Categorial Local Quantum Physics

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Abstract Categorial local quantum field theory was suggested as a new paradigm for quantum field theory by Brunetti, Fredenhagen and Verch in 2003 (Commun Math Phys 237:31–68, [7]). In this paradigm quantum field theory is defined to be a covariant functor from the category of certain spacetimes (with isometric embeddings as morphisms) into the category of C^*-algebras (with injective C^*-algebra homomorphisms as morphisms). Further properties of the functor are stipulated axiomatically on the basis of physical considerations. The present paper suggests an additional axiom on the functor that expresses independence of systems as morphism co-possibility. It is argued that this axiom is very natural because it has a direct physical interpretation. The relation of the axiom system containing the morphism co-possibility axiom to other axiom systems is investigated. It will be seen that this axiom system is strictly stronger than the axiom system originally formulated in Brunetti, Fredenhagen, Verch (Commun Math Phys 237:31–68, 2003, [7]), and it is conjectured that it is strictly weaker than the ones formulated in subsequent development of categorial quantum field theory in which the functor is required to be extensible to a tensor functor. Determining the precise status of the axiom system based on morphism co-possibility as independence needs further analysis.

Keywords Quantum field theory · Category theory · Operator algebra theory

1 The Main Idea of the Categorial Paradigm for Quantum Field Theory

In their seminal paper, [7], Brunetti, Fredenhagen and Verch initiated a new approach to quantum field theory. The new approach is based on category theory. The theory

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M. Ozawa et al. (eds.), *Reality and Measurement in Algebraic Quantum Theory*,
Springer Proceedings in Mathematics & Statistics 261,
https://doi.org/10.1007/978-981-13-2487-1_5

was further developed in a series of papers [5, 6, 14, 18] (the recent papers [12, 13] give an overview of the framework).

This new, categorial approach generalizes substantially the Haag–Kastler algebraic axiomatization of quantum field theory (for monographic presentation of the Haag–Kastler axiomatization see [2, 15, 16]). The main motivation for the Brunetti-Fredenhagen-Verch generalization is the desire to develop quantum field theory in a general (curved) spacetime. To do this one needs a formalism that is flexible enough to accommodate any (physically reasonable) spacetime as background to quantum field theory. The Haag–Kastler approach is unsatisfactory from this perspective because it relies on certain axioms (e.g. covariance with respect to the group of symmetries of the spacetime, spectrum condition, existence of vacuum state) which are framed in terms of a preferred representation of the Poincaré group. In a curved spacetime, however, there are no non-trivial global symmetries; hence none of the standard axioms that rely on the existence of a global symmetry make sense in a quantum field theory on a general, curved spacetime. Brunetti, Fredenhagen and Verch summarize these motivations this way:

> Quantum field theory incorporates two main principles into quantum physics, locality and covariance. Locality expresses the idea that quantum processes can be localized in space and time (and, at the level observable quantities, that causally separated processes are exempt from any uncertainty relations restricting their commensurability). The principle of covariance within *special* relativity states that there are no preferred Lorentzian coordinates for the description of physical processes, and thereby the concept of an absolute space as an arena for physical phenomena is abandoned. Yet it is meaningful to speak of events in terms of spacetime points as entities of a given, fixed spacetime background, in the setting of special relativistic physics.
>
> In general relativity, however, spacetime points loose this a priori meaning. The principle of general covariance forces one to regard spacetime points simultaneously as members of several, locally diffeomorphic spacetimes. It is rather the relations between distinguished events that have physical interpretation.
>
> This principle should also be observed when quantum field theory in presence of gravitational fields is discussed.
>
> Quantum field theory ... is a *covariant functor* ... in the ... fundamental and physical sense of implementing the principles of locality and general covariance... [7] [p. 61–78]

The covariant functor Brunetti, Fredenhagen and Verch refer to in the above quotation is between two concrete categories (see Sect. 2 for the properties of these two categories):

- $(\mathfrak{Man}, hom_{\mathfrak{Man}})$
  The category of spacetimes with isometric embeddings of spacetimes as morphisms.
- $(\mathfrak{Alg}, hom_{\mathfrak{Alg}})$
  The category of C^*-algebras with injective C^*-algebra homomorphisms as morphisms.

The properties of the functor are fixed axiomatically: one requires the functor to have certain features that express "locality" alluded to in the quotation above. It is a priori more or less clear that this can be done in more than one ways. It will be seen in

Sect. 2 that different axioms have indeed been formulated in the papers [5–7, 13, 14, 18]. The different axiomatizations differ in how they express independence of physical systems pertaining to causally disjoint spacetime regions.

The present paper suggests an axiomatization in which the axiom expressing independence of systems differs from the ones in the aforementioned papers. The independence axiom proposed here is the categorial morphism-co-possibility, introduced first in [20]. I will argue that the independence axiom suggested is very natural because it has a direct physical interpretation. Having different axiomatizations, the question of their relation emerges as a non-trivial problem. Some results will be recalled in Sect. 4 that clarify some of the relations but there remain open questions in this regard. It will be seen that the axiom system proposed in this paper is strictly stronger than the axiom system originally formulated in [7], and it is conjectured that it is strictly weaker than the ones formulated in subsequent developments of categorial quantum field theory in which the functor is required to be extensible to a tensor functor. Determining the status of the axiom system based on morphism–co-possibility as independence needs further analysis.

2 The Covariant Functor of Categorial Local Quantum Physics

The category $(\mathfrak{Man}, hom_{\mathfrak{Man}})$ is specified by the following stipulations (see [7] for more details):

- The objects in $Obj(\mathfrak{Man})$ are 4 dimensional C^∞ spacetimes (M, g) with a Lorentzian metric g and such that (M, g) is Hausdorff, connected, time oriented and globally hyperbolic.
- The morphisms in $hom_{\mathfrak{Man}}$:

$$\psi : (M_1, g_1) \to (M_2, g_2)$$

  are isometric smooth embeddings such that

  - ψ preserves the time orientation;
  - ψ is causal in the following sense:
    if the endpoints $\gamma(a), \gamma(b)$ of a timelike curve $\gamma : [a, b] \to M_2$ are in the image $\psi(M_1)$, then the whole curve is in the image: $\gamma(t) \in \psi(M_1)$ for all $t \in [a, b]$.
  - The composition of morphisms is the usual composition of maps.

The category $(\mathfrak{Alg}, hom_{\mathfrak{Alg}})$ is defined as:

- The objects in $Obj(\mathfrak{Alg})$ are unital C^*-algebras.
- The morphisms are injective, unit preserving C^*-algebra homomorphisms

$$\alpha : \mathscr{A}_1 \to \mathscr{A}_2$$

The composition of morphisms is the usual composition of C^*-algebra homomorphisms.

Categorial local quantum field theory is then defined as a functor:

Definition 1 A locally covariant quantum field theory is a covariant functor $\mathscr{F}$ between the categories $(\mathfrak{Man}, hom_{\mathfrak{Man}})$ and $(\mathfrak{Alg}, hom_{\mathfrak{Alg}})$:

- For any object (M, g) in $\mathfrak{Man}$ the $\mathscr{F}(M, g)$ is a C^*-algebra in $\mathfrak{Alg}$.
- For any homomorphism ψ in $hom_{\mathfrak{Man}}$ the $\mathscr{F}(\psi)$ is a C^*-algebra homomorphism in $hom_{\mathfrak{Alg}}$

such that the following hold

$$\mathscr{F}(\psi_1 \circ \psi_2) = \mathscr{F}(\psi_1) \circ \mathscr{F}(\psi_2)$$
$$\mathscr{F}(id_{\mathfrak{Man}}) = id_{\mathfrak{Alg}}$$

The physical interpretation of the elements in the definition is along the lines of local quantum physics as this is understood in the Haag–Kastler version of algebraic quantum field theory: The functor $\mathscr{F}$ assigns to a spacetime manifold (M, g) an operator algebra $\mathscr{F}(M, g)$ of observables measurable in M. This explicit association of the observables with a specific spacetime embodies an elementary but fundamental aspect of locality: the idea that any measurement, observation, and interaction can only take place at a particular location in spacetime. Following the terminology introduced in [20], I call this kind of locality "spatio-temporal locality".

3 Causal Locality Conditions on the Covariant Functor

The interpretation of $\mathscr{F}(M, g)$ as the algebra of observables measurable in M motivates imposing further conditions on the functor $\mathscr{F}$. The further conditions express "locality", understood as conditions ensuring harmony of the assignment $(M, g) \mapsto \mathscr{F}(M, g)$ with the causal structure of the spacetimes. Following the terminology introduced in [20], I call this kind of locality "causal locality" to distinguish it from "spatio-temporal locality", which does not involve causal content explicitly: Spatio-temporal locality expresses the fact that $\mathscr{F}$ explicitly specifies the spatio-temporal location of observables in such a way that spatio-temporal locality of observables is in harmony with the subsystem relation. That is to say, the content of spatio-temporal locality is that a physical system's set of observables are a subset of the set of observables of a system if the latter system's spatio-temporal locality region contains that of the former (this is expressed by the covariance property of the functor). While extremely natural, spatio-temporal locality is crucially important: it is a conceptual pre-condition without which causal locality cannot be formulated at all [20]; furthermore, as emphasized by [15], all the physical information is contained in the association of observables with spacetime regions. This is reflected by the fact that in the Haag–Kastler version of algebraic quantum field theory it holds that the local algebras pertaining to typical spacetime regions are all isomorphic [8], [15] [p. 225]; thus the physical content of the theory is contained in the way the isomorphic algebras are related to each other via the isotony relation constrained by the causal locality condition.

Causal locality so interpreted cannot be expressed as a single condition: A spacetime has a causal structure that specifies both causally *independent* and causally *dependent* spacetime regions. Accordingly, causal locality conditions to be imposed on the functor $\mathscr{F}$ should regulate the behavior of $\mathscr{F}$ from the perspective of both causally independent and dependent spacetime regions. The most basic stipulations were formulated in [7]: Einstein Locality and Time Slice axiom, they are recalled in the next subsection.

3.1 The BASIC Axioms: Einstein Locality and Time Slice

Definition 2 The covariant functor of categorial quantum field theory

$$\mathscr{F} : (\mathfrak{Man}, hom_{\mathfrak{Man}}) \to (\mathfrak{Alg}, hom_{\mathfrak{Alg}})$$

should satisfy

- Causal Locality – Independence: Einstein Causality:

$$\Big[\mathscr{F}(\psi_1)\Big(\mathscr{F}(M_1, g_1)\Big), \mathscr{F}(\psi_2)\Big(\mathscr{F}(M_2, g_2)\Big)\Big]_-^{\mathscr{F}(M,g)} = \{0\} \tag{1}$$

whenever

$$\psi_1 \ : \ (M_1, g_1) \to (M, g) \tag{2}$$
$$\psi_2 \ : \ (M_2, g_2) \to (M, g) \tag{3}$$

and $\psi_1(M_1)$ and $\psi_2(M_2)$ are spacelike in M, where $[\ ,\ ]_-^{\mathscr{F}(M,g)}$ in (1) denotes the commutator in the C^*-algebra $\mathscr{F}(M, g)$.

- Causal Locality – Dependence: Time slice axiom: If (M, g) and (M', g') and the embedding

$$\psi : (M, g) \to (M', g')$$

are such that $\psi(M, g)$ contains a Cauchy surface for (M', g') then

$$\mathscr{F}(\psi)\mathscr{F}(M, g) = \mathscr{F}(M', g')$$

I call the axiom system specified by Definition 2 BASIC.

In what sense is Einstein Causality a causal independence condition? The standard answer is that Einstein Causality entails no superluminal signaling with respect to non-selective operations represented by local Kraus operations: A completely positive, unite preserving map (operation) T on local algebra $\mathscr{F}(M, g)$ of the form

$$T(X) = \sum_i W_i^* X W_i \tag{4}$$

is called a local Kraus operation represented by the local Kraus operators W_i, if all W_i are in $\mathscr{F}(M, g)$ and sum up to the identity:

$$\sum_i W_i^* W_i = I \tag{5}$$

(See [3, 17] for the definition and elementary facts about operations, including operations that are not Kraus representable.) Given spacetimes (M_1, g_1), (M_2, g_2) and (M, g) with embeddings ψ_1, ψ_2 (2)–(3) such that $\psi_1(M_1)$ and $\psi_2(M_2)$ are spacelike in M, any Kraus operation T_1 on $\mathscr{F}(M_1, g_1)$ can be extended to the local algebra $\mathscr{F}(M, g)$ to a Kraus operation T by

$$T(A) \doteq \sum_i \mathscr{F}(\psi_1)(W_i^*) A \mathscr{F}(\psi_1)(W_i) \qquad A \in \mathscr{F}(M, g) \tag{6}$$

Einstein Locality together with (5) entails then that the restriction of T to the algebra $\mathscr{F}(\psi_2)(\mathscr{F}(M_2, g_2))$ is the identity map. Thus the state of system localized in spacetime (M_2, g_2) and viewed as a subsystem of $\mathscr{F}(M, g)$ remains unaffected by performing the operation T_1 on system in spacetime (M_1, g_1) (viewed as a subsystem of $\mathscr{F}(M, g)$). This is the content of no-signaling. A particular case of no-signaling is when the Kraus operators W_i in (4) are one dimensional projections, giving no-signaling with respect to the projection postulate.

Einstein Causality does not entail however no superluminal signaling with respect to *general* spatio-temporally local operations; i.e. with respect to operations T on $\mathscr{F}(M_1, g_1)$ that are not Kraus representable. An example of such an operation is the Accardi-Cecchini state-preserving conditional expectation [1] in the context of the Haag–Kastler quantum field theory (see [23] for details).

More generally: Einstein Causality does not, in and of itself, entail what is called operational subsystem independence: That any two (non-selective) operations performed on spacelike separated subsystems S_1, S_2 of system S are jointly implementable as a single operation on S [22]. Given the significance of the concept of subsystem independence in quantum field theory [24, 25], one should ensure that the axioms of the categorial approach to quantum field theory express subsystem independence. One way to do this is to formulate a categorial version of subsystem independence and postulate it axiomatically as a required feature of the functor $\mathscr{F}$. The natural independence notion in a concrete category is morphism co-possibility. This notion was introduced in [20] in the specific context of the category $(\mathfrak{Alg}, hom_{\mathfrak{Alg}})$ but I formulate it in an arbitrary concrete category in the next subsection.

3.2 Amending the Basic Axioms by Adding Morphism Co-possibility as Subsystem Independence

Let $(\mathfrak{C}, hom_{\mathfrak{C}})$ be a concrete category and $Mor_{\mathfrak{C}}$ be a class of morphisms between objects of $\mathfrak{C}$. The morphism class $Mor_{\mathfrak{C}}$ can be the same as $hom_{\mathfrak{C}}$, but this is not required: $Mor_{\mathfrak{C}}$ can be larger than $hom_{\mathfrak{C}}$. The class of morphisms $Mor_{\mathfrak{C}}$ should be viewed as a variable in the independence notion, specified by the following definition. Different choices of $Mor_{\mathfrak{C}}$ yield qualitatively different independence concepts.

Definition 3 Given objects C_1, C_2 and C in $\mathfrak{C}$ and homomorphisms $h_1 \colon C_1 \to C$ and $h_2 \colon C_2 \to C$ in $hom_{\mathfrak{C}}$, the objects $h_1(C_1)$ and $h_2(C_2)$ are said to be $Mor_{\mathfrak{C}}$-independent in C, if for any two morphisms $m_1 \colon h_1(C_1) \to h_1(C_1)$ and $m_2 \colon h_2(C_2) \to h_2(C_2)$ in $Mor_{\mathfrak{C}}$, there exists a morphism $m \colon C \to C$ in $Mor_{\mathfrak{C}}$ that coincides with m_1 on $h_1(C_1)$ and coincides with m_2 on $h_2(C_2)$.

It is intuitively clear why $Mor_{\mathfrak{C}}$-independence of objects $h_1(C_1)$ and $h_2(C_2)$ is an independence condition: fixing morphism m_1 on object $h_1(C_1)$ does not interfere with fixing any morphism m_2 on object $h_1(C_1)$ and vice versa. That is to say, morphisms can be independently chosen on these objects seen as parts of object C. This independence notion is a natural categorial generalization of the concept known as subsystem independence [24, 25]. One can recover all the major subsystem independence concepts that occur in algebraic quantum (field) theory by taking the category $(\mathfrak{Alg}, hom_{\mathfrak{Alg}})$ and choosing, as morphism class $Mor_{\mathfrak{Alg}}$, special subclasses of the class of all non-selective operations (unit preserving completely positive, linear maps) $Op_{\mathfrak{Alg}}$ [19].

Given the concept of $Op_{\mathfrak{Alg}}$-independence, it is natural to impose it on the covariant functor $\mathscr{F}$ representing quantum field theory in order to express causal locality in terms of it:

Definition 4 The covariant functor of categorial quantum field theory

$$\mathscr{F} : (\mathfrak{Man}, hom_{\mathfrak{Man}}) \to (\mathfrak{Alg}, hom_{\mathfrak{Alg}})$$

should satisfy

- Causal Locality – Independence: $Op_{\mathfrak{Alg}}$-independence:
  whenever

$$\psi_1 \ : \ (M_1, g_1) \to (M, g) \tag{7}$$

$$\psi_2 \ : \ (M_2, g_2) \to (M, g) \tag{8}$$

and $\psi_1(M_1)$ and $\psi_2(M_2)$ are spacelike in M, the objects $\mathscr{F}(M_1, g_1)$ and $\mathscr{F}(M_2, g_2)$ are $Op_{\mathfrak{Alg}}$-independent in the sense of Definition 3, taking $Op_{\mathfrak{Alg}}$ as $Mor_{\mathfrak{Alg}}$.

The axiom system that requires quantum field theory to be a covariant functor having the features of Einstein Locality, Time Slice axiom and $Op_{\mathfrak{Alg}}$-independence, is called **OPIND**.

One can strengthen $Op_{\mathfrak{Alg}}$-independence into $Op_{\mathfrak{Alg}}$-independence *in the product sense* by requiring the morphism m in Definition 3 that extends m_1 and m_2 to factorize across $h_1(C_1)$ and $h_2(C_2)$:

$$m(AB) = m(A)m(B) = m_1(A) = m_2(B) \qquad A \in h_1(C_1) \quad B \in h_2(C_2) \tag{9}$$

This leads to a natural strengthening of the axiom system **OPIND**: by requiring that the extension T in (6) factorizes across the algebras $\mathscr{F}(\psi_1)(\mathscr{F}(M_1, g_1))$ and $\mathscr{F}(\psi_2)(\mathscr{F}(\psi_2))$:

$$T(AB) = T(A)T(B) \qquad A \in \mathscr{F}(\psi_1)(\mathscr{F}(M_1, g_1)), \quad B \in \mathscr{F}(\psi_2)(\mathscr{F}(M_2, g_2)) \tag{10}$$

I call the strengthened axiom system **OPIND**$^{\times}$.

3.3 Amending the Basic Axioms by Adding the Categorial Split Property

Categorial split property was introduced in [6] (also see [11]). The categorial split property is the categorial version of what is known the funnel property of the Haag–Kastler net of local algebras:

Definition 5 The functor $\mathscr{F}$ has the categorial split property if the following two conditions hold:

1. For spacetimes $(M, g_M), (N, g_N)$ in $\mathfrak{Man}$ and morphism $\psi: (M, g_M) \to (N, g_N)$ such that the closure of $\psi(M, g_M)$ is compact, connected and in the interior of M, there exists a type I von Neumann factor $\mathscr{R}$ such that

$$\mathscr{F}(\psi)(\mathscr{F}(M, g_M)) \subset \mathscr{R} \subset \mathscr{F}(N, g_N) \tag{11}$$

2. σ-continuity of the $\mathscr{F}(\psi')$ with respect to the inclusion $\mathscr{R} \subset \mathscr{R}'$, where $\psi' : (M, g_M) \to (L, g_L)$ and

$$(\mathscr{F}(\psi') \circ \mathscr{F}(\psi))(\mathscr{F}(M, g_M)) \subset \mathscr{F}(\psi')(\mathscr{R}) \tag{12}$$

$$\subset \mathscr{F}(\psi')(\mathscr{F}(N, g_N)) \subset \mathscr{R}' \subset \mathscr{F}(L, g_L) \tag{13}$$

For later purposes I recall the notion of weak additivity of the functor $\mathscr{F}$:

Definition 6 The functor $\mathscr{F}$ satisfies weak additivity if for any spacetime (M, g) and any family of spacetimes (M_i, g_i) with morphisms $\psi_i : (M_i, g_i) \to (M, g)$ such that

$$M \subseteq \cup_i \psi_i(M_i) \tag{14}$$

we have

$$\mathscr{F}(M, g) = \overline{\cup_i \mathscr{F}(\psi_i)(\mathscr{F}(M_i, g_i))}^{norm} \tag{15}$$

I call **BASIC+SPLIT** the axiom system that requires of the covariant functor $\mathscr{F}$ to have weak additivity and the categorial split property, in addition to Einstein Locality and Time Slice axiom.

3.4 The Tensor Axiom

The axiom system BASIC was modified by Brunetti and Fredenhagen by replacing the Einstein Causality condition by an axiom that requires a tensorial property of $\mathscr{F}$ (Axiom 4 in [5]; also see [14]). To formulate this axiom one has to extend $(\mathfrak{Man}, hom_{\mathfrak{Man}})$ and $(\mathfrak{Alg}, hom_{\mathfrak{Alg}})$ to tensor categories.

The category $(\mathfrak{Man}^{\otimes}, hom^{\otimes}_{\mathfrak{Man}})$ has, by definition, as its objects *finite* disjoint unions of objects from $\mathfrak{Man}$, and the empty set as unit object. (Thus the objects in $\mathfrak{Man}^{\otimes}$ are no longer connected spacetimes.) By definition, the morphisms $\psi^{\otimes}$ in $hom^{\otimes}_{\mathfrak{Man}}$ are maps of the form

$$\psi^{\otimes} : M_1 \sqcup M_2 \sqcup \ldots \sqcup M_n \to M \tag{16}$$

$\sqcup$ denoting the disjoint union) such that

- the restriction of $\psi^{\otimes}$ to any M_i are morphisms in the category $(\mathfrak{Man}, hom_{\mathfrak{Man}})$;
- the images $\psi^{\otimes}(M_i)$ and $\psi^{\otimes}(M_j)$ of the spacetimes M_i are spacelike in M for $i \neq j$.

One can take $(\mathfrak{Alg}^{\otimes}, hom^{\otimes}_{\mathfrak{Alg}})$ to be the tensor category of C^*-algebras with respect to the minimal tensor product of C^*-algebras, with the set of complex numbers as unit object and with the homomorphisms $hom^{\otimes}_{\mathfrak{Alg}}$ being identical to $hom_{\mathfrak{Alg}}$: the class of injective C^*-algebra homomorphisms.

To define the tensorial features of the functor, we need some notation first. Let $\psi_i : M_i \to N_i$ be embeddings of disjoint spacetimes M_i $(i = 1, 2)$ such that the images $\psi_1(M_1)$ and $\psi_2(M_2)$ are causally disjoint in $N_1 \cup N_2$. Then $\psi_1 \otimes \psi_2$ denotes the map

$$(\psi_1 \otimes \psi_2) \; : \; M_1 \sqcup M_2 \to N_1 \cup N_2 \tag{17}$$

$$(\psi_1 \otimes \psi_2)(x) \doteq \begin{cases} \psi_1(x) & \text{if } x \in M_1 \\ \psi_2(x) & \text{if } x \in M_2 \end{cases} \tag{18}$$

Clearly, the map $(\psi_1 \otimes \psi_2)$ is a morphism in the category $(\mathfrak{Man}^{\otimes}, hom^{\otimes}_{\mathfrak{Man}})$. The tensor product $\alpha_1 \otimes \alpha_2$ of two injective C^*-algebra homomorphisms α_1 and α_2 on the tensor product $\mathscr{A}_1 \otimes \mathscr{A}_2$ of C^*-algebras $\mathscr{A}_1$ and $\mathscr{A}_2$ is defined in the usual way as the extension to $\mathscr{A}_1 \otimes \mathscr{A}_2$ of the map

$$(\mathscr{A}_1 \otimes \mathscr{A}_2) \ni A_1 \otimes A_2 \mapsto \alpha_1(A_1) \otimes \alpha_2(A_2) \tag{19}$$

Let $\iota_1 \colon M_1 \to M_1 \otimes M_2$ denote the trivial embedding of spacetime M_1 into the disjoint union $M_1 \otimes M_2$. One can then require that the covariant functor $\mathscr{F}$ be a tensor functor in the sense of the following definition:

Definition 7 The covariant functor

$$\mathscr{F}^{\otimes} \colon (\mathfrak{Man}^{\otimes}, hom^{\otimes}_{\mathfrak{Man}}) \to (\mathfrak{Alg}^{\otimes}, hom^{\otimes}_{\mathfrak{Alg}}) \tag{20}$$

is called a *tensor* functor if for any two spacetimes $M_1, M_2 \in \mathfrak{Man}$ with $M_1 \cap M_2 = \emptyset$ and embeddings $\psi_1 \colon M_1 \to N$ and $\psi_2 \colon M_2 \to N$ with causally disjoint images in N we have

$$\mathscr{F}^{\otimes}(\emptyset) = \mathbb{C} \tag{21}$$

$$\mathscr{F}^{\otimes}(\iota_1)(A_1) = A_1 \otimes I \qquad A_1 \in \mathscr{F}^{\otimes}(M_1) \tag{22}$$

$$\mathscr{F}^{\otimes}(\iota_2)(A_2) = I \otimes A_2 \qquad A_2 \in \mathscr{F}^{\otimes}(M_2) \tag{23}$$

$$\mathscr{F}^{\otimes}(M_1 \otimes M_2) = \mathscr{F}^{\otimes}(M_1) \otimes \mathscr{F}^{\otimes}(M_2) \tag{24}$$

$$\mathscr{F}^{\otimes}(\psi_1 \otimes \psi_2) = \mathscr{F}^{\otimes}(\psi_1) \otimes \mathscr{F}^{\otimes}(\psi_2) \tag{25}$$

I call **TENSOR** the axiom system that requires the functor $\mathscr{F}$ to be extendible to a tensor functor $\mathscr{F}^{\otimes}$ between the tensor categories $(\mathfrak{Man}^{\otimes}, hom^{\otimes}_{\mathfrak{Man}})$ and $(\mathfrak{Alg}^{\otimes}, hom^{\otimes}_{\mathfrak{Alg}})$, in addition to the Time Slice axiom.

4 Relation of Axiom Systems

Given the axiom systems **BASIC**, **OPIND**, **OPIND**$^{\times}$, **BASIC+SPLIT** and **TENSOR**, two questions arise:

- What is their logical relation?
- Assuming that they are not equivalent, which one is the most suitable one?

The next few propositions summarize some relations among the axiom systems. In the rest of this section (M_1, g_1), (M_2, g_2) and (M, g) are objects from $\mathfrak{Man}$, ψ_1 and

ψ_2 are morphisms from $hom_{\mathfrak{Man}}$ such that

$$\psi_1 \; : \; (M_1, g_1) \to (M, g) \tag{26}$$
$$\psi_2 \; : \; (M_2, g_2) \to (M, g) \tag{27}$$

and $\psi_1(M_1)$ and $\psi_2(M_2)$ are spacelike in M.

Proposition 1

$$\textbf{\textit{BASIC+SPLIT}} \Leftrightarrow \textbf{\textit{TENSOR}}$$

Proposition 1 is the combined content of Theorem 1 in [5] and Theorem 2.5 in [6]. The role of the split property and weak additivity in the equivalence claim is to pick out the minimal (also called: spatial) tensor product in the category of C^*-algebras from the other possible tensor products as the suitable one to define the extension of the functor $\mathscr{F}$ to the tensor functor $\mathscr{F}^{\otimes}$.

Proposition 10 in [22] entails that $Op_{\mathfrak{Alg}}$-independence in the product sense of algebras $\mathscr{F}(\psi_1)(\mathscr{F}(M_1, g_1))$ and $\mathscr{F}(\psi_2)(\mathscr{F}(M_1, g_1))$ is equivalent to the condition that the algebra in $\mathscr{F}(M, g)$ generated by algebras $\mathscr{F}(\psi_1)(\mathscr{F}(M_1, g_1))$ and $\mathscr{F}(\psi_2)(\mathscr{F}(M_1, g_1))$ is isomorphic to the tensor product $\mathscr{F}(\psi_1)(\mathscr{F}(M_1, g_1)) \otimes \mathscr{F}(\psi_2)(\mathscr{F}(M_1, g_1))$. So we have

Proposition 2

$$\textbf{\textit{BASIC+SPLIT}} \Leftrightarrow \textbf{\textit{TENSOR}} \Leftarrow \textbf{\textit{OPIND}}^{\times}$$

Does **TENSOR** entail **OPIND**$^{\times}$? Since the tensor product of two operations is again an operation [4] [p. 190], (see also Proposition 9 in [22]), it follows that if $\mathscr{F}$ can be extended to a tensor functor $\mathscr{F}^{\otimes}$, then the algebras $\mathscr{F}(\psi_1)(\mathscr{F}(M_1, g_1))$ and $\mathscr{F}(\psi_2)(\mathscr{F}(M_1, g_1))$ are $Op_{\mathfrak{Alg}}$-independent in the product sense *in the tensor product* $\mathscr{F}(\psi_1)(\mathscr{F}(M_1, g_1)) \otimes \mathscr{F}(\psi_2)(\mathscr{F}(M_1, g_1))$. Although this tensor product algebra is a C^*-subalgebra of $\mathscr{F}(M, g)$, since operations on subalgebras need not be extendible to operations to superalgebras, [3], the $Op_{\mathfrak{Alg}}$-independence of algebras $\mathscr{F}(\psi_1)(\mathscr{F}(M_1, g_1))$ and $\mathscr{F}(\psi_2)(\mathscr{F}(M_1, g_1))$ in the tensor product algebra $\mathscr{F}(\psi_1)(\mathscr{F}(M_1, g_1)) \otimes \mathscr{F}(\psi_2)(\mathscr{F}(M_1, g_1))$ does *not* entail, without further assumptions, the $Op_{\mathfrak{Alg}}$-independence of $\mathscr{F}(\psi_1)(\mathscr{F}(M_1, g_1))$ and $\mathscr{F}(\psi_2)(\mathscr{F}(M_1, g_1))$ in $\mathscr{F}(M, g)$. A C^*-algebra $\mathscr{A}$ is called injective just in case the following holds: Given any C^*-algebra $\mathscr{B}$ and a C^*-subalgebra $\mathscr{B}_0 \subseteq \mathscr{B}$, every completely positive map $T_0 \colon \mathscr{B}_0 \to \mathscr{A}$ has an extension to $\mathscr{B}$ to a completely positive map $T \colon \mathscr{B} \to \mathscr{A}$. Thus **TENSOR** entails **OPIND**$^{\times}$ holds if the local algebras $\mathscr{F}(M, g)$ are injective. In the Haag–Kastler quantum field theory the local algebras associated with double cones can be proved to be hyperfinite hence injective [8], [15] [p. 225], [9], [10] [Theorem 6]. But it is not clear to me whether injectivity of the algebras $\mathscr{F}(M, g)$ can be proved to be a consequence of **BASIC+SPLIT**. Thus the following problem seems to be open:

Problem 1

$$\textbf{BASIC+SPLIT} \Leftrightarrow \textbf{TENSOR} \overset{?}{\not\Rightarrow} \textbf{OPIND}^{\times}$$

OPIND$^{\times}$ obviously entails **OPIND**. I conjecture that the converse is not true. To prove this rigorously one would have to display a model of the axioms such that **OPIND** holds but **OPIND**$^{\times}$ does not. I am not aware of such models; however, this conjecture is supported by the following two facts: (i) C^*-independence is strictly weaker than C^*-independence in the product sense ([24], also see Proposition 1 in [22]); (ii) operational C^*-independence in the product sense is equivalent to C^*-independence in the product sense when the C^*-subalgebras commute (Proposition 10 in [22]).

We have seen in Sect. 3.1 that **BASIC** does not entail **OPIND**. Since **OPIND** entails **BASIC** by definition, the logical relationship of the different axioms can be summarized in the following diagram (question mark ? next to the arrows indicating open questions):

$$
\begin{array}{rc}
\textbf{BASIC+SPLIT} \Leftrightarrow & \textbf{TENSOR} \\
 & \Uparrow \ \Downarrow ? \\
 & \textbf{OPIND}^{\times} \\
 & \Uparrow ? \ \ \Downarrow \\
 & \textbf{OPIND} \\
 & \Downarrow \not\Uparrow \\
 & \textbf{BASIC}
\end{array}
$$

Assessing the suitability of the different axiom systems, one has to ask which of the axioms has a direct physical interpretation. From this perspective I regard **OPIND** as the most suitable one: $Op_{\mathfrak{A}\mathfrak{l}\mathfrak{g}}$-independence has a clear operational physical content. This content is not captured fully by **BASIC**, and both **OPIND**$^{\times}$ and the (possibly equivalent) **TENSOR** seem to require too much: a particular (product) form of independence that does not seem to be justifiable by some specific physical considerations.

What matters ultimately, however, is which of the axiom systems allows a sufficient number of models that describe physically relevant quantum fields. Axiom systems in physics, in quantum field theory in particular, should possess the right balance of two features that are pulling in different directions: restricting their models and, at the same time, being sufficiently non-categorical, allowing a large number of models describing physical systems [21].

Acknowledgements Research supported in part by the Hungarian Scientific Research Found (OTKA). Contract numbers: K-115593 and K100715.

References

1. L. Accardi and C. Cecchini. Conditional expectations in von Neumann algebras and a theorem of Takesaki. *Journal of Functional Analysis*, 45(245-273), 1982.
2. H. Araki. *Mathematical Theory of Quantum Fields*, volume 101 of *International Series of Monograps in Physics*. Oxford University Press, Oxford, 1999. Originally published in Japanese by Iwanami Shoten Publishers, Tokyo, 1993.
3. W. Arveson. Subalgebras of C^*-algebras. *Acta Mathematica*, 123:141–224, 1969.
4. B. Blackadar. *Operator Algebras: Theory of C*-Algebras and von Neumann Algebras*. Encyclopaedia of Mathematical Sciences. Springer, 1. edition, 2005.
5. R. Brunetti and K. Fredenhagen. Quantum field theory on curved backgrounds. In C. Bär and K. Fredenhagen, editors, *Quantum Field Theory on Curved Spacetimes*, volume 786 of *Lecture Notes in Physics*, chapter 5, pages 129–155. Springer, Dordrecht, Heidelberg, London, New York, 2009.
6. R. Brunetti, K. Fredenhagen, I. Paniz, and K. Rejzner. The locality axiom in quantum field theory and tensor products of C^*-algebras. *Reviews of Mathematical Physics*, 26:1450010, 2014. arXiv:1206.5484 [math-ph].
7. R. Brunetti, K. Fredenhagen, and R. Verch. The generally covariant locality principle – a new paradigm for local quantum field theory. *Communications in Mathematical Physics*, 237:31–68, 2003. arXiv:math-ph/0112041.
8. D. Buchholz, C. D'Antoni, and K. Fredenhagen. The universal structure of local algebras. *Commununications in Mathematical Physics*, 111:123–135, 1987.
9. A. Connes. Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$. *The Annals of Mathematics*, 104:73–115, 1976.
10. A. Connes. *Non-commutative Geometry*. Academic Press, San Diego and New York, 1994.
11. C.J. Fewster. The split property for quantum field theories in flat and curved spacetimes. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 86:153–175, 2016. arXiv:1601.06936 [math-ph].
12. C.J. Fewster and R. Verch. Algebraic quantum field theory in curved spacetimes. In R. Brunetti, C. Dappiaggi, K. Fredenhagen, and J. Yngvason, editors, *Advances in Algebraic Quantum Field Theory*, Mathematical Physics Studies, pages 125–189. Springer International Publishing, Switzerland, 2015.
13. K. Fredenhagen and K. Reijzner. Quantum field theory on curved spacetimes: Axiomatic framework and examples. *Journal of Mathematical Physics*, 57:031101, 2016.
14. K. Fredenhagen and K. Rejzner. Local covariance and background independence. In F. Finster, O. Müller, M. Nardmann, J. Tolksdorf, and E. Zeidler, editors, *Quantum Field Theory and Gravity. Conceptual and Mathematical Advances in the Search for a Unified Framework*, pages 15–24. Birkhäuser Springer Basel, Basel, 2012. arXiv:1102.2376 [math-ph].
15. R. Haag. *Local Quantum Physics: Fields, Particles, Algebras*. Springer Verlag, Berlin and New York, 1992.
16. S.S. Horuzhy. *Introduction to Algebraic Quantum Field Theory*. Kluwer Academic Publishers, Dordrecht, 1990.
17. K. Kraus. *States, Effects and Operations*, volume 190 of *Lecture Notes in Physics*. Springer, New York, 1983.
18. I. Paniz. Tensor pruducts of C^*-algebras and independent systems. Master's thesis, Institut für Theoretische Physik, Universität Hamburg, Hamburg, 2012.
19. M. Rédei. Operational independence and operational separability in algebraic quantum mechanics. *Foundations of Physics*, 40:1439–1449, 2010.
20. M. Rédei. A categorial approach to relativistic locality. *Studies in History and Philosophy of Modern Physics*, 48:137–146, 2014.
21. M. Rédei. Hilbert's 6th problem and axiomatic quantum field theory. *Perspectives on Science*, 22:80–97, 2014.
22. M. Rédei and S.J. Summers. When are quantum systems operationally independent? *International Journal of Theoretical Physics*, 49:3250–3261, 2010.

23. M. Rédei and G. Valente. How local are local operations in local quantum field theory? *Studies in History and Philosophy of Modern Physics*, 41:346–353, 2010.
24. S.J. Summers. On the independence of local algebras in quantum field theory. *Reviews in Mathematical Physics*, 2:201–247, 1990.
25. S.J. Summers. Subsystems and independence in relativistic microphysics. *Studies in History and Philosophy of Modern Physics*, 40:133–141, 2009. arXiv:0812.1517 [quant-ph].