Chapter 8
A Coupled Fixed Point Problem Under a Finite Number of Equality Constraints

Let $(E, \|\cdot\|)$ be a Banach space with a cone P. Let $F, \varphi_i : E \times E \to E$ $(i = 1, 2, \ldots, r)$ be a finite number of mappings. In this chapter, we provide sufficient conditions for the existence and uniqueness of solutions to the problem: Find $(x, y) \in E \times E$ such that

$$\begin{cases} F(x, y) = x, \\ F(y, x) = y, \\ \varphi_i(x, y) = 0_E, \ i = 1, 2, \ldots, r, \end{cases} \tag{8.1}$$

where 0_E is the zero vector of E. The main reference for this chapter is the paper [4].

8.1 Preliminaries

At first, let us recall some basic definitions and some preliminary results that will be used later. In this chapter, the considered Banach space $(E, \|\cdot\|)$ is supposed to be partially ordered by a cone P. Recall that a nonempty closed convex set $P \subset E$ is said to be a cone (see [2]) if it satisfies the following conditions:

(P1) $\lambda \geq 0, \ x \in P \Longrightarrow \lambda x \in P$;
(P2) $-x, x \in P \Longrightarrow x = 0_E$.

We define the partial order $\leq_P$ in E induced by the cone P by

$$(x, y) \in E \times E, \quad x \leq_P y \Longleftrightarrow y - x \in P.$$

Definition 8.1 ([1]) Let $\varphi : E \times E \to E$ be a given mapping. We say that φ is level closed from the right if for every $e \in E$, the set

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$$\text{lev}\varphi_{\leq_P}(e) := \{(x, y) \in E \times E : \varphi(x, y) \leq_P e\}$$

is closed.

Definition 8.2 Let $\varphi : E \times E \to E$ be a given mapping. We say that φ is level closed from the left if for every $e \in E$, the set

$$\text{lev}\varphi_{\geq_P}(e) := \{(x, y) \in E \times E : e \leq_P \varphi(x, y)\}$$

is closed.

We denote by Ψ the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the conditions:

(Ψ_1) ψ is nondecreasing;
(Ψ_2) For all $t > 0$, we have

$$\sum_{k=0}^{\infty} \psi^k(t) < \infty.$$

Here, ψ^k is the kth iterate of ψ.

The following properties are not difficult to prove.

Lemma 8.1 *Let $\psi \in \Psi$. Then*

(i) $\psi(t) < t, t > 0$;
(ii) $\psi(0) = 0$;
(iii) ψ *is continuous at* $t = 0$.

Example 8.1 As examples, the following functions belong to the set Ψ:

$\psi(t) = kt, k \in (0, 1)$.

$$\psi(t) = \begin{cases} t/2 & \text{if } 0 \leq t \leq 1, \\ 1/2 & \text{if } t > 1. \end{cases}$$

$$\psi(t) = \begin{cases} t/2 & \text{if } 0 \leq t < 1, \\ t - 1/3 & \text{if } t \geq 1. \end{cases}$$

Now, we are ready to state and prove the main results of this chapter. This is the aim of the next section.

8.2 Main Results

Through this chapter, $(E, \|\cdot\|)$ is a Banach space partially ordered by a cone P and 0_E denotes the zero vector of E.

Let us start with the case of one equality constraint.

8.2.1 A Coupled Fixed Point Problem Under One Equality Constraint

We are interested with the existence and uniqueness of solutions to the problem: Find $(x, y) \in E \times E$ such that

$$\begin{cases} F(x, y) = x, \\ F(y, x) = y, \\ \varphi(x, y) \ = 0_E, \end{cases} \tag{8.2}$$

where $F, \varphi : E \times E \to E$ are two given mappings.

The following theorem provides sufficient conditions for the existence and uniqueness of solutions to (8.2).

Theorem 8.1 *Let $F, \varphi : E \times E \to E$ be two given mappings. Suppose that the following conditions are satisfied:*

(i) φ is level closed from the right.
(ii) There exists $(x_0, y_0) \in E \times E$ such that $\varphi(x_0, y_0) \leq_P 0_E$.
(iii) For every $(x, y) \in E \times E$, we have

$$\varphi(x, y) \leq_P 0_E \Longrightarrow \varphi(F(x, y), F(y, x)) \geq_P 0_E.$$

(iv) For every $(x, y) \in E \times E$, we have

$$\varphi(x, y) \geq_P 0_E \Longrightarrow \varphi(F(x, y), F(y, x)) \leq_P 0_E.$$

(v) There exists some $\psi \in \Psi$ such that

$$\|F(u, v) - F(x, y)\| + \|F(y, x) - F(v, u)\| \leq \psi\left(\|u - x\| + \|v - y\|\right),$$

for all $(x, y), (u, v) \in E \times E$ with $\varphi(x, y) \leq_P 0_E$, $\varphi(u, v) \geq_P 0_E$.

Then (8.2) has a unique solution.

Proof Let $(x_0, y_0) \in E \times E$ be such that

$$\varphi(x_0, y_0) \leq_p 0_E.$$

Such a point exists from (ii). From (iii), we have

$$\varphi(x_0, y_0) \leq_P 0_E \Longrightarrow \varphi(F(x_0, y_0), F(y_0, x_0)) \geq_P 0_E.$$

Define the sequences $\{x_n\}$ and $\{y_n\}$ in E by

$$x_{n+1} = F(x_n, y_n), \ \ y_{n+1} = F(y_n, x_n), \quad n = 0, 1, 2, \ldots$$

Then we have

$$\varphi(x_1, y_1) \geq_P 0_E.$$

From (iv), we have

$$\varphi(x_1, y_1) \geq_P 0_E \Longrightarrow \varphi(F(x_1, y_1), F(y_1, x_1)) \leq_P 0_E,$$

that is,

$$\varphi(x_2, y_2) \leq_P 0_E.$$

Again, using (iii), we get from the above inequality that

$$\varphi(x_3, y_3) \geq_P 0_E.$$

Then, by induction, we obtain

$$\varphi(x_{2n}, y_{2n}) \leq_P 0_E, \ \varphi(x_{2n+1}, y_{2n+1}) \geq_P 0_E, \quad n = 0, 1, 2, \ldots \tag{8.3}$$

Using (v) and (8.3), by symmetry, we obtain

$$\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \leq \psi\left(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\right), \quad n = 1, 2, 3, \ldots \tag{8.4}$$

From (8.4), since ψ is a nondecreasing function, for every $n = 1, 2, 3, \ldots$, we have

$$\begin{aligned}\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| &\leq \psi\left(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\right)\\ &\leq \psi^2\left(\|x_{n-1} - x_{n-2}\| + \|y_{n-1} - y_{n-2}\|\right)\\ &\leq \cdots\\ &\leq \psi^n\left(\|x_1 - x_0\| + \|y_1 - y_0\|\right).\end{aligned} \tag{8.5}$$

Suppose that

$$\|x_1 - x_0\| + \|y_1 - y_0\| = 0.$$

In this case, we have

$$x_0 = x_1 = F(x_0, y_0) \quad \text{and} \quad y_0 = y_1 = F(y_0, x_0).$$

Moreover, from (iii), since $\varphi(x_0, y_0) \leq_P 0_E$, we obtain $\varphi(x_1, y_1) = \varphi(x_0, y_0) \geq 0_E$. Since P is a cone, the two inequalities $\varphi(x_0, y_0) \leq_P 0_E$ and $\varphi(x_0, y_0) \geq_P 0_E$ yield

$$\varphi(x_0, y_0) = 0_E.$$

Thus, we proved that in this case, $(x_0, y_0) \in E \times E$ is a solution to (8.2).

Now, we may suppose that $\|x_1 - x_0\| + \|y_1 - y_0\| \neq 0$. Set

$$\delta = \|x_1 - x_0\| + \|y_1 - y_0\| > 0.$$

From (8.5), we have

$$\|x_{n+1} - x_n\| \leq \psi^n(\delta), \quad n = 0, 1, 2, \ldots \tag{8.6}$$

Using the triangular inequality and (8.6), for all $m = 1, 2, 3, \ldots$, we have

$$\begin{aligned}
\|x_n - x_{n+m}\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + \cdots + \|x_{n+m-1} - x_{n+m}\| \\
&\leq \psi^n(\delta) + \psi^{n+1}(\delta) + \cdots + \psi^{n+m-1}(\delta) \\
&= \sum_{i=n}^{n+m-1} \psi^i(\delta) \\
&\leq \sum_{i=n}^{\infty} \psi^i(\delta).
\end{aligned}$$

On the other hand, since $\sum_{k=0}^{\infty} \psi^k(\delta) < \infty$, we have

$$\sum_{i=n}^{\infty} \psi^i(\delta) \to 0 \text{ as } n \to \infty,$$

which implies that $\{x_n\}$ is a Cauchy sequence in $(E, \|\cdot\|)$. The same argument gives us that $\{y_n\}$ is a Cauchy sequence in $(E, \|\cdot\|)$. As consequence, there exists a pair of points $(x^*, y^*) \in E \times E$ such that

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n\to\infty} \|y_n - y^*\| = 0. \tag{8.7}$$

From (8.3), we have

$$\varphi(x_{2n}, y_{2n}) \leq_P 0_E, \quad n = 0, 1, 2, \ldots,$$

that is,

$$(x_{2n}, y_{2n}) \in \text{lev}\varphi_{\leq_P}(0_E), \quad n = 0, 1, 2, \ldots,$$

Since φ is level closed from the right, passing to the limit as $n \to \infty$ and using (8.7), we obtain

$$(x^*, y^*) \in \text{lev}\varphi_{\leq_P}(0_E),$$

that is,

$$\varphi(x^*, y^*) \leq_P 0_E. \tag{8.8}$$

Now, using (8.3), (8.8), and (v), we obtain

$$\|F(x_{2n+1}, y_{2n+1}) - F(x^*, y^*)\| + \|F(y^*, x^*) - F(y_{2n+1}, x_{2n+1})\|$$
$$\leq \psi\left(\|x_{2n+1} - x^*\| + \|y_{2n+1} - y^*\|\right),$$

for all $n = 0, 1, 2, \ldots$, which implies that

$$\|x_{2n+2} - F(x^*, y^*)\| + \|F(y^*, x^*) - y_{2n+2}\| \leq \psi\left(\|x_{2n+1} - x^*\| + \|y_{2n+1} - y^*\|\right),$$

for all $n = 0, 1, 2, \ldots$ Passing to the limit as $n \to \infty$, using (8.7), the continuity of ψ at 0, and the fact that $\psi(0) = 0$ (see Lemma 8.1), we get

$$\|x^* - F(x^*, y^*)\| + \|F(y^*, x^*) - y^*\| = 0,$$

that is,

$$x^* = F(x^*, y^*) \quad \text{and} \quad y^* = F(y^*, x^*).$$

This proves that $(x^*, y^*) \in E \times E$ is a coupled fixed point of F. Finally, using (8.8) and the fact that (x^*, y^*) is a coupled fixed point of F, it follows from (iii) that

$$\varphi(x^*, y^*) \geq_P 0_E. \tag{8.9}$$

Then (8.8) and (8.9) yield

$$\varphi(x^*, y^*) = 0_E.$$

Thus, we proved that $(x^*, y^*) \in E \times E$ is a solution to (8.2). Suppose now that $(u^*, v^*) \in E \times E$ is a solution to (8.2) with $(x^*, y^*) \neq (u^*, v^*)$. Using (v), we obtain

$$\|u^* - x^*\| + \|y^* - v^*\| \leq \psi(\|u^* - x^*\| + \|y^* - v^*\|).$$

Since $\|u^* - x^*\| + \|y^* - v^*\| > 0$, from (i) of Lemma 8.1, we have

$$\psi(\|u^* - x^*\| + \|y^* - v^*\|) < \|u^* - x^*\| + \|y^* - v^*\|.$$

Then

$$\|u^* - x^*\| + \|y^* - v^*\| < \|u^* - x^*\| + \|y^* - v^*\|,$$

which is a contradiction. As consequence, (x^*, y^*) is the unique solution to (8.2).

Remark 8.1 Observe that the conclusion of Theorem 8.1 is still valid if we replace condition (i) by the following condition:
(i') φ is level closed from the left.
In fact, from (8.3), we have

$$\varphi(x_{2n+1}, y_{2n+1}) \geq_P 0_E, \quad n = 0, 1, 2, \ldots,$$

that is,

$$(x_{2n+1}, y_{2n+1}) \in \text{lev}\varphi_{\geq_P}, \quad n = 0, 1, 2, \ldots$$

Passing to the limit as $n \to \infty$ and using (8.7), we obtain

$$\varphi(x^*, y^*) \geq_P 0_E. \tag{8.10}$$

Using (8.3), (8.10) and (v), we obtain

$$\|F(x_{2n}, y_{2n}) - F(x^*, y^*)\| + \|F(y^*, x^*) - F(y_{2n}, x_{2n})\| \leq \psi\left(\|x_{2n} - x^*\| + \|y_{2n} - y^*\|\right),$$

for all $n = 0, 1, 2, \ldots$, which implies that

$$\|x_{2n+1} - F(x^*, y^*)\| + \|F(y^*, x^*) - y_{2n+1}\| \leq \psi\left(\|x_{2n} - x^*\| + \|y_{2n} - y^*\|\right),$$

for all $n = 0, 1, 2, \ldots$ Passing to the limit as $n \to \infty$, we get

$$\|x^* - F(x^*, y^*)\| + \|F(y^*, x^*) - y^*\| = 0,$$

which proves that $(x^*, y^*) \in E \times E$ is a coupled fixed point of F. Using (8.10) and the fact that (x^*, y^*) is a coupled fixed point of F, it follows from (iv) that

$$\varphi(x^*, y^*) \leq_P 0_E. \tag{8.11}$$

Then (8.10) and (8.11) yield

$$\varphi(x^*, y^*) = 0_E.$$

Thus, $(x^*, y^*) \in E \times E$ is a solution to (8.2).

8.2.2 A Coupled Fixed Point Problem Under Two Equality Constraints

Here, we are interested with the existence and uniqueness of solutions to the following problem: Find $(x, y) \in E \times E$ such that

$$\begin{cases} F(x, y) &= x, \\ F(y, x) &= y, \\ \varphi_1(x, y) &= 0_E, \\ \varphi_2(x, y) &= 0_E, \end{cases} \tag{8.12}$$

where $F, \varphi_1, \varphi_2 : E \times E \to E$ are three given mappings.

We have the following result.

Theorem 8.2 *Let $F, \varphi_1, \varphi_2 : E \times E \to E$ be three given mappings. Suppose that the following conditions are satisfied:*

(i) φ_i *($i = 1, 2$) is level closed from the right.*
(ii) *There exists $(x_0, y_0) \in E \times E$ such that $\varphi_i(x_0, y_0) \leq_P 0_E$ ($i = 1, 2$).*
(iii) *For every $(x, y) \in E \times E$, we have*

$$\varphi_i(x, y) \leq_P 0_E,\ i = 1, 2 \Longrightarrow \varphi_i(F(x, y), F(y, x)) \geq_P 0_E,\ i = 1, 2.$$

(iv) *For every $(x, y) \in E \times E$, we have*

$$\varphi_i(x, y) \geq_P 0_E,\ i = 1, 2 \Longrightarrow \varphi_i(F(x, y), F(y, x)) \leq_P 0_E,\ i = 1, 2.$$

(v) *There exists some $\psi \in \Psi$ such that*

$$\|F(u, v) - F(x, y)\| + \|F(y, x) - F(v, u)\| \leq \psi\left(\|u - x\| + \|v - y\|\right),$$

for all $(x, y), (u, v) \in E \times E$ with $\varphi_i(x, y) \leq_P 0_E, \varphi_i(u, v) \geq_P 0_E$, $i = 1, 2$.

Then (8.12) has a unique solution.

Proof Let $(x_0, y_0) \in E \times E$ be such that

$$\varphi_i(x_0, y_0) \leq_p 0_E, \quad i = 1, 2.$$

Then from (iii), we have

$$\varphi_i(F(x_0, y_0), F(y_0, x_0)) \geq_P 0_E, \quad i = 1, 2.$$

Define the sequences $\{x_n\}$ and $\{y_n\}$ in E by

$$x_{n+1} = F(x_n, y_n),\ \ y_{n+1} = F(y_n, x_n), \quad n = 0, 1, 2, \ldots$$

We have

$$\varphi_i(x_1, y_1) \geq_P 0_E, \quad i = 1, 2.$$

Then from (iv), we obtain

$$\varphi_i(x_2, y_2) \leq_P 0_E, \quad i = 1, 2.$$

Again, using (iii), we get from the above inequality that

$$\varphi_i(x_3, y_3) \geq_P 0_E, \quad i = 1, 2.$$

Then, by induction, we obtain

$$\varphi_i(x_{2n}, y_{2n}) \leq_P 0_E, \ \varphi_i(x_{2n+1}, y_{2n+1}) \geq_P 0_E, \quad i = 1, 2, \ n = 0, 1, 2, \ldots$$

Then, using (v), we obtain

$$\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \leq \psi\left(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\right), \quad n = 1, 2, 3, \ldots$$

Now, we argue exactly as in the proof of Theorem 8.1 to show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $(E, \|\cdot\|)$. As consequence, there exists a pair of points $(x^*, y^*) \in E \times E$ such that

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n\to\infty} \|y_n - y^*\| = 0.$$

On the other hand, we have

$$(x_{2n}, y_{2n}) \in \text{lev}\varphi_{i\,\leq_P}(0_E), \quad i = 1, 2, \ n = 0, 1, 2, \ldots,$$

Since φ_i $(i = 1, 2)$ is level closed from the right, passing to the limit as $n \to \infty$, we obtain

$$(x^*, y^*) \in \text{lev}\varphi_{i\,\leq_P}(0_E), \quad i = 1, 2,$$

that is,

$$\varphi_i(x^*, y^*) \leq_P 0_E, \quad i = 1, 2.$$

Then we have

$$\begin{aligned}&\|F(x_{2n+1}, y_{2n+1}) - F(x^*, y^*)\| + \|F(y^*, x^*) - F(y_{2n+1}, x_{2n+1})\|\\ &\leq \psi\left(\|x_{2n+1} - x^*\| + \|y_{2n+1} - y^*\|\right),\end{aligned}$$

for all $n = 0, 1, 2, \ldots$, which implies that

$$\|x_{2n+2} - F(x^*, y^*)\| + \|F(y^*, x^*) - y_{2n+2}\| \leq \psi\left(\|x_{2n+1} - x^*\| + \|y_{2n+1} - y^*\|\right),$$

for all $n = 0, 1, 2, \ldots$ Passing to the limit as $n \to \infty$, we get

$$\|x^* - F(x^*, y^*)\| + \|F(y^*, x^*) - y^*\| = 0,$$

that is,

$$x^* = F(x^*, y^*) \quad \text{and} \quad y^* = F(y^*, x^*).$$

This proves that $(x^*, y^*) \in E \times E$ is a coupled fixed point of F. Since $\varphi_i(x^*, y^*) \leq_P 0_E$ for $i = 1, 2$, from (iii) we have

$$\varphi_i(F(x^*, y^*), F(y^*, x^*)) \geq_P 0_E, \quad i = 1, 2,$$

that is,

$$\varphi_i(x^*, y^*) \geq_P 0_E, \quad i = 1, 2.$$

Finally, the two inequalities $\varphi_i(x^*, y^*) \leq_P 0_E$ and $\varphi_i(x^*, y^*) \geq_P 0_E$, $i = 1, 2$ yield $\varphi_i(x^*, y^*) = 0_E$, $i = 1, 2$. Then we proved that $(x^*, y^*) \in E \times E$ is a solution to (8.12). The uniqueness can be obtained using a similar argument as in the proof of Theorem 8.1.

Replace φ_2 in Theorem 8.2 by $-\varphi_2$, we obtain the following result.

Theorem 8.3 *Let $F, \varphi_1, \varphi_2 : E \times E \to E$ be three given mappings. Suppose that the following conditions are satisfied:*

(i) *φ_1 is level closed from the right and φ_2 is level closed from the left.*
(ii) *There exists $(x_0, y_0) \in E \times E$ such that $\varphi_1(x_0, y_0) \leq_P 0_E$ and $\varphi_2(x_0, y_0) \geq_p 0_E$.*
(iii) *For every $(x, y) \in E \times E$ with $\varphi_1(x, y) \leq_P 0_E$ and $\varphi_2(x, y) \geq_P 0_E$, we have*

$$\varphi_1(F(x, y), F(y, x)) \geq_P 0_E, \ \varphi_2(F(x, y), F(y, x)) \leq_P 0_E.$$

(iv) *For every $(x, y) \in E \times E$ with $\varphi_1(x, y) \geq_P 0_E$ and $\varphi_2(x, y) \leq_P 0_E$, we have*

$$\varphi_1(F(x, y), F(y, x)) \leq_P 0_E, \ \varphi_2(F(x, y), F(y, x)) \geq_P 0_E.$$

(v) *There exists some $\psi \in \Psi$ such that*

$$\|F(u, v) - F(x, y)\| + \|F(y, x) - F(v, u)\| \leq \psi\left(\|u - x\| + \|v - y\|\right),$$

for all $(x, y), (u, v) \in E \times E$ with $\varphi_1(x, y) \leq_P 0_E$, $\varphi_2(x, y) \geq_P 0_E$, $\varphi_1(u, v) \geq_P 0_E$, $\varphi_2(u, v) \leq_P 0_E$.

Then (8.12) has a unique solution.

Replace φ_1 in Theorem 8.3 by $-\varphi_1$, we obtain the following result.

Theorem 8.4 *Let $F, \varphi_1, \varphi_2 : E \times E \to E$ be three given mappings. Suppose that the following conditions are satisfied:*

(i) *φ_i $(i = 1, 2)$ is level closed from the left.*
(ii) *There exists $(x_0, y_0) \in E \times E$ such that $\varphi_i(x_0, y_0) \geq_P 0_E$ $(i = 1, 2)$.*
(iii) *For every $(x, y) \in E \times E$, we have*

$$\varphi_i(x, y) \leq_P 0_E, \ i = 1, 2 \Longrightarrow \varphi_i(F(x, y), F(y, x)) \geq_P 0_E, \ i = 1, 2.$$

(iv) *For every $(x, y) \in E \times E$, we have*

$$\varphi_i(x, y) \geq_P 0_E, \ i = 1, 2 \Longrightarrow \varphi_i(F(x, y), F(y, x)) \leq_P 0_E, \ i = 1, 2.$$

(v) There exists some $\psi \in \Psi$ such that

$$\|F(u, v) - F(x, y)\| + \|F(y, x) - F(v, u)\| \leq \psi(\|u - x\| + \|v - y\|),$$

for all $(x, y), (u, v) \in E \times E$ with $\varphi_i(x, y) \leq_P 0_E$, $\varphi_i(u, v) \geq_P 0_E$, $i = 1, 2$.

Then (8.12) has a unique solution.

8.2.3 A Coupled Fixed Point Problem Under r Equality Constraints

Now, we argue exactly as in the proof of Theorem 8.2 to obtain the following existence result for (8.1).

Theorem 8.5 *Let $F, \varphi_i : E \times E \to E$ $(i = 1, 2, \ldots, r)$ be $r + 1$ given mappings. Suppose that the following conditions are satisfied:*

(i) φ_i $(i = 1, 2, \ldots, r)$ is level closed from the right.
(ii) There exists $(x_0, y_0) \in E \times E$ such that $\varphi_i(x_0, y_0) \leq_P 0_E$ $(i = 1, 2, \ldots, r)$.
(iii) For every $(x, y) \in E \times E$, we have

$$\varphi_i(x, y) \leq_P 0_E,\ i = 1, 2, \ldots, r \Longrightarrow \varphi_i(F(x, y), F(y, x)) \geq_P 0_E,\ i = 1, 2, \ldots, r.$$

(iv) For every $(x, y) \in E \times E$, we have

$$\varphi_i(x, y) \geq_P 0_E,\ i = 1, 2, \ldots, r \Longrightarrow \varphi_i(F(x, y), F(y, x)) \leq_P 0_E,\ i = 1, 2, \ldots r.$$

(v) There exists some $\psi \in \Psi$ such that

$$\|F(u, v) - F(x, y)\| + \|F(y, x) - F(v, u)\| \leq \psi(\|u - x\| + \|v - y\|),$$

for all $(x, y), (u, v) \in E \times E$ with $\varphi_i(x, y) \leq_P 0_E$, $\varphi_i(u, v) \geq_P 0_E$, $i = 1, 2, \ldots, r$.

Then (8.1) has a unique solution.

8.3 Some Consequences

In this section, we present some consequences following from Theorem 8.5.

8.3.1 A Fixed Point Problem Under Symmetric Equality Constraints

Let X be a nonempty set and let $F : X \times X \to X$ be a given mapping. Recall that that $x \in X$ is said to be a fixed point of F if $F(x, x) = x$.

Let $F, \varphi : E \times E \to E$ be given mappings. We consider the problem: Find $x \in E$ such that

$$\begin{cases} F(x, x) = x, \\ \varphi(x, x) \ = 0_E. \end{cases} \tag{8.13}$$

We have the following result.

Corollary 8.1 *Let $F, \varphi : E \times E \to E$ be two given mappings. Suppose that the following conditions are satisfied:*

(i) φ is level closed from the right.
(ii) φ is symmetric, that is,

$$\varphi(x, y) = \varphi(y, x), \quad (x, y) \in E \times E.$$

(iii) There exists $(x_0, y_0) \in E \times E$ such that $\varphi(x_0, y_0) \leq_P 0_E$.
(iv) For every $(x, y) \in E \times E$, we have

$$\varphi(x, y) \leq_P 0_E \Longrightarrow \varphi(F(x, y), F(y, x)) \geq_P 0_E.$$

(v) For every $(x, y) \in E \times E$, we have

$$\varphi(x, y) \geq_P 0_E \Longrightarrow \varphi(F(x, y), F(y, x)) \leq_P 0_E.$$

(vi) There exists some $\psi \in \Psi$ such that

$$\|F(u, v) - F(x, y)\| + \|F(y, x) - F(v, u)\| \leq \psi\left(\|u - x\| + \|v - y\|\right),$$

for all $(x, y), (u, v) \in E \times E$ with $\varphi(x, y) \leq_P 0_E$ and $\varphi(u, v) \geq_P 0_E$.

Then (8.13) has a unique solution.

Proof From Theorem 8.1, we know that (8.2) has a unique solution $(x^*, y^*) \in E \times E$. Since φ is symmetric, (y^*, x^*) is also a solution to (8.2). By uniqueness, we get $x^* = y^*$. Then $x^* \in E$ is the unique solution to (8.13).

Let $F, \varphi_i : E \times E \to E$ $(i = 1, 2, \ldots, r)$ be $r + 1$ given mappings. We consider the problem: Find $x \in X$ such that

$$\begin{cases} F(x, x) = x, \\ \varphi_i(x, x) = 0_E, \ \ i = 1, 2, \ldots, r. \end{cases} \tag{8.14}$$

Similarly, from Theorem 8.5, we have the following result.

Corollary 8.2 *Let $F, \varphi_i : E \times E \to E$ $(i = 1, 2, \ldots, r)$ be $r + 1$ given mappings. Suppose that the following conditions are satisfied:*

(i) φ_i *$(i = 1, 2, \ldots, r)$ is level closed from the right.*
(ii) φ_i *$(i = 1, 2, \ldots, r)$ is symmetric.*
(iii) There exists $(x_0, y_0) \in E \times E$ such that $\varphi_i(x_0, y_0) \leq_P 0_E$ $(i = 1, 2, \ldots, r)$.
(iv) For every $(x, y) \in E \times E$, we have

$$\varphi_i(x, y) \leq_P 0_E,\ i = 1, 2, \ldots, r \Longrightarrow \varphi_i(F(x, y), F(y, x)) \geq_P 0_E,\ i = 1, 2, \ldots, r.$$

(v) For every $(x, y) \in E \times E$, we have

$$\varphi_i(x, y) \geq_P 0_E,\ i = 1, 2, \ldots, r \Longrightarrow \varphi_i(F(x, y), F(y, x)) \leq_P 0_E,\ i = 1, 2, \ldots r.$$

(vi) There exists some $\psi \in \Psi$ such that

$$\|F(u, v) - F(x, y)\| + \|F(y, x) - F(v, u)\| \leq \psi\left(\|u - x\| + \|v - y\|\right),$$

for all $(x, y), (u, v) \in E \times E$ with $\varphi_i(x, y) \leq_P 0_E$, $\varphi_i(u, v) \geq_P 0_E$, $i = 1, 2, \ldots, r$.

Then (8.14) has a unique solution.

8.3.2 A Common Coupled Fixed Point Result

We need the following definition.

Definition 8.3 Let X be a nonempty set, $F : X \times X \to X$ and $g : X \to X$ be two given mappings. We say that the pair of elements $(x, y) \in X \times X$ is a common coupled fixed point of F and g if

$$x = gx = F(x, y) \quad \text{and} \quad y = gy = F(y, x).$$

We have the following common coupled fixed point result.

Corollary 8.3 *Let $F : E \times E \to E$ and $g : E \to E$ be two given mappings. Suppose that the following conditions hold:*

(i) g is a continuous mapping.
(ii) There exists $(x_0, y_0) \in E \times E$ such that

$$gx_0 \leq_p x_0 \quad \textit{and} \quad gy_0 \leq_p y_0.$$

(iii) For every $(x, y) \in E \times E$*, we have*

$$gx \leq_P x,\; gy \leq_p y \Longrightarrow gF(x, y) \geq_P F(x, y),\; gF(y, x) \geq_P F(y, x).$$

(iv) For every $(x, y) \in E \times E$*, we have*

$$gx \geq_P x,\; gy \geq_P y \Longrightarrow gF(x, y) \leq_P F(x, y),\; gF(y, x) \leq_P F(y, x).$$

(v) There exists some $\psi \in \Psi$ *such that*

$$\|F(u, v) - F(x, y)\| + \|F(y, x) - F(v, u)\| \leq \psi\left(\|u - x\| + \|v - y\|\right),$$

for all $(x, y), (u, v) \in E \times E$ *with* $gx \leq_P x$, $gy \leq_P y$ *and* $gu \geq_P u$, $gv \geq_P v$.

Then F *and* g *have a unique common coupled fixed point.*

Proof Let us consider the mappings $\varphi_1, \varphi_2 : E \times E \to E$ defined by

$$\varphi_1(x, y) = gx - x, \quad (x, y) \in E \times E$$

and

$$\varphi_2(x, y) = gy - y, \quad (x, y) \in E \times E.$$

Observe that $(x, y) \in E \times E$ is a common coupled fixed point of F and g if and only if $(x, y) \in E \times E$ is a solution to (8.12). Note that since g is continuous, then φ_i is level closed from the right (also from the left) for all $i = 1, 2$. Now, applying Theorem 8.2, we obtain the desired result.

8.3.3 A Fixed Point Result

We denote by $\widetilde{\Psi}$ the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

($\widetilde{\Psi}_1$) $\psi \in \Psi$.
($\widetilde{\Psi}_2$) For all $a, b \in [0, \infty)$, we have

$$\psi(a) + \psi(b) \leq \psi(a + b).$$

Example 8.2 As example, let us consider the function

$$\psi(t) = \begin{cases} t/2 & \text{if } 0 \leq t < 1, \\ t - 1/3 & \text{if } t \geq 1. \end{cases}$$

It is not difficult to observe that $\psi \in \Psi$. Now, let us consider an arbitrary pair $(a, b) \in [0, \infty) \times [0, \infty)$. We discuss three possible cases.

Case 1. If $(a, b) \in [0, 1) \times [0, 1)$.
In this case, we have $\psi(a) + \psi(b) = (a + b)/2$. On the other hand, we have $a + b \in [0, 2)$. So, if $0 \le a + b < 1$, then $\psi(a) + \psi(b) = (a + b)/2 = \psi(a + b)$. However, if $1 \le a + b < 2$, then $\psi(a + b) - \psi(a) - \psi(b) = (a + b)/2 - 1/3 \ge 0$.

Case 2. If $(a, b) \in [0, 1) \times [1, \infty)$.
In this case, we have $\psi(a) + \psi(b) = a/2 + b - 1/3 \le a + b - 1/3 = \psi(a + b)$.

Case 3. If $(a, b) \in [1, \infty) \times [1, \infty)$.
In this case, we have $\psi(a) + \psi(b) = a + b - 2/3 \le a + b - 1/3 = \psi(a + b)$.
Therefore, we have $\psi \in \widetilde{\Psi}$.

Note that the set Ψ is more large than the set $\widetilde{\Psi}$. The following example illustrates this fact.

Example 8.3 Let us consider the function

$$\psi(t) = \begin{cases} t/2 & \text{if } 0 \le t \le 1, \\ 1/2 & \text{if } t > 1. \end{cases}$$

Clearly, we have $\psi \in \Psi$. However,

$$\psi(1 + 1) = 1/2 < 1 = \psi(1) + \psi(1),$$

which proves that $\psi \notin \widetilde{\Psi}$.

We have the following fixed point result.

Corollary 8.4 *Let $T : E \to E$ be a given mapping. Suppose that there exists some $\psi \in \widetilde{\Psi}$ such that*

$$\|Tu - Tx\| \le \psi(\|u - x\|), \quad (u, x) \in E \times E. \tag{8.15}$$

Then T has a unique fixed point.

Proof Let us define the mapping $F : E \times E \to E$ by

$$F(x, y) = Tx, \quad (x, y) \in E \times E.$$

Let $g : E \to E$ be the identity mapping, that is,

$$gx = x, \quad x \in E.$$

From (8.15), for all $(x, y), (u, v) \in E \times E$, we have

$$\|Tu - Tx\| \le \psi(\|u - x\|)$$

and

$$\|Ty - Tv\| \leq \psi(\|v - y\|).$$

Then

$$\|Tu - Tx\| + \|Ty - Tv\| \leq \psi(\|u - x\|) + \psi(\|v - y\|).$$

Using the property $(\widetilde{\Psi}_2)$, we obtain

$$\|Tu - Tx\| + \|Ty - Tv\| \leq \psi(\|u - x\| + \|v - y\|), \quad (x, y), (u, v) \in E \times E.$$

From the definitions of F and g, we obtain

$$\|F(u, v) - F(x, y)\| + \|F(y, x) - F(v, u)\| \leq \psi\left(\|u - x\| + \|v - y\|\right),$$

for all $(x, y), (u, v) \in E \times E$ with $gx \leq_P x$, $gy \leq_P y$ and $gu \geq_P u$, $gv \geq_P v$. By Corollary 3.5, there exists a unique $(x^*, y^*) \in E \times E$ such that

$$x^* = F(x^*, y^*) = Tx^* \quad \text{and} \quad y^* = F(y^*, x^*) = Ty^*.$$

Suppose that $x^* \neq y^*$. By (8.15), we have

$$\|x^* - y^*\| = \|Tx^* - Ty^*\| \leq \psi(\|x^* - y^*)) < \|x^* - y^*\|,$$

which is a contradiction. As consequence, $x^* \in E$ is the unique fixed point of T.

Remark 8.2 Taking

$$\psi(t) = kt, \quad t \geq 0,$$

where $k \in (0, 1)$ is a constant, we obtain from Corollary 8.4 the Banach contraction principle.

Finally, for other related results, we refer the reader to Jleli and Samet [3].

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