Chapter 6
Implicit Contractions on a Set Equipped with Two Metrics

Several classical fixed point theorems have been unified by considering general contractions expressed via an implicit inequality, see, for examples, Turinici [15], Popa [8, 9], Berinde [2], and references therein. In this chapter, we consider a class of mappings defined on a set equipped with two metrics and satisfying an implicit contraction involving two functions $F : [0, \infty)^6 \to \mathbb{R}$ and $\alpha : X \times X \to \mathbb{R}$. The existence of fixed points for this class of mappings is investigated. The main reference for this chapter is the paper [14].

6.1 Preliminaries

Let $\mathscr{F}$ be the set of functions $F : [0, +\infty)^6 \to \mathbb{R}$ satisfying the following conditions:

(I) F is continuous;
(II) F is nondecreasing in the first variable;
(III) F is decreasing in the fifth variable;
(IV) $\exists\, h \in (0, 1) : F(u, v, v, u, u + v, 0) \leq 0 \Longrightarrow u \leq hv$.

Let us give some examples of functions that belong to the set $\mathscr{F}$.

Example 6.1 The function $F : [0, \infty)^6 \to \mathbb{R}$ defined by

$$F(u_1, u_2, \ldots, u_6) = u_1 - \lambda u_2, \quad u_i \geq 0,\ i = 1, 2, \ldots, 6,$$

where $\lambda \in (0, 1)$ is a constant, belongs to the set $\mathscr{F}$. In this case, (IV) is satisfied with $h = \lambda$.

Example 6.2 The function $F : [0, \infty)^6 \to \mathbb{R}$ defined by

$$F(u_1, u_2, \ldots, u_6) = u_1 - \lambda u_2 - \gamma u_3, \quad u_i \geq 0,\ i = 1, 2, \ldots, 6,$$

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where $\lambda, \gamma \geq 0$ are constants with $\lambda + \gamma \in (0, 1)$, belongs to the set $\mathscr{F}$. In this case, (IV) is satisfied with $h = \lambda + \gamma$.

Example 6.3 The function $F : [0, \infty)^6 \to \mathbb{R}$ defined by

$$F(u_1, u_2, \ldots, u_6) = u_1 - \lambda \max\left\{u_2, u_3, u_4, \frac{u_5 + u_6}{2}\right\}, \quad u_i \geq 0,\ i = 1, 2, \ldots, 6,$$

where $\lambda \in (0, 1)$ is a constant, belongs to the set $\mathscr{F}$. In fact, (I)–(III) are obvious. Further, let $u, v \geq 0$ be such that $F(u, v, v, u, u + v, 0) \leq 0$. By the definition of F, we obtain

$$u - \lambda \max\left\{v, u, \frac{u + v}{2}\right\} = u - \lambda \max\{v, u\} \leq 0,$$

which yields

$$u \leq \lambda \max\{v, u\}.$$

Since $\lambda \in (0, 1)$, we obtain

$$u \leq \lambda v.$$

Therefore, (IV) is satisfied with $h = \lambda$.

Let X be a nonempty set endowed with two metrics d and d'. For $x_0 \in X$ and $r > 0$, let

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}.$$

We denote by $\overline{B(x_0, r)}^{d'}$ the d'-closure of $B(x_0, r)$ (the closure of $B(x_0, r)$ with respect to the topology of d').

Before stating and proving the main results of this chapter, we need to introduce the following concepts (some of them are introduced in the previous chapters).

Definition 6.1 Let $T : \overline{B(x_0, r)}^{d'} \to X$ and $\alpha : X \times X \to \mathbb{R}$. We say that T is α-admissible (see [13]) if the following condition holds: For all $x, y \in B(x_0, r)$, we have

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 6.2 We say that the set X satisfies the property (H) with respect to the metric d if the following condition holds: For every sequence $\{x_n\} \subset X$ satisfying

$$\lim_{n\to\infty} d(x_n, x) = 0, \quad x \in X$$

and

$$\alpha(x_n, x_{n+1}) \geq 1, \quad n \in \mathbb{N},$$

there exist a positive integer κ and a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n(k)}, x) \geq 1, \quad k \geq \kappa.$$

6.2 Fixed Point Results

The first main result is giving by the following theorem.

Theorem 6.1 *Let X be a nonempty set equipped with two metrics d and d' such that (X, d') is a complete metric space. Let $T : \overline{B(x_0, r)}^{d'} \to X$ be a given mapping, where $x_0 \in X$ and $r > 0$. Suppose that there exist two functions $F \in \mathscr{F}$ and $\alpha : X \times X \to \mathbb{R}$ such that for all $(x, y) \in \overline{B(x_0, r)}^{d'} \times \overline{B(x_0, r)}^{d'}$, we have*

$$F(\alpha(x, y)d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0. \tag{6.1}$$

In addition, assume that the following properties hold:

(i) $d(x_0, Tx_0) < (1 - h)r$ and $\alpha(x_0, Tx_0) \geq 1$;
(ii) T is α-admissible;
(iii) If $d \ngeq d'$, then T is uniformly continuous from $(B(x_0, r), d)$ into (X, d');
(iv) If $d = d'$, then the set X satisfies the property (H) with respect to the metric d;
(v) If $d \neq d'$, then T is continuous from $(\overline{B(x_0, r)}^{d'}, d')$ into (X, d').

Then T has a fixed point.

Proof Let $x_1 = Tx_0$. From (i), we have

$$d(x_0, x_1) = d(x_0, Tx_0) \leq (1 - h)r < r,$$

i.e., $x_1 \in B(x_0, r)$. Let $x_2 = Tx_1$. From (6.1), we have

$$F(\alpha(x_0, x_1)d(Tx_0, Tx_1), d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_0, x_2), 0) \leq 0.$$

On the other hand, by (i) we have

$$d(Tx_0, Tx_1) \leq \alpha(x_0, x_1)d(Tx_0, Tx_1).$$

Therefore, by the monotony property of F, we obtain that

$$F(d(x_1, x_2), d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_0, x_2), 0) \leq 0.$$

Using the fact that $d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2)$ and property (III) of F, we obtain that

$$F(d(x_1,x_2),d(x_0,x_1),d(x_0,x_1),d(x_1,x_2),d(x_0,x_1)+d(x_1,x_2),0)\le 0,$$

which implies from property (IV) that

$$d(x_1,x_2)\le hd(x_0,x_1)\le h(1-h)r<r.$$

Now, we have

$$d(x_0,x_2)\le d(x_0,x_1)+hd(x_0,x_1)=(1+h)d(x_0,x_1)\le (1+h)(1-h)r<r,$$

i.e., $x_2\in B(x_0,r)$. Again, let $x_3=Tx_2$. Since T is α-admissible and $\alpha(x_0,x_1)\ge 1$, we have

$$d(x_2,x_3)\le \alpha(x_1,x_2)d(Tx_1,Tx_2).$$

Then, from (6.1), we obtain that

$$F(d(x_2,x_3),d(x_1,x_2),d(x_1,x_2),d(x_2,x_3),d(x_1,x_3),0)\le 0.$$

Using property (III) of F, we get

$$F(d(x_2,x_3),d(x_1,x_2),d(x_1,x_2),d(x_2,x_3),d(x_1,x_2)+d(x_2,x_3),0)\le 0,$$

which implies from property (IV) that

$$d(x_2,x_3)\le hd(x_1,x_2)\le h^2(1-h)r<r.$$

Therefore, we have

$$d(x_0,x_3)\le d(x_0,x_2)+d(x_2,x_3)\le (1+h)(1-h)r+h^2(1-h)r=(1-h^3)r<r,$$

i.e., $x_3\in B(x_0,r)$. Continuing this process, by induction, we can define the sequence $\{x_n\}$ by

$$x_{n+1}=Tx_n,\quad n\in\mathbb{N}.$$

Such sequence satisfies the following property:

$$x_n\in B(x_0,r),\quad \alpha(x_n,x_{n+1})\ge 1,\quad \text{and}\quad d(x_n,x_{n+1})\le h^n(1-h)r,\quad n\in\mathbb{N}. \tag{6.2}$$

Since $h\in(0,1)$, it follows from (6.2) that $\{x_n\}$ is a Cauchy sequence with respect to the metric d. Now, we shall prove that $\{x_n\}$ is also a Cauchy sequence with respect to the metric d'. If $d\not\geq d'$, from (iii), given $\varepsilon>0$, there exists $\delta>0$ such that

$$(x,y)\in B(x_0,r)\times B(x_0,r),\ d(x,y)<\delta \implies d'(Tx,Ty)<\varepsilon. \tag{6.3}$$

On the other hand, since $\{x_n\}$ is Cauchy with respect to d, there exists a positive integer N such that

$$d(x_n, x_m) < \delta, \quad n, m \geq N.$$

Using (6.3), we obtain

$$d'(x_{n+1}, x_{m+1}) < \varepsilon, \quad n, m \geq N,$$

which proves that $\{x_n\}$ is Cauchy with respect to d'.

Since (X, d') is complete, there exists $z \in \overline{B(x_0, r)}^{d'}$ such that

$$\lim_{n\to\infty} d'(x_n, z) = 0. \tag{6.4}$$

We shall prove that z is a fixed point of T. We consider two cases.

Case 1. If $d = d'$.
From (iv), there exist a positive integer κ and a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n(k)}, z) \geq 1, \quad k \geq \kappa. \tag{6.5}$$

Using (6.1), for all $k \geq \kappa$, we have

$$F(\alpha(x_{n(k)}, z)d(Tx_{n(k)}, Tz), d(x_{n(k)}, z), d(x_{n(k)}, x_{n(k)+1}), d(z, Tz), d(x_{n(k)}, Tz), d(z, x_{n(k)+1})) \leq 0.$$

Next, by (6.5) and property (II) of F, for all $k \geq \kappa$, we have

$$F(d(x_{n(k)+1}, Tz), d(x_{n(k)}, z), d(x_{n(k)}, x_{n(k)+1}), d(z, Tz), d(x_{n(k)}, Tz), d(z, x_{n(k)+1})) \leq 0.$$

Passing to the limit as $k \to \infty$, using (6.4) and the continuity of F, we get

$$F(d(z, Tz), 0, 0, d(z, Tz), d(z, Tz), 0) \leq 0,$$

which implies from property (IV) that $d(z, Tz) = 0$.

Case 2. If $d \neq d'$.
In this case, using (v) and (6.4), we get

$$\lim_{n\to\infty} d'(Tx_n, Tz) = \lim_{n\to\infty} d'(x_{n+1}, Tz) = 0.$$

The uniqueness of the limit gives us that $z = Tz$.

Taking $d = d'$ in Theorem 6.1, we obtain the following result.

Theorem 6.2 *Let (X, d) be a complete metric space, and let $T : \overline{B(x_0, r)}^d \to X$ be a given mapping, where $x_0 \in X$ and $r > 0$. Suppose that there exist two functions*

$F \in \mathscr{F}$ and $\alpha : X \times X \to \mathbb{R}$ such that for all $(x, y) \in \overline{B(x_0, r)}^d \times \overline{B(x_0, r)}^d$, we have

$$F(\alpha(x, y)d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that the following properties hold:

(i) *$d(x_0, Tx_0) < (1 - h)r$ and $\alpha(x_0, Tx_0) \geq 1$;*
(ii) *T is α-admissible;*
(iii) *The set X satisfies the property (H) with respect to the metric d.*

Then T has a fixed point.

From Theorem 6.1, we can deduce the following global result.

Theorem 6.3 *Let X be a nonempty set equipped with two metrics d and d' such that (X, d') is a complete metric space. Let $T : X \to X$ be a given mapping. Suppose that there exist two functions $F \in \mathscr{F}$ and $\alpha : X \times X \to \mathbb{R}$ such that for all $(x, y) \in X \times X$, we have*

$$F(\alpha(x, y)d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that the following properties hold:

(i) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(ii) *T is α-admissible ($x, y \in X, \alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1$);*
(iii) *If $d \ngeq d'$, then T is uniformly continuous from (X, d) into (X, d');*
(iv) *If $d = d'$, then the set X satisfies the property (H) with respect to the metric d;*
(v) *If $d \neq d'$, then T is continuous from (X, d') into (X, d').*

Then T has a fixed point.

Proof We take $r > 0$ such that $d(x_0, Tx_0) < (1 - h)r$. Then, from Theorem 6.1, T has a fixed point in $\overline{B(x_0, r)}^{d'}$.

Taking $d = d'$ in Theorem 6.3, we obtain the following result.

Theorem 6.4 *Let (X, d) be a complete metric space, and let $T : X \to X$ be a given mapping. Suppose that there exist two functions $F \in \mathscr{F}$ and $\alpha : X \times X \to \mathbb{R}$ such that for all $(x, y) \in X \times X$, we have*

$$F(\alpha(x, y)d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that the following properties hold:

(i) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(ii) *T is α-admissible ($x, y \in X, \alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1$);*
(iii) *The set X satisfies the property (H) with respect to the metric d.*

Then T has a fixed point.

6.3 Some Consequences

We present in this section some interesting consequences that can be derived from the previous obtained results.

6.3.1 The Case $\alpha(x, y) = 1$

Taking $\alpha(x, y) = 1$ for all $x, y \in X$, from Theorems 6.1, 6.2, 6.3, and 6.4, we obtain the following results that are generalizations of the fixed point results in [1–4, 6, 8, 11].

Corollary 6.1 *Let (X, d') be a complete metric space, d another metric on X, $x_0 \in X$, $r > 0$, and $T : \overline{B(x_0, r)}^{d'} \to X$. Suppose that there exists $F \in \mathscr{F}$ such that for all $(x, y) \in \overline{B(x_0, r)}^{d'} \times \overline{B(x_0, r)}^{d'}$, we have*

$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that the following properties hold:

(i) $d(x_0, Tx_0) < (1 - h)r$;
(ii) If $d \ngeq d'$, then T is uniformly continuous from $(B(x_0, r), d)$ into (X, d');
(iii) If $d \neq d'$, then T is continuous from $(\overline{B(x_0, r)}^{d'}, d')$ into (X, d').

Then T has a fixed point.

Corollary 6.2 *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$, and $T : \overline{B(x_0, r)}^{d} \to X$. Suppose that there exists $F \in \mathscr{F}$ such that for all $(x, y) \in \overline{B(x_0, r)}^{d} \times \overline{B(x_0, r)}^{d}$, we have*

$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that $d(x_0, Tx_0) < (1 - h)r$. Then T has a fixed point.

Corollary 6.3 *Let (X, d') be a complete metric space, d another metric on X, and $T : X \to X$. Suppose that there exists $F \in \mathscr{F}$ such that for all $(x, y) \in X \times X$, we have*

$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that the following properties hold:

(i) If $d \ngeq d'$, then T is uniformly continuous from (X, d) into (X, d');
(ii) If $d \neq d'$, then T is continuous from (X, d') into (X, d').

Then T has a fixed point.

Corollary 6.4 *Let (X, d) be a complete metric space, and let $T : X \to X$. Suppose that there exists $F \in \mathscr{F}$ such that for all $(x, y) \in X \times X$, we have*

$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

Then T has a fixed point.

Remark 6.1 Corollary 6.4 is an enriched version of Popa [8] that unifies the most important metrical fixed point theorems for contraction-type mappings in Rhoades' classification [12].

6.3.2 The Case of Partially Ordered Sets

Let $\preceq$ be a partial order on X. Let $\triangleleft$ be the binary relation on X defined by

$$(x, y) \in X \times X, \quad x \triangleleft y \Longleftrightarrow x \preceq y \text{ or } y \preceq x.$$

We say that $(X, \triangleleft)$ satisfies the property (H) with respect to the metric d if the following condition holds: For every sequence $\{x_n\} \subset X$ satisfying

$$\lim_{n\to\infty} d(x_n, x) = 0, \quad x \in X$$

and

$$x_n \triangleleft x_{n+1}, \quad n \in \mathbb{N},$$

there exist a positive integer κ and a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$x_{n(k)} \triangleleft x, \quad k \geq \kappa.$$

From Theorems 6.1, 6.2, 6.3, and 6.4, we obtain the following results that are extensions and generalizations of the fixed point results in [7, 10].

At first, we denote by $\widetilde{\mathscr{F}}$ the set of functions $F : [0, +\infty)^6 \to \mathbb{R}$ satisfying the following conditions:

(j) $F \in \mathscr{F}$;
(jj) For every $u_i \geq 0$, $i = 2, \ldots, 6$, we have

$$F(0, u_2, \ldots, u_6) \leq 0.$$

We have the following fixed point result.

Corollary 6.5 *Let (X, d') be a complete metric space, d another metric on X, $x_0 \in X$, $r > 0$, and $T : \overline{B(x_0, r)}^{d'} \to X$. Suppose that there exists $F \in \widetilde{\mathscr{F}}$ such that for all $(x, y) \in \overline{B(x_0, r)}^{d'} \times \overline{B(x_0, r)}^{d'}$, we have*

$$x \lhd y \implies F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \le 0.$$

In addition, assume that the following properties hold:

(i) $d(x_0, Tx_0) < (1 - h)r$ *and* $x_0 \lhd Tx_0$*;*
(ii) $x, y \in \overline{B(x_0, r)}^{d'}$, $x \lhd y \implies Tx \lhd Ty$*;*
(iii) If $d \not\geq d'$*, then* T *is uniformly continuous from* $(B(x_0, r), d)$ *into* (X, d')*;*
(iv) If $d = d'$*, then* $(X, \lhd)$ *satisfies the property (H) with respect to the metric* d*;*
(v) If $d \neq d'$*, then* T *is continuous from* $(\overline{B(x_0, r)}^{d'}, d')$ *into* (X, d')*.*

Then T has a fixed point.

Proof It follows from Theorem 6.1 by taking

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x \lhd y; \\ 0 \text{ if } x \ntriangleleft y. \end{cases}$$

Similarly, from Theorem 6.2, we obtain the following result.

Corollary 6.6 *Let (X, d) be a complete metric space, and let $T : \overline{B(x_0, r)}^{d} \to X$ be a given mapping, where $x_0 \in X$ and $r > 0$. Suppose that there exists $F \in \widetilde{\mathscr{F}}$ such that for all $(x, y) \in \overline{B(x_0, r)}^{d} \times \overline{B(x_0, r)}^{d}$, we have*

$$x \lhd y \implies F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \le 0.$$

In addition, assume that the following properties hold:

(i) $d(x_0, Tx_0) < (1 - h)r$ *and* $x_0 \lhd Tx_0$*;*
(ii) $x, y \in \overline{B(x_0, r)}^{d'}$, $x \lhd y \implies Tx \lhd Ty$*;*
(iii) $(X, \lhd)$ *satisfies the property (H) with respect to the metric* d*.*

Then T has a fixed point.

From Theorem 6.3, we obtain the following global result.

Corollary 6.7 *Let (X, d') be a complete metric space, d another metric on X, and $T : X \to X$. Suppose that there exists $F \in \widetilde{\mathscr{F}}$ such that for all $(x, y) \in X \times X$, we have*

$$x \lhd y \implies F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \le 0.$$

In addition, assume that the following properties hold:

(i) *There exists* $x_0 \in X$ *such that* $x_0 \lhd Tx_0$*;*
(ii) $x, y \in X,\ x \lhd y \implies Tx \lhd Ty$*;*
(iii) *If* $d \not\geq d'$*, then* T *is uniformly continuous from* (X, d) *into* (X, d')*;*
(iv) *If* $d = d'$*, then* $(X, \lhd)$ *satisfies the property (H) with respect to the metric* d*;*
(v) *If* $d \neq d'$*, then* T *is continuous from* (X, d') *into* (X, d')*.*

Then T *has a fixed point.*

Finally, from Theorem 6.4, we obtain the following fixed point result.

Corollary 6.8 *Let* (X, d) *be a complete metric space, and let* $T : X \to X$*. Suppose that there exists* $F \in \widetilde{\mathscr{F}}$ *such that for all* $(x, y) \in X \times X$*, we have*

$$x \lhd y \implies F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that the following properties hold:

(i) *There exists* $x_0 \in X$ *such that* $x_0 \lhd Tx_0$*;*
(ii) $x, y \in X,\ x \lhd y \implies Tx \lhd Ty$*;*
(iii) $(X, \lhd)$ *satisfies the property (H) with respect to the metric* d*.*

Then T *has a fixed point.*

6.3.3 *The Case of Cyclic Mappings*

From Theorem 6.4, we obtain the following fixed point result that is a generalization of Theorem 1.1 in [5].

Corollary 6.9 *Let* (Y, d) *be a complete metric space,* $\{A, B\}$ *a pair of nonempty closed subsets of* Y*, and* $T : A \cup B \to A \cup B$*. Suppose that there exists* $F \in \widetilde{\mathscr{F}}$ *such that for all* $(x, y) \in A \times B$*, we have*

$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that $T(A) \subseteq B$ *and* $T(B) \subseteq A$*. Then* T *has a fixed point in* $A \cap B$*.*

Proof Let $X = A \cup B$. Clearly (since A and B are closed), (X, d) is a complete metric space. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 \text{ if } (x, y) \in (A \times B) \cup (B \times A); \\ \\ 0 \text{ if } (x, y) \notin (A \times B) \cup (B \times A). \end{cases}$$

Clearly (since $F \in \widetilde{\mathscr{F}}$), for all $x, y \in X$, we have

$$F(\alpha(x, y)d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

Taking any point $x_0 \in A$, since $T(A) \subseteq B$, we have $Tx_0 \in B$, which implies that $\alpha(x_0, Tx_0) \geq 1$. Now, let $(x, y) \in X \times X$ be such that $\alpha(x, y) \geq 1$. We have two cases.

Case 1. If $(x, y) \in A \times B$.
Since $T(A) \subseteq B$ and $T(B) \subseteq A$, we have $(Tx, Ty) \in B \times A$, which implies that $\alpha(Tx, Ty) \geq 1$.

Case 2. If $(x, y) \in B \times A$.
In this case, we have $(Tx, Ty) \in A \times B$, which implies that $\alpha(Tx, Ty) \geq 1$.
Therefore, we proved that the mapping T is α-admissible.

Next, we shall prove that X satisfies the property (H) with respect to the metric d. Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} d(x_n, x) = 0, \quad x \in X$$

and

$$\alpha(x_n, x_{n+1}) \geq 1, \quad n \in \mathbb{N}.$$

From the definition of α, we get

$$(x_n, x_{n+1}) \in (A \times B) \cup (B \times A), \quad n \in \mathbb{N}.$$

Since A and B are closed, we have $x \in A \cap B$. Therefore,

$$\alpha(x_n, x) = 1, \quad n \in \mathbb{N},$$

which proves that the set X satisfies the property (H) with respect to the metric d.

Now, from Theorem 6.4, the mapping T has a fixed point in X, i.e., there exists $z \in A \cup B$ such that $Tz = z$. Since $T(A) \subseteq B$ and $T(B) \subseteq A$, obviously, we have $z \in A \cap B$.

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