On the Integral Inequalities for Riemann–Liouville and Conformable Fractional Integrals

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Abstract An integral operator is sometimes called an integral transformation. In the fractional analysis, Riemann–Liouville integral operator (transformation) of fractional integral is defined as

$$S_{\alpha}(x) = \frac{1}{\Gamma(x)} \int_0^x (x-t)^{\alpha-1} f(t)dt$$

where $f(t)$ is any integrable function on [0, 1] and $\alpha > 0$, t is in domain of f.

1 Introduction

The history of fractional analysis goes back to the arising of classical differential theory. Despite the fact that history is based on extreme ages, the interpretation of classical analysis as a result of the complexity of its physical structure has not been postponed and the science has not been very popular in engineering. However, the fact that fractional derivatives and integrals are not local or punctate has made the matter of fractional analysis remarkable in terms of better expressing the reality of nature. Thus, making this more widespread in science and engineering will play an important role in better interpreting and expressing nature.

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Fractional analysis can be considered as an extension of classical analysis. Fractional analysis does not have the definition of a single derivative as it is in the classical analysis, but the presence of more than one derivative gives the opportunity to obtain the best solution to the problems.

Fractional analysis has been studied by many scholars, and they have expressed fractional derivatives and integrals in different forms with different notations. But although these expressions are transitions between each other, they differ in terms of definitions and physical interpretations of their definitions. For the first time in 1695, the notion of fractional derivative and integral was raised by asking whether it would be meaningful if the derivation order was $1/2$ in a letter sent by L'Hospital to Leibnitz. Thus, the origin of fractional analysis begins with the question of L'Hospital.

This question on fractional derivatives and integrals has been a subject of study by many famous mathematicians such as Liouville, Riemann, Weyl, Fourier, Laplace, Lagrange, Euler, Abel, Lacroix, Grünwald, and Letnikov for more than 300 years. Since then, fractional differential equations have found many application areas including the theory of transmission lines, chemical analysis of fluids, heat transfer, diffusion, Schrödinger equation, material science, fluids, electrochemistry, fractal processes. Much of the mathematical application of fractional computing techniques has been put into place before the end of the twentieth century, but it has only been possible within a hundred years to achieve exciting achievements in engineering and scientific applications.

The fractional differential calculation technique not only contributes to a new dimension to mathematical approaches to explain physical phenomena, but also contributes to the interpretation of physical phenomena. The ranks of the differential equations describing the physical phenomena determine the rate of change in the physical state involved. The fractional-order differential at this point plays a major role in understanding the character of the physical phenomenon as well as closing the weaknesses of differential equations of integer order to explain some physical phenomena.

There are many definitions in the literature of the fractional derivative and integrals. Many of these definitions make use of the integral form when making fractional derivative definitions. The most famous of these definitions is Riemann–Liouville.

Some authors discussed whether the fractional derivative is indeed a fractional operator. Today, this question is still open to debate. Perhaps this is a philosophical issue. Moreover, this new definition can be considered as a transformation for the solution of differential equations of fractional order even if there is no definition of a fractional derivative. Obviously, this discussion is an argument of what the new theory is to be given. It is always a matter of deserving to study the definition of this new fractional derivative and fractional integral.

Various types of fractional derivative and integral operator were studied: Riemann–Liouville, conformable fractional integral operators, Caputo, Hadamard, Erdelyi–Kober, Grünwald–Letnikov, Marchaud, and Riesz are just a few to name.

In the present chapter, we shall recall some of fractional integral operators, which generalizes the classical integrals. We shall start this chapter with some results and definitions to refresh our memories about some of the remarkable milestones in the theory of fractional calculus and recall some inequalities involving two kinds of fractional integral operators.

Finally, we will give inequalities of Hermite–Hadamard type, Grüss type, Ostrowski type involving other types of fractional integral operators. All of this will be presented chronologically.

2 Riemann–Liouville Fractional Integral Operators and Inequalities

The following definitions are well-known in the fractional calculus and have been used in many fields of mathematics (see the references [1–4]).

Definition 2.1 ([5]) Let $f \in L_1[a, b]$. The Riemann–Liouville integrals $J^{\alpha}_{a+} f$ and $J^{\alpha}_{b-} f$ of order $\alpha > 0$ are defined by

$$J^{\alpha}_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x)dx, \qquad t > a,$$

and

$$J^{\alpha}_{b-} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (x-t)^{\alpha-1} f(x)dx, \qquad t < b,$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$. Here $J^0_{a+} f(t) = J^0_{b-} f(t) = f(t)$.

In the case of $\alpha = 1$, the fractional integral reduces to classical integral.

In this paper, some new integral inequalities have been proved by using conformable fractional integrals for functions whose derivatives of absolute values are quasi-convex, s-convex and log-convex functions.

Several researches have proved different types of integral inequalities via Riemann–Liouville fractional integrals. We will start with the new representation of celebrated Montgomery identity for fractional calculus that was proved Anastassiou et al. in 2009.

Lemma 2.1 ([6]) *Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$, and $f' : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, then the following Montgomery identity for fractional integrals holds:*

$$f(x) = \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J^{\alpha}_a f(b) - J^{\alpha-1}_a (P_2(x, b) f(b)) + J^{\alpha}_a (P_2(x, b) f'(b)), \quad \alpha \geq 1$$

where $P_2(x, t)$ is the fractional Peano kernel defined by:

$$P_2(x, t) = \begin{cases} \frac{t-a}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & a \leq t \leq x, \\ \frac{t-b}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & x \leq t \leq b. \end{cases}$$

The authors have also extended Ostrowski's inequality and Gruss inequality to fractional calculus as follows.

Theorem 2.1 ([6]) *Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ and $|f'(x)| \leq M$, for every $x \in [a, b]$ and $\alpha \geq 1$. Then, the following Ostrowski fractional inequality holds:*

$$\left| f(x) - \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^{\alpha} f(b) + J_a^{\alpha-1} P_2(x, b) f(b) \right|$$
$$\leq \frac{M}{\alpha(\alpha+1)} \left[(b-x)\left(2\alpha\left(\frac{b-x}{b-a}\right) - \alpha - 1\right) + (b-a)^{\alpha}(b-x)^{1-\alpha} \right].$$

Proposition 1 ([6]) *Suppose that $f(x)$ and $g(x)$ are two integrable functions for all $x \in [a, b]$, and satisfy the conditions*

$$m \leq (b-x)^{\alpha-1} f(x) \leq M, \quad n \leq (b-x)^{\alpha-1} g(x) \leq N,$$

where $\alpha > 1/2$, and m, M, n, N are real constants. Then, the following Gruss fractional inequality holds:

$$\left| \frac{\Gamma(2\alpha-1)}{(b-a)\Gamma^2(\alpha)} J_a^{2\alpha-1}(fg)(b) - \frac{1}{(b-a)^2} J_a^{\alpha} f(b) J_a^{\alpha} g(b) \right|$$
$$\leq \frac{1}{4\Gamma^2(\alpha)}(M-m)(N-n).$$

Another important study on the Riemann–Liouville fractional integrals has been written by Dahmani in 2010. The following results are concerning with Minkowski inequality.

Theorem 2.2 ([7]) *Let $\alpha > 0$, $p \geq 1$ and let f, g be two positive functions on $[0, \infty)$ such that for all $t > 0$, $J^{\alpha} f^p(t) < \infty$, $J^{\alpha} g^p(t) < \infty$. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in [0, t]$, then we have*

$$\left[J^{\alpha} f^p(t)\right]^{\frac{1}{p}} + \left[J^{\alpha} g^p(t)\right]^{\frac{1}{p}} \leq \frac{1 + M(m+2)}{(m+1)(M+1)} \left[J^{\alpha}(f+g)^p(t)\right]^{\frac{1}{p}}.$$

Theorem 2.3 ([7]) *Let $\alpha > 0$, $p \geq 1$ and let f, g be two positive functions on $[0, \infty)$ such that for all $t > 0$, $J^{\alpha} f^p(t) < \infty$, $J^{\alpha} g^p(t) < \infty$. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in [0, t]$, then we have*

$$\left[J^{\alpha} f^p(t)\right]^{\frac{2}{p}} + \left[J^{\alpha} g^p(t)\right]^{\frac{2}{p}} \leq \left(\frac{(M+1)(m+1)}{M} - 2\right)$$
$$\left[J^{\alpha} f^p(t)\right]^{\frac{1}{p}} \left[J^{\alpha} g^p(t)\right]^{\frac{1}{p}}.$$

Theorem 2.4 ([7]) *Let $\alpha > 0$, $p \geq 1$ and let f, g be two positive functions on $[0, \infty)$. If f^p, g^p are two concave functions on $[0, \infty)$, then we have*

$$2^{-p-q}(f(0)+f(t))^p(g(0)+g(t))^q(J^\alpha(t^{\alpha-1}))^2 \\ \le J^\alpha(t^{\alpha-1}f^p(t))J^\alpha(t^{\alpha-1}g^q(t)).$$

We will remind an integral identity that was proved by Set in 2012.

Lemma 2.2 ([8]) *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then for all $x\in[a,b]$ and $\alpha>0$ we have:*

$$\left(\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{b-a}[J^\alpha_{x^-}f(a)+J^\alpha_{x^+}f(b)] \\ =\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^\alpha f'(tx+(1-t)a)dt-\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^\alpha f'(tx+(1-t)b)dt$$

where $\Gamma(\alpha)=\int_0^\infty e^{-1}u^{\alpha-1}du$.

By using this identity, the author has been given Ostrowski-type integral inequalities for s-convex functions where Γ is Euler gamma function.

Theorem 2.5 ([8]) *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L[a,b]$. If $|f'|$ is s-convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$ and $|f'(x)|\le M$, $x\in[a,b]$, then the following inequality for fractional integrals with $\alpha>0$ holds:*

$$\left|\left(\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}[J^\alpha_{x^-}f(a)+J^\alpha_{x^+}f(b)]\right| \\ \le\frac{M}{b-a}\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{\alpha+s+1}\right].$$

Theorem 2.6 ([8]) *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L[a,b]$. If $|f'|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$, $p,q>1$ and $|f'(x)|\le M$, $x\in[a,b]$, then the following inequality for fractional integrals holds:*

$$\left|\left(\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}[J^\alpha_{x^-}f(a)+J^\alpha_{x^+}f(b)]\right| \\ \le\frac{M}{(1+p\alpha)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right]$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 2.7 ([8]) *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L[a,b]$. If $|f'|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$, $q\ge1$, and $|f'(x)|\le M$, $x\in[a,b]$, then the following inequality for fractional integrals holds:*

$$\left| \left(\frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{b-a} \right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} [J_{x^-}^{\alpha} f(a) + J_{x^+}^{\alpha} f(b)] \right|$$
$$\leq M \left(\frac{1}{1+\alpha} \right)^{1-\frac{1}{q}} \left(\frac{1}{\alpha+s+1} \right)^{\frac{1}{q}}$$
$$\times \left(1 + \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)} \right)^{\frac{1}{q}} \left[\frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{b-a} \right]$$

where $\alpha > 0$.

Sarıkaya and Öğünmez have extended the Montgomery identities for the Riemann–Liouville fractional integrals by using a different proof method; they have used these Montgomery identities to establish some new integral inequalities. The authors have also developed some integral inequalities for the fractional integral using differentiable convex functions.

Lemma 2.3 ([9]) *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I^o *with* $a, b \in I (a < b)$ *and* $f' \in L_1[a, b]$, *then*

$$f(x) = \frac{\Gamma(\alpha)}{b-a} (b-x)^{1-\alpha} J_a^{\alpha} f(b) - J_a^{\alpha-1} (P_2(x, b) f(b)) + J_a^{\alpha} \left(P_2(x, b) f'(b) \right), \quad \alpha \geq 1,$$

where $P_2(x, t)$ *is as in Lemma 2.1*

$$P_2(x, t) := \begin{cases} \frac{(t-a)}{b-a} (b-x)^{1-\alpha} \Gamma(\alpha), \ a \leq t < x, \\ \frac{(t-b)}{b-a} (b-x)^{1-\alpha} \Gamma(\alpha), \ x \leq t \leq b. \end{cases}$$

Theorem 2.8 ([9]) *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I^o *with* $f' \in L_1[a, b]$, *then the following identity holds:*

$$(1-2\lambda) f(x) = \frac{\Gamma(\alpha)}{b-a} (b-x)^{1-\alpha} J_a^{\alpha} f(b) - \lambda \left(\frac{b-a}{b-x} \right)^{\alpha-1} f(a)$$
$$- J_a^{\alpha-1} (P_3(x, b) f(b)) + J_a^{\alpha} \left(P_3(x, b) f'(b) \right), \quad \alpha \geq 1,$$

where $P_3(x, t)$ *is the fractional Peano kernel defined by*

$$P_3(x, t) := \begin{cases} \frac{t-(1-\lambda)a-\lambda b}{b-a} (b-x)^{1-\alpha} \Gamma(\alpha), \ a \leq t < x, \\ \frac{t-(1-\lambda)b-\lambda a}{b-a} (b-x)^{1-\alpha} \Gamma(\alpha), \ x \leq t \leq b. \end{cases}$$

for $0 \leq \lambda \leq 1$.

Theorem 2.9 ([9]) *Let* $f : [a, b] \to \mathbb{R}$ *be differentiable on* (a, b) *such that* $f' \in L_1[a, b]$, *where* $a < b$. *If* $|f'(x)| \leq M$ *for every* $x \in [a, b]$ *and* $\alpha \geq 1$, *then the following inequality holds:*

$$\left|(1-2\lambda)f(x)-\frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^{\alpha}f(b)\right.$$
$$\left.+\lambda\left(\frac{b-a}{b-x}\right)^{\alpha-1}f(a)+J_a^{\alpha-1}\left(P_3(x,b)f(b)\right)\right|$$
$$\leq\frac{M}{\alpha(\alpha+1)}\Big\{(b-a)^{\alpha}(b-x)^{1-\alpha}\left[2\lambda^{\alpha+1}+2(1-\lambda)^{\alpha+1}+\lambda(b-a)-1\right]$$
$$+(b-x)\left[2\alpha\frac{b-x}{b-a}-(\alpha+1)\right]\Big\}.$$

Theorem 2.10 ([9]) *Let $f:[a,b]\to\mathbb{R}$ be a differentiable convex function on (a,b) and $f'\in L_1[a,b]$. Then for any $x\in(a,b)$, the following inequality holds:*

$$\frac{1}{\alpha(\alpha+1)}\left[\alpha\frac{(b-x)^2}{b-a}f'_+(x)-\left((b-a)^{\alpha}(b-x)^{1-\alpha}\right.\right.$$
$$\left.\left.+\alpha\frac{(b-x)^2}{b-a}-(\alpha+1)(b-x)\right)f'_-(x)\right]$$
$$\leq\frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^{\alpha}f(b)-J_a^{\alpha-1}\left(P_2(x,b)f(b)\right)-f(x),\quad\alpha\geq1$$

The fractional integral form of Hermite–Hadamard inequality was proved by Sarıkaya et al. in 2013 as follows.

Theorem 2.11 ([10]) *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is a convex function on [a,b], then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}\left[\mathrm{J}_{a^+}^{\alpha}f(b)+\mathrm{J}_{b^-}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2}$$

with $\alpha>0$.

In the same paper, the authors have given a new integral identity and generalized Dragomir and Agarwal's results to fractional calculus.

Lemma 2.4 ([10]) *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}\left[\mathrm{J}_{a^+}^{\alpha}f(b)+\mathrm{J}_{b^-}^{\alpha}f(a)\right]$$
$$=\frac{b-a}{2}\int_0^1[(1-t)^{\alpha}-t^{\alpha}]f'(ta+(1-t)b)dt.$$

Theorem 2.12 ([10]) *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is convex on [a,b], then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}\,\left[\mathrm{J}_{a^+}^{\alpha}f(b)+\mathrm{J}_{b^-}^{\alpha}f(a)\right]\right|$$
$$\leq \frac{b-a}{2(\alpha+1)}\big(1-\frac{1}{2^{\alpha}}\big)[f'(a)+f'(b)].$$

Tariboon et al. have proved some new Gruss-type inequalities involving Riemann–Liouville fractional integrals.

Theorem 2.13 ([11]) *Let f be integrable function on $[0,\infty)$. Assume that (H_1) there exist two integrable functions φ_1 and φ_2 on $[0,\infty)$ such that*

$$\varphi_1(t)\leq f(t)\leq \varphi_2(t),\quad \forall t\in[0,\infty),$$

Then, for $t>0,\ \alpha,\beta>0$, one has:

$$J^{\beta}\varphi_1(t)J^{\alpha}f(t)+J^{\alpha}\varphi_2(t)J^{\beta}f(t)\geq J^{\alpha}\varphi_2(t)J^{\beta}\varphi_1(t)+J^{\alpha}f(t)J^{\beta}f(t).$$

Theorem 2.14 ([11]) *Let f and g be two integrable functions on $[0,\infty)$. Suppose that (H_1) holds, and moreover, one assumes that (H_2) there exist ψ_1 and ψ_2 integrable functions on $[0,\infty)$ such that*

$$\psi_1(t)\leq g(t)\leq \psi_2(t),\quad \forall t\in[0,\infty),$$

Then for $t>0,\ \alpha,\beta>0$ the following inequalities hold:

$$(a)\quad J^{\beta}\psi_1(t)J^{\alpha}f(t)+J^{\alpha}\varphi_2(t)J^{\beta}g(t)\geq J^{\beta}\psi_1(t)J^{\alpha}\varphi_2(t)+J^{\alpha}f(t)J^{\beta}g(t),$$

$$(b)\quad J^{\beta}\varphi_1(t)J^{\alpha}g(t)+J^{\alpha}\psi_2(t)J^{\beta}f(t)\geq J^{\beta}\psi_1(t)J^{\alpha}\psi_2(t)+J^{\beta}f(t)J^{\beta}g(t),$$

$$(c)\quad J^{\alpha}\varphi_2(t)J^{\beta}\psi_2+J^{\alpha}f(t)J^{\beta}g(t)\geq J^{\alpha}\varphi_2(t)J^{\beta}g(t)+J^{\beta}\psi_2(t)J^{\alpha}f(t),$$

$$(d)\quad J^{\alpha}\varphi_1(t)J^{\beta}\psi_1+J^{\alpha}f(t)J^{\beta}g(t)\geq J^{\alpha}\varphi_1(t)J^{\beta}g(t)+J^{\beta}\psi_1(t)J^{\alpha}f(t).$$

Theorem 2.15 ([11]) *Let f and g be integrable functions on $[0,\infty)$ and let $\varphi_1,\varphi_2,\psi_1$ and ψ_2 be integrable functions on $[0,\infty)$, satisfying the conditions (H_1) and (H_2) on $[0,\infty)$. Then, for all $t>0,\alpha>0$, one has*

$$\left|\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}fg(t)-J^{\alpha}f(t)J^{\alpha}g(t)\right|\leq\sqrt{T(f,\varphi_1,\varphi_2)T(g,\psi_1,\psi_2)},$$

where $T(u, v, w)$ is defined by

$$\begin{aligned} T(u, v, w) &= (J^\alpha w(t) - J^\alpha u(t))(J^\alpha u(t) - J^\alpha v(t)) \\ &+ \frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha vu(t) - J^\alpha v(t) J^\alpha u(t) \\ &+ \frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha wu(t) - J^\alpha w(t) J^\alpha u(t) \\ &+ J^\alpha v(t) J^\alpha w(t) - \frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha vw(t). \end{aligned}$$

A new generalization of Montgomery identity has been given by Sarıkaya et al., and the authors have established new Ostrowski-type inequalities by using this identity as follows.

Throughout this study, we assume that Peano kernels defined by

$$K_1(x,t) = \begin{cases} \left[t - a - \frac{\lambda}{2}(x-a)\right], \ a \le t < x \\ \left[t - b + \frac{\lambda}{2}(b-x)\right], \ x \le t \le b \end{cases}$$

$$K_2(x,t) = \begin{cases} \frac{1}{b-a}\left[t - a - \frac{\lambda}{2}(x-a)\right](b-x)^{1-\alpha}\Gamma(\alpha), \ a \le t < x \\ \frac{1}{b-a}\left[t - b + \frac{\lambda}{2}(b-x)\right](b-x)^{1-\alpha}\Gamma(\alpha), \ x \le t \le b \end{cases}$$

$$h(x,t) = \begin{cases} \frac{1}{b-a}\left[t - a - \frac{\lambda}{2}(x-a)\right](b-x)^{1-\alpha}\Gamma(\alpha), \ a \le t < x \\ \frac{1}{b-a}\left[b - t + \frac{\lambda}{2}(b-x)\right](b-x)^{1-\alpha}\Gamma(\alpha), \ x \le t \le b. \end{cases}$$

Lemma 2.5 ([12]) *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^0 with $a, b \in I$ $(a < b)$, $\alpha \ge 1$, $0 \le \lambda \le 1$, and $f' \in L_1[a,b]$, then the generalization of Montgomery identity for fractional integral holds:*

$$\begin{aligned} \left(1 - \frac{\lambda}{2}\right) f(x) &= J_a^\alpha (K_2(x,b) f'(b)) + \frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha) J_a^\alpha f(b) \\ &= -J_a^{\alpha-1}(K_2(x,b) f(b)) - \frac{\lambda}{2}(b-a)^{\alpha-2}(x-a)(b-x)^{\alpha-1} f(a) \end{aligned}$$

Theorem 2.16 ([12]) *Let $f : [a,b] \to \mathbb{R}$ be differentiable on (a,b) such that $f' \in L_1[a,b]$, where $a < b$ and $0 \le \lambda \le 1$. If $|f'(x)| \le M$ for every $x \in [a,b]$ and $\alpha \ge 1$, then the following Ostrowski fractional inequality holds:*

$$\begin{aligned} &\left| \left(1 - \frac{\lambda}{2}\right) f(x) - \frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha) J_a^\alpha f(b) \right. \\ &\left. + J_a^{\alpha-1}(K_2(x,b) f(b)) + \frac{\lambda}{2}(b-a)^{\alpha-2}(x-a)(b-x)^{\alpha-1} f(a) \right| \\ &\le \frac{M}{\Gamma(\alpha)} A(x), \end{aligned}$$

where

$$
\begin{aligned}
&A(x)\\
&=\frac{\Gamma(\alpha)(b-x)^{1-\alpha}}{b-a}\Bigg\{(b-a)\left[\frac{2(b-a)+\lambda(x-a)}{2\alpha}-\frac{b-a}{\alpha+1}\right]\\
&\qquad+(b-x)^{\alpha}\left[\frac{2(b-x)}{\alpha+1}-\frac{(b-a)+\lambda(x-\frac{a+b}{2}}{\alpha}\right]\Bigg\}.
\end{aligned}
$$

Theorem 2.17 ([12]) *Let $f:[a,b]\to\mathbb{R}$ be differentiable on (a,b) such that $f'\in L_1[a,b]$, where $a<b$, $0\le\lambda\le 1$, and $\alpha\ge 1$. If the mapping $|f'|^q$ is convex on $[a,b]$, $q\ge 1$, then the following fractional inequality holds:*

$$
\begin{aligned}
&\Bigg|\left(1-\frac{\lambda}{2}\right)f(x)-\frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha)J_a^{\alpha}f(b)+J_a^{\alpha-1}(K_2(x,b)f(b))\\
&+\frac{\lambda}{2}(b-a)^{\alpha-2}(x-a)(b-x)^{\alpha-1}f(a)\Bigg|\\
&\le\frac{1}{\Gamma(\alpha)}(A(x))^{1-\frac{1}{q}}(|f'(a)|^{q}B(x)+|f'(b)|^{q}C(x))^{\frac{1}{q}}
\end{aligned}
$$

where

$$
\begin{aligned}
B(x)=\frac{\Gamma(\alpha)(b-a)^{1-\alpha}}{(b-a)^2}\Bigg\{&(b-a)^{\alpha+1}\left[\frac{2(b-a)+\lambda(x-a)}{2(\alpha+1)}-\frac{b-a}{\alpha+2}\right]\\
&+(b-x)^{\alpha+1}\left[\frac{2(b-x)}{\alpha+2}-\frac{(b-a)+\lambda(x-\frac{a+b}{2}}{\alpha+1}\right]\Bigg\}
\end{aligned}
$$

and

$$
\begin{aligned}
C(x)=\frac{\Gamma(\alpha)(b-x)^{1-\alpha}}{(b-a)}\Bigg\{&(b-a)^{\alpha}\left[\frac{2(b-a)+\lambda(x-a)}{2\alpha(\alpha+1)}-\left(\frac{b-a}{\alpha+1}-\frac{1}{\alpha+2}\right)\right]\\
&+2(b-x)^{\alpha+1}\left(\frac{1}{\alpha+1}-\frac{(b-x)}{(\alpha+2)(b-a)}\right)\\
&-(b-x)^{\alpha}\left((b-a)+\lambda\left(x-\frac{a+b}{2}\right)\right)\left(\frac{b-x}{(b-a)(\alpha+1)}-\frac{1}{\alpha}\right)\Bigg\}.
\end{aligned}
$$

Set et al. have given a new integral identity by using Riemann–Liouville fractional integrals and proved several new Simpson-type integral inequalities that generalize previous results.

Lemma 2.6 ([13]) *$f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with $a<b$. If $f'\in L[a,b]$, $n\ge 0$, and $\alpha>0$, then the following equality holds:*

$$
\begin{aligned}
I(a,b;n,\alpha) &= \frac{1}{6}\left[f(a)+f(b)+2f\left(\frac{a+nb}{n+1}\right)+2f\left(\frac{na+b}{n+1}\right)\right] \\
&\quad -\frac{\Gamma(\alpha+1)(n+1)^{\alpha}}{6(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f\left(\frac{na+b}{n+1}\right)+J_{b^{-}}^{\alpha}f\left(\frac{a+nb}{n+1}\right)\right] \\
&\quad -\frac{\Gamma(\alpha+1)(n+1)^{\alpha}}{3(b-a)^{\alpha}}\left[J_{\frac{a+nb}{n+1}^{+}}^{\alpha}f(b)+J_{\frac{na+b}{n+1}^{-}}^{\alpha}f(a)\right] \\
&= \frac{b-a}{2(n+1)}\left(\int_0^1\left[\frac{2(1-t)^{\alpha}-t^{\alpha}}{3}\right]f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)dt\right. \\
&\quad \left.+\int_0^1\left[\frac{t^{\alpha}-2(1-t)^{\alpha}}{3}\right]f'\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)dt\right)
\end{aligned}
$$

for all $x \in [a,b]$ *and where* $\Gamma(\alpha)=\int_0^{\infty}e^{-u}u^{\alpha-1}du$.

Proof By using integration by parts, we have

$$
\begin{aligned}
I_1 &= \int_0^1\left[\frac{2(1-t)^{\alpha}-t^{\alpha}}{3}\right]f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)dt \\
&= \frac{n+1}{3(b-a)}\left[f(a)+2f\left(\frac{na+b}{n+1}\right)\right] \\
&\quad -\frac{\alpha(n+1)^{\alpha+1}}{3(b-a)^{\alpha+1}}\int_a^{\frac{na+b}{n+1}}f(x)\left(\frac{na+b}{n+1}-x\right)^{\alpha-1}dx \\
&\quad -\frac{2\alpha(n+1)^{\alpha+1}}{3(b-a)^{\alpha+1}}\int_a^{\frac{na+b}{n+1}}f(x)(x-a)^{\alpha-1}dx
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \int_0^1\left[\frac{t^{\alpha}-2(1-t)^{\alpha}}{3}\right]f'\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)dt \\
&= \frac{n+1}{3(b-a)}\left[f(b)+2f\left(\frac{a+nb}{n+1}\right)\right] \\
&\quad -\frac{\alpha(n+1)^{\alpha+1}}{3(b-a)^{\alpha+1}}\int_{\frac{a+nb}{n+1}}^{b}f(x)\left(x-\frac{a+nb}{n+1}\right)^{\alpha-1}dx \\
&\quad -\frac{2\alpha(n+1)^{\alpha+1}}{3(b-a)^{\alpha+1}}\int_{\frac{a+nb}{n+1}}^{b}f(x)(b-x)^{\alpha-1}dx.
\end{aligned}
$$

By adding I_1 and I_2 and multiplying the both sides $\frac{b-a}{2(n+1)}$, we can write

$$I_1+I_2=\frac{1}{6}\left[f(a)+f(b)+2f\left(\frac{a+nb}{n+1}\right)+2f\left(\frac{na+b}{n+1}\right)\right]$$
$$-\frac{\alpha\,(n+1)^\alpha}{6(b-a)^\alpha}\int_a^{\frac{na+b}{n+1}} f\,(x)\left(\frac{na+b}{n+1}-x\right)^{\alpha-1}dx$$
$$-\frac{\alpha\,(n+1)^\alpha}{3(b-a)^\alpha}\int_a^{\frac{na+b}{n+1}} f\,(x)\,(x-a)^{\alpha-1}\,dx$$
$$-\frac{\alpha\,(n+1)^\alpha}{6(b-a)^\alpha}\int_{\frac{a+nb}{n+1}}^{b} f\,(x)\left(x-\frac{a+nb}{n+1}\right)^{\alpha-1}dx$$
$$-\frac{\alpha\,(n+1)^\alpha}{3(b-a)^\alpha}\int_{\frac{a+nb}{n+1}}^{b} f\,(x)\,(b-x)^{\alpha-1}\,dx.$$

From the facts that

$$\frac{1}{\Gamma(\alpha)}\int_a^{\frac{na+b}{n+1}} f\,(x)\,(x-a)^{\alpha-1}\,dx = J^\alpha_{\frac{na+b}{n+1}^-}f\,(a)$$
$$\frac{1}{\Gamma(\alpha)}\int_{\frac{a+nb}{n+1}}^{b} f\,(x)\,(b-x)^{\alpha-1}\,dx = J^\alpha_{\frac{a+nb}{n+1}^+}f\,(b)$$
$$\frac{1}{\Gamma(\alpha)}\int_a^{\frac{na+b}{n+1}} f\,(x)\left(\frac{na+b}{n+1}-x\right)^{\alpha-1}dx = J^\alpha_{a^+}f\left(\frac{na+b}{n+1}\right)$$
$$\frac{1}{\Gamma(\alpha)}\int_{\frac{a+nb}{n+1}}^{b} f\,(x)\left(x-\frac{a+nb}{n+1}\right)^{\alpha-1}dx = J^\alpha_{b^-}f\left(\frac{a+nb}{n+1}\right),$$

we get the result. □

Theorem 2.18 ([13]) *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on $[a,b]$. If $f'\in L[a,b]$ and $\left|f'(x)\right|$ is convex function, then the following inequality holds for fractional integrals with $\alpha>0$;*

$$\left|I\,(a,b;n,\alpha)\right| \le \frac{b-a}{2\,(n+1)}\left[\frac{3-2\left(\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1}-4\left(1-\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1}}{3\,(\alpha+1)}\right]\left(\left|f'(a)\right|+\left|f'(b)\right|\right)$$

where $\Gamma\,(\alpha)$ is Euler gamma function.

Proof From the integral identity given in Lemma 1 and by using the properties of modulus, we have

$$
\begin{aligned}
&|I(a,b;n,\alpha)| \\
&\leq \frac{b-a}{2(n+1)}\left(\int_0^1\left|\frac{2(1-t)^\alpha - t^\alpha}{3}\right|\left|f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|dt\right. \\
&\quad\left.+\int_0^1\left|\frac{t^\alpha - 2(1-t)^\alpha}{3}\right|\left|f'\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|dt\right).
\end{aligned}
$$

Since $|f'(x)|$ is convex function, we can write

$$
\begin{aligned}
&|I(a,b;n,\alpha)| \\
&\leq \frac{b-a}{2(n+1)}\left(\int_0^1\left|\frac{2(1-t)^\alpha - t^\alpha}{3}\right|\left(\frac{n+t}{n+1}|f'(a)|+\frac{1-t}{n+1}|f'(b)|\right)dt\right. \\
&\quad\left.+\int_0^1\left|\frac{t^\alpha - 2(1-t)^\alpha}{3}\right|\left(\frac{1-t}{n+1}|f'(a)|+\frac{n+t}{n+1}|f'(b)|\right)dt\right) \\
&= \frac{b-a}{2(n+1)}\left(\int_0^{\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}}\left(\frac{2(1-t)^\alpha - t^\alpha}{3}\right)\left(\frac{n+t}{n+1}|f'(a)|+\frac{1-t}{n+1}|f'(b)|\right)dt\right. \\
&\quad+\int_{\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}}^1\left(\frac{t^\alpha - 2(1-t)^\alpha}{3}\right)\left(\frac{n+t}{n+1}|f'(a)|+\frac{1-t}{n+1}|f'(b)|\right)dt \\
&\quad+\int_0^{\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}}\left(\frac{2(1-t)^\alpha - t^\alpha}{3}\right)\left(\frac{1-t}{n+1}|f'(a)|+\frac{n+t}{n+1}|f'(b)|\right)dt \\
&\quad\left.+\int_{\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}}^1\left(\frac{t^\alpha - 2(1-t)^\alpha}{3}\right)\left(\frac{1-t}{n+1}|f'(a)|+\frac{n+t}{n+1}|f'(b)|\right)dt\right).
\end{aligned}
$$

By a simple computation, we obtain the desired result. □

Theorem 2.19 ([13]) *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable function on* $[a,b]$. *If* $f'\in L[a,b]$ *and* $|f'(x)|^q$ *is convex function, then the following inequality holds for fractional integrals with* $\alpha>0$, $q>1$, *and* $p^{-1}+q^{-1}=1$;

$$
\begin{aligned}
&\left|I\left(a,b;n,\alpha\right)\right| \\
&\leq \frac{b-a}{2\left(n+1\right)}\left(\int_0^1\left|\frac{2\left(1-t\right)^{\alpha}-t^{\alpha}}{3}\right|^p dt\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{2n+1}{2\left(n+1\right)}\left|f'\left(a\right)\right|^q+\frac{1}{2\left(n+1\right)}\left|f'\left(b\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{1}{2\left(n+1\right)}\left|f'\left(a\right)\right|^q+\frac{2n+1}{2\left(n+1\right)}\left|f'\left(b\right)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

where $\Gamma\left(\alpha\right)$ is Euler gamma function.

Proof By using Lemma 1 and Hölder integral inequality, we can write

$$
\begin{aligned}
&\left|I\left(a,b;n,\alpha\right)\right| \\
&\leq \frac{b-a}{2\left(n+1\right)}\left(\int_0^1\left|\frac{2\left(1-t\right)^{\alpha}-t^{\alpha}}{3}\right|\left|f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|dt\right. \\
&\left.+\int_0^1\left|\frac{t^{\alpha}-2\left(1-t\right)^{\alpha}}{3}\right|\left|f'\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|dt\right) \\
&\leq \frac{b-a}{2\left(n+1\right)}\left(\left(\int_0^1\left|\frac{2\left(1-t\right)^{\alpha}-t^{\alpha}}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left|\frac{t^{\alpha}-2\left(1-t\right)^{\alpha}}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}}\right).
\end{aligned}
$$

Since $\left|f'\left(x\right)\right|^q$ is convex function, we can write

$$
\begin{aligned}
&\left|I\left(a,b;n,\alpha\right)\right| \\
&\leq \frac{b-a}{2\left(n+1\right)}\left(\int_0^1\left|\frac{2\left(1-t\right)^{\alpha}-t^{\alpha}}{3}\right|\left|f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|dt\right. \\
&\left.+\int_0^1\left|\frac{t^{\alpha}-2\left(1-t\right)^{\alpha}}{3}\right|\left|f'\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|dt\right) \\
&\leq \frac{b-a}{2\left(n+1\right)}\left(\left(\int_0^1\left|\frac{2\left(1-t\right)^{\alpha}-t^{\alpha}}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{n+t}{n+1}\left|f'\left(a\right)\right|^q\right.\right.\right. \\
&\qquad\left.\left.+\frac{1-t}{n+1}\left|f'\left(b\right)\right|^q\right)dt\right)^{\frac{1}{q}}
\end{aligned}
$$

$$+\left(\int_0^1 \left|\frac{t^\alpha - 2(1-t)^\alpha}{3}\right|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left(\frac{1-t}{n+1}\left|f'(a)\right|^q + \frac{n+t}{n+1}\left|f'(b)\right|^q\right) dt\right)^{\frac{1}{q}}\Bigg).$$

By taking into account,

$$\int_0^1 \left(\frac{n+t}{n+1}\left|f'(a)\right|^q + \frac{1-t}{n+1}\left|f'(b)\right|^q\right) dt = \frac{2n+1}{2(n+1)}\left|f'(a)\right|^q + \frac{1}{2(n+1)}\left|f'(b)\right|^q$$

$$\int_0^1 \left(\frac{1-t}{n+1}\left|f'(a)\right|^q + \frac{n+t}{n+1}\left|f'(b)\right|^q\right) dt = \frac{1}{2(n+1)}\left|f'(a)\right|^q + \frac{2n+1}{2(n+1)}\left|f'(b)\right|^q,$$

we obtain

$$\begin{aligned}
&\left|I(a,b;n,\alpha)\right| \\
&\leq \frac{b-a}{2(n+1)}\left(\int_0^1 \left|\frac{2(1-t)^\alpha - t^\alpha}{3}\right|^p dt\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{2n+1}{2(n+1)}\left|f'(a)\right|^q + \frac{1}{2(n+1)}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{1}{2(n+1)}\left|f'(a)\right|^q + \frac{2n+1}{2(n+1)}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

which completes the proof. □

Theorem 2.20 ([13]) *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on $[a,b]$. If $f' \in L[a,b]$ and $\left|f'(x)\right|^q$ is convex function, then the following inequality holds for fractional integrals with $\alpha > 0$ and $q \geq 1$;*

$$\begin{aligned}
&\left|I(a,b;n,\alpha)\right| \\
&\leq \frac{b-a}{2(n+1)}\left(\frac{3 - 2\left(\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1} - 4\left(1-\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1}}{3(\alpha+1)}\right)^{1-\frac{1}{q}} \\
&\times\left(\left(K_1(\alpha,n)\left|f'(a)\right|^q + K_2(\alpha,n)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(K_2(\alpha,n)\left|f'(a)\right|^q + K_1(\alpha,n)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right).
\end{aligned}$$

where $\Gamma(\alpha)$ is Euler gamma function and

$$K_1(\alpha,n)=\frac{(-4n-4)\left(1-\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1}+3n+2-2n\left(\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1}}{3(\alpha+1)(n+1)}$$
$$+\frac{4\left(1-\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+2}-2\left(\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+2}-1}{3(\alpha+2)(n+1)}$$
$$K_2(\alpha,n)=\frac{1-2\left(\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1}}{3(\alpha+1)(n+1)}+\frac{1-4\left(1-\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+2}+2\left(\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+2}}{3(\alpha+2)(n+1)}.$$

Proof By Lemma 1 and power-mean integral inequality, we can write

$$\begin{aligned}
&|I(a,b;n,\alpha)|\\
&\leq \frac{b-a}{2(n+1)}\left(\int_0^1\left|\frac{2(1-t)^\alpha-t^\alpha}{3}\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\left(\int_0^1\left|\frac{2(1-t)^\alpha-t^\alpha}{3}\right|\left|f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\int_0^1\left|\frac{t^\alpha-2(1-t)^\alpha}{3}\right|\left|f'\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

By taking into account convexity of $\left|f'(x)\right|^q$, we get

$$\begin{aligned}
&|I(a,b;n,\alpha)|\\
&\leq \frac{b-a}{2(n+1)}\left(\int_0^1\left|\frac{2(1-t)^\alpha-t^\alpha}{3}\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\left(\int_0^1\left|\frac{2(1-t)^\alpha-t^\alpha}{3}\right|\left(\frac{n+t}{n+1}\left|f'(a)\right|^q+\frac{1-t}{n+1}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\left|\frac{t^\alpha-2(1-t)^\alpha}{3}\right|\left(\frac{1-t}{n+1}\left|f'(a)\right|^q+\frac{n+t}{n+1}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Computing the above integrals, we get the result. □

Sarıkaya and Yıldırım have given a new refinement of Hermite–Hadamard inequality for Riemann–Liouville fractional integrals. They have proved an integral identity that gives some results for left side of Hermite–Hadamard inequality as follows.

Theorem 2.21 ([14]) *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for*

fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a)\right] \leq \frac{f(a)+f(b)}{2},$$

with $\alpha > 0$.

Lemma 2.7 ([14]) *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$. *If* $f' \in L[a,b]$, *then the following equality for fractional integrals holds:*

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a)\right] - f\left(\frac{a+b}{2}\right)\\
&= \frac{b-a}{4}\left\{\int_0^1 t^{\alpha} f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)dt - \int_0^1 t^{\alpha} f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right)dt\right\},
\end{aligned}$$

with $\alpha > 0$.

Theorem 2.22 ([14]) *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$. *If* $|f'|^q$ *is a convex function on* $[a,b]$ *for* $q \geq 1$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a)\right] - f\left(\frac{a+b}{2}\right)\right|\\
&\leq \frac{(b-a)}{4(\alpha+1)}\left(\frac{1}{2(\alpha+2)}\right)^{\frac{1}{q}}\left\{\left[(\alpha+1)|f'(a)|^q + (\alpha+3)|f'(b)|^q\right]^{\frac{1}{q}}\right.\\
&\left.+\left[(\alpha+3)|f'(a)|^q + (\alpha+1)|f'(b)|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 2.23 ([14]) *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$. *If* $|f'|^q$ *is a convex on* $[a,b]$ *for* $q > 1$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a)\right] - f(\frac{a+b}{2})\right|\\
&\leq \frac{(b-a)}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left\{\left[\frac{|f'(a)| + 3|f'(b)|}{4}\right]^{\frac{1}{q}} + \left[\frac{3|f'(a)| + |f'(b)|}{4}\right]^{\frac{1}{q}}\right\}\\
&\leq \frac{b-a}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left[|f'(a)| + |f'(b)|\right],
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

3 Conformable Fractional Integrals and Inequalities

The conformable fractional derivative attracts attention with conformity to classical derivative. Khalil et al. have introduced the conformable fractional derivative by the equation which has a limit form similar to the classical derivative. Khalil et al. have proved that this definition provides multiplication and division rules. They also express the Rolle theorem and the mean value theorem for functions which are differentiable with conformable fractional order.

The analysis of the conformable fractional was developed by Abdeljawad. In his work, he has presented left and right conformable fractional derivative concepts, fractional chain rule, and Gronwall inequality for a conformable fractional derivative. We will mention the beta function (see [5]):

$$B\left(a,b\right)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 t^{a-1}\left(1-t\right)^{b-1}dt,\quad a,b>0,$$

where $\Gamma\left(\alpha\right)=\int_0^\infty e^{-t}t^{\alpha-1}dt$ is gamma function.

Incomplete beta function is defined as:

$$B_x\left(a,b\right)=\int_0^x t^{a-1}\left(1-t\right)^{b-1}dt,\quad a,b>0.$$

In spite of its valuable contributions to mathematical analysis, the Riemann–Liouvile fractional integrals have deficiencies. For example, the solution of the differential equation is given as:

$$y^{(\frac{1}{2})}+y=x^{(\frac{1}{2})}+\frac{2}{\Gamma(2.5)}x^{(\frac{3}{2})},\quad y(0)=0$$

where $y^{(\frac{1}{2})}$ is the fractional derivative of y of order $\frac{1}{2}$.

The solution of the above differential equation has caused to imagine on a new and simple representation of the definition of fractional derivative. In [15] , Khalil et al. gave a new definition that is called "conformable fractional derivative." They not only proved further properties of these definitions but also gave the differences with the other fractional derivatives. Besides, another considerable study has been presented by Abdeljawad to discuss the basic concepts of fractional calculus.

In [16], Abdeljawad gave the following definitions of right–left conformable fractional integrals:

Definition 3.1 Let $\alpha\in(n,n+1], n=0,1,2,\ldots$ and set $\beta=\alpha-n$. Then, the left conformable fractional integral of any order $\alpha>0$ is defined by

$$(I_\alpha^a f)(t)=\frac{1}{n!}\int_a^t (t-x)^n(x-a)^{\beta-1}f(x)dx.$$

Definition 3.2 Analogously, the right conformable fractional integral of any order $\alpha > 0$ is defined by

$$({}^{b}I_{\alpha} f)(t) = \frac{1}{n!}\int_{t}^{b} (x-t)^{n}(b-x)^{\beta-1} f(x)dx.$$

Notice that if $\alpha = n+1$, then $\beta = \alpha - n = n+1-n = 1$; hence, $(I_{n+1}^{a} f)(t) = (J_{a+}^{n+1} f)(t)$ and $({}^{b}I_{n+1} f)(t) = \left(J_{b-}^{n+1} f\right)(t)$.

In [15, 16], the authors have pointed that the Riemann–Liouville derivatives are not valid for product of two functions. In this case, the inequalities that have been proved by Riemann–Liouville integrals are not valid. The results which are obtained by using the conformable fractional integrals have a wide range of validity. (Let us consider the function f defined as $f : \mathbb{R}^{+} \to \mathbb{R}$, $f = x^{2}e^{x}$ which is convex.)

Several researchers have focused on new integral inequalities involving conformable fractional integrals in recent years. In [17], Set et al. have given some more general Hadamard-type inequalities for convex functions. Set, Akdemir, and Mumcu have proved several Ostrowski-type inequalities by using conformable fractional integrals involving special functions in [18]. In [19–21], the authors have obtained new inequalities of Hermite–Hadamard type associated with conformable fractional integrals. In [22], several new integral inequalities have been established via conformable fractional integrals for pre-invex functions by Awan et al. In [23], Sarıkaya and Budak have proved some Opial-type inequalities.

Set, Akdemir, and Mumcu have established a new form of Hermite–Hadamard inequality via conformable fractional integrals and also proved an extension of Hermite–Hadamard inequality as follows.

Theorem 3.1 ([24]) *Let $f : [a,b] \to \mathbb{R}$ be a mapping with $0 \le a < b$ and $f \in L_1[a,b]$. If f is a convex mapping on $[a,b]$, then one can obtain the following inequalities for conformable fractional integrals:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}\left[(I_{\alpha}^{a} f)(b) + ({}^{b}I_{\alpha} f)(a)\right] \le \frac{f(a)+f(b)}{2}, \tag{3.1}$$

with $\alpha \in (n, n+1]$.

3.1 Extensions of HH-Inequality

Theorem 3.2 ([24]) *Assume that $f : [a,b] \to \mathbb{R}$ is a twice differentiable mapping with $a < b$ and $f \in L_1[a,b]$. If f'' is bounded on $[a,b]$, then we have*

$$\frac{m\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2$$
$$\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx$$
$$\leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I_{\alpha}^{a}f)(b)+({}^{b}I_{\alpha}f)(a)]-f\left(\frac{a+b}{2}\right) \tag{3.2}$$
$$\leq \frac{M\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2,$$
$$\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx,$$

and

$$\frac{-M\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}(x-a)(b-x)$$
$$\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx$$
$$\leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I_{\alpha}^{a}f)(b)+({}^{b}I_{\alpha}f)(a)]-\frac{f(a)+f(b)}{2} \tag{3.3}$$
$$\leq \frac{-m\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}(x-a)(b-x)$$
$$\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx,$$

with $\alpha \in (n, n+1]$, *where* $m = inf_{t\in[a,b]}f^{''}(t),\ \ M = sup_{t\in[a,b]}f^{''}(t)$.

It is obvious that $f^{''} \geq 0$ implies that $f^{'}$ non-decreasing. Therefore,

$$f^{'}(a+b-x) \geq f^{'}(x), \tag{3.4}$$

holds for all $x \in [a, \frac{a+b}{2}]$. So, we establish the following theorem using inequality of (3.4).

Theorem 3.3 ([24]) *Let* $f : [a, b] \to \mathbb{R}$ *be a positive, differentiable mapping with* $a < b$ *and* $f \in L_1[a, b]$. *If* $f^{'}(a+b-x) \geq f^{'}(x)$ *for all* $x \in [a, \frac{a+b}{2}]$. *Then, the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I_{\alpha}^{a}f)(b)+({}^{b}I_{\alpha}f)(a)] \leq \frac{f\ (a)+f\ (b)}{2}.$$

The following results have been obtained by Set et. al. involving Ostrowski-type inequalities for conformable fractional integrals.

Lemma 3.1 ([25]) *Let* $f : [a, b] \to \mathbb{R}$ *be a differentiable mapping in the interior* I° *on* (a, b) *with* $a < b$. *If* $f^{'} \in L[a, b]$, *then for all* $x \in [a, b]$ *and* $\alpha \in [n, n+1)$ *we have:*

$$\begin{aligned}&\frac{(x-a)^{\alpha+1}}{n!(b-a)}\int_0^1 B_t(n+1,\alpha-n)f^{'}(tx+(1-t)a)dt\\&-\frac{(b-x)^{\alpha+1}}{n!(b-a)}\int_0^1 B_t(n+1,\alpha-n)f^{'}(tx+(1-t)b)dt\\&=\frac{\Gamma(\alpha-n)[(x-a)^{\alpha}+(b-x)^{\alpha}]}{\Gamma(\alpha+1)(b-a)}f(x)-\frac{1}{b-a}[{}^{x}I_{\alpha}f(a)+I_{\alpha}^{x}f(b)],\end{aligned}\tag{3.5}$$

where $\Gamma(\alpha)=\int_0^1 e^{-t}u^{\alpha-1}du$.

Theorem 3.4 ([25]) *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$ *such that* $f^{'}\in L[a,b]$. *If* $|f^{'}|$ *is convex and* $|f^{'}(x)|\le M$, $x\in[a,b]$, *then the following inequality for fractional integrals with* $\alpha\in[n,n+1)$ *holds:*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha-n)[(x-a)^{\alpha}+(b-x)^{\alpha}]}{\Gamma(\alpha+1)(b-a)}f(x)-\frac{1}{b-a}[{}^{x}I_{\alpha}f(a)+I_{\alpha}^{x}f(b)]\right|\\&\le\frac{M\Gamma(\alpha-n+1)}{\Gamma(\alpha+2)(b-a)}[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}].\end{aligned}\tag{3.6}$$

Theorem 3.5 ([25]) *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$ *such that* $f^{'}\in L[a,b]$. *If* $|f^{'}|^{q}$ *is convex,* $p,q>1$, *and* $|f^{'}(x)|\le M$, $x\in[a,b]$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha-n)[(x-a)^{\alpha}+(b-x)^{\alpha}]}{\Gamma(\alpha+1)(b-a)}f(x)-\frac{1}{b-a}[{}^{x}I_{\alpha}f(a)+I_{\alpha}^{x}f(b)]\right|\\&\le\frac{M}{n!(b-a)}[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]\left(\int_0^1 B_t(n+1,\alpha-n)^{p}dt\right)^{\frac{1}{p}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha\in[n,n+1)$.

Theorem 3.6 ([25]) *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$ *such that* $f^{'}\in L[a,b]$. *If* $|f^{'}|^{q}$ *is convex,* $q\ge1$, *and* $|f^{'}(x)|\le M$, $x\in[a,b]$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha-n)[(x-a)^{\alpha}+(b-x)^{\alpha}]}{\Gamma(\alpha+1)(b-a)}f(x)-\frac{1}{b-a}[{}^{x}I_{\alpha}f(a)+I_{\alpha}^{x}f(b)]\right|\\&\le M\frac{\Gamma(\alpha-n+1)}{\Gamma(\alpha+2)(b-a)}\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right],\end{aligned}$$

where $\alpha\in[n,n+1)$.

Theorem 3.7 ([25]) *Let* $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ *be a differentiable mapping on (a,b) with* $a<b$ *such that* $f^{'}\in L[a,b]$. *If* $|f^{'}|^{q}$ *is a concave on* $[a,b]$ *and* $p,q>1$, *then the following inequality for conformable fractional integrals holds:*

$$\left|\frac{\Gamma(\alpha-n)[(x-a)^{\alpha}+(b-x)^{\alpha}]}{\Gamma(\alpha+1)(b-a)}f(x)-\frac{1}{b-a}[{}^{x}I_{\alpha}f(a)+I_{\alpha}^{x}f(b)]\right|$$
$$\leq\left(\int_{0}^{1}B_{t}(n+1,\alpha-n)^{p}dt\right)^{\frac{1}{p}}$$
$$\times\left[\frac{(x-a)^{\alpha+1}}{n!(b-a)}\left|f^{'}\left(\frac{x+a}{2}\right)\right|+\frac{(b-x)^{\alpha+1}}{n!(b-a)}\left|f^{'}\left(\frac{b+x}{2}\right)\right|\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha\in[n,n+1)$.

In [26], Akdemir, Ekinci, and Set have proved some inequalities involving conformable fractional integral operators as follows:

Theorem 3.8 *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$ *such that* $f'\in L[a,b]$. *Suppose that there exist two integrable functions* φ_1,φ_2 *on* $[a,b]$ *such that*

$$\varphi_1(t)\leq f(t)\leq\varphi_2(t),\quad\forall t\in[a,b]. \tag{3.7}$$

Then, the inequality

$${}^{x}I_{\alpha}\varphi_2(a)\,I_{\alpha}^{x}f(b)+I_{\alpha}^{x}\varphi_1(b)^{x}\,I_{\alpha}f(a)\geq I_{\alpha}^{x}\varphi_1(b)^{x}\,I_{\alpha}\varphi_2(a)+I_{\alpha}^{x}f(b)^{x}\,I_{\alpha}f(a)$$

holds true, where $x\in[a,b]$.

Proof From the inequality (3.7), for all $u,v\in[a,b]$, we have

$$(\varphi_2(u)-f(u))(f(v)-\varphi_1(v))\geq 0.$$

This implies that

$$\varphi_2(u)f(v)+\varphi_1(v)f(u)\geq\varphi_1(v)\varphi_2(u)+f(u)f(v).$$

For $x\in[a,b]$, if we use the change of variables $u=rx+(1-r)a$ and $v=sx+(1-s)b$ for $r,s\in[0,1]$ and multiply both sides of the above inequality by

$$\left[r^{n}(1-r)^{\alpha-n-1}\right]\left[s^{n}(1-s)^{\alpha-n-1}\right],$$

later by integrating the resulting expression with respect to r and s, we have the following equality for the first integral

$$\int_{0}^{1}\int_{0}^{1}\left[r^{n}(1-r)^{\alpha-n-1}\right]\left[s^{n}(1-s)^{\alpha-n-1}\right]\varphi_2(rx+(1-r)a)f(sx+(1-s)b)\,drds$$
$$=\int_{0}^{1}\left[s^{n}(1-s)^{\alpha-n-1}\right]f(sx+(1-s)b)\,ds\int_{0}^{1}\left[r^{n}(1-r)^{\alpha-n-1}\right]\varphi_2(rx+(1-r)a)\,dr.$$

By using the change of variables above, we get

$$\int_0^1 \left[s^n (1-s)^{\alpha-n-1}\right] f(sx+(1-s)b)\,ds \int_0^1 \left[r^n (1-r)^{\alpha-n-1}\right] \varphi_2(rx+(1-r)a)\,dr$$
$$= \left[\int_x^b \left(\frac{x-v}{x-b}\right)^n \left(\frac{v-b}{x-b}\right)^{\alpha-n-1} \frac{f(v)}{x-b}dv\right]$$
$$\left[\int_a^x \left(\frac{u-a}{x-a}\right)^n \left(\frac{x-u}{x-a}\right)^{\alpha-n-1} \frac{\varphi_2(u)}{x-a}du\right]$$
$$= \left[\frac{1}{(b-x)^\alpha}\int_x^b (x-v)^n (v-b)^{\alpha-n-1} f(v)\,dv\right]$$
$$\left[\frac{1}{(x-a)^\alpha}\int_a^x (u-a)^n (x-u)^{\alpha-n-1} \varphi_2(u)\,du\right]$$
$$= (n!)^2\, {}^x I_\alpha \varphi_2(a)\, I_\alpha^x f(b).$$

If we proceed the similar methods for the other integrals, we deduce the desired result. □

Theorem 3.9 *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L[a,b]$. Suppose that $m\le f(t)\le M$, for all $t\in[a,b]$ and for some $m, M\in\mathbb{R}$. Then, the following inequality holds:*

$$\frac{(n!)\, m I_\alpha^x f(b)}{(b-x)^\alpha} + \frac{(n!)\, M\, {}^x I_\alpha f(a)}{(x-a)^\alpha} \ge \frac{(n!)^2 I_\alpha^x f(b)\, {}^x I_\alpha f(a)}{(b-x)^\alpha (x-a)^\alpha} + B(n+1,\alpha-n)\, mM.$$

Proof Since

$$m \le f(t) \le M,$$

for all $t, u, v \in [a,b]$, we have

$$(m - f(u))(f(v) - M) \ge 0.$$

By using the above inequality and a similar argument to the proof of Theorem 3.8, we get the desired result. □

Theorem 3.10 *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L[a,b]$. Suppose that there exist two integrable functions φ_1, φ_2 on $[a,b]$ such that*

$$\varphi_1(t) \le f(t) \le \varphi_2(t), \quad \forall t \in [a,b].$$

Then, the inequality

$$^{x}I_{\alpha}\varphi_2(a)\, I_{\alpha}^{x} f(b) + I_{\alpha}^{x}\varphi_1(b)^{x} I_{\alpha} f(a) \geq I_{\alpha}^{x}\varphi_1(b)^{x} I_{\alpha}\varphi_2(a) + I_{\alpha}^{x} f(b)^{x} I_{\alpha} f(a)$$

holds true, where $x \in [a, b]$.

Proof From inequality (3.7), for all $u, v \in [a, b]$, we have

$$\left(\varphi_2(u) - f'(u)\right)\left(f'(v) - \varphi_1(v)\right) \geq 0.$$

This implies

$$\varphi_2(u)\, f'(v) + \varphi_1(v)\, f'(u) \geq \varphi_1(v)\,\varphi_2(u) + f'(u)\, f'(v).$$

For $x \in [a, b]$, if we use the change of variables $u = rx + (1-r)a$ and $v = sb + (1-s)x$ for $r, s \in [0, 1]$ and multiply both sides of the above inequality by

$$B_r(n+1, \alpha - n)\, B_s(n+1, \alpha - n),$$

and then integrate it with respect to r and s, for the first integral, we have

$$\int_0^1\int_0^1 B_r(n+1, \alpha-n)\, B_s(n+1, \alpha-n)\,\varphi_2(rx+(1-r)a)\, f'(sb+(1-s)x)\, dr ds$$
$$= \int_0^1 B_s(n+1, \alpha-n)\, f'(sb+(1-s)x)\, ds \int_0^1 B_r(n+1, \alpha-n)\,\varphi_2(rx+(1-r)a)\, dr.$$

By using integration by parts and the change of variables above, we get

$$\int_0^1 B(n+1, \alpha-n)\, f'(sb+(1-s)x)\, ds$$
$$= B_s(n+1, \alpha-n)\left.\frac{f(sb+(1-s)x)}{b-x}\right|_0^1 - \int_0^1 s^n (1-s)^{\alpha-n-1}\frac{f(sb+(1-s)x)}{b-x} ds$$
$$= B(n+1, \alpha-n)\frac{f(b)}{b-x} - \frac{1}{b-x}\int_x^b \left(\frac{v-x}{b-x}\right)^n \left(\frac{b-v}{b-x}\right)^{\alpha-n-1} f(v)\, dv$$
$$= \frac{\Gamma(n+1)\,\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\frac{f(b)}{b-x} - \frac{1}{(b-x)^{\alpha+1}}\int_x^b (v-x)^n (b-v)^{\alpha-n-1} f(v)\, dv$$
$$= B(n+1, \alpha-n)\frac{f(b)\, n!}{b-x} - \frac{n!}{(b-x)^{\alpha+1}} I_{\alpha}^{x} f(b),$$

and

$$\int_0^1 B_r(n+1,\alpha-n)\,\varphi_2(rx+(1-r)a)\,dr$$
$$= B(n+1,\alpha-n)\int_a^x \varphi_2(u)\,du.$$

By changing of the variables above, we get

$$\begin{aligned}
&= \left[\int_x^b \left(\frac{x-v}{x-b}\right)^n \left(\frac{v-b}{x-b}\right)^{\alpha-n-1} \frac{f(v)}{x-b}dv\right] \\
&\quad \left[\int_a^x \left(\frac{u-a}{x-a}\right)^n \left(\frac{x-u}{x-a}\right)^{\alpha-n-1} \frac{f(u)}{x-a}du\right] \\
&= \left[\frac{1}{(b-x)^\alpha}\int_x^b (x-v)^n (v-b)^{\alpha-n-1}\,dv\right] \\
&\quad \left[\frac{1}{(x-a)^\alpha}\int_a^x (u-a)^n (x-u)^{\alpha-n-1}\,du\right] \\
&= (n!)^{2\,x} I_\alpha \varphi_2(a)\, I_\alpha^x f(b).
\end{aligned}$$

By using the similar methods for the other integrals, we deduce the desired result.□

Theorem 3.11 *Let $f, g : [a,b] \to \mathbb{R}$ be two Lipschitzian mappings with the constants $L_1 > 0$ and $L_2 > 0$, i.e.,*

$$|f(x)-f(y)| \le L_1|x-y|, \quad |g(x)-g(y)| \le L_2|x-y|, \tag{3.8}$$

for all $x, y \in [a,b]$. Then, the following inequality holds for conformable fractional integrals

$$\begin{aligned}
&\Big|\Gamma(\alpha-n)\left[{}^x I_\alpha(fg)(a) + I_\alpha^x(fg)(b)\right] \\
&-\Gamma(n+1)\left[{}^x I_\alpha g(a)\, I_\alpha^x f(b) + I_\alpha^x g(b)\,{}^x I_\alpha f(a)\right]\Big| \\
&\le \frac{L_1 L_2}{\Gamma(n+1)}\left[\frac{B(n+1,\alpha-n)}{(x-a)^\alpha}K_1 + \frac{B(n+1,\alpha-n)}{(b-x)^\alpha}K_2\right. \\
&\left.-\frac{2}{(x-a)^\alpha(b-x)^\alpha}K_3K_4\right],
\end{aligned}$$

where

$$K_1 = \frac{(x-a)^\alpha}{\Gamma(\alpha+3)}\Big((n+1)(n+2)x^2 - 2ax(n+1)(n-\alpha) + a^2(n-\alpha)(n-\alpha-1)\Big),$$

$$K_2 = \frac{\Gamma(n+1)\Gamma(\alpha-n)(x-b)^\alpha}{\Gamma(\alpha+3)} \times \Big((n+1)(n+2)b^2 - 2bx(n+1)(\alpha-n) + x^2(n-\alpha)(n-\alpha-1)\Big),$$

$$K_3 = \frac{\Gamma(n+1)\Gamma(\alpha-n)(x-a)^\alpha}{\Gamma(\alpha+3)} \times \Big((n+1)(n+2)x^2 - 2ax(n+1)(n-\alpha) + a^2(n-\alpha)(n-\alpha-1)\Big),$$

$$K_4 = \frac{\Gamma(n+1)\Gamma(\alpha-n)(x-b)^\alpha}{\Gamma(\alpha+2)}((n+1)b - x(n-\alpha)).$$

Proof By (3.8), we can write

$$|(f(x)-f(y))(g(x)-g(y))| \le L_1L_2(x-y)^2$$

for all $x, y \in [a,b]$. For $x \in [a,b]$, if we use the change of variables $u = rx + (1-r)a$ and $v = sx + (1-s)b$ for $r, s \in [0,1]$ and multiply both sides of the above inequality by $\left[r^n(1-r)^{\alpha-n-1}\right]\left[s^n(1-s)^{\alpha-n-1}\right]$, we get

$$\begin{aligned}
&\left[r^n(1-r)^{\alpha-n-1}\right]\left[s^n(1-s)^{\alpha-n-1}\right][|f(rx+(1-r)a)g(rx+(1-r)a)\\
&f(sx+(1-s)b)g(sx+(1-s)b) - f(rx+(1-r)a)g(sx+(1-s)b)\\
&+f(sx+(1-s)b)g(rx+(1-r)a)|]\\
\le&\left[r^n(1-r)^{\alpha-n-1}\right]\left[s^n(1-s)^{\alpha-n-1}\right]L_1L_2((rx+(1-r)a)-(sx+(1-s)b))^2.
\end{aligned}$$

Then by integrating the resulting inequality with respect to r and s, we have

$$\begin{aligned}
&\int_0^1\int_0^1 \left[r^n(1-r)^{\alpha-n-1}\right]\left[s^n(1-s)^{\alpha-n-1}\right][|f(rx+(1-r)a)g(rx+(1-r)a)\\
&f(sx+(1-s)b)g(sx+(1-s)b) - f(rx+(1-r)a)g(sx+(1-s)b)\\
&+f(sx+(1-s)b)g(rx+(1-r)a)|]\,drds\\
&\le L_1L_2\int_0^1\int_0^1 \left[r^n(1-r)^{\alpha-n-1}\right]\left[s^n(1-s)^{\alpha-n-1}\right]\\
&\quad((rx+(1-r)a)-(sx+(1-s)b))^2\,drds.
\end{aligned}$$

By computing above integrals and by using the definition of conformable fractional integrals, we get the result. □

Theorem 3.12 *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a function such that $f \in L_1[a, b]$, where $a, b \in I$ with $a < b$. If f is GA-convex function on $[a, b]$, we have the following inequalities for conformable fractional integrals:*

$$f\left(\sqrt{ab}\right) \le \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^{\alpha}\Gamma(\alpha-n)}[I_{\alpha}^{a} f(b) +^{b} I_{\alpha} f(a)] \le \frac{f(a)+f(b)}{2}.$$

Proof Since f is GA-convex function on $[a, b]$, we have

$$f\left(\sqrt{xy}\right) \le \frac{f(x)+f(y)}{2}.$$

for all $x, y \in [a, b]$ (with $t = \frac{1}{2}$ in the definition of GA-convexity). By setting $x = a^t b^{1-t}$ and $y = b^t a^{1-t}$, we get

$$2f\left(\sqrt{ab}\right) \le f\left(a^t b^{1-t}\right) + f\left(b^t a^{1-t}\right).$$

By multiplying both sides of this inequality by $\frac{1}{n!}t^n(1-t)^{\alpha-n-1}$, then integrating the resulting inequality with respect to t over [0, 1], we obtain

$$\begin{aligned} &\frac{2}{n!} f\left(\sqrt{ab}\right)\int_0^1 t^n(1-t)^{\alpha-n-1}dt \\ &\le \frac{1}{n!}\int_0^1 t^n(1-t)^{\alpha-n-1} f\left(a^t b^{1-t}\right)dt \\ &+\frac{1}{n!}\int_0^1 t^n(1-t)^{\alpha-n-1} f\left(b^t a^{1-t}\right)dt. \end{aligned}$$

Namely,

$$f\left(\sqrt{ab}\right) \le \frac{\Gamma(\alpha+1)}{2(\ln\frac{b}{a})^{\alpha}\Gamma(\alpha-n)}[I_{\alpha}^{a} f(b) +^{b} I_{\alpha} f(a)],$$

which completes the proof of the first inequality. For the proof of the second inequality, we can write

$$f\left(a^t b^{1-t}\right) \le t f(a) + (1-t) f(b)$$

and

$$f\left(b^t a^{1-t}\right) \le t f(b) + (1-t) f(a).$$

By adding these inequalities, we have

$$f\left(a^t b^{1-t}\right) + f\left(b^t a^{1-t}\right) \le f(a) + f(b).$$

Multiplying both sides of this inequality by $\frac{1}{n!}t^n(1-t)^{\alpha-n-1}$, then integrating the resulting inequality with respect to t over [0, 1], we deduce

$$\frac{\Gamma(\alpha+1)}{(\ln b - \ln a)^{\alpha}\Gamma(\alpha-n)}[I_{\alpha}^{a} f(b) +^{b} I_{\alpha} f(a)] \le f(a) + f(b).$$

This completes the proof. □

Lemma 3.2 *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I^o such that $f' \in L_1[a,b]$, where $a, b \in I$ with $a < b$. Then, for all $x \in [a,b]$ and $\alpha \in (n, n+1]$, we have*

$$\begin{aligned}&\left(\ln \frac{x}{a}\right)^{\alpha} \int_0^1 B_t(n+1, \alpha-n)\, df\left(x^t a^{1-t}\right) \\ &+ \left(\ln \frac{b}{x}\right)^{\alpha} \int_0^1 B_t(n+1, \alpha-n)\, df\left(b^t x^{1-t}\right) \\ =& \frac{\Gamma(n+1)\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\left[\left(\ln\frac{x}{a}\right)^{\alpha} f(x) + \left(\ln\frac{b}{x}\right)^{\alpha} f(b)\right] \\ &- n![I_{\alpha}^{\ln a}(f\circ \exp)(\ln x) +^{\ln b} I_{\alpha}(f\circ\exp)(\ln x)].\end{aligned}$$

Proof By using integration by parts in the left-hand side of the above inequality, one can obtain the right-hand side. We omit the details. □

For simplicity, we will use following notation

$$\begin{aligned}&F_f(\alpha, n; x) \\ =& \frac{\Gamma(n+1)\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\left[\left(\ln\frac{x}{a}\right)^{\alpha} f(x) + \left(\ln\frac{b}{x}\right)^{\alpha} f(b)\right] \\ &- n![I_{\alpha}^{\ln a}(f\circ \exp)(\ln x) +^{\ln b} I_{\alpha}(f\circ\exp)(\ln x)].\end{aligned}$$

Theorem 3.13 *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I^o such that $f' \in L_1[a,b]$, where $a, b \in I$ with $a < b$. If $\left|f'\right|^q$ is quasi-geometrically convex on $[a,b]$ and $q > 1$, then we have the following inequality for conformable fractional integrals*

$$\begin{aligned}
&\left|F_f(\alpha,n;x)\right| \\
&\leq \left(\int_0^1 |B_t(n+1,\alpha-n)|^p\,dt\right)^{\frac{1}{p}} \left(\ln\frac{x}{a}\right)^{\alpha} L^{\frac{1}{q}}\left(a^q,x^q\right)\sup\left\{\left|f'(x)\right|,\left|f'(a)\right|\right\} \\
&\quad + \left(\int_0^1 |B_t(n+1,\alpha-n)|^p\,dt\right)^{\frac{1}{p}} \left(\ln\frac{b}{x}\right)^{\alpha} L^{\frac{1}{q}}\left(x^q,b^q\right)\sup\left\{\left|f'(b)\right|,\left|f'(x)\right|\right\},
\end{aligned}$$

for all $x\in[a,b]$, $p^{-1}+q^{-1}=1$ *and* $\alpha\in(n,n+1]$.

Proof By using Lemma 3.2 and by applying Hölder integral inequality, we can write

$$\begin{aligned}
&\left|F_f(\alpha,n;x)\right| \\
&\leq \left(\ln\frac{x}{a}\right)^{\alpha}\int_0^1 B_t(n+1,\alpha-n)\,df\left(x^t a^{1-t}\right) \\
&\quad + \left(\ln\frac{b}{x}\right)^{\alpha}\int_0^1 B_t(n+1,\alpha-n)\,df\left(b^t x^{1-t}\right) \\
&\leq \left(\ln\frac{x}{a}\right)^{\alpha}\left(\int_0^1 |B_t(n+1,\alpha-n)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 x^{qt}a^{q(1-t)}\left|f'\left(x^t a^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+ \left(\ln\frac{b}{x}\right)^{\alpha}\left(\int_0^1 |B_t(n+1,\alpha-n)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 b^{qt}x^{q(1-t)}\left|f'\left(b^t x^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|f'\right|^q$ is quasi-geometrically convex, we get

$$\begin{aligned}
&\left|F_f(\alpha,n;x)\right| \\
&\leq \left(\ln\frac{x}{a}\right)^{\alpha}\sup\left\{\left|f'(x)\right|,\left|f'(a)\right|\right\}\left(\int_0^1 |B_t(n+1,\alpha-n)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 x^{qt}a^{q(1-t)}dt\right)^{\frac{1}{q}} \\
&\quad + \left(\ln\frac{b}{x}\right)^{\alpha}\sup\left\{\left|f'(b)\right|,\left|f'(x)\right|\right\}\left(\int_0^1 |B_t(n+1,\alpha-n)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 b^{qt}x^{q(1-t)}dt\right)^{\frac{1}{q}}.
\end{aligned}$$

By computing the above integrals, one can easily obtain the desired inequality. □

Theorem 3.14 *Let* $f: I\subset(0,\infty)\to\mathbb{R}$ *be a differentiable function on* I^o *such that* $f'\in L_1[a,b]$, *where* $a,b\in I$ *with* $a<b$. *If* $\left|f'\right|^q$ *is quasi-geometrically convex on*

$[a, b]$, *then we have the following inequality for conformable fractional integrals*

$$\begin{aligned}
&\left|F_f(\alpha, n; x)\right| \\
&\leq \left(\ln \frac{x}{a}\right)^{\alpha} \sup\left\{\left|f'(x)\right|, \left|f'(a)\right|\right\} \\
&\quad \times \left(\frac{n!\Gamma(\alpha-n+1)}{\Gamma(\alpha+2)}\right)^{1-\frac{1}{q}} \left(\int_0^1 \left|B_t(n+1, \alpha-n)\right| x^{qt} a^{q(1-t)} dt\right)^{\frac{1}{q}} \\
&\quad + \left(\ln \frac{b}{x}\right)^{\alpha} \sup\left\{\left|f'(b)\right|, \left|f'(x)\right|\right\} \\
&\quad \times \left(\frac{n!\Gamma(\alpha-n+1)}{\Gamma(\alpha+2)}\right)^{1-\frac{1}{q}} \left(\int_0^1 \left|B_t(n+1, \alpha-n)\right| b^{qt} x^{q(1-t)} dt\right)^{\frac{1}{q}},
\end{aligned}$$

for all $x \in [a, b]$, $\alpha \in (n, n+1]$ *where* $q \geq 1$.

Proof From Lemma 3.2 and the power-mean integral inequality, we have

$$\begin{aligned}
&\left|F_f(\alpha, n; x)\right| \\
&\leq \left(\ln \frac{x}{a}\right)^{\alpha} \int_0^1 B_t(n+1, \alpha-n)\, df\left(x^t a^{1-t}\right) \\
&\quad + \left(\ln \frac{b}{x}\right)^{\alpha} \int_0^1 B_t(n+1, \alpha-n)\, df\left(b^t x^{1-t}\right) \\
&\leq \left(\ln \frac{x}{a}\right)^{\alpha} \left(\int_0^1 \left|B_t(n+1, \alpha-n)\right| dt\right)^{1-\frac{1}{q}} \\
&\quad \left(\int_0^1 \left|B_t(n+1, \alpha-n)\right| x^{qt} a^{q(1-t)} \left|f'\left(x^t a^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad + \left(\ln \frac{b}{x}\right)^{\alpha} \left(\int_0^1 \left|B_t(n+1, \alpha-n)\right| dt\right)^{1-\frac{1}{q}} \\
&\quad \left(\int_0^1 \left|B_t(n+1, \alpha-n)\right| b^{qt} x^{q(1-t)} \left|f'\left(b^t x^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

By taking into account quasi-geometrically convexity of $\left|f'\right|^q$, we obtain

$$
\begin{aligned}
&\left|F_f(\alpha, n; x)\right| \\
&\leq \tau^{1-\frac{1}{q}} \left(\ln \frac{x}{a}\right)^{\alpha} \left(\int_0^1 |B_t(n+1, \alpha-n)| \, x^{qt} a^{q(1-t)} dt \right)^{\frac{1}{q}} \\
&\quad \times \sup\left\{ \left|f'(x)\right|, \left|f'(a)\right| \right\} \\
&\quad + \tau^{1-\frac{1}{q}} \left(\ln \frac{b}{x}\right)^{\alpha} \left(\int_0^1 |B_t(n+1, \alpha-n)| \, b^{qt} x^{q(1-t)} dt \right)^{\frac{1}{q}} \\
&\quad \times \sup\left\{ \left|f'(b)\right|, \left|f'(x)\right| \right\}.
\end{aligned}
$$

Then, use the following formula:

$$
\begin{aligned}
\int_0^1 |B_t(n+1, \alpha-n)| \, dt &= B(n+1, \alpha-n) - B(n+2, \alpha-n) \\
&= \frac{n!\Gamma(\alpha-n+1)}{\Gamma(\alpha+2)}.
\end{aligned}
$$

This completes the proof. □

Corollary 3.1 *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on I^o such that $f' \in L_1[a, b]$, where $a, b \in I$ with $a < b$. If $\left|f'\right|^q$ is quasi-geometrically convex on $[a, b]$ and $q > 1$, then we have the following inequality for conformable fractional integrals*

$$
\begin{aligned}
&\left|F_f(\alpha, n; x)\right| \\
&\leq \left(\int_0^1 |B_t(n+1, \alpha-n)|^p \, dt \right)^{\frac{1}{p}} \left(\ln \frac{x}{a}\right)^{\alpha} L_q^{\frac{1}{q}}(a, x) \sup\left\{ \left|f'(x)\right|, \left|f'(a)\right| \right\} \\
&\quad + \left(\int_0^1 |B_t(n+1, \alpha-n)|^p \, dt \right)^{\frac{1}{p}} \left(\ln \frac{b}{x}\right)^{\alpha} L_q^{\frac{1}{q}}\left(x^q, b^q\right) \sup\left\{ \left|f'(b)\right|, \left|f'(x)\right| \right\},
\end{aligned}
$$

for all $x \in [a, b]$, $p^{-1} + q^{-1} = 1$ and $\alpha \in (n, n+1]$.

Proof By using a similar argument as in the proof of Theorem 3.14, we can write

$$\begin{aligned}&\left|F_f(\alpha,n;x)\right|\\&\leq\left(\ln\frac{x}{a}\right)^{\alpha}\int_0^1 B_t\left(n+1,\alpha-n\right)df\left(x^t a^{1-t}\right)\\&\quad+\left(\ln\frac{b}{x}\right)^{\alpha}\int_0^1 B_t\left(n+1,\alpha-n\right)df\left(b^t x^{1-t}\right).\end{aligned}$$

By using the general Cauchy inequality, we have

$$\begin{aligned}&\left|F_f(\alpha,n;x)\right|\\&\leq\left(\ln\frac{x}{a}\right)^{\alpha}\int_0^1 B_t\left(n+1,\alpha-n\right)(tx+(1-t)a)\left|f'\left(x^t a^{1-t}\right)\right|dt\\&\quad+\left(\ln\frac{b}{x}\right)^{\alpha}\int_0^1 B_t\left(n+1,\alpha-n\right)(tb+(1-t)x)\left|f'\left(b^t x^{1-t}\right)\right|dt.\end{aligned}$$

By applying the Hölder integral inequality and from quasi-geometrically convexity of $\left|f'\right|^q$, we obtain

$$\begin{aligned}&\left|F_f(\alpha,n;x)\right|\\&\leq\left(\ln\frac{x}{a}\right)^{\alpha}\sup\left\{\left|f'(x)\right|,\left|f'(a)\right|\right\}\left(\int_0^1\left|B_t\left(n+1,\alpha-n\right)\right|^p dt\right)^{\frac{1}{p}}\\&\quad\left(\int_0^1(tx+(1-t)a)^q\,dt\right)^{\frac{1}{q}}\\&\quad+\left(\ln\frac{b}{x}\right)^{\alpha}\sup\left\{\left|f'(b)\right|,\left|f'(x)\right|\right\}\left(\int_0^1\left|B_t\left(n+1,\alpha-n\right)\right|^p dt\right)^{\frac{1}{p}}\\&\quad\left(\int_0^1(tb+(1-t)x)^q\,dt\right)^{\frac{1}{q}}.\end{aligned}$$

By computing the above integrals, we get the result. □

Corollary 3.2 *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I^o such that $f' \in L_1[a,b]$, where $a, b \in I$ with $a < b$. If $\left|f'\right|^q$ is quasi-geometrically convex on $[a,b]$, then we have the following inequality for conformable fractional integrals*

$$
\begin{aligned}
&\left|F_f(\alpha,n;x)\right| \\
&\leq \left(\ln\frac{x}{a}\right)^{\alpha}\sup\left\{\left|f'(x)\right|,\left|f'(a)\right|\right\}\left(\frac{n!\Gamma(\alpha-n+1)}{\Gamma(\alpha+2)}\right)^{1-\frac{1}{q}} \\
&\times\left(B(n+1,\alpha-n)A\left(a^q,x^q\right)-\frac{x^q}{2}B(n+3,\alpha-n)-\frac{a^q}{2}\tau_1\right)^{\frac{1}{q}} \\
&+\left(\ln\frac{b}{x}\right)^{\alpha}\sup\left\{\left|f'(b)\right|,\left|f'(x)\right|\right\}\left(\frac{n!\Gamma(\alpha-n+1)}{\Gamma(\alpha+2)}\right)^{1-\frac{1}{q}} \\
&\times\left(B(n+1,\alpha-n)A\left(x^q,b^q\right)-\frac{b^q}{2}B(n+3,\alpha-n)-\frac{x^q}{2}\tau_1\right)^{\frac{1}{q}}
\end{aligned}
$$

for all $x\in[a,b]$*,* $\alpha\in(n,n+1]$ *where* $q\geq 1$ *and* $\tau_1=\frac{(2\alpha-n+2)\Gamma(n+2)\Gamma(\alpha-n)}{\Gamma(\alpha+3)}$.

Proof If we use the general Cauchy inequality and power-mean inequality in the proof of Theorem 3.14, we can write

$$
\begin{aligned}
&\left|F_f(\alpha,n;x)\right| \\
&\leq \left(\ln\frac{x}{a}\right)^{\alpha}\sup\left\{\left|f'(x)\right|,\left|f'(a)\right|\right\} \\
&\times\left(\frac{n!\Gamma(\alpha-n+1)}{\Gamma(\alpha+2)}\right)^{1-\frac{1}{q}}\left(\int_0^1\left|B_t(n+1,\alpha-n)\right|\left(tx^q+(1-t)a^q\right)dt\right)^{\frac{1}{q}} \\
&+\left(\ln\frac{b}{x}\right)^{\alpha}\sup\left\{\left|f'(b)\right|,\left|f'(x)\right|\right\} \\
&\times\left(\frac{n!\Gamma(\alpha-n+1)}{\Gamma(\alpha+2)}\right)^{1-\frac{1}{q}}\left(\int_0^1\left|B_t(n+1,\alpha-n)\right|\left(tb^q+(1-t)x^q\right)dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

By computing the above integrals, we get the desired result. □

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