Trends in Mathematics

Praveen Agarwal
Silvestru Sever Dragomir
Mohamed Jleli
Bessem Samet
Editors

Advances in Mathematical Inequalities and Applications

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Inequalities for the Generalized k-g-Fractional Integrals in Terms of Double Integral Means

Silvestru Sever Dragomir

Abstract In this chapter, we establish some inequalities for the k-g-fractional integrals of various subclasses of Lebesgue integrable functions in terms of double integral means. Some examples for the *generalized left-sided* and *right-sided Riemann–Liouville fractional integrals* of a function f with respect to another function g on $[a, b]$ and for general exponential fractional integrals are also given.

Keywords Generalized Riemann–Liouville fractional integrals · Hadamard fractional integrals · Functions of bounded variation · Ostrowski-type inequalities Trapezoid inequalities

1991 Mathematics Subject Classification 26D15 · 26D10 · 26D07 · 26A33

1 Introduction

Assume that the kernel k is defined either on $(0, \infty)$ or on $[0, \infty)$ with complex values and integrable on any finite subinterval. We define the function $K : [0, \infty) \to \mathbb{C}$ by

$$K(t) := \begin{cases} \int_0^t k(s)\, ds \text{ if } 0 < t, \\ 0 \text{ if } t = 0. \end{cases}$$

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As a simple example, if $k(t) = t^{\alpha-1}$, then for $\alpha \in (0, 1)$ the function k is defined on $(0, \infty)$ and $K(t) := \frac{1}{\alpha} t^{\alpha}$ for $t \in [0, \infty)$. If $\alpha \geq 1$, then k is defined on $[0, \infty)$ and $K(t) := \frac{1}{\alpha} t^{\alpha}$ for $t \in [0, \infty)$.

Let g be a strictly increasing function on (a, b), having a continuous derivative g' on (a, b). For the Lebesgue integrable function $f : (a, b) \to \mathbb{C}$, we define the *k-g-left-sided fractional integral* of f by

$$S_{k,g,a+} f(x) = \int_a^x k(g(x) - g(t)) g'(t) f(t) \, dt, \ x \in (a, b] \tag{1.1}$$

and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-} f(x) = \int_x^b k(g(t) - g(x)) g'(t) f(t) \, dt, \ x \in [a, b). \tag{1.2}$$

If we take $k(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$\begin{aligned} S_{k,g,a+} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x [g(x) - g(t)]^{\alpha-1} g'(t) f(t) \, dt \\ &=: I_{a+,g}^{\alpha} f(x), \ a < x \leq b \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} S_{k,g,b-} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b [g(t) - g(x)]^{\alpha-1} g'(t) f(t) \, dt \\ &=: I_{b-,g}^{\alpha} f(x), \ a \leq x < b, \end{aligned} \tag{1.4}$$

which are as defined in [24, p. 100].

For $g(t) = t$ in (1.4), we have the classical *Riemann–Liouville fractional integrals* while for the logarithmic function $g(t) = \ln t$, we have the *Hadamard fractional integrals* [24, p. 111]

$$H_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left[\ln\left(\frac{x}{t}\right) \right]^{\alpha-1} \frac{f(t) \, dt}{t}, \ 0 \leq a < x \leq b \tag{1.5}$$

and

$$H_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left[\ln\left(\frac{t}{x}\right) \right]^{\alpha-1} \frac{f(t) \, dt}{t}, \ 0 \leq a < x < b. \tag{1.6}$$

One can consider the function $g(t) = -t^{-1}$ and define the "*Harmonic fractional integrals*" by

$$R_{a+}^{\alpha} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}\, t^{\alpha+1}},\ 0 \le a < x \le b \tag{1.7}$$

and

$$R_{b-}^{\alpha} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(t)\,dt}{(t-x)^{1-\alpha}\, t^{\alpha+1}},\ 0 \le a < x < b. \tag{1.8}$$

Also, for $g(t) = \exp(\beta t)$, $\beta > 0$, we can consider the "*β -Exponential fractional integrals*"

$$E_{a+,\beta}^{\alpha} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_a^x \left[\exp(\beta x) - \exp(\beta t)\right]^{\alpha-1} \exp(\beta t)\, f(t)\, dt, \tag{1.9}$$

for $a < x \le b$ and

$$E_{b-,\beta}^{\alpha} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_x^b \left[\exp(\beta t) - \exp(\beta x)\right]^{\alpha-1} \exp(\beta t)\, f(t)\, dt, \tag{1.10}$$

for $a \le x < b$.

If we take $g(t) = t$ in (1.1) and (1.2), then we can consider the following *k-fractional integrals*

$$S_{k,a+} f(x) = \int_a^x k(x-t)\, f(t)\, dt,\ x \in (a, b] \tag{1.11}$$

and

$$S_{k,b-} f(x) = \int_x^b k(t-x)\, f(t)\, dt,\ x \in [a, b). \tag{1.12}$$

In [27], Raina studied a class of functions defined formally by

$$\mathcal{F}_{\rho,\lambda}^{\sigma}(x) := \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k,\ \ |x| < R,\ \text{with } R > 0 \tag{1.13}$$

for $\rho, \lambda > 0$ where the coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers. With the help of (1.13), Raina defined the following left-sided fractional integral operator

$$\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} f(x) := \int_a^x (x-t)^{\lambda-1}\, \mathcal{F}_{\rho,\lambda}^{\sigma}(w(x-t)^{\rho})\, f(t)\, dt,\ x > a \tag{1.14}$$

where $\rho, \lambda > 0$, $w \in \mathbb{R}$ and f is such that the integral on the right side exists.

In [1], the right-sided fractional operator was also introduced as

$$\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w} f(x) := \int_x^b (t-x)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda} (w(t-x)^{\rho}) f(t)\,dt, \ \ x < b \tag{1.15}$$

where $\rho, \lambda > 0$, $w \in \mathbb{R}$ and f is such that the integral on the right side exists. Several Ostrowski-type inequalities were also established.

We observe that for $k(t) = t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}(wt^{\rho})$, we re-obtain the definitions of (1.14) and (1.15) from (1.11) and (1.12).

In [25], Kirane and Torebek introduced the following *exponential fractional integrals*

$$\mathcal{T}^{\alpha}_{a+} f(x) := \frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\} f(t)\,dt, \ \ x > a \tag{1.16}$$

and

$$\mathcal{T}^{\alpha}_{b-} f(x) := \frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\} f(t)\,dt, \ \ x < b \tag{1.17}$$

where $\alpha \in (0, 1)$.

We observe that for $k(t) = \frac{1}{\alpha}\exp(-\frac{1-\alpha}{\alpha}t)$, $t \in \mathbb{R}$ we re-obtain the definitions of (1.16) and (1.17) from (1.11) and (1.12).

Let g be a strictly increasing function on (a, b), having a continuous derivative g' on (a, b). We can define the more general exponential fractional integrals

$$\mathcal{T}^{\alpha}_{g,a+} f(x) := \frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(g(x)-g(t))\right\} g'(t) f(t)\,dt, \ \ x > a \tag{1.18}$$

and

$$\mathcal{T}^{\alpha}_{g,b-} f(x) := \frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(g(t)-g(x))\right\} g'(t) f(t)\,dt, \ \ x < b \tag{1.19}$$

where $\alpha \in (0, 1)$.

Let g be a strictly increasing function on (a, b), having a continuous derivative g' on (a, b). Assume that $\alpha > 0$. We can also define the *logarithmic fractional integrals*

$$\mathcal{L}^{\alpha}_{g,a+} f(x) := \int_a^x (g(x)-g(t))^{\alpha-1} \ln(g(x)-g(t))\, g'(t) f(t)\,dt, \tag{1.20}$$

for $0 < a < x \le b$ and

$$\mathcal{L}^{\alpha}_{g,b-} f(x) := \int_x^b (g(t)-g(x))^{\alpha-1} \ln(g(t)-g(x))\, g'(t) f(t)\,dt, \tag{1.21}$$

for $0 < a \le x < b$, where $\alpha > 0$. These are obtained from (1.11) and (1.12) for the kernel $k(t) = t^{\alpha-1}\ln t$, $t > 0$.

For $\alpha = 1$, we get

$$\mathcal{L}_{g,a+} f(x) := \int_a^x \ln(g(x) - g(t)) g'(t) f(t) dt, \ 0 < a < x \leq b \tag{1.22}$$

and

$$\mathcal{L}_{g,b-} f(x) := \int_x^b \ln(g(t) - g(x)) g'(t) f(t) dt, \ 0 < a \leq x < b. \tag{1.23}$$

For $g(t) = t$, we have the simple forms

$$\mathcal{L}_{a+}^{\alpha} f(x) := \int_a^x (x-t)^{\alpha-1} \ln(x-t) f(t) dt, \ 0 < a < x \leq b, \tag{1.24}$$

$$\mathcal{L}_{b-}^{\alpha} f(x) := \int_x^b (t-x)^{\alpha-1} \ln(t-x) f(t) dt, \ 0 < a \leq x < b, \tag{1.25}$$

$$\mathcal{L}_{a+} f(x) := \int_a^x \ln(x-t) f(t) dt, \ 0 < a < x \leq b \tag{1.26}$$

and

$$\mathcal{L}_{b-} f(x) := \int_x^b \ln(t-x) f(t) dt, \ 0 < a \leq x < b. \tag{1.27}$$

For several Ostrowski-type inequalities for Riemann–Liouville fractional integrals see [2–19, 22–37] and the references therein.

For k and g as at the beginning of Introduction, we consider the mixed operator

$$\begin{aligned} & S_{k,g,a+,b-} f(x) \\ & := \frac{1}{2}\left[S_{k,g,a+} f(x) + S_{k,g,b-} f(x)\right] \\ & = \frac{1}{2}\left[\int_a^x k(g(x) - g(t)) g'(t) f(t) dt + \int_x^b k(g(t) - g(x)) g'(t) f(t) dt\right] \end{aligned} \tag{1.28}$$

for the Lebesgue integrable function $f : (a, b) \to \mathbb{C}$ and $x \in (a, b)$.

We also define the functions $\mathbf{K}_p : [0, \infty) \to [0, \infty)$ by

$$\mathbf{K}_p(t) := \begin{cases} \left(\int_0^t |k(s)|^p\right)^{1/p} ds \text{ if } 0 < t, \ p \geq 1 \\ 0 \text{ if } t = 0 \end{cases}$$

For $p = 1$, we put

$$\mathbf{K}(t) := \mathbf{K}_1(t) = \begin{cases} \int_0^t |k(s)|\, ds \text{ if } 0 < t, \\ \\ 0 \text{ if } t = 0. \end{cases}$$

Observe that

$$S_{k,g,x+} f(b) = \int_x^b k(g(b) - g(t))\, g'(t)\, f(t)\, dt,\ x \in [a, b) \tag{1.29}$$

and

$$S_{k,g,x-} f(a) = \int_a^x k(g(t) - g(a))\, g'(t)\, f(t)\, dt,\ x \in (a, b]. \tag{1.30}$$

We can define also the mixed operator

$$\begin{aligned} &\breve{S}_{k,g,a+,b-} f(x) \\ &:= \frac{1}{2}\left[S_{k,g,x+} f(b) + S_{k,g,x-} f(a)\right] \\ &= \frac{1}{2}\left[\int_x^b k(g(b) - g(t))\, g'(t)\, f(t)\, dt + \int_a^x k(g(t) - g(a))\, g'(t)\, f(t)\, dt\right] \end{aligned} \tag{1.31}$$

for any $x \in (a, b)$.

The following two parameters representation for the operators $S_{k,g,a+,b-}$ and $\breve{S}_{k,g,a+,b-}$ hold [21]:

Lemma 1 *Assume that the kernel k is defined either on* $(0, \infty)$ *or on* $[0, \infty)$ *with complex values and integrable on any finite subinterval. Let* $f : [a, b] \to \mathbb{C}$ *be an integrable function on* $[a, b]$ *and g be a strictly increasing function on* (a, b)*, having a continuous derivative* g' *on* (a, b)*. Then*

$$\begin{aligned} S_{k,g,a+,b-} f(x) &= \frac{1}{2}\left[\gamma K(g(b) - g(x)) + \lambda K(g(x) - g(a))\right] \\ &+ \frac{1}{2}\int_a^x k(g(x) - g(t))\, g'(t)\, [f(t) - \lambda]\, dt \\ &+ \frac{1}{2}\int_x^b k(g(t) - g(x))\, g'(t)\left[f(t) - \gamma\right] dt \end{aligned} \tag{1.32}$$

and

$$\begin{aligned} \breve{S}_{k,g,a+,b-} f(x) &= \frac{1}{2}\left[\gamma K(g(b) - g(x)) + \lambda K(g(x) - g(a))\right] \\ &+ \frac{1}{2}\int_a^x k(g(t) - g(a))\, g'(t)\, [f(t) - \lambda]\, dt \\ &+ \frac{1}{2}\int_x^b k(g(b) - g(t))\, g'(t)\left[f(t) - \gamma\right] dt \end{aligned} \tag{1.33}$$

for $x \in (a, b)$ *and for any* $\lambda, \gamma \in \mathbb{C}$.

In the recent paper [20], by using the above representations (1.32) and (1.33) we obtained the following result for functions of bounded variation:

Theorem 1 *Assume that the kernel k is defined either on* $(0, \infty)$ *or on* $[0, \infty)$ *with complex values and integrable on any finite subinterval. Let* $f : [a, b] \to \mathbb{C}$ *be a function of bounded variation on* $[a, b]$ *and g be a strictly increasing function on* (a, b), *having a continuous derivative* g' *on* (a, b). *Then we have the Ostrowski-type inequality*

$$\left| S_{k,g,a+,b-} f(x) - \frac{1}{2}\left[K(g(b) - g(x)) + K(g(x) - g(a))\right] f(x) \right| \tag{1.34}$$

$$\leq \frac{1}{2}\left[\int_x^b |k(g(t) - g(x))| \bigvee_x^t (f) g'(t)\, dt + \int_a^x |k(g(x) - g(t))| \bigvee_t^x (f) g'(t)\, dt\right]$$

$$\leq \frac{1}{2}\left[\mathbf{K}(g(b) - g(x)) \bigvee_x^b (f) + \mathbf{K}(g(x) - g(a)) \bigvee_a^x (f)\right] \tag{1.35}$$

$$\leq \frac{1}{2}\begin{cases} \max\{\mathbf{K}(g(b) - g(x)), \mathbf{K}(g(x) - g(a))\} \bigvee_a^b (f); \\ \\ \left[\mathbf{K}^p(g(b) - g(x)) + \mathbf{K}^p(g(x) - g(a))\right]^{1/p} \left(\left(\bigvee_a^x (f)\right)^q + \left(\bigvee_x^b (f)\right)^q\right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \left[\mathbf{K}(g(b) - g(x)) + \mathbf{K}(g(x) - g(a))\right]\left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right] \end{cases}$$

and the trapezoid-type inequality

$$\left| S_{k,g,a+,b-} f(x) - \frac{1}{2}\left[K(g(b) - g(x)) f(b) + K(g(x) - g(a)) f(a)\right] \right| \tag{1.36}$$

$$\leq \frac{1}{2}\left[\int_a^x |k(g(x) - g(t))| \bigvee_a^t (f) g'(t)\, dt + \int_x^b |k(g(t) - g(x))| \bigvee_t^b (f) g'(t)\, dt\right]$$

$$\leq \frac{1}{2}\left[\mathbf{K}(g(b) - g(x)) \bigvee_x^b (f) + \mathbf{K}(g(x) - g(a)) \bigvee_a^x (f)\right]$$

$$\le \frac{1}{2}\begin{cases} \max\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\}\bigvee_a^b(f); \\ \\ \left[\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))\right]^{1/p} \\ \times\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))\right] \\ \times\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right] \end{cases}$$

for any $x\in(a,b)$, *where* $\bigvee_c^d(f)$ *denoted the total variation on the interval* $[c,d]$.

In this chapter, we establish some inequalities for the k-g-fractional integrals of Lebesgue integrable function $f:[a,b]\to\mathbb{C}$ that provide error bounds in approximating the composite operators $S_{k,g,a+,b-}f$ and $\breve{S}_{k,g,a+,b-}f$ in terms of the *double integral means*

$$\frac{1}{2}\left[\frac{K(g(b)-g(x))}{b-x}\int_x^b f(t)\,dt+\frac{K(g(x)-g(a))}{x-a}\int_a^x f(t)\,dt\right],\ x\in(a,b).$$

Examples for the *generalized left-sided* and *right-sided Riemann–Liouville fractional integrals* of a function f with respect to another function g and a general exponential fractional integral are also provided.

2 The Main Results

We use the classical Lebesgue p-norms defined as

$$\|h\|_{[c,d],\infty}:=\operatorname*{essup}_{s\in[c,d]}|h(s)|$$

and

$$\|h\|_{[c,d],p}:=\left(\int_c^d|h(s)|^p\,ds\right)^{1/p},\ p\ge 1.$$

We have

Theorem 2 *Assume that the kernel k is defined either on* $(0,\infty)$ *or on* $[0,\infty)$ *with complex values and integrable on any finite subinterval. Let* $f:[a,b]\to\mathbb{C}$ *be an integrable function on* $[a,b]$ *and g be a strictly increasing function on* (a,b), *having a continuous derivative* g' *on* (a,b). *Then*

$$\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))\frac{1}{b-x}\int_x^b f(t)\,dt + K(g(x)-g(a))\frac{1}{x-a}\int_a^x f(t)\,dt\right]\right| \tag{2.1}$$

$$\leq \frac{1}{2}\begin{cases}\left\|f-\frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],\infty}\mathbf{K}(g(x)-g(a)) \\ \textit{if } f\in L_\infty[a,b]; \\ \\ \left\|f-\frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],q}\mathbf{K}_p(g(x)-g(a)) \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \textit{ if } f\in L_q[a,b]\end{cases} + \frac{1}{2}\begin{cases}\left\|f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[x,b],\infty}\mathbf{K}(g(b)-g(x)) \\ \textit{if } f\in L_\infty[a,b]; \\ \\ \left\|f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[a,x],q}\mathbf{K}_p(g(b)-g(x)) \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \textit{ if } f\in L_q[a,b]\end{cases} \tag{2.2}$$

and

$$\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))\frac{1}{b-x}\int_x^b f(t)\,dt + K(g(x)-g(a))\frac{1}{x-a}\int_a^x f(t)\,dt\right]\right| \tag{2.3}$$

$$\leq \frac{1}{2}\begin{cases}\left\|f-\frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],\infty}\mathbf{K}(g(x)-g(a)) \\ \textit{if } f\in L_\infty[a,b]; \\ \\ \left\|f-\frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],q}\mathbf{K}_p(g(x)-g(a)) \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \textit{ if } f\in L_q[a,b]\end{cases} \tag{2.4}$$

$$+\frac{1}{2}\begin{cases}\left\|f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[x,b],\infty}\mathbf{K}(g(b)-g(x)) \\ \textit{if } f\in L_\infty[a,b]; \\ \\ \left\|f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[a,x],q}\mathbf{K}_p(g(b)-g(x)) \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \textit{ if } f\in L_q[a,b]\end{cases}$$

for $x\in(a,b)$.

Proof If we write the equality (1.32) for $\gamma=\frac{1}{b-x}\int_x^b f(s)\,ds$ and $\lambda=\frac{1}{x-a}\int_a^x f(s)\,ds$, we get

$$
\begin{aligned}
&\left| S_{k,g,a+,b-} f(x) - \frac{1}{2}\left[K(g(b)-g(x)) \frac{1}{b-x}\int_x^b f(t)\,dt \right.\right. \\
&\qquad \left.\left. + \; K(g(x)-g(a)) \frac{1}{x-a}\int_a^x f(t)\,dt \right]\right| \\
&\quad \le \frac{1}{2}\left| \int_a^x k(g(x)-g(t))\, g'(t) \left[f(t) - \frac{1}{x-a}\int_a^x f(s)\,ds \right] dt \right| \\
&\quad + \frac{1}{2}\left| \int_x^b k(g(t)-g(x))\, g'(t) \left[f(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \right] dt \right| \\
&\quad \le \frac{1}{2}\int_a^x |k(g(x)-g(t))|\, g'(t) \left| f(t) - \frac{1}{x-a}\int_a^x f(s)\,ds \right| dt \\
&\quad + \frac{1}{2}\int_x^b |k(g(t)-g(x))|\, g'(t) \left| f(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \right| dt \\
&\quad =: B(x)
\end{aligned} \tag{2.5}
$$

for $x \in (a,b)$.

Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then by Hölder's integral inequality, we have

$$
\begin{aligned}
&\int_a^x |k(g(x)-g(t))|\, g'(t) \left| f(t) - \frac{1}{x-a}\int_a^x f(s)\,ds \right| dt \\
&\le \begin{cases} \left\| f - \frac{1}{x-a}\int_a^x f(s)\,ds \right\|_{[a,x],\infty} \int_a^x |k(g(x)-g(t))|\, g'(t)\,dt \\ \text{if } f \in L_\infty[a,b]; \\ \\ \left\| f - \frac{1}{x-a}\int_a^x f(s)\,ds \right\|_{[a,x],q} \left(\int_a^x |k(g(x)-g(t))|^p\, g'(t)\,dt\right)^{1/p} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f \in L_q[a,b] \end{cases}
\end{aligned} \tag{2.6}
$$

and

$$
\begin{aligned}
&\int_x^b |k(g(t)-g(x))|\, g'(t) \left| f(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \right| dt \\
&\le \begin{cases} \left\| f - \frac{1}{b-x}\int_x^b f(s)\,ds \right\|_{[x,b],\infty} \int_x^b |k(g(t)-g(x))|\, g'(t)\,dt \\ \text{if } f \in L_\infty[a,b]; \\ \\ \left\| f - \frac{1}{b-x}\int_x^b f(s)\,ds \right\|_{[a,x],q} \left(\int_a^x |k(g(t)-g(x))|^p\, g'(t)\,dt\right)^{1/p} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f \in L_q[a,b] \end{cases}
\end{aligned} \tag{2.7}
$$

for $x \in (a,b)$.

Observe that, by taking the derivative over t and using the chain rule we have

$$
(\mathbf{K}(g(x)-g(t)))' = -\mathbf{K}'(g(x)-g(t))\, g'(t) = -\left|k(g(x)-g(t))\right| g'(t)
$$

for $t \in (a, x)$ and

$$(\mathbf{K}(g(t) - g(x)))' = \mathbf{K}'(g(t) - g(x))\, g'(t) = |k(g(t) - g(x))|\, g'(t)$$

for $t \in (x, b)$.

Then

$$\int_a^x |k(g(x) - g(t))|\, g'(t)\, dt = -\int_a^x (\mathbf{K}(g(x) - g(t)))'\, dt = \mathbf{K}(g(x) - g(a))$$

and

$$\int_x^b |k(g(t) - g(x))|\, g'(t)\, dt = \int_x^b (\mathbf{K}(g(t) - g(x)))'\, dt = \mathbf{K}(g(x) - g(x))$$

where $x \in (a, b)$.

We also have for $p > 1$

$$\left(\mathbf{K}_p^p(g(x) - g(t))\right)' = -|k(g(x) - g(t))|^p\, g'(t)$$

for $t \in (a, x)$ and

$$\left(\mathbf{K}_p^p(g(t) - g(x))\right)' = |k(g(t) - g(x))|^p\, g'(t)$$

for $t \in (x, b)$.

These give

$$\int_a^x |k(g(x) - g(t))|^p\, g'(t)\, dt = -\int_a^x \left(\mathbf{K}_p^p(g(x) - g(t))\right)' dt = \mathbf{K}_p^p(g(x) - g(a))$$

and

$$\int_x^b |k(g(t) - g(x))|^p\, g'(t)\, dt = \int_x^b \left(\mathbf{K}_p^p(g(t) - g(x))\right)' dt = \mathbf{K}_p^p(g(b) - g(x)),$$

which provide

$$\left(\int_a^x |k(g(x) - g(t))|^p\, g'(t)\, dt\right)^{1/p} = \mathbf{K}_p(g(x) - g(a))$$

and

$$\left(\int_x^b |k(g(t) - g(x))|^p\, g'(t)\, dt\right)^{1/p} = \mathbf{K}_p(g(b) - g(x))$$

for $x \in (a, b)$.

By making use of (2.6) and (2.7), we get

$$B(x) \le \begin{cases} \left\| f - \frac{1}{x-a}\int_a^x f(s)\,ds \right\|_{[a,x],\infty} \mathbf{K}(g(x)-g(a)) \\ \text{if } f \in L_\infty[a,b]; \\ \\ \left\| f - \frac{1}{x-a}\int_a^x f(s)\,ds \right\|_{[a,x],q} \mathbf{K}_p(g(x)-g(a)) \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f \in L_q[a,b] \end{cases}$$
$$+ \begin{cases} \left\| f - \frac{1}{b-x}\int_x^b f(s)\,ds \right\|_{[x,b],\infty} \mathbf{K}(g(x)-g(x)) \\ \text{if } f \in L_\infty[a,b]; \\ \\ \left\| f - \frac{1}{b-x}\int_x^b f(s)\,ds \right\|_{[a,x],q} \mathbf{K}_p(g(b)-g(x)) \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f \in L_q[a,b] \end{cases}$$

and by (2.5) we get (2.1).

Further on, by utilizing the identity (1.33) for $\gamma = \frac{1}{b-x}\int_x^b f(s)\,ds$ and $\lambda = \frac{1}{x-a}\int_a^x f(s)\,ds$ we get

$$\begin{aligned} &\left| \breve{S}_{k,g,a+,b-} f(x) - \frac{1}{2}\left[K(g(b)-g(x))\frac{1}{b-x}\int_x^b f(s)\,ds \right.\right. \\ &\qquad \left.\left. + K(g(x)-g(a))\frac{1}{x-a}\int_a^x f(s)\,ds \right]\right| \\ &\le \frac{1}{2}\int_a^x |k(g(t)-g(a))|\, g'(t) \left| f(t) - \frac{1}{x-a}\int_a^x f(s)\,ds \right| dt \\ &+ \frac{1}{2}\int_x^b |k(g(b)-g(t))|\, g'(t) \left| f(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \right| dt \\ &\le \frac{1}{2}\begin{cases} \left\| f - \frac{1}{x-a}\int_a^x f(s)\,ds \right\|_{[a,x],\infty} \int_a^x |k(g(t)-g(a))|\, g'(t)\,dt \\ \text{if } f \in L_\infty[a,b]; \\ \\ \left\| f - \frac{1}{x-a}\int_a^x f(s)\,ds \right\|_{[a,x],q} \left(\int_a^x |k(g(t)-g(a))|^p g'(t)\,dt \right)^{1/p} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f \in L_q[a,b] \end{cases} \\ &\quad + \frac{1}{2}\begin{cases} \left\| f - \frac{1}{b-x}\int_x^b f(s)\,ds \right\|_{[x,b],\infty} \int_x^b |k(g(b)-g(t))|\, g'(t)\,dt \\ \text{if } f \in L_\infty[a,b]; \\ \\ \left\| f - \frac{1}{b-x}\int_x^b f(s)\,ds \right\|_{[a,x],q} \left(\int_x^b |k(g(b)-g(t))|^p g'(t)\,dt \right)^{1/p} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f \in L_q[a,b] \end{cases} \end{aligned} \tag{2.8}$$

for $x \in (a,b)$.

Since

$$\int_a^x |k(g(t)-g(a))| \, g'(t) \, dt = \int_a^x (\mathbf{K}(g(t)-g(a)))' \, dt = \mathbf{K}(g(x)-g(a)),$$

$$\int_a^x |k(g(t)-g(a))|^p \, g'(t) \, dt = \int_a^x \left(\mathbf{K}_p^p(g(t)-g(a))\right)' \, dt = \mathbf{K}_p^p(g(x)-g(a)),$$

$$\int_x^b |k(g(b)-g(t))| \, g'(t) \, dt = -\int_x^b (\mathbf{K}(g(b)-g(t)))' \, dt = \mathbf{K}(g(b)-g(x))$$

and

$$\int_x^b |k(g(b)-g(t))|^p \, g'(t) \, dt = -\int_x^b \left(\mathbf{K}_p^p(g(b)-g(t))\right)' \, dt = \mathbf{K}_p^p(g(b)-g(x)),$$

where $x \in (a,b)$, then by (2.8) we get the desired result (2.3).

Remark 1 We observe that

$$\mathbf{K}(t) \le t \, \|k\|_{[0,t]} \text{ for } t \ge 0,$$

which implies that

$$\mathbf{K}(g(x)-g(a)) \le (g(x)-g(a)) \, \|k\|_{[0,g(x)-g(a)]}$$

and

$$\mathbf{K}(g(b)-g(x)) \le (g(b)-g(x)) \, \|k\|_{[0,g(b)-g(x)]}$$

for $x \in (a,b)$.

Therefore by (2.1) and (2.3), we get

$$\begin{aligned} &\left| S_{k,g,a+,b-} f(x) - \frac{1}{2}\left[K(g(b)-g(x)) \frac{1}{b-x} \int_x^b f(t) \, dt \right.\right. \\ &\qquad \left.\left. + K(g(x)-g(a)) \frac{1}{x-a} \int_a^x f(t) \, dt \right] \right| \\ &\le \frac{1}{2} \left\| f - \frac{1}{x-a} \int_a^x f(s) \, ds \right\|_{[a,x],\infty} (g(x)-g(a)) \, \|k\|_{[0,g(x)-g(a)]} \\ &+ \frac{1}{2} \left\| f - \frac{1}{b-x} \int_x^b f(s) \, ds \right\|_{[x,b],\infty} (g(b)-g(x)) \, \|k\|_{[0,g(b)-g(x)]} \\ &\le \frac{1}{2} \|k\|_{[0,g(b)-g(a)]} \left[\left\| f - \frac{1}{x-a} \int_a^x f(s) \, ds \right\|_{[a,x],\infty} (g(x)-g(a)) \right. \\ &\qquad \left. + \left\| f - \frac{1}{b-x} \int_x^b f(s) \, ds \right\|_{[x,b],\infty} (g(b)-g(x)) \right] \end{aligned} \tag{2.9}$$

and

$$\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))\frac{1}{b-x}\int_x^b f(t)\,dt\right.\right. \tag{2.10}$$
$$\left.\left.+\;K(g(x)-g(a))\frac{1}{x-a}\int_a^x f(t)\,dt\right]\right|$$
$$\leq\frac{1}{2}\left\|f-\frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],\infty}(g(x)-g(a))\,\|k\|_{[0,g(x)-g(a)]}$$
$$+\frac{1}{2}\left\|f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[x,b],\infty}(g(b)-g(x))\,\|k\|_{[0,g(b)-g(x)]}$$
$$\leq\frac{1}{2}\|k\|_{[0,g(b)-g(a)]}\left[\left\|f-\frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],\infty}(g(x)-g(a))\right.$$
$$\left.+\left\|f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[x,b],\infty}(g(b)-g(x))\right]$$

for $x\in(a,b)$.

The following result for functions of bounded variation hold [13]:

Lemma 2 *Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation on $[a,b]$. Then*

$$\|f\|_{[a,b],\infty}\leq\frac{1}{b-a}\left|\int_a^b f(t)\,dt\right|+\bigvee_a^b(f). \tag{2.11}$$

The multiplicative constant 1 in front of $\bigvee_a^b(f)$ cannot be replaced by a smaller quantity.

Lemma 3 *Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation on $[a,b]$. Then for $p\geq 1$ one has the inequality*

$$\|f\|_{[a,b],p}\leq\frac{1}{(b-a)^{1-\frac{1}{p}}}\left|\int_a^b f(t)\,dt\right|+\frac{1}{2}\frac{(b-a)^{\frac{1}{p}}\left(2^{p+1}-1\right)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\bigvee_a^b(f). \tag{2.12}$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

The following result may be then stated:

Corollary 1 *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then*

$$\left|S_{k,g,a+,b-} f(x) - \frac{1}{2}\left[K(g(b)-g(x))\frac{1}{b-x}\int_x^b f(t)\,dt + K(g(x)-g(a))\frac{1}{x-a}\int_a^x f(t)\,dt\right]\right| \tag{2.13}$$
$$\leq \frac{1}{2}\begin{cases} \bigvee_a^x (f)\,\mathbf{K}(g(x)-g(a)) \\ \frac{1}{2}\frac{(x-a)^{\frac{1}{q}}\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\bigvee_a^x (f)\,\mathbf{K}_p(g(x)-g(a)) \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \end{cases} + \frac{1}{2}\begin{cases} \bigvee_x^b (f)\,\mathbf{K}(g(b)-g(x)) \\ \frac{1}{2}\frac{(b-x)^{\frac{1}{q}}\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\bigvee_x^b (f)\,\mathbf{K}_p(g(b)-g(x)) \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \end{cases}$$

and

$$\left|\breve{S}_{k,g,a+,b-} f(x) - \frac{1}{2}\left[K(g(b)-g(x))\frac{1}{b-x}\int_x^b f(t)\,dt + K(g(x)-g(a))\frac{1}{x-a}\int_a^x f(t)\,dt\right]\right| \tag{2.14}$$
$$\leq \frac{1}{2}\begin{cases} \bigvee_a^x (f)\,\mathbf{K}(g(x)-g(a)) \\ \frac{1}{2}\cdot\frac{(x-a)^{\frac{1}{q}}\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\bigvee_a^x (f)\,\mathbf{K}_p(g(x)-g(a)) \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \end{cases} + \frac{1}{2}\begin{cases} \bigvee_x^b (f)\,\mathbf{K}(g(b)-g(x)) \\ \frac{1}{2}\cdot\frac{(b-x)^{\frac{1}{q}}\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\bigvee_x^b (f)\,\mathbf{K}_p(g(b)-g(x)) \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \end{cases}$$

for $x \in (a,b)$.

Proof By using Lemma 2, we have

$$\begin{aligned}\left\|f - \frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],\infty} &\leq \frac{1}{x-a}\left|\int_a^x \left(f(t) - \frac{1}{x-a}\int_a^x f(s)\,ds\right)dt\right| \\ &\quad + \bigvee_a^x\left(f - \frac{1}{x-a}\int_a^x f(s)\,ds\right) \\ &= \bigvee_a^x (f)\end{aligned}$$

and

$$\left\| f - \frac{1}{b-x}\int_x^b f(s)\,ds \right\|_{[x,b],\infty} \le \frac{1}{b-x}\left| \int_x^b \left(f(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \right) dt \right| + \bigvee_x^b \left(f(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \right) = \bigvee_x^b (f)$$

for $x \in (a,b)$.

Also, by using Lemma 3 we have for $q > 1$ that

$$\left\| f - \frac{1}{x-a}\int_a^x f(s)\,ds \right\|_{[a,x],q} \le \frac{1}{(x-a)^{1-\frac{1}{q}}}\left| \int_a^x \left(f(t) - \frac{1}{x-a}\int_a^x f(s)\,ds \right) dt \right| + \frac{1}{2}\frac{(x-a)^{\frac{1}{q}}\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\bigvee_a^x \left(f - \frac{1}{x-a}\int_a^x f(s)\,ds \right) = \frac{1}{2}\frac{(x-a)^{\frac{1}{q}}\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\bigvee_a^x (f)$$

and

$$\left\| f - \frac{1}{b-x}\int_x^b f(s)\,ds \right\|_{[x,b],q} \le \frac{1}{(b-x)^{1-\frac{1}{q}}}\left| \int_a^b \left(f(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \right) dt \right| + \frac{1}{2}\frac{(b-x)^{\frac{1}{q}}\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\bigvee_x^b \left(f - \frac{1}{b-x}\int_x^b f(s)\,ds \right) = \frac{1}{2}\frac{(b-x)^{\frac{1}{q}}\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\bigvee_x^b (f)$$

for $x \in (a,b)$.

By using Theorem 2, we obtain the desired results (2.13) and (2.14).

Remark 2 With the assumptions of Corollary 1, we have

$$\left| S_{k,g,a+,b-} f(x) - \frac{1}{2}\left[K(g(b)-g(x)) \frac{1}{b-x}\int_x^b f(t)\,dt + K(g(x)-g(a)) \frac{1}{x-a}\int_a^x f(t)\,dt \right]\right| \tag{2.15}$$
$$\leq \frac{1}{2}\begin{cases} \max\{\mathbf{K}(g(x)-g(a)), \mathbf{K}(g(b)-g(x))\} \bigvee_a^b (f) \\ [\mathbf{K}(g(x)-g(a)) + \mathbf{K}(g(b)-g(x))] \max\left\{\bigvee_a^x (f), \bigvee_x^b (f)\right\} \end{cases}$$

and

$$\left| \breve{S}_{k,g,a+,b-} f(x) - \frac{1}{2}\left[K(g(b)-g(x)) \frac{1}{b-x}\int_x^b f(t)\,dt + K(g(x)-g(a)) \frac{1}{x-a}\int_a^x f(t)\,dt \right]\right| \tag{2.16}$$
$$\leq \frac{1}{2}\begin{cases} \max\{\mathbf{K}(g(x)-g(a)), \mathbf{K}(g(b)-g(x))\} \bigvee_a^b (f) \\ [\mathbf{K}(g(x)-g(a)) + \mathbf{K}(g(b)-g(x))] \max\left\{\bigvee_a^x (f), \bigvee_x^b (f)\right\} \end{cases}$$

for $x \in (a, b)$.

3 Applications for Generalized Riemann–Liouville Fractional Integrals

If we take $k(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$S_{k,g,a+} f(x) = I^{\alpha}_{a+,g} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x [g(x)-g(t)]^{\alpha-1} g'(t) f(t)\,dt$$

for $a < x \leq b$ and

$$S_{k,g,b-} f(x) = I^{\alpha}_{b-,g} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b [g(t)-g(x)]^{\alpha-1} g'(t) f(t)\,dt$$

for $a \leq x < b$, which are the *generalized left-sided* and *right-sided Riemann–Liouville fractional integrals* of a function f with respect to another function g on $[a, b]$ as defined in [24, p. 100].

We consider the mixed operators

$$I^{\alpha}_{g,a+,b-} f(x) := \frac{1}{2}\left[I^{\alpha}_{a+,g} f(x) + I^{\alpha}_{b-,g} f(x) \right] \tag{3.1}$$

and

$$\breve{I}^{\alpha}_{g,a+,b-}f(x) := \frac{1}{2}\left[I^{\alpha}_{x+,g}f(b) + I^{\alpha}_{x-,g}f(a)\right] \tag{3.2}$$

for $x \in (a, b)$.

We observe that for $\alpha > 0$, we have

$$\mathbf{K}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t s^{\alpha-1}ds = \frac{t^\alpha}{\alpha\Gamma(\alpha)} = \frac{t^\alpha}{\Gamma(\alpha+1)},\ t \geq 0$$

and for $\alpha > \frac{p-1}{p} > 0$, where $p > 1$, we have

$$\mathbf{K}_p(t) = \frac{1}{\Gamma(\alpha)}\left(\int_0^t s^{(\alpha-1)p}ds\right)^{1/p} = \frac{1}{(\alpha-1+1/p)\Gamma(\alpha)}t^{\alpha-1+1/p},\ t \geq 0.$$

Using Theorem 2, we can state the following inequalities for $\alpha > 0$

$$\begin{aligned}\Bigg|I^{\alpha}_{g,a+,b-}f(x) &- \frac{1}{2\Gamma(\alpha+1)}\left[\frac{(g(b)-g(x))^\alpha}{b-x}\int_x^b f(t)\,dt\right. \\ &\left.+\frac{(g(x)-g(a))^\alpha}{x-a}\int_a^x f(t)\,dt\right]\Bigg| \\ \leq \frac{1}{2\Gamma(\alpha+1)}&\left[\left\|f-\frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],\infty}(g(x)-g(a))^\alpha\right. \\ &\left.+\left\|f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[x,b],\infty}(g(b)-g(x))^\alpha\right]\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\Bigg|\breve{I}^{\alpha}_{g,a+,b-}f(x) &- \frac{1}{2\Gamma(\alpha+1)}\left[\frac{(g(b)-g(x))^\alpha}{b-x}\int_x^b f(t)\,dt\right. \\ &\left.+\frac{(g(x)-g(a))^\alpha}{x-a}\int_a^x f(t)\,dt\right]\Bigg| \\ \leq \frac{1}{2\Gamma(\alpha+1)}&\left[\left\|f-\frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],\infty}(g(x)-g(a))^\alpha\right. \\ &\left.+\left\|f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[x,b],\infty}(g(b)-g(x))^\alpha\right]\end{aligned} \tag{3.4}$$

for $x \in (a, b)$.

If $p, q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$ and $\alpha > \frac{p-1}{p} = \frac{1}{q} > 0$, then by Theorem 2 we can state the following inequalities as well

$$\left| I^{\alpha}_{g,a+,b-} f(x) - \frac{1}{2\Gamma(\alpha+1)} \left[\frac{(g(b)-g(x))^{\alpha}}{b-x} \int_x^b f(t)\,dt + \frac{(g(x)-g(a))^{\alpha}}{x-a} \int_a^x f(t)\,dt \right] \right| \leq \frac{1}{2} \frac{1}{(\alpha-1/q)\Gamma(\alpha)} \left[\left\| f - \frac{1}{x-a} \int_a^x f(s)\,ds \right\|_{[a,x],q} (g(x)-g(a))^{\alpha-1+1/p} + \left\| f - \frac{1}{b-x} \int_x^b f(s)\,ds \right\|_{[a,x],q} (g(b)-g(x))^{\alpha-1+1/p} \right] \tag{3.5}$$

and

$$\left| \breve{I}^{\alpha}_{g,a+,b-} f(x) - \frac{1}{2\Gamma(\alpha+1)} \left[\frac{(g(b)-g(x))^{\alpha}}{b-x} \int_x^b f(t)\,dt + \frac{(g(x)-g(a))^{\alpha}}{x-a} \int_a^x f(t)\,dt \right] \right| \leq \frac{1}{2} \frac{1}{(\alpha-1/q)\Gamma(\alpha)} \left[\left\| f - \frac{1}{x-a} \int_a^x f(s)\,ds \right\|_{[a,x],q} (g(x)-g(a))^{\alpha-1+1/p} + \left\| f - \frac{1}{b-x} \int_x^b f(s)\,ds \right\|_{[a,x],q} (g(b)-g(x))^{\alpha-1+1/p} \right] \tag{3.6}$$

for $x \in (a, b)$.

If we assume that $f : [a, b] \to \mathbb{C}$ is of bounded variation, then by Corollary 1 we have for $\alpha > 0$ that

$$\left| I^{\alpha}_{g,a+,b-} f(x) - \frac{1}{2\Gamma(\alpha+1)} \left[\frac{(g(b)-g(x))^{\alpha}}{b-x} \int_x^b f(t)\,dt + \frac{(g(x)-g(a))^{\alpha}}{x-a} \int_a^x f(t)\,dt \right] \right| \leq \frac{1}{2\Gamma(\alpha+1)} \left[\bigvee_a^x (f) (g(x)-g(a))^{\alpha} + \bigvee_x^b (f) (g(b)-g(x))^{\alpha} \right] \tag{3.7}$$

and

$$\left| \breve{I}^{\alpha}_{g,a+,b-} f(x) - \frac{1}{2\Gamma(\alpha+1)} \left[\frac{(g(b)-g(x))^{\alpha}}{b-x} \int_x^b f(t)\,dt + \frac{(g(x)-g(a))^{\alpha}}{x-a} \int_a^x f(t)\,dt \right] \right| \leq \frac{1}{2\Gamma(\alpha+1)} \left[\bigvee_a^x (f) (g(x)-g(a))^{\alpha} + \bigvee_x^b (f) (g(b)-g(x))^{\alpha} \right] \tag{3.8}$$

for $x \in (a, b)$.

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha > \frac{p-1}{p} = \frac{1}{q} > 0$, then by Corollary 1 we have

$$\left| I^{\alpha}_{g,a+,b-} f(x) - \frac{1}{2\Gamma(\alpha+1)} \left[\frac{(g(b) - g(x))^{\alpha}}{b-x} \int_x^b f(t)\,dt + \frac{(g(x) - g(a))^{\alpha}}{x-a} \int_a^x f(t)\,dt \right] \right| \leq \frac{1}{4} \frac{\left(2^{q+1} - 1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}} (\alpha - 1/q)\Gamma(\alpha)} \left[(x-a)^{\frac{1}{q}} \bigvee_a^x (f) (g(x) - g(a))^{\alpha - 1 + 1/p} + (b-x)^{\frac{1}{q}} \bigvee_x^b (f) (g(b) - g(x))^{\alpha - 1 + 1/p} \right] \tag{3.9}$$

and

$$\left| \breve{I}^{\alpha}_{g,a+,b-} f(x) - \frac{1}{2\Gamma(\alpha+1)} \left[\frac{(g(b) - g(x))^{\alpha}}{b-x} \int_x^b f(t)\,dt + \frac{(g(x) - g(a))^{\alpha}}{x-a} \int_a^x f(t)\,dt \right] \right| \leq \frac{1}{4} \frac{\left(2^{q+1} - 1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}} (\alpha - 1/q)\Gamma(\alpha)} \left[(x-a)^{\frac{1}{q}} \bigvee_a^x (f) (g(x) - g(a))^{\alpha - 1 + 1/p} + (b-x)^{\frac{1}{q}} \bigvee_x^b (f) (g(b) - g(x))^{\alpha - 1 + 1/p} \right] \tag{3.10}$$

for $x \in (a, b)$.

4 Example for an Exponential Kernel

For $\alpha \in \mathbb{R}$ we consider the kernel $k(t) := \exp(\alpha t)$, $t \in \mathbb{R}$. We have

$$|k(s)| = \exp(\alpha s) \text{ for } s \in \mathbb{R},$$

$$K(t) = \frac{\exp(\alpha t) - 1}{\alpha}, \text{ if } t \in \mathbb{R}$$

and for $p \geq 1$

$$\mathbf{K}_p(t) = \left(\int_0^t \exp(p\alpha s) \right)^{1/p} ds = \left(\frac{\exp(p\alpha t) - 1}{p\alpha} \right)^{1/p}$$

for $\alpha \neq 0$.

Let $f : [a, b] \to \mathbb{C}$ be an integrable function on $[a, b]$ and g be a strictly increasing function on (a, b), having a continuous derivative g' on (a, b). Define

$$\mathcal{H}^{\alpha}_{g,a+,b-} f(x) = \frac{1}{2}\int_x^b \exp\left[\alpha\left(g(t) - g(x)\right)\right] g'(t) f(t)\, dt + \frac{1}{2}\int_a^x \exp\left[\alpha\left(g(x) - g(t)\right)\right] g'(t) f(t)\, dt \tag{4.1}$$

for $x \in (a, b)$.

If $g = \ln h$ where $h : [a, b] \to (0, \infty)$ is a strictly increasing function on (a, b), having a continuous derivative h' on (a, b), then we can consider the following operator as well

$$\begin{aligned} &\kappa^{\alpha}_{h,a+,b-} f(x) \\ &:= \mathcal{H}^{\alpha}_{\ln h,a+,b-} f(x) \\ &= \frac{1}{2}\left[\int_x^b \left(\frac{h(t)}{h(x)}\right)^{\alpha} \frac{h'(t)}{h(t)} f(t)\, dt + \int_a^x \left(\frac{h(x)}{h(t)}\right)^{\alpha} \frac{h'(t)}{h(t)} f(t)\, dt\right], \end{aligned} \tag{4.2}$$

for $x \in (a, b)$.

Furthermore, let $f : [a, b] \to \mathbb{C}$ be an integrable function on $[a, b]$ and g be a strictly increasing function on (a, b), having a continuous derivative g' on (a, b). Also define

$$\begin{aligned} &\breve{\mathcal{H}}^{\alpha}_{g,a+,b-} f(x) \\ &:= \frac{1}{2}\int_x^b \exp\left[\alpha\left(g(b) - g(t)\right)\right] g'(t) f(t)\, dt \\ &+ \frac{1}{2}\int_a^x \exp\left[\alpha\left(g(t) - g(a)\right)\right] g'(t) f(t)\, dt \end{aligned} \tag{4.3}$$

for any $x \in (a, b)$.

If $g = \ln h$ where $h : [a, b] \to (0, \infty)$ is a strictly increasing function on (a, b), having a continuous derivative h' on (a, b), then we can consider the following operator as well

$$\begin{aligned} &\breve{\kappa}^{\alpha}_{h,a+,b-} f(x) \\ &:= \breve{\mathcal{H}}^{\alpha}_{\ln h,a+,b-} f(x) \\ &= \frac{1}{2}\left[\int_x^b \left(\frac{h(b)}{h(t)}\right)^{\alpha} \frac{h'(t)}{h(t)} f(t)\, dt + \int_a^x \left(\frac{h(t)}{h(a)}\right)^{\alpha} \frac{h'(t)}{h(t)} f(t)\, dt\right], \end{aligned} \tag{4.4}$$

for any $x \in (a, b)$.

Using Theorem 2, we have

$$\left|\mathcal{H}_{g,a+,b-}^{\alpha}f(x)-\frac{1}{2}\left[\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\frac{1}{b-x}\int_x^b f(t)\,dt\right.\right. \tag{4.5}$$
$$\left.\left.+\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\frac{1}{x-a}\int_a^x f(t)\,dt\right]\right|$$
$$\leq\frac{1}{2}\begin{cases}\left\|f-\frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],\infty}\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\\ \text{if } f\in L_\infty[a,b];\\ \\ \left\|f-\frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],q}\left(\frac{\exp(p\alpha(g(x)-g(a)))-1}{p\alpha}\right)^{1/p}\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f\in L_q[a,b]\end{cases}$$
$$+\frac{1}{2}\begin{cases}\left\|f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[x,b],\infty}\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\\ \text{if } f\in L_\infty[a,b];\\ \\ \left\|f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[a,x],q}\left(\frac{\exp(p\alpha(g(b)-g(x)))-1}{p\alpha}\right)^{1/p}\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f\in L_q[a,b]\end{cases}$$

and

$$\left|\breve{\mathcal{H}}_{g,a+,b-}^{\alpha}f(x)-\frac{1}{2}\left[\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\frac{1}{b-x}\int_x^b f(t)\,dt\right.\right. \tag{4.6}$$
$$\left.\left.+\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\frac{1}{x-a}\int_a^x f(t)\,dt\right]\right|$$
$$\leq\frac{1}{2}\begin{cases}\left\|f-\frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],\infty}\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\\ \text{if } f\in L_\infty[a,b];\\ \\ \left\|f-\frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],q}\left(\frac{\exp(p\alpha(g(x)-g(a)))-1}{p\alpha}\right)^{1/p}\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f\in L_q[a,b]\end{cases}$$
$$+\frac{1}{2}\begin{cases}\left\|f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[x,b],\infty}\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\\ \text{if } f\in L_\infty[a,b];\\ \\ \left\|f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[a,x],q}\left(\frac{\exp(p\alpha(g(b)-g(x)))-1}{p\alpha}\right)^{1/p}\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f\in L_q[a,b],\end{cases}$$

for any $x\in(a,b)$.

If we take in (4.5) and (4.6) $g=\ln h$ where $h:[a,b]\to(0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we have

$$\left| \kappa^{\alpha}_{h,a+,b-} f(x) - \frac{1}{2} \left[\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha} - 1}{\alpha} \frac{1}{b-x} \int_x^b f(t)\,dt \right. \right. \tag{4.7}$$

$$\left. \left. + \frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha} - 1}{\alpha} \frac{1}{x-a} \int_a^x f(t)\,dt \right] \right|$$

$$\leq \frac{1}{2} \begin{cases} \left\| f - \frac{1}{x-a} \int_a^x f(s)\,ds \right\|_{[a,x],\infty} \frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha} - 1}{\alpha} \\ \text{if } f \in L_\infty [a,b]; \\ \\ \left\| f - \frac{1}{x-a} \int_a^x f(s)\,ds \right\|_{[a,x],q} \left(\frac{\left(\frac{h(x)}{h(a)}\right)^{p\alpha} - 1}{p\alpha} \right)^{1/p} \\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \text{ if } f \in L_q [a,b] \end{cases}$$

$$+ \frac{1}{2} \begin{cases} \left\| f - \frac{1}{b-x} \int_x^b f(s)\,ds \right\|_{[x,b],\infty} \frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha} - 1}{\alpha} \\ \text{if } f \in L_\infty [a,b]; \\ \\ \left\| f - \frac{1}{b-x} \int_x^b f(s)\,ds \right\|_{[a,x],q} \left(\frac{\left(\frac{h(b)}{h(x)}\right)^{p\alpha} - 1}{p\alpha} \right)^{1/p} \\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \text{ if } f \in L_q [a,b] \end{cases}$$

and

$$\left| \breve{\kappa}^{\alpha}_{h,a+,b-} f(x) - \frac{1}{2} \left[\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha} - 1}{\alpha} \frac{1}{b-x} \int_x^b f(t)\,dt \right. \right. \tag{4.8}$$

$$\left. \left. + \frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha} - 1}{\alpha} \frac{1}{x-a} \int_a^x f(t)\,dt \right] \right|$$

$$\leq \frac{1}{2} \begin{cases} \left\| f - \frac{1}{x-a} \int_a^x f(s)\,ds \right\|_{[a,x],\infty} \frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha} - 1}{\alpha} \\ \text{if } f \in L_\infty [a,b]; \\ \\ \left\| f - \frac{1}{x-a} \int_a^x f(s)\,ds \right\|_{[a,x],q} \left(\frac{\left(\frac{h(x)}{h(a)}\right)^{p\alpha} - 1}{p\alpha} \right)^{1/p} \\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \text{ if } f \in L_q [a,b] \end{cases}$$

$$+\frac{1}{2}\begin{cases} \left\| f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[x,b],\infty}\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha} \\ \text{if } f\in L_\infty[a,b]; \\ \\ \left\| f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[a,x],q}\left(\frac{\left(\frac{h(b)}{h(x)}\right)^{p\alpha}-1}{p\alpha}\right)^{1/p} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f\in L_q[a,b] \end{cases}$$

for any $x\in(a,b)$.

Finally, if we assume that $f:[a,b]\to\mathbb{C}$ is of bounded variation, then by Corollary 1 we have

$$\left|\kappa_{h,a+,b-}^{\alpha}f(x)-\frac{1}{2}\left[\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\frac{1}{b-x}\int_x^b f(t)\,dt \right.\right. \tag{4.9}$$

$$\left.\left.+\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha}\frac{1}{x-a}\int_a^x f(t)\,dt\right]\right|$$

$$\leq\frac{1}{2}\begin{cases} \bigvee_a^x(f)\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha} \text{ if } f\in L_\infty[a,b]; \\ \\ \frac{1}{2}\frac{(x-a)^{\frac{1}{q}}\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\bigvee_a^x(f)\left(\frac{\left(\frac{h(x)}{h(a)}\right)^{p\alpha}-1}{p\alpha}\right)^{1/p} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f\in L_q[a,b] \end{cases}$$

$$+\frac{1}{2}\begin{cases} \bigvee_x^b(f)\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha} \text{ if } f\in L_\infty[a,b]; \\ \\ \frac{1}{2}\frac{(b-x)^{\frac{1}{q}}\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\bigvee_x^b(f)\left(\frac{\left(\frac{h(b)}{h(x)}\right)^{p\alpha}-1}{p\alpha}\right)^{1/p} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f\in L_q[a,b] \end{cases}$$

and

$$\left|\breve{\kappa}_{h,a+,b-}^{\alpha} f(x) - \frac{1}{2}\left[\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha} - 1}{\alpha}\frac{1}{b-x}\int_x^b f(t)\,dt + \frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha} - 1}{\alpha}\frac{1}{x-a}\int_a^x f(t)\,dt\right]\right| \tag{4.10}$$
$$\leq \frac{1}{2}\begin{cases} \bigvee_a^x (f)\,\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha} \text{ if } f\in L_\infty[a,b]; \\ \frac{1}{2}\frac{(x-a)^{\frac{1}{q}}\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\bigvee_a^x (f)\left(\frac{\left(\frac{h(x)}{h(a)}\right)^{p\alpha}-1}{p\alpha}\right)^{1/p} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f\in L_q[a,b] \end{cases}$$
$$+\frac{1}{2}\begin{cases} \bigvee_x^b (f)\,\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha} \text{ if } f\in L_\infty[a,b]; \\ \frac{1}{2}\frac{(b-x)^{\frac{1}{q}}\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\bigvee_x^b (f)\left(\frac{\left(\frac{h(b)}{h(x)}\right)^{p\alpha}-1}{p\alpha}\right)^{1/p} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f\in L_q[a,b] \end{cases}$$

for any $x\in(a,b)$.

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Existence Theory on Modular Metric Spaces

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1 Geraghty-Type Theorems and Application to Partial Differential Equation

Since the year 1922, Banach's contraction principle, due to its simplicity and usability, has become a popular tool in modern analytics, particularly in nonlinear analysis, including the use of equations, differential equations, variance, equilibrium problems, and much more (see, e.g., [1–10]).

Throughout this paper, let $\mathbb{R}^+$ denote the set of all positive real numbers and $\mathbb{R}_+$ denote the set of all nonnegative real numbers.

In 1973, Geraghty [11] gave an interesting generalization of the contraction principle by using the class $\mathcal{S}$ of the functions $\beta : \mathbb{R}_+ \to [0, 1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0.$$

Theorem 1.1 ([11]) *Let (X, d) be a complete metric space and f be a self-mapping on X such that there exists $\beta \in \mathcal{S}$ satisfying*

$$d(fx, fy) \leq \beta(d(x, y))d(x, y) \tag{1.1}$$

for all $x, y \in X$. Then the sequence $\{x_n\}$ defined by $x_n = fx_{n-1}$ for each $n \geq 1$ converges to the unique fixed point of z in X.

Later, Amini-Harandini et al. [12] extended Geraghty's fixed point theorem to the setting of partially ordered metric spaces as follows:

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Theorem 1.2 ([12]) *Let $(X, \sqsubseteq)$ be a partially ordered metric set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let f be a nondecreasing self-mapping on X which satisfies the inequality (1.1) whenever $x, y \in X$ are comparable. Assume that f is either continuous or*

$$\text{if a nondecreasing sequence } \{x_n\} \text{ converges to } x_*, \text{ then } x_n \sqsubseteq x_* \text{ for each } n \geq 1. \tag{1.2}$$

If, additionally, the following condition is satisfied:

$$\text{for any } x, y \in X, \text{ there exists } z \in X \text{ which is comparable to both } x \text{ and } y, \tag{1.3}$$

then the sequence $\{x_n\}$ converges to the unique fixed point of z in X.

Let Ψ denote the class of functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

(a) ψ is nondecreasing,
(b) ψ is continuous,
(c) $\psi(t) = 0$ if and only if $t = 0$.

By using this class, Eshaghi Gordji et al. [13] extended Theorem 1.2 as follows:

Theorem 1.3 ([13]) *Let $(X, \sqsubseteq)$ be a partially ordered metric set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let f be a nondecreasing self-mapping on X such that there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$. Suppose that there exist $\beta \in \mathcal{S}$ and $\psi \in \Psi$ such that*

$$\psi(d(fx, fy)) \leq \beta(\psi(d(x, y)))\psi(d(x, y)),$$

whenever $x, y \in X$ are comparable. Assume also that the condition (1.2) holds. Then f has a fixed point.

In 2010, Chistyakov [14] introduced the notion of a modular metric space which is raised in an attempt to avoid some restrictions of the concept of a modular space (for the literature of a modular space; see, e.g., [15–21] and references therein). Some of the early investigations on metric fixed point theory in this space refer to [22–24, 54–59].

For the rest of this section, we present some notions and basic facts of modular metric spaces.

Definition 1.4 ([14]) Let X be a nonempty set. A function $\omega : \mathbb{R}^+ \times X \times X \to \mathbb{R}_+ \cup \{\infty\}$ is said to be a *metric modular* on X if, for all $x, y, z \in X$, the following conditions hold:

(a) $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$.
(b) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$.
(c) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$.

For any $x_\iota \in X$, the set $X_\omega(x_\iota) = \{x \in X : \lim_{\lambda\to\infty} \omega_\lambda(x, x_\iota) = 0\}$ is called a *modular metric space* generated by x_ι and induced by ω. If its generator x_ι does not play any role in the situation (i.e., X_ω is independent of generators), we write X_ω instead of $X_\omega(x_\iota)$.

Observe that a metric modular ω on X is nonincreasing with respect to $\lambda > 0$. We can simply show this assertion by using the condition (c). For any $x, y \in X$ and $0 < \mu < \lambda$, we have

$$\omega_\lambda(x, y) \le \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y). \tag{1.4}$$

For any $x, y \in X$ and $\lambda > 0$, we set

$$\omega_{\lambda^+}(x, y) := \lim_{\epsilon\to 0} \omega_{\lambda+\epsilon}(x, y), \quad \omega_{\lambda^-}(x, y) := \lim_{\epsilon\to 0} \omega_{\lambda-\epsilon}(x, y).$$

Consequently, from (1.4), it follows that

$$\omega_{\lambda^+}(x, y) \le \omega_\lambda(x, y) \le \omega_{\lambda^-}(x, y).$$

For any $x, y \in X$, if a metric modular ω on X possesses a finite value and $\omega_\lambda(x, y) = \omega_\mu(x, y)$ for all $\lambda, \mu > 0$, then $d(x, y) := \omega_\lambda(x, y)$ is a metric on X.

Example 1.5 Let $X = \mathbb{R}$ and ω is defined by $\omega_\lambda(x, y) = \infty$ if $\lambda < 1$, and $\omega_\lambda(x, y) = |x - y|$ if $\lambda \ge 1$, it is easy to verify that ω is regular modular metric but not modular metric.

Later, Chaipunya et al. [23] have altered the notion of convergence and Cauchy sequence in modular metric spaces under the direction of Mongkolkeha et al. [24].

Definition 1.6 ([23, 24]) Let (X, ω) be a modular metric space and $\{x_n\}$ be a sequence in X_ω.

(1) A point $x \in X_\omega$ is called a *limit* of $\{x_n\}$ if, for each $\lambda, \epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda(x_n, x) < \epsilon$ for all $n \ge n_0$. A sequence that has a limit is said to be *convergent* (or *converges* to x), which is written as $\lim_{n\to\infty} x_n = x$.
(2) A sequence $\{x_n\}$ in X_ω is said to be a *Cauchy sequence* if, for each $\lambda, \epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda(x_n, x_m) < \epsilon$ for all $m, n \ge n_0$.
(3) If every Cauchy sequences in X converges, X is said to be *complete*.

We prove a generalization of Geraghty's theorem which also improves the result of Eshagi Gordji et al. [13] under the influence of a modular metric space.

Geraghty-type theorems

Before stating our main results, we first introduce the following classes for a more convenience of usage.

For each $n \in \mathbb{N}$, let $\mathcal{S}_n$ denote the class of n-tuples of functions $(\beta_1, \beta_2, \ldots, \beta_n)$, where for each $i \in \{1, 2, \ldots, n\}$, $\beta_i : \mathbb{R}_+ \cup \{\infty\} \to [0, 1)$ and the following implication holds:

$$\beta(t_k) := \beta_1(t_k) + \beta_2(t_k) + \cdots + \beta_n(t_k) \to 1 \text{ implies } t_k \to 0.$$

Actually, Geraghty's class $\mathcal{S}$ is equivalent to the class $\mathcal{S}_1$ when ∞ is not considered. It follows that for each $m \in \{1, 2, \ldots, n\}$, if $(\beta_1, \beta_2, \ldots, \beta_m) \in \mathcal{S}_m$, then $(\beta_1, \beta_2, \ldots, \beta_m, \underbrace{\theta, \theta, \ldots, \theta}_{n-m \text{entries}}) \in \mathcal{S}_n$, where θ denotes the zero function. Also, note that if $(\underbrace{\beta, \beta, \ldots, \beta}_{n \text{ entries}}) \in \mathcal{S}_n$, then we also have the following:

$$\beta(t_k) \to \frac{1}{n} \text{ implies } t_k \to 0.$$

Besides, if $(\beta_1, \beta_2 \ldots, \beta_n) \in \mathcal{S}_n$, then $\pi((\beta_1, \beta_2 \ldots, \beta_n)) \in \mathcal{S}_n$, where $\pi((\beta_1, \beta_2 \ldots, \beta_n))$ is a permutation of $(\beta_1, \beta_2 \ldots, \beta_n)$. It is also important to know that if $(\beta_1, \beta_2, \ldots, \beta_n) \in \mathcal{S}_n$, then $(\beta_{n_1}, \beta_{n_2}, \ldots, \beta_{n_m}) \in \mathcal{S}_m$ for each $m \in \{1, 2, \ldots, n\}$, where each β_{n_i} is selected from $\{\beta_1, \beta_2, \ldots, \beta_n\}$ and $\beta_{n_i} \neq \beta_{n_j}$.

Let $\overline{\Psi}$ denote the class of functions $\psi : \mathbb{R}_+ \cup \{\infty\} \to \mathbb{R}_+ \cup \{\infty\}$ satisfying the following conditions:

(a) If $0 < t < \infty$, then $\psi(t) < \infty$.
(b) $\psi|_{\mathbb{R}_+} \in \Psi$.

Now, we are ready to give our main results.

Theorem 1.7 ([25]) *Let (X, ω) be a complete modular metric space with a partial ordering $\sqsubseteq$ and f be a self-mapping on X_ω such that for each $\lambda > 0$, there exists $\eta(\lambda) \in (0, \lambda)$ such that*

$$\begin{aligned}\psi(\omega_\lambda(fx, fy)) \leq \alpha(\psi(\omega_\lambda(x, y)))\psi(\omega_{\lambda+\eta(\lambda)}(x, y)) + \beta(\psi(\omega_\lambda(x, y)))\psi(\omega_\lambda(x, fx)) \\ + \gamma(\psi(\omega_\lambda(x, y)))\psi(\omega_\lambda(y, fy)),\end{aligned}$$

where $\psi \in \overline{\Psi}$ and $(\alpha, \beta, \gamma) \in \mathcal{S}_3$ with $\alpha(t) + 2\max\{\sup_{t\geq 0}\beta(t), \sup_{t\geq 0}\gamma(t)\} < 1$. Assume also that the condition (1.2) holds. If there exists $x_0 \in X_\omega$ such that $\omega_\lambda(x_0, fx_0) < \infty$ for all $\lambda > 0$, then the following holds:

(1) f has a fixed point $x_\infty \in X_\omega$.
(2) The sequence $\{f^n x_0\}$ converges to x_∞.

Proof It is easy to see that the sequence $\{f^n x_0\}$ is nondecreasing. Suppose that for each $n \geq 1$, there exists $\lambda_n > 0$ such that $\omega_{\lambda_n}(f^n x_0, f^{n+1} x_0) \neq 0$. Otherwise, the proof is complete. For each $n \geq 1$, if $0 < \lambda \leq \lambda_n$, then we also have $\omega_\lambda(f^n x_0, f^{n+1} x_0) \neq 0$. Since $f^n x_0 \sqsubseteq f^{n+1} x_0$, for any $0 < \lambda \leq \lambda_n$, and see [25, Theorem 2.1] for more detail of proof. □

Theorem 1.8 ([25]) *Additional to the Theorem 1.7, if* ψ *is subadditive and the following condition holds:*

$$\begin{aligned} &\text{for any } x, y \in X_\omega, \text{ there exists } w \in X_\omega \text{ with } w \sqsubseteq fw \text{ and } \omega_\lambda(w, fw) < \infty \\ &\text{for all } \lambda > 0 \text{ such that } w \text{ is comparable to both } x \text{ and } y, \end{aligned} \tag{1.5}$$

then the fixed point in Theorem 1.7 is unique.

Corollary 1.9 ([25]) *Additional to Theorem 1.7, if* X_ω *is totally ordered, then the fixed point in Theorem 1.7 is unique.*

The following two corollaries nicely broaden the results in [24] (see Theorems 3.2 and 3.6 [24]).

Corollary 1.10 ([25]) *Let* (X, ω) *be a complete modular metric space with a partial ordering* $\sqsubseteq$ *and* f *be a self-mapping on* X_ω *such that for any* $\lambda > 0$*, there exists* $\eta(\lambda) \in (0, \lambda)$ *such that*

$$\psi(\omega_\lambda(fx, fy)) \le \alpha(\psi(\omega_\lambda(x, y)))\psi(\omega_{\lambda+\eta(\lambda)}(x, y)),$$

where $\alpha \in \mathcal{S}$ *and* $\psi \in \overline{\Psi}$*. Assume also that* f *is continuous or the condition (1.2) holds. Then* f *has a fixed point in* X_ω*. Moreover, if the condition (1.5) is satisfied, the fixed point is unique.*

Corollary 1.11 ([25]) *Let* (X, ω) *be a complete modular metric space with a partial ordering* $\sqsubseteq$ *and* f *be a self-mapping on* X_ω *such that for any* $\lambda > 0$*, there exist* $\zeta(\lambda), \mu(\lambda) \in (0, \lambda)$ *such that*

$$\psi(\omega_\lambda(fx, fy)) \le \beta(\psi(\omega_\lambda(x, y)))\psi(\omega_\lambda(x, fx)) + \gamma(\psi(\omega_\lambda(x, y)))\psi(\omega_\lambda(y, fy)),$$

where $\psi \in \overline{\Psi}$ *and* $(\beta, \gamma) \in \mathcal{S}_2$ *with* $\max\{\sup_{t\ge 0}\beta(t), \sup_{t\ge 0}\gamma(t)\} < 1$*. Assume also that* f *is continuous or that the condition (1.2) holds. Then* f *has a fixed point in* X_ω*. Moreover, if the condition (1.5) is satisfied, the fixed point is unique.*

A correction of the recent results of Mongkolkeha et al. [24]

In [24], Mongkolkeha et al. introduced the following theorems:

Theorem 1.12 ([24]) *Let* (X, ω) *be a complete modular metric space and* f *be a self-mapping on* X *satisfying the inequality*

$$\omega_\lambda(fx, fy) \le k\omega_\lambda(x, y), \tag{1.6}$$

for all $x, y \in X_\omega$*, where* $k \in [0, 1)$*. Then,* f *has a unique fixed point in* $x_* \in X_\omega$ *and the sequence* $\{f^n x\}$ *converges to* x_**.*

Theorem 1.13 ([24]) *Let (X, ω) be a complete modular metric space and f be a self-mapping on X satisfying the inequality*

$$\omega_\lambda(fx, fy) \le k[\omega_{2\lambda}(x, fx) + \omega_{2\lambda}(y, fy)],$$

for all $x, y \in X_\omega$, where $k \in \left[0, \frac{1}{2}\right)$. Then, f has a unique fixed point in $x_* \in X_\omega$ and the sequence $\{f^n x\}$ converges to x_*.

We now claim that the conditions in the above theorems are not sufficient to guarantee the existence and uniqueness of the fixed points. We state a counterexample to Theorem 1.12 in the following:

Example 1.14 ([25]) Let $X := \{0, 1\}$ and ω be given by

$$\omega_\lambda(x, y) = \begin{cases} \infty, & \text{if } 0 < \lambda < 1 \text{ and } x \neq y, \\ 0, & \text{if } \lambda \ge 1 \text{ or } x = y. \end{cases}$$

Thus, the modular metric space $X_\omega = X$. Now let f be a self-mapping on X defined by

$$\begin{cases} f(0) = 1, \\ f(1) = 0. \end{cases}$$

Then, f satisfies the inequality (1.6) with any $k \in [0, 1)$ but it possesses no fixed point after all.

Notice that this gap flaws the two above-mentioned theorems only when ∞ is involved.

In this section, we give corrections to both theorems above as follows:

Theorem 1.15 ([25]) *Let (X, ω) be a complete modular metric space and f be a self-mapping on X satisfying the inequality*

$$\omega_\lambda(fx, fy) \le k\omega_\lambda(x, y),$$

for all $x, y \in X_\omega$, where $k \in [0, 1)$. Suppose that there exists $x_0 \in X$ such that $\omega_\lambda(x_0, fx_0) < \infty$ for all $\lambda > 0$. Then, f has a unique fixed point in $x_* \in X_\omega$ and the sequence $\{f^n x_0\}$ converges to x_*.

Theorem 1.16 ([25]) *Let (X, ω) be a complete modular metric space and f be a self-mapping on X satisfying the inequality*

$$\omega_\lambda(fx, fy) \le k[\omega_{2\lambda}(x, fx) + \omega_{2\lambda}(y, fy)],$$

for all $x, y \in X_\omega$, where $k \in \left[0, \frac{1}{2}\right)$. Suppose that there exists $x_0 \in X$ such that $\omega_\lambda(x_0, fx_0) < \infty$ for all $\lambda > 0$. Then, f has a unique fixed point in $x_* \in X_\omega$ and the sequence $\{f^n x\}$ converges to x_*.

Applications

In this section, we give applications of our theorems to establish the existence and uniqueness of a solution to a nonhomogeneous linear parabolic partial differential equation satisfying a given initial condition.

Consider the following initial value problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + F(x,t,u(x,t),u_x(x,t)), & -\infty < x < \infty, 0 < t \le T \\ u(x,0) = \varphi(x) \ge 0, & -\infty < x < \infty, \end{cases} \tag{1.7}$$

where we assume φ to be continuously differentiable such that φ and φ' are bounded and F is continuous.

By a *solution* of the system (3.2), we meant a function $u \equiv u(x,t)$ defined on $\mathbb{R} \times I$, where $I := [0, T]$, satisfying the following conditions:

(a) $u, u_t, u_x, u_{xx} \in C(\mathbb{R} \times I)$.
(b) u and u_x are bounded in $\mathbb{R} \times I$.
(c) $u_t(x,t) = u_{xx}(x,t) + F(x,t,u(x,t),u_x(x,t))$ for all $(x,t) \in \mathbb{R} \times I$.
(d) $u(x,0) = \varphi(x)$ for all $x \in \mathbb{R}$.

Now, we consider the following space:

$$\Omega := \{u(x,t) : u, u_x \in C(\mathbb{R} \times I) \text{ and } \|u\| < \infty\},$$

where

$$\|u\| := \sup_{x\in\mathbb{R},t\in I} |u(x,t)| + \sup_{x\in\mathbb{R},t\in I} |u_x(x,t)| \,.$$

Obviously, the function $\omega : \mathbb{R}^+ \times \Omega \times \Omega \to \mathbb{R}_+$ given by

$$\omega_\lambda(x,y) := \frac{1}{1+\lambda} \|u - v\|$$

is a metric modular on Ω. Clearly, the set Ω_ω is a complete modular metric space independent of generators. Define a partial ordering $\sqsubseteq$ on Ω_ω by

$$u, v \in \Omega_\omega, u \sqsubseteq v \iff u(x,t) \le v(x,t) \text{ and } u_x(x,t) \le v_x(x,t) \text{ at each } (x,t) \in \mathbb{R} \times I.$$

Taking a nondecreasing sequence $\{u_n\}$ in Ω_ω converging to $u \in \Omega_\omega$. For any $(x,t) \in \mathbb{R} \times I$, we have

$$u_1(x,t) \le u_2(x,t) \le \cdots \le u_n(x,t) \le \cdots$$

and

$$(u_1)_x(x,t) \le (u_2)_x(x,t) \le \cdots \le (u_n)_x(x,t) \le \cdots .$$

Since the sequences $\{u_n(x,t)\}$ and $\{(u_n)_x(x,t)\}$ converge to $u(x,t)$ and $u_x(x,t)$, respectively, it follows that for any $(x,t) \in \mathbb{R} \times I$,

$$u_n(x,t) \leq u(x,t) \text{ and } (u_n)_x(x,t) \leq u_x(x,t)$$

for all $n \geq 1$. Therefore, $u_n \sqsubseteq u$ for all $n \geq 1$. So, the space Ω_ω satisfies the condition (1.2).

Theorem 1.17 ([25]) *Consider the problem (3.2) and assume the following:*

(1) For any $c > 0$ with $|s| < c$ and $|p| < c$, the function $F(x,t,s,p)$ is uniformly Hölder continuous in x and t for each compact subset of $\mathbb{R} \times I$.
(2) There exists a constant $c_F \leq (T + 2\pi^{-\frac{1}{2}}T^{\frac{1}{2}})^{-1}$ such that for any $\lambda > 0$, there exists $\eta(\lambda) \in (0,\lambda)$ such that

$$\begin{aligned} 0 &\leq \tfrac{1}{1+\lambda}[F(x,t,s_2,p_2) - F(x,t,s_1,p_1)] \\ &\leq c_F[\tfrac{1}{1+\lambda+\eta(\lambda)}\rho\left(\Xi\left(\frac{s_2 - s_1 + p_2 - p_1}{1+\lambda}\right)\right) \\ &\quad + \tfrac{1}{1+\lambda}\sigma\left(\Xi\left(\frac{s_2 - s_1 + p_2 - p_1}{1+\lambda}\right)\right)] \end{aligned}$$

for all $(s_1,p_1),(s_2,p_2) \in \mathbb{R} \times \mathbb{R}$ with $s_1 \leq s_2$ and $p_1 \leq p_2$, where $\Xi \in \overline{\Psi}$ is sublinear with $\Xi(x) \leq t$ and ρ,σ are nondecreasing functions on $\mathbb{R}_+$ such that $\rho(t) < (1-k)t$ and $\sigma(t) < (1-k)kt$ for all $t > 0$ and for some fixed $k \in (0,1)$.
(3) The two functions $\Gamma, \Upsilon : \mathbb{R}_+ \to [0,1)$ given by

$$\Gamma(t) = \begin{cases} 0 & if\, t = 0, \\ \frac{\rho(t)}{(1-k)t} & if\, t > 0, \end{cases} \qquad \Upsilon(t) = \begin{cases} 0 & if\, t = 0, \\ \frac{\sigma(t)}{(1-k)t} & if\, t > 0, \end{cases}$$

are such that $(\Gamma,\Upsilon,\Upsilon) \in \mathcal{S}_3$, $\Gamma + 2\Upsilon < 1$.
(4) $F(x,t,s,0) \geq \dfrac{s}{\int_0^t \int_\infty^\infty k(x-\xi,t-\tau)d\xi d\tau}$ for all $s \geq 0$.
(5) F is bounded for bounded s and p.

Then, the existence and uniqueness of the solution of the system (3.2) are affirmative.

It is essential to note that the problem (3.2) is equivalent (under the assumption of Theorem 3.11) to the integral equation:

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} k(x-\xi,t)\varphi(\xi)d\xi \\ &\quad + \int_0^t \int_\infty^\infty k(x-\xi,t-\tau)F(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau))d\xi \mathrm{d}\tau \end{aligned} \tag{1.8}$$

for all $x \in \mathbb{R}$ and $0 < t \leq T$, where

$$k(x,t) := \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$$

for all $x \in \mathbb{R}$ and $t > 0$. The system (3.2) possesses a unique solution if and only if Eq. (3.3) possesses a unique solution u such that u and u_x are both continuous and bounded for all $x \in \mathbb{R}$ and $0 < t \leq T$.

Define a mapping $\Lambda : \Omega_\omega \to \Omega_\omega$ by

$$(\Lambda u)(x,t) := \int_{-\infty}^{\infty} k(x-\xi,t)\varphi(\xi)d\xi + \int_0^t \int_{\infty}^{\infty} k(x-\xi,t-\tau)F(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau))d\xi\mathrm{d}\tau$$

for all $(x,t) \in \mathbb{R} \times I$. Then the problem of finding the solution to Eq. (3.3) is equivalent to the problem of finding the fixed point of Λ.

Proof It is easy to see that the mapping Λ is nondecreasing by the definition. Let $u, v \in \Omega_\omega$ with $u \sqsubseteq v$. Suppose that $u \neq v$ and see more detail [25, Theorem 2.1] for proof. □

2 Fixed Point Results Based on α-Type F-Contractions

Fixed point technique is one of the most important tools in terms of studying the existence and uniqueness of the solution of various mathematical methods that appear in practical problems. Specifically, Banach's reduction theory is a creative way to find specific solutions for models related to differential equations and integral equation. This principle is generalized by several authors in various directions; see [26–31]. Recently, Gopal et al. [32] introduced the concept of α-type F-contraction in metric space by combining the ideas given in [31] and obtained some fixed point results.

We introduce the concept of α-type F-contraction in the setting of modular metric spaces and establish fixed point and periodic point results for such contraction. Consequently, our results generalize and improve some known results from the literature.

Following [33, 34], we denote by $\mathcal{F}$ the family of all functions, $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

$(F1)$ F is strictly increasing on $\mathbb{R}^+$,
$(F2)$ for every sequence $\{s_n\}$ in $\mathbb{R}^+$, we have $\lim_{n\to\infty} s_n = 0$ if and only if $\lim_{n\to\infty} F(s_n) = -\infty$,
$(F3)$ there exists a number $k \in (0,1)$ such that $\lim_{s\to 0^+} s^k F(s) = 0$.

Example 2.1 The following function $F : \mathbb{R}^+ \to \mathbb{R}$ belongs to $\mathcal{F}$:

(i) $F(t) = \ln t$, with $t > 0$,
(ii) $F(t) = \ln t + t$, with $t > 0$.

Definition 2.2 ([31]) A mapping $T : X \to X$ is said to be α-admissible if there exists a function $\alpha : X \times X \to \mathbb{R}_+$ such that

$$x, y \in X,\ \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 2.3 ([34]) Let Δ_G denote the set of all functions $G : (\mathbb{R}^+)^4 \to \mathbb{R}^+$ satisfying the condition: (G) for all $t_1, t_2, t_3, t_4 \in \mathbb{R}^+$ with $t_1 t_2 t_3 t_4 = 0$, there exists $\tau > 0$ such that $G(t_1, t_2, t_3, t_4) = \tau$.

Example 2.4 The following function $G : (\mathbb{R}^+)^4 \to \mathbb{R}$ belongs to Δ_G :

(i) $G(t_1, t_2, t_3, t_4) = L\min(t_1, t_2, t_3, t_4) + \tau,$
(ii) $G(t_1, t_2, t_3, t_4) = \tau e^{L\min(t_1,t_2,t_3,t_4)}$. Where $L \in \mathbb{R}^+$

Definition 2.5 ([34]) Let (X, ω) be a modular metric space and T be a self-mapping on X_ω. Suppose that $\alpha, \eta : X_\omega \times X_\omega \to \mathbb{R}_+$ be two functions. We say T is an α-η-GF-contraction if for $x, y \in X_\omega$ with $\eta(x, Tx) \leq \alpha(x, y)$, $\omega_{\lambda/l}(Tx, Ty) > 0$, and $\lambda, l > 0$, we have

$$\begin{aligned}&F(\omega\lambda/l(x, y))\\&\geq G(\omega_{\lambda/l}(x, Tx), \omega_{\lambda/l}(y, Ty), \omega_{\lambda/l}(x, Ty), \omega_{\lambda/l}(y, Tx)) + F(\omega_{\lambda/l}(Tx, Ty))\end{aligned}$$

where $G \in \Delta_G$ and $F \in \mathcal{F}$.

Fixed point results based on α-type F-contractions

We begin with the following definitions:

Definition 2.6 ([35]) Let (X, ω) be a modular metric space. Let C be a nonempty subset of X_ω. A mapping $T : C \to C$ is said to be an α-type F-contraction if there exists $\tau > 0$ and two functions $F \in \mathcal{F}$, $\alpha : C \times C \to (0, \infty)$ such that for all $x, y \in C$, with $\omega_1(Tx, Ty) > 0$, the following inequality holds:

$$\tau + \alpha(x, y)F(\omega_1(Tx, Ty) \leq F(\omega_1(x, y)). \tag{2.1}$$

Definition 2.7 ([35]) Let (X, ω) be a modular metric space. Let C be a nonempty subset of X_ω. A mapping $T : C \to C$ is said to be an α-type F-weak contraction if there exists $\tau > 0$ and two functions $F \in \mathcal{F}$, $\alpha : C \times C \to (0, \infty)$ such that for all $x, y \in C$, with $\omega_1(Tx, Ty) > 0$, the following inequality holds:

$$\begin{aligned}&\tau + \alpha(x, y)F(\omega_1(Tx, Ty)\\&\leq F\left(\max\left\{\omega_1(x, y), \omega_1(x, Tx), \omega_1(y, Ty), \frac{\omega_2(x, Ty) + \omega_2(y, Tx)}{2}\right\}\right).\end{aligned} \tag{2.2}$$

Remark 2.8 ([35]) Every α-type F-contraction is an α-type F-weak contraction, but converse is not necessarily true.

Example 2.9 ([35]) Let $X_\omega = C = \left[0, \frac{9}{2}\right]$, $\omega_1 = |x - y|$ and $\omega_2 = |x - y|$. Define $T : C \to C$, $\alpha : C \times C \to (0, \infty)$ and $F : \mathbb{R}^+ \to \mathbb{R}$ by

$$T(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{2}{9}] \\ \frac{9}{2}, & \text{otherwise.} \end{cases}$$

Then, for $x = 0$ and $y = 1$, by putting $F(t) = \ln t$ with $t > 0$, we have

$$\tau + \alpha(0, 1)F(\omega_1(T(0), T(1)) = \tau + \alpha(0, 1)\ln\left(\frac{9}{2}\right)$$

and

$$F(\omega_1(0, 1)) = \ln(1).$$

Clearly, we have

$$e^\tau \left(\frac{9}{2}\right)^{\alpha(0,1)} \not\leq 1 \quad \text{for all } \tau > 0 \text{ and for all } \alpha \in (0, \infty).$$

However, since

$$\inf_{x\in[0,\frac{2}{9}], y\in(\frac{2}{9},\frac{9}{2}]} \left\{ \max\left\{ \omega_1(x, y), \omega_1(x, Tx), \omega_1(y, Ty), \frac{\omega_2(x, Ty) + \omega_2(y, Tx)}{2} \right\} \right\} = \frac{9}{4}$$

then T is an α-type F-weak contraction for the choice

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in \left[0, \frac{2}{9}\right] \text{ or } x, y \in \left(\frac{2}{9}, \frac{9}{2}\right] \\ \frac{\log 10 - \log 9}{\log 9 - \log 2}, & \text{otherwise} \end{cases}$$

and $\tau > 0$ such that $e^{-\tau} = \frac{8}{9}$.

Remark 2.10 Definition 2.6 (respectively, Definition 2.7) reduces to F-contraction (respectively, F-weak contraction) for $\alpha(x, y) = 1$.

The motivation of the following definition can be predicted from the last step of the proof of Cauchy sequence in our Theorems.

Definition 2.11 ([35]) Let (X, ω) be a modular metric space and C be a nonempty subset of X_ω. The sequence $(x_n)_{n\in\mathbb{N}}$ in C is said to satisfy Δ_M-condition if this is the case, i.e., $\lim_{m,n\to\infty} \omega_{m-(n+1)}(x_n, x_m) = 0$ for $(m, n \in \mathbb{N},\ m > n + 1)$ implies $\lim_{m,n\to\infty} \omega_\lambda(x_n, x_m) = 0$ for some $\lambda > 0$.

Next, we are ready to state our first theorem which generalizes the main theorem of Gopal et al. [32] for modular metric spaces.

Theorem 2.12 ([35]) *Let (X, ω) be a modular metric space. Assume that ω is regular and satisfies Δ_M-condition. Let C be a nonempty subset of X_ω. Assume that C is complete and bounded, i.e., $\delta_\omega(C) = \sup\{\omega_1(x, y) : x, y \in C\} < \infty$. Let $T : C \to C$ be an α-type F-weak contraction satisfying the following conditions:*

(i) *T is α-admissible,*
(ii) *there exists $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$,*
(iii) *T is continuous.*

Then T has a fixed point $x^* \in C$, and for every $x_0 \in C$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.

Proof See [35, Theorem 2.9] for proof. □

Theorem 2.13 ([35]) *Let (X, ω) be a modular metric space. Assume that ω is regular and satisfies Δ_M-condition. Let C be a nonempty subset of X_ω. Assume that C is complete modular metric space and bounded, i.e., $\delta_\omega(C) = \sup\{\omega_1(x, y) : x, y \in C\} < \infty$. Let $T : C \to C$ be an α-type F-weak contraction satisfying the following conditions:*

(i) *there exists $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$,*
(ii) *T is α-admissible,*
(iii) *if $\{x_n\}$ is a sequence in X_ω such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$,*
(iv) *F is continuous.*

Then T has a fixed point $x^* \in C$, and for every $x_0 \in C$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.

Proof See [35, Theorem 2.10] for proof. □

Indeed, uniqueness of the fixed point, we will consider the following hypothesis. (H): for all $x, y \in Fix(T), \alpha(x, y) \geq 1$.

Theorem 2.14 ([35]) *Adding condition (H) to the hypotheses of Theorem 2.12 (respectively, Theorem 6.5), the uniqueness of the fixed point is obtained.*

Proof See [35, Theorem 2.10] for proof. □

The following result improves the main theorem of F-contraction [36] for a modular metric space.

Corollary 2.15 ([35]) *Let (X, ω) be a modular metric space. Assume that ω is regular and satisfies Δ_M-condition. Let C be a nonempty subset of X_ω. Assume that C is complete and bounded, i.e., $\delta_\omega(C) = \sup\{\omega_1(x, y) : x, y \in C\} < \infty$. Let $T : C \to C$ be an α- type F-contraction satisfying the hypotheses of Theorem 2.14, then T has unique fixed point.*

From Example 2.1(i) and Corollary 2.15 (above), we obtain the following result given [37].

Theorem 2.16 ([35]) *Let (X, ω) be a modular metric space. Assume that ω is regular. Let C be a nonempty subset of X_ω. Assume that C is complete and bounded, i.e., $\delta_\omega(C) = \sup\{\omega_1(x, y); x, y \in C\} < \infty$. Let $T : C \to C$ be a contraction. Then T has a unique fixed point x_0. Moreover, the orbit $\{T^n(x)\}$ converges to x_0 for $x \in C$.*

Periodic point results

In this section, we prove some periodic point results for self-mappings on a modular metric space. In the sequel, we need the following definition.

Definition 2.17 ([32]) A mapping $T : C \to C$ is said to have the $property\ (P)$ if $Fix(T^n) = Fix(T)$ for every $n \in \mathbb{N}$, where $Fix(T) := \{x \in X_\omega : Tx = x\}$.

Theorem 2.18 ([35]) *Let (X, ω) be a modular metric space. Assume that ω is regular and satisfies Δ_M-condition. Let C be a nonempty subset of X_ω. Assume that C is complete and bounded, i.e., $\delta_\omega(C) = \sup\{\omega_1(x, y) : x, y \in C\} < \infty$. Let $T : C \to C$ be a mapping satisfying the following conditions:*

(i) there exists $\tau > 0$ and two functions $F \in \mathcal{F}$ and $\alpha : C \times C \to (0, \infty)$ such that

$$\tau + \alpha(x, Tx)F(\omega_1(Tx, T^2x)) \leq F(\omega_1(x, Tx))$$

holds for all $x \in C$ with $\omega_1(Tx, T^2x) > 0$,
(ii) there exists $x_0 \in C$ such that $\alpha(x_0, Tx_0) \geq 1$,
(iii) T is α-admissible,
(iv) if $\{x_n\}$ is a sequence in C such that $\alpha(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ and $\omega_1(x_n, x) \to 0$, as $n \to \infty$, then $\omega_1(Tx_n, Tx) \to 0$ as $n \to \infty$,
(v) if $z \in Fix(T^n)$ and $z \notin Fix(T)$, then $\alpha(T^{n-1}z, T^n z) \geq 1$. Then T has the $property\ (P)$.

Proof See [35, Theorem 3.2] for proof. □

Taking $\alpha(x, y) = 1$ for all $x, y \in C$ in Theorem 2.18, we get the following result:

Corollary 2.19 ([35]) *Let (X, ω) be a complete modular metric space. Assume that ω is regular and satisfies Δ_M-condition. Let C be a nonempty subset of X_ω. Assume that C is complete and bounded, i.e., $\delta_\omega(C) = \sup\{\omega_1(x, y) : x, y \in C\} < \infty$. Let $T : C \to C$ be a continuous mapping satisfying*

$$\tau + F(\omega_1(Tx, T^2x)) \leq F(\omega_1(x, Tx))$$

for some $\tau > 0$ and for all $x \in X_\omega$ such that $\omega_1(Tx, T^2x)) > 0$. Then T has $property\ (P)$.

3 Coincidence Point Results Endowed with a Graph

In 2007, Jachymski [38] using the language of graph theory introduced the concept of a G-contraction on a metric space endowed with a graph and proved a fixed point theorem which extends the results of Ran and Reurings [39].

Let (X, ω) be a modular metric space and D be a nonempty subset of X_ω. Let Δ denote the diagonal of the Cartesian product $D \times D$. Consider a directed graph (digraphs) G_ω such that the set $V(G_\omega)$ of its vertices coincides with D, and the set $E(G_\omega)$ of its edges contains all loops, i.e., $E(G_\omega) \supseteq \Delta$. We assume G_ω simple graph (opposite of multigraph), so we can identify G_ω with the pair $(V(G_\omega), E(G_\omega))$. Our graph theory notations and terminology are standard and can be found in all graph theory books, like [40, 41]. Moreover, we may treat G_ω as a weighted graph (see [41], p. 309) by assigning to each edge the distance between its vertices. By G^{-1}, we denote the reverse of a graph G, i.e., the graph obtained from G by reversing the direction of edges. Thus, we have

$$E(G^{-1}) = \{(y, x) | (x, y) \in E(G)\}.$$

A digraph G is a directed graph if whenever $(u, v) \in E(G)$, then $(v, u) \notin E(G)$. The letter $\tilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

We call (V', E') a subgraph of G if $V' \subseteq V(G)$, $E' \subseteq E(G)$, and for any edge $(x, y) \in E'$, $x, y \in V'$.

If x and y are vertices in a graph G, then a (directed) path in G from x to y of length N is a sequence $(x_i)_{i=1}^{i=N}$ of $N + 1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, \dots, N$. A graph G is connected if there is a directed path between any two vertices. G is weakly connected if $\tilde{G}$ is connected. If G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation $\mathcal{R}$ defined on $V(G)$ by the rule: $y\mathcal{R}z$ if there is a (directed) path in G from y to z. Clearly, G_x is connected.

We establish some coincidence and periodic point theorems concerning F-contractive mappings in modular metric space endowed with a graph. Our main result is a generalization of Gopal et al. [32] theorem and others. We also give an application of our main results to establish the existence of solution for a nonhomogeneous linear parabolic partial differential equation.

Coincidence point results

Throughout this section, we assume that (X, ω) is a modular metric space, D be a nonempty subset of X_ω and $\mathcal{G} := \{G_\omega$ is a directed graph with $V(G_\omega) = D$ and $\Delta \subseteq E(G_\omega)\}$.

Definition 3.1 [38, 42] The pair (D, G_ω) has Property (A) if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in D, with $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G_\omega)$, then $(x_n, x) \in E(G_\omega)$, for all n.

Definition 3.2 ([43]) Let $F \in \mathcal{F}$ and $G_\omega \in \mathcal{G}$. A mapping $T : D \to D$ is said to be F-G_ω-contraction with respect to $R : D \to D$ if

(i) $(Rx, Ry) \in E(G_\omega) \Rightarrow (Tx, Ty) \in E(G_\omega)$ for all $x, y \in D$, i.e., T preserves edges w.r.t. R,
(ii) there exists a number $\tau > 0$ such that

$$\omega_1(Tx, Ty) > 0 \Rightarrow \tau + F(\omega_1(Tx, Ty)) \leq F(\omega_1(Rx, Ry))$$

for all $x, y \in D$ with $(Rx, Ry) \in E(G_\omega)$.

Example 3.3 ([43]) Let $F \in \mathcal{F}$ be arbitrary. Then every F-contractive mapping w.r.t. R is an F-G_ω-contraction w.r.t. R for the graph G_ω given by $V(G_\omega) = D$ and $E(G_\omega) = D \times D$.

We denote by $C(T, R) := \{x \in D : Tx = Rx\}$ the set of all coincidence points of two self-mappings T and R, defined on D.

Now, we state our first theorem which generalizes the main theorem of Gopal et al. [32] for regular modular metric spaces.

Theorem 3.4 ([43]) *Let (X, ω) be a regular modular metric space with a graph G_ω. Assume that $D = V(G_\omega)$ is a nonempty bounded subset of X_ω and the pair (D, G_ω) has property (A) and also satisfy Δ_M-condition. Let $R, T : D \to D$ be two self-mappings satisfying the following conditions:*

(i) there exists $x_0 \in D$ such that $(Rx_0, Tx_0) \in E(G_\omega)$,
(ii) T is an F-G_ω-contraction w.r.t R,
(iii) $T(D) \subseteq R(D)$,
(iii) $R(D)$ is complete.

Then $C(R, T) \neq \emptyset$.

Proof See [43, Theorem 2.1] for proof. □

Periodic point results

In this section, we prove some periodic point results for self-mappings on a modular metric space endowed with a graph.

Definition 3.5 [44] Let (X, ω) be a modular metric space and $T : D \to D$ be a mapping. Then T is said to have the property (P) if $Fix(T^n) = Fix(T)$ for every $n \in \mathbb{N}$ where $Fix(T) := \{x \in D : Tx = x\}$.

Again, let (X, ω) be a modular metric space and $T : D \to D$ be a mapping. The set $\mathcal{O}(x) = \{x, Tx, T^2x, \ldots, T^nx, \ldots\}$ is called the orbit of x under T.

Definition 3.6 ([43]) A mapping $T : D \to D$ is called strong orbitally G_ω-at x if

$$\lim_{n\to\infty} T^n x = x_* \text{ and } (T^n x, T^{n+1}x) \in E(G_\omega) \Rightarrow \lim_{n\to\infty} T^{n+1}x = Tx_*.$$

A mapping T is called strongly G_ω-orbitally continuous on D if T is strongly orbitally G_ω-continuous for all $x \in D$.

We denote $D^T := \{x \in D : (x, Tx) \in E(G_\omega) \text{ or } (Tx, x) \in E(G_\omega)\}$.

Definition 3.7 ([43]) Let (X, ω) be a modular metric space. A mapping $T : D \to D$ is called an F-G_ω graphic contraction if

(i) T preserves edges, i.e., $(x, y) \in E(G_\omega) \Rightarrow (Tx, Ty) \in E(G_\omega)$,
(ii) there exists a number $\tau > 0$ such that

$$\omega_1(Tx, T^2x) > 0 \Rightarrow \tau + F(\omega_1(Tx, T^2x)) \leq F(\omega_1(x, Tx)) \tag{3.1}$$

for all $x \in D^T$ and $F \in \mathcal{F}$.

Remark 3.8 If we consider $F(s) = \ln s$ for all $s > 0$, then Definition 3.7 reduces to G_ω-graphic contractive given in [45].

Before stating the theorem of this section, we give the following lemma without proof.

Lemma 3.9 *Let (X, ω) be a modular metric space endowed with a graph G_ω. Let $T : D \to D$ be a G_ω-graphic contractive. Then T is a G_ω^{-1}-graphic contractive too.*

Theorem 3.10 ([43]) *Let (X, ω) be a regular modular metric space with a graph G_ω. Assume that $D = V(G_\omega)$ is complete, bounded (nonempty) subset of X_ω and the pair (D, G_ω) satisfy Δ_M-condition. Suppose that $T : D \to D$ is an F-G_ω-graphic contraction satisfying the following condition:*

(∗) *$(x, Tx) \in E(G_\omega)$ or $(Tx, x) \in E(G_\omega)$ for all $x \in D$.*

Then T has the property (P) provided that T is strongly G_ω-orbitally continuous on D.

Proof See [43, Theorem 3.2] for proof. □

Existence of solution for a nonhomogeneous linear parabolic partial differential equation

In this section, following the idea in [23], we discuss the application of coincidence (fixed) point techniques to the solution of the nonhomogeneous linear parabolic partial differential equation satisfying a given initial condition.
More precisely, we consider the following initial value problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + H(x,t,u(x,t),u_x(x,t)), & -\infty < x < \infty, 0 < t \leq T \\ u(x,0) = \varphi(x) \geq 0, & -\infty < x < \infty, \end{cases} \tag{3.2}$$

where H is continuous and φ assumes to be continuously differentiable such that φ and φ' are bounded.

By a *solution* of the problem (3.2), we mean a function $u \equiv u(x,t)$ defined on $\mathbb{R} \times I$, where $I := [0, T]$, satisfying the following conditions:

(i) $u, u_t, u_x, u_{xx} \in C(\mathbb{R} \times I)$. { $C(\mathbb{R} \times I)$ denote the space of all continuous real valued functions },
(ii) u and u_x are bounded in $\mathbb{R} \times I$,
(iii) $u_t(x,t) = u_{xx}(x,t) + H(x,t,u(x,t),u_x(x,t))$ for all $(x,t) \in \mathbb{R} \times I$,
(iv) $u(x,0) = \varphi(x)$ for all $x \in \mathbb{R}$.

It is important to note that the initial value problem (3.2) is equivalent to the following integral equation

$$\begin{aligned} u(x,t) = & \int_{-\infty}^{\infty} k(x-\xi,t)\varphi(\xi)d\xi \\ & + \int_0^t \int_{-\infty}^{\infty} k(x-\xi,t-\tau)H(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau))d\xi \mathrm{d}\tau \end{aligned} \tag{3.3}$$

for all $x \in \mathbb{R}$ and $0 < t \leq T$, where

$$k(x,t) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

The problem (3.2) admits a solution if and only if the corresponding integral equation (3.3) has a solution.

Let

$$\Omega := \{u(x,t); u, u_x \in C(\mathbb{R} \times I) \text{ and } \|u\| < \infty\},$$

where

$$\|u\| := \sup_{x\in\mathbb{R},t\in I} |u(x,t)| + \sup_{x\in\mathbb{R},t\in I} |u_x(x,t)|.$$

Obviously, the function $\omega : \mathbb{R}^+ \times \Omega \times \Omega \to \mathbb{R}_+$ given by

$$\omega_\lambda(u, v) := \frac{1}{\lambda}\|u - v\| = \frac{1}{\lambda}d(u, v)$$

is a metric modular on Ω. Clearly, the set Ω_ω is a complete modular metric space independent of generators.

Theorem 3.11 ([43]) *Consider the problem (3.2) and assume the followings:*

(i) for $c > 0$ with $|s| < c$ and $|p| < c$, the function $F(x, t, s, p)$ is uniformly Hölder continuous in x and t for each compact subset of $\mathbb{R} \times I$,
(ii) there exists a constant $c_H \leq (T + 2\pi^{-\frac{1}{2}} T^{\frac{1}{2}})^{-1} \leq q$, where $q \in (0, 1)$ such that

$$\begin{aligned} 0 &\leq \tfrac{1}{\lambda}[H(x, t, s_2, p_2) - H(x, t, s_1, p_1)] \\ &\leq c_H \left[\frac{s_2 - s_1 + p_2 - p_1}{\lambda}\right] \end{aligned}$$

for all $(s_1, p_1), (s_2, p_2) \in \mathbb{R} \times \mathbb{R}$ with $s_1 \leq s_2$ and $p_1 \leq p_2$,
(iii) H is bounded for bounded s and p.

Then the problem (3.2) admits a solution.

Proof See [43, Theorem 4.3] for proof. □

4 Coincidence Point Theorems Base on (CLR_T)-property

We consider important property for coincidence point theorems which is defined by Sintunavarat and Kumam [46], is called the (CLR_T)-property as follows:

Let (X, d) is a metric space and $S, T : X \to X$ be two mappings. The mappings S and T are said to satisfy the *common limit in the range of T* (shortly, (CLR_T)-property) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Tx$$

for some $x \in X$. The importance of (CLR_T)-property ensures that one does not require the closeness of range subspaces.

We study and prove the existence of some coincidence point theorems for generalized contraction mappings in modular metric spaces and give some applications on integral equations for our main results.

Lemma 4.1 *Let S and T be weakly compatible self-mappings of a set X_ω. If S and T have a unique coincidence point, i.e., $t = Sx = Tx$, then t is the common fixed point of S and T.*

Theorem 4.2 *Let (X, ω) be a modular metric space and $S, T : X_\omega \to X_\omega$ be weakly compatible mappings such that $S(X_\omega) \subset T(X_\omega)$. Suppose that there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in [0, \frac{1}{4})$ and $\sum_{i=1}^{5} \alpha_i < 1$ such that for all $x, y \in X_\omega$ and $\lambda > 0$,*

(a) there exists $x_0, x_1 \in X_\omega$ such that $\omega_\lambda(Sx_0, Tx_1) < \infty$;
(b) $\omega_\lambda(Sx, Sy) \leq \alpha_1\omega_\lambda(Sx, Tx) + \alpha_2\omega_\lambda(Sy, Ty) + \alpha_3\omega_\lambda(Sy, Tx) + \alpha_4\omega_\lambda(Sx, Ty) + \alpha_5\omega_\lambda(Ty, Tx)$.

If S and T satisfy (CLR_T)-property, then S and T have a unique common fixed point.

Proof It follows from condition (b) and S and T satisfy the (CLR_T)-property, there exists a sequence $\{x_n\}$ in X_ω, we have

$$\omega_\lambda(Sx_n, Sx) \leq a_1\omega_\lambda(Sx_n, Tx_n) + a_2\omega_\lambda(Sx, Tx) + a_3\omega_\lambda(Sx, Tx_n) + a_4\omega_\lambda(Sx_n, Tx) + a_5\omega_\lambda(Tx, Tx_n)$$

for all $n \geq 1$. By taking the limit $n \to \infty$, we get

$$\omega_\lambda(Tx, Sx) \leq (a_2 + a_3)\omega_\lambda(Sx, Tx).$$

This implies that $(1 - a_2 - a_3)\omega_\lambda(Sx, Tx) \leq 0$ for all $\lambda > 0$, which is a contradiction. Thus, $Sx = Tx$. Then, following the same argument in the proof of [47, Theorem 3]. □

By setting $T = I_{X_\omega}$, we deduce the following result of fixed point for one self-mapping from Theorem 4.2.

Corollary 4.3 *Let (X, ω) be an complete modular metric space and $S : X_\omega \to X_\omega$ such that for all $\lambda > 0$ and $x, y \in X_\omega$, $\omega_\lambda(x_0, Sx_0) < \infty$ and*

$$\omega_\lambda(Sx, Sy) \leq \alpha_1\omega_\lambda(Sx, x) + \alpha_2\omega_\lambda(Sy, y) + \alpha_3\omega_\lambda(Sy, x) + \alpha_4\omega_\lambda(Sx, y) + \alpha_5\omega_\lambda(x, y)$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in [0, \frac{1}{4})$ and $\sum_{i=1}^{5} \alpha_i < 1$. Then S has a unique fixed point z. Further, for any $x_0 \in X_\omega$, the Picard sequence $\{Sx_n\}$ with an initial point x_0 is convergent to the fixed point z.

Corollary 4.4 *Let (X, ω) be an complete modular metric space and $S : X_\omega \to X_\omega$ such that for all $\lambda > 0$ and $x, y \in X_\omega$, $\omega_\lambda(x_0, Sx_0) < \infty$ and*

$$\omega_\lambda(Sx, Sy) \leq \alpha_1\omega_\lambda(Sx, x) + \alpha_2\omega_\lambda(Sy, y) + \alpha_3\omega_\lambda(x, y)$$

where $\alpha_1, \alpha_2, \alpha_3 \in [0, \frac{1}{4})$ and $\sum_{i=1}^{3} \alpha_i < 1$. Then S has a unique fixed point.

Corollary 4.5 *Let (X, ω) be an complete modular metric space and $S : X_\omega \to X_\omega$ such that for all $\lambda > 0$ and $x, y \in X_\omega$, $\omega_\lambda(x_0, Sx_0) < \infty$ and*

$$\omega_\lambda(Sx, Sy) \leq \alpha\omega_\lambda(x, y)$$

where $0 \leq \alpha < 1$. *Then* S *has a unique fixed point.*

Now, we give some examples of the (CLR_T)-property as follows:

Example 4.6 Let $X_\omega = [0, \infty)$ be a modular metric space. Define two mappings $S, T : X_\omega \to X_\omega$ by $Sx = 5x - 4$ and $Tx = x$ for all $x \in X_\omega$, respectively. Now, we consider the sequence $\{x_n\}$ defined by $x_n = \{1 + \frac{1}{n+5}\}$ for each $n \geq 1$. Since

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 1 = T(1) \in X_\omega,$$

S and T satisfy the (CLR_T)-property.

Some applications to Fredholm integral equations

The purpose of this section is to show the existence and uniqueness of a solution of Fredholm integral equations in modular metric spaces with a function space $(C(I, \mathbb{R}), \omega_\lambda)$ and a contraction by using our main results.

Consider the integral equation:

$$Sx(t) - \mu \int_0^r K(t, s)hx(s)ds = T(t), \tag{4.1}$$

where $x : I \to \mathbb{R}$ is an unknown function, $T : I \to \mathbb{R}$ and $h, S : \mathbb{R} \to \mathbb{R}$ are two functions, and μ is a parameter. The kernel K of the integral equation is defined by $I \times \mathbb{R} \to \mathbb{R}$, where $I = [0, r]$.

Theorem 4.7 *Let* K, S, T, h *be continuous. Suppose that* $C \in \mathbb{R}$ *is such that for all* $t, s \in I$,

$$|K(t, s)| \leq C$$

and, for each $x \in (C(I, \mathbb{R}), \omega_\lambda)$*, there exists* $y \in (C(I, \mathbb{R}), \omega_\lambda)$ *such that*

$$(Sy)(t) = T(t) + \mu \int_0^r K(t, s)hx(s)ds$$

for all $r \in C(I, \mathbb{R})$. *If* S *is injective, there exists* $L \in \mathbb{R}$ *such that for all* $x, y \in \mathbb{R}$,

$$|hx - hy| \leq L|Sx - Sy|$$

and $\{Sx : x \in (C(I, \mathbb{R}), \omega_\lambda)\}$ *is complete, then for any* $\mu \in \left(-\frac{1}{CrL}, \frac{1}{CrL}\right)$, *there exists* $w \in (C(I, \mathbb{R}), \omega_\lambda)$ *such that for any* $x_0 \in (C(I, \mathbb{R}), \omega_\lambda)$,

$$Sw(t) = \lim_{x\to\infty} Sx_n(t) = \lim_{x\to\infty} \left[T(t) + \mu \int_0^r K(t, s)hx_{n-1}(s)ds\right] \tag{4.2}$$

and w *is the unique solution of Eq.* (4.1).

Proof See [47, Theorem 6] for proof. □

5 An Observation on Set-Valued Contraction Mappings

On the other hand, the set-valued alternative of the contraction principle was given in [23, 48]. Unfortunately, the proofs of the main results contain a small, but defective gap (we shall discuss this matter precisely in the forthcoming section). This leaves the problem of the set-valued contraction principle open.

This research is conducted to properly give sufficient conditions for a set-valued contraction to possess a fixed point. Our results also fix the slip found in [23, 48], under some additional assumptions.

Given a modular metric space X_ω. Suppose that $x \in X_\omega$ and $r > 0$, we define an open ball of radius r around x by

$$B(x; r) := \left\{ z \in X,\ \sup_{\lambda > 0} \omega_\lambda(x, z) < r \right\}.$$

Let $\mathfrak{B}$ be a set containing all open balls in X_ω. We may easily see that $\mathfrak{B}$ actually acts as a base determining a unique topology on X, namely τ. Always assume that X_ω is a given modular metric space equipped with the topology generated by $\mathfrak{B}$.

With the same elementary proofs (and so omitted) as in a classical metric space, we may obtain the following results:

Proposition 5.1 ([49]) *X_ω is Hausdorff separable.*

Proposition 5.2 ([49]) *In X_ω, the compactness and sequential compactness characterize each others.*

Proposition 5.3 ([49]) *A sequence (x_n) in X_ω converges to a point $x \in X$ if and only if for any given $\varepsilon > 0$, we have $\sup_{\lambda>0} \omega_\lambda(x, x_n) < \varepsilon$ for sufficiently large $n \in \mathbb{N}$.*

We may now define a Cauchy sequence in parallel to the characterization in Proposition 5.3.

Definition 5.4 ([49]) A sequence (x_n) in X_ω is Cauchy if for any $\varepsilon > 0$, there holds that $\sup_{\lambda>0} \omega_\lambda(x_m, x_n) < \varepsilon$ for sufficiently large $m, n \in \mathbb{N}$.

Naturally, each convergent sequence is Cauchy. If the converse is true for all sequence in X_ω, we say that X_ω is *complete*.

Definition 5.5 A set $Z \subset X_\omega$ is said to be bounded if $\sup_{x,y\in Z} \sup_{\lambda>0} \omega_\lambda(x, y) < \infty$.

We may note that a non-singleton finite set in a modular metric space is no need to be bounded (for instance, take any metric space (M, ρ), and the metric modular $(\lambda, x, y) \in \mathbb{R}^+ \times M \times M \mapsto \frac{\rho(x,y)}{\lambda}$). This fact gives an example of a compact set which is not bounded in contrast to metric spaces. However, a compact set is always closed by Proposition 5.1.

In accordance with Chaipunya et al. [23], for $x \in X_\omega$ and $Y, Z \subset X_\omega$, we write

$$\begin{cases} \omega_\lambda(x, Z) := \inf_{z\in Z} \omega_\lambda(x, z), \\ e_\lambda(Y, Z) := \sup_{y\in Y} \omega_\lambda(y, Z), \\ W_\lambda(Y, Z) := \max\{e_\lambda(Y, Z), e_\lambda(Z, Y)\}. \end{cases}$$

A number of fundamental properties of these functions for closed bounded sets can be found in [23]. In fact, such properties also work, with the same proofs, for closed (and not necessarily bounded) sets. Also note that if $Z \subset X_\omega$ is closed and $z \in X_\omega$, we have $z \in Z$ if and only if $\omega_\lambda(z, Z) = 0$ for all $\lambda > 0$.

A remark on set-valued contraction

Given a set-valued map $F : X_\omega \multimap X_\omega$, if there exists a constant $k \in (0, 1)$ such that

$$W_\lambda(F(x), F(y)) \le k\omega_\lambda(x, y), \tag{5.1}$$

for all $\lambda > 0$ and all $x, y \in X_\omega$, we say that F is a set-valued contraction.

The existence of fixed points for a set-valued contraction in modular metric space is first considered in [23, Theorem 3.3]. The proof exploited the existence of a sequence (x_n) such that for each $n \in \mathbb{N}$, $x_n \in F(x_n)$ and

$$\omega_s(x_n, x_{n+1}) \le k^n + W_s(F(x_{n-1}), F(x_n)), \tag{5.2}$$

where $s > 0$ is pre-given. Note that the property (5.2) is not preserved upon the change of s. Unfortunately, (5.2) is needed for all $s > 0$, and this leaves out a gap in this proof.

To fill this gap in, we need some additional definitions, lemmas, and assumptions. These materials will be discussed in the succeeding section.

Definition 5.6 ([49]) A nonempty subset $Z \subset X_\omega$ is said to be *reachable* from a point $x \in X_\omega$ if

$$\inf_{z\in Z} \sup_{\lambda>0} \omega_\lambda(x, z) = \sup_{\lambda>0} \inf_{z\in Z} \omega_\lambda(x, z) < \infty.$$

Remark 5.7 ([49]) To show the reachability, we only need to show that

$$\inf_{z\in Z} \sup_{\lambda>0} \omega_\lambda(x, z) \le \sup_{\lambda>0} \inf_{z\in Z} \omega_\lambda(x, z) < \infty,$$

since the reverse is always true.

An advantage of the notion of reachability is illustrated in the following lemma:

Lemma 5.8 ([49]) *Given two nonempty closed subsets $Y, Z \subset X_\omega$ and a point $z \in Z$. Suppose that Y is reachable from z. Then, to each $\varepsilon > 0$, there corresponds a point $y_\varepsilon \in Y$ such that* $\sup_{\lambda>0} \omega_\lambda(z, y_\varepsilon) \le \varepsilon + \sup_{\lambda>0} W_\lambda(Y, Z)$.

Proof Let $\varepsilon > 0$ be given. It is clear that we can find a point $y_\varepsilon \in Y$ such that $\sup_{\lambda>0} \omega_\lambda(z, y_\epsilon) \le \varepsilon + \inf_{y\in Y} \sup_{\lambda>0} \omega_\lambda(z, y)$. By the reachability of Y from z, we have

$$\inf_{y\in Y}\sup_{\lambda>0}\omega_\lambda(z,y)=\sup_{\lambda>0}\inf_{y\in Y}\omega_\lambda(z,y)=\sup_{\lambda>0}\omega_\lambda(z,Y)\le\sup_{\lambda>0}W_\lambda(Y,Z).$$

The conclusion thus follows. □

On the other hand, let us turn to a simple sufficient condition for a subset $Z\subset X_\omega$ to be reachable from $x\in X_\omega$.

Lemma 5.9 ([49]) *Given a point $x\in X_\omega$ and a nonempty compact subset $Z\subset X_\omega$. If the metric modular ω is l.s.c. in X and either $\inf_{z\in Z}\sup_{\lambda>0}\omega_\lambda(x,z)$ or $\sup_{\lambda>0}\inf_{z\in Z}\omega_\lambda(x,z)$ is finite, then Z is reachable from x.*

Proof For each $s>0$, we can find a sequence (z_n^s) such that

$$\omega_s(x,z_n^s)\to\inf_{z\in Z}\omega_s(x,z).$$

Since Z is compact, we may assume that (z_n^s) converges to some point $z^s\in Z$. Since ω is l.s.c. in X, we have

$$\omega_s(x,z^s)\le\liminf_{n\to\infty}\omega_s(x,z_n^s)=\inf_{z\in Z}\omega_s(x,z),$$

and therefore $\omega_s(x,z^s)=\inf_{z\in Z}\omega_s(x,z)$. Finally, we have

$$\inf_{z\in Z}\sup_{\lambda>0}\omega_\lambda(x,z)\le\sup_{\lambda>0}\omega_\lambda(x,z^s)=\sup_{\lambda>0}\inf_{z\in Z}\omega_s(x,z).$$

This completes the proof. □

Existence Theorems

At this stage, we exploit the notion of reachability and its supplementary results to deduce some fixed point theorems for set-valued contractions. The obtained result also fix the error in [23]. Additionally, assume through the rest of the paper that X_ω is complete.

Theorem 5.10 ([49]) *Suppose that F is a set-valued contraction (w.r.t. $k\in(0,1)$) on X_ω having compact values, and that the metric modular ω is l.s.c. in X. If there exist two points $x_0\in X_\omega$ and $x_1\in F(x_0)$ such that the set $\{x_0,x_1\}$ is bounded and $F(x_1)$ is reachable from x_1, then F has a fixed point.*

Proof See [49, Theorem 3.5] for proof. □

Along with the set-valued contraction (5.1), we may consider another class of maps: Let $F: X_\omega \multimap X_\omega$. If the inequality

$$W_\lambda(F(x),F(y))\le k[\omega_\lambda(x,F(x))+\omega_\lambda(y,F(y))]$$

is satisfied for all $\lambda>0$ and all $x,y\in X_\omega$, at some fixed $k\in(0,\frac{1}{2})$, we say that F is a *set-valued Kannan's contraction*. We close our paper with the following theorem which is similarly obtained to the preceding theorem.

Theorem 5.11 ([49]) *Suppose that F is a set-valued Kannan's contraction (w.r.t. $k \in (0, \frac{1}{2})$) on X_ω having compact values, and that the metric modular ω is l.s.c. in X. If there exist two points $x_0 \in X_\omega$ and $x_1 \in F(x_0)$ such that the set $\{x_0, x_1\}$ is bounded and $F(x_1)$ is reachable from x_1, then F has a fixed point.*

Proof See [49, Theorem 3.6] for proof. □

6 Fixed Point Results Based on Multivalued Mappings

We extended work of Nadler [50], Wardowski [36] and Sgroi [51] to modular metric spaces.

Let $CB(D) := \{C : C$ is nonempty closed and bounded subsets of $D\}$, $K(D) := \{C : C$ is nonempty compact subsets of $D\}$ and the Hausdorff metric modular defined on $CB(D)$ by

$$H_\omega(A, B) := \max\{\sup_{x\in A} \omega_1(x, B), \sup_{y\in B} \omega_1(A, y)\},$$

where $\omega_1(x, B) = \inf_{y\in B} \omega_1(x, y)$.

Lemma 6.1 ([37]) *Let (X, ω) be a modular metric space. Assume that ω satisfies Δ_2-condition. Let D be a nonempty subset of X_ω. Let A_n be a sequence of sets in $CB(D)$, and suppose $\lim_{n\to\infty} H_\omega(A_n, A_0) = 0$ where $A_0 \in CB(D)$. Then if $x_n \in A_n$ and $\lim_{n\to\infty} x_n = x_0$, it follows that $x_0 \in A_0$.*

Fixed point results based on multivalued F-contractions

Definition 6.2 ([52]) Let (X, ω) be a modular metric space. Let D be non empty bounded subset of X. A multivalued mapping $T : D \to CB(D)$ is called F-contraction on X if $F \in \mathcal{F}$, and $\tau \in \mathbb{R}^+$, for all $x, y \in D$ with $y \in Tx$ there exists $z \in Ty$ such that $\omega_1(y, z) > 0$, the following inequality holds:

$$\tau + F(\omega_1(y, z)) \le F(\mathcal{M}(x, y)) \tag{6.1}$$

where $\mathcal{M}(x, y) = \max\left\{\omega_1(x, y), \omega_1(x, Tx), \omega_1(y, Ty), \omega_1(y, Tx)\right\}$.

Definition 6.3 ([52]) Let (X, ω) be a modular metric space. Let D be a nonempty subset of X_ω. A multivalued mapping $T : D \to CB(D)$ is said to be F-contraction of Hardy–Rogers-type if $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that

$$2\tau + F(H_\omega(Tx, Ty)) \le F(\alpha\omega_1(x, y) + \beta\omega_1(x, Tx) + \gamma\omega_1(y, Ty) + L\omega_1(y, Tx)) \tag{6.2}$$

for all $x, y \in D$ with $H_\omega(Tx, Ty) > 0$, where $\alpha, \beta, \gamma, L \ge 0$, $\alpha + \beta + \gamma = 1$ and $\gamma \ne 1$.

Example 6.4 Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by $F(s) = \ln s$. It is clear that F satisfies $(F1) - (F3)$ for any $k \in (0, 1)$. Each mapping $T : D \to CB(D)$ satisfying equation (6.2) is an F-contraction such that

$$H_\omega(Tx, Ty) \le e^{-\tau} \omega_1(x, y), \quad \text{for all } x, y \in D, \;\; Tx \ne Ty$$

It is clear that for $x, y \in D$ such that $Tx = Ty$ the previous inequality also holds, and hence, T is a contraction.

Next, we give a fixed point result for multivalued F-contractions of Hardy–Rogers-type in modular metric space.

Theorem 6.5 *Let (X, ω) be a modular metric space. Assume that ω is a regular modular satisfying Δ_M-condition and Δ_2-condition. Let D be a nonempty bounded and complete subset of X_ω and $T : X \to K(D)$ be an F-contractions of Hardy–Rogers-type. Then T has a fixed point.*

Proof Let x_0 be an arbitrary point of D and $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T and the proof is completed. Assume that $x_1 \notin Tx_1$, then $Tx_0 \ne Tx_1$. Since F is continuous from the right, there exists a real number $h > 1$ and $\tau > 0$ such that

$$F(hH_\omega(Tx_0, Tx_1)) \le F(H_\omega(Tx_0, Tx_1)) + \tau.$$

Now, from $\omega_1(x_1, Tx_1) < hH_\omega(Tx_0, Tx_1)$, we deduce that there exists $x_2 \in Tx_1$ such that $\omega_1(x_1, x_2) \le hH_\omega(Tx_0, Tx_1)$. Consequently, we have

$$F(\omega_1(x_1, x_2)) \le F(hH_\omega(Tx_0, Tx_1)) < F(H_\omega(Tx_0, Tx_1)) + \tau,$$

which implies

$$2\tau + F(\omega_1(x_1, x_2)) \le F((\alpha + \beta + \delta)\omega_1(x_0, x_1) + (\gamma + \delta)\omega_1(x_1, x_2)) + \tau.$$

Thus,

$$F(\omega_1(x_1, x_2)) \le F((\alpha + \beta + \delta)\omega_1(x_0, x_1) + (\gamma + \delta)\omega_1(x_1, x_2)) - \tau.$$

Then, following the same argument in the proof of [52, Theorem 15]. □

Application to integral equations

Integral equations arise in many scientific and engineering problems. A large class of initial and boundary value problem can be converted to Volterra or Fredholm integral equation (see for instant [53]).

In this section, we consider the following integral equation:

$$u(t) = \beta A\big(u(t)\big) + \gamma B\big(u(t)\big) + g(t), \;\; t \in [0, T], \;\; T > 0 \tag{6.3}$$

where

$$A\big(u(t)\big) = \int_0^t K_1\big(t, s, u(s)\big)ds, \quad B\big(u(t)\big) = \int_0^t K_2\big(t, s, u(s)\big)ds \quad \text{and} \quad \beta, \gamma \geq 0$$

Let $C(I, \mathbb{R})$ be the space of all continuous functions on I, where $I = [0, T]$ with the norm $\|u\| = \sup_{t\in I} |u(t)|$ and the metric $\omega_\lambda(u, v) := \frac{1}{\lambda}\|u - v\| = \frac{1}{\lambda}d(u, v)$ for all $u, v \in C(I, \mathbb{R})$. For $r > 0$ and $u \in C(I, \mathbb{R})$, we denote by $B_\lambda(u, r) = \{v \in C(I, \mathbb{R}) : \omega_\lambda(u, v) \leq r\}$ the closed ball concerned at u and of radius r.

Theorem 6.6 *Let $r > 0$ be a fixed real number and the following conditions are satisfied:*

(i) $K : I \times I \times \mathbb{R} \to \mathbb{R}$ *and* $g : I \to \mathbb{R}$ *are continuous;*
(ii) *there exists* $u_0 \in C(I, \mathbb{R})$ *such that* $\beta A\big(u_0(t)\big) + \gamma B\big(u_0(t)\big) + g(t) \in B(u_0, r)$;
(iii) *if* $v \in B_\lambda(u, r)$, $\lambda > 0$, *then*

$$|K_i\big(t, s, u(s)\big) - K_i\big(t, s, v(s)\big)| \leq L_i(t, s, u(s), v(s))\frac{|u(s) - v(s)|}{\left(1 + \tau\sqrt{\frac{|u(s)-v(s)|}{\lambda}}\right)^2},$$

$i = 1, 2$ *for all* $t, s \in I$, $u, v \in \mathbb{R}$ *and for some continuous functions* $L_1, L_2 : I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$.

such that $L_i\big(t, s, u(s), v(s)\big)(\beta + \gamma)T \leq 1,\ i = 1, 2$ *for all* $s, t \in I$, *then the integral Equation (6.3) admit a solution.*

Proof See [52, Theorem 15] for proof. □

Now, we observe that the function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by $F(\alpha) = -\frac{1}{\sqrt{\alpha}}, \alpha > 0$ is in $\mathcal{F}$ and so we deduce that the mapping T satisfies all condition of Theorem 2.12 with $\mathcal{M}(u, v) = \omega_\lambda(u, v)$ for $\lambda = 1$. Hence, there exists a solution of the integral equation (6.3).

Remark 6.7 Our above Theorem 6.3 is an abstract application of F-contraction mapping which cannot be covered by Banach contraction principle.

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Lyapunov Inequalities for Some Differential Equations with Integral-Type Boundary Conditions

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Abstract In this work, we derive a Lyapunov-type inequality for a fractional problem depending on an integral boundary condition. We believe our results to be new even for the classical integer-order derivative case.

1 Introduction

In this work, we will be dealing with the following fractional boundary value problem:

$$D_a^\alpha x(t)+h(t,x(t))=0,\ a<t<b,\ 1<\alpha\leq 2, \tag{1}$$

$$x(a)=0, x(b)=\lambda\int_a^b x(s)ds,\ \lambda\geq 0. \tag{2}$$

We derive the Green function for the linear case and prove some results related to it.

In (1), the operator D_a^α stands for the Riemann–Liouville fractional derivative of order $1<\alpha\leq 2$: $(D_a^\alpha f)(t)=(D^2 I_a^{2-\alpha} f)(t)$, where

$$(I_a^\alpha f)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds,\quad \alpha>0,\quad t>a,$$

with $(I_a^0 f)(t)=f(t)$.

We are particularly interested in finding a Lyapunov-type inequality for the fractional boundary value problem (1)–(2) with $h(t,x)=q(t)x$. Let us recall the classical Lyapunov inequality:

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Theorem 1.1 *Let $a < b$ be two real numbers and suppose that $q \in C[a, b]$. If the boundary value problem*

$$\begin{aligned} x''(t) + q(t)x(t) = 0, \quad a < t < b, \\ x(a) = 0, x(b) = 0, \end{aligned}$$

has a nontrivial continuous solution x, then the following inequality holds,

$$\int_a^b |q(s)|ds > \frac{4}{b-a}.$$

The first generalization of Theorem 1.1 including fractional derivatives appeared in the literature in 2013 [3] and reads as follows:

Theorem 1.2 *Let $a < b$ be two real numbers and suppose that $q \in C[a, b]$. If the boundary value problem*

$$\begin{aligned} D_a^\alpha(t) + q(t)x(t) = 0, \quad a < t < b,\ 1 < \alpha \leq 2 \\ x(a) = 0, x(b) = 0, \end{aligned}$$

has a nontrivial continuous solution x, then the following inequality holds,

$$\int_a^b |q(s)|ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}.$$

One immediately observes that, when $\alpha = 2$, Theorem 1.2 becomes Theorem 1.1 and this proves our claim of a generalization of the classical Lyapunov inequality. After the publication of [3], many other works consisting essentially in finding Lyapunov-type inequalities where fractional derivatives are somehow involved appeared in the literature—the reader may consult [1–5, 7–14] and the references therein. Various kinds of problems were studied in the above-mentioned works, e.g., using the Caputo fractional derivative instead of the Riemann–Liouville one, involving higher-order derivatives in the differential equation, considering *sequential fractional derivatives*, and considering several types of boundary conditions. In all of the above papers, the technique used to derive these inequalities was essentially the same: rewriting the boundary value problem as an equivalent integral equation and then perform an analysis on the Green function in order to find the maximum value of its modulus in a square $[a, b] \times [a, b]$. Though this approach seems to be well suited for studying fractional boundary value problems, it might become, nevertheless, very much complex to analyze the corresponding Green's functions. But, to the best of our knowledge, there is no other known approach to obtain Lyapunov-type inequalities for boundary value problems which depend on fractional derivatives.

In this work, we consider boundary conditions as in (2) which means that we will have what is known in the literature by an *integral boundary condition*. This type of boundary condition was already considered before in two works [2, 12]. However, the exact form of the boundary condition is different: In [2], the authors considered integral boundary conditions depending also on the parameter α in such a way that, when $\alpha = 2$, then they get Theorem 1.1. In [12], the authors consider the boundary condition

$$x(b) = (I_a^\alpha hx)(b),$$

where $h \in C[a, b]$. Our motivation to consider the BVP (1)–(2) came mainly from the observation that

$$x''(t) + q(t)h(t, x(t)) = 0, \ a < t < b, \tag{3}$$

$$x(a) = 0, x(b) = \int_a^b x(s)ds, \tag{4}$$

is one of the simplest, yet most studied, boundary value problems depending on nonlocal boundary conditions. However, though we may find in the literature results considering existence of solutions (or stability of solutions, or existence of positive solutions and so on), we did not find any result regarding Lyapunov-type inequalities for the above-mentioned problem (3)–(4). Indeed, we believe that Corollary 2.1 stated in the next section is a novel result in the literature.

Summarizing, in the next section, we derive the Green function for the fractional boundary value problem (1)–(2) and prove some of its properties. As a consequence, we enunciate and present a proof of a Lyapunov-type inequality for the linear BVP. Moreover, we provide a criteria for existence and uniqueness of solution to (1)–(2).

2 Main Results

We start by transforming the fractional boundary value problem (1)–(2) into an equivalent integral equation.

Lemma 2.1 *Given $h \in C([a, b], \mathbb{R})$, $1 < \alpha \leq 2$ and $\lambda \in \mathbb{R}$ such that $\alpha - \lambda(b - a) \neq 0$, the unique solution of the fractional differential equation*

$$D_a^\alpha x(t) + h(t) = 0, \quad a < t < b, \tag{5}$$

with the following boundary conditions

$$x(a) = 0, \ x(b) = \lambda \int_a^b x(t)dt, \tag{6}$$

is

$$x(t) = \int_a^b G(t, s)h(s)ds,$$

where

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}\left(1+\frac{\lambda(s-a)}{\alpha-\lambda(b-a)}\right)-(t-s)^{\alpha-1}, & a\le s\le t\le b,\\ \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}\left(1+\frac{\lambda(s-a)}{\alpha-\lambda(b-a)}\right), & a\le t\le s\le b.\end{cases}$$

Proof It is well known that (5) can be represented by an equivalent integral equation

$$x(t)=-\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}h(s)ds+c_1(t-a)^{\alpha-1}+c_2(t-a)^{\alpha-2}, \tag{7}$$

for some $c_1, c_2\in\mathbb{R}$. By (6), it is clear that $c_2=0$ and

$$\begin{aligned}c_1&=\frac{1}{(b-a)^{\alpha-1}}\left[\lambda\int_a^b x(t)dt+\frac{1}{\Gamma(\alpha)}\int_a^b(b-s)^{\alpha-1}h(s)ds\right]\\&=\frac{A}{(b-a)^{\alpha-1}}+\frac{1}{(b-a)^{\alpha-1}\Gamma(\alpha)}\int_a^b(b-s)^{\alpha-1}h(s)ds,\end{aligned}$$

where we define $A=\lambda\int_a^b x(t)dt$. Therefore, it follows from (7) that

$$\begin{aligned}x(t)=&-\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}h(s)ds+\frac{A(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}\\&+\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}\Gamma(\alpha)}\int_a^b(b-s)^{\alpha-1}h(s)ds.\end{aligned} \tag{8}$$

Note that

$$\begin{aligned}A&=\lambda\int_a^b x(t)dt\\&=-\int_a^b\int_a^t\frac{\lambda(t-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}dsdt+\int_a^b\frac{\lambda A(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}dt\\&\quad+\int_a^b\int_a^b\frac{\lambda(t-a)^{\alpha-1}(b-s)^{\alpha-1}h(s)}{(b-a)^{\alpha-1}\Gamma(\alpha)}ds\\&=-\int_a^b\frac{\lambda(b-s)^{\alpha}h(s)}{\alpha\Gamma(\alpha)}ds+\frac{\lambda(b-a)A}{\alpha}+\frac{\lambda(b-a)}{\alpha\Gamma(\alpha)}\int_a^b(b-s)^{\alpha-1}h(s)ds,\end{aligned}$$

from where it follows that

$$\begin{aligned}\left(1-\frac{\lambda}{\alpha}(b-a)\right)A&=\frac{\lambda}{\alpha\Gamma(\alpha)}\int_a^b\left\{(b-a)(b-s)^{\alpha-1}-(b-s)^{\alpha}\right\}h(s)ds\\&=\frac{\lambda}{\alpha\Gamma(\alpha)}\int_a^b(s-a)(b-s)^{\alpha-1}h(s)ds.\end{aligned}$$

Hence,

$$A = \frac{\lambda}{(\alpha - \lambda(b-a))\,\Gamma(\alpha)} \int_a^b (s-a)(b-s)^{\alpha-1} h(s)ds.$$

Substituting A in (8), we get

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t \left[\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} \left\{ 1 + \frac{\lambda(s-a)}{\alpha - \lambda(b-a)} \right\} - (t-s)^{\alpha-1} \right] h(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_t^b \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} \left\{ 1 + \frac{\lambda(s-a)}{\alpha - \lambda(b-a)} \right\} h(s)ds \\ &= \int_a^b G(t,s)ds. \end{aligned}$$

The proof is now complete.

The following properties of the Green function will be used to derive our main results.

Theorem 2.1 *Let $a, b, \alpha, \lambda \in \mathbb{R}$ with $a < b$, $1 < \alpha \le 2$ and $\lambda \ge 0$ such that $\alpha - \lambda(b-a) > 0$. Then,*

1. *$G(t,s) \ge 0$ for all $(t,s) \in [a,b]^2$.*
2. *Define $g : [a,b] \to \mathbb{R}$ by*

$$g(x) = \frac{(x-a)^{\alpha-1}(b-x)^{\alpha-1}}{(b-a)^{\alpha-1}} \left(1 + \frac{\lambda(x-a)}{\alpha - \lambda(b-a)} \right).$$

Then,

$$\max_{(t,s)\in[a,b]^2} G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \left(\frac{b-a}{4}\right)^{\alpha-1}, & \lambda = 0, \\ \max\left\{ g(x^\star), \frac{(\alpha-1)^{\alpha-1}\lambda}{\alpha-\lambda(b-a)} \left(\frac{b-a}{\alpha}\right)^{\alpha} \right\}, & \lambda \ne 0, \end{cases}$$

where

$$\begin{aligned} x^\star &= \frac{-(3\alpha b\lambda - 2\alpha^2 - 2b\lambda + 2\alpha + a\alpha\lambda)}{2\lambda(1-2\alpha)} \\ &\quad - \frac{\sqrt{\begin{array}{l} (3\alpha b\lambda - 2\alpha^2 - 2b\lambda + 2\alpha + a\alpha\lambda)^2 \\ \quad -4\lambda(1-2\alpha)(\alpha b^2\lambda - \alpha^2 b - b^2\lambda + b\alpha - a\alpha^2 + a\alpha + a\alpha\lambda b) \end{array}}}{2\lambda(1-2\alpha)} \end{aligned}$$

3. *Define $f : [a,b] \to \mathbb{R}$ by*

$$f(x) = \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha+1)} \left(b - x + \frac{\lambda(b-a)^2}{[\alpha - \lambda(b-a)](\alpha+1)} \right),$$

and

$$t^\star = -\frac{1}{\alpha}\left[(1-\alpha)b - a - \frac{\lambda(\alpha-1)(b-a)^2}{[\alpha-\lambda(b-a)](\alpha+1)}\right].$$

We have,

$$\max_{t\in[a,b]}\int_a^b G(t,s)ds = \begin{cases} f(t^\star), & t^\star \le b,\\ f(b), & t^\star > b.\end{cases}$$

Proof We start by defining two functions:

$$g^1(t,s) = \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}\left(1+\frac{\lambda(s-a)}{\alpha-\lambda(b-a)}\right),\ a\le t\le s\le b,$$

and

$$g^2(t,s) = \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}\left(1+\frac{\lambda(s-a)}{\alpha-\lambda(b-a)}\right) - (t-s)^{\alpha-1},\ a\le s\le t\le b.$$

Given our hypothesis on the parameters, it is clear that $g^1 \ge 0$. Now,

$$\begin{aligned} g^2(t,s) &= \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}\left(1+\frac{\lambda(s-a)}{\alpha-\lambda(b-a)}\right)\\ &\quad - \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}\left(b-\left(a+\frac{(s-a)(b-a)}{t-a}\right)\right)^{\alpha-1}\\ &= \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}\left[(b-s)^{\alpha-1} + (b-s)^{\alpha-1}\frac{\lambda(s-a)}{\alpha-\lambda(b-a)} - \left(b-\left(a+\frac{(s-a)(b-a)}{t-a}\right)\right)^{\alpha-1}\right].\end{aligned}$$

But, as observed in the proof of [3, Lemma 2.2],

$$a + \frac{(s-a)(b-a)}{t-a} \ge s,$$

hence $g^2 \ge 0$. The proof of 1. is done.

We now proceed to prove item 2. Suppose that $t \le s$. Then, $g^1(t,s) \le g^1(s,s) := G_1(s)$. Then, after some calculations we get:

$$\begin{aligned} G_1'(s) =& \frac{(s-a)^{\alpha-2}(b-s)^{\alpha-2}}{(b-a)^{\alpha-1}(\alpha-\lambda(b-a))}\left[(\alpha-1)(b-s)\left(1+\frac{\lambda(s-a)}{\alpha-\lambda(b-a)}\right)\right.\\ &\left. -(\alpha-1)(s-a)\left(1+\frac{\lambda(s-a)}{\alpha-\lambda(b-a)}\right) + \frac{(s-a)(b-s)\lambda}{\alpha-\lambda(b-a)}\right],\ a<s<b.\end{aligned}$$

Now, defining f by

$$f(s) = (\alpha-1)(b-s)\left(1+\frac{\lambda(s-a)}{\alpha-\lambda(b-a)}\right) - (\alpha-1)(s-a)\left(1+\frac{\lambda(s-a)}{\alpha-\lambda(b-a)}\right) + \frac{(s-a)(b-s)\lambda}{\alpha-\lambda(b-a)},$$

we see that $f(a) = (\alpha-1)(b-a) > 0$ and $f(b) = -(\alpha-1)(b-a)\left(1+\frac{\lambda(b-a)}{\alpha-\lambda(b-a)}\right) < 0$. Hence, f has a zero in the interval (a, b). Moreover, after some rearrangements, we derive that:

$$f(s) = \lambda(1-2\alpha)s^2 + (3\alpha b\lambda - 2\alpha^2 - 2b\lambda + 2\alpha + a\alpha\lambda)s - (\alpha b^2\lambda - \alpha^2 b - b^2\lambda + b\alpha - a\alpha^2 + a\alpha + a\alpha\lambda b).$$

Since the coefficient of s^2 is negative (we assume from now on that $\lambda > 0$; the case $\lambda = 0$ gives immediately that $f(s) = 0 \Leftrightarrow s = \frac{a+b}{2}$, and this was studied in [3]), we conclude that the other zero of f must be less than a. Therefore, the zero of f on (a, b) is explicitly given by the quantity:

$$s^\star = \frac{-(3\alpha b\lambda - 2\alpha^2 - 2b\lambda + 2\alpha + a\alpha\lambda)}{2\lambda(1-2\alpha)} - \frac{\sqrt{\begin{array}{l}(3\alpha b\lambda - 2\alpha^2 - 2b\lambda + 2\alpha + a\alpha\lambda)^2 \\ \quad -4\lambda(1-2\alpha)(\alpha b^2\lambda - \alpha^2 b - b^2\lambda + b\alpha - a\alpha^2 + a\alpha + a\alpha\lambda b)\end{array}}}{2\lambda(1-2\alpha)}$$

Finally, since G_1 is continuous on $[a, b]$ and $G_1(a) = G_1(b) = 0$, then $\max_{s\in[a,b]} G_1(s) = G_1(s^\star)$.

We now consider the function $g^2(t, s)$ for $s \le t$. Differentiating with respect to t, we get

$$\begin{aligned} g_t^2(t,s) &= \frac{(\alpha-1)(t-a)^{\alpha-2}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}\left(1+\frac{\lambda(s-a)}{\alpha-\lambda(b-a)}\right) \\ &\quad - (\alpha-1)\frac{(t-a)^{\alpha-2}}{(b-a)^{\alpha-2}}\left(b-\left(a+\frac{(s-a)(b-a)}{t-a}\right)\right)^{\alpha-2} \\ &= \frac{(\alpha-1)(t-a)^{\alpha-2}}{(b-a)^{\alpha-2}}\left[\underbrace{(b-s)^{\alpha-1}\left(1+\frac{\lambda(s-a)}{\alpha-\lambda(b-a)}\right) - \left(b-\left(a+\frac{(s-a)(b-a)}{t-a}\right)\right)^{\alpha-2}}_{r(t,s)}\right]. \end{aligned}$$

Note that the sign of g_t^2 is the same of the function r. Fix a $s \in [a, t)$. It is easily seen that $r(t, s)$ has at most one zero in t. If r does not have a zero, then $r < 0$ since $\lim_{t\to s^+} r(t, s) = -\infty$. Suppose now that $r(t^\star, s) = 0$ for $t^\star \in (s, b)$. Again, by the fact that $\lim_{t\to s^+} r(t, s) = -\infty$, we know that $r(t, s) < 0$ for $t \in (s, t^\star)$. In the interval $(t^\star, b)$, the function r might be negative or positive. What we may conclude from the analysis done is that, since $g^2 \ge 0$, then $\max_{t\in[s,b]} g^2(t, s) =$

$\max\{g^2(s,s), g^2(b,s)\}$. Since $g^2(s,s) = g^1(s,s)$ which was analyzed before, we now calculate $\max_{s\in[a,b]} g^2(b,s)$. First, notice that

$$g^2(b,s) = \frac{(b-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}\left(1+\frac{\lambda(s-a)}{\alpha-\lambda(b-a)}\right)-(b-s)^{\alpha-1}$$
$$= (b-s)^{\alpha-1}\frac{\lambda(s-a)}{\alpha-\lambda(b-a)}.$$

From that it is easy to show that

$$\max_{s\in[a,b]} g^2(b,s) = g^2\left(b, \frac{a(\alpha-1)+b}{\alpha}\right) = \frac{(\alpha-1)^{\alpha-1}\lambda}{\alpha-\lambda(b-a)}\left(\frac{b-a}{\alpha}\right)^{\alpha}.$$

Finally, we prove item 3. We have,

$$\int_a^b G(t,s)ds =$$
$$\frac{1}{\Gamma(\alpha)}\left(\int_a^t\left[\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}\left\{1+\frac{\lambda(s-a)}{\alpha-\lambda(b-a)}\right\}-(t-s)^{\alpha-1}\right]ds\right.$$
$$\left.+\int_t^b\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}\left\{1+\frac{\lambda(s-a)}{\alpha-\lambda(b-a)}\right\}ds\right)$$
$$=\frac{1}{\Gamma(\alpha)}\left(\int_a^b\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}\left\{1+\frac{\lambda(s-a)}{\alpha-\lambda(b-a)}\right\}ds-\int_a^t(t-s)^{\alpha-1}ds\right)$$
$$=\frac{1}{\Gamma(\alpha)}\left(\int_a^b\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}ds+\int_a^b\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}\frac{\lambda(s-a)}{\alpha-\lambda(b-a)}ds\right.$$
$$\left.-\int_a^t(t-s)^{\alpha-1}ds\right)$$
$$=\frac{1}{\Gamma(\alpha)}\left(\frac{(t-a)^{\alpha-1}(b-a)}{\alpha}+\frac{(t-a)^{\alpha-1}\lambda}{\alpha-\lambda(b-a)}\frac{(b-a)^2}{\alpha(\alpha+1)}-\frac{(t-a)^{\alpha}}{\alpha}\right)$$
$$=\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha+1)}\left(b-t+\frac{\lambda(b-a)^2}{[\alpha-\lambda(b-a)](\alpha+1)}\right) := F(t).$$

Now,

$$F'(t) = \frac{(t-a)^{\alpha-2}}{\Gamma(\alpha+1)}\left(\underbrace{(\alpha-1)\left[b-t+\frac{\lambda(b-a)^2}{[\alpha-\lambda(b-a)](\alpha+1)}\right]-(t-a)}_{H(t)}\right).$$

If H does not have a zero on (a,b), then since

$$H(a) = \frac{\lambda(\alpha-1)(b-a)^2}{[\alpha-\lambda(b-a)](\alpha+1)} + (\alpha-1)(b-a) > 0,$$

the function F is increasing, i.e., $\max_{t\in[a,b]} F(t) = F(b) = \frac{\lambda(b-a)^{\alpha+1}}{\Gamma(\alpha+2)[\alpha-\lambda(b-a)]}$. If H has zeros, it is easily seen that it has only one and is given by

$$t^\star = -\frac{1}{\alpha}\left[(1-\alpha)b - a - \frac{\lambda(\alpha-1)(b-a)^2}{[\alpha-\lambda(b-a)](\alpha+1)}\right],$$

provided $t^\star \le b$ (it is easy to verify that $t^\star \ge a$). Moreover, $H(t) < 0$ for $t \in (t^\star, b)$ and $H(t) > 0$ for $t \in (a, t^\star)$. In this case, we finally conclude that $\max_{t\in[a,b]} F(t) = F(t^\star)$. The proof is done.

Remark 2.1 We would like to point out that, depending on the values of the parameters involved in Theorem 2.1, the expression that gives the

$$\max\left\{g(x^\star), \frac{(\alpha-1)^{\alpha-1}\lambda}{\alpha-\lambda(b-a)}\left(\frac{b-a}{\alpha}\right)^\alpha\right\}$$

varies. To see this, consider $a = 0$ and $b = 1$, and define for this purpose $h(\alpha,\lambda) = \frac{(\alpha-1)^{\alpha-1}\lambda}{\alpha-\lambda}\left(\frac{1}{\alpha}\right)^\alpha$. With the help of Maple Software, we find that $g(x^\star) < h\left(\frac{15}{10}, \frac{14}{10}\right)$ and $g(x^\star) > h\left(\frac{15}{10}, 1\right)$.

It follows the Lyapunov inequality for the linear fractional boundary value problem (1)–(2).

Theorem 2.2 *Suppose that $q \in C[a,b]$. If $x \in C[a,b]$ is a nontrivial solution of the following BVP*

$$D_a^\alpha x(t) + q(t)x(t) = 0,\ a < t < b,$$
$$x(a) = 0, x(b) = \lambda\int_a^b x(s)ds,$$

where $a, b, \alpha, \lambda \in \mathbb{R}$ with $a < b$, $1 < \alpha \le 2$ and $\lambda \ge 0$ such that $\alpha - \lambda(b-a) > 0$, then

$$\int_a^b |q(s)|ds > \frac{1}{C}, \tag{9}$$

where

$$C = \frac{1}{\Gamma(\alpha)}\begin{cases}\left(\frac{b-a}{4}\right)^{\alpha-1}, & \lambda = 0,\\ \max\left\{g(x^\star), \frac{(\alpha-1)^{\alpha-1}\lambda}{\alpha-\lambda(b-a)}\left(\frac{b-a}{\alpha}\right)^\alpha\right\}, & \lambda \neq 0,\end{cases}$$

with g and $x^\star$ defined as in 2. of Theorem 2.1.

Proof Let $\mathscr{B} = C[a,b]$ be the Banach space endowed with norm $\|x\| = \max_{t\in[a,b]} |x(t)|$.

It follows from Lemma 2.1 that a solution x to the BVP satisfies the integral equation

$$x(t) = \int_a^b G(t,s)q(s)x(s)ds, \quad t \in [a,b],$$

hence,

$$|x(t)| \leq \int_a^b |G(t,s)||q(s)||x(s)|ds, \ t \in [a,b].$$

Since x is nontrivial, there exists an interval $[c,d] \subset [a,b]$ such that $|q(s)| > 0$ on $[c,d]$. From the proof of Theorem 2.1, we know that $|G(t,s)| = G(t,s) < C$ for almost all $t \in [a,b]$ and $s \in [c,d]$. Therefore,

$$\|x\| < C \int_a^b |q(s)|ds\|x\|,$$

from which inequality in (9) follows.

As it was mentioned in the introduction, we believe that the previous result is new, even in the classical case, i.e., when $\alpha = 2$. For completeness, we enunciate it below.

Corollary 2.1 *Suppose that $q \in C[a,b]$. If $x \in C[a,b]$ is a nontrivial solution of the following BVP*

$$x''(t) + q(t)x(t) = 0, \ a < t < b,$$
$$x(a) = 0, x(b) = \lambda \int_a^b x(s)ds,$$

where $a, b, \lambda \in \mathbb{R}$ with $a < b$, and $\lambda \geq 0$ such that $2 - \lambda(b-a) > 0$, then

$$\int_a^b |q(s)|ds > \frac{1}{C}, \tag{10}$$

where

$$C = \begin{cases} \frac{b-a}{4}, & \lambda = 0, \\ \max\left\{g(x^\star), \frac{\lambda}{2-\lambda(b-a)}\left(\frac{b-a}{2}\right)^2\right\}, & \lambda \neq 0, \end{cases}$$

with g and $x^\star$ defined as in 2. of Theorem 2.1.

We end this work presenting a result that follows the same lines of the one recently obtained in [6, Theorem 2.3].

Theorem 2.3 *Assume $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a uniform Lipschitz condition with respect to the second variable on $[a,b] \times \mathbb{R}$ with Lipschitz constant K, that is,*

$$|h(t,x) - h(t,y)| \leq K|x-y|,$$

for all $(t,x), (t,y) \in [a,b] \times \mathbb{R}$*. Define*

$$M = \begin{cases} f(t^\star), & t^\star \leq b, \\ f(b), & t^\star > b, \end{cases}$$

with f *and* $t^\star$ *defined as in 3. of Theorem 2.1.*

If

$$KM < 1, \tag{11}$$

then the boundary value problem

$$D_a^\alpha x(t) = -h(t, x(t)), \quad a < t < b, \tag{12}$$

$$x(a) = 0, \ x(b) = \lambda \int_a^b x(s)ds, \tag{13}$$

where, as before, $\lambda \geq 0$ *and* $\alpha - \lambda(b-a) > 0$*, has a unique continuous solution.*

Proof Let $\mathscr{B}$ be the Banach space of continuous functions defined on $[a,b]$ with norm given by

$$\|x\| = \max_{t\in[a,b]} |x(t)|.$$

By Lemma 2.1, $x \in C[a,b]$ is a solution of the BVP (12)–(13) if and only if it is a solution of the integral equation

$$x(t) = \int_a^b G(t,s)h(s,x(s))ds.$$

Define the operator $T : \mathscr{B} \to \mathscr{B}$ by

$$Tx(t) = \int_a^b G(t,s)h(s,x(s))ds,$$

for $t \in [a,b]$. We will show that the operator T has a unique fixed point.

Let $x, y \in \mathscr{B}$. Then,

$$\begin{aligned} |Tx(t) - Ty(t)| &\leq \int_a^b |G(t,s)||h(s,x(s)) - h(s,y(s))|ds \\ &\leq \int_a^b |G(t,s)|K|x(s) - y(s)|ds \\ &\leq K \int_a^b G(t,s)ds\|x-y\| \\ &\leq KM\|x-y\|, \end{aligned}$$

where we have used Theorem 2.1. By (11), we conclude that T is a contracting mapping on $\mathscr{B}$, and by the Banach contraction mapping theorem, we get the desired result.

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A New Class of Generalized Convex Functions and Integral Inequalities

Mohamed Jleli, Donal O'Regan and Bessem Samet

Abstract In this chapter, we introduce the class of η_φ-convex functions which is larger than the class of η-convex functions introduced by Gordji et al. (Preprint Rgmia Res Rep Coll 1–14, 2015 [1]). Some Fejér type integral inequalities are established for this new class of functions. As consequences, we deduce some Hermite–Hadamard type inequalities involving different kinds of fractional integrals.

1 Introduction and Preliminaries

Convexity is a very important concept both in pure mathematics and in applications, especially in nonlinear programming and optimization. On the other hand, in many cases from real applications, the convexity property of the examined function is not satisfied. For this reason, several authors are concerned with a generalization of the convexity concept. For some works in this direction, see for example [1–10].

Many inequalities involving convex functions exist in the literature. The Hermite–Hadamard inequality is one of the fundamental results for convex functions having a natural geometrical interpretation and many applications. Recently, a great deal of attention has been paid to the study of such inequality for different kinds of convexity. For more details, we refer the reader to [1, 3, 11–22] and the references therein.

In this contribution, we introduce the notion of η_φ-convexity, which extends the concept of η-convexity proposed by Gordji et al. [1]. We establish some Fejér type

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integral inequalities for the suggested class of functions. Next, we deduce several Hermite–Hadamard type inequalities via different kinds of fractional integrals.

In [1], the authors introduced the concept of η-convexity as follows.
A function $f : [a, b] \to \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, is said to be η-convex iff for every $x, y \in [a, b]$ and $t \in [0, 1]$, we have

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y)),$$

where $\eta : f([a, b]) \times f([a, b]) \to \mathbb{R}$.
Observe that any convex function is an η-convex function with

$$\eta(x, y) = x - y.$$

Now, we introduce the following notion which extends the above concept.
A function $f : [a, b] \to \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, is said to be η_φ-convex iff for every $x, y \in [a, b]$ and $t \in [0, 1]$, we have

$$f(tx + (1-t)y) \leq f(y) + \varphi(t)\eta(f(x), f(y)), \tag{1}$$

where $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\varphi : [0, 1] \to [0, \infty)$.
It can be easily seen that any η-convex function is an η_φ-convex function withbreak $\varphi(t) = t$.

Lemma 1 *Let $f : [a, b] \to \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an η_φ-convex function. Then, for all $x, y \in [a, b]$, we have*

$$\varphi(0)\eta(f(x), f(y)) \geq 0, \tag{2}$$

$$\varphi(1)\eta(f(x), f(y)) \geq f(x) - f(y), \tag{3}$$

$$\varphi(t)\eta(f(x), f(x)) \geq 0, \quad t \in [0, 1]. \tag{4}$$

Proof Inequality (2) follows from (1) by taking $t = 0$. Taking $t = 1$ in (1), we obtain (3). Taking $x = y$ in (1), we get (4).

Remark 1 Observe that from (3), if $f : [a, b] \to \mathbb{R}$ is η_φ-convex with $\varphi(1) = 0$, then f is a constant function. Observe also that any constant function is η_φ-convex with $\varphi \equiv 0$.

The following example shows us that the set of η_φ-convex functions is larger than the set of η-convex functions.

Example 1 Let $f : [0, 1] \to \mathbb{R}$ be the function defined by

$$f(x) = \sqrt{x}, \quad x \in [0, 1].$$

Obviously, f is a concave function. Next, for all $x, y, t \in [0, 1]$, we have

$$f(tx+(1-t)y)=\sqrt{y+t(x-y)}\leq\sqrt{y}+\sqrt{t}\sqrt{|x-y|}$$
$$=\sqrt{y}+\sqrt{t}\sqrt{\left|(\sqrt{x})^2-(\sqrt{y})^2\right|}=f(y)+\varphi(t)\eta(f(x),f(y)),$$

where

$$\varphi(t)=\sqrt{t},\quad t\in[0,1]$$

and

$$\eta(u,v)=\sqrt{|u^2-v^2|},\quad u,v\in\mathbb{R}.$$

Therefore, f is an η_φ-convex function.

Note that there is no function $\eta:[0,1]\times[0,1]\to\mathbb{R}$ such that f is η-convex. Indeed, suppose that f is an η-convex function for some $\eta:[0,1]\times[0,1]\to\mathbb{R}$. Then, for all $x,y\in[0,1]$, we have

$$\sqrt{tx+(1-t)y}\leq\sqrt{y}+t\eta(\sqrt{x},\sqrt{y}),\quad t\in[0,1].$$

Let $x>0$ be fixed and $y=0$. Therefore, we get

$$\sqrt{t}\sqrt{x}\leq t\eta(\sqrt{x},0),\quad t\in[0,1],$$

which yields

$$\sqrt{x}\leq\sqrt{t}\eta(\sqrt{x},0),\quad t\in(0,1].$$

Passing to the limit as $t\to0^+$, we obtain $x=0$, which is a contradiction with $x>0$.

2 Fejér Type Integral Inequalities

In this section, some Fejér type integral inequalities involving η_φ-convex functions are presented.

Theorem 1 *Let $f:[a,b]\to\mathbb{R}$ and $g:(a,b)\to[0,\infty)$, $(a,b)\in\mathbb{R}^2$, $a<b$, be two given functions. Suppose that*

(i) f is η_φ-convex with η bounded above and $\varphi\in L^\infty[0,1]$;
(ii) $f\in L^\infty[a,b]$;
(iii) $g\in L^1(a,b)$;
(iv) $g(a+b-x)=g(x)$, for all $x\in(a,b)$.

Then

$$
\begin{aligned}
&\left(f\left(\frac{a+b}{2}\right)-\varphi\left(\frac{1}{2}\right)M_\eta\right)\int_a^b g(x)\,dx\\
&\le \int_a^b f(x)g(x)\,dx\\
&\le \left(\frac{f(a)+f(b)}{2}\right)\int_a^b g(x)\,dx\\
&+\left(\frac{\eta(f(a),f(b))+\eta(f(b),f(a))}{2}\right)\int_a^b \varphi\left(\frac{b-x}{b-a}\right)g(x)\,dx\\
&\le \left(\frac{f(a)+f(b)}{2}\right)\int_a^b g(x)\,dx+M_\eta\int_a^b \varphi\left(\frac{b-x}{b-a}\right)g(x)\,dx,
\end{aligned}
$$

where M_η is an upper bound of η.

Proof For all $t \in [0,1]$, we can write that

$$
f\left(\frac{a+b}{2}\right)=f\left(\frac{1}{2}\left(\frac{a+b+t(b-a)}{2}\right)+\left(1-\frac{1}{2}\right)\left(\frac{a+b-t(b-a)}{2}\right)\right).
$$

Since f is η_φ-convex, we have

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right)&\le f\left(\frac{a+b-t(b-a)}{2}\right)\\
&+\varphi\left(\frac{1}{2}\right)\eta\left(f\left(\frac{a+b+t(b-a)}{2}\right),f\left(\frac{a+b-t(b-a)}{2}\right)\right),
\end{aligned}
$$

for all $t \in [0,1]$. Taking into consideration that M_η is an upper bound of η, we obtain

$$
f\left(\frac{a+b}{2}\right)\le f\left(\frac{a+b-t(b-a)}{2}\right)+\varphi\left(\frac{1}{2}\right)M_\eta,\quad t\in[0,1],
$$

that is,

$$
f\left(\frac{a+b}{2}\right)-\varphi\left(\frac{1}{2}\right)M_\eta\le f\left(\frac{a+b-t(b-a)}{2}\right),\quad t\in[0,1]. \tag{5}
$$

Similarly, for all $t \in [0,1]$, we can write that

$$
f\left(\frac{a+b}{2}\right)=f\left(\frac{1}{2}\left(\frac{a+b-t(b-a)}{2}\right)+\left(1-\frac{1}{2}\right)\left(\frac{a+b+t(b-a)}{2}\right)\right).
$$

We argue as previously to get

$$
f\left(\frac{a+b}{2}\right)-\varphi\left(\frac{1}{2}\right)M_\eta\le f\left(\frac{a+b+t(b-a)}{2}\right),\quad t\in[0,1]. \tag{6}
$$

Adding (5) to (6), we obtain

$$f\left(\frac{a+b}{2}\right)-\varphi\left(\frac{1}{2}\right)M_\eta \le \frac{1}{2}f\left(\frac{a+b-t(b-a)}{2}\right)+\frac{1}{2}f\left(\frac{a+b+t(b-a)}{2}\right),\quad t\in[0,1]. \tag{7}$$

Multiplying (7) by

$$g\left(\frac{a+b+t(b-a)}{2}\right),$$

integrating over (0, 1) with respect to the variable t, using (iv) and a change of variable, we get

$$2\left(f\left(\frac{a+b}{2}\right)-\varphi\left(\frac{1}{2}\right)M_\eta\right)\int_{\frac{a+b}{2}}^{b} g(x)\,dx \le \int_a^b f(x)g(x)\,dx. \tag{8}$$

Similarly, multiplying (7) by

$$g\left(\frac{a+b-t(b-a)}{2}\right)$$

and integrating over (0, 1) with respect to the variable t, we get

$$2\left(f\left(\frac{a+b}{2}\right)-\varphi\left(\frac{1}{2}\right)M_\eta\right)\int_a^{\frac{a+b}{2}} g(x)\,dx \le \int_a^b f(x)g(x)\,dx. \tag{9}$$

Now, adding (8)–(9), we obtain

$$\left(f\left(\frac{a+b}{2}\right)-\varphi\left(\frac{1}{2}\right)M_\eta\right)\int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx,$$

which proves the first inequality.

In order to prove the second inequality, let $x\in[a,b]$ be an arbitrary element. We can write that

$$x=ta+(1-t)b,\ \ t=\frac{b-x}{b-a}.$$

Since f is η_φ-convex, we have

$$f(x)\le f(b)+\varphi\left(\frac{b-x}{b-a}\right)\eta(f(a),f(b)).$$

Multiplying the above inequality by $g(x)$ and integrating over (a,b) with respect to the variable x, we obtain

$$\int_a^b f(x)g(x)\,dx \leq f(b)\int_a^b g(x)\,dx + \eta(f(a), f(b))\int_a^b \varphi\left(\frac{b-x}{b-a}\right)g(x)\,dx. \tag{10}$$

Similarly, we can write that

$$x = tb + (1-t)a,\ \ t = \frac{x-a}{b-a}.$$

Since f is η_φ-convex, we have

$$f(x) \leq f(a) + \varphi\left(\frac{x-a}{b-a}\right)\eta(f(b), f(a)).$$

Multiplying the above inequality by $g(x)$, integrating over (a, b) with respect to the variable x, using a change of variable and (iv), we obtain

$$\int_a^b f(x)g(x)\,dx \leq f(a)\int_a^b g(x)\,dx + \eta(f(b), f(a))\int_a^b \varphi\left(\frac{b-x}{b-a}\right)g(x)\,dx. \tag{11}$$

Adding (10)–(11), we get

$$\begin{aligned} 2\int_a^b f(x)g(x)\,dx &\leq \Big(f(a)+f(b)\Big)\int_a^b g(x)\,dx \\ &+ \Big(\eta(f(a), f(b)) + \eta(f(b), f(a))\Big)\int_a^b \varphi\left(\frac{b-x}{b-a}\right)g(x)\,dx, \end{aligned}$$

that is,

$$\begin{aligned} \int_a^b f(x)g(x)\,dx &\leq \left(\frac{f(a)+f(b)}{2}\right)\int_a^b g(x)\,dx \\ &+ \left(\frac{\eta(f(a), f(b)) + \eta(f(b), f(a))}{2}\right)\int_a^b \varphi\left(\frac{b-x}{b-a}\right)g(x)\,dx, \end{aligned}$$

which proves the second inequality. Finally, using that M_η is an upper bound of η, we obtain immediately

$$\begin{aligned} &\left(\frac{f(a)+f(b)}{2}\right)\int_a^b g(x)\,dx + \left(\frac{\eta(f(a), f(b)) + \eta(f(b), f(a))}{2}\right)\int_a^b \varphi\left(\frac{b-x}{b-a}\right)g(x)\,dx \\ &\leq \left(\frac{f(a)+f(b)}{2}\right)\int_a^b g(x)\,dx + M_\eta\int_a^b \varphi\left(\frac{b-x}{b-a}\right)g(x)\,dx, \end{aligned}$$

which proves the third inequality.

Taking $\varphi(t) = t$ in Theorem 1, we obtain the following result for η-convex functions, which is due to Delavar and Dragomir [3].

Corollary 1 *Let $f : [a, b] \to \mathbb{R}$ and $g : (a, b) \to [0, \infty)$, $(a, b) \in \mathbb{R}^2$, $a < b$, be two given functions. Suppose that*

(i) f *is* η*-convex with* η *bounded above;*
(ii) $f \in L^\infty[a, b]$*;*
(iii) $g \in L^1(a, b)$*;*
(iv) $g(a + b - x) = g(x)$*, for all* $x \in (a, b)$*.*

Then

$$\left(f\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2}\right)\int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx$$
$$\le \left(\frac{f(a)+f(b)}{2}\right)\int_a^b g(x)\,dx + \left(\frac{\eta(f(a), f(b)) + \eta(f(b), f(a))}{2(b-a)}\right)\int_a^b (b-x)g(x)\,dx$$
$$\le \left(\frac{f(a)+f(b)}{2}\right)\int_a^b g(x)\,dx + \frac{M_\eta}{b-a}\int_a^b (b-x)g(x)\,dx,$$

where M_η *is an upper bound of* η*.*

Taking $g \equiv 1$ in Corollary 1, we obtain the following result, which is due to Gordji et al. [1].

Corollary 2 *Let $f : [a, b] \to \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be a given function. Suppose that*

(i) f *is* η*-convex with* η *bounded above;*
(ii) $f \in L^\infty[a, b]$*.*

Then

$$f\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2} \le \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$\le \frac{f(a)+f(b)}{2} + \frac{\eta(f(a), f(b)) + \eta(f(b), f(a))}{4}$$
$$\le \frac{f(a)+f(b)}{2} + \frac{M_\eta}{2},$$

where M_η *is an upper bound of* η*.*

Corollary 3 *Let $f : [a, b] \to \mathbb{R}$ and $w : (a, b) \to [0, \infty)$, $(a, b) \in \mathbb{R}^2$, $a < b$, be two given functions. Suppose that*

(i) f *is* η_φ*-convex with* η *bounded above and* $\varphi \in L^\infty[0, 1]$*;*
(ii) $f \in L^\infty[a, b]$*;*
(iii) $w \in L^1(a, b)$*.*

Then

$$
\begin{aligned}
&\left(f\left(\frac{a+b}{2}\right)-\varphi\left(\frac{1}{2}\right)M_\eta\right)\int_a^b w(x)\,dx \le \frac{1}{2}\int_a^b f(x)\Big(w(x)+w(a+b-x)\Big)\,dx\\
&\le \left(\frac{f(a)+f(b)}{2}\right)\int_a^b w(x)\,dx\\
&+\left(\frac{\eta(f(a),f(b))+\eta(f(b),f(a))}{4}\right)\int_a^b \left(\varphi\left(\frac{b-x}{b-a}\right)+\varphi\left(\frac{x-a}{b-a}\right)\right)w(x)\,dx\\
&\le \left(\frac{f(a)+f(b)}{2}\right)\int_a^b w(x)\,dx\\
&+\frac{M_\eta}{2}\int_a^b \left(\varphi\left(\frac{b-x}{b-a}\right)+\varphi\left(\frac{x-a}{b-a}\right)\right)w(x)\,dx,
\end{aligned}
$$

where M_η is an upper bound of η.

Proof Let $g : (a, b) \to \mathbb{R}$ be the function defined by

$$g(x) = w(x) + w(a + b - x), \quad x \in (a, b).$$

Observe that the function g satisfies all the assumptions of Theorem 1. In particular, we have

$$g(a + b - x) = g(x), \quad x \in (a, b).$$

Therefore, applying Theorem 1 with the function g defined as above, we get the desired result.

We will see later that Corollary 3 allows us to deduce several Hermite–Hadamard's type inequalities via different kinds of fractional integrals.

The following lemma will be useful later.

Lemma 2 *Let $f : I^\circ \to \mathbb{R}$ and $w : (a, b) \to \mathbb{R}$ be two given functions with $a, b \in I^\circ$, $a < b$. Suppose that*

(i) f is a differentiable mapping on I°;
(ii) w is continuous on (a, b);
(iii) $w \in L^1(a, b)$.

Let $\theta : [0, 1] \to \mathbb{R}$ be the function defined by

$$\theta(t) = \int_0^t w(sb + (1 - s)a)\,ds, \quad t \in [0, 1].$$

Then, the following equality holds:

$$\left(\frac{f(a)+f(b)}{2}\right)\int_a^b w(x)\,dx - \frac{1}{2}\int_a^b \Big(w(a+b-x)+w(x)\Big)f(x)\,dx$$
$$= \frac{(b-a)^2}{2}\int_0^1 \Big(\theta(1-t)-\theta(t)\Big)f'(ta+(1-t)b)\,dt \tag{12}$$
$$= \frac{(b-a)^2}{2}\int_0^1 \Big(\theta(t)-\theta(1-t)\Big)f'(tb+(1-t)a)\,dt. \tag{13}$$

Proof We have

$$\int_0^1 \Big(\theta(1-t)-\theta(t)\Big)f'(ta+(1-t)b)\,dt = \int_0^1 \theta(1-t)f'(ta+(1-t)b)\,dt - \int_0^1 \theta(t)f'(ta+(1-t)b)\,dt := I_1 - I_2. \tag{14}$$

Using an integration by parts, we obtain

$$\begin{aligned} I_1 &= \frac{1}{a-b}\left[f(ta+(1-t)b)\theta(1-t)\right]_{t=0}^1 + \frac{1}{a-b}\int_0^1 \theta'(1-t)f(ta+(1-t)b)\,dt \\ &= \frac{1}{a-b}\Big(f(a)\theta(0)-f(b)\theta(1)\Big) + \frac{1}{a-b}\int_0^1 w(ta+(1-t)b)f(ta+(1-t)b)\,dt \\ &= \frac{f(b)\theta(1)}{b-a} - \frac{1}{b-a}\int_0^1 w(ta+(1-t)b)f(ta+(1-t)b)\,dt \\ &= \frac{f(b)\theta(1)}{b-a} - \frac{1}{(b-a)^2}\int_a^b w(x)f(x)\,dx. \end{aligned}$$

Similarly, we have

$$\begin{aligned} I_2 &= \frac{1}{a-b}\left[f(ta+(1-t)b)\theta(t)\right]_{t=0}^1 - \frac{1}{a-b}\int_0^1 w(tb+(1-t)a)f(ta+(1-t)b)\,dt \\ &= \frac{-f(a)\theta(1)}{b-a} + \frac{1}{(b-a)^2}\int_a^b w(a+b-x)f(x)\,dx. \end{aligned}$$

Therefore,

$$I_1 - I_2 = \frac{\Big(f(a)+f(b)\Big)\theta(1)}{b-a} - \frac{1}{(b-a)^2}\int_a^b \Big(w(x)+w(a+b-x)\Big)f(x)\,dx.$$

Note that

$$\theta(1) = \int_0^1 w(sb+(1-s)a)\,ds, = \frac{1}{b-a}\int_a^b w(x)\,dx.$$

Hence, we have

$$I_1 - I_2 = \frac{1}{(b-a)^2}\left(\left(f(a)+f(b)\right)\int_a^b w(x)\,dx - \int_a^b \left(w(x)+w(a+b-x)\right) f(x)\,dx\right). \tag{15}$$

Combining (14) with (15), we obtain the equality (12). Finally, (13) follows from (12) by a change of variable.

Remark 2 If $w \equiv 1$ in Lemma 2, we obtain [13, Lemma 2.1].

As an application of identities (12) and (13), we have the following result.

Theorem 2 *Suppose that all the assumptions of Lemma 2 are satisfied. Moreover, suppose that* $|f'|$ *is* η_φ*-convex in* $[a, b]$ *with* $\varphi \in L^\infty[0, 1]$*. Then*

$$\frac{4}{(b-a)^2}\left|\left(\frac{f(a)+f(b)}{2}\right)\int_a^b w(x)\,dx - \frac{1}{2}\int_a^b \left(w(a+b-x)+w(x)\right) f(x)\,dx\right|$$
$$\leq \int_0^1 \left[|f'(a)|+|f'(b)|+\left(\eta(|f'(a)|,|f'(b)|)+\eta(|f'(b)|,|f'(a)|)\right)\varphi(t)\right] |\theta(1-t)-\theta(t)|\,dt.$$

Proof Using identity (12) and taking into consideration the η_φ convexity of $|f'|$ in $[a, b]$, we obtain

$$\frac{2}{(b-a)^2}\left|\left(\frac{f(a)+f(b)}{2}\right)\int_a^b w(x)\,dx - \frac{1}{2}\int_a^b \left(w(a+b-x)+w(x)\right) f(x)\,dx\right|$$
$$\leq \int_0^1 \left(|f'(b)|+\eta(|f'(a)|,|f'(b)|)\varphi(t)\right)|\theta(1-t)-\theta(t)|\,dt. \tag{16}$$

Similarly, using identity (13), we obtain

$$\frac{2}{(b-a)^2}\left|\left(\frac{f(a)+f(b)}{2}\right)\int_a^b w(x)\,dx - \frac{1}{2}\int_a^b \left(w(a+b-x)+w(x)\right) f(x)\,dx\right|$$
$$\leq \int_0^1 \left(|f'(a)|+\eta(|f'(b)|,|f'(a)|)\varphi(t)\right)|\theta(1-t)-\theta(t)|\,dt. \tag{17}$$

Adding (16)–(17), we obtain the desired inequality.

If η is bounded above, we obtain immediately from Theorem 2 the following result.

Corollary 4 *Suppose that all the assumptions of Lemma 2 are satisfied. Moreover, suppose that* $|f'|$ *is* η_φ*-convex in* $[a, b]$ *with* $\varphi \in L^\infty[0, 1]$ *and* η *bounded above. Then*

$$\frac{4}{(b-a)^2}\left|\left(\frac{f(a)+f(b)}{2}\right)\int_a^b w(x)\,dx - \frac{1}{2}\int_a^b \Big(w(a+b-x)+w(x)\Big)f(x)\,dx\right|$$
$$\leq \int_0^1 \Big(|f'(a)|+|f'(b)|+2M_\eta\varphi(t)\Big)|\theta(1-t)-\theta(t)|\,dt,$$

where M_η *is an upper bound of* η.

Now, suppose that in Theorem 2, we have $w \geq 0$ and $\varphi(t) = t$. In this case, θ is a nondecreasing function. Therefore,

$$|\theta(1-t)-\theta(t)| = \begin{cases} \theta(1-t)-\theta(t) & \text{if } 0 \leq t \leq 1/2, \\ \theta(t)-\theta(1-t) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Hence,

$$\int_0^1 \left[|f'(a)|+|f'(b)|+\Big(\eta(|f'(a)|,|f'(b)|)+\eta(|f'(b)|,|f'(a)|)\Big)\varphi(t)\right]$$
$$|\theta(1-t)-\theta(t)|\,dt$$
$$= K_1 + K_2,$$

where

$$K_1 = \Big(|f'(a)|+|f'(b)|\Big)\int_0^{\frac{1}{2}}\Big(\theta(1-t)-\theta(t)\Big)dt$$
$$+\; M(\eta,a,b)\left(\int_0^{\frac{1}{2}} t\theta(1-t)\,dt - \int_0^{\frac{1}{2}} t\theta(t)\,dt\right),$$
$$K_2 = \Big(|f'(a)|+|f'(b)|\Big)\int_{\frac{1}{2}}^1\Big(\theta(t)-\theta(1-t)\Big)dt$$
$$+\; M(\eta,a,b)\left(\int_{\frac{1}{2}}^1 t\theta(t)\,dt - \int_{\frac{1}{2}}^1 t\theta(1-t)\,dt\right),$$
$$M(\eta,a,b) = \eta(|f'(a)|,|f'(b)|)+\eta(|f'(b)|,|f'(a)|).$$

Via integration by parts, we obtain

$$\begin{aligned}K_1 &= \Big(|f'(a)|+|f'(b)|\Big)\left(\widetilde{\theta}(1)-2\widetilde{\theta}\left(\frac{1}{2}\right)\right)\\&\quad+ M(\eta,a,b)\left(-\widetilde{\theta}\left(\frac{1}{2}\right)+\int_0^1\widetilde{\theta}(t)\,dt\right),\\K_2 &= \Big(|f'(a)|+|f'(b)|\Big)\left(\widetilde{\theta}(1)-2\widetilde{\theta}\left(\frac{1}{2}\right)\right)\\&\quad+ M(\eta,a,b)\left(\widetilde{\theta}(1)-\widetilde{\theta}\left(\frac{1}{2}\right)-\int_0^1\widetilde{\theta}(t)\,dt\right)\end{aligned}$$

and

$$K_1+K_2=\left(\widetilde{\theta}(1)-2\widetilde{\theta}\left(\frac{1}{2}\right)\right)\Big(2(|f'(a)|+|f'(b)|)+M(\eta,a,b)\Big),$$

where

$$\widetilde{\theta}(t)=\int_0^t\theta(\tau)\,d\tau.$$

Therefore, we deduce the following result.

Corollary 5 *Suppose that all the assumptions of Lemma 2 are satisfied. Moreover, suppose that $w\geq 0$ and $|f'|$ is η-convex in $[a,b]$. Then*

$$\begin{aligned}&\frac{4}{(b-a)^2}\left|\left(\frac{f(a)+f(b)}{2}\right)\int_a^b w(x)\,dx-\frac{1}{2}\int_a^b\Big(w(a+b-x)+w(x)\Big)f(x)\,dx\right|\\&\leq\left(\widetilde{\theta}(1)-2\widetilde{\theta}\left(\frac{1}{2}\right)\right)\Big(2(|f'(a)|+|f'(b)|)+\eta(|f'(a)|,|f'(b)|)+\eta(|f'(b)|,|f'(a)|)\Big),\end{aligned}$$

where

$$\widetilde{\theta}(t)=\int_0^t\theta(\tau)\,d\tau,\quad t\in[0,1].$$

Taking $w\equiv 1$ in Corollary 5, we obtain the following result.

Corollary 6 *Let $f:I^\circ\to\mathbb{R}$ be a given function with $a,b\in I^\circ$, $a<b$. Suppose that*

(i) f is a differentiable mapping on I°;
(ii) $|f'|$ is η-convex in $[a,b]$.

Then

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\leq\frac{(b-a)}{4}\left(\frac{|f'(a)|+|f'(b)|}{2}+\frac{\eta(|f'(a)|,|f'(b)|)+\eta(|f'(b)|,|f'(a)|)}{4}\right).\end{aligned}$$

Taking $\eta(x, y) = x - y$ in Corollary 6, we obtain the following estimate due to Dragomir and Agarwal [13].

Corollary 7 *Let $f : I^\circ \to \mathbb{R}$ be a given function with $a, b \in I^\circ$, $a < b$. Suppose that*

(i) *f is a differentiable mapping on I°;*
(ii) *$|f'|$ is convex in $[a, b]$.*

Then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \le \frac{(b-a)}{8} \Big(|f'(a)| + |f'(b)| \Big).$$

3 Applications to Fractional Integral Inequalities

In this section, from the previous obtained results, we deduce several Hermite–Hadamard's integral inequalities involving different kinds of fractional integrals. For more details on fractional calculus, we refer the reader to [23].

3.1 Hermite–Hadamard's Inequalities Via Riemann–Liouville Fractional Integrals

In the following, we recall the definition of the Riemann–Liouville fractional integral. Let $f : [a, b] \to \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be a given function.

The left-sided Riemann–Liouville fractional integral $J_{a^+}^\alpha$ of order $\alpha > 0$ of f is defined by

$$J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad x > a,$$

provided that the integral exists, where $\Gamma(\cdot)$ is the Gamma function.

The right-sided Riemann–Liouville fractional integral $J_{b^-}^\alpha$ of order $\alpha > 0$ of f is defined by

$$J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau - x)^{\alpha-1} f(\tau)\, d\tau, \quad x < b,$$

provided that the integral exists.

Now, let us define the function $w : (a, b) \to \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, by

$$w(x) = \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)}, \quad x \in (a, b), \tag{18}$$

where $\alpha > 0$. Observe that the function w satisfies all the assumptions of Corollary 3. Let $f : [a, b] \to \mathbb{R}$ be a function that satisfies the assumptions of Corollary 3. Simple computations yield

$$\int_a^b \left(\varphi\left(\frac{b-x}{b-a}\right) + \varphi\left(\frac{x-a}{b-a}\right)\right) w(x)\,dx = J_{a^+}^{\alpha}\widetilde{\varphi}(b),$$
$$\int_a^b f(x)\Big(w(x) + w(a+b-x)\Big)\,dx = J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a),$$
$$\int_a^b w(x)\,dx = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)},$$

where

$$\widetilde{\varphi}(x) = \varphi\left(\frac{b-x}{b-a}\right) + \varphi\left(\frac{x-a}{b-a}\right), \quad x \in [a, b].$$

Therefore, from Corollary 3, we deduce the following result.

Corollary 8 *Let $f : [a, b] \to \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be a given function. Suppose that*

(i) f is η_{φ}-convex with η is bounded above and $\varphi \in L^{\infty}[0, 1]$;
(ii) $f \in L^{\infty}[a, b]$.

Then

$$f\left(\frac{a+b}{2}\right) - \varphi\left(\frac{1}{2}\right) M_{\eta} \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]$$
$$\le \frac{f(a)+f(b)}{2} + \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(\frac{\eta(f(a), f(b)) + \eta(f(b), f(a))}{4}\right) J_{a^+}^{\alpha}\widetilde{\varphi}(b)$$
$$\le \frac{f(a)+f(b)}{2} + \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} M_{\eta} J_{a^+}^{\alpha}\widetilde{\varphi}(b),$$

where M_{η} is an upper bound of η, $\alpha > 0$, and

$$\widetilde{\varphi}(x) = \varphi\left(\frac{b-x}{b-a}\right) + \varphi\left(\frac{x-a}{b-a}\right), \quad x \in [a, b].$$

Remark 3 It can be easily seen that $J_{a^+}^{\alpha}\widetilde{\varphi}(b) = J_{b^-}^{\alpha}\widetilde{\varphi}(a)$.

If $\varphi(t) = t$, $t \in [0, 1]$, then

$$\widetilde{\varphi}(x) = 1, \quad x \in [a, b].$$

In this case, we have

$$J_{a^+}^{\alpha}\widetilde{\varphi}(b) = J_{b^-}^{\alpha}\widetilde{\varphi}(a) = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}.$$

Therefore, taking $\varphi(t) = t$ in Corollary 8, we deduce the following result for η-convex functions.

Corollary 9 *Let $f : [a, b] \to \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be a given function. Suppose that*

(i) f is η-convex with η bounded above;
(ii) $f \in L^\infty[a, b]$.

Then

$$f\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2} \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)\right]$$
$$\le \frac{f(a)+f(b)}{2} + \frac{\eta(f(a), f(b)) + \eta(f(b), f(a))}{4}$$
$$\le \frac{f(a)+f(b)}{2} + \frac{M_\eta}{2},$$

where M_η is an upper bound of η and $\alpha > 0$.

Now, taking in Corollary 5 the function w defined by (18), simple computations yield

$$\theta(t) = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}\Big(1 - (1-t)^\alpha\Big)$$

and

$$\widetilde{\theta}(t) = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}\left(t + \frac{1}{\alpha+1}\Big((1-t)^{\alpha+1} - 1\Big)\right).$$

Therefore, from Corollary 5, we deduce the following result.

Corollary 10 *Let $f : I^\circ \to \mathbb{R}$ be a given function with $a, b \in I^\circ$, $a < b$. Suppose that*

(i) f is a differentiable mapping on I°;
(ii) $|f'|$ is η-convex in $[a, b]$.

Then

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+} f(b) + J^\alpha_{b-} f(a)\right]\right|$$
$$\le \frac{(b-a)}{4(\alpha+1)}\left(1 - \frac{1}{2^\alpha}\right)\Big(2(|f'(a)| + |f'(b)|) + \eta(|f'(a)|, |f'(b)|) + \eta(|f'(b)|, |f'(a)|)\Big),$$

where $\alpha > 0$.

Taking $\eta(x, y) = x - y$ in Corollary 10, we obtain the following result which is due to Sarikaya et al. [21].

Corollary 11 *Let $f : I^\circ \to \mathbb{R}$ be a given function with $a, b \in I^\circ$, $a < b$. Suppose that*

(i) f *is a differentiable mapping on* I°;
(ii) $|f'|$ *is convex in* $[a, b]$.

Then

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a) \right] \right| \leq \frac{(b-a)}{2(\alpha+1)} \left(1 - \frac{1}{2^\alpha}\right) \Big(|f'(a)| + |f'(b)| \Big),$$

where $\alpha > 0$.

3.2 Hermite–Hadamard's Inequalities Via Hadamard Fractional Integrals

In the following, we recall the definition of the Hadamard fractional integral. Let $f : [a, b] \to \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $0 < a < b$, be a given function.

The left-sided Hadamard fractional integral $I^\alpha_{a^+}$ of order $\alpha > 0$ of f is defined by

$$I^\alpha_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{\tau} \right)^{\alpha-1} \frac{f(\tau)}{\tau} \, d\tau, \quad x > a,$$

provided that the integral exists.

The right-sided Hadamard fractional integral $I^\alpha_{b^-}$ of order $\alpha > 0$ of f is defined by

$$I^\alpha_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{\tau}{x} \right)^{\alpha-1} \frac{f(\tau)}{\tau} \, d\tau, \quad x < b,$$

provided that the integral exists.

Now, let us define the function $w : (a, b) \to \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $0 < a < b$, by

$$w(x) = \frac{1}{\Gamma(\alpha)} \left(\frac{\left(\ln \frac{b}{x}\right)^{\alpha-1}}{x} + \frac{\left(\ln \frac{x}{a}\right)^{\alpha-1}}{x} \right), \quad x \in (a, b),$$

where $\alpha > 0$. Observe that the function w satisfies all the assumptions of Corollary 3. Let $f : [a, b] \to \mathbb{R}$ be a function that satisfies the assumptions of Corollary 3. Simple computations yield

$$\int_a^b \left(\varphi\left(\frac{b-x}{b-a}\right)+\varphi\left(\frac{x-a}{b-a}\right)\right)w(x)\,dx = I_{a^+}^{\alpha}\widetilde{\varphi}(b)+I_{b^-}^{\alpha}\widetilde{\varphi}(a),$$

$$\int_a^b f(x)\Big(w(x)+w(a+b-x)\Big)\,dx = I_{a^+}^{\alpha}F(b)+I_{b^-}^{\alpha}F(a),$$

$$\int_a^b w(x)\,dx = \frac{2}{\Gamma(\alpha+1)}\left(\ln\frac{b}{a}\right)^{\alpha},$$

where

$$\widetilde{\varphi}(x)=\varphi\left(\frac{b-x}{b-a}\right)+\varphi\left(\frac{x-a}{b-a}\right),\quad x\in[a,b]$$

and

$$F(x)=f(a+b-x)+f(x),\quad x\in[a,b].$$

Therefore, from Corollary 3, we deduce the following result.

Corollary 12 *Let* $f:[a,b]\to\mathbb{R}$, $(a,b)\in\mathbb{R}^2$, $0<a<b$, *be a given function. Suppose that*

(i) f *is* η_{φ}*-convex with* η *bounded above and* $\varphi\in L^{\infty}[0,1]$;
(ii) $f\in L^{\infty}[a,b]$.

Then

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)-\varphi\left(\frac{1}{2}\right)M_{\eta}\le\frac{\Gamma(\alpha+1)}{4\left(\ln\frac{b}{a}\right)^{\alpha}}\left[I_{a^+}^{\alpha}F(b)+I_{b^-}^{\alpha}F(a)\right]\\
&\le\frac{f(a)+f(b)}{2}+\frac{\Gamma(\alpha+1)}{8\left(\ln\frac{b}{a}\right)^{\alpha}}\left(\eta(f(a),f(b))+\eta(f(b),f(a))\right)\left[I_{a^+}^{\alpha}\widetilde{\varphi}(b)+I_{b^-}^{\alpha}\widetilde{\varphi}(a)\right]\\
&\le\frac{f(a)+f(b)}{2}+\frac{\Gamma(\alpha+1)}{4\left(\ln\frac{b}{a}\right)^{\alpha}}\left[I_{a^+}^{\alpha}\widetilde{\varphi}(b)+I_{b^-}^{\alpha}\widetilde{\varphi}(a)\right]M_{\eta},
\end{aligned}$$

where M_{η} *is an upper bound of* η, $\alpha>0$,

$$\widetilde{\varphi}(x)=\varphi\left(\frac{b-x}{b-a}\right)+\varphi\left(\frac{x-a}{b-a}\right),\quad x\in[a,b],$$

and

$$F(x)=f(a+b-x)+f(x),\quad x\in[a,b].$$

Taking $\varphi(t)=t$ in Corollary 12, we obtain the following result for η-convex functions.

Corollary 13 *Let* $f:[a,b]\to\mathbb{R}$, $(a,b)\in\mathbb{R}^2$, $0<a<b$, *be a given function. Suppose that*

(i) f *is* η*-convex with* η *bounded above;*
(ii) $f \in L^\infty[a,b]$.

Then

$$
\begin{aligned}
& f\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2} \le \frac{\Gamma(\alpha+1)}{4\left(\ln\frac{b}{a}\right)^\alpha}\left[I^\alpha_{a^+}F(b) + I^\alpha_{b^-}F(a)\right] \\
& \le \frac{f(a)+f(b)}{2} + \frac{\eta(f(a),f(b)) + \eta(f(b),f(a))}{4} \\
& \le \frac{f(a)+f(b)}{2} + \frac{M_\eta}{2},
\end{aligned}
$$

where M_η *is an upper bound of* η*,* $\alpha > 0$*, and*

$$
F(x) = f(a+b-x) + f(x), \quad x \in [a,b].
$$

4 Conclusion

In this chapter, a new convexity concept is introduced. This concept generalizes different types of convexity from the literature, including the η-convexity notion introduced in [1]. We established different Fejér type integral inequalities involving functions satisfying our convexity notion. Moreover, we showed that via particular choices of the weight function w in Corollary 3, we can deduce easily fractional versions of the obtained inequalities. From this fact, we can observe that many fractional integral inequalities established recently by many authors are not real generalizations of existing standard inequalities, but just particular cases of those results. For further discussions on this subject, we refer the reader to the recent paper [24].

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Redheffer-Type Inequalities for the Fox–Wright Functions

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Abstract In this chapter, new sharpened Redheffer-type inequalities related to the Fox–Wright functions are established. As consequence, we show new Redheffer-type inequalities for hypergeometric functions and for the four-parametric Mittag-Leffler functions with best possible exponents.

Keywords Fox–Wright function · Sharpening Redheffer-type inequalities
Hypergeometric function · Four-parametric Mittag-Leffler function

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1 Introduction and Main Results

In 1969, Redheffer [1] posed the problem of proving the inequality

$$\frac{\pi^2 - x^2}{\pi^2 + x^2} \leq \frac{\sin x}{x}, \quad x \in (0, \pi]. \tag{1}$$

Williams [2] proved this inequality. Motivated by this inequality recently, Zhu and Sun, [3] using the fact that the hyperbolic functions $\sinh x$ and $\cosh x$ have no zeros in $(0, \infty)$, established the following Redheffer-type inequalities:

$$\left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\alpha} \leq \frac{\sinh x}{x} \leq \left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\beta} \tag{2}$$

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and

$$\left(\frac{r^2+x^2}{r^2-x^2}\right)^{\alpha} \leq \cosh x \leq \left(\frac{r^2+x^2}{r^2-x^2}\right)^{\beta_1}, \tag{3}$$

where $0 < x < r$, $\alpha \leq 0, \beta \geq \frac{r^2}{12}$, and $\beta_1 \geq \frac{r^2}{4}$.

Recently, some extensions of inequalities (2) and (3) involving modified Bessel function have been shown by Zhu [4] and Mehrez [5], as follows:

Theorem A *Let* $0 < x < r$ *and* $\nu > -1$, *then the following inequalities*

$$\left(\frac{r^2+z^2}{r^2-z^2}\right)^{\alpha} \leq \mathcal{I}_\nu(z) \leq \left(\frac{r^2+z^2}{r^2-z^2}\right)^{\beta} \tag{4}$$

hold, if and only if $\alpha \leq 0$ *and* $\beta \geq \frac{r^2}{8(\nu+1)}$, *where* $\mathcal{I}_\nu(z)$ *is the normalized modified Bessel function of the first kind, defined by*

$$\mathcal{I}_\nu(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\nu+1)z^{2k}}{2^{2k}k!\Gamma(\nu+k+1)}.$$

Moreover, the author of this paper extended and sharpened the inequalities (4), as follows [6]:

Theorem A *Let* $r > 0$ *and* $\alpha, \beta > 0$. *Then the following inequalities*

$$\left(\frac{r+z}{r-z}\right)^{\sigma_{\alpha,\beta}} \leq \mathcal{W}_{\alpha,\beta}(z) \leq \left(\frac{r+z}{r-z}\right)^{\gamma_{\alpha,\beta}} \tag{5}$$

hold for all $0 < z < r$, *where* $\sigma_{\alpha,\beta} = 0$ *and* $\gamma_{\alpha,\beta} = \frac{r\Gamma(\beta)}{2\Gamma(\beta+\alpha)}$ *are the best possible constants, and* $\mathcal{W}_{\alpha,\beta}(z)$ *is the normalized Wright function defined by*

$$\mathcal{W}_{\alpha,\beta}(z) = \Gamma(\beta)\sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\beta+k\alpha)}, \ \alpha > -1, \ \beta \in \mathbb{C}.$$

The Fox–Wright function ${}_p\Psi_q$ is a generalization of the familiar hypergeometric ${}_pF_q$ function with p numerator and q denominator parameters (see [7]), defined by (cf., e.g.,[8, p. 4, Eq. (2.4)]

$${}_p\Psi_q\left[{}^{(\alpha_1,A_1),\ldots,(\alpha_p,A_p)}_{(\beta_1,B_1),\ldots,(\beta_q,B_q)}\Big| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{p}\Gamma(\alpha_l+kA_l)}{\prod_{l=1}^{q}\Gamma(\beta_l+kB_l)}\frac{z^k}{k!}, \tag{6}$$

where $A_l \geq 0, l = 1, \ldots, p$; $B_l \geq 0, l = 1, \ldots, q$; such that $1 + \sum_{l=1}^{q} B_l - \sum_{l=1}^{p} A_l > 0$, for suitably bounded values of $|z|$. The generalized hypergeometric function ${}_pF_q$ is defined by

$$ {}_pF_q\left[\begin{smallmatrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{smallmatrix}\Big| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{p}(\alpha_l)_k}{\prod_{l=1}^{q}(\beta_l)_k}\frac{z^k}{k!} \tag{7} $$

where, as usual, we make use of the following notation:

$$ (\tau)_0 = 1, \text{ and } \; (\tau)_k = \tau(\tau+1)\cdots(\tau+k-1) = \frac{\Gamma(\tau+k)}{\Gamma(\tau)},\; k \in \mathbb{N}, $$

to denote the shifted factorial or the Pochhammer symbol. Obviously, we find from the Definitions (6) and (7) that

$$ {}_p\Psi_q\left[\begin{smallmatrix}(\alpha_1,1),\ldots,(\alpha_p,1)\\ (\beta_1,1),\ldots,(\beta_q,1)\end{smallmatrix}\Big| z\right] = \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)}{\Gamma(\beta_1)\cdots\Gamma(\beta_q)}\, {}_pF_q\left[\begin{smallmatrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{smallmatrix}\Big| z\right]. \tag{8} $$

The Mittag-Leffler functions with $2n$ parameters are defined for $B_j \in \mathbb{R}$ ($B_1^2 + \cdots + B_n^2 \neq 0$) and $\beta_j \in \mathbb{C}$ ($j = 1, \ldots, n \in \mathbb{N}$) by the series

$$ E_{(\beta,B)_n}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\prod_{j=1}^{n}\Gamma(\beta_j + kB_j)},\; z \in \mathbb{C}. \tag{9} $$

When $n = 1$, the definition in (9) coincides with the definition of the two-parametric Mittag-Leffler function [9–11]

$$ E_{(\beta,B)_1}(z) = E_{\beta,B}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + kB)},\; z \in \mathbb{C}, \tag{10} $$

and similarly for $n = 2$, where $E_{(\beta,B)_2}(z)$ coincides with the four-parametric Mittag-Leffler function

$$ E_{(\beta,B)_2}(z) = E_{\beta_1,B_1;\beta_2,B_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta_1 + kB_1)\Gamma(\beta_2 + kB_2)},\; z \in \mathbb{C}. \tag{11} $$

The generalized $2n-$parametric Mittag-Leffler function $E_{(\beta,B)_n}(z)$ can be represented in terms of the Fox–Wright hypergeometric function ${}_p\Psi_q(z)$ by

$$ E_{(\beta,B)_n}(z) = {}_1\Psi_n\left[\begin{smallmatrix}(1,1)\\ (\beta_1,B_1),\ldots,(\beta_q,B_q)\end{smallmatrix}\Big| z\right],\; z \in \mathbb{C}. \tag{12} $$

In the following, we define the function $\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)} : (0,\infty) \longrightarrow \mathbb{R}$ by

$$\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z) = \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)} {}_1\Psi_2\Big[{}^{(\alpha_1,1)}_{(\beta_1,B_1),(\beta_2,1)}\Big| z\Big],$$

where $\alpha_1, \beta_1, \beta_2 > 0$ and $B_1 > 0$.

In this paper, we shall extend and sharpen the inequalities (4) and (5) and obtain a general refinement of Redheffer-type inequality involving the normalized Fox–Wright functions $\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)$. As consequence, we show new Redheffer-type inequalities for the hypergeometric function ${}_1F_2$ and for the four-parametric Mittag-Leffler function $\tilde{E}_{\beta_1,B_1;\beta_2,1}(z) = \Gamma(\beta_1)\Gamma(\beta_2)E_{\beta_1,B_1;\beta_2,1}(z)$ as follows.

Theorem 1 *Let $r, \alpha_1, \beta_1, \beta_2, B_1 > 0$. If $\alpha_1 \geq \beta_2$, then the following inequalities*

$$\left(\frac{r+z}{r-z}\right)^{\lambda_{\alpha_1,\beta_2}^{(\beta_1,B_1)}} \leq \Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z) \leq \left(\frac{r+z}{r-z}\right)^{\mu_{\alpha_1,\beta_2}^{(\beta_1,B_1)}} \tag{13}$$

hold true for all $z \in (0,r)$, where $\lambda_{\alpha_1,\beta_2}^{(\beta_1,B_1)} = 0$, and $\mu_{\alpha_1,\beta_2}^{(\beta_1,B_1)} = \frac{\alpha_1\Gamma(\beta_1)r}{2\beta_2\Gamma(\beta_1+B_1)}$, are the best possible constants.

Taking in (13) the value $B_1 = 1$ and using the identities (8), we obtain the Redheffer-type inequalities for hypergeometric function ${}_1F_2$.

Corollary 1 *Let $r, \alpha_1, \beta_1, \beta_2 > 0$. If $\alpha_1 \geq \beta_2$, then the following inequalities*

$$\left(\frac{r+z}{r-z}\right)^{\lambda_{\alpha_1,\beta_2}^{(\beta_1,B_1)}} \leq {}_1F_2(\alpha_1;\beta_1,\beta_2;z) \leq \left(\frac{r+z}{r-z}\right)^{\mu_{\alpha_1,\beta_2}^{(\beta_1,1)}} \tag{14}$$

hold true for all $z \in (0,r)$, where $\lambda_{\alpha_1,\beta_2}^{(\beta_1,1)} = 0$, and $\mu_{\alpha_1,\beta_2}^{(\beta_1,1)} = \frac{\alpha_1 r}{2\beta_2\beta_1}$, are the best possible constants.

Letting in (13) the value $\alpha_1 = 1$ and using the identities (11), we obtain the Redheffer-type inequalities for the four-parametric Mittag-Leffler function $\tilde{E}_{\beta_1,B_1;\beta_2,1}(z)$.

Corollary 2 *Let $r, \beta_1, B_1 > 0$. If $0 < \beta_2 \leq 1$, then the following inequalities*

$$\left(\frac{r+z}{r-z}\right)^{\lambda_{1,\beta_2}^{(\beta_1,B_1)}} \leq \tilde{E}_{\beta_1,B_1;\beta_2,1}(z) \leq \left(\frac{r+z}{r-z}\right)^{\mu_{1,\beta_2}^{(\beta_1,B_1)}} \tag{15}$$

hold true for all $z \in (0,r)$, where $\lambda_{1,\beta_2}^{(\beta_1,B_1)} = 0$, and $\mu_{1,\beta_2}^{(\beta_1,B_1)} = \frac{r\Gamma(\beta_1)}{2\beta_2\Gamma(\beta_1+B_1)}$, are the best possible constants.

Remark:

1. We note that choosing $\alpha_1 = \beta_2,\ \beta_1 = \beta$ and $B_1 = \alpha$ in (13), we obtain Theorem B.
2. Taking in (13) the values $\alpha_1 = \beta_2,\ \beta_1 = \beta,\ B_1 = \alpha,\ z = x^2/4, \alpha = 1,\ \beta = \nu + 1$ where $\nu > -1$ and replacing r by $r^2/4$ in (5), we obtain Theorem A.

2 Proof of the Main Results

In the proof of the main result, we will need the following two lemmas. The first lemma is about the monotonicity of two power series. For more details, one may see [12].

Lemma 1 *Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be two sequences of real numbers, and let the power series $f(x) = \sum_{n\geq 0} a_n x^n$ and $g(x) = \sum_{n\geq 0} b_n x^n$ be convergent for $|x| < r$. If $b_n > 0$ for $n \geq 0$ and if the sequence $\{a_n/b_n\}_{n\geq 0}$ is (strictly) increasing (decreasing), then the function $x \mapsto f(x)/g(x)$ is (strictly) increasing (decreasing) on $(0, r)$.*

The second lemma is the so-called monotone form of l'Hospital's rule, see [13] for a proof.

Lemma 2 *Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be two continuous functions which are differentiable on (a, b). Further, let $g' \neq 0$ on (a, b). If f'/g' is increasing (decreasing) on (a, b), then the functions*

$$x \mapsto \frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad x \mapsto \frac{f(x) - f(b)}{g(x) - g(b)}$$

are also increasing (decreasing) on (a, b).

Now, we are ready to prove the main result.

Proof of Theorem 1. By using the definition of the function $\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)$, we have

$$\left(\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)\right)' = \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha_1 + k + 1) z^k}{k!\Gamma(\beta_2 + k + 1)\Gamma(\beta_1 + (k+1)B_1)}. \tag{16}$$

Let

$$K(z) = \frac{\log \Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)}{\log\left(\frac{r+z}{r-z}\right)} = \frac{f(z)}{g(z)},$$

where $f(z) = \log \Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)$ and $g(z) = \log\left(\frac{r+z}{r-z}\right)$. Then

$$\frac{f'(z)}{g'(z)} = \frac{(r^2 - z^2)\left(\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)\right)'}{2r\,\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)} = \frac{A(z)}{2r\,B(z)},$$

where $A(z) = (r^2 - z^2)\left(\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)\right)'$ and $B(z) = \Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)$. By computation, we get

$$\begin{aligned}
A(z) &= (r^2 - z^2)\left(\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)\right)' \\
&= \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)}(r^2 - z^2)\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+k+1)z^k}{k!\Gamma(\beta_2+k+1)\Gamma(\beta_1+(k+1)B_1)} \\
&= \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)}\left(\sum_{k=0}^{\infty}\frac{r^2\Gamma(\alpha_1+k+1)z^k}{k!\Gamma(\beta_2+k+1)\Gamma(\beta_1+(k+1)B_1)}\right. \\
&\qquad\left. - \sum_{k=2}^{\infty}\frac{\Gamma(\alpha_1+k-1)z^k}{(k-2)!\Gamma(\beta_2+k-1)\Gamma(\beta_1+(k-1)B_1)}\right) \\
&= \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1+1)r^2}{\Gamma(\alpha_1)\Gamma(\beta_2+1)\Gamma(\beta_1+B_1)} + \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1+2)r^2}{\Gamma(\beta_2+2)\Gamma(\alpha_1)\Gamma(\beta_1+2B_1)}z \\
&\quad + \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)}\sum_{k=2}^{\infty}\left(\frac{r^2\Gamma(\alpha_1+k+1)}{k!\Gamma(\beta_2+k+1)\Gamma(\beta_1+(k+1)B_1)}\right. \\
&\qquad\left. - \frac{\Gamma(\alpha_1+k-1)}{(k-2)!\Gamma(\beta_2+k-1)\Gamma(\beta_1+(k-1)B_1)}\right)z^k \\
&:= \sum_{k=0}^{\infty} a_k z^k,
\end{aligned} \tag{17}$$

where $a_0 = \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1+1)r^2}{\Gamma(\alpha_1)\Gamma(\beta_2+1)\Gamma(\beta_1+B_1)}$ and $a_1 = \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1+2)r^2}{\Gamma(\beta_2+2)\Gamma(\alpha_1)\Gamma(\beta_1+2B_1)}$ and a_k is defined for $k \geq 2$ by

$$\begin{aligned}
a_k = \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)}&\left(\frac{r^2\Gamma(\alpha_1+k+1)}{k!\Gamma(\beta_2+k+1)\Gamma(\beta_1+(k+1)B_1)}\right. \\
&\left. - \frac{\Gamma(\alpha_1+k-1)}{(k-2)!\Gamma(\beta_2+k-1)\Gamma(\beta_1+(k-1)B_1)}\right).
\end{aligned}$$

On the other hand, we write $B(z)$ in the following form:

$$B(z) = \sum_{k=0}^{\infty} b_k z^k,$$

where

$$b_k = \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1+k)}{k!\Gamma(\alpha_1)\Gamma(\beta_2+k)\Gamma(\beta_1+kB_1)}, \text{ for all } k \geq 0.$$

Now, we consider the sequence $u_k = a_k/b_k$ by $u_0 = a_0,\ u_1 = a_1/b_1$ and for $k \geq 2$

$$u_k = \frac{\Gamma(\beta_2+k)\Gamma(\beta_1+kB_1)\Gamma(\alpha_1+k+1)r^2}{\Gamma(\alpha_1+k)\Gamma(\beta_2+k+1)\Gamma(\beta_1+(k+1)B_1)} - \frac{k!\Gamma(\alpha_1+k-1)\Gamma(\beta_2+k)\Gamma(\beta_1+kB_1)}{(k-2)!\Gamma(\beta_2+k-1)\Gamma(\alpha_1+k)\Gamma(\beta_1+(k-1)B_1)}.$$

Since $\alpha_1 \geq \beta_2$, we have

$$\begin{aligned} u_1 - u_0 &= \frac{\Gamma(\alpha_1+2)\Gamma(\beta_2+1)\Gamma(\beta_1+B_1)r^2}{\Gamma(\alpha_1+1)\Gamma(\beta_2+2)\Gamma(\beta_1+2B_1)} - \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1+1)r^2}{\Gamma(\alpha_1)\Gamma(\beta_2+1)\Gamma(\beta_1+B_1)} \\ &= \frac{(\alpha_1+1)\Gamma(\beta_1+B_1)}{(\beta_2+1)\Gamma(\beta_1+2B_1)} - \frac{\alpha_1\Gamma(\beta_1)}{\beta_2\Gamma(\beta_1+B_1)} \\ &\leq \frac{\alpha_1}{\beta_2}\left(\frac{\Gamma(\beta_1+B_1)}{\Gamma(\beta_1+2B_1)} - \frac{\Gamma(\beta_1)}{\Gamma(\beta_1+B_1)}\right) \\ &\leq 0. \end{aligned} \tag{18}$$

Indeed, due to log-convexity property of the Gamma function $\Gamma(z)$, the ratio $z \mapsto \Gamma(z+a)/\Gamma(z)$ is increasing on $(0,\infty)$, when $a > 0$. Thus implies that the following inequality:

$$\frac{\Gamma(z+a)}{\Gamma(z)} \leq \frac{\Gamma(z+a+b)}{\Gamma(z+b)}, \tag{19}$$

holds for all $a, b, z > 0$. Let $z = \beta_1$ and $a = b = B_1$ in (19) and using the inequality (18), we deduce that $u_1 \leq u_0$. On the other hand, we have

$$\begin{aligned} u_2 - u_1 &= \frac{\Gamma(\alpha_1+2)\Gamma(\beta_2+1)}{\Gamma(\alpha_1+1)\Gamma(\beta_2+2)}\left[\frac{(\alpha_1+2)(\beta_2+1)\Gamma(\beta_1+2B_1)}{(\alpha_1+1)(\beta_2+2)\Gamma(\beta_1+3B_1)} - \frac{\Gamma(\beta_1+B_1)}{\Gamma(\beta_1+2B_1)}\right] \\ &\quad - \frac{2\Gamma(\alpha_1+1)\Gamma(\beta_2+2)\Gamma(\beta_1+2B_1)}{\Gamma(\alpha_1+2)\Gamma(\beta_2+1)} \\ &\leq \frac{\Gamma(\alpha_1+2)\Gamma(\beta_2+1)}{\Gamma(\alpha_1+1)\Gamma(\beta_2+2)}\left[\frac{\Gamma(\beta_1+2B_1)}{\Gamma(\beta_1+3B_1)} - \frac{\Gamma(\beta_1+B_1)}{\Gamma(\beta_1+2B_1)}\right] \\ &\quad - \frac{2\Gamma(\alpha_1+1)\Gamma(\beta_2+2)\Gamma(\beta_1+2B_1)}{\Gamma(\alpha_1+2)\Gamma(\beta_2+1)} \\ &\leq 0. \end{aligned} \tag{20}$$

Indeed, in view of inequality (19) when $z = \beta_1 + B_1,\ a = b = B_1$ and inequality (20), we deduce that $u_2 \leq u_1$. Now, let $k \geq 2$, we have

$$\begin{aligned} u_{k+1} - u_k &= r^2 \left[\frac{(\alpha_1 + k + 1)\Gamma(\beta_1 + (k+1)B_1)}{(\beta_2 + k + 1)\Gamma(\beta_1 + (k+2)B_1)} - \frac{(\alpha_1 + k)\Gamma(\beta_1 + kB_1)}{(\beta_2 + k)\Gamma(\beta_1 + (k+1)B_1)} \right] \\ &+ \frac{k!}{(k-2)!} \left[\frac{(\beta_2 + k - 1)\Gamma(\beta_1 + kB_1)}{(\alpha_1 + k - 1)\Gamma(\beta_1 + (k-1)B_1)} \right. \\ &\qquad \left. - \frac{(k+1)(\beta_2 + k)\Gamma(\beta_1 + (k+1)B_1)}{(k-1)(\alpha_1 + k)\Gamma(\beta_1 + kB_1)} \right] \\ &\leq \frac{(\alpha_1 + k)r^2}{(\beta_2 + k)} \left[\frac{\Gamma(\beta_1 + (k+1)B_1)}{\Gamma(\beta_1 + (k+2)B_1)} - \frac{\Gamma(\beta_1 + kB_1)}{\Gamma(\beta_1 + (k+1)B_1)} \right] \\ &+ \frac{k!(\beta_2 + k - 1)}{(k-2)!(\alpha_1 + k - 1)} \left[\frac{\Gamma(\beta_1 + kB_1)}{\Gamma(\beta_1 + (k-1)B_1)} - \frac{\Gamma(\beta_1 + (k+1)B_1)}{\Gamma(\beta_1 + kB_1)} \right]. \end{aligned} \tag{21}$$

Setting in (19) the values $z = \beta_1 + kB_1$ and $a = b = B_1$, we obtain the following Turán type inequality for the gamma function $\Gamma(z)$

$$\Gamma(\beta_1 + kB_1)\Gamma(\beta_1 + (k+2)B_1) - \Gamma^2(\beta_1 + (k+1)B_1) \geq 0. \tag{22}$$

Similarly, letting in (19) the values $z = \beta_1 + (k-1)B_1$ and $a = b = B_1$, we get

$$\Gamma(\beta_1 + (k-1)B_1)\Gamma(\beta_1 + (k+1)B_1) - \Gamma^2(\beta_1 + kB_1) \geq 0. \tag{23}$$

In view of (18), (20), (21), (22), and (23), we deduce that the sequence $(u_k)_{k\geq 0}$ is decreasing. By using Lemma 1, we clearly have that f'/g' is decreasing on $(0, r)$, and consequently the function $K(z)$ is also decreasing $(0, r)$, by means of Lemma 2. On the other hand, by using the Bernoulli–l'Hospital's rule, we obtain

$$\lim_{z\to 0} K(z) = \frac{u_0}{2r} = \frac{\alpha_1 \Gamma(\beta_1) r}{2\beta_2 \Gamma(\beta_1 + B_1)}, \quad \text{and} \quad \lim_{z \to r} K(z) = 0.$$

It is important to mention here that there is another proof of the inequalities (13). Namely, if we consider the function $\chi : (0, r) \longrightarrow \mathbb{R}$, defined by

$$\chi(z) = \frac{\alpha_1 \Gamma(\beta_1) r}{2\beta_2 \Gamma(\beta_1 + B_1)} \log\left(\frac{r+z}{r-z}\right) - \log \Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z).$$

Then,

$$\begin{aligned}
\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)\chi'(z) &= \frac{\alpha_1\Gamma(\beta_1)r^2}{\beta_2\Gamma(\beta_1+B_1)(r^2-z^2)}\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z) - \left(\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)\right)' \\
&= \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)}\left[\frac{\alpha_1\Gamma(\beta_1)r^2}{\beta_2\Gamma(\beta_1+B_1)(r^2-z^2)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+k)z^k}{k!\Gamma(\beta_1+kB_1)\Gamma(\beta_2+k)}\right. \\
&\qquad \left. -\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+k+1)z^k}{k!\Gamma(\beta_1+kB_1+B_1)\Gamma(\beta_2+k+1)}\right] \\
&\geq \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)}\left[\frac{\alpha_1\Gamma(\beta_1)}{\beta_2\Gamma(\beta_1+B_1)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+k)z^k}{k!\Gamma(\beta_1+kB_1)\Gamma(\beta_2+k)}\right. \\
&\qquad \left. -\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+k+1)z^k}{k!\Gamma(\beta_1+kB_1+B_1)\Gamma(\beta_2+k+1)}\right] \\
&= \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+k)z^k}{k!\Gamma(\beta_2+k)}\left(\frac{\alpha_1\Gamma(\beta_1)}{\beta_2\Gamma(\beta_1+B_1)\Gamma(\beta_1+kB_1)}\right. \\
&\qquad \left. -\frac{\alpha_1+k}{(\beta_2+k)\Gamma((\beta_1+kB_1+B_1)}\right)z^k.
\end{aligned}$$

On the other hand, using the fact that $\alpha_1 \geq \beta_2$, we have $\frac{\alpha_1}{\beta_2} \geq \frac{\alpha_1+k}{\beta_2+k}$ for each $k \geq 0$, and consequently,

$$\begin{aligned}
\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)\chi'(z) \geq \frac{\alpha_1\Gamma(\beta_1)\Gamma(\beta_2)}{\beta_2\Gamma(\alpha_1)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+k)z^k}{k!\Gamma(\beta_2+k)}&\left(\frac{\Gamma(\beta_1)}{\Gamma(\beta_1+B_1)\Gamma(\beta_1+kB_1)}\right. \\
&\left. -\frac{1}{\Gamma((\beta_1+kB_1+B_1)}\right)z^k.
\end{aligned} \tag{24}$$

Now, taking in (19) the values $z = \beta_1,\ a = B_1$ and $b = kB_1$, we obtain

$$\Gamma(\beta_1)\Gamma(\beta_1+kB_1+B_1) \geq \Gamma(\beta_1+B_1)\Gamma(\beta_1+kB_1). \tag{25}$$

In view of inequalities (24) and (25), we deduce that the function $\chi(z)$ is increasing on $(0, r)$, and hence $\chi(z) \geq \chi(0) = 0$, which implies the right-hand side of inequalities (13). To prove the left-hand side of (13), by using (16), we deduce that the function $\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z)$ is increasing on $(0, \infty)$, and hence

$$\Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(z) \geq \Phi_{\alpha_1,\beta_2}^{(\beta_1,B_1)}(0) = 1.$$

This completes the proof of Theorem 1. ∎

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Relations of the Extended Voigt Function with Other Families of Polynomials and Numbers

M. A. Pathan

Abstract Author presents a new family of generalized Voigt functions related to recently introduced k-Fibonacci–Hermite numbers, $h(x)$-Fibonacci–Hermite polynomials, Lucas–Hermite numbers and $h(x)$-Lucas–Hermite polynomials where $h(x)$ is a polynomial with real coefficients. The multivariable extensions of these results provide a natural generalization and unification of integral representations which may be viewed as a new relationship for the product of two different families of Lucas and Hermite polynomials. Some interesting explicit series representations, integrals and identities are obtained. The resulting formulas allow a considerable unification of various special results which appear in the literature.

1 Introduction

The integral form of a generalization of the Voigt functions $K(x, y)$ and $L(x, y)$ is (see, Srivastava and Miller [21]; Klusch [10])

$$V_{\mu,\nu}(x, y, z) = \sqrt{\frac{x}{2}} \int_0^\infty t^\mu e^{-yt-zt^2} J_\nu(xt)\, dt \tag{1.1}$$

$$= \frac{z^{-\alpha} x^{\nu+\frac{1}{2}}}{2^{\nu+\frac{1}{2}}\Gamma(\nu+1)} \left\{ \Gamma(\alpha)\psi_2\left[\alpha;\ \nu+1, \frac{1}{2};\ -\frac{x^2}{4z}, -\frac{y^2}{4z}\right] - \frac{y}{\sqrt{z}}\Gamma(\alpha+\frac{1}{2})\,\psi_2\left[\alpha+\frac{1}{2};\ \nu+1, \frac{3}{2};\ -\frac{x^2}{4z}, -\frac{y^2}{4z}\right]\right\} \tag{1.2}$$

$$(\alpha = (\mu+\nu+1)/2,\ x, y, z \in \mathbb{R}^+,\ R(\mu+\nu) > -1)$$

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where ψ_2 denotes one of Humbert's confluent hypergeometric function of two variables, defined by [20, p. 59]

$$\psi_2[\alpha;\ \gamma, \gamma';\ x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)_m\,(\gamma')_n}\,\frac{x^m\,y^n}{m!\,n!},\quad \max\{|x|, |y|\} < \infty \tag{1.3}$$

and the classical Bessel function $J_\nu(z)$ is defined by [20]

$$J_\nu(z) \;=\; \sum_{m=0}^{\infty} \frac{(-1)^m\,\left(\frac{z}{2}\right)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)},\quad (z \in \mathbb{C} \setminus (-\infty, 0)) \tag{1.4}$$

Note that

$$K(x, y) \;=\; V_{1/2,-1/2}\left(x, y, \frac{1}{4}\right) \text{ and } L(x, y) \;=\; V_{1/2,1/2}\left(x, y, \frac{1}{4}\right) \tag{1.5}$$

For a number of specializations of Voigt functions $V_{\mu,\nu}(x, y, z)$ and their generalizations in multivariables, we refer [14, 15, 17, 22].

The Fibonacci numbers F_n [5, 8, 9, 11, 19, 23] are the terms of the sequence $0, 1, 2, 3, 5, \ldots,$ where $F_n = F_{n-1} + F_{n-2}, n \geq 2,\ F_0 = 0$ and $F_1 = 1$. Falcon and Plaza [8] introduced a general Fibonacci sequence that generalizes among others both the classical Fibonacci sequence and the Pell sequence. These general k-Fibonacci numbers $F_{k,n}$ are defined by $F_{k,n} = kF_{k,n-1} + F_{k,n-2}, n \geq 2,\ F_0 = 0$ and $F_1 = 1$. The Pell numbers are the 2-Fibonacci numbers. In [9] the k-Fibonacci numbers were defined in explicit way and many properties were given. In particular, the k-Fibonacci numbers were related with the so called Pascal 2-triangle.
The polynomials $F_n(x)$ studied by Catalan are defined by the recurrence relation

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x);\ n \geq 3 \tag{1.6}$$

where $F_1(x) = 1,\ F_2(x) = x$. The Fibonacci polynomials studied by P. F. Byrd are defined by

$$\phi_n(x) = 2x\phi_{n-1}(x) + \phi_{n-2}(x);\ n \geq 2 \tag{1.7}$$

where $\phi_0(x) = 0,\ \phi_1(x) = 1$. The Lucas polynomials $L_n(x)$ originally studied in 1970 by Bicknell are defined by

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x);\ n \geq 2 \tag{1.8}$$

where $L_0(x) = 2,\ L_1(x) = x$.

In [12], Nalli et al. introduced the $h(x)$-Fibonacci polynomials. That generalize Catalan's Fibonacci polynomials $F_n(x)$ and the k-Fibonacci numbers $F_{k,n}$. The $h(x)$-Fibonacci polynomials are

$$\frac{t}{1 - h(x)t - t^2} = \sum_{n=0}^{\infty} F_{h,n}(x)t^n \tag{1.9}$$

For $h(x) = x$, we obtain Catalan's Fibonacci polynomials, and for $h(x) = 2x$, we obtain Byrd's Fibonacci polynomials. For $h(x) = k$, we obtain the k-Fibonacci numbers. For $k = 1$ and $k = 2$, we obtain the usual Fibonacci numbers and the Pell numbers.

The 2-variable Kampe de Feriet generalization of the Hermite polynomials (see Dattoli et al. [1–4]) reads

$$H_n(x, y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!} \tag{1.10}$$

These polynomials are usually defined by the generating function

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y)\frac{t^n}{n!} \tag{1.11}$$

and reduce to the ordinary Hermite polynomials $H_n(x)$ (see [2]) when $y = -1$ and x is replaced by $2x$.
We recall that the Hermite numbers H_n are the values of the Hermite polynomials $H_n(x)$ at zero argument, that is, $H_n(0) = 0$. A closed formula for H_n is given by

$$H_n = \begin{cases} 0, & \text{if n is odd} \\ \frac{(-1)^{n/2} n!}{(\frac{n}{2})!}, & \text{if n is even} \end{cases} \tag{1.12}$$

Recently in a paper [16], Pathan and Khan introduced k-Fibonacci–Hermite numbers, $h(x)$-Fibonacci–Hermite polynomials, Lucas–Hermite numbers and $h(x)$-Lucas–Hermite polynomials and obtained new sums and identities. Their definition for k-Fibonacci–Hermite numbers is given by means of the following generating function

$$\frac{t}{1 - kt - t^2} e^{-t^2} = \sum_{n=0}^{\infty} {}_H F_{k,n} t^n \tag{1.13}$$

where ${}_H F_{k,n}$ are k-Fibonacci–Hermite numbers. When $k = 1$ and $k = 2$, we obtain the usual Fibonacci–Hermite numbers ${}_H F_{1,n}$ and the Pell–Hermite numbers ${}_H F_{2,n}$, respectively. Let $h(x)$ be a polynomial with real coefficients. We recall the definition of $h(x)$-Fibonacci–Hermite polynomials [16] given by the generating function

$$\frac{t}{1 - h(x)t - t^2} e^{yt+zt^2} = \sum_{n=0}^{\infty} {}_H F_{h,n}(x, y, z)t^n \tag{1.14}$$

so that

$$ {}_H F_{h,n}(x, y, z) = \sum_{m=0}^{n} \frac{1}{(n-m)!} F_{h,m}(x) H_{n-m}(y, z) \tag{1.15} $$

In this work, we will consider various generalizations of Voigt function. We will show that generalized Voigt function is expressible in terms of a combination of Fibonacci–Hermite polynomials and Kampé de Fériet's functions. In the final section, we give further generalizations (involving multivariables) of Voigt functions in terms of multiple series and integrals.

2 Generalized Voigt Function $\Omega^{\alpha}_{\mu,\nu,h}(x, y, z)$

In an attempt to generalize (1.1), we first investigate here the generalized Voigt function $\Omega^{\alpha}_{\mu,\nu,h}(x, y, z)$.

Definition Let $h(\omega)$ be a polynomial with real coefficients and $\alpha \geq 0$. Then

$$ \Omega^{\alpha}_{\mu,\nu,h} = \Omega^{\alpha}_{\mu,\nu,h}(x, y, z) = \sqrt{\frac{x}{2}} \int_0^{\infty} \frac{t^{\mu} e^{-yt-zt^2}}{(1 - h(\omega)t - t^2)^{\alpha}} J_{\nu}(xt)\, dt \tag{2.1} $$

where $x, y, z \in \mathbb{R}^+$ and $\mathrm{Re}(\mu + \nu) > -1$.
Clearly, the case $\alpha = 0$ corresponds to (1.1) and (1.2) and we have

$$ \Omega^{(\alpha)}_{\mu,\nu,h}(x, y, z) \mid_{\alpha=0} = V_{\mu,\nu}(x, y, z) \tag{2.2} $$

Moreover, $\Omega^{\alpha}_{\mu,\nu,h}(x, y, 0) \mid_{\alpha=0}$ is the classical Laplace transform of $t^{\mu} J_{\nu}(xt)$. The case when $z = 1/4$ in (2.2) yields

$$ \Omega^{(\alpha)}_{1/2,-1/2.h}\left(x, y, \frac{1}{4}\right) \mid_{\alpha=0} = K(x, y) \text{ and } \Omega^{(\alpha)}_{1/2,1/2,h}\left(x, y, \frac{1}{4}\right) \mid_{\alpha=0} = L(x, y) $$

Using the Definition (2.1) and some manipulation in the integral results in a connection between $V_{\mu,\nu}$ and $\Omega_{\mu,\nu,h}$

$$ \Omega^{\alpha}_{\mu,\nu,h} = \Omega^{\alpha+1}_{\mu,\nu,h} - h(w)\Omega^{\alpha+1}_{\mu+1,\nu,h} - \Omega^{\alpha+1}_{\mu+2,\nu,h} $$

which on setting $\alpha = 0$ reduces to

$$ V_{\mu,\nu,h} = \Omega_{\mu,\nu,h} - h(w)\Omega_{\mu+1,\nu,h} - \Omega_{\mu+2,\nu,h} $$

where $\Omega^1_{\mu,\nu,h} = \Omega_{\mu,\nu,h}$

Relationships (1.2) and (1.9) can indeed be used to obtain an interesting connection between $\Omega^\alpha_{\mu,\nu}(x, y, z)$, $V_{\mu,\nu}(x, y, z)$ and $F_{n,h}(w)$ when $\alpha = 1$ by writing

$$\Omega^1_{\mu,\nu,h}(x, y, z) = \Omega_{\mu,\nu,h}(x, y, z) = \sqrt{\frac{x}{2}} \sum_{n=0}^{\infty} F_{h,n}(w) \int_0^\infty t^{n+\mu-1} e^{-yt-zt^2} J_\nu(xt)\, dt \tag{2.3}$$

which yields the formula

$$\Omega_{\mu,\nu,h}(x, y, z) = \sum_{n=0}^{\infty} F_{h,n}(w) V_{\mu+n-1,\nu}(x, y, z) \tag{2.4}$$

where $V_{\mu+n+1,\nu}(x, y, z)$ is given by (1.2). This result seems in some way related to the formula mentioned below

$$\Omega_{\mu,\nu,h}(x, y, z) = \sum_{n=0}^{\infty} {}_H F_{h,n}(w, u, v) V_{\mu+n-1,\nu}(x, y+u, z+v) \tag{2.5}$$

where ${}_H F_{h,n}(x, u, v)$ is given by (1.14). This formula when $u = v = 0$ yields the result (2.4).
The formula given by (2.1) for $\Omega^\alpha_{\mu,\nu,h}$ may be converted to generate

$$\Omega^\alpha_{\mu,\nu,h}(x, y, z) = \sum_{n=0}^{\infty} F_{h,n}(w) \Omega^{\alpha-1}_{\mu+n-1,\nu,h}(x, y, z) \tag{2.6}$$

which reduces to (2.4) when $\alpha = 1$.

3 Explicit Representations for $\Omega^\alpha_{\mu,\nu,h}(x, y, z)$

The use of (1.11) can be exploited to obtain the series representations of (2.1). We have indeed

$$\begin{aligned}\Omega^\alpha_{\mu,\nu,h}(x, y, z) &== \sqrt{\frac{x}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} H_n(u, v) \int_0^\infty \frac{t^{\mu+n} e^{-(y+u)t-(z+v)t^2}}{(1 - h(\omega)t - t^2)^\alpha} J_\nu(xt)\, dt \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} H_n(u, v) \Omega^\alpha_{\mu+n,\nu,h}(x, y+u, z+v)\end{aligned} \tag{3.1}$$

by applying (1.11) to the integral on the right of (2.1).

Since

$$\lim_{x\to 0} x^{-\nu} J_\nu(x) = \frac{1}{2^\nu \Gamma(\nu+1)},$$

we may write a limiting case of (2.1) in the form

$$\lim_{x\to 0} \frac{\Omega^{\alpha}_{\mu,\nu,h}(x,y,z)}{x^{\nu+1/2}} = \frac{2^{-\nu-1/2}}{\Gamma(\nu+1)} \int_0^\infty \frac{t^{\mu+\nu} e^{-yt-zt^2}}{(1-h(\omega)t-t^2)^\alpha}\, dt$$
$$= \frac{2^{-\nu-1/2}}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{1}{n!} H_n(u,v) \int_0^\infty \frac{t^{\mu+\nu+n} e^{-(y+u)t-(z+v)t^2}}{(1-h(\omega)t-t^2)^\alpha}\, dt \tag{3.2}$$

It may be of interest to observe here that result (3.2) yields a number of new expressions. One class of expansions is given by setting $\alpha = 0$ in (3.2) and using [6, 146(24)]

$$\int_0^\infty t^\sigma e^{-yt-zt^2}\, dt = 2^{(\sigma+1)/2}\, \Gamma(\sigma+1)\, e^{y^2/8z}\, D_{-\sigma-1}\left(\sqrt{\frac{y}{2z}}\right) \quad (\mathrm{Re}(\sigma+1) > 0,\ \mathrm{Re}\, y > 0) \tag{3.3}$$

where $D_{-\nu}(x)$ is parabolic cylinder function [20]. Thus, we will be able to obtain

$$\lim_{x\to 0} \frac{V_{\mu,\nu}(x,y,z)}{x^{\nu+1/2}}$$
$$= \frac{2^{\frac{\mu-\nu}{2}} e^{(y+u)^2/8(z+v)}}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{2^{n/2}(\mu+\nu+1)_n}{n!} H_n(u,v) D_{-\mu-\nu-n-1}\left(\sqrt{\frac{y+u}{2(z+v)}}\right) \tag{3.4}$$

A reduction of interest involves the case of replacing y by $y-u$, z by $z-v$ and μ by $\mu-\nu$, and thus we obtain the following result (see [24])

$$\int_0^\infty t^\mu e^{-(y-u)t-(z-v)t^2}\, dt$$
$$= \Gamma(\mu+1) e^{\frac{y^2}{8z}} \sum_{n=0}^{\infty} \frac{2^{\frac{\mu+n+1}{2}}(\mu+1)_n}{n!} H_n(u,v) D_{-\mu-n-1}\left(\sqrt{\frac{y}{2z}}\right) \tag{3.5}$$

A second class of expansions, a consequence of (3.2) may be obtained by setting $\alpha = 1$.

$$\lim_{x\to 0} \frac{\Omega_{\mu,\nu,h}(x,y,z)}{x^{\nu+1/2}} = \frac{2^{-\nu-1/2}}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{1}{n!} H_n(u,v) \int_0^\infty \frac{t^{\mu+\nu+n} e^{-(y+u)t-(z+v)t^2}}{(1-h(\omega)t-t^2)}\, dt$$

$$= \frac{2^{-\nu-1/2}}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{1}{n!} H_n(u,v) \sum_{m=0}^{\infty} F_{h,m}(w) \int_0^{\infty} t^{\mu+\nu+n+m-1} e^{-(y+u)t-(z+v)t^2}\, dt$$

$$= \frac{2^{-\nu-1/2}\Gamma(\sigma)}{\Gamma(\nu+1)} e^{\frac{(y+u)^2}{8(z+v)}} \sum_{n,m=0}^{\infty} (\sigma)_{m+n} 2^{\frac{\sigma+m+n}{2}} \frac{1}{n!} H_n(u,v) F_{h,m}(w) D_{-\sigma-m-n}\left(\sqrt{\frac{y+u}{2(z+v)}}\right)$$

where $\sigma = \mu + \nu$.

4 Representation of $\Omega_{\mu,\nu,h}(x, y, z)$

Theorem 4.1 *For* $m \geq 1$

$$\Omega_{\mu,\nu,h}(x,y,z) = \sum_{m=0}^{\infty} \sum_{i=0}^{[\frac{m-1}{2}]} \binom{m-i-1}{i} h^{m-2i-1}(w) V_{\mu+m-1,\nu}(x,y,z) \tag{4.1}$$

Proof Let

$$G(h(w),t) = \frac{t}{1-h(w)t-t^2} = t\sum_{n=0}^{\infty}(h(w)t+t^2)^n$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} (h(w)t)^{n-i} (t^{n+i+1})$$

On writing $n+i+1 = m$ in R.H.S of the above equation, we get

$$G(h(w),t) = \sum_{m=0}^{\infty} \left[\sum_{i=0}^{[\frac{m-1}{2}]} \binom{m-i-1}{i} h^{m-2i-1}(w) \right] t^m \tag{4.2}$$

Now from (2.1), we can write

$$\Omega_{\mu,\nu,h}(x,y,z) = \sqrt{\frac{x}{2}} \int_0^{\infty} t^{\mu-1} G(h(w),t) e^{-yt-zt^2} J_\nu(xt)\, dt$$

which on using (1.1) and (4.2) gives (4.1).
Next we write (1.14) in the form

$$\frac{t}{1-h(x)t-t^2} e^{yt+zt^2} = G(h(x),t)e^{yt+zt^2} = \sum_{n=0}^{\infty} {}_H F_{h,n}(x,y,z) t^n \tag{4.3}$$

and using (1.11) and (4.2), we have

$$\sum_{n=0}^{\infty} {}_HF_n(x,y,z)t^n = \sum_{n=0}^{\infty} H_n(y,z)\frac{t^n}{n!}\sum_{m=0}^{\infty}\left[\sum_{i=0}^{[\frac{m-1}{2}]}\binom{m-i-1}{i} h^{m-2i-1}(x)\right]t^m$$

Replacing n by $n-m$ and comparing the coefficients of t^n, we get the following theorem.

Theorem 4.2 *For* $m \geq 1$

$${}_HF_n(x,y,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{H_{n-m}(y,z)}{(n-m)!}\left[\sum_{i=0}^{[\frac{m-1}{2}]}\binom{m-i-1}{i} h^{m-2i-1}(x)\right] \tag{4.4}$$

For $y = z = 0$ in equation (4.4), the result reduces to known result of Nalli and Haukkanen [12, p. 3181(2.8)].

5 Another Representation for $\Omega_{\mu,\nu,h}(x, y, z)$

Applying the result [18, p. 101(2)]

$$\int_0^{\infty} t^{\alpha} e^{-zt^2} J_{\nu}(xt)\,dt = \frac{\sin\nu\pi\,\Gamma(\alpha+1)/2}{2\nu\pi\, z^{(\alpha+1)/2}}\,{}_2F_2\left[1, \frac{\alpha+1}{2};\ 1+\frac{\nu}{2}, 1-\frac{\nu}{2};\ -\frac{x^2}{4z}\right]$$
$$-\frac{x\sin\nu\pi\,\Gamma(\alpha+2)/2}{2\pi(1-\nu^2)\,z^{(\alpha+2)/2}}\,{}_2F_2\left[1, \frac{\alpha+2}{2};\ \frac{3+\nu}{2}, \frac{3-\nu}{2};\ -\frac{x^2}{4z}\right] \tag{5.1}$$
$$[\mathrm{Re}\,z,\ \mathrm{Re}(\alpha+1) > 0],$$

we derive another class of representations of generalized Voigt function $\Omega_{\mu,\nu,h}(x, y, z)$ associated with the product of Fibonacci polynomials $F_{h,n}(w)$ and hypergeometric functions ${}_2F_2$ (see, e.g., [20, p. 42]) in the following form

$$\Omega_{\mu,\nu,h}(x,y,z) = \sqrt{\frac{x}{2}}\sum_{n,m=0}^{\infty} F_{h,n}(w)\frac{y^m}{m!}\{\frac{\sin\nu\pi\,\Gamma(\mu+n+m)/2}{2\nu\pi\, z^{(\mu+n+m)/2}}$$
$${}_2F_2\left[1, \frac{\mu+n+m}{2};\ 1+\frac{\nu}{2}, 1-\frac{\nu}{2};\ -\frac{x^2}{4z}\right]$$
$$-\frac{x\sin\nu\pi\,\Gamma(\mu+n+m+1)/2}{2\pi(1-\nu^2)\,z^{(\mu+n+m+1)/2}}\,{}_2F_2\left[1, \frac{\mu+n+m+1}{2};\ \frac{3+\nu}{2}, \frac{3-\nu}{2};\ -\frac{x^2}{4z}\right]\} \tag{5.2}$$

To prove (5.2), we expand the exponential function e^{-yt} in (2.1) and then use (1.9) and (5.1).
For $y = 0$, (5.2) reduces to

$$\Omega_{\mu,\nu,h}(x,0,z) = \sqrt{\frac{x}{2}} \sum_{n=0}^{\infty} F_{h,n}(w)\{\frac{\sin\nu\pi\ \Gamma(\mu+n)/2}{2\nu\pi\ z^{(\mu+n)/2}}\ {}_2F_2\left[1, \frac{\mu+n}{2};\ 1+\frac{\nu}{2}, 1-\frac{\nu}{2};\ -\frac{x^2}{4z}\right]$$
$$-\frac{x\sin\nu\pi\ \Gamma(\mu+n+1)/2}{2\pi(1-\nu^2)\ z^{(\mu+n+1)/2}}\ {}_2F_2\left[1, \frac{\mu+n+1}{2};\ \frac{3+\nu}{2}, \frac{3-\nu}{2};\ -\frac{x^2}{4z}\right]\} \tag{5.3}$$

Now we examine consequences of (5.2) for special values of y, z and $h(w)$ in $\Omega_{\mu,\nu,h}(x, y, z)$. Let $h(w) = k$ and $z = 1$. Then using (1.13), we have

$$\Omega_{\mu,\nu,k}(x,y,1) = \sqrt{\frac{x}{2}} \sum_{n=0}^{\infty} F_{k,n} \int_0^{\infty} t^{n+\mu-1} e^{-yt} J_\nu(xt)\,dt \tag{5.4}$$

where $F_{k,n}$ are k-Fibonacci–Hermite numbers.
Solving Laplace transform involving in (5.4) with the help of [7, p. 29(7)], we get

$$\Omega_{\mu,\nu,k}(x,y,1) = \frac{x^\nu}{2^\nu y^{\mu+\nu}\Gamma(\nu+1)} \sum_{n=0}^{\infty} F_{k,n} \frac{\Gamma(\mu+\nu+n)}{y^n}$$
$${}_2F_1\left[\frac{\mu+\nu+n}{2}; \frac{\mu+\nu+n+1}{2}; \nu+1; -\frac{x^2}{y^2}\right] \tag{5.5}$$

where ${}_2F_1$ is Gauss hypergeometric function [20].
With again $y = 0$ and the use of the integral [7, p. 22(7)] for solving the Hankel transform in (5.4) gives

$$\Omega_{\mu,\nu,k}(x,0,1) = \sum_{n=0}^{\infty} F_{k,n} \frac{2^{\mu-\frac{3}{2}+n}\Gamma(\frac{\nu-\mu+n-2}{2})}{x^{\mu-\frac{1}{2}+n}\Gamma(\frac{\nu-\mu+2-n}{2})} \tag{5.6}$$

6 Further Extension $\Omega^{\alpha,j}_{\mu,\nu,h}(x, y, z)$ and Related Functions

A natural generalization of (2.1) is accomplished by defining the generalized Voigt function $\Omega^{\alpha,j}_{\mu,\nu,h}(x, y, z)$ by the introduction of a positive integer j.
Definition Let $h(\omega)$ be a polynomial with real coefficients, $\alpha \geq 0$ and j be a positive integer. Then

$$\Omega^{\alpha,j}_{\mu,\nu,h} = \Omega^{\alpha,j}_{\mu,\nu,h}(x,y,z) = \sqrt{\frac{x}{2}} \int_0^{\infty} \frac{t^\mu e^{-yt-zt^j}}{(1-h(\omega)t-t^2)^\alpha} J_\nu(xt)\,dt \tag{6.1}$$

where $x, y, z \in \mathbb{R}^+$ and $\mathrm{Re}(\mu+\nu) > -1$.
Clearly, the case $j = 2$ corresponds to (2.1) and we have
$\Omega^{\alpha,2}_{\mu,\nu,h}(x, y, z) = \Omega^{\alpha}_{\mu,\nu,h}(x, y, z)$ and $\Omega^{\alpha,2}_{\mu,\nu,h}(x, y, z)\,|_{\alpha=0} = V_{\mu,\nu}(x, y, z)$
To obtain the various explicit representations for the generalized Voigt function $\Omega^{\alpha,j}_{\mu,\nu,h}$, our starting point is (6.1). Making use of the series representation (1.4) and (1.9) in (6.1) and integrating the resulting series term by term with the help of the result

$$\int_0^\infty t^\mu e^{-pt-\beta t^\lambda}\,dt = \sum_{r=0}^\infty \frac{(-\beta)^r\,\Gamma(\mu+1+\lambda r)}{r!\,p^{\mu+1+\lambda r}}, \tag{6.2}$$

$$(\mathrm{Re}(\mu+1) > 0,\ \mathrm{Re}\,p > 0 \text{ and } \lambda > 0),$$

we obtain

$$\Omega^{\alpha,j}_{\mu,\nu}(x, y, z) = \frac{x^{\nu+\frac{1}{2}}}{2^{\nu+\frac{1}{2}}\,y^{\mu+\nu}} \sum_{n=0}^\infty \frac{F_{h,n}(w)}{y^n} \sum_{r,m=0}^\infty \frac{\Gamma(\mu+\nu+2m+jr+n)}{\Gamma(\nu+m+1)\,m!\,r!} \left(\frac{-z}{y^j}\right)^r \left(\frac{-x^2}{4y^2}\right)^m \tag{6.3}$$

Formula (6.3) is an interesting generalization of a representation [13, p. 13,(4.3)] in terms of Kampé de Feriét series $F^{p:q;r}_{l:m;n}$ [see (20, p. 63)] given by

$$\Omega_{\mu,\nu}(x, y, z) = \frac{x^{\nu+\frac{1}{2}}\,\Gamma(\mu+\nu+1)}{2^{\nu+\frac{1}{2}}\,y^{\mu+\nu+1}\,\Gamma(\nu+1)} \times F^{2:0;0}_{0:1;0}\left[\begin{matrix} \frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2} : \_\_\_ ; \_ ; \\ \qquad\qquad\qquad\qquad\qquad \frac{-x^2}{y^2}, \frac{-4z}{y^2} \\ \_\_\_ \qquad : \nu+1 ; \_ ; \end{matrix}\right], \tag{6.4}$$

which is given by recently, though in a slightly specialized form (for $z = \frac{1}{4}$), by Pathan and Shahwan [17].
The representation (6.4) is derivable from (6.3) by setting $j = 2$, $\alpha = 0$ and then using Legendre's duplication formula [20, p. 23(26)].

7 The Multivariable Extension of the Voigt Function

The definition of generalized Voigt function given in the preceding section may be extended slightly to include the multivariable extension of the Voigt functions. Just as in the previous sections, in order to obtain representations for the multivariable Voigt functions we propose to make use of the multivariable Hermite polynomials $H^{(m)}_n(\{x\}^m_1)$ which are specified by the generating function [4]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(m)}(\{x\}_1^m) = e^{\sum_{j=1}^{m} x_j t^j} \tag{7.1}$$

where $\{x\}_1^m = x_1, x_2, \ldots, x_m$.
We begin by recalling the relationship

$$H_n^{(1)}(x) = H_n^{(2)}(2x, -1) = H_n(x) \tag{7.2}$$

where $H_n(x_1, x_2)$ are two variable Hermite–Kampé de Feriét polynomials given by (1.10).

A three variable generalized Hermite polynomials $H_n(x, y, z)$ defined by the generating function [see (1, p. 511)] is

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y, z) = e^{2xt - yt^2 + zt^3} \tag{7.3}$$

where

$$H_n^{(3)}(2x, -y, z) = H_n(x, y, z)$$

Among the numerous specializations of (1.10), considered by Dattoli et al. [3], we mention the following generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x, y; \xi) = e^{2xt - t^2 + 2y\xi t - \xi^2 t^2} \tag{7.4}$$

straightforwardly yielding the following expansion in terms of ordinary Hermite polynomials

$$H_n^{(2)}(2x + 2y\xi, -\xi^2 - 1) = h_n(x, y; \xi) = \sum_{s=0}^{n} \binom{n}{s} \xi^s H_{n-s}(x) H_s(y) \tag{7.5}$$

which can also be written as

$$h_n(x, y; \xi) = (1 + \xi^2)^{n/2} H_n\left(\frac{x + \xi y}{\sqrt{1 + \xi^2}}\right) \tag{7.6}$$

Definition Let $h(\omega)$ be a polynomial with real coefficients, $\alpha \geq 0$ and j be a positive integer. Then multivariable extension of the Voigt function is

$$\Lambda_{\mu,\nu,h}^{\alpha,m} = \Lambda_{\mu,\nu,h}^{\alpha,m}(x, y, z, x_1, \ldots, x_m) = \sqrt{\frac{x}{2}} \int_0^{\infty} \frac{t^{\mu} e^{-yt - zt^2 + \sum_{j=1}^{m} x_j t^j}}{(1 - h(\omega)t - t^2)^{\alpha}} J_{\nu}(xt)\, dt, \tag{7.7}$$

where m is a positive integer, $x, y, z, x_1, \ldots, x_m \in \mathbb{R}^+$ and $\mathrm{Re}(\mu+\nu) > -1$. By comparing the definitions (1.2), (2.1), (2.2) and (6.1) with (7.7), we obtain the following relationships

$$\Lambda^{\alpha,m}_{\mu,\nu,h}(x, y, 0, 0, 0, \ldots, -z) = \Omega^{\alpha,m}_{\mu,\nu,h}(x, y, z)$$

$$\Lambda^{\alpha,m}_{\mu,\nu,h}(x, y, z, 0, \ldots, 0) = \Omega^{\alpha,m}_{\mu,\nu,h}(x, y, z)$$

$$\Lambda^{\alpha}_{\mu,\nu,h}(x, y, z, x_1) = \Omega^{\alpha}_{\mu,\nu,h}(x, y - x_1, z),$$

$$\Lambda_{\mu,\nu}(x, y, z, x_1, x_2) = \Omega_{\mu,\nu}(x, y - x_1, z - x_2).$$

Additionally, we record here

$$\lim_{x\to 0} \Lambda^{\alpha,m}_{\mu,\nu,h}(x, y, z, x_1, \ldots, x_m) = \frac{1}{\Gamma(\nu+1)} \int_0^\infty \frac{t^{\mu+\nu} e^{-yt-zt^2+\sum_{j=1}^{m} x_j t^j}}{(1-h(\omega)t-t^2)^\alpha}\, dt, \quad (7.8)$$

which for $\alpha = 0$ reduces to [13, p. 10(1.9)]

$$\lim_{x\to 0} \Lambda_{\mu,\nu}(x, y, z, x_1, \ldots, x_m) = \frac{1}{\Gamma(\nu+1)} \int_0^\infty t^{\mu+\nu}\, e^{-yt-zt^2+\sum_{j=1}^{m} x_j t^j}\, dt \quad (7.9)$$

The use of (7.1) can be exploited to obtain the series representations of (7.7). We have indeed

$$\Lambda^{\alpha,m}_{\mu,\nu}(x, y, z, x_1, \ldots, x_m) = \sqrt{\frac{x}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} H^{(m)}_n(\{x\}^m_1)$$

$$\int_0^\infty \frac{t^{\mu+n}\, e^{-yt-zt^2}}{(1-h(\omega)t-t^2)^\alpha}\, J_\nu(xt)\, dt \quad (7.10)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} H^{(m)}_n(\{x\}^m_1)\, \Omega^{\alpha}_{\mu+n,\nu}(x, y, z) \quad (7.11)$$

by applying (7.1) to the integral on the right of (7.7)
Making use of $\alpha = 0$ in the above result yields a known result [13, p. 14(5.2)]

$$\Lambda_{\mu,\nu}(x, y, z, x_1, \ldots, x_m) = \sum_{n=0}^{\infty} \frac{1}{n!} H^{(m)}_n(\{x\}^m_1)\, \Omega_{\mu+n,\nu}(x, y, z) \quad (7.12)$$

It may be of interest to observe here that by letting $x \to 0$ in the above result, we have

$$\int_0^\infty t^\mu\, e^{-yt-zt^2+\sum_{j=1}^{m} x_j t^s}\, dt = \sum_{n=0}^{\infty} \frac{2^{n/2}(\mu+1)_n}{n!}\, H_n^{(m)}\left(\{x\}_1^m\right) D_{-\mu-n-1}\left(\sqrt{\frac{y}{2z}}\right) \tag{7.13}$$

where $D_{-\nu}(x)$ is parabolic cylinder function [20].
Set $m = 2$, $x_1 = 2x$ and $x_2 = -1$ in (7.10) to get

$$\Lambda^{\alpha}_{\mu,\nu}(x, y, z, 2x, -1) \;=\; \Omega^{\alpha}_{\mu,\nu}(x, y-2x, z+1) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)\, \Omega^{\alpha}_{\mu,\nu}(x, y, z) \tag{7.14}$$

More generally, for $m = 3$ in (7.10) yields

$$\Lambda^{\alpha}_{\mu,\nu}(x, y, z, x_1, x_2, x_3) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n^{(3)}(x_1, x_2, x_3)\Omega^{\alpha}_{\mu,\nu}(x, y, z) \tag{7.15}$$

where $H_n^{(3)}(x_1, x_2, x_3)$ is defined by (7.3).
For $m = 2$, $x_1 = 2x + 2y\xi$ and $x_2 = -\xi - 1$, (7.10) gives

$$\sum_{n=0}^{\infty} \frac{1}{n!} h_n(x, y; \xi)\, \Omega^{\alpha}_{\mu,\nu}(x, y, z) \;=\; \Omega^{\alpha}_{\mu,\nu}(x, y-2x-2y\xi, z+\xi^2+1) \tag{7.16}$$

where $h_n(x, y; \xi)$ is defined by (7.4) (or its equivalent form (7.5)).

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Nonlinear Dynamical Model for DNA

Haci Mehmet Baskonus and Carlo Cattani

Abstract This chapter deals with a nonlinear dynamical system arising in the analysis of the double-chain model in deoxyribonucleic acid (DNA). Bernoulli sub-equation function method and modified exp $(-\Omega(\xi))$-expansion function method to obtain some novel dynamical structures to the nonlinear dynamical system are used. We construct some new exponential, hyperbolic and complex periodic wave solutions to this model. Under some suitable values of parameters, we plot the 2D and 3D graphics of the solutions obtained in this study. All the solutions found in this study satisfy the nonlinear dynamical system. Moreover, these solutions can be used to explain some new significant physical meanings of the nonlinear dynamical model for DNA.

Keywords The new double-chain model
Bernoulli sub-equation function method · Exponential · Rational
Complex function solutions

1 Introduction

Nonlinear partial differential equations and nonlinear mathematical models are playing an important role in many phenomena arising in health applications such as biology, biosciences, biochemical, physics, water sciences, fluid mechanics, hydrodynamics, nonlinear dynamical system, plasma physics [1–11]. The significance of differential equations and inequalities in the investigation of mathematical equations has been almost completely studied especially during recent years.

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Many differential inequalities such as the delay integral inequality, the differential inequalities, the impulsive differential inequalities, and the Halanay inequalities and so on [12–19] have been constructed. However, the linear differential inequalities are not sufficient to describe nonlinear partial differential equations along with fractional differential equations. Therefore, new nonlinear inequalities have been developed by Xu and Wang [20]. These new mathematical inequalities have been used to discover new significant properties of real-world problems. For example, the socio-territorial inequalities have been investigated by Jorge Blanco in the field of Geography [21]. Yuhua Sun has studied on nonnegative solutions to the system of differential inequalities on manifolds which geodesically complete connected non-compact Riemannian manifold [22]. Bin Qian has presented a study that they obtained differential Harnack inequalities for the positive solutions of the Schrödinger equation associated with subelliptic operator with potential under the generalized curvature-dimension inequality recently introduced by Baudoin and Garofalo [23]. Moreover, they have derived the corresponding parabolic Harnack inequality along with the Perelman-type entropy. Lyapunov-type inequalities for partial differential equations have been submitted by de Nápoli and Pinasco [24].

When it comes to the real-world problems, nonlinear problems are usually related to phenomena having a strong impact and consequences on human beings. For example, the tsunami by earthquakes in 2011 in Japan, mostly related to short-distance seismic activities (earthquakes from Mw 6.9 on 8.8) [25], is one of them. Moreover, a study handling with first evidences of natural disasters which is namely "First Evidence of Aleo-Tsunami Deposits of A Major Historic Event in Ecuador" presented by Chunga and Toulkeridis has been submitted to the literature [25].

Another important real-world problem is the correct interpretation of the hidden properties of the deoxyribonucleic acid (DNA). In recent years, Kong et al. [26], Alka et al. [27] and Abdelrahman et al. [28] have introduced that a new double-chain DNA consisting two long elastic homogeneous strands physical mathematical model for a two polynucleotide chains. In this model, DNA is made of connected with each other by an elastic membrane representing the hydrogen bonds between the base pair of the two chains.

In this chapter, we will study the nonlinear dynamical model of DNA discovered in [26–28]. Then by using the analytical methods of Bernoulli sub-equation function method (BSEFM) and modified exp $(-\Omega(\xi))$-expansion function method, we will find some new solutions of this model. These solutions depend on some parameter so that we will give, for the first time, a deep analysis of the parametric dependence and show the biological consequences of some thresholds on parameters.

2 Preliminary Remarks on DNA

DNA is one the most complicated and comprehensive molecules in life. Many different models of DNA in the general properties of the DNA dynamics are very complicated because there are many various items in everyone [26].

First experimental evidence of resonant microwave absorption in DNA was studied by Webb and Booth [29]. After then, microwave absorption of DNA has been investigated by Swicord and Davis [30, 31]. However, Gabriel et al. [32], Yakushevich [33], Bixon et al. [34], Henderson [35] and Bruinsma [36] have mentioned that such results and findings are still controversial. As a result, many different approaches have been proposed to express of the model of DNA. Ludmila V. Yakushevich has comprehensively studied on nonlinear properties of physics of DNA [33]. Some models of DNA have been based on linear model [37–39] and other models expressed by using nonlinear models [40–42].

Muto et al. have firstly submitted the nonlinear mathematical model of the interaction of DNA with an external microwave field which was proposed as

$$u_{tt} = C^2 u_{zz} - \left(\varepsilon / C^2\right) u_{zztt} + \delta\left(u_z^2\right)_z,$$

where $u(z, t)$ describes longitudinal displacements in DNA [40, 41]. After then, they have developed this model by adding two extra terms as

$$u_{tt} = C^2 u_{zz} - \left(\varepsilon / C^2\right) u_{zztt} + \delta\left(u_z^2\right)_z - A u_t + F(z)\cos(\Omega t),$$

where $u(z, t)$ is the longitudinal displacement, C is the sound wave velocity, and $\varepsilon,\ \delta$ are the dispersive and anharmonic parameters, respectively [33, 40, 41].

Later, Zhang [42] has improved the model of Muto et al. He took into account both longitudinal and torsional degrees of freedom. Consequently, he proposed two coupled equations:

$$\begin{aligned} u_{tt} &= C^2 u_{zz} - \left(\varepsilon / C^2\right) u_{zztt} + \delta\left(u_z^2\right)_z + \chi_1\left(\varphi_z^2\right)_z + \chi_2\left(\varphi_z u_z\right)_z, \\ \varphi_{tt} &= v^2 \varphi_{zz} - w_0^2 \varphi + s\chi_2\left(u_z^2\right)_z + 4s\chi_1\left(\varphi_z u_z\right)_z, \end{aligned}$$

where $u(z, t)$, $\varphi(z, t)$ are the longitudinal and rotational displacements, respectively; C and ν are the torsional and longitudinal acoustic velocities; ε and δ are the dispersive and anharmonic parameters; w_0 and s are the frequency parameter and the parameter for dimensional transform; χ_1 and χ_2 are the coupling parameters.

The simplest model that describes the motions in this range of the timescale has been presented by Barkley and Zimm [43]. They have settled the theory of twisting and bending of chain macromolecules along with analysis of the fluorescence depolarization of DNA [43]. Many authors have been applied different methods for obtaining soliton and travelling wave solutions to these models. Especially, Mahmoud A. E. Abdelrahman, Emad H. M. Zahran and Mostafa M. A. Khater have applied the exp(-ϕ(xi))-expansion method to the nonlinear dynamical system of double-chain model in DNA [28].

In this chapter, the Bernoulli sub-equation function method (BSEFM) and modified exp $(-\Omega(\xi))$-expansion function method are used to solve analytically the nonlinear dynamical system arising in a new double-chain model of DNA. This model is based on the following nonlinear dynamical system [26–28]:

$$u_{tt} - c_1^2 u_{xx} = \lambda_1 u + \gamma_1 uv + \mu_1 u^3 + \beta_1 uv^2, \tag{1a}$$

$$v_{tt} - c_2^2 v_{xx} = \lambda_2 v + \gamma_2 u^2 + \mu_2 u^2 v + \beta_2 v^3 + c_0, \tag{1b}$$

where

$$c_1 = \pm\frac{Y}{\rho}, c_2 = \pm\frac{F}{\rho}, \lambda_1 = \frac{-2\mu}{\rho\sigma h}(c - l_0), \lambda_2 = \frac{-2\mu}{\rho\sigma h},$$
$$\gamma_1 = 2\gamma_2 = \frac{2\sqrt{2}\mu l_0}{\rho\sigma h^2}, \mu_1 = \mu_2, \beta_1 = \beta_2 = \frac{4\mu l_0}{\rho\sigma h^3}, c_0 = \frac{\sqrt{2}\mu(h - l_0)}{\rho\sigma}.$$

Here, ρ, σ, Y and F denote, respectively, the mass density, the area of transverse cross-section, the Young's modulus and tension density of each strand; μ is the rigidity of the elastic membrance; h is the distance between the two strands, and l_0 is the height of the membrance in the equilibrium positive [26–28]. In the system of Eq. (1a, b), $u(x, t)$ is the difference of the longitudinal displacements of the bottom and top strands while $v(x, t)$ is the difference of the transverse displacements of the bottom and top strands [26–28]. In the nonlinear dynamics of DNA Eq. (1a, b), this model consists of two long elastic homogeneous strands connected with each other by an elastic membrane for longitudinal and transverse motions [27]. It is also in the framework of the microscopic model of Peyrard and Bishop [27].

3 General Structures of Methods

3.1 Bernoulli Sub-equation Function Method (BSEFM)

In this section, an approach to the nonlinear partial differential equations (NLPDE) will be given. In order to apply this method to the NLPDE, we consider the following steps [44].

Step 1. We consider the partial differential equation in two variables such as x, t and a dependent variable u

$$P(u_x, u_t, u_{xt}, u_{xx}, \ldots) = 0, \tag{2}$$

and take the wave transformation

$$u(x, t) = U(\xi),\ \ \xi = kx + wt, \tag{3}$$

where $k \neq 0$, $w \neq 0$. Substituting Eq. (3) in Eq. (2), it gives us the following nonlinear ordinary differential equation:

$$N\big(U, U', U'', U''', \ldots\big) = 0. \tag{4}$$

Step 2. Take trial equation as follows:

$$U(\xi) = \sum_{i=0}^{n} a_i F^i(\xi) = a_0 + a_1 F(\xi) + a_2 F^2(\xi) + \cdots + a_n F^n(\xi), \tag{5}$$

being $F(\xi)$ the solution of the following Bernoulli differential equation

$$F'(\xi) = \alpha F(\xi) + \beta F^M(\xi), \tag{6}$$

where $\alpha \neq 0, \;\; \beta \neq 0, \;\; M \in \mathbb{R} - \{0, 1, 2\}$. Substituting in Eq. (5), we can obtain the following polynomial equation $\Omega(F)$ as a function of F :

$$\Omega(F(\xi)) = \rho_s F(\xi)^s + \cdots + \rho_1 F(\xi) + \rho_0 = 0. \tag{7}$$

According to the balance principle, we can get the relationship between n and M.

Step 3. Let us consider the coefficients of $\Omega(F(\xi))$ all be zero, and we will obtain an algebraic equations system:

$$\rho_i = 0, \;\; i = 0, \ldots, s. \tag{8}$$

By solving this system, we obtain $a_0, a_1, a_2, \ldots, a_n$.

Step 4. When we solve the nonlinear Bernoulli Eq. (6) by known methods, we obtain following two situations according to α and β:

$$F(\xi) = \left[\frac{-\beta}{\alpha} + \frac{c}{e^{\alpha(M-1)\xi}}\right]^{\frac{1}{1-M}}, \quad \alpha \neq \beta, \tag{9}$$

$$F(\xi) = \left[\frac{(c-1) + (c+1)\tanh\left(\frac{\alpha(1-M)\xi}{2}\right)}{1 - \tanh\left(\frac{\alpha(1-M)\xi}{2}\right)}\right]^{\frac{1}{1-M}}, \quad \alpha = \beta, \tag{10}$$

where $c \in \mathbb{R}$. By using a complete discrimination system for polynomials to classify the roots of $F(\xi)$, we solve Eq. (8) with the help of some computational software and classify the exact solutions of Eq. (2). For a better interpretation of the obtained results, we can plot two- and three-dimensional surfaces of solutions by choosing suitable parameter.

3.2 Modified $\exp(-\Omega(\xi))$-Expansion Function Method (MEFM)

The general properties of MEFM have been proposed in this section. MEFM is based on the $\exp(-\Omega(\xi))$-expansion function method [45–55]. In order to apply this method to the nonlinear partial differential equations, we consider it as following:

$$P(u, u_x, u_t, u_{xx}, u_{tt}, \ldots) = 0, \tag{11}$$

where $u = u(x, t)$ is an unknown function, P is a polynomial in $u(x, t)$, and its derivative and the subscripts stand for the partial derivatives.

Step 1: Let us consider the following travelling wave transformation defined by

$$u(x, t) = U(\xi), \quad \xi = kx - ct. \tag{12}$$

Using Eq. (12), we convert Eq. (11) into nonlinear ordinary differential equation (NODE) defined by

$$\text{NODE}\big(U, U', U'', U''', \ldots\big) = 0. \tag{13}$$

where NODE is a polynomial of U, and its derivatives and the superscripts indicate the ordinary derivatives with respect to ξ.

Step 2: Suppose the travelling wave solution of Eq. (13) can be rewritten as follows:

$$U(\xi) = \frac{\sum_{i=0}^{N} A_i \big[\exp(-\Omega(\xi))\big]^i}{\sum_{j=0}^{M} B_j \big[\exp(-\Omega(\xi))\big]^j} = \frac{A_0 + A_1 \exp(-\Omega) + \cdots + A_N \exp(\mathrm{N}(-\Omega))}{B_0 + B_1 \exp(-\Omega) + \cdots + B_M \exp(\mathrm{M}(-\Omega))}, \tag{14}$$

where A_i , B_j , $(0 \le i \le N, 0 \le j \le M)$ are constants to be determined later, such that $A_N \neq 0, B_M \neq 0$, and $\Omega = \Omega(\xi)$ verify the following ordinary differential equation

$$\Omega'(\xi) = \exp(-\Omega(\xi)) + \mu \exp(\Omega(\xi)) + \lambda. \tag{15}$$

Equation (15) has the following solution families [55, 56]:
Family 1: When $\mu \neq 0,\ \lambda^2 - 4\mu > 0$,

$$\Omega(\xi) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu}}{2\mu} \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + c_1)\right) - \frac{\lambda}{2\mu}\right). \tag{16}$$

Family 2: When $\mu \neq 0,\ \lambda^2 - 4\mu < 0$,

$$\Omega(\xi) = \ln\left(\frac{\sqrt{-\lambda^2+4\mu}}{2\mu}\tan\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}(\xi+c_1)\right)-\frac{\lambda}{2\mu}\right). \tag{17}$$

Family 3: When $\mu = 0,\ \lambda \neq 0$, and $\lambda^2 - 4\mu > 0$,

$$\Omega(\xi) = -\ln\left(\frac{\lambda}{\exp(\lambda(\xi+c_1))-1}\right). \tag{18}$$

Family 4: When $\mu \neq 0,\ \lambda \neq 0$, and $\lambda^2 - 4\mu = 0$,

$$\Omega(\xi) = \ln\left(-\frac{2\lambda(\xi+E)+4}{\lambda^2(\xi+c_1)}\right). \tag{19}$$

Family 5: When $\mu = 0,\ \lambda = 0$, and $\lambda^2 - 4\mu = 0$

$$\Omega(\xi) = \ln(\xi+c_1), \tag{20}$$

being $A_0, A_1, A_2, \ldots A_N, B_0, B_1, B_2, \ldots B_M, c_1, \lambda, \mu$ constants to be determined later. The positive integer N and M can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms occurring in (13).

Step 3: Setting (14) and (15) in (13), we get a polynomial of $\exp(-\Omega(\xi))$. We equate all the coefficients of same power of $\exp(-\Omega(\xi))$ to zero. This procedure yields a system of equations that can be solved to find $A_0, A_1, A_2, \ldots A_N, B_0, B_1, B_2, \ldots B_M, c_1, \lambda, \mu$. Substituting these values in (13), the general solutions of (13) complete the determination of the solution of (12).

4 Implementations of the Methods

4.1 Implementation of BSEFM

In this section, we obtain some new complex and exponential function solutions to the DNA model by using BSEFM.

Application
Firstly, we assume that $v(x,t) = au(x,t) + b$ [27, 28], so that from (1) we obtain the following nonlinear partial differential equation [26–28]:

$$u_{tt} - r^2u_{xx} - mu^3 - nu^2 - pu = 0, \tag{21}$$

where

$$m = \frac{\kappa}{h^3}\left(-2 + 4a^2\right), n = \frac{6\sqrt{2}a\kappa}{h^2}, p = \frac{6\kappa}{h} - \frac{2\kappa}{l_0},$$
$$\kappa = \frac{\mu l_0}{\rho\sigma}, c_1^2 = \frac{Y}{\rho}, b = \frac{h}{\sqrt{2}}, F = Y. \tag{22}$$

If we are looking for travelling wave solutions of (21), we have:

$$u(x,t) = U(\xi),\ \ \xi = kx + wt, \tag{23}$$
$$\left(w^2 - k^2r^2\right)U'' - mU^3 - nU^2 - pU = 0, \tag{24}$$

where k, w are real constants and not zero [26–28]. So that from (5) and (6) by the balance principle, we get the following relationships between n and M;

$$\begin{aligned}
&U = \sum_{i=0}^{n} a_i F^i = a_0 + a_1F + a_2F^2 + \cdots + a_nF^n,\\
&U \approx F^n,\\
&U' \approx F^{n+M-1},\\
&U'' \approx F^{n+2M-2},\\
&U'' \approx U^3\\
&F^{n+2M-2} \approx F^{3n}\\
&n + 2M - 2 = 3n,\\
&M = n + 1.
\end{aligned} \tag{25}$$

According to suitable values of M and n, we can obtain some novel different cases as follows:

Case 1: If we take as $M = 3$ and $n = 2$ via (25), then we can write the following equations

$$U = a_0 + a_1F + a_2F^2, \tag{26}$$
$$U' = a_1F' + 2a_2FF' = \alpha a_1F + 2\alpha a_2F^2 + \beta a_1F^3 + 2a_2\beta F^4, \tag{27}$$
$$U'' = \alpha a_1F' + 4\alpha a_2FF' + 3\beta a_1F^2F' + 8a_2\beta F^3F', \tag{28}$$

where $F' = \alpha F + \beta F^3,\ \alpha \neq 0,\ \beta \neq 0$. By using (26), (28) in (24), we can get an equation including various power of F. By setting the same power of F to zero, we can find a system of equations. By solving this system, we obtain the following coefficients.

Case 1.1. For $\alpha \neq \beta$, we have the following coefficients:

$$a_0 = \frac{\sqrt{2p}}{\sqrt{m}}, a_1 = 0, \alpha = \frac{\sqrt{p}}{2\sqrt{w^2 - k^2r^2}}, \beta = \frac{a_2\sqrt{m}}{2\sqrt{2}\sqrt{w^2 - k^2r^2}}, n = \frac{-3\sqrt{mp}}{\sqrt{2}}, \tag{29}$$

where $m = A, n = B, p = C$. Substituting (29) in (26) along with (3), we obtain the exponential function solution to the nonlinear dynamical system of double-chain model in DNA in the following form:

$$u_1(x,t) = \frac{\sqrt{2p}}{\sqrt{m}} + \frac{a_2}{-\frac{a_2\sqrt{m}}{\sqrt{2p}} + Ee^{-\frac{\sqrt{p}(kx+wt)}{\sqrt{w^2-k^2r^2}}}},$$
$$v_1(x,t) = au_1(x,t) + b. \tag{30}$$

In the system (1), $u(x,t)$ is the difference of the longitudinal displacements of the bottom and top strands while $v(x,t)$ is the difference of the transverse displacements of the bottom and top strands [26–28]. In the nonlinear dynamics of DNA Eq. (1a, b), describe the model which consists of two long elastic homogeneous strands connected with each other by an elastic membrane is for longitudinal and transverse motions [27].

Case 1.2. For $\alpha \neq \beta$, we have the following coefficients:

$$a_0 = \frac{-i\sqrt{p}a_2}{2\beta\sqrt{w^2 - k^2r^2}}, a_1 = 0, \alpha = \frac{-i\sqrt{p}}{2\sqrt{-w^2 + k^2r^2}}, m = \frac{8}{a_2^2}\left(w^2 - k^2r^2\right),$$
$$n = \frac{-6i\beta}{a_2}\sqrt{p\left(k^2r^2 - w^2\right)}. \tag{31}$$

Substituting (31) in (26) along with (3), we obtain the new complex function solution to the nonlinear dynamical system of double-chain model in DNA in the following form:

$$u_2(x,t) = \frac{-i\sqrt{p}a_2}{2\beta\sqrt{k^2r^2 - w^2}} + \frac{a_2}{-\frac{i\beta}{\sqrt{p}}\sqrt{4k^2r^2 - 4w^2} + Ee^{\frac{2i\sqrt{p}(kx+wt)}{\sqrt{4k^2r^2-4w^2}}}},$$
$$v_2(x,t) = au_2(x,t) + b. \tag{32}$$

Case 1.3. For $\alpha \neq \beta$, we have the following coefficients

$$a_0 = \frac{i\sqrt{p}a_2}{2\beta\sqrt{k^2r^2 - w^2}}, a_1 = 0, \alpha = \frac{i\sqrt{p}}{\sqrt{4k^2r^2 - 4w^2}}, m = \frac{8\beta^2}{a_2^2}\left(w^2 - k^2r^2\right),$$
$$n = \frac{6i\beta}{a_2}\sqrt{p\left(k^2r^2 - w^2\right)}. \tag{33}$$

Considering (33) in (26) along with (3), we obtain another complex function solution to the nonlinear dynamical system of double-chain model in DNA in the following form:

$$u_3(x,t) = \frac{i\sqrt{p}a_2}{2\beta\sqrt{k^2r^2 - w^2}} + \frac{a_2}{Ee^{\frac{-2i\sqrt{p}(kx+wt)}{\sqrt{4k^2r^2-4w^2}}} + \frac{i\beta}{\sqrt{p}}\sqrt{4k^2r^2 - 4w^2}},$$
$$v_3(x,t) = au_3(x,t) + b. \tag{34}$$

Case 2: If we take as $M = 4$ and $n = 3$ via (25), then we can write the following equations

$$U = a_0 + a_1F + a_2F^2 + a_3F^3, \tag{35}$$
$$U' = a_1F' + 2a_2FF' + 3a_3F^2F' = \alpha a_1F + 2\alpha a_2F^2 + \beta a_1F^3 + 2a_2\beta F^4 + 3a_3F^2F', \tag{36}$$
$$U'' = \alpha a_1F' + 4\alpha a_2FF' + 3\beta a_1F^2F' + 8a_2\beta F^3F' + 9\alpha a_3F^2F' + 15\beta a_3F^4F', \tag{37}$$

where $F' = \alpha F + \beta F^4,\ \alpha \neq 0,\ \beta \neq 0$. When we use (35), (37) in Eq. (24), we get an equation having various power of F. By setting all the same power of F to zero, we can find a system of equations. By solving this system, we obtain the following coefficients.

Case 2.1. For $\alpha \neq \beta$, it can be considered that the following coefficients:

$$a_0 = \frac{-2n}{3m}, a_1 = a_2 = 0, a_3 = \frac{3\beta\sqrt{2\left(w^2 - k^2r^2\right)}}{\sqrt{m}},$$
$$\alpha = \frac{-n\sqrt{2}}{9\sqrt{m\left(w^2 - k^2r^2\right)}}, p = \frac{2n^2}{9m}. \tag{38}$$

Taking (38) in (35) along with (3), we obtain another exponential function solution to the nonlinear dynamical system of double-chain model in DNA in the form

$$u_4(x,t) = \frac{-2n}{3m} + \frac{3\beta\sqrt{2w^2 - 2k^2r^2}}{E\sqrt{m}e^{\frac{n\sqrt{2}(kx+wt)}{3\sqrt{m}\sqrt{w^2-k^2r^2}}} + \frac{9\beta\sqrt{mw^2-mk^2r^2}}{n\sqrt{2}}},$$
$$v_4(x,t) = au_4(x,t) + b. \tag{39}$$

Case 2.2. For $\alpha \neq \beta$, we have the following coefficients:

$$a_0 = \frac{-i\sqrt{p}a_3}{3\beta\sqrt{w^2 - k^2r^2}}, a_1 = a_2 = 0, \alpha = \frac{-i\sqrt{p}}{\sqrt{9k^2r^2 - 9w^2}}, m = \frac{18}{a_3^2}\beta^2\left(w^2 - k^2r^2\right),$$
$$n = \frac{-9i\beta}{a_3}\sqrt{p\left(k^2r^2 - w^2\right)}. \tag{40}$$

By considering (40) in (35) along with (3), we obtain another complex function solution to the nonlinear dynamical system of double-chain model in DNA in the following form:

$$u_5(x,t) = \frac{-ia_3\sqrt{p}}{3\beta\sqrt{k^2r^2 - w^2}} + \frac{a_3}{Ee^{i\frac{\sqrt{p}}{\sqrt{k^2r^2-w^2}}(kx+wt)} - \frac{i\beta}{\sqrt{p}}\sqrt{9k^2r^2 - 9w^2}},$$
$$v_5(x,t) = au_5(x,t) + b. \tag{41}$$

Case 2.3. For $\alpha \neq \beta$, we have the following coefficients:

$$a_0 = \frac{i\sqrt{p}a_3}{3\beta\sqrt{-w^2 + k^2r^2}}, a_1 = a_2 = 0, \alpha = \frac{i\sqrt{p}}{\sqrt{9k^2r^2 - 9w^2}}, m = \frac{18}{a_3^2}\beta^2\left(w^2 - k^2r^2\right),$$
$$n = \frac{9i\beta}{a_3}\sqrt{p\left(k^2r^2 - w^2\right)}. \tag{42}$$

With the help of (42), (35) along with (3), we get another complex function solution of the nonlinear dynamical system of double-chain model in DNA in the form

$$u_6(x,t) = \frac{ia_3\sqrt{p}}{3\beta\sqrt{k^2r^2 - w^2}} + \frac{a_3\sqrt{p}}{E\sqrt{p}e^{\frac{-3\sqrt{p}}{\sqrt{k^2r^2-w^2}}(kx+wt)} + i\beta\sqrt{9k^2r^2 - 9w^2}},$$
$$v_6(x,t) = au_6(x,t) + b. \tag{43}$$

Case 2.4. For $\alpha \neq \beta$, we have the following coefficients:

$$a_0 = \frac{9\alpha\kappa\sqrt{m} - 3|\alpha\kappa|\sqrt{m}}{2m\sqrt{2\kappa}}, a_1 = a_2 = 0, \beta = \frac{a_3\sqrt{m}}{\sqrt{18\kappa}}, n = \frac{9\left(-\kappa\alpha\sqrt{m} + |\alpha\kappa|\sqrt{m}\right)}{2\sqrt{2\kappa}},$$
$$p = \frac{9}{4}\alpha\left(\alpha\kappa - \frac{3|\alpha\kappa|\sqrt{m}}{\sqrt{m}}\right), \tag{44}$$

where $\kappa = w^2 - k^2r^2$. Considering (44) in (35) along with (3), we get the exponential function solution to the DNA model in the form

$$u_7(x,t)=\frac{9\alpha\kappa\sqrt{m}-3|\alpha\kappa|\sqrt{m}}{2m\sqrt{2\kappa}}+\frac{a_3}{Ee^{-3\alpha(kx+wt)}+\frac{a_3\sqrt{m}}{\alpha\sqrt{18\kappa}}},$$
$$v_7(x,t)=au_7(x,t)+b, \tag{45}$$

where $\kappa = w^2 - k^2r^2$.

Case 3: If we take as $M = 5$ and $n = 4$ via (25), then we get the equations:

$$U = a_0 + a_1F + a_2F^2 + a_3F^3 + a_4F^4, \tag{46}$$
$$U' = a_1F' + 2a_2FF' + 3a_3F^2F' = \alpha a_1F + 2\alpha a_2F^2 + \beta a_1F^3 + 2a_2\beta F^4 + 3a_3F^2F' + 4a_4F^3F',$$
$$U'' = \cdots, \tag{47}$$

where $F' = \alpha F + \beta F^5,\ \alpha \neq 0,\ \beta \neq 0$. When we use (46), (47) in (24), we get an equation with various power of F. By setting all the same power of F to zero, we get a system of equations. By solving this system, we obtain the following coefficients.

Case 3.1. For $\alpha \neq \beta$, we have the following coefficients:

$$a_0=\frac{-i\sqrt{(pkr-pw)(2kr+2w)}}{\sqrt{-mk^2r^2+mw^2}}, a_1=a_2=0, a_3=0, \alpha=\frac{-i\sqrt{p}}{4\sqrt{-w^2+k^2r^2}},$$
$$\beta=\frac{a_4\sqrt{m}}{4\sqrt{2w^2-2k^2r^2}}, n=\frac{3i\sqrt{mp}\sqrt{(kr-w)(kr+w)}}{\sqrt{2}\sqrt{-k^2r^2+w^2}}. \tag{48}$$

Taking (48) in (46) along with (3), we obtain the complex exponential function solution of the nonlinear dynamical system of double-chain model in DNA in the form

$$u_8(x,t)=\frac{-i\sqrt{(pkr-pw)(2kr+2w)}}{\sqrt{-mk^2r^2+mw^2}}+\frac{a_4}{Ee^{\frac{i\sqrt{p}(kx+wt)}{\sqrt{-w^2+k^2r^2}}}+\frac{-ia_4\sqrt{(mkr-mw)(kr+w)}}{\sqrt{-2pk^2r^2+2pw^2}}},$$
$$v_8(x,t)=au_8(x,t)+b. \tag{49}$$

Case 3.2. For $\alpha \neq \beta$, we have the following coefficients:

$$a_0=\frac{\sqrt{2}\left(3i\sqrt{m}\kappa\alpha+\sqrt{-m(-\kappa)^2\alpha^2}\right)}{m\sqrt{\kappa}}, a_1=a_2=a_3=0, \beta=\frac{-ia_4\sqrt{m}}{\sqrt{32\kappa}},$$
$$n=\frac{3\sqrt{2}\left(i\alpha\kappa\sqrt{m}+|\kappa\alpha|\sqrt{-m}\right)}{\sqrt{\kappa}}, p=4\alpha\left(-\alpha\kappa+\frac{3i|\kappa\alpha|\sqrt{-m}}{\sqrt{m}}\right), \tag{50}$$

where $\kappa = k^2r^2 - w^2$. If we consider (50) in (46) along with (3), we obtain the complex exponential function solution of the nonlinear dynamical system of double-chain model in DNA as follows:

$$u_9(x,t) = \frac{\sqrt{2}\big(3i\kappa\alpha\sqrt{m} + |\alpha\kappa|\sqrt{-m}\big)}{m\sqrt{\kappa}} + \frac{a_4}{Ee^{-4\alpha(kx+wt)} + \frac{ia_4\sqrt{m}}{\alpha\sqrt{32\kappa}}},$$
$$v_9(x,t) = au_9(x,t) + b, \tag{51}$$

where $\kappa = w^2 - k^2r^2$.

Case 3.3. For $\alpha \neq \beta$, we have the following coefficients:

$$a_0 = \frac{-2n}{3m}, a_1 = a_2 = 0, a_3 = 0, \alpha = \frac{n}{6\sqrt{2m\big(w^2 - k^2r^2\big)}}, \beta = \frac{-a_4m}{4\sqrt{2m\big(w^2 - k^2r^2\big)}},$$
$$p = \frac{2n^2}{9m} \quad . \tag{52}$$

Taking (52) in (46) with (3), we obtain the exponential function solution of the DNA model in the form

$$u_{10}(x,t) = \frac{-2n}{3m} + \frac{a_4}{Ee^{\frac{-\sqrt{2}n(kx+wt)}{3\sqrt{mw^2-mk^2r^2}}} + \frac{3ma_4}{2n}},$$
$$v_{10}(x,t) = au_{10}(x,t) + b. \tag{53}$$

4.2 Implementation of MEFM

If we reconsider (14) and (15) along with the help of balance principle between U'' and U^3, we obtain the following relationship between M and N;

$$N = M + 1. \tag{54}$$

This relationship gives us some new exact solutions for the DNA model in (1a) as follows:

Case 1: If we choose $M = 1$ and $N = 2$, we can write

$$U = \frac{A_0 + A_1\exp(-\Omega) + A_2\exp(2(-\Omega))}{B_0 + B_1\exp(-\Omega)} = \frac{\Upsilon}{\Psi}, \tag{55}$$

and

$$U' = \frac{\Upsilon'\Psi - \Psi'\Upsilon}{\Psi^2}, \tag{56}$$

$$U'' = \frac{\Upsilon''\Psi^3 - \Psi^2\Upsilon'\Psi' - (\Psi''\Upsilon + \Psi'\Upsilon')\Psi^2 + 2(\Psi')^2\Upsilon\Psi}{\Psi^4},$$
$$\vdots, \tag{57}$$

where $A_2 \neq 0$ and $B_1 \neq 0$. When we use (55) and (57) in (24), we get an algebraic equation for the coefficients of polynomial of $\exp(-\Omega(\xi))$. By solving, this algebraic equation yields the following coefficients.

Case 1.1:

$$A_0 = \frac{\lambda}{4A_1}\left(A_1^2 - \frac{2pB_1^2}{m}\right), A_2 = \frac{A_1}{\lambda}, B_0 = \frac{\lambda B_1}{6}\left(3 + \frac{2nB_1}{mA_1}\right),$$
$$\mu = \frac{\lambda^2}{4}\left(1 - \frac{2pB_1^2}{mA_1^2}\right), r = -\frac{i\sqrt{3mA_1 + nB_1} - \sqrt{mA_1^2 - 2w^2\lambda^2B_1^2}}{\sqrt{2}\sqrt{k^2\lambda^2B_1^2\left(3mA_1 + nB_1\right)}}. \tag{58}$$

Substituting (58) in (55) along with (16), we obtain the hyperbolic function solution of the nonlinear dynamical system of double-chain model in DNA in the form

$$u_{11}(x,t) = \frac{3p\sec h^2(f_1(x,t))\left(-mA_1^2 + 2pB_1^2\right)}{\left[\sqrt{m}A_1 + \sqrt{2p}B_1\tanh(f_1(x,t))\right][A_1f_2(x,t) + 2B_1f_3(x,t)]},$$
$$v_{11}(x,t) = au_{11}(x,t) + b, \tag{59}$$

where $f_1(x,t) = \frac{\lambda B_1\sqrt{p}(E+kx+wt)}{\sqrt{2m}A_1}$, $f_2(x,t) = 2n\sqrt{m} + 3m\sqrt{2p}\tanh(f_1(x,t))$, $f_3(x,t) = 3p\sqrt{m} + n\sqrt{2p}\tanh(f_1(x,t))$.

Case 1.2:

$$A_2 = \frac{-(\lambda\kappa - 2\mu)A_0 + \kappa\mu A_1}{2\mu^2}, B_1 = \frac{-B_0}{2\mu A_0}(\kappa A_0 - 2\mu A_1),$$
$$m = \frac{1}{A_0^2}\left(k^2r^2 - w^2\right)B_0^2(\lambda(\kappa - 2\lambda) + 2\mu), p = -\left(k^2r^2 - w^2\right)\left(\lambda^2 - 4\mu\right),$$
$$n = \frac{-3}{2A_0}B_0\left(k^2r^2 - w^2\right)(\lambda(\kappa - 2\lambda) + 4\mu), \tag{60}$$

where $\kappa = \lambda + \sqrt{\lambda^2 - 4\mu}$. When we substitute (60) into (55) along with (16), we obtain the soliton solution to the nonlinear dynamical system of double-chain model in DNA in the form:

$$u_{12}(x,t) = \frac{\sqrt{\lambda^2 - 4\mu}A_0\left(-1 + \tanh\left(\frac{1}{2}(E + kx + wt)\sqrt{\lambda^2 - 4\mu}\right)\right)}{B_0\left(\lambda + \sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{1}{2}(E + kx + wt)\sqrt{\lambda^2 - 4\mu}\right)\right)},$$
$$v_{12}(x,t) = au_{12}(x,t) + b. \tag{61}$$

Case 1.3:

$$
\begin{aligned}
A_1 &= \frac{1}{4}\left(\lambda A_2 - \kappa\sqrt{A_2}\right),\ B_1 = \frac{-3\lambda A_2 B_0 + \kappa B_0\sqrt{A_2}}{4(A_0 - \mu A_2)},\\
m &= \frac{1}{4A_2}(A_0 - \mu A_2)^{-2}\left(k^2r^2 - w^2\right)\left(-24A_0 + \left(-5\lambda^2 + 8\mu\right)A_2 + 3\lambda\kappa B_0^2\sqrt{A_2}\right),\\
n &= \frac{-3}{2A_2(-A_0 + \mu A_2)}B_0\left(k^2r^2 - w^2\right)\left(8A_0 + \lambda^2 A_2 - \lambda\kappa\sqrt{A_2}\right),\\
p &= -\frac{1}{4A_2}\left(k^2r^2 - w^2\right)\left(24A_0 + \left(\lambda^2 + 8\mu\right)A_2 - 3\lambda\kappa\sqrt{A_2}\right),
\end{aligned}
\tag{62}
$$

where $\kappa = \sqrt{48A_0 + \left(\lambda^2 - 16\mu\right)A_2}$. If we consider (62) in (55) along with (16), we obtain the new hyperbolic and rational function solution to the nonlinear dynamical system of double-chain model in DNA in the form:

$$
\begin{aligned}
u_{13}(x,t) =& \frac{2}{B_0}\left(A_0 + \frac{\mu}{2f_2(x,t)^2}\left(-\sqrt{A_2}\kappa f_2(x,t) - A_2\left(\lambda^2 - 8\mu + f_2(x,t)\right)\right)\right)\\
&\times\left(2 + \frac{\mu\left(3\lambda A_2 - \sqrt{A_2}\kappa\right)}{(A_0 - \mu A_2)f_2(x,t)}\right)^{-1},\\
v_{13}(x,t) =& a u_{13}(x,t) + b,
\end{aligned}
\tag{63}
$$

where $f_1(x,t) = \tanh\left(\frac{1}{2}(E + kx + wt)\sqrt{\lambda^2 - 4\mu}\right)$, $\kappa = \sqrt{48A_0 + \left(\lambda^2 - 16\mu\right)A_2}$, $f_2(x,t) = \lambda + \sqrt{\lambda^2 - 4\mu}\, f_1(x,t)$.

Case 1.4:

$$
\begin{aligned}
A_1 &= \frac{1}{6}\left(-\lambda A_1 + \frac{2A_1^2}{A_2} + 2A_2\right),\ B_1 = \frac{6A_2(-A_1 + \lambda A_2)B_0}{-2A_1^2 + \lambda A_1A_2 + 4\mu A_2^2},\\
m &= -72\left(k^2r^2 - w^2\right)(A_1 - \lambda A_2)B_0^2\left(-2A_1^2 + \lambda A_1A_2 + 4\mu A_2^2\right)^{-2},\\
n &= \frac{-6}{A_2}\left(k^2r^2 - w^2\right)\left(4A_1^2 - 8\lambda A_1A_2 + \left(3\lambda^2 + 8\mu\right)A_2^2\right)B_0\left(-2A_1^2 + \lambda A_1A_2 + 4\mu A_2^2\right)^{-1},\\
p &= -A_2^{-2}\left(k^2r^2 - w^2\right)\left(2A_1^2 - 4\lambda A_1A_2 + \left(\lambda^2 + 4\mu\right)A_2^2\right).
\end{aligned}
\tag{64}
$$

Substituting (64) in (55) along with (16), we obtain the rational function solution to the DNA model in the form:

$$
\begin{aligned}
u_{14}(x,t) &= \frac{\frac{A_1^2}{A_2} + \mu A_2\left(1 + \frac{12\mu}{f_1^2(x,t)}\right) + A_1\left(-\frac{\lambda}{2} - \frac{6\mu}{f_1(x,t)}\right)}{3B_0\left(1 + \frac{12\mu A_2(A_1 - \lambda A_2)}{\left(-2A_1^2 + \lambda A_1A_2 + 4\mu A_2^2\right)f_1(x,t)}\right)},\\
v_{14}(x,t) &= a u_{14}(x,t) + b,
\end{aligned}
\tag{65}
$$

where $f_1(x,t) = \lambda + \sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{1}{2}(E + kx + wt)\sqrt{\lambda^2 - 4\mu}\right)$.

Case 1.5:

$$
\begin{aligned}
A_1 &= \frac{1}{4}(\lambda A_2 - \varpi),\ B_1 = \frac{-(3\lambda A_2 + \varpi)B_0}{4(A_0 - \mu A_2)},\\
m &= \frac{-1}{4A_2(A_0 - \mu A_2)^2}\left(k^2r^2 - w^2\right)\left(24A_0 + \left(5\lambda^2 - 8\mu\right)A_2 + 3\lambda\varpi\right)B_0^2,\\
n &= \frac{-3}{2A_2}\left(k^2r^2 - w^2\right)\left(8A_0 + \lambda^2 A_2 + \lambda\varpi\right)B_0(-A_0 + \mu A_2)^{-1},\\
p &= -\frac{1}{4}A_2^{-1}\left(k^2r^2 - w^2\right)\left(24A_0 + \left(\lambda^2 + 8\mu\right)A_2 + 3\lambda\varpi\right),
\end{aligned} \tag{66}
$$

where $\varpi = \sqrt{A_2}\sqrt{48A_0 + \left(\lambda^2 - 16\mu\right)A_2}$. Considering (66) in (55) along with (16), we obtain the following rational function solution to the nonlinear dynamical system of double-chain model in DNA in the form:

$$
\begin{aligned}
u_{15}(x,t) &= \frac{A_0 + \frac{\mu}{2}\left(\varpi f_1(x,t) - A_2\left(\lambda^2 - 8\mu + \lambda(f_1(x,t) - \lambda)\right)\right)f_1^{-2}(x,t)}{B_0 + \frac{\mu(3\lambda A_2 + \varpi)B_0}{2(A_0 - \mu A_2)f_1(x,t)}},\\
v_{15}(x,t) &= au_{15}(x,t) + b,
\end{aligned} \tag{67}
$$

where $\varpi = \sqrt{A_2}\sqrt{48A_0 + \left(\lambda^2 - 16\mu\right)A_2}$, $f_1(x,t) = \lambda + \sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{1}{2}(E + kx + wt)\sqrt{\lambda^2 - 4\mu}\right)$.

5 A Biological Perspective Point of View on the Obtained Results

As we mentioned in section "Preliminary Remarks on DNA", $u(x,t)$ is the difference of the longitudinal displacements of the bottom and top strands while $v(x,t)$ is the difference of the transverse displacements of the bottom and top strands [26–28]. In this chapter, we have found several interesting solutions to the nonlinear dynamics of DNA. This model consists of two long elastic homogeneous strands connected with each other by an elastic membrane. In this model, we study the longitudinal and transverse motions [27]. Therefore, it is estimated that the u_1, u_7, u_{11}, u_{12}, u_{14}, and u_{15} solutions are new positions of longitudinal displacements of strands. Moreover, we can observe the corresponding simulations in Figs. 1, 7, 8, 11, 14 and 15. We also found that the u_2, u_3, u_5, u_6, u_8 and u_9 solutions are complex so that we can observe new positions of two long elastic homogeneous strands. As can be seen also from simulations in Figs. 2, 3, 5, 6, 8 and 9, it is also predicted that the u_4 and u_{10} solutions along with Figs. 4 and 19 are almost basic and smooth longitudinal displacements

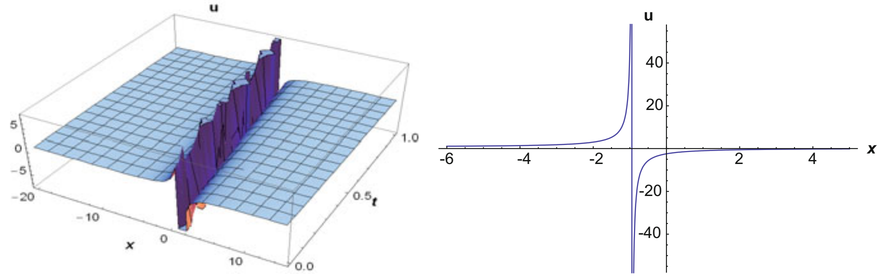

Fig. 1 3D and 2D surfaces of the exponential solution (30) by considering the values $p = 2, m = 5, E = 1, a_2 = 0.6, w = 3, r = 0.5, k = 0.7$ for 3D graphics and $t = 0.5$ for 2D surfaces

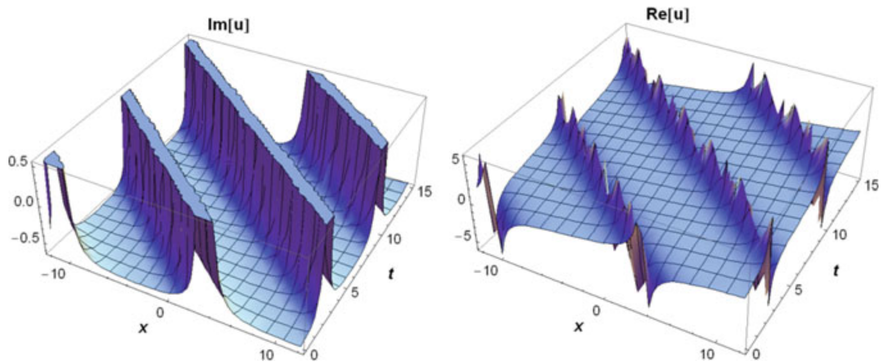

Fig. 2 3D surfaces of the complex exponential function solution (32) by considering the values $p = 1, E = 2, a_2 = 3, w = 0.2, r = 3, k = 0.1, \beta = 5, -12 < x < 12, 0 < t < 15$.

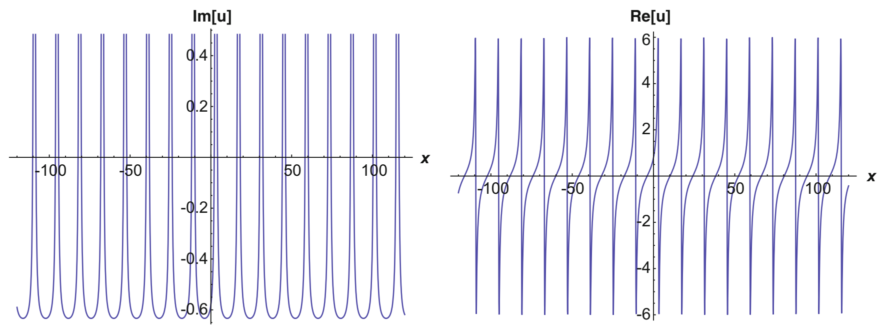

Fig. 3 2D surfaces of the complex exponential function solution (32) by considering the values $p = 1, E = 2, a_2 = 3, w = 0.2, r = 3, k = 0.1, \beta = 5, t = 0.2, -12 < x < 12$.

of two long elastic homogeneous strands. When it comes to the u_{13} solution with Fig. 13, it is made inferences that the longitudinal displacements of two long elastic homogeneous strands are very strict and, even, break off both strands each other.

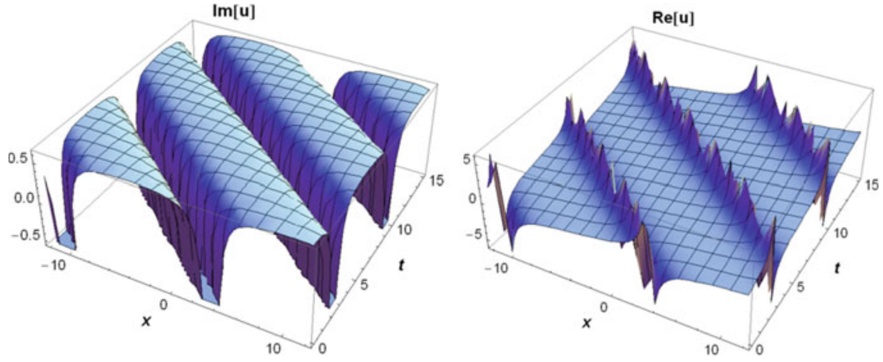

Fig. 4 3D surfaces of the complex exponential function solution (34) by considering the values $p = 1$, $E = 2$, $a_2 = 3$, $w = 0.2$, $r = 3$, $k = 0.1$, $\beta = 5$, $-12 < x < 12$, $0 < t < 15$.

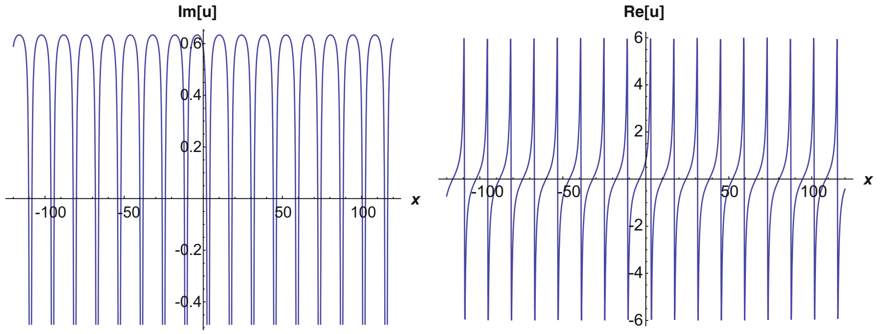

Fig. 5 2D surfaces of the complex exponential function solution (34) by considering the values $p = 1$, $E = 2$, $a_2 = 3$, $w = 0.2$, $r = 3$, $k = 0.1$, $\beta = 5$, $t = 0.2$, $-120 < x < 120$.

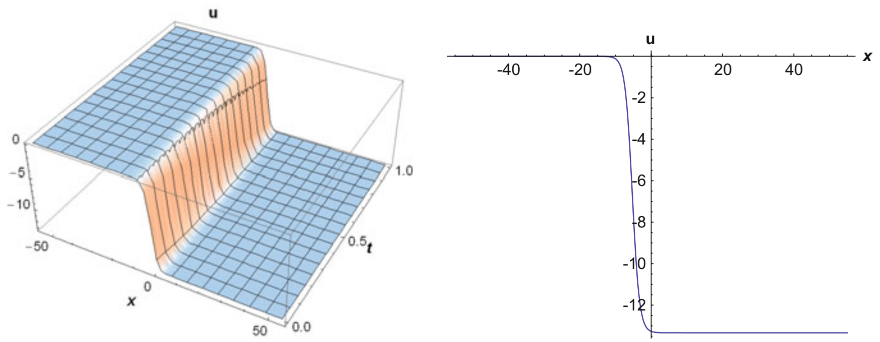

Fig. 6 3D and 2D surfaces of the exponential solution (39) by considering the values $n = 4$, $m = 0.2$, $\beta = 0.3$, $E = 4$, $w = 3$, $r = 0.5$, $k = 0.7$, $-55 < x < 55$, $0 < t < 1$ and $t = 0.8$ for 2D surfaces

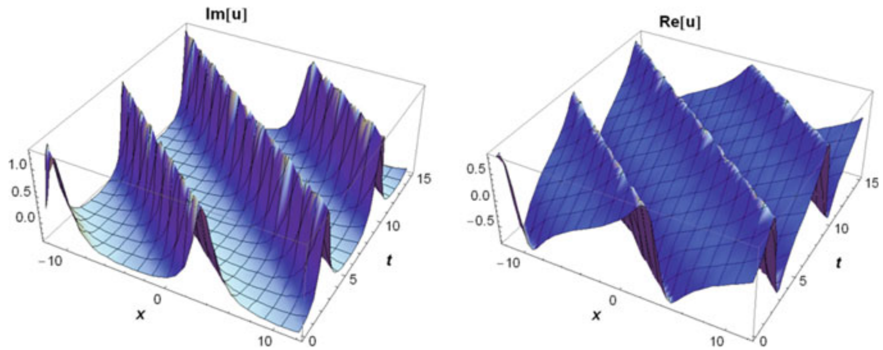

Fig. 7 3D surfaces of the complex exponential function solution (41) by considering the values $p = 1, E = 2, a_2 = 3, w = 0.2, r = 3, k = 0.1, \beta = 5, -12 < x < 12, 0 < t < 15$.

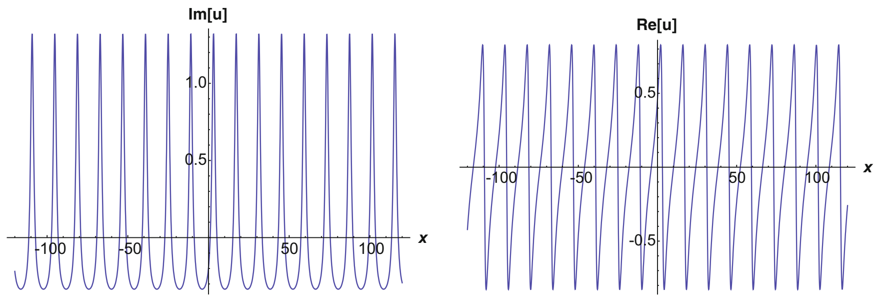

Fig. 8 2D surfaces of the complex exponential function solution (41) by considering the values $p = 1, E = 2, a_2 = 3, w = 0.2, r = 3, k = 0.1, \beta = 5, t = 0.2, -12 < x < 12$.

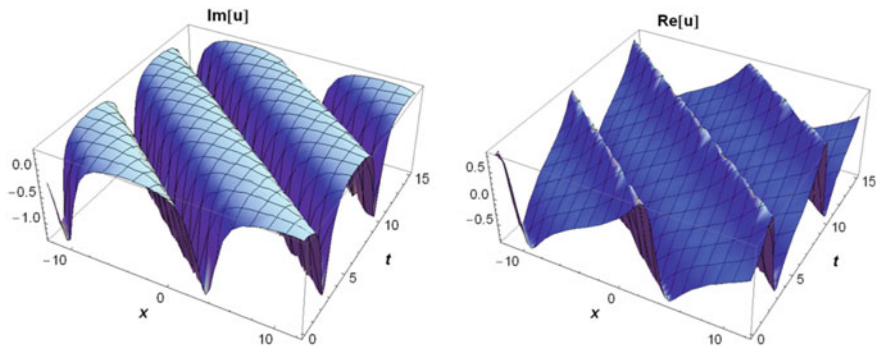

Fig. 9 3D surfaces of the complex exponential function solution (43) by considering the values $p = 1, E = 2, a_3 = 3, w = 0.2, r = 3, k = 0.1, \beta = 5, -12 < x < 12, 0 < t < 15$.

6 Discussions, Remarks and Conclusions

Two powerful mathematical and analytical tools described as Bernoulli sub-equation function method and modified exp $(-\Omega(\xi))$-expansion function method have been applied to the nonlinear dynamical system arising in a new double-chain model in DNA. We have obtained several hundred coefficients which are giving novel solutions to the model considered. By choosing several cases of them, we have found some new exponential, rational and complex function solutions to the nonlinear dynamical system arising in a new double-chain model in DNA. It has been observed that all solutions fulfil (21) obtained by considering suitable transformations from (1a, b).

When we compare our results obtained by using two methods, first of all, we can start with BSEFM. We can say that the solutions of $u_2(x,t),\ u_3(x,t), u_5(x,t),\ u_6(x,t), u_8(x,t)$ and $u_9(x,t)$ are new complex exponential rational function solutions to the nonlinear dynamical system arising in a new double-chain model in DNA comparing with the results in [28]. The solutions of $u_1(x,t),\ u_4(x,t), u_7(x,t)$ and $u_{10}(x,t)$ are exponential rational function solutions to the nonlinear dynamical system arising in a new double-chain model in DNA. Furthermore, if we can consider more values of M and N, of course, we can find more different solutions to the model considered in this manuscript.

Secondly, when it comes to the modified exp $(-\Omega(\xi))$-expansion function method, firstly, this method is based on extended version of the exp $(-\Omega(\xi))$-expansion function method [28]. Therefore, it has one more parameter like M in Eq. (14). This gives much more coefficient for the system obtained by putting necessary derivations of the solution form being Eq. (14) into Eq. (24). For example, if we choose $M = 2$ and $N = 3$, we can write as follows:

$$U = \frac{A_0 + A_1\exp(-\Omega) + A_2\exp(2(-\Omega)) + A_3\exp(3(-\Omega))}{B_0 + B_1\exp(-\Omega) + B_2\exp(-2\Omega)} = \frac{\Upsilon}{\Psi}, \tag{68}$$

and

$$U' = \frac{\Upsilon'\Psi - \Psi'\Upsilon}{\Psi^2}, \tag{69}$$

$$U'' = \frac{\Upsilon''\Psi^3 - \Psi^2\Upsilon'\Psi' - (\Psi''\Upsilon + \Psi'\Upsilon')\Psi^2 + 2(\Psi')^2\Upsilon\Psi}{\Psi^4},$$

$$\vdots, \tag{70}$$

where $A_3 \neq 0$ and $B_2 \neq 0$. When we use (68) and (70) in (24), we get an algebraic equation from the coefficients of polynomial of $\exp(-\Omega(\xi))$. By solving this algebraic equation, we can obtain much more different solutions to the nonlinear dynamical system arising in a new double-chain model in DNA. This parameter M is one of the most fundamental properties of MEFM when we reconsider exp $(-\Omega(\xi))$-expansion function method.

Thirdly, comparing with the results found by Mahmoud A. E. Abdelrahman et al. in [28], they have gained some exponential solutions to the model considered in this chapter. However, the hyperbolic function solutions such as $u_{11}(x,t)$, $u_{12}(x,t)$, $u_{13}(x,t)$, $u_{14}(x,t)$ and $u_{15}(x,t)$ found by MEFM in this chapter are fully different and new. Comparing the results obtained by the two methods, the solutions from u_1 to u_{10} obtained by using IBSEFM are exponential and complex function solutions including new replacement position of longitudinal displacements of two long elastic homogeneous strands. On the other hand, the solutions obtained with the help of MEFM are hyperbolic and complex function solutions to the DNA model.

Consequently, as can be seen on Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19 and 20, BSEFM and MEFM are a powerful and reliable tools for obtaining novel soliton hyperbolic and complex function solutions of such systems. Therefore, we think that these methods can also be conducted to other nonlinear evaluation equations and inequalities.

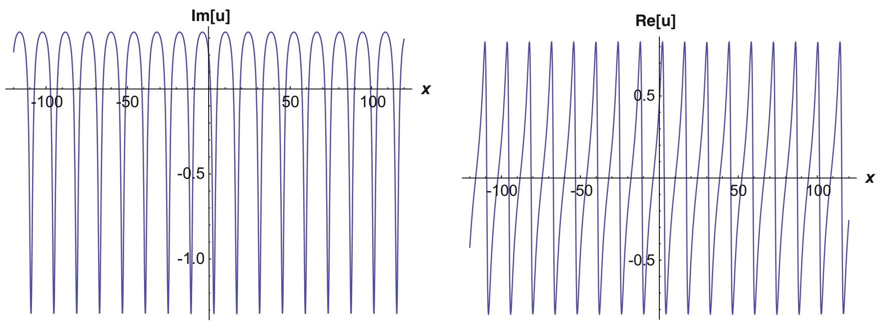

Fig. 10 2D surfaces of the complex exponential function solution (43) by considering the values $p = 1, E = 2, a_3 = 3, w = 0.2, r = 3, k = 0.1, \beta = 5, t = 0.2, -120 < x < 120$.

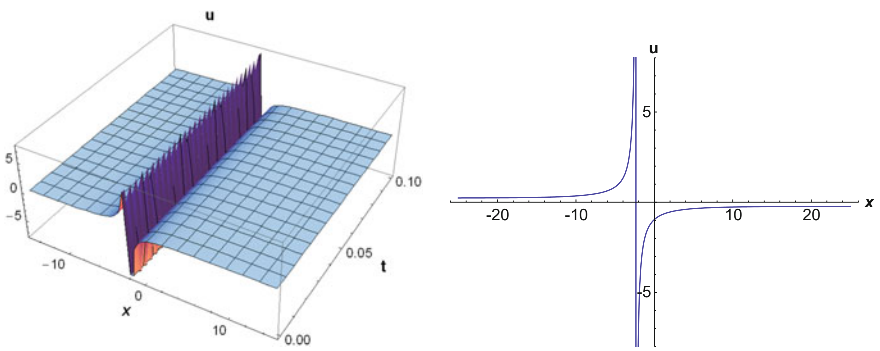

Fig. 11 3D and 2D surfaces of the exponential solution (45) by considering the values $\alpha = 0.1, m = 0.2, a_3 = 0.3, E = 0.4, w = 0.6, r = 0.5, k = 0.7, -15 < x < 15, 0 < t < 0.1$ and $t = 0.04$ for 2D surfaces

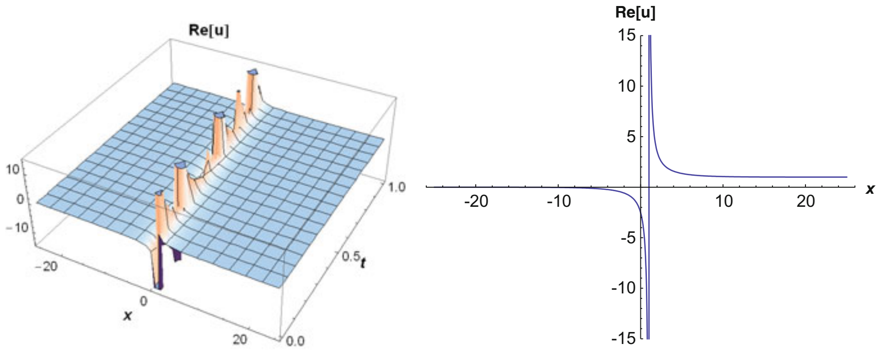

Fig. 12 3D and 2D surfaces of the exponential solution (49) by considering the values $p = 1, m = 2, k = 3, E = 5, w = 15, r = 4, a_4 = 7, -25 < x < 25, 0 < t < 1$ and $t = 0.001$ for 2D surfaces

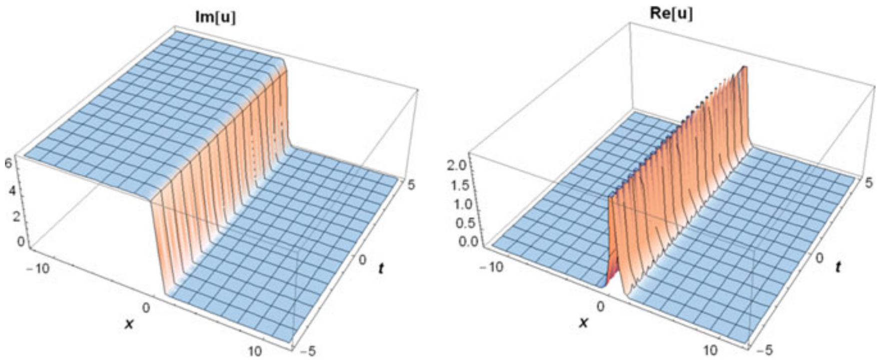

Fig. 13 3D surfaces of the complex exponential function solution (51) by considering the values $m = 9, E = 0.2, a_4 = 0.3, w = 0.04, r = 3, k = 2, \alpha = 0.6, -12 < x < 12, -5 < t < 5.$

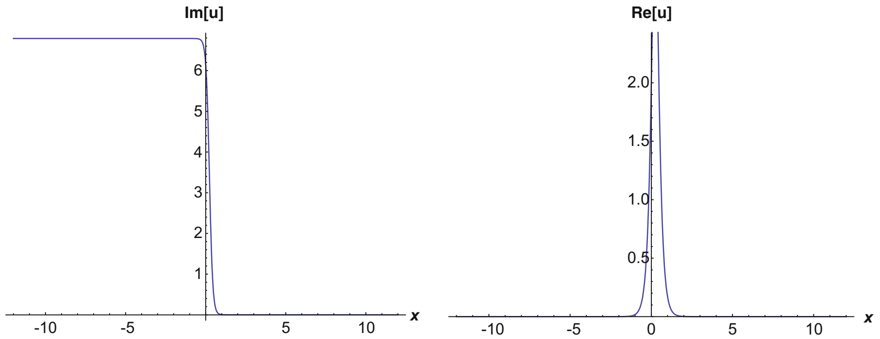

Fig. 14 2D surfaces of the complex exponential function solution (51) by considering the values $m = 9, E = 0.2, a_4 = 0.3, w = 0.04, r = 3, k = 2, \alpha = 0.6, t = 4, -12 < x < 12.$

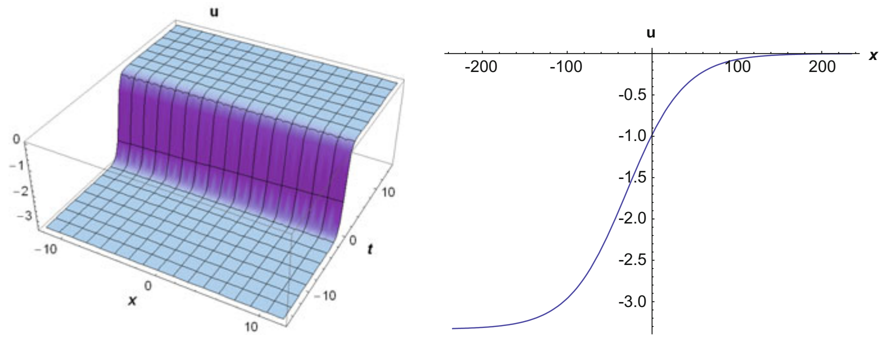

Fig. 15 3D and 2D surfaces of the exponential solution (53) by considering the values $m = 1$, $E = 2$, $a_4 = 3$, $w = 8$, $r = 2$, $k = 0.1$, $n = 5$, $-12 < x < 12$, $-15 < t < 15$. and $t = 0.71$ and $-235 < x < 235$ for 2D surfaces

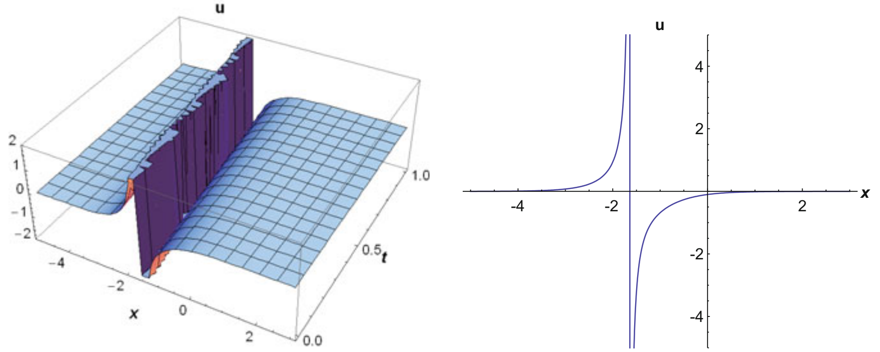

Fig. 16 3D and 2D surfaces of the exponential solution (59) by considering the values $m = 5$, $n = 3$, $p = 2$, $k = 4$, $E = 1$, $A_1 = 0.6$, $w = 3$, $B_1 = 0.5$, $\lambda = 0.7$, $-5 < x < 3$, $0 < t < 1$ for 3D graphics and $t = 0.01$ for 2D surfaces

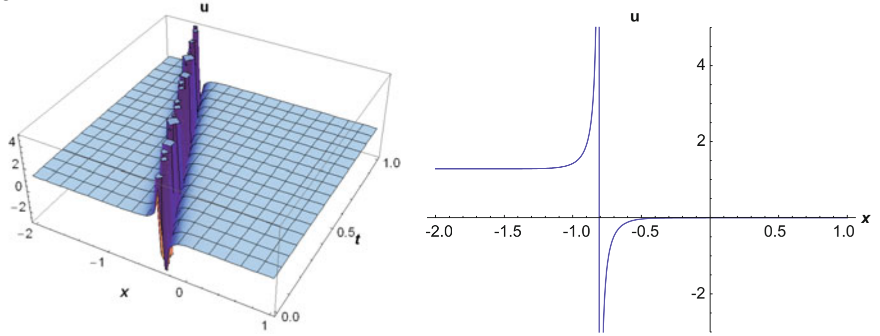

Fig. 17 3D and 2D surfaces of the exponential solution (61) by considering the values $\lambda = 1$, $\mu = -2$, $A_0 = 3$, $k = 4$, $w = 5$, $E = 1$, $A_1 = 6$, $B_0 = 7$, $-2 < x < 1$, $0 < t < 1$ for 3D graphics and $t = 0.4$ for 2D surfaces

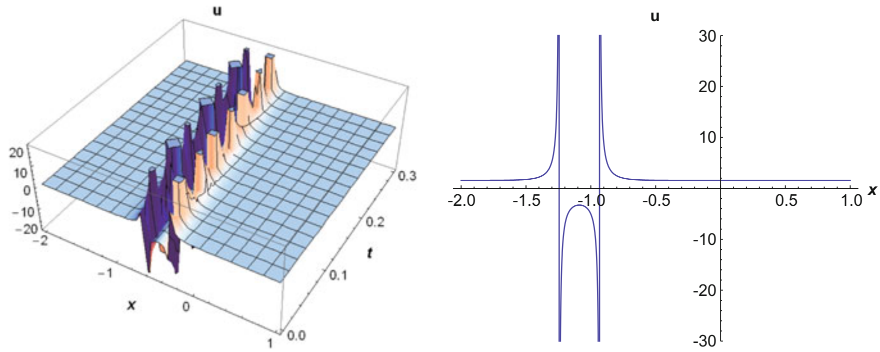

Fig. 18 3D and 2D surfaces of the exponential solution (63) by considering the values $\lambda = 1$, $\mu = -2$, $A_0 = 3$, $k = 4$, $w = 5$, $E = 1$, $A_1 = 6$, $B_0 = 7$, $A_2 = 4$, $-2 < x < 1$, $0 < t < 0.3$ for 3D graphics and $t = 0.5$ for 2D surfaces

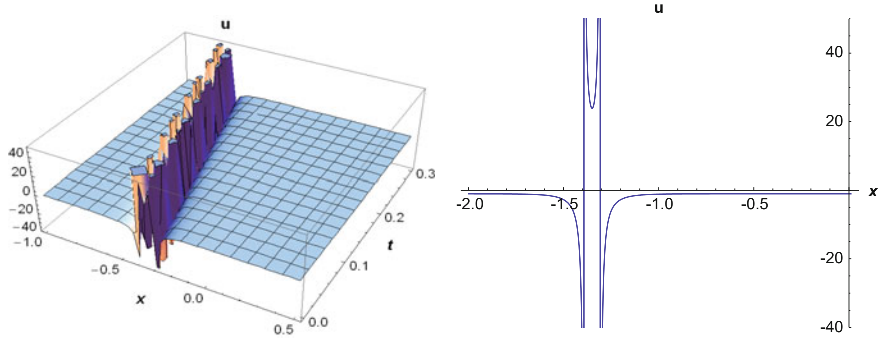

Fig. 19 3D and 2D surfaces of the exponential solution (65) by considering the values $\lambda = 1$, $\mu = -2$, $A_0 = 3$, $k = 4$, $w = 5$, $E = 1$, $A_1 = 6$, $B_0 = 7$, $A_2 = 4$, $-2 < x < 1$, $0 < t < 0.3$ for 3D graphics and $t = 0.5$ for 2D surfaces

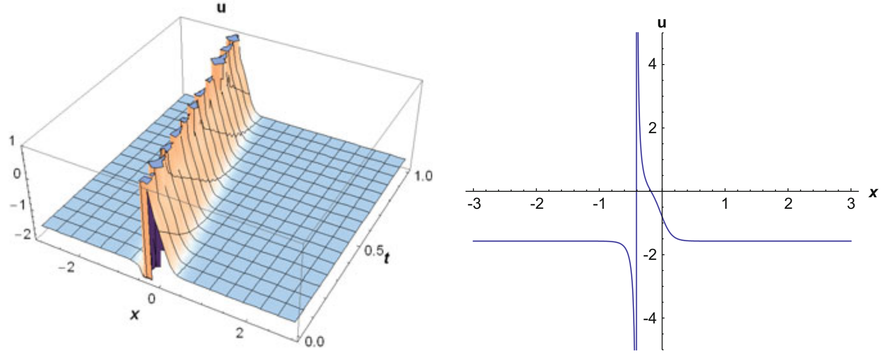

Fig. 20 3D and 2D surfaces of the exponential solution (67) by considering the values $\lambda = 1$, $\mu = -2$, $A_0 = 3$, $k = 4$, $w = 5$, $E = 1$, $B_0 = -7$, $A_2 = 4$, $-3 < x < 3$, $0 < t < 0.3$ for 3D graphics and $t = 0.08$ for 2D surfaces

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A Variety of Nonlinear Retarded Integral Inequalities of Gronwall Type and Their Applications

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1 Introduction

Integral inequalities that provide explicit bounds on unknown functions have proved to be useful in the study of qualitative properties of the solutions of differential, integral, and integro-differential equations. The Gronwall inequality [1] states that if f and u are real-valued nonnegative continuous functions defined on $I = [0, \infty)$, with a positive constant u_0, then

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds, \forall t \in I, \tag{1.1}$$

implies

$$u(t) \leq u_0 \exp\left(\int_0^t f(s)ds\right), \forall t \in I.$$

As a generalization of (1.1), Bellman [2] proved that: If $u,\ f,\ a \in C(I, I)$ and a is a nondecreasing, then the inequality

$$u(t) \leq a(t) + \int_0^t f(s)u(s)ds, \forall t \in I, \tag{1.2}$$

implies

$$u(t) \leq a(t) \exp\left(\int_0^t f(s)ds\right), \forall t \in I.$$

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Because of its fundamental importance, over the years, many generalizations and analogous results of Gronwall–Bellman inequality have been established, such as [3–13].

A fairly general linear version of the Gronwall–Bellman inequality established by Pachpatte [14] is given in the following theorem:

Theorem 1.1 *Let u, f and g be nonnegative continuous functions defined on I, for which the inequality*

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s g(\sigma)u(\sigma)d\sigma\right)ds, \quad t \in I,$$

holds, where u_0 is a nonnegative constant. Then

$$u(t) \leq u_0\left[1 + \int_0^t f(s)\exp\left(\int_0^s [f(\sigma) + g(\sigma)]d\sigma\right)ds\right], \quad t \in I.$$

Remark 1.1 It is interesting to note that, in the special case when $g(t) = 0$, the above inequality reduces to Bellman's inequality (1.1).

In the following two theorems, we present some useful generalizations of Theorem 1.1 given by Pachpatte [15–17].

Theorem 1.2 *Let u, f, g, h and p be nonnegative continuous functions defined on I, and u_0 be a nonnegative constant.*

(1) *If the inequality*

$$u(t) \leq u_0 + \int_0^t [f(s)u(s) + p(s)]ds + \int_0^t f(s)\left(\int_0^s g(\sigma)u(\sigma)d\sigma\right)ds, \quad (1.3)$$

holds for $t \in I$, then

$$u(t) \leq u_0 + \int_0^t \left[p(s) + f(s)\left\{u_0 \exp\left(\int_0^s [f(\sigma) + g(\sigma)]d\sigma\right) + \int_0^s p(\sigma)\exp\left(\int_\sigma^s [f(\tau) + g(\tau)]d\tau\right)d\sigma\right\}\right]ds,$$

for $t \in I$.

(2) *If the inequality*

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s [g(\sigma)u(\sigma) + p(t)]d\sigma\right)ds,$$

holds for $t \in I$, then

$$u(t) \le u_0 + \int_0^t f(s)\Big\{u_0 \exp\Big(\int_0^s [f(\sigma)+g(\sigma)]d\sigma\Big) + \int_0^t p(\sigma)\exp\Big(\int_\sigma^s [f(\tau)+g(\tau)]d\tau\Big)d\sigma\Big\}ds,$$

for $t \in I$.

(3) *If the inequality*

$$u(t) \le u_0 + \int_0^t f(s)u(s)ds + \int_0^t g(s)\Big(u(s) + \int_0^s h(\sigma)u(\sigma)d\sigma\Big)ds,$$

holds for $t \in I$, then

$$u(t) \le u_0\Big[\exp\Big(\int_0^t f(\sigma)d\sigma\Big) + \int_0^t g(s) \times \exp\Big(\int_0^s [f(\sigma)+g(\sigma)+h(\sigma)]d\sigma\Big)\exp\Big(\int_s^t f(\sigma)d\sigma\Big)ds\Big],$$

for $t \in I$.

(4) *If the inequality*

$$u(t) \le h(t) + p(t)\Big[\int_0^t f(s)u(s)ds + \int_0^t f(s)p(s)\Big(\int_0^s g(\sigma)u(\sigma)d\sigma\Big)ds\Big],$$

holds for $t \in I$, then

$$u(t) \le h(t) + p(t)\Big[\int_0^t f(s)\Big\{h(s) + p(s)\int_0^s h(\sigma)[f(\sigma)+g(\sigma)] \times \exp\Big(\int_\sigma^s p(\tau)[f(\tau)+g(\tau)]d\tau\Big)d\sigma\Big\}ds\Big],$$

for $t \in I$.

In some certain problems, the bounds obtained by the inequalities mentioned above are not directly applicable, and it is desirable to prove some new integral inequalities that will be equally important in order to achieve a diversity of desired goals. In the present chapter, we prove the retarded version of Bellman and Pachpatte-like inequalities mentioned above. We introduce some applications of some of our inequalities to study the qualitative properties of nonlinear retarded differential, integral, and integro-differential equations.

Throughout this chapter, unless otherwise stated, all the functions which appear in the inequalities are assumed to be real-valued in their domains of definitions. $\mathbb{R}$

denotes the set of real numbers, $I = [0, \infty)$, $\mathbb{R}_+ = (0, \infty)$, $I_1 = [a, b]$, $\mathbb{R}_1 = [1, \infty)$ and $\mathcal{C}(I, I)$ denotes the set of all nonnegative real-valued continuous functions from I into I and $\mathcal{C}^1(I, I)$ denotes the set of all nonnegative real-valued continuously differentiable functions from I into I. The ordinary first-order derivative of a function u defined for $t \in \mathbb{R}$ is denoted by u' and $\dot{u}$ or $\frac{du}{dt}$, and the higher-order derivatives are denoted in the usual way. The notations, definitions, and symbols used throughout this thesis are standard and explained if necessary at appropriate places.

2 Nonlinear Retarded Integral Inequalities

We prove the following useful nonlinear retarded generalization of Gronwall–Bellman's inequality. The results in this section are adapted from [5, 18].

Theorem 2.1 *Let u, g, $h \in \mathcal{C}(I_1, \mathbb{R}_+)$, and $f \in \mathcal{C}(I_1, \mathbb{R}_+)$, $\alpha \in \mathcal{C}^1(I_1, I_1)$ be nondecreasing functions with $\alpha(a) = a$, $\alpha(t) \leq t$ on I_1. If the inequality*

$$u(t) \leq f(t) + \int_a^{\alpha(t)} g(s)u(s)ds + \int_a^{\alpha(t)} g(s)u(s)\left[u(s) + \int_a^{\alpha(s)} h(\lambda)u(\lambda)d\lambda\right]ds, \tag{2.1}$$

holds for all $t \in I_1$. Then

$$u(t) \leq f(t)\exp\left(\int_a^{\alpha(t)} g(s)(1 + f(s)\Theta_1(s))ds\right), \forall\, t \in I_1, \tag{2.2}$$

where

$$\Theta_1(t) = \frac{\exp\left(\int_a^{\alpha(t)}[g(s) + h(s)]ds\right)}{1 - \int_a^{\alpha(t)} g(s)f(s)\exp\left(\int_a^s[g(\tau) + h(\tau)]d\tau\right)ds}, \forall\, t \in I_1, \tag{2.3}$$

such that

$$\int_a^{\alpha(t)} g(s)f(s)\exp\left(\int_a^{\alpha(s)}[g(\tau) + h(\tau)]d\tau\right)ds < 1, \forall\, t \in I_1.$$

Proof Since f is a positive, monotonic, nondecreasing function, we observe from (2.1) that

$$\begin{aligned}\frac{u(t)}{f(t)} \leq 1 + \int_a^{\alpha(t)} g(s)\frac{u(s)}{f(s)}ds + \int_a^{\alpha(t)} g(s)f(s)\frac{u(s)}{f(s)}\bigg[\frac{u(s)}{f(s)}\\ + \int_a^{\alpha(s)} h(\lambda)\frac{u(\lambda)}{f(\lambda)}d\lambda\bigg]ds,\end{aligned}$$

for all $t \in I_1$. Let

$$r_2(t) = \frac{u(t)}{f(t)}, \forall\, t \in I_1 \qquad r_2(a) \leq 1, \tag{2.4}$$

hence

$$r_2(t) \leq 1 + \int_a^{\alpha(t)} g(s) r_2(s) ds + \int_a^{\alpha(t)} g(s) f(s) r_2(s) \Bigg[r_2(s) + \int_a^{\alpha(s)} h(\lambda) r_2(\lambda) d\lambda \Bigg] ds,$$

for all $t \in I_1$. Let V equals the right-hand side in the above inequality, we have

$$r_2(t) \leq V(t),\; r_2(\alpha(t)) \leq V(\alpha(t)) \leq V(t),\; V(a) = 1, \forall\, t \in I_1. \tag{2.5}$$

Differentiating V with respect to t, and using (2.5), we obtain

$$V'(t) \leq g(\alpha(t))\alpha'(t) V(t)[1 + f(\alpha(t))\gamma(t)], \forall\, t \in I_1, \tag{2.6}$$

where $\gamma(t) = V(t) + \int_a^{\alpha(t)} h(s)V(s)ds$, hence $\gamma(a) = 1$, and $V(t) \leq \gamma(t)$. Differentiating $\gamma(t)$ with respect to t, and using (2.6), we get

$$\gamma'(t) \leq [g(\alpha(t)) + h(\alpha(t))]\alpha'(t)\gamma(t) + g(\alpha(t))\alpha'(t) f(\alpha(t))\gamma^2(t), \forall\, t \in I_1,$$

but $\gamma(t) > 0$, thus from the above inequality, we get

$$\gamma^{-2}(t)\gamma'(t) - [g(\alpha(t)) + h(\alpha(t))]\alpha'(t)\gamma^{-1}(t) \leq g(\alpha(t))\alpha'(t) f(\alpha(t)), \forall\, t \in I_1. \tag{2.7}$$

If we let

$$l(t) = \gamma^{-1}(t), \forall\, t \in I_1, \tag{2.8}$$

then we get $l(a) = 1$ and $\gamma^{-2}\gamma'(t) = -l'(t)$, thus from (2.7) we have

$$l'(t) + [g(\alpha(t)) + h(\alpha(t))]\alpha'(t) l(t) \geq -g(\alpha(t))\alpha'(t) f(\alpha(t)).$$

The above inequality implies the estimation for $l(t)$ such that

$$l(t) \geq \frac{1 - \int_a^{\alpha(t)} g(s) f(s) \exp\left(\int_a^s [g(\tau) + h(\tau)] d\tau \right) ds}{\exp\left(\int_a^{\alpha(t)} [g(s) + h(s)] ds \right)}, \forall\, t \in I_1.$$

Then from the above inequality in (2.8), we have

$$\gamma(t) \le \Theta_1(t), \forall\, t \in I_1,$$

where $\Theta_1(t)$ as defined in (2.3), thus from (2.6) and the above inequality, we have

$$V'(t) \le g(\alpha(t))\alpha'(t)V(t)[1 + f(t)\Theta_1(t)], \forall\, t \in I_1.$$

Integrating the above inequality from a to t, and making the change of variable yield

$$V(t) \le \exp\left(\int_a^{\alpha(t)} g(s)(1 + f(s)\Theta_1(s)ds\right), \forall\, t \in I_1.$$

Using the above inequality and (2.5) in (2.4), we get the required inequality in (2.2). The proof is complete. □

We prove the following generalization of Theorem 2.1.

Theorem 2.2 *Let u, g, $h \in \mathcal{C}(I, I)$ be nonnegative functions, and f be a positive, monotonic, nondecreasing function. We suppose that φ, φ', $\alpha \in \mathcal{C}^1(I, I)$ are increasing functions and $\frac{\varphi(u(t))}{f(t)} \le \varphi(\frac{u(t)}{f(t)})$, with $\varphi'(t) \le k, \alpha(t) \le t, \alpha(0) = 0$, for all $t \in I$; k, u_0 be positive constants. If the inequality*

$$\begin{aligned} u(t) \le & f(t) + \int_0^{\alpha(t)} g(s)\varphi(u(s))ds \\ & + \int_0^{\alpha(t)} g(s)\varphi(u(s))\left[\varphi(u(s)) + \int_0^s h(\lambda)\varphi(u(\lambda))d\lambda\right]ds, \end{aligned} \tag{2.9}$$

holds for all $t \in I$. Then

$$u(t) \le f(t)\Phi^{-1}\left(\Phi(1) + \int_0^{\alpha(t)} g(s)[1 + f(s)\Theta(\alpha^{-1}(s))]ds\right), \tag{2.10}$$

for all $t \in I$, where Φ as defined in (2.40) and

$$\Theta(t) = \frac{\exp\left(\int_0^{\alpha(t)}[kg(s) + h(s)]ds\right)}{\varphi^{-1}(1) - \int_0^{\alpha(t)} kg(s)f(s)\exp\left(\int_0^s[kg(\tau) + h(\tau)]d\tau\right)ds}, \forall\, t \in I, \tag{2.11}$$

such that

$$\int_0^{\alpha(t)} g(s)f(s)\exp\left(\int_0^s [g(\tau) + h(\tau)]d\tau\right)ds < \varphi^{-1}(1), \forall\, t \in I.$$

Proof Since f is a positive, monotonic, nondecreasing function, we observe from (2.9) that

$$\frac{u(t)}{f(t)} \leq 1 + \int_0^{\alpha(t)} g(s)\frac{\varphi(u(s))}{f(s)}ds + \int_0^{\alpha(t)} g(s)f(s)\frac{\varphi(u(s))}{f(s)}\left[\frac{\varphi(u(s))}{f(s)} + \int_0^s h(\lambda)\frac{\varphi(u(\lambda))}{f(\lambda)}d\lambda\right]ds,$$

for all $t \in I$. By the relation $\frac{\varphi(u(t))}{f(t)} \leq \varphi(\frac{u(t)}{f(t)})$, then the above inequality can be written as

$$\frac{u(t)}{f(t)} \leq 1 + \int_0^{\alpha(t)} g(s)\varphi(\frac{u(t)}{f(t)})ds + \int_0^{\alpha(t)} g(s)f(s)\varphi(\frac{u(t)}{f(t)})\left[\varphi(\frac{u(t)}{f(t)}) + \int_0^s h(\lambda)\varphi(\frac{u(\lambda)}{f(\lambda)})d\lambda\right]ds,$$

for all $t \in I$. Let

$$r(t) = \frac{u(t)}{f(t)}, \forall\, t \in I, \qquad r(0) \leq 1, \tag{2.12}$$

hence

$$\begin{aligned} r(t) \leq & 1 + \int_0^{\alpha(t)} g(s)\varphi(r(s))ds \\ & + \int_0^{\alpha(t)} g(s)f(s)\varphi(r(s))\left[\varphi(r(s)) + \int_0^s h(\lambda)\varphi(r(\lambda))d\lambda\right]ds, \end{aligned} \tag{2.13}$$

for all $t \in I$. Let V denotes the function on the right-hand side of (2.13), which is a nonnegative and nondecreasing function on I with $V(0) = 1$. Then (2.13) is equivalent to

$$r(t) \leq V(t), r(\alpha(t)) \leq V(\alpha(t)) \leq V(t), \qquad \forall t \in I. \tag{2.14}$$

Differentiating V with respect to t, and using (2.14), we get

$$V'(t) \leq g(\alpha(t))\alpha'(t)\varphi(V(t))[1 + f(\alpha(t))\gamma(t)], \forall\, t \in I, \tag{2.15}$$

where $\gamma(t) = \varphi(V(t)) + \int_0^{\alpha(t)} h(s)\varphi(V(s))ds$, hence $\gamma(0) = \varphi(1)$, and $\varphi(V(t)) \leq \gamma(t)$, $\gamma(t)$ is a nonnegative and nondecreasing function on I. By the monotonicity of φ, φ', V and $\alpha(t) \leq t$, we have $\varphi(V(t)) \leq \gamma(t)$, $\varphi'(V(t)) \leq k$. Differentiating $\gamma(t)$ with respect to t, and using (2.15), we get

$$\gamma'(t) \leq [kg(\alpha(t)) + h(\alpha(t))]\alpha'(t)\gamma(t) + kg(\alpha(t))\alpha'(t)f(\alpha(t))\gamma^2(t), \forall\, t \in I,$$

but $\gamma(t) > 0$, thus from the above inequality, we get

$$\gamma^{-2}(t)\gamma'(t) - [kg(\alpha(t)) + h(\alpha(t))]\alpha'(t)\gamma^{-1}(t) \leq kg(\alpha(t))\alpha'(t)f(\alpha(t)), \forall\, t \in I. \tag{2.16}$$

If we let

$$l(t) = \gamma^{-1}(t), \forall\, t \in I, \tag{2.17}$$

then we get $l(0) = \varphi^{-1}(1)$ and $\gamma^{-2}\gamma'(t) = -l'(t)$, thus from (2.16), we have

$$l'(t) + [kg(\alpha(t)) + h(\alpha(t))]\alpha'(t)l(t) \geq -kg(\alpha(t))\alpha'(t)f(\alpha(t)), \forall\, t \in I.$$

The above inequality implies the estimation for l such that

$$l(t) \geq \frac{\varphi^{-1}(1) - k\int_0^{\alpha(t)} g(s)f(s)\exp\left(\int_0^s [kg(\tau) + h(\tau)]d\tau\right)ds}{\exp\left(\int_0^{\alpha(t)}[kg(s) + h(s)]ds\right)}, \forall\, t \in I.$$

Then from the above inequality in (2.17), we have

$$\gamma(t) \leq \Theta(t), \forall\, t \in I,$$

where Θ as defined in (2.11), thus from (2.15) and the above inequality, we obtain

$$V'(t) \leq g(\alpha(t))\alpha'(t)\varphi(V(t))[1 + f(\alpha(t))\Theta(t)], \forall\, t \in I. \tag{2.18}$$

Since $\varphi(V(t)) > 0$, for all $t > 0$, then from (2.18), we have

$$\frac{V'(t)}{\varphi(V(t))} \leq g(\alpha(t))\alpha'(t)[1 + f(\alpha(t))\Theta(t)],$$

for all $t \in I$. By taking $t = s$ in the above inequality and integrating it from 0 to t, and using the definition of Φ in (2.40), we get

$$\Phi(V(t)) \leq \Phi(1) + \int_0^{\alpha(t)} g(s)[1 + f(s)\Theta(\alpha^{-1}(s))], \tag{2.19}$$

for all $t \in I$, where Φ is defined by (2.40), from (2.19), we have

$$V(t) \leq \Phi^{-1}\left(\Phi(1) + \int_0^{\alpha(t)} g(s)[1 + f(s)\Theta(\alpha^{-1}(s))]ds\right), \tag{2.20}$$

for all $t \in I$, from (2.12), (2.14), and (2.20), we get the required inequality in (2.10). This completes the proof. □

Remark 2.1 Theorem 2.2 gives the explicit estimation in Theorem 2.1, when $\varphi(u(t)) = u(t)$.

In the following two theorems, we prove some new nonlinear retarded integral inequalities.

Theorem 2.3 *Let $u, g, f \in \mathcal{C}(I, I)$ be nonnegative functions. We suppose that φ_1, $\varphi_2, \alpha \in \mathcal{C}^1(I, I)$ are increasing functions with $\alpha(t) \le t$, $\varphi_i(t) > 0$, $i = 1, 2$, $\alpha(0) = 0$ and $\varphi_1'(t) = \varphi_2(t)$, for all $t \in I$; u_0 be positive constant. If the inequality*

$$\varphi_1(u(t)) \le u_0 + \int_0^{\alpha(t)} f(s)\varphi_2(u(s))\left[u(s) + \int_0^s g(\lambda)\varphi_1(u(\lambda))d\lambda\right]^p ds, \forall t \in I, \tag{2.21}$$

holds, for all $t \in I$. Then

$$u(t) \le \varphi_1^{-1}(u_0) + \int_0^{\alpha(t)} f(s)\beta_1(\alpha^{-1})(s)ds, \forall t < T_1, \tag{2.22}$$

where

$$\beta_1(t) = \Omega^{-1}\left\{\Omega\left(\left[\varphi_1^{p-1}(u_0) + (1-p)\int_0^{\alpha(t)} f(s)ds\right]^{\frac{1}{1-p}}\right) + \int_0^{\alpha(t)} g(s)ds\right\}, \tag{2.23}$$

$$\Omega(t) = \int_1^t \frac{ds}{\varphi_1(s)}, \forall t > 0, \tag{2.24}$$

Ω^{-1}, φ_1^{-1} are the inverse functions of Ω, φ_1, respectively, and T_1 is the largest number such that

$$\Omega\left(\left[\varphi_1^{p-1}(u_0) + (1-p)\int_0^{\alpha(t)} f(s)ds\right]^{\frac{1}{1-p}}\right) + \int_0^{\alpha(t)} g(s)ds \le \int_1^{\infty} \frac{ds}{\varphi_1(s)}, \tag{2.25}$$

for all $t < T_1$.

Proof Let $\varphi_1(J)$ denotes the function on the right-hand side of (2.21), which is a nonnegative and nondecreasing function on I with $J(0) = \varphi_1^{-1}(u_0)$. Then (2.21) is equivalent to

$$u(t) \le J(t), u(\alpha(t)) \le J(\alpha(t)) \le J(t), \qquad \forall t \in I. \tag{2.26}$$

Differentiating $\varphi_1(J)$ with respect to t, and using (3.34), we get

$$\begin{aligned}\varphi_1'(J(t))\frac{dJ}{dt}(t) &= \alpha'(t)f(\alpha(t))\varphi_2(u(\alpha(t)))\left[u(\alpha(t)) + \int_0^{\alpha(t)} g(\lambda)\varphi_1(u(\lambda))d\lambda\right]^p \\ &\le \alpha'(t)f(\alpha(t))\varphi_2(J(t))\left[J(t) + \int_0^{\alpha(t)} g(\lambda)\varphi_1(J(\lambda))d\lambda\right]^p, \forall t \in I.\end{aligned}$$

Using the relation $\varphi_1'(J(t)) = \varphi_2(J(t))$, then from the above inequality, we obtain

$$\frac{dJ}{dt}(t) \le \alpha'(t) f(\alpha(t)) w^p(t), \tag{2.27}$$

where $w(t) = J(t) + \int_0^{\alpha(t)} g(s)\varphi_1(J(s))ds$, $w(0) = J(0) = \varphi_1^{-1}(u_0)$ and $J(t) \le w(t)$, w is a nonnegative and nondecreasing function on I. Differentiating w with respect to t, and using (3.30), we have

$$\begin{aligned}\frac{dw}{dt}(t) &\le \alpha'(t) f(\alpha(t)) w^p(t) + \alpha'(t) g(\alpha(t))\varphi_1(J(\alpha(t)))\\ &\le \alpha'(t) f(\alpha(t)) w^p(t) + \alpha'(t) g(\alpha(t))\varphi_1(w(\alpha(t))), \forall t \in I.\end{aligned} \tag{2.28}$$

By $w(t) > 0$, from (2.28), we get

$$\frac{dw}{w^p}(t) \le \alpha'(t) f(\alpha(t))dt + \alpha'(t) g(\alpha(t))\frac{\varphi_1(w(\alpha(t)))}{w^p(\alpha(t))}dt, \forall t \in I. \tag{2.29}$$

Integrating (2.29) from 0 to t, we have

$$w^{1-p}(t) \le \varphi_1^{p-1}(u_0) + (1-p)\int_0^{\alpha(t)} f(s)ds + (1-p)\int_0^{\alpha(t)} g(s)\frac{\varphi_1(w(s))}{w^p(s)}ds, \tag{2.30}$$

for all $t \in I$, from (2.30), we have

$$w^{1-p}(t) \le \varphi_1^{p-1}(u_0) + (1-p)\int_0^{\alpha(T)} f(s)ds + (1-p)\int_0^{\alpha(t)} g(s)\frac{\varphi_1(w(s)}{w^p(s)}ds, \tag{2.31}$$

for all $t \le T$, where $0 \le T < T_1$ is chosen arbitrarily, T_1 is defined by (2.25). Let $m^{1-p}(t)$ denotes the function on the right-hand side of (2.31), which is a positive and nondecreasing function on I with $m(0) = \left[\varphi_1^{p-1}(u_0) + (1-p)\int_0^{\alpha(T)} f(s)ds\right]^{\frac{1}{1-p}}$ and

$$w(t) \le m(t), \forall t < T. \tag{2.32}$$

Differentiating m^{1-p} with respect to t, and using (2.32), we get

$$\frac{dm}{\varphi_1(m)}(t) \le \alpha'(t) g(\alpha(t)), \forall t < T, \tag{2.33}$$

by the definition of Ω in (2.24), then from (2.33), we obtain

$$\begin{aligned}\Omega(m(t)) &\leq \Omega(m(0)) + \int_0^{\alpha(t)} g(s)ds \\ &\leq \Omega\left(\left[\varphi_1^{p-1}(u_0) + (1-p)\int_0^{\alpha(T)} f(s)ds\right]^{\frac{1}{1-p}}\right) + \int_0^{\alpha(t)} g(s)ds,\end{aligned}$$

for all $t < T$. Let $t = T$, then from the above inequality, we get

$$\Omega(m(t)) \leq \Omega\left(\left[\varphi_1^{p-1}(u_0) + (1-p)\int_0^{\alpha(T)} f(s)ds\right]^{\frac{1}{1-p}}\right) + \int_0^{\alpha(T)} g(s)ds. \quad (2.34)$$

Since $0 < T < T_1$ is chosen arbitrary, then from (2.34) in (2.32), we obtain

$$w(t) \leq \beta_1(t), \forall t < T_1, \quad (2.35)$$

where β_1 as defined in (2.23), thus from (2.27) and (2.35), we obtain

$$\frac{dJ}{dt}(t) \leq \alpha'(t) f(\alpha(t))\beta_1(t), \forall t < T_1. \quad (2.36)$$

By taking $t = s$ in (2.36) and integrating it from 0 to t we have

$$J(t) \leq \varphi_1^{-1}(u_0) + \int_0^{\alpha(t)} f(s)\beta_1(\alpha^{-1})(s)ds, \forall t < T_1. \quad (2.37)$$

Using (2.37) in (2.26), we get the required inequality in (2.22). This completes the proof. □

Theorem 2.4 *Let u, g, $f \in \mathcal{C}(I, I)$ be nonnegative functions. We suppose that $\varphi, \varphi', \alpha \in \mathcal{C}^1(I, I)$ are increasing functions, with $\varphi'(t) \leq k$, $\varphi > 0$, $\alpha(t) \leq t$, $\alpha(0) = 0$, for all $t \in I$; k, u_0 be positive constants. If the inequality*

$$u(t) \leq u_0 + \int_0^{\alpha(t)} f(s)\varphi(u(s))\left[\varphi(u(s)) + \int_0^s g(\lambda)\varphi(u(\lambda))d\lambda\right]ds, \quad (2.38)$$

holds, for all $t \in I$. Then

$$u(t) \leq \Phi^{-1}\left(\Phi(u_0) + \int_0^{\alpha(t)} f(s)\beta(\alpha^{-1}(s))ds\right), \forall t \in I, \quad (2.39)$$

where

$$\Phi(r) = \int_1^r \frac{dt}{\varphi(t)}, r > 0, \quad (2.40)$$

and

$$\beta(t)=\exp\biggl(\int_0^{\alpha(t)} g(s)ds\biggr)\biggl((\varphi^{-1}(u_0))-k\int_0^{\alpha(t)} f(s)\exp\biggl(\int_0^s g(\lambda)d\lambda\biggr)ds\biggr)^{-1}, \tag{2.41}$$

for all $t\in I$*, such that* $(\varphi^{-1}(u_0))-k\int_0^{\alpha(t)} f(s)\exp\Bigl(\int_0^s g(\lambda)d\lambda\Bigr)ds>0, \forall t\in I.$

Proof Let z denotes the function on the right-hand side of (2.38), which is a nonnegative and nondecreasing function on I with $z(0)=u_0$. Then (2.38) is equivalent to

$$u(t)\le z(t), u(\alpha(t))\le z(\alpha(t))\le z(t), \qquad \forall t\in I. \tag{2.42}$$

Differentiating z with respect to t, we get

$$\frac{dz}{dt}(t)=\alpha'(t)f(\alpha(t))\varphi(u(\alpha(t)))\biggl[\varphi(u(\alpha(t)))+\int_0^{\alpha(t)} g(s)\varphi(u(s))ds\biggr], \forall t\in I.$$

Using (2.42), we get

$$\frac{dz}{dt}(t)\le\alpha'(t)f(\alpha(t))\varphi(z(\alpha(t)))y(t), \forall t\in I, \tag{2.43}$$

where $y(t)=\varphi(z(t))+\int_0^{\alpha(t)} g(s)\varphi(z(s))ds$, $y(0)=\varphi(z(0))=\varphi(u_0)$, y is a nonnegative and nondecreasing function on I. By the monotonicity φ, φ', z and $\alpha(t)\le t$ we have $\varphi(z(t))\le y(t)$, $\varphi'(z(t))\le k$. Differentiating y with respect to t, and using (2.43), we have

$$\begin{aligned}\frac{dy}{dt}(t)&\le\varphi'(z(t))\alpha'(t)f(\alpha(t))y^2(t)+\alpha'(t)g(\alpha(t))\varphi(z(t))\\&\le k\alpha'(t)f(\alpha(t))y^2(t)+\alpha'(t)g(\alpha(t))y(t), \forall t\in I.\end{aligned} \tag{2.44}$$

But $y(t)>0$, from (2.44) we get

$$y^{-2}(t)\frac{dy}{dt}(t)-\alpha'(t)g(\alpha(t))y^{-1}(t)\le k\alpha'(t)f(\alpha(t)), \forall t\in I. \tag{2.45}$$

If we let

$$v(t)=y^{-1}(t), \forall t\in I, \tag{2.46}$$

then we get $v(0)=\varphi^{-1}(u_0)$ and $y^{-2}(t)\frac{dy}{dt}(t)=-\frac{dv}{dt}(t)$, thus from (2.45) and (2.46), we have

$$\frac{dv}{dt}(t)+\alpha'(t)g(\alpha(t))\ge -k\alpha'(t)f(\alpha(t)), \forall t\in I.$$

The above inequality implies an estimation for v as in the following

$$v(t) \geq \exp\bigg(- \int_0^{\alpha(t)} g(s)ds\bigg)\bigg((\varphi^{-1}(u_0)) - k\int_0^{\alpha(t)} f(s)\exp\bigg(\int_0^s g(\lambda)d\lambda\bigg)ds\bigg), \tag{2.47}$$

for all $t \in I$, from (2.41), (2.46), and (2.47), we get $y(t) \leq \beta(t)$, where β as defined in (2.41). Thus from (2.43), we have

$$\frac{dz}{dt}(t) \leq \alpha'(t)f(\alpha(t))\varphi(z(t))\beta(t), \forall t \in I. \tag{2.48}$$

By taking $t = s$ in (2.48) and integrating it from 0 to t, using (2.40), we obtain

$$\begin{aligned} z(t) &\leq \Phi^{-1}\bigg(\Phi(u_0) + \int_0^t \alpha'(s)f(\alpha(s))\beta(s)\bigg)ds, \\ &\leq \Phi^{-1}\bigg(\Phi(u_0) + \int_0^{\alpha(t)} f(s)\beta(\alpha^{-1}(s))\bigg)ds, \forall t \in I. \end{aligned} \tag{2.49}$$

Using (2.49) in (2.42) , we get the required inequality in (2.39). This completes the proof. □

3 More Nonlinear Retarded Integral Inequalities

In this section, we state and prove some new retarded nonlinear integral inequalities of Gronwall–Bellman–Pachpatte type, which can be used in the analysis of various problems in the theory of retarded nonlinear differential equations. The results proved in this section are adapted from [4].

Theorem 3.1 *Let $u, g, f \in \mathcal{C}(I, I)$, $\alpha \in \mathcal{C}^1(I, I)$ be nondecreasing with $\alpha(t) \leq t$ on I with $\alpha(0) = 0$ and u_0 be a nonnegative constant. If the inequality*

$$u(t) \leq u_0 + \int_0^{\alpha(t)} [f(s)u(s) + q(s)]ds + \int_0^{\alpha(t)} f(s)\bigg(\int_0^s g(s)u(s)ds\bigg)ds, \tag{3.1}$$

for all $t \in I$. Then

$$\begin{aligned} u(t) \leq& u_0 + \int_0^t \bigg(\alpha'(s)p(\alpha(s)) + \alpha'(s)f(\alpha(s))\exp\bigg(\int_0^{\alpha(s)} [f(\tau) + g(\tau)]d\tau\bigg)\bigg[u_0 \\ &+ \int_0^{\alpha(s)} p(\sigma)\exp\bigg(\int_0^{\sigma} [f(\tau) + g(\tau)]d\tau\bigg)d\sigma\bigg]\bigg), \end{aligned} \tag{3.2}$$

for all $t \in I$.

Proof Define a function z by the right-hand side of (3.1) which is a nonnegative and nondecreasing function on I with $z(0) = u_0$. Then (3.1) is equivalent to

$$u(t) \leq z(t), u(\alpha(t)) \leq z(\alpha(t)) \leq z(t), \qquad \forall t \in I. \tag{3.3}$$

Differentiating z with respect to t, and using (3.3), we get

$$\begin{aligned} z'(t) &= \alpha'(t)[f(\alpha(t))u(\alpha(t)) + p(\alpha(t))] + \alpha'(t)f(\alpha(t))\int_0^{\alpha(t)} g(\sigma)u(\sigma)d\sigma \\ &\leq \alpha'(t)p(\alpha(t)) + \alpha'(t)f(\alpha(t))\left[z(t) + \int_0^{\alpha(t)} g(\sigma)z(\sigma)d\sigma\right], \end{aligned} \tag{3.4}$$

for all $t \in I$. Define a function v by

$$v(t) = z(t) + \int_0^{\alpha(t)} g(\sigma)z(\sigma)d\sigma, \forall t \in I, \tag{3.5}$$

then $v(0) = z(0) = u_0$, $z'(t) \leq \alpha'(t)(p(\alpha(t)) + f(t)v(t))$ from (3.4), and from (3.5) $z(t) \leq v(t), z(\alpha(t)) \leq v(\alpha(t)) \leq v(t)$. Differentiating v, with respect to t, we get

$$\begin{aligned} v'(t) &= z'(t) + \alpha'(t)g(\alpha(t))z(\alpha(t)) \\ &\leq \alpha'(t)p(\alpha(t)) + \alpha'(t)[f(\alpha(t)) + g(\alpha(t))]v(t), \forall t \in I. \end{aligned} \tag{3.6}$$

Integrating the inequality (3.6) from 0 to t implies the estimation

$$v(t) \leq \exp\left(\int_0^{\alpha(t)} [f(\tau) + g(\tau)]d\tau\right)\left[u_0 + \int_0^{\alpha(t)} p(\sigma)\exp\left(\int_0^{\sigma} [f(\tau) + g(\tau)]d\tau\right)d\sigma\right], \tag{3.7}$$

for all $t \in I$. Using (3.7) in (3.4), we have

$$\begin{aligned} z'(t) \leq \alpha'(t)p(\alpha(t)) + \alpha'(t)f(\alpha(t))\exp\left(\int_0^{\alpha(t)} [f(\tau) + g(\tau)]d\tau\right)\Bigg[u_0 \\ + \int_0^{\alpha(t)} p(\sigma)\exp\left(\int_0^{\sigma} [f(\tau) + g(\tau)]d\tau\right)d\sigma\Bigg], \forall t \in I. \end{aligned} \tag{3.8}$$

Now by sitting $t = s$ in (3.8) and integrating it from 0 to t and substituting the bound on z in (3.3), we obtain the required inequality in (3.2). This completes the proof. □

Remark 3.1 Theorem 3.1 gives the explicit estimation (3.2) for the inequality (3.1), which is just the inequality (1.3) in Theorem 1.2 when $\alpha(t) = t$.

Theorem 3.2 *Let $u, g, f \in \mathcal{C}(I, I)$, be nonnegative functions. We suppose that φ, φ', $\alpha, \in \mathcal{C}^1(I, I)$ are increasing functions, with $\varphi'(t) \leq k$, $\varphi > 0, \alpha(t) \leq t, \alpha(0) = 0$, for all $t \in I$; k, u_0 be positive constants. If the inequality*

$$u(t) \le u_0 + \int_0^{\alpha(t)} \varphi(u(s))[f(s)\varphi(u(s)) + q(s)]ds + \int_0^{\alpha(t)} \varphi(u(s))f(s)\left(\int_0^s g(s)\varphi(u(s))ds\right)ds, \tag{3.9}$$

holds, for all $t \in I$. Then

$$u(t) \le \Phi^{-1}\left(\Phi(u_0) + \int_0^{\alpha(t)} [p(s) + f(s)\beta(s)]ds\right), \forall t \in I, \tag{3.10}$$

where Φ as defined in (2.40), and

$$\beta(t) \le \frac{\exp\left(\int_0^{\alpha(t)}[kp(s) + g(s)]ds\right)}{\left[\varphi^{-1}(u_0) - \int_0^{\alpha(t)} kf(s)\exp\left(\int_0^s[kp(\sigma) + g(\sigma)]d\sigma\right)\right]}, \forall t \in I. \tag{3.11}$$

Proof Define a function z_1 by the right-hand side of (3.9), which is a nonnegative and nondecreasing function on I with $z_1(0) = u_0$. Then (3.9) is equivalent to

$$u(t) \le z_1(t), u(\alpha(t)) \le z_1(\alpha(t)) \le z_1(t), \qquad \forall t \in I. \tag{3.12}$$

Differentiating z_1 with respect to t, and using (3.12), we get

$$\begin{aligned} z_1'(t) &= \alpha'(t)\varphi(u(\alpha(t)))[f(\alpha(t))\varphi(u(\alpha(t))) + p(\alpha(t))] \\ &\quad + \alpha'(t)f(\alpha(t))\varphi(u(\alpha(t)))\int_0^{\alpha(t)} g(\sigma)\varphi(u(\sigma))d\sigma \\ &\le \alpha'(t)p(\alpha(t))\varphi(z_1(t)) + \alpha'(t)f(\alpha(t))\varphi(z_1(t))\Bigg[\varphi(z_1(t)) \\ &\quad + \int_0^{\alpha(t)} g(\sigma)\varphi(z_1(\sigma))d\sigma\Bigg] \\ &\le \alpha'(t)p(\alpha(t))\varphi(z_1(t)) + \alpha'(t)f(\alpha(t))\varphi(z_1(t)(t))v(t), \end{aligned} \tag{3.13}$$

for all $t \in I$, where

$$v_1(t) = \varphi(z_1(t) + \int_0^{\alpha(t)} g(\sigma)\varphi(z_1(\sigma))d\sigma, \forall t \in I. \tag{3.14}$$

Hence $v(0) = \varphi(z(0)) = \varphi(u_0)$, and

$$\varphi(z_1(t)) \le v(t), \forall t \in I. \tag{3.15}$$

Differentiating (3.14) with respect to t, and using the relation $\varphi'(z_1(t) < k$ and (3.13), (3.15), we get

$$
\begin{aligned}
v'(t) &= \varphi'(z_1(t)z_1'(t) + \alpha'(t)g(\alpha(t))\varphi(z_1(t)) \\
&\leq k\alpha'(t)p(\alpha(t))\varphi(z_1(t)) + k\alpha'(t)f(\alpha(t))v(t)\varphi(z_1(t)) + \alpha'(t)g(\alpha(t))\varphi(z_1(t)) \\
&\leq [k\alpha'(t)p(\alpha(t)) + \alpha'(t)g(\alpha(t))]v(t) + k\alpha'(t)f(\alpha(t))v^2(t), \qquad (3.16)
\end{aligned}
$$

for all $t \in I$, since $v(t) > 0$ then we can write the inequality (3.16) in the following form

$$v^{-2}(t)v'(t) - [k\alpha'(t)p(\alpha(t)) + \alpha'(t)g(\alpha(t))]v^{-1}(t) \leq k\alpha'(t)f(\alpha(t)), \forall t \in I. \qquad (3.17)$$

If we let

$$v^{-1}(t) = S(t), \forall t \in I. \qquad (3.18)$$

We have $S(0) = v^{-1}(0) = \varphi^{-1}(u_0)$, and $v^{-2}(t)v'(t) = -S'(t)$, then we can write the inequality (3.17) as follows

$$S'(t) + [k\alpha'(t)p(\alpha(t)) + \alpha'(t)g(\alpha(t))]S(t) \geq -k\alpha'(t)f(\alpha(t)), \forall t \in I. \qquad (3.19)$$

The inequality (3.19) implies an estimation for $S(t)$ as in the following

$$S(t) \geq \frac{\left[\varphi^{-1}(u_0) - \int_0^{\alpha(t)} kf(s)\exp\left(\int_0^s [kp(\sigma) + g(\sigma)]d\sigma\right)\right]}{\exp\left(\int_0^{\alpha(t)} [kp(s) + g(s)]ds\right)}, \qquad (3.20)$$

for all $t \in I$, then from (3.18) and (3.20), we have

$$v(t) \leq \beta(t), \forall t \in I, \qquad (3.21)$$

where β as defined in (3.11), thus from (3.21) in (3.13), we have

$$z_1'(t) \leq \alpha'(t)p(\alpha(t))\varphi(z_1(t)) + \alpha'(t)f(\alpha(t))\varphi(z_1(t)(t))\beta(t)\forall t \in I. \qquad (3.22)$$

Hence, we can write the inequality (3.22) as follows

$$\frac{z_1'(t)}{\varphi(z_1(t))} \leq \alpha'(t)p(\alpha(t)) + \alpha'(t)f(\alpha(t))\beta(t)\forall t \in I. \qquad (3.23)$$

By taking $t = s$ in (3.23) and integrating it from 0 to t, using (2.40), we have

$$z_1(t) \leq \Phi^{-1}\left(\Phi(u_0) + \int_0^{\alpha(t)} [p(s) + f(s)\beta(s)]ds\right), \forall t \in I. \qquad (3.24)$$

Using (3.24) in (3.12) , we get the required inequality in (3.10). This completes the proof. □

Theorem 3.3 *Let u, g, $f \in \mathcal{C}(I, I)$, be nonnegative functions. We suppose that φ_1, φ_2, $\alpha \in \mathcal{C}^1(I, I)$ are increasing functions with $\alpha(t) \leq t$, $\varphi_i(t) > 0$, $i = 1, 2$, $\alpha(0) = 0$, $\varphi_1'(t) = \varphi_2(t)$, and $\varphi_1^{-1}(t)$ is a submultiplicative function for all $t \in I$; u_0 be positive constants. If the inequality*

$$\varphi_1(u(t)) \leq u_0 + \int_0^{\alpha(t)} g(s)\varphi_1(u(s))ds + \int_0^{\alpha(t)} h(s)\varphi_2(u(s))ds, \forall t \in I, \tag{3.25}$$

holds, for all $t \in I$. Then

$$u(t) \leq \left(\varphi_1^{-1}(u_0) + \int_0^{\alpha(t)} h(s)ds\right)\varphi_1^{-1}\left(\exp(\int_0^{\alpha(t)} g(s)ds)\right), \tag{3.26}$$

for all $t \in [0, T_1]$, where φ_1^{-1} is the inverse function of φ_1 and T_1 is the largest number such that

$$\varphi_1^{-1}(u_0) + \int_0^{\alpha(T)} h(s)ds > 0, \forall t \in [0, T_1]. \tag{3.27}$$

Proof Let $\varphi_1(J)$ denotes the function on the right-hand side of (2.21), which is a nonnegative and nondecreasing function on I with $J(0) = \varphi_1^{-1}(u_0)$. Then (2.21) is equivalent to

$$u(t) \leq J(t), u(\alpha(t)) \leq J(\alpha(t)) \leq J(t), \qquad \forall t \in I. \tag{3.28}$$

Differentiating $\varphi_1(J)$, with respect to t, we get

$$\begin{aligned} \varphi_1'(J(t))\frac{dJ}{dt})(t) &= \alpha'(t)g(\alpha(t))\varphi_1(u(\alpha(t))) + \alpha'(t)h(\alpha(t))\varphi_2(u(\alpha(t))) \\ &\leq \alpha'(t)g(\alpha(t))\varphi_1(J(t)) + \alpha'(t)h(\alpha(t))\varphi_2(J(t)). \end{aligned} \tag{3.29}$$

Using the relation $\varphi_1'(J(t)) = \varphi_2(J(t))$, from (3.29) we obtain

$$\frac{dJ}{dt}(t) \leq \alpha'(t)h(\alpha(t)) + g(\alpha(t))\frac{\varphi_1(J(t))}{\varphi_1'(J(t))}, \forall t \in I. \tag{3.30}$$

Integrating both sides of (3.30) from 0 to t, we get

$$\begin{aligned} J(t) &= \varphi_1^{-1}(u_0) + \int_0^{\alpha(t)} h(s)ds + \int_0^{\alpha(t)} g(s)\frac{\varphi_1(J(s))}{\varphi_1'(J(s))}ds, \\ &\leq \varphi_1^{-1}(u_0) + \int_0^{\alpha(T)} h(s)ds + \int_0^{\alpha(t)} g(s)\frac{\varphi_1(J(s))}{\varphi_1'(J(s))}ds, \forall t \in [0, T], \end{aligned} \tag{3.31}$$

where $T \in [0, T_1]$ is a positive constant chosen arbitrary, T_1 is defined by (3.27). Let

$$R(t) = \varphi_1^{-1}(u_0) + \int_0^{\alpha(T)} h(s)ds + \int_0^{\alpha(t)} g(s)\frac{\varphi_1(J(s))}{\varphi_1'(J(s))}ds, \forall t \in [0, T], \tag{3.32}$$

then R is a nonnegative and nondecreasing function on I with $R(0) = \varphi_1^{-1}(u_0) + \int_0^{\alpha(T)} h(s)ds$, then (3.32) is equivalent to

$$J(t) \leq R(t), \forall t \in [0, T]. \tag{3.33}$$

Differentiating (3.32) with respect to t, and using (3.33), we obtain

$$\begin{aligned} \frac{dR}{dt}(t) &= g(t)\frac{\varphi_1(J(t))}{\varphi_1'(J(t))}, \\ &\leq g(t)\frac{\varphi_1(R(t))}{\varphi_1'(R(t))}, \forall t \in [0, T]. \end{aligned} \tag{3.34}$$

The inequality given in (3.34) can be written as

$$\frac{\varphi_1'(R)dR}{\varphi_1(R)}(t) \leq g(t)dt, \forall t \in [0, T]. \tag{3.35}$$

Integrating both sides of (3.35) from 0 to t and using the multiplicity of the inverse function φ_1^{-1}, we have

$$R(t) \leq \left(\varphi_1^{-1}(u_0) + \int_0^{\alpha(T)} h(s)ds\right)\varphi_1^{-1}\left(\exp(\int_0^{\alpha(t)} g(s)ds)\right), \tag{3.36}$$

for all $t \in [0, T]$. Letting $t = T$ in (3.36), from (3.28), (3.33), and (3.36), we get

$$u(T) \leq \left(\varphi_1^{-1}(u_0) + \int_0^{\alpha(T)} h(s)ds\right)\varphi_1^{-1}\left(\exp(\int_0^{\alpha(T)} g(s)ds)\right). \tag{3.37}$$

Because $T \in [0, T_1]$ is chosen arbitrarily. This completes the proof. □

4 Applications

In this section, we present some applications for the results which we have established above and apply them to qualitative and quantitative analysis of solutions of certain delay differential equations to which the inequalities available in the literature do not apply directly.

4.1 Differential Equations with Delay

We apply our result obtained in Theorem 2.4 to study the boundedness and the existence of the solutions of the initial value problem for nonlinear delay differential equation of the form:

$$\begin{cases} \frac{du}{dt}(t) = M(t, u(\alpha(t)), H(t, u(\alpha(t)))), \forall t \in I, \\ \\ u(0) = u_0, \end{cases} \tag{4.1}$$

where u_0 is a constant, $M \in \mathcal{C}(I^3, \mathbb{R}), H \in \mathcal{C}(I \times I, \mathbb{R})$, satisfy the following conditions:

$$|M(t, u, H)| \le f(\alpha(t))\varphi(u(\alpha(t)))\left[|u(\alpha(t))| + \int_0^t |K(s, u(\alpha(s)))|ds\right], \tag{4.2}$$

$$|K(t, u(\alpha(t)))| \le g(\alpha(t))\varphi(u(\alpha(t))), \tag{4.3}$$

where f, g as defined in Theorem 2.4.

Theorem 4.1 *Consider nonlinear system (4.1) and suppose that M, H satisfy the conditions (4.2) and (4.3). We suppose that $\varphi, \varphi', \alpha \in \mathcal{C}^1(I, I)$ are increasing functions with $\varphi_1'(t) \le k$, $\alpha(t) \le t$, $\alpha(0) = 0$, for all $t \in I$; k, u_0 are positive constants, then each solution u of (4.1) under discussion verifies the following estimation:*

$$u(t) \le \Phi^{-1}\left(\Phi(u_0) + \int_0^{\alpha(t)} \frac{f(s)}{\alpha'(\alpha^{-1}(s))}\beta_2(\alpha^{-1}(s))ds\right), \qquad \forall t \in I, \tag{4.4}$$

where Φ as defined in (2.40), and

$$\beta_2(t) = \exp\left(\int_0^{\alpha(t)} \frac{g(s)}{\alpha'(\alpha^{-1}(s))}ds\right) \times\left((\varphi^{-1}(u_0)) - k\int_0^{\alpha(t)} \frac{f(s)}{\alpha'(\alpha^{-1}(s))}\exp\left(\int_0^s \frac{g(\lambda)}{\alpha'(\alpha^{-1}(\lambda))}d\lambda\right)ds\right)^{-1}, \forall t \in I. \tag{4.5}$$

Proof Integrating both sides of (4.1) from 0 to t, we have

$$u(t) = u_0 + \int_0^t M(s, u(\alpha(s)), H(s, u(\alpha(s))))ds, \forall t \in I, \tag{4.6}$$

using the conditions (4.2) and (4.3), then from (4.6), we get

$$|u(t)| \leq u_0 + \int_0^t f(s)|\varphi(u(\alpha(s)))|\left[|\varphi(u(\alpha(s)))| + \int_0^s g(\alpha(\lambda))|\varphi(u(\alpha(\lambda)))|d\lambda\right]ds$$
$$\leq u_0 + \int_0^{\alpha(t)} \frac{f(s)|\varphi(u(s))|}{\alpha'(\alpha^{-1}(s))}\left[|\varphi(u(s))| + \int_0^s \frac{g(\lambda)|\varphi(u(\lambda))}{\alpha'(\alpha^{-1}(s))}|d\lambda\right]ds,$$

for all $t \in I$, applying Theorem 2.4 to the above inequality, we obtain the required estimation (4.4). This completes the proof. □

4.2 Retarded Integro-Differential Equations

We show that our main results are useful in showing the global existence of solutions to certain integro-differential equations of the form:

$$u'(t) = F\left(t, u(\alpha(t)), \int_0^t h(s, u(\alpha(s)))ds\right), \tag{4.7}$$

for any $t \in I$ with the initial condition

$$u(0) = u_0, \tag{4.8}$$

where $h \in \mathcal{C}(R^2, R)$, $F \in \mathcal{C}(R^3, R)$, and $u_0 \geq 0$ is constant.
Assume that

$$\int_0^t |F(s, u(\alpha(s)), \nu)|ds \leq \int_0^t (\varphi(|u(s)|)[f(s)\varphi(|u(s)|) + p(s)] + \varphi(|u(s)|)f(s)|\nu|)ds, \tag{4.9}$$

$$h(t, u(t)) \leq g(t)(\varphi(|u(t)|) \tag{4.10}$$

where the functions f, α, and g are defined as in Theorem 3.2. If u is a solution of the equation (4.7) with (4.8), then the solution u can be written as

$$u(t) = u_0 + \int_0^t F\left(s, u(\alpha(s)), \int_0^s f(\sigma, u(\alpha(\sigma)))d\sigma\right)ds, \tag{4.11}$$

for any $t \in I$. Using (4.9) and (4.10) in (4.11) and making the change of variables, we get

$$
\begin{aligned}
|u(t)| &\leq u_0 + \int_0^t \varphi(|u(\alpha(s))|)[f(s)\varphi(|u(\alpha(s))|) + q(s)]ds \\
&\quad + \int_0^{\alpha(t)} \varphi(|u(\alpha(s))|) f(s) \left(\int_0^s g(\sigma)\varphi(|u(\alpha(\sigma))|)d\sigma \right) ds \\
&\leq u_0 + \int_0^{\alpha(t)} \frac{\varphi(|u(s)|)}{\alpha'(\alpha^{-1}(s))}[f(s)\varphi(|u(s)|) + q(s)]ds \\
&\quad + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\varphi(|u(s)|) \left(\int_0^s g(\sigma)\varphi(|u(\sigma)|)d\sigma \right) ds, \qquad (4.12)
\end{aligned}
$$

holds, for all $t \in I$. Now, a suitable application of Theorem 3.2 to (4.12) yields

$$
u(t) \leq \Phi^{-1}\left(\Phi(u_0) + \int_0^{\alpha(t)} \frac{1}{\alpha'(\alpha^{-1}(s))}[p(s) + f(s)\beta(s)]ds \right), \forall t \in I, \qquad (4.13)
$$

which implies that u is bounded, where Φ is defined as in Theorem 3.2 and

$$
\beta(t) \leq \frac{\exp\left(\int_0^{\alpha(t)} \frac{1}{\alpha'(\alpha^{-1}(s))}[kp(s) + g(s)]ds \right)}{\left[\varphi^{-1}(u_0) - \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \exp\left(\int_0^s [kp(\sigma) + g(\sigma)]d\sigma \right) \right]}, \forall t \in I. \qquad (4.14)
$$

Remark 4.1 Gronwall-like inequality can be applied to the analysis of the behavior of the solutions of some retarded nonlinear differential equations. Our results also can be used to prove the global existence, uniqueness, stability, and other properties of the solutions of various nonlinear retarded differential and integral equations. The importance of these inequalities stems from the fact that it is applicable in certain situations in which other available inequalities do not apply directly.

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On the Integral Inequalities for Riemann–Liouville and Conformable Fractional Integrals

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Abstract An integral operator is sometimes called an integral transformation. In the fractional analysis, Riemann–Liouville integral operator (transformation) of fractional integral is defined as

$$S_\alpha(x) = \frac{1}{\Gamma(x)} \int_0^x (x-t)^{\alpha-1} f(t)dt$$

where $f(t)$ is any integrable function on [0, 1] and $\alpha > 0$, t is in domain of f.

1 Introduction

The history of fractional analysis goes back to the arising of classical differential theory. Despite the fact that history is based on extreme ages, the interpretation of classical analysis as a result of the complexity of its physical structure has not been postponed and the science has not been very popular in engineering. However, the fact that fractional derivatives and integrals are not local or punctate has made the matter of fractional analysis remarkable in terms of better expressing the reality of nature. Thus, making this more widespread in science and engineering will play an important role in better interpreting and expressing nature.

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Fractional analysis can be considered as an extension of classical analysis. Fractional analysis does not have the definition of a single derivative as it is in the classical analysis, but the presence of more than one derivative gives the opportunity to obtain the best solution to the problems.

Fractional analysis has been studied by many scholars, and they have expressed fractional derivatives and integrals in different forms with different notations. But although these expressions are transitions between each other, they differ in terms of definitions and physical interpretations of their definitions. For the first time in 1695, the notion of fractional derivative and integral was raised by asking whether it would be meaningful if the derivation order was $1/2$ in a letter sent by L'Hospital to Leibnitz. Thus, the origin of fractional analysis begins with the question of L'Hospital.

This question on fractional derivatives and integrals has been a subject of study by many famous mathematicians such as Liouville, Riemann, Weyl, Fourier, Laplace, Lagrange, Euler, Abel, Lacroix, Grünwald, and Letnikov for more than 300 years. Since then, fractional differential equations have found many application areas including the theory of transmission lines, chemical analysis of fluids, heat transfer, diffusion, Schrödinger equation, material science, fluids, electrochemistry, fractal processes. Much of the mathematical application of fractional computing techniques has been put into place before the end of the twentieth century, but it has only been possible within a hundred years to achieve exciting achievements in engineering and scientific applications.

The fractional differential calculation technique not only contributes to a new dimension to mathematical approaches to explain physical phenomena, but also contributes to the interpretation of physical phenomena. The ranks of the differential equations describing the physical phenomena determine the rate of change in the physical state involved. The fractional-order differential at this point plays a major role in understanding the character of the physical phenomenon as well as closing the weaknesses of differential equations of integer order to explain some physical phenomena.

There are many definitions in the literature of the fractional derivative and integrals. Many of these definitions make use of the integral form when making fractional derivative definitions. The most famous of these definitions is Riemann–Liouville.

Some authors discussed whether the fractional derivative is indeed a fractional operator. Today, this question is still open to debate. Perhaps this is a philosophical issue. Moreover, this new definition can be considered as a transformation for the solution of differential equations of fractional order even if there is no definition of a fractional derivative. Obviously, this discussion is an argument of what the new theory is to be given. It is always a matter of deserving to study the definition of this new fractional derivative and fractional integral.

Various types of fractional derivative and integral operator were studied: Riemann–Liouville, conformable fractional integral operators, Caputo, Hadamard, Erdelyi–Kober, Grünwald–Letnikov, Marchaud, and Riesz are just a few to name.

In the present chapter, we shall recall some of fractional integral operators, which generalizes the classical integrals. We shall start this chapter with some results and definitions to refresh our memories about some of the remarkable milestones in the theory of fractional calculus and recall some inequalities involving two kinds of fractional integral operators.

Finally, we will give inequalities of Hermite–Hadamard type, Grüss type, Ostrowski type involving other types of fractional integral operators. All of this will be presented chronologically.

2 Riemann–Liouville Fractional Integral Operators and Inequalities

The following definitions are well-known in the fractional calculus and have been used in many fields of mathematics (see the references [1–4]).

Definition 2.1 ([5]) Let $f \in L_1[a,b]$. The Riemann–Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ are defined by

$$J^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1} f(x)dx, \qquad t > a,$$

and

$$J^{\alpha}_{b-}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (x-t)^{\alpha-1} f(x)dx, \qquad t < b,$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$. Here $J^0_{a+}f(t) = J^0_{b-}f(t) = f(t)$.

In the case of $\alpha = 1$, the fractional integral reduces to classical integral.

In this paper, some new integral inequalities have been proved by using conformable fractional integrals for functions whose derivatives of absolute values are quasi-convex, s-convex and log-convex functions.

Several researches have proved different types of integral inequalities via Riemann–Liouville fractional integrals. We will start with the new representation of celebrated Montgomery identity for fractional calculus that was proved Anastassiou et al. in 2009.

Lemma 2.1 ([6]) *Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$, and $f' : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, then the following Montgomery identity for fractional integrals holds:*

$$f(x) = \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J^{\alpha}_{a} f(b) - J^{\alpha-1}_{a}(P_2(x,b)f(b)) + J^{\alpha}_{a}(P_2(x,b)f'(b)), \quad \alpha \geq 1$$

where $P_2(x,t)$ is the fractional Peano kernel defined by:

$$P_2(x,t) = \begin{cases} \frac{t-a}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & a \leq t \leq x, \\ \frac{t-b}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & x \leq t \leq b. \end{cases}$$

The authors have also extended Ostrowski's inequality and Gruss inequality to fractional calculus as follows.

Theorem 2.1 ([6]) *Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ and $|f'(x)| \leq M$, for every $x \in [a, b]$ and $\alpha \geq 1$. Then, the following Ostrowski fractional inequality holds:*

$$\left| f(x) - \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^{\alpha} f(b) + J_a^{\alpha-1} P_2(x, b) f(b) \right|$$
$$\leq \frac{M}{\alpha(\alpha+1)} \left[(b-x)\left(2\alpha\left(\frac{b-x}{b-a}\right) - \alpha - 1\right) + (b-a)^{\alpha}(b-x)^{1-\alpha} \right].$$

Proposition 1 ([6]) *Suppose that $f(x)$ and $g(x)$ are two integrable functions for all $x \in [a, b]$, and satisfy the conditions*

$$m \leq (b-x)^{\alpha-1} f(x) \leq M, \quad n \leq (b-x)^{\alpha-1} g(x) \leq N,$$

where $\alpha > 1/2$, and m, M, n, N are real constants. Then, the following Gruss fractional inequality holds:

$$\left| \frac{\Gamma(2\alpha-1)}{(b-a)\Gamma^2(\alpha)} J_a^{2\alpha-1}(fg)(b) - \frac{1}{(b-a)^2} J_a^{\alpha} f(b) J_a^{\alpha} g(b) \right|$$
$$\leq \frac{1}{4\Gamma^2(\alpha)}(M-m)(N-n).$$

Another important study on the Riemann–Liouville fractional integrals has been written by Dahmani in 2010. The following results are concerning with Minkowski inequality.

Theorem 2.2 ([7]) *Let $\alpha > 0$, $p \geq 1$ and let f, g be two positive functions on $[0, \infty)$ such that for all $t > 0$, $J^{\alpha} f^p(t) < \infty$, $J^{\alpha} g^p(t) < \infty$. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in [0, t]$, then we have*

$$\left[J^{\alpha} f^p(t)\right]^{\frac{1}{p}} + \left[J^{\alpha} g^p(t)\right]^{\frac{1}{p}} \leq \frac{1 + M(m+2)}{(m+1)(M+1)} \left[J^{\alpha}(f+g)^p(t)\right]^{\frac{1}{p}}.$$

Theorem 2.3 ([7]) *Let $\alpha > 0$, $p \geq 1$ and let f, g be two positive functions on $[0, \infty)$ such that for all $t > 0$, $J^{\alpha} f^p(t) < \infty$, $J^{\alpha} g^p(t) < \infty$. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in [0, t]$, then we have*

$$\left[J^{\alpha} f^p(t)\right]^{\frac{2}{p}} + \left[J^{\alpha} g^p(t)\right]^{\frac{2}{p}} \leq \left(\frac{(M+1)(m+1)}{M} - 2\right)$$
$$\left[J^{\alpha} f^p(t)\right]^{\frac{1}{p}} \left[J^{\alpha} g^p(t)\right]^{\frac{1}{p}}.$$

Theorem 2.4 ([7]) *Let $\alpha > 0$, $p \geq 1$ and let f, g be two positive functions on $[0, \infty)$. If f^p, g^p are two concave functions on $[0, \infty)$, then we have*

$$\begin{aligned}&2^{-p-q}(f(0)+f(t))^p(g(0)+g(t))^q(J^\alpha(t^{\alpha-1}))^2\\ &\leq J^\alpha(t^{\alpha-1}f^p(t))J^\alpha(t^{\alpha-1}g^q(t)).\end{aligned}$$

We will remind an integral identity that was proved by Set in 2012.

Lemma 2.2 ([8]) *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then for all $x\in[a,b]$ and $\alpha>0$ we have:*

$$\begin{aligned}&\left(\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{b-a}[J^\alpha_{x^-}f(a)+J^\alpha_{x^+}f(b)]\\ &=\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^\alpha f'(tx+(1-t)a)dt-\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^\alpha f'(tx+(1-t)b)dt\end{aligned}$$

where $\Gamma(\alpha)=\int_0^\infty e^{-1}u^{\alpha-1}du$.

By using this identity, the author has been given Ostrowski-type integral inequalities for s-convex functions where Γ is Euler gamma function.

Theorem 2.5 ([8]) *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L[a,b]$. If $|f'|$ is s-convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$ and $|f'(x)|\leq M$, $x\in[a,b]$, then the following inequality for fractional integrals with $\alpha>0$ holds:*

$$\begin{aligned}&\left|\left(\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}[J^\alpha_{x^-}f(a)+J^\alpha_{x^+}f(b)]\right|\\ &\leq\frac{M}{b-a}\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{\alpha+s+1}\right].\end{aligned}$$

Theorem 2.6 ([8]) *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L[a,b]$. If $|f'|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$, $p,q>1$ and $|f'(x)|\leq M$, $x\in[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\left(\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}[J^\alpha_{x^-}f(a)+J^\alpha_{x^+}f(b)]\right|\\ &\leq\frac{M}{(1+p\alpha)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right]\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 2.7 ([8]) *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L[a,b]$. If $|f'|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$, $q\geq 1$, and $|f'(x)|\leq M$, $x\in[a,b]$, then the following inequality for fractional integrals holds:*

$$\left|\left(\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}[J_{x^-}^\alpha f(a)+J_{x^+}^\alpha f(b)]\right|$$
$$\le M\left(\frac{1}{1+\alpha}\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha+s+1}\right)^{\frac{1}{q}}$$
$$\times\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)^{\frac{1}{q}}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right]$$

where $\alpha>0$.

Sarıkaya and Öğünmez have extended the Montgomery identities for the Riemann–Liouville fractional integrals by using a different proof method; they have used these Montgomery identities to establish some new integral inequalities. The authors have also developed some integral inequalities for the fractional integral using differentiable convex functions.

Lemma 2.3 ([9]) *Let* $f: I\subset\mathbb{R}\to\mathbb{R}$ *be a differentiable function on* I^o *with* $a,b\in I\,(a<b)$ *and* $f'\in L_1[a,b]$, *then*

$$f(x)=\frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^\alpha f(b)-J_a^{\alpha-1}\left(P_2(x,b)f(b)\right)+J_a^\alpha\left(P_2(x,b)f'(b)\right),\quad \alpha\ge 1,$$

where $P_2(x,t)$ *is as in Lemma* 2.1

$$P_2(x,t):=\begin{cases}\frac{(t-a)}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & a\le t<x,\\ \frac{(t-b)}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & x\le t\le b.\end{cases}$$

Theorem 2.8 ([9]) *Let* $f: I\subset\mathbb{R}\to\mathbb{R}$ *be a differentiable function on* I^o *with* $f'\in L_1[a,b]$, *then the following identity holds:*

$$\begin{aligned}(1-2\lambda)f(x)=&\frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^\alpha f(b)-\lambda\left(\frac{b-a}{b-x}\right)^{\alpha-1}f(a)\\&-J_a^{\alpha-1}\left(P_3(x,b)f(b)\right)+J_a^\alpha\left(P_3(x,b)f'(b)\right),\quad \alpha\ge 1,\end{aligned}$$

where $P_3(x,t)$ *is the fractional Peano kernel defined by*

$$P_3(x,t):=\begin{cases}\frac{t-(1-\lambda)a-\lambda b}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & a\le t<x,\\ \frac{t-(1-\lambda)b-\lambda a}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & x\le t\le b.\end{cases}$$

for $0\le\lambda\le 1$.

Theorem 2.9 ([9]) *Let* $f:[a,b]\to\mathbb{R}$ *be differentiable on* (a,b) *such that* $f'\in L_1[a,b]$, *where* $a<b$. *If* $|f'(x)|\le M$ *for every* $x\in[a,b]$ *and* $\alpha\ge 1$, *then the following inequality holds:*

$$\Bigg|(1-2\lambda)f(x)-\frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^{\alpha}f(b)$$
$$+\lambda\left(\frac{b-a}{b-x}\right)^{\alpha-1}f(a)+J_a^{\alpha-1}\left(P_3(x,b)f(b)\right)\Bigg|$$
$$\leq\frac{M}{\alpha(\alpha+1)}\Bigg\{(b-a)^{\alpha}(b-x)^{1-\alpha}\left[2\lambda^{\alpha+1}+2(1-\lambda)^{\alpha+1}+\lambda(b-a)-1\right]$$
$$+(b-x)\left[2\alpha\frac{b-x}{b-a}-(\alpha+1)\right]\Bigg\}.$$

Theorem 2.10 ([9]) *Let $f:[a,b]\to\mathbb{R}$ be a differentiable convex function on (a,b) and $f'\in L_1[a,b]$. Then for any $x\in(a,b)$, the following inequality holds:*

$$\frac{1}{\alpha(\alpha+1)}\Bigg[\alpha\frac{(b-x)^2}{b-a}f'_{+}(x)-\Bigg((b-a)^{\alpha}(b-x)^{1-\alpha}$$
$$+\alpha\frac{(b-x)^2}{b-a}-(\alpha+1)(b-x)\Bigg)f'_{-}(x)\Bigg]$$
$$\leq\frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^{\alpha}f(b)-J_a^{\alpha-1}\left(P_2(x,b)f(b)\right)-f(x),\quad \alpha\geq 1$$

The fractional integral form of Hermite–Hadamard inequality was proved by Sarıkaya et al. in 2013 as follows.

Theorem 2.11 ([10]) *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is a convex function on [a,b], then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}\left[\mathrm{J}_{a^+}^{\alpha}f(b)+\mathrm{J}_{b^-}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2}$$

with $\alpha>0$.

In the same paper, the authors have given a new integral identity and generalized Dragomir and Agarwal's results to fractional calculus.

Lemma 2.4 ([10]) *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}\left[\mathrm{J}_{a^+}^{\alpha}f(b)+\mathrm{J}_{b^-}^{\alpha}f(a)\right]$$
$$=\frac{b-a}{2}\int_0^1[(1-t)^{\alpha}-t^{\alpha}]f'(ta+(1-t)b)dt.$$

Theorem 2.12 ([10]) *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is convex on [a,b], then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}\,\left[\mathrm{J}_{a^+}^{\alpha}f(b)+\mathrm{J}_{b^-}^{\alpha}f(a)\right]\right|$$
$$\leq \frac{b-a}{2(\alpha+1)}\big(1-\frac{1}{2^{\alpha}}\big)[f'(a)+f'(b)].$$

Tariboon et al. have proved some new Gruss-type inequalities involving Riemann–Liouville fractional integrals.

Theorem 2.13 ([11]) *Let f be integrable function on $[0,\infty)$. Assume that (H_1) there exist two integrable functions φ_1 and φ_2 on $[0,\infty)$ such that*

$$\varphi_1(t)\leq f(t)\leq \varphi_2(t),\quad \forall t\in[0,\infty),$$

Then, for $t>0,\ \alpha,\beta>0$, one has:

$$J^{\beta}\varphi_1(t)J^{\alpha}f(t)+J^{\alpha}\varphi_2(t)J^{\beta}f(t)\geq J^{\alpha}\varphi_2(t)J^{\beta}\varphi_1(t)+J^{\alpha}f(t)J^{\beta}f(t).$$

Theorem 2.14 ([11]) *Let f and g be two integrable functions on $[0,\infty)$. Suppose that (H_1) holds, and moreover, one assumes that (H_2) there exist ψ_1 and ψ_2 integrable functions on $[0,\infty)$ such that*

$$\psi_1(t)\leq g(t)\leq \psi_2(t),\quad \forall t\in[0,\infty),$$

Then for $t>0,\ \alpha,\beta>0$ the following inequalities hold:

$$(a)\quad J^{\beta}\psi_1(t)J^{\alpha}f(t)+J^{\alpha}\varphi_2(t)J^{\beta}g(t)\geq J^{\beta}\psi_1(t)J^{\alpha}\varphi_2(t)+J^{\alpha}f(t)J^{\beta}g(t),$$

$$(b)\quad J^{\beta}\varphi_1(t)J^{\alpha}g(t)+J^{\alpha}\psi_2(t)J^{\beta}f(t)\geq J^{\beta}\psi_1(t)J^{\alpha}\psi_2(t)+J^{\beta}f(t)J^{\beta}g(t),$$

$$(c)\quad J^{\alpha}\varphi_2(t)J^{\beta}\psi_2+J^{\alpha}f(t)J^{\beta}g(t)\geq J^{\alpha}\varphi_2(t)J^{\beta}g(t)+J^{\beta}\psi_2(t)J^{\alpha}f(t),$$

$$(d)\quad J^{\alpha}\varphi_1(t)J^{\beta}\psi_1+J^{\alpha}f(t)J^{\beta}g(t)\geq J^{\alpha}\varphi_1(t)J^{\beta}g(t)+J^{\beta}\psi_1(t)J^{\alpha}f(t).$$

Theorem 2.15 ([11]) *Let f and g be integrable functions on $[0,\infty)$ and let $\varphi_1,\varphi_2,\psi_1$ and ψ_2 be integrable functions on $[0,\infty)$, satisfying the conditions (H_1) and (H_2) on $[0,\infty)$. Then, for all $t>0,\alpha>0$, one has*

$$\left|\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}fg(t)-J^{\alpha}f(t)J^{\alpha}g(t)\right|\leq\sqrt{T(f,\varphi_1,\varphi_2)T(g,\psi_1,\psi_2)},$$

where $T(u, v, w)$ *is defined by*

$$\begin{aligned} T(u, v, w) &= (J^\alpha w(t) - J^\alpha u(t))(J^\alpha u(t) - J^\alpha v(t)) \\ &+ \frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha vu(t) - J^\alpha v(t) J^\alpha u(t) \\ &+ \frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha wu(t) - J^\alpha w(t) J^\alpha u(t) \\ &+ J^\alpha v(t) J^\alpha w(t) - \frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha vw(t). \end{aligned}$$

A new generalization of Montgomery identity has been given by Sarıkaya et al., and the authors have established new Ostrowski-type inequalities by using this identity as follows.

Throughout this study, we assume that Peano kernels defined by

$$K_1(x,t) = \begin{cases} \left[t - a - \frac{\lambda}{2}(x-a)\right], \ a \le t < x \\ \left[t - b + \frac{\lambda}{2}(b-x)\right], \ x \le t \le b \end{cases}$$

$$K_2(x,t) = \begin{cases} \frac{1}{b-a}\left[t - a - \frac{\lambda}{2}(x-a)\right](b-x)^{1-\alpha}\Gamma(\alpha), \ a \le t < x \\ \frac{1}{b-a}\left[t - b + \frac{\lambda}{2}(b-x)\right](b-x)^{1-\alpha}\Gamma(\alpha), \ x \le t \le b \end{cases}$$

$$h(x,t) = \begin{cases} \frac{1}{b-a}\left[t - a - \frac{\lambda}{2}(x-a)\right](b-x)^{1-\alpha}\Gamma(\alpha), \ a \le t < x \\ \frac{1}{b-a}\left[b - t + \frac{\lambda}{2}(b-x)\right](b-x)^{1-\alpha}\Gamma(\alpha), \ x \le t \le b. \end{cases}$$

Lemma 2.5 ([12]) *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I^0 *with* $a, b \in I\ (a < b),\ \alpha \ge 1,\ 0 \le \lambda \le 1$, *and* $f' \in L_1[a, b]$, *then the generalization of Montgomery identity for fractional integral holds:*

$$\begin{aligned} \left(1 - \frac{\lambda}{2}\right) f(x) &= J_a^\alpha (K_2(x,b) f'(b)) + \frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha) J_a^\alpha f(b) \\ &= -J_a^{\alpha-1}(K_2(x,b) f(b)) - \frac{\lambda}{2}(b-a)^{\alpha-2}(x-a)(b-x)^{\alpha-1} f(a) \end{aligned}$$

Theorem 2.16 ([12]) *Let* $f : [a, b] \to \mathbb{R}$ *be differentiable on* (a, b) *such that* $f' \in L_1[a, b]$, *where* $a < b$ *and* $0 \le \lambda \le 1$. *If* $|f'(x)| \le M$ *for every* $x \in [a, b]$ *and* $\alpha \ge 1$, *then the following Ostrowski fractional inequality holds:*

$$\begin{aligned} &\Bigg| \left(1 - \frac{\lambda}{2}\right) f(x) - \frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha) J_a^\alpha f(b) \\ &+ J_a^{\alpha-1}(K_2(x,b) f(b)) + \frac{\lambda}{2}(b-a)^{\alpha-2}(x-a)(b-x)^{\alpha-1} f(a) \Bigg| \\ &\le \frac{M}{\Gamma(\alpha)} A(x), \end{aligned}$$

where

$$A(x) = \frac{\Gamma(\alpha)(b-x)^{1-\alpha}}{b-a}\left\{(b-a)\left[\frac{2(b-a)+\lambda(x-a)}{2\alpha} - \frac{b-a}{\alpha+1}\right] + (b-x)^{\alpha}\left[\frac{2(b-x)}{\alpha+1} - \frac{(b-a)+\lambda(x-\frac{a+b}{2})}{\alpha}\right]\right\}.$$

Theorem 2.17 ([12]) *Let $f : [a,b] \to \mathbb{R}$ be differentiable on (a,b) such that $f' \in L_1[a,b]$, where $a < b$, $0 \leq \lambda \leq 1$, and $\alpha \geq 1$. If the mapping $|f'|^q$ is convex on $[a,b]$, $q \geq 1$, then the following fractional inequality holds:*

$$\left|\left(1-\frac{\lambda}{2}\right)f(x) - \frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha)J_a^{\alpha}f(b) + J_a^{\alpha-1}(K_2(x,b)f(b)) + \frac{\lambda}{2}(b-a)^{\alpha-2}(x-a)(b-x)^{\alpha-1}f(a)\right|$$
$$\leq \frac{1}{\Gamma(\alpha)}(A(x))^{1-\frac{1}{q}}(|f'(a)|^q B(x) + |f'(b)|^q\, C(x))^{\frac{1}{q}}$$

where

$$B(x) = \frac{\Gamma(\alpha)(b-a)^{1-\alpha}}{(b-a)^2}\left\{(b-a)^{\alpha+1}\left[\frac{2(b-a)+\lambda(x-a)}{2(\alpha+1)} - \frac{b-a}{\alpha+2}\right] + (b-x)^{\alpha+1}\left[\frac{2(b-x)}{\alpha+2} - \frac{(b-a)+\lambda(x-\frac{a+b}{2})}{\alpha+1}\right]\right\}$$

and

$$C(x) = \frac{\Gamma(\alpha)(b-x)^{1-\alpha}}{(b-a)}\left\{(b-a)^{\alpha}\left[\frac{2(b-a)+\lambda(x-a)}{2\alpha(\alpha+1)} - \left(\frac{b-a}{\alpha+1} - \frac{1}{\alpha+2}\right)\right] + 2(b-x)^{\alpha+1}\left(\frac{1}{\alpha+1} - \frac{(b-x)}{(\alpha+2)(b-a)}\right) - (b-x)^{\alpha}\left((b-a)+\lambda\left(x-\frac{a+b}{2}\right)\right)\left(\frac{b-x}{(b-a)(\alpha+1)} - \frac{1}{\alpha}\right)\right\}.$$

Set et al. have given a new integral identity by using Riemann–Liouville fractional integrals and proved several new Simpson-type integral inequalities that generalize previous results.

Lemma 2.6 ([13]) *$f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $f' \in L[a,b]$, $n \geq 0$, and $\alpha > 0$, then the following equality holds:*

$$
\begin{aligned}
I(a,b;n,\alpha) &= \frac{1}{6}\left[f(a)+f(b)+2f\left(\frac{a+nb}{n+1}\right)+2f\left(\frac{na+b}{n+1}\right)\right] \\
&\quad -\frac{\Gamma(\alpha+1)(n+1)^{\alpha}}{6(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(\frac{na+b}{n+1}\right)+J_{b^-}^{\alpha}f\left(\frac{a+nb}{n+1}\right)\right] \\
&\quad -\frac{\Gamma(\alpha+1)(n+1)^{\alpha}}{3(b-a)^{\alpha}}\left[J_{\frac{a+nb}{n+1}^+}^{\alpha}f(b)+J_{\frac{na+b}{n+1}^-}^{\alpha}f(a)\right] \\
&= \frac{b-a}{2(n+1)}\left(\int_0^1\left[\frac{2(1-t)^{\alpha}-t^{\alpha}}{3}\right]f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)dt\right. \\
&\quad \left.+\int_0^1\left[\frac{t^{\alpha}-2(1-t)^{\alpha}}{3}\right]f'\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)dt\right)
\end{aligned}
$$

for all $x \in [a,b]$ *and where* $\Gamma(\alpha)=\int_0^{\infty}e^{-u}u^{\alpha-1}du$.

Proof By using integration by parts, we have

$$
\begin{aligned}
I_1 &= \int_0^1\left[\frac{2(1-t)^{\alpha}-t^{\alpha}}{3}\right]f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)dt \\
&= \frac{n+1}{3(b-a)}\left[f(a)+2f\left(\frac{na+b}{n+1}\right)\right] \\
&\quad -\frac{\alpha(n+1)^{\alpha+1}}{3(b-a)^{\alpha+1}}\int_a^{\frac{na+b}{n+1}}f(x)\left(\frac{na+b}{n+1}-x\right)^{\alpha-1}dx \\
&\quad -\frac{2\alpha(n+1)^{\alpha+1}}{3(b-a)^{\alpha+1}}\int_a^{\frac{na+b}{n+1}}f(x)(x-a)^{\alpha-1}dx
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \int_0^1\left[\frac{t^{\alpha}-2(1-t)^{\alpha}}{3}\right]f'\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)dt \\
&= \frac{n+1}{3(b-a)}\left[f(b)+2f\left(\frac{a+nb}{n+1}\right)\right] \\
&\quad -\frac{\alpha(n+1)^{\alpha+1}}{3(b-a)^{\alpha+1}}\int_{\frac{a+nb}{n+1}}^{b}f(x)\left(x-\frac{a+nb}{n+1}\right)^{\alpha-1}dx \\
&\quad -\frac{2\alpha(n+1)^{\alpha+1}}{3(b-a)^{\alpha+1}}\int_{\frac{a+nb}{n+1}}^{b}f(x)(b-x)^{\alpha-1}dx.
\end{aligned}
$$

By adding I_1 and I_2 and multiplying the both sides $\frac{b-a}{2(n+1)}$, we can write

$$
\begin{aligned}
I_1 + I_2 = \frac{1}{6}\left[f(a) + f(b) + 2f\left(\frac{a+nb}{n+1}\right) + 2f\left(\frac{na+b}{n+1}\right)\right] \\
-\frac{\alpha\,(n+1)^\alpha}{6(b-a)^\alpha}\int_a^{\frac{na+b}{n+1}} f\,(x)\left(\frac{na+b}{n+1}-x\right)^{\alpha-1} dx \\
-\frac{\alpha\,(n+1)^\alpha}{3(b-a)^\alpha}\int_a^{\frac{na+b}{n+1}} f\,(x)\,(x-a)^{\alpha-1}\,dx \\
-\frac{\alpha\,(n+1)^\alpha}{6(b-a)^\alpha}\int_{\frac{a+nb}{n+1}}^{b} f\,(x)\left(x-\frac{a+nb}{n+1}\right)^{\alpha-1} dx \\
-\frac{\alpha\,(n+1)^\alpha}{3(b-a)^\alpha}\int_{\frac{a+nb}{n+1}}^{b} f\,(x)\,(b-x)^{\alpha-1}\,dx.
\end{aligned}
$$

From the facts that

$$
\begin{aligned}
\frac{1}{\Gamma\,(\alpha)}\int_a^{\frac{na+b}{n+1}} f\,(x)\,(x-a)^{\alpha-1}\,dx &= J^\alpha_{\frac{na+b}{n+1}^-} f\,(a) \\
\frac{1}{\Gamma\,(\alpha)}\int_{\frac{a+nb}{n+1}}^{b} f\,(x)\,(b-x)^{\alpha-1}\,dx &= J^\alpha_{\frac{a+nb}{n+1}^+} f\,(b) \\
\frac{1}{\Gamma\,(\alpha)}\int_a^{\frac{na+b}{n+1}} f\,(x)\left(\frac{na+b}{n+1}-x\right)^{\alpha-1} dx &= J^\alpha_{a^+} f\left(\frac{na+b}{n+1}\right) \\
\frac{1}{\Gamma\,(\alpha)}\int_{\frac{a+nb}{n+1}}^{b} f\,(x)\left(x-\frac{a+nb}{n+1}\right)^{\alpha-1} dx &= J^\alpha_{b^-} f\left(\frac{a+nb}{n+1}\right),
\end{aligned}
$$

we get the result. □

Theorem 2.18 ([13]) *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on $[a,b]$. If $f' \in L[a,b]$ and $\left|f'(x)\right|$ is convex function, then the following inequality holds for fractional integrals with $\alpha > 0$;*

$$
\begin{aligned}
&|I\,(a,b;n,\alpha)| \\
&\le \frac{b-a}{2\,(n+1)}\left[\frac{3-2\left(\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1} - 4\left(1-\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1}}{3\,(\alpha+1)}\right]\left(\left|f'(a)\right| + \left|f'(b)\right|\right)
\end{aligned}
$$

where $\Gamma\,(\alpha)$ is Euler gamma function.

Proof From the integral identity given in Lemma 1 and by using the properties of modulus, we have

$$\begin{aligned}
&|I(a,b;n,\alpha)| \\
&\le \frac{b-a}{2(n+1)}\left(\int_0^1\left|\frac{2(1-t)^\alpha-t^\alpha}{3}\right|\left|f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|dt\right. \\
&\quad\left.+\int_0^1\left|\frac{t^\alpha-2(1-t)^\alpha}{3}\right|\left|f'\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|dt\right).
\end{aligned}$$

Since $\left|f'(x)\right|$ is convex function, we can write

$$\begin{aligned}
&|I(a,b;n,\alpha)| \\
&\le \frac{b-a}{2(n+1)}\left(\int_0^1\left|\frac{2(1-t)^\alpha-t^\alpha}{3}\right|\left(\frac{n+t}{n+1}\left|f'(a)\right|+\frac{1-t}{n+1}\left|f'(b)\right|\right)dt\right. \\
&\quad\left.+\int_0^1\left|\frac{t^\alpha-2(1-t)^\alpha}{3}\right|\left(\frac{1-t}{n+1}\left|f'(a)\right|+\frac{n+t}{n+1}\left|f'(b)\right|\right)dt\right) \\
&= \frac{b-a}{2(n+1)}\left(\int_0^{\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}}\left(\frac{2(1-t)^\alpha-t^\alpha}{3}\right)\left(\frac{n+t}{n+1}\left|f'(a)\right|+\frac{1-t}{n+1}\left|f'(b)\right|\right)dt\right. \\
&\quad+\int_{\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}}^1\left(\frac{t^\alpha-2(1-t)^\alpha}{3}\right)\left(\frac{n+t}{n+1}\left|f'(a)\right|+\frac{1-t}{n+1}\left|f'(b)\right|\right)dt \\
&\quad+\int_0^{\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}}\left(\frac{2(1-t)^\alpha-t^\alpha}{3}\right)\left(\frac{1-t}{n+1}\left|f'(a)\right|+\frac{n+t}{n+1}\left|f'(b)\right|\right)dt \\
&\quad\left.+\int_{\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}}^1\left(\frac{t^\alpha-2(1-t)^\alpha}{3}\right)\left(\frac{1-t}{n+1}\left|f'(a)\right|+\frac{n+t}{n+1}\left|f'(b)\right|\right)dt\right).
\end{aligned}$$

By a simple computation, we obtain the desired result. □

Theorem 2.19 ([13]) *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on $[a,b]$. If $f'\in L[a,b]$ and $\left|f'(x)\right|^q$ is convex function, then the following inequality holds for fractional integrals with $\alpha>0$, $q>1$, and $p^{-1}+q^{-1}=1$;*

$$\begin{aligned}
&|I(a,b;n,\alpha)| \\
&\le \frac{b-a}{2(n+1)}\left(\int_0^1 \left|\frac{2(1-t)^\alpha - t^\alpha}{3}\right|^p dt\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{2n+1}{2(n+1)}\left|f'(a)\right|^q + \frac{1}{2(n+1)}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{1}{2(n+1)}\left|f'(a)\right|^q + \frac{2n+1}{2(n+1)}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

where $\Gamma(\alpha)$ is Euler gamma function.

Proof By using Lemma 1 and Hölder integral inequality, we can write

$$\begin{aligned}
&|I(a,b;n,\alpha)| \\
&\le \frac{b-a}{2(n+1)}\left(\int_0^1 \left|\frac{2(1-t)^\alpha - t^\alpha}{3}\right|\left|f'\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right)\right| dt\right. \\
&\left.+\int_0^1 \left|\frac{t^\alpha - 2(1-t)^\alpha}{3}\right|\left|f'\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right)\right| dt\right) \\
&\le \frac{b-a}{2(n+1)}\left(\left(\int_0^1 \left|\frac{2(1-t)^\alpha - t^\alpha}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 \left|\frac{t^\alpha - 2(1-t)^\alpha}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Since $\left|f'(x)\right|^q$ is convex function, we can write

$$\begin{aligned}
&|I(a,b;n,\alpha)| \\
&\le \frac{b-a}{2(n+1)}\left(\int_0^1 \left|\frac{2(1-t)^\alpha - t^\alpha}{3}\right|\left|f'\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right)\right| dt\right. \\
&\left.+\int_0^1 \left|\frac{t^\alpha - 2(1-t)^\alpha}{3}\right|\left|f'\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right)\right| dt\right) \\
&\le \frac{b-a}{2(n+1)}\left(\left(\int_0^1 \left|\frac{2(1-t)^\alpha - t^\alpha}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left(\frac{n+t}{n+1}\left|f'(a)\right|^q\right.\right.\right. \\
&\left.\left.+\frac{1-t}{n+1}\left|f'(b)\right|^q\right) dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_0^1\left|\frac{t^\alpha-2(1-t)^\alpha}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{1-t}{n+1}\left|f'(a)\right|^q\right.\right.$$
$$\left.\left.+\frac{n+t}{n+1}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\Bigg).$$

By taking into account,

$$\int_0^1\left(\frac{n+t}{n+1}\left|f'(a)\right|^q+\frac{1-t}{n+1}\left|f'(b)\right|^q\right)dt=\frac{2n+1}{2(n+1)}\left|f'(a)\right|^q+\frac{1}{2(n+1)}\left|f'(b)\right|^q$$
$$\int_0^1\left(\frac{1-t}{n+1}\left|f'(a)\right|^q+\frac{n+t}{n+1}\left|f'(b)\right|^q\right)dt=\frac{1}{2(n+1)}\left|f'(a)\right|^q+\frac{2n+1}{2(n+1)}\left|f'(b)\right|^q,$$

we obtain

$$\begin{aligned}&|I(a,b;n,\alpha)|\\&\le\frac{b-a}{2(n+1)}\left(\int_0^1\left|\frac{2(1-t)^\alpha-t^\alpha}{3}\right|^p dt\right)^{\frac{1}{p}}\\&\times\left[\left(\frac{2n+1}{2(n+1)}\left|f'(a)\right|^q+\frac{1}{2(n+1)}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.\\&\left.+\left(\frac{1}{2(n+1)}\left|f'(a)\right|^q+\frac{2n+1}{2(n+1)}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

which completes the proof. □

Theorem 2.20 ([13]) *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on $[a,b]$. If $f'\in L[a,b]$ and $\left|f'(x)\right|^q$ is convex function, then the following inequality holds for fractional integrals with $\alpha>0$ and $q\ge 1$;*

$$\begin{aligned}&|I(a,b;n,\alpha)|\\&\le\frac{b-a}{2(n+1)}\left(\frac{3-2\left(\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1}-4\left(1-\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1}}{3(\alpha+1)}\right)^{1-\frac{1}{q}}\\&\times\left(\left(K_1(\alpha,n)\left|f'(a)\right|^q+K_2(\alpha,n)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.\\&\left.+\left(K_2(\alpha,n)\left|f'(a)\right|^q+K_1(\alpha,n)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right).\end{aligned}$$

where $\Gamma(\alpha)$ is Euler gamma function and

$$K_1(\alpha, n) = \frac{(-4n-4)\left(1-\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1} + 3n + 2 - 2n\left(\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1}}{3(\alpha+1)(n+1)} + \frac{4\left(1-\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+2} - 2\left(\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+2} - 1}{3(\alpha+2)(n+1)}$$

$$K_2(\alpha, n) = \frac{1-2\left(\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1}}{3(\alpha+1)(n+1)} + \frac{1-4\left(1-\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+2} + 2\left(\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+2}}{3(\alpha+2)(n+1)}.$$

Proof By Lemma 1 and power-mean integral inequality, we can write

$$\begin{aligned} &|I(a, b; n, \alpha)| \\ &\leq \frac{b-a}{2(n+1)} \left(\int_0^1 \left|\frac{2(1-t)^\alpha - t^\alpha}{3}\right| dt\right)^{1-\frac{1}{q}} \\ &\times \left(\left(\int_0^1 \left|\frac{2(1-t)^\alpha - t^\alpha}{3}\right| \left|f'\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\ &\left. + \int_0^1 \left|\frac{t^\alpha - 2(1-t)^\alpha}{3}\right| \left|f'\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}}. \end{aligned}$$

By taking into account convexity of $\left|f'(x)\right|^q$, we get

$$\begin{aligned} &|I(a, b; n, \alpha)| \\ &\leq \frac{b-a}{2(n+1)} \left(\int_0^1 \left|\frac{2(1-t)^\alpha - t^\alpha}{3}\right| dt\right)^{1-\frac{1}{q}} \\ &\times \left(\left(\int_0^1 \left|\frac{2(1-t)^\alpha - t^\alpha}{3}\right| \left(\frac{n+t}{n+1}\left|f'(a)\right|^q + \frac{1-t}{n+1}\left|f'(b)\right|^q\right) dt\right)^{\frac{1}{q}}\right. \\ &\left. + \left(\int_0^1 \left|\frac{t^\alpha - 2(1-t)^\alpha}{3}\right| \left(\frac{1-t}{n+1}\left|f'(a)\right|^q + \frac{n+t}{n+1}\left|f'(b)\right|^q\right) dt\right)^{\frac{1}{q}}\right). \end{aligned}$$

Computing the above integrals, we get the result. □

Sarıkaya and Yıldırım have given a new refinement of Hermite–Hadamard inequality for Riemann–Liouville fractional integrals. They have proved an integral identity that gives some results for left side of Hermite–Hadamard inequality as follows.

Theorem 2.21 ([14]) *Let* $f : [a, b] \to \mathbb{R}$ *be a positive function with* $0 \leq a < b$ *and* $f \in L_1[a, b]$. *If* f *is a convex function on* $[a, b]$, *then the following inequalities for* *fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+b}{2})^{+}}f(b)+J^{\alpha}_{(\frac{a+b}{2})^{-}}f(a)\right] \leq \frac{f(a)+f(b)}{2},$$

with $\alpha > 0$.

Lemma 2.7 ([14]) *Let* $f : [a, b] \to \mathbb{R}$ *be a differentiable mapping on* (a, b) *with* $a < b$. *If* $f' \in L[a, b]$, *then the following equality for fractional integrals holds:*

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+b}{2})^{+}}f(b)+J^{\alpha}_{(\frac{a+b}{2})^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\\&=\frac{b-a}{4}\left\{\int_0^1 t^{\alpha}f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt-\int_0^1 t^{\alpha}f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right\},\end{aligned}$$

with $\alpha > 0$.

Theorem 2.22 ([14]) *Let* $f : [a, b] \to \mathbb{R}$ *be a differentiable mapping on* (a, b) *with* $a < b$. *If* $|f'|^q$ *is a convex function on* $[a, b]$ *for* $q \geq 1$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+b}{2})^{+}}f(b)+J^{\alpha}_{(\frac{a+b}{2})^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|\\&\leq\frac{(b-a)}{4(\alpha+1)}\left(\frac{1}{2(\alpha+2)}\right)^{\frac{1}{q}}\left\{\left[(\alpha+1)|f'(a)|^q+(\alpha+3)|f'(b)|^q\right]^{\frac{1}{q}}\right.\\&\left.+\left[(\alpha+3)|f'(a)|^q+(\alpha+1)|f'(b)|^q\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Theorem 2.23 ([14]) *Let* $f : [a, b] \to \mathbb{R}$ *be a differentiable mapping on* (a, b) *with* $a < b$. *If* $|f'|^q$ *is a convex on* $[a, b]$ *for* $q > 1$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+b}{2})^{+}}f(b)+J^{\alpha}_{(\frac{a+b}{2})^{-}}f(a)\right]-f(\frac{a+b}{2})\right|\\&\leq\frac{(b-a)}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left\{\left[\frac{|f'(a)|+3|f'(b)|}{4}\right]^{\frac{1}{q}}+\left[\frac{3|f'(a)|+|f'(b)|}{4}\right]^{\frac{1}{q}}\right\}\\&\leq\frac{b-a}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left[|f'(a)|+|f'(b)|\right],\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

3 Conformable Fractional Integrals and Inequalities

The conformable fractional derivative attracts attention with conformity to classical derivative. Khalil et al. have introduced the conformable fractional derivative by the equation which has a limit form similar to the classical derivative. Khalil et al. have proved that this definition provides multiplication and division rules. They also express the Rolle theorem and the mean value theorem for functions which are differentiable with conformable fractional order.

The analysis of the conformable fractional was developed by Abdeljawad. In his work, he has presented left and right conformable fractional derivative concepts, fractional chain rule, and Gronwall inequality for a conformable fractional derivative. We will mention the beta function (see [5]):

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 t^{a-1}(1-t)^{b-1}\,dt,\quad a,b>0,$$

where $\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}dt$ is gamma function.

Incomplete beta function is defined as:

$$B_x(a,b)=\int_0^x t^{a-1}(1-t)^{b-1}\,dt,\quad a,b>0.$$

In spite of its valuable contributions to mathematical analysis, the Riemann–Liouvile fractional integrals have deficiencies. For example, the solution of the differential equation is given as:

$$y^{(\frac{1}{2})}+y=x^{(\frac{1}{2})}+\frac{2}{\Gamma(2.5)}x^{(\frac{3}{2})},\quad y(0)=0$$

where $y^{(\frac{1}{2})}$ is the fractional derivative of y of order $\frac{1}{2}$.

The solution of the above differential equation has caused to imagine on a new and simple representation of the definition of fractional derivative. In [15] , Khalil et al. gave a new definition that is called “conformable fractional derivative.” They not only proved further properties of these definitions but also gave the differences with the other fractional derivatives. Besides, another considerable study has been presented by Abdeljawad to discuss the basic concepts of fractional calculus.

In [16], Abdeljawad gave the following definitions of right–left conformable fractional integrals:

Definition 3.1 Let $\alpha\in(n,n+1]$, $n=0,1,2,\ldots$ and set $\beta=\alpha-n$. Then, the left conformable fractional integral of any order $\alpha>0$ is defined by

$$(I_\alpha^a f)(t)=\frac{1}{n!}\int_a^t (t-x)^n(x-a)^{\beta-1}f(x)dx.$$

Definition 3.2 Analogously, the right conformable fractional integral of any order $\alpha > 0$ is defined by

$$({}^bI_\alpha f)(t) = \frac{1}{n!}\int_t^b (x-t)^n (b-x)^{\beta-1} f(x)dx.$$

Notice that if $\alpha = n+1$, then $\beta = \alpha - n = n+1-n = 1$; hence, $(I^a_{n+1}f)(t) = (J^{n+1}_{a+}f)(t)$ and $({}^bI_{n+1}f)(t) = \left(J^{n+1}_{b-}f\right)(t)$.

In [15, 16], the authors have pointed that the Riemann–Liouville derivatives are not valid for product of two functions. In this case, the inequalities that have been proved by Riemann–Liouville integrals are not valid. The results which are obtained by using the conformable fractional integrals have a wide range of validity. (Let us consider the function f defined as $f : \mathbb{R}^+ \to \mathbb{R}$, $f = x^2e^x$ which is convex.)

Several researchers have focused on new integral inequalities involving conformable fractional integrals in recent years. In [17], Set et al. have given some more general Hadamard-type inequalities for convex functions. Set, Akdemir, and Mumcu have proved several Ostrowski-type inequalities by using conformable fractional integrals involving special functions in [18]. In [19–21], the authors have obtained new inequalities of Hermite–Hadamard type associated with conformable fractional integrals. In [22], several new integral inequalities have been established via conformable fractional integrals for pre-invex functions by Awan et al. In [23], Sarıkaya and Budak have proved some Opial-type inequalities.

Set, Akdemir, and Mumcu have established a new form of Hermite–Hadamard inequality via conformable fractional integrals and also proved an extension of Hermite–Hadamard inequality as follows.

Theorem 3.1 ([24]) *Let $f : [a, b] \to \mathbb{R}$ be a mapping with $0 \le a < b$ and $f \in L_1[a, b]$. If f is a convex mapping on $[a, b]$, then one can obtain the following inequalities for conformable fractional integrals:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)}[(I^a_\alpha f)(b) + ({}^bI_\alpha f)(a)] \le \frac{f(a)+f(b)}{2}, \tag{3.1}$$

with $\alpha \in (n, n+1]$.

3.1 Extensions of HH-Inequality

Theorem 3.2 ([24]) *Assume that $f : [a, b] \to \mathbb{R}$ is a twice differentiable mapping with $a < b$ and $f \in L_1[a, b]$. If f'' is bounded on $[a, b]$, then we have*

$$
\begin{aligned}
&\frac{m\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2\\
&\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx\\
\leq\ &\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I_{\alpha}^{a}f)(b)+({}^{b}I_{\alpha}f)(a)]-f\left(\frac{a+b}{2}\right) \qquad (3.2)\\
\leq\ &\frac{M\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2,\\
&\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{-M\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}(x-a)(b-x)\\
&\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx\\
\leq\ &\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I_{\alpha}^{a}f)(b)+({}^{b}I_{\alpha}f)(a)]-\frac{f(a)+f(b)}{2} \qquad (3.3)\\
\leq\ &\frac{-m\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}(x-a)(b-x)\\
&\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx,
\end{aligned}
$$

with $\alpha \in (n, n+1]$, *where* $m = inf_{t\in[a,b]} f^{''}(t),\ \ M = sup_{t\in[a,b]} f^{''}(t)$.

It is obvious that $f^{''} \geq 0$ implies that $f^{'}$ non-decreasing. Therefore,

$$f^{'}(a+b-x) \geq f^{'}(x), \qquad (3.4)$$

holds for all $x \in [a, \frac{a+b}{2}]$. So, we establish the following theorem using inequality of (3.4).

Theorem 3.3 ([24]) *Let* $f : [a, b] \to \mathbb{R}$ *be a positive, differentiable mapping with* $a < b$ *and* $f \in L_1[a, b]$. *If* $f^{'}(a+b-x) \geq f^{'}(x)$ *for all* $x \in [a, \frac{a+b}{2}]$. *Then, the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I_{\alpha}^{a}f)(b)+({}^{b}I_{\alpha}f)(a)] \leq \frac{f(a)+f(b)}{2}.$$

The following results have been obtained by Set et. al. involving Ostrowski-type inequalities for conformable fractional integrals.

Lemma 3.1 ([25]) *Let* $f : [a, b] \to \mathbb{R}$ *be a differentiable mapping in the interior* I° *on* (a, b) *with* $a < b$. *If* $f^{'} \in L[a, b]$, *then for all* $x \in [a, b]$ *and* $\alpha \in [n, n+1)$ *we have:*

$$\begin{aligned}
&\frac{(x-a)^{\alpha+1}}{n!(b-a)}\int_0^1 B_t(n+1,\alpha-n)f^{'}(tx+(1-t)a)dt \\
&-\frac{(b-x)^{\alpha+1}}{n!(b-a)}\int_0^1 B_t(n+1,\alpha-n)f^{'}(tx+(1-t)b)dt \\
&=\frac{\Gamma(\alpha-n)[(x-a)^{\alpha}+(b-x)^{\alpha}]}{\Gamma(\alpha+1)(b-a)}f(x)-\frac{1}{b-a}[{}^{x}I_{\alpha}f(a)+I_{\alpha}^{x}f(b)],
\end{aligned} \tag{3.5}$$

where $\Gamma(\alpha)=\int_0^1 e^{-t}u^{\alpha-1}du$.

Theorem 3.4 ([25]) *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$ *such that* $f^{'}\in L[a,b]$. *If* $|f^{'}|$ *is convex and* $|f^{'}(x)|\le M$, $x\in[a,b]$, *then the following inequality for fractional integrals with* $\alpha\in[n,n+1)$ *holds:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha-n)[(x-a)^{\alpha}+(b-x)^{\alpha}]}{\Gamma(\alpha+1)(b-a)}f(x)-\frac{1}{b-a}[{}^{x}I_{\alpha}f(a)+I_{\alpha}^{x}f(b)]\right| \\
&\le\frac{M\Gamma(\alpha-n+1)}{\Gamma(\alpha+2)(b-a)}[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}].
\end{aligned} \tag{3.6}$$

Theorem 3.5 ([25]) *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$ *such that* $f^{'}\in L[a,b]$. *If* $|f^{'}|^{q}$ *is convex,* $p,q>1$, *and* $|f^{'}(x)|\le M$, $x\in[a,b]$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha-n)[(x-a)^{\alpha}+(b-x)^{\alpha}]}{\Gamma(\alpha+1)(b-a)}f(x)-\frac{1}{b-a}[{}^{x}I_{\alpha}f(a)+I_{\alpha}^{x}f(b)]\right| \\
&\le\frac{M}{n!(b-a)}[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]\left(\int_0^1 B_t(n+1,\alpha-n)^{p}dt\right)^{\frac{1}{p}},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha\in[n,n+1)$.

Theorem 3.6 ([25]) *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$ *such that* $f^{'}\in L[a,b]$. *If* $|f^{'}|^{q}$ *is convex,* $q\ge1$, *and* $|f^{'}(x)|\le M$, $x\in[a,b]$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha-n)[(x-a)^{\alpha}+(b-x)^{\alpha}]}{\Gamma(\alpha+1)(b-a)}f(x)-\frac{1}{b-a}[{}^{x}I_{\alpha}f(a)+I_{\alpha}^{x}f(b)]\right| \\
&\le M\frac{\Gamma(\alpha-n+1)}{\Gamma(\alpha+2)(b-a)}\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right],
\end{aligned}$$

where $\alpha\in[n,n+1)$.

Theorem 3.7 ([25]) *Let* $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ *be a differentiable mapping on (a,b) with* $a<b$ *such that* $f^{'}\in L[a,b]$. *If* $|f^{'}|^{q}$ *is a concave on* $[a,b]$ *and* $p,q>1$, *then the following inequality for conformable fractional integrals holds:*

$$\left|\frac{\Gamma(\alpha-n)[(x-a)^{\alpha}+(b-x)^{\alpha}]}{\Gamma(\alpha+1)(b-a)}f(x)-\frac{1}{b-a}[{}^{x}I_{\alpha}f(a)+I_{\alpha}^{x}f(b)]\right|$$
$$\leq\left(\int_0^1 B_t(n+1,\alpha-n)^p dt\right)^{\frac{1}{p}}$$
$$\times\left[\frac{(x-a)^{\alpha+1}}{n!(b-a)}\left|f^{'}\left(\frac{x+a}{2}\right)\right|+\frac{(b-x)^{\alpha+1}}{n!(b-a)}\left|f^{'}\left(\frac{b+x}{2}\right)\right|\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha\in[n,n+1)$.

In [26], Akdemir, Ekinci, and Set have proved some inequalities involving conformable fractional integral operators as follows:

Theorem 3.8 *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$ *such that* $f'\in L[a,b]$. *Suppose that there exist two integrable functions* φ_1,φ_2 *on* $[a,b]$ *such that*

$$\varphi_1(t)\leq f(t)\leq\varphi_2(t),\quad\forall t\in[a,b]. \tag{3.7}$$

Then, the inequality

$${}^{x}I_{\alpha}\varphi_2(a)\,I_{\alpha}^{x}f(b)+I_{\alpha}^{x}\varphi_1(b)^{x}\,I_{\alpha}f(a)\geq I_{\alpha}^{x}\varphi_1(b)^{x}\,I_{\alpha}\varphi_2(a)+I_{\alpha}^{x}f(b)^{x}\,I_{\alpha}f(a)$$

holds true, where $x\in[a,b]$.

Proof From the inequality (3.7), for all $u,v\in[a,b]$, we have

$$(\varphi_2(u)-f(u))(f(v)-\varphi_1(v))\geq 0.$$

This implies that

$$\varphi_2(u)f(v)+\varphi_1(v)f(u)\geq\varphi_1(v)\varphi_2(u)+f(u)f(v).$$

For $x\in[a,b]$, if we use the change of variables $u=rx+(1-r)a$ and $v=sx+(1-s)b$ for $r,s\in[0,1]$ and multiply both sides of the above inequality by

$$\left[r^n(1-r)^{\alpha-n-1}\right]\left[s^n(1-s)^{\alpha-n-1}\right],$$

later by integrating the resulting expression with respect to r and s, we have the following equality for the first integral

$$\int_0^1\int_0^1\left[r^n(1-r)^{\alpha-n-1}\right]\left[s^n(1-s)^{\alpha-n-1}\right]\varphi_2(rx+(1-r)a)f(sx+(1-s)b)\,drds$$
$$=\int_0^1\left[s^n(1-s)^{\alpha-n-1}\right]f(sx+(1-s)b)\,ds\int_0^1\left[r^n(1-r)^{\alpha-n-1}\right]\varphi_2(rx+(1-r)a)\,dr.$$

By using the change of variables above, we get

$$\int_0^1 \left[s^n (1-s)^{\alpha-n-1}\right] f(sx+(1-s)b)\,ds \int_0^1 \left[r^n (1-r)^{\alpha-n-1}\right] \varphi_2(rx+(1-r)a)\,dr$$
$$= \left[\int_x^b \left(\frac{x-v}{x-b}\right)^n \left(\frac{v-b}{x-b}\right)^{\alpha-n-1} \frac{f(v)}{x-b}dv\right]$$
$$\left[\int_a^x \left(\frac{u-a}{x-a}\right)^n \left(\frac{x-u}{x-a}\right)^{\alpha-n-1} \frac{\varphi_2(u)}{x-a}du\right]$$
$$= \left[\frac{1}{(b-x)^\alpha}\int_x^b (x-v)^n (v-b)^{\alpha-n-1} f(v)\,dv\right]$$
$$\left[\frac{1}{(x-a)^\alpha}\int_a^x (u-a)^n (x-u)^{\alpha-n-1} \varphi_2(u)\,du\right]$$
$$= (n!)^2\, {}^x I_\alpha \varphi_2(a)\, I_\alpha^x f(b).$$

If we proceed the similar methods for the other integrals, we deduce the desired result. □

Theorem 3.9 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ such that $f' \in L[a,b]$. Suppose that $m \le f(t) \le M$, for all $t \in [a,b]$ and for some $m, M \in \mathbb{R}$. Then, the following inequality holds:*

$$\frac{(n!)\, m I_\alpha^x f(b)}{(b-x)^\alpha} + \frac{(n!)\, M {}^x I_\alpha f(a)}{(x-a)^\alpha} \ge \frac{(n!)^2 I_\alpha^x f(b) {}^x I_\alpha f(a)}{(b-x)^\alpha (x-a)^\alpha} + B(n+1, \alpha-n)\, mM.$$

Proof Since

$$m \le f(t) \le M,$$

for all $t, u, v \in [a,b]$, we have

$$(m - f(u))(f(v) - M) \ge 0.$$

By using the above inequality and a similar argument to the proof of Theorem 3.8, we get the desired result. □

Theorem 3.10 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ such that $f' \in L[a,b]$. Suppose that there exist two integrable functions φ_1, φ_2 on $[a,b]$ such that*

$$\varphi_1(t) \le f(t) \le \varphi_2(t), \quad \forall t \in [a,b].$$

Then, the inequality

$$^{x}I_{\alpha}\varphi_{2}(a)\,I_{\alpha}^{x}f(b)+I_{\alpha}^{x}\varphi_{1}(b)^{x}I_{\alpha}f(a)\geq I_{\alpha}^{x}\varphi_{1}(b)^{x}I_{\alpha}\varphi_{2}(a)+I_{\alpha}^{x}f(b)^{x}I_{\alpha}f(a)$$

holds true, where $x\in[a,b]$.

Proof From inequality (3.7), for all $u,v\in[a,b]$, we have

$$\left(\varphi_{2}(u)-f'(u)\right)\left(f'(v)-\varphi_{1}(v)\right)\geq 0.$$

This implies

$$\varphi_{2}(u)\,f'(v)+\varphi_{1}(v)\,f'(u)\geq\varphi_{1}(v)\,\varphi_{2}(u)+f'(u)\,f'(v).$$

For $x\in[a,b]$, if we use the change of variables $u=rx+(1-r)a$ and $v=sb+(1-s)x$ for $r,s\in[0,1]$ and multiply both sides of the above inequality by

$$B_{r}(n+1,\alpha-n)\,B_{s}(n+1,\alpha-n),$$

and then integrate it with respect to r and s, for the first integral, we have

$$\int_{0}^{1}\int_{0}^{1}B_{r}(n+1,\alpha-n)\,B_{s}(n+1,\alpha-n)\,\varphi_{2}(rx+(1-r)a)\,f'(sb+(1-s)x)\,drds$$
$$=\int_{0}^{1}B_{s}(n+1,\alpha-n)\,f'(sb+(1-s)x)\,ds\int_{0}^{1}B_{r}(n+1,\alpha-n)\,\varphi_{2}(rx+(1-r)a)\,dr.$$

By using integration by parts and the change of variables above, we get

$$\int_{0}^{1}B(n+1,\alpha-n)\,f'(sb+(1-s)x)\,ds$$
$$=B_{s}(n+1,\alpha-n)\left.\frac{f(sb+(1-s)x)}{b-x}\right|_{0}^{1}-\int_{0}^{1}s^{n}(1-s)^{\alpha-n-1}\,\frac{f(sb+(1-s)x)}{b-x}ds$$
$$=B(n+1,\alpha-n)\,\frac{f(b)}{b-x}-\frac{1}{b-x}\int_{x}^{b}\left(\frac{v-x}{b-x}\right)^{n}\left(\frac{b-v}{b-x}\right)^{\alpha-n-1}f(v)\,dv$$
$$=\frac{\Gamma(n+1)\,\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\frac{f(b)}{b-x}-\frac{1}{(b-x)^{\alpha+1}}\int_{x}^{b}(v-x)^{n}(b-v)^{\alpha-n-1}f(v)\,dv$$
$$=B(n+1,\alpha-n)\,\frac{f(b)\,n!}{b-x}-\frac{n!}{(b-x)^{\alpha+1}}I_{\alpha}^{x}f(b),$$

and

$$\int_0^1 B_r\left(n+1,\alpha-n\right)\varphi_2\left(rx+(1-r)a\right)dr$$
$$=B\left(n+1,\alpha-n\right)\int_a^x \varphi_2\left(u\right)du.$$

By changing of the variables above, we get

$$=\left[\int_x^b\left(\frac{x-v}{x-b}\right)^n\left(\frac{v-b}{x-b}\right)^{\alpha-n-1}\frac{f\left(v\right)}{x-b}dv\right]$$
$$\left[\int_a^x\left(\frac{u-a}{x-a}\right)^n\left(\frac{x-u}{x-a}\right)^{\alpha-n-1}\frac{f\left(u\right)}{x-a}du\right]$$
$$=\left[\frac{1}{(b-x)^\alpha}\int_x^b (x-v)^n(v-b)^{\alpha-n-1}dv\right]$$
$$\left[\frac{1}{(x-a)^\alpha}\int_a^x (u-a)^n(x-u)^{\alpha-n-1}du\right]$$
$$=(n!)^2\,{}^xI_\alpha\varphi_2\left(a\right)I_\alpha^x f\left(b\right).$$

By using the similar methods for the other integrals, we deduce the desired result.□

Theorem 3.11 *Let $f,g:[a,b]\to\mathbb{R}$ be two Lipschitzian mappings with the constants $L_1>0$ and $L_2>0$, i.e.,*

$$\left|f\left(x\right)-f\left(y\right)\right|\le L_1\left|x-y\right|,\ \ \left|g\left(x\right)-g\left(y\right)\right|\le L_2\left|x-y\right|, \tag{3.8}$$

for all $x,y\in[a,b]$. Then, the following inequality holds for conformable fractional integrals

$$\Big|\Gamma\left(\alpha-n\right)\left[{}^xI_\alpha\left(fg\right)\left(a\right)+I_\alpha^x\left(fg\right)\left(b\right)\right]$$
$$-\Gamma\left(n+1\right)\left[{}^xI_\alpha g\left(a\right)I_\alpha^x f\left(b\right)+I_\alpha^x g\left(b\right){}^xI_\alpha f\left(a\right)\right]\Big|$$
$$\le\frac{L_1L_2}{\Gamma\left(n+1\right)}\Bigg[\frac{B\left(n+1,\alpha-n\right)}{(x-a)^\alpha}K_1+\frac{B\left(n+1,\alpha-n\right)}{(b-x)^\alpha}K_2$$
$$-\frac{2}{(x-a)^\alpha(b-x)^\alpha}K_3K_4\Bigg],$$

where

$$K_1 = \frac{(x-a)^\alpha}{\Gamma(\alpha+3)} \Big((n+1)(n+2)x^2 - 2ax(n+1)(n-\alpha) + a^2(n-\alpha)(n-\alpha-1) \Big),$$

$$K_2 = \frac{\Gamma(n+1)\Gamma(\alpha-n)(x-b)^\alpha}{\Gamma(\alpha+3)} \times \Big((n+1)(n+2)b^2 - 2bx(n+1)(\alpha-n) + x^2(n-\alpha)(n-\alpha-1) \Big),$$

$$K_3 = \frac{\Gamma(n+1)\Gamma(\alpha-n)(x-a)^\alpha}{\Gamma(\alpha+3)} \times \Big((n+1)(n+2)x^2 - 2ax(n+1)(n-\alpha) + a^2(n-\alpha)(n-\alpha-1) \Big),$$

$$K_4 = \frac{\Gamma(n+1)\Gamma(\alpha-n)(x-b)^\alpha}{\Gamma(\alpha+2)} ((n+1)b - x(n-\alpha)).$$

Proof By (3.8), we can write

$$|(f(x) - f(y))(g(x) - g(y))| \le L_1 L_2 (x-y)^2$$

for all $x, y \in [a,b]$. For $x \in [a,b]$, if we use the change of variables $u = rx + (1-r)a$ and $v = sx + (1-s)b$ for $r, s \in [0,1]$ and multiply both sides of the above inequality by $\left[r^n(1-r)^{\alpha-n-1}\right]\left[s^n(1-s)^{\alpha-n-1}\right]$, we get

$$\begin{aligned} &\left[r^n(1-r)^{\alpha-n-1}\right]\left[s^n(1-s)^{\alpha-n-1}\right][|f(rx+(1-r)a)g(rx+(1-r)a) \\ &f(sx+(1-s)b)g(sx+(1-s)b) - f(rx+(1-r)a)g(sx+(1-s)b) \\ &+ f(sx+(1-s)b)g(rx+(1-r)a)|] \\ \le &\left[r^n(1-r)^{\alpha-n-1}\right]\left[s^n(1-s)^{\alpha-n-1}\right] L_1L_2((rx+(1-r)a) - (sx+(1-s)b))^2. \end{aligned}$$

Then by integrating the resulting inequality with respect to r and s, we have

$$\begin{aligned} &\int_0^1\int_0^1 \left[r^n(1-r)^{\alpha-n-1}\right]\left[s^n(1-s)^{\alpha-n-1}\right][|f(rx+(1-r)a)g(rx+(1-r)a) \\ &f(sx+(1-s)b)g(sx+(1-s)b) - f(rx+(1-r)a)g(sx+(1-s)b) \\ &+ f(sx+(1-s)b)g(rx+(1-r)a)|]\,drds \\ &\le L_1L_2\int_0^1\int_0^1 \left[r^n(1-r)^{\alpha-n-1}\right]\left[s^n(1-s)^{\alpha-n-1}\right] \\ &\quad ((rx+(1-r)a) - (sx+(1-s)b))^2\,drds. \end{aligned}$$

By computing above integrals and by using the definition of conformable fractional integrals, we get the result. □

Theorem 3.12 *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a function such that $f \in L_1[a, b]$, where $a, b \in I$ with $a < b$. If f is GA-convex function on $[a, b]$, we have the following inequalities for conformable fractional integrals:*

$$f\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^{\alpha}\Gamma(\alpha-n)}[I_{\alpha}^{a} f(b) +^{b} I_{\alpha} f(a)] \leq \frac{f(a)+f(b)}{2}.$$

Proof Since f is GA-convex function on $[a, b]$, we have

$$f\left(\sqrt{xy}\right) \leq \frac{f(x)+f(y)}{2}.$$

for all $x, y \in [a, b]$ (with $t = \frac{1}{2}$ in the definition of GA-convexity). By setting $x = a^t b^{1-t}$ and $y = b^t a^{1-t}$, we get

$$2f\left(\sqrt{ab}\right) \leq f\left(a^t b^{1-t}\right) + f\left(b^t a^{1-t}\right).$$

By multiplying both sides of this inequality by $\frac{1}{n!}t^n(1-t)^{\alpha-n-1}$, then integrating the resulting inequality with respect to t over [0, 1], we obtain

$$\begin{aligned}
&\frac{2}{n!} f\left(\sqrt{ab}\right) \int_0^1 t^n (1-t)^{\alpha-n-1} dt \\
&\leq \frac{1}{n!} \int_0^1 t^n (1-t)^{\alpha-n-1} f\left(a^t b^{1-t}\right) dt \\
&+ \frac{1}{n!} \int_0^1 t^n (1-t)^{\alpha-n-1} f\left(b^t a^{1-t}\right) dt.
\end{aligned}$$

Namely,

$$f\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{2(\ln \frac{b}{a})^{\alpha}\Gamma(\alpha-n)}[I_{\alpha}^{a} f(b) +^{b} I_{\alpha} f(a)],$$

which completes the proof of the first inequality. For the proof of the second inequality, we can write

$$f\left(a^t b^{1-t}\right) \leq t f(a) + (1-t) f(b)$$

and

$$f\left(b^t a^{1-t}\right) \leq t f(b) + (1-t) f(a).$$

By adding these inequalities, we have

$$f\left(a^t b^{1-t}\right) + f\left(b^t a^{1-t}\right) \le f(a) + f(b).$$

Multiplying both sides of this inequality by $\frac{1}{n!}t^n(1-t)^{\alpha-n-1}$, then integrating the resulting inequality with respect to t over $[0, 1]$, we deduce

$$\frac{\Gamma(\alpha+1)}{(\ln b - \ln a)^\alpha \Gamma(\alpha - n)}[I_\alpha^a f(b) +^b I_\alpha f(a)] \le f(a) + f(b).$$

This completes the proof. □

Lemma 3.2 *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on I^o such that $f' \in L_1[a, b]$, where $a, b \in I$ with $a < b$. Then, for all $x \in [a, b]$ and $\alpha \in (n, n+1]$, we have*

$$\begin{aligned}
&\left(\ln \frac{x}{a}\right)^\alpha \int_0^1 B_t(n+1, \alpha - n)\, df\left(x^t a^{1-t}\right) \\
&+ \left(\ln \frac{b}{x}\right)^\alpha \int_0^1 B_t(n+1, \alpha - n)\, df\left(b^t x^{1-t}\right) \\
&= \frac{\Gamma(n+1)\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\left[\left(\ln \frac{x}{a}\right)^\alpha f(x) + \left(\ln \frac{b}{x}\right)^\alpha f(b)\right] \\
&\quad - n![I_\alpha^{\ln a}(f \circ \exp)(\ln x) +^{\ln b} I_\alpha (f \circ \exp)(\ln x)].
\end{aligned}$$

Proof By using integration by parts in the left-hand side of the above inequality, one can obtain the right-hand side. We omit the details. □

For simplicity, we will use following notation

$$\begin{aligned}
&F_f(\alpha, n; x) \\
&= \frac{\Gamma(n+1)\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\left[\left(\ln \frac{x}{a}\right)^\alpha f(x) + \left(\ln \frac{b}{x}\right)^\alpha f(b)\right] \\
&\quad - n![I_\alpha^{\ln a}(f \circ \exp)(\ln x) +^{\ln b} I_\alpha (f \circ \exp)(\ln x)].
\end{aligned}$$

Theorem 3.13 *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on I^o such that $f' \in L_1[a, b]$, where $a, b \in I$ with $a < b$. If $\left|f'\right|^q$ is quasi-geometrically convex on $[a, b]$ and $q > 1$, then we have the following inequality for conformable fractional integrals*

$$
\begin{aligned}
&\left|F_f(\alpha, n; x)\right| \\
&\leq \left(\int_0^1 \left|B_t\left(n+1, \alpha-n\right)\right|^p dt\right)^{\frac{1}{p}} \left(\ln\frac{x}{a}\right)^{\alpha} L^{\frac{1}{q}}\left(a^q, x^q\right) \sup\left\{\left|f'(x)\right|, \left|f'(a)\right|\right\} \\
&\quad + \left(\int_0^1 \left|B_t\left(n+1, \alpha-n\right)\right|^p dt\right)^{\frac{1}{p}} \left(\ln\frac{b}{x}\right)^{\alpha} L^{\frac{1}{q}}\left(x^q, b^q\right) \sup\left\{\left|f'(b)\right|, \left|f'(x)\right|\right\},
\end{aligned}
$$

for all $x \in [a, b]$, $p^{-1} + q^{-1} = 1$ *and* $\alpha \in (n, n+1]$.

Proof By using Lemma 3.2 and by applying Hölder integral inequality, we can write

$$
\begin{aligned}
&\left|F_f(\alpha, n; x)\right| \\
&\leq \left(\ln\frac{x}{a}\right)^{\alpha} \int_0^1 B_t\left(n+1, \alpha-n\right) df\left(x^t a^{1-t}\right) \\
&\quad + \left(\ln\frac{b}{x}\right)^{\alpha} \int_0^1 B_t\left(n+1, \alpha-n\right) df\left(b^t x^{1-t}\right) \\
&\leq \left(\ln\frac{x}{a}\right)^{\alpha} \left(\int_0^1 \left|B_t\left(n+1, \alpha-n\right)\right|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 x^{qt} a^{q(1-t)} \left|f'\left(x^t a^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad + \left(\ln\frac{b}{x}\right)^{\alpha} \left(\int_0^1 \left|B_t\left(n+1, \alpha-n\right)\right|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 b^{qt} x^{q(1-t)} \left|f'\left(b^t x^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

Since $\left|f'\right|^q$ is quasi-geometrically convex, we get

$$
\begin{aligned}
&\left|F_f(\alpha, n; x)\right| \\
&\leq \left(\ln\frac{x}{a}\right)^{\alpha} \sup\left\{\left|f'(x)\right|, \left|f'(a)\right|\right\} \left(\int_0^1 \left|B_t\left(n+1, \alpha-n\right)\right|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 x^{qt} a^{q(1-t)} dt\right)^{\frac{1}{q}} \\
&\quad + \left(\ln\frac{b}{x}\right)^{\alpha} \sup\left\{\left|f'(b)\right|, \left|f'(x)\right|\right\} \left(\int_0^1 \left|B_t\left(n+1, \alpha-n\right)\right|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 b^{qt} x^{q(1-t)} dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

By computing the above integrals, one can easily obtain the desired inequality.□

Theorem 3.14 *Let* $f : I \subset (0, \infty) \to \mathbb{R}$ *be a differentiable function on* I^o *such that* $f' \in L_1[a, b]$, *where* $a, b \in I$ *with* $a < b$. *If* $\left|f'\right|^q$ *is quasi-geometrically convex on*

$[a, b]$, *then we have the following inequality for conformable fractional integrals*

$$
\begin{aligned}
&\left|F_f(\alpha, n; x)\right| \\
\leq &\left(\ln \frac{x}{a}\right)^{\alpha} \sup\left\{\left|f'(x)\right|, \left|f'(a)\right|\right\} \\
&\times \left(\frac{n!\Gamma(\alpha-n+1)}{\Gamma(\alpha+2)}\right)^{1-\frac{1}{q}} \left(\int_0^1 \left|B_t(n+1, \alpha-n)\right| x^{qt} a^{q(1-t)} dt\right)^{\frac{1}{q}} \\
&+ \left(\ln \frac{b}{x}\right)^{\alpha} \sup\left\{\left|f'(b)\right|, \left|f'(x)\right|\right\} \\
&\times \left(\frac{n!\Gamma(\alpha-n+1)}{\Gamma(\alpha+2)}\right)^{1-\frac{1}{q}} \left(\int_0^1 \left|B_t(n+1, \alpha-n)\right| b^{qt} x^{q(1-t)} dt\right)^{\frac{1}{q}},
\end{aligned}
$$

for all $x \in [a, b]$, $\alpha \in (n, n+1]$ *where* $q \geq 1$.

Proof From Lemma 3.2 and the power-mean integral inequality, we have

$$
\begin{aligned}
&\left|F_f(\alpha, n; x)\right| \\
\leq &\left(\ln \frac{x}{a}\right)^{\alpha} \int_0^1 B_t(n+1, \alpha-n)\, df\left(x^t a^{1-t}\right) \\
&+ \left(\ln \frac{b}{x}\right)^{\alpha} \int_0^1 B_t(n+1, \alpha-n)\, df\left(b^t x^{1-t}\right) \\
\leq &\left(\ln \frac{x}{a}\right)^{\alpha} \left(\int_0^1 \left|B_t(n+1, \alpha-n)\right| dt\right)^{1-\frac{1}{q}} \\
&\left(\int_0^1 \left|B_t(n+1, \alpha-n)\right| x^{qt} a^{q(1-t)} \left|f'\left(x^t a^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+ \left(\ln \frac{b}{x}\right)^{\alpha} \left(\int_0^1 \left|B_t(n+1, \alpha-n)\right| dt\right)^{1-\frac{1}{q}} \\
&\left(\int_0^1 \left|B_t(n+1, \alpha-n)\right| b^{qt} x^{q(1-t)} \left|f'\left(b^t x^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

By taking into account quasi-geometrically convexity of $\left|f'\right|^q$, we obtain

$$
\begin{aligned}
&\left|F_f(\alpha, n; x)\right| \\
&\leq \tau^{1-\frac{1}{q}} \left(\ln \frac{x}{a}\right)^{\alpha} \left(\int_0^1 \left|B_t\left(n+1, \alpha-n\right)\right| x^{qt} a^{q(1-t)} dt\right)^{\frac{1}{q}} \\
&\quad \times \sup\left\{\left|f'(x)\right|, \left|f'(a)\right|\right\} \\
&\quad + \tau^{1-\frac{1}{q}} \left(\ln \frac{b}{x}\right)^{\alpha} \left(\int_0^1 \left|B_t\left(n+1, \alpha-n\right)\right| b^{qt} x^{q(1-t)} dt\right)^{\frac{1}{q}} \\
&\quad \times \sup\left\{\left|f'(b)\right|, \left|f'(x)\right|\right\}.
\end{aligned}
$$

Then, use the following formula:

$$
\begin{aligned}
\int_0^1 \left|B_t\left(n+1, \alpha-n\right)\right| dt &= B\left(n+1, \alpha-n\right) - B\left(n+2, \alpha-n\right) \\
&= \frac{n!\Gamma\left(\alpha-n+1\right)}{\Gamma\left(\alpha+2\right)}.
\end{aligned}
$$

This completes the proof. □

Corollary 3.1 *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on I^o such that $f' \in L_1[a, b]$, where $a, b \in I$ with $a < b$. If $\left|f'\right|^q$ is quasi-geometrically convex on $[a, b]$ and $q > 1$, then we have the following inequality for conformable fractional integrals*

$$
\begin{aligned}
&\left|F_f(\alpha, n; x)\right| \\
&\leq \left(\int_0^1 \left|B_t\left(n+1, \alpha-n\right)\right|^p dt\right)^{\frac{1}{p}} \left(\ln \frac{x}{a}\right)^{\alpha} L_q^{\frac{1}{q}}\left(a, x\right) \sup\left\{\left|f'(x)\right|, \left|f'(a)\right|\right\} \\
&\quad + \left(\int_0^1 \left|B_t\left(n+1, \alpha-n\right)\right|^p dt\right)^{\frac{1}{p}} \left(\ln \frac{b}{x}\right)^{\alpha} L_q^{\frac{1}{q}}\left(x^q, b^q\right) \sup\left\{\left|f'(b)\right|, \left|f'(x)\right|\right\},
\end{aligned}
$$

for all $x \in [a, b]$, $p^{-1} + q^{-1} = 1$ and $\alpha \in (n, n+1]$.

Proof By using a similar argument as in the proof of Theorem 3.14, we can write

$$\begin{aligned} &\left|F_f(\alpha, n; x)\right| \\ &\leq \left(\ln \frac{x}{a}\right)^{\alpha} \int_0^1 B_t\left(n+1, \alpha-n\right) df\left(x^t a^{1-t}\right) \\ &\quad + \left(\ln \frac{b}{x}\right)^{\alpha} \int_0^1 B_t\left(n+1, \alpha-n\right) df\left(b^t x^{1-t}\right). \end{aligned}$$

By using the general Cauchy inequality, we have

$$\begin{aligned} &\left|F_f(\alpha, n; x)\right| \\ &\leq \left(\ln \frac{x}{a}\right)^{\alpha} \int_0^1 B_t\left(n+1, \alpha-n\right)(tx+(1-t)a)\left|f'\left(x^t a^{1-t}\right)\right| dt \\ &\quad + \left(\ln \frac{b}{x}\right)^{\alpha} \int_0^1 B_t\left(n+1, \alpha-n\right)(tb+(1-t)x)\left|f'\left(b^t x^{1-t}\right)\right| dt. \end{aligned}$$

By applying the Hölder integral inequality and from quasi-geometrically convexity of $\left|f'\right|^q$, we obtain

$$\begin{aligned} &\left|F_f(\alpha, n; x)\right| \\ &\leq \left(\ln \frac{x}{a}\right)^{\alpha} \sup\left\{\left|f'(x)\right|, \left|f'(a)\right|\right\} \left(\int_0^1 \left|B_t\left(n+1, \alpha-n\right)\right|^p dt\right)^{\frac{1}{p}} \\ &\quad \left(\int_0^1 (tx+(1-t)a)^q \, dt\right)^{\frac{1}{q}} \\ &\quad + \left(\ln \frac{b}{x}\right)^{\alpha} \sup\left\{\left|f'(b)\right|, \left|f'(x)\right|\right\} \left(\int_0^1 \left|B_t\left(n+1, \alpha-n\right)\right|^p dt\right)^{\frac{1}{p}} \\ &\quad \left(\int_0^1 (tb+(1-t)x)^q \, dt\right)^{\frac{1}{q}}. \end{aligned}$$

By computing the above integrals, we get the result. □

Corollary 3.2 *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on I^o such that $f' \in L_1[a, b]$, where $a, b \in I$ with $a < b$. If $\left|f'\right|^q$ is quasi-geometrically convex on $[a, b]$, then we have the following inequality for conformable fractional integrals*

$$\begin{aligned}
&\left|F_f(\alpha,n;x)\right| \\
&\leq \left(\ln\frac{x}{a}\right)^{\alpha}\sup\left\{\left|f'(x)\right|,\left|f'(a)\right|\right\}\left(\frac{n!\Gamma(\alpha-n+1)}{\Gamma(\alpha+2)}\right)^{1-\frac{1}{q}} \\
&\times\left(B(n+1,\alpha-n)A\left(a^q,x^q\right)-\frac{x^q}{2}B(n+3,\alpha-n)-\frac{a^q}{2}\tau_1\right)^{\frac{1}{q}} \\
&+\left(\ln\frac{b}{x}\right)^{\alpha}\sup\left\{\left|f'(b)\right|,\left|f'(x)\right|\right\}\left(\frac{n!\Gamma(\alpha-n+1)}{\Gamma(\alpha+2)}\right)^{1-\frac{1}{q}} \\
&\times\left(B(n+1,\alpha-n)A\left(x^q,b^q\right)-\frac{b^q}{2}B(n+3,\alpha-n)-\frac{x^q}{2}\tau_1\right)^{\frac{1}{q}}
\end{aligned}$$

for all $x\in[a,b]$, $\alpha\in(n,n+1]$ *where* $q\geq 1$ *and* $\tau_1=\frac{(2\alpha-n+2)\Gamma(n+2)\Gamma(\alpha-n)}{\Gamma(\alpha+3)}$.

Proof If we use the general Cauchy inequality and power-mean inequality in the proof of Theorem 3.14, we can write

$$\begin{aligned}
&\left|F_f(\alpha,n;x)\right| \\
&\leq \left(\ln\frac{x}{a}\right)^{\alpha}\sup\left\{\left|f'(x)\right|,\left|f'(a)\right|\right\} \\
&\times\left(\frac{n!\Gamma(\alpha-n+1)}{\Gamma(\alpha+2)}\right)^{1-\frac{1}{q}}\left(\int_0^1\left|B_t(n+1,\alpha-n)\right|\left(tx^q+(1-t)a^q\right)dt\right)^{\frac{1}{q}} \\
&+\left(\ln\frac{b}{x}\right)^{\alpha}\sup\left\{\left|f'(b)\right|,\left|f'(x)\right|\right\} \\
&\times\left(\frac{n!\Gamma(\alpha-n+1)}{\Gamma(\alpha+2)}\right)^{1-\frac{1}{q}}\left(\int_0^1\left|B_t(n+1,\alpha-n)\right|\left(tb^q+(1-t)x^q\right)dt\right)^{\frac{1}{q}}.
\end{aligned}$$

By computing the above integrals, we get the desired result. □

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Weighted Integral Inequalities in Terms of Omega-Fractional Integro-Differentiation

P. Agarwal, A. M. Jerbashian and J. E. Restrepo

Abstract Some generalizations of fractional integro-differentiation operators containing a functional parameter ω are introduced. These operators are used to get some new inequalities including ω-weighted Pólya–Szegö type inequalities, ω-weighted Chebyshev-type integral inequalities, ω-weighted Minkowskis reverse integral inequalities, ω-weighted Hölder reverse integral inequalities, ω-weighted integral inequalities for arithmetic and geometric means. The majority of the obtained inequalities becomes the classical or the well-known ones in some particular cases of the weights.

Keywords Integral inequalities · Fractional integro-differentiation · Weighted classes

Mathematics Subject Classification Number: 26A33, 35A23

1 Introduction

In many contemporary investigations, different types of fractional integro-differentiation operators are being used, while the use of the general ω-functional-parametered M. M. Djrbashian integro-differentiation [43, Sect. 18.6], which led him to some

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exhaustive results [15], is staying an essential gap in fractional calculus since 1974, except very few works, [21–27, 41].
In this work, a multivariable analog of M. M. Djrbashian's operator

$$L_{\omega} f(z) = -\int_0^1 f(tz) d\omega(t), \quad |z| < 1,$$

is used to get some ω-weighted generalizations of several classical fractional integral inequalities. The ω-generalizations of the following inequalities are established: Pólya–Szegö inequality, Chebyshev integral inequality, Minkowski reverse integral inequality, Hölder reverse integral inequality, integral inequalities for arithmetic and geometric means, and some other integral inequalities.

The considered multivariable analog of the operator L_{ω} turns to many well-known fractional integro-differentiation operators under some particular choices of the functional parameter ω. Under these choices, the established in this work inequalities become the above-mentioned classical inequalities, and some unexpected results are obtained, which perhaps are new.

In Sect. 2, we introduce our general operator and give some necessary remarks. In Sects. 3 and 4, some new Pólya–Szegö-type integral inequalities are proved. Then, these inequalities are used to establish some fractional integral inequalities of Chebyshev type. In Sect. 5, we establish some ω-weighted Chebyshev fractional integral inequalities. In Sects. 6 and 7, several ω-weighted reverse inequalities are proved. Namely, weighted Minkowski reverse fractional integral inequalities and weighted Hölder reverse fractional integral inequalities. Sections 8 and 9 are devoted to some ω-weighted integral inequalities for arithmetic and geometric means and some other integral inequalities.

2 The Generalized M. M. Djrbashian Fractional Integral

Everywhere below, we assume that the function ω is of the class Ω, i.e., $\omega : \mathbb{R}^{n+2} \to [0, \infty)$ is an integrable function with respect to the second variable. Further, if $\omega \in \Omega$ and $f(t)$ is a real-valued, integrable function on $\mathbb{R}$, we formally define the operator

$$I_{a,\omega} f(t) := \int_a^t \omega(t, \tau, x_1, \dots, x_n) f(\tau) d\tau, \quad -\infty < a < t < +\infty, \tag{2.1}$$

where $x_1, \dots, x_n$ are real parameters. Note that this operator is a real, multivariable generalization of the M. M. Djrbashian fractional integral considered in [15], [43, Sect. 18.6] (see also [23–27, 39–41]).
Before giving some remarks on the form of the operator $I_{a,\omega}$ for some particular cases of the functional parameter ω, we remind one extension of the classical Gamma function called the generalized k-gamma function and introduced in [14]:

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \quad k > 0,\ x \in \mathbb{C} \setminus k\mathbb{Z}^-$$

(not to be confused with a similar notation of a different object in [18]), where $(x)_{n,k}$ is the Pochhammer k-symbol defined as $(x)_{n,k} = x(x+k)(x+2k)\ldots(x+(n-1)k) \quad (n \geq 1)$. Also, if $\operatorname{Re} x > 0$, we can write the function Γ_k as:

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt.$$

The k-gamma function has the following properties: $\Gamma(x) = \lim_{k\to 1} \Gamma_k(x)$, $\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right)$, $\Gamma_k(x+k) = x\Gamma_k(x)$.

Remark 2.1 If $\omega(t,\tau,k,\alpha) = (t-\tau)^{\alpha/k-1}[k\Gamma_k(\alpha)]^{-1}$ with $b \geq t \geq \tau \geq a \geq 0$ and $k > 0$, then

$$I_{a,\omega} f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^t (t-\tau)^{\alpha/k-1} f(\tau) d\tau, \quad t \in [a,b], \tag{2.2}$$

is the Riemann–Liouville k-fractional integral of order $\alpha > 0$ for a real-valued continuous function $f(t)$ [2, 46]. Besides, for $k = 1$, this operator becomes to the classical Riemann–Liouville fractional integral. Also, if $r \in \mathbb{R} \setminus \{-1\}$, then

$$I_{a,\omega} f(t) = \frac{(1+r)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{r+1} - \tau^{r+1})^{\frac{\alpha}{k}-1} f(\tau) d\tau, \quad t \in [a,b], \tag{2.3}$$

is a generalization of an Erdélyi–Kober fractional integral [17, 31, 44].

Remark 2.2 If $\omega(t,\tau,x,y,k) = \tau^{x/k-1}(1-\tau)^{y/k-1}[kf(\tau)]^{-1}$ with $0 \leq \tau \leq t \leq 1$, $x > 0$, $y > 0$, $k > 0$ and f is a continuous function on $[0, 1]$, then

$$I_{a,\omega} f(t) = \frac{1}{k} \int_0^t \tau^{x/k-1}(1-\tau)^{y/k-1} d\tau = \beta_k^{[0,t]}(x,y),$$

is the k-beta function of [14] for $t = 1$.

Remark 2.3 If $\omega(t,\tau,s,k,\alpha) = (1+s)^{1-\alpha/k}[k\Gamma_k(\alpha)]^{-1}(t^{s+1} - \tau^{s+1})^{\alpha/k-1}\tau^s$ with $t \in [a,b]$, $t \geq \tau \geq a$, $k > 0$, $\alpha > 0$, $s \in \mathbb{R} \setminus \{-1\}$ and f is a continuous function on $[a,b]$, then

$$I_{a,\omega} f(t) = \frac{(1+s)^{1-\alpha/k}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - \tau^{s+1})^{\alpha/k-1} \tau^s f(\tau) d\tau = {}^s_k J_a^\alpha f(t),$$

is the $(k;s)$-Riemann–Liouville fractional integral of f of the order $\alpha > 0$ [44].

Remark 2.4 If $\omega(t,\tau,x,\eta,\alpha) = x^{\eta}\left(1-\frac{\tau}{t}\right)^{\frac{\eta}{1-\alpha}}$ with $\eta > 0$, $x > 0$, $a > 0$ and $\alpha < 1$, then, $I_{a,\omega} f\left(\frac{x}{a(1-\alpha)}\right)$ is the pathway fractional integral operator for the functions $f(t) \in L(0,b)$ [36].

Remark 2.5 If $\omega(t,\tau,\alpha) = \frac{1}{\tau}\left(\log\frac{t}{\tau}\right)^{\alpha-1}$ with $\alpha > 0$ and $t \geq \tau \in [a,b]$ $(a \geq 1)$, then

$$I_{a,\omega} f(t) = \int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{f(\tau)}{\tau} d\tau, \quad t \in [a,b],$$

is the classical left-sided Hadamard integral of the fractional order α [35].

Remark 2.6 If $\lambda(t,\tau,\alpha) = [h(t) - h(\tau)]^{\alpha-1} h'(\tau)[\Gamma(\alpha)]^{-1}$ $(\tau \in (a,t))$ with $\alpha > 0$, and $h(\tau)$ is an increasing, positive, monotone, continuously differentiable function on (a,b), then $I_{a,\omega} f(t)$ becomes the operator $J^{\alpha}_{a^+,h} f$ considered in [30].

Remark 2.7 If $\omega(t,\tau,\rho,\alpha) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \frac{\tau^{\rho-1}}{(t^{\rho}-\tau^{\rho})^{1-\alpha}}$ with $\alpha > 0$ and $\rho \in \mathbb{R} \neq \{-1\}$, then

$$I_{a,\omega} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^{\rho}-\tau^{\rho})^{1-\alpha}} f(\tau) d\tau = ({}^{\rho}I^{\alpha}_{a+} f)(t), \quad t > a,$$

is the left-sided Katugampola fractional integral [28, 29].

Remark 2.8 If $\omega(t,\tau,\alpha,\beta,\eta) = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)}(t-\tau)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\tau}{t}\right)$ $(t \geq \tau \geq 0)$ with $\alpha > 0$ and $\beta, \eta \in \mathbb{R} \setminus \mathbb{Z}^-$, then $I_{a,\omega} f(t)$ becomes the Saigo generalized fractional integral [42].

The next two remarks show that the operator $I_{a,\omega}$ turns to the most common fractional derivatives for some particular weights.

Remark 2.9 If $\omega(t,\tau,\alpha) = (t-\tau)^{-\alpha}/\Gamma(1-\alpha)$ $(t \geq \tau)$ with $0 < \alpha < 1$, then the operator

$$(D^{\alpha}_{a^+} f)(x) = \frac{d}{dt} I_{a,\omega} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha}} d\tau,$$

is the Riemann–Liouville fractional derivative [43].

Remark 2.10 If $\omega(t,\tau,\alpha) = (t-\tau)^{m-\alpha-1}/\Gamma(m-\alpha)$ $(t \geq \tau)$ with $m-1 < \alpha < m$ and $m \in \mathbb{N}$, then

$$I_{0,\omega} f^{(m)}(t) = D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau,$$

and $D^{\alpha} f(t) = f^{(m)}(t)$ for $m = \alpha$. This operator is being called Caputo fractional derivative [8].

3 Some ω-Weighted Pólya–Szegö-Type Inequalities

In this section, we established some ω-weighted generalizations of the well-known inequality

$$\frac{\int_a^b f^2(x)dx \int_a^b g^2(x)dx}{\left(\int_a^b f(x)g(x)dx\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{MN}{mn}} + \sqrt{\frac{mn}{MN}}\right)^2, \tag{3.1}$$

where it is assumed that f and g are positive, integrable functions which are synchronous on $[a, b]$, i.e., $(f(x) - f(y))(g(x) - g(y)) \geq 0$ for any $x, y \in [a, b]$ and such that $0 < m \leq f(x) \leq M < \infty$, $0 < n \leq g(y) \leq N < \infty$ for any $x, y \in [a, b]$ and some $m, M, n, N \in \mathbb{R}$.
Note that the inequality (3.1) mainly is being attributed to Pólya and Szegö [38] in the literature, in spite of the fact that first it was established by Pal Schweitzer in 1914 [45]. For such kind of inequalities see also [4].
Now, we proceed to our ω-weighted inequalities.

Theorem 3.1 *Let $\omega \in \Omega$ and let f and g be positive, square-integrable functions on an interval $(a, +\infty)$ with $a > -\infty$. If φ_1, φ_2, ψ_1 and ψ_2 are integrable functions on $(a, +\infty)$, such that*

$$0 < \varphi_1(\tau) \leq f(\tau) \leq \varphi_2(\tau) \quad \text{and} \quad 0 < \psi_1(\tau) \leq g(\tau) \leq \psi_2(\tau), \quad \tau \in (a, t), \tag{I}$$

for some $t \in (a, +\infty)$. Then

$$\frac{I_{a,\omega}\left(\psi_1\psi_2 f^2\right)(t) I_{a,\omega}\left(\varphi_1\varphi_2 g^2\right)(t)}{\left(I_{a,\omega}\left((\varphi_1\psi_1 + \varphi_2\psi_2) fg\right)(t)\right)^2} \leq \frac{1}{4}. \tag{3.2}$$

Proof If $f(\tau) \leq \varphi_2(\tau)$ and $\psi_1(\tau) \leq g(\tau)$ for $\tau \in (a, t)$, then

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right) \geq 0, \quad \tau \in (a, t).$$

Similarly, if $\varphi_1(\tau) \leq f(\tau)$ and $g(\tau) \leq \psi_2(\tau)$, then

$$\left(\frac{f(\tau)}{g(\tau)} - \frac{\varphi_1(\tau)}{\psi_2(\tau)}\right) \geq 0, \quad \tau \in (a, t).$$

Multiplying these two inequalities, we get

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\tau)} + \frac{\varphi_1(\tau)}{\psi_2(\tau)}\right) \frac{f(\tau)}{g(\tau)} \geq \frac{f^2(\tau)}{g^2(\tau)} + \frac{\varphi_1(\tau)\varphi_2(\tau)}{\psi_1(\tau)\psi_2(\tau)}, \quad \tau \in (a, t),$$

or, what is the same,

$$\big(\varphi_1(\tau)\psi_1(\tau)+\varphi_2(\tau)\psi_2(\tau)\big)f(\tau)g(\tau) \geq \psi_1(\tau)\psi_2(\tau)f^2(\tau)+\varphi_1(\tau)\varphi_2(\tau)g^2(\tau), \quad \tau \in (a,t).$$

Further, multiplying both sides of this inequality by $\omega(t,\tau,x_1,\ldots,x_n)$ and integrating with respect to τ over (a,t), we come to

$$I_{a,\omega}\big((\varphi_1\psi_1+\varphi_2\psi_2)fg\big)(t) \geq I_{a,\omega}\big(\psi_1\psi_2 f^2\big)(t) + I_{a,\omega}\big(\varphi_1\varphi_2 g^2\big)(t)$$

and applying the inequality $a+b \geq 2\sqrt{ab}$ $(a,b \geq 0)$, we get

$$I_{a,\omega}\big((\varphi_1\psi_1+\varphi_2\psi_2)fg\big)(t) \geq 2\sqrt{I_{a,\omega}\big(\psi_1\psi_2 f^2\big)(t) I_{a,\omega}\big(\varphi_1\varphi_2 g^2\big)(t)}$$

which implies (3.2).

Corollary 3.1 *Let $\omega \in \Omega$ and let f and g be two positive square-integrable functions on $(a,+\infty)$ $(a > -\infty)$, such that*

$$0 < m \leq f(\tau) \leq M \quad \text{and} \quad 0 < n \leq g(\tau) \leq N \tag{J}$$

for $\tau \in (a,t)$ with some $t \in (a,+\infty)$. Then

$$\frac{I_{a,\omega}f^2(t)I_{a,\omega}g^2(t)}{\big(I_{a,\omega}(fg)(t)\big)^2} \leq \frac{1}{4}\left(\frac{\sqrt{mn}}{\sqrt{MN}}+\frac{\sqrt{MN}}{\sqrt{mn}}\right)^2.$$

Remark 3.1 The well-known inequality (3.1) follows by Corollary 3.1 for $\omega \equiv 1$.

From Corollary 3.1 and Remark 2.3, we get the following statement.

Corollary 3.2 *Let $\omega \in \Omega$ and let f and g be two positive square-integrable functions on $(a,+\infty)$ $(a > -\infty)$, which satisfy the condition (J) for some $\tau \in (a,t)$ with $t \in (a,+\infty)$. Then*

$$\frac{{}^s_k J^\alpha_a f^2(t)\, {}^s_k J^\alpha_a g^2(t)}{\big({}^s_k J^\alpha_a (fg)(t)\big)^2} \leq \frac{1}{4}\left(\frac{\sqrt{mn}}{\sqrt{MN}}+\frac{\sqrt{MN}}{\sqrt{mn}}\right)^2.$$

Lemma 3.1 *Let $\omega_{1,2} \in \Omega$ and let f and g be positive, square-integrable functions on $(a,+\infty)$ with $a > -\infty$. If φ_1, φ_2, ψ_1 and ψ_2 are integrable functions on $(a,+\infty)$, which satisfy (I) with some $t \in (a,+\infty)$, then*

$$\frac{I_{a,\omega_1}(\varphi_1\varphi_2)(t)I_{a,\omega_2}(\psi_1\psi_2)(t)I_{a,\omega_1}f^2(t)I_{a,\omega_2}g^2(t)}{\big(I_{a,\omega_1}(\varphi_1 f)(t)I_{a,\omega_2}(\psi_1 g)(t)+I_{a,\omega_1}(\varphi_2 f)(t)I_{a,\omega_2}(\psi_2 g)(t)\big)^2} \leq \frac{1}{4}. \tag{3.3}$$

Proof By (I), it follows that

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\rho)} - \frac{f(\tau)}{g(\rho)}\right) \geq 0 \quad \text{and} \quad \left(\frac{f(\tau)}{g(\rho)} - \frac{\varphi_1(\tau)}{\psi_2(\rho)}\right) \geq 0 \quad \text{for} \quad \tau, \rho \in (a, t).$$

These inequalities imply

$$\left(\frac{\varphi_1(\tau)}{\psi_2(\rho)} + \frac{\varphi_2(\tau)}{\psi_1(\rho)}\right)\frac{f(\tau)}{g(\rho)} \geq \frac{f^2(\tau)}{g^2(\rho)} + \frac{\varphi_1(\tau)\varphi_2(\tau)}{\psi_1(\rho)\psi_2(\rho)}, \quad \tau, \rho \in (a, t).$$

Multiplying both sides of this inequality by $\psi_1(\rho)\psi_2(\rho)g^2(\rho)$, we obtain

$$\varphi_1(\tau)f(\tau)\psi_1(\rho)g(\rho) + \varphi_2(\tau)f(\tau)\psi_2(\rho)g(\rho) \geq \psi_1(\rho)\psi_2(\rho)f^2(\tau) + \varphi_1(\tau)\varphi_2(\tau)g^2(\rho).$$

Then, multiplying both sides by $\omega_1(t, \tau, x_1, \ldots, x_n)\omega_2(t, \rho, x_1, \ldots, x_n)$ and integrating with respect to τ and ρ over (a, t), we get

$$\begin{aligned} &I_{a,\omega_1}(\varphi_1 f)(t)I_{a,\omega_2}(\psi_1 g)(t) + I_{a,\omega_1}(\varphi_2 f)(t)I_{a,\omega_2}(\psi_2 g)(t) \\ &\qquad \geq I_{a,\omega_1} f^2(t)I_{a,\omega_2}(\psi_1\psi_2)(t) + I_{a,\omega_1}(\varphi_1\varphi_2)(t)I_{a,\omega_2} g^2(t), \end{aligned}$$

and an application of the inequality $a + b \geq 2\sqrt{ab}$ $(a, b \geq 0)$ yields (3.3).

Lemma 3.2 *Let $\omega_{1,2} \in \Omega$ and let f and g be positive, square-integrable functions on $(a, +\infty)$ with $a > -\infty$. If φ_1, φ_2, ψ_1 and ψ_2 are integrable functions on $(a, +\infty)$, which satisfy (I) with some $t \in (a, +\infty)$, then*

$$I_{a,\omega_1} f^2(t)I_{a,\omega_2} g^2(t) \leq I_{a,\omega_1}\big((\varphi_2 fg)/\psi_1\big)(t)I_{a,\omega_2}\big((\psi_2 fg)/\varphi_1\big)(t). \tag{3.4}$$

Proof Obviously $f(\tau)\psi_1(\tau) \leq g(\tau)\varphi_2(\tau)$ $(\tau \in (a, t))$, and hence

$$\int_a^t \omega_1(t, \tau, x_1, \ldots, x_n)f^2(t)d\tau \leq \int_a^t \omega_1(t, \tau, x_1, \ldots, x_n)\frac{\varphi_2(\tau)}{\psi_1(\tau)}f(\tau)g(\tau)d\tau$$

which implies

$$I_{a,\omega_1} f^2(t) \leq I_{a,\omega_1}\big((\varphi_2 fg)/\psi_1\big)(t).$$

Similarly $\varphi_1(\tau) \leq f(\tau)$ and $g(\tau) \leq \psi_2(\tau)$ for $\tau \in (a, t)$, and hence

$$I_{a,\omega_2} g^2(t) \leq I_{a,\omega_2}\big((\psi_2 fg)/\varphi_1\big)(t).$$

These two inequalities imply (3.4).

Corollary 3.3 *Let $\omega_{1,2} \in \Omega$ and let f and g be positive, square-integrable functions on $(a, +\infty)$, which satisfy (J). Then*

$$\frac{I_{a,\omega_1} f^2(t)I_{a,\omega_2} g^2(t)}{I_{a,\omega_1}(fg)(t)I_{a,\omega_2}(fg)(t)} \leq \frac{MN}{mn}. \tag{3.5}$$

4 ω-Weighted Chebyshev Functional Inequalities

In this section, some ω-weighted Chebyshev-type integral inequalities are obtained by the use of Chebyshev's well-known functional [11]

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right),$$

where f and g are supposed to be integrable functions. Note that $T(f,g)\geq 0$, if additionally the functions f and g are synchronous on $[a,b]$. This functional has been used for many researchers to find some new inequalities and their applications, see e.g., [2, 5, 13, 33, 37, 48, 50]. One of the most famous ones is that of Grüss [20]: $|T(f,g)|\leq (M-m)(N-m)/4$, where f and g are supposed to be integrable, synchronous on $[a,b]$ and such that $m\leq f(x)\leq M$, $n\leq g(x)\leq N$ for any $x\in[a,b]$ and some $m,M,n,N\in\mathbb{R}$. Besides, Dragomir and Diamond [16] proved that

$$|T(f,g)|\leq\frac{(M-m)(N-n)}{4(b-a)^2\sqrt{MmNn}}\int_a^b f(x)dx\int_a^b g(x)dx. \tag{4.1}$$

Below, we establish our ω-weighted Chebyshev-type integral inequalities by an application of the ω-weighted Pólya–Szegö fractional integral inequality of Theorem 3.1. Then, we present a particular case of the inequality (4.1).

Theorem 4.1 *Let $\omega_{1,2}\in\Omega$ and let f and g be positive, square-integrable functions on $(a,+\infty)$ with $a>-\infty$. If φ_1, φ_2 are integrable functions on $(a,+\infty)$, such that $\varphi_1(\tau)\leq f(\tau)\leq\varphi_2(\tau)$ and $\varphi_1(\tau)\leq g(\tau)\leq\varphi_2(\tau)$ for $\tau\in(a,t)$, then*

$$\begin{aligned}&\left|I_{a,\omega_1}(fg)(t)I_{a,\omega_2}1(t)+I_{a,\omega_2}(fg)(t)I_{a,\omega_1}1(t)-I_{a,\omega_1}f(t)I_{a,\omega_2}g(t)-I_{a,\omega_1}g(t)I_{a,\omega_2}f(t)\right|\\&\qquad\leq 2\big(T_{\omega_1,\omega_2}(f,\varphi_1,\varphi_2)(t)T_{\omega_1,\omega_2}(g,\varphi_1,\varphi_2)(t)\big)^{1/2},\end{aligned}$$

where

$$\begin{aligned}T_{\omega_1,\omega_2}(x,y,z)(t)=&\frac{1}{8}\frac{\big(I_{a,\omega_1}((y+z)x)(t)\big)^2}{I_{a,\omega_1}(yz)(t)}I_{a,\omega_2}1(t)\\&+\frac{1}{8}\frac{\big(I_{a,\omega_2}((y+z)x)(t)\big)^2}{I_{a,\omega_2}(yz)(t)}I_{a,\omega_1}1(t)-I_{a,\omega_1}x(t)I_{a,\omega_2}x(t).\end{aligned}$$

Proof Setting

$$R(\tau,\rho)=f(\tau)g(\tau)+f(\rho)g(\rho)-f(\tau)g(\rho)-f(\rho)g(\tau),\quad \tau,\rho\in(a,t),$$

and multiplying both sides of this equality by $\omega_1(t,\tau,x_1,\ldots,x_n)\omega_2(t,\rho,x_1,\ldots,x_n)$ with $\omega_{1,2}\in\Omega$ and integrating with respect to τ and ρ over (a,t) we get

$$\int_a^t \int_a^t \omega_1(t,\tau,x_1,\dots,x_n)\omega_2(t,\rho,x_1,\dots,x_n)R(\tau,\rho)d\tau d\rho \tag{4.2}$$
$$= I_{a,\omega_1}(fg)(t)I_{a,\omega_2}1(t) + I_{a,\omega_2}(fg)(t)I_{a,\omega_1}1(t) - I_{a,\omega_1}f(t)I_{a,\omega_2}g(t) - I_{a,\omega_1}g(t)I_{a,\omega_2}f(t).$$

Then, setting $\mu(\tau) = \int_a^\tau \omega_1(t,\tau_1,x_1,\dots,x_n)d\tau_1$ and $\nu(\rho) = \int_a^\rho \omega_2(t,\rho_1,x_1,\dots,x_n)d\rho_1$, by the Cauchy–Schwartz inequality, we obtain

$$\left|\int_a^t \int_a^t R(\tau,\rho)d\mu(\tau)d\nu(\rho)\right|$$
$$\le \left(\int_a^t \int_a^t \big(f(\tau)-f(\rho)\big)^2\omega_1(t,\tau,x_1,\dots,x_n)\omega_2(t,\rho,x_1,\dots,x_n)d\tau d\rho\right)^{1/2}$$
$$\times \left(\int_a^t \int_a^t \big(g(\tau)-g(\rho)\big)^2\omega_1(t,\tau,x_1,\dots,x_n)\omega_2(t,\rho,x_1,\dots,x_n)d\tau d\rho\right)^{1/2}$$
$$\le 2\big(1/2I_{a,\omega_1}f^2(t)I_{a,\omega_2}1(t) + 1/2I_{a,\omega_2}f^2(t)I_{a,\omega_1}1(t) - I_{a,\omega_1}f(t)I_{a,\omega_2}f(t)\big)^{1/2}$$
$$\times \big(1/2I_{a,\omega_1}g^2(t)I_{a,\omega_2}1(t) + 1/2I_{a,\omega_2}g^2(t)I_{a,\omega_1}1(t) - I_{a,\omega_1}g(t)I_{a,\omega_2}g(t)\big)^{1/2}.$$

Applying Lemma 3.2, where we set $\psi_1(t) = \psi_2(t) = g(t) = 1$, we get

$$I_{a,\omega_{1,2}}f^2(t) \le \frac{1}{4}\frac{\big(I_{a,\omega_{1,2}}\big((\varphi_1+\varphi_2)f\big)(t)\big)^2}{I_{a,\omega_{1,2}}(\varphi_1\varphi_2)(t)},$$

and hence

$$1/2I_{a,\omega_1}f^2(t)I_{a,\omega_2}1(t) + 1/2I_{a,\omega_2}f^2(t)I_{a,\omega_1}1(t) - I_{a,\omega_1}f(t)I_{a,\omega_2}f(t) \tag{4.3}$$
$$\le \frac{1}{8}\frac{\big(I_{a,\omega_1}((\varphi_1+\varphi_2)f)(t)\big)^2}{I_{a,\omega_1}(\varphi_1\varphi_2)(t)}I_{a,\omega_2}1(t) + \frac{1}{8}\frac{\big(I_{a,\omega_2}((\varphi_1+\varphi_2)f)(t)\big)^2}{I_{a,\omega_2}(\varphi_1\varphi_2)(t)}I_{a,\omega_1}1(t)$$
$$- I_{a,\omega_1}f(t)I_{a,\omega_2}f(t) = T_{\omega_1,\omega_2}(f,\varphi_1,\varphi_2)(t).$$

Similarly, we get

$$1/2I_{a,\omega_1}g^2(t)I_{a,\omega_2}1(t) + 1/2I_{a,\omega_2}g^2(t)I_{a,\omega_1}1(t) - I_{a,\omega_1}g(t)I_{a,\omega_2}g(t) \tag{4.4}$$
$$\le \frac{1}{8}\frac{\big(I_{a,\omega_1}((\varphi_1+\varphi_2)g)(t)\big)^2}{I_{a,\omega_1}(\varphi_1\varphi_2)(t)}I_{a,\omega_2}1(t) + \frac{1}{8}\frac{\big(I_{a,\omega_2}((\varphi_1+\varphi_2)g)(t)\big)^2}{I_{a,\omega_2}(\varphi_1\varphi_2)(t)}I_{a,\omega_1}1(t)$$
$$- I_{a,\omega_1}g(t)I_{a,\omega_2}g(t) = T_{\omega_1,\omega_2}(g,\varphi_1,\varphi_2)(t).$$

The desired inequality holds by (4.2), (4.3), and (4.4).

Corollary 4.1 *Let $\omega \in \Omega$ and let f and g be positive, square-integrable functions on $(a,+\infty)$ $(a > -\infty)$. If φ_1, φ_2 are any functions which are integrable on $(a,+\infty)$ and such that $\varphi_1(\tau) \le f(\tau) \le \varphi_2(\tau)$ and $\varphi_1(\tau) \le g(\tau) \le \varphi_2(\tau)$ for $\tau \in (a,t)$, then*

$$\left|I_{a,\omega}(fg)(t)I_{a,\omega}1(t) - I_{a,\omega}g(t)I_{a,\omega}f(t)\right| \le \left(T_{\omega,\omega}(f,\varphi_1,\varphi_2)(t)T_{\omega,\omega}(g,\varphi_1,\varphi_2)(t)\right)^{1/2}.$$

Remark 4.1 If $\omega = 1$ and $m \le f, g \le M$, then the inequality of Corollary 4.1 becomes the inequality (4.1) in [16].

Remark 4.2 When the parameter-function $\omega = \omega_1 = \omega_2$ is that of formula (2.3), the results of this section become almost the same as those in [2].

5 ω-Weighted Chebyshev Fractional Inequalities

In this section, some ω-weighted Chebyshev integral inequalities are obtained, which extend and generalize some recent [1, 13, 44] and classical [11, 20] results in the field, see also [6, 9, 10]. For some special case of the parameter-function ω, our inequalities become the classical Chebyshev integral inequality [19]

$$\int_a^b f_1(x)dx \int_a^b f_2(x)dx \cdots \int_a^b f_n(x)dx \le (b-a)^{n-1} \int_a^b f_1(x)f_2(x)\cdots f_n(x)dx,$$

where $f_1, f_2, \ldots, f_n$ are supposed to be nonnegative, integrable, all monotone increasing or decreasing functions on $[a, b]$.

Theorem 5.1 *Let f and g be any synchronous functions on $(a, +\infty)$ $(a > -\infty)$. Then for any $\omega \in \Omega$*

$$I_{a,\omega}1(t)\, I_{a,\omega}(fg)(t) \ge I_{a,\omega}f(t)I_{a,\omega}g(t), \quad t \in (a + \infty).$$

Proof As f and g are synchronous on $(a, +\infty)$, we get

$$f(\tau)g(\tau) + f(\rho)g(\rho) \ge f(\tau)g(\rho) + f(\rho)g(\tau), \quad \tau, \rho \in (a, +\infty).$$

Multiplying both sides of this inequality by $\omega(t, \tau, x_1, \ldots, x_n)$ with $t \ge \tau$ and integrating with respect to τ over (a, t), we obtain

$$I_{a,\omega}(fg)(t) + f(\rho)g(\rho)I_{a,\omega}1(t) \ge g(\rho)I_{a,\omega}f(t) + f(\rho)I_{a,\omega}g(t). \tag{5.1}$$

Multiplication of both sides of this inequality by $\omega(t, \rho, x_1, \ldots, x_n)$ with $\rho \in (a, t)$ and integration with respect to ρ over (a, t) complete the proof.

Theorem 5.2 *Let f and g be any synchronous functions on $(a, +\infty)$ $(a > -\infty)$. Then for any $t > a$ and $\omega_{1,2} \in \Omega$*

$$I_{a,\omega_1}(fg)(t)I_{a,\omega_2}1(t) + I_{a,\omega_1}1(t)I_{a,\omega_2}(fg)(t) \ge I_{a,\omega_1}f(t)I_{a,\omega_2}g(t) + I_{a,\omega_1}g(t)I_{a,\omega_2}f(t).$$

Proof Taking $\omega_1 \in \Omega$ in the equality (5.1), then multiplying its both sides by $\omega_2(t, \rho, x_1, \ldots, x_n)$ with $\rho \in (a, t)$ and integrating with respect to ρ over (a, t), we come to the desired inequality.

Theorem 5.3 *Let* $\{f_i\}_{i=1}^n$ *be positive, increasing functions on* $(a, +\infty)$ $(a > -\infty)$*, and let* $\omega \in \Omega$*. Then*

$$I_{a,\omega}\left(\prod_{i=1}^{n} f_i\right)(t) \geq \left(I_{a,\omega}1(t)\right)^{1-n} \prod_{i=1}^{n} I_{a,\omega} f_i(t), \quad t \in (a, +\infty). \tag{5.2}$$

Proof The inequality is obvious for $n = 1$. Now, assume

$$I_{a,\omega}\left(\prod_{i=1}^{n-1} f_i\right)(t) \geq \left(I_{a,\omega}1(t)\right)^{2-n} \prod_{i=1}^{n-1} I_{a,\omega} f_i(t), \quad t \in (a, +\infty). \tag{5.3}$$

The functions $\{f_i\}_{i=1}^n$ are positive and increase on $(a, +\infty)$, therefore also $\prod_{i=1}^{n-1} f_i$ is an increasing function. By Theorem 5.1 with $g = \prod_{i=1}^{n-1} f_i$ and $f = f_n$, we get

$$I_{a,\omega}\left(\prod_{i=1}^{n} f_i\right)(t) \geq \left(I_{a,\omega}1(t)\right)^{-1} I_{a,\omega}\left(\prod_{i=1}^{n-1} f_i\right)(t)\, I_{a,\omega} f_n(t), \quad t \in (a, +\infty). \tag{5.4}$$

The inequalities (5.3) and (5.4) imply (5.2).

Corollary 5.1 *Chebyshev's classical integral inequality follows from* (5.2)*. Namely, for* $\omega \equiv 1$ *and* $t = b$*, the inequality* (5.2) *takes the form*

$$\int_a^b f_1(x)dx \int_a^b f_2(x)dx \cdots \int_a^b f_n(x)dx \leq (b-a)^{n-1} \int_a^b f_1(x) f_2(x) \cdots f_n(x)dx.$$

6 ω-Weighted Minkowski Reverse Fractional Inequalities

In this section, some generalizations of the Minkowski reverse fractional integral inequalities are established. Note that sometimes such inequalities are called reverse Cauchy–Bunyakowsky inequalities, see [4].

Theorem 6.1 *Let* $a \in \mathbb{R}$*,* $p \geq 1$*,* $\omega \in \Omega$*, and let* f *and* g *be any positive functions on* $(a, +\infty)$*, such that*

$$I_{a,\omega} f^p(t) < \infty \quad \text{and} \quad I_{a,\omega} g^p(t) < \infty, \quad t \in (a, +\infty),$$

If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M < +\infty$ *for* $\tau \in (a, t)$*, then*

$$\left(I_{a,\omega} f^p(t)\right)^{1/p} + \left(I_{a,\omega} g^p(t)\right)^{1/p} \leq \frac{M(m+2)+1}{(m+1)(M+1)} \left(I_{a,\omega}(f+g)^p(t)\right)^{1/p}. \quad (6.1)$$

Proof As $f(\tau) \leq Mg(\tau)$ for $\tau \in (a,t)$, we have $(M+1)^p f^p(\tau) \leq M^p (f+g)^p(\tau)$. Multiplying both sides of this inequality by $\omega(t,\tau,x_1,\ldots,x_n)$, integrating with respect to τ over (a,t) and reminding the definition (2.1) of the operator $I_{a,\omega}$, we obtain

$$(M+1)^p I_{a,\omega} f^p(t) \leq M^p I_{a,\omega}(f+g)^p(t).$$

Hence

$$\left(I_{a,\omega} f^p(t)\right)^{1/p} \leq \frac{M}{M+1} \left(I_{a,\omega}(f+g)^p(t)\right)^{1/p}. \quad (6.2)$$

Further, by $mg(\tau) \leq f(\tau)$, $\tau \in (a,t)$ it follows that

$$\left(1+\frac{1}{m}\right)^p g^p(\tau) \leq \frac{1}{m^p}\left(f(\tau)+g(\tau)\right)^p, \quad \tau \in (a,t).$$

Multiplying both sides of this inequality by $\omega(t,\tau,x_1,\ldots,x_n)$ and integrating with respect to τ over (a,t), we get

$$\left(I_{a,\omega} g^p(t)\right)^{1/p} \leq \frac{1}{m+1} \left(I_{a,\omega}(f+g)^p(t)\right)^{1/p}. \quad (6.3)$$

By (6.2) and (6.3), we arrive at (6.1).

Remark 6.1 For $\omega \equiv 1$, Theorem 6.1 becomes the statement of Theorem 1.2 in [7] for $[a,t]$. For $\omega(t,\tau,\alpha) = (t-\tau)^{\alpha-1}/\Gamma(\alpha)$ $(t \geq \tau > 0)$ with $\alpha > 0$, Theorem 6.1 becomes the statement of Theorem 2.1 in [12] for $(0,t)$.

Theorem 6.2 *Let $a \in \mathbb{R}$, $p \geq 1$, $\omega \in \Omega$, and let f and g be any positive functions on $(a,+\infty)$, such that $I_{a,\omega} f^p(t) < \infty$ and $I_{a,\omega} g^p(t) < \infty$ for all $t \in (a,+\infty)$. If $0 < c < m \leq \frac{f(\tau)}{g(\tau)} \leq M < +\infty$ for all $\tau \in (a,t)$, then*

$$\frac{M+1}{M-c}\left(I_{a,\omega}(f-cg)^p(t)\right)^{\frac{1}{p}} \leq \left(I_{a,\omega} f^p(t)\right)^{\frac{1}{p}} + \left(I_{a,\omega} g^p(t)\right)^{\frac{1}{p}}$$
$$\leq \frac{m+1}{m-c}\left(I_{a,\omega}(f-cg)^p(t)\right)^{\frac{1}{p}}.$$

Proof Under our hypotheses

$$\frac{f(\tau)-cg(\tau)}{M-c} \leq g(\tau) \leq \frac{f(\tau)-cg(\tau)}{m-c}, \quad \tau \in (a,t).$$

Multiplying this inequality by $\omega(t,\tau,x_1,\ldots,x_n)$ and integrating with respect τ over (a,t), we obtain

$$\frac{1}{M-c}\big(I_{a,\omega}(f-cg)^p(t)\big)^{1/p} \le \big(I_{a,\omega}g^p(t)\big)^{1/p} \le \frac{1}{m-c}\big(I_{a,\omega}(f-cg)^p(t)\big)^{1/p}. \tag{6.4}$$

On the other hand, by a straightforward calculation, we get

$$\frac{M}{M-c}(f(\tau)-cg(\tau)) \le f(\tau) \le \frac{m}{m-c}(f(\tau)-cg(\tau)), \quad \tau \in (a,t).$$

Multiplying this inequalities by $\omega(t,\tau,x_1,\dots,x_n)$ and integrating with respect to τ over (a,t), we get

$$\frac{M}{M-c}\big(I_{a,\omega}(f-cg)^p(t)\big)^{1/p} \le \big(I_{a,\omega}f^p(t)\big)^{1/p} \le \frac{m}{m-c}\big(I_{a,\omega}(f-cg)^p(t)\big)^{1/p}. \tag{6.5}$$

Now, by (6.4) and (6.5), we come to the desired inequalities.

Remark 6.2 Taking $\omega \equiv 1$ in Theorem 6.2, we get Theorem 2.2 of [47]. And if, in addition, $c = 1$, then we get Theorem 1.1 of [49].

7 ω-Weighted Hölder Reverse Fractional Inequality

In this section, two weighted Hölder reverse fractional integral inequalities are established, which differ from the known ones [32].

Theorem 7.1 *Let $a \in \mathbb{R}$, $p > 1$, $1/p + 1/q = 1$, $\omega \in \Omega$ and let f and g be any positive functions on (a,∞), such that $I_{a,\omega}f(t) < +\infty$ and $I_{a,\omega}g(t) < +\infty$ for all $t > a$. If $0 < m \le \frac{f(\tau)}{g(\tau)} \le M < +\infty$ for all $\tau \in (a,t)$, then*

$$\big(I_{a,\omega}f(t)\big)^{1/p}\big(I_{a,\omega}g(t)\big)^{1/q} \le \left(\frac{M}{m}\right)^{\frac{1}{pq}} I_{a,\omega}(f^{1/p}g^{1/q})(t). \tag{7.1}$$

Proof Using the inequality $f(\tau) \le Mg(\tau)$, $\tau \in (a,t)$, one can be convinced that

$$M^{-1/q}f(\tau) \le \big(f(\tau)\big)^{1/p}\big(g(\tau)\big)^{1/q}, \quad \tau \in (a,t).$$

Multiplying this inequality by $\omega(t,\tau,x_1,\dots,x_n)$ and integrating with respect to τ over (a,t), we get

$$\big(I_{a,\omega}(f^{1/p}g^{1/q})(t)\big)^{1/p} \ge M^{-\frac{1}{pq}}\big(I_{a,\omega}f(t)\big)^{1/p}. \tag{7.2}$$

Further, the inequality $m\,g(\tau) \le f(\tau)$, $\tau \in (a,t)$, yields

$$\big(f(\tau)\big)^{1/p}\big(g(\tau)\big)^{1/q} \ge m^{1/p}g(\tau), \quad \tau \in (a,t).$$

Multiplying this inequality by $\omega(t, \tau, x_1, \ldots, x_n)$ and integrating with respect to τ over (a, t), we get

$$\left(I_{a,\omega}(f^{1/p}g^{1/q})(t)\right)^{1/q} \geq m^{\frac{1}{pq}}\left(I_{a,\omega}g(t)\right)^{1/q}. \tag{7.3}$$

The inequality (7.1) follows by multiplying (7.2) by (7.3).

As a corollary to the above theorem, we get the following ω-weighted Hölder's reverse fractional integral inequality

Corollary 7.1 *Let $a \in \mathbb{R}$, $p > 1$, $1/p + 1/q = 1$, $\omega \in \Omega$, and let f and g be any positive functions on $(a, +\infty)$ $(a > -\infty)$, such that $I_{a,\omega}f^p(t) < +\infty$ and $I_{a,\omega}g^q(t) < +\infty$ for some $t > a$. If $0 < m \leq \frac{f(\tau)^p}{g(\tau)^q} \leq M < +\infty$ for any $\tau \in (a, t)$, then*

$$\left(I_{a,\omega}f^p(t)\right)^{1/p}\left(I_{a,\omega}g^q(t)\right)^{1/q} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} I_{a,\omega}(fg)(t).$$

8 ω-Weighted Arithmetic and Geometric Mean Inequalities

In this section, some ω-weighted integral inequalities of arithmetic and geometric means are proved.

Theorem 8.1 *Let $a \in \mathbb{R}$, $p \geq 1$, $\omega \in \Omega$, and let f and g be any positive functions on $(a, +\infty)$ $(a > -\infty)$, such that $I_{a,\omega}f^p(t) < +\infty$ and $I_{a,\omega}g^p(t) < +\infty$ for all $t > a$. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M < +\infty$ for all $\tau \in (a, t)$, then*

$$\left(\frac{(M+1)(m+1)}{M} - 2\right)\left(I_{a,\omega}f^p(t)\right)^{1/p}\left(I_{a,\omega}g^p(t)\right)^{1/p} \leq \left(I_{a,\omega}f^p(t)\right)^{2/p} + \left(I_{a,\omega}g^p(t)\right)^{2/p}.$$

Proof Multiplying the inequalities (6.2) and (6.3), we obtain

$$\frac{(M+1)(m+1)}{M}\left(I_{a,\omega}f^p(t)\right)^{1/p}\left(I_{a,\omega}g^p(t)\right)^{1/p} \leq \left(I_{a,\omega}(f+g)^p(t)\right)^{2/p}.$$

Applying Minkowski's inequality to the right-hand side of this inequality, we get

$$\left(I_{a,\omega}(f+g)^p(t)\right)^{2/p} \leq \left(\left(I_{a,\omega}f^p(t)\right)^{1/p} + \left(I_{a,\omega}g^p(t)\right)^{1/p}\right)^2.$$

The below two inequalities yield the desired one.

Theorem 8.2 *Let $a \in \mathbb{R}$, $p > 1$, $1/p + 1/q = 1$, $\omega \in \Omega$, and let f and g be any positive, continuous functions on $(a, +\infty)$ $(a > -\infty)$, such that $0 < m < \frac{f(\tau)}{g(\tau)} < M < +\infty$ for all $\tau \in (a, t)$. Then*

$$I_{a,\omega}(fg)(t) \le \frac{2^{p-1}M^p}{p(M+1)^p} I_{a,\omega}(f^p + g^p)(t) + \frac{2^{q-1}}{q(m+1)^q} I_{a,\omega}(f^q + g^q)(t). \tag{8.1}$$

Proof Obviously $(M+1)f(\tau) \le M(f+g)(\tau)$, $\tau \in (a,t)$. Multiplying the p-th orders of both sides of this inequality by $\omega(t, \tau, x_1, \dots, x_n)$ and integrating with respect τ over (a,t), we arrive at the inequality

$$I_{a,\omega} f^p(t) \le \frac{M^p}{(M+1)^p} I_{a,\omega}(f+g)^p(t). \tag{8.2}$$

Further, $(m+1)g(\tau) \le (f+g)(\tau)$, $\tau \in (a,t)$. Multiplying this inequality by $\omega(t, \tau, x_1, \dots, x_n)$ and integrating with respect to τ over (a,t), we get

$$I_{a,\omega} g^q(t) \le \frac{1}{(m+1)^q} I_{a,\omega}(f+g)^q(t). \tag{8.3}$$

Now observe that by the Young inequality

$$f(\tau)g(\tau) \le \frac{f^p(\tau)}{p} + \frac{g^q(\tau)}{q}, \quad \tau \in (a,t).$$

Multiplying both sides of this inequality by $\omega(t, \tau, x_1, \dots, x_n)$ and integrating with respect to τ over (a,t), we obtain

$$I_{a,\omega}(fg)(t) \le \frac{1}{p} I_{a,\omega} f^p(t) + \frac{1}{q} I_{a,\omega} g^q(t). \tag{8.4}$$

By (8.2), (8.3) and (8.4), we get

$$I_{a,\omega}(fg)(t) \le \frac{M^p}{p(M+1)^p} I_{a,\omega}(f+g)^p(t) + \frac{1}{q(m+1)^q} I_{a,\omega}(f+g)^q(t). \tag{8.5}$$

Further, the next two estimates follow by the inequality $(a+b)^p \le 2^{p-1}(a^p + b^p)$ $(p > 1, a, b \ge 0)$:

$$I_{a,\omega}(f+g)^p(t) \le 2^{p-1} I_{a,\omega}(f^p + g^p)(t), \tag{8.6}$$

$$I_{a,\omega}(f+g)^q(t) \le 2^{q-1} I_{a,\omega}(f^q + g^q)(t). \tag{8.7}$$

The desired inequality (8.1) follows by applying (8.6) and (8.7) to (8.5).

9 Some Other Integral Inequalities

The integral inequalities proved in this section are the ω-weighted generalizations of some results of [3, 34] and become some classical inequalities for particular cases of the parameter-function ω.

Theorem 9.1 *Let $\{f_i\}_{i=1}^n$ be positive, continuous, decreasing functions on $(a,+\infty)$ $(a>-\infty)$ and let $\omega\in\Omega$. Then, for any fixed $p\in\{1,\cdots,n\}$, $\mu>0$, $\sigma\geq\lambda_p>0$*

$$I_{a,\omega}\left((t-a)^{\mu}\prod_{i=1}^{n}f_i^{\lambda_i}(t)\right)I_{a,\omega}\left(\prod_{i\neq p}^{n}f_i^{\lambda_i}f_p^{\sigma}(t)\right) \tag{9.1}$$
$$\geq I_{a,\omega}\left((t-a)^{\mu}\prod_{i\neq p}^{n}f_i^{\lambda_i}f_p^{\sigma}(t)\right)I_{a,\omega}\left(\prod_{i=1}^{n}f_i^{\lambda_i}(t)\right),\quad t\in(a,+\infty).$$

Proof As f_p is a decreasing function for $p\in\{1,\cdots,n\}$,

$$\big((\rho-a)^{\mu}-(\tau-a)^{\mu}\big)\big(f_p^{\sigma-\lambda_p}(\tau)-f_p^{\sigma-\lambda_p}(\rho)\big)\geq 0,\quad \rho,\tau\in(a,t),$$

for any $t\in(a,+\infty)$ and $\sigma\geq\lambda_p$. Further, $\{f_i\}_{i=1}^n$ are positive functions on $(a,+\infty)$ and $\omega(t,\tau,x_1,\ldots,x_n)\geq 0$ for any $\tau\in(a,t)$, and therefore

$$\omega(t,\tau,x_1,\ldots,x_n)\prod_{i=1}^{n}f_i^{\lambda_i}(\tau)\big((\rho-a)^{\mu}-(\tau-a)^{\mu}\big)\big(f_p^{\sigma-\lambda_p}(\tau)-f_p^{\sigma-\lambda_p}(\rho)\big)\geq 0.$$

Integrating this inequality with respect to τ over (a,t), we obtain

$$(\rho-a)^{\mu}f_p^{\sigma-\lambda_p}(\rho)I_{a,\omega}\left(\prod_{i=1}^{n}f_i^{\lambda_i}(t)\right)+I_{a,\omega}\left((t-a)^{\mu}\prod_{i\neq p}^{n}f_i^{\lambda_i}f_p^{\sigma}(t)\right)$$
$$\leq(\rho-a)^{\mu}I_{a,\omega}\left(\prod_{i\neq p}^{n}f_i^{\lambda_i}f_p^{\sigma}(t)\right)+f_p^{\sigma-\lambda_p}(\rho)I_{a,\omega}\left((t-a)^{\mu}\prod_{i=1}^{n}f_i^{\lambda_i}(t)\right).$$

At last, multiplying both sides of this inequality by $\omega(t,\rho,x_1,\ldots,x_n)\prod_{i=1}^n f_i^{\lambda_i}(\rho)$, integrating with respect to ρ over (a,t) we come to the inequality (9.1).

Remark 9.1 If in Theorem 9.1 $\{f_i\}_{i=1}^n$ are supposed to be increasing functions on $(a,+\infty)$, then the converse to (9.1) inequality is true. Besides, for $\omega\equiv 1$, Theorem 9.1 becomes Theorem 3 of [34], and by Remark 2.3, Theorem 9.1 becomes Theorem 2.1 of [3] with $s\in\mathbb{R}^+$.

Theorem 9.2 *Let* $\{f_i\}_{i=1}^n$ *be positive, continuous, decreasing functions on* $(a, +\infty)$ $(a > -\infty)$*, let* g *be a positive, continuous, increasing function on* $(a, +\infty)$ *and let* $\omega \in \Omega$*. Then for any fixed* $p \in \{1, \cdots, n\}$*,* $\mu > 0$ *and* $\sigma \geq \lambda_p > 0$

$$I_{a,\omega}\left(g^{\mu}(t)\prod_{i=1}^{n} f_i^{\lambda_i}(t)\right) I_{a,\omega}\left(\prod_{i\neq p}^{n} f_i^{\lambda_i} f_p^{\sigma}(t)\right) \geq I_{a,\omega}\left(g^{\mu}(t)\prod_{i\neq p}^{n} f_i^{\lambda_i} f_p^{\sigma}(t)\right) I_{a,\omega}\left(\prod_{i=1}^{n} f_i^{\lambda_i}(t)\right), \quad t \in (a, +\infty). \tag{9.2}$$

Proof Evidently

$$\big(g^{\mu}(\rho) - g^{\mu}(\tau)\big)\big(f_p^{\sigma-\lambda_p}(\tau) - f_p^{\sigma-\lambda_p}(\rho)\big) \geq 0, \quad \rho, \tau \in (a, t),$$

for any $t \in (a, +\infty)$. Further, since $f_1, \ldots, f_n$ are positive functions on $(a, +\infty)$ and $\omega(t, \tau, x_1, \ldots, x_n) \geq 0$

$$\omega(t, \tau, x_1, \ldots, x_n)\prod_{i=1}^{n} f_i^{\lambda_i}(\tau)\big(g^{\mu}(\rho) - g^{\mu}(\tau)\big)\big(f_p^{\sigma-\lambda_p}(\tau) - f_p^{\sigma-\lambda_p}(\rho)\big) \geq 0$$

for any $\tau \in (a, t)$. Integrating this inequality with respect to τ over (a, t), we get

$$g^{\mu}(\rho) f_p^{\sigma-\lambda_p}(\rho) I_{a,\omega}\left(\prod_{i=1}^{n} f_i^{\lambda_i}(t)\right) + I_{a,\omega}\left(g^{\mu}(t)\prod_{i\neq p}^{n} f_i^{\lambda_i} f_p^{\sigma}(t)\right) \leq g^{\mu}(\rho) I_{a,\omega}\left(\prod_{i\neq p}^{n} f_i^{\lambda_i} f_p^{\sigma}(t)\right) + f_p^{\sigma-\lambda_p}(\rho) I_{a,\omega}\left(g^{\mu}(t)\prod_{i=1}^{n} f_i^{\lambda_i}(t)\right).$$

At last, multiplying both sides of the last inequality by $\omega(t, \rho, x_1, \ldots, x_n)\prod_{i=1}^{n} f_i^{\lambda_i}(\rho)$, then integrating with respect to ρ over (a, t), we come to the inequality (9.2).

Remark 9.2 For $\omega \equiv 1$, Theorem 9.2 becomes Theorem 4 of [34]. Besides, by Remark 2.3, Theorem 9.2 becomes Theorem 2.5 of [3] with $s \in \mathbb{R}^+$.

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On Sherman Method to Deriving Inequalities for Some Classes of Functions Related to Convexity

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1 Introduction, Notation, and Summary

In this chapter, we show the usefulness of Sherman method in deriving inequalities for convex, strongly convex, uniformly convex, and superquadratic functions. The inequality due to Sherman [32] generalizes the well-known inequality by Hardy, Littlewood, and Pólya in majorization theory [17, 21]. In addition, the HLP inequality includes the celebrated Jensen's inequality. These results have been extensively studied by many researchers.

The theory of majorization has many applications in linear algebra, convex analysis, probability, statistics, geometry, optimization, approximation, numerical analysis, statistical mechanics, econometrics, etc [21]. So, Sherman method gives further perspectives to find some nice applications. Therefore, this research topic is important and intriguing.

In this work, we provide a unified framework for generalizations of some classical results. First, we demonstrate the method by giving alternative unified proofs for some known inequalities involving convex functions. To do so, we use suitable column stochastic matrices. In particular, we deal with (i) the converse of Jensen's inequality [28], (ii) the monotonicity property of the Jensen's functional [16], (iii) an extension of Jensen's inequality by Mitroi-Symeonidis and Minculete [22], and (iv) Simić's

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results on global upper bounds for Jensen's functional [30, 31]. Also, we discuss (v) Csiszár f-divergence [13]. This divergence, which is used for appropriate convex functions f, leads to many important notions as the Kullback–Leibler distance, α-order Rényi entropy, Hellinger distance, variational distance, and χ^2-distance [14]. We interpret Csiszár–Körner's inequality for Csiszár f-divergence as a specialization of Sherman's inequality.

Next, we extend the above-mentioned results to uniformly convex, strongly convex, and superquadratic functions. Firstly, we establish a Sherman-like inequality for each of these classes of functions. Secondly, by using such inequalities, we derive some relevant results for the above-mentioned problems (i)–(v). For instance, we obtain the converse of Jensen's inequality generated by a strongly convex function (respectively by a uniformly convex or by a superquadratic function). The remaining problems (ii)–(v) are also explored in this context. Some results are new, and the others generalize some known inequalities.

The presented approach is very general and innovative. Its implementation in concrete situation depends on the possibility of construction of suitable column stochastic matrix generating the required inequality. We deliver such matrix separately for each of the mentioned problems (i)–(v).

It is known that φ-uniformly convex and c-strongly convex functions are convex as well. Likewise, nonnegative superquadratic functions are also convex. Therefore, for such functions the (known) results of Sect. 2 and the (new) ones of Sects. 3–7 are applicable. However, it is worth emphasizing that the new inequalities derived for uniformly convex, strongly convex, and nonnegative superquadratic functions are the refinements of the corresponding inequalities related to convex functions.

In the sequel, the cited theorems will be numbered by capital letters like Theorem A, Theorem B, ... , Theorem H. The original author's results are numbered like Theorem 3.4, Theorem 4.1, Theorem 5.1.

We begin our discussion with quoting some notation, definitions, and basic facts.

Definition 1.1 ([10, pp. 72–73]) A function $f : I \to \mathbb{R}$ is said to be *convex* on interval $I \subset \mathbb{R}$, if

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i) \tag{1}$$

for all $x_1, x_2, \ldots, x_n \in I$ and $p_1, p_2, \ldots, p_n \geq 0$ with $\sum_{i=1}^{n} p_i = 1$. Statement (1) is called *Jensen's inequality*.

Definition 1.2 ([21, pp. 29–30]) A $k \times n$ real matrix $S = (s_{ij})$ is said to be *column stochastic* if $s_{ij} \geq 0$ for $i = 1, \ldots, k,\ j = 1, \ldots, n$, and all column sums of S are equal to 1, i.e., $\sum_{i=1}^{k} s_{ij} = 1$ for $j = 1, \ldots, n$.

An $n \times n$ real matrix $S = (s_{ij})$ is called *doubly stochastic* if $s_{ij} \geq 0$ for $i, j = 1, \ldots, n$, and all column and row sums of S are equal to 1, i.e., $\sum_{i=1}^{n} s_{ij} = 1 = \sum_{j=1}^{n} s_{ij}$ for $i, j = 1, \ldots, n$.

Definition 1.3 ([21, p. 8]) We say that a vector $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ is *majorized* by a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, in symbols $\mathbf{y} \prec \mathbf{x}$, if

$$\sum_{i=1}^{l} y_{[i]} \leq \sum_{i=1}^{l} x_{[i]} \quad \text{for } l = 1, 2, \ldots, n$$

with equality for $l = n$. Here the symbols $x_{[i]}$ and $y_{[i]}$ stand for the ith largest entry of $\mathbf{x}$ and $\mathbf{y}$, respectively.

It is known by Birkhoff's and Rado's Theorems [21, pp. 10, 34, 162] that $\mathbf{y} \prec \mathbf{x}$ if and only if $\mathbf{y} = \mathbf{x}S$ for some $n \times n$ doubly stochastic matrix S.

We are now in a position to present Hardy–Littlewood–Pólya–Karamata's theorem showing a relationship between majorization and convexity [17, 18, 21].

Theorem A (HLPK's inequality [17, p. 75], [21, p. 92]) *Let $f : I \to \mathbb{R}$ be a convex continuous function on an interval $I \subset \mathbb{R}$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in I^n$.*

If $\mathbf{y} \prec \mathbf{x}$, then

$$\sum_{i=1}^{n} f(y_i) \leq \sum_{i=1}^{n} f(x_i). \tag{2}$$

If f is concave, then the inequality (2) is reversed.

We now demonstrate Sherman's inequality (4) (cf. [32], see also [9, 11, 24, 26]).

Theorem B (Sherman's inequality [9, 11, 32]) *Let $f : I \to \mathbb{R}$ be a convex function defined on an interval $I \subset \mathbb{R}$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in I^k$, $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in I^n$, $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{R}_+^k$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}_+^n$.*

If

$$\mathbf{y} = \mathbf{x}S \quad \textit{and} \quad \mathbf{a} = \mathbf{b}S^T \tag{3}$$

for some $k \times n$ column stochastic matrix $S = (s_{ij})$, then

$$\sum_{j=1}^{n} b_j f(y_j) \leq \sum_{i=1}^{k} a_i f(x_i). \tag{4}$$

If f is concave, then the inequality (4) is reversed.

The relation (3) is called *weighted majorization* of pairs $(\mathbf{x}, \mathbf{b})$ and $(\mathbf{y}, \mathbf{a})$ (see [9, 11]).

Remark 1.4 The proof of Theorem B requires Jensen's inequality. Conversely, Jensen's inequality is a special form of Sherman's inequality (4) with $n = 1$ and $b_1 = 1$.

Moreover, in the case when S is an $n \times n$ doubly stochastic matrix and $\mathbf{b} = (1, 1, \ldots, 1) \in \mathbb{R}^n_+$, then Theorem B reduces to Theorem A.

This work is organized as follows. In the next section, our aim is to demonstrate alternative unified proofs, based on Sherman inequality, of some known results for convex functions. In Sect. 2.1, we point out that the converse of Jensen's inequality [28] is a special case of Sherman's inequality. In a similar manner, we prove that the monotonicity property of the Jensen's functional [16] is a consequence of Sherman's inequality (see Sect. 2.2). Section 2.3 is devoted to analyzing an extension of Jensen's inequality [22] via Theorem B. Next, in Sect. 2.4 we show that some Simić's results [30, 31] follow from Theorems C and D. Finally, in Sect. 2.5, we conclude our discussion by interpreting Csiszár–Körner's inequality for Csiszár's f-divergence [13] as a specialization of Sherman's inequality.

In the subsequent sections we extend the mentioned results (i)-(v) from convex functions to uniformly convex, strongly convex, and superquadratic functions, respectively. In doing so, we begin with the version of the Sherman's inequality for these three classes of functions (see Theorem H). The difference with the standard case of convex functions (see Theorem B) lies in the existence of an extra term R in the new Sherman-like inequality. The term R plays an essential role in all applications of Theorem H in Sections 3–7. Each of these sections deals with one of the problems (i)–(v) for all of the above-mentioned classes of functions. In the described situations, we give concrete form of R.

In summary, we establish some new generalizations of the statements (i)–(v). The used method is based on Theorem H. The obtained inequalities for uniformly convex, strongly convex, and nonnegative superquadratic functions are the refinements of the corresponding inequalities for convex functions.

2 Proving Inequalities for Convex Functions

Theorems discussed in this section are known and are due to Pečarić et al. [28], Dragomir et al. [16], Kian [19], Mitroi-Symeonidis and Minculete [22], Simić [30, 31], and Csiszár and Körner [13], respectively. However, we give some alternative unified proofs by using the Sherman method described in Theorem B.

2.1 Converse of Jensen's Inequality

In [28], Pečarić et al. showed the following result (see also [8]).

Theorem C ([28, p. 105], [8, p. 513]) *Let $f : I \to \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$, $x_i \in [m, M] \subset I$, $-\infty < m < M < \infty$, and $p_i \geq 0$, $i = 1, \dots, n$, be such that $P_n = \sum_{i=1}^{n} p_i > 0$.*

Then the following converse of Jensen's inequality holds:

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) \leq \frac{M - \overline{x}}{M - m} f(m) + \frac{\overline{x} - m}{M - m} f(M), \tag{5}$$

where $\overline{x} = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i$.

Proof Since $x_i \in [m, M]$, we have the identity

$$x_i = \frac{M - x_i}{M - m} m + \frac{x_i - m}{M - m} M$$

with $\alpha_i = \frac{M - x_i}{M - m} \geq 0$, $\beta_i = \frac{x_i - m}{M - m} \geq 0$ and $\alpha_i + \beta_i = 1$ for $i = 1, \dots, n$.

In other words, in matrix notation we obtain

$$(x_1, x_2, \dots, x_n) = (m, M) \cdot \begin{pmatrix} \alpha_1, & \alpha_2, & \cdots, & \alpha_n \\ \beta_1, & \beta_2, & \cdots, & \beta_n \end{pmatrix}.$$

It is clear that the above $2 \times n$ matrix, denoted S, is column stochastic.

According to Sherman's inequality (see Theorem B), we get

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) \leq a_1 f(m) + a_2 f(M), \tag{6}$$

where the coefficients a_1 and a_2 can be derived from the formula (see (3))

$$(a_1, a_2) = \frac{1}{P_n} (p_1, p_2, \dots, p_n) \cdot \begin{pmatrix} \alpha_1, & \beta_1 \\ \alpha_2, & \beta_2 \\ \cdots, & \cdots \\ \alpha_n, & \beta_n \end{pmatrix}.$$

Hence

$$a_1 = \frac{1}{P_n} \sum_{i=1}^{n} p_i \frac{M - x_i}{M - m} = \frac{1}{P_n} \frac{P_n M - P_n \overline{x}}{M - m} = \frac{M - \overline{x}}{M - m}, \tag{7}$$

$$a_2 = \frac{1}{P_n} \sum_{i=1}^{n} p_i \frac{x_i - m}{M - m} = \frac{1}{P_n} \frac{P_n \overline{x} - P_n m}{M - m} = \frac{\overline{x} - m}{M - m}. \tag{8}$$

Now, the required result (5) is due to (6). ■

2.2 Monotonicity of Jensen Functional

Given a function $f : I \to \mathbb{R}$ on an interval $I \subset \mathbb{R}$, $\mathbf{x} = (x_1, x_2, \dots, x_n) \in I^n$ and $\mathbf{p} \in \mathcal{P}_n^0$, where

$$\mathcal{P}_n^0 = \{\mathbf{p} = (p_1, p_2, \dots, p_n) \in \mathbb{R}^n : p_i \geq 0, \quad P_n > 0\} \text{ with } P_n = \sum_{i=1}^n p_i, \tag{9}$$

the *Jensen's functional* is defined by

$$J(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^n p_i f(x_i) - P_n f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \tag{10}$$

(see [16]).

In light of Jensen's inequality (1),

$$J(f, \mathbf{x}, \mathbf{p}) \geq 0 \quad \text{whenever } f \text{ is a convex function.}$$

Theorem D [16] *If $f : I \to \mathbb{R}$ is a convex function, then the function $\mathbf{p} \to J(f, \mathbf{x}, \mathbf{p})$ is monotone for any fixed $\mathbf{x} = (x_1, \dots, x_n) \in I^n$, i.e., for $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^0$,*

$$\mathbf{p} \leq \mathbf{q} \quad \textit{implies} \quad J(f, \mathbf{x}, \mathbf{p}) \leq J(f, \mathbf{x}, \mathbf{q}). \tag{11}$$

Proof Fix any $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^0$ such that $\mathbf{p} \leq \mathbf{q}$. Then $q_i - p_i \geq 0$ for $i = 1, \dots, n$.

It is not hard to check that

$$\frac{1}{Q_n}\sum_{i=1}^n q_i x_i = \frac{P_n}{Q_n}\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) + \sum_{i=1}^n \frac{q_i - p_i}{Q_n} x_i. \tag{12}$$

By denoting

$$y = \frac{1}{Q_n}\sum_{i=1}^n q_i x_i, \quad x_0 = \frac{1}{P_n}\sum_{i=1}^n p_i x_i, \quad \alpha_0 = \frac{P_n}{Q_n} \quad \text{and} \quad \alpha_i = \frac{q_i - p_i}{Q_n} \quad \text{for } i = 1, \dots, n,$$

we can rewrite (12) in the matrix form

$$y = (x_0, x_1, x_2, \dots, x_n) \cdot \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix},$$

where the above $n \times 1$ matrix is column stochastic.

In this case, Sherman's inequality becomes the following (see Theorem B):

$$f(y) \le a_0 f(x_0) + \sum_{i=1}^{n} a_i f(x_i) \tag{13}$$

with

$$(a_0, a_1, a_2, \ldots, a_n) = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n)$$

(see (3)), i.e.,

$$a_0 = \alpha_0 = \frac{P_n}{Q_n} \quad \text{and} \quad a_i = \alpha_i = \frac{q_i - p_i}{Q_n} \quad \text{for } i = 1, \ldots, n. \tag{14}$$

It is now sufficient to apply (13) and (14) in order to get

$$f\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right) \le \frac{P_n}{Q_n} f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) + \sum_{i=1}^{n} \frac{q_i - p_i}{Q_n} f(x_i). \tag{15}$$

We complete the proof of (11) with the observation that (15) is equivalent to $J(f, \mathbf{x}, \mathbf{p}) \le J(f, \mathbf{x}, \mathbf{q})$. ■

As a consequence of the last theorem, we present a result by Kian [19, Corollary 2.4]. Namely, if $f : I \to \mathbb{R}$ is a convex function, then

$$r\left(\sum_{i=1}^{n} f(x_i) - nf\left(\sum_{i=1}^{n} \frac{x_i}{n}\right)\right) \le \sum_{i=1}^{n} t_i f(x_i) - f\left(\sum_{i=1}^{n} t_i x_i\right) \tag{16}$$

$$\le \sum_{i=1}^{n} f(x_i) - nf\left(\sum_{i=1}^{n} \frac{x_i}{n}\right)$$

for all $x_i \in I$, $t_i \in [0, 1]$ with $\sum_{i=1}^{n} t_i = 1$, where $r = \min_{i \in \{1,\ldots,n\}} t_i$.

In fact, to see this, it is enough to appeal to Theorem D, because

$$(r, r, \ldots, r) \preceq (t_1, t_2, \ldots, t_n) \preceq (1, 1, \ldots, 1).$$

■

2.3 An extension of Jensen's Inequality

The special case of [22, Theorem 1] for convex functions is incorporated in Theorem E. We shall analyze it from the point of view of Sherman method.

Theorem E *Let $f : I \to \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$, $x_i \in I$, and $p_i \in (0, 1)$, $i = 1, \dots, n$, with $\sum_{i=1}^{n} p_i = 1$, and $\lambda, \mu \in [0, 1]$.*

Then

$$\sum_{i=1}^{n} p_i f((1-\lambda\mu)\overline{x} + \lambda\mu x_i) \le (1-\lambda) f(\overline{x}) + \lambda \sum_{i=1}^{n} p_i f((1-\mu)\overline{x} + \mu x_i), \tag{17}$$

where $\overline{x} = \sum_{i=1}^{n} p_i x_i$.

Proof We denote

$$A_i = (1-\mu)\overline{x} + \mu x_i \quad \text{and} \quad B_i = (1-\lambda\mu)\overline{x} + \lambda\mu x_i \quad \text{for } i = 1, \dots, n.$$

But $\lambda \in [0, 1]$, so we have the identity

$$B_i = \frac{A_i - B_i}{A_i - \overline{x}}\overline{x} + \frac{B_i - \overline{x}}{A_i - \overline{x}} A_i \quad \text{for } i = 1, \dots, n$$

with $\alpha_i = \frac{A_i - B_i}{A_i - \overline{x}} = 1 - \lambda \ge 0, \beta_i = \frac{B_i - \overline{x}}{A_i - \overline{x}} = \lambda \ge 0$ and $\alpha_i + \beta_i = 1$ for $i = 1, \dots, n$.

In matrix notation, the above can be restated as follows:

$$(B_1, B_2, \dots, B_n) = (\overline{x}, A_1, \overline{x}, A_2, \dots, \overline{x}, A_n) \cdot \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ \beta_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \\ 0 & 0 & \cdots & \beta_n \end{pmatrix}.$$

Obviously, the above $2n \times n$ matrix, denoted by S, is column stochastic.

By virtue of Sherman's inequality (see Theorem B), we find that

$$\sum_{i=1}^{n} p_i f(B_i) \le \sum_{i=1}^{n} (a_i f(\overline{x}) + b_i f(A_i)), \tag{18}$$

where the coefficients a_i and b_i satisfy

$$(a_1, b_1, a_2, b_2, \ldots, a_n, b_n) = (p_1, p_2, \ldots, p_n) \cdot \begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \alpha_2 & \beta_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_n & \beta_n \end{pmatrix}$$

(see (3)). A bit of algebra yields

$$a_i = p_i \alpha_i = p_i \frac{A_i - B_i}{A_i - \overline{x}} = (1-\lambda) p_i \quad \text{and} \quad b_i = p_i \beta_i = p_i \frac{B_i - \overline{x}}{A_i - \overline{x}} = \lambda p_i. \tag{19}$$

In consequence (17) follows from (18) and (19). This completes the proof of Theorem E. ■

2.4 Global Upper Bounds for Jensen's Inequality

Simić [30, 31] proved the following inequalities (20) and (21). We shall show that they are direct consequences of Theorems C and D.

Theorem F ([31, Theorem C], [30, Theorem 1]) *Let* $f : [m, M] \to \mathbb{R}$ *be a convex function on an interval* $[m, M] \subset \mathbb{R}$*,* $x_i \in [m, M]$*, and* $p_i \geq 0$*,* $i = 1, \ldots, n$*, with* $\sum_{i=1}^n p_i = 1$.

Then

$$\sum_{i=1}^n p_i f(x_i) \leq \max_{p\in[0,1]} \{pf(m) + qf(M) - f(pm + qM)\} \quad \textit{with } q = 1 - p, \tag{20}$$

$$\sum_{i=1}^n p_i f(x_i) \leq f(m) + f(M) - 2f\left(\frac{m+M}{2}\right). \tag{21}$$

Proof We proceed as in the proof of Theorem C to derive Sherman's inequality (see Theorem B) as follows

$$\sum_{i=1}^n p_i f(x_i) \leq a_1 f(m) + a_2 f(M), \tag{22}$$

where $\alpha_i = \frac{M - x_i}{M - m}$, $\beta_i = \frac{x_i - m}{M - m}$ and

$$a_1 = \sum_{i=1}^n p_i \alpha_i = \frac{M - \overline{x}}{M - m} \quad \text{and} \quad a_2 = \sum_{i=1}^n p_i \beta_i = \frac{\overline{x} - m}{M - m}. \tag{23}$$

Simultaneously, (22) is equivalent to

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \le a_1 f(m) + a_2 f(M) - f(a_1 m + a_2 M), \tag{24}$$

since $x_i = \alpha_i m + \beta_i M$, $i = 1, \ldots, n$, and

$$f\left(\sum_{i=1}^{n} p_i x_i\right) = f\left(\sum_{i=1}^{n} p_i(\alpha_i m + \beta_i M)\right)$$

$$= f\left(\sum_{i=1}^{n} p_i \alpha_i m + \sum_{i=1}^{n} p_i \beta_i M\right) = f(a_1 m + a_2 M).$$

Clearly, $a_1 + a_2 = 1$, $a_1, a_2 \ge 0$ by (23). For this reason the inequality (24) easily implies (20), as required.

On the other hand, to prove (21) it is sufficient to combine (24) with the inequality

$$a_1 f(m) + a_2 f(M) - f(a_1 m + a_2 M) \le f(m) + f(M) - 2f\left(\frac{m+M}{2}\right) \tag{25}$$

being a consequence of the property of monotonicity of Jensen's functional (see Theorem D), because $(a_1, a_2) \le (1, 1)$. ■

2.5 Csiszár–Körner's Inequality

We conclude our discussion by pointing out that Csiszár–Körner's inequality for Csiszár f-divergence can be derived via Sherman's inequality.

Given a convex function $f : (0, \infty) \to \mathbb{R}$ and two n-tuples of positive numbers $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$, the *Csiszár f-divergence* is defined by

$$I_f(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{n} p_i f\left(\frac{q_i}{p_i}\right) \tag{26}$$

(see [12–15, 25]).

Theorem G (Csiszár–Körner's inequality [13–15]) *Let $f : (0, \infty) \to \mathbb{R}$ be a convex function, and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ be two n-tuples of positive numbers.*

Then

$$\sum_{i=1}^{n} p_i f\left(\frac{\sum_{i=1}^{n} q_i}{\sum_{i=1}^{n} p_i}\right) \le I_f(\mathbf{p}, \mathbf{q}). \tag{27}$$

Proof It is easily seen that

$$\frac{\sum_{i=1}^{n} q_i}{\sum_{i=1}^{n} p_i} = \frac{p_1}{\sum_{i=1}^{n} p_i} \cdot \frac{q_1}{p_1} + \frac{p_2}{\sum_{i=1}^{n} p_i} \cdot \frac{q_2}{p_2} + \ldots + \frac{p_n}{\sum_{i=1}^{n} p_i} \cdot \frac{q_n}{p_n}. \tag{28}$$

By denoting

$$y = \frac{\sum_{i=1}^{n} q_i}{\sum_{i=1}^{n} p_i}, \quad x_i = \frac{q_i}{p_i}, \quad \text{and} \quad \alpha_i = \frac{p_i}{\sum_{i=1}^{n} p_i} \quad \text{for } i = 1, \ldots, n,$$

we can state (28) in the matrix form

$$y = (x_1, x_2, \ldots, x_n) \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

In consequence, Sherman's inequality reduces to the form (see Theorem B):

$$\left(\sum_{i=1}^{n} p_i \right) f(y) \le \sum_{i=1}^{n} a_i f(x_i), \tag{29}$$

where

$$(a_1, a_2, \ldots, a_n) = \left(\sum_{i=1}^{n} p_i \right) (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

(see (3)), i.e.,

$$a_i = \left(\sum_{i=1}^{n} p_i \right) \alpha_i = p_i \quad \text{for } i = 1, \ldots, n. \tag{30}$$

Now, the result (27) can be deduced from (29) and (30). ■

Remark 2.1 In light of (27) we see that if in addition $\sum_{k=1}^{n} p_k = \sum_{k=1}^{n} q_k$ and $f(1) = 0$, then we obtain a Shannon-type inequality:

$$0 \le I_f(\mathbf{p}, \mathbf{q}) \tag{31}$$

(see [14, Corollary 1]).

3 Converse of Jensen's Inequality for Some Functions

Some analogs of Jensen's inequality hold for certain classes of functions (see [27]; cf. also [2, 20, 23]). In this section, we establish some converse results. To do so, we begin with some relevant definitions.

Definition 3.1 ([33]) A function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}$ is said to be uniformly convex with modulus $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, if the following inequality holds for all points $x, y \in I$ and $p \in [0, 1]$:

$$f(px + (1-p)y) + p(1-p)\varphi(|x-y|) \leq pf(x) + (1-p)f(y). \tag{32}$$

In the case of differentiable f, condition (32) gives

$$f(x) - f(y) \geq f'(y)(x-y) + \varphi(|x-y|) \quad \text{for } x, y \in I. \tag{33}$$

Definition 3.2 ([5, 29]) A function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}$ is said to be *c-strongly convex* on I, where $c \in \mathbb{R}_+$, if the following inequality holds for all points $x, y \in I$ and $p \in [0, 1]$:

$$f(px + (1-p)y) + \frac{c}{2}p(1-p)|x-y|^2 \leq pf(x) + (1-p)f(y). \tag{34}$$

For differentiable f, condition (34) implies

$$f(x) - f(y) \geq f'(y)(x-y) + \frac{c}{2}|x-y|^2 \quad \text{for } x, y \in I$$

(see [5, p. 684]).

It is readily seen that if f is a c-strongly convex function, then f is uniformly convex with modulus $\varphi(t) = \frac{c}{2}t^2$.

Definition 3.3 ([1]) A function $f : \mathbb{R}_+ \to \mathbb{R}$ defined on the interval $I = \mathbb{R}_+$ is said to be *superquadratic*, if for each point $y \in \mathbb{R}_+$ there exists a number $C(y) \in \mathbb{R}$ such that the following condition is fulfilled:

$$f(x) - f(y) \geq C(y)(x-y) + f(|x-y|) \quad \text{for } x \in I. \tag{35}$$

An equivalent condition for the superquadracity of f is as follows (see [7, Theorem 9]):

$$f(px + (1-p)y) + pf((1-p)|x-y|) + (1-p)f(p|x-y|) \leq pf(x) + (1-p)f(y)$$

holds for $x, y \geq 0$ and $p \in [0, 1]$.

If f is superquadratic and differentiable with $f(0) = f'(0) = 0$, then $C(y) = f'(y)$ for $y > 0$ (see [7, p. 720]).

For further information on superquadratic functions, consult [1, 3, 4, 6].

We can now state Sherman's inequality for uniformly convex functions, c-strongly convex functions, and superquadratic functions [27].

Theorem H ([27, p. 4794, Theorems 6.2 and 7.2]) *Let $I \subset \mathbb{R}$ be an interval. Let $f : I \to \mathbb{R}$ be a differentiable function of type* (i)*,* (ii)*, or* (iii) *as described below. Let $\mathbf{x} = (x_1, \ldots, x_k) \in I^k$, $\mathbf{y} = (y_1, \ldots, y_n) \in I^n$, $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{R}^k_+$, and $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n_+$.*

If

$$\mathbf{y} = \mathbf{x}S \quad \textit{and} \quad \mathbf{a} = \mathbf{b}S^T \tag{36}$$

for some $k \times n$ column stochastic matrix $S = (s_{ij})$, then

$$\sum_{j=1}^{n} b_j f(y_j) + R \le \sum_{i=1}^{k} a_i f(x_i) \tag{37}$$

for R defined as follows:

(i) If f is uniformly convex with modulus φ, then

$$R = \sum_{j=1}^{n} b_j \sum_{i=1}^{k} s_{ij} \varphi(|x_i - y_j|). \tag{38}$$

(ii) If f is c-strongly convex, then

$$R = \frac{c}{2} \sum_{j=1}^{n} b_j \sum_{i=1}^{k} s_{ij} |x_i - y_j|^2. \tag{39}$$

(iii) If f is superquadratic on $I = \mathbb{R}_+$, then

$$R = \sum_{j=1}^{n} b_j \sum_{i=1}^{k} s_{ij} f(|x_i - y_j|). \tag{40}$$

The following theorem is a complement to a result of Pečarić et al. [28, p. 105] (see also [8, p. 513]) for uniformly convex, strongly convex, and superquadratic functions, respectively, in place of convex functions.

Theorem 3.4 *Let $f : I \to \mathbb{R}$ be a differentiable function on an interval $I \subset \mathbb{R}$, $-\infty < m < M < \infty$, $x_i \in [m, M] \subset I$, and $p_i \ge 0$, $i = 1, \ldots, n$, be such that $P_n = \sum_{i=1}^{n} p_i > 0$. Denote $\overline{x} = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i$.*

Then the following converses of Jensen-type inequalities hold.

(i) If f is uniformly convex with modulus φ, then

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) + \frac{1}{P_n}\sum_{i=1}^{n} p_i \left[\frac{M-x_i}{M-m}\,\varphi(|x_i-m|) + \frac{x_i-m}{M-m}\,\varphi(|M-x_i|)\right]$$
$$\le \frac{M-\overline{x}}{M-m} f(m) + \frac{\overline{x}-m}{M-m} f(M). \tag{41}$$

(ii) If f is c-strongly convex, then

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) + \frac{c}{2P_n}\sum_{i=1}^{n} p_i \left[\frac{M-x_i}{M-m}\,|x_i-m|^2 + \frac{x_i-m}{M-m}\,|M-x_i|^2\right]$$
$$\le \frac{M-\overline{x}}{M-m} f(m) + \frac{\overline{x}-m}{M-m} f(M). \tag{42}$$

(iii) If f is superquadratic on $I = \mathbb{R}_+$, then

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) + \frac{1}{P_n}\sum_{i=1}^{n} p_i \left[\frac{M-x_i}{M-m} f(|m-x_i|) + \frac{x_i-m}{M-m} f(|M-x_i|)\right]$$
$$\le \frac{M-\overline{x}}{M-m} f(m) + \frac{\overline{x}-m}{M-m} f(M). \tag{43}$$

Proof We begin as in the proof of Theorem C in the previous section.

Because $m \le x_i \le M$ for $i = 1, \ldots, n$, we obtain

$$x_i = \frac{M-x_i}{M-m} m + \frac{x_i-m}{M-m} M$$

with

$$\alpha_i = \frac{M-x_i}{M-m} \ge 0\,, \quad \beta_i = \frac{x_i-m}{M-m} \ge 0 \quad \text{for } i = 1, \ldots, n, \tag{44}$$

and $\alpha_i + \beta_i = 1$ for $i = 1, \ldots, n$.

Hence

$$(x_1, x_2, \ldots, x_n) = (m, M) \cdot \begin{pmatrix} \alpha_1, & \alpha_2, & \cdots, & \alpha_n \\ \beta_1, & \beta_2, & \cdots, & \beta_n \end{pmatrix},$$

where above $2 \times n$ matrix is column stochastic.

Let the coefficients a_1 and a_2 be defined by the following condition (see (3))

$$(a_1, a_2) = \frac{1}{P_n}(p_1, p_2, \dots, p_n) \cdot \begin{pmatrix} \alpha_1, & \beta_1 \\ \alpha_2, & \beta_2 \\ \vdots & \vdots \\ \alpha_n, & \beta_n \end{pmatrix}.$$

Some algebra (cf. (7) and (8)) gives

$$a_1 = \frac{M - \overline{x}}{M - m} \quad \text{and} \quad a_2 = \frac{\overline{x} - m}{M - m}. \tag{45}$$

(i). According to Sherman's inequality (37) with R given by (38) (see Theorem H), we find

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) + \frac{1}{P_n}\sum_{i=1}^{n} p_i \left[\alpha_i \varphi(|m - x_i|) + \beta_i \varphi(|M - x_i|)\right]$$
$$\leq a_1 f(m) + a_2 f(M). \tag{46}$$

It now follows from (44) and (45) that (41) is proved, as required.

(ii). If f is a c-strongly convex function, then f is a uniformly convex function with modulus $\varphi(t) = \frac{c}{2}t^2$. Therefore, (42) is an easy consequence of (41).

(iii). It follows from (37) and (40) that

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) + \frac{1}{P_n}\sum_{i=1}^{n} p_i \left[\alpha_i f(|m - x_i|) + \beta_i f(|M - x_i|)\right]$$
$$\leq a_1 f(m) + a_2 f(M). \tag{47}$$

Combining this with (44) and (45) leads to (43).

This completes the proof. ■

Remark 3.5 In Theorem 3.4, the case (iii) for superquadratic functions is in the same line as a result for linear isotonic functionals by Banić and Varošanec [7, Theorem 15].

Remark 3.6 We have some observations concerning Theorem 3.4.

(i) Let f be φ-uniformly convex. Since the modulus φ is nonnegative, f is convex and the term on the left-hand side of inequality (41) is nonnegative, i.e.,

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i \left[\frac{M - x_i}{M - m}\varphi(|x_i - m|) + \frac{x_i - m}{M - m}\varphi(|M - x_i|)\right] \geq 0.$$

In consequence, if f is φ-uniformly convex, then inequality (41) is a refinement of the converse of Jensen's inequality (5).

(ii) Let f be c-strongly convex with $c \geq 0$. Then f is convex. Moreover, the term

$$\frac{c}{2P_n}\sum_{i=1}^{n} p_i \left[\frac{M-x_i}{M-m}|x_i-m|^2+\frac{x_i-m}{M-m}|M-x_i|^2\right] \geq 0$$

on the left-hand side of inequality (42) is nonnegative.
Therefore, if f is c-strongly convex, then inequality (42) is a refinement of the converse of Jensen's inequality (5).

(iii) Let f be nonnegative superquadratic on $I = \mathbb{R}_+$. Then f must be convex. Furthermore, the term

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i \left[\frac{M-x_i}{M-m} f(|m-x_i|)+\frac{x_i-m}{M-m} f(|M-x_i|)\right] \geq 0$$

on the left-hand side of inequality (43) is nonnegative.
So, we deduce that if f is both nonnegative and superquadratic, then the inequality (43) is a refinement of the converse of Jensen's inequality (5).

4 Generalized Monotonicity of Jensen's Functional

In this section, we study Jensen's functional $J(f, \mathbf{x}, \mathbf{p})$ (see (10)) defined for uniformly convex, strongly convex, and superquadratic functions $f : I \to \mathbb{R}$, respectively, with an interval $I \subset \mathbb{R}$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\mathbf{p} \in \mathcal{P}_n^0$, where $\mathcal{P}_n^0$ is defined by (9).

The next theorem corresponds to a result due to Dragomir, Pečarić, and Persson [16] on monotonicity of the Jensen's functional for a convex function.

Theorem 4.1 *Let $f : I \to \mathbb{R}$ be a differentiable function, $\mathbf{p} = (p_1, \ldots, p_n) \in \mathcal{P}_n^0$, and $\mathbf{q} = (q_1, \ldots, q_n) \in \mathcal{P}_n^0$ be such that $\mathbf{p} \leq \mathbf{q}$.*

(i) If f is uniformly convex with modulus φ, then

$$P_n\varphi\left(\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i - \frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right|\right) + \sum_{i=1}^{n}(q_i-p_i)\varphi\left(\left|x_i - \frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right|\right) \leq J(f,\mathbf{x},\mathbf{q}) - J(f,\mathbf{x},\mathbf{p}). \tag{48}$$

(ii) If f is c-strongly convex, then

$$\frac{c}{2}P_n\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i - \frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right|^2 + \frac{c}{2}\sum_{i=1}^{n}(q_i-p_i)\left|x_i - \frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right|^2$$

$$\leq J(f, \mathbf{x}, \mathbf{q}) - J(f, \mathbf{x}, \mathbf{p}). \tag{49}$$

(iii) If f is superquadratic on $I = \mathbb{R}_+$, then

$$P_n f\left(\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i - \frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right|\right) + \sum_{i=1}^{n}(q_i - p_i) f\left(\left|x_i - \frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right|\right)$$
$$\leq J(f, \mathbf{x}, \mathbf{q}) - J(f, \mathbf{x}, \mathbf{p}). \tag{50}$$

Proof Similarly as in the proof of Theorem D, we find

$$\frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i = \frac{P_n}{Q_n}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) + \sum_{i=1}^{n}\frac{q_i - p_i}{Q_n} x_i \tag{51}$$

with $q_i - p_i \geq 0$ for $i = 1, \ldots, n$.

By making use of the notation

$$y = \frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i \quad \text{and} \quad x_0 = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i, \tag{52}$$

$$\alpha_0 = \frac{P_n}{Q_n} \quad \text{and} \quad \alpha_i = \frac{q_i - p_i}{Q_n} \quad \text{for } i = 1, \ldots, n, \tag{53}$$

we can restate (51) as

$$y = (x_0, x_1, x_2, \ldots, x_n) \cdot \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

Evidently, the above $n \times 1$ matrix S is column stochastic.

In light of (3) we define

$$(a_0, a_1, a_2, \ldots, a_n) = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n)$$

i.e.,

$$a_0 = \alpha_0 = \frac{P_n}{Q_n} \quad \text{and} \quad a_i = \alpha_i = \frac{q_i - p_i}{Q_n} \quad \text{for } i = 1, \ldots, n. \tag{54}$$

(i). If f is uniformly convex with modulus φ, then Sherman-type inequality (see Theorem H) with $b_1 = 1$, $y = y_1$ and $s_{i1} = \alpha_i$ for $i = 0, 1, \ldots, n$ yields

$$f(y) + \sum_{i=0}^{n} \alpha_i \varphi(|x_i - y|) \leq \sum_{i=0}^{n} a_i f(x_i). \tag{55}$$

By employing (55) and (54) we get

$$f\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right) + \sum_{i=0}^{n} \alpha_i \varphi\left(\left|x_i - \frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right|\right)$$
$$\leq \frac{P_n}{Q_n} f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) + \sum_{i=1}^{n} \frac{q_i - p_i}{Q_n} f(x_i). \tag{56}$$

Equivalently,

$$\frac{P_n}{Q_n}\varphi\left(\left|x_0 - \frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right|\right) + \sum_{i=1}^{n} \frac{q_i - p_i}{Q_n}\varphi\left(\left|x_i - \frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right|\right)$$
$$\leq \frac{P_n}{Q_n} f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) + \sum_{i=1}^{n} \frac{q_i - p_i}{Q_n} f(x_i) - f\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right). \tag{57}$$

It is easy to check that the right-hand side of the inequality (30) is equal to $\frac{1}{Q_n}[J(f, \mathbf{x}, \mathbf{q}) - J(f, \mathbf{x}, \mathbf{p})]$. Multiplying both the sides of the above inequality (57) by $Q_n > 0$ leads to (48). This completes the proof of (i).

(ii). If f is a c-strongly convex function, then f is a uniformly convex function with modulus $\varphi(t) = \frac{c}{2}t^2$. Therefore, (49) is an easy consequence of (48).

(iii). If f is superquadratic, then we obtain three inequalities as in (55)–(57) with φ replaced by f. In summary, the last of them is equivalent to (50), as desired. ∎

Remark 4.2 Here we give some comments on Theorem 4.1.

(i) Let f be φ-uniformly convex with (nonnegative) modulus φ. Then f is convex and the left-hand side of inequality (41) is nonnegative, that is

$$P_n\varphi\left(\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i - \frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right|\right) + \sum_{i=1}^{n}(q_i - p_i)\varphi\left(\left|x_i - \frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right|\right) \geq 0.$$

For this reason, if f is φ-uniformly convex, then inequality (48) is a refinement of the monotonicity property (11) of the Jensen's functional for convex functions.

(ii) Let f be c-strongly convex with $c \geq 0$. Then f is convex. It is easy to see that the left-hand side of inequality (49) is nonnegative, i.e.,

$$\frac{c}{2}P_n\left|\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i-\frac{1}{Q_n}\sum_{i=1}^{n}q_ix_i\right|^2+\frac{c}{2}\sum_{i=1}^{n}(q_i-p_i)\left|x_i-\frac{1}{Q_n}\sum_{i=1}^{n}q_ix_i\right|^2\geq 0.$$

Therefore, the following implication is valid. If f is c-strongly convex, then inequality (49) is a refinement of the monotonicity property (11) of the Jensen's functional for convex functions.

(iii) Let f be nonnegative superquadratic on $I=\mathbb{R}_+$. Then f is convex, and the left-hand side of inequality (50) is nonnegative as follows:

$$P_nf\left(\left|\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i-\frac{1}{Q_n}\sum_{i=1}^{n}q_ix_i\right|\right)+\sum_{i=1}^{n}(q_i-p_i)f\left(\left|x_i-\frac{1}{Q_n}\sum_{i=1}^{n}q_ix_i\right|\right)\geq 0.$$

So, if f is both nonnegative and superquadratic, then inequality (50) is a refinement of the monotonicity of Jensen's functional (11).

5 An Extension of Jensen's Inequality

Theorem E in Sect. 2 is a special case of [22, Theorem 1] (see also item (ii) below) and is devoted to convex functions. The following result is a development of Theorem E established for other classes of functions related to convexity.

Theorem 5.1 *Let $f: I\to\mathbb{R}$ be a differentiable function on an interval $I\subset\mathbb{R}$, $x_i\in I$, and $p_i\in(0,1)$, $i=1,\ldots,n$, with $\sum_{i=1}^{n}p_i=1$, and $\lambda,\mu\in[0,1]$. Denote $\overline{x}=\sum_{i=1}^{n}p_ix_i$.*

(i) If f is uniformly convex with modulus φ, then

$$\begin{aligned}&\sum_{i=1}^{n}p_if((1-\lambda\mu)\overline{x}+\lambda\mu x_i)+(1-\lambda)\sum_{i=1}^{n}p_i\varphi(\lambda\mu|\overline{x}-x_i|)\\&\quad+\lambda\sum_{i=1}^{n}p_i\varphi(\mu(1-\lambda)|x_i-\overline{x}|)\\&\quad\leq(1-\lambda)f(\overline{x})+\lambda\sum_{i=1}^{n}p_if((1-\mu)\overline{x}+\mu x_i).\end{aligned}\tag{58}$$

(ii) If f is c-strongly convex, then

$$\sum_{i=1}^{n} p_i f((1-\lambda\mu)\overline{x} + \lambda\mu x_i) + \frac{c}{2}\lambda(1-\lambda)\mu^2 \sum_{i=1}^{n} p_i (x_i - \overline{x})^2$$
$$\leq (1-\lambda) f(\overline{x}) + \lambda \sum_{i=1}^{n} p_i f((1-\mu)\overline{x} + \mu x_i). \tag{59}$$

(iii) If f is superquadratic on $I = \mathbb{R}_+$, then

$$\sum_{i=1}^{n} p_i f((1-\lambda\mu)\overline{x} + \lambda\mu x_i) + (1-\lambda) \sum_{i=1}^{n} p_i f(\lambda\mu|\overline{x} - x_i|)$$
$$+ \lambda \sum_{i=1}^{n} p_i f(\mu(1-\lambda)|x_i - \overline{x}|)$$
$$\leq (1-\lambda) f(\overline{x}) + \lambda \sum_{i=1}^{n} p_i f((1-\mu)\overline{x} + \mu x_i). \tag{60}$$

Proof Analogously as in the proof of Theorem E, we use the notation

$$A_i = (1-\mu)\overline{x} + \mu x_i \quad \text{and} \quad B_i = (1-\lambda\mu)\overline{x} + \lambda\mu x_i \quad \text{for } i = 1, \ldots, n.$$

It is readily seen that

$$B_i = \frac{A_i - B_i}{A_i - \overline{x}}\overline{x} + \frac{B_i - \overline{x}}{A_i - \overline{x}} A_i \quad \text{for } i = 1, \ldots, n$$

with

$$\alpha_i = \frac{A_i - B_i}{A_i - \overline{x}} = 1 - \lambda \geq 0 \quad \text{and} \quad \beta_i = \frac{B_i - \overline{x}}{A_i - \overline{x}} = \lambda \geq 0 \tag{61}$$

and $\alpha_i + \beta_i = 1$ for $i = 1, \ldots, n$.

The above can be rewritten as

$$(B_1, B_2, \ldots, B_n) = (\overline{x}, A_1, \overline{x}, A_2, \ldots, \overline{x}, A_n) \cdot \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ \beta_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \\ 0 & 0 & \cdots & \beta_n \end{pmatrix}. \tag{62}$$

It is not hard to verify that the above $2n \times n$ matrix is column stochastic.

We introduce coefficients a_i and b_i by

$$(a_1, b_1, a_2, b_2, \dots, a_n, b_n) = (p_1, p_2, \dots, p_n) \cdot \begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \alpha_2 & \beta_2 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_n & \beta_n \end{pmatrix} \tag{63}$$

(see (3)). Moreover, we have

$$a_i = p_i\alpha_i = p_i\frac{A_i - B_i}{A_i - \overline{x}} = (1-\lambda)p_i \quad \text{and} \quad b_i = p_i\beta_i = p_i\frac{B_i - \overline{x}}{A_i - \overline{x}} = \lambda p_i. \tag{64}$$

(i). Assume that f is φ-uniformly convex. On account of Sherman's inequality (see Theorem H) with the help of (61)–(63), we infer that

$$\sum_{i=1}^n p_i f(B_i) + \sum_{i=1}^n p_i[\alpha_i\varphi(|\overline{x} - B_i|) + \beta_i\varphi(|A_k - B_i|)]$$

$$\leq \sum_{i=1}^n (a_i f(\overline{x}) + b_i f(A_i)) = \sum_{i=1}^n a_i f(\overline{x}) + \sum_{i=1}^n b_i f(A_i) \tag{65}$$

$$= \sum_{i=1}^n (1-\lambda)p_i f(\overline{x}) + \sum_{i=1}^n \lambda p_i f(A_i) = (1-\lambda)f(\overline{x}) + \lambda\sum_{i=1}^n p_i f(A_i).$$

Thus we obtain the desired result (i) of Theorem 5.1.

(ii). If f is c-strongly convex, then f is φ-uniformly convex for $\varphi(t) = \frac{c}{2}t^2$. In this situation we employ (58) to get (59).

(iii). Let f be superquadratic. It is enough to apply the same argument as for (i) with φ replaced by f. ■

Remark 5.2 In Theorem 5.1, item (ii) for strongly convex functions is due to Mitroi-Symeonidis and Minculete [22, Theorem 1].

Remark 5.3 We now present some remarks on Theorem 5.1.

(i) Let f be φ-uniformly convex with (nonnegative) modulus φ. Then f is convex and the left-hand side of inequality (41) is nonnegative, that is

$$(1-\lambda)\sum_{i=1}^n p_i\varphi(\lambda\mu|\overline{x} - x_i|) + \lambda\sum_{i=1}^n p_i\varphi(\mu(1-\lambda)|x_i - \overline{x}|) \geq 0.$$

If f is φ-uniformly convex, then inequality (58) refines inequality (17), which is adequate for convex functions.

(ii) Let f be c-strongly convex with $c \geq 0$. Then f is convex. It is easy to see that the left-hand side of inequality (49) is nonnegative, i.e.,

$$\frac{c}{2}(1-\lambda)\sum_{i=1}^{n} p_i(\lambda\mu|\overline{x}-x_i|)^2 + \frac{c}{2}\lambda\sum_{i=1}^{n} p_i(\mu(1-\lambda)|x_i-\overline{x}|)^2 \geq 0.$$

In the case f is c-strongly convex, inequality (59) refines inequality (17).

(iii) Let f be nonnegative superquadratic on $I = \mathbb{R}_+$. Then f is convex, and the left-hand side of inequality (50) is nonnegative as follows:

$$(1-\lambda)\sum_{i=1}^{n} p_i f(\lambda\mu|\overline{x}-x_i|) + \lambda\sum_{i=1}^{n} p_i f(\mu(1-\lambda)|x_i-\overline{x}|) \geq 0.$$

In the case f is both nonnegative and superquadratic, inequality (60) is a refinement of inequality (17).

6 Refined Global Upper Bounds for Jensen's Functional

In this section, we extend the results of Simić [30, 31] (see Sect. 2.4) for uniformly convex, strongly convex, and superquadratic functions, respectively.

Theorem 6.1 *Let $f : [m, M] \to \mathbb{R}$ be a differentiable convex function on an interval $[m, M] \subset \mathbb{R}$, $x_i \in [m, M]$, and $p_i \geq 0$, $i = 1, \dots, n$, with $\sum_{i=1}^{n} p_i = 1$. Denote $a_1 = \frac{M-\overline{x}}{M-m}$ and $a_2 = \frac{\overline{x}-m}{M-m}$, where $\overline{x} = \frac{1}{P_n}\sum\limits_{i=1}^{n} p_i x_i$.*

Then

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) + T$$

$$\leq \max_{p\in[0,1]}\{pf(m) + qf(M) - f(pm+qM)\} \quad \text{with } q = 1-p, \tag{66}$$

and

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) + T \leq f(m) + f(M) - 2f\left(\frac{m+M}{2}\right) - U, \tag{67}$$

where T and U are defined as follows:

(i) If f is uniformly convex with modulus φ, then

$$T = \frac{1}{P_n}\sum_{i=1}^{n} p_i\left[\frac{M-x_i}{M-m}\varphi(|x_i-m|) + \frac{x_i-m}{M-m}\varphi(|M-x_i|)\right],$$

$$U = \varphi\left(\left|\sum_{i=1}^{2}\left(\frac{1}{2} - a_i\right)x_i\right|\right) + \sum_{i=1}^{2}(1-a_i)\varphi\left(\left|x_i - \frac{1}{2}\sum_{i=1}^{2}x_i\right|\right).$$

(ii) If f is c-strongly convex, then

$$T = \frac{c}{2P_n}\sum_{i=1}^{n}p_i\left[\frac{M-x_i}{M-m}|x_i - m|^2 + \frac{x_i - m}{M-m}|M - x_i|^2\right],$$

$$U = \frac{c}{2}\left|\sum_{i=1}^{2}\left(\frac{1}{2} - a_i\right)x_i\right|^2 + \frac{c}{2}\sum_{i=1}^{2}(1-a_i)\left|x_i - \frac{1}{2}\sum_{i=1}^{2}x_i\right|^2.$$

(iii) If f is superquadratic on $I = \mathbb{R}_+$, then

$$T = \frac{1}{P_n}\sum_{i=1}^{n}p_i\left[\frac{M-x_i}{M-m}f(|m - x_i|) + \frac{x_i - m}{M-m}f(|M - x_i|)\right],$$

$$U = f\left(\left|\sum_{i=1}^{2}\left(\frac{1}{2} - a_i\right)x_i\right|\right) + \sum_{i=1}^{2}(1-a_i)f\left(\left|x_i - \frac{1}{2}\sum_{i=1}^{2}x_i\right|\right).$$

Proof By Theorem 3.4 we have

$$\sum_{i=1}^{n}p_i f(x_i) + T \le \frac{M-\overline{x}}{M-m}f(m) + \frac{\overline{x}-m}{M-m}f(M). \quad (68)$$

As in the proof of Theorem 3.4, we introduce $\alpha_i = \frac{M-x_i}{M-m}$, $\beta_i = \frac{x_i-m}{M-m}$. So, we have

$$a_1 = \frac{M-\overline{x}}{M-m} = \sum_{i=1}^{n}p_i\alpha_i \quad \text{and} \quad a_2 = \frac{\overline{x}-m}{M-m} = \sum_{i=1}^{n}p_i\beta_i. \quad (69)$$

Because $x_i = \alpha_i m + \beta_i M$, $i = 1, \ldots, n$, we get

$$f\left(\sum_{i=1}^{n}p_i x_i\right) = f\left(\sum_{i=1}^{n}p_i\alpha_i m + \sum_{i=1}^{n}p_i\beta_i M\right) = f(a_1 m + a_2 M).$$

Therefore, (68) is equivalent to

$$\sum_{i=1}^{n}p_i f(x_i) + T - f\left(\sum_{i=1}^{n}p_i x_i\right) \le a_1 f(m) + a_2 f(M) - f(a_1 m + a_2 M). \quad (70)$$

It is obvious by (69) that $a_1 + a_2 = 1$ with $a_1, a_2 \geq 0$. In consequence, the inequality (70) gives (66).

In order to prove (21), we notice that $0 \leq a_1 \leq 1$ and $0 \leq a_2 \leq 1$, and hence $(a_1, a_2) \leq (1, 1)$. By making use of Theorem 4.1 for $n = 2$, $\mathbf{p} = (a_1, a_2)$, $\mathbf{q} = (1, 1)$, $P_2 = 1$, and $Q_2 = 2$, we obtain the inequality

$$a_1 f(m) + a_2 f(M) - f(a_1 m + a_2 M) \leq f(m) + f(M) - 2f\left(\frac{m+M}{2}\right) - U. \tag{71}$$

It is now sufficient to combine (24) with (71). ■

Remark 6.2 In Theorem 6.1, since the components T and U are nonnegative, in all the three cases of f being φ-uniformly convex or c-strongly convex or nonnegative superquadratic, inequalities (66) and (67) are refinements of Simič's inequalities (20) and (21) corresponding to convex functions.

7 Csiszár–Körner's Inequality for Some Functions

Here we use Sherman-like inequalities (see Theorem H) to establish some extensions of Csiszár–Körner's inequality (see Theorem G) for the Csiszár f-divergence $I_f(\mathbf{p}, \mathbf{q})$, where $f : (0, \infty) \to \mathbb{R}$ is a function. In what follows, we consider f to be uniformly convex, strongly convex, and superquadratic, respectively.

Theorem 7.1 *[Csiszár–Körner-type inequalities] Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function, and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ be two n-tuples of positive numbers.*

Then

$$\sum_{i=1}^{n} p_i f\left(\frac{\sum_{i=1}^{n} q_i}{\sum_{i=1}^{n} p_i}\right) + R \leq \sum_{i=1}^{n} p_i f\left(\frac{q_i}{p_i}\right), \tag{72}$$

where R is defined as follows:

(i) If f is uniformly convex with modulus φ, then

$$R = \sum_{i=1}^{n} p_i \varphi\left(\left|\frac{q_i}{p_i} - \frac{\sum_{i=1}^{n} q_i}{\sum_{i=1}^{n} p_i}\right|\right).$$

(ii) If f is c-strongly convex, then

$$R = \frac{c}{2}\sum_{i=1}^{n} p_i \left| \frac{q_i}{p_i} - \frac{\sum_{i=1}^{n} q_i}{\sum_{i=1}^{n} p_i} \right|^2 .$$

(iii) If f is superquadratic on $I = \mathbb{R}_+$, then

$$R = \sum_{i=1}^{n} p_i f \left(\left| \frac{q_i}{p_i} - \frac{\sum_{i=1}^{n} q_i}{\sum_{i=1}^{n} p_i} \right| \right).$$

Proof As in the proof of Theorem G (see Sect. 2.5), with the aid of the notation

$$y = \frac{\sum_{i=1}^{n} q_i}{\sum_{i=1}^{n} p_i}, \quad x_i = \frac{q_i}{p_i}, \quad \text{and} \quad \alpha_i = \frac{p_i}{\sum_{i=1}^{n} p_i} \quad \text{for } i = 1, \dots, n,$$

we find

$$y = (x_1, x_2, \dots, x_n) \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix},$$

where the above $n \times 1$ matrix is column stochastic.

Furthermore, according to (3), we define

$$(a_1, a_2, \dots, a_n) = \left(\sum_{i=1}^{n} p_i \right) (\alpha_1, \alpha_2, \dots, \alpha_n).$$

For this reason we obtain

$$a_i = \left(\sum_{i=1}^{n} p_i \right) \alpha_i = p_i \quad \text{for } i = 1, \dots, n. \tag{73}$$

In this context, Sherman-type inequality (see Theorem H with n and k replaced by 1 and n, respectively, and $b_1 = \sum_{i=1}^{n} p_i$, $y_1 = y$, $s_{i1} = \alpha_i = \frac{p_i}{\sum_{i=1}^{n} p_i}$ and $a_i = p_i$ for $i = 1, \dots, n$) yields

$$\left(\sum_{i=1}^{n} p_i\right) f(y) + R \leq \sum_{i=1}^{n} a_i f(x_i), \tag{74}$$

It now follows from (74) and (73) that (72) holds. ■

Remark 7.2 In Theorem 7.1, since the expression R is nonnegative, in all the three cases of f being φ-uniformly convex or c-strongly convex or nonnegative superquadratic, the inequality (72) refines the standard Csiszár–Körner inequality (27) devoted to convex functions.

Remark 7.3 Inequality (72) in the version for superquadratic functions has been shown in [20, Corollary 2.5].

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Divisibility of Class Numbers of Quadratic Fields: Qualitative Aspects

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Abstract Class numbers of quadratic fields have been the object of attention for many years, and there exist a large number of interesting results. This is a survey aimed at reviewing results concerning the divisibility of class numbers of both real and imaginary quadratic fields. More precisely, to review the question 'do there exist infinitely many real (respectively imaginary) quadratic fields whose class numbers are divisible by a given integer?' This survey also contains the current status of a quantitative version of this question.

Keywords Quadratic fields · Discriminant · Class number · Hilbert class field

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1 Introduction

A number field K is a finite extension of the field of rational numbers $\mathbb{Q}$. A degree 2 extension of $\mathbb{Q}$ is called quadratic field. Every quadratic field K is of the form $\mathbb{Q}(\sqrt{d})$, where d is a square-free integer. The field $K = \mathbb{Q}(\sqrt{d})$ is real (respectively imaginary) if d is positive (respectively negative). A complex number α is called an algebraic integer if it is a root of a nonzero, monic polynomial over $\mathbb{Z}$. The ring of integers of K is the set of all algebraic integers in K, and is traditionally denoted by $\mathcal{O}_K$. When $K = \mathbb{Q}(\sqrt{d})$ with d square-free integer, then $\mathcal{O}_K$ is given by

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$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\frac{1+\sqrt{d}}{2}] \text{ if } d \equiv 1 \pmod 4, \\ \mathbb{Z}[\sqrt{d}] \text{ if } d \equiv 2, 3 \pmod 4. \end{cases}$$

Unlike in $\mathbb{Z}$, the unique factorization of (algebraic) integers into primes (irreducibles) does not hold in general in $\mathcal{O}_K$. That is, $\mathcal{O}_K$ is not a principal ideal domain (PID) in general. For example, let $K = \mathbb{Q}(\sqrt{-6})$, then $\mathcal{O}_K = \mathbb{Z}[\sqrt{-6}]$. In $\mathcal{O}_K$, we have

$$-2 \times 3 = -6 = (\sqrt{-6})^2.$$

These are two distinct factorizations of -6 since -2, 3 and $\sqrt{-6}$ are irreducible in $\mathcal{O}_K$, and thus, $\mathcal{O}_K$ is not a unique factorization domain (UFD).

It is interesting and of considerable importance too, to understand 'how far' the ring of (algebraic) integers of a number field fails to be a PID. As a result, the concept of the ideal class group or simply class group of a number field was introduced to measure this failure. A fractional ideal in a number field K is a nonzero $\mathcal{O}_K$-module $\mathcal{I} \subset K$ such that for some nonzero element $\alpha \in \mathcal{O}_K,\ \alpha\mathcal{I} \subset \mathcal{O}_K$. The ideal class group (in short, class group) $\mathfrak{C}_K$ of K is defined as

$$\mathfrak{C}_K := \frac{\text{The group of all fractional ideals of K}}{\text{The group of all principal fractional ideals of K}}.$$

Thus, one can say that $\mathfrak{C}_K$ is the group of all non-principal fractional ideals of K. It is a well-known fact in algebraic number theory that the order of this quotient group is finite. The class number of K is defined to be the order of $\mathfrak{C}_K$. Thus, class number is 1 if and only if the ring of integers of that particular field is a PID. We refer the reader [1, 2] for detail information on class group of number fields.

The class number of quadratic fields is one of the fundamental and mysterious objects in algebraic number theory. Starting from Gauss, this topic has been studied extensively by many authors, and thus, there exist a large number of research articles. In this survey article, we discuss some interesting results concerning the divisibility of class numbers of real as well as imaginary quadratic fields. We also provide the outlines of the proof of some of these results. Due to the versatility of the question, this survey may miss out some interesting references and thus some interesting results too, and thus, this article is never claimed to be a complete one. It is interesting to ask the following two questions while one investigates the class numbers of quadratic fields:

(I) **Qualitative Aspect**: Do there exist infinitely many real (respectively imaginary) quadratic fields whose class number is divisible by a given integer $n \geq 2$?

(II) **Quantitative Aspect**: Find a good lower bound on the number of real (respectively imaginary) quadratic fields whose class number is divisible by a given integer $n \geq 2$ and whose absolute discriminant is bounded by a large real number?

We intend mainly to focus on the first question. Let us fix some notations before we proceed further which will be used throughout the article.

Notations:
$K \to$ a number field (mostly quadratic number field);
$\Delta_K \to$ the discriminant of K;
$\mathcal{O}_K \to$ the ring of integers of K;
$d \to$ a square-free integer;
$h(d) \to$ the class number of $\mathbb{Q}(\sqrt{d})$;
$N_{K/\mathbb{Q}} \to$ the norm map of K;
$T_{K/\mathbb{Q}} \to$ the trace map of K;
$\left(\frac{a}{b}\right) \to$ Legendre symbol;
$v_p(n) \to$ the greatest exponent μ of p such that $p^\mu \mid n$ for an integer n and for a prime number p.

2 Imaginary Quadratic Fields

It is well known as mentioned earlier that in general the ring of integers of quadratic fields are not unique factorization domain, and the class number, which is the order of the ideal class group of the quadratic fields, measures how far this unique factorization fails in the corresponding ring of integers. It is proven that only nine imaginary quadratic fields have class number 1 and thus admits unique factorization, whereas in case of real quadratic fields, it is a folk-lore conjecture due to Gauss that there exist infinitely many real quadratic fields with class number 1. Thus, it is important to get information about class number of a given quadratic field, and in this quest, the divisibility questions assume considerable significance.

The answer to the first question is well understood in case when $n = 2$ since the beginning of the nineteenth century. In fact, if the discriminant of a quadratic field contains more than two prime factors, then 2 divides its class number. Gut [3] generalized this result to show that there exist infinitely many quadratic imaginary fields each with class number divisible by 3. In 1970, Yamamoto [4] gave an affirmative answer to this question. Here, we revisit some of the important results towards the answer to this question. More precisely, we discuss some infinite families of imaginary quadratic fields each with class number divisible by a given integer $n \geq 2$.

2.1 The Family $\mathbb{Q}(\sqrt{x^2 - y^n})$

We start with the family

$$K_{x,y,n} := \mathbb{Q}(\sqrt{x^2 - y^n}),$$

where x, y and n are positive integers. In 1922, Nagell [5] proved the following:

Theorem 2.1 (Nagel [5], Satz V) *Let $n > 0$ be an odd integer. Let x and y be two positive integers satisfying:*

(Ni) $\gcd(x, y) = 1$,
(Nii) $x^2 < y^n$,
(Niii) $2 \nmid y$,
(Niv) $q \mid\mid y$ *(i.e. no higher power of* q *divides* y*), for any prime divisor* q *of* n.

Let $y = \prod_i q_i^{e_i}$ *be the prime decomposition of* y. *Then* $\left(\frac{x^2-y^n}{q_i}\right) = 1$ *and* $\mathcal{Q}_i = (q_i, x + \sqrt{x^2 - y^n})$ *is a prime ideal of* $\mathcal{O}_{K_{x,y,n}}$ *over* q_i. *Set* $\mathfrak{A} = \prod_i \mathcal{Q}_i^{e_i}$. *Then the ideal class of* $\mathfrak{A}$ *is of order* n *(i.e.* n *divides the class number).*

On the other hand, for an even integer $n \geq 2$, N. C. Ankeny and S. Chowla proved the divisibility by n of the class numbers of a similar family $K_{x,3,n}$. Namely, they proved:

Theorem 2.2 (Ankeny and Chowla [6], Theorem 1) *The class number of* $K_{x,3,n}$ *is divisible by* n *if* x *and* n *satisfy the following:*

(Ai) x *is even and* $0 < x < (2 \times 3^{n-1})^{1/2}$,
(Aii) n *is even and sufficiently large,*
(Aiii) $x^2 - 3^n$ *is square-free.*

To show the existence of infinitely many imaginary quadratic fields each with class number divisible by n, we recall the following lemma.

Lemma 2.1 (Ankeny and Chowla [6], Lemma 1) *Let* x *and* n *be the integers as in Theorem 2.2 satisfying the conditions (Ai) and (Aii). Then the number of square-free integers of the form* $x^2 - 3^n$ *is at least* $\frac{1}{25} \times 3^{n/2}$.

Theorem 2.2 and Lemma 2.1 clearly shows that there are at least $\frac{1}{25} \times 3^{n/2}$ imaginary quadratic fields each with class number divisible by n. Set $n_1 = n^t$ ($t > 0$ integer) in such a way that the class number of none of these fields are divisible by n_1. Then, as in the earlier case, one can find at least $\frac{1}{25} \times 3^{n_1/2}$ fields each with class number divisible by n_1. In fact, all these fields are distinct from the previous fields. Repeating this one concluded that there exist infinitely many imaginary quadratic fields with class number divisible by n.

Another particular case of the family $K_{x,y,n}$ was considered by J. H. E. Cohn. He proved the following result.

Theorem 2.3 (Cohn [7], Theorem) $(n - 2)$ *divides the class number of* $K_{1,2,n}$, *for an integer* $n > 2$, *except for the case* $n = 6$.

Y. Kishi in 2009 was able to remove the conditions *'even'* and *'square-free'* in Theorem 2.2 by putting $x = 2^{2t}$. His result is:

Theorem 2.4 (Kishi [8], Theorem 1.2) *For any positive integers* t *and* n *with* $2^{2t} < 3n$, *the class number of* $K_{2^t,3,n}$ *is divisible by* n, *except the case* $(t, n) \neq (2, 3)$.

In the case when $t = 1$, he also proved:

Theorem 2.5 (Kishi [9], Theorem 1) *Let $n \geq 3$ be an odd integer. Then the class number of $K_{2,3,n}$ is divisible by* 3.

A. Ito gave a generalized version of Theorem 2.4. More precisely, she proved the following:

Theorem 2.6 (Ito [10], Theorem 1) *Let q be an odd prime, n and k be positive integers with $2^{2k} < q^n$. Then the following hold:*

(I1) For the case $q \equiv 3 \pmod 8$, if n and k satisfy either $k \equiv 1 \pmod 2$ or $n \not\equiv 3 \pmod 6$, then the class number of $K_{2^k,q,n}$ is divisible by n, except for the case $K_{2^k,q,n} \neq \mathbb{Q}(\sqrt{-3})$.

(I2) For the case $q \equiv 1 \pmod 4$, the class number of $K_{2^k,q,n}$ is divisible by n, except for the case $K_{2^k,q,n} = \mathbb{Q}(\sqrt{-1})$.

(I3) For the case $q \equiv 7 \pmod 8$, the class number of $K_{2^k,q,n}$ is divisible by n, except for the case $K_{2^k,q,n} = \mathbb{Q}(\sqrt{-3})$.

One can see that for the primes $q \equiv 11, 23 \pmod{24}$ that $K_{2^k,q,n} \neq \mathbb{Q}(\sqrt{-3})$. In this case, both (I1) and (I3) hold without the exception. Similarly, for even n and the primes $q \equiv 3 \pmod 4$, one can easily show $K_{2^k,q,n} \neq \mathbb{Q}(\sqrt{-3})$. Thus, in this case also both (I1) and (I3) hold without the exception. She also proved another result in [11] along the similar line by utilizing some strong conditions. Recently, Chakraborty et al. [12] gave a more general result along this line.

Theorem 2.7 (Chakraborty et al. [12], Theorem 1.1) *Let $n \geq 3$ be an odd integer and p, q be distinct odd primes with $q^2 < p^n$. Let d be the square-free part of $q^2 - p^n$. Assume that $q \not\equiv \pm 1 \pmod{|d|}$. Moreover, we assume $p^{n/3} \neq (2q+1)/3, (q^2+2)/3$ whenever both $d \equiv 1 \pmod 4$ and $3 \mid n$. Then the class number of $K_{p,q,n} = \mathbb{Q}(\sqrt{d})$ is divisible by n.*

We intend to sketch the proof of Theorem 2.7. Here, d is the square-free part of $q^2 - p^n$ and thus $q^2 - p^n = m^2 d$ for some positive integer m. Let $\alpha = q + m\sqrt{d}$, and thus $(\alpha) = \mathfrak{A}^n$ for some integral ideal $\mathfrak{A}$ (usual ideal in the ring of integers) of $K_{p,q,n}$. The idea here is to produce an element of order n in the class group. Thus, one proves more than just divisibility.

Claim: $O([\mathfrak{A}]) = n$ (i.e. the order of $\mathfrak{A}$ is exactly n).

Suppose on the contrary that the claim is not true. Then there exist an odd prime divisor ℓ of n and an integer β in $K_{p,q,n}$ such that $(\alpha) = (\beta)^\ell$. The condition '$q \not\equiv \pm 1 \pmod{|d|}$' ensures that $d < -3$ since q and p are distinct odd primes. Thus ± 1 are the only units in the ring of integers of $K_{p,q,n}$, and therefore, we can write $\alpha = \gamma^\ell$ for some integer γ in $K_{p,q,n}$. This contradicts the following:

Proposition 2.1 (Chakraborty et al. [12]) *Let n, q, p, d be as in Theorem 2.7, and let m be the positive integer with $q^2 - p^n = m^2 d$. Then the element $\alpha = q + m\sqrt{d}$ is not an ℓth power of an element in the ring of integers of $K_{p,q}$ for any prime divisor ℓ of n.*

To prove this proposition, the main ingredient used is an important result by Bugeaud and Shorey [13] on the number of solutions in positive integers of the generalized Ramanujan–Nagell equation. We need to introduce further definitions and notations before stating their result.

Let F_k denote the kth term in the Fibonacci sequence defined by $F_0 = 0,\ F_1 = 1$ and $F_{k+2} = F_k + F_{k+1}$ for $k \geq 0$. Similarly L_k denotes the kth term in the Lucas sequence defined by $L_0 = 2,\ L_1 = 1$ and $L_{k+2} = L_k + L_{k+1}$ for $k \geq 0$. For $\lambda \in \{1, \sqrt{2}, 2\}$, we define the subsets $\mathcal{F},\ \mathcal{G}_\lambda,\ \mathcal{H}_\lambda \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ by

$$\mathcal{F} := \{(F_{k-2\epsilon}, L_{k+\epsilon}, F_k) \mid k \geq 2, \epsilon \in \{\pm 1\}\},$$
$$\mathcal{G}_\lambda := \{(1, 4p^r - 1, p) \mid p \text{ is an odd prime}, r \geq 1\},$$
$$\mathcal{H}_\lambda := \left\{ (D_1, D_2, p) \;\middle|\; \begin{array}{l} D_1, D_2 \text{ and } p \text{ are mutually co-prime positive integers} \\ \text{with } p \text{ an odd prime and there exist positive integers} \\ r, s \text{ such that } D_1 s^2 + D_2 = \lambda^2 p^r \text{ and } 3D_1 s^2 - D_2 = \pm\lambda^2 \end{array} \right\},$$

except when $\lambda = 2$, in which case the condition 'odd' on the prime p should be removed in the definitions of $\mathcal{G}_\lambda$ and $\mathcal{H}_\lambda$.

Theorem 2.8 (Bugeaud and Shorey [13], Theorem 1) *Given* $\lambda \in \{1, \sqrt{2}, 2\}$*, a prime* p *and positive co-prime integers* D_1 *and* D_2*, the number of positive integer solutions* (x, y) *of the Diophantine equation*

$$D_1 x^2 + D_2 = \lambda^2 p^y \tag{1}$$

is at most one except for

$$(\lambda, D_1, D_2, p) \in \mathcal{E} := \left\{ \begin{array}{l} (2, 13, 3, 2), (\sqrt{2}, 7, 11, 3), (1, 2, 1, 3), (2, 7, 1, 2), \\ (\sqrt{2}, 1, 1, 5), (\sqrt{2}, 1, 1, 13), (2, 1, 3, 7) \end{array} \right\}$$

and $(D_1, D_2, p) \in \mathcal{F} \cup \mathcal{G}_\lambda \cup \mathcal{H}_\lambda$.

The proof of the Proposition 2.1 uses the above-mentioned result by exhibiting a Diophantine equation of the form (1) with two distinct positive integer solutions after sorting out all the exceptions in Theorem 2.8 and thus gets the required contradiction. The following two propositions help in getting the result.

Proposition 2.2 (Chakraborty et al. [12]) *Let* $d \equiv 5 \pmod 8$ *be an integer and* ℓ *be a prime. For odd integers* a, b *we have*

$$\left(\frac{a + b\sqrt{d}}{2}\right)^\ell \in \mathbb{Z}[\sqrt{d}] \text{ if and only if } \ell = 3.$$

Proposition 2.3 (Cohn [14]) *The only perfect squares appearing in the Lucas sequence are* $L_1 = 1$ *and* $L_3 = 4$.

One can easily prove, by reading modulo 4, that the condition '$p^{n/3} \neq (2q + 1)/3, (q^2 + 2)/3$' in Theorem 2.7 holds whenever $p \equiv 3 \pmod 4$. Further, if one fixes an odd prime q, then the condition '$q \not\equiv \pm 1 \pmod{|d|}$' in Theorem 2.7 holds almost always, and this can be proved using the celebrated Siegel's theorem on integral points on affine curves. More precisely, the following result was proved to show the existence of infinitely many imaginary quadratic fields in Theorem 2.7.

Theorem 2.9 (Chakraborty et al. [12], Theorem 1.2) *Let $n \geq 3$ be an odd integer not divisible by 3. For each odd prime q the class number of $K_{p,q,n}$ is divisible by n for all but finitely many p's. Furthermore, for each q there are infinitely many fields $K_{p,q,n}$.*

The following question naturally arise in the opposite direction from Theorem 2.9.
Question 1: Are there infinitely many choices of distinct odd primes p and q such that the class number of $K_{p,q,n}$ is **not divisible** by a given integer n?

2.2 *The Family* $\mathbb{Q}(\sqrt{x^2 - 4y^n})$

We move into the next family of quadratic fields:

$$L_{x,y,n} := \mathbb{Q}(\sqrt{x^2 - 4y^n}),$$

where x, y and n are positive integers. The notation $[a]$ would mean the principal ideal in $\mathcal{O}_{L_{x,y,n}}$(the ring of integers in the field $L_{x,y,n}$) generated by the element a. For any two elements a and b in $\mathcal{O}_{L_{x,y,n}}$ and an ideal $\mathfrak{A} \subset \mathcal{O}_{L_{x,y,n}}$; $[a] = [b]\mathfrak{A}$ implies that there exists an element $c \in \mathcal{O}_{L_{x,y,n}}$ such that $a = bc$. Therefore, $[a] = [b]\mathfrak{A}$ can be written as $[b][c] = [b]$ which gives $[c] = \mathfrak{A}$. Applying this fact, M. J. Cowles proved the following result.

Theorem 2.10 (Cowles [15], Theorem) *Let y and n be odd primes. Suppose $x > 0$ is an integer satisfying:*

(C1) $\gcd(x, y) = 1$,
(C2) $x^2 - y^n$ *is odd and square-free,*
(C3) $x^2 < y^n$,
(C4) *atleast one of the prime divisors of* $[y]$ *is not principal.*

Then the class number of $L_{x,y,n}$ is divisible by n.

Here, the conditions (C1)–(C4) are very strong which lead to a almost trivial proof. Gross and Rohrlick [16] were able to prove that with $x = 1$, the conditions '*square-free*' in (C2) and '*q a prime*' can be eliminated. In fact, their result holds for any odd integer n. In 2002, Cohn [7] entered into this family and proved that $n \neq 4$, be it odd or even, always divides the class number of the field $L_{1,2,n}$. This generalized the result ([16], Theorem 5.3) of B. Gross and D. Rohrlick and provides a simple proof of the existence of infinitely many imaginary quadratic fields whose class number is divisible by any given integer $n \geq 2$. S. R. Louboutin in 2009 revisited this family of imaginary quadratic fields with the following objectives:

(L1) to expound the proof of B. Gross and D. Rohrlich ((Li) of Theorem 2.11),
(L2) to prove a result stronger than [[16], Theorem 5.3] ((Lii) of Theorem 2.11).

Theorem 2.11 (Louboutin [17], Theorem 1) *Let $n > 1$ be an integer.*

(Li) If n is odd, then for any integer $y \geq 2$ the ideal class groups of $L_{1,y,n}$ contain an element of order n.
(Lii) If at least one of the prime divisors of an odd integer $y \geq 3$ is equal to $3 \pmod 4$, then the ideal class groups of $L_{1,y,n}$ contain an element of order n.

He used an interesting and indigenous idea to prove this theorem. The main ingredients of his proof of Theorem 2.11 are the following three propositions.

Proposition 2.4 (Louboutin [17], Lemma 2) *For a quadratic field K, let $\alpha \in \mathcal{O}_K$. If $k \geq 1$ is odd, then $T_{K/\mathbb{Q}}(\alpha)$ divides $T_{K/\mathbb{Q}}(\alpha k)$. If p is an odd prime, then $T_{K/\mathbb{Q}}(\alpha p) \equiv T_{K/\mathbb{Q}}(\alpha) \pmod p$. Hence, if $T_{K/\mathbb{Q}}(\alpha) = 1$ and if α is a p-th power in $\mathcal{O}_K$, then α is a unit of $\mathcal{O}_K$.*

Proposition 2.5 (Louboutin [17], Proposition 3) *Let K be an imaginary quadratic field. If $\alpha \in \mathcal{O}_K$ with $T_{K/\mathbb{Q}}(\alpha) = 1$ is 'associated' with a p-th power for some odd prime p, then α is a unit of $\mathcal{O}_K$.*

Proposition 2.6 (Louboutin [17], Lemma 4) *Let K be a quadratic field. If $\alpha \in \mathcal{O}_K$, then α is a square in $\mathcal{O}_K$ if and only if there exists $A \in \mathbb{Z}$ such that $N_{K/\mathbb{Q}}(\alpha) = A^2$ and such that $T_{K/\mathbb{Q}}(\alpha) + 2A$ is a square in $\mathbb{Z}$. If K is an imaginary quadratic field, then we may assume that $A \geq 0$.*

In 2011, K. Ishii proved a stronger version for 'even' n of Theorem 2.11 by adapting the method of S. R. Louboutin in [17]. He proved:

Theorem 2.12 (Ishii [18], Theorem) *Let $y > 1$ be an integer. If n is even with $n \geq 6$, then the class number of $L_{1,y,n}$ is divisible by n, except $(y, n) = (13, 8)$.*

Recently, A. Ito derived some interesting results for certain cases of the family $L_{x,y,n}$ in [19]. She cleverly used Theorem 2.8 and applied the method of Yamomoto [4]. Here, we revisit her results in brief. We first treat the case where y is an odd integer including the case where all prime divisors of y are congruent to 1 modulo 4.

Theorem 2.13 (Ito [19], Theorem 1.4) *Let $n > 1$ be an integer and $y > 1$ be an odd integer. Then the following hold:*

(I4) Assume $y \neq 5, 13$. Then the class number of $L_{1,y,n}$ is divisible by n, except for at most one n. The exceptional case is either $n = 2$ or $n = 4$ and then the class number of the field is divisible by $n/2$.
(I5) Assume $y = 5$. Then the class number of $L_{1,y,n}$ is divisible by n, except for the two cases $n = 2$ and $n = 4$. The class numbers of the fields $L_{1,5,2} = \mathbb{Q}(\sqrt{-11})$ and $L_{1,5,4} = \mathbb{Q}(\sqrt{-51})$ are 1 and 2, respectively. These class numbers are divisible by $n/2$ but are not divisible by n.

(I6) *Assume* $y = 13$. *Then the class number of* $L_{1,y,n}$ *is divisible by* n, *except for the two cases* $n = 2$ *and* $n = 8$. *The class numbers of the fields* $L_{1,13,2} = \mathbb{Q}(\sqrt{-3})$ *and* $L_{1,13,8} = \mathbb{Q}(\sqrt{-6347})$ *are* 1 *and* 28, *respectively. These class numbers are divisible by* $n/2$ *but are not divisible by* n.

In Theorem 2.13 (I4), there exists an imaginary quadratic field whose class number is divisible by $n/2$ but not divisible by n. That field corresponds to $n = 4$, viz. $L_{1,29,4} = \mathbb{Q}(\sqrt{-187})$. The class number of this field is 2. She mainly used Propositions 2.4–2.6 in proving Theorem 2.13.

We now treat the case where y is a prime and x is a power of 3. A. Ito used the methods of Kishi [8] and herself [10] to prove the next result. The main ingredient used here is Theorem 2.8.

Theorem 2.14 (Ito [19], Theorem 1.6) *Let* y *be a prime other than* 3, *and let* n *and* e *be positive integers with* $3^{2e} < 4y^n$. *Then the following hold:*

(I7) *Assume* $y \equiv 1 \pmod 3$ *or* $n \not\equiv 2 \pmod 4$. *Then the class number of* $L_{3^e,y,n}$ *is divisible by* n.

(I8) *Assume* $y \equiv 2 \pmod 3$ *with* $y \neq 2$ *and* $n \equiv 2 \pmod 4$.

- (I8.1) *If* $2y^{n/2} - (-3)^e \neq \square$, *then the class number of* $L_{3^e,y,n}$ *is divisible by* n.
- (I8.2) *If* $2y^{n/2} - (-3)^e = \square$, *then the class number of* $L_{3^e,y,n}$ *is divisible by* $n/2$.

(I9) *Assume* $y = 2$ *and* $n \equiv 2 \pmod 4$.

- (I9.1) *When* $(n, e) \neq (6, 2)$, *we have the following:*
  - (I9.1.1) *If* e *is even, then the class number of* $L_{3^e,y,n}$ *is divisible by* n.
  - (I9.1.2) *If* e *is odd and* $2^{(n/2)+1} - 3^e \neq \square$, *then the class number of* $L_{3^e,y,n}$ *is divisible by* n.
  - (I9.1.3) *If* e *is odd and* $2^{(n/2)+1} - 3^e = \square$, *then the class number of* $L_{3^e,y,n}$ *is divisible by* $n/2$.
- (I9.2) *When* $(n, e) = (6, 2)$, *we have* $L_{3^e,y,n} = \mathbb{Q}(\sqrt{-7})$ *whose class number is* 1.

Further we give existence of imaginary quadratic fields satisfying Theorem 2.14 (I8.2) and (I9.1.3) where the class numbers are divisible by $n/2$ but not by n. Let $(y, n, e) = (5, 2, 2)$, then $L_{3,y,n} = \mathbb{Q}(\sqrt{-19})$ and $2y^{n/2} - (-3)^e = 1$, a perfect square. The class number in this case, $h(-19) = 1$. Again if we choose $(y, n, e) = (2, 2, 1)$, then $L_{3,y,n} = \mathbb{Q}(\sqrt{-7})$ and $2^{(n/2)+1} - 3^e = 1$, a perfect square. Here also $h(-7) = 1$. One of the extended versions of Theorem 2.14 is the following result:

Theorem 2.15 (Ito [19], Theorem 4.1) *Let* $n > 2$ *be an integer,* $y > 1$ *an integer and* $x > 0$ *an odd integer such that* $\gcd(x, y) = 1$ *and* $x^2 < 4y^n$. *Assume* $x^2 - 4y^n = a^2 d$, *where* a *is a positive integer and* d *is a square-free integer less than* -3. *If* $y^n < \frac{(1-d)^2}{16}$, *then the class number of* $L_{x,y,n} = \mathbb{Q}(\sqrt{d})$ *is divisible by* n.

2.3 Other Families of Imaginary Quadratic Fields

In 1974, P. Hartung proved the following general result to show the existence of infinitely many imaginary quadratic fields each with class number divisible by 3. This is an existential result, i.e. here the family is not explicit as in the other cases we have mentioned so far.

Theorem 2.16 (Hartung [20], Theorem) *Let d be a positive integer satisfying:*

(HA1) $d \equiv 7 \pmod{12}$,
(HA2) *d is square-free,*
(HA3) *d is of the form $(t^2 - 4)/27$, where t is an integer.*

Then the class number of $\mathbb{Q}(\sqrt{-d})$ is divisible by 3.

A. Hoque and H. K. Saikia along the similar lines provided a more specific family of imaginary quadratic fields whose class number is divisible by 3. Namely, they proved:

Theorem 2.17 (Hoque and Saikia [21], Theorem 3.1) *If $d = 3^m p^{2n} + r$ is square-free, where p is an odd prime, r is either* 4 *or* -2, *$m > 1$ is an odd integer and n is a positive integer, then the class number of $\mathbb{Q}(\sqrt{-d})$ is a multiple of* 3.

The proof of this theorem is based on the following identity of Ankeny et al. [22]:

$$h(q) \equiv -\frac{u}{t} h(d) \pmod{3},$$

where $q \equiv 1 \pmod{3}$ is a square-free positive integer such that $d = 3q$, and u and v are the coefficients of the fundamental unit in $\mathbb{Q}(\sqrt{d})$.

Recently, Chakraborty and Hoque [23] extended Theorem 2.17. More precisely, they proved the following:

Theorem 2.18 (Chakraborty and Hoque [23], Theorem 2.3) *Let $m > 1$ and t be odd integers and n be any positive integer. Let d be the square-free part of $3^m t^{2n} + r$ with* $r \in \{-2, 4\}$. *Then the class number of $K = \mathbb{Q}(\sqrt{-d})$ is divisible by* 3.

This is an existential result, i.e. without explicitly constructing the family as in the other cases we have mentioned so far.

To prove this theorem, the authors mainly constructed a cubic unramified cyclic extension of K. Before proceeding further, we clarify the term 'unramification'. Let K be a number field, and let p be a prime number. Then we can write

$$p\mathcal{O}_K = \prod_{i=1}^{g} \mathfrak{P}_i^{e_i}, \quad (e_i \in \mathbb{N}),$$

where the $\mathfrak{P}_i$'s are district prime ideals in $\mathcal{O}_K$. The number e_i is called the ramification index of $\mathfrak{P}_i$ (over p) in $\mathcal{O}_K$ and is denoted by $e_{K/\mathbb{Q}}(\mathfrak{P}_i)$. Here, p is said to be ramified in $\mathcal{O}_K$ if $e_{K/\mathbb{Q}}(\mathfrak{P}_i) > 1$, for some i and the ideals $\mathfrak{P}_i$ those satisfy the condition $e_{K/\mathbb{Q}}(\mathfrak{P}_i) > 1$ are called ramified in $\mathcal{O}_K$. Further, p is said to be completely ramified or totally ramified in $\mathcal{O}_K$ if $e_{K/\mathbb{Q}}(\mathfrak{P}_{\mathrm{i}})$ is equal to the degree of K for some $i = 1, 2, 3, \ldots, g$. Furthermore, p is said to be unramified (also called completely split) in $\mathcal{O}_K$ if $e_{K/\mathbb{Q}}(\mathfrak{P}_i) = 1$, for all i. In this case, g is equal to the degree of K. Moreover, p is said to be inert in $\mathcal{O}_K$ if it remains prime in $\mathcal{O}_K$. The number g is called the decomposition number of p in $\mathcal{O}_K$, denoted by $g_{K/\mathbb{Q}}(p)$. The field K is said to be unramified if every prime number is unramified in $\mathcal{O}_K$. Otherwise, it is ramified.

For example, let $K = \mathbb{Q}(\sqrt{10})$. Then $\mathcal{O}_K = \mathbb{Z}[\sqrt{10}]$. Now, $2\mathcal{O}_K = \mathfrak{P}^2$, where $\mathfrak{P} = (2,\ \sqrt{10})$. Thus we have the following:

- 2 is (completely) ramified in $\mathbb{Z}[\sqrt{10}]$.
- $e_{K/\mathbb{Q}}(\mathfrak{P}) = 2$, and hence, prime ideal $\mathfrak{P}$ itself is ramified in $\mathbb{Z}[\sqrt{10}]$.
- $g_{K/\mathbb{Q}}(\mathfrak{P}) = 1$.

Again $3\mathcal{O}_K = \mathfrak{P}\mathfrak{Q}$, where $\mathfrak{P} = (3,\ 1 + \sqrt{10})$ and $\mathfrak{Q} = (3,\ 1 - \sqrt{10})$ are the prime ideals in $\mathcal{O}_K$. Here, we have the following:

- 3 is unramified (split completely) in $\mathbb{Z}[\sqrt{10}]$.
- $e_{K/\mathbb{Q}}(\mathfrak{P}) = 1$, and hence, $\mathfrak{P}$ in $\mathbb{Z}[\sqrt{10}]$ is unramified in $\mathbb{Z}[\sqrt{10}]$. Similarly, $\mathfrak{Q}$ is also unramified in $\mathbb{Z}[\sqrt{10}]$.
- $g_{K/\mathbb{Q}}(p) = 2$.

Lastly 7 remains prime in $\mathbb{Z}[\sqrt{10}]$, and thus, 7 is inert in $\mathbb{Z}[\sqrt{10}]$.

Let us now look at the construction of a cubic extension of K which is cyclic and unramified everywhere that is involved in the proof. Firstly, one choses an element $\alpha \in \mathcal{O}_K$ satisfying $N_{K/\mathbb{Q}}(\alpha) \in \mathbb{Z}^3$, i.e. a cube, and then considers the following cubic equation involving norm and trace of α:

$$f_\alpha(X) := X^3 - 3[N_{K/\mathbb{Q}}(\alpha)]^{1/3}X - T_{K/\mathbb{Q}}(\alpha).$$

Then used the following lemmas to construct the required unramified extension.

Lemma 2.2 (Chakraborty and Hoque [23], Lemma 2.1) *Let* $K = \mathbb{Q}(\sqrt{d})$. *Suppose* $\alpha = \frac{a+b\sqrt{d}}{2} \in \mathcal{O}_K$ *with* $N_{K/\mathbb{Q}}(\alpha) \in \mathbb{Z}^3$. *Then* $f_\alpha(X)$ *is reducible over* $\mathbb{Q}$ *if and only if* α *is a cube in* K.

Let $d (\neq 1, -3)$ be a square-free integer and

$$D = \begin{cases} -d/3 & \text{if } d \text{ is a multiple of } 3, \\ -3d & \text{otherwise.} \end{cases}$$

Let $K = \mathbb{Q}(\sqrt{d})$ and $L = \mathbb{Q}(\sqrt{D})$. Also,

$$R_d := \{\alpha \in \mathcal{O}_K : \alpha \text{ is not a cube in } K \text{ and } N_{K/\mathbb{Q}}(\alpha) \text{ is a cube in } \mathbb{Z}\}$$

and

$$R_D := \{\alpha \in \mathcal{O}_L : \alpha \text{ is not a cube in } L \text{ and } N_{L/\mathbb{Q}}(\alpha) \text{ is a cube in } \mathbb{Z}\}.$$

It is clear that the subset R_d (respectively R_D) contains all those units in K which are not cubes in K (respectively in L). Further let,

$$R_d^* := \{\alpha \in R_d : \gcd(N_{K/\mathbb{Q}}(\alpha), T_{K/\mathbb{Q}}(\alpha)) = 1\}$$

and

$$R_D^* := \{\alpha \in R_D : \gcd(N_{L/\mathbb{Q}}(\alpha), T_{L/\mathbb{Q}}(\alpha)) = 1\}.$$

Lemma 2.3 (Kishi [24], Proposition 6.5) *Let $\alpha \in R_D^*$ (respectively $\alpha \in R_d^*$). Then $S_{\mathbb{Q}}(f_\alpha)$ is an S_3-field containing $K = \mathbb{Q}(\sqrt{d})$ (respectively $L = \mathbb{Q}(\sqrt{D})$) which is a cyclic cubic extension of K (respectively L) unramified outside 3 and contains a cubic subfield K' with $v_3(\Delta_{K'}) \neq 5$. Conversely, every S_3-field containing K (respectively L) which is unramified outside 3 over K (respectively L) and contains a cubic subfield K' satisfying $v_3(\Delta_{K'}) \neq 5$ is given by $S_{\mathbb{Q}}(f_\alpha)$ with $\alpha \in R_D^*$ (respectively $\alpha \in R_d^*$).*

One is now left with verifying the ramification at 3, and the following result of Llorente and Nart ([25], Theorem 1) talks about the ramification at $p = 3$.

Lemma 2.4 *Suppose that*

$$g(X) := X^3 - aX - b \in \mathbb{Z}[X]$$

is irreducible over $\mathbb{Q}$ and that either $v_3(a) < 2$ or $v_3(b) < 3$ holds. Let θ be a root of $g(X)$. Then 3 is totally ramified in $\mathbb{Q}(\theta)/\mathbb{Q}$ if and only if one of the following conditions holds:

(LN-1) $1 \leq v_3(b) \leq v_3(a)$,
(LN-2) $3 \mid a,\ a \not\equiv 3 \pmod 9,\ 3 \nmid b$ *and* $b^2 \not\equiv a + 1 \pmod 9$,
(LN-3) $a \equiv 3 \pmod 9,\ 3 \nmid b$ *and* $b^2 \not\equiv a + 1 \pmod{27}$.

In [26], A. Hoque and H. K. Saikia studied prime numbers of the form

$$M_{p,q} = p^q - p + 1,$$

where p and q are positive integers. Some authors do use the terminology generalized Mersenne primes (GMP) for these primes. In [27], the authors proved that if p is a prime and q is a positive integer, then $p^{q-1}M_{p,q}$ is $(p-1)$-hyperperfect number. The authors in [26] used these primes to construct an infinite family of imaginary quadratic fields whose class numbers are divisible by 3. More precisely, they proved:

Theorem 2.19 (Hoque and Saikia [26], Theorem 2.8) *The class number of $\mathbb{Q}\left(\sqrt{-(3(M_{p,q}+2)^2+4)}\right)$, where p is prime and q is an odd positive integer, is divisible by 3.*

Recently, Chakraborty and Hoque [23] proved by producing an element of order n that the class number of $\mathbb{Q}(\sqrt{1-2m^n})$, where m is an odd integer and n is an odd prime, is divisible by n. They also proved a more general version of this result in [28].

3 Real Quadratic Fields

We deal with the divisibility questions in the real quadratic set-up here. Real quadratic fields are relatively harder to handle due to the existence of non-trivial units, and thus, there exist very few results in this direction for real quadratic fields. Yamamoto [4] in 1970 first gave an affirmative answer to the question of existence of infinitely many real quadratic fields with class number divisible by a given integer n. However, a couple of years later T. Honda proved the following result to showcase the existence of infinitely many real quadratic fields with class number 3 divisibility property.

Theorem 3.1 (Honda [29], Proposition) *Let m and n be two integers satisfying* $\gcd(m, 3n) = 1$. *If m cannot be represented in a form $\frac{n+h^3}{h}$ with $h \in \mathbb{Z}$, then class number of $K_{m,n} = \mathbb{Q}(\sqrt{4m^3 - 27n^2})$ is a multiple of* 3.

Weinberger [30] in 1973 also gave an affirmative answer to the question (I) for real quadratic fields. He considered discriminants of the type $d = m^{2n} + 4$ with $m > n$ a prime. Then the fundamental unit of $\mathbb{Q}(\sqrt{d})$ is $\frac{m^n+\sqrt{d}}{2}$ (one needs to avoid $d = 5$). Now suppose that $t^k - 4$ is irreducible in $\mathrm{F_m}[t]$ for all $k|n$. In this set-up, one considers the ideal $\mathfrak{A} = (m^2, 2 + \sqrt{d})$. Clearly, the order of $\mathfrak{A}$ in the class group of $\mathbb{Q}(\sqrt{d})$ is a divisor of n. Then it is not difficult to show that the order of $\mathfrak{A}$ is exactly n or $n/2$ according as n is odd or even respectively with the above assumptions. Here, one uses the fundamental unit. Next, one applies some density result (more precisely, Chebotarev density theorem) to conclude that there exist infinitely many primes m such that $t^k - 4$ is irreducible in $\mathrm{F}_m[t]$ for all $k|n$.

Now repetitions of the fields possible only when

$$\mathbb{Q}(\sqrt{d_1}) = \mathbb{Q}(\sqrt{m^{2n} + 4}).$$

It is well known that the Diophantine equation $x^{2n} + 4 = d_1 y^2$ has only finitely many solutions. This implies that repetitions of the resulting fields are possible only for finitely many m. This shows the existence of infinitely many such fields.

H. Ichimura in [31] showed that the conditions assumed in Weinberger's proof are not necessary and proved that for all integers $n \geq 2$ and each odd integer $m \geq 3$, the class number of $\mathbb{Q}(\sqrt{m^{2n} + 4})$ is divisible by n.

We now consider the polynomial

$$f(x) = x^3 - M_{p,q}x + p,$$

where p is an odd prime and q is an odd integer. The discriminant of $f(x)$ is $D_f = 4M_{p,q}^3 - 27p^2$. In [32], A. Hoque and H. K. Saikia proved that the class number of $\mathbb{Q}(\sqrt{D_f})$ is divisible by 3. In fact, there are infinitely many such real quadratic fields each with class number divisible by 3.

In [33], Y. Kishi and K. Miyake gave a parametrization of quadratic fields whose class numbers are divisible by 3. Namely, they proved:

Theorem 3.2 (Kishi and Miyake [33], Theorem 1) *Let u and v be two integers and*

$$f_{u,v}(x) = x^3 - uvx - u^2.$$

If

(K1) u and v are relatively prime;
(K2) $f_{u,v}(x)$ is irreducible over $\mathbb{Q}$;
(K3) discriminant $D_{f_{u,v}}$ of $f_{u,v}(x)$ is not a perfect square in $\mathbb{Z}$;
(K4) one of the following conditions hold:

(K4.1) $3 \nmid v$,
(K4.2) $3 \mid v,\ uv \not\equiv 3 \pmod 9$ *and* $u \equiv v \pm 1 \pmod 9$,
(K4.3) $3 \mid v,\ uv \equiv 3 \pmod 9$ *and* $u \equiv v \pm 1 \pmod{27}$,

then 3 divides the class number of $\mathbb{Q}(\sqrt{D_{f_{u,v}}})$. Conversely, every quadratic number field $\mathbb{Q}(\sqrt{D_{f_{u,v}}})$ with class number divisible by 3 arises in the above way from a suitable choices of integers u and v.

Recently, K. Chakraborty and A. Hoque used this parametrization to prove:

Theorem 3.3 (Chakraborty and Hoque [23], Theorem 3.1 (II)) *Let $m \equiv 4 \pmod{15}$ be an odd positive integer. Then for any odd integer $n \geq 3$, 3 divides the class number of the field $\mathbb{Q}(\sqrt{3(2m^{3n} - 1}))$.*

Along the same line, they also produced a simple family of real quadratic fields having infinitely many members each with class number divisible by 3. They then used this family of fields to understand the class numbers of associated cyclotomic fields. More precisely, they proved the following:

Theorem 3.4 (Chakraborty and Hoque [34], Theorem 3.1) *For a positive integer n satisfying $n \equiv 0 \pmod 3$, the class number*

$$k = \mathbb{Q}(\sqrt{3(4 \times 3^n - 1)})$$

is divisible by 3. In fact, there are infinitely many such real quadratic fields with class number divisible by 3.

4 Concluding Remarks

A lot more work has been done towards the second question, i.e. in the quantitative direction. Here, we give the current status of this question. Let us denote by

$$N_n(X) = \#\{d \le X : n|h(d)\},$$

where X is a large real number. Thus, the problem is to find a good (non-trivial) lower bound of $N_n(X)$ in terms of X.

The famous Cohen-Lenstra heuristics [35] predict that quadratic fields (in fact, for any number field of degree larger than 1) with class number divisible by n should have 'positive density' among all quadratic fields (respectively for all number fields of degree larger than 1). Thus, the prediction is

$$N_n(X) \sim c_n X$$

for a positive constant c_n. For odd primes n, it predicts

$$c_n = \begin{cases} \frac{6}{\pi^2}(1 - \prod_{i=2}^{\infty}(1 - \frac{1}{n^i})) & \text{(In the real quadratic field case)} \\ \frac{6}{\pi^2}(1 - \prod_{i=1}^{\infty}(1 - \frac{1}{n^i})) & \text{(In the imaginary quadratic field case).} \end{cases}$$

This implies a positive proportion of quadratic fields contain a non-trivial p-part in the class group. So far a very little progress has been made towards settling this conjecture.

Murty [36, 37] was the first who considered getting a lower bound in the case of imaginary field, and he proved the following result.

Theorem 4.1 (Murty [37], Theorem) *For any integer $n \ge 3$,*

$$N_n(X) >> X^{1/2+1/n}.$$

Soundararajan [38] improved this bound, and this is the best known bound till the date. More precisely, he proved:

Theorem 4.2 (Soundararajan [38], Theorem 1) *For large X, we have*

$$N_n(X) >> \begin{cases} X^{\frac{1}{2}+\frac{2}{n}-\epsilon} & if n \equiv 0 \pmod 4, \\ X^{\frac{1}{2}+\frac{3}{n+2}-\epsilon} & if n \equiv 2 \pmod 4. \end{cases}$$

It is to be noted that Theorem 4.2 contains bounds for $N_n(X)$ when n is odd since $N_n(X) \ge N_{even}(X)$. Also to be noted that Cohen-Lenstra heuristics predict the bound should be a constant times X.

Murty [37] once again was the first to consider getting a lower bound in real quadratic case and he proved:

Theorem 4.3 (Murty [37], Theorem 2) *For any odd integer n,*

$$N_n(X) \gg X^{\frac{1}{2n}-\epsilon},$$

for any $\epsilon > 0$.

In [39], G. Yu used Yamamoto's construction [4] of discriminants and by quantifying it improved Murty's bound in Theorem 4.3. He proved the following result which is the best known bound in the general case in this direction.

Theorem 4.4 (Yu [39], Theorem 2) *Let n be an odd integer. Then for any* $\epsilon > 0$

$$N_n(X) >> X^{\frac{1}{n}-\epsilon}.$$

Analogously, Chakraborty et al. [40] proved that the number of real quadratic fields K of discriminant $\Delta_K < X$ whose class group has an element of order n (with n even) is $\geq \frac{X^{1/n}}{5}$ if $X > X_0$, uniformly for positive integers $n \leq \frac{\log\log X}{8\log\log\log X}$. They used this result to find real quadratic number fields whose class numbers have many prime factors.

In another work, Chakraborty and Murty [41] gave the following improvement in the case when $n = 3$.

Theorem 4.5 (Chakraborty and Murty [41], Theorem 1) *For a large real number X,*

$$N_3(X) \gg X^{\frac{5}{6}}.$$

Recently, A. Hoque and H. K. Saikia further improved the above bound and that is the best known bound in this case.

Theorem 4.6 (Hoque and Saikia [42], Theorem 3.2) *For a large real number X,*

$$N_3(X) \gg X^{\frac{15}{16}}.$$

In [43], D. Byeon considered the case for $n = 5, 7$ and he could managed to derive the following:

Theorem 4.7 (Byeon [43], Theorem 1.1) *For* $n = 5, 7$,

$$N_n(X) \gg X^{\frac{1}{2}}.$$

This is the best known bound till the date for $n = 5, 7$. It is worth mentioning that his method can give as strong as $N_n(X) \gg X^{\frac{2}{3}-\epsilon}$ for $n = 5, 7$.

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Some Identities on Derangement and Degenerate Derangement Polynomials

Taekyun Kim and Dae San Kim

Abstract In combinatorics, a derangement is a permutation that has no fixed points. The number of derangements of an n-element set is called the nth derangement number. In this paper, as natural companions to derangement numbers and degenerate versions of the companions we introduce derangement polynomials and degenerate derangement polynomials. We give some of their properties, recurrence relations, and identities for those polynomials which are related to some special numbers and polynomials.

Keywords Derangement polynomials · Degenerate derangement polynomials

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1 Introduction

It is known that the Fubini polynomials are defined by the generating function

$$\frac{1}{1-y(e^t-1)} = \sum_{n=0}^{\infty} F_n(y)\frac{t^n}{n!}, \quad \text{(see [7, 11])}. \tag{1.1}$$

Thus, by (1.1), we get

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$$F_n(y) = \sum_{k=0}^{n} S_2(n,k)k!y^k, \quad \text{(see [7, 11])}. \tag{1.2}$$

Here $S_2(n,k)$ is the Stirling number of the second kind which is defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \ \ (n \geq 0), \tag{1.3}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\ldots(x-n+1)$, $(n \geq 1)$.

As is well known, the Bell polynomials are given by the generating function as follows:

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}, \quad \text{(see [5, 6, 12])}. \tag{1.4}$$

When $x = 1$, $Bel_n = Bel_n(1)$ are called the Bell numbers. For $\lambda \in \mathbb{R}$, the partially degenerate Bell polynomials were introduced by Kim–Kim–Dolgy as

$$e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} = \sum_{n=0}^{\infty} Bel_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [12])}. \tag{1.5}$$

Note that $\lim_{\lambda\to 0} Bel_{n,\lambda}(x) = Bel_n(x)$, $(n \geq 0)$. When $x = 1$, $Bel_{n,\lambda} = Bel_{l,\lambda}(1)$ are called the partially degenerate Bell numbers.

From (1.5), we have

$$Bel_{n,\lambda}(x) = \sum_{k=0}^{n}\sum_{m=0}^{k} S_2(k,m)S_1(n,k)\lambda^{n-k}x^m, \tag{1.6}$$

where $S_1(n,k)$ is the Stirling number of the first kind given by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \ \ (n \geq 0), \quad \text{(see [8])}. \tag{1.7}$$

In [1], Carlitz introduced the degenerate Bernoulli and Euler polynomials which are defined by

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!}, \tag{1.8}$$

and

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.9}$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$, $\mathcal{E}_{n,\lambda} = \mathcal{E}_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers and degenerate Euler numbers.

Recently, the degenerate Stirling numbers of the second kind are defined by

$$S_{2,\lambda}(n+1,k) = kS_{2,\lambda}(n,k) + S_{2,\lambda}(n,k-1) - n\lambda S_{2,\lambda}(n,k), \tag{1.10}$$

where $n \geq 0$ (see [10]).

Note that $\lim_{\lambda\to 0} S_{2,\lambda}(n,k) = S_2(n,k)$. For $\lambda \in \mathbb{R}$, the λ-analogue of falling factorial sequence is defined by

$$(x)_{0,\lambda} = 1,\ (x)_{n,\lambda} = x(x-\lambda)\ldots(x-(n-1)\lambda),\ (n \geq 1),\quad \text{(see [6, 8])}. \tag{1.11}$$

Note that $\lim_{\lambda\to 1}(x)_{n,\lambda} = (x)_n$, $(n \geq 0)$, (see [14]).

A derangement is a permutation with no fixed points. In other words, a derangement of a set leaves no elements in the original place. The number of derangements of a set of size n, denoted d_n, is called the nth derangement number (see [9, 15, 16]).

For $n \geq 0$, it is well known that the recurrence relation of derangement numbers is given by

$$d_n = \sum_{k=0}^{n} \binom{n}{k}(n-k)!(-1)^k = n!\sum_{k=0}^{n} \frac{(-1)^k}{k!},\quad \text{(see [9])}. \tag{1.12}$$

It is not difficult to show that

$$\sum_{n=0}^{\infty} d_n \frac{t^n}{n!} = \frac{1}{1-t}e^{-t},\quad \text{(see [2, 3, 4, 5, 9])}. \tag{1.13}$$

From (1.13), we note that

$$d_n = n \cdot d_{n-1} + (-1)^n,\ (n \geq 1),\quad \text{(see [9, 13, 14, 16, 17])}. \tag{1.14}$$

and

$$d_n = (n-1)(d_{n-1} + d_{n-2}),\ (n \geq 2). \tag{1.15}$$

In this paper, as natural companions to derangement numbers and degenerate versions of the companions we introduce derangement polynomials and degenerate derangement polynomials. We give some of their properties, recurrence relations, and identities for those polynomials which are related to some special numbers and polynomials.

2 Derangement Polynomials

Now, we define the derangement polynomials which are given by the generating function

$$\frac{1}{1-xt}e^{-t}=\sum_{n=0}^{\infty}d_n(x)\frac{t^n}{n!}. \tag{2.1}$$

When $x=1$, $d_n(1)=d_n$ are the derangement numbers.
From (1.1), we note that

$$\begin{aligned}\frac{1}{1-yt}&=\sum_{m=0}^{\infty}F_m(y)\frac{1}{m!}\big(\log(1+t)\big)^m\\&=\sum_{m=0}^{\infty}F_m(y)\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}F_m(y)S_1(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.2}$$

On the other hand,

$$\frac{1}{1-yt}=\sum_{n=0}^{\infty}y^n n!\frac{t^n}{n!}. \tag{2.3}$$

Therefore, by (2.2) and (2.3), we obtain the following lemma.

Lemma 2.1 *For $n\geq 0$, we have*

$$y^n=\frac{1}{n!}\sum_{m=0}^{n}F_m(y)S_1(n,m).$$

We observe that

$$\begin{aligned}\frac{1}{1-yt}&=\left(\frac{1}{1-yt}e^{-t}\right)e^t=\left(\sum_{l=0}^{\infty}d_l(y)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\frac{t^m}{m!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}d_l(y)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

From (2.2) and (2.4), we obtain the following theorem.

Theorem 2.2 *For $n\geq 0$, we have*

$$\sum_{l=0}^{n}\binom{n}{l}d_l(y)=\sum_{m=0}^{n}F_m(y)S_1(n,m).$$

By (2.1), we get

$$\sum_{n=0}^{\infty} d_n(x)\frac{t^n}{n!} = \frac{1}{1-xt}e^{-t} = \left(\sum_{m=0}^{\infty} x^m t^m\right)\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}t^k\right) = \sum_{n=0}^{\infty}\left(n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}x^{n-k}\right)\frac{t^n}{n!}. \tag{2.5}$$

By comparing the coefficients on both sides of (2.5), we obtain the following theorem.

Theorem 2.3 *For $n \geq 0$, we have*

$$d_n(x) = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}x^{n-k}.$$

From (2.1), we have

$$e^{-t} = (1-xt)\sum_{n=0}^{\infty} d_n(x)\frac{t^n}{n!} = d_0(x) + \sum_{n=1}^{\infty}\left(d_n(x) - nxd_{n-1}(x)\right)\frac{t^n}{n!}. \tag{2.6}$$

On the other hand,

$$e^{-t} = \sum_{n=0}^{\infty}(-1)^n\frac{t^n}{n!}. \tag{2.7}$$

Thus, by (2.6) and (2.7), we get

$$d_0(x) = 1,\ \ d_n(x) = nxd_{n-1}(x) + (-1)^n,\ \ (n \geq 1). \tag{2.8}$$

From (2.8), we note that

$$\begin{aligned} d_n(x) &= (nx-1)d_{n-1}(x) + d_{n-1}(x) + (-1)^n \\ &= (nx-1)d_{n-1}(x) + (n-1)xd_{n-2}(x) + (-1)^{n-1} + (-1)^n \\ &= (nx-1)\left[d_{n-1}(x) + d_{n-2}(x)\right] + (1-x)d_{n-2}(x),\ \ (n \geq 2). \end{aligned} \tag{2.9}$$

Therefore, we obtain the following theorem.

Theorem 2.4 *For $n \geq 1$, we have*

$$d_n(x) = nxd_{n-1}(x) + (-1)^n.$$

In particular, for $n \geq 2$, we have

$$d_n(x) = (nx - 1)\left[d_{n-1}(x) + d_{n-2}(x)\right] + (1 - x)d_{n-2}(x).$$

Replacing t by $e^t - 1$ in (2.1), we get

$$\begin{aligned} \frac{1}{1 - x(e^t - 1)} e^{-(e^t-1)} &= \sum_{m=0}^{\infty} d_m(x) \frac{1}{m!}(e^t - 1)^m \\ &= \sum_{m=0}^{\infty} d_m(x) \sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} d_m(x) S_2(n, m) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.10}$$

By (2.10), we see that

$$\begin{aligned} \frac{1}{1 - x(e^t - 1)} &= e^{(e^t-1)} \sum_{k=0}^{\infty} \left(\sum_{m=0}^{k} d_m(x) S_2(k, m) \right) \frac{t^k}{k!} \\ &= \left(\sum_{l=0}^{\infty} Bel_l \frac{t^l}{l!} \right) \left(\sum_{k=0}^{\infty} \left(\sum_{m=0}^{k} d_m(x) S_2(k, m) \right) \frac{t^k}{k!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} d_m(x) S_2(k, m) Bel_{n-k} \right) \frac{t^n}{n!}. \end{aligned} \tag{2.11}$$

From (1.1), we note that

$$\frac{1}{1 - x(e^t - 1)} = \sum_{n=0}^{\infty} F_n(x) \frac{t^n}{n!}. \tag{2.12}$$

Therefore, by (2.11) and (2.12), we obtain the following theorem.

Theorem 2.5 *For $n \geq 0$, we have*

$$F_n(x) = \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} d_m(x) S_2(k, m) Bel_{n-k}.$$

From (1.1), we can derive the following Eq. (2.13):

$$\begin{aligned}\frac{1}{1-xt}e^{-t} &= \left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}F_m(x)S_1(k,m)\right)\frac{t^k}{k!}\right)e^{-t}\\ &= \left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}F_m(x)S_1(k,m)\right)\frac{t^k}{k!}\right)\left(\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}t^l\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}F_m(x)S_1(k,m)\frac{(-1)^{n-k}}{(n-k)!}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.13}$$

On the other hand,

$$\frac{1}{1-xt}e^{-t} = \sum_{n=0}^{\infty}d_n(x)\frac{t^n}{n!}. \tag{2.14}$$

Therefore, by (2.13) and (2.14), we obtain the following theorem.

Theorem 2.6 *For $n \geq 0$, we have*

$$d_n(x) = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}F_m(x)S_1(k,m)\frac{(-1)^{n-k}}{(n-k)!}.$$

As is known, Bernoulli polynomials are defined by the generating function

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \quad \text{(see [15])}. \tag{2.15}$$

When $x = 0$, $B_n = B_n(0)$ are Bernoulli numbers. By (2.15), we easily get

$$\begin{aligned}\sum_{k=0}^{m-1}e^{kt} &= \frac{1}{e^t-1}\left(e^{mt}-1\right) = \frac{1}{t}\left\{\frac{t}{e^t-1}e^{mt} - \frac{t}{e^t-1}\right\}\\ &= \sum_{n=0}^{\infty}\left(\frac{B_{n+1}(m)-B_{n+1}}{n+1}\right)\frac{t^n}{n!}, \quad (n \geq 1).\end{aligned} \tag{2.16}$$

By Taylor expansion, we get

$$\sum_{k=0}^{m-1}e^{kt} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{m-1}k^n\right)\frac{t^n}{n!}, \quad (m \geq 1). \tag{2.17}$$

From (2.16) and (2.17), we get

$$\sum_{k=0}^{m-1}k^n = \frac{B_{n+1}(m)-B_{n+1}}{n+1}. \tag{2.18}$$

By Lemma 2.1, we easily get

$$\sum_{k=0}^{m-1} k^n = \frac{1}{n!} \sum_{k=0}^{m-1} \sum_{l=0}^{n} F_l(k) S_1(n,l). \tag{2.19}$$

Therefore, by Theorem 2.2, (2.18), and (2.19), we obtain the following theorem.

Theorem 2.7 *For $m \geq 1$ and $n \geq 0$, we have*

$$\begin{aligned} \frac{B_{n+1}(m) - B_{n+1}}{n+1} &= \frac{1}{n!} \sum_{k=0}^{m-1} \sum_{l=0}^{n} F_l(k) S_1(n,l) \\ &= \frac{1}{n!} \sum_{k=0}^{m-1} \sum_{l=0}^{n} \binom{n}{l} d_l(k). \end{aligned}$$

3 Degenerate Derangement Polynomials

Here we consider the degenerate derangement polynomials which are given by

$$\frac{1}{1-xt}(1-\lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty} d_{n,\lambda}(x) \frac{t^n}{n!}, \quad (\lambda \in \mathbb{R}). \tag{3.1}$$

When $x = 1$, $d_{n,\lambda} = d_{n,\lambda}(1)$ are called the degenerate derangement numbers. From (3.1), we note that

$$\begin{aligned} (1-\lambda t)^{\frac{1}{\lambda}} &= \left(\sum_{n=0}^{\infty} d_{n,\lambda}(x) \frac{t^n}{n!} \right) (1-xt) \\ &= \sum_{n=0}^{\infty} d_{n,\lambda}(x) \frac{t^n}{n!} - \sum_{n=0}^{\infty} x d_{n,\lambda}(x) \frac{t^{n+1}}{n!} \\ &= d_{0,\lambda}(x) + \sum_{n=1}^{\infty} \left(d_{n,\lambda}(x) - x n d_{n-1,\lambda}(x) \right) \frac{t^n}{n!}. \end{aligned} \tag{3.2}$$

On the other hand,

$$(1-\lambda t)^{\frac{1}{\lambda}} = \sum_{m=0}^{\infty} \binom{\frac{1}{\lambda}}{m} (-\lambda)^m t^m = \sum_{m=0}^{\infty} (-1)^m (1)_{m,\lambda} \frac{t^m}{m!}. \tag{3.3}$$

Therefore, by (3.2) and (3.3), we obtain the following theorem.

Theorem 3.1 *For $n \geq 0$, we have*

$$d_{0,\lambda}(x) = 1,\ \ d_{n,\lambda}(x) = nxd_{n-1,\lambda}(x) + (-1)^n(1)_{n,\lambda},\ \ (n \geq 1).$$

Note that $\lim_{\lambda\to 0} d_{n,\lambda}(x) = d_n(x)$, $\lim_{\lambda\to 0} d_{n,\lambda} = d_n$, $(n \geq 0)$.
From (3.1), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} d_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{1}{1-xt}(1-\lambda t)^{\frac{1}{\lambda}} = \left(\sum_{m=0}^{\infty} x^m t^m\right)\left(\sum_{k=0}^{\infty}(-1)^k(1)_{k,\lambda}\frac{t^k}{k!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^k}{k!}(1)_{k,\lambda}x^{n-k}\right)t^n.\end{aligned} \tag{3.4}$$

Comparing the coefficients on both sides of (3.4), we obtain the following theorem.

Theorem 3.2 *For $n \geq 0$, we have*

$$d_{n,\lambda}(x) = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}(1)_{k,\lambda}x^{n-k}.$$

In particular, for $x = 1$,

$$d_{n,\lambda} = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}(1)_{k,\lambda}.$$

Now, we observe that

$$\begin{aligned}\frac{1}{1-xt} &= \left(\frac{1}{1-xt}\right)(1-\lambda t)^{\frac{1}{\lambda}}\cdot(1-\lambda t)^{-\frac{1}{\lambda}}\\ &= \left(\sum_{l=0}^{\infty} d_{l,\lambda}(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\binom{-\frac{1}{\lambda}}{m}(-\lambda)^m t^m\right)\\ &= \left(\sum_{l=0}^{\infty} d_{l,\lambda}(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}1(1+\lambda)\ldots(1+(m-1)\lambda)\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}d_{l,\lambda}(x)(1)_{n-l,-\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{3.5}$$

On the other hand,

$$\frac{1}{1-xt} = \sum_{n=0}^{\infty} x^n n!\frac{t^n}{n!}. \tag{3.6}$$

Therefore, by (3.5) and (3.6), we obtain the following theorem.

Theorem 3.3 *For $n \geq 0$, we have*

$$x^n = \frac{1}{n!}\sum_{l=0}^{n}\binom{n}{l} d_{l,\lambda}(x)(1)_{n-l,-\lambda}.$$

From Theorem 3.1, we have

$$\begin{aligned} d_{n,\lambda}(x) &= nxd_{n-1,\lambda}(x) + (-1)^n(1)_{n,\lambda} \\ &= (nx-1)d_{n-1,\lambda}(x) + d_{n-1,\lambda}(x) + (-1)^n(1)_{n,\lambda} \\ &= (nx-1)d_{n-1,\lambda}(x) + (n-1)xd_{n-2,\lambda}(x) \\ &\quad + (-1)^{n-1}(1)_{n-1,\lambda} + (-1)^n(1)_{n,\lambda} \\ &= (nx-1)\left[d_{n-1,\lambda}(x) + d_{n-2,\lambda}(x)\right] \\ &\quad + (1-x)d_{n-2,\lambda}(x) + (-1)^{n-1}(1)_{n-1,\lambda}(n-1)\lambda, \end{aligned} \tag{3.7}$$

where $n \geq 2$.

Therefore, by (3.7), we obtain the following theorem.

Theorem 3.4 *For $n \geq 2$, we have*

$$\begin{aligned} d_{n,\lambda}(x) &= (nx-1)\left[d_{n-1,\lambda}(x) + d_{n-2,\lambda}(x)\right] \\ &\quad + (1-x)d_{n-2,\lambda}(x) + (-1)^{n-1}(1)_{n-1,\lambda}(n-1)\lambda. \end{aligned}$$

In particular, $x = 1$,

$$d_{n,\lambda} = (n-1)\left[d_{n-1,\lambda} + d_{n-2,\lambda}\right] + \lambda(n-1)(-1)^{n-1}(1)_{n-1,\lambda}.$$

Note that

$$d_n = \lim_{\lambda \to 0} d_{n,\lambda} = (n-1)\left[d_{n-1} + d_{n-2}\right] \ (n \geq 2).$$

By using Taylor expansion, we get

$$\begin{aligned} (1-\lambda t)^{\frac{1}{\lambda}} = e^{\frac{1}{\lambda}\log(1-\lambda t)} &= \sum_{m=0}^{\infty}\lambda^{-m}\frac{1}{m!}\Big(\log(1-\lambda t)\Big)^m \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}(-1)^n S_1(n,m)\right)\frac{t^n}{n!}. \end{aligned} \tag{3.8}$$

On the other hand,

$$\begin{aligned}(1-\lambda t)^{\frac{1}{\lambda}} &= \frac{1}{1-xt}(1-\lambda t)^{\frac{1}{\lambda}}(1-xt) \\ &= \sum_{n=0}^{\infty} d_{n,\lambda}(x)\frac{t^n}{n!} - \sum_{n=1}^{\infty} nxd_{n-1,\lambda}(x)\frac{t^n}{n!} \\ &= d_{0,\lambda}(x) + \sum_{n=1}^{\infty}\left\{d_{n,\lambda}(x) - nxd_{n-1,\lambda}(x)\right\}\frac{t^n}{n!} \\ &= 1 + \sum_{n=1}^{\infty}\left(d_{n,\lambda}(x) - nxd_{n-1,\lambda}(x)\right)\frac{t^n}{n!}\end{aligned} \tag{3.9}$$

From (3.8) and (3.9), we have

$$(-1)^n \sum_{m=0}^{n} \lambda^{n-m} S_1(n,m) = d_{n,\lambda}(x) - nxd_{n-1,\lambda}(x) = (-1)^n (1)_{n,\lambda}, \ \ (n \geq 1). \tag{3.10}$$

Therefore, by (3.10), we obtain the following theorem.

Theorem 3.5 *For $n \geq 1$, we have*

$$\sum_{m=0}^{n} \lambda^{n-m} S_1(n,m) = (1)_{n,\lambda}.$$

By (1.13), we get

$$\begin{aligned}\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}+1}e^{(1+\lambda t)^{\frac{1}{\lambda}}} &= \sum_{m=0}^{\infty}(-1)^m d_m \frac{1}{m!}(1+\lambda t)^{\frac{m}{\lambda}} \\ &= \sum_{m=0}^{\infty}(-1)^m d_m \frac{1}{m!}\sum_{n=0}^{\infty}(m)_{n,\lambda}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}(-1)^m d_m \frac{(m)_{n,\lambda}}{m!}\right)\frac{t^n}{n!}.\end{aligned} \tag{3.11}$$

On the other hand,

$$\begin{aligned}\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}+1}e^{(1+\lambda t)^{\frac{1}{\lambda}}} &= \frac{e}{2}\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}e^{(1+\lambda t)^{\frac{1}{\lambda}}-1} \\ &= \frac{e}{2}\left(\sum_{l=0}^{\infty}\mathcal{E}_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}Bel_{m,\lambda}\frac{t^m}{m!}\right) \\ &= \frac{e}{2}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}Bel_{m,\lambda}\mathcal{E}_{n-m,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{3.12}$$

Therefore, by (3.11) and (3.12), we obtain the following theorem.

Theorem 3.6 *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} \binom{n}{m} Bel_{m,\lambda} \mathcal{E}_{n-m,\lambda} = \frac{2}{e} \sum_{m=0}^{\infty} (-1)^m d_m \frac{(m)_{n,\lambda}}{m!}.$$

From (3.11), we note that

$$\begin{aligned} e^{(1+\lambda t)^{\frac{1}{\lambda}}} &= \sum_{m=0}^{\infty} d_m \frac{(-1)^m}{m!} (1+\lambda t)^{\frac{m}{\lambda}} \left(1 + (1+\lambda t)^{\frac{1}{\lambda}}\right) \\ &= \sum_{m=0}^{\infty} d_m \frac{(-1)^m}{m!} (1+\lambda t)^{\frac{m}{\lambda}} + \sum_{m=0}^{\infty} d_m \frac{(-1)^m}{m!} (1+\lambda t)^{\frac{m+1}{\lambda}} \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{\infty} d_m \frac{(-1)^m}{m!} \left((m)_{n,\lambda} + (m+1)_{n,\lambda}\right) \right\} \frac{t^n}{n!}. \end{aligned} \tag{3.13}$$

On the other hand,

$$\begin{aligned} e^{(1+\lambda t)^{\frac{1}{\lambda}}} &= e \cdot e^{(1+\lambda t)^{\frac{1}{\lambda}} - 1} = e \sum_{k=0}^{\infty} \frac{1}{k!} \left((1+t)^{\frac{1}{\lambda}} - 1\right)^k \\ &= e \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} S_{2,\lambda}(n,k) \frac{t^n}{n!} = e \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} S_{2,\lambda}(n,k) \right) \frac{t^n}{n!}. \end{aligned} \tag{3.14}$$

Therefore, by (3.13) and (3.14), we obtain the following theorem.

Theorem 3.7 *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} S_{2,\lambda}(n,m) = \frac{1}{e} \sum_{m=0}^{\infty} d_m \frac{(-1)^m}{m!} \left((m)_{n,\lambda} + (m+1)_{n,\lambda}\right).$$

Indeed,

$$\begin{aligned} \sum_{n=0}^{\infty} Bel_{n,\lambda} \frac{t^n}{n!} &= e^{\left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)} = \sum_{m=0}^{\infty} \frac{1}{m!} \left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)^m \\ &= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} S_{2,\lambda}(n,m) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} S_{2,\lambda}(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{3.15}$$

Thus, by (3.15), we get

$$Bel_{n,\lambda} = \sum_{m=0}^{n} S_{2,\lambda}(n, m), \quad (n \geq 0). \tag{3.16}$$

Therefore, by (3.16), we obtain the following corollary.

Corollary 3.8 *For $n \geq 0$, we have*

$$Bel_{n,\lambda} = \frac{1}{e} \sum_{m=0}^{\infty} d_m \frac{(-1)^m}{m!} \left((m)_{n,\lambda} + (m+1)_{n,\lambda} \right).$$

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Some Perturbed Ostrowski Type Inequalities for Twice Differentiable Functions

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Abstract In this study, we first obtain an identity for twice differentiable functions. Then we establish some perturbed Ostrowski type integral inequalities for functions whose second derivatives are bounded. Moreover, some perturbed versions of Ostrowski type inequalities for mapping whose second derivatives are either of bounded variation or Lipschitzian.

Keywords Function of bounded variation · Ostrowski type inequalities

2000 Mathematics Subject Classification 26D15 · 26A45 · 26D10

1 Introduction

In 1938, Ostrowski [1] established the following useful inequality:

Theorem 1 *Let* $f : [a, b] \to \mathbb{R}$ *be a differentiable mapping on* (a, b) *whose derivative* $f' : (a, b) \to \mathbb{R}$ *is bounded on* (a, b), *i.e.* $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty$. *Then, we have the inequality*

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$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_\infty, \tag{1.1}$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is the best possible.

Definition 1 Let $P : a = x_0 < x_1 < \cdots < x_n = b$ be any partition of $[a, b]$ and let $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$. Then f is said to be of bounded variation if the sum

$$\sum_{i=1}^{m} |\Delta f(x_i)|$$

is bounded for all such partitions [2].

Definition 2 Let f be of bounded variation on $[a, b]$, and $\sum \Delta f(P)$ denote the sum $\sum_{i=1}^{n} |\Delta f(x_i)|$ corresponding to the partition P of $[a, b]$. The number

$$\bigvee_a^b (f) := \sup \left\{ \sum \Delta f(P) : P \in P([a, b]) \right\},$$

is called the total variation of f on $[a, b]$. Here $P([a, b])$ denotes the family of partitions of $[a, b]$ [2].

In [3], Dragomir proved the following Ostrowski type inequalities for functions of bounded variation:

Theorem 2 *Let* $f : [a, b] \to \mathbb{R}$ *be a mapping of bounded variation on* $[a, b]$. *Then*

$$\left| \int_a^b f(t)dt - (b-a) f(x) \right| \leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (f) \tag{1.2}$$

holds for all $x \in [a, b]$. *The constant* $\frac{1}{2}$ *is the best possible.*

In [4], Dragomir and Barnett obtained the following Ostrowski type inequalities for functions whose second derivatives are bounded:

Theorem 3 *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and twice differentiable on* (a, b), *whose second derivative* $f'' : (a, b) \to \mathbb{R}$ *is bounded on* $(a; b)$. *Then we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right|$$
$$\leq \frac{1}{2}\left\{\left[\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} + \frac{1}{4}\right]^2 + \frac{1}{12}\right\}(b-a)^2 \left\| f''\right\|_\infty$$
$$\leq \frac{\left\| f''\right\|_\infty}{6}(b-a)^2$$

for all $x \in [a, b]$.

Ostrowski inequality has potential applications in Mathematical Sciences. It has applications in numerical integration, probability and optimization theory, stochastic, statistics, information and integral operator theory. In the past, many authors have worked on Ostrowski type inequalities for functions (bounded, of bounded variation, etc.) see for example [3–28]. Moreover, Dragomir proved some perturbed Ostrowski type inequalities for bounded functions and functions of bounded variation, please refer to [29–35]. In this study, we establish some perturbed Ostrowski type inequalities for twice differentiable functions whose second derivatives are either bounded or of bounded variation.

2 Some Identities

Before we start our main results, we state and prove the following lemma:

Lemma 1 *Let* $f : [a, b] \to \mathbb{C}$ *be a twice differantiable function on* (a, b). *Then for any* $\lambda_1(x)$ *and* $\lambda_2(x)$ *complex number the following identity holds*

$$\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt$$
$$- \frac{1}{2(b-a)}\left[\frac{\lambda_1(x)(x-a)^3 + \lambda_2(x)(b-x)^3}{3}\right]$$
$$= \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2\left[f''(t) - \lambda_1(x)\right]dt + \frac{1}{b-a}\int_x^b (t-b)^2\left[f''(t) - \lambda_2(x)\right]dt\right], \tag{2.1}$$

for all $x \in [a, b]$, *where the integrals in the right hand side are taken in the Lebesgue sense.*

Proof Using the integration by parts, we have

$$\begin{aligned}
&\int_a^x (t-a)^2 \left[f''(t) - \lambda_1(x)\right] dt \\
&= \int_a^x (t-a)^2 f''(t)dt - \lambda_1(x) \int_a^x (t-a)^2 dt \\
&= (t-a)^2 f'(t)\Big|_a^x - 2\int_a^x (t-a) f'(t)dt - \frac{\lambda_1(x)}{3} (t-a)^3\Big|_a^x \\
&= (x-a)^2 f'(x) - 2\left[(t-a) f(t)|_a^x - \int_a^x f(t)dt\right] - \frac{\lambda_1(x)}{3} (x-a)^3 \\
&= (x-a)^2 f'(x) - 2(x-a) f(x) + 2\int_a^x f(t)dt - \frac{\lambda_1(x)}{3} (x-a)^3 \qquad (2.2)
\end{aligned}$$

and

$$\begin{aligned}
&\int_x^b (t-b)^2 \left[f''(t) - \lambda_2(x)\right] dt \\
&= \int_x^b (t-b)^2 f''(t)dt - \lambda_2(x) \int_x^b (t-b)^2 dt \\
&= (t-b)^2 f'(t)\Big|_x^b - 2\int_x^b (t-b) f'(t)dt - \frac{\lambda_1(x)}{3} (t-b)^3\Big|_x^b \\
&= -(b-x)^2 f'(x) - 2\left[(t-b) f(t)|_x^b - \int_x^b f(t)dt\right] - \frac{\lambda_2(x)}{3} (b-x)^3 \\
&= -(b-x)^2 f'(x) - 2(b-x) f(x) + 2\int_x^b f(t)dt - \frac{\lambda_1(x)}{3} (x-a)^3 . \qquad (2.3)
\end{aligned}$$

If we add the equalities (2.2) and (2.3) and divide by $2(b-a)$, then we obtain required identity. □

Corollary 1 *Under assumption of Lemma 1 with* $\lambda_1(x) = \lambda_2(x) = \lambda(x)$, *we have*

$$\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt - \frac{\lambda(x)}{6(b-a)}\left[(x-a)^3 + (b-x)^3\right]$$
$$= \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2\left[f''(t) - \lambda(x)\right]dt + \frac{1}{b-a}\int_x^b (t-b)^2\left[f''(t) - \lambda(x)\right]dt\right] \quad (2.4)$$

for all $x \in [a, b]$.

Remark 1 If we choose $\lambda(x) = 0$ in (2.4), then for all $x \in [a, b]$ we have the following identity

$$\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt$$
$$= \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2 f''(t)dt + \frac{1}{b-a}\int_x^b (t-b)^2 f''(t)dt\right] \quad (2.5)$$

which is given by [16].

Corollary 2 *Under assumption of Lemma 1 with* $\lambda_1(x) = \lambda_1 \in \mathbb{C}$ *and* $\lambda_2(x) = \lambda_2 \in \mathbb{C}$, *we get*

$$\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt$$
$$- \frac{1}{6(b-a)}\left[\lambda_1(x-a)^3 + \lambda_2(b-x)^3\right]$$
$$= \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2\left[f''(t) - \lambda_1\right]dt + \frac{1}{b-a}\int_x^b (t-b)^2\left[f''(t) - \lambda_2\right]dt\right]. \quad (2.6)$$

for all $x \in [a, b]$.

In particular, taking $\lambda_1 = \lambda_2 = \lambda$ *we have*

$$\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt - \frac{\lambda}{6(b-a)}\left[(x-a)^3 + (b-x)^3\right]$$
$$= \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2\left[f''(t) - \lambda\right]dt + \frac{1}{b-a}\int_x^b (t-b)^2\left[f''(t) - \lambda\right]dt\right] \tag{2.7}$$

for all $x \in [a, b]$.

Corollary 3 *Under assumption of Lemma 1 with* $\lambda_1(x) = \lambda_2(x) = f''(x),\ x \in (a, b)$*, we have the equality*

$$\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt - \frac{f''(x)}{6(b-a)}\left[(x-a)^3 + (b-x)^3\right]$$
$$= \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2\left[f''(t) - f''(x)\right]dt + \frac{1}{b-a}\int_x^b (t-b)^2\left[f''(t) - f''(x)\right]dt\right] \tag{2.8}$$

for all $x \in [a, b]$.

Corollary 4 *Under assumption of Lemma 1, we assume that the lateral derivatives* $f''_+(a)$ *and* $f''_-(b)$ *exist and finite. If we take* $\lambda_1(x) = f''_+(a)$ *and* $\lambda_2(x) = f''_-(b)$ *in* (2.1)*, then we have*

$$\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt$$
$$- \frac{1}{2(b-a)}\left[\frac{f''_+(a)(x-a)^3 + f''_-(b)(b-x)^3}{3}\right]$$
$$= \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2\left[f''(t) - f''_+(a)\right]dt + \frac{1}{b-a}\int_x^b (t-b)^2\left[f''(t) - f''_-(b)\right]dt\right]. \tag{2.9}$$

for all $x \in [a, b]$.

In particular, we get

$$\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{48}\left[f''_+(a)+f''_-(b)\right]$$
$$= \frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}} (t-a)^2\left[f''(t)-f''_+(a)\right]dt + \int_{\frac{a+b}{2}}^b (t-b)^2\left[f''(t)-f''_-(b)\right]dt\right]. \tag{2.10}$$

Corollary 5 *Under assumption of Lemma 1, we assume that the derivatives $f''_+(a)$, $f''_-(b)$ and $f''(x)$ exist and finite. If we choose $\lambda_1(x) = \frac{f''_+(a)+f''(x)}{2}$ and $\lambda_2(x) = \frac{f''(x)+f''_-(b)}{2}$ in (2.1), then we have*

$$\left(x-\frac{a+b}{2}\right)f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt - \frac{(x-a)^3+(b-x)^3}{12(b-a)}f''(x)$$
$$- \frac{1}{12(b-a)}\left[(x-a)^3 f''_+(a) + (b-x)^3 f''_-(b)\right]$$
$$= \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2\left[f''(t)-\frac{f''_+(a)+f''(x)}{2}\right]dt\right.$$
$$\left. + \frac{1}{b-a}\int_x^b (t-b)^2\left[f''(t)-\frac{f''(x)+f''_-(b)}{2}\right]dt\right]. \tag{2.11}$$

for all $x \in [a,b]$.

In particular, we have

$$\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24}f''\left(\frac{a+b}{2}\right) - \frac{(b-a)}{48}\left[f''_+(a)+f''_-(b)\right]$$
$$= \frac{1}{2}\left[\frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)^2\left[f''(t)-\frac{f''_+(a)+f''\left(\frac{a+b}{2}\right)}{2}\right]dt\right.$$
$$\left. + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)^2\left[f''(t)-\frac{f''\left(\frac{a+b}{2}\right)+f''_-(b)}{2}\right]dt\right]. \tag{2.12}$$

3 Inequalities for Functions Whose Second Derivatives are Bounded

Recall the sets of complex-valued functions:

$$\overline{U}_{[a,b]}(\gamma,\Gamma)$$
$$:=\left\{f:[a,b]\to\mathbb{C}:\left[(\Gamma-f(t))\left(\overline{f(t)}\right)-\overline{\gamma}\right]\geq 0 \text{ for almast every } t\in[a,b]\right\}$$

and

$$\overline{\Delta}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}:\left|f(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}\left|\Gamma-\gamma\right| \text{ for a.e. } t\in[a,b]\right\}.$$

Proposition 1 *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that $\overline{U}_{[a,b]}(\gamma,\Gamma)$ and $\overline{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty and closed sets and*

$$\overline{U}_{[a,b]}(\gamma,\Gamma)=\overline{\Delta}_{[a,b]}(\gamma,\Gamma).$$

Theorem 4 *Let $f:[a,b]\to\mathbb{C}$ be a twice differantiable function on (a,b) and $x\in(a,b)$. Suppose that $\gamma_i,\Gamma_i\in\mathbb{C}$, $\gamma_i\neq\Gamma_i$, $i=1,2$ and $f''\in\overline{U}_{[a,x]}(\gamma_1,\Gamma_1)\cap\overline{U}_{[x,b]}(\gamma_2,\Gamma_2)$. Then we have the inequalities*

$$\begin{aligned}&\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt\right.\\&\left.-\frac{(\gamma_1+\Gamma_1)(x-a)^3+(\gamma_2+\Gamma_2)(b-x)^3}{12(b-a)}\right|\\&\leq\frac{(b-a)^2}{12}\left[\left|\Gamma_1-\gamma_1\right|\left(\frac{x-a}{b-a}\right)^3+\left|\Gamma_2-\gamma_2\right|\left(\frac{b-x}{b-a}\right)^3\right]\\&\leq\frac{(b-a)^2}{12}\begin{cases}\left[\left(\frac{x-a}{b-a}\right)^3+\left(\frac{b-x}{b-a}\right)^3\right]\max\left\{\left|\Gamma_1-\gamma_1\right|,\left|\Gamma_2-\gamma_2\right|,\right\}\\[2ex]\left[\left(\frac{x-a}{b-a}\right)^{3p}+\left(\frac{b-x}{b-a}\right)^{3p}\right]^{\frac{1}{p}}\left(\left|\Gamma_1-\gamma_1\right|^q+\left|\Gamma_2-\gamma_2\right|^q\right)^{\frac{1}{q}},\\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex]\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^3\left[\left|\Gamma_1-\gamma_1\right|+\left|\Gamma_2-\gamma_2\right|\right].\end{cases}\end{aligned}\tag{3.1}$$

Proof Taking the modulus identity (2.1) for $\lambda_1(x) = \frac{\gamma_1+\Gamma_1}{2}$ and $\lambda_2(x) = \frac{\gamma_2+\Gamma_2}{2}$, since $f'' \in \overline{U}_{[a,x]}(\gamma_1, \Gamma_2) \cap \overline{U}_{[x,b]}(\gamma_2, \Gamma_2)$, we have

$$\left| \left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt \right.$$
$$\left. - \frac{(\gamma_1+\Gamma_1)(x-a)^3 + (\gamma_2+\Gamma_2)(b-x)^3}{12(b-a)} \right|$$
$$\leq \frac{1}{2(b-a)} \left[\int_a^x (t-a)^2 \left| f''(t) - \frac{\gamma_1+\Gamma_1}{2} \right| dt \right.$$
$$\left. + \int_x^b (t-b)^2 \left| f''(t) - \frac{\gamma_2+\Gamma_2}{2} \right| dt \right]$$
$$\leq \frac{1}{2(b-a)} \left[\frac{|\Gamma_1-\gamma_1|}{2} \int_a^x (t-a)^2 dt + \frac{|\Gamma_2-\gamma_2|}{2} \int_x^b (t-b)^2 dt \right]$$
$$= \frac{(b-a)^2}{12} \left[|\Gamma_1-\gamma_1| \left(\frac{x-a}{b-a}\right)^3 + |\Gamma_2-\gamma_2| \left(\frac{b-x}{b-a}\right)^3 \right]$$

which completes the first inequality in (3.1).

The proofs of the first and third branches of the second inequality in (3.1) are obvious. Using Hölder's inequality

$$mn + pq \leq (m^\alpha + p^\alpha)^{\frac{1}{\alpha}} \left(n^\beta + q^\beta\right)^{\frac{1}{\beta}}, \; m, n, p, q \geq 0 \text{ and } \alpha > 1 \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1$$

we can easily obtain the second branch of second inequality in (3.1). □

Corollary 6 *Let $f : [a, b] \to \mathbb{C}$ be a twice differantiable function on (a, b) and $x \in (a, b)$. If $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$ and $f'' \in \overline{U}_{[a,b]}(\gamma, \Gamma)$, then we have*

$$\left| \left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt - \frac{\gamma+\Gamma}{12(b-a)}\left[(x-a)^3 + (b-x)^3\right] \right|$$
$$\leq \frac{|\Gamma-\gamma|}{12(b-a)}\left[(x-a)^3 + (b-x)^3\right]$$

for all $x \in [a, b]$.

Corollary 7 *Under assumption of Theorem 4 with* $x = \frac{a+b}{2}$, *we have*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{48}\left[\frac{\gamma_1+\Gamma_1}{2} + \frac{\gamma_2+\Gamma_2}{2}\right] \right|$$

$$\leq \frac{1}{96}\left[|\Gamma_1-\gamma_1| + |\Gamma_2-\gamma_2|\right](b-a)^2.$$

4 Inequalities for Functions Whose Second Derivatives are of Bounded Variation

Assume that $f : [a, b] \to \mathbb{C}$ be a twice differantiable function on I° (the interior of I) and $[a, b] \subset I^\circ$. Then, as in (2.11), we have the identity

$$\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt - \frac{(x-a)^3+(b-x)^3}{12(b-a)} f''(x)$$
$$- \frac{1}{12(b-a)}\left[(x-a)^3 f''(a) + (b-x)^3 f''(b)\right]$$
$$= \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2\left[f''(t) - \frac{f''(a)+f''(x)}{2}\right]dt\right.$$
$$\left. + \frac{1}{b-a}\int_x^b (t-b)^2\left[f''(t) - \frac{f''(x)+f''(b)}{2}\right]dt\right], \tag{4.1}$$

for any $x \in [a, b]$.

Theorem 5 *Let*: $f : [a, b] \to \mathbb{C}$ *be a twice differantiable function on* I° *and* $[a, b] \subset I^\circ$. *If the second derivative* f'' *is of bounded variation on* $[a, b]$, *then*

$$\left|\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt - \frac{(x-a)^3+(b-x)^3}{12(b-a)} f''(x)\right.$$
$$\left. - \frac{1}{12(b-a)}\left[(x-a)^3 f''(a) + (b-x)^3 f''(b)\right]\right|$$
$$\leq \frac{(b-a)^2}{12}\left[\left(\frac{x-a}{b-a}\right)^3 \bigvee_a^x (f'') + \left(\frac{b-x}{b-a}\right)^3 \bigvee_x^b (f'')\right]$$

$$\leq \frac{(b-a)^2}{12} \begin{cases} \left[\left(\frac{x-a}{b-a}\right)^3 + \left(\frac{b-x}{b-a}\right)^3\right]\left[\frac{1}{2}\bigvee_a^b(f'') + \frac{1}{2}\left|\bigvee_a^x(f'') - \bigvee_x^b(f'')\right|\right], \\ \left[\left(\frac{x-a}{b-a}\right)^{3p} + \left(\frac{b-x}{b-a}\right)^{3p}\right]^{\frac{1}{p}}\left[\left(\bigvee_a^x(f'')\right)^q + \left(\bigvee_x^b(f'')\right)^q\right]^{\frac{1}{q}} \\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^3 \bigvee_a^b(f''), \end{cases} \tag{4.2}$$

for any $x \in [a, b]$.

Proof Taking modulus (4.1), we get

$$\begin{aligned} &\left|\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt - \frac{(x-a)^3 + (b-x)^3}{12(b-a)} f''(x)\right. \\ &\left.- \frac{1}{12(b-a)}\left[(x-a)^3 f''(a) + (b-x)^3 f''(b)\right]\right| \\ \leq &\frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2 \left|f''(t) - \frac{f''(a)+f''(x)}{2}\right| dt\right. \\ &\left.+ \frac{1}{b-a}\int_x^b (t-b)^2 \left|f''(t) - \frac{f''(x)+f''(b)}{2}\right| dt\right]. \end{aligned} \tag{4.3}$$

Since f'' is of bounded variation on $[a, x]$, we get

$$\begin{aligned} \left|f''(t) - \frac{f''(a)+f''(x)}{2}\right| &\leq \frac{\left|2f''(t) - f''(a) - f''(x)\right|}{2} \\ &\leq \frac{\left|f''(t) - f''(a)\right| + \left|f''(x) - f''(t)\right|}{2} \\ &\leq \frac{1}{2}\bigvee_a^x(f''). \end{aligned}$$

Thus,

$$\int_a^x (t-a)^2 \left| f''(t) - \frac{f''(a)+f''(x)}{2} \right| dt \le \frac{1}{2} \bigvee_a^x (f'') \int_a^x (t-a)^2\, dt$$
$$\le \frac{(x-a)^3}{6} \bigvee_a^x (f''). \tag{4.4}$$

Similarly, since f'' is of bounded variation on $[x, b]$, we have

$$\left| f''(t) - \frac{f''(x)+f''(b)}{2} \right| \le \frac{1}{2} \bigvee_x^b (f'')$$

and thus,

$$\int_x^b (t-b)^2 \left| f''(t) - \frac{f''(x)+f''(b)}{2} \right| dt \le \frac{(b-x)^3}{6} \bigvee_x^b (f''). \tag{4.5}$$

If we substitute the inequalities (4.4) and (4.5) in (4.3), we obtain the first inequality in (4.2). The second inequality follows by Hölder's inequality

$$mn + pq \le (m^\alpha + p^\alpha)^{\frac{1}{\alpha}} \left(n^\beta + q^\beta\right)^{\frac{1}{\beta}},\ m, n, p, q \ge 0 \text{ and } \alpha > 1 \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

□

Corollary 8 *Under assumptions of Theorem 5 with $x = \frac{a+b}{2}$, we have the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{48} f''\left(\frac{a+b}{2}\right) - \frac{(b-a)}{96}\left[f''(a) + f''(b)\right] \right|$$
$$\le \frac{(b-a)^2}{96} \bigvee_a^b (f'').$$

5 Inequalities for Functions Whose Second Derivatives are Lipschitzian

Theorem 6 *Let : $f : [a, b] \to \mathbb{C}$ be a twice differantiable function on I° and $[a, b] \subset I^\circ$. If the second derivative f'' is Lipschitzian with the constant $L_1(x)$ on $[a, x]$ and $L_2(x)$ on $[x, b]$, then we have*

$$\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt-\frac{(x-a)^3+(b-x)^3}{12(b-a)}f''(x)\right.$$

$$\left.-\frac{1}{12(b-a)}\left[(x-a)^3 f''(a)+(b-x)^3 f''(b)\right]\right|$$

$$\leq\frac{(b-a)^3}{12}\left[\left(\frac{x-a}{b-a}\right)^4 L_1(x)+\left(\frac{b-x}{b-a}\right)^4 L_2(x)\right]$$

$$\leq\frac{(b-a)^3}{12}\begin{cases}\left[\left(\frac{x-a}{b-a}\right)^4+\left(\frac{b-x}{b-a}\right)^4\right]\max\{L_1(x),L_2(x)\}, \\ \left[\left(\frac{x-a}{b-a}\right)^{4p}+\left(\frac{b-x}{b-a}\right)^{4p}\right]^{\frac{1}{p}}\left[(L_1(x))^q+(L_1(x))^q\right]^{\frac{1}{q}} \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \max\left\{\left(\frac{x-a}{b-a}\right)^4,\left(\frac{b-x}{b-a}\right)^4\right\}[L_1(x)+L_2(x)],\end{cases} \tag{5.1}$$

for any $x\in[a,b]$.

Proof Since f'' is Lipschitzian with the costant $L_1(x)$ on $[a,x]$, we get

$$\begin{aligned}\left|f''(t)-\frac{f''(a)+f''(x)}{2}\right| &\leq\frac{\left|2f''(t)-f''(a)-f''(x)\right|}{2}\\ &\leq\frac{\left|f''(t)-f''(a)\right|+\left|f''(x)-f''(t)\right|}{2}\\ &\leq\frac{1}{2}L_1(x)\left[|t-a|+|x-t|\right]\\ &=\frac{1}{2}L_1(x)(x-a).\end{aligned}$$

Thus,

$$\begin{aligned}\int_a^x (t-a)^2\left|f''(t)-\frac{f''(a)+f''(x)}{2}\right|dt &\leq\frac{1}{2}L_1(x)(x-a)\int_a^x (t-a)^2\,dt\\ &\leq\frac{1}{6}(x-a)^4 L_1(x).\end{aligned} \tag{5.2}$$

Similarly, f'' is Lipschitzian with the costant $L_2(x)$ on $[x, b]$, we get

$$\left| f''(t) - \frac{f''(x) + f''(b)}{2} \right| \leq \frac{1}{2} L_2(x)(b - x)$$

and thus,

$$\int_x^b (t-b)^2 \left| f''(t) - \frac{f''(x) + f''(b)}{2} \right| dt \leq \frac{1}{6}(b-x)^4 L_2(x). \tag{5.3}$$

If we substitute the inequalities (5.2) and (5.3) in (4.3), we obtain the first inequality in (5.1). The second inequalities can be proved as in Theorems 4 and 5. □

Corollary 9 *Under assumption of Theorem 6 with $L_1(x) = L_2(x) = L$, we have*

$$\begin{aligned}
&\left| \left(x - \frac{a+b}{2} \right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t)dt - \frac{(x-a)^3 + (b-x)^3}{12(b-a)} f''(x) \right. \\
&\left. - \frac{1}{12(b-a)} \left[(x-a)^3 f''(a) + (b-x)^3 f''(b) \right] \right| \\
&= \frac{1}{12} \left[\left(\frac{x-a}{b-a} \right)^4 + \left(\frac{b-x}{b-a} \right)^4 \right] L(b-a)^3
\end{aligned} \tag{5.4}$$

for all $x \in [a, b]$.

Corollary 10 *If we choose $x = \frac{a+b}{2}$ in (5.4), we get the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2} \right) - \frac{(b-a)^2}{48} f''\left(\frac{a+b}{2} \right) - \frac{(b-a)}{96} \left[f''(a) + f''(b) \right] \right|$$
$$\leq \frac{1}{192} L(b-a)^3.$$

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Comprehensive Inequalities and Equations Specified by the Mittag-Leffler Functions and Fractional Calculus in the Complex Plane

Hüseyin Irmak and Praveen Agarwal

Abstract Inequalities *or* equations appertaining to (generalized) Mittag-Leffler functions and/or asserted by (generalized) fractional calculus play important roles in themselves and also in their diverse applications in nearly all sciences and engineering. Many inequalities *or* equations involving (one variable and three parameters of) the Mittag-Leffler (type) functions and also (generalized) fractional calculus have been established by several researchers in many different ways. In this investigation, many comprehensive results containing several differential inequalities *and/or* equations (in the complex plane $\mathbb{C}$) in relation with (one variable and three parameters of) the Mittag-Leffler (type) functions given by

$$E^{\gamma}_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!\,\Gamma(n\alpha+\beta)} z^n \quad \big(\beta, \gamma \in \mathbb{C}; \Re e(\alpha) > 0\big),$$

in its kernel, here throughout this investigation, $(\gamma)_n$ being the familiar Pochhammer symbol *or* the shifted factorial, and/or fractional calculus (i.e., differentiation and integration of an arbitrary real or complex order) are presented, for a function $f(z)$, by the familiar differ-integral operator ${}_c\mathcal{D}^{\mu}_z[\cdot]$, defined by

$${}_c\mathcal{D}^{\mu}_z\big[f(z)\big] := \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_c^z \frac{f(\tau)}{(z-\tau)^{1-\mu}} d\tau & \big(c \in \mathbb{R}; \Re e(\mu) < 0\big) \\ \dfrac{d^m}{dz^m}\Big({}_c\mathcal{D}^{\mu-m}_z\big[f(z)\big]\Big) & \big(m-1 \le \Re e(\mu) < m; m \in \mathbb{N}\big), \end{cases}$$

provided that the integral exists, are first established and several consequences of our results are then pointed out.

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1 Introduction, Definitions, and Preliminaries

The basic processes of diffusion, oscillations, relaxation, and wave propagation have been generalized by several researchers by applying *or* introducing fractional calculus in the governing (ordinary *or* partial differential) equations. This leads to super-slow or intermediate processes that, in mathematical physics, we may refer to as fractional phenomena. Recent advances in the theory and applications of fractional differential equations are stimulated by new examples of applications in the fluid mechanics, viscoelasticity, mathematical biology, electrochemistry, physics, and so on. As is known, fractional (differential) equations are very useful tools for modeling (*or* applications) many anomalous phenomena in nature and in the theory of complex systems. More particularly, the main physical purpose for adopting and investigating diffusion equations of fractional order is to describe phenomena of anomalous diffusion usually met in transport processes through complex or disordered systems involving (for instance) fractal media.

Motivated essentially by the success of the applications of (generalized) Mittag-Leffler functions in many areas of science and engineering, several authors present, in a unified manner, a detailed account or rather a brief survey of the Mittag-Leffler function, generalized Mittag-Leffler functions, (one variable and three parameters) of Mittag-Leffler (type) functions, and their interesting and also useful properties. Many applications of the Mittag-Leffler functions in certain areas of physical and applied sciences are also demonstrated. During the last two decades, this function has come into prominence after about nine decades of its discovery by a Swedish Mathematician Mittag-Leffler, due to the vast potential of its applications in solving the problems of physical, biological, engineering, and earth sciences, and so forth.

Our analysis of these phenomena carried out by means of certain equations *or* inequalities constituted by (generalized) fractional calculus leads to several special functions in one variable and several parameters of the Mittag-Leffler-type. (For the details and also, for example, see the works given in [3, 4, 6–8, 17, 18, 22–24].)

The aim of this investigation, as a novel investigation, certain complex equations and inequalities consisting of (one variable and three parameters of) the Mittag-Leffler (type) functions and/or (generalized) fractional calculus, that is, that derivative(s), are first presented and several useful consequences of the related complex equations and/or inequalities are also emphasized. For example, in the near time, certain interesting and comprehensive results consisting of several inequalities in relation with the (one variable and several parameters) Mittag-Leffler (type) functions were saved to the literature as novel investigation consisting of various theoretical *or* elementary results appertaining to relation with some applications of fractional calculus and Mittag-Leffler (type) functions in the complex plane. See, for their details, [5] and also [8].

For the main purpose indicated above, there is a need to introduce *or* recall certain well-known definitions and also notations.

Firstly, let us denote by the notations $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{U}$ the set of natural numbers, the set of real numbers, the set of complex numbers and unit open disk, namely $\big\{z \in \mathbb{C} : |z| < 1\big\}$, respectively.

Also let $\mathbb{D} := \mathbb{U} - \{0\}$, $\mathbb{C}^* := \mathbb{C} - \{0\}$, $\mathbb{R}^* := \mathbb{R} - \{0\}$ and $\mathbb{N}^* := \mathbb{N} - \{0\}$.

We now begin by recalling the (one variable and three parameters) Mittag-Leffler (type) function is denoted by $E^{\gamma}_{\alpha,\beta}(z)$ and also defined by

$$E^{\gamma}_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!\,\Gamma(n\alpha + \beta)} z^n \tag{1.1}$$

where $z \in \mathbb{C}$, $\beta \in \mathbb{C}$, $\gamma \in \mathbb{C}$, $\Re e(\alpha) > 0$, and $\Re e(\beta) > 0$ and (*here and throughout this work*) $(\omega)_\tau$ denotes the familiar Pochhammer symbol *or* the shifted factorial, since

$$(1)_n = n! \ \big(n \in \mathbb{N}_0\big),$$

defined (for $\omega \in \mathbb{C}$, $\tau \in \mathbb{C}$ and in terms of the familiar Gamma function) by

$$(\omega) := \frac{\Gamma(\omega + \tau)}{\Gamma(\tau)}$$

$$= \begin{cases} 1 & \text{when } \tau = 0 \text{ and } \omega \in \mathbb{C}^* \\ \omega(\omega + 1) \cdots (\omega + n - 1) & \text{when } \tau = n \in \mathbb{N} \text{ and } \omega \in \mathbb{C}. \end{cases}$$

For $\gamma = 1$, we then recover from (1.1) (one variable and two parameters) of the Mittag-Leffler function denoted by $E_{\alpha,\beta}(z)$ and also defined by

$$E_{\alpha,\beta}(z) := E^{1}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \tag{1.2}$$

where $z \in \mathbb{C}$, $\beta \in \mathbb{C}$, $\alpha \in \mathbb{C}$, $\Re e(\alpha) > 0$ and $\Re e(\beta) > 0$.

Moreover, for $\beta := 1$ and $\gamma := 1$, we also get the (classical) Mittag-Leffler function *or* (one variable and one parameter) of the Mittag-Leffler function denoted by $E_\alpha(z)$ and also defined by

$$E_\alpha(z) := E^1_{\alpha,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}, \tag{1.3}$$

where $z \in \mathbb{C}$, $\alpha \in \mathbb{C}$ and $\Re e(\alpha) > 0$.

For a detailed account of the various properties, generalizations, and applications of the Mittag-Leffler functions, the researcher may refer to the recent investigations, for example, Gorenflo et al. [3, 4] and Kilbas et al. [10–12]. The Mittag-Leffler function (1.3) and some of its various generalizations have only recently been determined numerically in the whole complex plane [12, 21, 24]. By means of the series representation, a generalization of the Mittag-Leffler function given by (1.1) was introduced by Prabhakar [16]. Indeed, for the various special results relating to the functions in (1.1)–(1.3), one looks overall works concerning to the (one variable and three parameters) Mittag-Leffler (type) functions in the references.

Next, the most frequently encountered tools in the theory of fractional calculus (i.e., differentiation and integration of an arbitrary real or complex order) are presented by the familiar differ-integral operator ${}_c\mathcal{D}^\mu_z[\cdot]$, defined by

$${}_c\mathcal{D}^\mu_z\big[f(z)\big] := \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_c^z \frac{f(\tau)}{(z-\tau)^{1-\mu}} d\tau & \text{when } \Re e(\mu) < 0 \\ \dfrac{d^m}{dz^m}\Big({}_c\mathcal{D}^{\mu-m}_z\big[f(z)\big]\Big) & \text{when } m-1 \le \Re e(\mu) < m, \end{cases}$$

provided that the integral exists, *where* $c \in \mathbb{R}$ and $m \in \mathbb{N}$.

For $c := 0$, the operator $\mathcal{D}^\mu_z[\cdot]$ given by

$$\mathcal{D}^\mu_z\big[f(z)\big] := {}_0\mathcal{D}^\mu_z\big[f(z)\big] \quad \big(\mu \in \mathbb{C}\big)$$

corresponds essentially to the classical Riemann–Liouville fractional derivative (*or* integral) of order $\mu\big(or - \mu\big)$.

In special, for the function $f(z) = z^\kappa$, of course, with $c := 0$, it is easily seen that

$$\mathcal{D}^\mu_z\big[z^\kappa\big] := \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\mu+1)} z^{\kappa-\mu} \quad \big(\Re e(\kappa) > -1\big), \tag{1.4}$$

and, more importantly, that

$$\Gamma(\gamma)\, E^\gamma_{\alpha,\beta}(z) = \mathcal{D}^{\gamma-1}_z\Big(z^{\gamma-1} E_{\alpha,\beta}(z)\Big) \tag{1.5}$$

and

$$\alpha\Big(E^\gamma_{\alpha,\beta}(z)\Big)' = E^\gamma_{\alpha,\beta-1}(z) + \big(1-\beta\big) E^\gamma_{\alpha,\beta}(z), \tag{1.6}$$

where $z, \alpha, \beta, \gamma \in \mathbb{C}$, $\Re e(\alpha) > 0$, $\Re e(\beta) > 0$, $\Re e(\gamma) > 0$, Moreover, there are several interesting and/or important relationships among the definitions identified by the series representation given by (1.1), (1.2), and (1.3). We want to point out a number of those special relationships that we find useful for certain special consequences of our main results. Some of them are in the following forms:

$$E_{\alpha}(z) = E^{1}_{\alpha,1}(z) \tag{1.7}$$

$$\big(z, \alpha \in \mathbb{C}\,;\ \Re e(\alpha) > 0\big),$$

$$E_{\alpha,\beta}(z) = E^{1}_{\alpha,\beta}(z) \tag{1.8}$$

$$\big(z, \alpha, \beta \in \mathbb{C}\,;\ \Re e(\alpha) > 0\,;\ \Re e(\beta) > 0\big),$$

$$E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)} \tag{1.9}$$

$$\big(z, \alpha, \beta \in \mathbb{C}\,;\ \Re e(\alpha) > 0\,;\ \Re e(\beta) > 0\big),$$

$$z^{m}E_{\alpha,\beta+m\alpha}(z) = E_{\alpha,\beta}(z) - \sum_{n=0}^{m-1} \frac{z^{n}}{\Gamma(\beta + n\alpha)} \tag{1.10}$$

$$\big(m \in \mathbb{N}^{*}\,;\ z, \alpha, \beta \in \mathbb{C}\,;\ \Re e(\alpha) > 0\,;\ \Re e(\beta) > 0\big),$$

$$E_{\alpha,\beta}(z) = \beta E_{\alpha,1+\beta}(z) + \alpha z \frac{d}{dz}\Big(E_{\alpha,1+\beta}(z)\Big) \tag{1.11}$$

$$\big(z, \alpha, \beta \in \mathbb{C}\,;\ \Re e(\alpha) > 0\,;\ \Re e(\beta) > 0\big),$$

$$\frac{d^{n}}{dz^{n}}\Big(E_{\alpha,\beta}\big(z^{\alpha}\big)\Big) = z^{\beta-n-1}E_{\alpha,\beta-n}\big(z^{\alpha}\big) \tag{1.12}$$

$$\big(\, n \in \mathbb{N}\,;\ z, \alpha, \beta \in \mathbb{C}\,;\ \Re e(\alpha) > 0\,;\ \Re e(\beta) > n\big),$$

$$\frac{d^{n}}{dz^{n}}\Big(z^{n-\beta}E_{n,\beta}\big(\kappa z^{-n}\big)\Big) = \kappa(-1)^{n}z^{-n-\beta}E_{n,\beta}\big(\kappa z^{-n}\big) \tag{1.13}$$

$$\big(n \in \mathbb{N}\,;\ \kappa \in \mathbb{C}^{*}\,;\ z, \alpha, \beta \in \mathbb{C}\,;\ \Re e(\alpha) > 0\,;\ \Re e(\beta) > 0\big),$$

$$\alpha z \frac{d}{dz}\Big(E_{\alpha,\beta}(z)\Big) = E_{\alpha,\beta-1}(z) + (1-\beta)E_{\alpha,\beta}(z) \tag{1.14}$$

$$\Big(z, \alpha, \beta \in \mathbb{C}\ ;\ \Re e(\alpha) > 0\ ;\ \Re e(\beta) > 0\Big),$$

$$\alpha\gamma E^{1+\gamma}_{\alpha,\beta}(z) = (1+\alpha\gamma-\beta)E^{\gamma}_{\alpha,\beta}(z) + E^{\gamma}_{\alpha,\beta-1}(z) \tag{1.15}$$

$$\Big(z \in \mathbb{C}^*\ ;\ z, \alpha, \beta, \gamma \in \mathbb{C}\ ;\ \Re e(\alpha) > 0\ ;\ \Re e(\beta) > 0\ ;\ \Re e(\gamma) > 0\Big),$$

$$\frac{d^n}{dz^n}\Big(E_{\alpha}(z)\Big) = n!E^{1+n}_{\alpha,1+n\alpha}(z) \tag{1.16}$$

$$\Big(n \in \mathbb{N}\ ;\ z \in \mathbb{C}^*\ ;\ \alpha \in \mathbb{C}\ ;\ \Re e(\alpha) > 0\Big),$$

$$\frac{d^n}{dz^n}\Big(E_{\alpha,\beta}(z)\Big) = n!E^{1+n}_{\alpha,\beta+n\alpha}(z) \tag{1.17}$$

$$\Big(n \in \mathbb{N}\ ;\ z \in \mathbb{C}^*\ ;\ \alpha, \beta \in \mathbb{C}\ ;\ \Re e(\alpha) > 0\ ;\ \Re e(\beta) > 0\Big),$$

$$\frac{d^n}{dz^n}\Big(z^{\beta-1}E_{\alpha,\beta}\big(\kappa z^{\alpha}\big)\Big) = z^{\beta-n-1}E^{\gamma}_{\alpha,\beta-n}\big(\kappa z^{\alpha}\big) \tag{1.18}$$

$$\Big(n \in \mathbb{N}\ ;\ \alpha, \beta \in \mathbb{C}\ ;\ z, \kappa \in \mathbb{C}^*\ ;\ \Re e(\alpha) > 0\ ;\ \Re e(\beta) > n\Big),$$

$$\frac{d^n}{dz^n}\Big(E^{\gamma}_{\alpha,\beta}(z)\Big) = (\gamma)_n E^{n+\gamma}_{\alpha,\beta+n\alpha}(z) \tag{1.19}$$

$$\Big(n \in \mathbb{N}\ ;\ z \in \mathbb{C}^*\ ;\ \alpha, \beta, \gamma \in \mathbb{C}\ ;\ \Re e(\alpha) > 0\ ;\ \Re e(\beta) > 0\ ;\ \Re e(\gamma) > 0\Big),$$

$$\frac{d^n}{dz^n}\Big(z^{\beta-1}E^{\gamma}_{\alpha,\beta}\big(\kappa z^{\alpha}\big)\Big) = z^{\beta-n-1}E^{\gamma}_{\alpha,\beta-n}\big(\kappa z^{\alpha}\big) \tag{1.20}$$

$$\Big(n \in \mathbb{N}\ ;\ \alpha, \beta \in \mathbb{C}\ ;\ z, \kappa \in \mathbb{C}^*\ ;\ \Re e(\alpha) > 0\ ;\ \Re e(\beta) > n\ ;\ \Re e(\gamma) > 0\Big),$$

and

$$\frac{d^n}{dz^n}\Big(z^{\beta-1}\Phi\big(\gamma, \beta; \kappa z\big)\Big) = \frac{\Gamma(\beta)}{\Gamma(\beta-n)}z^{\beta-n-1}\Phi\big(\gamma; \beta-n; \kappa z\big) \tag{1.21}$$

$$\big(n \in \mathbb{N}\ ;\ \alpha, \beta \in \mathbb{C}\ ;\ z, \kappa \in \mathbb{C}^*\ ;\ \Re e(\alpha) > 0\ ;\ \Re e(\beta) > n\ ;\ \Re e(\gamma) > 0\big).$$

For some of the earlier results, which are also consist of certain relationships indicated by (1.6)–(1.22), specially, see the recent works given by [5, 8].

Especially, for positive integer $m \in \mathbb{N}$, $E^{\gamma}_{\alpha,\beta}(z)$ coincides with the generalized hypergeometric function with $p = 1$ and $q = m$, apart from a constant multiplier given by

$$\Gamma(\beta)E^{\gamma}_{m,\beta}(z) = {}_1F_m\Big(\gamma; \frac{\beta}{m}, \frac{1+\beta}{m}, \ \cdots, \ \frac{m-1+\beta}{m}; \frac{z}{m^m}\Big),$$

and, as its special case, for $m := 1$, $E^{\gamma}_{1,\beta}(z)$ also coincides with the Kummer confluent hypergeometric function $\Phi(\gamma, \beta; z)$ that is that

$$\Gamma(\beta)E^{\gamma}_{1,\beta}(z) = \Phi(\gamma, \beta; z), \tag{1.22}$$

where $\gamma \in \mathbb{C}$, $\beta \in \mathbb{C}$, $z \in \mathbb{C}$, $\Re e(\gamma) > 0$ and $\Re e(\beta) > 0$.

As we indicated before, in recent years, a great deal of literature has appeared discussing the application of the aforementioned fractional calculus operators in a number of areas of mathematical analysis (cf., e.g., [1, 2, 7, 13, 17]; see also (for example) the results in [1, 2, 7]. We also note that the fractional calculus operator (1.3) was investigated earlier by Kilbas et al. [12] and its generalization involving a family of more general Mittag-Leffler-type functions than $E_{\alpha,\beta}(z)$ was studied recently by Srivastava and Tomovski [22]. (For more information and also, for example, see the results in [1, 7, 19, 22, 23].)

For the main results, there is a need to recall the well-known assertions, which are Lemma 1.1 given by [9] (see, also [14]) and Lemma 1.2 given by [15], below.

Lemma 1.1 *Let $w(z)$ defined by*

$$w(z) = a_n z^n + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots \quad (a_n \neq 0; n \in \mathbb{N})$$

be an analytic function in $\mathbb{U}$. If the maximum value of $|w(z)|$ on the circle $|z| = r$ is attained at $z = z_0 \in \mathbb{U}$, then

$$zw'(z)\big|_{z=z_0} = \rho w(z)\big|_{z=z_0}\ , \tag{1.23}$$

where $\rho \geq n$ and $n \in \mathbb{N}$.

Lemma 1.2 *Let $q(z)$ be an analytic function in $\mathbb{U}$ with $q(0) = 1$. If there exists a point z_0 in $\mathbb{U}$ such that*

$$\Re e\Big(q(z)\Big) > 0\ \Big(|z| < |z_0|\Big)\ ,\ \Re e\Big(q(z_0)\Big) = 0\ \ and\ \ q(z_0) \neq 0, \tag{1.24}$$

then

$$q(z_0) = ia \ \ and \ \ zq'(z)\big|_{z=z_0} = i\rho\Big(a + \frac{1}{a}\Big)q(z)\big|_{z=z_0}, \tag{1.25}$$

where $a \in \mathbb{R}^*$ *and* $\rho \geq 1/2$.

2 Main Results and Conclusions

We now begin by setting and then by proving the main results consisting of several comprehensive consequences dealing with certain complex equations and inequalities constituted by the (one variable and three parameters) Mittag-Leffler (type) functions and the (generalized) fractional calculus, which are given by theorems in the following forms.

Theorem 2.1 *Let* $\phi(z)$ *be an analytic function that satisfies the inequality:*

$$\big|\phi(z)\big| < \rho(1+\kappa) \quad \big(\rho > 0; \kappa \in \mathbb{N}; z \in \mathbb{U}\big) \tag{2.1}$$

and also let the function $W(z)$ *be in the form:*

$$W(z) = z^{\kappa} E^{\gamma}_{\alpha,\beta}(z) \quad \big(\kappa \in \mathbb{N}\big), \tag{2.2}$$

where $E^{\gamma}_{\alpha,\beta}(z)$ *is the (three parameters) Mittag-Leffler (type) function given as in (1.1).*

If an analytic function $W := W(z)$ is a any solution of the following (fractional type) complex equation:

$$z\mathcal{D}^{1+\mu}_z\big[W\big] + (1+\mu)\mathcal{D}^{\mu}_z\big[W\big] - \frac{\phi(z)}{z^{\mu}} = 0, \tag{2.3}$$

then

$$\Big|\mathcal{D}^{\mu}_z\big[W\big]\Big| < \frac{\rho}{|z^{\mu}|} \quad \big(\rho > 0; 0 \leq \mu < 1; z \in \mathbb{D}\big), \tag{2.4}$$

where and the values of all complex powers above are taken to be as their principal values.

Proof Firstly, let the functions $E^{\gamma}_{\alpha,\beta}(z)$ and $W(z)$ be defined by (1.1) and (2.2), respectively. By the help of (1.1) and (1.4), the following:

$$\begin{aligned}\mathcal{D}^{\mu}_z\big[W\big] \equiv \mathcal{D}^{\mu}_z\big[W(z)\big] &= z^{-\mu}\left(\sum_{n=0}^{\infty} \Omega^{\alpha,\beta,\gamma}_{\mu,\kappa}(n)z^{n+\kappa}\right)\\ &= z^{-\mu}\Big(\Omega^{\alpha,\beta,\gamma}_{\mu}(\kappa;0)z^{\kappa} + \Omega^{\alpha,\beta,\gamma}_{\mu}(\kappa;1)z^{1+\kappa} + \cdots\Big),\end{aligned}$$

is then determined, *where*

$$\Omega_{\mu}^{\alpha,\beta,\gamma}(\kappa;n) := \frac{(\gamma)_n \Gamma(n+\kappa+1)}{n!\Gamma(n\alpha+\beta),\ \Gamma(n+\kappa-\mu+1)}$$

for all $n \in \mathbb{N}_0$ and $\kappa \in \mathbb{N}$.

Let us now define a function $w(z)$ in the form:

$$z^{\mu}\mathcal{D}_z^{\mu}\big[W\big] = w(z) \quad \big(0 \leq \mu < 1; \kappa \in \mathbb{N}; z \in \mathbb{U}\big). \tag{2.5}$$

Since $\kappa \in \mathbb{N}$, it is obvious that the function $w(z)$ is analytic in the domain $\mathbb{U}$. It follows from (2.5), we then obtain

$$z\Big(z^{\mu}\mathcal{D}_z^{\mu}\big[W\big]\Big)' = \mu z^{\mu}\mathcal{D}_z^{\mu}\big[W\big] + z^{1+\mu}\mathcal{D}_z^{1+\mu}\big[W\big] = zw'(z). \tag{2.6}$$

By combining the identities given by (2.5) and (2.6), we also get that

$$\begin{aligned} &z^{1+\mu}\mathcal{D}_z^{1+\mu}\big[W\big] + (1+\mu)z^{\mu}\mathcal{D}_z^{\mu}\big[W\big] \\ &= w(z) + zw'(z)\big(=: \phi(z)\big). \quad (say.) \end{aligned} \tag{2.7}$$

It is clear that the function $\phi(z)$ satisfies the fractional complex-type equation given by (2.3), when one takes in consideration the function W as in the form given by (2.2).

We now suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max\Big\{|w(z)| : |z| \leq |z_0|\Big\} = |w(z_0)| = \rho \quad (\rho > 0),$$

then the assertion (1.15) (of Lemma 1.1) gives us

$$w(z_0) = \rho e^{i\theta} \;\; (0 \leq \theta < 2\pi) \quad \text{and} \quad z_0 w'(z_0) = cw(z_0),$$

where c is real and $c \geq \kappa$ ($\kappa \in \mathbb{N}$).

We thus obtain that

$$\begin{aligned} \big|\phi(z_0)\big| &= \big|w(z_0) + z_0 w'(z_0)\big| \\ &= (1+c)\big|w(z_0)\big| = \rho(1+c) \geq \rho(1+\kappa), \end{aligned}$$

which is a contradiction with the assumption given by (2.1). Therefore, there is not any z_0 in the domain $\mathbb{U}$ such that $|w(z_0)| = \rho$ ($\rho > 0$). This means that $|w(z)| < \rho$ ($\rho > 0$) for all $z \in \mathbb{U}$. Hence, the equality given in (2.5) immediately yields that

$$\left|z^{\mu}\mathcal{D}_z^{\mu}\big[W\big]\right| = |w(z)| < \rho \quad \big(\rho > 0; z \in \mathbb{U}\big),$$

which completes the proof of Theorem 2.1.

Theorem 2.2 *Let $\phi(z)$ be an analytic function that satisfies the inequality given by (2.1) and also let the function $\hat{W}(z)$ be in the form:*

$$\hat{W}(z) := z^{\kappa}\, E_{\alpha,\beta}(z) \quad \big(\kappa \in \mathbb{N}\big), \tag{2.8}$$

where $E_{\alpha,\beta}(z)$ is the (two parameters) Mittag-Leffler (type) function given as in (1.2).

If an analytic function $\hat{W} := \hat{W}(z)$ is a any solution of the following (fractional-type complex) equation:

$$z\mathcal{D}_z^{1+\mu}\big[\hat{W}\big] + (1+\mu)\mathcal{D}_z^{\mu}\big[\hat{W}\big] - \frac{\phi(z)}{z^{\mu}} = 0,$$

then

$$\left|\mathcal{D}_z^{\mu}\big[\hat{W}\big]\right| < \frac{\rho}{|z^{\mu}|} \quad \big(\rho > 0; 0 \le \mu < 1; z \in \mathbb{D}\big),$$

where the values of the related complex powers are taken to be as their principal values.

Proof For the proof of Theorem 2.2, it is enough to choose the value of the parameter γ as $\gamma := 1$ in the proof of Theorem 2.1 and to take into account the well-known identity given by (1.8).

Theorem 2.3 *Let $\phi(z)$ be an analytic function and satisfy the inequality given by (2.1) and also let the function $\tilde{W}(z)$ be in the form:*

$$\tilde{W}(z) := E_{\alpha,\beta}^{\gamma}(z), \tag{2.9}$$

where $E_{\alpha,\beta}^{\gamma}(z)$ is the (three parameters) Mittag-Leffler (type) function given as in (1.1).

If an analytic function $\tilde{W} := \tilde{W}(z)$ is a any solution of the following (fractional-type complex) equation:

$$z\Gamma(\gamma)\,\tilde{W}' + \Gamma(1+\gamma)\,\tilde{W} - \frac{\phi(z)}{z^{\gamma}} = 0,$$

then

$$\left|\Gamma(\gamma)z^{\gamma}\,\tilde{W}(z)\right| < \rho\,|z| \quad \big(\rho > 0; \gamma \in \mathbb{N}; z \in \mathbb{D}\big),$$

where the values of the related powers are taken to be as their principal values.

Proof By setting $\kappa := \gamma - 1$ and $\mu := \gamma - 1$ in Theorem 2.1 and also by taking in consideration the well-known identity given by (1.5), the desired proof of Theorem 2.3 can be easily obtained.

Theorem 2.4 *Let $\phi(z)$ be an analytic function that satisfies any one of the inequalities:*

$$\Im m\Big(\phi(z)\Big) = 0 \quad and \quad \Re e\Big(\phi(z)\Big) \geq -\frac{1}{2} \quad \big(z \in \mathbb{U}\big) \tag{2.10}$$

and also let the function $W(z)$ be defined as in (2.2) with $\kappa := 1$.

If the function $W := W(z)$ is any solution of the following (complex fractional type) equation:

$$z^{1+\mu}\mathcal{D}_z^{1+\mu}\big[W\big] + \mu z^{\mu}\mathcal{D}_z^{\mu}\big[W\big] - \Gamma(\beta)\Gamma(2-\mu)z\phi(z) = 0, \tag{2.11}$$

then

$$\Re e\left(\frac{z^{\mu-1}\mathcal{D}_z^{\mu}\big[W\big]}{\Gamma(\beta)\Gamma(2-\mu)}\right) > 0 \tag{2.12}$$

$$\big(0 \leq \mu < 1;\, \Re e(\beta) > 0;\, z \in \mathbb{D}\big),$$

where the values of the related complex powers here are taken to be as their principal values.

Proof Firstly, here and throughout the proof of this theorem, let the functions $E_{\alpha,\beta}^{\gamma}(z)$ and $W := W(z)$ be defined as in (1.1) and (2.2) with $\kappa := 1$. Then, by the help of (1.1) and (1.4), we easily calculate that

$$\begin{aligned}\mathcal{D}_z^{\mu}\big[W\big] &= z^{-\mu}\left(\sum_{n=0}^{\infty}\Omega_{\mu}^{\alpha,\beta,\gamma}(n)z^{n+1}\right)\\ &= z^{-\mu}\Big(\Omega_{\mu}^{\alpha,\beta,\gamma}(0)z + \Omega_{\mu}^{\alpha,\beta,\gamma}(1)z^2 + \cdots\Big),\end{aligned}$$

where

$$\Omega_{\mu}^{\alpha,\beta,\gamma}(n) := \frac{(\gamma)_n\Gamma(n+1)}{n!\,\Gamma(n\alpha+\beta)\Gamma(n-\mu+2)}$$

for all $n \in \mathbb{N}_0$.

Next, we define a function $q(z)$ in the form:

$$z^{\mu-1}\mathcal{D}_z^{\mu}\Big(zE_{\alpha,\beta}^{\gamma}(z)\Big) = \Gamma(\beta)\Gamma(2-\mu)q(z)z^{1-\mu}, \tag{2.13}$$

where $0 \leq \mu < 1$, $\Re e\big(\beta\big) > 0$ and $z \in \mathbb{U}$. Obviously, $q(z)$ is an analytic function in $\mathbb{U}$ with $q(0) = 1$. From the statement (2.13), we also obtain

$$z\Big(\mathcal{D}_z^{\mu}\big[W\big]\Big)' = \Gamma(\beta)\Gamma(2-\mu)z^{1-\mu}\Big((1-\mu)q(z) + zq'(z)\Big) \tag{2.14}$$

After simple calculations, (2.13) and (2.14) give us

$$z\mathcal{D}_z^{1+\mu}\big[W\big] + \mu\mathcal{D}_z^{\mu}\big[W\big] = \Gamma(\beta)\Gamma(2-\mu)z^{1-\mu}\Big(zq'(z) + q(z)\Big) \tag{2.15}$$

$$\big(=: \Gamma(\beta)\Gamma(2-\mu)z^{1-\mu}\phi(z)\big).\ (say.)$$

Clearly, the function $\phi(z)$ satisfies the (fractional complex) equation given by (2.11).

We now assume that there exists a point $z_0 \in \mathbb{U}$ satisfying the hypotheses given by (1.24). Under the conditions $\rho \geq \frac{1}{2}$ and $a \in \mathbb{R}^*$, from (1.24) of Lemma 1.2, we then get

$$q(z_0) = ia \quad \text{and} \quad zq'(z)\big|_{z=z_0} = i\rho\big(a + 1/a\big)q(z)\big|_{z=z_0}.$$

If we take into consideration the hypotheses (above) in (2.15), since

$$\phi(z)\big|_{z=z_0} = zq'(z) + q(z)\big|_{z=z_0} = ia - \rho\big(1 + a^2\big),$$

we easily obtain that

$$\Im m\Big(\phi(z_0)\Big) = a \neq 0$$

and

$$\Re e\Big(\phi(z_0)\Big) = -\rho\big(1 + a^2\big) \leq -\frac{1 + a^2}{2} < -\frac{1}{2},$$

which are contradictions with the assumptions given by (2.10), respectively. Hence, the equality in (2.13) yields that

$$\Re e\Big(q(z)\Big) = \Re e\left(\frac{z^{\mu-1}\mathcal{D}_z^{\mu}\big(z\,E_{\alpha,\beta}^{\gamma}(z)\big)}{\Gamma(\beta)\,\Gamma(2-\mu)}\right) > 0,$$

where $0 \leq \mu < 1$, $\Re e(\beta) > 0$, $z \in \mathbb{D}$. This completes the proof of Theorem 2.4.

Theorem 2.5 *Let $\phi(z)$ be an analytic function that satisfies any one of the inequalities given by (2.7) and also let the function $\hat{W}(z)$ be defined as in the form (2.8) with* $\kappa := 1$.

If the function $\hat{W} := \hat{W}(z)$ is any solution of the following (fractional-type complex) equation:

$$z^{1+\mu}\mathcal{D}_z^{1+\mu}\big[\hat{W}\big] + \mu z^{\mu}\mathcal{D}_z^{\mu}\big[\hat{W}\big] - \Gamma(\beta)\,\Gamma(2-\mu)\,z\phi(z) = 0,$$

then

$$\Re e\left(\frac{z^{\mu-1}\mathcal{D}_z^{\mu}\big[\hat{W}\big]}{\Gamma(\beta)\,\Gamma(2-\mu)}\right) > 0$$

$$\big(0 \leq \mu < 1; \Re e(\beta) > 0; z \in \mathbb{D}\big),$$

where the values of the complex powers there are taken to be as their principal values.

Proof By setting $\gamma := 1$ in the proof of Theorem 2.4, the proof of Theorem 2.5 can be easily achieved.

By a simple investigation, it can be easily seen the complex-valued functions defined by the special series given in (1.1), (1.2), and (1.3) contain several specific complex (elementary) functions which are analytic in the open unit disk $\mathbb{U}$, in the punctured open unit disk $\mathbb{D}$ *or* in the complex plane $\mathbb{C}$. All right, it is not possible to reveal all of them. But, especially, we want to bring out a number of them into the open for the readers. Namely, some particular cases of the Mittag-Leffler (type) functions are presented by the following forms.

$$E_0(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \big(z \in \{z \in \mathbb{C} : |z| < 1\}\big), \tag{2.16}$$

$$E_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp(z) \quad (z \in \mathbb{C}), \tag{2.17}$$

$$E_1\big(-z^2\big) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{n!} = \exp\big(-z^2\big) \quad (z \in \mathbb{C}), \tag{2.18}$$

$$E_2(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!} = \cos z \quad (z \in \mathbb{C}), \tag{2.19}$$

$$E_2(z^2) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = \cosh z \quad \big(z \in \mathbb{C}\big) \tag{2.20}$$

and

$$\begin{aligned} E_{1/2}\big(\pm\sqrt{z}\big) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{n/2}}{\Gamma(n/2+1)} \\ &= \exp(z)\big[1 + \text{erf}\big(\pm\sqrt{z}\big)\big] \\ &= \exp(z)\text{erfc}\big(\pm\sqrt{z}\big), \end{aligned} \tag{2.21}$$

where $\text{erfc}(z)$ denotes the complementary error function and the error function $\text{erf}(z)$ defined by

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-t^2} dt \quad (z \in \mathbb{C}).$$

We also note that

$$E^1_{1,1}\left(z^3\right) = E_{1,1}\left(z^3\right) = E_1\left(z^3\right) = \exp\left(z^3\right) \quad (z \in \mathbb{C}), \tag{2.22}$$

$$E^1_{1,2}(z) = E_{1,2}(z) = \frac{e^z - 1}{z} \quad (z \in \mathbb{C} - \{0\}), \tag{2.23}$$

$$E_{2,2}\left(z^2\right) = \frac{\sinh z}{z} \quad (z \in \mathbb{C} - \{0\}), \tag{2.24}$$

$$E^1_{1,3}(z) = E_{1,3}\left(z\right) = \frac{e^z - z - 1}{z^2} \quad (z \in \mathbb{C} - \{0\}), \tag{2.25}$$

$$E^1_{2,2}(z) = E_{2,2}\left(-z^2\right) = \frac{\sin z}{z} \quad (z \in \mathbb{C} - \{0\}) \tag{2.26}$$

and so on.

In addition, as various interpretations *or* applications of the (generalized) fractional calculus to the Mittag-Leffler (type) functions, by looking over all theorems, which are Theorems 2.1–2.5, it is easily observed that it also includes several comprehensive results relating to some connections between certain analytic functions and certain complex (differential) equations constituted by generalized fractional derivative operators and Mittag-Leffler (type) functions. Namely, they also involve several consequences consisting of the (three parameters) Mittag-Leffler (type) functions and some types of certain complex equations and inequalities connecting with fractional type functions. Particularly, certain special results of those consequences containing results dealing with elementary complex functions (and also their applications to all theorems) will be interesting for the researchers who have been working on the theory and applications of complex (fractional) differential equation. Accordingly, for example, we want to present only one of them. The other possible consequences of the main results (and also their possible applications which can be related to (1.5)–(1.22) and also certain elementary-special type functions like (2.16)–(2.26), which are here omitted are presented to the attention of the researchers.

Proposition *Let* $\rho > 0$, $z \in \mathbb{D}$ *and let the function* $\tilde{W}$ *be also defined as in (2.9).*

If the inequality:

$$\left| z\frac{d\tilde{W}}{dz} + 2\tilde{W} \right| < \frac{2\rho}{|z|}$$

is satisfied, then the equality:

$$\left|\tilde{W}\right| < \frac{\rho}{|z|}$$

holds, or, equivalently, if the inequality:

$$\left|2 + \Gamma(\beta)\sum_{n=1}^{\infty}\frac{(n+2)(\gamma)_n}{n!\Gamma(n\alpha+\beta)}z^n\right| < \left|\frac{\Gamma(\beta)}{z}\right| 2\rho$$

is also satisfied, then the inequality:

$$\left|\sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!\Gamma(n\alpha+\beta)}z^n\right| < \frac{\rho}{|z|}$$

is also true.

Proof By setting $\kappa := 1$ and $\mu := 0$ in Theorem 2.1, the proof of proposition can be then obtained.

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Novel Results on Hermite–Hadamard Kind Inequalities for η-Convex Functions by Means of (k, r)-Fractional Integral Operators

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Abstract We establish new integral inequalities of Hermite–Hadamard type for the recent class of η-convex functions. This is done via generalized (k, r)-Riemann–Liouville fractional integral operators. Our results generalize some known theorems in the literature. By choosing different values for the parameters k and r, one obtains interesting new results.

Keywords Hermite–Hadamard inequalities · η-convexity · Riemann–Liouville integrals

2010 Mathematics Subject Classification 26A51 · 26D15

1 Introduction

Throughout this work, $I \subset \mathbb{R}$ shall denote an interval and I° the interior of I. We say that a function $g : I \to \mathbb{R}$ is convex if, for every $x, y \in I$ and $\beta \in [0, 1]$, one has

$$g(\beta x + (1 - \beta)y) \leq \beta g(x) + (1 - \beta)g(y). \tag{1}$$

Let $a, b \in I$. For a function g satisfying (1), the following inequalities hold:

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b g(x)\,dx \leq \frac{g(a)+g(b)}{2}. \tag{2}$$

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Result (2) was proved by Hadamard in 1893 [6] and is celebrated in the literature as the Hermite–Hadamard integral inequality for convex functions [2]. Along the years, it has been extended to different classes of convex functions: see, e.g., [3, 8, 15] and references therein.

In 2016, the so-called φ-convexity was introduced [5], subsequently denoted as η-convexity [4, 12]. Let us recall its definition here.

Definition 1 (*See* [5]) A function $g : I \to \mathbb{R}$ is called convex with respect to η (for short, η-convex), if

$$g(\beta x + (1 - \beta)y) \leq g(y) + \beta\eta(g(x), g(y))$$

for all $x, y \in I$ and $\beta \in [0, 1]$.

By taking $\eta(x, y) = x - y$, Definition 1 reduces to the classical notion (1) of convexity. It was further shown in [5] that for every convex function g there exists some η, different from $\eta(x, y) = x - y$, for which the function g is η-convex. The converse is, however, not necessarily true, that is, there are η-convex functions that are not convex.

Example 2 Consider function $g : \mathbb{R} \to \mathbb{R}$ defined piecewise by

$$g(x) = \begin{cases} -x, & x \geq 0, \\ x, & x < 0, \end{cases}$$

and let $\eta : [-\infty, 0] \times [-\infty, 0] \to \mathbb{R}$ be given by $\eta(x, y) = -x - y$. Function g is clearly not convex, but it is easy to see that it is η-convex. Indeed, in [12, Remark 4], it is noted that an η-convex function $g : [a, b] \to \mathbb{R}$ is integrable if η is bounded from above on $g([a, b]) \times g([a, b])$.

For the class of η-convex functions, the following theorem was obtained as an analog of (2).

Theorem 3 (See [5]) *Suppose that $g : I \to \mathbb{R}$ is an η-convex function such that η is bounded from above on $g(I) \times g(I)$. Then, for any $a, b \in I$ with $a < b$,*

$$2g\left(\frac{a+b}{2}\right) - M_\eta \leq \frac{1}{b-a}\int_a^b g(x)\,dx \leq f(b) + \frac{\eta(g(a), g(b))}{2},$$

where M_η is an upper bound of η on $g([a, b]) \times g([a, b])$.

Recently, Rostamian Delavar and De La Sen obtained, among other results, the following theorem associated to η-convex functions [12].

Theorem 4 (See [12]) *Suppose $g : [a, b] \to \mathbb{R}$ is a differentiable function and $|g'|$ is an η-convex function with η bounded from above on $[a, b]$. Then,*

$$\left|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(x)\,dx\right|\leq\frac{1}{8}(b-a)K,$$

where $K=\min\left\{|g'(b)|+\frac{|\eta(g'(a),g'(b))|}{2},\ |g'(a)|+\frac{|\eta(g'(b),g'(a))|}{2}\right\}$.

Still in the same spirit, Khan et al. established in 2017 the following result for η-convex functions via Riemann–Liouville fractional integral operators [9].

Theorem 5 (See [9]) *Let* $g:[a,b]\to\mathbb{R}$ *be a differentiable function on* (a,b) *with* $a<b$. *If* $|g'|$ *is an* η*-convex function on* $[a,b]$, *then for* $\alpha>0$ *the inequality*

$$\left|\frac{g(a)+g(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a_+}^\alpha g(b)+J_{b_-}^\alpha g(a)\right]\right|$$
$$\leq\frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\left(2|g'(b)|+\eta(|g'(a)|,|g'(b)|)\right)$$

holds, where

$$J_{a_+}^\alpha g(x)=\frac{1}{\Gamma_1(\alpha)}\int_a^x(x-t)^{\alpha-1}g(t)\,dt$$

is the left Riemann–Liouville fractional integral and

$$J_{b_-}^\alpha g(x)=\frac{1}{\Gamma_1(\alpha)}\int_x^b(t-x)^{\alpha-1}g(t)\,dt$$

is the right Riemann–Liouville fractional integral.

Fractional calculus is an area under strong development [11, 13]. Sarikaya et al. proposed the following broader definition of the Riemann–Liouville fractional integral operators.

Definition 6 (*See* [13]) The (k,r)-Riemann–Liouville fractional integral operators ${}_k^r\mathcal{J}_{a^+}^\alpha$ and ${}_k^r\mathcal{J}_{b^-}^\alpha$ of order $\alpha>0$, for a real-valued continuous function $g(x)$, are defined as

$${}_k^r\mathcal{J}_{a^+}^\alpha g(x)=\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x(x^{r+1}-t^{r+1})^{\frac{\alpha}{k}-1}t^r g(t)\,dt,\quad x>a,\qquad(3)$$

and

$${}_k^r\mathcal{J}_{b^-}^\alpha g(x)=\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_x^b\left(t^{r+1}-x^{r+1}\right)^{\frac{\alpha}{k}-1}t^r g(t)\,dt,\quad x<b,\qquad(4)$$

where $k>0$, $r\in\mathbb{R}\setminus\{-1\}$, and Γ_k is the k-gamma function given by

$$\Gamma_k(x):=\int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}\,dt,\quad Re(x)>0,$$

with the properties $\Gamma_k(x+k)=x\Gamma_k(x)$ and $\Gamma_k(k)=1$.

For some results related to the operators (3) and (4), we refer the interested readers to [7, 10, 14, 16]. Using these operators, Agarwal et al. established the following Hermite–Hadamard type result for convex functions [1].

Theorem 7 (See [1]) *Let* $\alpha,\ k > 0$ *and* $r \in \mathbb{R} \setminus \{-1\}$. *If* g *is a convex function on* $[a, b]$, *then*

$$g\left(\frac{a+b}{2}\right) \le \frac{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}\left[{}_k^r\mathcal{J}_{a^+}^{\alpha}G(b) + {}_k^s\mathcal{J}_{b^-}^{\alpha}G(a)\right] \le \frac{g(a)+g(b)}{2},$$

where function G *is defined by* (5) *below.*

Inspired by the above works, it is our purpose to obtain here more general integral inequalities associated to η-convex functions via the (k, r)-Riemann–Liouville fractional operators. Theorems 8 and 12 generalize Theorems 7 and 5, respectively (see Remarks 9 and 13). In addition, two more fractional Hermite–Hadamard type inequalities are also established (see Theorems 14 and 15).

2 Main Results

We establish four new results. For this, we start by making the following observations. Let g be a function defined on I with $[a, b] \subset I^{\circ}$ and define functions $G,\ \tilde{g} : [a, b] \to \mathbb{R}$ by

$$\tilde{g}(x) := g(a+b-x) \ \text{ and } \ G(x) := g(x) + \tilde{g}(x). \tag{5}$$

For the fractional operators to be well defined, we shall assume $g \in L_{\infty}[a, b]$. By making use of the substitutions $w = \frac{t-a}{x-a}$ and $w = \frac{b-t}{b-x}$ in (3) and (4), respectively, one gets that

$${}_k^r\mathcal{J}_{a^+}^{\alpha}g(x) = (x-a)\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^1 \frac{(wx+(1-w)a)^r g(wx+(1-w)a)}{\left[x^{r+1}-(wx+(1-w)a)^{r+1}\right]^{1-\frac{\alpha}{k}}}\,dw \tag{6}$$

and

$${}_k^r\mathcal{J}_{b^-}^{\alpha}g(x) = (b-x)\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^1 \frac{(wx+(1-w)b)^r g(wx+(1-w)b)}{\left[(wx+(1-w)b)^{r+1}-x^{r+1}\right]^{1-\frac{\alpha}{k}}}\,dw. \tag{7}$$

Noting that $\tilde{g}\left((1-w)a+wb\right) = g\left(wa+(1-w)b\right)$, we also obtain

$${}_k^r\mathcal{J}_{a^+}^{\alpha}\tilde{g}(x) = (x-a)\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^1 \frac{(wx+(1-w)a)^r g((1-w)x+wa)}{\left[x^{r+1}-(wx+(1-w)a)^{r+1}\right]^{1-\frac{\alpha}{k}}}\,dw \tag{8}$$

and

$$ {}_k^r\mathcal{J}_{b^-}^{\alpha}\tilde{g}(x) = (b-x)\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^1 \frac{(wx+(1-w)b)^r g((1-w)x+wb)}{\left[(wx+(1-w)b)^{r+1}-x^{r+1}\right]^{1-\frac{\alpha}{k}}}\,dw. \tag{9} $$

We are now ready to formulate and prove our first result.

Theorem 8 *Let* $\alpha,\ k > 0$*,* $r \in \mathbb{R}\setminus\{-1\}$*, and* $g : I \to \mathbb{R}$ *be a positive function on* $[a, b] \subset I^\circ$ *with* $a < b$*. If, in addition,* g *is* η*-convex on* $[a, b]$ *with* η *bounded on* $g([a, b]) \times g([a, b])$*, then the* (k, r)*-fractional integral inequality*

$$ \frac{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}\left[{}_k^r\mathcal{J}_{a^+}^{\alpha}G(b) + {}_k^r\mathcal{J}_{b^-}^{\alpha}G(a)\right] \le g(b) + \frac{\eta(g(a),g(b))}{2} $$

holds.

Proof Function g is η-convex on $[a, b]$, which implies, by definition, the following inequalities for $t \in [0, 1]$:

$$ g(ta+(1-t)b) \le g(b) + t\eta(g(a),g(b)) \tag{10} $$

and

$$ g((1-t)a+tb) \le g(b) + (1-t)\eta(g(a),g(b)). \tag{11} $$

Adding inequalities (10) and (11), we get

$$ g(ta+(1-t)b) + g((1-t)a+tb) \le 2g(b) + \eta(g(a),g(b)). \tag{12} $$

Multiplying both sides of (12) by

$$ (b-a)\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\frac{(tb+(1-t)a)^r}{\left[b^{r+1}-(tb+(1-t)a)^{r+1}\right]^{1-\frac{\alpha}{k}}}, $$

and integrating over [0, 1] with respect to t, we get

$$ \begin{aligned} &(b-a)\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^1 \frac{(tb+(1-t)a)^r g((1-t)b+ta)}{\left[b^{r+1}-(tb+(1-t)a)^{r+1}\right]^{1-\frac{\alpha}{k}}}\,dt \\ &+ (b-a)\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^1 \frac{(tb+(1-t)a)^r g(tb+(1-t)a)}{\left[b^{r+1}-(tb+(1-t)a)^{r+1}\right]^{1-\frac{\alpha}{k}}}\,dt \\ &\le [2g(b)+\eta(g(a),g(b))]\,(b-a)\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^1 \frac{(tb+(1-t)a)^r}{\left[b^{r+1}-(tb+(1-t)a)^{r+1}\right]^{1-\frac{\alpha}{k}}}\,dt. \end{aligned} $$

Now, using (6) and (8) in the above inequality, we get

$$ {}_k^r\mathcal{J}_{a^+}^{\alpha}\tilde{g}(b) +{}_k^r \mathcal{J}_{a^+}^{\alpha}g(b) \le \frac{(s+1)^{1-\frac{\alpha}{k}}(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}{(s+1)\alpha\Gamma_k(\alpha)}\left[2g(b)+\eta(g(a),g(b))\right], $$

that is,

$$
{}_k^r\mathcal{J}_{a^+}^{\alpha} G(b) \le \frac{(b^{r+1} - a^{r+1})^{\frac{\alpha}{k}}}{(r+1)^{\frac{\alpha}{k}} \Gamma_k(\alpha + k)} \left[2g(b) + \eta(g(a), g(b))\right]. \tag{13}
$$

Similarly, multiplying again both sides of (12) by

$$
(b-a)\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \frac{(tb + (1-t)a)^r}{\left[(tb + (1-t)a)^{r+1} - a^{r+1}\right]^{1-\frac{\alpha}{k}}}
$$

and integrating with respect to t over $[0, 1]$, we obtain that

$$
{}_k^r\mathcal{J}_{b^-}^{\alpha} G(a) \le \frac{(b^{r+1} - a^{r+1})^{\frac{\alpha}{k}}}{(r+1)^{\frac{\alpha}{k}} \Gamma_k(\alpha + k)} \left[2g(b) + \eta(g(a), g(b))\right]. \tag{14}
$$

Hence, the intended inequality follows by adding (13) and (14).

Remark 9 By taking $\eta(x, y) = x - y$ in our Theorem 8, we recover the right-hand side of the inequalities in Theorem 7.

For the rest of our results, we will need the following two lemmas.

Lemma 10 (See [1]) *Let $\alpha,\ k > 0$ and $r \in \mathbb{R} \setminus \{-1\}$. If $g : I \to \mathbb{R}$ is differentiable on I° and $a, b \in I^\circ$ such that $g' \in L[a, b]$ with $a < b$, then the following identity holds:*

$$
\begin{aligned}
&\frac{g(a) + g(b)}{2} - \frac{(r+1)^{\frac{\alpha}{k}} \Gamma_k(\alpha + k)}{4(b^{r+1} - a^{r+1})^{\frac{\alpha}{k}}} \left[{}_k^r\mathcal{J}_{a^+}^{\alpha} G(b) + {}_k^s\mathcal{J}_{b^-}^{\alpha} G(a)\right] \\
&\quad = \frac{b-a}{4(b^{r+1} - a^{r+1})^{\frac{\alpha}{k}}} \int_0^1 \Theta_{\alpha,r}(t) g'(ta + (1-t)b)\, dt,
\end{aligned}
$$

where $\Theta_{\alpha,r} : [0, 1] \to \mathbb{R}$ is defined by

$$
\begin{aligned}
\Theta_{\alpha,r}(t) := &\left[(ta + (1-t)b)^{r+1} - a^{r+1}\right]^{\frac{\alpha}{k}} - \left[(tb + (1-t)a)^{r+1} - a^{r+1}\right]^{\frac{\alpha}{k}} \\
&+ \left[b^{r+1} - (tb + (1-t)a)^{r+1}\right]^{\frac{\alpha}{k}} - \left[b^{r+1} - (ta + (1-t)b)^{r+1}\right]^{\frac{\alpha}{k}}.
\end{aligned}
$$

Lemma 11 *Under the conditions of Lemma 10, we have that*

$$
\int_0^1 |\Theta_{\alpha,r}(t)|\, dt = \frac{1}{b-a} \left(\Re_1 + \Re_2 + \Re_3 + \Re_4\right),
$$

where

$$
\Re_1 = \int_{\frac{a+b}{2}}^{b} \left(w^{r+1} - a^{r+1}\right)^{\frac{\alpha}{k}} dw - \int_a^{\frac{a+b}{2}} \left(w^{r+1} - a^{r+1}\right)^{\frac{\alpha}{k}} dw,
$$

$$\Re_2 = \int_{\frac{a+b}{2}}^{b} \left[b^{r+1} - (b+a-w)^{r+1}\right]^{\frac{\alpha}{k}} \, dw - \int_{a}^{\frac{a+b}{2}} \left[b^{r+1} - (b+a-w)^{r+1}\right]^{\frac{\alpha}{k}} \, dw,$$

$$\Re_3 = \int_{a}^{\frac{a+b}{2}} \left(b^{r+1} - w^{r+1}\right)^{\frac{\alpha}{k}} \, dw - \int_{\frac{a+b}{2}}^{b} \left(b^{r+1} - w^{r+1}\right)^{\frac{\alpha}{k}} \, dw,$$

and

$$\Re_4 = \int_{a}^{\frac{a+b}{2}} \left[(b+a-w)^{r+1} - a^{r+1}\right]^{\frac{\alpha}{k}} \, dw - \int_{\frac{a+b}{2}}^{b} \left[(b+a-w)^{r+1} - a^{r+1}\right]^{\frac{\alpha}{k}} \, dw.$$

Proof Using the substitution $w = ta + (1-t)b$, we get

$$\int_0^1 |\Theta_{\alpha,r}(t)| \, dt = \frac{1}{b-a} \int_a^b |\wp(w)| \, dw, \tag{15}$$

where

$$\wp(w) = \left(w^{r+1} - a^{r+1}\right)^{\frac{\alpha}{k}} - \left[(b+a-w)^{r+1} - a^{r+1}\right]^{\frac{\alpha}{k}} + \left[b^{r+1} - (b+a-w)^{r+1}\right]^{\frac{\alpha}{k}} - \left(b^{r+1} - w^{r+1}\right)^{\frac{\alpha}{k}}.$$

The required result follows from (15) and by observing that $\wp$ is a nondecreasing function on $[a, b]$, $\wp(a) = -2(b^{r+1} - a^{r+1})^{\frac{\alpha}{k}} < 0$, $\wp\left(\frac{a+b}{2}\right) = 0$, and thus

$$\begin{cases} \wp(w) \le 0 & \text{if} \quad a \le w \le \frac{a+b}{2}, \\ \wp(w) > 0 & \text{if} \quad \frac{a+b}{2} < w \le b. \end{cases}$$

This concludes the proof.

Theorem 12 *Let α, $k > 0$, $r \in \mathbb{R} \setminus \{-1\}$, $g : I \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$. Suppose $|g'|$ is η-convex on $[a, b]$ with η bounded on $|g'|([a, b]) \times |g'|([a, b])$. Then the following (k, r)-fractional integral inequality holds:*

$$\left| \frac{g(a) + g(b)}{2} - \frac{(r+1)^{\frac{\alpha}{k}} \Gamma_k(\alpha + k)}{4(b^{r+1} - a^{r+1})^{\frac{\alpha}{k}}} \left[{}_k^r\mathcal{J}_{a^+}^{\alpha} G(b) + {}_k^r\mathcal{J}_{b^-}^{\alpha} G(a) \right] \right| \le \frac{1}{4(b^{r+1} - a^{r+1})^{\frac{\alpha}{k}}} \left[\Re |g'(b)| + \frac{\Xi}{b-a} \eta(|g'(a)|, |g'(b)|) \right],$$

where $\Re = \Re_1 + \Re_2 + \Re_3 + \Re_4$ (see Lemma 11) and $\Xi = \xi_1 + \xi_2 + \xi_3 + \xi_4$ with

$$\xi_1 = \int_a^{\frac{a+b}{2}} (b-w)(b^{r+1} - w^{r+1})^{\frac{\alpha}{k}} \, dw - \int_{\frac{a+b}{2}}^{b} (b-w)(b^{r+1} - w^{r+1})^{\frac{\alpha}{k}} \, dw,$$

$$\xi_2 = \int_{\frac{a+b}{2}}^{b} (b-w)(w^{r+1} - a^{r+1})^{\frac{\alpha}{k}}\, dw - \int_{a}^{\frac{a+b}{2}} (b-w)(w^{r+1} - a^{r+1})^{\frac{\alpha}{k}}\, dw,$$

$$\begin{aligned}\xi_3 = &\int_{a}^{\frac{a+b}{2}} (b-w)((b+a-w)^{r+1} - a^{r+1})^{\frac{\alpha}{k}}\, dw \\ &- \int_{\frac{a+b}{2}}^{b} (b-w)((b+a-w)^{r+1} - a^{r+1})^{\frac{\alpha}{k}}\, dw,\end{aligned}$$

$$\begin{aligned}\xi_4 = &\int_{\frac{a+b}{2}}^{b} (b-w)(b^{r+1} - (b+a-w)^{r+1})^{\frac{\alpha}{k}}\, dw \\ &- \int_{a}^{\frac{a+b}{2}} (b-w)(b^{r+1} - (b+a-w)^{r+1})^{\frac{\alpha}{k}}\, dw.\end{aligned}$$

Proof Since $|f'|$ is η-convex, it follows, by definition, that

$$\left|g'(ta + (1-t)b)\right| \leq |g'(b)| + t\eta\left(|g'(a)|, |g'(b)|\right) \tag{16}$$

for $t \in [0, 1]$. From [1, p. 9], we have

$$\int_0^1 t|\Theta_{\alpha,r}(t)|\, dt = \frac{\xi_1 + \xi_2 + \xi_3 + \xi_4}{(b-a)^2}. \tag{17}$$

Using Lemmas 10 and 11, inequality (16), identity (17), and properties of the modulus, we obtain

$$\begin{aligned}
&\left|\frac{g(a)+g(b)}{2} - \frac{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{r+1} - a^{r+1})^{\frac{\alpha}{k}}}\left[{}^r_k\mathcal{J}^{\alpha}_{a^+}G(b) + {}^r_k\mathcal{J}^{\alpha}_{b^-}G(a)\right]\right| \\
&\leq \frac{b-a}{4(b^{r+1} - a^{r+1})^{\frac{\alpha}{k}}}\int_0^1 |\Theta_{\alpha,r}(t)||g'(ta+(1-t)b)|\, dt \\
&\leq \frac{b-a}{4(b^{r+1} - a^{r+1})^{\frac{\alpha}{k}}}\int_0^1 |\Theta_{\alpha,r}(t)|\left[|g'(b)| + t\eta(|g'(a)|, |g'(b)|)\right] dt \\
&= \frac{b-a}{4(b^{r+1} - a^{r+1})^{\frac{\alpha}{k}}}\left(|g'(b)|\int_0^1 |\Theta_{\alpha,r}(t)|\, dt + \eta(|g'(a)|, |g'(b)|)\int_0^1 t|\Theta_{\alpha,r}(t)|\, dt\right) \\
&= \frac{b-a}{4(b^{s+1} - a^{s+1})^{\frac{\alpha}{k}}}\left[|g'(b)|\frac{1}{b-a}(\Re_1 + \Re_2 + \Re_3 + \Re_4)\right. \\
&\quad \left. + \eta(|g'(a)|, |g'(b)|)\frac{\xi_1 + \xi_2 + \xi_3 + \xi_4}{(b-a)^2}\right].
\end{aligned}$$

The desired result follows.

Remark 13 By taking $r = 0$ and $k = 1$ in Theorem 12, we recover Theorem 5. In this case,

$$\Re = \frac{4}{\alpha+1}(b-a)^{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)$$

and

$$\Xi = \frac{2}{\alpha+1}(b-a)^{\alpha+2}\left(1-\frac{1}{2^{\alpha}}\right).$$

Theorem 14 *Let g be differentiable on I° with $a, b \in I^{\circ}$. If $|g'|^q$ is η-convex on $[a, b]$ and $q > 1$ with η bounded on $|g'|^q([a, b]) \times |g'|^q([a, b])$, then the (k, r)-fractional integral inequality*

$$\left|\frac{g(a)+g(b)}{2} - \frac{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}\left[{}_k^r\mathcal{J}_{a^+}^{\alpha}G(b) + {}_k^r\mathcal{J}_{b^-}^{\alpha}G(a)\right]\right|$$
$$\leq \frac{b-a}{4(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}\left(|g'(b)|^q + \frac{\eta(|g'(a)|^q, |g'(b)|^q)}{2}\right)^{\frac{1}{q}} ||\Theta_{\alpha,r}||_p$$

holds, where $\frac{1}{p} + \frac{1}{q} = 1$ and $||\Theta_{\alpha,r}||_p = \left(\int_0^1 |\Theta_{\alpha,r}(t)|^p\, dt\right)^{\frac{1}{p}}$.

Proof Function $|g'|^q$ is η-convex, which implies

$$|g'(ta+(1-t)b)|^q \leq |g'(b)|^q + t\eta(|g'(a)|^q, |g'(b)|^q), \tag{18}$$

$t \in [0, 1]$. Using Lemma 10, inequality (18), Hölder's inequality, and the properties of modulus, we get

$$\left|\frac{g(a)+g(b)}{2} - \frac{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}\left[{}_k^r\mathcal{J}_{a^+}^{\alpha}G(b) + {}_k^r\mathcal{J}_{b^-}^{\alpha}G(a)\right]\right|$$
$$\leq \frac{b-a}{4(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}\int_0^1 |\Theta_{\alpha,r}(t)||g'(ta+(1-t)b)|\, dt$$
$$\leq \frac{b-a}{4(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}\left(\int_0^1 |\Theta_{\alpha,r}(t)|^p\, dt\right)^{\frac{1}{p}}\left(\int_0^1 |g'(ta+(1-t)b)|^q\, dt\right)^{\frac{1}{q}}$$
$$\leq \frac{b-a}{4(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}\left(\int_0^1 |\Theta_{\alpha,r}(t)|^p\, dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[|g'(b)|^q + t\eta(|g'(a)|^q, |g'(b)|^q)\right] dt\right)^{\frac{1}{q}}$$
$$= \frac{b-a}{4(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}\left(\int_0^1 |\Theta_{\alpha,r}(t)|^p\, dt\right)^{\frac{1}{p}}\left(|g'(b)|^q + \frac{\eta(|g'(a)|^q, |g'(b)|^q)}{2}\right)^{\frac{1}{q}}.$$

This completes the proof.

Theorem 15 *Let g be differentiable on I° with $a, b \in I^\circ$. If $|g'|^q$ is η-convex on $[a, b]$ and $q > 1$ with η bounded on $|g'|^q([a, b]) \times |g'|^q([a, b])$, then the (k, r)-fractional integral inequality*

$$\left|\frac{g(a)+g(b)}{2} - \frac{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}\left[{}_k^r\mathcal{J}_{a^+}^{\alpha}G(b) + {}_k^r\mathcal{J}_{b^-}^{\alpha}G(a)\right]\right|$$
$$\leq \frac{\Re^{\frac{1}{p}}}{4(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}\left[\left|g'(b)\right|^q + \frac{\Xi}{b-a}\eta\left(|g'(a)|^q, |g'(b)|^q\right)\right]^{\frac{1}{q}}$$

holds, where $\frac{1}{p}+\frac{1}{q}=1$ and $\Re$ and Ξ are defined as in Theorem 12.

Proof Following a similar approach as in the proof of Theorem 14, we have, by using Lemmas 10 and 11 combined with the power mean inequality plus inequality (18), that

$$\left|\frac{g(a)+g(b)}{2} - \frac{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}\left[{}_k^r\mathcal{J}_{a^+}^{\alpha}G(b) + {}_k^r\mathcal{J}_{b^-}^{\alpha}G(a)\right]\right|$$
$$\leq \frac{b-a}{4(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}\int_0^1 |\Theta_{\alpha,r}(t)||g'(ta+(1-t)b)|\,dt$$
$$\leq \frac{b-a}{4(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}\left(\int_0^1 |\Theta_{\alpha,r}(t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |\Theta_{\alpha,r}(t)||g'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}$$
$$\leq \frac{b-a}{4(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}\left(\int_0^1 |\Theta_{\alpha,r}(t)|\,dt\right)^{1-\frac{1}{q}}$$
$$\left(\int_0^1 |\Theta_{\alpha,r}(t)|\left[|g'(b)|^q + t\eta(|g'(a)|^q, |g'(b)|^q)\right]\,dt\right)^{\frac{1}{q}}.$$

The required inequality follows.

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A Family of Integral Inequalities on the Interval $[-1, 1]$

Ali Hafidi, Moulay Rchid Sidi Ammi and Praveen Agarwal

Abstract We study the heat semigroup $(P_t^n)_{t\geq 0} = \{\exp(tL_n)\}_{t\geq 0}$ generated by the Gegenbauer operator $L_n := (1-x^2)\frac{d^2}{dx^2} - nx\frac{d}{dx}$, on the interval $[-1, 1]$ equipped with the probability measure $\mu_n(dx) := c_n(1-x^2)^{\frac{n}{2}-1}$, where c_n the normalization constant and n is a strictly positive real number. By means of a simple method involving essentially a commutation property between the semigroup and derivation, we describe a large family of optimal integral inequalities with logarithmic Sobolev and Poincaré inequalities as particular cases.

Keywords Heat semigroup · Gegenbauer operator · Spectral gap · Poincaré's inequality · Sobolev's inequality · Logarithmic Sobolev inequality · φ-entropy inequality

Mathematics Subject Classification 2010 39B62 · 39B72 · 44A15 · 46E35 60J25

1 Introduction

Let γ_d be the standard Gaussian measure on $\mathbb{R}^d$. The celebrated logarithmic Sobolev inequality [5] states that for all nonnegative smooth functions f on $\mathbb{R}^d$

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$$\int_{\mathbb{R}^d} f log f \, d\gamma_d - \left(\int_{\mathbb{R}^d} f \, d\gamma_d\right) log \left(\int_{\mathbb{R}^d} f \, d\gamma_d\right) \leq \frac{1}{2} \int_{\mathbb{R}^d} \frac{|\nabla f|^2}{f} \, d\gamma_d, \quad (1)$$

where $|\nabla f|$ is the length of the usual gradient of. This inequality is in fact a reinforced form of the classical Poincaré inequality:

$$\int_{\mathbb{R}^d} f^2 \, d\gamma_d - \left(\int_{\mathbb{R}^d} f \, d\gamma_d\right)^2 \leq \int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma_d. \quad (2)$$

Recently, Bentaleb, Fahlaoui, and Hafidi proposed in ([2], Sect. 2) a generalization of the inequality (2) and obtained, under some assumptions on function ψ, the following inequality: for all smooth functions on $\mathbb{R}^d$,

$$\mathbb{E}nt_t^{\psi}(f) := \int_{\mathbb{R}^d} \psi(f) \, d\gamma_d - \int_{\mathbb{R}^d} \psi(P_t f) \, d\gamma_d \leq \frac{1 - e^{-2t}}{2}$$
$$\int_{\mathbb{R}^d} \psi''(f) |\nabla f|^2 \, d\gamma_d, \quad t \in [0, +\infty].$$

Similar investigation on this kind of inequalities for general probability measure generated by diffusion has been done by many authors (see, for instance, [1, 3, 4, 8]).

As mentioned in the abstract, the main investigation of this paper is to establish similar inequalities for the probability measure $\mu_n(dx) := c_n(1-x^2)^{\frac{n}{2}-1}$ on $[-1, 1]$ related to Dirichlet form $\int_{-1}^{1} (1-x^2) f'^2(x) \, \mathrm{d}\mu_n(x)$. These types of integral inequalities are deeply connected to the aspects of the large-time behavior of prabolic PDEs (see for example [7]).

2 Preliminaries

In the present section, we recall briefly some needed spectral properties of the Gegenbauer operator. We denote the Gegenbauer operator L acting on $C^2([-1, 1])$ by:

$$L_n := (1-x^2) \frac{d^2}{dx^2} - nx \frac{d}{dx}, \quad (x \in I := [-1, 1]).$$

Note that if $n \in \mathbb{N}^*$ the operator L_n may be obtained as the projection of the Laplacian on the unit sphere $\mathbf{S^n}$.

The classical Gegenbauer polynomials $(G_k^n)_{k \in \mathbb{N}}$ are defined by the Rodrigues formula (see for instance, [6])

$$2^k (-1)^k k! G_k^n = \frac{1}{(1-x^2)^{\frac{n}{2}-1}} \frac{d^k}{dx^k} (1-x^2)^{\frac{n}{2}+k-1}, \quad (x \in]-1, 1[).$$

These polynomials $(G_k^n)_{k\in\mathbb{N}}$ are orthonormal with the respective probability measure $\mu_n(dx) := c_n(1-x^2)^{\frac{n}{2}-1}$, where c_n is the normalization constant. Each Gegenbauer polynomial G_k^n is an eigenfunction of the differential operator $-L_n$ with corresponding eigenvalue

$$k(n+k-1) \quad k = 0, 1, 2, \ldots$$

In fact, the distribution μ_n is symmetrizing for L and the sequence $(-k(k+n-1), Vect(G_k)_k)$ forms the spectral decomposition of the minimal self-adjoint extension of this operator on $\mathrm{L}^2([-1, 1], \mu_n)$. By integration by parts, it is easy to establish the symmetry and dissipativity formulas: for all $f, g \in C^2([-1, 1])$,

$$\int_{-1}^{1} (-L_n f) g \, \mathrm{d}\mu_n = \int_{-1}^{1} f(-L_n g) \, \mathrm{d}\mu_n = \int_{-1}^{1} (1-x^2) f'(x) g'(x) \, \mathrm{d}\mu_n(x), \quad (3)$$

The Gegenbauer semigroup $P_t^n := \mathrm{e}^{-tL_n}$ for $t \geq 0$ applied to $f = \sum_{k=0}^{\infty} a_k G_k^n$ in $\mathrm{L}^2([-1, 1], \mu_n)$ is given by:

$$P_t^n f = \sum_{k=0}^{\infty} a_k \mathrm{e}^{-tk(k+n-1)} G_k^n \quad (4)$$

$(P_t^n)_{t\geq 0 \geq 0}$ defines thus a Markovian semigroup of positive contractions in all $\mathrm{L}^p([-1, 1], \mu_n)$,
$(p \in [1, +\infty])$ with the measure μ_n as the symmetric (and invariant) measure:

$$\int_{-1}^{1} (P_t^n f) g \, \mathrm{d}\mu_n = \int_{-1}^{1} f(P_t^n g) \, \mathrm{d}\mu_n, \quad f, g \in \mathrm{L}^2([-1, 1], \mu_n).$$

According to (4), P_t^n is ergodic,i.e, $P_t^n \longrightarrow \int_{-1}^{1} f \, \mathrm{d}\mu_n$ in $\mathrm{L}^2([-1, 1], \mu_n)$ as $t \longrightarrow \infty$

The commutation relation between the action of the operator L and the derivation is given as

$$\frac{d}{dx} L_n = \tilde{L}\left(\frac{d}{dx}\right) - n\frac{d}{dx},$$

where

$$\tilde{L} := (1-x^2)\frac{d^2}{dx^2} - (2+n)x\frac{d}{dx} = L_{n+2}.$$

This commutation formula translates for the semigroup $(P_t^n)_{t\geq 0}$ by

$$\frac{d}{dx}P_t^n = \mathrm{e}^{-nt}\tilde{P}_t\left(\frac{d}{dx}\right), \tag{5}$$

where $\tilde{P}_t$ designates the heat semigroup generated by $\tilde{L}$. Notice that $\tilde{P}_t = P_t^{n+2}$ so it is symmetric (and invariant) with respect to the probability measure $\tilde{\mu} = \mu_{n+2}$ The generator $\tilde{L}$ satisfies the following dissipativity formula

$$\int (-\tilde{L}f)g\,d\tilde{\mu} = \int (1-x^2)f'g'\,d\tilde{\mu}, \tag{6}$$

for f, g sufficiently smooth functions on $[-1, 1]$.

3 φ Entropy Inequalities

Our objective in this section is to establish a family of integral inequalities on $I = [-1, 1]$ which provide interpolation between Sobolev and Poincaré inequalities.

Let $\varphi : [0, +\infty[\longrightarrow \mathbb{R}$ be a strictly convex function such that $\varphi(0) = 0$. We define the φ-entropy functional of a nonnegative smooth function $f : \mathbb{R}^d \longrightarrow [0, +\infty[$ by:

$$\mathbb{E}nt_{\mu_n}^{(t,\varphi)}(f) = \int \varphi(f)d\mu_n - \int \varphi(P_t^n f)\,d\mu_n, \quad t \in [0, +\infty].$$

The quantity $\mathbb{E}nt_{\mu_n}^{(t,\varphi)}(f)$ is always nonnegative since P_t^n is invariant for the probability measure μ_n. By the ergodic property of the semigroup, we have

$$\mathbb{E}nt_{\mu_n}^{(\infty,\varphi)}(f) := \int \varphi(f)\,d\mu_n - \varphi\left(\int f\,d\mu_n\right).$$

When $\varphi(x) = x^2$, $\mathbb{E}nt_{\mu_n}^{(\infty,\varphi)}(f)$ coincides with the classical notion of variance,

$$\mathbb{E}nt_{\mu_n}^{(\infty,\varphi)}(f) := Var_{\mu_n}(f) = \int f^2\,d\mu_n - \left(\int f\,d\mu_n\right)^2.$$

When $\varphi(x) = xlogx$, we have

$$\mathbb{E}nt_{\mu_n}^{(\infty,\varphi)}(f) := \mathbb{E}nt_{\mu_n}(f) = \int flogf\,d\mu_n - \int f\,d\mu_n \log\left(\int f\,d\mu_n\right).$$

In the sequel, we shall restrict ourself to the class $\mathcal{C}_n$ of real functions $\varphi \in C^\infty(\mathbb{R}^+)$: $\varphi \in \mathcal{C}_n$ means that $\varphi(0) = 0$, φ'' is strictly positive on $\mathbb{R}^+$ and

$$\frac{(2n+1)^2}{2n(n+2)}(\varphi''')^2 \le \varphi''\psi^{(IV)} \quad \text{on} \quad \mathbb{R}^+.$$

Having in our disposal enough basic background, we are now ready to prove the following estimate of φ-entropy functional $\mathbb{E}nt_{\mu_n}^{(t,\varphi)}$

Theorem 3.1 *Let $\varphi \in \mathcal{C}_n$. Then, for all nonnegative smooth function $f : [-1, 1] \longrightarrow [0, +\infty[$ and $t \in [0, +\infty]$, we have*

$$\mathbb{E}nt_{\mu_n}^{(t,\varphi)}(f) \leq \frac{1}{2n}(1 - e^{-2nt}) \int \varphi''(f)\Gamma(f, f)\, d\mu_n. \tag{7}$$

Moreover, the numeric constant at the right-hand side of inequality (7) is optimal.

To illustrate this theorem, let analyze some practical applications. The most important examples of the class $\mathcal{C}_n$ when $n > \dfrac{1}{4}$ in our mind are

$$\varphi_p = \frac{-x^{\frac{2}{p}} + x}{p - 2} \quad for \;\; p \in \left[1, \frac{2n^2 + 1}{(n-1)^2}\right], \quad p \neq 2; n \neq 1$$

and

$$\varphi_2 = \frac{1}{2} x log x,$$

which corresponds to the limiting case of φ_p as $p \longrightarrow 2$. If $\varphi = \varphi_p$, inequality (7), written for $t = +\infty$, describes the Sobolev inequality: for all nonnegative smooth function $f : [-1, 1] \longrightarrow [0, +\infty[$

$$\frac{\|f\|_p^2 - \|f\|_2^2}{p - 2} \leq \frac{1}{n} \int \Gamma(f, f)\, d\mu_n. \tag{8}$$

For $\varphi = \varphi_2$ and $t = +\infty$, (7) is exactly the Sobolev logarithmic inequality. Replacing f positive by f^2, it yields for all smooth function $f : [-1, 1] \longrightarrow \mathbb{R}$, that

$$\mathbb{E}nt(f^2) \leq \frac{2}{n} \int \Gamma(f, f)\, d\mu_n. \tag{9}$$

Let $D_2(L_n)$ denotes the domain of the generator L_n of $(P_t^n)_{t\geq 0}$ in $L^2([-1, 1], \mu_n)$. Taking into account that

$$\int \Gamma(|f|, |f|)\, d\mu_n \leq \int \Gamma(f, f)\, d\mu_n,$$

and using the fact that set of bounded functions in $C^2([-1, 1])$ is dense in $D_2(L_n)$, we can extend inequalities (8) and (9) to $D_2(L_n)$. The above inequality (9) is equivalent to the hypercontractivity estimate for the semigroup $(P_t^n)_{t\geq 0}$: whenever $1 < p < q < +\infty$ and $t > 0$ satisfy $e^{2nt} \geq \frac{q-1}{p-1}$, then, for all function $f \in L^p([-1, 1], \mu_n)$, we have

$$\|P_t^n f\|_q \leq \|f\|_p.$$

In other words, P_t maps $L^p([-1,1],\mu_n)$ in $L^q([-1,1],\mu_n)$ $(q>p)$ with norm one.

Proof of Theorem 3.1 By the Fubini theorem, it follows from the definition of $\mathbb{E}nt_{\mu_n}^{(t,\varphi)}(f)$ that for any $t>0$,

$$\begin{aligned}
\mathbb{E}nt_{\mu_n}^{(t,\varphi)}(f) &= -\int \varphi(P_t^n f) - \varphi(P_0^n f), d\mu_n \\
&= \int_0^t \frac{d}{ds}\left[\int \varphi(P_s^n f) - \varphi(P_0^n f), d\mu_n\right] ds \\
&= \int_0^t \left(\int -(L_n P_s^n f)\varphi'(P_s^n f) d\mu_n\right) ds \\
&= \int_0^t \left(\int (1-x^2)(P_s^n f)'^2 \varphi''(P_s^n f) d\mu_n\right) ds \\
&= \int_0^t \mathrm{e}^{-2ns} \left(\int (1-x^2)(\tilde{P}_s f')^2 \varphi''(P_s^n f) d\mu_n\right) ds.
\end{aligned}$$

The last two equalities follow from the dissipativity property (3). An integration by parts over the time variable yields

$$\begin{aligned}
\mathbb{E}nt_{\mu_n}^{(t,\varphi)}(f) = &-\frac{1}{2n}\mathrm{e}^{-2nt}\int (1-x^2)(\tilde{P}_t f')^2\varphi''(P_t^n f) d\mu_n \\
&+ \frac{1}{4}\int (1-x^2) f'^2 \varphi''(f) d\mu_n \\
&+ \frac{1}{2n}\int_0^t \mathrm{e}^{-2ns}\frac{d}{ds}\left[\int (1-x^2)(\tilde{P}_s f')^2 \varphi''(P_s^n f) d\mu_n\right] ds.
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^t \frac{d}{ds}\left[\int (1-x^2)(\tilde{P}_s f')^2\varphi''(P_s^n f) d\mu_n\right] ds = &\int (1-x^2)(\tilde{P}_t f')^2 \varphi''(P_t^n f) d\mu_n \\
&- \int (1-x^2) f'^2 \varphi''(f) d\mu_n,
\end{aligned}$$

we get

$$\begin{aligned}
\mathbb{E}nt_{\mu_n}^{(t,\varphi)}(f) = &\frac{1}{2n}(1-\mathrm{e}^{-2nt})\int (1-x^2) f'^2\varphi''(f) d\mu_n \\
&+ \frac{1}{2n}\int_0^t (\mathrm{e}^{-2ns} - \mathrm{e}^{-2nt})\frac{d}{ds}\left[\int (1-x^2)(\tilde{P}_s f')^2 \varphi''(P_s^n f) d\mu_n\right] ds.
\end{aligned}$$

We also have

$$\begin{aligned} &\mathrm{e}^{-2ns}\frac{d}{ds}\left[\int (1-x^2)(\tilde{P}_s f')^2\varphi''(P_s^n f)d\mu_n\right]ds \\ &= 2\int (1-x^2)\tilde{L}(P_s f)'\varphi''(P_s^n f)(P_s^n f)'d\mu_n \\ &\quad + \int (1-x^2)L_n(P_s^n f)'\varphi'''(P_s^n f)(P_s^n f)'^2 d\mu_n. \end{aligned}$$

Applying successively (3) and (6), the first integral in this sum is reduced to

$$\begin{aligned} &-2\int (1-x^2)^2(P_s^n f)''^2\varphi''(P_s^n f)d\mu_n \\ &-2\int (1-x^2)^2(P_s^n f)''(P_s^n f)'^2\varphi'''(P_s^n f)(P_s^n f)'d\mu_n, \end{aligned}$$

while the second integral is equal to

$$\begin{aligned} &-2\int (1-x^2)^2(P_s^n f)'^2(P_s^n f)''\varphi'''(P_s^n f)d\mu_n \\ &-\int (1-x^2)^2(P_s^n f)'^4\varphi''''(P_s^n f)d\mu_n \\ &+2\int x(1-x^2)(P_s^n f)'^3\varphi''(P_s^n f)d\mu_n. \end{aligned}$$

Replacing x by $\frac{-\tilde{L}(x)}{(n+2)}$ and invoking again the dissipativity formula (6), the last member in the preceding sum becomes

$$\begin{aligned} &\frac{6}{(n+2)}\int (1-x^2)^2(P_s^n f)'^2(P_s^n f)''\varphi'''(P_s^n f)d\mu_n \\ &+\frac{2}{(n+2)}\int (1-x^2)^2(P_s^n f)'^4\varphi''''(P_s^n f)d\mu_n. \end{aligned}$$

As a consequence, after gathering the different terms, we find

$$\begin{aligned} \frac{\mathbb{E}nt_{\mu_n}^{(t,\varphi)}(f)}{1-\mathrm{e}^{-2nt}} &= \frac{1}{2n}\int (1-x^2)f'^2\varphi''(f)d\mu_n - \frac{1}{2n}\int_0^t \frac{1-\mathrm{e}^{-2n(t-s)}}{1-\mathrm{e}^{-2nt}} \\ &\quad \times \left(\int (1-x^2)^2\xi(s,f,\varphi)d\mu_n\right)ds, \end{aligned} \tag{10}$$

with

$$\xi(s, f, \varphi) = 2f_s''^2\varphi''(f_s) + \left(\frac{4n+2}{n+2}\right) f_s'^2 f_s'' \varphi'''(f_s) + \left(\frac{n}{n+2}\right) f_s'^4 \varphi''''(f_s)$$
$$= 2\left[f_s'' \sqrt{\varphi''(f_s)} + \left(\frac{4n+2}{2n+4}\right) \frac{f_s'^2 \varphi'''(f_s)}{\sqrt{\varphi''(f_s)}} \right]^2$$
$$+ \left(\frac{n}{n+2}\right) \frac{f_s'^4}{\varphi''(f_s)} \left[\varphi''''(f_s)\varphi''(f_s) - \frac{(2n+1)^2}{2n(n+2)} (\varphi'''(f_s))^2 \right],$$

where we have posed $f_s = P_s^n f$. The positivity of $\xi(s, f, \varphi)$ then allows us to obtain the desired inequality (7) from (10). It remains to show that the numeric constant $\frac{1}{2n}(1 - e^{-2nt})$ at the right-hand side of inequality (7) is optimal. As usual, let us consider $c \in]0, +\infty[$ such that $\varphi''(c) > 0$. If f is replaced by $c + \varepsilon f$ in (7), and if we pass to limit as ε tends to 0^+, we easily recover the Poincaré inequality with best constant

$$\int f^2 d\mu_n - \int (P_t^n f)^2 d\mu_n \leq \frac{(1 - e^{-2nt})}{n} \int \Gamma(f, f) d\mu_n.$$

$\forall t \in [0, +\infty]$. The proof is now complete. We end the paper by the following concluding remark.

Remark 3.1 Of course letting $t = +\infty$, inequality (7) in Theorem 3.1 gives rise to

$$\mathbb{E}nt_{\mu_n}^{(t,\infty)}(f) \leq \frac{1}{2n} \int \Gamma(f, f) d\mu_n. \tag{11}$$

Moreover, it is easy to observe that (7) provides a smooth nonincreasing interpolation for inequality (11)

$$\mathbb{E}nt_{\mu_n}^{(t,\infty)}(f) \leq \frac{\mathbb{E}nt_{\mu_n}^{(t,\varphi)}(f)}{1 - e^{-2nt}} \leq \frac{1}{2n} \int \Gamma(f, f) d\mu_n.$$

By (10), we point out that, if $\frac{(2n+1)^2}{2n(n+2)}(\varphi''')^2 \leq \varphi''\psi^{(IV)}$, the equality holds in (7) if and only if f is constant. In particular, inequalities (8) and (9) do not admit nonconstant extremal functions.

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A Generalization of Cauchy–Bunyakovsky Integral Inequality Via Means with Max and Min Values

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Abstract In the paper, we give a brief survey of a method for constructing generalizations of Cauchy–Bunyakovsky integral inequality using abstract mean values. One special inequality of this type is considered in details in terms of min and max functions. Some direct proofs of this inequality are given and application to inequalities for special functions. Also related recent references are briefly considered.

1 Means and Generalizations of Cauchy–Bunyakovsky Integral Inequality

1.1 Introduction

Cauchy–Bunyakovsky inequalities for finite sums, series and integrals are among the most important inequalities with many applications in different fields of mathematics and applied sciences. For references, we mention just well-known general books on inequalities [1–4] and very informative and concise the specialized ones of Dragomir [5] and Steel [6], cf. also the survey [7].

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The Cauchy–Bunyakovsky inequality for sums was found by Cauchy in 1821 [8] and for integrals by Bunyakovsky in 1859 [9]. It was rediscovered for integrals 26 years later in 1885 by Schwarz [10]. Different important applications derived from original text of Bunyakovsky's paper [9] were considered recently by J.Sándor in [11, 12]. Inequality for inner product spaces oppositely to general opinion ascribed it to Schwarz was in fact first published only in 1932 by von Neumann in his book on mathematical foundations of quantum mechanics [13].

A new method for generalization of Cauchy–Bunyakovsky inequalities for finite sums, series and integrals was proposed by the third named author in early 1990s, these results are summed up in the survey [7], cf. also references to previous papers on this method in this survey. Shortly, an idea of this method is that every mean of two numbers defined by natural axioms leads to some classes of generalizations of Cauchy–Bunyakovsky inequalities.

Augustin Louis Cauchy (1789–1857)

Viktor Yakovlevich Bunyakovsky (1804–1889)

1.2 Means

To demonstrate the results from [7] let us remind well-known definitions for arithmetic, geometric, quadratic and harmonic means for two positive numbers $x > 0, y > 0$:

$$A(x, y) = \frac{x+y}{2}, G(x, y) = \sqrt{x, y}, Q(x, y) = \sqrt{\frac{x^2+y^2}{2}}, x, y \geq 0,$$

$$H(x, y) = A(\frac{1}{x}, \frac{1}{y}) = \frac{2xy}{x+y}, \ x, y > 0.$$

These classical means are:
(1) intermediate

$$\min(x, y) \leq M(x, y) \leq \max(x, y),$$

(2) homogenic

$$M(\lambda x, \lambda y) = \lambda\, M(x, y), \lambda > 0,$$

(3) monotonic

$$x_2 > x_1 \Rightarrow M(x_2, y) > M(x_1, y),\ \ y_2 > y_1 \Rightarrow M(x, y_2) > M(x, y_1);$$

(4) symmetric

$$M(x, y) = M(y, x).$$

For general mean, it is natural to define it as a function $M(x, y)$ for which all conditions (1)–(4) (or just some of them) are valid. This approach is not new and was used in many papers started from Cauchy.

From now to the end of the paper, we fix a condition for numbers in mean values $M(x, y)$ to be strictly positive, $x > 0$, $y > 0$.

We need also a notion of complementary mean.

Definition 1 For a mean $M(x, y)$ a complementary mean is defined by

$$M^*(x, y) = \frac{xy}{M(x, y)};$$

The most known classes of means are power means and Radó means. The power means are defined by cf. [1–3, 14, 15]

$$M(x, y) = M_\alpha(x, y) = \left(\frac{x^\alpha + y^\alpha}{2}\right)^{\frac{1}{\alpha}}, -\infty \leq \alpha \leq \infty\ , \alpha \neq 0\ ;$$

$$M_{-\infty}(x, y) = \min(x, y)\ ,\ \ M_0 = \sqrt{xy}\ ,\ \ M_\infty(x, y) = \max(x, y).$$

They form a parametric scale

$$\alpha_1 > \alpha_2 \Rightarrow M_{\alpha_1}(x, y) \geq M_{\alpha_2}(x, y),\ \forall\, x, y.$$

Three exceptional values $\alpha = -\infty, 0, +\infty$ are defined by limits.

So for classical means

$$M_{-1}(x, y) = H(x, y) \leq M_0(x, y) = G(x, y)$$

$$\leq M_1(x, y) = A(x, y) \leq M_2(x, y) = Q(x, y).$$

For power mean the complementary mean is

$$(M_\alpha)^* = M_{-\alpha}.$$

Power means have many applications in different fields of mathematics and other sciences. And even two simplest arithmetical operations $+, \times$ are expressed via them:

$$x + y = 2M_1(x, y), x \cdot y = (M_0(x, y))^2,$$

the same is true for max and min values.

The second important class are the Radó means:

$$R_\beta(x, y) = \left(\frac{x^{\beta+1} - y^{\beta+1}}{(\beta + 1)(x - y)}\right)^{\frac{1}{\beta}}, -\infty \le \beta \le \infty,\ \beta \ne 0, -1;$$

$$R_{-\infty}(x, y) = \min(x, y), R_\infty(x, y) = \max(x, y).$$

Obviously

$$R_{-2}(x, y) = M_0(x, y), R_1(x, y) = M_1(x, y).$$

Exceptional values gives logarithmic mean

$$R_{-1}(x, y) = L(x, y) = \frac{y - x}{\ln y - \ln x}$$

and identric mean (the author prefer a name "multi–floored")

$$R_0(x, y) = \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}}.$$

Tibor Radó (1895–1965)

The Radó means also form a parametric scale

$$\beta_1 > \beta_2 \Rightarrow R_{\beta_1}(x, y) \ge R_{\beta_2}(x, y),\ \forall x, y;$$

with four exceptional values $\beta = \{-\infty, -1, 0, +\infty\}$. The complementary mean now is

$$R^*_\beta(x, y) = \frac{1}{R_\beta(\frac{1}{x}, \frac{1}{y})}.$$

The Radó means were introduced explicitly and studied by Tibor Radó in 1935 in [16], cf. also [3, 7]. Radó applied these means in his study of subharmonic functions. He proved important and non-trivial results on comparing of these means and power means. His results may be formulated in two main theorems.

Theorem R1 (T. Radó, 1935). *Radó and power means coincides only for five values of parameters:*

$$M_{-\infty} = R_{-\infty}, M_0 = R_{-2}, M_{\frac{1}{2}} = R_{\frac{1}{2}}, M_1 = R_1, M_\infty = R_\infty.$$

So the most popular classical means are in both scales.

His another important result was a finding of parameter sets (α, β) *for which the next inequalities are valid:*

$$M_{\alpha_1} \leq R_\beta \leq M_{\alpha_2}, R_{\beta_1} \leq M_\alpha \leq R_{\beta_2},$$

Theorem R2 (T. Radó, 1935). *The next best possible two-sided inequalities are valid for Radó means via power means:*

$$M_{\frac{\alpha+2}{3}} \leq R_\alpha \leq M_0 \text{ , for } \alpha \in (-\infty, -2],$$

$$M_0 \leq R_\alpha \leq M_{\frac{\alpha+2}{3}}, \text{ for } \alpha \in [-2, -1],$$

$$M_{\frac{\alpha \ln 2}{\ln(1+\alpha)}} \leq R_\alpha \leq M_{\frac{\alpha+2}{3}} \text{ , for } \alpha \in (-1, -1/2],$$

$$M_{\frac{\alpha+2}{3}} \leq R_\alpha \leq M_{\frac{\alpha \ln 2}{\ln(1+\alpha)}} \text{ , for } \alpha \in [-1/2, 1),$$

(for $\alpha = 0$ *the above inequality is understood in the limiting sense* $M_{\frac{2}{3}} \leq R_0 \leq M_{\ln 2}$*),*

$$M_{\frac{\alpha \ln 2}{\ln(1+\alpha)}} \leq R_\alpha \leq M_{\frac{\alpha+2}{3}}, \text{ for } \alpha \in [1, \infty].$$

Note that all indices in Theorem R2 are sharp and cannot be improved.

As spectacular consequences of T. Radó inequalities from Theorem R2 two folklore inequalities follow. First, it follows that beside the well-known inequality for logarithmic mean $M_0 \leq L \leq M_1$, which is as old as published by V. Bunyakovsky in his seminal work [9], the next inequality is valid

$$M_0(x, y) = \sqrt{xy} \leq L(x, y) = \frac{x - y}{\ln x - \ln y}$$

$$\leq M_{\frac{1}{3}}(x, y) = \left(\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{2} \right)^3,$$

and mean indices 0 ?1/3 are the best ones. Also for identric mean Theorem R2 gives:

$$M_{\frac{2}{3}}(x, y) = \left(\frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{2} \right)^{\frac{2}{3}} \leq R_0(x, y)$$

$$= \frac{1}{e} \left(\frac{y^y}{x^x} \right)^{\frac{1}{y-x}} \leq M_{\ln 2}(x, y) = \left(\frac{x^{\ln 2} + y^{\ln 2}}{2} \right)^{\frac{1}{\ln 2}},$$

and again indices are the best possible. Note that the simple estimate $M_0 \leq R_0 \leq M_1$ was also published by Bunyakovsky in [9].

Tibor Radó was a prominent mathematician in many fields, and his results on inequalities are very important too. They were many times reopened by and attributed to other researchers.

The world of means is very large and rich. Among other means just mention:

(1) Gini means, introduced by Corradó Gini in 1938

$$Gi_{u,v}(x, y) = \left(\frac{x^u + y^u}{x^v + y^v} \right)^{\frac{1}{u-v}}, u \neq v, u, v \in \mathbb{R},$$

$$Gi_{u,v}(x, y) = \exp \left(\frac{x^u \ln x + y^u \ln y}{x^u + y^u} \right), u = v \neq 0,$$

$$Gi_{u,v}(x, y) = G(x, y), u = v = 0.$$

(2) Special case of Gini means — Lehmer means

$$Le_u(x, y) = \frac{x^{u+1} + y^{u+1}}{x^u + y^u}, u \in \mathbb{R}.$$

(3) Quasi-arithmetic means for non-negative values $x = (x_1, x_2, \ldots, x_n)$ and weights $p = (p_1, p_2, \ldots, p_n)$

$$K_p(x) = f^{-1} \left(\sum_{k=1}^{n} p_k f(x_k) \right), \sum_{k=1}^{n} p_k = 1.$$

(4) Iterated means starting from values x_0, y_0 and defined by a pair of means (M, N) and a limit process:

$$x_{n+1} = M(x_n, y_n), \; y_{n+1} = N(x_n, y_n),$$

$$\mu(M, N|\, x_0, y_0) = \mu(x_0, y_0) = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n.$$

The most celebrated of iterated means is of course Gauss AGM one for a choice $M = M_1$, $N = M_0$, it equals

$$\mu(M, N|\, x_0, y_0) = \frac{\frac{\pi}{2}\, x_0}{K\left(\sqrt{1-\left(\frac{y_0}{x_0}\right)^2}\right)}, 0 < y_0 < x_0,$$

$K(x)$ — the complete Legendre elliptic integral of the first kind.

Now let us stop here. There are much more known means of different kinds. In the survey [7] even there are general theorems characterizing all possible means. Our aim is to demonstrate that there are many concrete examples for constructions using general means. Let consider a method of generalizing Cauchy–Bunyakovsky inequalities using any means.

1.3 Means Method for Generalizations of Cauchy–Bunyakovsky Inequalities

Before formulations of our results let fix the next conditions: all functions below are continuous and integrals are of Riemann type. Further such restrictions will be omitted.

Now let us list some main results from the survey [7].

Theorem 1 *Let M be any abstract mean for which above formulated properties (1)–(4) are fulfilled, M^*—its complimentary mean. Then the next generalization of Cauchy–Bunyakovsky inequality is valid*

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \le \int_a^b (M(f,g))^2\,dx \cdot \int_a^b (M^*(f,g))^2\,dx \le$$

$$\le \int_a^b (f(x))^2\,dx \cdot \int_a^b (g(x))^2\,dx,$$

So we may state that an integral analogue of *sufficient part* of Carlitz–Daykin–Eliezer theorem (CDE theorem) is valid reformulated via means and complimentary means as proposed in [7]. Suddenly enough the *necessary part* of this theorem as it was proved in [7] is not valid contrary to the discrete version. For the discrete version Carlitz–Daykin–Eliezer theorem is necessary and sufficient, cf. [5]. This is a difference of discrete and integral generalizations of Cauchy–Bunyakovsky inequalities.

Note that only RHS of the Theorem 1 is non-trivial inequality, the LHS is trivial and being Cauchy–Bunyakovsky inequality itself.

Now a choice of any known means and its complimentary ones generates by the Theorem 1 a number of generalizations of the Cauchy–Bunyakovsky LHS inequality. For example a choice of power means leads to the next

Theorem 2 *For any positive functions $f(x)$, $g(x)$, $x \in [a, b]$, the next generalization of the Cauchy–Bunyakovsky inequality holds:*

$$\left(\int_a^b f(x)g(x)dx\right)^2 \le \int_a^b [M_\alpha(f,g)]^2 dx \cdot \int_a^b [M_{-\alpha}(f,g)]^2 dx =$$

$$= \int_a^b (f^\alpha + g^\alpha)^{2/\alpha} dx \cdot \int_a^b f^2 g^2 (f^\alpha + g^\alpha)^{-2/\alpha} dx \le \int_a^b f^2 dx \cdot \int_a^b g^2 dx.$$

Consider special cases.

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \le \int_a^b \left(\sqrt{f(x)} + \sqrt{g(x)}\right)^4 dx \cdot$$

$$\cdot \int_a^b f^2 g^2 / \left(\sqrt{f(x)} + \sqrt{g(x)}\right)^4 dx \le \int_a^b f^2(x)\,dx \cdot \int_a^b g^2(x)\,dx,$$

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \le \int_a^b (f(x) + g(x))^2\,dx \cdot$$

$$\cdot \int_a^b f^2 g^2 / (f(x) + g(x))^2\,dx \le \int_a^b f^2(x)\,dx \cdot \int_a^b g^2(x)\,dx,$$

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \le \int_a^b \left(f^2(x) + g^2(x)\right)\,dx \cdot$$

$$\cdot \int_a^b f^2 g^2 / \left(f^2(x) + g^2(x)\right), dx \le \int_a^b f^2(x)\,dx \cdot \int_a^b g^2(x)\,dx.$$

The case $\alpha = 2$ is an integral variant of Milne inequality, cf. [1, 5, 7].

A choice of Radó means leads to the next

Theorem 3 *For non-negative continuous functions $f(x)$, $g(x)$, $f(x)$, $g(x) \neq 0$, $x \in [a, b]$, the next generalization of the Cauchy–Bunyakovsky inequality holds:*

$$\left(\int_a^b f(x)g(x)dx\right)^2 \le \int_a^b \left[R_\beta(f,g)\right]^2 dx \cdot \int_a^b \left[\frac{1}{R_\beta\left(\frac{1}{f}, \frac{1}{g}\right)}\right]^2 dx$$

$$\le \int_a^b f^2(x)dx \int_a^b g^2(x)dx.$$

Special cases:

$$\left(\int_a^b f(x)g(x)dx\right)^2 \le \int_a^b \left(\frac{f-g}{\ln\frac{f}{g}}\right)^2 dx \cdot \int_a^b f^2g^2/\left(\frac{f-g}{\ln\frac{f}{g}}\right)^2 dx$$
$$\le \int_a^b f^2(x)\,dx \cdot \int_a^b g^2(x)\,dx,$$
$$\left(\int_a^b f(x)g(x)dx\right)^2 \le \int_a^b \left[\frac{f^f}{g^g}\right]^{\frac{2}{f-g}} dx \cdot \int_a^b f^2g^2/\left(\frac{f^f}{g^g}\right)^{\frac{2}{f-g}} dx$$
$$\le \int_a^b f^2(x)\,dx \cdot \int_a^b g^2(x)\,dx,$$
$$\left(\int_a^b f(x)g(x)dx\right)^2 \le \int_a^b \left(f^2+fg+g^2\right) dx \cdot \int_a^b \frac{f^2g^2}{f^2+fg+g^2}\,dx$$
$$\le \int_a^b f^2(x)\,dx \cdot \int_a^b g^2(x)\,dx.$$

A choice of AGM mean leads to the next wonderful inequality

$$\left(\int_a^b f(x)g(x)dx\right)^2 \le \int_a^b \left[\frac{\max(f,g)}{K\left(\sqrt{1-\left(\frac{\min(f,g)}{\max(f,g)}\right)^2}\right)}\right]^2 dx\,\cdot$$
$$\cdot \int_a^b (\min(f,g))^2\left(K\left(\sqrt{1-\left(\frac{\min(f,g)}{\max(f,g)}\right)^2}\right)\right)^2 dx \le \int_a^b f^2\,dx \int_a^b g^2\,dx,$$

$K(x)$ — the complete Legendre elliptic integral of the first kind. In the last inequality, arbitrary functions are arguments of a concrete special function — the complete Legendre elliptic integral of the first kind!

2 Generalization of Cauchy–Bunyakovsky Inequality with Max–Min Values

Now let consider the central inequality of this paper. It is a consequence of Theorem 1 (or of Theorem 2) for a choice $\alpha = +\infty$.

In this case, a mean is a maximum and its complimentary mean is a minimum.

Theorem 4 *Let functions $f(x,y)$, $g(x,y)$ be nonnegative on $[a,b]$. Then the next generalization of Cauchy–Bunyakovsky inequality holds*

$$\left(\int_a^b f(x)g(x)dx\right)^2 \le \int_a^b [\max(f,g)]^2dx \cdot \int_a^b [\min(f,g)]^2dx \le$$

$$\leq \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx. \qquad (***)$$

From the first glance, there is a chance that all three parts in Theorem 4 are just equal. But it is not the case. For example for functions $f(x) = x$, $g(x) = 1 - x$ *and limits* $a = 0$, $b = 1$ *it reduces to the numerical inequality* $\frac{1}{36} < \frac{7}{144} < \frac{1}{9}$.

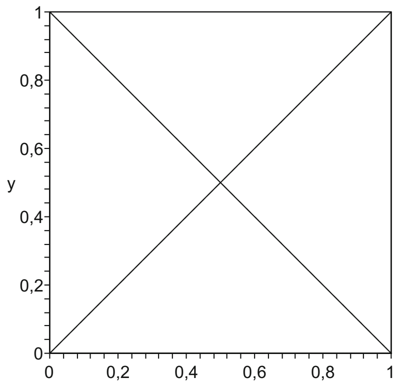
It is hard to believe — but the simple inequality from Theorem 4 is new, it was somehow missed by researches in inequalities. It was first published in 1995 by S.M.Sitnik as a special case of the Theorem 1 above, cf. [7] and references to earlier 1990s papers therein. In turn, it would be interesting to prove directly this surprising result. Here we give three direct proofs of the mentioned inequality independently of Theorems 1 and 2.

Further in the paper, we consider real-valued continuous Riemann-integrable functions. These restrictions may be weaker, but it needs careful special considerations.

2.1 Proofs of Min–Max Inequality

First proof. It is an application of above Theorems 1 or 2. This is an original indirect proof of 1995, included with proper references in [7].

Second proof.



Example of “envelope” $f(x)$, $g(x) = 1 - x$, $a = 0$, $b = 1$

Let introduce functions

$$E_1 = \{x : f(x) \geq g(x)\},\ E_2 = \{x : f(x) < g(x)\};$$

$$p_i = \int_{E_i} f(x)\,dx,\ q_i = \int_{E_i} g(x)\,dx,\ i = 1, 2.$$

It follows

$$\int_{E_1} (f(x) - g(x))\, dx \cdot \int_{E_2} (f(x) - g(x))\, dx \le 0.$$

Now

$$\int_{E_1} (f(x) - g(x))\, dx \cdot \int_{E_2} (f(x) - g(x))\, dx \le 0 \Leftrightarrow$$

$$\Leftrightarrow (p_1 - q_1)(p_2 - q_2) \le 0 \Leftrightarrow (p_1 + q_2)(p_2 + q_1) \le (p_1 + p_2)(q_1 + q_2) \Leftrightarrow$$

$$\Leftrightarrow \int_a^b [\max(f, g)]\, dx \cdot \int_a^b [\min(f, g)]\, dx \le \int_a^b f(x)dx \cdot \int_a^b g(x)dx.$$

For non-negative functions we may use equalities

$$[\max(f, g)]^2 = \max(f^2, g^2), [\min(f, g)]^2 = \min(f^2, g^2),$$

and so substituting in the last inequality f, g to f^2, g^2 we derive the inequality (***) and prove the Theorem 4.

□

Third proof.

It is well-known that the next formulas are valid for max and min as consequences of Vieta's theorem

$$m = \frac{a + b - |a - b|}{2}, M = \frac{a + b + |a - b|}{2}, \ m = \min(a, b), M = \max(a, b).$$

So to prove (***) denote

$$I(f) = \int_a^b f(x)\, dx.$$

Then the LHS of (***) equals

$$\begin{aligned} & I((\min(f, g))^2) I((\max(f, g))^2) \\ &= I\left(\left(\frac{f + g - |f - g|}{2}\right)^2\right) \cdot I\left(\left(\frac{f + g + |f - g|}{2}\right)^2\right) \\ &= \frac{1}{16} I\left((f + g)^2 - 2(f + g)|f - g| + (f - g)^2\right) \cdot \\ & \quad I\left((f + g)^2 + 2(f + g)|f - g| + (f - g)^2\right) \\ &= \frac{1}{16} I\left(2f^2 + 2g^2 - 2(f + g)|f - g|\right) \cdot I\left(2f^2 + 2g^2 + 2(f + g)|f - g|\right) \\ &= \frac{1}{4} I\left(f^2 + g^2 - (f + g)|f - g|\right) \cdot I\left(f^2 + g^2 + 2(f + g)|f - g|\right). \end{aligned}$$

After some obvious simplification it follows

$$\frac{1}{4}\left[(I(f^2))^2+(I(g^2))^2-(I\,((f+g)|f-g|))^2+2I(f^2)I(g^2)\right].$$

Then the difference of the RHS and LHS in (***) is represented as

$$\begin{aligned}
&\text{RHS}-\text{LHS}=I(f^2)I(g^2)\\
&-\frac{1}{4}\left[(I(f^2))^2+(I(g^2))^2-(I\,((f+g)|f-g|))^2+2I(f^2)I(g^2)\right]\\
&=\frac{1}{4}\left[I((f+g)|f-g|)\right]^2-\frac{1}{4}\left[I(f^2)-I(g^2)\right]^2\\
&=\frac{1}{4}\cdot\left[I((f+g)|f-g|)+I((f+g)(f-g))\right]\cdot\\
&\cdot\left[I((f+g)|f-g|)-I((f+g)(f-g))\right]\\
&=\frac{1}{4}\cdot I\,((f+g)((f-g)+|f-g|))\cdot I\,((f+g)((f-g)-|f-g|))\\
&=\int_{E(f\ge g)}(f+g)(f-g)\,dx\cdot\int_{E(f\le g)}(f+g)(g-f)\,dx\\
&=\int_{E(f\ge g)}(f^2-g^2)\,dx\cdot\int_{E(f\le g)}(g^2-f^2)\,dx.
\end{aligned}$$

Obviously, the last expression is non-negative, and so the inequality (***) is proved.

□

Corollary 1 *Let $f(x)$, $g(x)$ be functions of ANY signs, not necessary positive. Then the next identity is valid for the difference of RHS and LHS of (***)*

$$\begin{aligned}
&\textit{RHS of (***)} - \textit{LHS of (***)} =\\
&\int_a^b f^2(x)dx\cdot\int_a^b g^2(x)\,dx-\int_a^b[\max(f,g)]^2dx\cdot\int_a^b[\min(f,g)]^2\,dx=\\
&=\int_{E(f\ge g)}(f^2-g^2)\,dx\cdot\int_{E(f\le g)}(g^2-f^2)\,dx.
\end{aligned}$$

We emphasize that the last identity is valid for functions of arbitrary sign. So some generalizations of Cauchy–Bunyakovsky inequality of the form (***) are valid for functions of any sign.

Corollary 2 *Let for functions $f(x)$, $g(x)$ the condition $f(x)+g(x)\ge 0$ is fulfilled. Then the inequality-chain (***) holds true.*

It follows from the above identity that if $f(x)+g(x)\le 0$, then (***) is reversed, and equality conditions in (***) also follows.

3 Application to Inequalities for Special Functions with Mellin Transform Representations

The inequality (***) from Theorem 4 may be applied to many special functions, especially represented as Mellin transform. Just as an example of such applications consider gamma and incomplete gamma functions, respectively [17]:

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t}\, dt, \Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t}\, dt, \gamma(a, x) = \int_0^x t^{a-1} e^{-t}\, dt,$$

Now, specify in (***):

$$f(x) = x^{\frac{a+1}{2}} e^{-\frac{x}{2}}, g(x) = x^{\frac{a-1}{2}} e^{-\frac{x}{2}}, a > 0.$$

It follows that

$$\max(f, g) = \begin{cases} f(x), x \geq 1, \\ g(x), x \leq 1, \end{cases} \quad \min(f, g) = \begin{cases} g(x), x \geq 1, \\ f(x), x \leq 1. \end{cases}$$

So by (***) we infer

$$\Gamma^2(a+1) \leq (\gamma(a+1, 1) + \Gamma(a-1, 1)) \cdot (\gamma(a-1, 1) + \Gamma(a+1, 1)) \leq \\ \leq \Gamma(a+2) \cdot \Gamma(a)$$

which is a Turán type inequality with respect to the argument a, cf. [18–21]. Therefore the following result is proved.

Theorem 5 *Let* $a > 0$*, then the above inequality for gamma and incomplete gamma functions is valid.*

Note that this is a stronger result than log-convexity of classical gamma function. It may be combined with known results on log-convexity of some special functions [18, 19] *to derive new inequalities. In fact, the above inequality in the same way may be proved for any functions represented as Mellin transform.*

4 Concluding Remarks and Bibliography Comments

The above proofs of refinements of Cauchy–Bunyakovsky inequalities may be also applied in more general settings: infinite domains of integration, Lebesgue integrals, multivariate functions and its domains. But accurate proofs for such generalizations are not always direct and easy.

Our results are also generalized to integral Minkowski and discrete Cauchy–Bunyakovsky and Minkowski inequalities.

It seems that before publications in 1990s summed up in the survey [7] inequality (***) from Theorem 4 was never published. But for recent years it became popular and was reopened not once and also generalized in interesting directions. Let us mention some of them.

In 2013 the inequality (***) was reopened in [22], the proofs are based on an equality like Corollary 1 to Theorem 4. Starting from the discrete case the author went to integral case by limits. For the discrete case, an interesting combinatorial proof is proposed. The author also considered some examples to demonstrate non-triviality of (***), but instead of our 'envelope' piecewise constant functions are used. In this paper (***) is considered in equivalent form

$$\left(\int_a^b f(x)g(x)dx\right)^2 \le \int_a^b [\max(f^2, g^2)]\, dx \cdot \int_a^b [\min(f^2, g^2)]\, dx$$

$$\le \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx,$$

as for positive functions

$$\max(f^2, g^2) = (\max(f, g))^2, \min(f^2, g^2) = (\min(f, g))^2.$$

Here, input functions $f, g, f(x) + g(x) \ge 0$ also can be considered in the manner exposed previously.

In interesting papers of Pinelis [23, 24] also inequality (***) was considered with proper references to the original result in [7]. The author proved an intriguing result, that generalizations of the form (***) exist only for the Cauchy–Bunyakovsky inequality and do not exist for Rogers–Hölder–Riesz inequality. Exactly it is proved that refinements to Rogers–Hölder–Riesz inequality of the form

$$\int_a^b f(x)g(x)\, dx \le \left(\int_a^b (\max(f, g))^p\, dx\right)^{\frac{1}{p}} \cdot \left(\int_a^b (\min(f, g))^q\, dx\right)^{\frac{1}{q}}$$

$$\le \left(\int_a^b f^p(x)\, dx\right)^{\frac{1}{p}} \cdot \left(\int_a^b g^q(x)\, dx\right)^{\frac{1}{q}}$$

does not hold for $p > 1, 1/p + 1/q = 1$, except for $p = q = 2$ which turns out to be the Cauchy–Bunyakovsky inequality's interpolation. Moreover, those refinements do not hold in the dual form

$$\int_a^b f(x)g(x)\, dx \le \left(\int_a^b (\max(f, g))^q\, dx\right)^{\frac{1}{q}} \cdot \left(\int_a^b (\min(f, g))^p\, dx\right)^{\frac{1}{p}}$$

$$\le \left(\int_a^b f^p(x)\, dx\right)^{\frac{1}{p}} \cdot \left(\int_a^b g^q(x)\, dx\right)^{\frac{1}{q}}.$$

And more, even a compound of last two inequalities in the weakest form do not hold, it means that the next inequality do not hold

$$\min\left\{\left(\int_a^b (\max(f,g))^p\,dx\right)^{\frac{1}{p}}\cdot\left(\int_a^b (\min(f,g))^q\,dx\right)^{\frac{1}{q}},\right.$$

$$\left.\left(\int_a^b (\max(f,g))^q\,dx\right)^{\frac{1}{q}}\cdot\left(\int_a^b (\min(f,g))^p\,dx\right)^{\frac{1}{p}}\right\}$$

$$\leq \max\left\{\left(\int_a^b f^p(x)\,dx\right)^{\frac{1}{p}}\cdot\left(\int_a^b g^q(x)\,dx\right)^{\frac{1}{q}},\right.$$

$$\left.\left(\int_a^b f^q(x)\,dx\right)^{\frac{1}{q}}\cdot\left(\int_a^b g^p(x)\,dx\right)^{\frac{1}{p}}\right\}$$

for some choice of $f(x)$, $g(x)$. A counterexample is found in [23, 24] that even such minimized variant of LHS can be greater than maximized variant of RHS for Rogers–Hölder–Riesz inequality. It is worth to mention that in the case of Young inequality this kind of optimization is successfully used in [7].

So the case we consider in this paper of refinement of Cauchy–Bunyakovsky inequality in the form (***) is exceptional! Due to it it is even more interesting.

The author in [23, 24] also considered generalizations not only with max and min values, but also with more general transformations. In turn, it seems that these are covered by the famous Carlitz–Daykin–Eliezer theorem (CDE theorem), (cf. [3, 5] for classical formulation and [7] for the formulation via means).

Mention also papers of 2015–2016 [25, 26] which do not contain original new results and are compiled of consequences of Carlitz–Daykin–Eliezer theorem (authors do not mention this theorem), known results from [7] and further trivial applications.

Another important connected line of results is the reverse Cauchy–Bunyakovsky inequality, including Schweitzer, Kantorovich, Pólya–Szegó, Shisha–Mond, Diaz–Metcalf, Rennie and similar inequalities, see e.g. [5, 27–29].

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