An Interest-Matrix-Based Mechanism for Selfish Bin Packing

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Abstract. Selfish bin packing considers a cost-sharing system of the classical bin packing problem, where each item is controlled by a selfish agent and aims to minimize the sharing cost. In this paper we study an incentive mechanism: Interest-Matrix-based (IM-based) mechanism, a new perspective that focuses on the interest or the satisfaction between any pair of items rather than personal sharing cost. Under the IM-based mechanism, we show that $PoA \leq 1.7$ for general instances with item size inside $(1/n_0, 1]$, where n_0 is an arbitrary large integer. In special, when $n_0 = 4$, the PoA of the IM-based mechanism does not exceed 1.5.

Keywords: Selfish bin packing · Mechanism · Nash equilibrium

1 Introduction

Selfish bin packing, originating from bin packing problem [6,13], previously considers a cost-sharing game system. Given a set of items and sufficiently unit-capacity bins. Each item has its size inside the interval $(0, 1]$ and it is controlled by a selfish agent who chooses bins actively and aims to minimize the sharing cost rather than the social cost (the number of consumed bins in total). Note that each bin has unit cost 1, which is shared by all items packed in it. In order to minimize the social cost in this game circumstance, designing incentive mechanisms to lead the agent's actions is necessary. The quality of the mechanism is commonly evaluated by the price of anarchy (PoA), which is the ratio between the social welfare that derives from the worst Nash equilibrium and that of the social optimum.

Selfish bin packing was first introduced in 2006 by Bilò [1] with the first mechanism, proportional weight mechanism. He showed that a pure Nash equilibrium always exists and the PoA is between 1.6 and 1.6667. Epstein et al. [4] did further research of proportional weight mechanism, they proved that the PoA fell into $[1.6416, 1.6428]$, which is the currently best result of proportional weight mechanism.

Until 2013, Han et al. [8] present another mechanism for selfish bin packing, unit weight mechanism. Dosà and Epstein [3] proved that for this mechanism, $PoA \in [1.6966, 1.6994]$. It is the best-known result of unit weight mechanism.

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Recently, a novel mechanism proposed by Wang et al. [14] for selfish bin packing was proposed which is a generalization of several well-known mechanisms, such as, proportional-weight and unit-weight mechanisms. This new mechanism introduces an interest matrix and it focuses on various interests between any pair of items while they shares the same bin. In fact, the motivation of this new mechanism for selfish bin packing is to express that items choose bins not only considering packing-cost, but they always care about the interest or the satisfaction of current situations (sharing bins with items). Therefore, it is a natural idea to take into account the interest between any two items, i.e. interest matrix, in selfish bin packing [14], Wang et al. showed that (1) there exists Nash equilibrium when the matrix is symmetric; (2) the PoA is bounded in several special matrices while in general it can be arbitrary large.

Under the framework of the interest-matrix-based mechanism proposed by Wang et al. [14], we construct a new interest matrix based on the idea: leading the agent to choose bins with the most possible number of items and the largest possible total size.

The organization of the paper is as follows. In Sect. 2, we introduce the related definitions. In Sect. 3, we present the new mechanism, Interest-Matrix-based Mechanism (IM-based mechanism), and give some properties. Section 4 is dedicated to the analysis of PoA of the IM-based mechanism for a special kind of instances, where the size of the items belongs to $(1/4, 1]$. In Sect. 5, we extend the analysis in Sect. 4 to general instances where the size of the items belongs to $(1/n_0, 1]$, and show that the PoA of the IM-based mechanism falls into $[1.623, 1.7]$.

2 Preliminaries

2.1 Selfish Bin Packing and Previous Mechanisms

Selfish bin packing considers a cost-sharing game system [10]. An instance of this game, denoted as $\mathcal{I} = \{L, S\}$, consists of a list $L = \{a_1, \ldots, a_n\}$ of items and the sizes of the items $S = \{s(a_1), \ldots, s(a_n)\}$, $s(a_i) \in (0, 1]$. Assume that each item is controlled by a selfish agent whose strategy is to choose bins under the capacity constraint and aims to minimize the sharing cost rather than the social cost, i.e., the number of consumed bins. In order to minimize the social cost in this game system, designing reasonable payoff rules, i.e., mechanisms is a valid method.

Denote by B a unit-capacity bin that contains a set of items. For a consumed bin B, denote by $s(B)$ the total size of items, $|B|$ the number of items. Some known mechanisms, such as proportional weight mechanism and unit weight mechanism, are extensively studied [2,5,11,15].

– Proportional weight mechanism: item a_i's payoff is proportional to the total size of items sharing the same bin, i.e.,

$$p(a_i) = s(a_i)/s(B), \forall a_i \in B.$$

– Unit weight mechanism: item a_i's payoff is proportional to the number of items sharing the same bin, i.e.,

$$p(a_i) = 1/|B|, \forall a_i \in B.$$

– General weight mechanism: item a_i's payoff is proportional to the total weight of items sharing the same bin, i.e.,

$$p(a_i) = w(a_i)/w(B), \forall a_i \in B,$$

which is the generalization of proportional and unit weight mechanisms.

2.2 Nash Equilibrium and Price of Anarchy

Given any instance of selfish bin packing and a specific mechanism, selfish items actively choose bins or constantly change strategies for minimizing their costs. A Nash equilibrium (NE) [9] is a feasible packing and a stable state in this game system that (1) No item can benefit (decrease its cost) by changing only its own strategy (moving to anther bin) while the other items keep their unchanged. (2) A Nash equilibrium is not necessarily a optimal packing.

The price of anarchy (PoA) [7,12] is a metric to measure the quality of mechanisms, which is defined the ratio between the social cost of the worst NE and that of the optimal solution. Formally, given an instance of selfish bin packing $\mathcal{I}$ and a specific mechanism $\mathcal{M}$, denote by $NE(\mathcal{M}_{\mathcal{I}})$ the social cost of an NE under the mechanism $\mathcal{M}$ and denote by $OPT(\mathcal{I})$ the social cost of the optimal solution. The PoA of mechanism $\mathcal{M}$ is defined as

$$PoA(\mathcal{M}_{\mathcal{I}}) = \limsup_{OPT(\mathcal{I})\to\infty} \max_{\forall NE} \left\{ \frac{NE(\mathcal{M}_{\mathcal{I}})}{OPT(\mathcal{I})} \right\}.$$

3 Interest-Matrix-Based (IM-Based) Mechanism

In this section, we present an interest-matrix-based mechanism for selfish bin packing. We define the interests between each pair of items as the sum of the sizes of both items. The personal goal of each item is to maximize the total interest that derives from other items packed in the same bin.

Given an instance of selfish bin packing $\mathcal{I} = \{L, S\}$: $L = \{a_1, \cdots, a_n\}$, $S = \{s(a_1), \cdots, s(a_n)\}$ and given an interest matrix $A_{n\times n} = [a_{ij}]$, a_{ij} is defined as $a_{ij} = s(a_i) + s(a_j)$.

Our mechanism $\mathcal{M}^\flat$ is as following:

Interest-Matrix-Based (IM-Based) Mechanism $\mathcal{M}^\flat$
For any item a_i in the instance $\mathcal{I}$, assume that a_i chooses the bin B in the current situation. The payoff of item a_i is

$$p(a_i) = \sum_{j\in B} a_{ij} = \sum_{j\in B} (s(a_i) + s(a_j)) = |B| \cdot s(a_i) + s(B).$$

Remark. Under the IM-based mechanism, each item would like to choose the bin that contains the most possible items and the largest possible total size. However it is difficult to decide, between the number of items and the total size of the items in a bin, which is more influential for item's strategy.

We first show that, under the IM-based mechanism, a Nash equilibrium always exists and it can be obtained in finite steps from any packing.

Lemma 1. *Under the IM-based mechanism $\mathcal{M}^\flat$, an NE always exists and it can be obtained in finite steps from any feasible packing.*

The proof of the lemma is omitted, which follows directly from the result in [14] by making use of the method of potential functions.

The following proposition describes an important property of IM-based mechanism. Denote by $\mathcal{B}^i = \{B|B \in NE, |B| = i\}$.

Proposition 1. *Under the IM-based mechanism, there exists at most one bin $B_0^i \in \mathcal{B}^i$ in an NE packing such that $s(B_0^i) \leq \frac{i}{i+1}$.*

Proof. For an NE packing induced by IM-based mechanism, if there exists two bins $B_1, B_2 \in \mathcal{B}^i$ such that $s(B_k) \leq \frac{i}{i+1}$, $k = 1, 2$, we will show the contradiction at the end. Without loss of generality, assume that $s(B_1) \geq s(B_2)$. Consider the smallest item $a_{2\min}$ in the bin B_2, we have

$$s(a_{2\min}) \leq \frac{s(B_2)}{|B_2|} \leq \frac{\frac{i}{i+1}}{i} = \frac{1}{i+1} \leq (1 - s(B_1)),$$

implying that item $a_{2\min}$ can move to the bin B_1. Observe that item $a_{2\min}$'s payoff is

$$\begin{aligned} p(a_{2\min}) &= |B_2| \cdot s(a_{2\min}) + s(B_2) \\ &= i \cdot s(a_{2\min}) + s(B_2), \qquad [a_{2\min} \text{ is packed in } B_2] \end{aligned}$$

and if $a_{2\min}$ moves to the bin B_1, the payoff will be

$$\begin{aligned} p^{'}(a_{2\min}) &= (|B_1| + 1) \cdot s(a_{2\min}) + s(B_1) + s(a_{2\min}) \\ &= (i+2) \cdot s(a_{2\min}) + s(B_1) \\ &> i \cdot s(a_{2\min}) + s(B_2) \\ &= p(a_{2\min}), \qquad [a_{2\min} \text{ moves to } B_1] \end{aligned}$$

Therefore item $a_{2\min}$ has an incentive to move from B_2 to B_1, which contradicts the property of NE. The proof is complete. □

4 The Bounds of PoA While $s(a_{\min}) > 1/4$

In this section, we consider a special case $\mathcal{I}_s = \{L, S\}$ with $s(a_{\min}) > 1/4$, where $a_{\min}$ is the item of smallest size. We show that under the IM-based mechanism, the PoA falls into the interval $[1.333, 1.5]$.

4.1 Upper Bound of the PoA

Given an instance $\mathcal{I}_s = \{L, S\}$ where $L = \{a_1, \cdots, a_n\}$, $S = \{s(a_1), \cdots, s(a_n)\}$ and an interest matrix $A_{n\times n} = [a_{ij}]$ where $a_{ij} = s(a_i) + s(a_j)$. We first discuss the upper bound of the PoA.

Theorem 1. *Given a special instance $\mathcal{I}_s$ with $s(a_{\min}) > 1/4$. Under the IM-based mechanism for selfish bin packing, we have*

$$PoA(\mathcal{M}_{\mathcal{I}_s}) \leq \frac{3}{2}.$$

Proof. Consider an NE packing $NE = \mathcal{B}^1 \cup \mathcal{B}^2 \cup \mathcal{B}^3$ for the instance $\mathcal{I}_s$, where

$$\begin{aligned}\mathcal{B}^1 &= \{B \mid B \in NE, |B| = 1\};\\ \mathcal{B}^2 &= \{B \mid B \in NE, |B| = 2\};\\ \mathcal{B}^3 &= \{B \mid B \in NE, |B| = 3\}.\end{aligned}$$

Note that there exists no bins with four or more items in any feasible packing since $s(a_{\min}) > 1/4$ and the capacity constraint of bins. Denote by $NE(\mathcal{M}_{\mathcal{I}_s})$ and $OPT(\mathcal{I}_s)$ the number of bins in an NE packing and the optimal packing. The following will show the relation between $NE(\mathcal{M}_{\mathcal{I}_s})$ and $OPT(\mathcal{I}_s)$ by the bridge $S(\mathcal{I}_s) = \sum_{i\in\mathcal{I}_s} s(a_i)$, the total size of items in the instance.

We consider $\mathcal{B}^2 = \mathcal{B}^2_\ell \cup \mathcal{B}^2_s$, where

$$\begin{aligned}\mathcal{B}^2_\ell &= \{B|B \in \mathcal{B}^2, \exists a_i \in B, s(a_i) > \frac{1}{2}\};\\ \mathcal{B}^2_s &= \{B|B \in \mathcal{B}^2, \forall a_i \in B, s(a_i) \leq \frac{1}{2}\}.\end{aligned}$$

Observe that there exists $3|\mathcal{B}^3| + 2|\mathcal{B}^2_s| + |\mathcal{B}^2_\ell|$ small items (with size smaller than 1/2) in the item list. The following discusses whether all these small items can share bins with the large items (with size strictly larger than 1/2).

(1) If $3|\mathcal{B}^3| + 2|\mathcal{B}^2_s| \leq |\mathcal{B}^1|$, then $|\mathcal{B}^3| + |\mathcal{B}^2_s| \leq \frac{1}{2}|\mathcal{B}^1|$. Thus we obtain

$$\frac{NE(\mathcal{M}^\flat_{\mathcal{I}_s})}{OPT(\mathcal{I}_s)} \leq \frac{\frac{3}{2}|\mathcal{B}^1| + |\mathcal{B}^2_\ell|}{|\mathcal{B}^1| + |\mathcal{B}^2_\ell|} < \frac{3}{2},$$

which implies that $PoA \leq 3/2$.

(2) If $3|\mathcal{B}^3| + 2|\mathcal{B}^2_s| > |\mathcal{B}^1|$, we assume that $3|\mathcal{B}^3| + 2|\mathcal{B}^2_s| = |\mathcal{B}^1| + x, (x > 0)$. Since there are $|\mathcal{B}^1| + x + |\mathcal{B}^2_l|$ small items in the list, they need at least another $x/3$ bins besides sharing bins with large items. Then we obtain

$$\begin{aligned}OPT(\mathcal{I}_s) &\geq |\mathcal{B}^1| + |\mathcal{B}^2_\ell| + \frac{1}{3}\cdot x,\\ NE(\mathcal{M}^\flat_{\mathcal{I}_s}) &= |\mathcal{B}^1| + |\mathcal{B}^2_\ell| + |\mathcal{B}^2_s| + |\mathcal{B}^3| \leq \frac{3}{2}(|\mathcal{B}^1| + \frac{2}{3}|\mathcal{B}^2_\ell| + \frac{1}{3}\cdot x).\end{aligned}$$

Thus

$$\frac{NE(\mathcal{M}^\flat_{\mathcal{I}_s})}{OPT(\mathcal{I}_s)} \leq \frac{\frac{3}{2}(|\mathcal{B}^1| + |\mathcal{B}^2_\ell| + \frac{1}{3}\cdot x)}{|\mathcal{B}^1| + |\mathcal{B}^2_\ell| + \frac{1}{3}\cdot x} \leq \frac{3}{2},$$

implying that $PoA \leq 3/2$. □

4.2 Lower Bound of the PoA

In this subsection, we construct a worst case example for the IM-based mechanism $\mathcal{M}^\flat$ to give the lower bound of PoA.

Theorem 2. *Under the IM-based mechanism, for the special case $s(a_{\min}) > 1/4$, we have*

$$PoA(\mathcal{M}^\flat) \geq \frac{4}{3} > 1.333.$$

Proof. We construct an instance as follows. Let N be an arbitrary large integer and let $\varepsilon = 1/N$. There are N large items with size $1/2 + \varepsilon$ and N small items with size $1/4 + \varepsilon$.

We present an NE packing that consists of $4/3 \cdot N$ bins as illustrated in Fig. 1, where each large item monopoly occupies one bin and every three small items share one bin. It is clear that no item can benefit from moving alone:

1. Each large item can not move to other bins since the capacity constraint;
2. Each small item can not move to other three-item bins since the capacity constraint and each small item would not like to move to monopoly bins since its payoff will change from $6/4 + 6\varepsilon$ to $5/4 + 4\varepsilon$ (decreasing).

However, the optimal packing (showed in Fig. 1) only consumes N bins, one large item is packed with one small item in each bin. Therefore,

$$PoA(\mathcal{M}^\flat) \geq \frac{4}{3}.$$

□

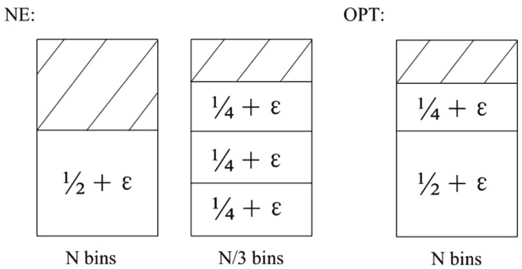

Fig. 1. A worst case of IM-based mechanism $\mathcal{M}^\flat$ while $s(a_{\min}) > 1/4$.

5 The Bounds of PoA While $s(a_{\min}) > 1/n_0$

In this section, we consider more general instances of selfish bin packing problem. Given an instance $\mathcal{I}$ of selfish bin packing, $\mathcal{I} = \{L, S\}$, where $L = \{a_1, \cdots, a_n\}$, $S = \{s(a_1), \cdots, s(a_n)\}$ and an interest matrix $A_{n\times n} = [a_{ij}]$ where $a_{ij} = s(a_i) + s(a_j)$. Note that $s(a_i) > 1/n_0$, $\forall a_i \in L$ and n_0 is an arbitrary large integer. We show that under the IM-based mechanism, the PoA falls into the interval $[1.623, 1.7]$.

5.1 Upper Bound of PoA

The following theorem illustrates that under the IM-based mechanism $\mathcal{M}^\flat$, the upper bound of the PoA is 17/10.

Theorem 3. *For any instance $\mathcal{I}$ with $s(a_{\min}) > 1/n_0$, we show that under the IM-based mechanism $\mathcal{M}_\mathcal{I}$,*

$$PoA(\mathcal{M}_\mathcal{I}) \leq \frac{17}{10}.$$

For the purpose of showing the upper bound of PoA, we introduce a weight function to be the bridge connecting the number of bins in optimal packing and an NE packing. For each item $a \in \mathcal{I}$, the weight $w(a)$ is defined as

$$w(a) = \frac{6}{5}s(a) + v(a),$$

where

$$v(a) = \begin{cases} 0, & 0 < s(a) \leq \frac{1}{6}; \\ \frac{3}{5}(s(a) - \frac{1}{6}), & \frac{1}{6} < s(a) \leq \frac{1}{3}; \\ \frac{1}{10}, & \frac{1}{3} < s(a) \leq \frac{1}{2}; \\ \frac{4}{10}, & \frac{1}{2} < s(a) \leq 1. \end{cases}$$

Denote by $w(\mathcal{I}) = \sum_{a\in\mathcal{I}} w(a)$ the total weight of all items in the instance. Recall that $NE(\mathcal{M}_\mathcal{I})$ and $OPT(\mathcal{I})$ are the number of bins in an NE packing and the optimal packing respectively.

Sketch of the Proof of Theorem 3. For any instance $\mathcal{I}$ and any NE packing, based on the weight function, we focus on illustrating both inequalities:

$$w(\mathcal{I}) \leq \frac{17}{10} \cdot OPT(\mathcal{I}) \quad \text{[total weight \& the optimal packing]} \tag{1}$$

$$w(\mathcal{I}) \geq NE(\mathcal{M}^\flat_\mathcal{I}) - n_0 \quad \text{[total weight \& an NE packing]} \tag{2}$$

If these two inequalities hold, we obtain

$$\frac{NE(\mathcal{M}^\flat_\mathcal{I}) - n_0}{OPT(\mathcal{I})} \leq \frac{17}{10}. \quad [n_0 \text{ is a constant}]$$

When $OPT(\mathcal{I}) \to \infty$,

$$PoA(\mathcal{M}_\mathcal{I}) \leq \frac{17}{10}.$$

In the rest of section we will show the both inequalities (1) and (2).

Lemma 2. *For any instance $\mathcal{I}$ with $s(a_{\min}) > 1/n_0$, we show that*

$$w(\mathcal{I}) \leq \frac{17}{10} \cdot OPT(\mathcal{I}).$$

Proof. To show the lemma, it is sufficient to show the following conclusion:

$$w(B) = \sum_{i=1}^{m} w(a_i) \leq \frac{17}{10}, \quad \forall B \in OPT(\mathcal{I}),$$

where $B = \{a_1, \cdots, a_m\}$ is an arbitrary bin in the optimal packing. Without loss of generality, assume that $s(a_1) \geq \cdots \geq s(a_m)$.

Case 1. If $s(a_1) \in (\frac{1}{2}, 1], s(a_2) \in (\frac{1}{3}, \frac{1}{2}], s(a_3), \cdots, s(a_m) \in (0, \frac{1}{6}]$. Based on the weight function, we have

$$w(B) = \frac{6}{5}s(B) + v(B) \leq \frac{6}{5} \cdot 1 + \frac{4}{10} + \frac{1}{10} = \frac{17}{10}.$$

Case 2. If $s(a_1) \in (\frac{1}{2}, 1], s(a_2), s(a_3) \in (\frac{1}{6}, \frac{1}{3}], s(a_4), \cdots, s(a_m) \in (0, \frac{1}{6}]$, then we obtain

$$\begin{aligned} w(B) &= \frac{6}{5}s(B) + v(B) = \frac{6}{5}s(B) + \frac{4}{10} + \frac{3}{5}\left[s(a_2) + s(a_3) - \frac{2}{6}\right] \\ &\leq \frac{6}{5} + \frac{4}{10} + \frac{3}{5}\left(\frac{1}{2} - \frac{2}{6}\right) = \frac{17}{10}. \end{aligned}$$

Case 3. If $s(a_1) \in (\frac{1}{2}, 1], s(a_2) \in (\frac{1}{6}, \frac{1}{3}], s(a_3), \cdots, s(a_m) \in (0, \frac{1}{6}]$, then

$$\begin{aligned} w(B) &= \frac{6}{5}s(B) + v(B) = \frac{6}{5}s(B) + \frac{4}{10} + \frac{3}{5}\left[s(a_2) - \frac{1}{6}\right] \\ &\leq \frac{6}{5} + \frac{4}{10} + \frac{3}{5}\left(\frac{1}{3} - \frac{1}{6}\right) = \frac{17}{10}. \end{aligned}$$

Case 4. If $s(a_1) \in (\frac{1}{2}, 1], s(a_2), \cdots, s(a_m) \in (0, \frac{1}{6}]$, then we obtain

$$w(B) = \frac{6}{5}s(B) + v(B) \leq \frac{6}{5} \cdot 1 + \frac{4}{10} = \frac{8}{5}.$$

Case 5. If $s(a_i) \in (0, \frac{1}{2}], \forall a_i \in B$. Observe that there exists at most five items with size inside $(1/6, 1]$ in B since capacity constraint. That is B contains at most five items with $v(a_i) > 0$ and $v(a_i) \leq 1/10$. Thus,

$$w(B) = \frac{6}{5}s(B) + v(B) \leq \frac{6}{5} \cdot 1 + \frac{1}{10} \cdot 5 = \frac{17}{10}.$$

Thus, the lemma is proved. □

Lemma 3. *For any instance $\mathcal{I}$ with $s(a_{\min}) > 1/n_0$, under IM-based mechanism $\mathcal{M}^\flat$, we show that*

$$w(\mathcal{I}) \geq NE(\mathcal{M}^\flat_{\mathcal{I}}) - n_0.$$

Proof. Consider any NE packing contains bins $\mathcal{B}^1 \cup \mathcal{B}^2 \cup \cdots \cup \mathcal{B}^{n_0-1}$, where

$$\begin{aligned}\mathcal{B}^1 &= \{B \mid B \in NE, |B| = 1\};\\ \mathcal{B}^2 &= \{B \mid B \in NE, |B| = 2\};\\ &\cdots\\ \mathcal{B}^{n_0-1} &= \{B \mid B \in NE, |B| = n_0 - 1\}.\end{aligned}$$

Note that there exists no bins with n_0 or more items since $s(a_{\min}) > 1/n_0$. To clearly illustrate the property of NE, we give an order of bins in the NE: sorting bins in non-decreasing order by the number of items in a bin.

Case 1. For one-item bins $\mathcal{B}^1$. Based on Proposition 1, there exists at most one bin B_0^1 belonging to $\mathcal{B}^1$ such that $s(B_0^1) \leq 1/2$. Thus,

$$w(B) = \frac{6}{5} \cdot s(B) + v(B) > \frac{6}{5} \cdot \frac{1}{2} + \frac{4}{10} = 1, \quad \forall B \in \mathcal{B}^1 \backslash B_0^1,$$

implying that

$$w(\mathcal{B}^1) = \sum_{B \in \mathcal{B}^1} w(B) \geq |\mathcal{B}^1| - 1.$$

Case 2. For two-item bins $\mathcal{B}^2$. Based on Proposition 1, there exists at most one bin B_0^2 belonging to $\mathcal{B}^2$ such that $s(B_0^2) \leq 2/3$. Let B_1, $B_2 = \{a_1, a_2\}$ be two bins in $\mathcal{B}^2 \backslash B_0^2$. Without loss of generality, assume that B_1 is arranged before B_2 in the order of NE and $s(B_1) \geq s(B_2)$. Note that $s(B_1), s(B_2) > 2/3$.

(1) If $s(B_1) \geq 5/6$, then we have

$$\frac{6}{5} s(B_1) + v(B_2) \geq \frac{6}{5} \cdot \frac{5}{6} = 1.$$

(2) If $s(B_1) < 5/6$, assume that $s(B_1) = 5/6 - x$, $x \in (0, 1/6)$. By Proposition 1, we obtain

$$s(a_1), s(a_2) > 1 - s(B_1) = \frac{1}{6} + x.$$

Then

$$v(a_i) \geq \frac{3}{5}\left(s(a_i) - \frac{1}{6}\right) > \frac{3}{5}\left(\frac{1}{6} + x - \frac{1}{6}\right) = \frac{3}{5} \cdot x, \quad i = 1, 2.$$

Thus we have

$$\frac{6}{5} s(B_1) + v(B_2) \geq \frac{6}{5}\left(\frac{5}{6} - x\right) + \frac{3}{5} \cdot x \cdot 2 = 1,$$

implying that

$$w(\mathcal{B}^2) = \sum_{B \in \mathcal{B}^2} s(B) + v(B) \geq \sum_{i=1, B_i \in \mathcal{B}^2}^{|\mathcal{B}^2|-1} s(B_i) + v(B_{i+1}) \geq |\mathcal{B}^2| - 2.$$

Case 3. For three-item bins $\mathcal{B}^3$. Based on Proposition 1, there exists at most one bin B_0^3 belonging to $\mathcal{B}^3$ such that $s(B_0^3) \leq 3/4$. Let $B = \{a_1, a_2, a_3\}$ be any bin belonging to $\mathcal{B}^3 \backslash B_0^3$. Note that $s(B) > 3/4$.

Based on the case-by-case analysis analogous to the proof of Lemma 2, we obtain that $w(B) \geq 1$.

Thus, we obtain

$$w(\mathcal{B}^3) \geq |\mathcal{B}^3| - 1.$$

Case 4. For four-item bins $\mathcal{B}^4$. Based on Proposition 1, there exists at most one bin B_0^4 belonging to $\mathcal{B}^4$ such that $s(B_0^4) \leq 4/5$. Let $B = \{a_1, a_2, a_3, a_4\}$ be any bin belonging to $\mathcal{B}^4 \backslash B_0^4$. Note that $s(B) > 4/5$.

Also by similar analysis as in Case 3, we have $w(B) \geq 1$.

Thus, we obtain

$$w(\mathcal{B}^4) \geq |\mathcal{B}^4| - 1.$$

Case 5. For five-plus-item bins $\mathcal{B}^j$, $5 \leq j \leq n_0 - 1$. Based on Proposition 1, there exists at most one bin B_0^j belonging to $\mathcal{B}^j$ such that $s(B_0^j) \leq j/(j+1)$. Let $B = \{a_1, \cdots, a_j\}$ be any bin belonging to $\mathcal{B}^j \backslash B_0^j$. Note that $s(B) > j/(j+1) \geq 5/6$. We have

$$w(B) = \frac{6}{5}s(B) + v(B) > \frac{6}{5} \cdot \frac{5}{6} = 1.$$

Then

$$w(\mathcal{B}^j) \geq |\mathcal{B}^j| - 1, \quad 5 \leq j \leq n_0 - 1.$$

In summary,

$$w(\mathcal{I}) = \sum_{i=1}^{n_0-1} w(\mathcal{B}^i) \geq \sum_{i=1}^{n_0-1} |\mathcal{B}^i| - n_0 = NE(\mathcal{M}_{\mathcal{I}}^{\flat}) - n_0.$$

□

5.2 Lower Bound of PoA

Given an instance of selfish bin packing with $s(a_i) \in (1/n_0, 1](i = 1, 2, \cdots, n)$, where n_0 is an arbitrary large integer. In this subsection, we discuss the lower bound of PoA for the IM-based mechanism.

Theorem 4. *For any instance $\mathcal{I}$ with $s(a_{\min}) > 1/n_0$, we show that under the IM-based mechanism $\mathcal{M}^{\flat}$,*

$$PoA(\mathcal{M}^{\flat}) \geq \frac{211}{130} > 1.623.$$

Proof. Consider the following instance: let N be an arbitrary large integer and let $\varepsilon = 1/N$,

$$\mathcal{I} = \left(\frac{1}{2} + \varepsilon, N\right), \left(\frac{1}{4} + \varepsilon, N\right), \left(\frac{1}{6} + \varepsilon, N\right), \left(\frac{1}{13} + \varepsilon, N\right), \left(\frac{1}{156} - 4\varepsilon, N\right),$$

which corresponding to five types of items: type I, II, III, IV, and V. Note that the first number in $(,)$ is the size, second one is the number of items occurred.

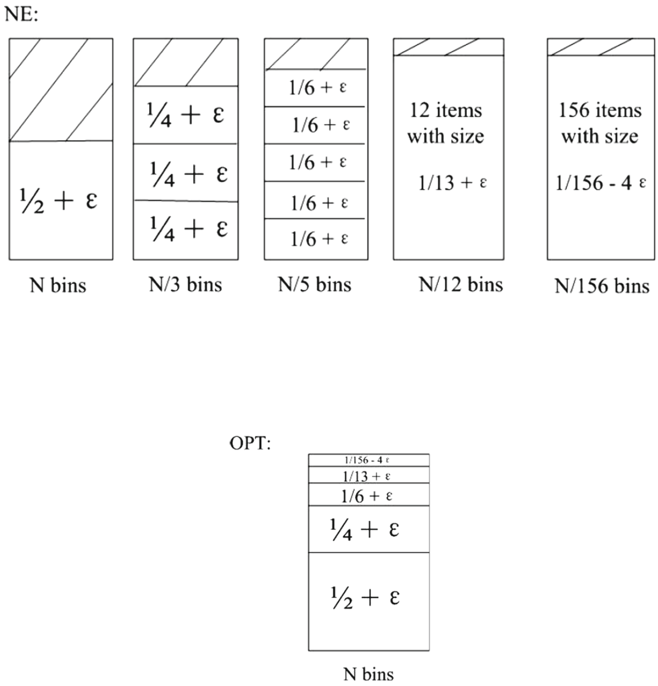

Fig. 2. A worst case of IM-based mechanism $\mathcal{M}^\flat$ while $s(a_{\min}) > 1/n_0$.

We present an NE packing that consists of $211/130 \cdot N$ bins as illustrated in Fig. 2. All these bins can be divided into five types: type 1, 2, 3, 4, and 5. More exactly, Each type-I item monopoly occupies one bin; Every three type-II items share one same bin; Every five type-III items share one same bin; Every 12 type-IV items share one same bin; Every 156 type-V items share one same bin. Clearly, no item can benefit by moving alone under this packing:

(1) Each type-I item can not move to other bins since the capacity constraint;
(2) Each type-II item can not move to other type 2–5 bins since the capacity constraint and each type-II item would not like to move to type-1 bins since its payoff will change from $6/4 + 6\varepsilon$ to $5/4 + 5\varepsilon$ (decreasing);
(3) Each type-III item can not move to other type 3–5 bins since the capacity constraint and type-III item would not like to move to type-1 or type-2 bins since its payoff will change from $20/12 + 10\varepsilon$ to at most $19/12 + 8\varepsilon$ (decreasing);
(4) Each type-IV item can not move to other type 4 or type 5 bins since the capacity constraint and type-IV item would not like to move to type 1–3 bins since its payoff will change from $144/78 + 24\varepsilon$ to at most $107/78 + 12\varepsilon$ (decreasing);
(5) Each type-V item would not like to move to other bins since it currently stays in the bin with the largest size and the most number of items.

However, the optimal packing (showed in Fig. 2), only consumes N bins, in which each bin is full and consists of five items that belongs to five different types. Therefore,

$$PoA(\mathcal{M}^\flat) \geq \frac{211}{130} > 1.623.$$

□

6 Conclusion and Extension

In this paper, we present an interest-matrix-based mechanism for selfish bin packing that focuses on the interest or the satisfaction between each pair of items rather than the sharing cost. Under this mechanism, we show that for a general case where $s(a_{\min}) > 1/n_0$ (n_0 is an larger integer), the PoA falls into the interval $[1.623, 1.7]$. Specially, when $n_0 = 4$, the $PoA \in [1.333, 1.5]$. Based on our discussion, there are still several open problems.

1. When the smallest item's size tends to zero, i.e., $n_0 \to \infty$ is not a fixed number, is the upper bound of PoA still 1.7?
2. Does there exist any more incentive mechanism with interest matrix for selfish bin packing?
3. Is it possible to consider a new social goal, such as the total interest of all items? Correspondingly, is it desirable to discuss the PoA that is the ratio between the total interest that derives from the worst NE and the optimal solution?

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