Chapter 7
On Fixed Points That Belong to the Zero Set of a Certain Function

Let $T : X \to X$ be a given mapping. The set $\text{Fix}(T)$ is said to be φ-admissible with respect to a certain mapping $\varphi : X \to [0, \infty)$, if $\emptyset \neq \text{Fix}(T) \subseteq Z_\varphi$, where Z_φ denotes the zero set of φ, i.e., $Z_\varphi = \{x \in X : \varphi(x) = 0\}$. In this chapter, we present the class of extended simulation functions recently introduced by Roldán and Samet [13], which is more large than the class of simulation functions, introduced by Khojasteh et al. [8]. We obtain a φ-admissibility result involving extended simulation functions, for a new class of mappings $T : X \to X$, with respect to a lower semi-continuous function $\varphi : X \to [0, \infty)$, where X is a set equipped with a certain metric d. From the obtained results, some fixed point theorems in partial metric spaces are derived, including Matthews fixed point theorem [9]. Moreover, we answer to three open problems posed by Ioan A. Rus in [16]. The main references for this chapter are the papers [7, 13, 17].

7.1 Partial Metric Spaces

In 1994, Matthews [9] introduced the concept of partial metric spaces as a part of the study of denotational semantics of dataflow networks and showed that Banach contraction principle can be generalized to the partial metric context for applications in program verification. Later on, many authors studied fixed point theorems on partial metric spaces (see, e.g., [1, 2, 5, 6, 10, 11, 14, 15, 18, 19] and references therein).

We start this section by recalling some basic definitions and properties of partial metric spaces (see [9] for more details).

Definition 7.1 A partial metric on a nonempty set X is a mapping $p : X \times X \to [0, \infty)$ satisfying the following axioms: For all $x, y, z \in X$, we have

(i) $p(x, x) = p(y, y) = p(x, y) \Longleftrightarrow x = y$;

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(ii) $p(x, x) \leq p(x, y)$;
(iii) $p(x, y) = p(y, x)$;
(iv) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

In this case, the pair (X, p) is said to be a partial metric space.

Remark 7.1 It is clear that, if $p(x, y) = 0$, then $x = y$; but if $x = y$, $p(x, y)$ may not be 0.

Example 7.1 A basic example of a partial metric space is the pair $([0, \infty), p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in [0, \infty)$. Other examples of partial metric spaces which are interesting from a computational point of view may be found in [9].

The next definitions generalize the metric space notions of convergent sequences and Cauchy sequences to partial metric spaces.

Definition 7.2 A sequence $\{x_n\}$ of points in a partial metric space (X, p) converges to $x \in X$ if

$$\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n) = p(x, x).$$

Definition 7.3 A sequence $\{x_n\}$ of points in a partial metric space (X, p) is Cauchy if $\lim_{m,n\to\infty} p(x_n, x_m)$ exists and is finite.

Definition 7.4 A partial metric space (X, p) is complete if every Cauchy sequence converges.

The following result can be shown easily.

Lemma 7.1 *Let X be a nonempty set and $p : X \times X \to [0, \infty)$ be a partial metric on X. Let $d_p : X \times X \to [0, \infty)$ be the mapping defined by*

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \quad (x, y) \in X \times X.$$

Then d_p is a metric on X.

Lemma 7.2 (see [10]) *Let (X, p) be a partial metric space. Then*

(i) $\{x_n\}$ is Cauchy in (X, p) if and only if $\{x_n\}$ is Cauchy in the metric space (X, d_p).
(ii) The partial metric space (X, p) is complete if and only if the metric space (X, d_p) is complete. Furthermore, $\lim_{n\to\infty} d_p(x_n, x) = 0$ if and only if

$$\lim_{n\to\infty} p(x_n, x) = p(x, x) = \lim_{m,n\to\infty} p(x_n, x_m).$$

In [9], Matthews obtained a partial metric version of Banach contraction principle as follows.

Theorem 7.1 (Matthews fixed point theorem) *Let (X, p) be a complete partial metric space. Let $T : X \to X$ be a contraction; i.e., there exists some constant $k \in (0, 1)$ such that*

$$p(Tx, Ty) \leq k\, p(x, y), \quad (x, y) \in X \times X. \tag{7.1}$$

Then T has a unique fixed point $x^* \in X$. Moreover, we have $p(x^*, x^*) = 0$.

Under the assumptions of Theorem 7.1, we observe easily that

$$\emptyset \neq \mathrm{Fix}(T) \subseteq Z_\varphi,$$

where Z_φ denotes the zero set of $\varphi(x) = p(x, x)$. A point $x \in X$ satisfying $p(x, x) = 0$ is called a total element (see [16]).

7.2 Three Open Questions of I.A. Rus

In [16], Ioan A. Rus presented three interesting open problems. Let (X, p) be a complete partial metric space.

Problem 1 If $T : (X, p) \to (X, p)$ is a contraction, which condition satisfies T with respect to the metric d_p?

Problem 2 It consists to give fixed point theorems for these new classes of operators on the metric space (X, d_p).

Problem 3 Use the results for the above problems to give fixed point theorems in a partial metric space.

The purpose of this chapter is to study the φ-admissibility for a new class of mappings $T : X \to X$, with respect to a lower semi-continuous function $\varphi : X \to [0, \infty)$, where X is a set equipped with a certain metric d. Next, from the obtained results, some fixed point theorems in partial metric spaces are derived, including Matthews fixed point theorem [9]. This contribution presents answers to the above problems of Ioan A. Rus.

7.3 The Class of Extended Simulation Functions

The class of simulation functions was introduced recently in [8] as follows.

Definition 7.5 Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a given map. We say that ζ is a simulation function if it satisfies the following conditions:

(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < s - t$, for every $t, s > 0$;
(ζ_3) For any sequences $\{t_n\}, \{s_n\} \subset (0,\infty)$, we have

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0 \Longrightarrow \limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Several examples of simulation functions were given in [8]. Let us denote by $\mathscr{Z}$ the st of all simulation functions.

Definition 7.6 ([8]) Let $T : X \to X$ be a given map, where X is endowed with a certain metric d. We say that T is a $\mathscr{Z}$-contraction with respect to a certain simulation function $\zeta \in \mathscr{Z}$ if

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0, \quad (x, y) \in X \times X.$$

The main result in [8] is the following fixed point theorem that generalizes and unifies several previous fixed point results from the literature including Banach contraction principle.

Theorem 7.2 ([8]) *Let $T : X \to X$ be a given map, where X is a set endowed with a certain metric d such that (X, d) is complete. If T is a $\mathscr{Z}$-contraction with respect to a certain simulation function $\zeta \in \mathscr{Z}$, then T has a unique fixed point. Moreover, for any $x \in X$, the Picard sequence $\{T^n x\}$ converges to this fixed point.*

The following concept was introduced in [13].

Definition 7.7 An extended simulation function (for short, an e-simulation function) is a function $\theta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following axioms:

(θ_1) $\theta(t,s) < s - t$, for every $t, s > 0$;
(θ_2) For any sequences $\{t_n\}, \{s_n\} \subset (0,\infty)$, we have

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0,\infty),\ s_n > \ell,\ n \in \mathbb{N} \Longrightarrow \limsup_{n\to\infty} \theta(t_n, s_n) < 0;$$

(θ_3) For any sequence $\{t_n\} \subset (0,\infty)$, we have

$$\lim_{n\to\infty} t_n = \ell \in [0,\infty),\ \theta(t_n, \ell) \geq 0,\ n \in \mathbb{N} \Longrightarrow \ell = 0.$$

Let us denote by $\mathscr{E}_Z$ the set of all e-simulation functions. In the following, we compare the set $\mathscr{E}_Z$ with the set $\mathscr{Z}$.

Proposition 7.1 *Every simulation function is an e-simulation function.*

Proof Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a simulation function. We have just to prove that the function ζ satisfies axiom (θ_3). Let $\{t_n\} \subset (0,\infty)$ be a sequence converging to $\ell \geq 0$, and such that

$$\zeta(t_n, \ell) \geq 0, \quad n \in \mathbb{N}. \tag{7.2}$$

Suppose that $\ell > 0$. Let us consider the sequence $\{s_n\} \subset (0, \infty)$ given by

$$s_n = \ell, \quad n \in \mathbb{N}.$$

Using axiom (ζ_3), we obtain

$$\limsup_{n\to\infty} \zeta(t_n, \ell) = \limsup_{n\to\infty} \zeta(t_n, s_n) < 0,$$

which is a contradiction with (7.2). Therefore, $\ell = 0$, and (θ_3) holds.

The converse of Proposition 7.1 is not true as it is shown by the following example.

Example 7.2 Let us consider the function $\theta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ defined by

$$\theta(t, s) = \begin{cases} 1 - t & \text{if } s = 0, \\ \dfrac{s}{2} - t & \text{if } s > 0. \end{cases}$$

At first, observe that $\theta \notin \mathscr{Z}$. In fact, $\theta(0, 0) = 1 \neq 0$, so axiom (ζ_1) is not satisfied. Let us prove now that $\theta \in \mathscr{E}_Z$. For all $t, s > 0$, we have

$$\theta(t, s) = \frac{s}{2} - t < s - t,$$

which yields (θ_1). Let $\{t_n\}$ and $\{s_n\}$ be two sequences in $(0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0, \infty).$$

We have

$$\theta(t_n, s_n) = \frac{s_n}{2} - t_n, \quad n \in \mathbb{N}.$$

Therefore,

$$\limsup_{n\to\infty} \theta(t_n, s_n) = -\frac{\ell}{2} < 0,$$

which proves (θ_2). Finally, let $\{t_n\}$ be a sequence in $(0, \infty)$ that converges to some $\ell \geq 0$, and such that

$$\theta(t_n, \ell) \geq 0, \quad n \in \mathbb{N}.$$

Suppose that $\ell > 0$. Then

$$\theta(t_n, \ell) = \frac{\ell}{2} - t_n \geq 0, \quad n \in \mathbb{N},$$

i.e.,

$$t_n \leq \frac{\ell}{2}, \quad n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$, we obtain

$$\ell \leq \frac{\ell}{2},$$

which is a contradiction with $\ell > 0$. Therefore, $\ell = 0$, and (θ_3) follows. As a consequence, $\theta \in \mathscr{E}_Z$.

For technical reasons, it is convenient to point that if we had considered the closed interval $[0, \infty)$ in Definition 7.7, then we would have obtained the same notion. The following result shows this fact.

Proposition 7.2 *Given a function* $\theta : [0, \infty) \times [0, \infty) \to \mathbb{R}$, *condition* (θ_2) *is equivalent to:*

(θ_2') *For any sequences* $\{t_n\}, \{s_n\} \subset [0, \infty)$, *we have*

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0, \infty), \ s_n > \ell, \ n \in \mathbb{N} \Longrightarrow \limsup_{n\to\infty} \theta(t_n, s_n) < 0.$$

Furthermore, property (θ_3) *is equivalent to:*

(θ_3') *For any sequence* $\{t_n\} \subset [0, \infty)$, *we have*

$$\lim_{n\to\infty} t_n = \ell \in [0, \infty), \ \theta(t_n, \ell) \geq 0, \ n \in \mathbb{N} \Longrightarrow \ell = 0.$$

Proof Clearly, we have $(\theta_2') \Longrightarrow (\theta_2)$. Let us prove the converse. Suppose that (θ_2) holds. Let $\{t_n\}$ and $\{s_n\}$ be two sequences in $[0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0, \infty), \ s_n > \ell, \ n \in \mathbb{N}.$$

Since $\ell > 0$, there exists some $N \in \mathbb{N}$ such that

$$t_n > 0, \ s_n > 0, \quad n \geq N + 1.$$

Let us define the sequences $\{T_n\}$ and $\{S_n\}$ by

$$T_0 = T_1 = \cdots = T_N = 1, \ T_n = t_n, \ n \geq N + 1$$

and

$$S_0 = S_1 = \cdots = S_N = \ell + 1, \ S_n = s_n, \ n \geq N + 1.$$

Then $\{T_n\}$ and $\{S_n\}$ are two sequences in $(0, \infty)$ converging to $\ell \in (0, \infty)$ with

$$S_n > \ell, \quad n \in \mathbb{N}.$$

By (θ_2), we obtain

$$\limsup_{n\to\infty} \theta(t_n, s_n) = \limsup_{n\to\infty} \theta(T_n, S_n) < 0,$$

from which (θ_2') follows. On the other hand, the implication $(\theta_3') \Longrightarrow (\theta_3)$ is obvious. Let us prove the converse. Suppose that (θ_3) holds true. Let $\{t_n\}$ be a sequence in $[0, \infty)$ converging to some $\ell \geq 0$, and such that

$$\theta(t_n, \ell) \geq 0, \quad n \in \mathbb{N}.$$

We have to prove that $\ell = 0$. Suppose that $\ell > 0$. Then there exists some $N \in \mathbb{N}$ such that

$$t_n > 0, \quad n \geq N + 1.$$

Define the sequence $\{T_n\}$ by

$$T_0 = T_1 = \cdots = T_N = t_{N+1}, \; T_n = t_n, \; n \geq N + 2.$$

Then $\{T_n\}$ is a sequence in $(0, \infty)$ converging to ℓ, and such that

$$\theta(T_n, \ell) \geq 0, \quad n \in \mathbb{N}.$$

By (θ_3), we obtain $\ell = 0$, which is a contradiction. Therefore, $\ell = 0$, and (θ_3') follows.

Remark 7.2 Properties (θ_2) and (θ_3) are easier to prove when we want to check that a given function is an e-simulation function. However, conditions (θ_2') and (θ_3') are useful when we assume that a given function is an e-simulation function.

Let Ψ be the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(ψ_1) ψ is upper semi-continuous from the right;
(ψ_2) $\psi(t) < t, t > 0$.

Lemma 7.3 *Given* $\psi \in \Psi$, *let* $\theta_\psi : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *be the function given by*

$$\theta_\psi(t, s) = \psi(s) - t, \quad t, s \geq 0. \tag{7.3}$$

Then θ_ψ *is an e-simulation function.*

Proof Let us check axiom (θ_1). For all $t, s > 0$, from property (ψ_2), we have

$$\theta_\psi(t, s) = \psi(s) - t < s - t,$$

which proves (θ_1). Let us consider two sequences $\{t_n\}$ and $\{s_n\}$ in $(0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0, \infty),\ s_n > \ell,\ n \in \mathbb{N}.$$

We have

$$\theta_\psi(t_n, s_n) = \psi(s_n) - t_n, \quad n \in \mathbb{N}.$$

Since from (ψ_1), the function ψ is upper semi-continuous form the right, we have

$$\psi(\ell) \geq \limsup_{n\to\infty} \psi(s_n),$$

which implies from (ψ_2) that

$$\limsup_{n\to\infty} \theta_\psi(t_n, s_n) \leq \psi(\ell) - \ell < 0.$$

Therefore, (θ_2) holds. Finally, we have to check axiom (θ_3). Let $\{t_n\}$ be a sequence in $(0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \ell \in [0, \infty),\ \theta_\psi(t_n, \ell) \geq 0,\ n \in \mathbb{N}.$$

Suppose that $\ell > 0$. We have

$$\psi(\ell) - t_n \geq 0, \quad n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$, we obtain

$$\psi(\ell) \geq \ell.$$

On the other hand, from (ψ_2), we have

$$\psi(\ell) < \ell,$$

which is a contradiction. Then $\ell = 0$, and (θ_3) holds. As a consequence, θ_ψ is an e-simulation function.

Remark 7.3 In general, if $\psi \in \Psi$, θ_ψ is not a simulation function. This fact can be shown by Example 7.2 with

$$\psi(s) = \begin{cases} 1 & \text{if } s = 0, \\ \dfrac{s}{2} & \text{if } s > 0. \end{cases}$$

However, if ψ is upper semi-continuous (rather than upper semi-continuous from the right), then we can modify θ_ψ to transform it in a simulation function. The next result shows this fact.

Proposition 7.3 *If $\psi \in \Psi$, then the function $\widetilde{\theta_\psi} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ given by*

$$\widetilde{\theta_\psi}(t,s) = \begin{cases} 0 \text{ if } t = s = 0, \\ \psi(s) - t \text{ otherwise} \end{cases}$$

is an e-simulation function. Furthermore, if $\psi \in \Psi$ is upper semi-continuous, then $\widetilde{\theta_\psi}$ is a simulation function.

Proof Let us prove first that $\widetilde{\theta_\psi}$ is an e-simulation function. For all $t, s > 0$, we have

$$\widetilde{\theta_\psi}(t,s) = \psi(s) - t < s - t,$$

which yields (θ_1). Let us consider two sequences $\{t_n\}$ and $\{s_n\}$ in $(0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0,\infty),\ s_n > \ell,\ n \in \mathbb{N}.$$

Then

$$\limsup_{n\to\infty} \widetilde{\theta_\psi}(t_n, s_n) = \limsup_{n\to\infty} \psi(s_n) - \ell \le \psi(\ell) - \ell < 0.$$

Therefore, (θ_2) holds. Finally, let $\{t_n\}$ be a sequence in $(0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \ell \in [0,\infty),\ \widetilde{\theta_\psi}(t_n, \ell) \ge 0,\ n \in \mathbb{N}.$$

Suppose that $\ell > 0$. Then

$$\widetilde{\theta_\psi}(t_n, \ell) = \psi(\ell) - t_n \ge 0, \quad n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$, and using axiom (ψ_2), we obtain

$$\ell \le \psi(\ell) < \ell,$$

which is a contradiction. Then $\ell = 0$, and (θ_3) follows. As a consequence, $\widetilde{\theta_\psi}$ is an e-simulation function.

Suppose now that $\psi \in \psi$ is upper semi-continuous. Let us prove that $\widetilde{\theta_\psi}$ is a simulation function. Observe that

$$\widetilde{\theta_\psi}(0,0) = 0,$$

which yields (ζ_1). Axiom (ζ_2) follows from the fact that $\widetilde{\theta_\psi}$ is an e-simulation function. Axiom (ζ_3) follows by using point by point the proof of (θ_2), and using the upper semi-continuity of ψ. Therefore, under the upper semi-continuity of $\psi \in \Psi$, $\widetilde{\theta_\psi}$ is a simulation function.

Remark 7.4 By (θ_1), if θ is an e-simulation function, then

$$\theta(r, r) < 0, \quad r > 0.$$

7.4 φ-Admissibility Results

The concept of φ-admissibility was introduced recently by Karapinar, Samet, and O'Regan in [7].

Definition 7.8 Let $T : X \to X$ be a given mapping. The set $\mathrm{Fix}(T)$ is said to be φ-admissible with respect to a certain mapping $\varphi : X \to [0, \infty)$, if

$$\emptyset \neq \mathrm{Fix}(T) \subseteq Z_\varphi,$$

where Z_φ denotes the zero set of φ, i.e.,

$$Z_\varphi = \{x \in X : \varphi(x) = 0\}.$$

Let $\mathscr{F}$ be the set of functions $F : [0, \infty)^3 \to [0, \infty)$ satisfying the following axioms:

(F_1) $\max\{a, b\} \leq F(a, b, c)$, for every $a, b, c \geq 0$;
(F_2) $F(a, 0, 0) = a$, for every $a \geq 0$;
(F_3) F is continuous.

The set $\mathscr{F}$ is nonempty. For instance, the following functions belong to $\mathscr{F}$:

- $F(a, b, c) = a + b + c$,
- $F(a, b, c) = \max\{a, b\} + \ln(c + 1)$,
- $F(a, b, c) = a + b + c(c + 1)$,
- $F(a, b, c) = (a + b)e^c$,
- $F(a, b, c) = (a + b)(c + 1)^n,\ n \in \mathbb{N}$.

Let (X, d) be a metric space, $\varphi : X \to [0, \infty)$, $F \in \mathscr{F}$, and $\theta \in \mathscr{E}_Z$. We denote by $\mathscr{T}(\varphi, F, \theta)$ the set of mappings $T : X \to X$ satisfying

$$\theta\big(F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)), M_F^\varphi(x, y)\big) \geq 0, \quad (x, y) \in X \times X, \tag{7.4}$$

where

$$M_F^\varphi(x, y) = \max\{F(d(x, y), \varphi(x), \varphi(y)), F(d(x, Tx), \varphi(x), \varphi(Tx)), F(d(y, Ty), \varphi(y), \varphi(Ty))\}. \tag{7.5}$$

The main result of this chapter is the following one.

Theorem 7.3 *Let (X, d) be a complete metric space. Let $T : X \to X$ be a mapping that belongs to $\mathscr{T}(\varphi, F, \theta)$, for some $\varphi : X \to [0, \infty)$, $F \in \mathscr{F}$, and $\theta \in \mathscr{E}_Z$. If φ is lower semi-continuous, then*

(i) For every $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point of T.
(ii) T has a unique fixed point.
(iii) $\mathrm{Fix}(T)$ *is φ-admissible.*

Proof First of all, we show that $\mathrm{Fix}(T) \subseteq Z_\varphi$. Indeed, let $\omega \in \mathrm{Fix}(T)$. Since

$$\begin{aligned} M_F^\varphi(\omega, \omega) &= \max\{F(d(\omega, \omega), \varphi(\omega), \varphi(\omega)), F(d(\omega, T\omega), \varphi(\omega), \varphi(T\omega)), \\ &\qquad F(d(\omega, T\omega), \varphi(\omega), \varphi(T\omega))\} \\ &= \max\{F(0, \varphi(\omega), \varphi(\omega)), F(0, \varphi(\omega), \varphi(\omega)), F(0, \varphi(\omega), \varphi(\omega))\} \\ &= F(0, \varphi(\omega), \varphi(\omega)), \end{aligned}$$

then (7.4) guarantees that

$$\begin{aligned} 0 &\le \theta\big(F(d(T\omega, T\omega), \varphi(T\omega), \varphi(T\omega)), M_F^\varphi(\omega, \omega)\big) \\ &= \theta\big(F(0, \varphi(\omega), \varphi(\omega)), F(0, \varphi(\omega), \varphi(\omega))\big). \end{aligned}$$

By Remark 7.4, we deduce that

$$F(0, \varphi(\omega), \varphi(\omega)) = 0.$$

It follows from condition (F_1) that

$$0 \le \varphi(\omega) = \max\{0, \varphi(\omega)\} \le F(0, \varphi(\omega), \varphi(\omega)) = 0,$$

which means that $\varphi(\omega) = 0$, and $\omega \in Z_\varphi$. Therefore, $\mathrm{Fix}(T) \subseteq Z_\varphi$.

Next, let us prove (i). Let $x_0 \in X$ be an arbitrary point and let $\{x_n\}$ be the Picard sequence defined by

$$x_n = T^n x_0, \quad n \in \mathbb{N}.$$

If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T (and $\{x_n\}$ converges to x_{n_0}). On the contrary case, suppose that

$$d(x_n, x_{n+1}) > 0, \quad n \in \mathbb{N}.$$

If there exists some $m_0 \in \mathbb{N}$ such that $F(d(x_{m_0}, x_{m_0+1}), \varphi(x_{m_0}), \varphi(x_{m_0+1})) = 0$, then we could deduce from condition (F_1) that

$$\begin{aligned} 0 &< d(x_{m_0}, x_{m_0+1}) \le \max\left\{d(x_{m_0}, x_{m_0+1}), \varphi(x_{m_0})\right\} \\ &\le F(d(x_{m_0}, x_{m_0+1}), \varphi(x_{m_0}), \varphi(x_{m_0+1})) = 0, \end{aligned}$$

which is impossible. Hence,

$$F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) > 0, \quad n \in \mathbb{N}.$$

For simplicity, let us denote

$$a_n = F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) > 0, \quad n \in \mathbb{N}.$$

Notice that, for all $n \in \mathbb{N}$,

$$\begin{aligned} M_F^{\varphi}(x_n, x_{n+1}) &= \max\big\{F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})), F(d(x_n, Tx_n), \varphi(x_n), \varphi(Tx_n)), \\ &\qquad F(d(x_{n+1}, Tx_{n+1}), \varphi(x_{n+1}), \varphi(Tx_{n+1}))\big\} \\ &= \max\big\{F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})), F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})), \\ &\qquad F(d(x_{n+1}, x_{n+2}), \varphi(x_{n+1}), \varphi(x_{n+2}))\big\} \\ &= \max\big\{a_n,\ a_n,\ a_{n+1}\big\} \\ &= \max\big\{a_n, a_{n+1}\big\} > 0. \end{aligned}$$

Using (7.4) and property (θ_2), we deduce that, for all $n \in \mathbb{N}$,

$$\begin{aligned} 0 &\le \theta\big(F(d(Tx_n, Tx_{n+1}), \varphi(Tx_n), \varphi(Tx_{n+1})), M_F^{\varphi}(x_n, x_{n+1})\big) \\ &= \theta\big(F(d(x_{n+1}, x_{n+2}), \varphi(x_{n+1}), \varphi(x_{n+2})), \max\{a_n, a_{n+1}\}\big) \\ &= \theta\big(a_{n+1}, \max\{a_n, a_{n+1}\}\big) \\ &< \max\{a_n, a_{n+1}\} - a_{n+1}, \end{aligned}$$

which means that $a_{n+1} < a_n$, for all $n \in \mathbb{N}$. As $\{a_n\}$ is a decreasing sequence of nonnegative real numbers, it has a limit. Let

$$L = \lim_{n\to\infty} a_n \ge 0.$$

As $\{a_n\}$ is strictly decreasing, then $L < a_n$, for all $n \in \mathbb{N}$. In order to prove that $L = 0$, suppose that $L > 0$. In such a case, we have

$$\lim_{n\to\infty} a'_n = \lim_{n\to\infty} b'_n = L,$$

where $a'_n = a_{n+1}$ and $b'_n = \max\{a_n, a_{n+1}\} = a_n$. Moreover, we have

$$L < b'_n, \quad n \in \mathbb{N}.$$

Thus, condition (θ_3) implies that

$$\limsup_{n\to\infty} \theta(a'_n, b'_n) < 0,$$

which contradicts the fact that

$$\theta(a_n', b_n') = \theta\big(a_{n+1}, \max\{a_n, a_{n+1}\}\big) \geq 0, \quad n \in \mathbb{N}.$$

This contradiction guarantees that

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) = 0. \tag{7.6}$$

Furthermore, by condition (F_1),

$$\begin{aligned}
&0 \leq \varphi(x_n) \leq \max\big\{d(x_n, x_{n+1}), \varphi(x_n)\big\} \leq F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) = a_n, \quad \text{and}\\
&0 \leq d(x_n, x_{n+1}) \leq \max\big\{d(x_n, x_{n+1}), \varphi(x_n)\big\} \leq F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) = a_n,
\end{aligned}$$

for all $n \in \mathbb{N}$. So,

$$\lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{7.7}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence reasoning by contradiction. Suppose that $\{x_n\}$ is not a Cauchy sequence in (X, d). In this case, it is well known (see, for instance, [12, Lemma 16], [3, Lemma 13]) that there exist $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$,

$$k \leq n(k) < m(k) < n(k+1) \quad \text{and} \quad d(x_{n(k)}, x_{m(k)-1}) \leq \varepsilon_0 < d(x_{n(k)}, x_{m(k)}), \tag{7.8}$$

and also

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon_0. \tag{7.9}$$

Let $\ell = \varepsilon_0 > 0$ and let us define

$$\begin{aligned}
a_k'' &= F(d(x_{n(k)+1}, x_{m(k)+1}), \varphi(x_{n(k)+1}), \varphi(x_{m(k)+1})), \quad \text{and}\\
b_k'' &= M_F^{\varphi}(x_{n(k)}, x_{m(k)}),
\end{aligned}$$

for all $k \in \mathbb{N}$. As F is continuous, it follows from (7.7), (7.9), and (F_2) that

$$\lim_{k\to\infty} a_k'' = \lim_{k\to\infty} F(d(x_{n(k)+1}, x_{m(k)+1}), \varphi(x_{n(k)+1}), \varphi(x_{m(k)+1})) = F\,(\varepsilon_0, 0, 0) = \varepsilon_0 = \ell.$$

On the other hand, for all $k \in \mathbb{N}$,

$$\begin{aligned}
b_k'' &= M_F^{\varphi}(x_{n(k)}, x_{m(k)})\\
&= \max\big\{F(d(x_{n(k)}, x_{m(k)}), \varphi(x_{n(k)}), \varphi(x_{m(k)})), F(d(x_{n(k)}, Tx_{n(k)}), \varphi(x_{n(k)}), \varphi(Tx_{n(k)})),\\
&\qquad F(d(x_{m(k)}, Tx_{m(k)}), \varphi(x_{m(k)}), \varphi(Tx_{m(k)}))\big\}\\
&= \max\big\{F(d(x_{n(k)}, x_{m(k)}), \varphi(x_{n(k)}), \varphi(x_{m(k)})), F(d(x_{n(k)}, x_{n(k)+1}), \varphi(x_{n(k)}), \varphi(x_{n(k)+1})),\\
&\qquad F(d(x_{m(k)}, x_{m(k)+1}), \varphi(x_{m(k)}), \varphi(x_{m(k)+1}))\big\}.
\end{aligned} \tag{7.10}$$

In particular, by (F_1) and (7.8), for all $n \in \mathbb{N}$,

$$\begin{aligned} b_k'' &\geq F(d(x_{n(k)}, x_{m(k)}), \varphi(x_{n(k)}), \varphi(x_{m(k)})) \geq \max\left\{d(x_{n(k)}, x_{m(k)}), \varphi(x_{n(k)})\right\} \\ &\geq d(x_{n(k)}, x_{m(k)}) > \varepsilon = \ell. \end{aligned} \tag{7.11}$$

Letting $k \to \infty$ in (7.10), we obtain

$$\begin{aligned} &\lim_{k\to\infty} b_k'' = \max\{F(\varepsilon_0, 0, 0), F(0, 0, 0), F(0, 0, 0)\} \\ &= F(\varepsilon_0, 0, 0) = \varepsilon_0 = \ell. \end{aligned}$$

As a consequence, $\{a_k''\}$ and $\{b_k''\}$ are sequences of positive real numbers converging to the same positive limit ℓ satisfying

$$\ell < b_k'', \quad k \in \mathbb{N}.$$

It follows from (θ_3) that

$$\limsup_{k\to\infty} \theta(a_k'', b_k'') < 0. \tag{7.12}$$

However, (7.4) ensures us that, for all $k \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq \theta\big(F(d(Tx_{n(k)}, Tx_{m(k)}), \varphi(Tx_{n(k)}), \varphi(Tx_{m(k)})), M_F^{\varphi}(x_{n(k)}, x_{m(k)})\big) \\ &\leq \theta\big(F(d(x_{n(k)+1}, x_{m(k)+1}), \varphi(x_{n(k)+1}), \varphi(x_{m(k)+1})),\ M_F^{\varphi}(x_{n(k)}, x_{m(k)})\big) \\ &= \theta(a_k'', b_k''), \end{aligned}$$

which contradicts (7.12). This contradiction guarantees that $\{x_n\}$ is a Cauchy sequence in (X, d). As it is complete, there exists $\omega \in X$ such that $\{x_n\} \to \omega$. As φ is lower semi-continuous, we have

$$0 \leq \varphi(\omega) \leq \limsup_{n\to\infty} \varphi(x_n) = 0,$$

so $\varphi(\omega) = 0$, that is, $\omega \in Z_\varphi$. ω is a fixed point of T reasoning by contradiction. Suppose that $d(\omega, T\omega) > 0$. Let us define

$$\begin{aligned} &r = F(d(\omega, T\omega), 0, \varphi(T\omega)), \\ &a_n''' = F(d(x_{n+1}, T\omega), \varphi(x_{n+1}), \varphi(T\omega)) \quad \text{and} \quad b_n''' = M_F^{\varphi}(x_n, \omega), \end{aligned}$$

for all $n \in \mathbb{N}$. By (F_1),

$$r = F(d(\omega, T\omega), 0, \varphi(T\omega)) \geq \max\{d(\omega, T\omega), 0\} = d(\omega, T\omega) > 0. \tag{7.13}$$

As F is continuous,

$$\lim_{n\to\infty} a_n''' = \lim_{n\to\infty} F(d(x_{n+1}, T\omega), \varphi(x_{n+1}), \varphi(T\omega)) = F(d(\omega, T\omega), 0, \varphi(T\omega)) = r.$$

On the other hand,

$$\begin{aligned} b_n''' = M_F^{\varphi}(x_n, \omega) = \max\{&F(d(x_n, \omega), \varphi(x_n), \varphi(\omega)), F(d(x_n, Tx_n), \varphi(x_n), \varphi(Tx_n)), \\ &F(d(\omega, T\omega), \varphi(\omega), \varphi(T\omega))\} \\ = \max\{&F(d(x_n, \omega), \varphi(x_n), 0), F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})), \\ &F(d(\omega, T\omega), \varphi(\omega), \varphi(T\omega))\}. \end{aligned}$$

Since F is continuous,

$$\begin{aligned} &\lim_{n\to\infty} F(d(x_n, \omega), \varphi(x_n), 0) = F(0,0,0) = 0, \\ &\lim_{n\to\infty} F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) = F(0,0,0) = 0. \end{aligned}$$

As a consequence, there exists $n_0 \in \mathbb{N}$ such that

$$b_n''' = F(d(\omega, T\omega), 0, \varphi(T\omega)) = r, \quad n \geq n_0.$$

In particular, $\{a_n'''\}_{n\geq n_0} \subset [0, \infty)$ is a sequence converging to $r > 0$ and such that, for all $n \geq n_0$,

$$\begin{aligned} \theta\left(a_n''', r\right) = \theta\left(a_n''', b_n'''\right) &= \theta\left(F(d(x_{n+1}, T\omega), \varphi(x_{n+1}), \varphi(T\omega)), M_F^{\varphi}(x_n, \omega)\right) \\ &= \theta\left(F(d(Tx_n, T\omega), \varphi(Tx_n), \varphi(T\omega)), M_F^{\varphi}(x_n, \omega)\right) \geq 0, \end{aligned}$$

by virtue of (7.4). Thus, condition (θ_3) guarantees that $r = 0$, which contradicts (7.13). This contradiction shows that $d(\omega, T\omega) = 0$; that is, ω is a fixed point of T. In particular, $\mathrm{Fix}(T)$ is nonempty, so $\emptyset \neq \mathrm{Fix}(T) \subseteq Z_{\varphi}$, and the set $\mathrm{Fix}(T)$ is φ-admissible. Furthermore, we have just proved that every Picard sequence of T converges to a fixed point of T. Therefore, (i) and (iii) hold.

Finally, let us show that T has a unique fixed point. By contradiction, assume that $(x, y) \in \mathrm{Fix}(T) \times \mathrm{Fix}(T)$, with $d(x, y) > 0$. In such a case, taking into account that $\mathrm{Fix}(T) \subseteq Z_{\varphi}$, we derive that $\varphi(x) = \varphi(y) = 0$. Furthermore, as

$$\begin{aligned} M_F^{\varphi}(x, y) &= \max\{F(d(x, y), \varphi(x), \varphi(y)), F(d(x, Tx), \varphi(x), \varphi(Tx)), \\ &\qquad F(d(y, Ty), \varphi(y), \varphi(Ty))\} \\ &= \max\{F(d(x, y), 0, 0), F(0,0,0), F(0,0,0)\} \\ &= F(d(x, y), 0, 0) \\ &= d(x, y), \end{aligned}$$

condition (7.4) yields

$$\begin{aligned}0 &\leq \theta\big(F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)),\, M_F^{\varphi}(x,y)\big)\\ &= \theta\big(F(d(x,y),\varphi(x),\varphi(y)),\, d(x,y)\big)\\ &= \theta\big(F(d(x,y),0,0),\, d(x,y)\big)\\ &= \theta\big(d(x,y),\, d(x,y)\big),\end{aligned}$$

which contradicts, by Remark 7.4, the fact that $\theta(d(x,y),d(x,y)) < 0$ (because $d(x,y) > 0$). Thus, $x = y$ and (ii) follows. The proof is complete.

The following result is similar to Theorem 7.3 and its proof follows, point by point, and in an easier way, repeating the arguments we have just shown in the proof of Theorem 7.3. However, there is not a direct relationship between both results because an e-simulation function does not have to be monotone in its second argument.

Theorem 7.4 *Let (X,d) be a metric space, and let $T : X \to X$ be a mapping. Assume that for some $\theta \in \mathscr{E}_Z$, $F \in \mathscr{F}$, and $\varphi : X \to [0,\infty)$, we have*

$$\theta\big(F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)),\ F(d(x,y),\varphi(x),\varphi(y))\big) \geq 0, \quad (x,y) \in X \times X. \tag{7.14}$$

If φ is lower semi-continuous, then

(i) For every $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point of T.
(ii) T has a unique fixed point.
(iii) Fix(T) *is φ-admissible.*

Let (X,d) be a metric space. For given functions $\varphi : X \to [0,\infty)$, $F \in \mathscr{F}$, and $\psi \in \Psi$, we denote by $\mathscr{T}(\varphi, F, \psi)$ the class of operators $T : X \to X$ satisfying

$$F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \leq \psi(F(d(x,y),\varphi(x),\varphi(y))), \quad (x,y) \in X \times X. \tag{7.15}$$

The following result due to Karapinar, O'Regan, and Samet [7] follows from Theorem 7.4.

Corollary 7.1 *Let (X,d) be a complete metric space and $T : X \to X$ be a given operator. Suppose that the following conditions hold:*

(i) There exist $\varphi : X \to [0,\infty)$, $F \in \mathscr{F}$, and $\psi \in \Psi$ such that $T \in \mathscr{T}(\varphi, F, \psi)$;
(ii) φ is lower semi-continuous.

Then the set Fix(T) *is φ-admissible. Moreover, the operator T has a unique fixed point.*

Proof Under the considered assumptions, let θ_ψ be the function defined by (7.3). Lemma 7.3 guarantees that θ_ψ is an e-simulation function. Moreover, condition (7.15) is equivalent to

$$\begin{aligned}&\theta_\psi\big(F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)),\, F(d(x,y),\varphi(x),\varphi(y))\big)\\ &= \psi\,(F(d(x,y),\varphi(x),\varphi(y))) - F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \geq 0, \quad (x,y) \in X \times X,\end{aligned}$$

which means that T satisfies (7.14) with $\theta = \theta_\psi$. Thus, Theorem 7.4 is applicable.

In the following example, we show that Theorem 7.3 improves Corollary 7.1.

Example 7.3 Let $X = [-3, 3]$. We endow X with the Euclidean metric

$$d(x, y) = |x - y|, \quad (x, y) \in X \times X.$$

Obviously, (X, d) is a complete metric space. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} -2 & \text{if} \quad x = 1, \\ -\dfrac{x}{12} & \text{if} \ \ x \in X \backslash \{1\}. \end{cases}$$

We will show that using the functions

$$\varphi : X \to [0, \infty), \quad \varphi(x) = 0, \quad \text{for all } x \in X, \qquad \text{and} \tag{7.16}$$
$$F : [0, \infty)^3 \to [0, \infty), \quad F(t, s, r) = t + s + r, \quad \text{for all } t, s, r \in [0, \infty), \tag{7.17}$$

Theorem 7.3 is applicable but Corollary 7.1 is not. Indeed, assume that there is $\psi \in \Psi$ such that (7.15) holds. Therefore, for all $x, y \in X$,

$$\begin{aligned} d(Tx, Ty) &= d(Tx, Ty) + 0 + 0 = d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) \\ &= F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq \psi\left(F(d(x, y), \varphi(x), \varphi(y))\right) \\ &= \psi\left(d(x, y) + 0 + 0\right) = \psi\left(d(x, y)\right). \end{aligned}$$

However, if $x_0 = 0$ and $y_0 = 1$, then

$$\begin{aligned} &d(T(0), T(1)) = d(0, -2) = 2, \qquad \text{but} \\ &\psi\left(d(0, 1)\right) = \psi(1) < 1, \end{aligned}$$

which contradicts the previous inequality. As a consequence, it is impossible to find $\psi \in \Psi$ such that (7.15) holds, so Corollary 7.1 is not applicable. Nevertheless, let us consider the function $\theta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ defined by

$$\theta(t, s) = \frac{3}{4}s - t, \quad t, s \geq 0.$$

Then θ is a simulation function (see [8], Example 2.2, (i)). By Proposition 7.1, it is also an e-simulation function. As φ and F are given by (7.16) and (7.17), we have to prove that

$$\theta\big(d(Tx, Ty), M_F^\varphi(x, y)\big) \geq 0, \quad (x, y) \in X \times X, \tag{7.18}$$

where

$$\begin{aligned} M_F^{\varphi}(x,y) &= \max\{F(d(x,y),\varphi(x),\varphi(y)), F(d(x,Tx),\varphi(x),\varphi(Tx)),\\ &\qquad F(d(y,Ty),\varphi(y),\varphi(Ty))\}\\ &= \max\{d(x,y), d(x,Tx), d(y,Ty)\}. \end{aligned}$$

Indeed, we consider two cases.

- If $x, y \in X\backslash\{1\}$, then

$$\begin{aligned} \theta\big(d(Tx,Ty), M_F^{\varphi}(x,y)\big) &= \frac{3}{4}M_F^{\varphi}(x,y) - d\left(-\frac{x}{12}, -\frac{y}{12}\right) \geq \frac{3}{4}d(x,y) - d\left(\frac{x}{12}, \frac{y}{12}\right)\\ &= \frac{3}{4}|x-y| - \frac{1}{12}|x-y| = \frac{2}{3}|x-y| \geq 0. \end{aligned}$$

- If $x \in X\backslash\{1\}$ and $y = 1$, taking into account that $x/12 \in [-1/4, 1/4]$, we deduce that

$$\begin{aligned} & d(Tx,Ty) = d\left(-\frac{x}{12}, -2\right) = d\left(\frac{x}{12}, 2\right) = \left|2 - \frac{x}{12}\right| = 2 - \frac{x}{12},\\ & d(y,Ty) = d(1,-2) = 3, \text{ and}\\ & M_F^{\varphi}(x,y) = \max\{d(x,y),\ d(x,Tx),\ d(y,Ty)\} \geq 3. \end{aligned}$$

  Therefore,

$$\begin{aligned} \theta\big(d(Tx,Ty), M_F^{\varphi}(x,y)\big) &= \frac{3}{4}M_F^{\varphi}(x,y) - \left(2 - \frac{x}{12}\right) \geq \frac{3}{4}3 - \left(2 - \frac{x}{12}\right)\\ &= \frac{x+3}{12} \geq 0. \end{aligned}$$

Thus, in all cases, (7.18) is satisfied. Therefore, Theorem 7.3 is applicable, and we conclude that T has a unique fixed point.

7.5 Some Consequences

In this section, some fixed point theorems in metric and partial metric spaces are deduced from the above results.

7.5.1 Fixed Point Results in Partial Metric Spaces via Extended Simulation Functions

In this part, some fixed point theorems in partial metric spaces are deduced from the above results. Therefore, we answer to all the questions of I.A. Rus presented in Sect. 7.2.

The following result will be useful later.

Lemma 7.4 *Let (X, p) be a partial metric space. Let $\varphi : X \to [0, \infty)$ be the function defined by*

$$\varphi(x) = p(x, x), \quad x \in X.$$

Then φ is continuous with respect to the topology induced by the metric d_p.

Proof Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} d_p(x_n, x) = 0,$$

for some $x \in X$. From (ii), Lemma 7.2, we have

$$\lim_{n\to\infty} p(x_n, x_n) = p(x, x),$$

i.e.,

$$\lim_{n\to\infty} \varphi(x_n) = \varphi(x),$$

which proves the continuity of φ with respect to d_p.

We have the following fixed point result in a complete partial metric space.

Corollary 7.2 *Let (X, p) be a complete metric space, and let $T : X \to X$ be a given mapping. Suppose that there exists some $\theta \in \mathscr{E}_Z$ such that*

$$\theta\big(p(Tx, Ty), \max\{p(x, y), p(x, Tx), p(y, Ty)\}\big) \geq 0, \quad (x, y) \in X \times X. \tag{7.19}$$

Then T has a unique fixed point $x^* \in X$. For all $x \in X$, the Picard sequence $\{T^n x\}$ converges to x^*. Moreover, $p(x^*, x^*) = 0$.

Proof Observe that (7.19) is equivalent to (7.4) with

$$\begin{aligned} F(a, b, c) &= a + b + c, \quad a, b, c \geq 0, \\ \varphi(x) &= \frac{p(x, x)}{2}, \quad x \in X, \\ d(x, y) &= \frac{d_p(x, y)}{2}, \quad (x, y) \in X \times X. \end{aligned}$$

On the other hand, from (ii), Lemma 7.2, since the partial metric space (X, p) is complete, then the metric space (X, d) is complete. Moreover, from Lemma 7.4, the function $\varphi : X \to [0, \infty)$ is continuous with respect to the metric d. Therefore, the desired result follows from Theorem 7.3.

Corollary 7.3 *Let (X, p) be a complete metric space, and let $T : X \to X$ be a given mapping. Suppose that there exists some $\psi \in \Psi$ such that*

$$p(Tx, Ty) \leq \psi\left(\max\{p(x, y), p(x, Tx), p(y, Ty)\}\right), \quad (x, y) \in X \times X. \tag{7.20}$$

Then T has a unique fixed point $x^* \in X$. For all $x \in X$, the Picard sequence $\{T^n x\}$ converges to x^*. Moreover, $p(x^*, x^*) = 0$.

Proof Taking $\theta = \theta_\psi$ in (7.19), we obtain (7.20). Using Lemma 7.3 and Corollary 7.2, the desired result follows.

Corollary 7.4 *Let (X, p) be a complete metric space, and let $T : X \to X$ be a given mapping. Suppose that there exists a lower semi-continuous function $\mu : [0, \infty) \to [0, \infty)$ with $\mu^{-1}(\{0\}) = \{0\}$, such that*

$$p(Tx, Ty) \leq \max\{p(x, y), p(x, Tx), p(y, Ty)\} - \mu\left(\max\{p(x, y), p(x, Tx), p(y, Ty)\}\right), \tag{7.21}$$

for all $(x, y) \in X \times X$. Then T has a unique fixed point $x^* \in X$. For all $x \in X$, the Picard sequence $\{T^n x\}$ converges to x^*. Moreover, $p(x^*, x^*) = 0$.

Proof Taking in (7.19), $\theta(t, s) = s - \mu(s) - t$, for all $t, s \geq 0$, we obtain (7.21). On the other hand, it was proved in [8] that the function θ defined above is a simulation function. Therefore, by Corollary 7.2 and Proposition 7.1, the result follows.

Remark 7.5 Observe that if a mapping $T : X \to X$ satisfies (7.1), then it satisfies (7.20) with $\psi(t) = kt$, $t \geq 0$. Therefore, Corollary 7.3 is a generalization of Matthews result given by Theorem 7.1.

7.5.2 Fixed Point Results in Metric Spaces via Extended Simulation Functions

As any metric space is a partial metric space, the following results follow immediately from the above corollaries.

From Corollary 7.2, we deduce the following result.

Corollary 7.5 *Let (X, d) be a complete metric space, and let $T : X \to X$ be a given mapping. Suppose that there exists some $\theta \in \mathcal{E}_Z$ such that*

$$\theta\big(d(Tx, Ty), \max\{d(x, y), d(x, Tx), d(y, Ty)\}\big) \geq 0, \quad (x, y) \in X \times X.$$

Then T has a unique fixed point $x^* \in X$. For all $x \in X$, the Picard sequence $\{T^n x\}$ converges to x^*.

From Corollary 7.3, we deduce the following result.

Corollary 7.6 *Let (X, d) be a complete metric space, and let $T : X \to X$ be a given mapping. Suppose that there exists some $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi\left(\max\{d(x, y), d(x, Tx), d(y, Ty)\}\right), \quad (x, y) \in X \times X.$$

Then T has a unique fixed point $x^* \in X$. For all $x \in X$, the Picard sequence $\{T^n x\}$ converges to x^*.

Finally, from Corollary 7.4, we deduce the following result.

Corollary 7.7 *Let (X, d) be a complete metric space, and let $T : X \to X$ be a given mapping. Suppose that there exists a lower semi-continuous function $\mu : [0, \infty) \to [0, \infty)$ with $\mu^{-1}(\{0\}) = \{0\}$, such that*

$$d(Tx, Ty) \leq \max\{d(x, y), d(x, Tx), d(y, Ty)\} - \mu\left(\max\{d(x, y), d(x, Tx), d(y, Ty)\}\right),$$

for all $(x, y) \in X \times X$. Then T has a unique fixed point $x^* \in X$. For all $x \in X$, the Picard sequence $\{T^n x\}$ converges to x^*.

Remark 7.6 Corollary 7.6 is an extension of Boyd–Wong fixed point theorem [4].

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