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Preface

Fixed point theory is a fundamental tool in nonlinear analysis and many other branches of modern mathematics. In particular, when we deal with the solvability of a certain functional equation (differential equation, fractional differential equation, integral equation, matrix equation, etc.), we formulate the problem in terms of finding a fixed point of a certain mapping. This theory has many applications, particularly in biology, chemistry, economics, game theory, optimization theory, physics, etc.

A fixed point problem can be stated as follows:

Let X be a given set, and let (M, N) be a pair of nonempty subsets of X such that $M \cap N \neq \emptyset$. For a given mapping $T : M \to N$, when does a point $x \in M$ such that $Tx = x$, also called a fixed point of T, exist? And if such a point exists, is it unique and how can we approximate it?

We can distinguish three major approaches in fixed point theory: metric approach, topological approach, and discrete approach. Historically, these approaches were initiated by the discovery of three major theorems: Banach fixed point theorem, Brouwer fixed point theorem, and Tarski fixed point theorem. In this book, we are concerned with the first approach, that is, metric fixed point theory.

Metric fixed point theory is an important mathematical discipline because of its applications in different areas such as variational and linear inequalities, optimization theory, boundary value problems. The aim of this book is to present some recent advances in this theory with some applications in nonlinear analysis, including matrix equations, integral equations, and polynomial approximations.

Most of the results presented in this book are up to date. In order to make easy the lecture of this monograph, in each chapter, the basic definitions and mathematical preliminaries are provided before presenting and proving the main results. This monograph should be of interest to graduate students seeking a field of interest, to mathematicians interested in learning about the subject, and to specialists.

The book is organized in ten chapters where in Chap. 1, we discuss Banach contraction principle and its converse. Some applications of this famous principle, including mixed Volterra–Fredholm-type integral equations and systems of nonlinear matrix equations, are presented. In Chap. 2, we are concerned with Ran–Reurings fixed point theorem and its applications to nonlinear matrix equations.

In Chap. 3, we investigate the existence of fixed points for the class of (α, ψ)-contractions. Three fixed point theorems are established for this class of operators. The results extend well-known fixed point theorems due to Banach, Kannan, Chatterjee, Zamfirescu, Berinde, Suzuki, Ćirić, Nieto, López, and many others. We show that (α, ψ)-contractions unify large classes of contraction-type operators, whose fixed points can be obtained by means of Picard iteration. Moreover, some applications to quadratic integral equations are provided. In Chap. 4, we are concerned with the study of fixed points for the class of cyclic mappings. An improvement result is presented by weakening the closure assumption that is usually supposed in the literature. As applications, we study the existence of solutions to certain systems of functional equations. In Chap. 5, we present a recent generalization of Banach contraction principle on the setting of Branciari metric spaces, which is due to [2]. In Chap. 6, we are concerned with the existence of fixed points for a class of mappings defined on a set equipped with two metrics, satisfying an implicit contraction. In Chap. 7, we introduce a class of extended simulation functions, which is larger than the class of simulation functions, recently introduced by Khojasteh et al. [3]. We prove a φ-admissibility result involving extended simulation functions, for a new class of mappings, with respect to a lower semi-continuous function. Next, some fixed point theorems in partial metric spaces are deduced, including Matthews fixed point theorem. Moreover, we answer three open problems posed by Rus [4]. In Chap. 8, we deal with the solvability of a coupled fixed point problem under a finite number of equality constraints. In Chap. 9, we discuss a recent concept of generalized metric spaces due to [1], for which we extend some well-known fixed point results. In Chap. 10, we establish a new fixed point theorem, which will be used to establish Kelisky–Rivlin-type results for q-Bernstein polynomials and modified q-Bernstein polynomials.

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Contents

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Praveen Agarwal is Professor in the Department of Mathematics, Anand International College of Engineering, Jaipur, India. He has published over 200 articles related to special functions, fractional calculus, and mathematical physics in several leading mathematics journals. His latest research focused on partial differential equations, fixed point theory, and fractional differential equations. He has been on the editorial boards of several journals, including *SCI*, *SCIE*, and *SCOPUS*, and he has been involved in a number of conferences. He has received numerous international research grants. Recently, he received the Most Outstanding Researcher Award 2018 for his contribution to mathematics from the Union Minister of Human Resource Development of India, Prakash Javadekar.

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Chapter 1
Banach Contraction Principle and Applications

Banach contraction principle is a fundamental result in Metric Fixed Point Theory. It is a very popular and powerful tool in solving the existence problems in pure and applied sciences. In this chapter, Banach contraction principle and its converse are presented. Moreover, various applications of this famous principle, including mixed Volterra-Fredholm-type integral equations and systems of nonlinear matrix equations, are provided. Some results of this chapter appeared in [3, 5, 13, 19].

1.1 Introduction

Let X be a given set and let $T : M \to N$ be a given mapping, where M and N are nonempty subsets of X such that $M \cap N \neq \emptyset$. Let us consider the problem

$$\begin{cases} \text{Find } x \in M \text{ such that} \\ Tx = x. \end{cases} \tag{1.1}$$

We denote by $\mathrm{Fix}(T)$ the subset of M defined by

$$\mathrm{Fix}(T) = \{x \in M : x \text{is a solution to (1.1)}\}.$$

Then any element of the set $\mathrm{Fix}(T)$ is said to be a fixed point of the mapping T. Observe that if $M \cap N = \emptyset$, then $\mathrm{Fix}(T) = \emptyset$. In Fixed Point Theory, we are interested in solving Problem (1.1). More precisely, we are interested on the following questions:

- Existence: When does Problem (1.1) have at least one solution?
- Uniqueness: If Problem (1.1) has a solution, when is such solution unique?

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- Approximation: In the case of uniqueness, provide a numerical algorithm that converges to the solution to Problem (1.1).

Fixed Point Theory is one of the most useful tools in Nonlinear Analysis. In particular, when we deal with the solvability of a functional equation (differential equation, fractional differential equation, integral or integro-differential equation, etc), we formulate the problem in terms of finding a fixed point of a certain mapping. This theory has several applications, particularly in biology, chemistry, economics, game theory, optimization theory, physics, etc.

In Fixed Point Theory, we can distinguish three main approaches:

- Metric Fixed Point Theory,
- Topological Fixed Point Theory,
- Discrete Fixed Point Theory.

Historically, the above approaches were initiated by the discovery of three major theorems:

- Banach contraction principle [1],
- Brouwer's fixed point theorem [6],
- Tarski's fixed point theorem [24].

In this book, we will focus mainly on the first approach, that is, Metric Fixed Point Theory approach.

1.2 Banach Contraction Principle

Banach contraction principle is a very important tool in the theory of metric spaces. It provides sufficient conditions for the existence and uniqueness of fixed points of certain classes of self-mappings and provides a numerical algorithm to approximate those fixed points. The theorem is named after Banach (1892–1945) and was first stated by him in 1922 [1]. Before presenting this famous result, let us recall briefly some topological tools of metric spaces. For more details, we refer the reader to the books [12, 23, 25].

Definition 1.1 Let X be a nonempty set, and let $d : X \times X \to [0, \infty)$ be a given mapping. We say that d is a metric on X if the following conditions are satisfied:

(d1)$d(x, y) = 0 \Leftrightarrow x = y$, for all $(x, y) \in X \times X$.
(d2)$d(x, y) = d(y, x)$, for all $(x, y) \in X \times X$.
(d3)$d(x, y) \leq d(x, z) + d(z, y)$. for all $(x, y, z) \in X \times X \times X$.

In this case, the pair (X, d) is said to be a metric space.

Further, let us give some standard examples of metric spaces.

Example 1.1 Let X be a nonempty set. Define $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Then d is a metric on X, and (X, d) is a discrete metric space.

Example 1.2 Let (X, d) be a metric space and let Y be a nonempty subset of X. The restriction of d: $X \times X \to [0, \infty)$ to $d : Y \times Y \to [0, \infty)$ induces a metric on Y.

Example 1.3 Let $X = \mathbb{R}^N$ $(N \geq 1)$. Define the mapping $d : X \times X \to [0, \infty)$ by

$$d((x_1, x_2, \ldots, x_N), (y_1, y_2, \ldots, y_N)) = \sum_{i=1}^{N} |x_i - y_i|.$$

Then d is a metric on X.

Example 1.4 Let $X = \mathbb{R}^N$ $(N \geq 1)$. Define the mapping $d : X \times X \to [0, \infty)$ by

$$d((x_1, x_2, \ldots, x_N), (y_1, y_2, \ldots, y_N)) = \max\{|x_i - y_i| : i = 1, 2, \ldots, N\}.$$

Then d is a metric on X.

Example 1.5 Let $(X, \|\cdot\|)$ be a normed space. Define the mapping $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \|x - y\|.$$

Then d is a metric on X.

Example 1.6 Given a metric space (X, d) and an increasing concave function $f : [0, \infty) \to [0, \infty)$ such that $f(x) = 0$ if and only if $x = 0$, then $f \circ d$ is also a metric on X.

Example 1.7 If G is an undirected connected graph, then the set V of vertices of G can be turned into a metric space by defining $d(x, y)$ to be the length of the shortest path connecting the vertices x and y.

Definition 1.2 Let (X, d) be a metric space. Let $\{x_n\}$ be a sequence in X. We say that the sequence $\{x_n\}$ converges to $x \in X$ if

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

In this case, we say that x is the limit of $\{x_n\}$.

Remark 1.1 Observe that if $\{x_n\} \subset X$ converges to $x \in X$, then x is the unique limit of $\{x_n\}$. Indeed, suppose that there exists a pair of elements $(x, y) \in X \times X$ such that

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x_n, y) = 0.$$

From the inequality

$$d(x, y) \leq d(x_n, x) + d(x_n, y), \text{ for all } n,$$

passing to the limit as $n \to \infty$, we obtain $d(x, y) = 0$, which yields $x = y$.

Definition 1.3 Let (X, d) be a metric space. A nonempty subset A of X is said to be closed if for every sequence $\{x_n\} \subset A$, we have

$$\lim_{n\to\infty} d(x_n, x) = 0, \ x \in X \implies x \in A.$$

Definition 1.4 Let (X, d_X) and (Y, d_Y) be two metric spaces. A mapping $f : X \to Y$ is continuous at a point $x \in X$ if for every sequence $\{x_n\} \subset X$, we have

$$\lim_{n\to\infty} d_X(x_n, x) = 0 \implies \lim_{n\to\infty} d_Y(fx_n, fx) = 0.$$

The mapping f is continuous if it is continuous at every element of X.

Let us denote by $\mathbb{N}$ the set of all natural numbers, i.e.,

$$\mathbb{N} = \{0, 1, 2, \ldots\}.$$

Definition 1.5 A sequence $\{x_n\}$ of points in a metric space (X, d) is a Cauchy sequence if

$$(\forall\, \varepsilon > 0)(\exists\, N \in \mathbb{N})[(m, n \geq N) \implies (d(x_n, x_m) < \varepsilon)].$$

Definition 1.6 A subset E of a metric space (X, d) is complete if for any Cauchy sequence of points $\{x_n\}$ in E there exists $x \in E$ such that $\lim_{n\to\infty} d(x_n, x) = 0$.

Proposition 1.1 *Let (X, d) be a metric space, and let E be a subset of X.*

(i) If E is complete then E is closed.
(ii) If X is complete, and E is closed, then E is complete.

Example 1.8 The set of real numbers $\mathbb{R}$ with the usual metric $d(x, y) = |x - y|$ is a complete metric space.

Example 1.9 Let $X = C([a, b]; \mathbb{R})$ be the set of real-valued and continuous functions in $[a, b]$ $(a < b)$. Define the mapping $d : X \times X \to [0, \infty)$ by

$$d(f, g) = \max\{|f(t) - g(t)| : t \in [a, b]\}.$$

Then (X, d) is a complete metric space.

Example 1.10 Let $(X, \|\cdot\|)$ be a Banach space. Define the mapping $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \|x - y\|.$$

Then (X, d) is a complete metric space.

In order to present Banach contraction principle, we need to introduce the concept of Lipschitz mappings.

Definition 1.7 Let (X, d) be a metric space. The mapping $T : X \to X$ is said to be Lipschitzian if there exists a constant $k > 0$ (called Lipschitz constant) such that

$$d(Tx, Ty) \leq kd(x, y), \quad (x, y) \in X \times X.$$

A Lipschitzian mapping with a Lipschitz constant $k < 1$ is called contraction.

Banach contraction principle can be stated as follows.

Theorem 1.1 (Banach contraction principle) *Let (X, d) be a complete metric space. Let $T : X \to X$ be a contraction mapping, with Lipschitz constant $k < 1$. Then*

(i) T has a unique fixed point $x^* \in X$.
(ii) For every $x \in X$, the Picard sequence $\{T^n x\}$ converges to x^*.
(iii) We have the following estimate: For every $x \in X$,

$$d(T^n x, x^*) \leq \frac{k^n}{1-k} d(x, Tx), \quad n \in \mathbb{N}.$$

Proof Let $x \in X$ be an arbitrary point. Using the fact that T is a Lipschitzian mapping with Lipschitz constant k, we obtain

$$d(T^2 x, Tx) \leq kd(Tx, x).$$

Again, we have

$$d(T^3 x, T^2 x) \leq kd(T^2 x, Tx) \leq k^2 d(Tx, x).$$

Continuing this process, by induction we obtain

$$d(T^{n+1} x, T^n x) \leq k^n d(Tx, x), \quad n \in \mathbb{N}. \tag{1.2}$$

Using (1.2), for $(n, m) \in \mathbb{N} \times \mathbb{N}\backslash\{0\}$, we have

$$\begin{aligned} d(T^n x, T^{n+m} x) &\leq d(T^n x, T^{n+1} x) + d(T^{n+1} x, T^{n+2} x) + \cdots + d(T^{n+m-1} x, T^{n+m} x) \\ &\leq (k^n + k^{n+1} + \cdots + k^{n+m-1}) d(x, Tx) \\ &= k^n \left(\frac{1 - k^m}{1 - k} \right) d(x, Tx) \\ &\leq \frac{k^n}{1 - k} d(x, Tx) \to 0 \text{ as } n \to \infty \text{ (since } 0 < k < 1). \end{aligned} \tag{1.3}$$

Therefore $\{T^n x\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $x^* \in X$ such that

$$\lim_{n \to \infty} d(T^n x, x^*) = 0. \tag{1.4}$$

On the other hand, we have

$$d(Tx^*, x^*) \leq d(Tx^*, T^{n+1} x) + d(T^{n+1} x, x^*) \leq k d(x^*, T^n x) + d(T^{n+1} x, x^*).$$

Passing to the limit as $n \to \infty$ and using (1.4), we obtain $d(Tx^*, x^*) = 0$, i.e., $Tx^* = x^*$. Therefore, $x^* \in X$ is a fixed point of T. Suppose now that $y^* \in X$ is another fixed point of T, that is,

$$Ty^* = y^* \quad \text{and} \quad d(x^*, y^*) > 0.$$

In this case, we obtain

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq k d(x^*, y^*) < d(x^*, y^*),$$

which is a contradiction. Then $x^* \in X$ is the unique fixed point of T. Therefore, we proved (i) and (ii). Finally, in order to obtain the estimate (iii), we have just to let $m \to \infty$ in (1.3).

1.3 The Converse of Banach Contraction Principle

In 1959, Bessaga [5] established the following converse of Banach contraction principle.

Theorem 1.2 (Bessaga) *Let X be a nonempty set, $T : X \to X$ and $k \in (0, 1)$. Then*

(a) If T^n has at most one fixed point for every $n \in \mathbb{N}$, then there exists a metric d such that $d(Tx, Ty) \leq k d(x, y)$, for all $x, y \in X$.

(b) If, in addition, some T^n has a fixed point, then there is a complete metric d such that $d(Tx, Ty) \leq k d(x, y)$, for all $x, y \in X$.

In order to prove the above result, Bessaga [5] used a special form of the Axiom of Choice. Note that there are many other proofs of this result involving different techniques. For more details, we refer the reader to Deimling's book [9], Wong [28], Janos [14], and Jachymski [13].

In this section, we give the proof of part (b) of Theorem 1.2 following Jachymski's technique [13].

1.3.1 A Technical Lemma

Lemma 1.1 *Let T be a self-mapping of a set X and $k \in (0,1)$. The following statements are equivalent:*

(i) There exists a complete metric d such that $d(Tx, Ty) \le kd(x, y)$, for all $x, y \in X$.
(ii) There exists a function $\varphi : X \to [0, \infty)$ such that $\varphi^{-1}(\{0\})$ is a singleton and

$$\varphi(Tx) \le k\varphi(x), \quad x \in X.$$

Proof (i) $\Longrightarrow$ (ii). By Banach contraction principle, T has a unique fixed point $x^* \in X$. Define the function $\varphi : X \to [0, \infty)$ by

$$\varphi(x) = d(x, x^*), \quad x \in X.$$

Then

$$\varphi(x) = 0 \Longleftrightarrow x = x^*.$$

Therefore, $\varphi^{-1}(\{0\}) = \{x^*\}$. On the other hand, for every $x \in X$, we have

$$\varphi(Tx) = d(Tx, Tx^*) \le kd(x, s^*) = k\varphi(x), \quad x \in X.$$

(ii) $\Longrightarrow$ (i). Define the mapping $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \begin{cases} \varphi(x) + \varphi(y) & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

It is not difficult to check that d is a metric on X. Moreover, for every $(x, y) \in X \times X$, we have

$$d(Tx, Ty) \le kd(x, y).$$

Now, we shall prove that (X, d) is a complete metric space. In order to do this, let us take a Cauchy sequence $\{x_n\} \subset X$. Without loss of the generality, we may assume that the set $\{x_n : n \in \mathbb{N}\}$ is infinite (otherwise, $\{x_n\}$ contains a constant subsequence and then $\{x_n\}$ converges). Then $\{x_n\}$ admits a subsequence $\{x_{n_k}\}$ such that

$$x_{n_p} \neq x_{n_q}, \quad p \neq q.$$

Therefore,

$$d(x_{n_p}, x_{n_q}) = \varphi(x_{n_p}) + \varphi(x_{n_q}), \quad p \neq q$$

which yields

$$\lim_{p \to \infty} \varphi(x_{n_p}) = 0 = \varphi(z),$$

for some $z \in X$. Then

$$\lim_{p \to \infty} d(x_{n_p}, z) = 0,$$

which implies that also $\{x_n\}$ converges to z.

1.3.2 Proof of Part (b) of Theorem 1.2

At first, let us recall the following result, which is known as Kuratowski–Zorn Lemma.

Lemma 1.2 (Kuratowski–Zorn Lemma) *Suppose a partially ordered set P has the property that every chain has an upper bound in P. Then the set P contains at least one maximal element.*

For the proof of the above result, we refer to [26]. Now, we give the proof of part (b) of Theorem 1.2.

Proof By hypothesis, some T^n has a unique fixed point $z \in X$. Therefore, we have

$$T^n(Tz) = TT^n z = Tz.$$

Then Tz is a fixed point of T^n. By uniqueness, we obtain $Tz = z$. Hence by (a), z is a unique fixed point of each iterate of T. Now, using the Kuratowski–Zorn Lemma we will show that there exists $\varphi : X \to [0, \infty)$ such that $\varphi^{-1}(\{0\}) = \{z\}$ and

$$\varphi(Tx) \leq k\varphi(x), \quad x \in X. \tag{1.5}$$

Define

$$\phi = \{\varphi : D_\varphi \to [0, \infty) : \{z\} \subset D_\varphi \subset X, \varphi^{-1}(\{0\}) = \{z\}, T(D_\varphi) \subset D_\varphi, (1.5) \text{ holds in } D_\varphi\}.$$

Observe that $\phi \neq \emptyset$. In fact, setting $D_{\varphi^*} = \{z\}$ and $\varphi^*(z) = 0$, we have $\varphi^* \in \phi$. We endow ϕ with the following partial ordering:

$$\varphi_1 \preceq \varphi_2 \Leftrightarrow D_{\varphi_1} \subset D_{\varphi_2},\ \varphi_{2|D_{\varphi_1}} = \varphi_1.$$

Suppose that ϕ_0 is a chain in $(\phi, \preceq)$. Let us consider the set

$$D = \bigcup_{\varphi \in \phi_0} D_\varphi.$$

We claim that

$$TD \subset D.$$

In fact, let $x \in D$. Then $x \in D_\varphi$ for some $\varphi \in \phi_0$. Since $T(D_\varphi) \subset D_\varphi$, we have $Tx \in D_\varphi \subset D$. Therefore, $Tx \in D$, and our claim is proved. Let $\psi : D \to [0, \infty)$ be the function defined by

$$\psi(x) = \varphi(x), \quad x \in D_\varphi,\ \varphi \in \phi_0.$$

Let us prove that ψ is an upper bound for ϕ_0. It is clear that $\psi \in \phi$. Now, let $\varphi_0 \in \phi_0$. We have

$$D_{\varphi_0} \subset D$$

and

$$\psi_{|D_{\varphi_0}} = \varphi_0.$$

Therefore, by the definition of the partial ordering $\preceq$, we have

$$\varphi_0 \preceq \psi.$$

This proves that ψ is an upper bound for ϕ_0. By the Kuratowski–Zorn Lemma, there exists a maximal element $\theta_0 : D_0 \to [0, \infty)$ in $(\phi, \preceq)$. Hence, by Lemma 1.1, It suffices to show that $D_0 = X$. We argue by contradiction. Suppose, on the contrary, that there is an $x_0 \in X \backslash D_0$. Set

$$O(x_0) = \{T^{n-1}x_0 : n \in \mathbb{N}\backslash\{0\}\}.$$

We claim that

$$D_0 \cap O(x_0) \neq \emptyset.$$

In order to prove our claim, we argue by contradiction. So, let us suppose that $D_0 \cap O(x_0) = \emptyset$. Then the elements $T^{n-1}(x_0)$ for $n \in \mathbb{N}\backslash\{0\}$ are distinct. In fact, suppose that

$$T^{p-1}(x_0) = T^{p+q-1}(x_0), \quad \text{for some } (p, q) \in \mathbb{N}\backslash\{0\} \times \mathbb{N}\backslash\{0\}.$$

Therefore, $T^{p-1}x_0$ is the unique fixed point of T^q, which yields $z = T^{p-1}x_0 \in O(x_0) \cap D_0$, a contradiction. Define

$$D_\varphi = D_0 \cup O(x_0),\ \varphi_{|D_0} = \theta_0,\ \varphi(T^{n-1}x_0) = k^{n-1},\ n \in \mathbb{N}\backslash\{0\}.$$

Then $\varphi \in \phi$ and $\varphi \neq \theta_0$. Moreover, we have $\theta_0 \preceq \varphi$, which is a contradiction with the fact that θ_0 is a maximal element in $(\phi, \preceq)$. Therefore, our claim is proved. Set

$$\mathcal{N} = \{n \in \mathbb{N}\backslash\{0\} :\ T^n x_0 \in D_0\}.$$

From the previous step, it is clear that $\mathcal{N} \neq \emptyset$. So, we can define

$$m = \min \mathcal{N}.$$

Then $m - 1 \notin \mathcal{N}$, i.e., $T^{m-1}x_0 \notin D_0$. Define

$$D_\varphi = \{T^{m-1}x_0\} \cup D_0.$$

Then

$$T D_\varphi = \{T^m x_0\} \cup T D_0 \subset D_0 \subset D_\varphi.$$

Now, we will define a function $\varphi : D_\varphi \to [0, \infty)$. Set

$$\varphi_{|D_0} = \theta_0.$$

We have two possible cases.

Case 1. If $T^m x_0 = z$.
In this case, we set

$$\varphi(T^{m-1}x_0) = 1.$$

Observe that $\varphi \in \phi$. Moreover, we have $\theta_0 \preceq \varphi$ and $\theta_0 \neq \varphi$, which is a contradiction with the fact that θ_0 is a maximal element in $(\phi, \preceq)$.

Case 2. If $T^m x_0 \neq z$.
In this case, we set

$$\varphi(T^{m-1}x_0) = \frac{\theta_0(T^m x_0)}{k}.$$

As in the previous case, we observe easily that $\varphi \in \phi$, $\theta_0 \preceq \varphi$ and $\theta_0 \neq \varphi$, which is a contradiction with the fact that θ_0 is a maximal element in $(\phi, \preceq)$.

As consequence, we infer that $D_0 = X$, which proves the desired result.

1.4 Some Applications

Some applications of Banach contraction principle are presented in this section. To be more precise, we study the solvability of a class of integral equations and of systems of nonlinear matrix equations involving Lipschitzian mappings.

1.4.1 Solvability of a Mixed Volterra–Fredholm-Type Integral Equation

In this part, we deal with the existence of solutions to the following general mixed Volterra–Fredholm-type integral equation

$$u(t,x) = f(t,x) + \int_0^t \int_\Omega F(t,x,s,y,u(s,y))\,dy\,ds, \quad (t,x) \in D, \tag{1.6}$$

where $f : D \to \mathbb{R}^N$, $F : D \times D \times \mathbb{R}^N \to \mathbb{R}^N$, $D = [0,T] \times \Omega$, $T > 0$ and Ω is a nonempty bounded and closed subset of the Euclidean space $\mathbb{R}^N$ equipped with convenient norm $\|\cdot\|$. Equations of the type (1.6) arise from the theory of nonlinear parabolic boundary value problems, the mathematical modeling of the spatiotemporal development of an epidemic [10, 20], and various physical and biological models. Following the techniques used in Pachpatte [19] and using Banach contraction principle, an existence result will be established for (1.6).

Let S be the set of functions $\phi : D \to \mathbb{R}^N$, which are continuous in D and satisfying the condition

$$\|\phi(t,x)\| = O\left(\exp\left(\mu(t + \|x\|)\right)\right), \quad (t,x) \in D, \tag{1.7}$$

where $\mu > 0$ is a constant. We endow the space S with the norm

$$|\phi| = \sup_{(t,x)\in D} \left[\|\phi(t,x)\| \exp\left(-\mu(t + \|x\|)\right)\right], \quad \phi \in S.$$

Then $(S, |\cdot|)$ is a Banach space (see [7]). Note that from (1.7), there exists some constant $M > 0$ such that

$$\|\phi(t,x)\| \le M \exp\left(\mu(t + \|x\|)\right), \quad (t,x) \in D.$$

Therefore, we have

$$|\phi| \le M, \quad \phi \in S. \tag{1.8}$$

Equation (1.6) is investigated under the following assumptions:

(A1) The functions $f : D \to \mathbb{R}^N$ and $F : D \times D \times \mathbb{R}^N \to \mathbb{R}^N$ are continuous.

(A2) There exists a continuous function $h : D \times D \to [0,\infty)$ such that

$$\|F(t,x,s,y,u_1) - F(t,x,s,y,u_2)\| \leq h(t,x,s,y)\|u_1 - u_2\|,$$

for all $(t,x,s,y,u_i) \in D \times D \times \mathbb{R}^N$, $i = 1, 2$.
(A3) There exists a constant $Q \in (0,1)$ such that

$$\int_0^t \int_\Omega h(t,x,s,y) \exp\left(\mu(s + \|y\|)\right) \, dy \, ds \leq Q \exp\left(\mu(t + \|x\|)\right), \quad (t,x) \in D.$$

(A4) There exists a constant $N > 0$ such that

$$\|f(t,x)\| + \int_0^t \int_\Omega \|F(t,x,s,y,0)\| \, dy \, ds \leq N \exp\left(\mu(t + \|x\|)\right), \quad (t,x) \in D.$$

We have the following existence result.

Theorem 1.3 *Under Assumptions (A1)–(A4),* (1.6) *has a unique solution $u^* \in S$. Moreover, for any $u_0 \in S$, the Picard sequence $\{u_n\}$ defined by*

$$u_{n+1}(t,x) = f(t,x) + \int_0^t \int_\Omega F(t,x,s,y,u_n(s,y)) \, dy \, ds, \quad (t,x) \in D$$

converges with respect to the norm $|\cdot|$ to u^*.

Proof Set

$$(Tu)(t,x) = f(t,x) + \int_0^t \int_\Omega F(t,x,s,y,u(s,y)) \, dy \, ds, \quad u \in S, \ (t,x) \in D.$$

We shall prove that T maps S into itself. So, let u be an element of S. It is easy to observe that $Tu : D \to \mathbb{R}^N$ is a continuous mapping. We have to check that (1.7) is satisfied. Using the considered assumptions, for all $(t,x) \in D$, we have

$$\begin{aligned}
\|(Tu)(t,x)\| &\leq \|f(t,x)\| + \int_0^t \int_\Omega \|F(t,x,s,y,u(s,y))\| \, dy \, ds \\
&\leq \|f(t,x)\| + \int_0^t \int_\Omega \|F(t,x,s,y,u(s,y)) - F(t,x,s,y,0)\| \, dy \, ds \\
&\quad + \int_0^t \int_\Omega \|F(t,x,s,y,0)\| \, dy \, ds \\
&\leq \int_0^t \int_\Omega h(t,x,s,y)\|u(s,y)\| \, dy \, ds + N \exp\left(\mu(t + \|x\|)\right) \\
&\leq M \int_0^t \int_\Omega h(t,x,s,y) \exp\left(\mu(s + \|y\|)\right) \, dy \, ds + N \exp\left(\mu(t + \|x\|)\right) \\
&\leq (MQ + N) \exp\left(\mu(t + \|x\|)\right).
\end{aligned}$$

Therefore, (1.7) holds and $T : S \to S$ is well defined.

Now, we verify that $T : S \to S$ is a contraction. So, let (u, v) be a pair of elements in S. For all $(t, x) \in D$, we have

$$\begin{aligned}
\|(Tu - Tv)(t, x)\| &\leq \int_0^t \int_\Omega \|F(t, x, s, y, u(s, y)) - F(t, x, s, y, v(s, y))\| \, dy \, ds \\
&\leq \int_0^t \int_\Omega h(t, x, s, y) \|u(s, y) - v(s, y)\| \, dy \, ds \\
&\leq \left(\int_0^t \int_\Omega h(t, x, s, y) \exp\left(\mu(s + \|y\|)\right) \, dy \, ds \right) |u - v| \\
&\leq Q \exp\left(\mu(t + \|x\|)\right) |u - v|.
\end{aligned}$$

Therefore,

$$\|(Tu - Tv)(t, x)\| \exp\left(-\mu(t + \|x\|)\right) \leq Q|u - v|.$$

Hence, we obtain

$$|Tu - Tv| \leq Q|u - v|, \quad (u, v) \in S \times S.$$

Now, by Banach contraction principle, the mapping T has a unique fixed point $u^* \in S$. Moreover, for any $u_0 \in S$, the Picard sequence $\{T^n u_0\}$ converges to u^* in $(S, |\cdot|)$. This completes the proof of the theorem.

1.4.2 Solving Systems of Nonlinear Matrix Equations Involving Lipschitzian Mappings

In this part, both theoretical results and numerical methods are derived for solving different classes of systems of nonlinear matrix equations involving Lipschitzian mappings. Our main tool in this study is Banach contraction principle. The main reference for this work is the paper Berzig and Samet [3].

We first review the Thompson metric on the open convex cone $P(n)$ $(n \geq 2)$, the set of all $n \times n$ Hermitian positive definite matrices. We endow $P(n)$ with the Thompson metric defined by:

$$d(A, B) = \max\left\{\ln M(A/B), \ln M(B/A)\right\},$$

where $M(A/B) = \inf\{\lambda > 0 : A \leq \lambda B\} = \lambda^+(B^{-1/2}AB^{-1/2})$, the maximal eigenvalue of $B^{-1/2}AB^{-1/2}$. Here, $X \leq Y$ means that $Y - X$ is positive semi-definite and $X < Y$ means that $Y - X$ is positive definite. Thompson [27] (cf. [17, 18]) has proved that $P(n)$ is a complete metric space with respect to the Thompson metric d and $d(A, B) = \|\log(A^{-1/2}BA^{-1/2})\|$, where $\|\cdot\|$ stands for the spectral norm. The Thompson metric exists on any open normal convex cones of real Banach spaces [17, 27], in particular, the open convex cone of positive definite operators of a Hilbert space. It is invariant under the matrix inversion and congruence transformations, that is,

$$d(A, B) = d(A^{-1}, B^{-1}) = d(MAM^*, MBM^*)$$

for any nonsingular matrix M. The other useful result is the nonpositive curvature property of the Thompson metric, that is,

$$d(X^r, Y^r) \leq r\, d(X, Y), \quad r \in [0, 1].$$

By the invariant properties of the metric, we then have

$$d(MX^rM^*, MY^rM^*) \leq |r|\, d(X, Y), \quad r \in [-1, 1]$$

for any $X, Y \in P(n)$ and nonsingular matrix M.

Lemma 1.3 *For all $A, B, C, D \in P(n)$, we have*

$$d(A + B, C + D) \leq \max\{d(A, C), d(B, D)\}.$$

In particular,

$$d(A + B, A + C) \leq d(B, C).$$

We refer to [16] for the proof of the above lemma.

In the last few years, there has been a constantly increasing interest in developing the theory and numerical approaches for Hermitian positive definite (HPD) solutions to different classes of nonlinear matrix equations (see [2–4, 8, 11, 15, 16, 21, 22, 29]). In this study, we consider the following problem: Find $(X_1, X_2, \ldots, X_m) \in (P(n))^m$ solution to the system of nonlinear matrix equations

$$X_i^{r_i} = Q_i + \sum_{j=1}^{m} \left(A_j^* F_{ij}(X_j) A_j\right)^{\alpha_{ij}}, \quad i = 1, 2, \ldots, m, \tag{1.9}$$

where $r_i \geq 1$, $0 < |\alpha_{ij}| \leq 1$, $Q_i \geq 0$, A_i are nonsingular matrices and $F_{ij} : P(n) \to P(n)$ are Lipschitzian mappings, that is,

$$\sup_{X,Y \in P(n), X \neq Y} \frac{d(F_{ij}(X), F_{ij}(Y))}{d(X, Y)} = k_{ij} < \infty.$$

If $m = 1$ and $\alpha_{11} = 1$, then (1.9) reduces to the problem: Find $X \in P(n)$ solution to

$$X^r = Q + A^* F(X) A.$$

Such equation was studied by Liao et al. [15]. Now, we introduce the following definition.

Definition 1.8 We say that System (1.9) is Banach admissible if the following hypothesis is satisfied:

$$\max_{1\le i\le m}\left\{\max_{1\le j\le m}\{|\alpha_{ij}|k_{ij}/r_i\}\right\} < 1.$$

Our main result is the following.

Theorem 1.4 *Suppose that* (1.9) *is Banach admissible. Then*

(i) (1.9) *admits one and only one solution* $(X_1^*, X_2^*, \dots, X_m^*) \in (P(n))^m$.

(ii) *For any* $(X_1(0), X_2(0), \dots, X_m(0)) \in (P(n))^m$, *the sequences* $(X_i(k))_{k\ge 0}$, $1 \le i \le m$, *defined by*

$$X_i(k+1) = \left(Q_i + \sum_{j=1}^{m}(A_j^* F_{ij}(X_j(k))A_j)^{\alpha_{ij}}\right)^{1/r_i},$$

converge respectively to $X_1^*, X_2^*, \dots, X_m^*$.

(iii) *The following estimate:*

$$\max\left\{d(X_1(k), X_1^*), d(X_2(k), X_2^*), \dots, d(X_m(k), X_m^*)\right\}$$
$$\le \frac{q_m^k}{1-q_m}\max\{d(X_1(1), X_1(0)), d(X_2(1), X_2(0)), \dots, d(X_m(1), X_m(0))\}$$

holds, where

$$q_m = \max_{1\le i\le m}\left\{\max_{1\le j\le m}\{|\alpha_{ij}|k_{ij}/r_i\}\right\}.$$

Proof Define the mapping $G : (P(n))^m \to (P(n))^m$ by

$$G(X_1, X_2, \dots, X_m) = (G_1(X_1, X_2, \dots, X_m), G_2(X_1, X_2, \dots, X_m), \dots, G_m(X_1, X_2, \dots, X_m)),$$

for all $X = (X_1, X_2, \dots, X_m) \in (P(n))^m$, where

$$G_i(X) = \left(Q_i + \sum_{j=1}^{m}(A_j^* F_{ij}(X_j)A_j)^{\alpha_{ij}}\right)^{1/r_i}, \quad i = 1, 2, \dots, m.$$

We endow $(P(n))^m$ with the metric d_m defined by

$$d_m((X_1, X_2, \dots, X_m), (Y_1, Y_2, \dots, Y_m)) = \max\{d(X_1, Y_1), d(X_2, Y_2), \dots, d(X_m, Y_m)\},$$

for all $X = (X_1, X_2, \dots, X_m), Y = (Y_1, Y_2, \dots, Y_m) \in (P(n))^m$. Obviously, $((P(n))^m, d_m)$ is a complete metric space. For all $X, Y \in (P(n))^m$, we have

$$d_m(G(X), G(Y)) = \max_{1\le i\le m}\{d(G_i(X), G_i(Y))\}. \tag{1.10}$$

On the other hand, for a fixed $i \in \{1, 2, \ldots, m\}$ and $X, Y \in (P(n))^m$,

$$\begin{aligned} &d(G_i(X), G_i(Y)) \\ &= d\left(\left(Q_i + \sum_{j=1}^{m}(A_j^* F_{ij}(X_j)A_j)^{\alpha_{ij}}\right)^{1/r_i}, \left(Q_i + \sum_{j=1}^{m}(A_j^* F_{ij}(Y_j)A_j)^{\alpha_{ij}}\right)^{1/r_i}\right). \end{aligned}$$

Using the properties of the Thompson metric, after some computations (see [3] for the details), we obtain

$$d(G_i(X), G_i(Y)) \le \max_{1\le j\le m}\{|\alpha_{ij}|k_{ij}/r_i\}\, d_m(X, Y).$$

Now, using the above inequality and (1.10), we deduce that

$$d_m(G(X), G(Y)) \le q_m\, d_m(X, Y), \text{ for all } X, Y \in (P(n))^m.$$

Applying Banach contraction principle to the mapping G, the desired result follows.

Now, we present some examples and numerical results in order to illustrate our obtained result.

The Matrix Equation: $\mathbf{X} = \left(\left((\mathbf{X}^{1/2} + \mathbf{B}_1)^{-1/2} + \mathbf{B}_2\right)^{1/3} + \mathbf{B}_3\right)^{1/2}$

We consider the problem: Find $X \in P(n)$ solution to

$$X = \left(\left((X^{1/2} + B_1)^{-1/2} + B_2\right)^{1/3} + B_3\right)^{1/2}, \tag{1.11}$$

where $B_i \ge 0$, for all $i = 1, 2, 3$.

Solving (1.11) is equivalent to: Find $X_1 \in P(n)$ solution to

$$X_1^{r_1} = Q_1 + (A_1^* F_{11}(X_1)A_1)^{\alpha_{11}}, \tag{1.12}$$

where $r_1 = 2$, $Q_1 = B_3$, $A_1 = I_n$ (the identity matrix), $\alpha_{11} = 1/3$ and $F_{11} : P(n) \to P(n)$ is given by

$$F_{11}(X) = (X^{1/2} + B_1)^{-1/2} + B_2.$$

Proposition 1.2 *F_{11} is a Lipschitzian mapping with $k_{11} \le 1/4$.*

Proof Using the properties of the Thompson metric, for all $X, Y \in P(n)$, we have

$$\begin{aligned} d(F_{11}(X), F_{11}(Y)) &= d((X^{1/2} + B_1)^{-1/2} + B_2, (Y^{1/2} + B_1)^{-1/2} + B_2) \\ &\leq d((X^{1/2} + B_1)^{-1/2}, (Y^{1/2} + B_1)^{-1/2}) \\ &\leq \frac{1}{2}\, d(X^{1/2} + B_1, Y^{1/2} + B_1) \\ &\leq \frac{1}{2}\, d(X^{1/2}, Y^{1/2}) \leq \frac{1}{4}\, d(X, Y), \end{aligned}$$

which yields the desired result.

Proposition 1.3 (1.12) *is Banach admissible.*

Proof We have

$$\frac{|\alpha_{11}|k_{11}}{r_1} \leq \frac{\frac{1}{3}\,\frac{1}{4}}{2} = \frac{1}{24} < 1,$$

which implies that (1.12) is Banach admissible.

We have the following solvability result for (1.12).

Theorem 1.5 *Equation* (1.12) *has one and only one solution* $X_1^* \in P(n)$. *For any* $X_1(0) \in P(n)$, *the sequence* $(X_1(k))_{k\geq 0}$ *defined by*

$$X_1(k+1) = \left(\left((X_1(k)^{1/2} + B_1)^{-1/2} + B_2\right)^{1/3} + B_3 \right)^{1/2}, \tag{1.13}$$

converges to X_1^*. *Moreover, the following estimate:*

$$d(X_1(k), X_1^*) \leq \frac{q_1^k}{1 - q_1}\, d(X_1(1), X_1(0))$$

holds, where $q_1 = 1/4$.

Proof It follows immediately from Propositions 1.2, 1.3 and Theorem 1.4.

Now, we give a numerical example to illustrate our result given by Theorem 1.5. We consider the 5×5 positive matrices B_1, B_2, and B_3 given by

$$B_1 = \begin{pmatrix} 1.0000 & 0.5000 & 0.3333 & 0.2500 & 0 \\ 0.5000 & 1.0000 & 0.6667 & 0.5000 & 0 \\ 0.3333 & 0.6667 & 1.0000 & 0.7500 & 0 \\ 0.2500 & 0.5000 & 0.7500 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1.4236 & 1.3472 & 1.1875 & 1.0000 & 0 \\ 1.3472 & 1.9444 & 1.8750 & 1.6250 & 0 \\ 1.1875 & 1.8750 & 2.1181 & 1.9167 & 0 \\ 1.0000 & 1.6250 & 1.9167 & 1.8750 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$B_3 = \begin{pmatrix} 2.7431 & 3.3507 & 3.3102 & 2.9201 & 0 \\ 3.3507 & 4.6806 & 4.8391 & 4.3403 & 0 \\ 3.3102 & 4.8391 & 5.2014 & 4.7396 & 0 \\ 2.9201 & 4.3403 & 4.7396 & 4.3750 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We use the iterative algorithm (1.13) to solve (1.11) for different values of $X_1(0)$:

$$X_1(0) = M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}, \quad X_1(0) = M_2 = \begin{pmatrix} 0.02 & 0.01 & 0 & 0 & 0 \\ 0.01 & 0.02 & 0.01 & 0 & 0 \\ 0 & 0.01 & 0.02 & 0.01 & 0 \\ 0 & 0 & 0.01 & 0.02 & 0.01 \\ 0 & 0 & 0 & 0.01 & 0.02 \end{pmatrix}$$

and

$$X_1(0) = M_3 = \begin{pmatrix} 30 & 15 & 10 & 7.5 & 6 \\ 15 & 30 & 20 & 15 & 12 \\ 10 & 20 & 30 & 22.5 & 18 \\ 7.5 & 15 & 22.5 & 30 & 24 \\ 6 & 12 & 18 & 24 & 30 \end{pmatrix}.$$

For $X_1(0) = M_1$, after nine iterations, we get the unique positive definite solution

$$X_1(9) = \begin{pmatrix} 1.6819 & 0.69442 & 0.61478 & 0.51591 & 0 \\ 0.69442 & 1.9552 & 0.96059 & 0.84385 & 0 \\ 0.61478 & 0.96059 & 2.0567 & 0.9785 & 0 \\ 0.51591 & 0.84385 & 0.9785 & 1.9227 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and its residual error is given by

$$R(X_1(9)) = \left\| X_1(9) - \left(\left(\left(X_1(9)^{1/2} + B_1 \right)^{-1/2} + B_2 \right)^{1/3} + B_3 \right)^{1/2} \right\| = 6.346 \times 10^{-13}.$$

For $X_1(0) = M_2$, after nine iterations, the residual error is

$$R(X_1(9)) = 1.5884 \times 10^{-12}.$$

For $X_1(0) = M_3$, after nine iterations, the residual error is

$$R(X_1(9)) = 1.1123 \times 10^{-12}.$$

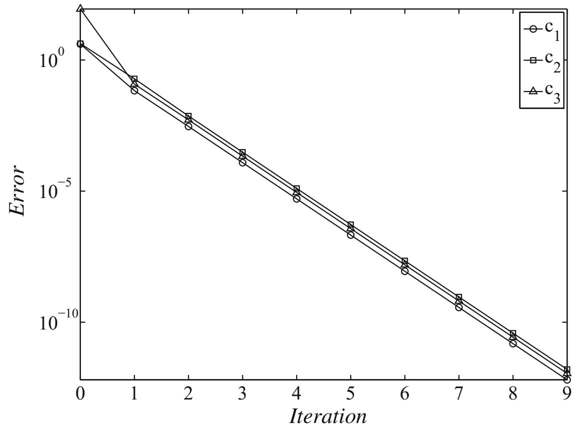

Fig. 1.1 Convergence history for Eq. (1.11)

The convergence history of the algorithm for different values of $X_1(0)$ is given by Fig. 1.1, where c_1 corresponds to $X_1(0) = M_1$, c_2 corresponds to $X_1(0) = M_2$, and c_3 corresponds to $X_1(0) = M_3$.

System of Three Nonlinear Matrix Equations

We consider the problem: Find $(X_1, X_2, X_3) \in (P(n))^3$ solution to

$$(S): \begin{cases} X_1 = I_n + A_1^*(X_1^{1/3} + B_1)^{1/2}A_1 + A_2^*(X_2^{1/4} + B_2)^{1/3}A_2 + A_3^*(X_3^{1/5} + B_3)^{1/4}A_3, \\ X_2 = I_n + A_1^*(X_1^{1/5} + B_1)^{1/4}A_1 + A_2^*(X_2^{1/3} + B_2)^{1/2}A_2 + A_3^*(X_3^{1/4} + B_3)^{1/3}A_3, \\ X_3 = I_n + A_1^*(X_1^{1/4} + B_1)^{1/3}A_1 + A_2^*(X_2^{1/5} + B_2)^{1/4}A_2 + A_3^*(X_3^{1/3} + B_3)^{1/2}A_3, \end{cases}$$

where A_i are $n \times n$ nonsingular matrices.

Solving (S) is equivalent to: Find $(X_1, X_2, X_3) \in (P(n))^3$ solution to

$$X_i^{r_i} = Q_i + \sum_{j=1}^{3}(A_j^* F_{ij}(X_j)A_j)^{\alpha_{ij}}, \quad i = 1, 2, 3, \tag{1.14}$$

where $r_1 = r_2 = r_3 = 1$, $Q_1 = Q_2 = Q_3 = I_n$ and for all $i, j \in \{1, 2, 3\}$, $\alpha_{ij} = 1$,

$$F_{ij}(X_j) = (X_j^{\theta_{ij}} + B_j)^{\gamma_{ij}}, \; \theta = (\theta_{ij}) = \begin{pmatrix} 1/3 & 1/4 & 1/5 \\ 1/5 & 1/3 & 1/4 \\ 1/4 & 1/5 & 1/3 \end{pmatrix}, \gamma = (\gamma_{ij}) = \begin{pmatrix} 1/2 & 1/3 & 1/4 \\ 1/4 & 1/2 & 1/3 \\ 1/3 & 1/4 & 1/2 \end{pmatrix}.$$

Proposition 1.4 *For all* $i, j \in \{1, 2, 3\}$*,* $F_{ij} : P(n) \to P(n)$ *is a Lipschitzian mapping with* $k_{ij} \le \gamma_{ij}\theta_{ij}$.

Proof For all $X, Y \in P(n)$, since $\theta_{ij}, \gamma_{ij} \in (0, 1)$, we have

$$\begin{aligned} d(F_{ij}(X), F_{ij}(Y)) &= d((X^{\theta_{ij}} + B_j)^{\gamma_{ij}}, (Y^{\theta_{ij}} + B_j)^{\gamma_{ij}}) \\ &\leq \gamma_{ij} d(X^{\theta_{ij}} + B_j, Y^{\theta_{ij}} + B_j) \\ &\leq \gamma_{ij} d(X^{\theta_{ij}}, Y^{\theta_{ij}}) \\ &\leq \gamma_{ij}\theta_{ij} d(X, Y), \end{aligned}$$

which implies that F_{ij} is a Lipschitzian mapping with $k_{ij} \leq \gamma_{ij}\theta_{ij}$.

We have the following Banach admissibility property.

Proposition 1.5 *System* (1.14) *is Banach admissible.*

Proof We have

$$\begin{aligned} \max_{1\leq i\leq 3} \left\{ \max_{1\leq j\leq 3} \{|\alpha_{ij}|k_{ij}/r_i\} \right\} &= \max_{1\leq i,j\leq 3} k_{ij} \\ &\leq \max_{1\leq i,j\leq 3} \gamma_{ij}\theta_{ij} \\ &= 1/6 < 1, \end{aligned}$$

which implies that (1.14) is Banach admissible.

Next, we deduce the following existence result for System (S).

Theorem 1.6 *System (S) has one and only one solution* $(X_1^*, X_2^*, X_3^*) \in (P(n))^3$. *For any* $(X_1(0), X_2(0), X_3(0)) \in (P(n))^3$, *the sequences* $(X_i(k))_{k\geq 0}$, $1 \leq i \leq 3$, *defined by*

$$X_i(k+1) = I_n + \sum_{j=1}^{3} A_j^* F_{ij}(X_j(k)) A_j, \tag{1.15}$$

converge respectively to X_1^*, X_2^*, X_3^*. *Moreover, the error estimate is given by*

$$\begin{aligned} &\max \left\{ d(X_1(k), X_1^*), d(X_2(k), X_2^*), d(X_3(k), X_3^*) \right\} \\ &\leq \frac{q_3^k}{1-q_3} \max \left\{ d(X_1(1), X_1(0)), d(X_2(1), X_2(0)), d(X_3(1), X_3(0)) \right\}, \end{aligned}$$

where $q_3 = 1/6$.

Proof It follows from Propositions 1.4, 1.5 and Theorem 1.4.

Now, we give a numerical example to illustrate our obtained result given by Theorem 1.6.

We consider the 3×3 positive matrices B_1, B_2, and B_3 given by

$$B_1 = \begin{pmatrix} 1. & 0.5 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1.25 & 1 & 0 \\ 1 & 1.25 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } B_3 = \begin{pmatrix} 1.75 & 1.625 & 0 \\ 1.625 & 1.75 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We consider the 3×3 nonsingular matrices A_1, A_2, and A_3 given by

$$A_1 = \begin{pmatrix} 0.3107 & -0.5972 & 0.7395 \\ 0.9505 & 0.1952 & -0.2417 \\ 0 & -0.7780 & -0.6282 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1.5 & -2 & 0.5 \\ 0.5 & 0 & -0.5 \\ -0.5 & 2 & -1.5 \end{pmatrix}$$

and

$$A_3 = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & -1 \end{pmatrix}.$$

We use the iterative algorithm (1.15) to solve (S) for different values of $(X_1(0), X_2(0), X_3(0))$:

$$X_1(0) = X_2(0) = X_3(0) = M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$X_1(0) = X_2(0) = X_3(0) = M_2 = \begin{pmatrix} 0.02 & 0.01 & 0 \\ 0.01 & 0.02 & 0.01 \\ 0 & 0.01 & 0.02 \end{pmatrix}$$

and

$$X_1(0) = X_2(0) = X_3(0) = M_3 = \begin{pmatrix} 30 & 15 & 10 \\ 15 & 30 & 20 \\ 10 & 20 & 30 \end{pmatrix}.$$

The error at the iteration k is given by

$$R(X_1(k), X_2(k), X_3(k)) = \max_{1 \le i \le 3} \left\| X_i(k) - I_3 - \sum_{j=1}^{3} A_j^* F_{ij}(X_j(k)) A_j \right\|.$$

For $X_1(0) = X_2(0) = X_3(0) = M_1$, after 15 iterations, we obtain

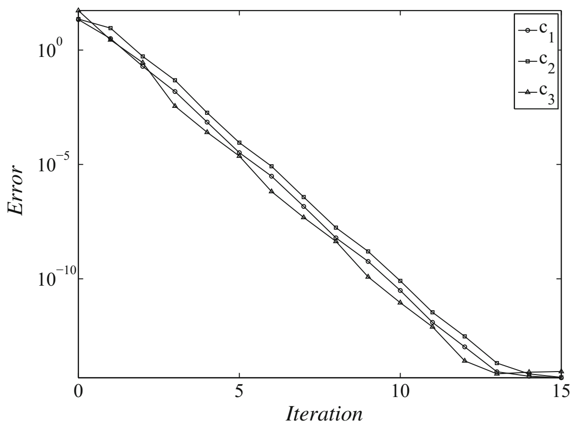

Fig. 1.2 Convergence history for (S)

$$X_1(15) = \begin{pmatrix} 10.565 & -4.4081 & 2.7937 \\ -4.4081 & 16.883 & -6.6118 \\ 2.7937 & -6.6118 & 9.7152 \end{pmatrix}, \quad X_2(15) = \begin{pmatrix} 11.512 & -5.8429 & 3.1922 \\ -5.8429 & 19.485 & -7.9308 \\ 3.1922 & -7.9308 & 10.68 \end{pmatrix}$$

and

$$X_3(15) = \begin{pmatrix} 11.235 & -3.5241 & 3.2712 \\ -3.5241 & 17.839 & -7.8035 \\ 3.2712 & -7.8035 & 11.618 \end{pmatrix}.$$

The residual error is given by

$$R(X_1(15), X_2(15), X_3(15)) = 4.722 \times 10^{-15}.$$

For $X_1(0) = X_2(0) = X_3(0) = M_2$, after 15 iterations, the residual error is given by

$$R(X_1(15), X_2(15), X_3(15)) = 4.911 \times 10^{-15}.$$

For $X_1(0) = X_2(0) = X_3(0) = M_3$, after 15 iterations, the residual error is given by

$$R(X_1(15), X_2(15), X_3(15)) = 8.869 \times \times 10^{-15}.$$

The convergence history of the algorithm for different values of $X_1(0)$, $X_2(0)$, and $X_3(0)$ is given by Fig. 1.2, where c_1 corresponds to $X_1(0) = X_2(0) = X_3(0) = M_1$, c_2 corresponds to $X_1(0) = X_2(0) = X_3(0) = M_2$, and c_3 corresponds to $X_1(0) = X_2(0) = X_3(0) = M_3$.

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Chapter 2
On Ran–Reurings Fixed Point Theorem

In order to study the existence of solutions to a certain class of nonlinear matrix equations, Ran and Reurings [38] established an extension of Banach contraction principle to metric spaces equipped with a partial order. In this chapter, we present another proof of Ran–Reurings fixed point theorem using Banach contraction principle. Next, we present some applications of this result to the solvability of some classes of matrix equations.

2.1 Preliminaries

In this section, we present some basic definitions that will be used later.

Definition 2.1 Let X be a nonempty set. Any nonempty subset $\mathscr{R}$ of the product set $X \times X$ is said to be a binary relation on X.

For $(x, y) \in X \times X$, the notation $x\mathscr{R}y$ means that the pair of points (x, y) belongs to $\mathscr{R}$.

Let X be a nonempty set and $\mathscr{R}$ be a binary relation on X.

Definition 2.2 We say that $\mathscr{R}$ is reflexive if

$$x\mathscr{R}x, \quad x \in X.$$

Definition 2.3 We say that $\mathscr{R}$ is transitive if

$$x\mathscr{R}y,\ y\mathscr{R}z \implies x\mathscr{R}z, \quad (x, y, z) \in X \times X \times X.$$

Definition 2.4 We say that $\mathscr{R}$ is antisymmetric if

$$x\mathscr{R}y,\ y\mathscr{R}x \implies x = y, \quad (x, y) \in X \times X.$$

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Definition 2.5 We say that $\mathscr{R} :=\preceq$ is a partial order on X if it is reflexive, antisymmetric, and transitive. In this case, the pair $(X, \preceq)$ is said to be a partially ordered set.

Example 2.1 The set of real numbers $\mathbb{R}$ equipped with the standard order $\leq$ is a partially ordered set.

Example 2.2 Let Y be a nonempty set. Let $X = \mathscr{P}(Y)$ be the set of all the subsets of Y. Define the binary relation $\preceq$ on X by

$$A, B \in X, \quad A \preceq B \Longleftrightarrow A \subseteq B.$$

Then $(X, \preceq)$ is a partially ordered set.

Example 2.3 Let $X = \mathbb{R}^2$. Define the binary relation $\preceq$ on X by

$$(x, y), (z, w) \in X, \quad (x, y) \preceq (z, w) \Longleftrightarrow x \leq z, \ y \leq w.$$

Then $(X, \preceq)$ is a partially ordered set.

Example 2.4 Let $X = C([a, b]; \mathbb{R})$ be the set of real-valued and continuous functions in $[a, b]$ $(a < b)$. Define the binary relation $\preceq$ on X by

$$f, g \in X, \quad f \preceq g \Longleftrightarrow f(t) \leq g(t), \ t \in [a, b].$$

Then $(X, \preceq)$ is a partially ordered set.

Definition 2.6 Let $(X, \preceq)$ be a partially ordered set, and let $T : X \to X$ be a given mapping.

(i) T is said to be a nondecreasing mapping if

$$(x, y) \in X \times X, \ x \preceq y \implies Tx \preceq Ty.$$

(ii) T is said to be a decreasing mapping if

$$(x, y) \in X \times X, \ x \preceq y \implies Ty \preceq Tx.$$

(iii) T is said to be a monotone mapping if it is a decreasing or nondecreasing mapping.

Definition 2.7 Let $(X, \preceq)$ be a partially ordered set, and let $F : X \times X \to X$ be a given mapping. Then F is said to be a mixed monotone mapping if

$$(x, y), (z, w) \in X \times X, \ x \preceq z, \ y \succeq w \implies F(x, y) \preceq F(z, w).$$

Remark 2.1 Let $(X, \preceq)$ be a partially ordered set, and let $F : X \times X \to X$ be a given mapping. We endow the product set $Z := X \times X$ with the binary relation $\preceq_2$ defined by

$$(x, y), (z, w) \in Z, \quad (x, y) \preceq_2 (z, w) \Leftrightarrow x \preceq z,\ y \succeq w.$$

Then it can be easily checked that $(Z, \preceq_2)$ is a partially ordered set. Moreover, the following statements are equivalent:

(i) F is a mixed monotone mapping.
(ii) The mapping $T : (Z, \preceq_2) \to (Z, \preceq_2)$ defined by

$$T(x, y) = (F(x, y), F(y, x)), \quad (x, y) \in Z$$

is nondecreasing.

2.2 Ran–Reurings Fixed Point Theorem

In this section, we state and prove Ran–Reurings fixed point theorem [38]. We give a nonstandard proof of this theorem, based on an application of Banach contraction principle, which is due to Samet [40].

Theorem 2.1 (Ran–Reurings fixed point) *Let $(X, \preceq)$ be a partially ordered set, and let d be a metric on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a given mapping. We suppose that the following conditions are satisfied:*

(i) *T is continuous.*
(ii) *T is nondecreasing.*
(iii) *There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.*
(iv) *There exists a constant $\lambda \in (0, 1)$ such that*

$$(x, y) \in X \times X,\ x \preceq y \implies d(Tx, Ty) \leq \lambda d(x, y).$$

Then the Picard sequence $\{T^n x_0\}$ converges to a fixed point of T.

Proof Let us consider the subset $\Lambda_T(x_0)$ of X defined by

$$\Lambda_T(x_0) = \{T^n x_0 : n = 0, 1, 2, \ldots\}.$$

Let

$$\mathscr{Z} = \overline{\Lambda_T(x_0)}$$

be the closure of $\Lambda_T(x_0)$ with respect to the metric d. Clearly, $(\mathscr{Z}, d)$ is a complete metric space. We claim that

$$T(\mathscr{X}) \subseteq \mathscr{X}.$$

Let $z \in \mathscr{X}$. From the definition of $\mathscr{X}$, there exists a sequence $\{T^{n_k}x_0\}_k$ that converges to z with respect to the metric d. The continuity of T yields $\{T^{n_k+1}x_0\}_k$ converges to Tz with respect to the metric d. Since $\{T^{n_k+1}x_0\}_k \subseteq \mathscr{X}$ and $\mathscr{X}$ is closed, then $Tz \in \mathscr{X}$, which proves our claim.

Now, let (x, y) be an arbitrary pair of points in $\mathscr{X} \times \mathscr{X}$. From the definition of $\mathscr{X}$, there exists a sequence $\{T^{n_k}x_0\}_k$ that converges to x with respect to the metric d. Similarly, there exists a sequence $\{T^{n_p}x_0\}_p$ that converges to y with respect to the metric d. On the other hand, the monotony of T yields

$$x_0 \preceq Tx_0 \preceq T^2x_0 \preceq \cdots \preceq T^nx_0 \preceq T^{n+1}x_0 \preceq \cdots$$

Then $T^{n_k}x_0$ and $T^{n_p}x_0$ are comparable with respect to the partial order $\preceq$ for every natural numbers p and k. Thus, we have

$$d(T^{n_k+1}x_0, T^{n_p+1}x_0) \leq \lambda d(T^{n_k}x_0, T^{n_p}x_0), \quad \text{for all } k, p.$$

Letting $k \to \infty$ and $p \to \infty$ in the above inequality, using the continuity of T and the metric d, we obtain

$$d(Tx, Ty) \leq \lambda d(x, y).$$

As consequence, we have

$$d(Tx, Ty) \leq \lambda d(x, y), \quad (x, y) \in \mathscr{X} \times \mathscr{X}.$$

Finally, since $x_0 \in \mathscr{X}$, by Banach contraction principle, the Picard sequence $\{T^nx_0\}$ converges to some $x^* \in \mathscr{X}$, which is the unique fixed point of T in $\mathscr{X}$. Note that the uniqueness is obtained just in the subspace $\mathscr{X}$ of X. So, T has at least one fixed point in the hole space X. This ends the proof.

Remark 2.2 It is not difficult to observe that Theorem 2.1 holds true if we replace Assumptions (ii) and (iii) by

(ii)' T is a decreasing mapping.
(iii)' There exists $x_0 \in X$ such that $x_0 \succeq Tx_0$.

2.3 An Extension of Ran–Reurings Fixed Point Theorem to Noncontinuous Mappings

Nieto and Rodríguez-López [33] extended Theorem 2.1 to the class of noncontinuous mappings. Before stating and proving Nieto–Rodríguez-López fixed point theorem, we need to introduce the following concept.

Definition 2.8 Let (X, d) be a metric space, and let $\preceq$ be a partial order on X. We say that (X, d) is $\preceq$-regular if the following condition is satisfied: If $\{x_n\}$ is a nondecreasing sequence (with respect to $\preceq$) in X such that $\{x_n\}$ converges to some $x \in X$, then $x_n \preceq x$, for all $n \in \mathbb{N}$.

Example 2.5 Let $X = C([a, b]; \mathbb{R})$ be the set of real-valued and continuous functions in $[a, b]$ $(a < b)$. Define the partial order $\preceq$ on X by

$$f, g \in X, \quad f \preceq g \iff f(t) \leq g(t),\ t \in [a, b].$$

We endow X with the metric d defined by

$$d(f, g) = \max\{|f(t) - g(t)| : t \in [a, b]\}, \quad (f, g) \in X \times X.$$

Then (X, d) is $\preceq$-regular.

Theorem 2.2 (Nieto–Rodríguez-López fixed point theorem) *Let $(X, \preceq)$ be a partially ordered set, and let d be a metric on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a given mapping. We suppose that the following conditions are satisfied:*

(i) *(X, d) is $\preceq$-regular.*
(ii) *T is nondecreasing.*
(iii) *There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.*
(iv) *There exists a constant $\lambda \in (0, 1)$ such that*

$$(x, y) \in X \times X,\ x \preceq y \implies d(Tx, Ty) \leq \lambda d(x, y).$$

Then the Picard sequence $\{T^n x_0\}$ converges to a fixed point of T.

Proof Using the considered assumptions, we have

$$T^n x_0 \preceq T^{n+1} x_0, \quad n \in \mathbb{N}.$$

Therefore, by (iv), we have

$$d(T^{n+1} x_0, T^n x_0) \leq \lambda^n d(x_0, Tx_0), \quad n \in \mathbb{N}.$$

Since $\lambda \in (0, 1)$, the Picard sequence $\{T^n x_0\}$ is Cauchy in the complete metric space (X, d). Then there exists some $x^* \in X$ such that $\{T^n x_0\}$ converges to x^*. The $\preceq$-regularity of (X, d) implies that

$$T^n x_0 \preceq x^*, \quad n \in \mathbb{N}.$$

Using the above inequality and (iv), we obtain

$$d(x^*, Tx^*) \le d(x^*, T^{n+1}x_0) + d(T^{n+1}x_0, Tx^*) \le d(x^*, T^{n+1}x_0) + \lambda d(T^n x_0, x^*), \quad n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$, we obtain $d(x^*, Tx^*) = 0$, i.e., x^* is a fixed point of T.

As it was proved in [33], under an additional assumption on the partially ordered set $(X, \preceq)$, the fixed point of T is unique in both Theorems 2.1 and 2.2.

Definition 2.9 Let $(X, \preceq)$ be a partially ordered set. We say that $(X, \preceq)$ satisfies the property (H) if the following condition is satisfied:

$$\forall\,(x, y) \in X \times X,\ \exists\, z \in X : x \preceq z,\ y \preceq z.$$

Example 2.6 Let $X = C([a, b]; \mathbb{R})$, and consider the partial order $\preceq$ on X defined by

$$f, g \in X, \quad f \preceq g \iff f(t) \le g(t),\ t \in [a, b].$$

Then $(X, \preceq)$ satisfies the property (H).

Theorem 2.3 *In addition to the assumptions of Theorem 2.1 (resp. Theorem 2.2) suppose that $(X, \preceq)$ satisfies the property (H). Then T has a unique fixed point in X.*

Proof Suppose that x^* and y^* are two fixed points of T, i.e.,

$$x^* = Tx^* \quad \text{and} \quad y^* = Ty^*.$$

From the property (H), there exists some $z \in X$ such that $x^* \preceq z$ and $y^* \preceq z$. From the monotony of T, we have

$$x^* \preceq T^n z, \quad n \in \mathbb{N}.$$

Therefore,

$$d(x^*, T^n z) = d(Tx^*, T^n z) \le \lambda d(Tx^*, T^{n-1}z) \le \cdots \le \lambda^n d(x^*, z),$$

for all $n \in \mathbb{N}$. Passing to the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} d(x^*, T^n z) = 0.$$

Using a similar argument, we obtain

$$\lim_{n\to\infty} d(y^*, T^n z) = 0.$$

By the uniqueness of the limit, we deduce that $x^* = y^*$.

2.4 Some Consequences: Fixed Point Results for Mixed Monotone Mappings

In this section, as consequences of the theorems presented previously, some existence results for the system of functional equations

$$\begin{cases} F(x, y) = x \\ F(y, x) = y \end{cases} \tag{2.1}$$

are presented, where $F : X \times X \to X$ is a given mapping.

Definition 2.10 Any pair of points $(x, y) \in X \times X$ satisfying (2.1) is said to be a coupled fixed point of the mapping F.

The following straightforward result proves the equivalence between the existence of a coupled fixed points for a given mapping and of fixed points for another related mapping.

Lemma 2.1 *Let X be a nonempty set, and let $F : X \times X \to X$ be a given mapping. Then $(x, y) \in X \times X$ is a coupled fixed point of F if and only if $(x, y) \in X \times X$ is a fixed point of the mapping $T : X \times X \to X \times X$ defined by*

$$T(x, y) = (F(x, y), F(y, x)), \quad (x, y) \in X \times X.$$

Definition 2.11 Let X be a nonempty set, and let $F : X \times X \to X$ be a given mapping. Any element $x \in X$ satisfying

$$x = F(x, x)$$

is said to be a fixed point of the mapping F.

The coupled fixed point's concept was introduced by Opoitsev [34, 35] and then by Guo and Lakshmikantham [19] in connection with coupled quasisolutions of an initial value problem for ordinary differential equations. Various existence results of coupled fixed points for different classes of operators were obtained by many authors. The motivation of such contributions is the usefulness of the coupled fixed point approach in studying the existence of solutions to nonlinear functional equations. For more details on coupled fixed point theory, we refer the reader to [6–8, 19–22, 39, 42, 43] and the references therein.

We shall use Ran–Reurings fixed point theorem in order to prove the following result which is due to Bhaskar and Lakshmikantham [8]. Our proof [41] is different to the one in [8].

Theorem 2.4 *Let $(X, \preceq)$ be a partially ordered set, and let d be a metric on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a given mapping. We suppose that the following conditions are satisfied:*

(i) *F is continuous.*
(ii) *F is mixed monotone.*
(iii) *There exists $(x_0, y_0) \in X \times X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$.*
(iv) *There exists a constant $\lambda \in (0, 1)$ such that*

$$(x, y), (u, v) \in X \times X,\ x \preceq u,\ y \succeq v \implies d(F(x, y), F(u, v)) \leq \frac{\lambda}{2}[d(x, u) + d(y, v)].$$

Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$x_{n+1} = F(x_n, y_n),\ \ y_{n+1} = F(y_n, x_n), \quad n \in \mathbb{N}$$

converge respectively to $x^* \in X$ and $y^* \in X$, where $(x^*, y^*) \in X \times X$ is a coupled fixed point of F.

Proof Let $Z = X \times X$. We define the partial order $\preceq_2$ on Z by

$$(x, y), (z, w) \in Z, \quad (x, y) \preceq_2 (z, w) \Leftrightarrow x \preceq z,\ y \succeq w.$$

Define the mapping $T : Z \to Z$ by

$$T(x, y) = (F(x, y), F(y, x)), \quad (x, y) \in Z.$$

By Remark 2.1, the mapping T is nondecreasing with respect to the partial order $\preceq_2$ on Z. From (iii), we have $z_0 \preceq_2 Tz_0$, where $z_0 = (x_0, y_0) \in Z$. Next, we endow Z with the metric d_2 defined by

$$d_2((x, y), (z, w)) = d(x, z) + d(y, w), \quad (x, y), (z, w) \in Z.$$

Obviously, (Z, d_2) is a complete metric space. Take two elements $z_1 = (x, y)$ and $z_2 = (u, v)$ in Z such that $z_1 \preceq_2 z_2$, i.e.,

$$x \preceq u \quad \text{and} \quad y \succeq v.$$

From (iv), we have

$$d(F(x, y), F(u, v)) \leq \frac{\lambda}{2}[d(x, u) + d(y, v)]. \tag{2.2}$$

Similarly, we have

$$d(F(v, u), F(y, x)) \leq \frac{\lambda}{2}[d(v, y) + d(u, x)]. \tag{2.3}$$

Adding (2.2) to (2.3), we get

$$d(F(x,y),F(u,v))+d(F(y,x),F(v,u))\le \lambda[d(x,u)+d(y,v)],$$

which implies that

$$d_2(Tz_1,Tz_2)\le \lambda d_2(z_1,z_2),\quad (z_1,z_2)\in Z\times Z,\ z_1\preceq_2 z_2.$$

Note that form (i), the mapping $T:Z\to Z$ is continuous. Now, using Ran–Reurings fixed point theorem and Lemma 2.1, we obtain the desired result.

Definition 2.12 Let (X,d) be a metric space, and let $\preceq$ be a partial order on X. We say that (X,d) is $\preceq_2$-regular if the metric space $(Z,\preceq_2)$ is $\preceq_2$-regular.

Using Theorem 2.2, we deduce the following coupled fixed point result for non-continuous mappings.

Theorem 2.5 *Let $(X,\preceq)$ be a partially ordered set, and let d be a metric on X such that (X,d) is a complete metric space. Let $F:X\times X\to X$ be a given mapping. We suppose that the following conditions are satisfied:*

(i) (X,d) *is* $\preceq_2$*-regular.*
(ii) F *is mixed monotone.*
(iii) *There exists* $(x_0,y_0)\in X\times X$ *such that* $x_0\preceq F(x_0,y_0)$ *and* $y_0\succeq F(y_0,x_0)$.
(iv) *There exists a constant* $\lambda\in(0,1)$ *such that*

$$(x,y),(u,v)\in X\times X,\ x\preceq u,\ y\succeq v\implies d(F(x,y),F(u,v))\le\frac{\lambda}{2}[d(x,u)+d(y,v)].$$

Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$x_{n+1}=F(x_n,y_n),\ \ y_{n+1}=F(y_n,x_n),\quad n\in\mathbb{N}$$

converge respectively to $x^*\in X$ and $y^*\in X$, where $(x^*,y^*)\in X\times X$ is a coupled fixed point of F.

We can introduce a condition similar to property (H) in order to ensure the uniqueness of a coupled fixed point.

Definition 2.13 Let $(X,\preceq)$ be a partially ordered set. We say that $(X,\preceq)$ satisfies the property (H2) if the following condition is satisfied:

$$\forall\,((x,y),(u,v))\in Z\times Z,\ \exists\,(z,w)\in Z:\ (x,y)\preceq_2(z,w),\ (u,v)\preceq_2(z,w).$$

Remark 2.3 Note that the above definition implies that we demand the existence of lower and upper bounds for any two elements in $(X,\preceq)$.

Using Theorem 2.3, we deduce the following uniqueness result.

Theorem 2.6 *In addition to the assumptions of Theorem 2.4 (resp. Theorem 2.5) suppose that* $(X, \preceq)$ *satisfies the property (H2). Then* F *has a unique coupled fixed point in* $X \times X$.

An immediate consequence of Theorem 2.6 is the following fixed point result.

Theorem 2.7 *In addition to the assumptions of Theorem 2.4 (resp. Theorem 2.5) suppose that* $(X, \preceq)$ *satisfies the property (H2). Then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *defined by*

$$x_{n+1} = F(x_n, y_n), \;\; y_{n+1} = F(y_n, x_n), \quad n \in \mathbb{N}$$

converge to $x^* \in X$*, which is the unique fixed point of* F.

Proof From Theorem 2.6, F has a unique coupled fixed point $(x^*, y^*) \in X \times X$. However, (y^*, x^*) is also a coupled fixed point of F. Therefore, by uniqueness, we have $x^* = y^*$.

2.5 Positive Definite Solution to the Matrix Equation $X = Q - A^*X^{-1}A + B^*X^{-1}B$

The main reference for this section is the paper [4].

We consider the matrix equation

$$X = Q - A^*X^{-1}A + B^*X^{-1}B, \tag{2.4}$$

where Q is an $n \times n$ Hermitian positive definite matrix, and A and B are arbitrary $n \times n$ matrices. Equation (2.4) is a special stochastic rational Riccati equation arising in stochastic control theory, and it can be described below. Some stochastic control problems lead to computing the positive definite solution of the following stochastic rational Riccati equation [45]:

$$\begin{aligned} &C^*XC - X + S + \Pi_1(X) \\ &- (L + C^*XP + \Pi_{12}(X))(R + P^*XP + \Pi_2(X))^+(L + C^*XP + \Pi_{12}(X))^* = 0, \end{aligned} \tag{2.5}$$

where Z^+ stands for the Moore–Penrose inverse of a matrix Z and C, P, S, R, and L are given matrices of size $n \times n$, $n \times m$, $n \times n$, $m \times m$, and $n \times m$, respectively, such that

$$T = \begin{pmatrix} S & L \\ L^* & R \end{pmatrix}$$

is a Hermitian matrix, and the operator

$$\Pi(X) = \begin{pmatrix} \Pi_1(X) & \Pi_{12}(X) \\ \Pi_{12}(X)^* & \Pi_2(X) \end{pmatrix}$$

is positive, i.e., $X \geq 0$ implies $\Pi(X) \geq 0$. Consider the following case: C is the identity matrix, P is an $n \times n$ nonsingular matrix, S is an $n \times n$ positive definite matrix, L is the zero matrix, and $\Pi_{12}(X) = \Pi_2(X) = 0$, $\Pi_1(X) = (R + P^* X P)^{-1}$, where $R + P^* X P$ is positive definite for all positive semi-definite matrices X. Meanwhile, the stochastic rational Riccati Equation (2.5) has the form

$$S + (R + P^* X P)^{-1} - X P (R + P^* X P)^{-1} P^* X = 0. \tag{2.6}$$

Set

$$Y = R + P^* X P, \tag{2.7}$$

then

$$P^{-*}(Y - R) = X P. \tag{2.8}$$

By Eqs. (2.6)–(2.8), we have

$$S + Y^{-1} - P^{-*}(Y - R) Y^{-1} (Y - R) P^{-1} = 0,$$

which implies that

$$Y + R^* Y^{-1} R - P^* Y^{-1} P = 2R + P^* S P.$$

Set

$$Q = 2R + P^* S P, \quad A = R, \quad B = P,$$

then Eq. (2.6) can be equivalently written as Eq. (2.4). Therefore, Eq. (2.4) is a special stochastic rational Riccati equation (2.5). Moreover, some special cases of Eq. (2.4) are also problems of practical importance, such as the matrix equation $X + M^* X^{-1} M = Q$ that arises in the control theory, ladder networks, dynamic programming, stochastic filtering, statistics, and so on [15, 23, 46]. The matrix equation $X - M^* X^{-1} M = Q$ arises in the analysis of stationary Gaussian reciprocal processes over a finite interval [2, 16].

Since 1993, the matrix equations $X + M^* X^{-1} M = Q$ and $X - M^* X^{-1} M = Q$ have been extensively studied, and the research results mainly concentrated on the following:

(a) sufficient conditions and necessary conditions for the existence of a (unique) positive definite solution [14–16, 47];
(b) numerical methods for computing the (unique) positive definite solution [2, 16–18, 27, 31, 32, 46];
(c) properties of the positive definite solution [15, 46]; and
(d) perturbation bound for the positive definite solution [23, 44].

In addition, other nonlinear matrix equations such as $AX^2 + BX + C = 0$ [1], $X^s \pm A^* X^{-t} A = Q$ [10, 29], $X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = I$ [25, 30], $X \pm A^* X^{-q} A = Q$ [5, 11, 23, 24, 26, 28, 36, 37], $X - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q$ [12], $X + A^* F(X) A = Q$ [3, 13] have been investigated by many authors. However, results on the general nonlinear matrix equation (2.4) are few as far as we know.

In this section, using Bhaskar–Lakshmikantham fixed point theorem, a sufficient condition for the existence of a unique positive definite solution to Eq. (2.4) is derived. Moreover, an iterative method is constructed to compute the unique Hermitian positive definite solution, and the error estimation formal is also given. In the end, we use some numerical examples to illustrate that the proposed iterative method is feasible to compute the unique positive definite solution to Eq. (2.4).

Throughout this section, we denote by $\mathscr{M}(N)$ and $\mathscr{H}(N)$ the set of $N \times N$ complex and $N \times N$ Hermitian matrices, respectively. For $A, B \in \mathscr{H}(N)$, $A \geq 0$ ($A > 0$) means that A is positive semi-definite (positive definite). Moreover, $A \geq B$ ($A > B$) means that $A - B \geq 0$ ($A - B > 0$), and $X \in [A, B]$ means $A \leq X \leq B$. A^* and $r(A)$ denote the complex conjugate transpose and the spectral radius of A, respectively. We denote by $\|\cdot\|$ the spectral norm, i.e., $\|A\| = \sqrt{\lambda^+(A^*A)}$, where $\lambda^+(A^*A)$ is the largest eigenvalue of A^*A. The $N \times N$ identity matrix will be written as I. We denote by $\|\cdot\|_{\mathrm{tr}}$ the trace norm. Recall that this norm is given by

$$\|A\|_{\mathrm{tr}} = \sum_{j=1}^{N} \sigma_j(A),$$

where $\sigma_j(A)$, $j = 1, \ldots, N$ are the singular values of A.

The following lemmas will be useful later.

Lemma 2.2 (See [38]) *Let $A \geq 0$ and $B \geq 0$ be $N \times N$ matrices, then*

$$0 \leq \mathrm{tr}(AB) \leq \|A\| \, \mathrm{tr}(B).$$

Lemma 2.3 (See [9]) *If $0 < \theta \leq 1$, and P and Q are positive definite matrices of the same order with $P, Q \geq bI > 0$, then for every unitarily invariant norm*

$$|||P^\theta - Q^\theta||| \leq \theta b^{\theta-1} |||P - Q|||$$

and

$$|||P^{-\theta} - Q^{-\theta}||| \leq \theta b^{-(\theta+1)} |||P - Q|||.$$

Lemma 2.4 (See [9]) *Let $A \in \mathscr{H}(N)$ satisfying $-I < A < I$, then $\|A\| < 1$.*

Suppose that there exist $a > 0$, $b > 0$ (real numbers), such that the following assumptions are satisfied:

(1) $a^{-1}A^*A + aI \leq Q \leq bI$
(2) $bA^*A - aB^*B \leq ab(Q - aI)$
(3) $bB^*B - aA^*A \leq ab(bI - Q)$
(4) $2A^*A < a^2I, 2B^*B < a^2I$.

We denote by Ω the set of matrices defined by

$$\Omega = \{X \in \mathscr{H}(N) : X \geq aI\}.$$

Our main result is discussed below.

Theorem 2.8 *Under the assumptions (1)–(4), we have*

(I) *Equation* (2.4) *has a unique solution* $\overline{X} \in \Omega$.
(II) $\overline{X} \in [Q + b^{-1}B^*B - a^{-1}A^*A, Q + a^{-1}B^*B - b^{-1}A^*A]$.
(III) *The sequences* $\{X_n\}$ *and* $\{Y_n\}$ *defined by*

$$\begin{cases} X_0 &= aI \\ X_{n+1} &= Q - A^*X_n^{-1}A + B^*Y_n^{-1}B \end{cases} ; \quad \begin{cases} Y_0 &= bI \\ Y_{n+1} &= Q - A^*Y_n^{-1}A + B^*X_n^{-1}B \end{cases}$$

converge to $\overline{X}$, *that is,*

$$\lim_{n\to\infty} \|X_n - \overline{X}\|_{\text{tr}} = \lim_{n\to\infty} \|Y_n - \overline{X}\|_{\text{tr}} = 0,$$

and the error estimation is given by

$$\max\left\{\|X_n - \widehat{X}\|_{\text{tr}}, \|Y_n - \widehat{X}\|_{\text{tr}}\right\} \leq \frac{\delta^n}{1-\delta} \max\left\{\|X_1 - X_0\|_{\text{tr}}, \|Y_1 - Y_0\|_{\text{tr}}\right\},$$

where $0 < \delta < 1$.

Proof For all $X, Y \in \mathscr{H}(N)$, let

$$F(X, Y) = Q - A^*X^{-1}A + B^*Y^{-1}B.$$

We claim that $F(\Omega \times \Omega) \subset \Omega$. Indeed, let $X, Y \in \Omega$, that is, $X \geq aI$ and $Y \geq aI$. This implies that

$$Q - A^*X^{-1}A + B^*Y^{-1}B \geq Q - A^*X^{-1}A \geq Q - a^{-1}A^*A.$$

On the other hand, from (1), we have

$$Q - A^*X^{-1}A \geq aI.$$

Thus, we have

$$F(X, Y) = Q - A^*X^{-1}A + B^*Y^{-1}B \geq aI,$$

which implies that $F(X, Y) \in \Omega$. Then, our claim holds.

Now, the mapping $F : \Omega \times \Omega \to \Omega$ is well defined. Let $X, Y, U, V \in \Omega$ be such that $X \geq U$ and $Y \leq V$. We have

$$\begin{aligned}
&\|F(X, Y) - F(U, V)\|_{\text{tr}} = \|A^*(U^{-1} - X^{-1})A + B^*(Y^{-1} - V^{-1})B\|_{\text{tr}} \\
&\leq \|A^*(U^{-1} - X^{-1})A\|_{\text{tr}} + \|B^*(Y^{-1} - V^{-1})B\|_{\text{tr}} \\
&= \text{tr}\left(A^*(U^{-1} - X^{-1})A\right) + \text{tr}\left(B^*(Y^{-1} - V^{-1})B\right) \\
&= \text{tr}\left(AA^*(U^{-1} - X^{-1})\right) + \text{tr}\left(BB^*(Y^{-1} - V^{-1})\right).
\end{aligned}$$

Since $U^{-1} - X^{-1} \geq 0$ and $Y^{-1} - V^{-1} \geq 0$, using Lemma 2.2, we get

$$\|F(X, Y) - F(U, V)\|_{\text{tr}} \leq \|AA^*\|\text{tr}(U^{-1} - X^{-1}) + \|BB^*\|\text{tr}(Y^{-1} - V^{-1}).$$

On the other hand, since $X, Y, U, V \geq aI$, using Lemma 2.3, we have

$$\text{tr}(U^{-1} - X^{-1}) \leq a^{-2}\text{tr}(X - U)$$

and

$$\text{tr}(Y^{-1} - V^{-1}) \leq a^{-2}\text{tr}(V - Y).$$

Thus, we get

$$\|F(X, Y) - F(U, V)\|_{\text{tr}} \leq \frac{\|AA^*\|}{a^2}\|X - U\|_{\text{tr}} + \frac{\|BB^*\|}{a^2}\|V - Y\|_{\text{tr}}.$$

This implies that

$$\|F(X, Y) - F(U, V)\|_{\text{tr}} \leq \frac{\delta}{2}\left(\|X - U\|_{\text{tr}} + \|V - Y\|_{\text{tr}}\right),$$

where

$$\delta = \frac{2}{a^2}\max\left\{\|AA^*\|, \|BB^*\|\right\}.$$

From (4) and Lemma 2.4, we can easily show that $0 \leq \delta < 1$. Now, taking $X_0 = aI$ and $Y_0 = bI$, from (2) and (3), we can easily show that $X_0 \leq F(X_0, Y_0)$ and $Y_0 \geq F(Y_0, X_0)$. On the other hand, for every $X, Y \in \mathcal{H}(N)$, there is a greatest lower bound and a least upper bound. Note also that F is a continuous mapping. Now, (I) and (III) follow immediately from Theorem 2.7. Let $\overline{X}$ be the unique solution to Eq. (2.4) in Ω.

To prove (II), we shall use Schauder fixed point theorem. We define the mapping $G : [F(aI, bI), F(bI, aI)] \to \Omega$ by

$$G(X) = F(X, X), \text{ for all } X \in [F(aI, bI), F(bI, aI)].$$

We claim that

$$G([F(aI, bI), F(bI, aI)]) \subseteq [F(aI, bI), F(bI, aI)].$$

Let $X \in [F(aI, bI), F(bI, aI)]$, that is,

$$F(aI, bI) \leq X \leq F(bI, aI).$$

Using the mixed monotone property of F, we get

$$F(F(aI, bI), F(bI, aI)) \leq F(X, X) = G(X) \leq F(F(bI, aI), F(aI, bI)). \tag{2.9}$$

On the other hand, from (2) and (3), we have

$$aI \leq F(aI, bI) \quad \text{and} \quad bI \geq F(bI, aI).$$

Again, using the mixed monotone property of F, we get

$$F(F(bI, aI), F(aI, bI)) \leq F(bI, aI) \quad \text{and} \quad F(F(aI, bI), F(bI, aI)) \geq F(aI, bI). \tag{2.10}$$

From (2.9) and (2.10), it follows that

$$F(aI, bI) \leq G(X) \leq F(bI, aI).$$

Thus, our claim that $G([F(aI, bI), F(bI, aI)]) \subseteq [F(aI, bI), F(bI, aI)]$ holds. Now, G maps the compact convex set $[F(aI, bI), F(bI, aI)]$ into itself. Since G is continuous, it follows from Schauder fixed point theorem that G has at least one fixed point in this set. However, fixed points of G are solutions of Eq. (2.4), and we proved already that Eq. (2.4) has a unique solution in Ω. Thus, this solution must be in the set $[F(aI, bI), F(bI, aI)]$, that is,

$$\overline{X} \in [Q + b^{-1}B^*B - a^{-1}A^*A, Q + a^{-1}B^*B - b^{-1}A^*A].$$

Thus, we proved (II). This makes end to the proof.

The following results are immediate consequences of Theorem 2.8.

Theorem 2.9 *Consider Eq.* (2.4) *with* $Q = I$. *Suppose that*

(1) $0 < a \leq \frac{1}{2}$, $b \geq 1 + \frac{a}{2}$; *and*

(2) $A^*A < \frac{a^2}{2}I$, $B^*B < \frac{a^2}{2}I$.

Then, (I)–(III) of Theorem 2.8 hold.

Theorem 2.10 *Consider Eq.* (2.4) *with A and B which are unitary matrices. Suppose that*

(1) $\sqrt{2} < a < b$; *and*
(2) $(a^{-1} + a)I \leq Q \leq (b + b^{-1} - a^{-1})I$.

Then, (I)–(III) of Theorem 2.8 hold.

Theorem 2.11 *Consider Eq.* (2.4) *with* $A = 0$. *Suppose that*

(1) $aI \leq Q \leq bI$;
(2) $B^*B \leq a(bI - Q)$; *and*
(3) $B^*B < \dfrac{a^2}{2}I$.

Then, (I)–(III) of Theorem 2.8 hold.

Theorem 2.12 *Consider Eq.* (2.4) *with* $B = 0$. *Suppose that*

(1) $a^{-1}A^*A + aI \leq Q \leq bI$;
(2) $A^*A \leq a(Q - aI)$; *and*
(3) $A^*A < \dfrac{a^2}{2}I$.

Then, (I)–(III) of Theorem 2.8 hold.

Now, we present some numerical results in order to illustrate the above theorems. All programs are written in MATLAB version 7.1.

Example 2.7 In this example, we consider Eq. (2.4) with

$$Q = \begin{pmatrix} 7 & -0 & 1 \\ -0 & 7 & 1 \\ 1 & 1 & 8 \end{pmatrix}, \quad A = \begin{pmatrix} 2.11 & 0.01 & 0.01 \\ -0.05 & 1.98 & -0.18 \\ 0.1 & 0.19 & 2.38 \end{pmatrix}, \quad B = \begin{pmatrix} -3.09 & 0.01 & 0.01 \\ -0.01 & -3.15 & -0.09 \\ 0.04 & 0.1 & -2.94 \end{pmatrix}.$$

All the hypotheses of Theorem 2.8 are satisfied with $a = 5$ and $b = 14$. We consider the sequences $\{X_n\}$ and $\{Y_n\}$ defined in item (III) of Theorem 2.8 with $X_0 = aI$ and $Y_0 = bI$. For each iteration k, we consider the errors

$$R(X_k) = \|X_k - (Q - A^*X_k^{-1}A + B^*X_k^{-1}B)\|,$$

$$R(Y_k) = \|Y_k - (Q - A^*Y_k^{-1}A + B^*Y_k^{-1}B)\|$$

and

$$R_k = \max\{R(X_k), R(Y_k)\}.$$

After 23 iterations, we get

$$\overline{X} \approx X_{23} = Y_{23} = \begin{pmatrix} 7.68020112227005 & 0.02950633669680 & 0.88917486612500 \\ 0.02950633669680 & 7.79693817383459 & 0.92560452577454 \\ 0.88917486612500 & 0.92560452577454 & 8.34452699090856 \end{pmatrix}$$

with

$$R_{23} = 2.42861287e - 017.$$

Example 2.8 In this example, we consider Eq. (2.4) with

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0.3 & 0.01 & 0.01 \\ 0 & 0.28 & -0.02 \\ 0.02 & 0.03 & 0.34 \end{pmatrix}, \quad B = \begin{pmatrix} -0.34 & 0 & 0 \\ 0 & -0.34 & 0 \\ 0.01 & 0.01 & -0.32 \end{pmatrix}.$$

All the hypotheses of Theorem 2.9 are satisfied with $a = 0.5$ and $b = 5$. After 20 iterations, we get $\overline{X} \approx X_{20} = Y_{20} =$

$$\begin{pmatrix} 1.02444745949421 & -0.003561623099836826 & -0.01296282338345968 \\ -0.003561623099836826 & 1.034823675282171 & -0.008218578980308637 \\ -0.01296282338345968 & -0.008218578980308639 & 0.9861513844061653 \end{pmatrix}$$

with

$$R_{20} = 2.09918957e - 016.$$

Example 2.9 We consider Eq. (2.4) with

$$Q = \begin{pmatrix} 10 & 5 & 3.4 \\ 5 & 10 & 6.7 \\ 3.4 & 6.7 & 10 \end{pmatrix}, \quad A = \begin{pmatrix} 0.0591 & 0.0737 & 0.0328 \\ 0.0737 & -0.0328 & -0.0591 \\ 0.0328 & -0.0591 & 0.0737 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 0.591 & 0.737 & 0.328 \\ 0.737 & -0.328 & -0.591 \\ 0.328 & -0.591 & 0.737 \end{pmatrix}.$$

In this case, A and B are unitary matrices. All the hypotheses of Theorem 2.10 are satisfied with $a = 1.514$ and $b = 101.5$. After 7 iterations, we get

$$\overline{X} \approx X_7 = Y_7 = \begin{pmatrix} 10.06412689941009 & 5.013263723550349 & 3.345079324929884 \\ 5.013263723550349 & 10.13999944657551 & 6.719887939894802 \\ 3.345079324929884 & 6.719887939894802 & 10.29931432720346 \end{pmatrix}$$

with

$$R_7 = 1.77635684e - 015.$$

Example 2.10 We consider Eq. (2.4) with

$$Q = \begin{pmatrix} 100 & 50 & 34 \\ 50 & 100 & 67 \\ 34 & 67 & 100 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0.5 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & 0.5 & 1.5 \end{pmatrix}.$$

All the hypotheses of Theorem 2.11 are satisfied with $a = 3.5$ and $b = 300$. After 3 iterations, we get

$$\overline{X} \approx X_3 = Y_3 = \begin{pmatrix} 100.0104629987089 & 50.00450680062249 & 34.00435076795997 \\ 50.00450680062249 & 100.0105221759655 & 66.99538011209222 \\ 34.00435076795997 & 66.99538011209222 & 100.0407917033456 \end{pmatrix}$$

with

$$R_3 = 3.00990733e - 014.$$

Example 2.11 We consider Eq. (2.4) with

$$Q = \begin{pmatrix} 10 & 5 & 3.4 \\ 5 & 10 & 6.7 \\ 3.4 & 6.7 & 10 \end{pmatrix}, \quad A = \begin{pmatrix} 0.5 & 0.25 & 0 \\ 0.25 & 0.5 & 0 \\ 0.25 & 0.25 & 0.75 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

All the hypotheses of Theorem 2.12 are satisfied with $a = 2$ and $b = 100$. After 10 iterations, we get

$$\overline{X} \approx X_{10} = Y_{10} = \begin{pmatrix} 9.973738915336433 & 4.988761264228204 & 3.388819129012571 \\ 4.988761264228204 & 9.973542675753565 & 6.712061714363009 \\ 3.388819129012571 & 6.712061714363009 & 9.89541012219485 \end{pmatrix}$$

with

$$R_{10} = 1.32107728e - 014.$$

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Chapter 3
The Class of (α, ψ)-Contractions and Related Fixed Point Theorems

The class of (α, ψ)-contractions was introduced by Samet et al. [26]. In this chapter, we prove three fixed point theorems for this class of mappings. The presented results are extensions of those obtained in [26]. Moreover, we show that the class of (α, ψ)-contractions includes as special cases several types of contraction-type mappings, whose fixed points can be obtained by means of Picard iteration. As an application, the existence and uniqueness of solutions to a certain class of quadratic integral equations is discussed. The main references of this chapter are the papers [24, 26].

3.1 Introduction

In [26], Samet et al. introduced the class of (α, ψ)-contractions and studied the existence of fixed points for this class of mappings. Let us recall the main results obtained in [26].

Let Ψ be the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following properties:

(P1) ψ is nondecreasing;
(P2) $\sum_{k=0}^{\infty} \psi^k(t) < \infty$, for all $t > 0$, where ψ^k is the kth iterate of ψ.

Definition 3.1 Let (X, d) be a metric space and $T : X \to X$ be a given mapping. Let $\psi \in \Psi$ and $\alpha : X \times X \to \mathbb{R}$ be a given function. We say that T is an (α, ψ)-contraction if

$$\alpha(x, y)d(Tx, Ty) \le \psi(d(x, y)), \quad (x, y) \in X \times X. \tag{3.1}$$

Definition 3.2 Let X be a nonempty set, $T : X \to X$ be a given mapping and $\alpha : X \times X \to \mathbb{R}$. We say that T is α-admissible if

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$$(x, y) \in X \times X,\ \alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1. \tag{3.2}$$

The obtained results in [26] can be summarized as follows.

Theorem 3.1 *Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi$ such that*

(i) (3.1) *is satisfied;*
(ii) *T is α-admissible;*
(iii) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iv) *T is continuous; or*
(v) *For every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ satisfying $\alpha(x_n, x_{n+1}) \geq 1$ for $n \in \mathbb{N}$, if $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in X$, then $\alpha(x_n, x) \geq 1$ for $n \in \mathbb{N}$.*

Then T has a fixed point. Moreover, if in addition we suppose that for every pair $(u, v) \in X \times X$, there exists $w \in X$ such that $\alpha(u, w) \geq 1$ and $\alpha(v, w) \geq 1$, we have a unique fixed point.

For other related results, we refer the reader to [14–16, 18, 22, 25] and the references therein.

In [26], it was shown that some fixed point results in a metric space with a partial order can be deduced from Theorem 3.1. In this chapter, an extension of Theorem 3.1, without condition (3.2), is proposed. Moreover, we show that the presented results unify the most existing fixed point theorems in the literature, where the fixed points can be obtained by means of Picard iteration. As an application, we discuss the existence and uniqueness of solutions to a certain class of quadratic integral equations.

3.2 Main Results

If $T : X \to X$ is a given mapping, we denote by $\mathrm{Fix}(T)$ the set of its fixed points; that is,

$$\mathrm{Fix}(T) = \{x \in X : x = Tx\}.$$

The following lemma will be useful later.

Lemma 3.1 ([4]) *Let $\psi \in \Psi$. Then*

(i) $\psi(t) < t$, $t > 0$.
(ii) $\psi(0) = 0$.
(iii) *ψ is continuous at $t = 0$.*

For a given $\psi \in \Psi$, let Σ_ψ be the set defined by

$$\Sigma_\psi = \{\sigma \in (0, \infty) : \sigma\psi \in \Psi\}.$$

Proposition 3.1 *Let (X, d) be a metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi$ such that T is an (α, ψ)-contraction. Suppose that there exists $\sigma \in \Sigma_\psi$, and for some positive integer p, there exists a finite sequence $\{\xi_i\}_{i=0}^p \subset X$ such that*

$$\xi_0 = x_0, \quad \xi_p = Tx_0, \quad \alpha(T^n\xi_i, T^n\xi_{i+1}) \geq \sigma^{-1}, \quad n \in \mathbb{N}, \ i = 0, \ldots, p-1. \tag{3.3}$$

Then $\{T^n x_0\}$ is a Cauchy sequence in (X, d).

Proof Let $\varphi = \sigma\psi$. By definition of Σ_ψ, we have $\varphi \in \Psi$. Let $\{\xi_i\}_{i=0}^p$ be a finite sequence in X satisfying (3.3). Consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ in X defined by $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. We claim that

$$d(T^r\xi_i, T^r\xi_{i+1}) \leq \varphi^r(d(\xi_i, \xi_{i+1})), \quad r \in \mathbb{N}, \ i = 0, \ldots, p-1. \tag{3.4}$$

Let $i \in \{0, 1, \ldots, p-1\}$. From (3.3), we have

$$\sigma^{-1}d(T\xi_i, T\xi_{i+1}) \leq \alpha(\xi_i, \xi_{i+1})d(T\xi_i, T\xi_{i+1}) \leq \psi(d(\xi_i, \xi_{i+1})),$$

which implies that

$$d(T\xi_i, T\xi_{i+1}) \leq \varphi(d(\xi_i, \xi_{i+1})). \tag{3.5}$$

Again, we have

$$\sigma^{-1}d(T^2\xi_i, T^2\xi_{i+1}) \leq \alpha(T\xi_i, T\xi_{i+1})d(T(T\xi_i), T(T\xi_{i+1})) \leq \psi(d(T\xi_i, T\xi_{i+1})),$$

which implies that

$$d(T^2\xi_i, T^2\xi_{i+1}) \leq \varphi(d(T\xi_i, T\xi_{i+1})). \tag{3.6}$$

Since φ is a nondecreasing function (from property (Ψ_1)), from (3.5) and (3.6), we obtain that

$$d(T^2\xi_i, T^2\xi_{i+1}) \leq \varphi^2(d(\xi_i, \xi_{i+1})).$$

Continuing this process, by induction, we obtain (3.4). Now, using the triangle inequality and (3.4), for every $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(T^n x_0, T^{n+1}x_0) \\ &\leq d(T^n\xi_0, T^n\xi_1) + d(T^n\xi_1, T^n\xi_2) + \cdots + d(T^n\xi_{p-1}, T^n\xi_p) \\ &= \sum_{i=0}^{p-1} d(T^n\xi_i, T^n\xi_{i+1}) \\ &\leq \sum_{i=0}^{p-1} \varphi^n(d(\xi_i, \xi_{i+1})). \end{aligned}$$

Thus, we proved that

$$d(x_n, x_{n+1}) \le \sum_{i=0}^{p-1} \varphi^n(d(\xi_i, \xi_{i+1})), \quad n \in \mathbb{N},$$

which implies that for $n < m$,

$$\begin{aligned} d(x_n, x_m) &\le \sum_{j=n}^{m-1} d(x_j, x_{j+1}) \\ &\le \sum_{j=n}^{m-1} \sum_{i=0}^{p-1} \varphi^j(d(\xi_i, \xi_{i+1})) \\ &= \sum_{i=0}^{p-1} \sum_{j=n}^{m-1} \varphi^j(d(\xi_i, \xi_{i+1})). \end{aligned}$$

On the other hand, from property (P2), we have

$$\sum_{i=0}^{p-1} \sum_{j=n}^{m-1} \varphi^j(d(\xi_i, \xi_{i+1})) \to 0 \quad \text{as } n, m \to \infty.$$

Then we proved that $d(x_n, x_m) \to 0$ as $n, m \to \infty$; that is, $\{x_n\}$ is a Cauchy sequence in the metric space (X, d).

The first main theorem is the following fixed point result obtained under the continuity assumption of the mapping T.

Theorem 3.2 *Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi$ such that T is an (α, ψ)-contraction. Suppose also that (3.3) is satisfied. Then $\{T^n x_0\}$ converges to some $x^* \in X$. Moreover, if T is continuous, then x^* is a fixed point of T.*

Proof From Proposition 3.1, we know that $\{T^n x_0\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(T^n x_0, x^*) = 0.$$

Since T is continuous, we have

$$\lim_{n\to\infty} d(T^{n+1} x_0, T x^*) = 0.$$

By the uniqueness of the limit, we obtain $x^* = T x^*$.

The next theorem does not require the continuity assumption of T.

Theorem 3.3 *Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi$ such that T is an (α, ψ)-contraction. Suppose also that (3.3) is satisfied. Then $\{T^n x_0\}$ converges to some $x^* \in X$. Moreover, if there exists a subsequence $\{T^{\gamma(n)} x_0\}$ of $\{T^n x_0\}$ such that*

$$\lim_{n\to\infty} \alpha(T^{\gamma(n)} x_0, x^*) = \ell \in (0, \infty),$$

then x^* is a fixed point of T.

Proof From Proposition 3.1 and the completeness of the metric space (X, d), we know that $\{T^n x_0\}$ converges to some $x^* \in X$. Suppose now that there exists a subsequence $\{T^{\gamma(n)} x_0\}$ of $\{T^n x_0\}$ such that

$$\lim_{n\to\infty} \alpha(T^{\gamma(n)} x_0, x^*) = \ell \in (0, \infty). \tag{3.7}$$

Since T is an (α, ψ)-contraction, we have

$$\alpha(T^{\gamma(n)} x_0, x^*) d(T^{\gamma(n)+1} x_0, T x^*) \le \psi(d(T^{\gamma(n)} x_0, x^*)), \quad n \in \mathbb{N}.$$

Letting $n \to \infty$ in the above inequality, using (3.7), properties (ii) and (iii) of Lemma 3.1, we obtain

$$\ell \, d(x^*, T x^*) \le \psi(0) = 0,$$

which implies that x^* is a fixed point of T.

The next theorem gives us a sufficient condition for the uniqueness of the fixed point.

Theorem 3.4 *Let (X, d) be a metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi$ such that T is an (α, ψ)-contraction. Suppose also that*

(i) $\mathrm{Fix}(T) \neq \emptyset$;
(ii) *For every pair $(x, y) \in \mathrm{Fix}(T) \times \mathrm{Fix}(T)$ with $x \neq y$, if $\alpha(x, y) < 1$, then there exists $\eta \in \Sigma_\psi$ and for some positive integer q, there is a finite sequence $\{\zeta_i(x, y)\}_{i=0}^q \subset X$ such that*

$$\zeta_0(x, y) = x, \quad \zeta_q(x, y) = y, \quad \alpha(T^n \zeta_i(x, y), T^n \zeta_{i+1}(x, y)) \ge \eta^{-1},$$

for $n \in \mathbb{N}$ and $i = 0, \ldots, q-1$.

Then T has a unique fixed point.

Proof Let $\varphi = \eta\psi \in \Psi$. Suppose that $u, v \in X$ are two fixed points of T such that $d(u, v) > 0$. We consider two cases.

Case 1. If $\alpha(u, v) \ge 1$.

Since T is an (α, ψ)-contraction, we have

$$d(u, v) \leq \alpha(u, v) d(Tu, Tv) \leq \psi(d(u, v)).$$

From property (i) of Lemma 3.1, we have $\psi(d(u, v)) < d(u, v)$, which yields $d(u, v) < d(u, v)$, leading to a contradiction.

Case 2. If $\alpha(u, v) < 1$.
By assumption, there exists a finite sequence $\{\zeta_i(u, v)\}_{i=0}^{q}$ in X such that

$$\zeta_0(u, v) = u, \quad \zeta_q(u, v) = v, \quad \alpha(T^n \zeta_i(u, v), T^n \zeta_{i+1}(u, v)) \geq \eta^{-1},$$

for $n \in \mathbb{N}$ and $i = 0, \ldots, q - 1$. As in the proof of Proposition 3.1, we can establish that

$$d(T^r \zeta_i(u, v), T^r \zeta_{i+1}(u, v)) \leq \varphi^r(d(\zeta_i(u, v), \zeta_{i+1}(u, v))), \quad r \in \mathbb{N}, \quad i = 0, \ldots, q - 1. \tag{3.8}$$

Using the triangle inequality and (3.8), we have

$$\begin{aligned} d(u, v) &= d(T^n u, T^n v) \\ &\leq \sum_{i=0}^{q-1} d(T^n \zeta_i(u, v), T^n \zeta_{i+1}(u, v)) \\ &\leq \sum_{i=0}^{q-1} \varphi^n(d(\zeta_i(u, v), \zeta_{i+1}(u, v))) \to 0 \quad \text{as } n \to \infty \end{aligned}$$

Then $u = v$, which contradicts the assumption $d(u, v) > 0$.

3.3 Consequences

In this section, we will show that the most existing fixed point results in the literature, where the fixed points can be obtained by means of Picard iteration are particular cases of the main theorems established in the previous section.

3.3.1 The Class of ψ-Contractions

The class of ψ-contractions is defined as follows.

Definition 3.3 Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a ψ-contraction, if there exists a function $\psi \in \Psi$ such that

$$d(Tx, Ty) \leq \psi(d(x, y)), \quad (x, y) \in X \times X. \tag{3.9}$$

Theorem 3.5 *Let (X, d) be a metric space and $T : X \to X$ be a given mapping. Suppose that there exists $\psi \in \Psi$ such that T is a ψ-contraction. Then there exists $\alpha : X \times X \to \mathbb{R}$ such that T is an (α, ψ)-contraction.*

Proof Consider the function $\alpha : X \times X \to \mathbb{R}$ defined by

$$\alpha(x, y) = 1, \quad (x, y) \in X \times X. \tag{3.10}$$

Clearly, from (3.9), T is an (α, ψ)-contraction.

Corollary 3.1 ([4, Theorem 2.8]) *Let (X, d) be a complete metric space and $T : X \to X$ be a ψ-contraction for some $\psi \in \Psi$. Then T has a unique fixed point.*

Proof From (i) Lemma 3.1, we have

$$d(Tx, Ty) \leq d(x, y), \quad (x, y) \in X \times X,$$

which implies that T is a continuous mapping. From Theorem 3.5, T is an (α, ψ)-contraction, where α is defined by (3.10). Clearly, (3.3) is satisfied with $p = 1$ and $\sigma = 1$. By Theorem 3.2, T has a fixed point. The uniqueness follows immediately from (3.10) and Theorem 3.4.

Remark 3.1 Note that Banach contraction principle follows immediately from Corollary 3.1 with $\psi(t) = kt$, $t \geq 0$, $k \in (0, 1)$.

3.3.2 *The Class of Rational Contractions*

3.3.2.1 Dass–Gupta Contraction

Definition 3.4 Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a Dass–Gupta contraction, if there exist constants $\lambda, \mu \geq 0$ with $\lambda + \mu < 1$ such that

$$d(Tx, Ty) \leq \mu d(y, Ty) \frac{1 + d(x, Tx)}{1 + d(x, y)} + \lambda d(x, y), \quad (x, y) \in X \times X. \tag{3.11}$$

Theorem 3.6 *Let (X, d) be a metric space and $T : X \to X$ be a Dass–Gupta contraction. Then there exist $\psi \in \Psi$ and $\alpha : X \times X \to \mathbb{R}$ such that T is an (α, ψ)-contraction.*

Proof From (3.11), for all $x, y \in X$, we have

$$d(Tx, Ty) - \mu d(y, Ty) \frac{1 + d(x, Tx)}{1 + d(x, y)} \leq \lambda d(x, y),$$

which yields

$$\left(1-\mu\frac{d(y,Ty)(1+d(x,Tx))}{(1+d(x,y))d(Tx,Ty)}\right)d(Tx,Ty)\le \lambda d(x,y),\quad (x,y)\in X\times X,\ Tx\neq Ty. \tag{3.12}$$

Consider the functions $\psi : [0,\infty) \to [0,\infty)$ and $\alpha : X \times X \to \mathbb{R}$ defined by

$$\psi(t)=\lambda t,\quad t\ge 0 \tag{3.13}$$

and

$$\alpha(x,y)=\begin{cases}1-\mu\frac{d(y,Ty)(1+d(x,Tx))}{(1+d(x,y))d(Tx,Ty)}, & \text{if } Tx\neq Ty,\\ 0, & \text{otherwise.}\end{cases} \tag{3.14}$$

From (3.12), we have

$$\alpha(x,y)d(Tx,Ty)\le \psi(d(x,y)),\quad (x,y)\in X\times X.$$

Then T is an (α,ψ)-contraction.

Corollary 3.2 (Dass–Gupta [7]) *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exist constants $\lambda,\mu \ge 0$ with $\lambda+\mu<1$ such that (3.11) is satisfied. Then T has a unique fixed point.*

Proof Let x_0 be an arbitrary point in X. If for some $r \in \mathbb{N}$, $T^r x_0 = T^{r+1}x_0$, then $T^r x_0$ will be a fixed point of T. So, we can suppose that $T^r x_0 \neq T^{r+1}x_0$, for all $r \in \mathbb{N}$. From (3.14), for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\alpha(T^n x_0, T^{n+1}x_0) &= 1-\mu\frac{d(T^{n+1}x_0,T^{n+2}x_0)(1+d(T^n x_0,T^{n+1}x_0))}{(1+d(T^n x_0,T^{n+1}x_0))d(T^{n+1}x_0,T^{n+2}x_0)}\\ &= 1-\mu>0.\end{aligned}$$

On the other hand, from (3.13), we have

$$(1-\mu)^{-1}\psi(t)=\frac{\lambda}{1-\mu}t,\quad t\ge 0.$$

Since $\lambda+\mu<1$, we have $(1-\mu)^{-1}\psi \in \Psi$; that is, $(1-\mu)^{-1}\in \Sigma_\psi$. Then (3.3) is satisfied with $p=1$ and $\sigma=(1-\mu)^{-1}$. From the first part of Theorem 3.3, the sequence $\{T^n x_0\}$ converges to some $x^* \in X$. Without loss of generality, we can suppose that there exists $N \in \mathbb{N}$ such that

$$T^{n+1}x_0\neq Tx^*,\ n\ge N.$$

Otherwise, x^* will be a fixed point of T. From (3.14), for all $n \ge N$, we have

$$\alpha(T^n x_0,x^*)=1-\mu\frac{d(x^*,Tx^*)(1+d(T^n x_0,T^{n+1}x_0))}{(1+d(T^n x_0,x^*))d(T^{n+1}x_0,Tx^*)}\to 1-\mu\quad \text{as } n\to\infty.$$

From the second part of Theorem 3.3 (with $\ell = 1 - \mu$), we deduce that x^* is a fixed point of T. For the uniqueness, observe that for every pair $(x, y) \in \mathrm{Fix}(T) \times \mathrm{Fix}(T)$ with $x \neq y$, we have $\alpha(x, y) = 1$. By Theorem 3.4, x^* is the unique fixed point of T.

3.3.2.2 Jaggi Contraction

Definition 3.5 Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a Jaggi contraction, if there exist constants $\lambda, \mu \geq 0$ with $\lambda + \mu < 1$ such that

$$d(Tx, Ty) \leq \mu \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \lambda d(x, y), \quad (x, y) \in X \times X, \ x \neq y. \tag{3.15}$$

Theorem 3.7 *Let (X, d) be a metric space and $T : X \to X$ be a given mapping. Suppose that T is a Jaggi contraction. Then there exist $\psi \in \Psi$ and $\alpha : X \times X \to \mathbb{R}$ such that T is an (α, ψ)-contraction.*

Proof From (3.15), for all $x, y \in X$ with $x \neq y$, we have

$$d(Tx, Ty) - \mu \frac{d(x, Tx)d(y, Ty)}{d(x, y)} \leq \lambda d(x, y),$$

which yields

$$\left(1 - \mu \frac{d(x, Tx)d(y, Ty)}{d(x, y)d(Tx, Ty)}\right) d(Tx, Ty) \leq \lambda d(x, y), \quad (x, y) \in X \times X, \ Tx \neq Ty. \tag{3.16}$$

Consider the functions $\psi : [0, \infty) \to [0, \infty)$ and $\alpha : X \times X \to \mathbb{R}$ defined by

$$\psi(t) = \lambda t, \quad t \geq 0 \tag{3.17}$$

and

$$\alpha(x, y) = \begin{cases} 1 - \mu \frac{d(x,Tx)d(y,Ty)}{d(x,y)d(Tx,Ty)}, & \text{if } Tx \neq Ty, \\ 0, & \text{otherwise.} \end{cases} \tag{3.18}$$

From (3.16), we have

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)), \quad (x, y) \in X \times X.$$

Then T is an (α, ψ)-contraction.

Corollary 3.3 (Jaggi [12]) *Let (X, d) be a complete metric space and $T : X \to X$ be a continuous mapping. Suppose that there exist constants $\lambda, \mu \geq 0$ with $\lambda + \mu < 1$ such that (3.15) is satisfied. Then T has a unique fixed point.*

Proof Let x_0 be an arbitrary point in X. without loss of generality, we can suppose that $T^r x_0 \neq T^{r+1} x_0$, for all $r \in \mathbb{N}$. From (3.18), for all $n \in \mathbb{N}$, we have

$$\alpha(T^n x_0, T^{n+1} x_0) = 1 - \mu \frac{d(T^n x_0, T^{n+1} x_0) d(T^{n+1} x_0, T^{n+2} x_0)}{d(T^n x_0, T^{n+1} x_0) d(T^{n+1} x_0, T^{n+2} x_0)} = 1 - \mu > 0.$$

On the other hand, from (3.17), for all $t \geq 0$, we have

$$(1-\mu)^{-1} \psi(t) = \frac{\lambda}{1-\mu} t.$$

Since $\lambda + \mu < 1$, we have $(1-\mu)^{-1}\psi \in \Psi$; that is, $(1-\mu)^{-1} \in \Sigma_\psi$. Then (3.3) is satisfied with $p = 1$ and $\sigma = (1-\mu)^{-1}$. By the first part of Theorem 3.2, $\{T^n x_0\}$ converges to some $x^* \in X$. Since T is continuous, by the second part of Theorem 3.2, x^* is a fixed point of T. Moreover, for every pair $(x, y) \in \mathrm{Fix}(T) \times \mathrm{Fix}(T)$ with $x \neq y$, we have $\alpha(x, y) = 1$. Then by Theorem 3.4, x^* is the unique fixed point of T.

3.3.3 *The Class of Berinde Mappings*

In [3], Berinde introduced the concept of weak contractions and studied the existence of fixed points for this class of mappings. Moreover, he proved that several contraction-type mappings (Kannan contraction [13], Chatterjee contraction [5], Zamfirescu contraction [29], Hardy–Rogers contraction [10], and many others) are weakly contraction-type mappings. In this section, we will show that any weak contraction is an (α, ψ)-contraction. Moreover, we will show that Berinde fixed point theorem can be deduced immediately from Theorem 3.3.

Definition 3.6 Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a weak contraction, if there exist $\lambda \in (0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \lambda\, d(x, y) + L\, d(y, Tx), \quad (x, y) \in X \times X. \tag{3.19}$$

Theorem 3.8 *Let (X, d) be a metric space and $T : X \to X$ be a given mapping. If T is a weak contraction, then there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi$ such that T is an (α, ψ)-contraction.*

Proof From (3.19), we have

$$d(Tx, Ty) - L\, d(y, Tx) \leq \lambda d(x, y), \quad (x, y) \in X \times X,$$

which yields

$$\left(1 - L\frac{d(y, Tx)}{d(Tx, Ty)}\right)d(Tx, Ty) \le \lambda d(x, y), \quad (x, y) \in X \times X, \ Tx \ne Ty. \quad (3.20)$$

Consider the functions $\psi : [0, \infty) \to [0, \infty)$ and $\alpha : X \times X \to \mathbb{R}$ defined by

$$\psi(t) = \lambda t, \quad t \ge 0,$$

and

$$\alpha(x, y) = \begin{cases} 1 - L\frac{d(y,Tx)}{d(Tx,Ty)}, & \text{if } Tx \ne Ty, \\ 0, & \text{otherwise.} \end{cases} \quad (3.21)$$

From (3.20), we have

$$\alpha(x, y)d(Tx, Ty) \le \psi(d(x, y)), \quad (x, y) \in X \times X.$$

Then T is an (α, ψ)-contraction.

Corollary 3.4 (Berinde [3]) *Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exist constants $\lambda \in (0, 1)$ and $L \ge 0$ such that (3.19) is satisfied. Then T has a fixed point.*

Proof Let x_0 be an arbitrary point in X. Without loss of generality, we can suppose that $T^r x_0 \ne T^{r+1}x_0$, for all $r \in \mathbb{N}$. From (3.21), for all $n \in \mathbb{N}$, we have

$$\alpha(T^n x_0, T^{n+1}x_0) = 1 - L\frac{d(T^{n+1}x_0, T^{n+1}x_0)}{d(T^{n+1}x_0, T^{n+2}x_0)} = 1.$$

Then (3.3) holds with $\sigma = 1$ and $p = 1$. From the first part of Theorem 3.3, the sequence $\{T^n x_0\}$ converges to some $x^* \in X$. Without loss of generality, we can suppose that there exists some $N \in \mathbb{N}$ such that

$$T^{n+1}x_0 \ne Tx^*, \quad n \ge N.$$

From (3.21), for all $n \ge N$, we have

$$\alpha(T^n x_0, x^*) = 1 - L\frac{d(x^*, T^{n+1}x_0)}{d(T^{n+1}x_0, Tx^*)} \to 1 \quad \text{as } n \to \infty.$$

By the second part of Theorem 3.3 (with $\ell = 1$), we deduce that x^* is a fixed point of T.

Remark 3.2 Note that in general, we don't have uniqueness for the fixed points of Berinde mappings (see [4, Example 2.11]).

3.3.4 Ćirić Mappings with a Nonunique Fixed Point

Definition 3.7 Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a Ćirić mapping, if there exists $\lambda \in (0, 1)$ such that for all $x, y \in X$, we have

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(y, Tx)\} \leq \lambda\, d(x, y). \tag{3.22}$$

Theorem 3.9 *Let (X, d) be a metric space and $T : X \to X$ be a given mapping. Suppose that there exists $\lambda \in (0, 1)$ such that (3.22) is satisfied. Then there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi$ such that T is an (α, ψ)-contraction.*

Proof Consider the functions $\psi : [0, \infty) \to [0, \infty)$ and $\alpha : X \times X \to \mathbb{R}$ defined by

$$\psi(t) = \lambda t, \quad t \geq 0 \tag{3.23}$$

and

$$\alpha(x, y) = \begin{cases} \min\left\{1, \frac{d(x,Tx)}{d(Tx,Ty)}, \frac{d(y,Ty)}{d(Tx,Ty)}\right\} - \min\left\{\frac{d(x,Ty)}{d(Tx,Ty)}, \frac{d(y,Tx)}{d(Tx,Ty)}\right\}, & \text{if } Tx \neq Ty, \\ 0, & \text{otherwise.} \end{cases} \tag{3.24}$$

From (3.22), we have

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)), \quad (x, y) \in X \times X, \tag{3.25}$$

which implies that T is an (α, ψ)-contraction.

Corollary 3.5 (Ćirić [6]) *Let (X, d) be a complete metric space and let $T : X \to X$ be a continuous Ćirić mapping. Then T has a fixed point.*

Proof Let $x_0 \in X$ be an arbitrary point. Without loss of generality, we can suppose that $T^r x_0 \neq T^{r+1}x_0$, for all $r \in \mathbb{N}$. From (3.24), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \alpha(T^n x_0, T^{n+1}x_0) &= \min\left\{1, \frac{d(T^n x_0, T^{n+1}x_0)}{d(T^{n+1}x_0, T^{n+2}x_0)}, \frac{d(T^{n+1}x_0, T^{n+2}x_0)}{d(T^{n+1}x_0, T^{n+2}x_0)}\right\} \\ &\quad - \min\left\{\frac{d(T^n x_0, T^{n+2}x_0)}{d(T^{n+1}x_0, T^{n+2}x_0)}, \frac{d(T^{n+1}x_0, T^{n+1}x_0)}{d(T^{n+1}x_0, T^{n+2}x_0)}\right\} \\ &= \min\left\{1, \frac{d(T^n x_0, T^{n+1}x_0)}{d(T^{n+1}x_0, T^{n+2}x_0)}\right\}. \end{aligned}$$

Suppose that for some $n \in \mathbb{N}$, we have

$$\alpha(T^n x_0, T^{n+1}x_0) = \frac{d(T^n x_0, T^{n+1}x_0)}{d(T^{n+1}x_0, T^{n+2}x_0)}.$$

In this case, from (3.23) and (3.25), we have

$$d(T^n x_0, T^{n+1} x_0) \le \lambda d(T^n x_0, T^{n+1} x_0).$$

This implies (from the assumption $T^r x_0 \neq T^{r+1} x_0$, for all $r \in \mathbb{N}$) that $\lambda \ge 1$, which leads to a contradiction. Then

$$\alpha(T^n x_0, T^{n+1} x_0) = 1, \quad \text{for all } n \in \mathbb{N}.$$

Then (3.3) is satisfied with $p = 1$ and $\sigma = 1$. By Theorem 3.3, we deduce that the sequence $\{T^n x_0\}$ converges to a fixed point of T.

Remark 3.3 Note that in general, a Ćirić mapping does not have a unique fixed point (see [6]).

3.3.5 *The Class of Suzuki Mappings*

We define Suzuki mappings as follows.

Definition 3.8 Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a Suzuki mapping, if there exists $r \in (0, 1)$ such that

$$(1 + r)^{-1} d(x, Tx) \le d(x, y) \Longrightarrow d(Tx, Ty) \le r\, d(x, y), \quad (x, y) \in X \times X. \tag{3.26}$$

Theorem 3.10 *Let (X, d) be a metric space and $T : X \to X$ be a given mapping. Suppose that there exists $r \in (0, 1)$ such that (3.26) is satisfied. Then there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi$ such that T is an (α, ψ)-contraction.*

Proof Consider the functions $\psi : [0, \infty) \to [0, \infty)$ and $\alpha : X \times X \to \mathbb{R}$ defined by

$$\psi(t) = rt, \quad t \ge 0$$

and

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (1 + r)^{-1} d(x, Tx) \le d(x, y), \\ 0, & \text{otherwise.} \end{cases} \tag{3.27}$$

From (3.26), we have

$$\alpha(x, y) d(Tx, Ty) \le \psi(d(x, y)), \quad (x, y) \in X \times X.$$

Then T is an (α, ψ)-contraction.

Corollary 3.6 (Suzuki [27]) *Let (X, d) be a complete metric space, and suppose that $T : X \to X$ is a Suzuki mapping. Then T has a unique fixed point.*

Proof Let $x_0 \in X$ be an arbitrary point. For all $n \in \mathbb{N}$, we have

$$(1+r)^{-1} d(T^n x_0, T(T^n x_0)) \leq d(T^n x_0, T^{n+1} x_0), \tag{3.28}$$

which implies that $\alpha(T^n x_0, T^{n+1} x_0) = 1$, for all $n \in \mathbb{N}$, where α is defined by (3.27). Then (3.3) is satisfied with $p = 1$ and $\sigma = 1$. From the first part of Theorem 3.3, the sequence $\{T^n x_0\}$ converges to some $x^* \in X$. From (3.26) and (3.28), we have

$$d(T(T^n x_0), T^2(T^n x_0)) \leq r\, d(T^n x_0, T(T^n x_0)), \quad \text{for all } n \in \mathbb{N},$$

which implies from [28, Lemma 2.1] that there exists a subsequence $\{\gamma(n)\}$ of $\{n\}$ such that

$$(1+r)^{-1} d(T^{\gamma(n)} x_0, T^{\gamma(n)+1} x_0) \leq d(T^{\gamma(n)} x_0, x^*), \quad \text{for all } n \in \mathbb{N}.$$

From (3.27), we have

$$\alpha(T^{\gamma(n)} x_0, x^*) = 1, \quad \text{for all } n \in \mathbb{N}.$$

By the second part of Theorem 3.3 (with $\ell = 1$), x^* is a fixed point of T. On the other hand, from (3.27), for every pair $(x, y) \in \text{Fix}(T) \times \text{Fix}(T)$ with $x \neq y$, we have $\alpha(x, y) = 1$. By Theorem 3.4, x^* is the unique fixed point of T.

3.3.6 The Class of Cyclic Mappings

In [23] (see also [17]), the following notion was introduced.

Definition 3.9 Let (X, d) be a metric space, m be a positive integer, and $T : X \to X$ be an operator. We say that $X = \cup_{i=1}^m X_i$ is a cyclic representation of X with respect to T if

(i) $X_i, i = 1, \ldots, m$ are nonempty sets;
(ii) $T(X_1) \subseteq X_2, \ldots, T(X_{m-1}) \subseteq X_m, T(X_m) \subset X_1$.

Definition 3.10 Let (X, d) be a metric space, $A_1, \ldots, A_m \in P_{cl}(X)$, $Y = \cup_{i=1}^m A_i$, with m a positive integer, and $T : Y \to Y$ be an operator. We say that T is a cyclic ψ-contraction for some $\psi \in \Psi$, if

(i) $\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T;
(ii) for all $i = 1, \ldots, m$, we have

$$d(Tx, Ty) \leq \psi(d(x, y)), \quad (x, y) \in A_i \times A_{i+1},$$

where $A_{m+1} = A_1$.

Here, $P_{cl}(X)$ denotes the collection of nonempty closed subsets of (X, d).

We have the following result.

Theorem 3.11 *Let (X, d) be a metric space, m be a positive integer, $A_1, \ldots, A_m \in P_{cl}(X)$, $Y = \cup_{i=1}^m A_i$ and $T : Y \to Y$ be a cyclic ψ-contraction for some $\psi \in \Psi$. Then there exists a function $\alpha : Y \times Y \to \mathbb{R}$ such that*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)), \quad (x, y) \in Y \times Y. \tag{3.29}$$

Proof Define the function $\alpha : Y \times Y \to \mathbb{R}$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in A_i \times A_{i+1} \text{ for some } i = 1, \ldots, m, \\ 0, & \text{otherwise.} \end{cases} \tag{3.30}$$

From (ii) Definition 3.10, we obtain (3.29).

Corollary 3.7 (Păcurar and Rus [20]) *Let (X, d) be a complete metric space, m be a positive integer, $A_1, \ldots, A_m \in P_{cl}(X)$, $Y = \cup_{i=1}^m A_i$, $\psi \in \Psi$ and $T : Y \to Y$ be an operator. Suppose that*

(i) $\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T;
(ii) T is a cyclic ψ-contraction.

Then T has a unique fixed point $x^* \in \cap_{i=1}^m A_i$.

Proof Let $x_0 \in A_1$ be an arbitrary point. From condition (i) and (3.30), we have

$$\alpha(T^n x_0, T^{n+1} x_0) = 1, \quad \text{for all } n \in \mathbb{N}.$$

Then (3.3) is satisfied with $p = 1$ and $\sigma = 1$. By the first part of Theorem 3.3, the sequence $\{T^n x_0\}$ converges to some $x^* \in Y$. By (i), the sequence $\{T^n x_0\}$ has an infinite number of terms in each A_i, $i = 1, \ldots, m$, so from each A_i, $i = 1, \ldots, m$, one can extract a subsequence $\{T^{\gamma_i(n)} x_0\} \subset A_i$ of $\{T^n x_0\}$. Since $\{A_i\}_{i=1}^m \subset P_{cl}(X)$, it follows that $x^* \in \cap_{i=1}^m A_i$. Then by (3.30), for a fixed $j = 1 \ldots, m$, we have $\alpha(T^{\gamma_j(n)} x_0, x^*) = 1$, for all $n \in \mathbb{N}$. By the second part of Theorem 3.3 (with $\ell = 1$), we deduce that x^* is a fixed point of T. On the other hand, observe that

$$\text{Fix}(T) \times \text{Fix}(T) \subset \cap_{i=1}^m A_i \times \cap_{i=1}^m A_i,$$

which implies from (3.30) that

$$\alpha(x, y) = 1, \quad \text{for all } (x, y) \in \text{Fix}(T) \times \text{Fix}(T).$$

By Theorem 3.4, we deduce that x^* is the unique fixed point of T.

3.3.7 Edelstein Fixed Point Theorem

Another consequence of the main results presented in this chapter is the following generalized version of Edelstein fixed point theorem [9].

Corollary 3.8 *Let (X, d) be complete and ε-chainable for some $\varepsilon > 0$; i.e., given $x, y \in X$, there exist a positive integer N and a sequence $\{x_i\}_{i=0}^N \subset X$ such that*

$$x_0 = x, \quad x_N = y, \quad d(x_i, x_{i+1}) < \varepsilon, \quad \textit{for } i = 0, \ldots, N-1. \tag{3.31}$$

Let $T : X \to X$ be a given mapping such that

$$(x, y) \in X \times X, \quad d(x, y) < \varepsilon \Longrightarrow d(Tx, Ty) \le \psi(d(x, y)), \tag{3.32}$$

for some $\psi \in \Psi$. Then T has a unique fixed point.

Proof It is clear that from (3.32), the mapping T is continuous. Now, consider the function $\alpha : X \times X \to \mathbb{R}$ defined by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } d(x, y) < \varepsilon, \\ 0, & \text{otherwise.} \end{cases} \tag{3.33}$$

From (3.32), we have

$$\alpha(x, y) d(Tx, Ty) \le \psi(d(x, y)), \quad (x, y) \in X \times X.$$

Let $x_0 \in X$. For $x = x_0$ and $y = Tx_0$, from (3.31) and (3.33), for some positive integer p, there exists a finite sequence $\{\xi_i\}_{i=0}^p \subset X$ such that

$$x_0 = \xi_0, \quad \xi_p = Tx_0, \quad \alpha(\xi_i, \xi_{i+1}) \ge 1, \quad \text{for } i = 0, \ldots, p-1.$$

Now, let $i \in \{0, \ldots, p-1\}$ be fixed. From (3.33) and (3.32), we have

$$\begin{aligned} \alpha(\xi_i, \xi_{i+1}) \ge 1 &\Longrightarrow d(\xi_i, \xi_{i+1}) < \varepsilon \\ &\Longrightarrow d(T\xi_i, T\xi_{i+1}) \le \psi(d(\xi_i, \xi_{i+1})) \le d(\xi_i, \xi_{i+1}) < \varepsilon \\ &\Longrightarrow \alpha(T\xi_i, T\xi_{i+1}) \ge 1. \end{aligned}$$

Again,

$$\begin{aligned} \alpha(T\xi_i, T\xi_{i+1}) \ge 1 &\Longrightarrow d(T\xi_i, T\xi_{i+1}) < \varepsilon \\ &\Longrightarrow d(T^2\xi_i, T^2\xi_{i+1}) \le \psi(d(T\xi_i, T\xi_{i+1})) \le d(T\xi_i, T\xi_{i+1}) < \varepsilon \\ &\Longrightarrow \alpha(T^2\xi_i, T^2\xi_{i+1}) \ge 1. \end{aligned}$$

By induction, we obtain

$$\alpha(T^n \xi_i, T^{n+1} \xi_{i+1}) \geq 1, \quad \text{for all } n \in \mathbb{N}.$$

Then (3.3) is satisfied with $\sigma = 1$. From Theorem 3.2, the sequence $\{T^n x_0\}$ converges to a fixed point of T. Using a similar argument, we can see that condition (ii) of Theorem 3.4 is satisfied, which implies that T has a unique fixed point.

3.3.8 Fixed Point Theorems in Partially Ordered Sets

In this section, we use the main results of this chapter to establish some fixed point theorems in a metric space with a partial order. Let (X, d) be a metric space and $\preceq$ be a partial order on X. Let

$$\Delta = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}.$$

Corollary 3.9 *Let $T : X \to X$ be a given mapping. Suppose that there exists $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(d(x, y)), \quad (x, y) \in \Delta. \tag{3.34}$$

Suppose also that

(i) T is continuous;
(ii) For some positive integer p, there exists a finite sequence $\{\xi_i\}_{i=0}^{p} \subset X$ such that

$$\xi_0 = x_0, \quad \xi_p = Tx_0, \quad (T^n \xi_i, T^n \xi_{i+1}) \in \Delta, \quad n \in \mathbb{N}, \; i = 0, \ldots, p-1. \tag{3.35}$$

Then $\{T^n x_0\}$ converges to a fixed point of T.

Proof Consider the function $\alpha : X \times X \to \mathbb{R}$ defined by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \Delta, \\ 0, & \text{otherwise.} \end{cases} \tag{3.36}$$

From (3.34), we have

$$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y)), \quad (x, y) \in X \times X.$$

Then the result follows from Theorem 3.2 with $\sigma = 1$.

Corollary 3.10 *Let $T : X \to X$ be a given mapping. Suppose that*

(i) There exists $\psi \in \Psi$ such that (3.34) holds;
(ii) Condition (3.35) holds.

Then $\{T^n x_0\}$ converges to some $x^* \in X$. Moreover, if *(iii) There exist a subsequence* $\{T^{\gamma(n)}x_0\}$ *of* $\{T^n x_0\}$ *and* $N \in \mathbb{N}$ *such that*

$$(T^{\gamma(n)}x_0, x^*) \in \Delta, \ \ n \geq N,$$

then x^* *is a fixed point of* T.

Proof We continue to use the same function α defined by (3.36). From the first part of Theorem 3.3, the sequence $\{T^n x_0\}$ converges to some $x^* \in X$. From (iii) and (3.36), we have

$$\lim_{n\to\infty} \alpha(T^{\gamma(n)}x_0, x^*) = 1.$$

By the second part of Theorem 3.3 (with $\ell = 1$), we deduce that x^* is a fixed point of T.

Corollary 3.11 *Let* $T : X \to X$ *be a given mapping. Suppose that*

(i) There exists $\psi \in \Psi$ *such that* (3.34) *holds;*
(ii) $\mathrm{Fix}(T) \neq \emptyset$*;*
(iii) For every pair $(x, y) \in \mathrm{Fix}(T) \times \mathrm{Fix}(T)$ *with* $x \neq y$*, if* $(x, y) \notin \Delta$*, there exist a positive integer* q *and a finite sequence* $\{\zeta_i(x, y)\}_{i=0}^{q} \subset X$ *such that*

$$\zeta_0(x, y) = x, \quad \zeta_q(x, y) = y, \quad (T^n\zeta_i(x, y), T^n\zeta_{i+1}(x, y)) \in \Delta,$$

for $n \in \mathbb{N}$ *and* $i = 0, \ldots, q-1$.

Then T *has a unique fixed point.*

Remark 3.4 The above corollary follows from Theorem 3.4 with $\eta = 1$. Observe that in the above results, it is not supposed that T is monotone or T preserves order, as it was assumed in many papers (see, e.g., [11, 19, 21]).

3.4 Existence Results for a Class of Nonlinear Quadratic Integral Equations

Quadratic integral equations are often applicable in the theory of radiative transfer, in the kinetic theory of gases, in the theory of neutron transport, and in the traffic theory. The quadratic integral equations can be very often encountered in many applications (see, e.g., [1, 2, 8]).

Here, we are concerned with the nonlinear quadratic integral equation

$$x(t) = a(t) + \lambda \int_0^t k_1(t,s) f_1(s, x(s))\, ds \int_0^t k_2(t,s) f_2(s, x(s))\, ds, \quad t \in [0, T], \ T > 0. \tag{3.37}$$

Let $X = C([0, T]; \mathbb{R}^N)$ be the set of continuous functions from $[0, T]$ to $\mathbb{R}^N$. We endow X with the metric

$$d(x, y) = \max\left\{|x(t) - y(t)| : t \in [0, T]\right\}, \quad (x, y) \in X \times X.$$

It is well known that (X, d) is a complete metric space. We consider the norm on X defined by

$$\|x\|_\infty = \max\left\{|x(t)| : t \in [0, T]\right\}, \quad x \in X.$$

We endow $\mathbb{R}^N$ with the partial order

$$u = (u_1, u_2, \ldots, u_N) \leq_{\mathbb{R}^N} v = (v_1, v_2, \ldots, v_N) \Longleftrightarrow u_i \leq v_i, \quad i = 1, 2, \ldots, N.$$

We consider the following assumptions:

(i) $a : [0, T] \to \mathbb{R}^N$ is continuous;
(ii) $f_i : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$ are continuous;
(iii) For almost all $t \in [0, T]$, we have

$$|f_i(t, u) - f_i(t, v)| \leq L\,|u - v|, \quad u \leq_{\mathbb{R}^N} v,$$

where $L > 0$ is a constant;
(iv) There exist two functions $m_i : [0, T] \to \mathbb{R}$ such that $m_i \in L^1[0, T]$ and

$$|f_i(t, u)| \leq m_i(t),\ \ t \in [0, T],\ u \in \mathbb{R}^N;$$

(v) For all $t \in [0, T]$, we have

$$u, v \in \mathbb{R}^N,\ \ u \leq_{\mathbb{R}^N} v \Longrightarrow f(t, u) \leq_{\mathbb{R}^N} f(t, v);$$

(vi) $k_i : [0, T] \times [0, T] \to [0, \infty)$ are continuous,

$$K_i = \max\{k_i(t, s) : (t, s) \in [0, T] \times [0, T]\};$$

(vii) There exists a constant $K > 0$ such that

$$\int_0^t k_i(t, s) m_i(s)\, ds \leq K, \quad t \in [0, T];$$

(viii) There exists $x_0 \in X$ such that

$$x_0(t) \leq_{\mathbb{R}^N} a(t) + \lambda \int_0^t k_1(t, s) f_1(s, x_0(s))\, ds \int_0^t k_2(t, s) f_2(s, x_0(s))\, ds, \quad t \in [0, T].$$

We have the following result.

Theorem 3.12 *Suppose conditions (i)–(viii) are satisfied. If*

$$0 < \lambda < (LKT(K_1 + K_2))^{-1},$$

then the quadratic integral Eq. (3.37) *has a unique solution* $x^* \in C([0, T]; \mathbb{R}^N)$.

Proof We introduce the mapping T defined by

$$Tx(t) = a(t) + \lambda \int_0^t k_1(t, s) f_1(s, x(s))\, ds \int_0^t k_2(t, s) f_2(s, x(s))\, ds, \;\; x \in X, \; t \in [0, T].$$

We consider several steps for the proof.

Step 1. The operator T maps X into itself. Let $x \in X$, let $t_1, t_2 \in [0, T]$ be such that $t_1 < t_2$. After simple manipulation, we obtain

$$\begin{aligned} &|Tx(t_2) - Tx(t_1)| \\ &\le |a(t_2) - a(t_1)| + \lambda K \Big(\int_0^{t_2} |k_2(t_2, s) - k_2(t_1, s)| m_2(s)\, ds + \int_{t_1}^{t_2} k_2(t_1, s) m_2(s)\, ds \Big) \\ &\quad \times \lambda K \Big(\int_0^{t_2} |k_1(t_2, s) - k_1(t_1, s)| m_1(s)\, ds + \int_{t_1}^{t_2} k_1(t_1, s) m_1(s)\, ds \Big). \end{aligned}$$

Using the dominated convergence theorem and assumptions (i)–(viii), we obtain

$$\lim_{|t_2 - t_1| \to 0} |Tx(t_2) - Tx(t_1)| = 0,$$

which implies the continuity of Tx in $[0, T]$. This proves that $T : X \to X$.

Step 2. T is an (α, ψ)-contraction. Let $\alpha : X \times X \to \mathbb{R}$ be the function defined by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x(t) \preceq_{\mathbb{R}^N} y(t), \; t \in [0, T], \\ 0, & \text{otherwise.} \end{cases}$$

Consider the function $\psi : [0, \infty) \to [0, \infty)$ defined by

$$\psi(t) = \lambda KLT(K_1 + K_2)\, t, \;\; t \ge 0.$$

It is easy to show that $\psi \in \Psi$. We shall prove that T is an (α, ψ)-contraction; that is,

$$\alpha(x, y) d(Tx, Ty) \le \psi(d(x, y)), \quad (x, y) \in X \times X.$$

Let $x, y \in X$. If the condition $x(t) \preceq_{\mathbb{R}^N} y(t)$ is not satisfied, then the above inequality holds immediately. So we can suppose that $x(t) \preceq_{\mathbb{R}^N} y(t)$, for all $t \in [0, T]$. In this case, for all $t \in [0, T]$, we have

$$
\begin{aligned}
&|Tx(t) - Ty(t)| \\
&\leq \lambda \int_0^t k_1(t,s)|f_1(s,x(s))|\,ds \int_0^t k_2(t,s)|f_2(s,x(s)) - f_2(s,y(s))|\,ds \\
&\quad + \lambda \int_0^t k_2(t,s)|f_2(s,y(s))|\,ds \int_0^t k_1(t,s)|f_1(s,x(s)) - f_1(s,y(s))|\,ds \\
&\leq \lambda KL\Big(\int_0^t k_2(t,s)|x(s) - y(s)|\,ds + \int_0^t k_1(t,s)|x(s) - y(s)|\,ds\Big) \\
&\leq \lambda KLT(K_1 + K_2)d(x,y) = \psi(d(x,y)).
\end{aligned}
$$

Then T is an (α, ψ)-contraction.

Step 3. $\alpha(T^n x_0, T^{n+1} x_0) = 1$, $n \in \mathbb{N}$. From (viii), we have $\alpha(x_0, Tx_0) = 1$. Then our claim holds for $n = 0$. On the other hand, from condition (v), we have

$$\alpha(x, y) = 1 \Longrightarrow \alpha(Tx, Ty) = 1, \quad (x, y) \in X \times X.$$

Then by induction, we obtain easily our claim.

Step 4. Convergence of the Picard sequence $\{T^n x_0\}$. Using Theorem 3.3, we obtain the existence of $x^* \in X$ such that the Picard sequence $\{T^n x_0\}$ converges to x^* with respect to the metric d. Then from the previous step, we obtain

$$\alpha(T^n x_0, x^*) = 1, \quad n \in \mathbb{N}.$$

Step 5. Existence of a solution. Now, we can apply Theorem 3.3 to deduce that x^* is a fixed point of T; that is, $x^* \in X$ is a solution to the integral Eq. (3.37).

Step 6. Uniqueness of the solution. Let us consider an arbitrary pair $(x, y) \in X \times X$ given by

$$x(t) = (x_1(t), x_2(t), \ldots, x_N(t)), \quad y(t) = (y_1(t), y_2(t), \ldots, y_N(t)), \quad t \in [0, T].$$

For every $i = 1, 2, \ldots, N$, let

$$z_i(t) = \max\{x_i(t), y_i(t)\}, \ t \in [0, T].$$

Clearly, we have $\alpha(x, z) = \alpha(y, z) = 1$. Therefore, the uniqueness follows immediately from Theorem 3.4.

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Chapter 4
Cyclic Contractions: An Improvement Result

In this chapter, we give an improvement fixed point result for cyclic contractions by weakening the closure assumption that is usually supposed in the literature. As applications, we discuss the existence of solutions to certain systems of functional equations. The main reference of this chapter is the paper [4].

4.1 Introduction

In [6], Kirk et al. proved the following result.

Theorem 4.1 *Let (X, d) be a complete metric space and $(A_i)_{i=1}^p$ be a finite number of nonempty closed subsets of X. Let $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ be a given mapping. Suppose that the following conditions are satisfied:*

(i) $TA_i \subseteq A_{i+1}$, for $i = 1, 2, \ldots, p$, with $A_{p+1} = A_1$.
(ii) The mapping T satisfies a cyclic contraction; i.e., there exists some constant $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y), \quad (x, y) \in A_i \times A_{i+1},\ i = 1, 2, \ldots, p.$$

Then

(I) $\bigcap_{i=1}^p A_i$ is nonempty.
(II) T has a unique fixed point in $\bigcap_{i=1}^p A_i$.

Observe that Banach contraction principle follows immediately from Theorem 4.1 by taking $A_i = X$, for every $i = 1, 2, \ldots, p$. Observe also that (II) is an immediate consequence of (I) and Banach contraction principle. More precisely, if $\bigcap_{i=1}^p A_i$ is nonempty, from (i), we have $T\left(\bigcap_{i=1}^p A_i\right) \subseteq \bigcap_{i=1}^p A_i$. Moreover, from (ii), we have

$$d(Tx, Ty) \leq kd(x, y), \quad (x, y) \in \bigcap_{i=1}^p A_i \times \bigcap_{i=1}^p A_i.$$

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Since A_i is closed for every $i = 1, 2, \ldots, p$, and (X, d) is complete, then $\left(\bigcap_{i=1}^p A_i, d\right)$ is a complete metric space. Therefore, applying Banach contraction principle to the mapping $T : \bigcap_{i=1}^p A_i \to \bigcap_{i=1}^p A_i$, (II) follows.

In this chapter, we address the following question: Is it possible to obtain (I) and (II) of Theorem 4.1 without supposing that A_i is closed for every $i = 1, 2, \ldots, p$? Observe that in this case, if $\bigcap_{i=1}^p A_i$ is nonempty, we cannot obtain (II) via Banach contraction principle applied to the mapping $T : \bigcap_{i=1}^p A_i \to \bigcap_{i=1}^p A_i$, since $\bigcap_{i=1}^p A_i$ is not necessarily complete. We obtain an affirmative answer to the addressed question by supposing only that A_1 is closed. Moreover, we consider mappings satisfying a φ-contraction, which is a contraction involving a (c)-comparison function $\varphi : [0, \infty) \to [0, \infty)$. An example is provided to illustrate the obtained result. As applications, we give existence results to certain systems of functional equations.

Recall that a functions $\varphi : [0, \infty) \to [0, \infty)$ is said to be a (c)-comparison function if it satisfies the following conditions:

(φ_1) φ is a nondecreasing function.
(φ_2) There exists $k_0 = 1, 2, \ldots$ and $\lambda \in (0, 1)$ such that

$$\varphi^{k+1}(t) \leq \lambda \varphi^k(t) + v_k, \quad k = k_0, k_0 + 1, \ldots,$$

for all $t > 0$, where $\sum_{k=0}^{\infty} v_k$ is a convergent series of nonnegative terms. Here, φ^n denotes the nth iterate of φ.

We have the following properties (see [8]).

Lemma 4.1 *Let $\varphi : [0, \infty) \to [0, \infty)$ be a (c)-comparison function. Then*

(i) $\varphi(t) < t$, $t > 0$.
(ii) *φ is continuous at* 0.
(iii) $\varphi(0) = 0$.
(iv) $\displaystyle\sum_{n=0}^{\infty} \varphi^n(t) < \infty$, $t > 0$.

In [8], the authors extended Theorem 4.1 to the class of cyclic φ-contractions, where $\varphi : [0, \infty) \to [0, \infty)$ is a (c)-comparison function. Moreover, they provided error estimates for approximating the fixed point. The obtained result in [8] is the following.

Theorem 4.2 *Let (X, d) be a complete metric space and $(A_i)_{i=1}^p$ be a finite number of nonempty closed subsets of X. Let $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ be a given mapping. Suppose that the following conditions are satisfied:*

(i) *$TA_i \subseteq A_{i+1}$, for $i = 1, 2, \ldots, p$, with $A_{p+1} = A_1$.*
(ii) *The mapping T satisfies a cyclic φ-contraction; i.e., there exists a (c)-comparison function $\varphi : [0, \infty) \to [0, \infty)$ such that*

$$d(Tx, Ty) \leq \varphi(d(x, y)), \quad (x, y) \in A_i \times A_{i+1}, \ i = 1, 2, \ldots, p.$$

Then

(I) *T has a unique fixed point $x^* \in \bigcap_{i=1}^p A_i$. For any $x_0 \in \bigcup_{i=1}^p A_i$, the Picard sequence $\{T^n x_0\}$ converges to x^*.*

(II) *The following estimates hold:*

$$d(T^n x_0, x^*) \le s\left(\varphi^n(d(x_0, Tx_0))\right), \ n = 1, 2, \ldots,$$
$$d(T^n x_0, x^*) \le s\left(\varphi(d(T^n x_0, T^{n+1} x_0))\right), \ n = 1, 2, \ldots.$$

(III) *For any $x \in \bigcup_{i=1}^p A_i$,*

$$d(x, x^*) \le s(d(x, Tx)),$$

where $s(t) = \sum_{k=0}^{\infty} \varphi^k(t)$, $t \ge 0$.

In this chapter, we shall prove that the results of Theorem 4.2 hold true by assuming only that A_1 is closed. We present also an example where our result can be used; however, Theorem 4.2 cannot be applied.

Remark 4.1 Theorem 4.2 is a cyclical-type generalization of the following ordinary fixed point theorem.

Theorem 4.3 *Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exists a (c)-comparison function $\varphi : [0, \infty) \to [0, \infty)$ such that*

$$d(Tx, Ty) \le \varphi(d(x, y)), \quad (x, y) \in X \times X.$$

Then

(I) *T has a unique fixed point $x^* \in X$. For any $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to x^*.*

(II) *The following estimates hold:*

$$d(T^n x_0, x^*) \le s\left(\varphi^n(d(x_0, Tx_0))\right), \ n = 1, 2, \ldots,$$
$$d(T^n x_0, x^*) \le s\left(d(T^n x_0, T^{n+1} x_0)\right), \ n = 1, 2, \ldots.$$

(III) *For any $x \in X$,*

$$d(x, x^*) \le s(d(x, Tx)).$$

Note that in [11], the author claimed that Theorems 4.2 and 4.3 are equivalent. In fact, he claimed that by applying Theorem 4.3 to the mapping $T : \bigcap_{i=1}^p A_i \to \bigcap_{i=1}^p A_i$, we retrieve the results in Theorem 4.2. Obviously, such claim is not true. At first, in Theorem 4.2(I), for any $x_0 \in \bigcup_{i=1}^p A_i$, the Picard sequence $\{T^n x_0\}$ converges to the fixed point of T. However, by applying Theorem 4.3 with $X = \bigcap_{i=1}^p A_i$, the convergence holds only for $x_0 \in \bigcap_{i=1}^p A_i$. The same remark holds for the estimates given by (II) and (III) in Theorem 4.2.

For other results related to cyclic and generalized cyclic contractions, we refer the reader to [1–3, 5, 7, 9, 10, 12] and the references therein.

4.2 Main Result

We deal with the following problem: Find $x \in X$ such that

$$\begin{cases} Tx = x, \\ x \in \bigcap_{i=1}^{p} A_i, \end{cases} \tag{4.1}$$

where (X, d) is a complete metric space, A_1 is a nonempty closed subset of X, A_i, $i = 2, 3, \ldots, p$ are arbitrary nonempty subsets of X (nonnecessarily closed), and $T : X \to X$ is a mapping satisfying a cyclic φ-contraction.

We have the following result.

Theorem 4.4 *Let (X, d) be a complete metric space. Let $(A_i)_{i=1}^{p}$ be a finite number of nonempty subsets of X. Let $T : X \to X$ be a given mapping. Suppose that the following conditions are satisfied:*

(i) A_1 is closed.
(ii) $TA_i \subseteq A_{i+1}$ for all $i = 1, 2, \ldots, p$ with $A_{p+1} = A_1$.
(iii) There exists a (c)-comparison function $\varphi : [0, \infty) \to [0, \infty)$ such that

$$d(Tx, Ty) \leq \varphi(d(x, y)), \quad (x, y) \in A_i \times A_{i+1},\ i = 1, 2, \ldots, p.$$

Then

(I) For any $x_0 \in \bigcup_{i=1}^{p} A_i$, the Picard sequence $\{T^n x_0\}$ converges to $x^* \in X$, the unique solution to (4.1).
(II) The following estimates hold:

$$d(T^n x_0, x^*) \leq s\left(\varphi^n(d(x_0, Tx_0))\right),\ n \in \mathbb{N},$$
$$d(T^n x_0, x^*) \leq s\left(d(T^n x_0, T^{n+1} x_0)\right),\ n \in \mathbb{N}.$$

(III) For any $x \in \bigcup_{i=1}^{p} A_i$, we have

$$d(x, x^*) \leq s(d(x, Tx)).$$

Proof Let $x_0 \in \bigcup_{i=1}^{p} A_i$ be an arbitrary point. Without restriction of the generality, we may assume that $x_0 \in A_1$. Let $\{x_n\} \subset X$ be the Picard sequence defined by

$$x_n = T^n x_0, \quad n \in \mathbb{N}.$$

We argue exactly as in the proof of Theorem 4.2 in [8] to obtain that

$$d(x_n, x_{n+m}) \leq \sum_{i=n}^{\infty} \varphi^i(d(x_0, x_1)),$$

for any $(n, m) \in \mathbb{N} \times \mathbb{N}\backslash\{0\}$. From Lemma 4.1, the series $\sum_{i=0}^{\infty} \varphi^i(d(x_0, x_1))$ is convergent, which implies that

$$\sum_{i=n}^{\infty} \varphi^i(d(x_0, x_1)) \to 0 \text{ as } n \to \infty.$$

As consequence, $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, there exists some $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{4.2}$$

On the other hand, from (ii), we obtain

$$x_{sp+r-1} \in A_r, \quad r \in \{1, 2, \ldots, p\}, \ \ s \in \mathbb{N}. \tag{4.3}$$

Using (4.3), we obtain

$$\{x_{np}\} \subset A_1.$$

Since A_1 is closed, it follows from (4.2) that

$$x^* \in A_1. \tag{4.4}$$

Again, from (4.3), we know that

$$\left\{x_{np+1}\right\} \subset A_2. \tag{4.5}$$

Now, (4.4) and (4.5) yield

$$(x^*, x_{np+1}) \in A_1 \times A_2, \quad n \in \mathbb{N}.$$

Then by (iii), we obtain

$$d(Tx^*, x_{np+2}) = d(Tx^*, Tx_{np+1}) \leq \varphi(d(x^*, x_{np+1})), \quad n \in \mathbb{N}.$$

Note that from Lemma 4.1, we know that

$$\lim_{t\to 0^+} \varphi(t) = 0.$$

Using this fact, and passing to the limit as $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} d(x_{np+2}, Tx^*) = 0. \tag{4.6}$$

Now, it follows immediately from (4.2), (4.6) and the uniqueness of the limit that

$$x^* = Tx^*. \tag{4.7}$$

Next, from (ii) and (4.7), we obtain

$$x^* \in \bigcap_{i=1}^{p} A_i. \tag{4.8}$$

Then from (4.7) and (4.8), we deduce that $x^* \in X$ is a solution to (4.1). In order to prove the uniqueness of solutions to (4.1), suppose that $y^* \in X$ is a solution to (4.1) with $d(x^*, y^*) > 0$. Using (iii), we get

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq \varphi(d(x^*, y^*)).$$

Since $d(x^*, y^*) > 0$, using Lemma 4.1, we have

$$\varphi(d(x^*, y^*)) < d(x^*, y^*).$$

Then

$$d(x^*, y^*) < d(x^*, y^*),$$

which is a contradiction. As consequence, $d(x^*, y^*) = 0$, i.e., $x^* = y^*$. This proves that $x^* \in X$ is the unique solution to (4.1). Therefore, (I) is proved. The estimates given by (II) and (III) follow using exactly the same arguments as in the proof of Theorem 4.2 in [8].

Using the same argument as that used in the proof of Theorem 4.4, we obtain the following result.

Theorem 4.5 *Let (X, d) be a complete metric space. Let $(A_i)_{i=1}^p$ be a finite number of nonempty subsets of X. Let $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ be a given mapping. Suppose that the following conditions are satisfied:*

(i) *A_1 is closed.*
(ii) *$TA_i \subseteq A_{i+1}$ for all $i = 1, 2, \ldots, p$ with $A_{p+1} = A_1$.*
(iii) *There exists a (c)-comparison function $\varphi : [0, \infty) \to [0, \infty)$ such that*

$$d(Tx, Ty) \leq \varphi(d(x, y)), \quad (x, y) \in A_i \times A_{i+1},\ i = 1, 2, \ldots, p.$$

Then

(I) T *has a unique fixed point* $x^* \in \bigcap_{i=1}^{p} A_i$*. For any* $x_0 \in \bigcup_{i=1}^{p} A_i$*, the Picard sequence* $\{T^n x_0\}$ *converges to* x^**.*

(II) The following estimates hold:

$$d(T^n x_0, x^*) \le s\left(\varphi^n(d(x_0, Tx_0))\right), \quad n \in \mathbb{N},$$
$$d(T^n x_0, x^*) \le s\left(\varphi(d(T^n x_0, T^{n+1} x_0))\right), \quad n \in \mathbb{N}.$$

(III) For any $x \in \bigcup_{i=1}^{p} A_i$*,*

$$d(x, x^*) \le s(d(x, Tx)),$$

where $s(t) = \sum_{k=0}^{\infty} \varphi^k(t)$*,* $t \ge 0$*.*

The following simple example shows that Theorem 4.5 is more general than Theorem 4.2.

Example 4.1 Let $X = \mathbb{R}$. The set X is equipped with the standard metric

$$d(x, y) = |x - y|, \quad (x, y) \in X \times X. \tag{4.9}$$

Then (X, d) is a complete metric space. Let us consider the two subsets $A_1 = [0, 2]$ and $A_2 = (1, \infty)$. Define the mapping $T : A_1 \cup A_2 = [0, \infty) \to A_1 \cup A_2$ by

$$Tx = 2, \quad x \in A_1 \cup A_2.$$

Clearly, we have

$$TA_1 = \{2\} \subset A_2 \quad \text{and} \quad TA_2 = \{2\} \subset A_1.$$

Moreover, for any (c)-comparison function $\varphi : [0, \infty) \to [0, \infty)$, we have

$$d(Tx, Ty) \le \varphi(d(x, y)), \quad (x, y) \in A_1 \times A_2.$$

Therefore, by Theorem 4.5, T has a unique fixed point $x^* \in A_1 \cap A_2 = (1, 2]$. In this case, we have $x^* = 2$. Observe that Theorem 4.2 cannot be applied in this case since $A_2 = (1, \infty)$ is an open subset of X.

Note that under the assumptions of Theorem 4.4, the mapping $T : X \to X$ has at least one fixed point in X, which means that a fixed point of T in X is not necessarily unique. But from result (I), the mapping T has a unique fixed point in $\bigcap_{i=1}^{p} A_i$. The following simple example illustrates this fact.

Example 4.2 Let $X = \mathbb{R}$ and d be the metric on X given by (4.9). Let $T : X \to X$ be the mapping defined by

$$Tx = \begin{cases} -1 & \text{if } x < 0, \\ 2 & \text{if } x \geq 0. \end{cases}$$

Let $A_1 = [0, 2]$ and $A_2 = (1, \infty)$. Observe that

$$TA_1 = \{2\} \subset A_2 \quad \text{and} \quad TA_2 = \{2\} \subset A_1.$$

Moreover, for all $(x, y) \in A_1 \times A_2$, we have

$$d(Tx, Ty) = d(2, 2) = 0 \leq \varphi(d(x, y)),$$

for any (c)-comparison function $\varphi : [0, \infty) \to [0, \infty)$. Then all the required conditions of Theorem 4.4 are satisfied. In this case, we observe that $x^* = 2$ is the unique solution to (4.1) with $p = 2$. However, the mapping T has two fixed points in $X = \mathbb{R}$, $x^* = 2$, and $y^* = -1$.

4.3 Applications

Motivated by the suggestion of Kirk et al. [6] "Of course it would even be nicer to have applications," we present in this section some possible applications of the main result of this chapter.

Let (X, d) be a complete metric space, $T : X \to X$ be a given mapping, and $\alpha : X \to \mathbb{R}$ be a given function. We are concerned with the study of the existence of solutions to the following problem: Find $x \in X$ such that

$$\begin{cases} Tx &= x, \\ \alpha(x) &= 0. \end{cases} \tag{4.10}$$

We have the following result.

Theorem 4.6 *Suppose that the following conditions are satisfied:*

(i) α *is lower semi-continuous.*
(ii) There exists some $x_0 \in X$ *such that* $\alpha(x_0) \leq 0$.
(iii) For every $x \in X$*, we have*

$$\alpha(x)\alpha(Tx) \leq 0.$$

(iv) There exists a (c)-comparison function $\varphi : [0, \infty) \to [0, \infty)$ *such that*

$$\alpha(x)\alpha(y) \leq 0 \Longrightarrow d(Tx, Ty) \leq \varphi(d(x, y)).$$

Then (4.10) *has a unique solution.*

Proof Set

$$A_1 = \{x \in X : \alpha(x) \le 0\}$$

and

$$A_2 = \{x \in X : \alpha(x) \ge 0\}.$$

From (ii), the set A_1 is nonempty (since $x_0 \in A_1$). From (iii), we have $TA_1 \subseteq A_2$ and $TA_2 \subseteq A_1$. Moreover, since α is lower semi-continuous, then A_1 is a closed subset of X. Now, from (iv), for every pair of elements $(x, y) \in A_1 \times A_2$, we have

$$d(Tx, Ty) \le \varphi(d(x, y)).$$

Applying Theorem 4.4, we obtain the existence of a unique solution to (4.10).

Remark 4.2 The result of Theorem 4.6 is still valid if we replace condition (i) by (i') α is upper semi-continuous. In this case, we set

$$A_1 = \{x \in X : \alpha(x) \ge 0\}$$

and

$$A_2 = \{x \in X : \alpha(x) \le 0\}.$$

Since α is upper semi-continuous, then A_1 is a closed subset of X.

Next, we consider the problem: Find $x \in X$ such that

$$\begin{cases} Tx &= x, \\ \alpha(x) &= 0, \\ \beta(x) &= 0, \end{cases} \tag{4.11}$$

where $\alpha, \beta : X \to \mathbb{R}$ are given functions.

Theorem 4.7 *Suppose that the following conditions are satisfied:*

(i) *α and β are lower semi-continuous.*
(ii) *There exists some $x_0 \in X$ such that $\alpha(x_0) \le 0$ and $\beta(x_0) \le 0$.*
(iii) *For every $x \in X$, we have*

$$\alpha(x) \le 0,\ \beta(x) \le 0 \Longrightarrow \alpha(Tx) \ge 0,\ \beta(Tx) \ge 0.$$

(iv) *For every $x \in X$, we have*

$$\alpha(x) \ge 0,\ \beta(x) \ge 0 \Longrightarrow \alpha(Tx) \le 0,\ \beta(Tx) \le 0.$$

(v) *There exists a (c)-comparison function $\varphi : [0, \infty) \to [0, \infty)$ such that*

$$\alpha(x) \le 0,\ \beta(x) \le 0,\ \alpha(y) \ge 0,\ \beta(y) \ge 0 \Longrightarrow d(Tx, Ty) \le \varphi(d(x, y)).$$

Then (4.11) *has a unique solution.*

Proof We argue as in the proof of Theorem 4.6 by considering the sets

$$A_1 = \{x \in X : \alpha(x) \leq 0,\ \beta(x) \leq 0\}$$

and

$$A_2 = \{x \in X : \alpha(x) \geq 0,\ \beta(x) \geq 0\}.$$

Remark 4.3 The result of Theorem 4.7 is still valid if we replace condition (i) by (i') α and β are upper semi-continuous.

Theorem 4.8 *Suppose that the following conditions are satisfied:*

(i) *α is lower semi-continuous and β is upper semi-continuous.*
(ii) *There exists some $x_0 \in X$ such that $\alpha(x_0) \leq 0$ and $\beta(x_0) \geq 0$.*
(iii) *For every $x \in X$, we have*

$$\alpha(x) \leq 0,\ \beta(x) \geq 0 \Longrightarrow \alpha(Tx) \geq 0,\ \beta(Tx) \leq 0.$$

(iv) *For every $x \in X$, we have*

$$\alpha(x) \geq 0,\ \beta(x) \leq 0 \Longrightarrow \alpha(Tx) \leq 0,\ \beta(Tx) \geq 0.$$

(v) *There exists a (c)-comparison function $\varphi : [0, \infty) \to [0, \infty)$ such that*

$$\alpha(x) \leq 0,\ \beta(x) \geq 0,\ \alpha(y) \geq 0,\ \beta(y) \leq 0 \Longrightarrow d(Tx, Ty) \leq \varphi(d(x, y)).$$

Then (4.11) *has a unique solution.*

Proof We argue as in the proof of Theorem 4.6 by considering the sets

$$A_1 = \{x \in X : \alpha(x) \leq 0,\ \beta(x) \geq 0\}$$

and

$$sA_2 = \{x \in X : \alpha(x) \geq 0,\ \beta(x) \leq 0\}.$$

Note that since α is lower semi-continuous and β is upper semi-continuous, then A_1 is a closed subset of X.

We end this chapter with the following example.

Example 4.3 Let $X = [-1, 1]$ be the set endowed with the standard metric

$$d(x, y) = |x - y|, \quad (x, y) \in X \times X.$$

Let us consider the function $\alpha : X \to \mathbb{R}$ defined by

$$\alpha(x) = \begin{cases} -3 & \text{if } x = -1, \\ x(x^2+1) & \text{if } x \in (-1, 1]. \end{cases}$$

Clearly, α is a lower semi-continuous function, since

$$\alpha(-1) = -3 \le \liminf_{x \to -1} \alpha(x) = -2.$$

Let us consider the mapping $T : X \to X$ defined by

$$Tx = -\frac{x}{3}, \quad x \in X.$$

For $x_0 = -1$, we have $\alpha(x_0) = -3 < 0$.

For $x = -1$, we have

$$\alpha(x)\alpha(Tx) = \alpha(-1)\alpha(T(-1)) = -\left(\frac{1}{9} + 1\right) < 0.$$

For $x \in (-1, 1]$, we have

$$\alpha(x)\alpha(Tx) = -\frac{x^2}{3}(x^2+1)\left(\frac{x^2}{9} + 1\right) \le 0.$$

Moreover, for all $(x, y) \in X \times X$, we have

$$d(Tx, Ty) \le \varphi(d(x, y)),$$

where $\varphi(t) = \frac{t}{3}$, $t \ge 0$. Therefore, all the assumptions of Theorem 4.6 are satisfied. Then there is a unique $x^* \in X$ such that

$$\begin{cases} Tx^* &= x^*, \\ \varphi(x^*) &= 0. \end{cases}$$

Obviously, in this example, we have $x^* = 0$.

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Chapter 5
The Class of JS-Contractions in Branciari Metric Spaces

Banach contraction principle has been generalized in many ways over the years. In some generalizations, the contraction is weakened; see [3, 6, 12, 16, 20, 21, 24, 30] and others. In other generalizations, the topology is weakened; see [1, 4, 5, 8, 9, 11, 13, 14, 22, 23, 27–29] and others. In [18], Nadler extended Banach fixed point theorem from single-valued maps to set-valued maps. Other fixed point results for set-valued maps can be found in [2, 7, 15, 17, 19] and references therein. In 2000, Branciari [4] introduced the concept of generalized metric spaces, where the triangle inequality is replaced by the inequality $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for all pairwise distinct points $x, y, u, v \in X$. Various fixed point results were established on such spaces; see, for example [1, 8, 13, 14, 22, 23, 28] and references therein. In this chapter, we present a recent generalization of Banach contraction principle on the setting of Branciari metric spaces, which is due to Jleli and Samet [10].

5.1 Main Results

We denote by Θ the set of functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:
(Θ_1) θ is nondecreasing;
(Θ_2) For each sequence $\{t_n\} \subset (0, \infty)$, we have

$$\lim_{n\to\infty} \theta(t_n) = 1 \Longleftrightarrow \lim_{n\to\infty} t_n = 0^+;$$

(Θ_3) There exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \ell$.

Before stating and proving the main result of this chapter, we recall the following definitions introduced in [4].

P. Agarwal et al., *Fixed Point Theory in Metric Spaces*,
https://doi.org/10.1007/978-981-13-2913-5_5

Definition 5.1 Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, one has
(i) $d(x, y) = 0 \Longleftrightarrow x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.
Then (X, d) is called a generalized metric space (or for short g.m.s).

Definition 5.2 Let (X, d) be a g.m.s, $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ is convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$. We denote this by $x_n \to x$.

Definition 5.3 Let (X, d) be a g.m.s and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Definition 5.4 Let (X, d) be a g.m.s. We say that (X, d) is complete if and only if every Cauchy sequence in X converges to some element in X.

The following result was established in [10] (Lemma 1.10) (see also Kirk and Shahzad [13]).

Lemma 5.1 *Let (X, d) be a g.m.s, $\{x_n\}$ be a Cauchy sequence in (X, d), and $x, y \in X$. Suppose that there exist a positive integer N such that*

(i) $x_n \neq x_m$, *for all* $n, m > N$;
(ii) x_n *and* x *are distinct points in* X, *for all* $n > N$;
(iii) x_n *and* y *are distinct points in* X, *for all* $n > N$;
(iv) $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x_n, y)$.

Then we have $x = y$.

For more results on the topological properties of g.m.s, we refer to Suzuki [25].
The main result of this chapter is giving by the following theorem.

Theorem 5.1 *Let (X, d) be a complete g.m.s and $T : X \to X$ be a given map. Suppose that there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that*

$$(x, y) \in X \times X,\ d(Tx, Ty) \neq 0 \Longrightarrow \theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k. \tag{5.1}$$

Then T has a unique fixed point.

Proof Let $x \in X$ be an arbitrary point in X. If for some $p \in \mathbb{N}$, we have $T^p x = T^{p+1} x$, then $T^p x$ will be a fixed point of T. So, without restriction of the generality, we can suppose that $d(T^n x, T^{n+1} x) > 0$ for all $n \in \mathbb{N}$. Now, from (5.1), for all $n \in \mathbb{N}$, we have

$$\theta(d(T^n x, T^{n+1} x)) \leq [\theta(d(T^{n-1} x, T^n x))]^k \leq \cdots \leq [\theta(d(x, Tx))]^{k^n}.$$

Thus, we have

$$1 \le \theta(d(T^n x, T^{n+1}x)) \le [\theta(d(x, Tx))]^{k^n}, \text{ for all } n \in \mathbb{N}. \tag{5.2}$$

Letting $n \to \infty$ in (5.2), we obtain

$$\theta(d(T^n x, T^{n+1}x)) \to 1 \text{ as } n \to \infty,$$

which implies from (Θ_2) that

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0. \tag{5.3}$$

From (Θ_3), there exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta\left(d\left(T^n x, T^{n+1}x\right)\right) - 1}{\left[d(T^n x, T^{n+1}x)\right]^r} = \ell.$$

Suppose that $\ell < \infty$. In this case, let $B = \ell/2 > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left|\frac{\theta(d(T^n x, T^{n+1}x)) - 1}{[d(T^n x, T^{n+1}x)]^r} - \ell\right| \le B, \text{ for all } n \ge n_0.$$

This implies that

$$\frac{\theta(d(T^n x, T^{n+1}x)) - 1}{[d(T^n x, T^{n+1}x)]^r} \ge \ell - B = B, \text{ for all } n \ge n_0.$$

Then,

$$n[d(T^n x, T^{n+1}x)]^r \le An[\theta(d(T^n x, T^{n+1}x)) - 1], \text{ for all } n \ge n_0,$$

where $A = 1/B$.

Suppose now that $\ell = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta(d(T^n x, T^{n+1}x)) - 1}{[d(T^n x, T^{n+1}x)]^r} \ge B, \text{ for all } n \ge n_0.$$

This implies that

$$n[d(T^n x, T^{n+1}x)]^r \le An[\theta(d(T^n x, T^{n+1}x)) - 1], \text{ for all } n \ge n_0,$$

where $A = 1/B$.

Thus, in all cases, there exists $A > 0$ and $n_0 \in \mathbb{N}$ such that

$$n[d(T^n x, T^{n+1}x)]^r \leq An[\theta(d(T^n x, T^{n+1}x)) - 1], \text{ for all } n \geq n_0.$$

Using (5.2), we obtain

$$n[d(T^n x, T^{n+1}x)]^r \leq An\left([\theta(d(x, Tx))]^{k^n} - 1\right), \text{ for all } n \geq n_0.$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n[d(T^n x, T^{n+1}x)]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$d(T^n x, T^{n+1}x) \leq \frac{1}{n^{1/r}}, \text{ for all } n \geq n_1. \tag{5.4}$$

Now, we shall prove that T has a periodic point. Suppose that it is not the case, then $T^n x \neq T^m x$ for every $n, m \in \mathbb{N}$ such that $n \neq m$. Using (5.1), we obtain

$$\theta(d(T^n x, T^{n+2}x)) \leq [\theta(d(T^{n-1}x, T^{n+1}x))]^k \leq \cdots \leq [\theta(d(x, T^2x))]^{k^n}.$$

Letting $n \to \infty$ in the above inequality and using (Θ_2), we obtain

$$\lim_{n\to\infty} d(T^n x, T^{n+2}x) = 0. \tag{5.5}$$

Similarly, from (Θ_3), there exists $n_2 \in \mathbb{N}$ such that

$$d(T^n x, T^{n+2}x) \leq \frac{1}{n^{1/r}}, \text{ for all } n \geq n_2. \tag{5.6}$$

Let $N = \max\{n_0, n_1\}$. We consider two cases.
Case 1. If $m > 2$ is odd, then writing $m = 2L+1$, $L \geq 1$, using (5.4), for all $n \geq N$, we obtain that

$$\begin{aligned}
d(T^n x, T^{n+m}x) &\leq d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x) + \cdots + d(T^{n+2L}x, T^{n+2L+1}x)\\
&\leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}}\\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}.
\end{aligned}$$

Case 2. If $m > 2$ is even, then writing $m = 2L$, $L \geq 2$, using (5.4) and (5.6), for all $n \geq N$, we obtain

$$\begin{aligned} d(T^n x, T^{n+m} x) &\le d(T^n x, T^{n+2} x) + d(T^{n+2} x, T^{n+3} x) + \cdots + d(T^{n+2L-1} x, T^{n+2L} x) \\ &\le \frac{1}{n^{1/r}} + \frac{1}{(n+2)^{1/r}} + \cdots + \frac{1}{(n+2L-1)^{1/r}} \\ &\le \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}. \end{aligned}$$

Thus, combining all the cases we have

$$d(T^n x, T^{n+m} x) \le \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}, \text{ for all } n \ge N,\ m \in \mathbb{N}.$$

From the convergence of the series $\sum_i \frac{1}{i^{1/r}}$ (since $1/r > 1$), we deduce that $\{T^n x\}$ is a Cauchy sequence. Since (X, d) is complete, there is $z \in X$ such that $T^n x \to z$. On the other hand, observe that T is continuous, indeed, if $Tx \neq Ty$, then from (5.1) we have

$$\ln[\theta(d(Tx, Ty))] \le k \ \ln[\theta(d(x, y))] \le \ln[\theta(d(x, y))],$$

which implies from (Θ_1) that

$$d(Tx, Ty) \le d(x, y), \text{ for all } x, y \in X.$$

Further, for all $n \in \mathbb{N}$, we have

$$d(T^{n+1} x, Tz) \le d(T^n x, z).$$

Letting $n \to \infty$ in the above inequality, we obtain $T^{n+1} x \to Tz$. From Lemma 5.1, we obtain $z = Tz$, which is a contradiction with the assumption: T does not have a periodic point. Thus, T has a periodic point say z of period q. Suppose that the set of fixed points of T is empty. Then, we have

$$q > 1 \text{ and } d(z, Tz) > 0.$$

Using (5.1), we obtain

$$\theta(d(z, Tz)) = \theta(d(T^n z, T^{n+1} z)) \le [\theta(d(z, Tz))]^{k^n} < \theta(d(z, Tz)),$$

which is a contradiction. Thus, the set of fixed points of T is nonempty, i.e., T has at least one fixed point. Now, suppose that $z, u \in X$ are two fixed points of T such that $d(z, u) = d(Tz, Tu) > 0$. Using (5.1), we get

$$\theta(d(z, u)) = \theta(d(Tz, Tu)) \le [\theta(d(z, u))]^k < \theta(d(z, u)),$$

which is a contradiction. Then, we have one and only one fixed point.

5.2 Particular Cases

Since a metric space is a g.m.s, from Theorem 5.1, we deduce immediately the following result.

Corollary 5.1 *Let (X, d) be a complete metric space and $T : X \to X$ be a given map. Suppose that there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that*

$$(x, y) \in X \times X,\ d(Tx, Ty) \neq 0 \Longrightarrow \theta(d(Tx, Ty)) \leq [\theta(d(x, y)]^k.$$

Then T has a unique fixed point.

Remark 5.1 Observe that Banach contraction principle follows immediately from Corollary 5.1. Indeed, if T is a contraction, i.e., there exists $\lambda \in (0, 1)$ such that

$$d(Tx, Ty) \leq \lambda\, d(x, y), \quad (x, y) \in X \times X,$$

then, we have

$$e^{d(Tx,Ty)} \leq [e^{d(x,y)}]^k, \quad (x, y) \in X \times X.$$

Clearly, the function $\theta : (0, \infty) \to (1, \infty)$ defined by $\theta(t) := e^{\sqrt{t}}$ belongs to Θ. So, the existence and uniqueness of the fixed point follows from Corollary 5.1.

In the following example (inspired by [30]), we show that Corollary 5.1 is a real generalization of Banach contraction principle.

Example 5.1 Let X be the set defined by

$$X := \{\tau_n : n \in \mathbb{N}\},$$

where

$$\tau_n := \frac{n(n+1)}{2}, \text{ for all } n \in \mathbb{N}.$$

We endow X with the standard metric d given by $d(x, y) := |x - y|$ for all $x, y \in X$. Let $T : X \to X$ be the map defined by

$$T\tau_1 = \tau_1,\ T\tau_n = \tau_{n-1},\ \text{for all } n \geq 2.$$

Clearly, T is not a contraction. Indeed, we can check easily that

$$\lim_{n\to\infty} \frac{d(T\tau_n, T\tau_1)}{d(\tau_n, \tau_1)} = 1.$$

Now, consider the function $\theta : (0, \infty) \to (1, \infty)$ defined by

$$\theta(t) := e^{\sqrt{te^t}}.$$

It is not difficult to show that $\theta \in \Theta$. We shall prove that T satisfies (5.1), i.e.,

$$d(T\tau_n, T\tau_m) \neq 0 \Longrightarrow e^{\sqrt{d(T\tau_n,T\tau_m)e^{d(T\tau_n,T\tau_m)}}} \leq e^{k\sqrt{d(\tau_n,\tau_m)e^{d(\tau_n,\tau_m)}}},$$

for some $k \in (0, 1)$. The above condition is equivalent to

$$d(T\tau_n, T\tau_m) \neq 0 \Longrightarrow d(T\tau_n, T\tau_m)e^{d(T\tau_n,T\tau_m)} \leq k^2 d(\tau_n, \tau_m)e^{d(\tau_n,\tau_m)}.$$

So, we have to check that

$$d(T\tau_n, T\tau_m) \neq 0 \Longrightarrow \frac{d(T\tau_n, T\tau_m)e^{d(T\tau_n,T\tau_m)-d(\tau_n,\tau_m)}}{d(\tau_n, \tau_m)} \leq k^2, \tag{5.7}$$

for some $k \in (0, 1)$. We consider two cases.
Case 1. If $n = 1$ and $m > 2$. In this case, we have

$$\begin{aligned}
&\frac{d(T\tau_1, T\tau_m)e^{d(T\tau_1,T\tau_m)-d(\tau_1,\tau_m)}}{d(\tau_1, \tau_m)} \\
&= \frac{m^2 - m - 2}{m^2 + m - 2}e^{-m} \\
&\leq e^{-1}.
\end{aligned}$$

Case 2. If $m > n > 1$. In this case, we have

$$\begin{aligned}
&\frac{d(T\tau_m, T\tau_n)e^{d(T\tau_m,T\tau_n)-d(\tau_m,\tau_n)}}{d(\tau_m, \tau_n)} \\
&= \frac{m + n - 1}{m + n + 1}e^{n-m} \\
&\leq e^{-1}.
\end{aligned}$$

Thus, (5.7) is satisfied with $k = e^{-1/2}$. Theorem 5.1 (or Corollary 5.1) implies that T has a unique fixed point. In this example, τ_1 is the unique fixed point of T.

Note that Θ contains a large class of functions. For example, if

$$\theta(t) := 2 - \frac{2}{\pi} \arctan\left(\frac{1}{t^\alpha}\right), \ 0 < \alpha < 1, \ t > 0,$$

we obtain from Theorem 5.1 the following result.

Corollary 5.2 *Let* (X, d) *be a complete g.m.s and* $T : X \to X$ *be a given map. Suppose that there exist* $\alpha, k \in (0, 1)$ *such that*

$$2-\frac{2}{\pi}\arctan\left(\frac{1}{[d(Tx,Ty)]^{\alpha}}\right)\le\left[2-\frac{2}{\pi}\arctan\left(\frac{1}{[d(x,y)]^{\alpha}}\right)\right]^{k},\quad (x,y)\in X\times X,\ Tx\neq Ty.$$

Then T has a unique fixed point.

For other related results, we refer to Suzuki [26].

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Chapter 6
Implicit Contractions on a Set Equipped with Two Metrics

Several classical fixed point theorems have been unified by considering general contractions expressed via an implicit inequality, see, for examples, Turinici [15], Popa [8, 9], Berinde [2], and references therein. In this chapter, we consider a class of mappings defined on a set equipped with two metrics and satisfying an implicit contraction involving two functions $F : [0, \infty)^6 \to \mathbb{R}$ and $\alpha : X \times X \to \mathbb{R}$. The existence of fixed points for this class of mappings is investigated. The main reference for this chapter is the paper [14].

6.1 Preliminaries

Let $\mathscr{F}$ be the set of functions $F : [0, +\infty)^6 \to \mathbb{R}$ satisfying the following conditions:

(I) F is continuous;
(II) F is nondecreasing in the first variable;
(III) F is decreasing in the fifth variable;
(IV) $\exists\, h \in (0, 1) : F(u, v, v, u, u + v, 0) \le 0 \Longrightarrow u \le hv$.

Let us give some examples of functions that belong to the set $\mathscr{F}$.

Example 6.1 The function $F : [0, \infty)^6 \to \mathbb{R}$ defined by

$$F(u_1, u_2, \ldots, u_6) = u_1 - \lambda u_2, \quad u_i \ge 0,\ i = 1, 2, \ldots, 6,$$

where $\lambda \in (0, 1)$ is a constant, belongs to the set $\mathscr{F}$. In this case, (IV) is satisfied with $h = \lambda$.

Example 6.2 The function $F : [0, \infty)^6 \to \mathbb{R}$ defined by

$$F(u_1, u_2, \ldots, u_6) = u_1 - \lambda u_2 - \gamma u_3, \quad u_i \ge 0,\ i = 1, 2, \ldots, 6,$$

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where $\lambda, \gamma \geq 0$ are constants with $\lambda + \gamma \in (0, 1)$, belongs to the set $\mathscr{F}$. In this case, (IV) is satisfied with $h = \lambda + \gamma$.

Example 6.3 The function $F : [0, \infty)^6 \to \mathbb{R}$ defined by

$$F(u_1, u_2, \ldots, u_6) = u_1 - \lambda \max \left\{u_2, u_3, u_4, \frac{u_5 + u_6}{2}\right\}, \quad u_i \geq 0,\ i = 1, 2, \ldots, 6,$$

where $\lambda \in (0, 1)$ is a constant, belongs to the set $\mathscr{F}$. In fact, (I)–(III) are obvious. Further, let $u, v \geq 0$ be such that $F(u, v, v, u, u + v, 0) \leq 0$. By the definition of F, we obtain

$$u - \lambda \max \left\{v, u, \frac{u + v}{2}\right\} = u - \lambda \max\{v, u\} \leq 0,$$

which yields

$$u \leq \lambda \max\{v, u\}.$$

Since $\lambda \in (0, 1)$, we obtain

$$u \leq \lambda v.$$

Therefore, (IV) is satisfied with $h = \lambda$.

Let X be a nonempty set endowed with two metrics d and d'. For $x_0 \in X$ and $r > 0$, let

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}.$$

We denote by $\overline{B(x_0, r)}^{d'}$ the d'-closure of $B(x_0, r)$ (the closure of $B(x_0, r)$ with respect to the topology of d').

Before stating and proving the main results of this chapter, we need to introduce the following concepts (some of them are introduced in the previous chapters).

Definition 6.1 Let $T : \overline{B(x_0, r)}^{d'} \to X$ and $\alpha : X \times X \to \mathbb{R}$. We say that T is α-admissible (see [13]) if the following condition holds: For all $x, y \in B(x_0, r)$, we have

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 6.2 We say that the set X satisfies the property (H) with respect to the metric d if the following condition holds: For every sequence $\{x_n\} \subset X$ satisfying

$$\lim_{n\to\infty} d(x_n, x) = 0, \quad x \in X$$

and

$$\alpha(x_n, x_{n+1}) \geq 1, \quad n \in \mathbb{N},$$

there exist a positive integer κ and a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n(k)}, x) \geq 1, \quad k \geq \kappa.$$

6.2 Fixed Point Results

The first main result is giving by the following theorem.

Theorem 6.1 *Let X be a nonempty set equipped with two metrics d and d' such that (X, d') is a complete metric space. Let $T : \overline{B(x_0, r)}^{d'} \to X$ be a given mapping, where $x_0 \in X$ and $r > 0$. Suppose that there exist two functions $F \in \mathscr{F}$ and $\alpha : X \times X \to \mathbb{R}$ such that for all $(x, y) \in \overline{B(x_0, r)}^{d'} \times \overline{B(x_0, r)}^{d'}$, we have*

$$F(\alpha(x, y)d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0. \tag{6.1}$$

In addition, assume that the following properties hold:

(i) $d(x_0, Tx_0) < (1 - h)r$ and $\alpha(x_0, Tx_0) \geq 1$;
(ii) T is α-admissible;
(iii) If $d \ngeq d'$, then T is uniformly continuous from $(B(x_0, r), d)$ into (X, d');
(iv) If $d = d'$, then the set X satisfies the property (H) with respect to the metric d;
(v) If $d \neq d'$, then T is continuous from $(\overline{B(x_0, r)}^{d'}, d')$ into (X, d').

Then T has a fixed point.

Proof Let $x_1 = Tx_0$. From (i), we have

$$d(x_0, x_1) = d(x_0, Tx_0) \leq (1 - h)r < r,$$

i.e., $x_1 \in B(x_0, r)$. Let $x_2 = Tx_1$. From (6.1), we have

$$F(\alpha(x_0, x_1)d(Tx_0, Tx_1), d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_0, x_2), 0) \leq 0.$$

On the other hand, by (i) we have

$$d(Tx_0, Tx_1) \leq \alpha(x_0, x_1)d(Tx_0, Tx_1).$$

Therefore, by the monotony property of F, we obtain that

$$F(d(x_1, x_2), d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_0, x_2), 0) \leq 0.$$

Using the fact that $d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2)$ and property (III) of F, we obtain that

$$F(d(x_1,x_2),d(x_0,x_1),d(x_0,x_1),d(x_1,x_2),d(x_0,x_1)+d(x_1,x_2),0)\le 0,$$

which implies from property (IV) that

$$d(x_1,x_2)\le hd(x_0,x_1)\le h(1-h)r<r.$$

Now, we have

$$d(x_0,x_2)\le d(x_0,x_1)+hd(x_0,x_1)=(1+h)d(x_0,x_1)\le (1+h)(1-h)r<r,$$

i.e., $x_2\in B(x_0,r)$. Again, let $x_3=Tx_2$. Since T is α-admissible and $\alpha(x_0,x_1)\ge 1$, we have

$$d(x_2,x_3)\le \alpha(x_1,x_2)d(Tx_1,Tx_2).$$

Then, from (6.1), we obtain that

$$F(d(x_2,x_3),d(x_1,x_2),d(x_1,x_2),d(x_2,x_3),d(x_1,x_3),0)\le 0.$$

Using property (III) of F, we get

$$F(d(x_2,x_3),d(x_1,x_2),d(x_1,x_2),d(x_2,x_3),d(x_1,x_2)+d(x_2,x_3),0)\le 0,$$

which implies from property (IV) that

$$d(x_2,x_3)\le hd(x_1,x_2)\le h^2(1-h)r<r.$$

Therefore, we have

$$d(x_0,x_3)\le d(x_0,x_2)+d(x_2,x_3)\le (1+h)(1-h)r+h^2(1-h)r=(1-h^3)r<r,$$

i.e., $x_3\in B(x_0,r)$. Continuing this process, by induction, we can define the sequence $\{x_n\}$ by

$$x_{n+1}=Tx_n,\quad n\in\mathbb{N}.$$

Such sequence satisfies the following property:

$$x_n\in B(x_0,r),\quad \alpha(x_n,x_{n+1})\ge 1,\quad \text{and}\quad d(x_n,x_{n+1})\le h^n(1-h)r,\quad n\in\mathbb{N}. \tag{6.2}$$

Since $h\in(0,1)$, it follows from (6.2) that $\{x_n\}$ is a Cauchy sequence with respect to the metric d. Now, we shall prove that $\{x_n\}$ is also a Cauchy sequence with respect to the metric d'. If $d\not\geq d'$, from (iii), given $\varepsilon>0$, there exists $\delta>0$ such that

$$(x,y)\in B(x_0,r)\times B(x_0,r),\ d(x,y)<\delta\implies d'(Tx,Ty)<\varepsilon. \tag{6.3}$$

On the other hand, since $\{x_n\}$ is Cauchy with respect to d, there exists a positive integer N such that

$$d(x_n, x_m) < \delta, \quad n, m \geq N.$$

Using (6.3), we obtain

$$d'(x_{n+1}, x_{m+1}) < \varepsilon, \quad n, m \geq N,$$

which proves that $\{x_n\}$ is Cauchy with respect to d'.

Since (X, d') is complete, there exists $z \in \overline{B(x_0, r)}^{d'}$ such that

$$\lim_{n\to\infty} d'(x_n, z) = 0. \tag{6.4}$$

We shall prove that z is a fixed point of T. We consider two cases.

Case 1. If $d = d'$.
From (iv), there exist a positive integer κ and a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n(k)}, z) \geq 1, \quad k \geq \kappa. \tag{6.5}$$

Using (6.1), for all $k \geq \kappa$, we have

$$F(\alpha(x_{n(k)}, z)d(Tx_{n(k)}, Tz), d(x_{n(k)}, z), d(x_{n(k)}, x_{n(k)+1}), d(z, Tz), d(x_{n(k)}, Tz), d(z, x_{n(k)+1}))$$
$$\leq 0.$$

Next, by (6.5) and property (II) of F, for all $k \geq \kappa$, we have

$$F(d(x_{n(k)+1}, Tz), d(x_{n(k)}, z), d(x_{n(k)}, x_{n(k)+1}), d(z, Tz), d(x_{n(k)}, Tz), d(z, x_{n(k)+1})) \leq 0.$$

Passing to the limit as $k \to \infty$, using (6.4) and the continuity of F, we get

$$F(d(z, Tz), 0, 0, d(z, Tz), d(z, Tz), 0) \leq 0,$$

which implies from property (IV) that $d(z, Tz) = 0$.

Case 2. If $d \neq d'$.
In this case, using (v) and (6.4), we get

$$\lim_{n\to\infty} d'(Tx_n, Tz) = \lim_{n\to\infty} d'(x_{n+1}, Tz) = 0.$$

The uniqueness of the limit gives us that $z = Tz$.

Taking $d = d'$ in Theorem 6.1, we obtain the following result.

Theorem 6.2 *Let (X, d) be a complete metric space, and let $T : \overline{B(x_0, r)}^{d} \to X$ be a given mapping, where $x_0 \in X$ and $r > 0$. Suppose that there exist two functions*

$F \in \mathscr{F}$ and $\alpha : X \times X \to \mathbb{R}$ such that for all $(x, y) \in \overline{B(x_0, r)}^d \times \overline{B(x_0, r)}^d$, we have

$$F(\alpha(x, y)d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that the following properties hold:

(i) $d(x_0, Tx_0) < (1 - h)r$ *and* $\alpha(x_0, Tx_0) \geq 1$*;*
(ii) T *is* α*-admissible;*
(iii) The set X *satisfies the property (H) with respect to the metric* d*.*

Then T has a fixed point.

From Theorem 6.1, we can deduce the following global result.

Theorem 6.3 *Let X be a nonempty set equipped with two metrics d and d' such that (X, d') is a complete metric space. Let $T : X \to X$ be a given mapping. Suppose that there exist two functions $F \in \mathscr{F}$ and $\alpha : X \times X \to \mathbb{R}$ such that for all $(x, y) \in X \times X$, we have*

$$F(\alpha(x, y)d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that the following properties hold:

(i) There exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*
(ii) T *is* α*-admissible (*$x, y \in X, \alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1$*);*
(iii) If $d \ngeq d'$*, then* T *is uniformly continuous from* (X, d) *into* (X, d')*;*
(iv) If $d = d'$*, then the set* X *satisfies the property (H) with respect to the metric* d*;*
(v) If $d \neq d'$*, then* T *is continuous from* (X, d') *into* (X, d')*.*

Then T has a fixed point.

Proof We take $r > 0$ such that $d(x_0, Tx_0) < (1 - h)r$. Then, from Theorem 6.1, T has a fixed point in $\overline{B(x_0, r)}^{d'}$.

Taking $d = d'$ in Theorem 6.3, we obtain the following result.

Theorem 6.4 *Let (X, d) be a complete metric space, and let $T : X \to X$ be a given mapping. Suppose that there exist two functions $F \in \mathscr{F}$ and $\alpha : X \times X \to \mathbb{R}$ such that for all $(x, y) \in X \times X$, we have*

$$F(\alpha(x, y)d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that the following properties hold:

(i) There exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*
(ii) T *is* α*-admissible (*$x, y \in X, \alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1$*);*
(iii) The set X *satisfies the property (H) with respect to the metric* d*.*

Then T has a fixed point.

6.3 Some Consequences

We present in this section some interesting consequences that can be derived from the previous obtained results.

6.3.1 The Case $\alpha(x, y) = 1$

Taking $\alpha(x, y) = 1$ for all $x, y \in X$, from Theorems 6.1, 6.2, 6.3, and 6.4, we obtain the following results that are generalizations of the fixed point results in [1–4, 6, 8, 11].

Corollary 6.1 *Let (X, d') be a complete metric space, d another metric on X, $x_0 \in X$, $r > 0$, and $T : \overline{B(x_0, r)}^{d'} \to X$. Suppose that there exists $F \in \mathscr{F}$ such that for all $(x, y) \in \overline{B(x_0, r)}^{d'} \times \overline{B(x_0, r)}^{d'}$, we have*

$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that the following properties hold:

(i) $d(x_0, Tx_0) < (1 - h)r$;
(ii) If $d \ngeq d'$, then T is uniformly continuous from $(B(x_0, r), d)$ into (X, d');
(iii) If $d \neq d'$, then T is continuous from $(\overline{B(x_0, r)}^{d'}, d')$ into (X, d').

Then T has a fixed point.

Corollary 6.2 *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$, and $T : \overline{B(x_0, r)}^{d} \to X$. Suppose that there exists $F \in \mathscr{F}$ such that for all $(x, y) \in \overline{B(x_0, r)}^{d} \times \overline{B(x_0, r)}^{d}$, we have*

$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that $d(x_0, Tx_0) < (1 - h)r$. Then T has a fixed point.

Corollary 6.3 *Let (X, d') be a complete metric space, d another metric on X, and $T : X \to X$. Suppose that there exists $F \in \mathscr{F}$ such that for all $(x, y) \in X \times X$, we have*

$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that the following properties hold:

(i) If $d \ngeq d'$, then T is uniformly continuous from (X, d) into (X, d');
(ii) If $d \neq d'$, then T is continuous from (X, d') into (X, d').

Then T has a fixed point.

Corollary 6.4 *Let (X, d) be a complete metric space, and let $T : X \to X$. Suppose that there exists $F \in \mathscr{F}$ such that for all $(x, y) \in X \times X$, we have*

$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

Then T has a fixed point.

Remark 6.1 Corollary 6.4 is an enriched version of Popa [8] that unifies the most important metrical fixed point theorems for contraction-type mappings in Rhoades' classification [12].

6.3.2 The Case of Partially Ordered Sets

Let $\preceq$ be a partial order on X. Let $\lhd$ be the binary relation on X defined by

$$(x, y) \in X \times X, \quad x \lhd y \Longleftrightarrow x \preceq y \text{ or } y \preceq x.$$

We say that $(X, \lhd)$ satisfies the property (H) with respect to the metric d if the following condition holds: For every sequence $\{x_n\} \subset X$ satisfying

$$\lim_{n\to\infty} d(x_n, x) = 0, \quad x \in X$$

and

$$x_n \lhd x_{n+1}, \quad n \in \mathbb{N},$$

there exist a positive integer κ and a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$x_{n(k)} \lhd x, \quad k \geq \kappa.$$

From Theorems 6.1, 6.2, 6.3, and 6.4, we obtain the following results that are extensions and generalizations of the fixed point results in [7, 10].

At first, we denote by $\widetilde{\mathscr{F}}$ the set of functions $F : [0, +\infty)^6 \to \mathbb{R}$ satisfying the following conditions:

(j) $F \in \mathscr{F}$;
(jj) For every $u_i \geq 0$, $i = 2, \ldots, 6$, we have

$$F(0, u_2, \ldots, u_6) \leq 0.$$

We have the following fixed point result.

Corollary 6.5 *Let (X, d') be a complete metric space, d another metric on X, $x_0 \in X$, $r > 0$, and $T : \overline{B(x_0, r)}^{d'} \to X$. Suppose that there exists $F \in \widetilde{\mathscr{F}}$ such that for all $(x, y) \in \overline{B(x_0, r)}^{d'} \times \overline{B(x_0, r)}^{d'}$, we have*

$$x \lhd y \implies F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that the following properties hold:

(i) $d(x_0, Tx_0) < (1 - h)r$ *and* $x_0 \lhd Tx_0$;
(ii) $x, y \in \overline{B(x_0, r)}^{d'}, x \lhd y \implies Tx \lhd Ty$;
(iii) If $d \not\geq d'$, *then* T *is uniformly continuous from* $(B(x_0, r), d)$ *into* (X, d');
(iv) If $d = d'$, *then* $(X, \lhd)$ *satisfies the property (H) with respect to the metric* d;
(v) If $d \neq d'$, *then* T *is continuous from* $(\overline{B(x_0, r)}^{d'}, d')$ *into* (X, d').

Then T has a fixed point.

Proof It follows from Theorem 6.1 by taking

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x \lhd y; \\ 0 \text{ if } x \not\lhd y. \end{cases}$$

Similarly, from Theorem 6.2, we obtain the following result.

Corollary 6.6 *Let (X, d) be a complete metric space, and let $T : \overline{B(x_0, r)}^{d} \to X$ be a given mapping, where $x_0 \in X$ and $r > 0$. Suppose that there exists $F \in \widetilde{\mathscr{F}}$ such that for all $(x, y) \in \overline{B(x_0, r)}^{d} \times \overline{B(x_0, r)}^{d}$, we have*

$$x \lhd y \implies F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that the following properties hold:

(i) $d(x_0, Tx_0) < (1 - h)r$ *and* $x_0 \lhd Tx_0$;
(ii) $x, y \in \overline{B(x_0, r)}^{d'}, x \lhd y \implies Tx \lhd Ty$;
(iii) $(X, \lhd)$ *satisfies the property (H) with respect to the metric* d.

Then T has a fixed point.

From Theorem 6.3, we obtain the following global result.

Corollary 6.7 *Let (X, d') be a complete metric space, d another metric on X, and $T : X \to X$. Suppose that there exists $F \in \widetilde{\mathscr{F}}$ such that for all $(x, y) \in X \times X$, we have*

$$x \lhd y \implies F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that the following properties hold:

(i) *There exists* $x_0 \in X$ *such that* $x_0 \lhd Tx_0$;
(ii) $x, y \in X,\ x \lhd y \implies Tx \lhd Ty$;
(iii) *If* $d \not\geq d'$, *then* T *is uniformly continuous from* (X, d) *into* (X, d');
(iv) *If* $d = d'$, *then* $(X, \lhd)$ *satisfies the property (H) with respect to the metric* d;
(v) *If* $d \neq d'$, *then* T *is continuous from* (X, d') *into* (X, d').

Then T *has a fixed point.*

Finally, from Theorem 6.4, we obtain the following fixed point result.

Corollary 6.8 *Let* (X, d) *be a complete metric space, and let* $T : X \to X$. *Suppose that there exists* $F \in \widetilde{\mathscr{F}}$ *such that for all* $(x, y) \in X \times X$, *we have*

$$x \lhd y \implies F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that the following properties hold:

(i) *There exists* $x_0 \in X$ *such that* $x_0 \lhd Tx_0$;
(ii) $x, y \in X,\ x \lhd y \implies Tx \lhd Ty$;
(iii) $(X, \lhd)$ *satisfies the property (H) with respect to the metric* d.

Then T *has a fixed point.*

6.3.3 The Case of Cyclic Mappings

From Theorem 6.4, we obtain the following fixed point result that is a generalization of Theorem 1.1 in [5].

Corollary 6.9 *Let* (Y, d) *be a complete metric space,* $\{A, B\}$ *a pair of nonempty closed subsets of* Y, *and* $T : A \cup B \to A \cup B$. *Suppose that there exists* $F \in \widetilde{\mathscr{F}}$ *such that for all* $(x, y) \in A \times B$, *we have*

$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

In addition, assume that $T(A) \subseteq B$ *and* $T(B) \subseteq A$. *Then* T *has a fixed point in* $A \cap B$.

Proof Let $X = A \cup B$. Clearly (since A and B are closed), (X, d) is a complete metric space. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 \text{ if } (x, y) \in (A \times B) \cup (B \times A); \\ \\ 0 \text{ if } (x, y) \notin (A \times B) \cup (B \times A). \end{cases}$$

Clearly (since $F \in \widetilde{\mathscr{F}}$), for all $x, y \in X$, we have

$$F(\alpha(x, y)d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0.$$

Taking any point $x_0 \in A$, since $T(A) \subseteq B$, we have $Tx_0 \in B$, which implies that $\alpha(x_0, Tx_0) \geq 1$. Now, let $(x, y) \in X \times X$ be such that $\alpha(x, y) \geq 1$. We have two cases.

Case 1. If $(x, y) \in A \times B$.
Since $T(A) \subseteq B$ and $T(B) \subseteq A$, we have $(Tx, Ty) \in B \times A$, which implies that $\alpha(Tx, Ty) \geq 1$.

Case 2. If $(x, y) \in B \times A$.
In this case, we have $(Tx, Ty) \in A \times B$, which implies that $\alpha(Tx, Ty) \geq 1$.
Therefore, we proved that the mapping T is α-admissible.

Next, we shall prove that X satisfies the property (H) with respect to the metric d. Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} d(x_n, x) = 0, \quad x \in X$$

and

$$\alpha(x_n, x_{n+1}) \geq 1, \quad n \in \mathbb{N}.$$

From the definition of α, we get

$$(x_n, x_{n+1}) \in (A \times B) \cup (B \times A), \quad n \in \mathbb{N}.$$

Since A and B are closed, we have $x \in A \cap B$. Therefore,

$$\alpha(x_n, x) = 1, \quad n \in \mathbb{N},$$

which proves that the set X satisfies the property (H) with respect to the metric d.

Now, from Theorem 6.4, the mapping T has a fixed point in X, i.e., there exists $z \in A \cup B$ such that $Tz = z$. Since $T(A) \subseteq B$ and $T(B) \subseteq A$, obviously, we have $z \in A \cap B$.

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Chapter 7
On Fixed Points That Belong to the Zero Set of a Certain Function

Let $T : X \to X$ be a given mapping. The set $\text{Fix}(T)$ is said to be φ-admissible with respect to a certain mapping $\varphi : X \to [0, \infty)$, if $\emptyset \neq \text{Fix}(T) \subseteq Z_\varphi$, where Z_φ denotes the zero set of φ, i.e., $Z_\varphi = \{x \in X : \varphi(x) = 0\}$. In this chapter, we present the class of extended simulation functions recently introduced by Roldán and Samet [13], which is more large than the class of simulation functions, introduced by Khojasteh et al. [8]. We obtain a φ-admissibility result involving extended simulation functions, for a new class of mappings $T : X \to X$, with respect to a lower semi-continuous function $\varphi : X \to [0, \infty)$, where X is a set equipped with a certain metric d. From the obtained results, some fixed point theorems in partial metric spaces are derived, including Matthews fixed point theorem [9]. Moreover, we answer to three open problems posed by Ioan A. Rus in [16]. The main references for this chapter are the papers [7, 13, 17].

7.1 Partial Metric Spaces

In 1994, Matthews [9] introduced the concept of partial metric spaces as a part of the study of denotational semantics of dataflow networks and showed that Banach contraction principle can be generalized to the partial metric context for applications in program verification. Later on, many authors studied fixed point theorems on partial metric spaces (see, e.g., [1, 2, 5, 6, 10, 11, 14, 15, 18, 19] and references therein).

We start this section by recalling some basic definitions and properties of partial metric spaces (see [9] for more details).

Definition 7.1 A partial metric on a nonempty set X is a mapping $p : X \times X \to [0, \infty)$ satisfying the following axioms: For all $x, y, z \in X$, we have

(i) $p(x, x) = p(y, y) = p(x, y) \Longleftrightarrow x = y$;

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(ii) $p(x, x) \leq p(x, y)$;
(iii) $p(x, y) = p(y, x)$;
(iv) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

In this case, the pair (X, p) is said to be a partial metric space.

Remark 7.1 It is clear that, if $p(x, y) = 0$, then $x = y$; but if $x = y$, $p(x, y)$ may not be 0.

Example 7.1 A basic example of a partial metric space is the pair $([0, \infty), p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in [0, \infty)$. Other examples of partial metric spaces which are interesting from a computational point of view may be found in [9].

The next definitions generalize the metric space notions of convergent sequences and Cauchy sequences to partial metric spaces.

Definition 7.2 A sequence $\{x_n\}$ of points in a partial metric space (X, p) converges to $x \in X$ if

$$\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n) = p(x, x).$$

Definition 7.3 A sequence $\{x_n\}$ of points in a partial metric space (X, p) is Cauchy if $\lim_{m,n\to\infty} p(x_n, x_m)$ exists and is finite.

Definition 7.4 A partial metric space (X, p) is complete if every Cauchy sequence converges.

The following result can be shown easily.

Lemma 7.1 *Let X be a nonempty set and $p : X \times X \to [0, \infty)$ be a partial metric on X. Let $d_p : X \times X \to [0, \infty)$ be the mapping defined by*

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \quad (x, y) \in X \times X.$$

Then d_p is a metric on X.

Lemma 7.2 (see [10]) *Let (X, p) be a partial metric space. Then*

(i) $\{x_n\}$ is Cauchy in (X, p) if and only if $\{x_n\}$ is Cauchy in the metric space (X, d_p).
(ii) The partial metric space (X, p) is complete if and only if the metric space (X, d_p) is complete. Furthermore, $\lim_{n\to\infty} d_p(x_n, x) = 0$ if and only if

$$\lim_{n\to\infty} p(x_n, x) = p(x, x) = \lim_{m,n\to\infty} p(x_n, x_m).$$

In [9], Matthews obtained a partial metric version of Banach contraction principle as follows.

Theorem 7.1 (Matthews fixed point theorem) *Let (X, p) be a complete partial metric space. Let $T : X \to X$ be a contraction; i.e., there exists some constant $k \in (0, 1)$ such that*

$$p(Tx, Ty) \leq k\, p(x, y), \quad (x, y) \in X \times X. \tag{7.1}$$

Then T has a unique fixed point $x^* \in X$. Moreover, we have $p(x^*, x^*) = 0$.

Under the assumptions of Theorem 7.1, we observe easily that

$$\emptyset \neq \mathrm{Fix}(T) \subseteq Z_\varphi,$$

where Z_φ denotes the zero set of $\varphi(x) = p(x, x)$. A point $x \in X$ satisfying $p(x, x) = 0$ is called a total element (see [16]).

7.2 Three Open Questions of I.A. Rus

In [16], Ioan A. Rus presented three interesting open problems. Let (X, p) be a complete partial metric space.

Problem 1 If $T : (X, p) \to (X, p)$ is a contraction, which condition satisfies T with respect to the metric d_p?

Problem 2 It consists to give fixed point theorems for these new classes of operators on the metric space (X, d_p).

Problem 3 Use the results for the above problems to give fixed point theorems in a partial metric space.

The purpose of this chapter is to study the φ-admissibility for a new class of mappings $T : X \to X$, with respect to a lower semi-continuous function $\varphi : X \to [0, \infty)$, where X is a set equipped with a certain metric d. Next, from the obtained results, some fixed point theorems in partial metric spaces are derived, including Matthews fixed point theorem [9]. This contribution presents answers to the above problems of Ioan A. Rus.

7.3 The Class of Extended Simulation Functions

The class of simulation functions was introduced recently in [8] as follows.

Definition 7.5 Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a given map. We say that ζ is a simulation function if it satisfies the following conditions:

(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < s - t$, for every $t, s > 0$;
(ζ_3) For any sequences $\{t_n\}, \{s_n\} \subset (0,\infty)$, we have

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0 \Longrightarrow \limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Several examples of simulation functions were given in [8]. Let us denote by $\mathscr{Z}$ the st of all simulation functions.

Definition 7.6 ([8]) Let $T : X \to X$ be a given map, where X is endowed with a certain metric d. We say that T is a $\mathscr{Z}$-contraction with respect to a certain simulation function $\zeta \in \mathscr{Z}$ if

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0, \quad (x,y) \in X \times X.$$

The main result in [8] is the following fixed point theorem that generalizes and unifies several previous fixed point results from the literature including Banach contraction principle.

Theorem 7.2 ([8]) *Let $T : X \to X$ be a given map, where X is a set endowed with a certain metric d such that (X,d) is complete. If T is a $\mathscr{Z}$-contraction with respect to a certain simulation function $\zeta \in \mathscr{Z}$, then T has a unique fixed point. Moreover, for any $x \in X$, the Picard sequence $\{T^n x\}$ converges to this fixed point.*

The following concept was introduced in [13].

Definition 7.7 An extended simulation function (for short, an e-simulation function) is a function $\theta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following axioms:

(θ_1) $\theta(t,s) < s - t$, for every $t, s > 0$;
(θ_2) For any sequences $\{t_n\}, \{s_n\} \subset (0,\infty)$, we have

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0,\infty),\ s_n > \ell,\ n \in \mathbb{N} \Longrightarrow \limsup_{n\to\infty} \theta(t_n, s_n) < 0;$$

(θ_3) For any sequence $\{t_n\} \subset (0,\infty)$, we have

$$\lim_{n\to\infty} t_n = \ell \in [0,\infty),\ \theta(t_n, \ell) \geq 0,\ n \in \mathbb{N} \Longrightarrow \ell = 0.$$

Let us denote by $\mathscr{E}_Z$ the set of all e-simulation functions. In the following, we compare the set $\mathscr{E}_Z$ with the set $\mathscr{Z}$.

Proposition 7.1 *Every simulation function is an e-simulation function.*

Proof Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a simulation function. We have just to prove that the function ζ satisfies axiom (θ_3). Let $\{t_n\} \subset (0,\infty)$ be a sequence converging to $\ell \geq 0$, and such that

$$\zeta(t_n, \ell) \geq 0, \quad n \in \mathbb{N}. \tag{7.2}$$

Suppose that $\ell > 0$. Let us consider the sequence $\{s_n\} \subset (0, \infty)$ given by

$$s_n = \ell, \quad n \in \mathbb{N}.$$

Using axiom (ζ_3), we obtain

$$\limsup_{n\to\infty} \zeta(t_n, \ell) = \limsup_{n\to\infty} \zeta(t_n, s_n) < 0,$$

which is a contradiction with (7.2). Therefore, $\ell = 0$, and (θ_3) holds.

The converse of Proposition 7.1 is not true as it is shown by the following example.

Example 7.2 Let us consider the function $\theta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ defined by

$$\theta(t, s) = \begin{cases} 1 - t & \text{if } s = 0, \\ \frac{s}{2} - t & \text{if } s > 0. \end{cases}$$

At first, observe that $\theta \notin \mathscr{Z}$. In fact, $\theta(0, 0) = 1 \neq 0$, so axiom (ζ_1) is not satisfied. Let us prove now that $\theta \in \mathscr{E}_Z$. For all $t, s > 0$, we have

$$\theta(t, s) = \frac{s}{2} - t < s - t,$$

which yields (θ_1). Let $\{t_n\}$ and $\{s_n\}$ be two sequences in $(0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0, \infty).$$

We have

$$\theta(t_n, s_n) = \frac{s_n}{2} - t_n, \quad n \in \mathbb{N}.$$

Therefore,

$$\limsup_{n\to\infty} \theta(t_n, s_n) = -\frac{\ell}{2} < 0,$$

which proves (θ_2). Finally, let $\{t_n\}$ be a sequence in $(0, \infty)$ that converges to some $\ell \geq 0$, and such that

$$\theta(t_n, \ell) \geq 0, \quad n \in \mathbb{N}.$$

Suppose that $\ell > 0$. Then

$$\theta(t_n, \ell) = \frac{\ell}{2} - t_n \geq 0, \quad n \in \mathbb{N},$$

i.e.,

$$t_n \leq \frac{\ell}{2}, \quad n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$, we obtain

$$\ell \leq \frac{\ell}{2},$$

which is a contradiction with $\ell > 0$. Therefore, $\ell = 0$, and (θ_3) follows. As a consequence, $\theta \in \mathscr{E}_Z$.

For technical reasons, it is convenient to point that if we had considered the closed interval $[0, \infty)$ in Definition 7.7, then we would have obtained the same notion. The following result shows this fact.

Proposition 7.2 *Given a function* $\theta : [0, \infty) \times [0, \infty) \to \mathbb{R}$, *condition* (θ_2) *is equivalent to:*

(θ_2') For any sequences $\{t_n\}, \{s_n\} \subset [0, \infty)$, *we have*

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0, \infty),\ s_n > \ell,\ n \in \mathbb{N} \Longrightarrow \limsup_{n\to\infty} \theta(t_n, s_n) < 0.$$

Furthermore, property (θ_3) *is equivalent to:*

(θ_3') For any sequence $\{t_n\} \subset [0, \infty)$, *we have*

$$\lim_{n\to\infty} t_n = \ell \in [0, \infty),\ \theta(t_n, \ell) \geq 0,\ n \in \mathbb{N} \Longrightarrow \ell = 0.$$

Proof Clearly, we have $(\theta_2') \Longrightarrow (\theta_2)$. Let us prove the converse. Suppose that (θ_2) holds. Let $\{t_n\}$ and $\{s_n\}$ be two sequences in $[0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0, \infty),\ s_n > \ell,\ n \in \mathbb{N}.$$

Since $\ell > 0$, there exists some $N \in \mathbb{N}$ such that

$$t_n > 0,\ s_n > 0, \quad n \geq N + 1.$$

Let us define the sequences $\{T_n\}$ and $\{S_n\}$ by

$$T_0 = T_1 = \cdots = T_N = 1,\ T_n = t_n,\ n \geq N + 1$$

and

$$S_0 = S_1 = \cdots = S_N = \ell + 1,\ S_n = s_n,\ n \geq N + 1.$$

Then $\{T_n\}$ and $\{S_n\}$ are two sequences in $(0, \infty)$ converging to $\ell \in (0, \infty)$ with

$$S_n > \ell, \quad n \in \mathbb{N}.$$

By (θ_2), we obtain

$$\limsup_{n\to\infty} \theta(t_n, s_n) = \limsup_{n\to\infty} \theta(T_n, S_n) < 0,$$

from which (θ_2') follows. On the other hand, the implication $(\theta_3') \Longrightarrow (\theta_3)$ is obvious. Let us prove the converse. Suppose that (θ_3) holds true. Let $\{t_n\}$ be a sequence in $[0, \infty)$ converging to some $\ell \geq 0$, and such that

$$\theta(t_n, \ell) \geq 0, \quad n \in \mathbb{N}.$$

We have to prove that $\ell = 0$. Suppose that $\ell > 0$. Then there exists some $N \in \mathbb{N}$ such that

$$t_n > 0, \quad n \geq N + 1.$$

Define the sequence $\{T_n\}$ by

$$T_0 = T_1 = \cdots = T_N = t_{N+1}, \; T_n = t_n, \; n \geq N + 2.$$

Then $\{T_n\}$ is a sequence in $(0, \infty)$ converging to ℓ, and such that

$$\theta(T_n, \ell) \geq 0, \quad n \in \mathbb{N}.$$

By (θ_3), we obtain $\ell = 0$, which is a contradiction. Therefore, $\ell = 0$, and (θ_3') follows.

Remark 7.2 Properties (θ_2) and (θ_3) are easier to prove when we want to check that a given function is an e-simulation function. However, conditions (θ_2') and (θ_3') are useful when we assume that a given function is an e-simulation function.

Let Ψ be the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(ψ_1) ψ is upper semi-continuous from the right;
(ψ_2) $\psi(t) < t, t > 0$.

Lemma 7.3 *Given $\psi \in \Psi$, let $\theta_\psi : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be the function given by*

$$\theta_\psi(t, s) = \psi(s) - t, \quad t, s \geq 0. \tag{7.3}$$

Then θ_ψ is an e-simulation function.

Proof Let us check axiom (θ_1). For all $t, s > 0$, from property (ψ_2), we have

$$\theta_\psi(t, s) = \psi(s) - t < s - t,$$

which proves (θ_1). Let us consider two sequences $\{t_n\}$ and $\{s_n\}$ in $(0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0,\infty),\ s_n > \ell,\ n \in \mathbb{N}.$$

We have

$$\theta_\psi(t_n, s_n) = \psi(s_n) - t_n, \quad n \in \mathbb{N}.$$

Since from (ψ_1), the function ψ is upper semi-continuous form the right, we have

$$\psi(\ell) \geq \limsup_{n\to\infty} \psi(s_n),$$

which implies from (ψ_2) that

$$\limsup_{n\to\infty} \theta_\psi(t_n, s_n) \leq \psi(\ell) - \ell < 0.$$

Therefore, (θ_2) holds. Finally, we have to check axiom (θ_3). Let $\{t_n\}$ be a sequence in $(0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \ell \in [0,\infty),\ \theta_\psi(t_n, \ell) \geq 0,\ n \in \mathbb{N}.$$

Suppose that $\ell > 0$. We have

$$\psi(\ell) - t_n \geq 0, \quad n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$, we obtain

$$\psi(\ell) \geq \ell.$$

On the other hand, from (ψ_2), we have

$$\psi(\ell) < \ell,$$

which is a contradiction. Then $\ell = 0$, and (θ_3) holds. As a consequence, θ_ψ is an e-simulation function.

Remark 7.3 In general, if $\psi \in \Psi$, θ_ψ is not a simulation function. This fact can be shown by Example 7.2 with

$$\psi(s) = \begin{cases} 1 & \text{if } s = 0, \\ \dfrac{s}{2} & \text{if } s > 0. \end{cases}$$

However, if ψ is upper semi-continuous (rather than upper semi-continuous from the right), then we can modify θ_ψ to transform it in a simulation function. The next result shows this fact.

Proposition 7.3 *If $\psi \in \Psi$, then the function $\widetilde{\theta_\psi} : [0, \infty) \times [0, \infty) \to \mathbb{R}$ given by*

$$\widetilde{\theta_\psi}(t, s) = \begin{cases} 0 \text{ if } t = s = 0, \\ \psi(s) - t \text{ otherwise} \end{cases}$$

is an e-simulation function. Furthermore, if $\psi \in \Psi$ is upper semi-continuous, then $\widetilde{\theta_\psi}$ is a simulation function.

Proof Let us prove first that $\widetilde{\theta_\psi}$ is an e-simulation function. For all $t, s > 0$, we have

$$\widetilde{\theta_\psi}(t, s) = \psi(s) - t < s - t,$$

which yields (θ_1). Let us consider two sequences $\{t_n\}$ and $\{s_n\}$ in $(0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0, \infty),\ s_n > \ell,\ n \in \mathbb{N}.$$

Then

$$\limsup_{n\to\infty} \widetilde{\theta_\psi}(t_n, s_n) = \limsup_{n\to\infty} \psi(s_n) - \ell \le \psi(\ell) - \ell < 0.$$

Therefore, (θ_2) holds. Finally, let $\{t_n\}$ be a sequence in $(0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \ell \in [0, \infty),\ \widetilde{\theta_\psi}(t_n, \ell) \ge 0,\ n \in \mathbb{N}.$$

Suppose that $\ell > 0$. Then

$$\widetilde{\theta_\psi}(t_n, \ell) = \psi(\ell) - t_n \ge 0, \quad n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$, and using axiom (ψ_2), we obtain

$$\ell \le \psi(\ell) < \ell,$$

which is a contradiction. Then $\ell = 0$, and (θ_3) follows. As a consequence, $\widetilde{\theta_\psi}$ is an e-simulation function.

Suppose now that $\psi \in \psi$ is upper semi-continuous. Let us prove that $\widetilde{\theta_\psi}$ is a simulation function. Observe that

$$\widetilde{\theta_\psi}(0, 0) = 0,$$

which yields (ζ_1). Axiom (ζ_2) follows from the fact that $\widetilde{\theta_\psi}$ is an e-simulation function. Axiom (ζ_3) follows by using point by point the proof of (θ_2), and using the upper semi-continuity of ψ. Therefore, under the upper semi-continuity of $\psi \in \Psi$, $\widetilde{\theta_\psi}$ is a simulation function.

Remark 7.4 By (θ_1), if θ is an e-simulation function, then

$$\theta(r, r) < 0, \quad r > 0.$$

7.4 φ-Admissibility Results

The concept of φ-admissibility was introduced recently by Karapinar, Samet, and O'Regan in [7].

Definition 7.8 Let $T : X \to X$ be a given mapping. The set $\text{Fix}(T)$ is said to be φ-admissible with respect to a certain mapping $\varphi : X \to [0, \infty)$, if

$$\emptyset \neq \text{Fix}(T) \subseteq Z_\varphi,$$

where Z_φ denotes the zero set of φ, i.e.,

$$Z_\varphi = \{x \in X : \varphi(x) = 0\}.$$

Let $\mathscr{F}$ be the set of functions $F : [0, \infty)^3 \to [0, \infty)$ satisfying the following axioms:

(F_1) $\max\{a, b\} \leq F(a, b, c)$, for every $a, b, c \geq 0$;
(F_2) $F(a, 0, 0) = a$, for every $a \geq 0$;
(F_3) F is continuous.

The set $\mathscr{F}$ is nonempty. For instance, the following functions belong to $\mathscr{F}$:

- $F(a, b, c) = a + b + c$,
- $F(a, b, c) = \max\{a, b\} + \ln(c + 1)$,
- $F(a, b, c) = a + b + c(c + 1)$,
- $F(a, b, c) = (a + b)e^c$,
- $F(a, b, c) = (a + b)(c + 1)^n, \; n \in \mathbb{N}$.

Let (X, d) be a metric space, $\varphi : X \to [0, \infty)$, $F \in \mathscr{F}$, and $\theta \in \mathscr{E}_Z$. We denote by $\mathscr{T}(\varphi, F, \theta)$ the set of mappings $T : X \to X$ satisfying

$$\theta\big(F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)), M_F^\varphi(x, y)\big) \geq 0, \quad (x, y) \in X \times X, \tag{7.4}$$

where

$$M_F^\varphi(x, y) = \max\{F(d(x, y), \varphi(x), \varphi(y)), F(d(x, Tx), \varphi(x), \varphi(Tx)), F(d(y, Ty), \varphi(y), \varphi(Ty))\}. \tag{7.5}$$

The main result of this chapter is the following one.

Theorem 7.3 *Let (X, d) be a complete metric space. Let $T : X \to X$ be a mapping that belongs to $\mathscr{T}(\varphi, F, \theta)$, for some $\varphi : X \to [0, \infty)$, $F \in \mathscr{F}$, and $\theta \in \mathscr{E}_Z$. If φ is lower semi-continuous, then*

(i) For every $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point of T.
(ii) T has a unique fixed point.
(iii) Fix(T) *is φ-admissible.*

Proof First of all, we show that $\text{Fix}(T) \subseteq Z_\varphi$. Indeed, let $\omega \in \text{Fix}(T)$. Since

$$\begin{aligned} M_F^\varphi(\omega, \omega) &= \max\{F(d(\omega, \omega), \varphi(\omega), \varphi(\omega)), F(d(\omega, T\omega), \varphi(\omega), \varphi(T\omega)), \\ &\qquad F(d(\omega, T\omega), \varphi(\omega), \varphi(T\omega))\} \\ &= \max\{F(0, \varphi(\omega), \varphi(\omega)), F(0, \varphi(\omega), \varphi(\omega)), F(0, \varphi(\omega), \varphi(\omega))\} \\ &= F(0, \varphi(\omega), \varphi(\omega)), \end{aligned}$$

then (7.4) guarantees that

$$\begin{aligned} 0 &\le \theta\big(F(d(T\omega, T\omega), \varphi(T\omega), \varphi(T\omega)), M_F^\varphi(\omega, \omega)\big) \\ &= \theta\big(F(0, \varphi(\omega), \varphi(\omega)), F(0, \varphi(\omega), \varphi(\omega))\big). \end{aligned}$$

By Remark 7.4, we deduce that

$$F(0, \varphi(\omega), \varphi(\omega)) = 0.$$

It follows from condition (F_1) that

$$0 \le \varphi(\omega) = \max\{0, \varphi(\omega)\} \le F(0, \varphi(\omega), \varphi(\omega)) = 0,$$

which means that $\varphi(\omega) = 0$, and $\omega \in Z_\varphi$. Therefore, $\text{Fix}(T) \subseteq Z_\varphi$.

Next, let us prove (i). Let $x_0 \in X$ be an arbitrary point and let $\{x_n\}$ be the Picard sequence defined by

$$x_n = T^n x_0, \quad n \in \mathbb{N}.$$

If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T (and $\{x_n\}$ converges to x_{n_0}). On the contrary case, suppose that

$$d(x_n, x_{n+1}) > 0, \quad n \in \mathbb{N}.$$

If there exists some $m_0 \in \mathbb{N}$ such that $F(d(x_{m_0}, x_{m_0+1}), \varphi(x_{m_0}), \varphi(x_{m_0+1})) = 0$, then we could deduce from condition (F_1) that

$$\begin{aligned} 0 &< d(x_{m_0}, x_{m_0+1}) \le \max\big\{d(x_{m_0}, x_{m_0+1}), \varphi(x_{m_0})\big\} \\ &\le F(d(x_{m_0}, x_{m_0+1}), \varphi(x_{m_0}), \varphi(x_{m_0+1})) = 0, \end{aligned}$$

which is impossible. Hence,

$$F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) > 0, \quad n \in \mathbb{N}.$$

For simplicity, let us denote

$$a_n = F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) > 0, \quad n \in \mathbb{N}.$$

Notice that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
M_F^{\varphi}(x_n, x_{n+1}) &= \max\big\{F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})), F(d(x_n, Tx_n), \varphi(x_n), \varphi(Tx_n)),\\
&\qquad F(d(x_{n+1}, Tx_{n+1}), \varphi(x_{n+1}), \varphi(Tx_{n+1}))\big\}\\
&= \max\big\{F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})), F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})),\\
&\qquad F(d(x_{n+1}, x_{n+2}), \varphi(x_{n+1}), \varphi(x_{n+2}))\big\}\\
&= \max\big\{a_n,\ a_n,\ a_{n+1}\big\}\\
&= \max\big\{a_n, a_{n+1}\big\} > 0.
\end{aligned}$$

Using (7.4) and property (θ_2), we deduce that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
0 &\le \theta\big(F(d(Tx_n, Tx_{n+1}), \varphi(Tx_n), \varphi(Tx_{n+1})), M_F^{\varphi}(x_n, x_{n+1})\big)\\
&= \theta\big(F(d(x_{n+1}, x_{n+2}), \varphi(x_{n+1}), \varphi(x_{n+2})), \max\{a_n, a_{n+1}\}\big)\\
&= \theta\big(a_{n+1}, \max\{a_n, a_{n+1}\}\big)\\
&< \max\{a_n, a_{n+1}\} - a_{n+1},
\end{aligned}$$

which means that $a_{n+1} < a_n$, for all $n \in \mathbb{N}$. As $\{a_n\}$ is a decreasing sequence of nonnegative real numbers, it has a limit. Let

$$L = \lim_{n\to\infty} a_n \ge 0.$$

As $\{a_n\}$ is strictly decreasing, then $L < a_n$, for all $n \in \mathbb{N}$. In order to prove that $L = 0$, suppose that $L > 0$. In such a case, we have

$$\lim_{n\to\infty} a'_n = \lim_{n\to\infty} b'_n = L,$$

where $a'_n = a_{n+1}$ and $b'_n = \max\{a_n, a_{n+1}\} = a_n$. Moreover, we have

$$L < b'_n, \quad n \in \mathbb{N}.$$

Thus, condition (θ_3) implies that

$$\limsup_{n\to\infty} \theta(a'_n, b'_n) < 0,$$

which contradicts the fact that

$$\theta(a'_n, b'_n) = \theta\big(a_{n+1}, \max\{a_n, a_{n+1}\}\big) \geq 0, \quad n \in \mathbb{N}.$$

This contradiction guarantees that

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) = 0. \tag{7.6}$$

Furthermore, by condition (F_1),

$$\begin{aligned} &0 \leq \varphi(x_n) \leq \max\big\{d(x_n, x_{n+1}), \varphi(x_n)\big\} \leq F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) = a_n, \quad \text{and} \\ &0 \leq d(x_n, x_{n+1}) \leq \max\big\{d(x_n, x_{n+1}), \varphi(x_n)\big\} \leq F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) = a_n, \end{aligned}$$

for all $n \in \mathbb{N}$. So,

$$\lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{7.7}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence reasoning by contradiction. Suppose that $\{x_n\}$ is not a Cauchy sequence in (X, d). In this case, it is well known (see, for instance, [12, Lemma 16], [3, Lemma 13]) that there exist $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$,

$$k \leq n(k) < m(k) < n(k+1) \quad \text{and} \quad d(x_{n(k)}, x_{m(k)-1}) \leq \varepsilon_0 < d(x_{n(k)}, x_{m(k)}), \tag{7.8}$$

and also

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon_0. \tag{7.9}$$

Let $\ell = \varepsilon_0 > 0$ and let us define

$$\begin{aligned} a''_k &= F(d(x_{n(k)+1}, x_{m(k)+1}), \varphi(x_{n(k)+1}), \varphi(x_{m(k)+1})), \quad \text{and} \\ b''_k &= M_F^{\varphi}(x_{n(k)}, x_{m(k)}), \end{aligned}$$

for all $k \in \mathbb{N}$. As F is continuous, it follows from (7.7), (7.9), and (F_2) that

$$\lim_{k\to\infty} a''_k = \lim_{k\to\infty} F(d(x_{n(k)+1}, x_{m(k)+1}), \varphi(x_{n(k)+1}), \varphi(x_{m(k)+1})) = F(\varepsilon_0, 0, 0) = \varepsilon_0 = \ell.$$

On the other hand, for all $k \in \mathbb{N}$,

$$\begin{aligned} b''_k &= M_F^{\varphi}(x_{n(k)}, x_{m(k)}) \\ &= \max\big\{F(d(x_{n(k)}, x_{m(k)}), \varphi(x_{n(k)}), \varphi(x_{m(k)})), F(d(x_{n(k)}, Tx_{n(k)}), \varphi(x_{n(k)}), \varphi(Tx_{n(k)})), \\ &\qquad F(d(x_{m(k)}, Tx_{m(k)}), \varphi(x_{m(k)}), \varphi(Tx_{m(k)}))\big\} \\ &= \max\big\{F(d(x_{n(k)}, x_{m(k)}), \varphi(x_{n(k)}), \varphi(x_{m(k)})), F(d(x_{n(k)}, x_{n(k)+1}), \varphi(x_{n(k)}), \varphi(x_{n(k)+1})), \\ &\qquad F(d(x_{m(k)}, x_{m(k)+1}), \varphi(x_{m(k)}), \varphi(x_{m(k)+1}))\big\}. \end{aligned} \tag{7.10}$$

In particular, by (F_1) and (7.8), for all $n \in \mathbb{N}$,

$$\begin{aligned} b_k'' &\geq F(d(x_{n(k)}, x_{m(k)}), \varphi(x_{n(k)}), \varphi(x_{m(k)})) \geq \max\left\{d(x_{n(k)}, x_{m(k)}), \varphi(x_{n(k)})\right\} \\ &\geq d(x_{n(k)}, x_{m(k)}) > \varepsilon = \ell. \end{aligned} \tag{7.11}$$

Letting $k \to \infty$ in (7.10), we obtain

$$\begin{aligned} \lim_{k\to\infty} b_k'' &= \max\{F(\varepsilon_0, 0, 0), F(0, 0, 0), F(0, 0, 0)\} \\ &= F(\varepsilon_0, 0, 0) = \varepsilon_0 = \ell. \end{aligned}$$

As a consequence, $\{a_k''\}$ and $\{b_k''\}$ are sequences of positive real numbers converging to the same positive limit ℓ satisfying

$$\ell < b_k'', \quad k \in \mathbb{N}.$$

It follows from (θ_3) that

$$\limsup_{k\to\infty} \theta(a_k'', b_k'') < 0. \tag{7.12}$$

However, (7.4) ensures us that, for all $k \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq \theta\big(F(d(Tx_{n(k)}, Tx_{m(k)}), \varphi(Tx_{n(k)}), \varphi(Tx_{m(k)})), M_F^{\varphi}(x_{n(k)}, x_{m(k)})\big) \\ &\leq \theta\big(F(d(x_{n(k)+1}, x_{m(k)+1}), \varphi(x_{n(k)+1}), \varphi(x_{m(k)+1})),\ M_F^{\varphi}(x_{n(k)}, x_{m(k)})\big) \\ &= \theta(a_k'', b_k''), \end{aligned}$$

which contradicts (7.12). This contradiction guarantees that $\{x_n\}$ is a Cauchy sequence in (X, d). As it is complete, there exists $\omega \in X$ such that $\{x_n\} \to \omega$. As φ is lower semi-continuous, we have

$$0 \leq \varphi(\omega) \leq \limsup_{n\to\infty} \varphi(x_n) = 0,$$

so $\varphi(\omega) = 0$, that is, $\omega \in Z_\varphi$. ω is a fixed point of T reasoning by contradiction. Suppose that $d(\omega, T\omega) > 0$. Let us define

$$\begin{aligned} &r = F(d(\omega, T\omega), 0, \varphi(T\omega)), \\ &a_n''' = F(d(x_{n+1}, T\omega), \varphi(x_{n+1}), \varphi(T\omega)) \quad \text{and} \quad b_n''' = M_F^{\varphi}(x_n, \omega), \end{aligned}$$

for all $n \in \mathbb{N}$. By (F_1),

$$r = F(d(\omega, T\omega), 0, \varphi(T\omega)) \geq \max\{d(\omega, T\omega), 0\} = d(\omega, T\omega) > 0. \tag{7.13}$$

As F is continuous,

$$\lim_{n\to\infty} a_n''' = \lim_{n\to\infty} F(d(x_{n+1}, T\omega), \varphi(x_{n+1}), \varphi(T\omega)) = F(d(\omega, T\omega), 0, \varphi(T\omega)) = r.$$

On the other hand,

$$\begin{aligned} b_n''' = M_F^\varphi(x_n, \omega) = \max \{&F(d(x_n, \omega), \varphi(x_n), \varphi(\omega)),\, F(d(x_n, Tx_n), \varphi(x_n), \varphi(Tx_n)), \\ &F(d(\omega, T\omega), \varphi(\omega), \varphi(T\omega))\} \\ = \max \{&F(d(x_n, \omega), \varphi(x_n), 0),\, F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})), \\ &F(d(\omega, T\omega), \varphi(\omega), \varphi(T\omega))\}. \end{aligned}$$

Since F is continuous,

$$\begin{aligned} &\lim_{n\to\infty} F(d(x_n, \omega), \varphi(x_n), 0) = F(0,0,0) = 0, \\ &\lim_{n\to\infty} F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) = F(0,0,0) = 0. \end{aligned}$$

As a consequence, there exists $n_0 \in \mathbb{N}$ such that

$$b_n''' = F(d(\omega, T\omega), 0, \varphi(T\omega)) = r, \quad n \geq n_0.$$

In particular, $\{a_n'''\}_{n\geq n_0} \subset [0, \infty)$ is a sequence converging to $r > 0$ and such that, for all $n \geq n_0$,

$$\begin{aligned} \theta\left(a_n''', r\right) = \theta\left(a_n''', b_n'''\right) &= \theta\left(F(d(x_{n+1}, T\omega), \varphi(x_{n+1}), \varphi(T\omega)), M_F^\varphi(x_n, \omega)\right) \\ &= \theta\left(F(d(Tx_n, T\omega), \varphi(Tx_n), \varphi(T\omega)), M_F^\varphi(x_n, \omega)\right) \geq 0, \end{aligned}$$

by virtue of (7.4). Thus, condition (θ_3) guarantees that $r = 0$, which contradicts (7.13). This contradiction shows that $d(\omega, T\omega) = 0$; that is, ω is a fixed point of T. In particular, $\text{Fix}(T)$ is nonempty, so $\emptyset \neq \text{Fix}(T) \subseteq Z_\varphi$, and the set $\text{Fix}(T)$ is φ-admissible. Furthermore, we have just proved that every Picard sequence of T converges to a fixed point of T. Therefore, (i) and (iii) hold.

Finally, let us show that T has a unique fixed point. By contradiction, assume that $(x, y) \in \text{Fix}(T) \times \text{Fix}(T)$, with $d(x, y) > 0$. In such a case, taking into account that $\text{Fix}(T) \subseteq Z_\varphi$, we derive that $\varphi(x) = \varphi(y) = 0$. Furthermore, as

$$\begin{aligned} M_F^\varphi(x, y) &= \max \{F(d(x, y), \varphi(x), \varphi(y)),\, F(d(x, Tx), \varphi(x), \varphi(Tx)), \\ &\qquad F(d(y, Ty), \varphi(y), \varphi(Ty))\} \\ &= \max \{F(d(x, y), 0, 0),\, F(0,0,0),\, F(0,0,0)\} \\ &= F(d(x, y), 0, 0) \\ &= d(x, y), \end{aligned}$$

condition (7.4) yields

$$\begin{aligned}0 &\le \theta\big(F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)), M_F^{\varphi}(x,y)\big)\\ &= \theta\big(F(d(x,y),\varphi(x),\varphi(y)), d(x,y)\big)\\ &= \theta\big(F(d(x,y),0,0), d(x,y)\big)\\ &= \theta\big(d(x,y), d(x,y)\big),\end{aligned}$$

which contradicts, by Remark 7.4, the fact that $\theta(d(x,y), d(x,y)) < 0$ (because $d(x,y) > 0$). Thus, $x = y$ and (ii) follows. The proof is complete.

The following result is similar to Theorem 7.3 and its proof follows, point by point, and in an easier way, repeating the arguments we have just shown in the proof of Theorem 7.3. However, there is not a direct relationship between both results because an e-simulation function does not have to be monotone in its second argument.

Theorem 7.4 *Let (X,d) be a metric space, and let $T : X \to X$ be a mapping. Assume that for some $\theta \in \mathscr{E}_Z$, $F \in \mathscr{F}$, and $\varphi : X \to [0,\infty)$, we have*

$$\theta\big(F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)),\ F(d(x,y),\varphi(x),\varphi(y))\big) \ge 0, \quad (x,y) \in X \times X. \tag{7.14}$$

If φ is lower semi-continuous, then

(i) For every $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point of T.
(ii) T has a unique fixed point.
(iii) $\mathrm{Fix}(T)$ *is φ-admissible.*

Let (X,d) be a metric space. For given functions $\varphi : X \to [0,\infty)$, $F \in \mathscr{F}$, and $\psi \in \Psi$, we denote by $\mathscr{T}(\varphi, F, \psi)$ the class of operators $T : X \to X$ satisfying

$$F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \le \psi(F(d(x,y),\varphi(x),\varphi(y))), \quad (x,y) \in X \times X. \tag{7.15}$$

The following result due to Karapinar, O'Regan, and Samet [7] follows from Theorem 7.4.

Corollary 7.1 *Let (X,d) be a complete metric space and $T : X \to X$ be a given operator. Suppose that the following conditions hold:*

(i) There exist $\varphi : X \to [0,\infty)$, $F \in \mathscr{F}$, and $\psi \in \Psi$ such that $T \in \mathscr{T}(\varphi, F, \psi)$;
(ii) φ is lower semi-continuous.

Then the set $\mathrm{Fix}(T)$ *is φ-admissible. Moreover, the operator T has a unique fixed point.*

Proof Under the considered assumptions, let θ_ψ be the function defined by (7.3). Lemma 7.3 guarantees that θ_ψ is an e-simulation function. Moreover, condition (7.15) is equivalent to

$$\begin{aligned}&\theta_\psi\big(F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)), F(d(x,y),\varphi(x),\varphi(y))\big)\\ &= \psi\left(F(d(x,y),\varphi(x),\varphi(y))\right) - F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \ge 0, \quad (x,y) \in X \times X,\end{aligned}$$

which means that T satisfies (7.14) with $\theta = \theta_\psi$. Thus, Theorem 7.4 is applicable.

In the following example, we show that Theorem 7.3 improves Corollary 7.1.

Example 7.3 Let $X = [-3, 3]$. We endow X with the Euclidean metric

$$d(x, y) = |x - y|, \quad (x, y) \in X \times X.$$

Obviously, (X, d) is a complete metric space. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} -2 & \text{if} \quad x = 1, \\ -\dfrac{x}{12} & \text{if} \ x \in X \backslash \{1\}. \end{cases}$$

We will show that using the functions

$$\varphi : X \to [0, \infty), \quad \varphi(x) = 0, \quad \text{for all } x \in X, \qquad \text{and} \tag{7.16}$$

$$F : [0, \infty)^3 \to [0, \infty), \quad F(t, s, r) = t + s + r, \quad \text{for all } t, s, r \in [0, \infty), \tag{7.17}$$

Theorem 7.3 is applicable but Corollary 7.1 is not. Indeed, assume that there is $\psi \in \Psi$ such that (7.15) holds. Therefore, for all $x, y \in X$,

$$\begin{aligned} d(Tx, Ty) &= d(Tx, Ty) + 0 + 0 = d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) \\ &= F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq \psi\left(F(d(x, y), \varphi(x), \varphi(y))\right) \\ &= \psi\left(d(x, y) + 0 + 0\right) = \psi\left(d(x, y)\right). \end{aligned}$$

However, if $x_0 = 0$ and $y_0 = 1$, then

$$\begin{aligned} & d(T(0), T(1)) = d(0, -2) = 2, \qquad \text{but} \\ & \psi\left(d(0, 1)\right) = \psi(1) < 1, \end{aligned}$$

which contradicts the previous inequality. As a consequence, it is impossible to find $\psi \in \Psi$ such that (7.15) holds, so Corollary 7.1 is not applicable. Nevertheless, let us consider the function $\theta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ defined by

$$\theta(t, s) = \frac{3}{4} s - t, \quad t, s \geq 0.$$

Then θ is a simulation function (see [8], Example 2.2, (i)). By Proposition 7.1, it is also an e-simulation function. As φ and F are given by (7.16) and (7.17), we have to prove that

$$\theta\big(d(Tx, Ty), M_F^\varphi(x, y)\big) \geq 0, \quad (x, y) \in X \times X, \tag{7.18}$$

where

$$\begin{aligned} M_F^{\varphi}(x,y) &= \max\{F(d(x,y),\varphi(x),\varphi(y)), F(d(x,Tx),\varphi(x),\varphi(Tx)),\\ &\qquad F(d(y,Ty),\varphi(y),\varphi(Ty))\}\\ &= \max\{d(x,y),d(x,Tx),d(y,Ty)\}. \end{aligned}$$

Indeed, we consider two cases.

- If $x, y \in X\backslash\{1\}$, then

$$\begin{aligned} \theta\big(d(Tx,Ty),M_F^{\varphi}(x,y)\big) &= \frac{3}{4}M_F^{\varphi}(x,y) - d\left(-\frac{x}{12},-\frac{y}{12}\right) \geq \frac{3}{4}d(x,y) - d\left(\frac{x}{12},\frac{y}{12}\right)\\ &= \frac{3}{4}|x-y| - \frac{1}{12}|x-y| = \frac{2}{3}|x-y| \geq 0. \end{aligned}$$

- If $x \in X\backslash\{1\}$ and $y = 1$, taking into account that $x/12 \in [-1/4, 1/4]$, we deduce that

$$\begin{aligned} &d(Tx,Ty) = d\left(-\frac{x}{12},-2\right) = d\left(\frac{x}{12},2\right) = \left|2-\frac{x}{12}\right| = 2-\frac{x}{12},\\ &d(y,Ty) = d(1,-2) = 3, \text{ and}\\ &M_F^{\varphi}(x,y) = \max\{d(x,y),\ d(x,Tx),\ d(y,Ty)\} \geq 3. \end{aligned}$$

  Therefore,

$$\begin{aligned} \theta\big(d(Tx,Ty),M_F^{\varphi}(x,y)\big) &= \frac{3}{4}M_F^{\varphi}(x,y) - \left(2-\frac{x}{12}\right) \geq \frac{3}{4}3 - \left(2-\frac{x}{12}\right)\\ &= \frac{x+3}{12} \geq 0. \end{aligned}$$

Thus, in all cases, (7.18) is satisfied. Therefore, Theorem 7.3 is applicable, and we conclude that T has a unique fixed point.

7.5 Some Consequences

In this section, some fixed point theorems in metric and partial metric spaces are deduced from the above results.

7.5.1 Fixed Point Results in Partial Metric Spaces via Extended Simulation Functions

In this part, some fixed point theorems in partial metric spaces are deduced from the above results. Therefore, we answer to all the questions of I.A. Rus presented in Sect. 7.2.

The following result will be useful later.

Lemma 7.4 *Let (X, p) be a partial metric space. Let $\varphi : X \to [0, \infty)$ be the function defined by*

$$\varphi(x) = p(x, x), \quad x \in X.$$

Then φ is continuous with respect to the topology induced by the metric d_p.

Proof Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} d_p(x_n, x) = 0,$$

for some $x \in X$. From (ii), Lemma 7.2, we have

$$\lim_{n\to\infty} p(x_n, x_n) = p(x, x),$$

i.e.,

$$\lim_{n\to\infty} \varphi(x_n) = \varphi(x),$$

which proves the continuity of φ with respect to d_p.

We have the following fixed point result in a complete partial metric space.

Corollary 7.2 *Let (X, p) be a complete metric space, and let $T : X \to X$ be a given mapping. Suppose that there exists some $\theta \in \mathscr{E}_Z$ such that*

$$\theta\big(p(Tx, Ty), \max\{p(x, y), p(x, Tx), p(y, Ty)\}\big) \geq 0, \quad (x, y) \in X \times X. \tag{7.19}$$

Then T has a unique fixed point $x^* \in X$. For all $x \in X$, the Picard sequence $\{T^n x\}$ converges to x^*. Moreover, $p(x^*, x^*) = 0$.

Proof Observe that (7.19) is equivalent to (7.4) with

$$F(a, b, c) = a + b + c, \quad a, b, c \geq 0,$$
$$\varphi(x) = \frac{p(x, x)}{2}, \quad x \in X,$$
$$d(x, y) = \frac{d_p(x, y)}{2}, \quad (x, y) \in X \times X.$$

On the other hand, from (ii), Lemma 7.2, since the partial metric space (X, p) is complete, then the metric space (X, d) is complete. Moreover, from Lemma 7.4, the function $\varphi : X \to [0, \infty)$ is continuous with respect to the metric d. Therefore, the desired result follows from Theorem 7.3.

Corollary 7.3 *Let (X, p) be a complete metric space, and let $T : X \to X$ be a given mapping. Suppose that there exists some $\psi \in \Psi$ such that*

$$p(Tx, Ty) \leq \psi\left(\max\{p(x, y), p(x, Tx), p(y, Ty)\}\right), \quad (x, y) \in X \times X. \tag{7.20}$$

Then T has a unique fixed point $x^* \in X$. For all $x \in X$, the Picard sequence $\{T^n x\}$ converges to x^*. Moreover, $p(x^*, x^*) = 0$.

Proof Taking $\theta = \theta_\psi$ in (7.19), we obtain (7.20). Using Lemma 7.3 and Corollary 7.2, the desired result follows.

Corollary 7.4 *Let (X, p) be a complete metric space, and let $T : X \to X$ be a given mapping. Suppose that there exists a lower semi-continuous function $\mu : [0, \infty) \to [0, \infty)$ with $\mu^{-1}(\{0\}) = \{0\}$, such that*

$$p(Tx, Ty) \leq \max\{p(x, y), p(x, Tx), p(y, Ty)\} - \mu\left(\max\{p(x, y), p(x, Tx), p(y, Ty)\}\right), \tag{7.21}$$

for all $(x, y) \in X \times X$. Then T has a unique fixed point $x^* \in X$. For all $x \in X$, the Picard sequence $\{T^n x\}$ converges to x^*. Moreover, $p(x^*, x^*) = 0$.

Proof Taking in (7.19), $\theta(t, s) = s - \mu(s) - t$, for all $t, s \geq 0$, we obtain (7.21). On the other hand, it was proved in [8] that the function θ defined above is a simulation function. Therefore, by Corollary 7.2 and Proposition 7.1, the result follows.

Remark 7.5 Observe that if a mapping $T : X \to X$ satisfies (7.1), then it satisfies (7.20) with $\psi(t) = kt$, $t \geq 0$. Therefore, Corollary 7.3 is a generalization of Matthews result given by Theorem 7.1.

7.5.2 *Fixed Point Results in Metric Spaces via Extended Simulation Functions*

As any metric space is a partial metric space, the following results follow immediately from the above corollaries.

From Corollary 7.2, we deduce the following result.

Corollary 7.5 *Let (X, d) be a complete metric space, and let $T : X \to X$ be a given mapping. Suppose that there exists some $\theta \in \mathscr{E}_Z$ such that*

$$\theta\big(d(Tx, Ty), \max\{d(x, y), d(x, Tx), d(y, Ty)\}\big) \geq 0, \quad (x, y) \in X \times X.$$

Then T has a unique fixed point $x^* \in X$. For all $x \in X$, the Picard sequence $\{T^n x\}$ converges to x^*.

From Corollary 7.3, we deduce the following result.

Corollary 7.6 *Let (X, d) be a complete metric space, and let $T : X \to X$ be a given mapping. Suppose that there exists some $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi\left(\max\{d(x, y), d(x, Tx), d(y, Ty)\}\right), \quad (x, y) \in X \times X.$$

Then T has a unique fixed point $x^* \in X$. For all $x \in X$, the Picard sequence $\{T^n x\}$ converges to x^*.

Finally, from Corollary 7.4, we deduce the following result.

Corollary 7.7 *Let (X, d) be a complete metric space, and let $T : X \to X$ be a given mapping. Suppose that there exists a lower semi-continuous function $\mu : [0, \infty) \to [0, \infty)$ with $\mu^{-1}(\{0\}) = \{0\}$, such that*

$$d(Tx, Ty) \leq \max\{d(x, y), d(x, Tx), d(y, Ty)\} - \mu\left(\max\{d(x, y), d(x, Tx), d(y, Ty)\}\right),$$

for all $(x, y) \in X \times X$. Then T has a unique fixed point $x^* \in X$. For all $x \in X$, the Picard sequence $\{T^n x\}$ converges to x^*.

Remark 7.6 Corollary 7.6 is an extension of Boyd–Wong fixed point theorem [4].

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Chapter 8
A Coupled Fixed Point Problem Under a Finite Number of Equality Constraints

Let $(E, \|\cdot\|)$ be a Banach space with a cone P. Let $F, \varphi_i : E \times E \to E$ $(i = 1, 2, \ldots, r)$ be a finite number of mappings. In this chapter, we provide sufficient conditions for the existence and uniqueness of solutions to the problem: Find $(x, y) \in E \times E$ such that

$$\begin{cases} F(x, y) = x, \\ F(y, x) = y, \\ \varphi_i(x, y) = 0_E, \; i = 1, 2, \ldots, r, \end{cases} \tag{8.1}$$

where 0_E is the zero vector of E. The main reference for this chapter is the paper [4].

8.1 Preliminaries

At first, let us recall some basic definitions and some preliminary results that will be used later. In this chapter, the considered Banach space $(E, \|\cdot\|)$ is supposed to be partially ordered by a cone P. Recall that a nonempty closed convex set $P \subset E$ is said to be a cone (see [2]) if it satisfies the following conditions:

(P1) $\lambda \geq 0, \; x \in P \Longrightarrow \lambda x \in P$;
(P2) $-x, x \in P \Longrightarrow x = 0_E$.

We define the partial order $\leq_P$ in E induced by the cone P by

$$(x, y) \in E \times E, \quad x \leq_P y \Longleftrightarrow y - x \in P.$$

Definition 8.1 ([1]) Let $\varphi : E \times E \to E$ be a given mapping. We say that φ is level closed from the right if for every $e \in E$, the set

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$$\text{lev}\varphi_{\leq_P}(e) := \{(x, y) \in E \times E : \varphi(x, y) \leq_P e\}$$

is closed.

Definition 8.2 Let $\varphi : E \times E \to E$ be a given mapping. We say that φ is level closed from the left if for every $e \in E$, the set

$$\text{lev}\varphi_{\geq_P}(e) := \{(x, y) \in E \times E : e \leq_P \varphi(x, y)\}$$

is closed.

We denote by Ψ the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the conditions:

(Ψ_1) ψ is nondecreasing;
(Ψ_2) For all $t > 0$, we have

$$\sum_{k=0}^{\infty} \psi^k(t) < \infty.$$

Here, ψ^k is the kth iterate of ψ.

The following properties are not difficult to prove.

Lemma 8.1 *Let $\psi \in \Psi$. Then*

(i) $\psi(t) < t$, $t > 0$*;*
(ii) $\psi(0) = 0$*;*
(iii) *ψ is continuous at $t = 0$.*

Example 8.1 As examples, the following functions belong to the set Ψ:

$\psi(t) = k\,t$, $k \in (0, 1)$.

$$\psi(t) = \begin{cases} t/2 & \text{if } 0 \leq t \leq 1, \\ 1/2 & \text{if } t > 1. \end{cases}$$

$$\psi(t) = \begin{cases} t/2 & \text{if } 0 \leq t < 1, \\ t - 1/3 & \text{if } t \geq 1. \end{cases}$$

Now, we are ready to state and prove the main results of this chapter. This is the aim of the next section.

8.2 Main Results

Through this chapter, $(E, \|\cdot\|)$ is a Banach space partially ordered by a cone P and 0_E denotes the zero vector of E.

Let us start with the case of one equality constraint.

8.2.1 A Coupled Fixed Point Problem Under One Equality Constraint

We are interested with the existence and uniqueness of solutions to the problem: Find $(x, y) \in E \times E$ such that

$$\begin{cases} F(x, y) = x, \\ F(y, x) = y, \\ \varphi(x, y) = 0_E, \end{cases} \tag{8.2}$$

where $F, \varphi : E \times E \to E$ are two given mappings.

The following theorem provides sufficient conditions for the existence and uniqueness of solutions to (8.2).

Theorem 8.1 *Let $F, \varphi : E \times E \to E$ be two given mappings. Suppose that the following conditions are satisfied:*

(i) φ is level closed from the right.
(ii) There exists $(x_0, y_0) \in E \times E$ such that $\varphi(x_0, y_0) \leq_P 0_E$.
(iii) For every $(x, y) \in E \times E$, we have

$$\varphi(x, y) \leq_P 0_E \Longrightarrow \varphi(F(x, y), F(y, x)) \geq_P 0_E.$$

(iv) For every $(x, y) \in E \times E$, we have

$$\varphi(x, y) \geq_P 0_E \Longrightarrow \varphi(F(x, y), F(y, x)) \leq_P 0_E.$$

(v) There exists some $\psi \in \Psi$ such that

$$\|F(u, v) - F(x, y)\| + \|F(y, x) - F(v, u)\| \leq \psi\left(\|u - x\| + \|v - y\|\right),$$

for all $(x, y), (u, v) \in E \times E$ with $\varphi(x, y) \leq_P 0_E$, $\varphi(u, v) \geq_P 0_E$.

Then (8.2) has a unique solution.

Proof Let $(x_0, y_0) \in E \times E$ be such that

$$\varphi(x_0, y_0) \leq_p 0_E.$$

Such a point exists from (ii). From (iii), we have

$$\varphi(x_0, y_0) \leq_P 0_E \Longrightarrow \varphi(F(x_0, y_0), F(y_0, x_0)) \geq_P 0_E.$$

Define the sequences $\{x_n\}$ and $\{y_n\}$ in E by

$$x_{n+1} = F(x_n, y_n), \quad y_{n+1} = F(y_n, x_n), \quad n = 0, 1, 2, \ldots$$

Then we have

$$\varphi(x_1, y_1) \geq_P 0_E.$$

From (iv), we have

$$\varphi(x_1, y_1) \geq_P 0_E \Longrightarrow \varphi(F(x_1, y_1), F(y_1, x_1)) \leq_P 0_E,$$

that is,

$$\varphi(x_2, y_2) \leq_P 0_E.$$

Again, using (iii), we get from the above inequality that

$$\varphi(x_3, y_3) \geq_P 0_E.$$

Then, by induction, we obtain

$$\varphi(x_{2n}, y_{2n}) \leq_P 0_E, \ \varphi(x_{2n+1}, y_{2n+1}) \geq_P 0_E, \quad n = 0, 1, 2, \ldots \tag{8.3}$$

Using (v) and (8.3), by symmetry, we obtain

$$\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \leq \psi\left(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\right), \quad n = 1, 2, 3, \ldots \tag{8.4}$$

From (8.4), since ψ is a nondecreasing function, for every $n = 1, 2, 3, \ldots$, we have

$$\begin{aligned} \|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| &\leq \psi\left(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\right) \\ &\leq \psi^2\left(\|x_{n-1} - x_{n-2}\| + \|y_{n-1} - y_{n-2}\|\right) \\ &\leq \cdots \\ &\leq \psi^n\left(\|x_1 - x_0\| + \|y_1 - y_0\|\right). \end{aligned} \tag{8.5}$$

Suppose that

$$\|x_1 - x_0\| + \|y_1 - y_0\| = 0.$$

In this case, we have

$$x_0 = x_1 = F(x_0, y_0) \quad \text{and} \quad y_0 = y_1 = F(y_0, x_0).$$

Moreover, from (iii), since $\varphi(x_0, y_0) \leq_P 0_E$, we obtain $\varphi(x_1, y_1) = \varphi(x_0, y_0) \geq 0_E$. Since P is a cone, the two inequalities $\varphi(x_0, y_0) \leq_P 0_E$ and $\varphi(x_0, y_0) \geq_P 0_E$ yield

$$\varphi(x_0, y_0) = 0_E.$$

Thus, we proved that in this case, $(x_0, y_0) \in E \times E$ is a solution to (8.2).

Now, we may suppose that $\|x_1 - x_0\| + \|y_1 - y_0\| \neq 0$. Set

$$\delta = \|x_1 - x_0\| + \|y_1 - y_0\| > 0.$$

From (8.5), we have

$$\|x_{n+1} - x_n\| \leq \psi^n(\delta), \quad n = 0, 1, 2, \ldots \tag{8.6}$$

Using the triangular inequality and (8.6), for all $m = 1, 2, 3, \ldots$, we have

$$\begin{aligned}
\|x_n - x_{n+m}\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + \cdots + \|x_{n+m-1} - x_{n+m}\| \\
&\leq \psi^n(\delta) + \psi^{n+1}(\delta) + \cdots + \psi^{n+m-1}(\delta) \\
&= \sum_{i=n}^{n+m-1} \psi^i(\delta) \\
&\leq \sum_{i=n}^{\infty} \psi^i(\delta).
\end{aligned}$$

On the other hand, since $\sum_{k=0}^{\infty} \psi^k(\delta) < \infty$, we have

$$\sum_{i=n}^{\infty} \psi^i(\delta) \to 0 \text{ as } n \to \infty,$$

which implies that $\{x_n\}$ is a Cauchy sequence in $(E, \|\cdot\|)$. The same argument gives us that $\{y_n\}$ is a Cauchy sequence in $(E, \|\cdot\|)$. As consequence, there exists a pair of points $(x^*, y^*) \in E \times E$ such that

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n\to\infty} \|y_n - y^*\| = 0. \tag{8.7}$$

From (8.3), we have

$$\varphi(x_{2n}, y_{2n}) \leq_P 0_E, \quad n = 0, 1, 2, \ldots,$$

that is,

$$(x_{2n}, y_{2n}) \in \text{lev}\varphi_{\leq_P}(0_E), \quad n = 0, 1, 2, \ldots,$$

Since φ is level closed from the right, passing to the limit as $n \to \infty$ and using (8.7), we obtain

$$(x^*, y^*) \in \text{lev}\varphi_{\leq_P}(0_E),$$

that is,

$$\varphi(x^*, y^*) \leq_P 0_E. \tag{8.8}$$

Now, using (8.3), (8.8), and (v), we obtain

$$\|F(x_{2n+1}, y_{2n+1}) - F(x^*, y^*)\| + \|F(y^*, x^*) - F(y_{2n+1}, x_{2n+1})\|$$
$$\leq \psi\left(\|x_{2n+1} - x^*\| + \|y_{2n+1} - y^*\|\right),$$

for all $n = 0, 1, 2, \ldots$, which implies that

$$\|x_{2n+2} - F(x^*, y^*)\| + \|F(y^*, x^*) - y_{2n+2}\| \leq \psi\left(\|x_{2n+1} - x^*\| + \|y_{2n+1} - y^*\|\right),$$

for all $n = 0, 1, 2, \ldots$ Passing to the limit as $n \to \infty$, using (8.7), the continuity of ψ at 0, and the fact that $\psi(0) = 0$ (see Lemma 8.1), we get

$$\|x^* - F(x^*, y^*)\| + \|F(y^*, x^*) - y^*\| = 0,$$

that is,

$$x^* = F(x^*, y^*) \quad \text{and} \quad y^* = F(y^*, x^*).$$

This proves that $(x^*, y^*) \in E \times E$ is a coupled fixed point of F. Finally, using (8.8) and the fact that (x^*, y^*) is a coupled fixed point of F, it follows from (iii) that

$$\varphi(x^*, y^*) \geq_P 0_E. \tag{8.9}$$

Then (8.8) and (8.9) yield

$$\varphi(x^*, y^*) = 0_E.$$

Thus, we proved that $(x^*, y^*) \in E \times E$ is a solution to (8.2). Suppose now that $(u^*, v^*) \in E \times E$ is a solution to (8.2) with $(x^*, y^*) \neq (u^*, v^*)$. Using (v), we obtain

$$\|u^* - x^*\| + \|y^* - v^*\| \leq \psi(\|u^* - x^*\| + \|y^* - v^*\|).$$

Since $\|u^* - x^*\| + \|y^* - v^*\| > 0$, from (i) of Lemma 8.1, we have

$$\psi(\|u^* - x^*\| + \|y^* - v^*\|) < \|u^* - x^*\| + \|y^* - v^*\|.$$

Then

$$\|u^* - x^*\| + \|y^* - v^*\| < \|u^* - x^*\| + \|y^* - v^*\|,$$

which is a contradiction. As consequence, (x^*, y^*) is the unique solution to (8.2).

Remark 8.1 Observe that the conclusion of Theorem 8.1 is still valid if we replace condition (i) by the following condition:
(i') φ is level closed from the left.
In fact, from (8.3), we have

$$\varphi(x_{2n+1}, y_{2n+1}) \geq_P 0_E, \quad n = 0, 1, 2, \ldots,$$

that is,

$$(x_{2n+1}, y_{2n+1}) \in \text{lev}\varphi_{\geq_P}, \quad n = 0, 1, 2, \ldots$$

Passing to the limit as $n \to \infty$ and using (8.7), we obtain

$$\varphi(x^*, y^*) \geq_P 0_E. \tag{8.10}$$

Using (8.3), (8.10) and (v), we obtain

$$\|F(x_{2n}, y_{2n}) - F(x^*, y^*)\| + \|F(y^*, x^*) - F(y_{2n}, x_{2n})\| \leq \psi\left(\|x_{2n} - x^*\| + \|y_{2n} - y^*\|\right),$$

for all $n = 0, 1, 2, \ldots$, which implies that

$$\|x_{2n+1} - F(x^*, y^*)\| + \|F(y^*, x^*) - y_{2n+1}\| \leq \psi\left(\|x_{2n} - x^*\| + \|y_{2n} - y^*\|\right),$$

for all $n = 0, 1, 2, \ldots$ Passing to the limit as $n \to \infty$, we get

$$\|x^* - F(x^*, y^*)\| + \|F(y^*, x^*) - y^*\| = 0,$$

which proves that $(x^*, y^*) \in E \times E$ is a coupled fixed point of F. Using (8.10) and the fact that (x^*, y^*) is a coupled fixed point of F, it follows from (iv) that

$$\varphi(x^*, y^*) \leq_P 0_E. \tag{8.11}$$

Then (8.10) and (8.11) yield

$$\varphi(x^*, y^*) = 0_E.$$

Thus, $(x^*, y^*) \in E \times E$ is a solution to (8.2).

8.2.2 A Coupled Fixed Point Problem Under Two Equality Constraints

Here, we are interested with the existence and uniqueness of solutions to the following problem: Find $(x, y) \in E \times E$ such that

$$\begin{cases} F(x, y) &= x, \\ F(y, x) &= y, \\ \varphi_1(x, y) &= 0_E, \\ \varphi_2(x, y) &= 0_E, \end{cases} \tag{8.12}$$

where $F, \varphi_1, \varphi_2 : E \times E \to E$ are three given mappings.

We have the following result.

Theorem 8.2 *Let $F, \varphi_1, \varphi_2 : E \times E \to E$ be three given mappings. Suppose that the following conditions are satisfied:*

(i) φ_i *($i = 1, 2$) is level closed from the right.*
(ii) There exists $(x_0, y_0) \in E \times E$ *such that* $\varphi_i(x_0, y_0) \leq_P 0_E$ *($i = 1, 2$).*
(iii) For every $(x, y) \in E \times E$*, we have*

$$\varphi_i(x, y) \leq_P 0_E,\ i = 1, 2 \Longrightarrow \varphi_i(F(x, y), F(y, x)) \geq_P 0_E,\ i = 1, 2.$$

(iv) For every $(x, y) \in E \times E$*, we have*

$$\varphi_i(x, y) \geq_P 0_E,\ i = 1, 2 \Longrightarrow \varphi_i(F(x, y), F(y, x)) \leq_P 0_E,\ i = 1, 2.$$

(v) There exists some $\psi \in \Psi$ *such that*

$$\|F(u, v) - F(x, y)\| + \|F(y, x) - F(v, u)\| \leq \psi\left(\|u - x\| + \|v - y\|\right),$$

for all $(x, y), (u, v) \in E \times E$ *with* $\varphi_i(x, y) \leq_P 0_E, \varphi_i(u, v) \geq_P 0_E$*,* $i = 1, 2$.

Then (8.12) has a unique solution.

Proof Let $(x_0, y_0) \in E \times E$ be such that

$$\varphi_i(x_0, y_0) \leq_p 0_E, \quad i = 1, 2.$$

Then from (iii), we have

$$\varphi_i(F(x_0, y_0), F(y_0, x_0)) \geq_P 0_E, \quad i = 1, 2.$$

Define the sequences $\{x_n\}$ and $\{y_n\}$ in E by

$$x_{n+1} = F(x_n, y_n),\ \ y_{n+1} = F(y_n, x_n), \quad n = 0, 1, 2, \ldots$$

We have

$$\varphi_i(x_1, y_1) \geq_P 0_E, \quad i = 1, 2.$$

Then from (iv), we obtain

$$\varphi_i(x_2, y_2) \leq_P 0_E, \quad i = 1, 2.$$

Again, using (iii), we get from the above inequality that

$$\varphi_i(x_3, y_3) \geq_P 0_E, \quad i = 1, 2.$$

Then, by induction, we obtain

$$\varphi_i(x_{2n}, y_{2n}) \leq_P 0_E, \; \varphi_i(x_{2n+1}, y_{2n+1}) \geq_P 0_E, \quad i = 1, 2, \; n = 0, 1, 2, \ldots$$

Then, using (v), we obtain

$$\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \leq \psi\left(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\right), \quad n = 1, 2, 3, \ldots$$

Now, we argue exactly as in the proof of Theorem 8.1 to show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $(E, \|\cdot\|)$. As consequence, there exists a pair of points $(x^*, y^*) \in E \times E$ such that

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n\to\infty} \|y_n - y^*\| = 0.$$

On the other hand, we have

$$(x_{2n}, y_{2n}) \in \text{lev}\varphi_{i \leq_P}(0_E), \quad i = 1, 2, \; n = 0, 1, 2, \ldots,$$

Since φ_i $(i = 1, 2)$ is level closed from the right, passing to the limit as $n \to \infty$, we obtain

$$(x^*, y^*) \in \text{lev}\varphi_{i \leq_P}(0_E), \quad i = 1, 2,$$

that is,

$$\varphi_i(x^*, y^*) \leq_P 0_E, \quad i = 1, 2.$$

Then we have

$$\begin{aligned}&\|F(x_{2n+1}, y_{2n+1}) - F(x^*, y^*)\| + \|F(y^*, x^*) - F(y_{2n+1}, x_{2n+1})\| \\ &\leq \psi\left(\|x_{2n+1} - x^*\| + \|y_{2n+1} - y^*\|\right),\end{aligned}$$

for all $n = 0, 1, 2, \ldots$, which implies that

$$\|x_{2n+2} - F(x^*, y^*)\| + \|F(y^*, x^*) - y_{2n+2}\| \leq \psi\left(\|x_{2n+1} - x^*\| + \|y_{2n+1} - y^*\|\right),$$

for all $n = 0, 1, 2, \ldots$ Passing to the limit as $n \to \infty$, we get

$$\|x^* - F(x^*, y^*)\| + \|F(y^*, x^*) - y^*\| = 0,$$

that is,

$$x^* = F(x^*, y^*) \quad \text{and} \quad y^* = F(y^*, x^*).$$

This proves that $(x^*, y^*) \in E \times E$ is a coupled fixed point of F. Since $\varphi_i(x^*, y^*) \leq_P 0_E$ for $i = 1, 2$, from (iii) we have

$$\varphi_i(F(x^*, y^*), F(y^*, x^*)) \geq_P 0_E, \quad i = 1, 2,$$

that is,

$$\varphi_i(x^*, y^*) \geq_P 0_E, \quad i = 1, 2.$$

Finally, the two inequalities $\varphi_i(x^*, y^*) \leq_P 0_E$ and $\varphi_i(x^*, y^*) \geq_P 0_E$, $i = 1, 2$ yield $\varphi_i(x^*, y^*) = 0_E$, $i = 1, 2$. Then we proved that $(x^*, y^*) \in E \times E$ is a solution to (8.12). The uniqueness can be obtained using a similar argument as in the proof of Theorem 8.1.

Replace φ_2 in Theorem 8.2 by $-\varphi_2$, we obtain the following result.

Theorem 8.3 *Let $F, \varphi_1, \varphi_2 : E \times E \to E$ be three given mappings. Suppose that the following conditions are satisfied:*

(i) *φ_1 is level closed from the right and φ_2 is level closed from the left.*
(ii) *There exists $(x_0, y_0) \in E \times E$ such that $\varphi_1(x_0, y_0) \leq_P 0_E$ and $\varphi_2(x_0, y_0) \geq_p 0_E$.*
(iii) *For every $(x, y) \in E \times E$ with $\varphi_1(x, y) \leq_P 0_E$ and $\varphi_2(x, y) \geq_P 0_E$, we have*

$$\varphi_1(F(x, y), F(y, x)) \geq_P 0_E,\ \varphi_2(F(x, y), F(y, x)) \leq_P 0_E.$$

(iv) *For every $(x, y) \in E \times E$ with $\varphi_1(x, y) \geq_P 0_E$ and $\varphi_2(x, y) \leq_P 0_E$, we have*

$$\varphi_1(F(x, y), F(y, x)) \leq_P 0_E,\ \varphi_2(F(x, y), F(y, x)) \geq_P 0_E.$$

(v) *There exists some $\psi \in \Psi$ such that*

$$\|F(u, v) - F(x, y)\| + \|F(y, x) - F(v, u)\| \leq \psi\left(\|u - x\| + \|v - y\|\right),$$

for all $(x, y), (u, v) \in E \times E$ with $\varphi_1(x, y) \leq_P 0_E$, $\varphi_2(x, y) \geq_P 0_E$, $\varphi_1(u, v) \geq_P 0_E$, $\varphi_2(u, v) \leq_P 0_E$.

Then (8.12) has a unique solution.

Replace φ_1 in Theorem 8.3 by $-\varphi_1$, we obtain the following result.

Theorem 8.4 *Let $F, \varphi_1, \varphi_2 : E \times E \to E$ be three given mappings. Suppose that the following conditions are satisfied:*

(i) *φ_i $(i = 1, 2)$ is level closed from the left.*
(ii) *There exists $(x_0, y_0) \in E \times E$ such that $\varphi_i(x_0, y_0) \geq_P 0_E$ $(i = 1, 2)$.*
(iii) *For every $(x, y) \in E \times E$, we have*

$$\varphi_i(x, y) \leq_P 0_E,\ i = 1, 2 \Longrightarrow \varphi_i(F(x, y), F(y, x)) \geq_P 0_E,\ i = 1, 2.$$

(iv) *For every $(x, y) \in E \times E$, we have*

$$\varphi_i(x, y) \geq_P 0_E,\ i = 1, 2 \Longrightarrow \varphi_i(F(x, y), F(y, x)) \leq_P 0_E,\ i = 1, 2.$$

(v) *There exists some* $\psi \in \Psi$ *such that*

$$\|F(u,v) - F(x,y)\| + \|F(y,x) - F(v,u)\| \le \psi\left(\|u-x\| + \|v-y\|\right),$$

for all $(x,y),(u,v) \in E \times E$ *with* $\varphi_i(x,y) \le_P 0_E$, $\varphi_i(u,v) \ge_P 0_E$, $i = 1, 2$.

Then (8.12) *has a unique solution.*

8.2.3 A Coupled Fixed Point Problem Under r Equality Constraints

Now, we argue exactly as in the proof of Theorem 8.2 to obtain the following existence result for (8.1).

Theorem 8.5 *Let* $F, \varphi_i : E \times E \to E$ $(i = 1, 2, \ldots, r)$ *be* $r+1$ *given mappings. Suppose that the following conditions are satisfied:*

(i) φ_i $(i = 1, 2, \ldots, r)$ *is level closed from the right.*
(ii) *There exists* $(x_0, y_0) \in E \times E$ *such that* $\varphi_i(x_0, y_0) \le_P 0_E$ $(i = 1, 2, \ldots, r)$.
(iii) *For every* $(x,y) \in E \times E$, *we have*

$$\varphi_i(x,y) \le_P 0_E,\ i = 1, 2, \ldots, r \Longrightarrow \varphi_i(F(x,y), F(y,x)) \ge_P 0_E,\ i = 1, 2, \ldots, r.$$

(iv) *For every* $(x,y) \in E \times E$, *we have*

$$\varphi_i(x,y) \ge_P 0_E,\ i = 1, 2, \ldots, r \Longrightarrow \varphi_i(F(x,y), F(y,x)) \le_P 0_E,\ i = 1, 2, \ldots r.$$

(v) *There exists some* $\psi \in \Psi$ *such that*

$$\|F(u,v) - F(x,y)\| + \|F(y,x) - F(v,u)\| \le \psi\left(\|u-x\| + \|v-y\|\right),$$

for all $(x,y),(u,v) \in E \times E$ *with* $\varphi_i(x,y) \le_P 0_E$, $\varphi_i(u,v) \ge_P 0_E$, $i = 1, 2, \ldots, r$.

Then (8.1) *has a unique solution.*

8.3 Some Consequences

In this section, we present some consequences following from Theorem 8.5.

8.3.1 A Fixed Point Problem Under Symmetric Equality Constraints

Let X be a nonempty set and let $F : X \times X \to X$ be a given mapping. Recall that that $x \in X$ is said to be a fixed point of F if $F(x, x) = x$.

Let $F, \varphi : E \times E \to E$ be given mappings. We consider the problem: Find $x \in E$ such that

$$\begin{cases} F(x, x) = x, \\ \varphi(x, x) \ = 0_E. \end{cases} \tag{8.13}$$

We have the following result.

Corollary 8.1 *Let $F, \varphi : E \times E \to E$ be two given mappings. Suppose that the following conditions are satisfied:*

(i) φ is level closed from the right.
(ii) φ is symmetric, that is,

$$\varphi(x, y) = \varphi(y, x), \quad (x, y) \in E \times E.$$

(iii) There exists $(x_0, y_0) \in E \times E$ such that $\varphi(x_0, y_0) \leq_P 0_E$.
(iv) For every $(x, y) \in E \times E$, we have

$$\varphi(x, y) \leq_P 0_E \Longrightarrow \varphi(F(x, y), F(y, x)) \geq_P 0_E.$$

(v) For every $(x, y) \in E \times E$, we have

$$\varphi(x, y) \geq_P 0_E \Longrightarrow \varphi(F(x, y), F(y, x)) \leq_P 0_E.$$

(vi) There exists some $\psi \in \Psi$ such that

$$\|F(u, v) - F(x, y)\| + \|F(y, x) - F(v, u)\| \leq \psi\left(\|u - x\| + \|v - y\|\right),$$

for all $(x, y), (u, v) \in E \times E$ with $\varphi(x, y) \leq_P 0_E$ and $\varphi(u, v) \geq_P 0_E$.

Then (8.13) has a unique solution.

Proof From Theorem 8.1, we know that (8.2) has a unique solution $(x^*, y^*) \in E \times E$. Since φ is symmetric, (y^*, x^*) is also a solution to (8.2). By uniqueness, we get $x^* = y^*$. Then $x^* \in E$ is the unique solution to (8.13).

Let $F, \varphi_i : E \times E \to E$ $(i = 1, 2, \ldots, r)$ be $r + 1$ given mappings. We consider the problem: Find $x \in X$ such that

$$\begin{cases} F(x, x) = x, \\ \varphi_i(x, x) = 0_E, \ \ i = 1, 2, \ldots, r. \end{cases} \tag{8.14}$$

Similarly, from Theorem 8.5, we have the following result.

Corollary 8.2 *Let $F, \varphi_i : E \times E \to E$ $(i = 1, 2, \ldots, r)$ be $r + 1$ given mappings. Suppose that the following conditions are satisfied:*

(i) φ_i $(i = 1, 2, \ldots, r)$ *is level closed from the right.*
(ii) φ_i $(i = 1, 2, \ldots, r)$ *is symmetric.*
(iii) *There exists* $(x_0, y_0) \in E \times E$ *such that* $\varphi_i(x_0, y_0) \leq_P 0_E$ $(i = 1, 2, \ldots, r)$.
(iv) *For every* $(x, y) \in E \times E$, *we have*

$$\varphi_i(x, y) \leq_P 0_E,\ i = 1, 2, \ldots, r \Longrightarrow \varphi_i(F(x, y), F(y, x)) \geq_P 0_E,\ i = 1, 2, \ldots, r.$$

(v) *For every* $(x, y) \in E \times E$, *we have*

$$\varphi_i(x, y) \geq_P 0_E,\ i = 1, 2, \ldots, r \Longrightarrow \varphi_i(F(x, y), F(y, x)) \leq_P 0_E,\ i = 1, 2, \ldots r.$$

(vi) *There exists some* $\psi \in \Psi$ *such that*

$$\|F(u, v) - F(x, y)\| + \|F(y, x) - F(v, u)\| \leq \psi\left(\|u - x\| + \|v - y\|\right),$$

for all $(x, y), (u, v) \in E \times E$ *with* $\varphi_i(x, y) \leq_P 0_E$, $\varphi_i(u, v) \geq_P 0_E$, $i = 1, 2, \ldots, r$.

Then (8.14) *has a unique solution.*

8.3.2 A Common Coupled Fixed Point Result

We need the following definition.

Definition 8.3 Let X be a nonempty set, $F : X \times X \to X$ and $g : X \to X$ be two given mappings. We say that the pair of elements $(x, y) \in X \times X$ is a common coupled fixed point of F and g if

$$x = gx = F(x, y) \quad \text{and} \quad y = gy = F(y, x).$$

We have the following common coupled fixed point result.

Corollary 8.3 *Let $F : E \times E \to E$ and $g : E \to E$ be two given mappings. Suppose that the following conditions hold:*

(i) g *is a continuous mapping.*
(ii) *There exists* $(x_0, y_0) \in E \times E$ *such that*

$$gx_0 \leq_p x_0 \quad \text{and} \quad gy_0 \leq_p y_0.$$

(iii) For every $(x, y) \in E \times E$*, we have*

$$gx \leq_P x,\ gy \leq_p y \Longrightarrow gF(x, y) \geq_P F(x, y),\ gF(y, x) \geq_P F(y, x).$$

(iv) For every $(x, y) \in E \times E$*, we have*

$$gx \geq_P x,\ gy \geq_P y \Longrightarrow gF(x, y) \leq_P F(x, y),\ gF(y, x) \leq_P F(y, x).$$

(v) There exists some $\psi \in \Psi$ *such that*

$$\|F(u, v) - F(x, y)\| + \|F(y, x) - F(v, u)\| \leq \psi\left(\|u - x\| + \|v - y\|\right),$$

for all $(x, y), (u, v) \in E \times E$ *with* $gx \leq_P x$*,* $gy \leq_P y$ *and* $gu \geq_P u$*,* $gv \geq_P v$*.*

Then F *and* g *have a unique common coupled fixed point.*

Proof Let us consider the mappings $\varphi_1, \varphi_2 : E \times E \to E$ defined by

$$\varphi_1(x, y) = gx - x, \quad (x, y) \in E \times E$$

and

$$\varphi_2(x, y) = gy - y, \quad (x, y) \in E \times E.$$

Observe that $(x, y) \in E \times E$ is a common coupled fixed point of F and g if and only if $(x, y) \in E \times E$ is a solution to (8.12). Note that since g is continuous, then φ_i is level closed from the right (also from the left) for all $i = 1, 2$. Now, applying Theorem 8.2, we obtain the desired result.

8.3.3 A Fixed Point Result

We denote by $\widetilde{\Psi}$ the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

($\widetilde{\Psi}_1$) $\psi \in \Psi$.

($\widetilde{\Psi}_2$) For all $a, b \in [0, \infty)$, we have

$$\psi(a) + \psi(b) \leq \psi(a + b).$$

Example 8.2 As example, let us consider the function

$$\psi(t) = \begin{cases} t/2 & \text{if } 0 \leq t < 1, \\ t - 1/3 & \text{if } t \geq 1. \end{cases}$$

It is not difficult to observe that $\psi \in \Psi$. Now, let us consider an arbitrary pair $(a, b) \in [0, \infty) \times [0, \infty)$. We discuss three possible cases.

Case 1. If $(a, b) \in [0, 1) \times [0, 1)$.
In this case, we have $\psi(a) + \psi(b) = (a + b)/2$. On the other hand, we have $a + b \in [0, 2)$. So, if $0 \le a + b < 1$, then $\psi(a) + \psi(b) = (a + b)/2 = \psi(a + b)$. However, if $1 \le a + b < 2$, then $\psi(a + b) - \psi(a) - \psi(b) = (a + b)/2 - 1/3 \ge 0$.

Case 2. If $(a, b) \in [0, 1) \times [1, \infty)$.
In this case, we have $\psi(a) + \psi(b) = a/2 + b - 1/3 \le a + b - 1/3 = \psi(a + b)$.

Case 3. If $(a, b) \in [1, \infty) \times [1, \infty)$.
In this case, we have $\psi(a) + \psi(b) = a + b - 2/3 \le a + b - 1/3 = \psi(a + b)$.
Therefore, we have $\psi \in \widetilde{\Psi}$.

Note that the set Ψ is more large than the set $\widetilde{\Psi}$. The following example illustrates this fact.

Example 8.3 Let us consider the function

$$\psi(t) = \begin{cases} t/2 & \text{if } 0 \le t \le 1, \\ 1/2 & \text{if } t > 1. \end{cases}$$

Clearly, we have $\psi \in \Psi$. However,

$$\psi(1 + 1) = 1/2 < 1 = \psi(1) + \psi(1),$$

which proves that $\psi \notin \widetilde{\Psi}$.

We have the following fixed point result.

Corollary 8.4 *Let $T : E \to E$ be a given mapping. Suppose that there exists some $\psi \in \widetilde{\Psi}$ such that*

$$\|Tu - Tx\| \le \psi(\|u - x\|), \quad (u, x) \in E \times E. \tag{8.15}$$

Then T has a unique fixed point.

Proof Let us define the mapping $F : E \times E \to E$ by

$$F(x, y) = Tx, \quad (x, y) \in E \times E.$$

Let $g : E \to E$ be the identity mapping, that is,

$$gx = x, \quad x \in E.$$

From (8.15), for all $(x, y), (u, v) \in E \times E$, we have

$$\|Tu - Tx\| \le \psi(\|u - x\|)$$

and

$$\|Ty - Tv\| \leq \psi(\|v - y\|).$$

Then

$$\|Tu - Tx\| + \|Ty - Tv\| \leq \psi(\|u - x\|) + \psi(\|v - y\|).$$

Using the property $(\widetilde{\Psi}_2)$, we obtain

$$\|Tu - Tx\| + \|Ty - Tv\| \leq \psi(\|u - x\| + \|v - y\|), \quad (x, y), (u, v) \in E \times E.$$

From the definitions of F and g, we obtain

$$\|F(u, v) - F(x, y)\| + \|F(y, x) - F(v, u)\| \leq \psi\left(\|u - x\| + \|v - y\|\right),$$

for all $(x, y), (u, v) \in E \times E$ with $gx \leq_P x$, $gy \leq_P y$ and $gu \geq_P u$, $gv \geq_P v$. By Corollary 3.5, there exists a unique $(x^*, y^*) \in E \times E$ such that

$$x^* = F(x^*, y^*) = Tx^* \quad \text{and} \quad y^* = F(y^*, x^*) = Ty^*.$$

Suppose that $x^* \neq y^*$. By (8.15), we have

$$\|x^* - y^*\| = \|Tx^* - Ty^*\| \leq \psi(\|x^* - y^*)) < \|x^* - y^*\|,$$

which is a contradiction. As consequence, $x^* \in E$ is the unique fixed point of T.

Remark 8.2 Taking

$$\psi(t) = kt, \quad t \geq 0,$$

where $k \in (0, 1)$ is a constant, we obtain from Corollary 8.4 the Banach contraction principle.

Finally, for other related results, we refer the reader to Jleli and Samet [3].

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Chapter 9
JS-Metric Spaces and Fixed Point Results

In this chapter, we present a recent concept of generalized metric spaces due to Jleli and Samet [12], for which we extend some well-known fixed point results including Banach contraction principle, Ćirićs fixed point theorem, a fixed point result due to Ran and Reurings, and a fixed point result due to Nieto and Rodriguez-Lopez. This new concept of generalized metric spaces recovers various topological spaces including standard metric spaces, b-metric spaces, dislocated metric spaces, and modular spaces.

9.1 Introduction

The concept of standard metric spaces is a fundamental tool in topology, functional analysis, and nonlinear analysis. This structure has attracted a considerable attention from mathematicians because of the development of the fixed point theory in standard metric spaces.

In recent years, several generalizations of standard metric spaces have appeared. In 1993, Czerwik [6] introduced the concept of b-metric spaces. Since then, several works have dealt with fixed point theory in such spaces; see [3, 4, 7, 17, 22] and references therein. In 2000, Hitzler and Seda [11] introduced the notion of dislocated metric spaces in which self-distance of a point need not be equal to zero. Such spaces play a very important role in topology and logical programming. For fixed point theory in dislocated metric spaces, see [1, 2, 10, 13] and references therein. The theory of modular spaces was initiated by Nakano [20] in connection with the theory of order spaces and was redefined and generalized by Musielak and Orlicz [19]. By defining a norm, particular Banach spaces of functions can be considered. Metric fixed theory for these Banach spaces of functions has been widely studied. Even though a metric is not defined, many problems in fixed point theory can be reformulated in modular spaces (see [8, 9, 15, 16, 18, 24] and references therein).

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In this chapter, we present a new generalization of metric spaces that recovers a large class of topological spaces including standard metric spaces, b-metric spaces, dislocated metric spaces, and modular spaces. In such spaces, we establish new versions of some known fixed point theorems in standard metric spaces including Banach contraction principle, Ćirić's fixed point theorem [5], a fixed point result due to Ran and Reurings [23], and a fixed point result due to Nieto and Rodiguez-Lopez [21].

9.2 JS-Metric Spaces

Let X be a nonempty set and $D : X \times X \to [0, \infty]$ be a given mapping. For every $x \in X$, let us define the set

$$C(D, X, x) = \left\{ \{x_n\} \subset X : \lim_{n\to\infty} D(x_n, x) = 0 \right\}.$$

9.2.1 General Definition

Definition 9.1 (*Jleli and Samet* [12]) We say that D is a JS-metric on X if it satisfies the following conditions:

(D_1) $D(x, y) = 0 \implies x = y$, for all $(x, y) \in X \times X$.
(D_2) $D(x, y) = D(y, x)$, for all $(x, y) \in X \times X$.
(D_3) There exists $C > 0$ such that

$$(x, y) \in X \times X,\ \{x_n\} \in C(D, X, x) \implies D(x, y) \leq C \limsup_{n\to\infty} D(x_n, y).$$

In this case, we say the pair (X, D) is a JS-metric space.

Remark 9.1 Obviously, if the set $C(D, X, x)$ is empty for every $x \in X$, then (X, D) is a JS-metric space if and only if (D_1) and (D_2) are satisfied.

Example 9.1 ([14]) Let $X = \{0, 1\}$ be endowed with the function $D : X \times X \to [0, \infty]$ defined by

$$D(0, 0) = 0,\ D(1, 0) = D(0, 1) = D(1, 1) = \infty.$$

Let us show that (X, D) is a JS-metric space. Properties (D_1) and (D_2) are apparent. Let $x \in X$ and $\{x_n\}$ be a sequence that belongs to $C(D, X, x)$. Therefore, there exists some $N \in \mathbb{N}$ such that

$$D(x_n, x) < \frac{1}{2}, \quad n \geq N.$$

From the definition of D, we obtain

$$D(x_n, x) = 0, \quad n \geq N,$$

which implies from property (D_1) that

$$x_n = x, \quad n \geq N.$$

Then, for every $y \in X$, we have

$$D(x, y) = D(x_n, y), \quad n \geq N.$$

Therefore, (D_3) is satisfied with $C = 1$.

For many other examples of JS-metric spaces, we refer to the next sections.

9.2.2 Topological Concepts

Definition 9.2 Let (X, D) be a JS-metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. We say that the sequence $\{x_n\}$ D-converges to x if

$$\{x_n\} \in C(D, X, x).$$

Proposition 9.1 *Let (X, D) be a JS-metric space. Let $\{x_n\}$ be a sequence in X and $(x, y) \in X \times X$. If $\{x_n\}$ D-converges to x and $\{x_n\}$ D-converges to y, then $x = y$.*

Proof Using property (D_3), we obtain

$$D(x, y) \leq C \limsup_{n\to\infty} D(x_n, y) = 0,$$

which implies from property (D_1) that $x = y$.

Definition 9.3 Let (X, D) be a JS-metric space. Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is a D-Cauchy sequence if

$$\lim_{n,m\to\infty} D(x_n, x_m) = 0.$$

Definition 9.4 Let (X, D) be a JS-metric space. It is said to be D-complete if every D-Cauchy sequence in X is D-convergent to some element in X.

9.2.3 Examples

Now, we present several examples of JS-metric spaces. We will see that this new concept of generalized metric spaces recovers a large class of existing metrics in the literature.

9.2.3.1 Standard Metric Spaces

It is obvious that any metric space is a JS-metric space.

9.2.3.2 b-Metric spaces

In 1993, Czerwik [6] introduced the concept of b-metric spaces as follows.

Definition 9.5 Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a given mapping. We say that d is a b-metric on X if it satisfies the following conditions:

(b_1) $d(x, y) = 0 \Leftrightarrow x = y$, for all $(x, y) \in X \times X$.
(b_2) $d(x, y) = d(y, x)$, for all $(x, y) \in X \times X$.
(b_3) There exists $s \geq 1$ such that, for every $(x, y, z) \in X \times X \times X$, we have

$$d(x, y) \leq s[d(x, z) + d(z, y)].$$

In this case, (X, d) is said to be a b-metric space.

The concept of convergence in such spaces is similar to that of standard metric spaces.

Proposition 9.2 *Any b-metric on X is a JS-metric on X.*

Proof Let d be a b-metric on X. We have just to proof that d satisfies the property (D_3). Let $x \in X$ and $\{x_n\} \in C(d, X, x)$. For every $y \in X$, by the property (b_3), we have

$$d(x, y) \leq sd(x, x_n) + sd(x_n, y), \quad n \in \mathbb{N}.$$

Thus, we have

$$d(x, y) \leq s \limsup_{n\to\infty} d(x_n, y).$$

The property (D_3) is then satisfied with $C = s$.

9.2.3.3 Hitzler–Seda Metric Spaces

Hitzler and Seda [11] introduced the notion of dislocated metric spaces as follows.

Definition 9.6 Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a given mapping. We say that d is a dislocated metric on X if it satisfies the following conditions:

(HS1) $d(x, y) = 0 \implies x = y$, for all $(x, y) \in X \times X$.
(HS2) $d(x, y) = d(y, x)$, for all $(x, y) \in X \times X$.
(HS3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $(x, y, z) \in X \times X \times X$.

In this case, (X, d) is said to be a dislocated metric space.

The motivation of defining this new notion is to get better results in logic programming semantics.

The concept of convergence in such spaces is similar to that of standard metric spaces. The following result can easily be established, so we omit its proof.

Proposition 9.3 *Any dislocated metric on X is a JS-metric on X.*

9.2.3.4 Modular Spaces with Fatou Property

Let us recall briefly some basic concepts of modular spaces. For more details of modular spaces, the reader is advised to consult [18] and the references therein.

Definition 9.7 Let X be a linear space over $\mathbb{R}$. A functional $\rho : X \to [0, \infty]$ is said to be modular if the following conditions hold:

(ρ_1) $\rho(x) = 0 \Leftrightarrow x = 0$, for all $x \in X$.
(ρ_2) $\rho(-x) = \rho(x)$, for all $(x, y) \in X \times X$.
(ρ_3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, whenever $\alpha, \beta \geq 0$, and $\alpha + \beta = 1$.

Definition 9.8 If ρ is a modular on X, then the set

$$X_\rho = \left\{ x \in X : \lim_{\lambda \to 0} \rho(\lambda x) = 0 \right\}$$

is called a modular space.

The concept of convergence in such spaces is defined as follows.

Definition 9.9 Let X_ρ be a modular space.

(i) A sequence $\{x_n\} \subset X_\rho$ is said to be ρ-convergent to $x \in X$ if

$$\lim_{n \to \infty} \rho(x_n - x) = 0.$$

(ii) A sequence $\{x_n\} \subset X_\rho$ is said to be ρ-Cauchy if

$$\lim_{n,m \to \infty} \rho(x_n - x_{n+m}) = 0.$$

(iii) X_ρ is said to be ρ-complete if any ρ-Cauchy sequence is ρ-convergent.

Definition 9.10 The modular ρ has the Fatou property if, for every $y \in X_\rho$, we have

$$\rho(x-y) \leq \liminf_{n\to\infty} \rho(x_n - y),$$

whenever $\{x_n\} \subset X_\rho$ is ρ-convergent to $x \in X_\rho$.

Let X_ρ be a modular space. Define the mapping $D_\rho : X_\rho \times X_\rho \to [0, \infty]$ by

$$D(x, y) = \rho(x-y), \quad (x, y) \in X \times X.$$

We have the following result.

Proposition 9.4 *If ρ has the Fatou property, then D_ρ is a JS-metric on X_ρ.*

Proof We have just to proof that D_ρ satisfies property (D_3). Let $x \in X_\rho$ and $\{x_n\} \in C(D_\rho, X_\rho, x)$, which means that

$$\lim_{n\to\infty} \rho(x_n - x) = 0.$$

Using Fatou property, for all $y \in X_\rho$, we have

$$\rho(x-y) \leq \liminf_{n\to\infty} \rho(x_n - y),$$

which yields

$$D_\rho(x, y) \leq \liminf_{n\to\infty} D_\rho(x_n, y) \leq \limsup_{n\to\infty} D_\rho(x_n, y).$$

Then (D_3) is satisfied with $C = 1$, and D_ρ is a JS-metric on X_ρ.

The following result is immediate.

Proposition 9.5 *Let ρ be a modular on X having the Fatou property. Then*

(i) $\{x_n\} \subset X_\rho$ is ρ-convergent to $x \in X_\rho$ if and only if $\{x_n\}$ is D_ρ-convergent to x.
(ii) $\{x_n\} \subset X_\rho$ is ρ-Cauchy if and only if $\{x_n\}$ is D_ρ-Cauchy.
(iii) X_ρ is ρ-complete if and only if (X_ρ, D_ρ) is D_ρ-complete.

9.3 Banach Contraction Principle in JS-Metric Spaces

In this section, we present an extension of Banach contraction principle to the setting of JS-metric spaces.

Let (X, D) be a JS-metric space, and let $T : X \to X$ be a giving mapping.

Definition 9.11 Let $k \in (0,1)$. We say that T is a k-contraction if

$$D(Tx, Ty) \leq kD(x, y), \quad (x, y) \in X \times X.$$

Observe that

Proposition 9.6 *Suppose that T is a k-contraction for some $k \in (0,1)$. Then any fixed point $\omega \in X$ of T satisfies*

$$D(\omega, \omega) < \infty \implies D(\omega, \omega) = 0.$$

Proof Let $\omega \in X$ be a fixed point of T such that $D(\omega, \omega) < \infty$. Since T is a k-contraction, we have

$$D(\omega, \omega) = D(T\omega, T\omega) \leq kD(\omega, \omega),$$

which implies that $D(\omega, \omega) = 0$, since $k \in (0,1)$ and $D(\omega, \omega) < \infty$.

For every $x \in X$, let

$$\delta(D, T, x) = \sup\{D(T^i x, T^j x) : i, j \in \mathbb{N}\}.$$

We have the following extension of Banach contraction principle.

Theorem 9.1 *Suppose that the following conditions hold:*

(i) *(X, D) is complete.*
(ii) *T is a k-contraction for some $k \in (0,1)$.*
(iii) *There exists $x_0 \in X$ such that $\delta(D, T, x_0) < \infty$.*

Then $\{T^n x_0\}$ converges to $\omega \in X$, a fixed point of T. Moreover, if $\omega' \in X$ is another fixed point of T such that $D(\omega, \omega') < \infty$, then $\omega = \omega'$.

Proof Let $n \in \mathbb{N}$ (≥ 1). Since T is a k-contraction, for all $i, j \in \mathbb{N}$, we have

$$D(T^{n+i}x_0, T^{n+j}x_0) \leq kD(T^{n-1+i}x_0, T^{n-1+j}x_0),$$

which implies that

$$\delta(D, T, T^n x_0) \leq k\delta(D, T, T^{n-1}x_0).$$

Then, for every $n \in \mathbb{N}$, we have

$$\delta(D, T, T^n x_0) \leq k^n \delta(D, T, x_0).$$

Using the above inequality, for every $n, m \in \mathbb{N}$, we have

$$D(T^n x_0, T^{n+m}x_0) \leq \delta(D, T, T^n x_0) \leq k^n \delta(D, T, x_0).$$

Since $\delta(D, T, x_0) < \infty$ and $k \in (0, 1)$, we obtain

$$\lim_{n,m\to\infty} D(T^n x_0, T^{n+m} x_0) = 0,$$

which implies that $\{T^n x_0\}$ is a D-Cauchy sequence. Since (X, D) is D-complete, there exists some $\omega \in X$ such that $\{T^n x_0\}$ is D-convergent to ω. On the other hand, since T is a k-contraction, for all $n \in \mathbb{N}$, we have

$$D(T^{n+1} x_0, T\omega) \leq k D(T^n x_0, \omega).$$

Passing to the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} D(T^{n+1} x_0, T\omega) = 0.$$

Then $\{T^n x_0\}$ is D-convergent to $T\omega$. By the uniqueness of the limit (see Proposition 9.1), we get

$$\omega = T\omega,$$

that is, ω is a fixed point of T. Now, suppose that $\omega' \in X$ is a fixed point of T such that $D(\omega, \omega') < \infty$. Since T is a k-contraction, we have

$$D(\omega, \omega') = D(T\omega, T\omega') \leq k D(\omega, \omega'),$$

which implies by property (D_1) that $\omega = \omega'$.

The following result (see Kirk and Shahzad [17]) is an immediate consequence of Proposition 9.2 and Theorem 9.1.

Corollary 9.1 *Let (X, d) be a complete b-metric space, and let $T : X \to X$ be a giving mapping. Suppose that for some $k \in (0, 1)$, we have*

$$d(Tx, Ty) \leq k d(x, y), \quad (x, y) \in X \times X.$$

If there exists $x_0 \in X$ such that

$$\sup\{d(T^i x_0, T^j x_0) : i, j \in \mathbb{N}\} < \infty,$$

then the sequence $\{T^n x_0\}$ converges to a fixed point of T. Moreover, T has one and only one fixed point.

Note that in [6], there is a better result than this given by Corollary 9.1.

The next result is an immediate consequence of Proposition 9.3 and Theorem 9.1.

Corollary 9.2 *Let (X, d) be a complete dislocated metric space, and let $T : X \to X$ be a giving mapping. Suppose that for some $k \in (0, 1)$, we have*

$$d(Tx, Ty) \leq kd(x, y), \quad (x, y) \in X \times X.$$

If there exists $x_0 \in X$ *such that*

$$\sup\{d(T^i x_0, T^j x_0) : i, j \in \mathbb{N}\} < \infty,$$

then the sequence $\{T^n x_0\}$ *converges to a fixed point of* T*. Moreover,* T *has one and only one fixed point.*

The following result is an immediate consequence of Proposition 9.4, Proposition 9.5, and Theorem 9.1.

Corollary 9.3 *Let* (X_ρ, ρ) *be a complete modular space, and let* $T : X \to X$ *be a giving mapping. Suppose that for some* $k \in (0, 1)$*, we have*

$$\rho(Tx - Ty) \leq k\rho(x - y), \quad (x, y) \in X_\rho \times X_\rho.$$

Suppose also that ρ *satisfies the Fatou property. If there exists* $x_0 \in X_\rho$ *such that*

$$\sup\{\rho(T^i x_0 - T^j x_0) : i, j \in \mathbb{N}\} < \infty,$$

then the sequence $\{T^n x_0\}$ ρ*-converges to some* $\omega \in X_\rho$*, a fixed point of* T*. Moreover, if* $\omega' \in X_\rho$ *is another fixed point of* T *such that* $\rho(\omega - \omega') < \infty$*, then* $\omega = \omega'$*.*

Observe that in the above result, no Δ_2-condition is supposed.

9.4 Ćirić's Quasicontraction in JS-Metric Spaces

In this section, we extend Ćirić's fixed point theorem to quasicontraction-type mappings [5] in the setting of JS-metric spaces.

Let (X, D) be a JS-metric space, and let $T : X \to X$ be a mapping.

Definition 9.12 Let $k \in (0, 1)$. We say that T is a k-quasicontraction if

$$D(Tx, Ty) \leq k \max\{D(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\}, \quad (x, y) \in X \times X.$$

Proposition 9.7 *Suppose that* T *is a* k*-quasicontraction for some* $k \in (0, 1)$*. Then any fixed point* $\omega \in X$ *of* T *satisfies*

$$D(\omega, \omega) < \infty \implies D(\omega, \omega) = 0.$$

Proof Let $\omega \in X$ be a fixed point of T such that $D(\omega, \omega) < \infty$. Since T is a k-quasicontraction, we have

$$D(\omega, \omega) = D(T\omega, T\omega) \leq kD(\omega, \omega).$$

Since $k \in (0, 1)$, we get $D(\omega, \omega) = 0$.

Theorem 9.2 *Suppose that the following conditions hold:*

(i) (X, D) *is complete.*
(ii) T *is a* k*-quasicontraction for some* $k \in (0, 1/C)$, $C \geq 1$.
(iii) *There exists* $x_0 \in X$ *such that* $\delta(D, T, x_0) < \infty$.

Then $\{T^n x_0\}$ *converges to some* $\omega \in X$. *If* $D(x_0, T\omega) < \infty$ *and* $D(\omega, T\omega) < \infty$, *then* ω *is a fixed point of* T. *Moreover, if* $\omega' \in X$ *is another fixed point of* T *such that* $D(\omega, \omega') < \infty$ *and* $D(\omega', \omega') < \infty$, *then* $\omega = \omega'$.

Proof Let $n \in \mathbb{N}$ $(n \geq 1)$. Since T is a k-quasicontraction, for all $i, j \in \mathbb{N}$, we have

$$\begin{aligned} D(T^{n+i}x_0, T^{n+j}x_0) \leq k \max\{&D(T^{n-1+i}x_0, T^{n-1+j}x_0), D(T^{n-1+i}x_0, T^{n+i}x_0), \\ &D(T^{n-1+i}x_0, T^{n+j}x_0), D(T^{n-1+j}x_0, T^{n+j}x_0), \\ &D(T^{n-1+j}x_0, T^{n+i}x_0)\}, \end{aligned}$$

which implies that

$$\delta(D, T, T^n x_0) \leq k\delta(D, T, T^{n1}x_0).$$

Hence, we have

$$\delta(D, T, T^n x_0) \leq k^n \delta(D, T, x_0), \quad n \geq 1.$$

Using the above inequality, for every $n, m \in \mathbb{N}$, we have

$$D(T^n x_0, T^{n+m}x_0) \leq \delta(D, T, T^n x_0) \leq k^n \delta(D, T, x_0).$$

Since $(D, T, x_0) < \infty$ and $k \in (0, 1)$, we obtain

$$\lim_{n,m\to\infty} D(T^n x_0, T^{n+m}x_0) = 0,$$

which implies that $\{T^n x_0\}$ is a D-Cauchy sequence. Since (X, D) is D-complete, there exists some $\omega \in X$ such that $\{T^n x_0\}$ is D-convergent to ω. Now, we suppose that $D(x_0, T\omega) < \infty$. Using the inequality

$$D(T^n x_0, T^{n+m}x_0) \leq k^n \delta(D, T, x_0), \quad n, m \in \mathbb{N}, \tag{9.1}$$

by property (D_3),

$$D(\omega, T^n x_0) \leq C \limsup_{m\to\infty} D(T^n x_0, T^{n+m}x_0) \leq Ck^n \delta(D, T, x_0), \quad n \in \mathbb{N}. \tag{9.2}$$

On the other hand, we have

$$D(Tx_0, T\omega) \leq k \max\{D(x_0, \omega), D(x_0, Tx_0), D(\omega, T\omega), D(Tx_0, \omega), D(x_0, T\omega)\}.$$

Using (9.1) and (9.2), we get

$$D(Tx_0, T\omega) \leq \max\{kC\delta(D, T, x_0), k\delta(D, T, x_0), kD(\omega, T\omega), kD(x_0, T\omega)\}.$$

Again, using the above inequality, we have

$$D(T^2x_0, T\omega) \leq \max\{k^2C\delta(D, T, x_0), k^2\delta(D, T, x_0), kD(\omega, T\omega), k^2D(x_0, T\omega)\}.$$

Continuing this process, by induction, we get

$$D(T^nx_0, T\omega) \leq \max\{k^nC\delta(D, T, x_0), k^n\delta(D, T, x_0), kD(\omega, T\omega), k^nD(x_0, T\omega)\}, \quad n \geq 1.$$

Therefore, we have

$$\limsup_{n\to\infty} D(T^nx_0, T\omega) \leq kD(\omega, T\omega),$$

since $D(x_0, T\omega) < \infty$ and $\delta(D, T, x_0) < \infty$. Using property (D_3), we get

$$D(T\omega, \omega) \leq C \limsup_{n\to\infty} D(T^nx_0, T\omega) \leq kCD(\omega, T\omega),$$

which implies that $D(T\omega, \omega) = 0$, since $D(\omega, T\omega) < \infty$ and $kC \in (0, 1)$. Then ω is a fixed point of T. By Proposition 9.7, we have $D(\omega, \omega) = 0$. Finally, suppose that $\omega' \in X$ is another fixed point of T such that $D(\omega, \omega') < \infty$ and $D(\omega', \omega') < \infty$. By Proposition 9.7, we have $D(\omega', \omega') = 0$. Since T is a k-quasicontraction, we get

$$D(\omega, \omega') = D(T\omega, T\omega') \leq kD(\omega, \omega'),$$

which implies that $\omega = \omega'$.

9.5 Banach Contraction Principle in a JS-Metric Space with a Partial Order

In this section, we extend Banach contraction principle to the class of JS-metric spaces with a partial order.

Let (X, D) be a JS-metric space, and let $T : X \to X$ be a giving mapping. Let $\preceq$ be a partial order on X. We denote by $E_{\preceq}$ the subset of $X \times X$ defined by

$$E_{\preceq} = \{(x, y) \in X \times X : x \preceq y\}.$$

Now, let us introduce some concepts.

Definition 9.13 We say that T is weak continuous if the following condition holds: if $\{x_n\} \subset X$ is D-convergent to $x \in X$, then there exists a subsequence $\{x_{n_q}\}$ of $\{x_n\}$ such that $\{Tx_{n_q}\}$ is D-convergent to Tx (as $q \to \infty$).

Definition 9.14 We say that$T$ is $\preceq$-monotone if the following condition holds:

$$(x, y) \in E_{\preceq} \implies (Tx, Ty) \in E_{\preceq}.$$

Definition 9.15 We say that the pair (X, D) is D-regular if the following condition holds: For every sequence, $\{x_n\} \subset X$ satisfying

$$(x_n, x_{n+1}) \in E_{\preceq}, \quad n \text{ large enough},$$

if $\{x_n\}$ is D-convergent to $x \in X$, then there exists a subsequence $\{x_{n_q}\}$ of $\{x_n\}$ such that

$$(x_{n_q}, x) \in E_{\preceq}, \quad q \text{ large enough}.$$

Definition 9.16 We say that T is a weak k-contraction for some $k \in (0, 1)$ if the following condition holds:

$$(x, y) \in E_{\preceq} \implies D(Tx, Ty) \leq kD(x, y).$$

The first result holds under the weak continuity assumption.

Theorem 9.3 *Suppose that the following conditions hold:*

(i) (X, D) *is complete.*
(ii) T *is weak continuous.*
(iii) T *is a weak* k*-contraction for some* $k \in (0, 1)$.
(iv) *There exists* $x_0 \in X$ *such that* $\delta(D, T, x_0) < \infty$ *and* $(x_0, Tx_0) \in E_{\preceq}$.
(v) T *is* $\preceq$*-monotone.*

Then $\{T^n x_0\}$ *converges to some* $\omega \in X$ *such that* ω *is a fixed point of* T*. Moreover, if* $D(\omega, \omega) < \infty$*, then* $D(\omega, \omega) = 0$.

Proof Since T is $E_{\preceq}$-monotone and $(x_0, Tx_0) \in E$, then

$$(T^n x_0, T^{n+1} x_0) \in E_{\preceq}, \quad n \in \mathbb{N}.$$

Since $\preceq$ is a partial order (so it is transitive), then

$$(p, q) \in \mathbb{N} \times \mathbb{N}, \quad p \leq q \implies T^p x_0 \preceq T^q x_0.$$

Let $n \in \mathbb{N}$ ($n \geq 1$). Since T is a weak k-contraction and D is symmetric, for all $i, j \in \mathbb{N}$, we have

$$D(T^{n+i}x_0, T^{n+j}x_0) \le kD(T^{n-1+i}x_0, T^{n-1+j}x_0),$$

which implies that

$$\delta(D, T, T^n x_0) \le k\delta(D, T, T^{n-1}x_0).$$

Then,

$$\delta(D, T, T^n x_0) \le k^n \delta(D, T, x_0), \quad n \in \mathbb{N}.$$

Using the above inequality, for every $n, m \in \mathbb{N}$, we have

$$D(T^n x_0, T^{n+m}x_0) \le \delta(D, T, T^n x_0) \le k^n \delta(D, T, x_0).$$

Since $\delta(D, T, x_0) < \infty$ and $k \in (0, 1)$, we obtain

$$\lim_{n,m\to\infty} D(T^n x_0, T^{n+m}x_0) = 0,$$

which implies that $\{T^n x_0\}$ is a D-Cauchy sequence. Since (X, D) is D-complete, there exists some $\omega \in X$ such that $\{T^n x_0\}$ is D-convergent to ω. Since T is weak continuous, there exists a subsequence $\{T^{n_q}x_0\}$ of $\{T^n x_0\}$ such that $\{T^{n_q+1}x_0\}$ is D-convergent to $T\omega$ (as $q \to \infty$). By the uniqueness of the limit, we get $\omega = T\omega$, that is, ω is a fixed point of T. Suppose now that $D(\omega, \omega) < \infty$, since $(\omega, \omega) \in E_{\preceq}$, we have

$$D(\omega, \omega) = D(T\omega, T\omega) \le kD(\omega, \omega),$$

which implies that $D(\omega, \omega) = 0$ (since $k \in (0, 1)$).

Remark 9.2 Theorem 9.3 is an extension of Ran and Reurings fixed point result [23] established in the setting of metric spaces under the continuity of the mapping T.

Now, we replace the weak continuity assumption by the D-regularity of the pair (X, D). We have the following result.

Theorem 9.4 *Suppose that the following conditions hold:*

(i) (X, D) *is complete.*
(ii) (X, D) *is* D*-regular.*
(iii) T *is a weak* k*-contraction for some* $k \in (0, 1)$.
(iv) There exists $x_0 \in X$ *such that* $\delta(D, T, x_0) < \infty$ *and* $(x_0, Tx_0) \in E_{\preceq}$.
(v) T *is* $\preceq$*-monotone.*

Then $\{T^n x_0\}$ *converges to some* $\omega \in X$ *such that* ω *is a fixed point of* T*. Moreover, if* $D(\omega, \omega) < \infty$*, then* $D(\omega, \omega) = 0$.

Proof Following the proof of the previous theorem, we know that $\{T^n x_0\}$ is D-convergent to some $\omega \in X$ and

$$(T^n x_0, T^{n+1}x_0) \in E_{\preceq}, \quad n \in \mathbb{N}.$$

Since (X, D) is D-regular, there exists a subsequence $\{T^{n_q}x_0\}$ of $\{T^n x_0\}$ such that

$$(T^{n_q}x_0, \omega) \in E_{\preceq}, \quad q \text{ large enough.}$$

On the other hand, T is a weak k-contraction, so we have

$$D(T^{n_q+1}x_0, T\omega) \leq kD(T^{n_q}x_0, \omega), \quad q \text{ large enough.}$$

Passing to the limit as $q \to \infty$, we get

$$\lim_{q\to\infty} D(T^{n_q+1}x_0, T\omega) = 0,$$

which implies that $\{T^{n_q+1}x_0\}$ is D-convergent to $T\omega$. By uniqueness of the limit, we get $\omega = T\omega$. Similar to the proof in the previous theorem, we have $D(\omega, \omega) = 0$.

Remark 9.3 Theorem 9.4 is an extension of Nieto and Rodiguez-Lopez fixed point result ([21], Theorem 4), which was obtained in the setting of metric spaces.

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Chapter 10
Iterated Bernstein Polynomial Approximations

Kelisky and Rivlin [7] proved that each Bernstein operator B_n is a weaky Picard operator (WPO). Moreover, given $n \in \mathbb{N}$ and $\varphi \in C([0,1];\mathbb{R})$,

$$\lim_{j\to\infty} (B_n^j \varphi)(t) = \varphi(0) + (\varphi(1) - \varphi(0))t, \quad t \in [0,1].$$

In their opinion, the study of iterates of B_n is considerably simplified if one uses the language of Linear Algebra. Nevertheless, their proof is not easy: In particular, it involves the Stirling numbers of the second kind, and eigenvalues and eigenvectors of some matrices. A simple proof of the Kelisky–Rivlin Theorem was given by Rus [11] with the help of some trick with contraction principle. Another proof, which is based on a fixed point theorem for linear operators on a Banach space, was presented by Jachymski in [6]. In this chapter, we establish a new fixed point theorem, which will be used to establish a Kelisky–Rivlin type result for q-Bernstein polynomials and modified q-Bernstein polynomials. Note that the techniques used by Jachymski in [6] require linear operators defined on a certain Banach space, which is not the case for modified q-Bernstein polynomials.

10.1 A Fixed Point Theorem

In this section, we establish a fixed point theorem that will be used later.

Theorem 10.1 *Let E be a group with respect to a certain operation $+$. Let X be a subset of E endowed with a certain metric d such that (X, d) is complete. Let $X_0 \subset X$ be a closed subset of X such that X_0 is a subgroup of E. Let $T : X \to X$ be a given mapping satisfying*

$$(x, y) \in X \times X,\ x - y \in X_0 \implies d(Tx, Ty) \le kd(x, y), \tag{10.1}$$

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where $k \in (0, 1)$ *is a constant. Suppose that the operation mapping* $\pm : X \times X \to X$ *defined by*

$$\pm(x, y) = x \pm y, \quad (x, y) \in X \times X$$

is continuous with respect to the metric d*. Moreover, suppose that*

$$x - Tx \in X_0, \quad x \in X. \tag{10.2}$$

Then

(i) For every $x \in X$*, the Picard sequence* $\{T^n x\}$ *converges to a fixed point of* T*.*
(ii) For every $x \in X$*,*

$$(x + X_0) \cap \text{Fix}(T) = \left\{ \lim_{n\to\infty} T^n x \right\}.$$

Proof Let $x \in X$ be fixed. From (10.2), we have

$$x - Tx \in X_0.$$

Using (10.1), we obtain

$$d(Tx, T^2x) \leq kd(x, Tx).$$

Again, using (10.2), we obtain

$$Tx - T^2x = Tx - T(Tx) \in X_0,$$

which implies from (10.1) that

$$d(T^2x, T^3x) \leq kd(Tx, T^2x) \leq k^2 d(x, Tx).$$

Therefore, by induction we obtain

$$T^n x - T^{n+1}x \in X_0, \quad n \in \mathbb{N} \tag{10.3}$$

and

$$d(T^n x, T^{n+1}x) \leq k^n d(x, Tx), \quad n \in \mathbb{N}.$$

Since $k \in (0, 1)$, from the above inequality we deduce that the Picard sequence $\{T^n x\}$ is Cauchy in the complete metric space (X, d). Then there is some $\omega \in X$ such that

$$\lim_{n\to\infty} d(T^n x, \omega) = 0. \tag{10.4}$$

On the other hand, observe that for $n, p \geq 1$,

$$T^n x - T^{n+p}x = (T^n x - T^{n+1}x) + (T^{n+1}x - T^{n+2}x) + \cdots + (T^{n+p-1}x - T^{n+p}x).$$

Therefore, by (10.3) and using the fact that $(X_0, +)$ is a group, we deduce that

$$T^n x - T^{n+p} x \in X_0, \quad n, p \geq 1.$$

Passing to the limit as $p \to \infty$, using (10.4), the continuity of the operation mapping $\pm$, and the closure of X_0, we obtain

$$T^n x - \omega \in X_0, \quad n \in \mathbb{N}. \tag{10.5}$$

Therefore, by (10.1) we have

$$d(T^{n+1} x, T\omega) \leq k d(T^n x, \omega), \quad n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$ and using (10.4), we get

$$\lim_{n\to\infty} d(T^{n+1} x, T\omega) = 0.$$

The uniqueness of the limit yields $\omega = T\omega$; that is, ω is a fixed point of T. Then (i) is proved. In order to prove (ii), let $x \in X$ be fixed. We know that the Picard sequence $\{T^n x\}$ converges to $\omega \in X$, a fixed point of T. Moreover, from (10.2) and (10.5), we have $\omega - x \in X_0$, that is, $\omega \in x + X_0$. Therefore, we have

$$\left\{\lim_{n\to\infty} T^n x\right\} \subset (x + X_0) \cap \mathrm{Fix}(T).$$

Now, let $z \in (x + X_0) \cap \mathrm{Fix}(T)$ be fixed. Then

$$Tz = z \quad \text{and} \quad z - x \in X_0.$$

Therefore, we have

$$z - Tx = Tz - Tx = (Tz - z) + (x - Tx) + (z - x) \in X_0.$$

Again,

$$z - T^2 x = T^2 z - T^2 x = (T^2 z - Tz) + (Tx - T^2 x) + (z - Tx) \in X_0.$$

Hence, by induction we obtain

$$z - T^n x \in X_0, \quad n \in \mathbb{N}.$$

Using (10.1), we get

$$d(z, T^{n+1} x) = d(Tz, T^{n+1} x) \leq k d(z, T^n x) \leq k^2 d(z, T^{n-1} x) \leq \cdots \leq k^{n+1} d(z, x), \quad n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} d(z, T^n x) = 0,$$

which yields $z \in \left\{\lim_{n\to\infty} T^n x\right\}$. Then we proved that

$$(x + X_0) \cap \mathrm{Fix}(T) \subset \left\{\lim_{n\to\infty} T^n x\right\}.$$

The proof is complete.

10.2 Kelisky–Rivlin Theorem for Bernstein Polynomials

The Bernstein operator of order n ($n \geq 1$) associates with every function $f \in C([0, 1]; \mathbb{R})$ (the space of all continuous and real functions on the interval $[0, 1]$) the nth Bernstein polynomial

$$B_n(f)(t) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right)\binom{n}{i} t^i (1-t)^{n-i}, \quad t \in [0, 1].$$

These polynomials were introduced in 1912 in Bernstein's constructive proof of the Weierstrass approximation theorem [2]. Since then, they have been the object of multiple investigations, serving many times as a guide for several theorems in approximation theory (see, e.g., [3–5, 8, 13, 14]).

Using Linear Algebra tools, Kelisky and Rivlin [7] have proved that the iterates of the Bernstein operator (of fixed order) converge to L, the operator of linear interpolation at the endpoints of the interval $[0, 1]$. Using Theorem 10.1, we prove the following theorem due to Kelisky and Rivlin.

For a fixed $n \in \mathbb{N}$, $n \geq 1$, we denote by $(B_n^j)_{j\in\mathbb{N}}$ the sequence of the iterates of B_n.

Theorem 10.2 (Kelisky–Rivlin Theorem) *Let $n \in \mathbb{N}$, $n \geq 1$, be fixed. Then, for every $f \in C([0, 1]; \mathbb{R})$,*

$$\lim_{j\to\infty} B_n^j(f)(t) = f(0) + [f(1) - f(0)]t, \quad t \in [0, 1].$$

Proof Let $X = E = C([0, 1]; \mathbb{R})$. We endow X with the metric d defined by

$$d(f, g) = \max\{|f(t) - g(t)| : t \in [0, 1]\}, \quad (f, g) \in X \times X.$$

Then (X, d) is a complete metric space. Let X_0 be the subset of X defined by

$$X_0 = \{f \in X : f(0) = f(1) = 0\}.$$

Then X_0 is a closed linear subspace of X. Let $(f, g) \in X \times X$ be such that $f - g \in X_0$, that is,

$$(f, g) \in X \times X \quad \text{and} \quad f(0) = g(0),\ f(1) = g(1).$$

Let $t \in [0, 1]$ be fixed. Then we have

$$\begin{aligned}
|B_n(f)(t) - B_n(g)(t)| &= \left| \sum_{i=0}^{n} f\left(\frac{i}{n}\right) \binom{n}{i} t^i (1-t)^{n-i} - \sum_{i=0}^{n} g\left(\frac{i}{n}\right) \binom{n}{i} t^i (1-t)^{n-i} \right| \\
&\le \sum_{i=0}^{n} \left| f\left(\frac{i}{n}\right) - g\left(\frac{i}{n}\right) \right| \binom{n}{i} t^i (1-t)^{n-i} \\
&= \sum_{i=1}^{n-1} \left| f\left(\frac{i}{n}\right) - g\left(\frac{i}{n}\right) \right| \binom{n}{i} t^i (1-t)^{n-i} \\
&\le \sum_{i=1}^{n-1} \binom{n}{i} t^i (1-t)^{n-i} d(f, g) \\
&= (1 - t^n - (1-t)^n) d(f, g) \\
&\le \left(1 - \frac{1}{2^{n-1}}\right) d(f, g).
\end{aligned}$$

Therefore, we have

$$(f, g) \in X \times X,\ f - g \in X_0 \implies d(B_n(f), B_n(g)) \le \left(1 - \frac{1}{2^{n-1}}\right) d(f, g).$$

Now, let $f \in X$ be fixed. We have

$$f(0) - B_n(f)(0) = f(0) - f(0) = 0$$

and

$$f(1) - B_n(f)(1) = f(1) - f(1) = 0.$$

Therefore, we have

$$f - B_n(f) \in X_0, \quad f \in X.$$

Applying Theorem 10.1, we deduce that

$$(f + X_0) \cap \mathrm{Fix}(B_n) = \left\{ \lim_{j \to \infty} B_n^j(f) \right\}, \quad f \in X.$$

Let $f \in X$. It is not difficult to observe that the function $\omega : [0, 1] \to \mathbb{R}$ defined by

$$\omega(t) = f(0)(1-t) + f(1)t, \quad t \in [0, 1]$$

belongs to $\mathrm{Fix}(B_n)$. Moreover, for all $t \in [0, 1]$,

$$\theta(t) := \omega(t) - f(t) = f(0)(1-t) + f(1)t - f(t).$$

Observe that

$$\theta(0) = f(0) - f(0) = 0$$

and

$$\theta(1) = f(1) - f(1) = 0.$$

Therefore, $\omega \in f + X_0$. As consequence, we get

$$\lim_{j\to\infty} d(B_n^j(f), \omega) = 0,$$

which yields the desired result.

10.2.1 A Kelisky–Rivlin Type Result for q-Bernstein Polynomials

In this section, we are interested in establishing a Kelisky–Rivlin type result for q-Bernstein polynomials. To formulate our result, we need the following definitions.

Let $q > 0$. For any $n \in \mathbb{N}$, the q-integer $[n]_q$ is defined by

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1} \ (n \geq 1), \ [0]_q = 0.$$

The q-factorial $[n]_q!$ is defined by

$$[n]_q! = [1]_q[2]_q \cdots [n]_q \ (n \geq 1), \ [0]_q! = 1.$$

For integers $0 \leq k \leq n$, the q-binomial is defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}.$$

It is clear that for $q = 1$, we have

$$[n]_1 = n, \ [n]_1! = n!, \ \binom{n}{k}_1 = \binom{n}{k}.$$

Definition 10.1 (Phillips [10]) The q-Bernstein operator of order n ($n \geq 1$) associates with every function $f \in C([0, 1]; \mathbb{R})$ the nth q-Bernstein polynomial

$$B_n(q, f)(t) = \sum_{i=0}^{n} f\left(\frac{[i]_q}{[n]_q}\right)\binom{n}{i}_q t^i \prod_{s=0}^{n-1-i}(1 - q^s t), \quad t \in [0, 1].$$

From here on an empty product is taken to be equal to 1.

Theorem 10.3 *Let $n \in \mathbb{N}$, $n \geq 1$, and $0 < q \leq 1$. Then, for every $f \in C([0, 1]; \mathbb{R})$,*

$$\lim_{j\to\infty} B_n^j(q, f)(t) = f(0) + [f(1) - f(0)]t, \quad t \in [0, 1].$$

Proof We argue as in the proof of Theorem 10.2. Let $X = E = C([0, 1]; \mathbb{R})$. We endow X with the metric d defined by

$$d(f, g) = \max\{|f(t) - g(t)| : t \in [0, 1]\}, \quad (f, g) \in X \times X.$$

Then (X, d) is a complete metric space. Let X_0 be the subset of X defined by

$$X_0 = \{f \in X : f(0) = f(1) = 0\}.$$

Then X_0 is a closed linear subspace of X. Let $(f, g) \in X \times X$ be such that $f - g \in X_0$, that is,

$$(f, g) \in X \times X \quad \text{and} \quad f(0) = g(0),\ f(1) = g(1).$$

Let $t \in [0, 1]$ be fixed. Then we have

$$\begin{aligned}
&|B_n(q, f)(t) - B_n(q, g)(t)| \\
&= \left|\sum_{i=0}^{n} f\left(\frac{[i]_q}{[n]_q}\right)\binom{n}{i}_q t^i \prod_{s=0}^{n-1-i}(1 - q^s t) - \sum_{i=0}^{n} g\left(\frac{[i]_q}{[n]_q}\right)\binom{n}{i}_q t^i \prod_{s=0}^{n-1-i}(1 - q^s t)\right| \\
&\leq \sum_{i=0}^{n}\left|f\left(\frac{[i]_q}{[n]_q}\right) - g\left(\frac{[i]_q}{[n]_q}\right)\right|\binom{n}{i}_q t^i \prod_{s=0}^{n-1-i}(1 - q^s t) \\
&= \sum_{i=1}^{n-1}\left|f\left(\frac{[i]_q}{[n]_q}\right) - g\left(\frac{[i]_q}{[n]_q}\right)\right|\binom{n}{i}_q t^i \prod_{s=0}^{n-1-i}(1 - q^s t) \\
&\leq \sum_{i=1}^{n-1}\binom{n}{i}_q t^i \prod_{s=0}^{n-1-i}(1 - q^s t) d(f, g).
\end{aligned}$$

Note that (see [9])

$$\sum_{i=0}^{n}\binom{n}{i}_q t^i \prod_{s=0}^{n-1-i}(1 - q^s t) = 1.$$

Then, for $q \leq 1$, it is easy to observe that

$$\sum_{i=1}^{n-1} \binom{n}{i}_q t^i \prod_{s=0}^{n-1-i} (1 - q^s t) \leq \left(1 - \frac{1}{2^{n-1}}\right).$$

Therefore, we have

$$(f, g) \in X \times X,\ \ f - g \in X_0 \implies d(B_n(q, f), B_n(q, g)) \leq \left(1 - \frac{1}{2^{n-1}}\right) d(f, g).$$

The rest of the proof is similar to that of Theorem 10.2.

Remark 10.1 Taking $q = 1$ in Theorem 10.3, we obtain the result of Theorem 10.2.

10.2.2 A Kelisky–Rivlin Type Result for Modified q-Bernstein Polynomials

In this section, we are interested in establishing a Kelisky–Rivlin type result for modified q-Bernstein polynomials.

Definition 10.2 (see [1]) The modified q-Bernstein operator of order n ($n \geq 1$) associates with every function $f \in C([0, 1]; \mathbb{R})$ the nth modified q-Bernstein polynomial

$$T_n(q, f)(t) = \sum_{i=0}^{n} \left| f\left(\frac{[i]_q}{[n]_q}\right)\right| \binom{n}{i}_q t^i \prod_{s=0}^{n-1-i} (1 - q^s t), \quad t \in [0, 1].$$

Theorem 10.4 *Let $n \in \mathbb{N}$, $n \geq 1$, and $0 < q \leq 1$. Then, for every $f \in C([0, 1]; \mathbb{R})$,*

$$\lim_{j \to \infty} T_n^j(q, f)(t) = f(0) + [f(1) - f(0)]t, \quad t \in [0, 1].$$

Proof Let $E = C([0, 1]; \mathbb{R})$ and X be the subset of E defined by

$$X = \{f \in E :\ f(0) \geq 0,\ f(1) \geq 0\}.$$

We endow X with the metric d defined by

$$d(f, g) = \max\{|f(t) - g(t)| : t \in [0, 1]\}, \quad (f, g) \in X \times X.$$

Then (X, d) is a complete metric space. Let X_0 be the subset of X defined by

$$X_0 = \{f \in E :\ f(0) = f(1) = 0\}.$$

Then X_0 is a closed subgroup of E. Let $(f, g) \in X \times X$ be such that $f - g \in X_0$, that is,

$$(f, g) \in X \times X \quad \text{and} \quad f(0) = g(0),\ f(1) = g(1).$$

Let $t \in [0, 1]$ be fixed. Then

$$\begin{aligned}
&|T_n(q, f)(t) - T_n(q, g)(t)| \\
&= \left| \sum_{i=0}^{n} \left| f\left(\frac{[i]_q}{[n]_q}\right) \right| \binom{n}{i}_q t^i \prod_{s=0}^{n-1-i} (1 - q^s t) - \sum_{i=0}^{n} \left| g\left(\frac{[i]_q}{[n]_q}\right) \right| \binom{n}{i}_q t^i \prod_{s=0}^{n-1-i} (1 - q^s t) \right| \\
&\le \sum_{i=0}^{n} \left| \left| f\left(\frac{[i]_q}{[n]_q}\right) \right| - \left| g\left(\frac{[i]_q}{[n]_q}\right) \right| \right| \binom{n}{i}_q t^i \prod_{s=0}^{n-1-i} (1 - q^s t) \\
&\le \sum_{i=1}^{n-1} \left| f\left(\frac{[i]_q}{[n]_q}\right) - g\left(\frac{[i]_q}{[n]_q}\right) \right| \binom{n}{i}_q t^i \prod_{s=0}^{n-1-i} (1 - q^s t) \\
&\le \left(1 - \frac{1}{2^{n-1}}\right) d(f, g).
\end{aligned}$$

Therefore, we have

$$(f, g) \in X \times X,\ \ f - g \in X_0 \implies d(T_n(q, f), T_n(q, g)) \le \left(1 - \frac{1}{2^{n-1}}\right) d(f, g).$$

Now, let $f \in X$ be fixed. We have

$$f(t) - T_n(q, f)(t) = \sum_{i=0}^{n} \left(f(t) - \left| f\left(\frac{[i]_q}{[n]_q}\right) \right| \right) \binom{n}{i}_q t^i \prod_{s=0}^{n-1-i} (1 - q^s t), \quad t \in [0, 1].$$

Observe that

$$f(0) - T_n(q, f)(0) = f(1) - T_n(q, f)(1) = 0.$$

Therefore,

$$f - T_n(q, f) \in X_0, \quad f \in X.$$

Further, the desired result follows from Theorem 10.1.

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