Some Identities on Derangement and Degenerate Derangement Polynomials

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Abstract In combinatorics, a derangement is a permutation that has no fixed points. The number of derangements of an n-element set is called the nth derangement number. In this paper, as natural companions to derangement numbers and degenerate versions of the companions we introduce derangement polynomials and degenerate derangement polynomials. We give some of their properties, recurrence relations, and identities for those polynomials which are related to some special numbers and polynomials.

Keywords Derangement polynomials · Degenerate derangement polynomials

2010 Mathematices Subject Classification 11B83 · 11B73 · 05A19

1 Introduction

It is known that the Fubini polynomials are defined by the generating function

$$\frac{1}{1-y(e^t-1)}=\sum_{n=0}^{\infty}F_n(y)\frac{t^n}{n!}, \quad \text{(see [7, 11])}. \tag{1.1}$$

Thus, by (1.1), we get

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P. Agarwal et al. (eds.), *Advances in Mathematical Inequalities and Applications*, Trends in Mathematics,
https://doi.org/10.1007/978-981-13-3013-1_13

$$F_n(y) = \sum_{k=0}^{n} S_2(n,k)k!y^k, \quad \text{(see [7, 11])}. \tag{1.2}$$

Here $S_2(n,k)$ is the Stirling number of the second kind which is defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \ \ (n \geq 0), \tag{1.3}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\dots(x-n+1)$, $(n \geq 1)$.

As is well known, the Bell polynomials are given by the generating function as follows:

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}, \quad \text{(see [5, 6, 12])}. \tag{1.4}$$

When $x = 1$, $Bel_n = Bel_n(1)$ are called the Bell numbers. For $\lambda \in \mathbb{R}$, the partially degenerate Bell polynomials were introduced by Kim–Kim–Dolgy as

$$e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} = \sum_{n=0}^{\infty} Bel_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [12])}. \tag{1.5}$$

Note that $\lim_{\lambda\to 0} Bel_{n,\lambda}(x) = Bel_n(x)$, $(n \geq 0)$. When $x = 1$, $Bel_{n,\lambda} = Bel_{l,\lambda}(1)$ are called the partially degenerate Bell numbers.

From (1.5), we have

$$Bel_{n,\lambda}(x) = \sum_{k=0}^{n}\sum_{m=0}^{k} S_2(k,m)S_1(n,k)\lambda^{n-k}x^m, \tag{1.6}$$

where $S_1(n,k)$ is the Stirling number of the first kind given by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \ \ (n \geq 0), \quad \text{(see [8])}. \tag{1.7}$$

In [1], Carlitz introduced the degenerate Bernoulli and Euler polynomials which are defined by

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!}, \tag{1.8}$$

and

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.9}$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$, $\mathcal{E}_{n,\lambda} = \mathcal{E}_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers and degenerate Euler numbers.

Recently, the degenerate Stirling numbers of the second kind are defined by

$$S_{2,\lambda}(n+1,k) = kS_{2,\lambda}(n,k) + S_{2,\lambda}(n,k-1) - n\lambda S_{2,\lambda}(n,k), \tag{1.10}$$

where $n \geq 0$ (see [10]).

Note that $\lim_{\lambda\to 0} S_{2,\lambda}(n,k) = S_2(n,k)$. For $\lambda \in \mathbb{R}$, the λ-analogue of falling factorial sequence is defined by

$$(x)_{0,\lambda} = 1, \ (x)_{n,\lambda} = x(x-\lambda)\dots(x-(n-1)\lambda), \ (n \geq 1), \quad \text{(see [6, 8])}. \tag{1.11}$$

Note that $\lim_{\lambda\to 1}(x)_{n,\lambda} = (x)_n$, $(n \geq 0)$, (see [14]).

A derangement is a permutation with no fixed points. In other words, a derangement of a set leaves no elements in the original place. The number of derangements of a set of size n, denoted d_n, is called the nth derangement number (see [9, 15, 16]).

For $n \geq 0$, it is well known that the recurrence relation of derangement numbers is given by

$$d_n = \sum_{k=0}^{n} \binom{n}{k}(n-k)!(-1)^k = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}, \quad \text{(see [9])}. \tag{1.12}$$

It is not difficult to show that

$$\sum_{n=0}^{\infty} d_n \frac{t^n}{n!} = \frac{1}{1-t}e^{-t}, \quad \text{(see [2, 3, 4, 5, 9])}. \tag{1.13}$$

From (1.13), we note that

$$d_n = n \cdot d_{n-1} + (-1)^n, \ (n \geq 1), \quad \text{(see [9, 13, 14, 16, 17])}. \tag{1.14}$$

and

$$d_n = (n-1)(d_{n-1} + d_{n-2}), \ (n \geq 2). \tag{1.15}$$

In this paper, as natural companions to derangement numbers and degenerate versions of the companions we introduce derangement polynomials and degenerate derangement polynomials. We give some of their properties, recurrence relations, and identities for those polynomials which are related to some special numbers and polynomials.

2 Derangement Polynomials

Now, we define the derangement polynomials which are given by the generating function

$$\frac{1}{1-xt}e^{-t}=\sum_{n=0}^{\infty}d_n(x)\frac{t^n}{n!}. \tag{2.1}$$

When $x=1$, $d_n(1)=d_n$ are the derangement numbers.

From (1.1), we note that

$$\begin{aligned}\frac{1}{1-yt}&=\sum_{m=0}^{\infty}F_m(y)\frac{1}{m!}\big(\log(1+t)\big)^m\\&=\sum_{m=0}^{\infty}F_m(y)\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}F_m(y)S_1(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.2}$$

On the other hand,

$$\frac{1}{1-yt}=\sum_{n=0}^{\infty}y^n n!\frac{t^n}{n!}. \tag{2.3}$$

Therefore, by (2.2) and (2.3), we obtain the following lemma.

Lemma 2.1 *For $n\geq 0$, we have*

$$y^n=\frac{1}{n!}\sum_{m=0}^{n}F_m(y)S_1(n,m).$$

We observe that

$$\begin{aligned}\frac{1}{1-yt}&=\left(\frac{1}{1-yt}e^{-t}\right)e^t=\left(\sum_{l=0}^{\infty}d_l(y)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\frac{t^m}{m!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}d_l(y)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

From (2.2) and (2.4), we obtain the following theorem.

Theorem 2.2 *For $n\geq 0$, we have*

$$\sum_{l=0}^{n}\binom{n}{l}d_l(y)=\sum_{m=0}^{n}F_m(y)S_1(n,m).$$

By (2.1), we get

$$\begin{aligned}\sum_{n=0}^{\infty} d_n(x)\frac{t^n}{n!} &= \frac{1}{1-xt}e^{-t} = \left(\sum_{m=0}^{\infty} x^m t^m\right)\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}t^k\right) \\ &= \sum_{n=0}^{\infty}\left(n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}x^{n-k}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.5}$$

By comparing the coefficients on both sides of (2.5), we obtain the following theorem.

Theorem 2.3 *For $n \geq 0$, we have*

$$d_n(x) = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}x^{n-k}.$$

From (2.1), we have

$$\begin{aligned}e^{-t} &= (1-xt)\sum_{n=0}^{\infty} d_n(x)\frac{t^n}{n!} \\ &= d_0(x) + \sum_{n=1}^{\infty}\left(d_n(x) - nxd_{n-1}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

On the other hand,

$$e^{-t} = \sum_{n=0}^{\infty}(-1)^n\frac{t^n}{n!}. \tag{2.7}$$

Thus, by (2.6) and (2.7), we get

$$d_0(x) = 1,\ \ d_n(x) = nxd_{n-1}(x) + (-1)^n,\ \ (n \geq 1). \tag{2.8}$$

From (2.8), we note that

$$\begin{aligned}d_n(x) &= (nx-1)d_{n-1}(x) + d_{n-1}(x) + (-1)^n \\ &= (nx-1)d_{n-1}(x) + (n-1)xd_{n-2}(x) + (-1)^{n-1} + (-1)^n \\ &= (nx-1)\left[d_{n-1}(x) + d_{n-2}(x)\right] + (1-x)d_{n-2}(x),\ \ (n \geq 2).\end{aligned} \tag{2.9}$$

Therefore, we obtain the following theorem.

Theorem 2.4 *For $n \geq 1$, we have*

$$d_n(x) = nxd_{n-1}(x) + (-1)^n.$$

In particular, for $n \geq 2$, we have

$$d_n(x) = (nx - 1)\left[d_{n-1}(x) + d_{n-2}(x)\right] + (1 - x)d_{n-2}(x).$$

Replacing t by $e^t - 1$ in (2.1), we get

$$\begin{aligned}\frac{1}{1 - x(e^t - 1)}e^{-(e^t-1)} &= \sum_{m=0}^{\infty} d_m(x)\frac{1}{m!}(e^t - 1)^m \\ &= \sum_{m=0}^{\infty} d_m(x)\sum_{n=m}^{\infty} S_2(n, m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} d_m(x)S_2(n, m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.10}$$

By (2.10), we see that

$$\begin{aligned}\frac{1}{1 - x(e^t - 1)} &= e^{(e^t-1)}\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} d_m(x)S_2(k, m)\right)\frac{t^k}{k!} \\ &= \left(\sum_{l=0}^{\infty} Bel_l\frac{t^l}{l!}\right)\left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} d_m(x)S_2(k, m)\right)\frac{t^k}{k!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}d_m(x)S_2(k, m)Bel_{n-k}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.11}$$

From (1.1), we note that

$$\frac{1}{1 - x(e^t - 1)} = \sum_{n=0}^{\infty} F_n(x)\frac{t^n}{n!}. \tag{2.12}$$

Therefore, by (2.11) and (2.12), we obtain the following theorem.

Theorem 2.5 *For $n \geq 0$, we have*

$$F_n(x) = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}d_m(x)S_2(k, m)Bel_{n-k}.$$

From (1.1), we can derive the following Eq. (2.13):

$$\begin{aligned}\frac{1}{1-xt}e^{-t} &= \left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}F_m(x)S_1(k,m)\right)\frac{t^k}{k!}\right)e^{-t} \\ &= \left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}F_m(x)S_1(k,m)\right)\frac{t^k}{k!}\right)\left(\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}t^l\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}F_m(x)S_1(k,m)\frac{(-1)^{n-k}}{(n-k)!}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.13}$$

On the other hand,

$$\frac{1}{1-xt}e^{-t} = \sum_{n=0}^{\infty}d_n(x)\frac{t^n}{n!}. \tag{2.14}$$

Therefore, by (2.13) and (2.14), we obtain the following theorem.

Theorem 2.6 *For $n \geq 0$, we have*

$$d_n(x) = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}F_m(x)S_1(k,m)\frac{(-1)^{n-k}}{(n-k)!}.$$

As is known, Bernoulli polynomials are defined by the generating function

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \quad \text{(see [15])}. \tag{2.15}$$

When $x = 0$, $B_n = B_n(0)$ are Bernoulli numbers. By (2.15), we easily get

$$\begin{aligned}\sum_{k=0}^{m-1}e^{kt} &= \frac{1}{e^t-1}\left(e^{mt}-1\right) = \frac{1}{t}\left\{\frac{t}{e^t-1}e^{mt} - \frac{t}{e^t-1}\right\} \\ &= \sum_{n=0}^{\infty}\left(\frac{B_{n+1}(m)-B_{n+1}}{n+1}\right)\frac{t^n}{n!}, \ (n \geq 1).\end{aligned} \tag{2.16}$$

By Taylor expansion, we get

$$\sum_{k=0}^{m-1}e^{kt} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{m-1}k^n\right)\frac{t^n}{n!}, \ (m \geq 1). \tag{2.17}$$

From (2.16) and (2.17), we get

$$\sum_{k=0}^{m-1}k^n = \frac{B_{n+1}(m)-B_{n+1}}{n+1}. \tag{2.18}$$

By Lemma 2.1, we easily get

$$\sum_{k=0}^{m-1} k^n = \frac{1}{n!} \sum_{k=0}^{m-1} \sum_{l=0}^{n} F_l(k) S_1(n,l). \tag{2.19}$$

Therefore, by Theorem 2.2, (2.18), and (2.19), we obtain the following theorem.

Theorem 2.7 *For $m \geq 1$ and $n \geq 0$, we have*

$$\begin{aligned}\frac{B_{n+1}(m) - B_{n+1}}{n+1} &= \frac{1}{n!} \sum_{k=0}^{m-1} \sum_{l=0}^{n} F_l(k) S_1(n,l) \\ &= \frac{1}{n!} \sum_{k=0}^{m-1} \sum_{l=0}^{n} \binom{n}{l} d_l(k).\end{aligned}$$

3 Degenerate Derangement Polynomials

Here we consider the degenerate derangement polynomials which are given by

$$\frac{1}{1-xt}(1-\lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty} d_{n,\lambda}(x) \frac{t^n}{n!}, \quad (\lambda \in \mathbb{R}). \tag{3.1}$$

When $x = 1$, $d_{n,\lambda} = d_{n,\lambda}(1)$ are called the degenerate derangement numbers. From (3.1), we note that

$$\begin{aligned}(1-\lambda t)^{\frac{1}{\lambda}} &= \left(\sum_{n=0}^{\infty} d_{n,\lambda}(x) \frac{t^n}{n!} \right) (1-xt) \\ &= \sum_{n=0}^{\infty} d_{n,\lambda}(x) \frac{t^n}{n!} - \sum_{n=0}^{\infty} x d_{n,\lambda}(x) \frac{t^{n+1}}{n!} \\ &= d_{0,\lambda}(x) + \sum_{n=1}^{\infty} \big(d_{n,\lambda}(x) - x n d_{n-1,\lambda}(x) \big) \frac{t^n}{n!}.\end{aligned} \tag{3.2}$$

On the other hand,

$$(1-\lambda t)^{\frac{1}{\lambda}} = \sum_{m=0}^{\infty} \binom{\frac{1}{\lambda}}{m} (-\lambda)^m t^m = \sum_{m=0}^{\infty} (-1)^m (1)_{m,\lambda} \frac{t^m}{m!}. \tag{3.3}$$

Therefore, by (3.2) and (3.3), we obtain the following theorem.

Theorem 3.1 *For $n \geq 0$, we have*

$$d_{0,\lambda}(x) = 1,\ d_{n,\lambda}(x) = nxd_{n-1,\lambda}(x) + (-1)^n (1)_{n,\lambda},\ \ (n \geq 1).$$

Note that $\lim_{\lambda\to 0} d_{n,\lambda}(x) = d_n(x)$, $\lim_{\lambda\to 0} d_{n,\lambda} = d_n$, $(n \geq 0)$.
From (3.1), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} d_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{1}{1-xt}(1-\lambda t)^{\frac{1}{\lambda}} = \left(\sum_{m=0}^{\infty} x^m t^m\right)\left(\sum_{k=0}^{\infty}(-1)^k (1)_{k,\lambda}\frac{t^k}{k!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^k}{k!}(1)_{k,\lambda}x^{n-k}\right)t^n.\end{aligned} \tag{3.4}$$

Comparing the coefficients on both sides of (3.4), we obtain the following theorem.

Theorem 3.2 *For $n \geq 0$, we have*

$$d_{n,\lambda}(x) = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}(1)_{k,\lambda}x^{n-k}.$$

In particular, for $x = 1$,

$$d_{n,\lambda} = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}(1)_{k,\lambda}.$$

Now, we observe that

$$\begin{aligned}\frac{1}{1-xt} &= \left(\frac{1}{1-xt}\right)(1-\lambda t)^{\frac{1}{\lambda}}\cdot(1-\lambda t)^{-\frac{1}{\lambda}} \\ &= \left(\sum_{l=0}^{\infty} d_{l,\lambda}(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\binom{-\frac{1}{\lambda}}{m}(-\lambda)^m t^m\right) \\ &= \left(\sum_{l=0}^{\infty} d_{l,\lambda}(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} 1(1+\lambda)\dots(1+(m-1)\lambda)\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}d_{l,\lambda}(x)(1)_{n-l,-\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{3.5}$$

On the other hand,

$$\frac{1}{1-xt} = \sum_{n=0}^{\infty} x^n n!\frac{t^n}{n!}. \tag{3.6}$$

Therefore, by (3.5) and (3.6), we obtain the following theorem.

Theorem 3.3 *For $n \geq 0$, we have*

$$x^n = \frac{1}{n!}\sum_{l=0}^{n}\binom{n}{l}d_{l,\lambda}(x)(1)_{n-l,-\lambda}.$$

From Theorem 3.1, we have

$$\begin{aligned} d_{n,\lambda}(x) &= nxd_{n-1,\lambda}(x) + (-1)^n(1)_{n,\lambda} \\ &= (nx-1)d_{n-1,\lambda}(x) + d_{n-1,\lambda}(x) + (-1)^n(1)_{n,\lambda} \\ &= (nx-1)d_{n-1,\lambda}(x) + (n-1)xd_{n-2,\lambda}(x) \\ &\quad + (-1)^{n-1}(1)_{n-1,\lambda} + (-1)^n(1)_{n,\lambda} \\ &= (nx-1)\left[d_{n-1,\lambda}(x) + d_{n-2,\lambda}(x)\right] \\ &\quad + (1-x)d_{n-2,\lambda}(x) + (-1)^{n-1}(1)_{n-1,\lambda}(n-1)\lambda, \end{aligned} \tag{3.7}$$

where $n \geq 2$.

Therefore, by (3.7), we obtain the following theorem.

Theorem 3.4 *For $n \geq 2$, we have*

$$\begin{aligned} d_{n,\lambda}(x) &= (nx-1)\left[d_{n-1,\lambda}(x) + d_{n-2,\lambda}(x)\right] \\ &\quad + (1-x)d_{n-2,\lambda}(x) + (-1)^{n-1}(1)_{n-1,\lambda}(n-1)\lambda. \end{aligned}$$

In particular, $x = 1$,

$$d_{n,\lambda} = (n-1)\left[d_{n-1,\lambda} + d_{n-2,\lambda}\right] + \lambda(n-1)(-1)^{n-1}(1)_{n-1,\lambda}.$$

Note that

$$d_n = \lim_{\lambda\to 0} d_{n,\lambda} = (n-1)\left[d_{n-1} + d_{n-2}\right] \quad (n \geq 2).$$

By using Taylor expansion, we get

$$\begin{aligned} (1-\lambda t)^{\frac{1}{\lambda}} = e^{\frac{1}{\lambda}\log(1-\lambda t)} &= \sum_{m=0}^{\infty}\lambda^{-m}\frac{1}{m!}\Big(\log(1-\lambda t)\Big)^m \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}(-1)^n S_1(n,m)\right)\frac{t^n}{n!}. \end{aligned} \tag{3.8}$$

On the other hand,

$$
\begin{aligned}
(1-\lambda t)^{\frac{1}{\lambda}} &= \frac{1}{1-xt}(1-\lambda t)^{\frac{1}{\lambda}}(1-xt) \\
&= \sum_{n=0}^{\infty} d_{n,\lambda}(x)\frac{t^n}{n!} - \sum_{n=1}^{\infty} nxd_{n-1,\lambda}(x)\frac{t^n}{n!} \\
&= d_{0,\lambda}(x) + \sum_{n=1}^{\infty}\left\{d_{n,\lambda}(x) - nxd_{n-1,\lambda}(x)\right\}\frac{t^n}{n!} \\
&= 1 + \sum_{n=1}^{\infty}\left(d_{n,\lambda}(x) - nxd_{n-1,\lambda}(x)\right)\frac{t^n}{n!}
\end{aligned}
\tag{3.9}
$$

From (3.8) and (3.9), we have

$$
(-1)^n \sum_{m=0}^{n} \lambda^{n-m} S_1(n,m) = d_{n,\lambda}(x) - nxd_{n-1,\lambda}(x) = (-1)^n (1)_{n,\lambda}, \ \ (n \geq 1).
\tag{3.10}
$$

Therefore, by (3.10), we obtain the following theorem.

Theorem 3.5 *For $n \geq 1$, we have*

$$
\sum_{m=0}^{n} \lambda^{n-m} S_1(n,m) = (1)_{n,\lambda}.
$$

By (1.13), we get

$$
\begin{aligned}
\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}+1} e^{(1+\lambda t)^{\frac{1}{\lambda}}} &= \sum_{m=0}^{\infty} (-1)^m d_m \frac{1}{m!}(1+\lambda t)^{\frac{m}{\lambda}} \\
&= \sum_{m=0}^{\infty} (-1)^m d_m \frac{1}{m!} \sum_{n=0}^{\infty} (m)_{n,\lambda}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} (-1)^m d_m \frac{(m)_{n,\lambda}}{m!}\right)\frac{t^n}{n!}.
\end{aligned}
\tag{3.11}
$$

On the other hand,

$$
\begin{aligned}
\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}+1} e^{(1+\lambda t)^{\frac{1}{\lambda}}} &= \frac{e}{2}\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} e^{(1+\lambda t)^{\frac{1}{\lambda}}-1} \\
&= \frac{e}{2}\left(\sum_{l=0}^{\infty} \mathcal{E}_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} Bel_{m,\lambda}\frac{t^m}{m!}\right) \\
&= \frac{e}{2}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} Bel_{m,\lambda}\mathcal{E}_{n-m,\lambda}\right)\frac{t^n}{n!}.
\end{aligned}
\tag{3.12}
$$

Therefore, by (3.11) and (3.12), we obtain the following theorem.

Theorem 3.6 *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} \binom{n}{m} Bel_{m,\lambda} \mathcal{E}_{n-m,\lambda} = \frac{2}{e} \sum_{m=0}^{\infty} (-1)^m d_m \frac{(m)_{n,\lambda}}{m!}.$$

From (3.11), we note that

$$\begin{aligned} e^{(1+\lambda t)^{\frac{1}{\lambda}}} &= \sum_{m=0}^{\infty} d_m \frac{(-1)^m}{m!} (1+\lambda t)^{\frac{m}{\lambda}} \left(1 + (1+\lambda t)^{\frac{1}{\lambda}}\right) \\ &= \sum_{m=0}^{\infty} d_m \frac{(-1)^m}{m!} (1+\lambda t)^{\frac{m}{\lambda}} + \sum_{m=0}^{\infty} d_m \frac{(-1)^m}{m!} (1+\lambda t)^{\frac{m+1}{\lambda}} \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{\infty} d_m \frac{(-1)^m}{m!} \left((m)_{n,\lambda} + (m+1)_{n,\lambda}\right) \right\} \frac{t^n}{n!}. \end{aligned} \tag{3.13}$$

On the other hand,

$$\begin{aligned} e^{(1+\lambda t)^{\frac{1}{\lambda}}} &= e \cdot e^{(1+\lambda t)^{\frac{1}{\lambda}} - 1} = e \sum_{k=0}^{\infty} \frac{1}{k!} \left((1+t)^{\frac{1}{\lambda}} - 1\right)^k \\ &= e \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} S_{2,\lambda}(n,k) \frac{t^n}{n!} = e \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} S_{2,\lambda}(n,k) \right) \frac{t^n}{n!}. \end{aligned} \tag{3.14}$$

Therefore, by (3.13) and (3.14), we obtain the following theorem.

Theorem 3.7 *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} S_{2,\lambda}(n,m) = \frac{1}{e} \sum_{m=0}^{\infty} d_m \frac{(-1)^m}{m!} \left((m)_{n,\lambda} + (m+1)_{n,\lambda}\right).$$

Indeed,

$$\begin{aligned} \sum_{n=0}^{\infty} Bel_{n,\lambda} \frac{t^n}{n!} &= e^{\left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)} = \sum_{m=0}^{\infty} \frac{1}{m!} \left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)^m \\ &= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} S_{2,\lambda}(n,m) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} S_{2,\lambda}(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{3.15}$$

Thus, by (3.15), we get

$$Bel_{n,\lambda} = \sum_{m=0}^{n} S_{2,\lambda}(n, m), \quad (n \geq 0). \tag{3.16}$$

Therefore, by (3.16), we obtain the following corollary.

Corollary 3.8 *For $n \geq 0$, we have*

$$Bel_{n,\lambda} = \frac{1}{e} \sum_{m=0}^{\infty} d_m \frac{(-1)^m}{m!} \left((m)_{n,\lambda} + (m+1)_{n,\lambda} \right).$$

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