we have*

$$\left\| P_n^{**} f - f \right\|_{\rho_3} \leq \| f \|_{\rho_3} \left(\frac{n^2}{\left(n-\mu^2\right)^2} - 1 \right) + \frac{n}{n-\mu^2} \left(\frac{\sqrt{n}}{\sqrt{n}-\mu} + 1 \right) \widetilde{\omega} \left(\frac{f}{\exp_\mu}; \frac{\sqrt{n}}{\left(\sqrt{n}-\mu\right)^2} \right)$$

Proof Since

$$\left(P_n^{**} \rho_3\right)(x) \leq e^{\mu|x|} \frac{n^2}{\left(n-\mu^2\right)^2} \left(1 + \frac{\sqrt{n}}{\sqrt{n}-\mu}\right),$$

$P_n^{**} f$ is a sequence of linear positive operators acting $C_{\rho_3}(\mathbb{R})$ into itself. From Lemma 1, we can write

$$\left(P_n^{**} f\right)(x) - f(x) = f(x)\left(\left(P_n^{**} e_0\right)(x) - 1\right) + \frac{\sqrt{n}}{2} \int_{\mathbb{R}} \left(e^{-\mu\left(\alpha_n^{**}(x)+t\right)} e^{\mu x} f\left(\alpha_n^{**}(x)+t\right) - f(x) \right) e^{-\sqrt{n}|t|} dt$$

Using (4.1), it is not difficult to deduce that

$$\begin{aligned} & \left| e^{-\mu\left(\alpha_n^{**}(x)+t\right)} f\left(\alpha_n^{**}(x)+t\right) - e^{-\mu x} f(x) \right| \\ \leq & \left| e^{-\mu\left(\alpha_n^{**}(x)+t\right)} f\left(\alpha_n^{**}(x)+t\right) - e^{-\mu\alpha_n^{**}(x)} f\left(\alpha_n^{**}(x)\right) \right| + \left| e^{-\mu\alpha_n^{**}(x)} f\left(\alpha_n^{**}(x)\right) - e^{-\mu x} f(x) \right| \\ \leq & e^{\mu\left|\alpha_n^{**}(x)\right|} \widetilde{\omega}\left(\frac{f}{\exp}; |t|\right) + \left| e^{-\mu\alpha_n^{**}(x)} f\left(\alpha_n^{**}(x)\right) - e^{-\mu x} f(x) \right| \end{aligned}$$

and then we conclude that from (4.2),

$$\begin{aligned} \left(P_n^{**} f\right)(x) - f(x) = & f(x)\left(\left(P_n^{**} e_0\right)(x) - 1\right) + e^{\mu\left|\alpha_n^{**}(x)\right|} \frac{\sqrt{n}}{2} \int_{\mathbb{R}} \widetilde{\omega}\left(\frac{f}{\exp}; |t|\right) e^{-\sqrt{n}|t|} dt \\ & + \left| e^{-\mu\alpha_n^{**}(x)} f\left(\alpha_n^{**}(x)\right) - e^{-\mu x} f(x) \right| \left(P_n^{**} e_0\right)(x) \\ = & f(x)\left(\left(P_n^{**} e_0\right)(x) - 1\right) + \widetilde{\omega}\left(\frac{f}{\exp}; \delta_n\right) e^{\mu\left|\alpha_n^{**}(x)\right|} \frac{\sqrt{n}}{2} \int_{\mathbb{R}} \left(1 + \frac{|t|}{\delta_n}\right) e^{\mu|t|} e^{-\sqrt{n}|t|} dt \\ & + \left| e^{-\mu\alpha_n^{**}(x)} f\left(\alpha_n^{**}(x)\right) - e^{-\mu x} f(x) \right| \left(P_n^{**} e_0\right)(x) \\ = & f(x)\left(\left(P_n^{**} e_0\right)(x) - 1\right) + \frac{n}{n-\mu^2} \widetilde{\omega}\left(\frac{f}{\exp}; \delta\right) e^{\mu|x|} \left(\frac{\sqrt{n}}{\sqrt{n}-\mu} + \frac{1}{\delta_n} \frac{\sqrt{n}}{\left(\sqrt{n}-\mu\right)^2} \right). \\ & + \left| e^{-\mu\alpha_n^{**}(x)} f\left(\alpha_n^{**}(x)\right) - e^{-\mu x} f(x) \right| \left(P_n^{**} e_0\right)(x). \end{aligned}$$

Choosing $\delta = \frac{\sqrt{n}}{\left(\sqrt{n}-\mu\right)^2}$, we have desired result. □

5 Voronovskaya-Type Theorem

Using exponential moments, we shall prove the Voronovskaya-type theorem for $\left(P_n^{**}\right)_{n\geq 1}$.

Theorem 3 *If $f \in C_{\rho_3}(\mathbb{R})$ has a second derivative at a point $x \in \mathbb{R}$, then we have*

$$\lim_{n\to\infty} n\left(P_n^{**} f\right)(x) - f(x) = f^{''}(x) - 3\mu f^{'}(x) + 2\mu^2 f(x). \tag{5.1}$$

Proof We can use Taylor formula in the form

$$\begin{aligned} f(x+t) &= \left(f \circ \log_\mu\right)\left(e^{\mu(x+t)}\right) \\ &= \left(f \circ \log_\mu\right)\left(e^{\mu x}\right) + \left(f \circ \log_\mu\right)^{'}\left(e^{\mu x}\right)\left(e^{\mu(x+t)} - e^{\mu x}\right) \\ &\quad + \frac{1}{2}\left(f \circ \log_\mu\right)^{''}\left(e^{\mu x}\right)\left(e^{\mu(x+t)} - e^{\mu x}\right)^2 + h_x(t)\left(e^{\mu(x+t)} - e^{\mu x}\right)^2, \end{aligned}$$

where $h_x(t)$ is a continuous function which vanishes at 0.

Replacing x with $\alpha_n^{**}(x)$ in above equality and applying the operator $\left(P_n^{**}\right)_{n>n_\mu}$, one has

$$\begin{aligned} \left(P_n^{**} f\right)(x) &= f\left(\alpha_n^{**}(x)\right)\left(P_n^{**} e_0\right)(x) + \left(f \circ \log_\mu\right)^{'}\left(e^{\mu\alpha_n^{**}(x)}\right)\left(\left(P_n^{**}\exp_\mu\right)(x) - e^{\mu\alpha_n^{**}(x)}\left(P_n^{**} e_0\right)(x)\right) \\ &\quad + \frac{1}{2}\left(f \circ \log_\mu\right)^{''}\left(e^{\mu\alpha_n^{**}(x)}\right) \\ &\quad \times \left(\left(P_n^{**}\exp_\mu^2\right)(x) - 2e^{\mu\alpha_n^{**}(x)}\left(P_n^{**}\exp_\mu\right)(x) + e^{2\mu\alpha_n^{**}(x)}\left(P_n^{**} e_0\right)(x)\right) \\ &\quad + \left(P_n^{**} h_x(t)\left(e^{\mu\left(\alpha_n^{**}(x)+t\right)} - e^{\mu x}\right)^2\right)(x). \end{aligned}$$

This equality can be arranged as

$$\begin{aligned} \left(P_n^{**} f\right)(x) &= f(x)\left(P_n^{**} e_0\right)(x) + \left[f\left(\alpha_n^{**}(x)\right) - f(x)\right]\left(P_n^{**} e_0\right)(x) \\ &\quad + \left[\left(f \circ \log_\mu\right)^{'}\left(e^{\mu\alpha_n^{**}(x)}\right) - \left(f \circ \log_\mu\right)^{'}\left(e^{\mu x}\right)\right] \\ &\quad \times \left(\left(P_n^{**}\exp_\mu\right)(x) - e^{\mu\alpha_n^{**}(x)}\left(P_n^{**} e_0\right)(x)\right) \\ &\quad + \left(f \circ \log_\mu\right)^{'}\left(e^{\mu x}\right)\left(\left(P_n^{**}\exp_\mu\right)(x) - e^{\mu\alpha_n^{**}(x)}\left(P_n^{**} e_0\right)(x)\right) \\ &\quad + \frac{1}{2}\left[\left(f \circ \log_\mu\right)^{''}\left(e^{\mu\alpha_n^{**}(x)}\right) - \left(f \circ \log_\mu\right)^{''}\left(e^{\mu x}\right)\right] \\ &\quad \times \left(\left(P_n^{**}\exp_\mu^2\right)(x) - 2e^{\mu\alpha_n^{**}(x)}\left(P_n^{**}\exp_\mu\right)(x) + e^{2\mu\alpha_n^{**}(x)}\left(P_n^{**} e_0\right)(x)\right) \\ &\quad + \frac{1}{2}\left(f \circ \log_\mu\right)^{''}\left(e^{\mu x}\right) \\ &\quad \times \left(\left(P_n^{**}\exp_\mu^2\right)(x) - 2e^{\mu\alpha_n^{**}(x)}\left(P_n^{**}\exp_\mu\right)(x) + e^{2\mu\alpha_n^{**}(x)}\left(P_n^{**} e_0\right)(x)\right) \end{aligned}$$

$$+\left(P_n^{**}h_x\left(e^{\mu\left(\alpha_n^{**}(x)+t\right)}-e^{\mu x}\right)^2\right)(x).$$

Since

$$\lim_{n\to\infty}\alpha_n^{**}(x)=x,\quad \lim_{n\to\infty}n\left(e^{\mu\alpha_n^{**}(x)}-e^{\mu x}\right)=-\mu^2e^{\mu x}$$
$$\lim_{n\to\infty}n\left(e^{2\mu\alpha_n^{**}(x)}-e^{2\mu x}\right)=-2\mu^2e^{2\mu x}$$

and

$$\lim_{n\to\infty}n\left(\left(P_n^{**}e_0\right)(x)-1\right)=2\mu^2,$$

we get

$$\begin{aligned}
&\lim_{n\to\infty}n\left(P_n^{**}f\right)(x)-f(x)\\
&=2\mu^2f(x)+\left(f\circ\log_\mu\right)'\left(e^{\mu x}\right)\lim_{n\to\infty}n\left[\left(P_n^{**}\exp_\mu\right)(x)-e^{\mu x}\left(P_n^{**}e_0\right)(x)\right]\\
&\quad+\frac{1}{2}\left(f\circ\log_\mu\right)''\left(e^{\mu x}\right)\\
&\quad\times\lim_{n\to\infty}n\left[\left(P_n^{**}\exp_\mu^2\right)(x)-2e^{\mu x}\left(P_n^{**}\exp_\mu\right)(x)+e^{2\mu x}\left(P_n^{**}e_0\right)(x)\right]\\
&\quad+\lim_{n\to\infty}n\left(P_n^{**}h_x\left(e^{\mu\left(\alpha_n^{**}(x)+t\right)}-e^{\mu x}\right)^2\right)(x)
\end{aligned}$$

Using (1.5), (1.6) and Lemma 1, one finds that

$$\begin{aligned}
\lim_{n\to\infty}n\left[\left(\left(P_n^{**}\exp_\mu\right)(x)-e^{\mu x}\left(P_n^{**}e_0\right)(x)\right)\right]&=e^{\mu x}\lim_{n\to\infty}n\left[1-\left(P_n^{**}e_0\right)(x)\right]\\
&=-2\mu^2e^{\mu x}
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty}n\left[\left(P_n^{**}\exp_\mu^2\right)(x)-2e^{\mu x}\left(P_n^{**}\exp_\mu\right)(x)+e^{2\mu x}\left(P_n^{**}e_0\right)(x)\right]\\
&=e^{2\mu x}\lim_{n\to\infty}n\left[\left(P_n^{**}e_0\right)(x)-1\right]=2\mu^2e^{2\mu x}.
\end{aligned}$$

Since

$$\left(f\circ\log_\mu\right)'\left(e^{\mu x}\right)=e^{-\mu x}\mu^{-1}f'(x)\ \text{ and }\ \left(f\circ\log_\mu\right)''\left(e^{\mu x}\right)=e^{-2\mu x}\left(\mu^{-2}f''(x)-\mu^{-1}f'(x)\right),$$

we have

$$\lim_{n\to\infty} n\left(P_n^{**}f\right)(x) - f(x) = f''(x) - 3\mu f'(x) + 2\mu^2 f(x) + \lim_{n\to\infty} n\left(P_n^{**}h_x\left(e^{\mu\left(\alpha_n^{**}(x)+t\right)} - e^{\mu x}\right)^2\right)(x).$$

The proof of the theorem will be over if we prove

$$\lim_{n\to\infty} n\left(P_n^{**}h_x\left(e^{\mu\left(\alpha_n^{**}(x)+t\right)} - e^{\mu x}\right)^2\right)(x) = 0.$$

From Cauchy–Schwarz inequality, we can write

$$n\left|\left(P_n^{**}h_x\left(e^{\mu\left(\alpha_n^{**}(x)+t\right)} - e^{\mu x}\right)^2\right)\right| \le \sqrt{\left(P_n^{**}h_x^2\right)(x)}\sqrt{n^2\left(P_n^{**}\exp_{\mu,x}^4\right)(x)}.$$

Since

$$\begin{aligned}\left(P_n^{**}\exp_{\mu,x}^4\right)(x) &= \left(P_n^{**}\exp_\mu^4\right)(x) - 4e^{\mu x}\left(P_n^{**}\exp_\mu^3\right)(x) + 6e^{2\mu x}\left(P_n^{**}\exp_\mu^2\right)(x) \\ &\quad -4e^{3\mu x}\left(P_n^{**}\exp_\mu\right)(x) + e^{4\mu x}\left(P_n^{**}e_0\right)(x) \\ &= e^{4\mu x}\left(\frac{\left(n-\mu^2\right)^3}{n^2\left(n-9\mu^2\right)} - 4\frac{\left(n-\mu^2\right)^2}{n\left(n-4\mu^2\right)} + 2 + \frac{n^2}{\left(n-\mu^2\right)^2}\right)\end{aligned}$$

and

$$\lim_{n\to\infty} n^2\left(P_n^{**}\exp_{\mu,x}^4\right)(x) = 24\mu^4 e^{4\mu x},$$

we have desired result. □

6 Shape Preserving Properties

In this section, we will present some shape preserving properties of the operator (1.3). Also we will give the global smoothness preservation properties of mentioned operators. First, we have the following simple results.

Let $f \in C_{\rho_3}^2(\mathbb{R})$, we consider the operators for $x \in \mathbb{R}$, $n \in \mathbb{N}$,

$$\frac{\left(P_n^{**}f\right)(x)}{e^{\mu x}} = \frac{\sqrt{n}}{2}\int_{\mathbb{R}} e^{-\mu\left(\alpha_n^{**}(x)+t\right)} f\left(\alpha_n^{**}(x)+t\right) K_n(t)\,dt.$$

With simple calculations, we have

$$\Delta_h\left(\frac{P_n^{**}f}{\exp_\mu}\right)(x)=\frac{\sqrt{n}}{2}\int_{\mathbb{R}}\Delta_h\left(\frac{f}{\exp_\mu}\right)\left(\alpha_n^{**}(x)+t\right)K_n(t)\,dt$$

and

$$\Delta_h^2\left(\frac{P_n^{**}f}{\exp_\mu}\right)(x)=\frac{\sqrt{n}}{2}\int_{\mathbb{R}}\Delta_h^2\left(\frac{f}{\exp_\mu}\right)\left(\alpha_n^{**}(x)+t\right)K_n^P(t)\,dt,$$

where $\Delta_h(f)(x)=f(x+h)-f(x)$ and $\Delta_h^2(f)(x)=f(x+2h)-2f(x+h)+f(x)$.

Thus from previous expression, since $K_n(t)$ is positive for all $t\in\mathbb{R}$, if $\frac{f}{\exp_\mu}$ is increasing $\left(\Delta_h\left(\frac{f}{\exp_\mu}\right)(x)\geq 0\right)$ then $\Delta_h\left(\frac{P_n^{**}f}{\exp_\mu}\right)(x)\geq 0$, and so $\frac{P_n^{**}f}{\exp_\mu}$ is also increasing. If $\frac{f}{\exp_\mu}$ is convex $\left(\Delta_h^2\left(\frac{f}{\exp_\mu}\right)(x)\geq 0\right)$, then $\Delta_h^2\left(\frac{P_n^{**}f}{\exp_\mu}\right)(x)\geq 0$ and so $\frac{P_n^{**}f}{\exp_\mu}$ is also convex.

We want to give the connection of the operators $\left(P_n^{**}\right)_{n>n_\mu}$ with generalized convexity. Now we recall the definition of generalized convexities with respect to the functions $\exp_\mu$ and $\exp_\mu^2$.

Definition 1 A function f defined on $\mathbb{R}$ is said to be convex with respect to $\left\{\exp_\mu\right\}$, denoted by $f\in\mathcal{F}\left(\exp_\mu\right)$, if

$$\begin{vmatrix} e^{\mu x_0} & e^{\mu x_1}\\ f(x_0) & f(x_1)\end{vmatrix}\geq 0,\quad x_0<x_1.$$

f is said to be convex with respect to $\left\{\exp_\mu,\exp_\mu^2\right\}$, denoted by $f\in\mathcal{F}\left(\exp_\mu,\exp_\mu^2\right)$, if

$$\begin{vmatrix} e^{\mu x_0} & e^{\mu x_1} & e^{\mu x_2}\\ e^{2\mu x_0} & e^{2\mu x_1} & e^{2\mu x_2}\\ f(x_0) & f(x_1) & f(x_2)\end{vmatrix}\geq 0,\quad x_0<x_1<x_2.$$

Proposition 1 (see [4]) *Let $f\in C_{\rho_3}^2(\mathbb{R})$. Then the following items hold.*

(1) $f\in\mathcal{F}\left(\exp_\mu\right)$ if and only if $f/\exp_\mu$ is increasing for $x\in\mathbb{R}$,
(2) $f\in\mathcal{F}\left(\exp_\mu,\exp_\mu^2\right)$ if and only if $f''(x)-3\mu f'(x)+2\mu^2 f(x)\geq 0$ for $x\in\mathbb{R}$.

Using above proposition, we have

Theorem 4 *Let $f\in C_{\rho_3}(\mathbb{R})$. Then the following items hold.*

(1) If $f\in\mathcal{F}\left(\exp_\mu,\exp_\mu^2\right)$, then $\left(P_n^{**}f\right)(x)\geq f(x)$ for $x\in\mathbb{R}$,
(2) If $f\in\mathcal{F}\left(\exp_\mu\right)$, then $\left(P_n^{**}f\right)\in\mathcal{F}\left(\exp_\mu\right)$ for $x\in\mathbb{R}$.

Theorem 5 *Let $f\in C_{\rho_3}^2(\mathbb{R})$. Suppose that there exists $n_0\in\mathbb{N}$ such that*

$$f(x)\leq\left(P_n^{**}f\right)(x)\leq\left(P_nf\right)(x),\ \textit{for all } n\geq n_0,\quad x\in\mathbb{R}.\tag{6.1}$$

Then

$$f^{''}(x) \geq 3\mu f^{'}(x) - 2\mu^2 f(x) \geq 0, \quad x \in \mathbb{R}. \tag{6.2}$$

In particular, $f^{''}(x) \geq 0$.

Conversely, if (6.2) holds with strictly inequalities at a given point $x \in \mathbb{R}$, *then there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$

$$f(x) \leq \left(P_n^{**} f\right)(x) \leq (P_n f)(x).$$

Proof From (6.1), we have that

$$0 \leq n\left(\left(P_n^{**} f\right)(x) - f(x)\right) \leq n\left((P_n f)(x) - f(x)\right).$$

We know from [9] that

$$\lim_{n\to\infty} n\left((P_n f)(x) - f(x)\right) = f^{''}(x). \tag{6.3}$$

Using (5.1) and (6.3), we have the desired result.

Conversely, if (6.2) holds with strict inequalities at a given point $x \in \mathbb{R}$, using again (5.1) and (6.3), we have

$$f(x) \leq \left(P_n^{**} f\right)(x) \leq (P_n f)(x)$$

for all $n \geq n_0$. □

By using the weighted modulus of continuity defined by (4.1), the result regarding global smoothness preservation properties for the operators of $\left(P_n^{**}\right)_{n>n_\mu}$ will be given as follows:

Theorem 6 *Let* $\delta > 0$, *we have*

$$\widetilde{\omega}\left(\frac{P_n^{**}(f)}{\exp_\mu}; \delta\right) \leq \left(\frac{n}{n-\mu^2}\right)\widetilde{\omega}\left(\frac{f}{\exp_\mu}; \delta\right). \tag{6.4}$$

Proof For $x \in \mathbb{R}$, we have

$$\begin{aligned} & e^{-\mu(x+h)} P_n^{**}(f; x+h) - e^{-\mu x} P_n^{**}(f; x) \\ = & \frac{\sqrt{n}}{2}\int_{\mathbb{R}}\left[e^{-\mu(\alpha_n^{**}(x+h)+t)} f\left(\alpha_n^{**}(x+h)+t\right) - e^{-\mu(\alpha_n^{**}(x+h)+t)} f\left(\alpha_n^{**}(x+h)+t\right)\right] K_n(t)\, dt \end{aligned}$$

Thus, we have for $n > n_\mu$

$$
\begin{aligned}
& e^{-\mu|x|} \left| e^{-\mu(x+h)} P_n^{**}(f; x+h) - e^{-\mu x} P_n^{**}(f; x) \right| \\
\leq & \left(\frac{n}{n-\mu^2} \right) \frac{\sqrt{n}}{2} \int_{\mathbb{R}} e^{-\mu\left|\alpha_n^{**}(x)\right|} \left| \frac{f\left(\alpha_n^{**}(x+h)+t\right)}{e^{\mu\left(\alpha_n^{**}(x+h)+t\right)}} - \frac{f\left(\alpha_n^{**}(x)+t\right)}{e^{\mu\left(\alpha_n^{**}(x)+t\right)}} \right| K_n(t)\, dt \\
\leq & \left(\frac{n}{n-\mu^2} \right) \widetilde{\omega} \left(\frac{f}{\exp_\mu}; \left| \alpha_n^{**}(x+h) - \alpha_n^{**}(x) \right| \right) \\
\leq & \left(\frac{n}{n-\mu^2} \right) \widetilde{\omega} \left(\frac{f}{\exp_\mu}; h \right).
\end{aligned}
$$

Thus, we get

$$
\widetilde{\omega} \left(\frac{P_n^{**}(f)}{\exp_\mu}; \delta \right) \leq \left(\frac{n}{n-\mu^2} \right) \widetilde{\omega} \left(\frac{f}{\exp_\mu}; \delta \right).
$$

□

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From Uniform to Statistical Convergence of Binomial-Type Operators

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Abstract Sequences of binomial operators introduced by using umbral calculus are investigated from the point of view of statistical convergence. This approach is based on a detailed presentation of delta operators and their associated basic polynomials. Bernstein–Sheffer linear positive operators are analyzed, and some particular cases are highlighted: Cheney–Sharma operators, Stancu operators, Lupaş operators.

Keywords Statistical convergence · Binomial sequence · Linear positive operator · Umbral calculus · Bernstein–Sheffer operator · Pincherle derivative

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1 Introduction

Let $(L_n)_{n\geq 1}$ be a sequence of linear positive operators acting on the space $C([a,b])$ of all real-valued and continuous functions defined on the interval $[a,b]$, equipped with the norm $\|\cdot\|$ of the uniform convergence, namely $\|h\| = \sup\limits_{a\leq t\leq b} |h(t)|$. Bohman–Korovkin's theorem asserts: If the operators L_n, $n \in \mathbb{N}$, map $C([a,b])$ into itself such that

$$\lim_{n\to\infty} \|L_n e_j - e_j\| = 0 \text{ for } j \in \{0, 1, 2\}, \tag{1.1}$$

then one has

$$\lim_{n\to\infty} \|L_n f - f\| = 0 \text{ for every } f \in C([a,b]). \tag{1.2}$$

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In the above, e_j represents the monomial of jth degree, $e_0(x) = 1$ and $e_j(x) = x^j$, $j \geq 1$.

A current subject in approximation theory is the approximation of continuous functions by using the statistical convergence, the first research of this topic being done by Gadjiev and Orhan [5]. This approach models and improves the technique of signals' approximation in different function spaces.

On the other hand, sequences of polynomials of binomial type have been the subject of many mathematical studies, drawing to light their role in approximation theory. Practically, the theory of the approximation operators of binomial type is based on the technique of the *umbral calculus*. In its modern form, this is a strong tool for calculations with polynomials representing a successful combination between the finite differences calculus and certain chapters of probability theory. The topic discussed in this chapter is at the confluence of the two concepts mentioned above, statistical convergence and binomial-type operators, from the point of view of the approximation of some function classes. The material is structured in three sections.

First of all, we recall the variant of Bohman–Korovkin theorem via statistical convergence and we present elementary facts about polynomial sequences of binomial type. Further on, we deal with delta operators and their basic polynomials. In the last section, we will analyze the approximation properties of some binomial operators in terms of the statistical convergence.

We mention that at the first sight this work seems disproportionate, dominated by a lot of notions introduced and results already achieved. The goal was to be self-contained paper. It will be seen that for a clear understanding of the last paragraph, it was necessary to structure the article in this way.

2 Preliminaries

The concept of statistical convergence was first defined by Steinhaus [13] and Fast [4]. It is based on the notion of the asymptotic density of subsets of $\mathbb{N}$. The density of $S \subseteq \mathbb{N}$ denoted by $\delta(S)$ is given by

$$\delta(S) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_S(j),$$

where χ_S stands for the characteristic function of the set S. Clearly, $0 \leq \delta(S) \leq \delta(\mathbb{N}) = 1$. A sequence $(x_n)_{n\geq 1}$ of real numbers is said to be statistically convergent to a real number l, if, for every $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : \ |x_n - l| \geq \varepsilon\}) = 0,$$

the limit being denoted by $st - \lim\limits_{n\to\infty} x_n = l$. It is known that any convergent sequence is statistically convergent but the converse of this statement is not true. Even though this notion was introduced in 1951, its application to the study of sequences of positive linear operators was attempted only in 2002. We refer to the A.D. Gadjiev and C. Orhan [5] result, which reads as follows.

Theorem 2.1 *If the sequence of positive linear operators $L_n : C([a, b]) \to B([a, b])$ satisfies the condition*

$$st - \lim_{n\to\infty} \|L_n e_j - e_j\| = 0, \ j \in \{0, 1, 2\}, \tag{2.1}$$

then one has

$$st - \lim_{n\to\infty} \|L_n f - f\| = 0 \textit{ for every function } f \in C([a, b]). \tag{2.2}$$

As usual, $B([a, b])$ stands for the space of all real-valued bounded functions defined on $[a, b]$, endowed with the sup norm. The identities (2.1) and (2.2) generalize, respectively, relations (1.1), (1.2). From this moment, the statistical convergence of positive linear operators represented a new direction in the study of so-called KAT—Korovkin-type approximation theory.

Set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any $n \in \mathbb{N}_0$, we denote by Π_n the linear space of polynomials of degree no greater than n and by Π_n^* the set of all polynomials of degree n. We also set

$$\Pi := \bigcup_{n\geq 0} \Pi_n,$$

representing the commutative algebra of polynomials with coefficients in $\mathbb{K}$, this symbol standing either for the field $\mathbb{R}$ or for the field $\mathbb{C}$.

A sequence $p = (p_n)_{n\geq 0}$ such that $p_n \in \Pi_n^*$ for every $n \in \mathbb{N}_0$ is called a *polynomial sequence*.

Definition 2.2 A polynomial sequence $b = (b_n)_{n\geq 0}$ is called of binomial type if for any $(x, y) \in \mathbb{K} \times \mathbb{K}$ the following identities hold

$$b_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} b_k(x) b_{n-k}(y), n \in \mathbb{N}_0. \tag{2.3}$$

Remark 2.3 Knowing that $\deg(b_0) = 0$, we get $b_0(x) = 1$ for any $x \in \mathbb{K}$ and by induction we easily obtain $b_n(0) = 0$ for any $n \in \mathbb{N}$.

The most common example of binomial sequence is $e = (e_n)_{n\geq 0}$ (the monomials). Some nontrivial examples are given below.

(a) *The generalized factorial power with the step a:* $p = (p_n)_{n\geq 0}$,

$$p_0(x) = x^{[0,a]} := 1 \text{ and } p_n(x) = x^{[n,a]} := x(x - a)\cdots(x - (n - 1)a), \ n \in \mathbb{N}.$$

The Vandermonde formula, i.e.,

$$(x+y)^{[n,a]} = \sum_{k=0}^{n} \binom{n}{k} x^{[k,a]} y^{[n-k,a]},$$

guarantees that this is a binomial-type sequence. There are two particular cases: For $a = 1$, we obtain the *lower factorials* which, usually, are denoted by $\langle x \rangle_n$; for $a = -1$, we obtain the *upper factorials* denoted by Pochhammer's symbol $(x)_n$. By convention, we consider

$$x^{[-n,a]} := 1/(x+na)^{[n,a]}.$$

(b) *Abel polynomials:* $\widetilde{a} = (a_n^{\langle a \rangle})_{n \ge 0}$,

$$a_0^{\langle a \rangle} = 1,\ a_n^{\langle a \rangle}(x) = x(x-na)^{n-1},\ n \in \mathbb{N},\ a \neq 0.$$

Rewriting the identity (2.3) for these polynomials, we obtain the Abel-Jensen (1902) combinatorial formula

$$(x+y)(x+y+na)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} xy(x+ka)^{k-1}(y+(n-k)a)^{n-1-k},\ n \in \mathbb{N}.$$

(c) *Gould polynomials:* $g = (g_n^{\langle a,b \rangle})_{n \ge 0}$,

$$g_0^{\langle a,b \rangle} = 1,\ g_n^{\langle a,b \rangle}(x) = \frac{x}{x-an} \left\langle \frac{x-an}{b} \right\rangle_n,\ n \in \mathbb{N},\ ab \neq 0.$$

The space of all linear operators $T : \Pi \to \Pi$ will be denoted by $\mathscr{L}$. Among these operators, an important role will be played by the *shift operator*, named E^a. For every $a \in \mathbb{K}$, E^a is defined by

$$(E^a p)(x) = p(x+a),\ \text{where } p \in \Pi.$$

An operator $T \in \mathscr{L}$ which switches with all shift operators, that is

$$TE^a = E^a T \text{ for every } a \in \mathbb{K},$$

is called a *shift-invariant operator*, and the set of these operators is denoted by $\mathscr{L}_s$.

3 On Delta Operators

Definition 3.1 An operator $Q : \Pi \to \Pi$ is called delta operator if $Q \in \mathscr{L}_s$ and Qe_1 is a nonzero constant.

Let $\mathscr{L}_\delta$ denote the set of all delta operators. For a better understanding, we present some examples of delta operators. In the following, the symbol I stands for the identity operator on the space Π.
(a) The *derivative operator*, denoted by D.
(b) The operators used in calculus of divided differences. Let h be a fixed number belonging to the field $\mathbb{K}$. We set

$$\Delta_h := E^h - I, \text{ the } \textit{forward difference operator},$$
$$\nabla_h := I - E^{-h}, \text{ the } \textit{backward difference operator},$$
$$\delta_h := E^{h/2} - E^{-h/2}, \text{ the } \textit{central difference operator}.$$

It is evident that $\nabla_h = \Delta_h E^{-h}$, $\delta_n = \Delta_h E^{-h/2} = \nabla_h E^{h/2}$. The properties of these operators as well as their usefulness can be found in [6].
(c) *Abel operator*, $A_a := DE^a$. For any $p \in \Pi$, $(A_a p)(x) = \dfrac{dp}{dx}(x+a)$.
Writing (symbolically) Taylor's series in the following manner

$$E^h = \sum_{\nu=0}^{\infty} \frac{h^\nu D^\nu}{\nu!} = e^{hD}, \tag{3.1}$$

we can also get $A_a = D(e^{aD})$.
(d) *Gould operator*, $G_{a,b} := \Delta_b E^a = E^{a+b} - E^a$, $ab \neq 0$.

Definition 3.2 Let Q be a delta operator. A polynomial sequence $p = (p_n)_{n\geq 0}$ is called the sequence of basic polynomials associated with Q if
(i) $p_0(x) = 1$ for any $x \in \mathbb{K}$.
(ii) $p_n(0) = 0$ for any $n \in \mathbb{N}$.
(iii) $(Qp_n)(x) = np_{n-1}(x)$ for any $n \in \mathbb{N}$ and $x \in \mathbb{K}$.

Remark 3.3 If $p = (p_n)_{n\geq 0}$ is a sequence of basic polynomials associated with Q, then $\{p_0, p_1, \ldots, p_{n-1}, e_n\}$ is a basis of the linear space Π_n. Taking this fact into account, by induction it can be proved that every delta operator has a unique sequence of basic polynomials; see [9, *Proposition 3*].

Here are some examples. The basic polynomials associated with the operators $Q = D$, $Q = \Delta_h$, and $Q = \nabla_h$, are, respectively, $(e_n)_{n\geq 0}$, $(x^{[n,h]})_{n\geq 0}$, and $((x + (n-1)h)^{[n,h]})_{n\geq 0}$. Also, we can easily prove that $\widetilde{a} = (a_n^{\langle a\rangle})_{n\geq 0}$ respectively $g = (g_n^{\langle a,b\rangle})_{n\geq 0}$ is the sequence of basic polynomials associated with Abel operator A_a, respectively Gould operator $G_{a,b}$.

The connection between the delta operator and the binomial-type sequences is given by the following result [9, *Theorem 1*].

Theorem 3.4 *Let $p = (p_n)_{n\geq 0}$ be a sequence of polynomials. It is a sequence of binomial type if and only if it is a basic sequence for some delta operator.*

The following statement generalizes the Taylor expansion theorem to delta operators and their basic polynomials.

Theorem 3.5 *Let T be a shift-invariant operator, and let Q be a delta operator with its basic sequence $(p_n)_{n\geq 0}$. Then, the following identity holds*

$$T = \sum_{k\geq 0} \frac{(Tp_k)(0)}{k!} Q^k. \tag{3.2}$$

Let Q be a delta operator, and let $(\mathcal{F}, +, \cdot)$ be the ring of the formal power series in the variable t over the same field. Here, the product means the Cauchy product between two series. Further, let $(\mathscr{L}_s, +, \cdot)$ be the ring of shift-invariant operators, the product being defined as usually: For any $P_1, P_2 \in \mathscr{L}_s$, we have $P_1 P_2 : \Pi \to \Pi$, $(P_1 P_2)(q) = P_1(P_2(q))$, $q \in \Pi$. Then, there exists an isomorphism ψ from $\mathcal{F}$ onto $\mathscr{L}_s$ such that

$$\psi(f(t)) = T, \text{ where } f(t) = \sum_{k\geq 0} \frac{a_k}{k!} t^k \text{ and } T = \sum_{k\geq 0} \frac{a_k}{k!} Q^k. \tag{3.3}$$

This isomorphism allows us to conclude: A shift-invariant operator T is invertible if and only if $Te_0 \neq 0$. Since for every $Q \in \mathscr{L}_\delta$ we have $Qe_0 = 0$, we deduce that any delta operator is not invertible. Also, we can write $T = \phi(D)$, where $T \in \mathscr{L}_s$ and $\phi(t)$ is a formal power series, to indicate that the operator T corresponds to the series $\phi(t)$ under the isomorphism defined by (3.3).

Remark 3.6 In relation (3.1), we choose $T = E^x$ and expand E^x in terms of Q. Due to the identity $(E^x p_k)(0) = p_k(x)$ and the relation (3.2), one obtains

$$e^{xD} = \sum_{k\geq 0} \frac{p_k(x)}{k!} \phi^k(D).$$

Substituting D by u, the series terms lead us to the following result [9, *Corollary 3*].

Theorem 3.7 *Let Q be a delta operator with $p = (p_n)_{n\geq 0}$ its sequence of basic polynomials. Let $\phi(D) = Q$ and $\varphi(t)$ be the inverse formal power series of $\phi(u)$. Then,*

$$e^{x\varphi(t)} = \sum_{n\geq 0} \frac{p_n(x)}{n!} t^n, \tag{3.4}$$

where $\varphi(t)$ has the form $c_1 t + c_2 t^2 + \cdots$ $(c_1 \neq 0)$.

Another characterization of delta operators was included in [9] without proof. For this reason, we prove the following statement.

Theorem 3.8 *$Q \in \mathscr{L}_s$ is a delta operator if and only if $Q = DP$ for some shift-invariant operator P, where the inverse operator P^{-1} exists.*

Proof If in (3.3) we substitute T by a delta operator Q, then we get $a_0 = Q(e_0) = 0$ and $a_1 = Q(e_1) = c \neq 0$. Consequently, we can write

$$Q = \sum_{k\geq 1} \frac{a_k}{k!} D^k. \tag{3.5}$$

Denoting $\sum_{k\geq 1} \frac{a_k}{k!} D^{k-1}$ by P, we have $P \in \mathscr{L}_s$ and $P(e_0) = a_1 \neq 0$; thus, P is invertible; see the conclusion that emerges from (3.3). So, Q can be written as DP.

Reciprocally, for every $P \in \mathscr{L}_s$ such that P is invertible, DP is a shift-invariant operator, $E^a(DP) = (DP)E^a$, and

$$(DP)(e_1) = P(D(e_1)) = P(e_0) = c \neq 0$$

thus $DP \in \mathscr{L}_\delta$. □

Now we are ready to analyze some binomial operators investigating their statistical convergence to the identity operator.

4 Classes of Binomial Operators

We consider a delta operator Q and its sequence of basic polynomials $p = (p_n)_{n\geq 0}$, under the assumption that $p_n(1) \neq 0$ for every $n \in \mathbb{N}$. Also, according to Theorem 3.7, we shall keep the same meaning of the functions ϕ and φ. For every $n \geq 1$, we consider $L_n^Q : C([0, 1]) \to C([0, 1])$ defined as follows

$$(L_n^Q f)(x) = \frac{1}{p_n(1)} \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(1-x) f\left(\frac{k}{n}\right). \tag{4.1}$$

They are called by P. Sablonnière [10] *Bernstein–Sheffer operators*. As D.D. Stancu and M.R. Occorsio motivated in [12], these operators can be named *Popoviciu operators*. T. Popoviciu [8] indicated the construction (4.1) in front of the sum appearing the factor d_n^{-1} from the identities

$$(1 + d_1 t + d_2 t^2 + \cdots)^x = e^{x\varphi(t)} = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!},$$

see (3.4). If we choose $x = 1$, it becomes obvious that $d_n = p_n(1)/n!$.

In the particular case $Q = D$, L_n^D becomes genuine Bernstein operator of degree n. An integral generalization of L_n^Q in Kantorovich sense was introduced and studied in [1].

The operators $L_n^Q, n \in \mathbb{N}$, are linear and reproduce the constants. Indeed, choosing in (2.3) $y := 1 - x$, we obtain $L_n^Q e_0 = e_0$. The positivity of these operators is given by the sign of the coefficients of the series $\varphi(t) = c_1 + c_2 t + \cdots \ (c_1 \neq 0)$. More precisely, in [8, 10], the authors established the following.

Lemma 4.1 *L_n^Q is a positive operator on $C([0,1])$ for every $n \geq 1$ if and only if $c_1 > 0$ and $c_n \geq 0$ for all $n \geq 2$.*

Moreover, if L_n^Q satisfies the above conditions, then one has

$$L_n^Q e_1 = e_1, \ n \in \mathbb{N}, \ \text{and } L_n^Q e_2 = e_2 + a_n(e_1 - e_2), \ n \geq 2, \tag{4.2}$$

where $a_n = \dfrac{1}{n}\left(1 + (n-1)\dfrac{r_{n-2}(1)}{p_n(1)}\right)$, the sequence $(r_n(x))_{n\geq 0}$ being generated by

$$\varphi''(t)\exp(x\varphi(t)) = \sum_{n\geq 0} r_n(x)\frac{t^n}{n!}.$$

Theorem 4.2 *Let the operators L_n^Q, $n \in \mathbb{N}$, be defined by (4.1) such that the hypothesis of Lemma 4.1 takes place.*

If $st - \lim\limits_{n\to\infty} \dfrac{r_{n-2}(1)}{p_n(1)} = 0$, then

$$st - \lim_{n\to\infty} \|L_n^Q f - f\| = 0, \ \ f \in C([0,1]). \tag{4.3}$$

Proof We apply Theorem 2.1. Based on algebraic operations with statistically convergent sequences of real numbers, our hypothesis guarantees the identity (4.3). For a profound documentation of operations with such sequences [2, *Theorem 3.1*] can be consulted. □

Further, choosing particular delta operators Q, we reobtain some classical linear positive operator of discrete type.

Example 4.3 If $Q = A_a$ with its basic sequence $\widetilde{a}$ and assuming that the parameter a depends on n, $a := t_n$, one obtains the *Cheney–Sharma operators* [3]. The corresponding operators Q_n , $n \in \mathbb{N}$, are defined by the equation

$$(Q_n f)(x) := (1 + nt_n)^{1-n} \sum_{k=0}^{n} \binom{n}{k} x(x + kt_n)^{k-1}(1 - x)[1 - x + (n-k)t_n]^{n-1-k}.$$

Clearly, $Q_n e_0 = e_0$. To compute $Q_n e_j$, $j \in \{1, 2\}$, we follow the same path as in [3, *Section 3*]. We can deduce: If the sequence $(nt_n)_{n\geq 1}$ is statistically convergent to zero, then (4.3) takes place.

Example 4.4 If $Q = \frac{1}{\alpha}\nabla_\alpha$, $\alpha \neq 0$, with its basic polynomials

$$p_n(x) = (x + (n-1)\alpha)^{[n,\alpha]},$$

L_n^Q becomes *Stancu operator* [11] denoted by $P_n^{[\alpha]}$,

$$(P_n^{[\alpha]} f)(x) = \sum_{k=0}^{n} w_{n,k}(x;\alpha) f\left(\frac{k}{n}\right),$$
$$w_{n,k}(x;\alpha) = \binom{n}{k} \frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}},$$

α being a parameter which may depend on a natural number n. One has

$$P_n^{[\alpha]} e_j = e_j,\ \ j \in \{0,1\},\ \text{and}\ (P_n^{[\alpha]} e_2)(x) = \frac{1}{1+\alpha}\left(\frac{x(1-x)}{n} + x(x+\alpha)\right),$$

in accordance with [11, *Lemma 4.1*].

If $0 \le \alpha := \alpha(n)$ and $st - \lim_{n\to\infty} \alpha(n) = 0$, then (4.3) takes place.

Indeed, it is enough to prove (2.1) for $j = 2$. We get

$$|(P_n^{[\alpha]} e_2)(x) - x^2| = \frac{1}{1+\alpha}\left|\frac{x(1-x)}{n} - \alpha x^2 + x\alpha\right|$$
$$\le \frac{1}{1+\alpha}\left(\frac{1}{4n} + 2\alpha\right) \le \frac{1}{4n} + 2\alpha$$

and the conclusion follows.

At this moment, we take a break in order to illustrate some further properties of the binomial sequences. Keeping the notations $Q \in \mathscr{L}_\delta$, $p = (p_n)_{n\ge 0}$, $Q = \phi(D)$, $\phi^{-1} = \varphi$, we assume that the conditions of Lemma 4.1 are fulfilled.

In [7], A. Lupaş proved new inequalities between the terms of the binomial sequences p. For any $x > 0$ and $n \ge 2$, one has

$$\begin{cases} 0 < c_1 \dfrac{p_{n-1}(x)}{x} \le (Q'^{-2} p_{n-2})(x) \le \dfrac{p_n(x)}{x^2}; \\ \dfrac{1}{n} \le \rho_n(Q) < 1,\ \text{where}\ \rho_n(Q) := 1 - \dfrac{n(n-1)}{p_n(n)} (Q'^{-2} p_{n-2})(x). \end{cases} \tag{4.4}$$

In the above, Q' represents Pincherle derivative of Q. The concept is detailed further.

Knowing that the operator $X : \Pi \to \Pi$, $(Xp)(x) = xp(x)$ is called *multiplication operator*, we recall that the Pincherle derivative of an operator $U \in \mathscr{L}$ is defined by the formula

$$U' = UX - XU.$$

For example, we get $I' = 0$, $D' = I$, $(D^k)' = kD^{k-1}$, $k \geq 1$.

Example 4.5 Following Lupaş, we can define the operators $\widetilde{L}_n^Q : C([0,1]) \to C([0,1])$, $n \in \mathbb{N}$,

$$(\widetilde{L}_n^Q f)(x) = \frac{1}{p_n(n)} \sum_{k=0}^{n} \binom{n}{k} p_k(nx) p_{n-k}(n-nx) f\left(\frac{k}{n}\right).$$

It is proved that

$$\widetilde{L}_n^Q e_i = e_i, \; i \in \{0,1\}, \text{ and } \widetilde{L}_n^Q e_2 = e_2 + (e_1 - e_2)\rho_n(Q),$$

see (4.4). Consequently, if

$$st - \lim_{n\to\infty} \frac{n(n-1)}{p_n(n)} (Q'^{-2} p_{n-2})(x) = 1,$$

then the operators $\widetilde{L}_n^Q$, $n \in \mathbb{N}$, satisfy identity (4.3).

Concluding remarks. The paper reintroduces some linear positive operators of discrete type by using umbral calculus. Relative to these operators have been studied approximation properties in Banach space $(C[0,1], \|\cdot\|)$. The approach was based on Bohman–Korovkin theorem via statistical convergence. The usefulness of this type of convergence can be summarized as follows: The statistical convergence of a sequence is that the majority, in a certain sense, of its elements converges and we are not interested in what happens to the remaining elements. The advantage of replacing the uniform convergence by statistical convergence consists in the fact that the second convergence is efficient in summing divergent sequences which may have unbounded subsequences. In short, it is more lax.

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Weighted Statistical Convergence of Bögel Continuous Functions by Positive Linear Operator

Fadime Dirik

Abstract In the present work, we have introduced a weighted statistical approximation theorem for sequences of positive linear operators defined on the space of all real-valued B-continuous functions on a compact subset of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Furthermore, we display an application which shows that our new result is stronger than its classical version.

Keywords Weighted uniform convergence · Double sequences · Statistical convergence · Korovkin-type approximation theorem

Mathematics Subject Classification 40A35 · 41A36

1 Introduction

The classical Korovkin theory is mostly connected with the approximation to continuous functions by means of positive linear operators (see, for instance, [1, 17]). In order to work up the classical Korovkin theory, the space of Bögel-type continuous (or, simply, B-continuous) functions instead of the classical theory has been studied in [2–4]. The concept of statistical convergence for sequences of real numbers was introduced by Fast [14] and Steinhaus [21] independently in the same year 1951. Some Korovkin-type theorems in the setting of a statistical convergence were given by [5, 6, 10–13, 22].

Now we recall some notations and definitions.

A double sequence $x = (x_{mn}),\ \ m, n \in \mathbb{N}$, is convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$ whenever $m, n > N$, then L is called the Pringsheim limit of x and is denoted by $P - \lim x = L$ (see [20]). Also, if there exists a positive number M such that $|x_{mn}| \leq M$ for all $(m, n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, then $x = (x_{mn})$ is said to be bounded. Note that in contrast to

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the case for single sequences, a convergent double sequence need not to be bounded.

Definition 1 ([19]) Let $K \subset \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. Then *density of* K, denoted by $\delta^2(K)$, is given by:

$$\delta^2(K) := P - \lim_{m,n} \frac{|\{j \le m, k \le n : (j,k) \in K\}|}{mn}$$

provided that the limit on the right-hand side exists in the Pringsheim sense by $|B|$ we mean the cardinality of the set $B \subset \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. A real double sequence $x = (x_{mn})$ is said to be statistically convergent to L if, for every $\varepsilon > 0$,

$$\delta^2\left(\left\{(m,n) \in \mathbb{N}^2 : |x_{mn} - L| \ge \varepsilon\right\}\right) = 0.$$

In this case, we write $st^2 - \lim x = L$.

The concept of weighted statistical convergence was defined by Karakaya and Chishti [16]. Recently, Mursaleen et al. [18] modified the definition of weighted statistical convergence. In [15], Ghosal showed that both definitions of weighted statistical convergence are not well defined in general. So Ghosal modified the definition of weighted statistical convergence as follows:

Definition 2 Let $\{p_j\}, \{q_k\}, j,k \in \mathbb{N}$ be sequences of nonnegative real numbers such that $p_1 > 0, \liminf_{j\to\infty} p_j > 0, q_1 > 0, \liminf_{k\to\infty} q_k > 0$ and $P_m = \sum_{j=1}^{m} p_j$ and $Q_n = \sum_{k=1}^{n} q_k$ where $n, m \in \mathbb{N}$, $P_m \to \infty$ as $m \to \infty$, $Q_n \to \infty$ as $n \to \infty$. The double sequence $x = (x_{jk})$ is said to be weighted statistical convergent (or $S_{\overline{N_2}}$-convergent) to L if for every $\varepsilon > 0$,

$$P - \lim_{m,n} \frac{1}{P_m Q_n} |\{j \le P_m, k \le Q_n : p_j q_k |x_{jk} - L| \ge \varepsilon\}| = 0.$$

In this case, we write $st_{\overline{N_2}} - \lim x = L$ and we denote the set of all weighted statistical convergent sequences by $S_{\overline{N_2}}$.

Remark 1 If $p_j = 1$, $q_k = 1$ for all j,k, then weighted statistical convergence is reduced to statistical convergence for double sequences.

Example 1 Let $x = (x_{mn})$ is a sequence defined by

$$x_{mn} := \begin{cases} mn, & m \text{ and } n \text{ are squares,} \\ 0, & \text{otherwise,} \end{cases}.$$

Let $p_j = j, q_k = k$ for all j,k. Then $P_m = \frac{m(m+1)}{2}$ and $Q_n = \frac{n(n+1)}{2}$. Since, for every $\varepsilon > 0$,

$$
\begin{aligned}
& P-\lim_{m,n}\frac{|\{j\le P_m, k\le Q_n : p_jq_k|x_{jk}-0|\ge\varepsilon\}|}{P_mQ_n} \\
\le\ & P-\lim_{m,n}\frac{|\{j\le P_m, k\le Q_n : p_jq_k|x_{jk}|\neq 0\}|}{P_mQ_n} \\
\le\ & P-\lim_{m,n}\frac{\sqrt{P_m}\sqrt{Q_n}}{P_mQ_n}=0
\end{aligned}
$$

So $x=(x_{mn})$ is weighted statistical convergent to 0 but not Pringsheim's sense convergent.

In [15], Ghosal showed that both convergences which are weighted statistical convergence and statistical convergence do not imply each other in general.

In the work, using the Definition 2, we prove Korovkin-type approximation theorem for double sequences of B-continuous functions defined on a compact subset of the real two-dimensional space. Finally, we give an application which shows that our new result is stronger than its classical version.

2 A Korovkin-Type Approximation Theorem

Bögel introduced the definition of B-continuity [7–9] as follows:

Let I be a compact subset of $\mathbb{R}^2=\mathbb{R}\times\mathbb{R}$. Then, a function $f:I\to\mathbb{R}$ is called a B-continuous at a point $(x,y)\in I$ if, for every $\varepsilon>0$, there exists a positive number $\delta=\delta(\varepsilon)$ such that

$$\left|\Delta_{xy}\left[f(u,v)\right]\right|<\varepsilon,$$

for any $(u,v)\in I$ with $|u-x|<\delta$ and $|v-y|<\delta$, where the symbol $\Delta_{xy}\left[f(u,v)\right]$ denotes the mixed difference of f defined by

$$\Delta_{xy}\left[f(u,v)\right]=f(u,v)-f(u,y)-f(x,v)+f(x,y).$$

By $C_b(I)$, we denote the space of all B-continuous functions on I. Recall that $C(I)$ and $B(I)$ denote the space of all continuous (in the usual sense) functions on I and the space of all bounded functions on I, respectively. Then, notice that $C(I)\subset C_b(I)$. Moreover, one can find an unbounded B-continuous function, which follows from the fact that, for any function of the type $f(u,v)=g(u)+h(v)$, we have $\Delta_{xy}\left[f(u,v)\right]=0$ for all $(x,y),(u,v)\in I$. $\|f\|$ denotes the supremum norm of f in $B(I)$.

Let L be a linear operator from $C_b(I)$ into $B(I)$. Then, as usual, we say that L is positive linear operator provided that $f\ge 0$ implies $L(f)\ge 0$. Also, we denote the value of $L(f)$ at a point $(x,y)\in I$ by $L(f(u,v);x,y)$ or, briefly, $L(f;x,y)$. Since

$$\Delta_{xy}\left[f(u,y)+f(x,v)-f(u,v)\right]=-\Delta_{xy}\left[f(u,v)\right]$$

holds for all (x, y), $(u, v) \in I$, the B-continuity of f implies the B-continuity of $F_{xy}(u, v) := f(u, y) + f(x, v) - f(u, v)$ for every fixed $(x, y) \in I$. We also use the following test functions

$$f_0(x, y) = 1, \quad f_1(x, y) = x, \quad f_2(x, y) = y \text{ and } f_3(x, y) = x^2 + y^2.$$

We recall that the following lemma for B-continuous functions was proved by Badea et al. [3].

Lemma 1 ([3]) *If* $f \in C_b(I)$, *then, for every* $\varepsilon > 0$, *there are two positive numbers* $\alpha_1(\varepsilon) = \alpha_1(\varepsilon, f)$ *and* $\alpha_2(\varepsilon) = \alpha_2(\varepsilon, f)$ *such that*

$$\Delta_{xy}\left[f\,(u, v)\right] \leq \frac{\varepsilon}{3} + \alpha_1(\varepsilon)(u - x)^2 + \alpha_2(\varepsilon)(v - y)^2$$

holds for all (x, y), $(u, v) \in I$.

Now we have the following main result.

Theorem 1 *Let* (L_{mn}) *be a double sequence of positive linear operators acting from* $C_b\,(I)$ *into* $B\,(I)$. *Assume that the following conditions hold:*

$$P - \lim_{m,n} \frac{1}{P_m Q_n} \left|\{j \leq P_m, k \leq Q_n : L_{jk}(f_0; x, y) = f_0(x, y) \textit{ for all } (x, y) \in I\}\right| = 1 \tag{2.1}$$

and

$$st_{\overline{N_2}} - \lim \left\| L_{mn}\,(f_i; x, y) - f_i(x, y) \right\| = 0, i = 1, 2, 3. \tag{2.2}$$

Then, for all $f \in C_b\,(I)$,*we have*

$$st_{\overline{N_2}} - \lim \left\| L_{mn}\,(f(u, y) + f(x, v) - f(u, v); x, y) - f(x, y) \right\| = 0. \tag{2.3}$$

Proof Let $(x, y) \in I$ and $f \in C_b\,(I)$ be fixed. Taking

$$A := \left\{ j \leq P_m, k \leq Q_n : L_{jk}(f_0; x, y) = f_0(x, y) = 1 \text{ for all } (x, y) \in I \right\}, \tag{2.4}$$

we obtain from (2.1) that

$$P - \lim_{m,n} \frac{1}{P_m Q_n} \left|\{j \leq P_m, k \leq Q_n : L_{jk}(f_0; x, y) \neq f_0(x, y) \text{ for all } (x, y) \in I\}\right| = 0. \tag{2.5}$$

Using the B-continuity of the function $F_{xy}(u, v) := f(u, y) + f(x, v) - f(u, v)$, Lemma 1 implies that, for every $\varepsilon > 0$, there exist two positive numbers $\alpha_1(\varepsilon)$ *and* $\alpha_2(\varepsilon)$ such that

$$\left|\Delta_{xy}\left[f(u, y) + f(x, v) - f(u, v)\right]\right| \leq \frac{\varepsilon}{3} + \alpha_1(\varepsilon)(u - x)^2 + \alpha_2(\varepsilon)(v - y)^2 \tag{2.6}$$

holds for every $(u, v) \in I$. Also, by (2.12), see that

$$L_{jk}\left(f(u,y)+f(x,v)-f(u,v);x,y\right)-f(x,y)=L_{jk}\left(\Delta_{xy}\left[f(u,y)+f(x,v)-f(u,v)\right];x,y\right) \tag{2.7}$$

holds for all $(j,k)\in A$. We can write for all $(m,n)\in A$ from (2.6) and (2.7),

$$\begin{aligned}\left|L_{jk}\left(f(u,y)+f(x,v)-f(u,v);x,y\right)-f(x,y)\right| &= \left|L_{jk}\left(\Delta_{xy}\left[f(u,y)+f(x,v)-f(u,v)\right];x,y\right)\right|\\ &\le L_{jk}\left(\left|\Delta_{xy}\left[f(u,y)+f(x,v)-f(u,v)\right]\right|;x,y\right)\\ &\le \frac{\varepsilon}{3}+\alpha_1(\varepsilon)L_{jk}\left((u-x)^2;x,y\right)\\ &\quad+\alpha_2(\varepsilon)L_{jk}\left((v-y)^2;x,y\right)\\ &\le \frac{\varepsilon}{3}+\alpha(\varepsilon)\{x^2+y^2+L_{jk}(f_3;x,y)\\ &\quad-2xL_{jk}(f_1;x,y)-2yL_{jk}(f_2;x,y)\},\end{aligned}$$

where $\alpha(\varepsilon)=\max\{\alpha_1(\varepsilon),\alpha_2(\varepsilon)\}$. It follows from the last inequality that

$$\left|L_{jk}\left(f(u,y)+f(x,v)-f(u,v);x,y\right)-f(x,y)\right|\le\frac{\varepsilon}{3}+\alpha(\varepsilon)\sum_{i=1}^{3}\left|L_{jk}\left(f_i;x,y\right)-f_i(x,y)\right| \tag{2.8}$$

holds for all $(j,k)\in A$. Taking supremum over $(x,y)\in I$ on both sides of inequality (2.8), we obtain, for all $(j,k)\in I$, that

$$\left\|L_{jk}\left(f(u,y)+f(x,v)-f(u,v);x,y\right)-f(x,y)\right\|\le\frac{\varepsilon}{3}+\alpha(\varepsilon)\sum_{i=1}^{3}\left\|L_{jk}\left(f_i;x,y\right)-f_i(x,y)\right\|.$$

Because of ε is arbitrary, we obtain

$$\left\|L_{jk}\left(f(u,y)+f(x,v)-f(u,v);x,y\right)-f(x,y)\right\|\le\alpha(\varepsilon)\sum_{i=1}^{3}\left\|L_{jk}\left(f_i;x,y\right)-f_i(x,y)\right\|.$$

Hence,

$$p_jq_k\left\|L_{jk}\left(f(u,y)+f(x,v)-f(u,v);x,y\right)-f(x,y)\right\|\le\alpha(\varepsilon)\sum_{i=1}^{3}p_jq_k\left\|L_{jk}\left(f_i;x,y\right)-f_i(x,y)\right\|. \tag{2.9}$$

Now for a given $r>0$, consider the following sets:

$$\begin{aligned}U&:=\left\{j\le P_m,k\le Q_n: p_jq_k\left\|L_{jk}\left(f(u,y)+f(x,v)-f(u,v);x,y\right)-f(x,y)\right\|\ge r\right\},\\ U_i&:=\left\{j\le P_m,k\le Q_n: p_jq_k\left\|L_{jk}\left(f_i;x,y\right)-f_i(x,y)\right\|\ge\frac{r}{3\alpha(\varepsilon)}\right\},\quad i=1,2,3,\end{aligned}$$

Hence, inequality (2.9) yields that

$$\frac{|U\cap A|}{P_mQ_n}\le\frac{|U_1\cap A|}{P_mQ_n}+\frac{|U_2\cap A|}{P_mQ_n}+\frac{|U_3\cap A|}{P_mQ_n},$$

which gives,

$$P-\lim \frac{|U\cap A|}{P_m Q_n} \leq \sum_{i=1}^{3}\left\{P-\lim \frac{|U_i\cap A|}{P_m Q_n}\right\} \leq \sum_{i=1}^{3}\left\{P-\lim \frac{|U_i|}{P_m Q_n}\right\} \quad (2.10)$$

Letting $m, n \to \infty$ (in any manner) and also using (2.13), we see from (2.10) that

$$P-\lim \frac{|U\cap A|}{P_m Q_n} = 0. \quad (2.11)$$

Furthermore, if we use the inequality

$$\begin{aligned}\frac{|U|}{P_m Q_n} &= \frac{|U\cap A|}{P_m Q_n} + \frac{\left|U\cap\left(\mathbb{N}^2\backslash A\right)\right|}{P_m Q_n}\\ &\leq \frac{|U\cap A|}{P_m Q_n} + \frac{\left|\mathbb{N}^2\backslash A\right|}{P_m Q_n}\end{aligned}$$

and if we take limit as $m, n \to \infty$, then it follows from (2.5) and (2.11) that

$$P-\lim \frac{|U|}{P_m Q_n} = 0,$$

which means

$$st_{\overline{N_2}} - \lim \|L_{mn}\left(f(u,y)+f(x,v)-f(u,v);x,y\right)-f(x,y)\| = 0 = 0.$$

This completes the proof. □

If $p_j = 1$ and $q_k = 1$ with $j, k \in \mathbb{N}$, then we obtain the statistical case of the Korovkin-type result for a double sequences on $C_b(I)$ introduced in [13],

Theorem 2 ([13]) *Let (L_{mn}) be a sequence of positive linear operators acting from $C_b(I)$ into $B(I)$. Assume that the following conditions hold:*

$$\delta^2\left\{(m,n)\in\mathbb{N}^2 : L_{mn}(f_0;x,y) = 1 \text{ for all } (x,y)\in I\right\} = 1 \quad (2.12)$$

and

$$st^2 - \lim_{m,n} \|L_{mn}\left(f_i;x,y\right) - f_i(x,y)\| = 0 \text{ for } i = 1,2,3. \quad (2.13)$$

Then, for all $f \in C_b(I)$, we have

$$st^2 - \lim_{m,n} \|L_{mn}(f(u,y)+f(x,v)-f(u,v);x,y)-f(x,y)\| = 0.$$

Now we present an example for double sequences of positive linear operators. The first one shows that Theorem 1 does not work but Theorem 2 works. The second one gives that our approximation theorem and Theorem 2 work.

Example 2 Let $I = [0, 1] \times [0, 1]$. Consider the double Bernstein polynomials

$$B_{mn}(f; x, y) = \sum_{s=0}^{m} \sum_{t=0}^{n} f\left(\frac{s}{m}, \frac{t}{n}\right) x^s y^t (1 - x)^{m-s} (1 - y)^{n-t}$$

on $C_b(I)$.

(a) Using these polynomials, we introduce the following positive linear operators on $C_b(I)$:

$$P_{mn}(f; x, y) = (1 + \alpha_{mn}) B_{mn}(f; x, y), \quad (x, y) \in I \text{ and } f \in C_b(I), \tag{2.14}$$

where $\alpha := (\alpha_{mn})$ is given by $\alpha_{mn} := \begin{cases} 1 & m, n \text{ are squares,} \\ \frac{1}{\sqrt{mn}} & \text{otherwise,} \end{cases}$. Let $p_j = 2j + 1$, $q_k = k$ for all j, k. Then $P_m = m^2$ and $Q_n = \frac{n(n+1)}{2}$. Note that $\alpha = (\alpha_{mn})$ statistical convergent to 0 but it is not convergent and weighted statistical convergent to 0. Then, observe that

$$\begin{aligned} P_{mn}(f_0; x, y) &= (1 + \alpha_{mn}) f_0(x, y), \\ P_{mn}(f_1; x, y) &= (1 + \alpha_{mn}) f_1(x, y), \\ P_{mn}(f_2; x, y) &= (1 + \alpha_{mn}) f_2(x, y), \\ P_{mn}(f_3; x, y) &= (1 + \alpha_{mn}) \left[f_3(x, y) + \frac{x - x^2}{m} + \frac{y - y^2}{n} \right]. \end{aligned}$$

Since $st^2 - \lim \alpha_{mn} = 0$, we conclude that

$$st^2 - \lim \| P_{mn}(f_i; x, y) - f_i(x, y) \| = 0 \text{ for each } i = 0, 1, 2.$$

However, since α is statistically convergent, the sequence $\{P_{mn}(f; x, y)\}$ given by (2.14) does satisfy the Theorem 2 for all $f \in C_b(I)$. But Theorem 1 does not work since $\alpha = (\alpha_{mn})$ is not weighted statistical convergent to 0.

(b) Now we consider the following positive linear operators on $C_b(I)$:

$$T_{mn}(f; x, y) = (1 + \beta_{mn}) B_{mn}(f; x, y), \quad (x, y) \in I \text{ and } f \in C_b(I), \tag{2.15}$$

where $\beta := (\beta_{mn})$ is given by $\beta_{mn} := \begin{cases} mn & m, n \text{ are squares,} \\ 0 & \text{otherwise,} \end{cases}$. Let $p_j = j$, $q_k = k$ for all j, k. Then $P_m = \frac{m(m+1)}{2}$ and $Q_n = \frac{n(n+1)}{2}$. Note that $\alpha = (\alpha_{mn})$ statistical and weighted statistical convergent to 0 but it is not convergent to 0. Then, observe that

$$\begin{aligned} T_{mn}(f_0; x, y) &= (1 + \beta_{mn}) f_0(x, y), \\ T_{mn}(f_1; x, y) &= (1 + \beta_{mn}) f_1(x, y), \\ T_{mn}(f_2; x, y) &= (1 + \beta_{mn}) f_2(x, y), \end{aligned}$$

$$T_{mn}(f_3; x, y) = (1 + \beta_{mn})\left[f_3(x, y) + \frac{x - x^2}{m} + \frac{y - y^2}{n}\right].$$

Since $st_{\overline{N_2}} - \lim \beta_{mn} = 0$, we conclude that

$$st_{\overline{N_2}} - \lim \|T_{mn}(f_i; x, y) - f_i(x, y)\| = 0 \text{ for each } i = 1, 2, 3.$$

So, by Theorem 1, we have

$$st_{\overline{N_2}} - \lim \|T_{mn}(f(u, y) + f(x, v) - f(u, v); x, y) - f(x, y)\| = 0 \text{ for all } f \in C_b(I).$$

However, since β is weighted statistical convergent to 0, we can say that Theorem 1 works for our operators defined by (2.15).

Therefore, this application clearly shows that our Theorem 1 is a non-trivial generalization of the classical case of the Korovkin result introduced in [3].

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Optimal Linear Approximation Under General Statistical Convergence

Daniel Cárdenas-Morales and Pedro Garrancho

Abstract This work deals with optimal approximation by sequences of linear operators. Optimality is meant here as asymptotic formulae and saturation results, a natural step beyond the establishment of both qualitative and quantitative results. The ordinary convergence is replaced by B -statistical $\mathscr{A}$-summability, where B is a regular infinite matrix with non-negative real entries and $\mathscr{A}$ is a sequence of matrices of the aforesaid type, in such a way that the new notion covers the famous concept of almost convergence introduced by Lorentz, as well as a new one that merits being called statistical almost convergence.

1 Introduction

Under the framework of the approximation of functions by means of linear operators, the idea of assigning a limit to sequences of such operators that do not converge in the ordinary sense is a well-settled subject. A walk along the papers that deal with it starts from the results of King and Swetits [13] and finds at least as many directions as different notions of convergence have been moved from pure mathematical analysis to linear approximation theory.

B-statistical A-summability, introduced in [7], is one of those notions. It covers a lot of particular cases but do not cover the one that starred [13], namely the famous almost convergence introduced by Lorentz [15] in 1948 and characterized as follows: a bounded sequence of real numbers $z = (z_j)$ is said to be almost convergent to ℓ if

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=n}^{n+k-1} z_j = \ell \quad \text{uniformly for } n \in \mathbb{N} = \{1, 2, \ldots\}.$$

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In 1973, Bell [3] introduced a summability method that helped Swetits [19] unify both classical convergence and almost convergence and then state a general result on the approximation of continuous functions by sequences of positive linear operators.

In this work, we present a sort of unification of that type which covers both B-statistical A-summability and almost convergence, and then we go through the topics within Korovkin-type conservative approximation theory usually tackled right after studying primary qualitative and quantitative results, namely asymptotic formulae and saturation. In order to specify our objectives, we need some definitions that the reader may find in [5]; we recover them here, however, for the sake of completeness. As a matter of fact, the current paper can be considered as a continuation of that one; firstly, because a more general setting is considered, and secondly, because it deals with topics that have not been studied earlier under such a generality. Some previous results of the authors in this very line can be found in [1, 11]. See also [6, 16, 17] for further papers dealing with this matter.

The contents are organized as follows. In the next section, we present the new definition. In Sect. 3, we make it explicit the conservative approximation theory setting we shall be dealing with. Section 4 is devoted to typical linear approximation results, given by a quantitative theorem and statements about asymptotic formulae and saturation, whereas Sect. 5 contains a quantitative version of that asymptotic expression. We end this work with some final remarks in Sect. 6.

2 B-Statistical $\mathscr{A}$-Summability

Given an infinite matrix of real entries $A = (a_{ij})$, $i, j = 1, 2, \ldots$, and a sequence of real numbers $z = (z_j)$, the A-transform of z is the sequence, denoted by Az or $A(z_j)$, whose elements are defined by (provided the series below converges for each i)

$$(Az)_i := \sum_{j=1}^{\infty} a_{ij} z_j .$$

A is said to be regular if the convergence of z towards ℓ (as $j \to +\infty$) implies the convergence of Az towards the same limit ℓ (as $i \to +\infty$).

Let $B = (b_{nk})$, $n, k = 1, 2, \ldots$ be a non-negative regular infinite matrix. For a given set of natural numbers $K \subseteq \mathbb{N}$, the B -density of K, denoted by $\delta_B(K)$, is defined by (provided the limit exists)

$$\delta_B(K) := \lim_{n\to\infty} \sum_{k \in K} b_{nk} .$$

Edely used in [7] the previous concept introduced by Freedman and Sember [9] to present the following.

Definition 1 Let $A = (a_{ij})$ and $B = (b_{nk})$ be two non-negative regular infinite matrices. A sequence $z = (z_j)$ is said to be B-statistically A-summable to ℓ, denoted by $\ell = st_B - \lim\ Az$, if for every $\varepsilon > 0$, one has that $\delta_B\left(\{i \in \mathbb{N} : |(Az)_i - \ell| \geq \varepsilon\}\right) = 0$.

Next proposition is a key element to present our new definition. The assertion that contains was already observed by Fast [8] and Fridy [10] in the particular case in which $A = I$ (I denotes the infinite identity matrix) and $B = (C, 1)$, the Cesàro matrix of order 1, that is to say, when B-statistical A-summability amounts to classical statistical convergence. By the way, notice that the general notion includes further elementary concepts: classical convergence appears when $A = B = I$; if only $A = I$, then one has B-statistical convergence; if only $B = I$, then ordinary matrix summability is recovered; and if only $B = (C, 1)$, then statistical A-summability appears.

Proposition 1 *Let $\ell \in \mathbb{R}$ and let $z = (z_j)$ be a sequence of real numbers. Then the following items are equivalent:*

(i) $\ell = st_B - \lim\ Az$*;*
(ii) there exists a strictly increasing sequence of positive integers $\sigma : \mathbb{N} \to \mathbb{N}$, such that $\delta_B\big(\sigma(\mathbb{N})\big) = 1$ and $(Az)_{\sigma(i)}$ converges to ℓ as $i \to \infty$ in the classical sense.

Proof We prove firstly that the first statement implies the second. For $\nu \in \mathbb{N}$, if we consider

$$I_\nu = \left[\ell - 2^{-\nu}, \ell + 2^{-\nu}\right], \quad K_\nu = \{i \in \mathbb{N} : (Az)_i \notin I_\nu\},$$

we have from the hypothesis that

$$\delta_B(K_\nu) = \lim_{n\to\infty} \sum_{k \in K_\nu} b_{nk} = 0.$$

As a consequence, for each $\nu \in \mathbb{N}$, there exists $n_\nu \in \mathbb{N}$ such that if $n > n_\nu$, then

$$\sum_{k \in K_\nu} b_{nk} < \frac{1}{\nu}.$$

We define a subsequence from Az by removing the terms $(Az)_i$ with $i \in K$, where

$$K := \{i \in \mathbb{N} : i > n_1 \text{ and } i \in K_\nu \text{ whenever } n_\nu < i \leq n_{\nu+1}\}.$$

Naturally, we denote that subsequence by $(Az)_{\sigma(i)}$, $i = 1, 2, \ldots$, where $\sigma : \mathbb{N} \to \mathbb{N}$ is understood to be strictly increasing.

Thus,

$$\lim_{i\to\infty} (Az)_{\sigma(i)} = \ell$$

because for $i \in \mathbb{N}$ such that $\sigma(i) > n_\nu$ one has that

$$\left|(Az)_{\sigma(i)} - \ell\right| \le 2^{-\nu}.$$

Finally, $\delta_B(\sigma(\mathbb{N})) = 1$ amounts to $\delta_B(K) = 0$, and this last equality follows readily after observing that for $n_\nu < n < n_{\nu+1}$, one has that $K \subset K_\nu$, so

$$\sum_{k\in K} b_{nk} \le \sum_{k\in K_\nu} b_{nk} < \frac{1}{\nu}.$$

We prove now that *(ii)* implies *(i)*. Let $\varepsilon > 0$ and let

$$K_c = \{i \in \mathbb{N} : |(Az)_i - \ell| \ge \varepsilon\}, \quad K_\varepsilon^\sigma = \{i \in \mathbb{N} : |(Az)_{\sigma(i)} - \ell| \ge \varepsilon\}.$$

After Definition 1, we have to prove that $\delta_B(K_\varepsilon) = 0$.

Notice that $K_\varepsilon \subset K_\varepsilon^\sigma \cup \mathbb{N} \setminus \sigma(\mathbb{N})$, and that, directly from the hypothesis and basic facts about B-density, $\delta_B(\mathbb{N} \setminus \sigma(\mathbb{N})) = 0$. Thus,

$$\delta_B(K_\varepsilon) \le \delta_B(K_\varepsilon^\sigma) + \delta_B(\mathbb{N} \setminus \sigma(\mathbb{N})) = \delta_B(K_\varepsilon^\sigma).$$

Finally, since $(Az)_{\sigma(i)}$ converges to ℓ in the classical sense, then K_ε^σ is finite, so, as a direct consequence of the necessary and sufficient conditions for B to be regular given by the well-known Silverman–Toeplitz theorem, one derives that

$$\delta_B(K_\varepsilon^\sigma) = \lim_{n\to\infty} \sum_{k\in K_\varepsilon^\sigma} b_{nk} = 0.$$

□

Definition 2 Let $\mathscr{A} = (A^{(p)}) = (a_{ij}^{(p)})$ be a sequence of non-negative regular infinite matrices and let B be a matrix of that type. A sequence of real numbers $z = (z_j)$ is said to be B-statistically $\mathscr{A}$-summable to $\ell \in \mathbb{R}$, denoted by $\ell = st_B - \lim \mathscr{A}z$, if for each $p \in \mathbb{N}$ there exists a strictly increasing sequence of positive integers $\sigma_p : \mathbb{N} \to \mathbb{N}$ such that $\delta_B\big(\sigma_p(\mathbb{N})\big) = 1$ and $(A^{(p)}z)_{\sigma_{p,i}}$, with $\sigma_{p,i} := \sigma_p(i)$, $i \in \mathbb{N}$, converges in the ordinary sense to ℓ as $i \to \infty$ uniformly in p.

Notice that if z is B-statistically $\mathscr{A}$-summable to ℓ, then for every single $p \in \mathbb{N}$, z is B-statistically $A^{(p)}$ -summable to ℓ.

3 Conservative Approximation Theory Setting

Here we reproduce the general framework and notation displayed in [5].

Let $J = [0, 1] \subset \mathbb{R}$, $J^\circ = (0, 1)$ and let $i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We denote by $C^i(J)$ the space of all real valued i-times continuously differentiable functions defined on J, and we denote by D^i the classical ith differential operator. Obviously, $C^0(J) = C(J)$

is the space of all continuous functions on J, $D^0 = \mathbb{I}$ is the identity operator and $C^\infty(J) = \cap_{i\in\mathbb{N}} C^i(J)$. As usual, for $f \in C(J)$, $\omega(f, \xi)$ denotes its classical modulus of continuity with argument ξ.

Let $\tau \in C^\infty(J)$ such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(t) > 0$ for $t \in J^\circ$. As a generalization of the usual notation for the monomials $e_i(t) = t^i$ and $e_i^x(t) = (t - x)^i$, we denote by $e_{\tau,i}$ and $e_{\tau,i}^x$ the functions $e_{\tau,i}(t) = \tau(t)^i$ and $e_{\tau,i}^x(t) = (\tau(t) - \tau(x))^i$. We also define the differential operator D_τ^i to be

$$D_\tau^i f(t) := D^i \left(f \circ \tau^{-1}\right)(\tau(t)). \tag{1}$$

Obviously, $D_\tau^0 = \mathbb{I}$, and if $\tau = e_1$, then $D_\tau^i = D^i$. Moreover, we notice that the operator D_τ^i coincides with the ith iterate of the operator $\frac{1}{\tau'}D^1$, denoted by $D^{i,\tau'}$ and defined recursively as follows:

$$D^{0,\tau'} = \mathbb{I}, \quad D^{1,\tau'} = \frac{1}{\tau'}D^1, \quad D^{i+1,\tau'} = D^{1,\tau'} \circ D^{i,\tau'}, \ i \in \mathbb{N}.$$

Besides, it can be easily checked that for $x \in J$,

$$D_\tau^i e_{\tau,j}^x = \begin{cases} \frac{j!}{(j-i)!} e_{\tau,j-i}^x, & \text{if } j \geq i; \\ 0, & \text{if } j < i. \end{cases} \tag{2}$$

Finally, given $f \in C^m(J)$ and $x \in J$, we are interested in the optimal B-statistical $\mathscr{A}$-summability of the sequence $\left(D_\tau^m L_j f(x)\right)$ towards $D_\tau^m f(x)$, where $L = (L_j)$ with $L_j : C^m(J) \longrightarrow C^m(J)$ is a sequence of linear operators. In this respect, we shall make use of the following notation:

$$\mathscr{A}_{L,i}^{\tau,m,p} f(x) := \sum_{j=1}^{\infty} a_{ij}^{(p)} D_\tau^m L_j f(x).$$

In the section that follows we present a quantitative result, an asymptotic formula and some saturation results. The proofs follow the usual pattern that one finds when studying the so-called Korovkin-type approximation theory [2, 14], so we omit them. We do prove, however, in a subsequent section a quantitative version of the asymptotic formula, which is a novelty under our general setting.

4 Typical Linear Approximation Results

Theorem 1 *Under the general conditions above, assume that the following (shape preserving) property is satisfied:*

$$D_\tau^m f(t) \geq 0,\ \forall t \in J \ \ \Rightarrow \ \ \mathscr{A}_{L,i}^{\tau,m,p} f(t) \geq 0,\ \forall t \in J,\ i = 1, 2, \ldots,\ p = 1, 2, \ldots. \tag{3}$$

Then, for $f \in C^m(J)$ *and* $x \in J$,

$$\left|\mathscr{A}_{L,i}^{\tau,m,p} f(x) - D_\tau^m f(x)\right| \leq \frac{|D_\tau^m f(x)|}{m!} \cdot \left|\mathscr{A}_{L,i}^{\tau,m,p} e_{\tau,m}(x) - D_\tau^m e_{\tau,m}(x)\right|$$
$$+\frac{1}{m!}\left|\mathscr{A}_{L,i}^{\tau,m,p} e_{\tau,m}(x) + D_\tau^m e_{\tau,m}(x)\right| \omega\left(D_\tau^m f \circ \tau^{-1}, \eta_{i,\tau,m}(x)\right),$$

where

$$\eta_{i,\tau,m}^2(x) = \frac{2}{(m+2)!}\left|\mathscr{A}_{L,i}^{\tau,m,p} e_{\tau,m+2}^x(x)\right|.$$

A natural uniform qualitative result follows directly from Theorem 1. The B-statistical $\mathscr{A}$-summability of the sequence $\left(D_\tau^m L_j f(x)\right)$ towards $D_\tau^m f(x)$, for any $f \in C^m(J)$, uniformly for $x \in J$, is guaranteed whenever it holds just for the functions $e_{\tau,m+\upsilon}$, with $\upsilon = 0, 1, 2$. This is a remark that was already point out in the pioneer quantitative result of Shisha and Mond [18] under the most basic framework.

Here, we merely observe that the B-statistical $\mathscr{A}$-summability of the sequence $\left(D_\tau^m L_j f(x)\right)$ towards $D_\tau^m f(x)$ for $f = e_{\tau,m+\upsilon}$, with $\upsilon = 0, 1, 2$, implies that for every $p \in \mathbb{N}$ and $\upsilon = 0, 1, 2$, there exists a strictly increasing sequence of non-negative integers $\sigma_p^\upsilon : \mathbb{N} \to \mathbb{N}$, $\sigma_p^\upsilon(i) =: \sigma_{p,i}^\upsilon$, such that $\delta_B(\sigma_p^\upsilon(\mathbb{N})) = 1$ and $(A^{(p)} D_\tau^m e_{\tau,m+\upsilon}(x))_{\sigma_{p,i}^\upsilon}$ converges in the ordinary sense to $D_\tau^m e_{\tau,m+\upsilon}(x)$ as $i \to \infty$ uniformly in p. In addition to that, we observe that for each single $p \in \mathbb{N}$, $\delta_B\left(\cap_{\upsilon=0}^2 \sigma_p^\upsilon(\mathbb{N})\right) = 1$.

Now we proceed with the statement of an asymptotic result. For the sake of brevity, we take into account the considerations above and make use of the following notation:

$$\tilde{L}_i^p f(\cdot) := \mathscr{A}_{L,\sigma_{p,i}}^{\tau,m,p} f(\cdot) = \sum_{j=1}^\infty a_{\sigma_{p,i} j}^{(p)} D_\tau^m L_j f(\cdot),$$

where $\sigma_{p,i} = \sigma_p(i)$ and $\sigma_p : \mathbb{N} \to \mathbb{N}$ are the strictly increasing sequence of non-negative integers associated with the B-statistical $\mathscr{A}$-summability of the sequence $\left(D_\tau^m L_j f(x)\right)$ towards $D_\tau^m f(x)$.

Thus, we shall write it in terms of the sequence of operators

$$\tilde{L}_i^p : C^m(J) \to C^m(J)$$

under the assumption that

$$D_\tau^m f \geq 0 \implies \tilde{L}_i^{(p)} f \geq 0. \tag{4}$$

This will be our starting assumption for the rest of the theorems of the paper.

Theorem 2 *Let $x \in J^\circ$ and let us assume that there exist a sequence of real positive numbers $\lambda_i \to +\infty$ and three strictly positive functions $w_0 \in C^2(J^\circ)$, $w_1 \in C^1(J^\circ)$ and $w_2 \in C(J^\circ)$ such that, for $s \in \{m, m+1, m+2, m+4\}$, the following limit holds uniformly in p:*

$$\lim_{i\to+\infty} \lambda_i \left(\tilde{L}_i^{(p)} e_{\tau,s}^x(x) - D_\tau^m e_{\tau,s}^x(x) \right) = w_2^{-1} D^1(w_1^{-1} D^1(w_0^{-1} D_\tau^m e_{\tau,s}^x))(x).$$

Then, for any $f \in C^m(J)$, $m+2$ times differentiable in some neighbourhood of x, next limit holds true uniformly in p:

$$\lim_{i\to+\infty} \lambda_i \left(\tilde{L}_i^{(p)} f(x) - D_\tau^m f(x) \right) = w_2^{-1} D^1(w_1^{-1} D^1(w_0^{-1} D_\tau^m f))(x).$$

Now, we pass on to present some saturation results. We assume therein that the sequence of operators fulfil the asymptotic formula we have just described. Here $o^{(p)}(1)$ stands for a sequence that converges to 0 uniformly in p.

Theorem 3 *(i) Let $f \in C^m(J)$. Then*

$$\lambda_i \left| \tilde{L}_i^{(p)} f(x) - D_\tau^m f(x) \right| \le M + o^{(p)}(1), \quad x \in J^\circ$$

if and only if, almost everywhere on J°,

$$\frac{1}{w_2} D^1 \left(\frac{1}{w_1} D^1 \left(\frac{1}{w_0} D_\tau^m f \right) \right) \le M.$$

(ii) $f \in C^m(J)$ is a solution of the differential equation (in the unknown y)

$$\frac{1}{w_2} D^1 \left(\frac{1}{w_1} D^1 \left(\frac{1}{w_0} y \right) \right) \equiv 0$$

in some neighbourhood of $x \in J^\circ$ if and only if

$$\lambda_i \left(\tilde{L}_i^{(p)} f(x) - D_\tau^m f(x) \right) = o^{(p)}(1).$$

Next result follows readily from the previous one. We recover our terminology of interest.

Theorem 4 *Let $f \in C^m(J)$. Then*

$$\lambda_i \left| \mathscr{A}_{L,i}^{\tau,m,p} f(x) - D_\tau^m f(x) \right| \le M + \beta, \quad x \in J^\circ$$

if and only if, almost everywhere on J°,

$$\frac{1}{w_2} D^1 \left(\frac{1}{w_1} D^1 \left(\frac{1}{w_0} D_\tau^m f \right) \right) \leq M,$$

where $st_B - \lim \mathscr{A}\beta = 0$.

5 Quantitative Version of the Asymptotic Formula

In this section, we present a quantitative version of the asymptotic expression stated in Theorem 2. We make the most of the approach displayed in [12]. For $f \in C(J)$, we shall consider the K-functional

$$K(\varepsilon, f; C(J), C^1(J)) = \inf\{\|f - g\| + \varepsilon \|g'^1(J)\} \quad (5)$$

and the usual least concave majorant of $\omega(f, \cdot)$ denoted by $\widetilde{\omega}(f, \cdot)$, both connected by the equality (see [12, Lemma 3.1] and the remarks and references therein)

$$K(\varepsilon/2, f; C(J), C^1(J)) = \frac{1}{2}\widetilde{\omega}(f, \varepsilon), \quad \varepsilon \geq 0. \quad (6)$$

Theorem 5 *Let* $f \in C^{m+2}(J)$*, then*

$$\left| \tilde{L}_i^{(p)} f(x) - D_\tau^m f(x) - \sum_{\nu=0}^{2} \frac{D_\tau^{m+\nu} f(x)}{(m+\nu)!} \left(\tilde{L}_i^{(p)} e_{\tau,m+\nu}^x(x) - D_\tau^m e_{\tau,m+\nu}^x(x) \right) \right| \leq$$

$$\frac{2}{(m+2)!} \frac{1}{2} \tilde{L}_i^{(p)} e_{\tau,m+2}^x(x) \widetilde{\omega} \left(D_\tau^{m+2} f, \frac{(m+2)!}{6} \frac{\tilde{L}_i^{(p)} \Psi(x)}{\tilde{L}_i^{(p)} e_{\tau,m+2}^x(x)} \right),$$

where $D_\tau^m \Psi(t) = |\tau(t) - \tau(x)|^3$.

Proof Given $f \in C^{m+2}(J)$, second-order Taylor's polynomial of the function $D_\tau^m f \circ \tau^{-1}$ at the point $\tau(x)$ is given as follows with the use of the definition of the operator D_τ^i:

$$\sum_{\nu=0}^{2} \frac{D^\nu (D_\tau^m f \circ \tau^{-1})(\tau(x))}{\nu!} (\tau(t) - \tau(x))^\nu = \sum_{\nu=0}^{2} \frac{D_\tau^{m+\nu} f(x)}{\nu!} (\tau(t) - \tau(x))^\nu.$$

Directly from [12, Theorem 3.2] and (6), we can write

$$\left| D_\tau^m f(t) - \sum_{\nu=0}^{2} \frac{D_\tau^{m+\nu} f(x)}{\nu!} (\tau(t) - \tau(x))^\nu \right|$$

$$\leq \frac{(\tau(t)-\tau(x))^2}{2}\widetilde{\omega}\left(D_\tau^{m+2}f, \frac{|(\tau(t)-\tau(x)|}{3}\right)$$

$$= (\tau(t)-\tau(x))^2 K\left(\frac{|(\tau(t)-\tau(x)|}{6}, D_\tau^{m+2}f; C(J), C^1(J)\right),$$

and by using (5), one has, for every $g \in C^1(J)$,

$$\left| D_\tau^m f(t) - \sum_{\nu=0}^{2} \frac{D_\tau^{m+\nu} f(x)}{\nu!}(\tau(t)-\tau(x))^\nu \right|$$

$$\leq (\tau(t)-\tau(x))^2 ||D_\tau^{m+2}f - g|| + \frac{|\tau(t)-\tau(x)|^3}{6}||g'||.$$

Equivalently, we have the two inequalities

$$D_\tau^m\left(f(t) - \sum_{\nu=0}^{2} D_\tau^{m+\nu} f(x)\frac{e^x_{\tau,m+\nu}(t)}{(m+\nu)!}\right)$$

$$\leq D_\tau^m\left(\frac{2}{(m+2)!}e^x_{\tau,m+2}(t)||D_\tau^{m+2}f - g|| + \frac{1}{6}\Psi(t)||g'||\right) \tag{7}$$

and

$$-D_\tau^m\left(\frac{2}{(m+2)!}e^x_{\tau,m+2}(t)||D_\tau^{m+2}f - g|| + \frac{1}{6}\Psi(t)||g'||\right)$$

$$\leq D_\tau^m\left(f(t) - \sum_{\nu=0}^{2} D_\tau^{m+\nu} f(x)\frac{e^x_{\tau,m+\nu}(t)}{(m+\nu)!}\right), \tag{8}$$

where $D_\tau^m \Psi(t) = |\tau(t)-\tau(x)|^3$.

We work with (7) and use linearity arguments and (4) to obtain

$$\tilde{L}_i^{(p)} f(x) - \sum_{\nu=0}^{2} \frac{D_\tau^{m+\nu} f(x)}{(m+\nu)!}\tilde{L}_i^{(p)} e^x_{\tau,m+\nu}(x)$$

$$\leq \frac{2}{(m+2)!}\tilde{L}_i^{(p)} e^x_{\tau,m+2}(x)||D_\tau^{m+2}f - g|| + \frac{1}{6}\tilde{L}_i^{(p)}\Psi(x)||g'||$$

which amounts to

$$\tilde{L}_i^{(p)} f(x) - \sum_{\nu=0}^{2} \frac{D_\tau^{m+\nu} f(x)}{(m+\nu)!}\tilde{L}_i^{(p)} e^x_{\tau,m+\nu}(x)$$

$$\leq \frac{2}{(m+2)!}\tilde{L}_i^{(p)} e^x_{\tau,m+2}(x)\left(||D_\tau^{m+2} f - g|| + \frac{(m+2)!}{12}\frac{\tilde{L}_i^{(p)}\Psi(x)}{\tilde{L}_i^{(p)} e^x_{\tau,m+2}(x)}||g'||\right).$$

Passing to the infimum over $g \in C^1(J)$, we have

$$\tilde{L}_i^{(p)} f(x) - \sum_{\nu=0}^{2}\frac{D_\tau^{m+\nu} f(x)}{(m+\nu)!}\tilde{L}_i^{(p)} e^x_{\tau,m+\nu}(x)$$

$$\leq \frac{2}{(m+2)!}\tilde{L}_i^{(p)} e^x_{\tau,m+2}(x) K\left(D_\tau^{m+2} f, \frac{(m+2)!}{12}\frac{\tilde{L}_i^{(p)}\Psi(x)}{\tilde{L}_i^{(p)} e^x_{\tau,m+2}(x)}; C(J), C^1(J)\right),$$

and taking again into account (6),

$$\tilde{L}_i^{(p)} f(x) - \sum_{\nu=0}^{2}\frac{D_\tau^{m+\nu} f(x)}{(m+\nu)!}\tilde{L}_i^{(p)} e^x_{\tau,m+\nu}(x)$$

$$\leq \frac{2}{(m+2)!}\frac{1}{2}\tilde{L}_i^{(p)} e^x_{\tau,m+2}(x)\widetilde{\omega}\left(D_\tau^{m+2} f, \frac{(m+2)!}{6}\frac{\tilde{L}_i^{(p)}\Psi(x)}{\tilde{L}_i^{(p)} e^x_{\tau,m+2}(x)}\right)$$

After inserting the term $-D_\tau^m f(x) + D_\tau^m f(x)$, it is derived that

$$\tilde{L}_i^{(p)} f(x) - D_\tau^m f(x) - \sum_{\nu=0}^{2}\frac{D_\tau^{m+\nu} f(x)}{(m+\nu)!}\left(\tilde{L}_i^{(p)} e^x_{\tau,m+\nu}(x) - D_\tau^m e^x_{\tau,m+\nu}(x)\right)$$

$$\leq \frac{2}{(m+2)!}\frac{1}{2}\tilde{L}_i^{(p)} e^x_{\tau,m+2}(x)\widetilde{\omega}\left(D_\tau^{m+2} f, \frac{(m+2)!}{6}\frac{\tilde{L}_i^{(p)}\Psi(x)}{\tilde{L}_i^{(p)} e^x_{\tau,m+2}(x)}\right).$$

Finally, the proof is over if one proceeds analogously with (8). □

Notice that the necessary comparison between Theorems 2 and 5 makes sense when there exist three strictly positive functions $w_0 \in C^2(J^\circ)$, $w_1 \in C^1(J^\circ)$ and $w_2 \in C(J^\circ)$ such that $w_2^{-1} D^1(w_1^{-1} D^1(w_0^{-1} D_\tau^m f))(x)$ coincides with

$$\sum_{\nu=0}^{2}\frac{D_\tau^{m+\nu} f(x)}{(m+\nu)!}\lim_{i\to+\infty}\lambda_i\left(\tilde{L}_i^{(p)} e^x_{\tau,m+\nu}(x) - D_\tau^m e^x_{\tau,m+\nu}(x)\right).$$

This has been a natural assumption in the previous papers that have dealt with particular situations.

6 Final Remarks

If $\mathscr{A} = (A^{(m)}) = (a_{ij}^{(m)})$ is such that

$$a_{ij}^{(m)} = \begin{cases} 1/i, \ m \leq j < i+m; \\ 0, \quad \text{otherwise,} \end{cases}$$

then B-statistically $\mathscr{A}$-summability amounts to almost convergence whenever $B = I$, and refers to a new notion that merits being called statistical almost convergence if $B = (C, 1)$.

On the other hand, the results of the paper cover previous ones of the authors proved in [1, 5, 11] under different specific settings. A particular sequence of operators that shows the applicability of the results showed here is described briefly to finish off. Details are left to the reader.

Let

$$\alpha_j = \begin{cases} \dfrac{j^2}{j+1}, & \text{if } \ j \text{ is odd,} \\ -\dfrac{(j-1)^2}{j}, & \text{if } \ j \text{ is even,} \end{cases}$$

$$A^{(n)} = (a_{ij}^{(n)}); \ a_{ij}^{(n)} = \begin{cases} \dfrac{1}{i}, & \text{if} \quad 2n-1 \leq j < i + (2n-1), \\ 0, & \text{otherwise} \end{cases}$$

and

$$B = (b_{nk}); \ b_{nk} = \begin{cases} \dfrac{1}{n}, & \text{if} \quad k = 2\upsilon; \ 1 \leq \upsilon \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Then we take τ as described in the paper and consider the sequence of operators $(1+\alpha_j)B_j^\tau$, where (B_j^τ) is the sequence studied in [4] and defined for $f \in C(J)$ by

$$B_j^\tau f(t) = \sum_{\nu=0}^{j} \binom{j}{\nu} \tau(t)^\nu (1-\tau(t))^{1-\nu} (f \circ \tau^{-1})(\nu/j).$$

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Statistical Deferred Cesàro Summability Mean Based on (p, q)-Integers with Application to Approximation Theorems

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Abstract This chapter consists of four sections. The first section is introductory in which a concept (presumably new) of statistical deferred Cesàro summability mean based on (p, q)-integers has been introduced and accordingly some basic terminologies are presented. In the second section, we have applied our proposed mean under the difference sequence of order r to prove a Korovkin-type approximation theorem for the set of functions 1, e^{-x} and e^{-2x} defined on a Banach space $C[0, \infty)$ and demonstrated that our theorem is a non-trivial extension of some well-known Korovkin-type approximation theorems. In the third section, we have established a result for the rate of our statistical deferred Cesàro summability mean with the help of the modulus of continuity. Finally, in the last section, we have given some concluding remarks and presented some interesting examples in support of our definitions and results.

Keywords Statistical convergence · Statistical deferred Cesàro summability
Delayed arithmetic mean · Difference sequence of order r · (p, q)-integers
Banach space · Positive linear operators · Korovkin-type approximation theorem
Rate of convergence

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1 Introduction

In the study of sequence spaces, classical convergence has got numerous applications where the convergence of a sequence requires that almost all elements are to satisfy the convergence condition. That is, all the elements of the sequence need to be in an arbitrarily small neighborhood of the limit. However, such restriction is relaxed in statistical convergence, where the validity of convergence condition is achieved only for a majority of elements. The notion of statistical convergence was introduced by Fast [13] and Steinhaus [29]. Recently, statistical convergence has been a dynamic research area due to the fact that it is more general than classical convergence, and such theory is discussed in the study of Fourier analysis, number theory, and approximation theory. For more details, see [7, 9–11, 14, 15, 17, 21, 24, 26–28].

Let ω be the set of all real-valued sequences, and suppose any subspace of ω be the sequence space. Let (x_k) be a sequence with real and complex terms. Suppose ℓ_∞ be the class of all bounded linear spaces, and let c, c_0 be the respective classes for convergent and null sequences with real and complex terms. We have

$$\|x\|_\infty = \sup_k |x_k| \quad (k \in \mathbb{N}),$$

and we recall here that under this norm, the above-mentioned spaces are all Banach spaces.

The notion of difference sequence space was initially studied by Kızmaz [18], and then, it was extended to the difference sequence of natural order r $\;(r \in \mathbb{N}_0 := \{0\} \cup \mathbb{N})$ by defining

$$\lambda(\Delta^r) = \left\{x = (x_k): \quad \Delta^r(x) \in \lambda, \quad \lambda \in (\ell_\infty, c_0, c)\right\};$$
$$\Delta^0 x = (x_k); \quad \Delta^r x = (\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1})$$

and

$$\Delta^r x_k = \sum_{i=0}^{r} (-1)^i \binom{r}{i} x_{k+i}$$

(see [18]). Also, these are all Banach spaces under the norm defined by

$$\|x\|_{\Delta^r} = \sum_{i=1}^{r} |x_i| + \sup_k |\Delta^r x_k|.$$

For more interest in this direction, see the current works [6, 12, 16].

Let $\mathbb{N}$ be the set of natural numbers, and let $K \subseteq \mathbb{N}$. Also let

$$K_n = \{k : k \leq n \;\text{ and }\; k \in K\}$$

and suppose that $|K_n|$ be the cardinality of K_n. Then, the natural density of K is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{|K_n|}{n} = \lim_{n\to\infty} \frac{1}{n}|\{k : k \leq n \text{ and } k \in K\}|,$$

provided the limit exists.

A given sequence (x_n) is said to be statistically convergent to ℓ if, for each $\epsilon > 0$, the set

$$K_\epsilon = \{k : k \in \mathbb{N} \text{ and } |x_k - \ell| \geq \epsilon\}$$

has zero natural density (see [13, 29]). That is, for each $\epsilon > 0$,

$$\delta(K_\epsilon) = \lim_{n\to\infty} \frac{|K_\epsilon|}{n} = \lim_{n\to\infty} \frac{1}{n}|\{k : k \leq n \text{ and } |x_k - \ell| \geq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat} \lim_{n\to\infty} x_n = \ell.$$

Now, we present an example to show that every convergent sequence is statistically convergent, but the converse is not true in general.

Example 1 Let us consider the sequence $x = (x_n)$ by

$$x_n = \begin{cases} n & \text{when } n = m^2, \text{ for all } m \in \mathrm{N} \\ \frac{1}{n} & \text{otherwise.} \end{cases}$$

Then, it is easy to see that the sequence (x_n) is divergent in the ordinary sense, while 0 is the statistical limit of (x_n) since $\delta(K) = 0$, where $K = \{m^2, \text{ for all } m = 1, 2, 3, \ldots\}$.

In 2002, Móricz [22], introduced the fundamental idea of statistical $(C, 1)$ summability and recently Mohiuddine et al. [20] has established statistical $(C, 1)$ summability as follows.

Let us consider a sequence $x = (x_n)$; the $(C, 1)$ mean of the sequence is given by

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} x_k,$$

and (x_n) is said to be statistical $(C, 1)$ summable to ℓ if, for each $\epsilon > 0$, the set

$$\{k : k \in \mathbb{N} \text{ and } |\sigma_k - \ell| \geq \epsilon\}$$

has zero Cesàro density. That is, for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k : k \leq n \text{ and } |\sigma_k - \ell| \geq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat} \lim_{n\to\infty} \sigma_n = \ell \ \text{ or } \ C_1(\text{stat}) \lim_{n\to\infty} x_n = \ell.$$

Subsequently, with the development of q-calculus, various researchers worked on certain new generalizations of positive linear operators based on q-integers (see [3, 5]). Recently, Mursaleen et al. [23] introduced the (p, q)-analogue of Bernstein operators in connection with (p, q)-integers, and later on, some approximation results for Baskakov operators and Bernstein-Schurer operators are studied for (p, q)-integers by [1].

We now recall some definitions and basic notations on (p, q)-integers for our present study:

For any $(n \in \mathbb{N})$, the (p, q)-integer $[n]_{p,q}$ is defined by,

$$[n]_{p,q} = \begin{cases} \frac{p^n - q^n}{p-q} & (n \geq 1) \\ 0 & (n = 0) \end{cases}$$

where $0 < q < p \leq 1$.

The (p, q)-factorial is defined by

$$[n]!_{p,q} = \begin{cases} [1]_{p,q}[2]_{p,q} \ldots [n]_{p,q} & (n \geq 1) \\ 1 & (n = 0). \end{cases}$$

The (p, q)-binomial coefficient is defined by,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]!_{p,q}}{[k]!_{p,q}\ [n-k]!_{p,q}} \qquad \text{for all } \ n, k \in \mathbb{N} \ \text{ and } \ n \geq k.$$

We also recall that suppose $0 < q < p \leq 1$ and r be a nonnegative integer. Then, the operator

$$\Delta^{[r]}_{p,q} : \omega \to \omega$$

is defined by

$$\Delta^{[r]}_{p,q}(x_n) = \sum_{i=0}^{r} (-1)^i \begin{bmatrix} r \\ i \end{bmatrix}_{p,q} x_{n-i}.$$

That is,

$$\begin{aligned}\Delta^{[r]}_{p,q}(x_n) &= \begin{bmatrix} r \\ 0 \end{bmatrix}_{p,q} x_n - \begin{bmatrix} r \\ 1 \end{bmatrix}_{p,q} x_{n-1} + \begin{bmatrix} r \\ 2 \end{bmatrix}_{p,q} x_{n-2} - \begin{bmatrix} r \\ 3 \end{bmatrix}_{p,q} x_{n-3} + \cdots + (-1)^r \begin{bmatrix} r \\ r \end{bmatrix}_{p,q} x_{n-r} \\ &= x_n - [r]_{p,q} x_{n-1} + \frac{[r]_{p,q}[r-1]_{p,q}}{[2]_{p,q}!} x_{n-2} - \frac{[r]_{p,q}[r-1]_{p,q}[r-2]_{p,q}}{[3]!} x_{n-3} + \cdots + (-1)^r x_{n-r} \\ &= x_n - \left(\frac{p^r - q^r}{p-q}\right) x_{n-1} + \left(\frac{(p^r - q^r)(p^{r-1} - q^{r-1})}{(p-q)^2(p+q)}\right) x_{n-2} \\ &\quad - \left(\frac{(p^r - q^r)(p^{r-1} - q^{r-1})(p^{r-2} - q^{r-2})}{(p-q)^3(p^2+pq+q^2)(p+q)}\right) x_{n-3} + \cdots + (-1)^m x_{n-r}.\end{aligned}$$

Now, we present an example to see that a sequence is not convergent; however, the associated difference sequence is convergent.

Example 2 Let us consider a sequence $(x_n) = n + 1$ $(n \in \mathbb{N})$. It is clear that the sequence (x_n) is not convergent in the ordinary sense.

Also, we see that

$$\Delta^{[3]}(x_n) = x_n - 3x_{n-1} + 3x_{n-2} - x_{n-3} \ \ (x_n = n + 1)$$

converges to 0 $(n \to \infty)$.

For $r = 3$, we obtain that

$$\begin{aligned}\Delta^{[3]}_{p,q}(x_n) &= x_n - [3]_{p,q} x_{n-1} + [3]_{p,q} x_{n-2} - x_{n-3} \ \ (x_n = n+1) \\ &= x_n - (p_n^2 + p_n q_n + q_n^2) x_{n-1} + (p_n^2 + p_n q_n + q_n^2) x_{n-2} - x_{n-3} \\ &= n + 1 - (p_n^2 + p_n q_n + q_n^2) n + (p_n^2 + p_n q_n + q_n^2)(n-1) - (n-2) \ \ (x_n = n+1) \\ &= 3 - (\beta^2 + \alpha\beta + \alpha^2).\end{aligned}$$

Clearly, depending on the choice of the values of p and q, the difference sequence $\Delta^{[3]}_{p,q}(x_n)$ of third order has different limits. This situation is due to the definition of (p, q)-integers. However, in order to obtain a convergence criterion for all values of p and q, belonging to the operator $\Delta^{[r]}_{p,q}$, we must have to overcome this difficulty. This type of difficulties can be avoided in the following two ways. The first one is taking $p = q = 1$, and thus, the operator reduces to the usual difference sequence. Next, the second way is to replace $p = p_n$ and $q = q_n$ under the limits, $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 \leq \alpha, \beta \leq 1)$ where $0 < q_n < p_n \leq 1$, for all $(n \in \mathbb{N})$. Afterward, the difference sequence $\Delta^{[3]}_{p,q}(x_n)$ of third order 3 converges to the value $3 - (\beta^2 + \alpha\beta + \alpha^2)$. Thus, if we take $q_n = \left(\frac{n+1}{n+1+s}\right) < \left(\frac{n+1}{n+1+t}\right) = p_n$ such that $0 < q_n < p_n \leq 1$ $(s > t > 0)$, then $\lim_n q_n = 1 = \lim_n p_n$ and hence $\Delta^{[3]}_{p,q}(x_n) \to 0 \quad (n \to \infty)$.

Remark 1 If $r = 1$, $\lim_n q_n = 1$, and $\lim_n p_n = 1$, then the difference operator $\Delta^{[r]}_{p,q}$ reduces to the $\Delta^{[1]}$. Also, if $r = 0$, $\lim_n q_n = 1$ and $\lim_n p_n = 1$, then the difference operator $\Delta^{[r]}_{p,q}$ reduces to the general sequence (x_n).

Here, we now present the notion of the statistical deferred Cesàro summability under the generalized difference sequence of order r involving (p, q)-integers:

Let (a_n) and (b_n) be sequences of nonnegative integers such that (i) $a_n < b_n$ and (ii) $\lim_{n\to\infty} b_n = \infty$,

then the deferred Cesàro $D(a_n, b_n)$ mean based on (p, q)-integers is defined by,

$$D_{p,q}(a_n, b_n) = D_{p,q}(x_n) = \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} \Delta^{[r]}_{p,q}(x_k). \tag{1}$$

It is well known that $D_{p,q}(a_n, b_n)$ is regular under conditions (i) and (ii) (see Agnew [2]).

Remark 2 If $p = q = 1$, then the deferred Cesàro mean under (p, q)-integers reduces to the deferred Cesàro mean (see [15]).

Let us now introduce the following definitions in support of our proposed work.

Definition 1 Let $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$, and let r is a nonnegative integer. Also, let (a_n) and (b_n) be sequences of integers (nonnegative). A sequence (x_n) is said to be statistical deferred Cesàro summable to ℓ with respect to difference sequence of order r based on (p, q)-integers if, for every $\epsilon > 0$, the set

$$\{k : a_n < k \leq b_n \text{ and } |D_{p,q}(x_n) - \ell| \geq \epsilon\}$$

has natural density zero, that is,

$$\lim_{n\to\infty} \frac{1}{b_n - a_n} |\{k : a_n < k \leq b_n \text{ and } |D_{p,q}(x_n) - \ell| \geq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat} \lim_{n\to\infty} D_{p,q}(x_n) = \ell \text{ or } \lim_{n\to\infty} \text{stat}^{p,q}_{DC} x_n = \ell.$$

Clearly, above definition can be viewed as the generalization of some existing definitions.

Remark 3 If $a_n = n - 1$, $b_n = n$, and $p_n = q_n = 1$, then $D(n - 1, n)$ reduces to the identity transformation, and also, if $a_n = 0$, $b_n = n$, and $p_n = q_n = 1$, then $D(0, n)$ reduces to $(C, 1)$ transformation of x_n, which is often denoted as σ_n. Furthermore, if $a_n = n - 1$, $b_n = n + t - 1$, and $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$ and let r is a nonnegative integer, then

$$D_{p,q}(n - 1, n + t - 1) = \sigma^{p,q}_{n,t} = \left(\frac{t + n}{t}\right) \sigma^{p,q}_{n+t-1} - \left(\frac{n}{t}\right) \sigma^{p,q}_{n-1}, \tag{2}$$

which is called the deferred delayed arithmetic mean. Finally, if $a_n = n - 1$, $b_n = n + t - 1$ and $p_n = q_n = 1$, then

$$D(n-1, n+t-1) = \sigma_{n,t} = \left(\frac{t+n}{t}\right)\sigma_{n+t-1} - \left(\frac{n}{t}\right)\sigma_{n-1},$$

which is called the delayed arithmetic mean (see [31], p. 80).

Definition 2 Let $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$, and let r is a nonnegative integer. Also, let (a_n) and (b_n) be sequences of integers (nonnegative). A sequence (x_n) is said to be statistical deferred delayed arithmetic summable to ℓ if, for every $\epsilon > 0$, the set

$$\{k : n-1 < k \leq n+t-1 \text{ and } |\sigma_{n,t}^{p,q} - \ell| \geq \epsilon\}$$

has zero natural density, that is,

$$\lim_{n\to\infty} \frac{1}{t}|\{k : n-1 < k \leq n+t-1 \text{ and } |\sigma_{n,t}^{p,q} - \ell| \geq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat} \lim_{n\to\infty} \sigma_{n,t}^{p,q} = \ell \text{ or } \text{stat}_{DA}^{p,q} x_n = \ell.$$

Now, we present below an example to show that a sequence is statistically deferred Cesàro summable, whenever it is not statistically Cesàro summable.

Example 3 For $\lim_n q_n = 1$, $\lim_n p_n = 1$, $a_n = 2n$ and $b_n = 4n$ $(\forall\, n \in \mathbb{N})$, consider a sequence $x = (x_n)$,

$$x_n = \begin{cases} \frac{1}{m^2} & (n = m^2 - m, m^2 - m + 1, \ldots, m^2 - 1) \\ -\frac{1}{m^3} & (n = m^2, \quad m > 1) \\ 0 & (\text{otherwise}). \end{cases}$$

We have,

$$\begin{aligned}
\Delta_{p,q}^{[r]}(x_k) &= \sum_{i=0}^{r} (-1)^i \begin{bmatrix} r \\ i \end{bmatrix}_{p,q} x_{n-i} \\
&= \left\{ x_n - \begin{bmatrix} r \\ 1 \end{bmatrix}_{p,q} x_{n-1} + \begin{bmatrix} r \\ 2 \end{bmatrix}_{p,q} x_{n-2} - \begin{bmatrix} r \\ 3 \end{bmatrix}_{p,q} x_{n-3} + \cdots + (-1)^r \begin{bmatrix} r \\ r \end{bmatrix}_{p,q} x_{n-r} \right\} \\
&= \left\{ x_n - [r]_{p,q} x_{n-1} + \frac{[r]_{p,q}[r-1]_{p,q}}{[2]_{p,q}!} x_{n-2} - \frac{[r]_{p,q}[r-1]_{p,q}[r-2]_{p,q}}{[3]!} x_{n-3} + \cdots + (-1)^r x_{n-r} \right\}.
\end{aligned}$$

Thus,

$$D_{p,q}(a_n, b_n) = D_{p,q}(x_n) = \frac{1}{4n-2n} \sum_{k=2n+1}^{4n} \Delta_{p,q}^{[r]}(x_k)$$

which implies that

$$D_{p,q}(x_n) \to 0.$$

Hence, (x_n) is not deferred Cesàro summable , even if it is statistical deferred Cesàro summable under the difference operator of order r based on (p, q)-integers.

In the year 2012, Mohiuddine et al. [20] established statistical summability $(C, 1)$ and a Korovkin-type approximation theorem, and then, Jena et al. [15] investigated a Korovkin-type approximation theorem for exponential functions via the statistical deferred Cesàro summability of the real sequence. Very recently, Srivastava et al. [26] has established generalized equi-statistical convergence of the deferred Nörlund summability and its applications to associated approximation theorems, and then, Srivastava et al. [27] established a certain class of weighted statistical convergence and associated Korovkin-type approximation theorems for trigonometric functions. Furthermore, Srivastava et al. [28] has proved some interesting results on deferred weighted $\mathcal{A}$-statistical convergence based on the (p, q)-Lagrange polynomials and its applications to approximation theorems.

The main object of this chapter is to establish some important approximation theorems over the Banach space based on statistical deferred Cesàro summability for (p, q)-integers under difference sequence of order r which will effectively extend and improve most (if not all) of the existing results depending on the choice of sequences of the simple statistical deferred Cesàro means. Furthermore, we intend to estimate the rate of our statistical deferred Cesàro summability and investigate Korovkin-type approximation results.

2 A Korovkin-Type Approximation Theorem

Several researchers have worked on extending or generalizing the Korovkin-type theorems in many ways and to several settings, including Function spaces, Banach Algebras, Banach spaces. This theory is very useful in real analysis, functional analysis, harmonic analysis, measure theory, probability theory, and summability theory. Recently, Jena et al. [15] have proved the Korovkin theorem via statistical deferred Cesàro summability on $\mathcal{C}[0, \infty)$ by using the test functions $1,\ e^{-x}$, and e^{-2x}. In this paper, we generalize the result of Jena, Paikray, and Misra via the notion of statistical deferred Cesàro summability based on difference sequence of order r including (p, q)-integers for the same test functions $1,\ e^{-x}$, and e^{-2x}. We also present an example to justify that our result is stronger than that of Jena, Paikray, and Misra (see [15]).

Let $\mathcal{C}(X)$, be the space of all real-valued continuous functions defined on $[0, \infty)$ under the norm $\|.\|_\infty$. Also, $\mathcal{C}[0, \infty)$ is a Banach space. We have, for $f \in \mathcal{C}[0, \infty)$, the norm of f denoted by $\|f\|$ is given by

$$\|f\|_\infty = \sup_{x\in[0,\infty)} \{|f(x)|\}$$

with

$$\omega(\delta, f) = \sup_{0\leq|h|\leq\delta} \|f(x+h) - f(x)\|_\infty, \ \ f \in \mathcal{C}[0,\infty).$$

The quantities $\omega(\delta, f)$ is called the modulus of continuity of f.

Let $L : \mathcal{C}[0,\infty) \to \mathcal{C}[0,\infty)$ be a linear operator. Then, as usual, we say that L is a positive linear operator provided that

$$f \geq 0 \ \text{ implies } \ L(f) \geq 0.$$

Also, we denote the value of $L(f)$ at a point $x \in [0,\infty)$ by $L(f(u); x)$ or, briefly, $L(f; x)$.

The classical Korovkin theorem states as follows [19]:

Let $L_n : \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ be a sequence of positive linear operators and let $f \in \mathcal{C}[0,\infty)$. Then

$$\lim_{n\to\infty} \|L_n(f;x) - f(x)\|_\infty = 0 \Longleftrightarrow \lim_{n\to\infty} \|L_n(f_i;x) - f_i(x)\|_\infty = 0 \ \ (i = 0,1,2),$$

where

$$f_0(x) = 1, \quad f_1(x) = x \ \text{ and } \ f_2(x) = x^2.$$

Now, we prove the following theorem by using the notion of statistical deferred Cesàro summability based on (p, q)-integers.

Theorem 1 *Let $L_m : \mathcal{C}[0,\infty) \to \mathcal{C}[0,\infty)$ be a sequence of positive linear operators. Then, for all $f \in \mathcal{C}[0,\infty)$*

$$stat_{DC}^{p,q} \lim_{m\to\infty} \|L_m(f;x) - f(x)\|_\infty = 0, \tag{3}$$

if and only if

$$stat_{DC}^{p,q} \lim_{m\to\infty} \|L_m(1;x) - 1\|_\infty = 0, \tag{4}$$

$$stat_{DC}^{p,q} \lim_{m\to\infty} \|L_m(e^{-s};x) - e^{-x}\|_\infty = 0 \tag{5}$$

and

$$stat_{DC}^{p,q} \lim_{m\to\infty} \|L_m(e^{-2s};x) - e^{-2x}\|_\infty = 0. \tag{6}$$

Proof Since each of $f_i(x) = \{1, e^{-x}, e^{-2x}\} \in \mathcal{C}(X)$ $(i = 0,1,2)$ is continuous, the implication (3)$\Longrightarrow$(4)–(6) is obvious. In order to complete the proof of the theorem, we first assume that (4)–(6) hold true. Let $f \in \mathcal{C}[X]$, then there exists a constant

$\mathcal{K} > 0$ such that $|f(x)| \leq \mathcal{K},\ \forall\, x \in X = [0, \infty)$. Thus,

$$|f(s) - f(x)| \leq 2\mathcal{K},\ \ s, x \in X. \tag{7}$$

Clearly, for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(s) - f(x)| < \epsilon \tag{8}$$

whenever $|e^{-s} - e^{-x}| < \delta$, for all $s, x \in X$.

Let us choose $\varphi_1 = \varphi_1(s, x) = (e^{-s} - e^{-x})^2$. If $|e^{-s} - x^{-x}| \geq \delta$, then we obtain:

$$|f(s) - f(x)| < \frac{2\mathcal{K}}{\delta^2}\varphi_1(s, x). \tag{9}$$

From Eqs. (8) and (9), we get

$$|f(s) - f(x)| < \epsilon + \frac{2\mathcal{K}}{\delta^2}\varphi_1(s, x),$$

$$\Rightarrow\ -\epsilon - \frac{2\mathcal{K}}{\delta^2}\varphi_1(s, x) \leq f(s) - f(x) \leq \epsilon + \frac{2\mathcal{K}}{\delta^2}\varphi_1(s, x). \tag{10}$$

Now since $L_m(1; x)$ is monotone and linear, so by applying the operator $L_m(1; x)$ to this inequality, we have

$$L_m(1; x)\left(-\epsilon - \frac{2\mathcal{K}}{\delta^2}\varphi_1(s, x)\right) \leq L_m(1; x)(f(s) - f(x)) \leq L_m(1; x)\left(\epsilon + \frac{2\mathcal{K}}{\delta^2}\varphi_1(s, x)\right). \tag{11}$$

Note that x is fixed and so $f(x)$ is a constant number. Therefore,

$$-\epsilon L_m(1; x) - \frac{2\mathcal{K}}{\delta^2}L_m(\varphi_1; x) \leq L_m(f; x) - f(x)L_m(1; x) \leq \epsilon L_m(1; x) + \frac{2\mathcal{K}}{\delta^2}L_m(\varphi_1; x). \tag{12}$$

But

$$L_m(f; x) - f(x) = [L_m(f; x) - f(x)L_m(1; x)] + f(x)[L_m(1; x) - 1]. \tag{13}$$

Using (12) and (13), we have

$$L_m(f; x) - f(x) < \epsilon L_m(1; x) + \frac{2\mathcal{K}}{\delta^2}L_m(\varphi_1; x) + f(x)[L_m(1; x) - 1]. \tag{14}$$

Now, estimate $L_m(\varphi_1; x)$ as,

$$\begin{aligned} L_m(\varphi_1; x) &= L_m((e^{-s} - e^{-x})^2; x) = L_m(e^{-2s} - 2e^{-x}e^{-s} + e^{-2x}; x) \\ &= L_m(e^{-2s}; x) - 2e^{-x}L_m(e^{-s}; x) + e^{-2s}L_m(1; x) \\ &= [L_m(e^{-2s}; x) - e^{-2x}] - 2e^{-x}[L_m(e^{-s}; x) - e^{-x}] + e^{-2x}[L_m(1; x) - 1]. \end{aligned}$$

Using (14), we obtain

$$\begin{aligned} L_m(f; x) - f(x) &< \epsilon L_m(1; x) + \frac{2\mathcal{K}}{\delta^2}\{[L_m(e^{-2s}; x) - e^{-2s}] - 2e^{-x}[L_m(e^{-s}; x) - e^{-x}] \\ &\quad + e^{-2s}[L_m(1; x) - 1]\} + f(x)[L_m(1; x) - 1]. \\ &= \epsilon[L_m(1; x) - 1] + \epsilon + \frac{2\mathcal{K}}{\delta^2}\{[L_m(e^{-2s}; x) - e^{-2x}] - 2e^{-x}[L_m(e^{-s}; x) - e^{-x}] \\ &\quad + e^{-2x}[L_m(1; x) - 1]\} + f(x)[L_m(1; x) - 1]. \end{aligned}$$

Since ϵ is arbitrary, we can write

$$\begin{aligned} |L_m(f; x) - f(x)| &\leq \epsilon + \left(\epsilon + \frac{2\mathcal{K}}{\delta^2} + \mathcal{K}\right)|L_m(1; x) - 1| \\ &\quad + \frac{4\mathcal{K}}{\delta^2}|L_m(e^{-s}; x) - e^{-x}| + \frac{2\mathcal{K}}{\delta^2}|L_m(e^{-2s}; x) - e^{-2x}| \\ &\leq B\left(|L_m(1; x) - 1| + |L_m(e^{-s}; x) - e^{-x}| + |L_m(e^{-2s}; x) - e^{-2x}|\right), \end{aligned} \tag{15}$$

where

$$B = \max\left(\epsilon + \frac{2\mathcal{K}}{\delta^2} + \mathcal{K}, \frac{4\mathcal{K}}{\delta^2}, \frac{2\mathcal{K}}{\delta^2}\right).$$

Now replace $L_m(f; x)$ by

$$D_{p,q}(x_n) = \frac{1}{b_n - a_n}\sum_{m=a_n+1}^{b_n} \Delta_{p,q}^{[r]}(T_m(f; x))$$

in Eq. (15).

We have for a given $r > 0$, there exists $\epsilon > 0$, such that $\epsilon < r$. Then, by setting

$$\Psi_m(x; r) = \left\{m : a_n < m \leq b_n \text{ and } \left|\frac{1}{b_n - a_n}\sum_{m=a_n+1}^{b_n} \Delta_{p,q}^{[r]}(T_m(f; x)) - f(x)\right| \geq r\right\}$$

and for $i = 0, 1, 2,$

$$\Psi_{i,m}(x; r) = \left\{m : a_n < m \leq b_n \text{ and } \left|\frac{1}{b_n - a_n}\sum_{m=a_n+1}^{b_n} \Delta_{p,q}^{[r]}(T_m(f_i; x)) - f_i(x)\right| \geq \frac{r - \epsilon}{3B}\right\},$$

we obtain

$$\Psi_m(x;r) \leq \sum_{i=0}^{2} \Psi_{i,m}(x;r).$$

Clearly,

$$\frac{\|\Psi_m(x;r)\|_{\mathcal{C}(X)}}{b_n - a_n} \leq \sum_{i=0}^{2} \frac{\|\Psi_{i,m}(x;r)\|_{\mathcal{C}(X)}}{b_n - a_n}. \tag{16}$$

Now, using the above assumption about the implications in (4)–(6) and by Definition 1, the right-hand side of (16) is seen to tend to zero as $n \to \infty$. Consequently, we get

$$\lim_{n\to\infty} \frac{\|\Psi_m(x;r)\|_{\mathcal{C}(X)}}{b_n - a_n} = 0 \ \ (r > 0).$$

Therefore, the implication (3) holds true.

This completes the proof of Theorem 1. □

Corollary 1 *Let $L_m : \mathcal{C}[0,\infty) \to \mathcal{C}[0,\infty)$ be a sequence of positive linear operators, and let $f \in \mathcal{C}[0,\infty)$. Then,*

$$stat_{DA}^{p,q} \lim_{m\to\infty} \|L_m(f;x) - f(x)\|_\infty = 0 \tag{17}$$

if and only if

$$stat_{DA}^{p,q} \lim_{m\to\infty} \|L_m(1;x) - 1\|_\infty = 0, \tag{18}$$

$$stat_{DA}^{p,q} \lim_{m\to\infty} \|L_m(e^{-s};x) - e^{-x}\|_\infty = 0 \tag{19}$$

and

$$stat_{DA}^{p,q} \lim_{m\to\infty} \|L_m(e^{-2s};x) - e^{-2x}\|_\infty = 0. \tag{20}$$

Proof By taking $a_n = n - 1, \ \forall\, n$ and, $b_n = n + k - 1, \ \forall\, n$ and proceeding in the similar line of Theorem 1, the proof of Corollary 1 is established. □

Remark 4 By taking $p_n = q_n = 1 \ \forall\, n$ in Theorem 1, one can obtain the statistical deferred Cesàro summability version of Korovkin-type approximation for the set of functions $1, \ e^{-x}$, and e^{-2x} established by Jena et al. [15].

Now we present below an illustrative example for the sequence of positive linear operators that does not satisfy the conditions of the Korovkin approximation theorems due to Jena et al. [15], Mohiuddine et al. [20], and Boyanov and Veselinov [8] but satisfies the conditions of our Theorem 1. Thus, our theorem is stronger than the results established by Jena et al. [15], Mohiuddine et al. [20] and Boyanov and Veselinov [8].

Example 3 Let $X = [0, 1]$ and consider the (p, q)-analogue of Bernstein operators $\mathcal{B}_{n,p,q}(f; x)$ on $\mathcal{C}[0, 1]$ given by (see [23])

$$\mathcal{B}_{n,p,q}(f; x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right) \quad (x \in [0, 1]).$$

Also, observe that

$$\mathcal{B}_{n,p,q}(f_0; x) = 1, \quad \mathcal{B}_{n,p,q}(f_1; x) = e^{-x} \text{ and } \mathcal{B}_{n,p,q}(f_2; x) = \frac{p^{n-1}}{[n]_{p,q}} e^{-x} + \frac{q[n-1]_{p,q}}{[n]_{p,q}} e^{-2x}.$$

Let us consider $L_n : \mathcal{C}[0, 1] \to \mathcal{C}[0, 1]$ be sequence of positive linear operators defined as follows:

$$L_n(f; x) = [1 + f_n(x)]x(1 + xD)\mathcal{B}_{n,p,q}(f; x) \quad (f \in \mathcal{C}[0, 1]), \tag{21}$$

where the operator given by

$$x(1 + xD) \quad \left(D = \frac{d}{dx}\right)$$

was used earlier by Al-Salam [4] and, more recently, by Viskov and Srivastava [30] (see also the monograph by Srivastava and Manocha [25] for various general families of operators of this kind). If we choose the sequence $f_n(x)$ of functions just as we considered in Example 2, then we have

$$\begin{aligned} L_n(f_0; x) &= [1 + f_n(x)]x(1 + xD) \cdot \mathcal{B}_{n,p,q}(f_0; x) \\ &= [1 + f_n(x)]x(1 + xD) \cdot 1 = [1 + f_n(x)]x, \end{aligned}$$

$$\begin{aligned} L_n(f_1; x) &= [1 + f_n(x)]x(1 + xD) \cdot \mathcal{B}_{n,p,q}(f_1; x) \\ &= [1 + f_n(x)]x(1 + xD) \cdot e^{-x} = [1 + f_n(x)]x(e^{-x} - xe^{-x}), \end{aligned}$$

and

$$\begin{aligned} L_n(f_2; x) &= [1 + f_n(x)]x(1 + xD) \cdot \mathcal{B}_{n,p,q}(f_2; x) \\ &= [1 + f_n(x)]x(1 + xD) \cdot \left\{ \frac{p^{n-1}}{[n]_{p,q}} e^{-x} + \frac{q[n-1]_{p,q}}{[n]_{p,q}} e^{-2x} \right\} \\ &= [1 + f_n(x)]x \left[\frac{p^{n-1}}{[n]_{p,q}} e^{-x} + \frac{q[n-1]_{p,q}}{[n]_{p,q}} e^{-2x} - xe^{-x} \frac{p^{n-1}}{[n]_{p,q}} - 2e^{-2x} \frac{q(n-1)_{p,q}}{[n]_{p,q}} \right]. \end{aligned}$$

So that, we obtain

$$\text{stat}_{DC}^{p,q} \lim_{m\to\infty} \|L_m(1;x) - 1\|_\infty = 0,$$
$$\text{stat}_{DC}^{p,q} \lim_{m\to\infty} \|L_m(e^{-s};x) - e^{-x}\|_\infty = 0$$

and

$$\text{stat}_{DC}^{p,q} \lim_{m\to\infty} \|L_m(e^{-2s};x) - e^{-2x}\|_\infty = 0,$$

that is, the sequence $L_m(f;x)$ satisfies the conditions (4)–(6). Therefore, by Theorem 1, we have

$$\text{stat}_{DC}^{p,q} \lim_{m\to\infty} \|L_m(f;x) - f\|_\infty = 0.$$

Hence, it is statistically deferred Cesàro summable under (p,q)-integers; however, since (x_m) is neither statistically Cesàro summable nor statistically deferred Cesàro summable, so we conclude that earlier works under [15, 20] is not valid for the operators defined by (21), while our Theorem 1 still works.

3 Rate of Statistical Deferred Cesàro Summability

In this section, we study the rates of statistical deferred Cesàro summability based on (p,q)-integers of a sequence of positive linear operators $L(f;x)$ defined on $\mathcal{C}[0,\infty)$ with the help of modulus of continuity.

We now presenting the following definition.

Definition 3 Let $0 < q_n < p_n \le 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \le 1)$, and let r is a nonnegative integer. Also, let (a_n) and (b_n) be sequences of integers (nonnegative). Let (u_n) be a positive non-increasing sequence. A given sequence $x = (x_m)$ is statistically deferred Cesàro summable to a number ℓ with rate $o(u_n)$, if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{u_n(b_n - a_n)} \left|\left\{m : a_n < m \le b_n \text{ and } |D_{p,q}(x_m) - \ell| \ge \epsilon\right\}\right| = 0.$$

In this case, we may write

$$x_m - \ell = stat_{DC}^{p,q} - o(u_n).$$

We now prove the following basic lemma.

Lemma 1 *Let (u_n) and (v_n) be two positive non-increasing sequences. Let $x = (x_m)$ and $y = (y_m)$ be two sequences such that*

$$x_m - L_1 = stat_{DC}^{p,q} - o(u_n)$$

and

$$y_m - L_2 = stat_{DC}^{p,q} - o(v_n)$$

respectively. Then, the following conditions hold true

(i) $(x_m + y_m) - (\ell_1 + \ell_2) = stat_{DC}^{p,q} - o(w_n)$;
(ii) $(x_m - \ell_1)(y_m - \ell_2) = stat_{DC}^{p,q} - o(u_n v_n)$;
(iii) $\lambda(x_m - \ell_1) = stat_{DC}^{p,q} - o(u_n)$ *(for any scalar* λ*)*;
(iv) $\sqrt{|x_m - \ell_1|} = stat_{DC}^{p,q} - o(u_n)$,
where

$$w_n = \max\{u_n, v_n\}.$$

Proof In order to prove the condition (i), for $\epsilon > 0$ and $x \in [0, \infty)$, we define the following sets:

$$\mathcal{A}_n(x; \epsilon) = \left|\{m : a_n < m \leq b_n \text{ and } |D_{p,q}(x_m) + D_{p,q}(y_m) - (\ell_1 + \ell_2)| \geq \epsilon\}\right|,$$
$$\mathcal{A}_{0,n}(x; \epsilon) = \left|\left\{m : a_n < m \leq b_n \text{ and } |D_{p,q}(x_m) - \ell_1| \geq \frac{\epsilon}{2}\right\}\right|,$$

and

$$\mathcal{A}_{1,n}(x; \epsilon) = \left|\left\{m : a_n < m \leq b_n \text{ and } |D_{p,q}(y_m) - \ell_2| \geq \frac{\epsilon}{2}\right\}\right|.$$

Clearly, we have

$$\mathcal{A}_n(x; \epsilon) \subseteq \mathcal{A}_{0,n}(x; \epsilon) \cup \mathcal{A}_{1,n}(x; \epsilon).$$

Moreover, since

$$w_n = \max\{u_n, v_n\},$$

by condition (3) of Theorem 1, we obtain

$$\frac{\|\mathcal{A}_m(x; \epsilon)\|_\infty}{w_n(b_n - a_n)} \leq \frac{\|\mathcal{A}_{0,n}(x; \epsilon)\|_\infty}{u_n(b_n - a_n)} + \frac{\|\mathcal{A}_{1,n}(x; \epsilon)\|_\infty}{v_n(b_n - a_n)}. \tag{22}$$

Now, by conditions (4)–(6) of Theorem 1, we obtain

$$\frac{\|\mathcal{A}_n(x; \epsilon)\|_\infty}{w_n(b_n - a_n)} = 0, \tag{23}$$

which establishes (i). Since the proofs of other conditions (ii)–(iv) are similar, we omit them. □

Further, we recall that the modulus of continuity of a function $f \in \mathcal{C}[0, \infty)$ is defined by

$$\omega(f, \delta) = \sup_{|y-x|\leq\delta: x, y\in X} |f(y) - f(x)| \quad (\delta > 0)$$

which implies that

$$|f(y) - f(x)| \leq \omega(f, \delta)\left(\frac{|x - y|}{\delta} + 1\right). \tag{24}$$

Now, we state and prove a result in the form of the following theorem.

Theorem 2 *Let* $[0, \infty) \subset \mathbb{R}$*, and let* $L_m : \mathcal{C}[0, \infty) \to \mathcal{C}[0, \infty)$ *be a sequence of positive linear operators. Assume that the following conditions hold true:*

(i) $\|L_m(1; x) - 1\|_\infty = stat_{DC}^{p,q} - o(u_n)$,
(ii) $\omega(f, \lambda_m) = stat_{DC}^{p,q} - o(v_n)$,

where

$$\lambda_m = \sqrt{L_m(\varphi^2; x)} \text{ and } \varphi_1(y, x) = (e^{-y} - x^{-x})^2.$$

Then, for all $f \in \mathcal{C}[0, \infty)$*, the following statement holds true:*

$$\|L_m(f; x) - f\|_\infty = stat_{DC}^{p,q} - o(w_n), \tag{25}$$

$w_n = \max\{u_n, v_n\}$.

Proof Let $f \in \mathcal{C}[0, \infty)$ and $x \in [0, \infty)$. Using (24), we have

$$\begin{aligned}
|L_m(f; x) - f(x)| &\leq L_m(|f(y) - f(x)|; x) + |f(x)||L_m(1; x) - 1| \\
&\leq L_m\left(\frac{|e^{-x} - e^{-y}|}{\lambda_m} + 1; x\right)\omega(f, \lambda_m) + |f(x)||L_m(1; x) - 1| \\
&\leq L_m\left(1 + \frac{1}{\lambda_m^2}(e^{-x} - e^{-y})^2; x\right)\omega(f, \lambda_m) + |f(x)||L_m(1; x) - 1| \\
&\leq \left(L_m(1; x) + \frac{1}{\lambda_m^2}L_m(\varphi_x; x)\right)\omega(f, \lambda_m) + |f(x)||L_m(1; x) - 1|.
\end{aligned}$$

Putting $\lambda_m = \sqrt{L_m(\varphi^2; x)}$, we get

$$\begin{aligned}
\|L_m(f; x) - f(x)\|_\infty &\leq 2\omega(f, \lambda_m) + \omega(f, \lambda_m)\|L_m(1; x) - 1\|_\infty + \|f(x)\|\|L_m(1; x) - 1\|_\infty \\
&\leq \mathcal{M}\{\omega(f, \lambda_m) + \omega(f, \lambda_m)\|L_m(1; x) - 1\|_\infty + \|L_m(1; x) - 1\|_\infty\},
\end{aligned}$$

where

$$\mathcal{M} = \{\|f\|_\infty, 2\}.$$

Thus,

$$\left\| \frac{1}{b_n - a_n} \sum_{m=a_n+1}^{b_n} L_m(f;x) - f(x) \right\|_\infty \\ \leq \mathcal{M} \left\{ \omega(f,\lambda_m) \frac{1}{b_n - a_n} + \omega(f,\lambda_m) \left\| \frac{1}{b_n - a_n} \sum_{m=a_n+1}^{b_n} L_m(f;x) - f(x) \right\|_\infty \right\} \\ + \mathcal{M} \left\{ \left\| \frac{1}{b_n - a_n} \sum_{m=a_n+1}^{b_n} L_m(f;x) - f(x) \right\|_\infty \right\}.$$

Now, by using the conditions (i) and (ii) of Theorem 2, in conjunction with Lemma 1, we arrive at the statement (25) of Theorem 2.

This completes the proof of Theorem 2. □

4 Concluding Remarks

In this concluding section of our investigation, we present several further remarks and observations concerning to various results which we have proved here.

Remark 5 Let $(x_m)_{m\in\mathbb{N}}$ be a sequence given in Example 3. Then, since

$$\text{stat}_{DC}^{p,q} - \lim_{m\to\infty} x_m \to 0 \text{ on } [0,\infty),$$

we have

$$\text{stat}_{DC}^{p,q} - \lim_{m\to\infty} \|L_m(f_i;x) - f_i(x)\|_\infty = 0 \quad (i = 0,1,2). \tag{26}$$

Thus, we can write (by Theorem 1)

$$\text{stat}_{DC}^{p,q} - \lim_{m\to\infty} \|L_m(f;x) - f(x)\|_\infty = 0, \ (i = 0,1,2), \tag{27}$$

where

$$f_0(x) = 1, \quad f_1(x) = e^{-x} \text{ and } f_2(x) = e^{-2x}.$$

However, since (x_m) is not ordinarily convergent, and so also it does not converge uniformly in the ordinary sense. Thus, the classical Korovkin theorem does not work here for the operators defined by (21). Hence, this application clearly indicates that our Theorem 1 is a non-trivial generalization of the classical Korovkin-type theorem (see [19]).

Remark 6 Let $(x_m)_{m\in\mathbb{N}}$ be a sequence as given in Example 3. Then, since

$$\text{stat}_{DC}^{p,q} - \lim_{m\to\infty} x_m \to 0 \text{ on } [0,\infty),$$

so (26) holds true. Now by applying (26) and Theorem 1, condition (27) holds true. However, since (x_m) does not statistical Cesàro summable, so Theorem 2.1 of Jena et al. (see [15]) does not work for our operator defined in (21). Thus, our Theorem 1 is also a non-trivial extension of Theorem 2.1 of Jena et al. [15] (see also [8, 19]). Based on the above results, it is concluded here that our proposed method has successfully worked for the operators defined in (21) and therefore it is stronger than the classical and statistical version of the Korovkin-type approximation (see [8, 19, 20]) established earlier.

Remark 7 Let us suppose that we replace the conditions (i) and (ii) in Theorem 2, by the following condition:

$$|L_m(f_i;x) - f_i| = DC_1(\text{stat}) - o(u_{n_i}) \quad (i = 0,1,2). \tag{28}$$

Then, since

$$L_m(\varphi^2;x) = e^{-2x}|L_m(1;x) - 1| - 2e^{-x}|L_m(e^{-x};x) - e^{-x}| + |L_m(e^{-2x};x) - e^{-2x}|,$$

we can write

$$L_m(\varphi^2;x) \leq M\sum_{i=0}^{2}|L_m(f_i;x) - f_i(x)|_\infty, \tag{29}$$

where

$$M = \{\|f_2\|_\infty + 2\|f_1\|_\infty + 1\}.$$

Now it follows from (28), (29) and Lemma 1 that

$$\lambda_m = \sqrt{L_m(\varphi^2)} = DC_1(\text{stat}) - o(d_n), \tag{30}$$

where

$$o(d_n) = \max\{u_{n_0}, u_{n_1}, u_{n_2}\}.$$

This implies

$$\omega(f,\delta) = DC_1(\text{stat}) - o(d_n).$$

Now using (30) in Theorem 2, we immediately see that for $f \in \mathcal{C}[0,\infty)$,

$$L_m(f;x) - f(x) = DC_1(\text{stat}) - o(d_n). \tag{31}$$

Therefore, if we use the condition (28) in Theorem 2 instead of (i) and (ii), then we obtain the rates of statistical deferred Cesàro summability of the sequence of positive linear operators in Theorem 1.

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Approximation Results for Urysohn-Type Nonlinear Bernstein Operators

Harun Karsli

Abstract In the present work, our aim of this study is generalization and extension of the theory of interpolation of functions to functionals or operators by means of Urysohn-type nonlinear operators. In accordance with this purpose, we introduce and study a new type of Urysohn-type nonlinear operators. In particular, we investigate the convergence problem for nonlinear operators that approximate the Urysohn-type operator. The starting point of this study is motivated by the important applications that approximation properties of certain families of nonlinear operators have in signal–image reconstruction and in other related fields. We construct our nonlinear operators by using a nonlinear forms of the kernels together with the Urysohn-type operator values instead of the sampling values of the function. As far as we know, this will be first use of such kind of operators in the theory of interpolation and approximation. Hence, the present study is a generalization and extension of some previous results.

Keywords Urysohn integral operators · Nonlinear Bernstein operators · Urysohn-type nonlinear Bernstein operators · Approximation.

AMS Subject Classification 41A25 · 41A35 · 47G10 · 47H30.

1 Introduction

For a function defined on the interval [0, 1], the Bernstein operators $(B_n f)$, $n \geq 1$, are defined by

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x) \quad , \quad n \geq 1, \tag{1}$$

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where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the well-known Binomial distribution and called Bernstein basis $(0 \leq x \leq 1)$. These polynomials were introduced by Bernstein [1] in 1912 to give the first constructive proof of the Weierstrass approximation theorem.

The first main approximation result related to pointwise convergence of the Bernstein polynomials reads; let f be a bounded function on [0, 1], then

$$\lim_{n\to\infty} (B_n f)(x) = f(x)$$

holds at each point of continuity x of $f(x)$ and that the relation holds uniformly on [0, 1] if $f(x)$ is uniformly continuous on the interval.

Undoubtedly that the most intensively studied discrete operator is the celebrated Bernstein polynomial, which provides an elegant proof and example to the famous Weierstrass first approximation theorem for continuous function defined on [0, 1]. For detailed approach to this operator, see the classical book of Lorentz [2].

It is worthwhile to note that the linear positive operators have been obtained by starting from the following well-known properties of the probability density functions; for discrete case

$$\sum_{k=0}^{n} p_{n,k}(x) = 1$$

and for continuous case

$$\int_a^b f(t)dt = 1$$

from the probability theory.

Now, in view of the theory of singular integrals, we will characterize the positive linear operators in terms of the singular integrals.

In general, a singular integral may be written in the form

$$(T_n f)(x) = \int_a^b f(t) K_n(x,t) dt \tag{2}$$

where $K_n(x,t)$ is the kernel, defined for $a \leq x \leq b, a \leq t \leq b$, which has the property that for functions f of a certain class and in a certain sense, $(T_n f)(x)$ converges to $f(x)$ as $n \to \infty$.

The Bernstein polynomial (1) is a finite sum of a type corresponding to the integral (2). It is easy to see that (1) and (2) are special cases of singular Stieltjes integrals and hence (1) may be written in the form of a Stieltjes integral in the variable t as follows:

$$(B_n f)(x) = \int_0^1 f(t)\, d_t K_n(x, t)$$

with the kernel

$$K_n(x,t) = \sum_{k \le nt} \binom{n}{k} x^k (1-x)^{n-k} \quad , \quad 0 < t \le 1$$
$$K_n(x,0) = 0$$

which is constant in any interval $k/n \le t < (k+1)/n$, $k = 0, 1, \ldots, n-1$.

At the beginning, the theory of approximation is strongly related to the linearity of the operators. But, thanks to the approach of the famous Polish mathematician Julian Musielak, see [3], and afterwards continuous works of C. Bardaro, G. Vinti and their research group, this theory can be extended to the nonlinear-type operators, under some specific assumptions on its kernel functions; see the fundamental book due to Bardaro, Musielak and Vinti [4]. For further reading, please see [5–11] as well as the monographs [12].

Especially, nonlinear integral operators of type

$$(T_\lambda f)(x) = \int_a^b K_\lambda(t - x, f(t))\, dt, \qquad x \in (a, b),$$

and its special cases were studied by Bardaro, Karsli and Vinti [13, 14] and Karsli [15, 16] in some functional spaces.

In view of the approach due to Musielak [3], recently, Karsli–Tiryaki and Altin [17] introduced the following type nonlinear counterpart of the well-known Bernstein operators:

$$(NB_n f)(x) = \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) \quad , \quad 0 \le x \le 1 \quad , \quad n \in \mathbb{N}, \tag{3}$$

acting on bounded functions f on an interval $[0, 1]$, where $P_{n,k}$ satisfy some suitable assumptions. They proved some existence and approximation theorems for the nonlinear Bernstein operators. In particular, they obtain some pointwise convergence for the nonlinear sequence of Bernstein operators (3) to some discontinuity point of the first kind of f, as $n \to \infty$.

As a continuation of the very recent paper of the author [18], the author and his PhD student estimated a Voronovskaya-type formula for this class nonlinear Bernstein operators on the interval [0, 1] (see [19]). Please see also very recent papers of the author [20, 21].

The most important and frequently investigated integral equations in nonlinear functional analysis are the Hammerstein equations

$$x(t) = y(t) + \int_a^b k(t,s)f(s,x(s))ds, \qquad t \in [a,b],$$

and the Urysohn equations

$$x(t) = y(t) + \int_a^b k(t,s,x(s))ds, \qquad t \in [a,b].$$

Consider the nonlinear operator equation

$$x = y + K(x)$$

where K is a completely continuous operator defined on a Banach space. An example of such an operator K is the Urysohn integral operator with a kernel function

$$Kx(t) = \int_\Omega k(t,s,x(s))ds, \qquad t \in \Omega,\ x \in D$$

with a closed bounded region Ω in $\mathbb{R}^m$ for some $m \geq 1$, which includes the Fredholm equations of the first and second kind.

In the present work, we will deal with the following Urysohn equation:

$$x(t) = y(t) + \int_a^b k(t,s,x(s))ds, \qquad t \in [a,b]$$

and corresponding Urysohn operator

$$Ux(t) = \int_a^b k(t,s,x(s))ds, \qquad t \in [a,b]$$

where k and y are known functions and x is the unknown function to be determined. In the above equation, $k(t,s,x)$ is called kernel function of the type of Green's function, which is defined on $[a,b] \times [a,b] \times \mathbb{R}$ into $\mathbb{R}$.

The goal of this study is generalization and extension of the theory of interpolation of functions to functionals and operators by introducing the Urysohn-type nonlinear counterpart of the Bernstein operators. Afterwards, we investigate the convergence problem for these nonlinear operators that approximate the Urysohn-type operator in some functional spaces. The main difference between the present work and convergence to a function lies in the use of the Urysohn type operator values instead of the sampling values of a function.

Let us consider a sequence $NBF = (NB_nF)$ of operators, which we call it Urysohn-type nonlinear counterpart of the Bernstein operators, having the form

$$(NB_nF)x\,(t) = \int_0^1 \left[\sum_{k=0}^{n} P_{k,n}\left(x(s), f\left(t, s, \frac{k}{n}\right)\right)\right] ds \ , \ \ 0 \le x(s) \le 1 \ \ , \quad n \in \mathbb{N},$$

acting on bounded functions f on an interval $[0, 1]$, where $P_{k,n}$ satisfy some suitable assumptions. In particular, we will put $Dom\ NBF = \bigcap_{n\in\mathbb{N}} Dom\ NB_nF$, where $Dom\ NB_nF$ is the set of all functions $f : [0, 1] \to \mathbb{R}$ for which the operator is well defined.

Since the theory of approximation is quite different from its linear counterpart, in same cases we can obtain only some estimates related to the convergence problems. Actually, in some cases, it is not possible to obtain exact estimates for nonlinear operators, because of the nonlinearity of their kernel functions.

2 Preliminaries and Auxiliary Results

This section is devoted to collecting some definitions and results which will be needed further on.

Here we consider the following type Urysohn integral operator,

$$Fx(t) = \int_0^1 f(t, s, x(s))ds, \qquad t \in [0, 1] \tag{4}$$

with unknown kernel f. If such a representation exists, then the kernel function $f(t, s, x)$ is called the Green's function, which is strongly related to the function x.

For a constant function $x(s) = a$, we set $Fa(t) := F(a)$.

Equation (4) was investigated by Urysohn in 1923–1924 in [22, 23]. This kind of equations appears in many problems. For example, it occurs in solving problems arising in economics, mathematics, engineering and physics (see [12, 24]).

It is well known that the solution of the following differential equation

$$DG(x, y) = \delta(x - y),$$

represents a Green function $G(x, y)$; here D is a differential operator, δ is the Dirac Delta function satisfying a boundary condition. Note that

$$\delta(x) = \frac{dH(x)}{dx},$$

is true, where

$$H(x)=\begin{cases}1\ ,\ x\geq 0\\ 0\ ,\ x<0\end{cases}$$

is the Heaviside function.

In view of these relations, we assume that continuous interpolation conditions hold:

$$Fx_i(t)=\int\limits_0^1 f(t,s,x_i(s))ds, \qquad t\in[0,1] \tag{5}$$

where $x_i(s)=\frac{i}{n}H(s-\xi)$; $\xi\in[0;1]$; and $i=0,1,2,\ldots$.

The following equalities are well known:

$$\frac{\partial Fx_i(t)}{\partial s}=\frac{\partial F\left(\frac{i}{n}H(t-s)\right)}{\partial s}=f(t,s,0)-f(t,s,\frac{i}{n}), \tag{6}$$

where $x_i(t)=\frac{i}{n}H(t-s)$; $s\in[0;1]$, and $i=0,1,2,\ldots$.

Say

$$F_1\left(t,s,\frac{i}{n}\right):=\frac{\partial Fx_i(t)}{\partial s}. \tag{7}$$

According to the above definition together with (6) and (7), it is possible to construct an approximation operator in order to generalization and extension of the theory of interpolation of functions to operators.

In 2000, Demkiv [25] defined and investigated some properties of the following type Bernstein operators, which is linear with respect to F defined by (5):

$$(B_nF)\,x\,(t)=\int\limits_0^1\sum_{k=0}^{n}f\left(t,s,\frac{k}{n}\right)p_{n,k}\,(x(s))\,ds$$

In 2012, Makarov and Demkiv [26] considered the problem of approximation to the Urysohn operator (4) by Stancu-type operators, which is based on Polya distribution $p^{\alpha}_{n,k}\,(x(s))$, defined as:

$$\left(P_n^{\alpha}F\right)x\,(t)=\int\limits_0^1\sum_{k=0}^{n}f\left(t,s,\frac{k}{n}\right)p^{\alpha}_{n,k}\,(x(s))\,ds,$$

where $\alpha\geq 0$.

In 2017, the author [21] defined the following Urysohn type Meyer-K önig and Zeller operators:

$$(M_n F)x(t) = \int_0^1 \left[\sum_{k=0}^{\infty} f\left(t, s, \frac{k}{k+n}\right) m_{n,k}\left(x(s)\right) \right] ds,$$
$$(M_n F)1(t) = F1(t) = F(1),$$

where
$$m_{n,k}\left(x(s)\right) = \binom{n+k-1}{k} (x(s))^k (1 - x(s))^n,$$

n is a non-negative integer and $0 \leq x(s) < 1$, and obtained some positive results about the convergence problem.

In view of (3) and (5), we introduce the following Urysohn-type nonlinear Bernstein operators:

$$(NB_n F)x(t) = \int_0^1 \left[\sum_{k=0}^{n} P_{k,n}\left(x(s), f\left(t, s, \frac{k}{n}\right)\right) \right] ds, \tag{8}$$

where n is a non-negative integer, $P_{k,n}$ satisfy some suitable assumptions and $0 \leq x(s) \leq 1$.

Now, we assemble the main definitions and notations which will be used throughout the paper.

Let X be the set of all bounded Lebesgue measurable functions $f : [0, 1] \to \mathbb{R}_0^+ = [0, \infty)$.

Let Ψ be the class of all functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that the function ψ is continuous and concave with $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$.

We now introduce a sequence of functions. Let $\left\{P_{k,n}\right\}_{n\in\mathbb{N}}$ be a sequence functions $P_{k,n} : [0, 1]$ x $\mathbb{R} \to \mathbb{R}$ defined by

$$P_{k,n}(t, u) = p_{k,n}(t) H_n(u) \tag{9}$$

for every $t \in [0, 1]$, $u \in \mathbb{R}$, where $H_n : \mathbb{R} \to \mathbb{R}$ is such that $H_n(0) = 0$ and $p_{k,n}(t)$ is the Bernstein basis.

Throughout the paper, we assume that $\mu : \mathbb{N} \to \mathbb{R}^+$ is an increasing and continuous function such that $\lim_{n\to\infty} \mu(n) = \infty$.

First of all, we assume that the following conditions hold:

(a) $H_n : \mathbb{R} \to \mathbb{R}$ *is such that*

$$\left|H_n(u) - H_n(v)\right| \leq |u - v|,$$

holds for every $u, v \in \mathbb{R}$, *for every* $n \in \mathbb{N}$. *That is,* H_n *satisfies a strong Lipschitz condition.*

(b) Denoting by $r_n(u) := H_n(u) - u$, $u \in \mathbb{R}$ *and* $n \in \mathbb{N}$, *such that*

$$\lim_{n\to\infty} r_n(u) = 0$$

uniformly with respect to u.

In other words, for n *sufficiently large*

$$\sup_u |r_n(u)| = \sup_u |H_n(u) - u| \leq \frac{1}{\mu(n)},$$

holds.

The symbol $[a]$ will denote the greatest integer not greater than a .

At first we recall the following results.

Lemma 1 *For* $(B_n t^s)(x)$, $s = 0, 1, 2,$ *one has*

$$(B_n 1)(x) = 1$$
$$(B_n t)(x) = x$$
$$(B_n t^2)(x) = x^2 + \frac{x(1-x)}{n}.$$

For proof of this Lemma, see [2].

By direct calculation, we find the following equalities:

$$(B_n\,(t-x)^2)(x) = \frac{x(1-x)}{n} \quad , \quad (B_n\,(t-x))(x) = 0\,.$$

Lemma 2 *For the central moments of order* $m \in N_0$

$$T_{n,m}(x) := \sum_{k=0}^{n} (k - nx)^m \; p_{k,n}\,(x)\,.$$

One has for each $m = 0, 1, \ldots$ *there is a constant* A_m *such that*

$$0 \leq T_{n,2m}(x) \leq A_m n^m.$$

The presented inequality is the well-known bound for the moments of the Bernstein polynomials, and it can be found in Chap. 10 in [27].

Lemma 3 *The first-order absolute moment for Bernstein polynomial is defined as*

$$M_1\left(p_{n,k}, x(s)\right) = \sum_{k=0}^{n} \left|\frac{k}{n} - x(s)\right| p_{n,k}\,(x(s))$$

and satisfies the following inequality

$$M_1\left(p_{n,k}, x(s)\right) \leq \left(\frac{2x(s)\left(1-x(s)\right)}{\pi}\right)^{\frac{1}{2}} \frac{1}{\sqrt{n}} + \frac{B}{n} \frac{1}{\sqrt{x(s)\left(1-x(s)\right)}},$$

where

$$B = \left(\frac{\pi}{2}\right)^{\frac{5}{2}} + \frac{4}{\pi} + \frac{\pi^{\frac{9}{2}}}{54\sqrt{2}} o\left(\frac{1}{\sqrt{n}}\right),$$

which can be found in [28]. Note that the above inequality can be written as

$$M_1\left(p_{n,k}, x(s)\right) \leq \left(\frac{2x(s)\left(1-x(s)\right)}{\pi}\right)^{\frac{1}{2}} \frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right).$$

Remark 1 By (6), (NB_nF) satisfies the following inequality:

$$|(NB_nF)x\,(t)| \leq |F(0)| + \left| \int_0^1 \sum_{k=0}^{n} p_{k,n}\left(x(s)\right) \frac{\partial F\left(\frac{k}{n} H(t-s)\right)}{\partial s} ds \right|.$$

We can prove the above equality as follows:

$$\begin{aligned}
(NB_nF)x\,(t) &= \int_0^1 \left[\sum_{k=0}^{n} P_{k,n}\left(x(s), f\left(t, s, \frac{k}{n}\right)\right)\right] ds \\
&= \int_0^1 \sum_{k=0}^{n} p_{k,n}\left(x(s)\right) H_n\left(f\left(t, s, \frac{k}{n}\right)\right) ds \\
&\leq \int_0^1 \sum_{k=0}^{n} p_{k,n}\left(x(s)\right) \left| f\left(t, s, \frac{k}{n}\right)\right| ds \\
&= \int_0^1 \sum_{k=0}^{n} p_{k,n}\left(x(s)\right) \left| f(t, s, 0) - \frac{\partial F\left(\frac{k}{n} H(t-s)\right)}{\partial s}\right| ds \\
&\leq |F(0)| + \left| \int_0^1 \sum_{k=0}^{n} p_{k,n}\left(x(s)\right) \frac{\partial F\left(\frac{k}{n} H(t-s)\right)}{\partial s} ds \right|.
\end{aligned}$$

3 Convergence Property

We now introduce some notations and structural hypotheses, which will be fundamental in proving our convergence theorems.

Here, as usual, we denote by $C[0,1]$ the Banach space of continuous functions $u:[0,1]\to R$ with norm

$$\|u\| = \sup\{|u(x)| : x \in [0,1]\}$$

Let Ψ be the set of all continuous, concave and non-decreasing functions
$\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with

$$\varphi(0) = 0,$$
$$\varphi(u) > 0 \text{ for all } u > 0$$

and

$$\lim_{u\to+\infty} \varphi(u) = +\infty$$

in the usual sense. Such function is called a φ-function.

Assume that the following condition holds:

$f : [0,1]\times[0,1]\times[0,1] \to \mathbb{R}$ is such that

$$|f(t,s,u) - f(t,s,v)| \leq \psi(|u-v|), \quad \psi \in \Psi,$$

holds for every $u, v \in [0,1]$. That is, f satisfies a $(L-\Psi)$ Lipschitz condition with respect to the third variable.

Let $f \in C\left([a,b]^3\right)$ and $\delta > 0$ be given. Then the modulus of continuity is given by:

$$\omega(f;\delta) = \omega(\delta) = \sup_{|u-v|\leq\delta\ ,\ t,s\in[a,b]} |f(t,s,u) - f(t,s,v)|. \tag{10}$$

Recall that $\omega(f;\delta)$ has the following properties:

(i) Let $\lambda \in \mathbb{R}^+$, then $\omega(f;\lambda\delta) \leq (\lambda+1)\,\omega(f;\delta)$,
(ii) $\lim_{\delta\to0^+} \omega(f;\delta) = 0$,
(iii) $|f(t) - f(x)| \leq \omega(|t-x|)$,
(iv) $|f(t) - f(x)| \leq \left(\frac{|t-x|}{\delta} + 1\right)\omega(\delta)$.

We mention that some additional properties and applications of this modulus of continuity given in [2] and some of its generalizations can be found in [4].

Definition 1 We will say that the sequence $(P_n)_{n\in\mathbb{N}}$ is $(\psi-\alpha)$−singular if the following assumptions are satisfied:

(P.1) For every $x \in I$ and $\delta > 0$, there holds

$$\psi\left(\sum_{\left|\frac{k}{n}-x\right|\geq\delta} \left|\frac{k}{n} - x\right| p_{n,k}(x)\right) = o\left(n^{-\alpha}\right), \quad (n\to\infty).$$

(P.2) For every $u \in \mathbb{R}$ and for every $x \in I$, we have

$$\lim_{n\to\infty} n^{\alpha}\left[\sum_{k=0}^{n} P_{n,k}(x,u) - u\right] = 0.$$

We are now ready to establish one of the main results of this study:

Theorem 1 *Let F be the Urysohn integral operator with $0 \le x(s) \le 1$. Then (NB_nF) converges to F uniformly in $x \in C[0,1]$. That is,*

$$\lim_{n\to\infty} \|(NB_nF)x\,(t) - Fx(t)\| = 0.$$

Proof In view of the definition of the operator (8), by considering (5), (9), (6) and (7), we have

$$|(NB_nF)x\,(t) - Fx(t)| = \left|\int_0^1 \left[\sum_{k=0}^{n} P_{k,n}\left(x(s), f\left(t,s,\frac{k}{n}\right)\right)\right] ds - Fx(t)\right|$$

$$\le \left|\int_0^1 \left[\sum_{k=0}^{n} p_{k,n}\,(x(s))\, H_n\left(f\left(t,s,\frac{k}{n}\right)\right)\right] ds - \int_0^1 \left[\sum_{k=0}^{n} p_{k,n}\,(x(s))\, H_n\left(f\,(t,s,x(s))\right)\right] ds\right|$$
$$+\left|\int_0^1 \left[\sum_{k=0}^{n} p_{k,n}\,(x(s))\, H_n\left(f\,(t,s,x(s))\right)\right] ds - \int_0^1 f(t,s,x(s))ds\right|$$

$$\le \left|\int_0^1 \sum_{k=0}^{n} p_{k,n}\,(x(s))\left[H_n\left(f\left(t,s,\frac{k}{n}\right)\right) - H_n\left(f\,(t,s,x(s))\right)\right] ds\right|$$
$$+\int_0^1 \sum_{k=0}^{n} p_{k,n}\,(x(s))\,|H_n\left(f\,(t,s,x(s))\right) - f\,(t,s,x(s))|\,ds$$
$$:\, = I_1 + I_2.$$

By assumption (b), the second term, namely I_2, tends to zero as $n \to \infty$. In fact,

$$
\begin{aligned}
I_2 &= \int_0^1 \sum_{k=0}^{n} p_{k,n}\left(x(s)\right) \left|H_n\left(f\left(t,s,x(s)\right)\right) - f\left(t,s,x(s)\right)\right| ds \\
&\leq \int_0^1 \sum_{k=0}^{n} p_{k,n}\left(x(s)\right) \frac{1}{\mu\left(n\right)} ds \\
&= \frac{1}{\mu\left(n\right)},
\end{aligned}
$$

which tends to zero as $n \to \infty$. Now, it is sufficient to evaluate the term I_1. Using the definition of the function $F_1\left(t, s, x(s)\right)$, we have

$$
\begin{aligned}
I_1 &\leq \int_0^1 \sum_{k=0}^{n} p_{k,n}\left(x(s)\right) \left| f\left(t, s, \frac{k}{n}\right) - f\left(t,s,x(s)\right)\right| ds \\
&= \int_0^1 \sum_{k=0}^{n} p_{k,n}\left(x(s)\right) \left| f(t,s,0) - f\left(t,s,x(s)\right) - \left[f(t,s,0) - f\left(t,s,\frac{k}{n}\right)\right]\right| ds \\
&= \int_0^1 \sum_{k=0}^{n} p_{k,n}\left(x(s)\right) \left| F_1\left(t,s,x(s)\right) - F_1\left(t,s,\frac{k}{n}\right)\right| ds.
\end{aligned}
$$

Let us divide the last term into two parts as:

$$
I_1 \leq I_{1,1} + I_{1,2},
$$

where

$$
I_{1,1} = \int_0^1 \sum_{\left|\frac{k}{n} - x(s)\right| < \delta} p_{n,k}\left(x(s)\right) \left| F_1\left(t,s,x(s)\right) - F_1\left(t,s,\frac{k}{n}\right)\right| ds
$$

and

$$
I_{1,2} = \int_0^1 \sum_{\left|\frac{k}{n} - x(s)\right| \geq \delta} p_{n,k}\left(x(s)\right) \left| F_1\left(t,s,x(s)\right) - F_1\left(t,s,\frac{k}{n}\right)\right| ds.
$$

Since $x \in C[0,1]$, then clearly

$$
\left| F_1\left(t,s,x(s)\right) - F_1\left(t,s,\frac{k}{n}\right)\right| < \epsilon
$$

holds true when $\left|\frac{k}{n} - x(s)\right| < \delta$, and

$$\begin{aligned}\left|F_1(t,s,x(s)) - F_1\left(t,s,\frac{k}{n}\right)\right| &\le |F_1(t,s,x(s))| + \left|F_1\left(t,s,\frac{k}{n}\right)\right| \\ &\le 2M\end{aligned}$$

holds true for some $M > 0$, when $\left|\frac{k}{n} - x(s)\right| \ge \delta$.

So

$$\begin{aligned}I_{1,1} &= \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|<\delta} p_{n,k}(x(s)) \left|F_1(t,s,x(s)) - F_1\left(t,s,\frac{k}{n}\right)\right| ds \\ &\le \epsilon \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|<\delta} p_{n,k}(x(s))\, ds \\ &\le \epsilon,\end{aligned}$$

and

$$\begin{aligned}I_{1,2} &= \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|\ge\delta} p_{n,k}(x(s)) \left|F_1(t,s,x(s)) - F_1\left(t,s,\frac{k}{n}\right)\right| ds \\ &\le 2M \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|\ge\delta} p_{n,k}(x(s))\, ds \\ &\le 2M \int_0^1 \left[\sum_{\left|\frac{k}{n}-x(s)\right|\ge\delta} \left(\frac{\frac{k}{n} - x(s)}{\delta}\right)^2 p_{n,k}(x(s))\right] ds \\ &= \frac{2M}{\delta^2} \int_0^1 \left[\sum_{\left|\frac{k}{n}-x(s)\right|\ge\delta} \left(\frac{k}{n} - x(s)\right)^2 p_{n,k}(x(s))\right] ds \\ &\le \frac{2M}{\delta^2} \int_0^1 \left[\sum_{k=0}^{n} \left(\frac{k}{n} - x(s)\right)^2 p_{n,k}(x(s))\right] ds.\end{aligned}$$

In view of Lemma 2, we obtain

$$I_{1,2} \le \frac{2M}{\delta^2}\frac{A_1}{n}.$$

Collecting these estimates, we have

$$|(NB_nF)x\,(t) - Fx(t)| \leq \epsilon + \frac{2MA_1}{n\delta^2} + \frac{1}{\mu\,(n)}.$$

That is,

$$\lim_{n\to\infty} \|(NB_nF)x\,(t) - Fx(t)\|_{C[0,1]} = 0.$$

This completes the proof.

Theorem 2 *Let F be the Urysohn integral operator with $x \in C[0,1]$, and $0 \leq x(s) \leq 1$. Then for every $\varepsilon > 0$*

$$|(NB_nF)x\,(t) - Fx(t)| \leq \psi\,(\varepsilon) + 2\omega\,(f;\delta) + \frac{1}{\mu\,(n)}$$

holds true, where $\delta = \sqrt{x(s)\,(1 - x(s))\,/n}$.

Proof Clearly, one has

$$\begin{aligned}|(NB_nF)x\,(t) - Fx(t)| &\leq \int_0^1 \left[\sum_{k=0}^{n} p_{k,n}\,(x(s)) \left| f\left(t, s, \frac{k}{n}\right) - f\,(t, s, x(s))\right|\right] ds + \frac{1}{\mu\,(n)} \\ &:= I_{n,1}\,(x) + \frac{1}{\mu\,(n)},\end{aligned} \tag{11}$$

say. Since $x \in C[0,1]$ we can rewrite (11) as follows:

$$\begin{aligned}I_{n,1}\,(x) &\leq \int_0^1 \sum_{\left|\frac{k}{n} - x(s)\right| < \delta} p_{n,k}\,(x(s))\,\psi\left(\left|x(s) - \frac{k}{n}\right|\right) ds \\ &\quad + \int_0^1 \sum_{\left|\frac{k}{n} - x(s)\right| \geq \delta} p_{n,k}\,(x(s)) \left| f\left(t, s, \frac{k}{n}\right) - f\,(t, s, x(s))\right| ds \\ &\leq \psi\,(\varepsilon) + I_{n,1,2}\,(x)\,.\end{aligned}$$

Taking into account that $\omega\,(f;\delta)$ is the modulus of continuity defined as (10), $I_{n,1,2}\,(x)$ can be written as

$$I_{n,1,2}\,(x) = \int_0^1 \sum_{\left|\frac{k}{n} - x(s)\right| \geq \delta} p_{n,k}\,(x(s)) \left| f\left(t, s, \frac{k}{n}\right) - f\,(t, s, x(s))\right| ds$$

$$\leq \int_0^1 \omega(f;\delta) \sum_{\left|\frac{k}{n}-x(s)\right|\geq\delta} \left(\frac{\left|\frac{k}{n}-x(s)\right|}{\delta} + 1 \right) ds$$

$$\leq \omega(f;\delta) \left\{ 1 + \delta^{-1} \int_0^1 \sum_{\left|\frac{k}{n}-x(s)\right|\geq\delta} \left| \frac{k}{n} - x(s) \right| p_{n,k}(x(s))\, ds \right\}$$

$$\leq \omega(f;\delta) \left\{ 1 + \delta^{-2} \int_0^1 \sum_{k=0}^{n} \left(\frac{k}{n} - x(s) \right)^2 p_{n,k}(x(s))\, ds \right\}$$

$$\leq \omega(f;\delta) \left\{ 1 + \frac{A_1}{\delta^2 n} \right\}.$$

If we choose

$$\delta = \sqrt{\frac{A_1}{n}},$$

then one can obtain the desired estimate, namely

$$|(M_n F)x(t) - Fx(t)| \leq \psi(\varepsilon) + 2\omega(f;\delta) + \frac{1}{\mu(n)}.$$

Thus, the proof is now complete.

Theorem 3 *Let F be the Urysohn integral operator with $x \in C[0,1]$, and $0 < x(s) < 1$. Then*

$$|(NB_n F)x(t) - Fx(t)| \leq \psi\left(\frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{2\pi}} + BA \right) \right) + \frac{1}{\mu(n)}$$

holds true for constants A and B, for which

$$\int_0^1 \frac{ds}{\sqrt{x(s)(1-x(s))}} = A < \infty,$$

and

$$B = \left(\frac{\pi}{2}\right)^{\frac{5}{2}} + \frac{4}{\pi} + \frac{\pi^{\frac{9}{2}}}{54\sqrt{2}}.$$

Proof Using the similar lines to the proof of Theorem 2, one has

$$|(NB_nF)x(t) - Fx(t)| \leq \int_0^1 \left[\sum_{k=0}^{n} p_{k,n}(x(s)) \left| F_1(t,s,x(s)) - F_1\left(t,s,\frac{k}{n}\right)\right|\right] ds + \frac{1}{\mu(n)}$$

$$\leq \int_0^1 \left[\sum_{k=0}^{n} p_{n,k}(x(s))\, \psi\left(\left|x(s) - \frac{k}{n}\right|\right)\right] ds + \frac{1}{\mu(n)}.$$

By concavity of the function ψ, and using Jensen's inequality, we obtain

$$|(NB_nF)x(t) - Fx(t)| \leq \psi\left(\int_0^1 \left[\sum_{k=0}^{n} \left|\left(\frac{k}{n} - x(s)\right)\right| p_{n,k}(x(s))\right] ds\right) + \frac{1}{\mu(n)}$$

Since ψ is non-decreasing, we apply the inequality of the first absolutely moment given in Remark 1; then we can write

$$|(NB_nF)x(t) - Fx(t)| \leq \psi\left(\int_0^1 \left[\left(\frac{2x(s)(1-x(s))}{\pi}\right)^{\frac{1}{2}} \frac{1}{\sqrt{n}} + \frac{B}{n}\frac{1}{\sqrt{x(s)(1-x(s))}}\right] ds\right)$$

$$\leq \psi\left(\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{2\pi}} + BA\right)\right) + \frac{1}{\mu(n)}.$$

So we get the desired estimate.

4 Practical Examples, Graphical Representations

In this section, we will apply the theory to the theory of interpolation of functions to functionals or operators by means of Urysohn-type nonlinear operators.

We note that in Figs. 1 and 2, the graph with the red line belongs to the original function, the graph with the green line to the operators with $n = 2$, and finally the graph consisting of blue line to the operators with $n = 10$.

Example 1 Let us consider the operator $Fx(t) = \int_0^1 x^3(t)dt$, and we take its corresponding nonlinear Bernstein operator $(NB_nF)x(t)$; then one has for $n = 2$ and for $n = 10$.

The corresponding numerical evaluation on the left-hand side yields numerically, for $n = 10, 20, 30, 40$,

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$(NB_nF)(0.3)$	0.04674	0.03666	0.0333933	0.0317775 ,
$f(0.3)$	0.027	0.027	0.027	0.027

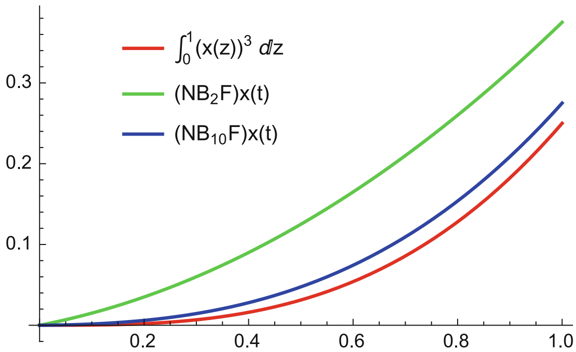

Fig. 1 Approximation of $Fx(t) = \int_0^1 x^3(t)dt$ by Urysohn-type nonlinear Bernstein operator, for $n = 2$ and $n = 10$

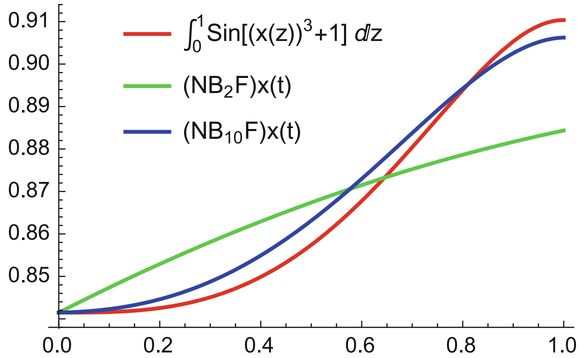

Fig. 2 Approximation of $Fx(t) = \int_0^1 \sin\left[x^3(t) + 1\right] dt$ by Urysohn-type nonlinear Bernstein operator, for $n = 2$ and $n = 10$

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$(NB_nF)\,(0.5)$	0.1625	0.14375	0.1375	0.134375 ,
$f\,(0.5)$	0.125	0.125	0.125	0.125

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$(NB_nF)\,(0.8)$	0.54944	0.53096	0.524693	0.52154 .
$f\,(0.8)$	0.512	0.512	0.512	0.512

Example 2 Let us consider the operator $Fx(t) = \int_0^1 \sin\left[x^3(t) + 1\right] dt$, and we take its corresponding nonlinear Bernstein operator $(NB_nF)x\,(t)$; then one has for $n = 2$ and for $n = 10$.

Finally, numerically for $n = 10, 20, 30, 40$,

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$(NB_nF)\,(0.3)$	0.864157	0.860156	0.858726	0.857995 ,
$f\,(0.3)$	0.855751	0.855751	0.855751	0.855751

	$n=10$	$n=20$	$n=30$	$n=40$
$(NB_nF)\,(0.5)$	0.908705	0.906317	0.905157	0.904507,
$f(0.5)$	0.902268	0.902268	0.902268	0.902268

	$n=10$	$n=20$	$n=30$	$n=40$
$(NB_nF)\,(0.8)$	0.9723	0.984958	0.98933	0.991542.
$f(0.8)$	0.998272	0.998272	0.998272	0.998272

The situation is similar for other examples studied.

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