Homomorphism Between Fuzzy Set-Valued Information Systems

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Abstract Communication between various information systems (IS) has been an urgent issue that needs to be discussed in the granular world. Compatible and consistent homomorphism is an important analytical tool to study communication among various IS. Fuzzy set-valued information systems (FSIS) are those IS that contain fuzzy set-values for some attribute. This paper aims to discuss important properties related to communication between FSIS. Fuzzy relation mapping from the perspective of FSIS is discussed. The proposed approach proved that feature selection and other properties of the original FSIS and the corresponding image FSIS are assured under consistent and compatible homomorphism. Finally, a real-life example demonstrated the utilization of the proposed work.

Keywords Fuzzy set-valued information systems · Homomorphism
Feature selection · Fuzzy relation mapping

1 Introduction

Rough set theory (RST) is an influential soft computing tool in the area of information technology and has attracted much attention and interest [1–5]. Continuous or real-valued IS cannot be handled directly by the traditional rough set because it requires discretization of real attributes before feature selection, so to handle this, fuzzy rough set (FRS) is introduced [2]. In FRS, fuzzy similarity relation is used to handle real-valued attributes. RST begins with a single-valued information system

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[2–5]; however, in real-life situations, some attribute values may contain set-values, resulting in the formation of SIS [6, 7]. Dai and Tian [6] investigated FRS model for SIS. SIS studied so far contains crisp set-values [6, 7] and does not consider the case of fuzzy set-valued attributes. To handle these types of attributes, a FSIS was introduced [1].

In today's Information era, information system (IS) is the realm of information technology, and the major concern here is communication between IS. Due to different nature of IS, it becomes necessary to transfer information within these IS [3, 5]. This motivates us to examine the communication among various IS's. In this work, important properties related to communication between FSIS using FRS model are discussed. Communication means transformation of information between IS, alongside keeping unblemished its prime properties and functions.

From mathematical viewpoint, mapping is an efficient way to communicate between IS by analyzing their properties [5]. Homomorphism mapping plays a vital role to accomplish this task [3, 5]. In this paper, important properties related to communication between FSIS under FRS model are discussed.

The proposed approach proved that feature selection and other properties of the original FSIS and the corresponding image FSIS are assured under consistent and compatible homomorphism.

The structure of the remaining paper is defined below:

A brief description about FSIS is given in Sect. 2. In Sect. 3, fuzzy relation mapping for FSIS is formalized. Section 4 discusses the notion of homomorphism between FSIS. In Sect. 5, the conclusion is outlined.

2 Preliminaries

2.1 Introduction to FSIS

Definition 1 A FSIS = (S, C, V, f) is an IS that contains real or fuzzy set-values for some of the attribute [1].

Where S and C are set of sample and set of fuzzy multivalued attributes, respectively. $f: S \times C \rightarrow V$, where for $m \in S$, $c \in C$, $f(\mathrm{m, c}) \in V_c$ assigns fuzzy set-values to samples. Also, V_c is the set of pairs $(x, \mu_{Vc}(\mathrm{x}))$ such that $\mu_{Vc}(x)$ is the membership value of x in V_c between [0, 1].

Example 2 FSIS is given in Table 1. Here, $S = \{m_1, m_2, m_3, m_4\}$ is a set of objects; $C = \{R, W, S\}$ is a set of attributes; $V = \{\text{French, German, English}\}$ is domain values. For ease, French, German, and English are denoted by f, g, and e, respectively.

If suppose, 'c' is an attribute 'Reading'= $\{f, g, e\}$. Then $c(m) = \{(f, 0.8), (g, 0.6), (e, 0.9)\}$ illustrates that reading abilities of 'm' in French, German, and English are 0.8, 0.6, and 0.9, respectively.

Table 1 An example of FSIS

FSIS	Reading (R)	Writing (W)	Speaking (S)
m_1	{(f, 0.8), (g, 0.6), (e, 0.9)}	{(f, 0.3), (g, 0.7), (e, 0.5)}	{(f, 0.6), (g, 0.9), (e, 0.7)}
m_2	{(f, 0.8), (g, 0.6), (e, 0.9)}	{(f, 0.3), (g, 0.7), (e, 0.5)}	{(f, 0.6), (g, 0.9), (e, 0.7)}
m_3	{(f, 1), (g, 0.3), (e, 0)}	{(f, 0.8), (g, 0.3), (e, 0)}	{(f, 0.9), (g, 0.6), (e, 0.3)}
m_4	{(f, 0.5), (g, 0.9), (e, 0.4)}	{(f, 0.7), (g, 0.7), (e, 0)}	{(f, 0.5), (g, 0.8), (e, 0.7)}

2.2 Fuzzy Similarity Relation for FSIS

Definition 3 For FSIS = (S, C) and $a \in C$ and $m, n \in S,$ the fuzzy similarity relation R_a is defined as:

$$\mu_{R_a}(m, n) = \frac{\sum \inf(a(m), a(n))}{\sum \sup(a(m), a(n))}$$

3 Fuzzy Relation Mapping for FSIS

A definition of fuzzy relation mapping to communicate between FSIS using Zadeh's extension principle is given below:

For two universal sets S_1 and S_2, let $R(S_1 \times S_1)$ and $R(S_2 \times S_2)$ represent classes of all fuzzy binary relation on S_1 and S_2, respectively.

Definition 4 Let $f: S_1 \to S_2$ be a mapping. 'f' generate a mapping from $R(S_1 \times S_1)$ to $R(\mathrm{S}_2 \times S_2)$ as:

$$f(R)(z_1, z_2) = \begin{cases} \sup_{m_1 \in f^{-1}(z_1)} \sup_{m_2 \in f^{-1}(z_2)} R(m_1, m_2), & (z_1, z_2) \in f(S_1) \times f(S_1) \\ 0, & (z_1, z_2) \notin f(S_1) \times f(S_1) \end{cases}$$

3.1 Consistent Functions

Definition 5 Let S_1 and S_2 are universe of discourse, $f: S_1 \to S_2$ is their mapping from S_1 to S_2, and $B, B_1, B_2 \in R(S_1 \times S_1)$. Let $[a]_f = \{b \in U: f(a) = f(b)\}$;

For any $m_1, m_2 \in [a]_f$, and $z_1, z_2 \in [b]_f$, we say 'f' is compatible with relation B, if $B(m_1, z_1) = B(m_2, z_2)$.

For any a, $b \in S_1$, if any of the condition holds:

(1) $B_1(m_1, z_1) \leq B_2(m_1, z_1)$ for any $(m_1, z_1) \in [a]_f \times [b]_f$
(2) $B_1(m_1, z_1) \geq B_2(m_1, z_1)$ for any $(m_1, z_1) \in [a]_f \times [b]_{f,}$ then the mapping 'f' is consistent to relation B_1 and B_2.

Theorem 6 *Let f: $S_1 \to S_2$, B, B_1, $B_2 \in R(S_1 \times S_1)$; then, we have the following:*

(1) $f(B_1 \cup B_2) = f(B_1) \cup f(B_2)$
(2) $f(B_1 \cap B_2) \subseteq f(B_1) \cap f(B_2)$; if '$f$' is a consistent mapping, then they are equal.

Proof (1) For any $z_1, z_2 \in S_2$

$$f(B_1 \cup B_2)(z_1, z_2) = \sup_{m_1 \in f^{-1}(z_1)} \sup_{m_2 \in f^{-1}(z_2)} (B_1 \cup B_2)(m_1, m_2)$$
$$= \sup_{m_1 \in f^{-1}(z_1)} \sup_{m_2 \in f^{-1}(z_2)} (B_1(m_1, m_2) \cup B_2(m_1, m_2))$$
$$= (f(B_1) \cup f(B_2))(z_1, z_2).$$

(2)

$$f(B_1 \cap B_2)(z_1, z_2) = \sup_{m_1 \in f^{-1}(z_1)} \sup_{m_2 \in f^{-1}(z_2)} (B_1 \cap B_2)(m_1, m_2)$$
$$= \sup_{m_1 \in f^{-1}(z_1)} \sup_{m_2 \in f^{-1}(z_2)} (B_1(m_1, m_2) \cap B_2(m_1, m_2))$$
$$\leq \sup_{m_1 \in f^{-1}(z_1)} \sup_{m_2 \in f^{-1}(z_2)} B_1(m_1, m_2) \cap \sup_{m_1 \in f^{-1}(z_1)} \sup_{m_2 \in f^{-1}(z_2)} B_2(m_1, m_2)$$
$$= (f(B_1) \cap f(B_2))(z_1, z_2).$$

Now, it will be proved that the equality holds, if the mapping 'f' is consistent mapping:

Now, since 'f' is consistent mapping to B_1 and B_2, it follows from Definition 5 that it satisfies one of the following conditions:

1. $B_1(m_1, m_2) \leq B_2(m_1, m_2)$
2. $B_1(m_1, m_2) \geq B_2(m_1, m_2)$.

For any $(m_1, m_2) \in f^{-1}(z_1) \times f^{-1}(z_2)$
For case 1,

$$f(B_1 \cap B_2)(z_1, z_2) = \sup_{m_1 \in f^{-1}(z_1)} \sup_{m_2 \in f^{-1}(z_2)} (B_1 \cap B_2)(m_1, m_2)$$
$$= \sup_{m_1 \in f^{-1}(z_1)} \sup_{m_2 \in f^{-1}(z_2)} (B_1(m_1, m_2) \cap B_2(m_1, m_2))$$
$$= \sup_{m_1 \in f^{-1}(z_1)} \sup_{m_2 \in f^{-1}(z_2)} B_1(m_1, m_2)$$
$$= f(B_1)(z_1, z_2)$$

Now taking RHS,

$$(f(B_1) \cap f(B_2)(z_1, z_2) = f(B_1)(z_1, z_2) \cap f(B_2)(z_1, z_2)$$
$$= (\sup_{m_1 \in f^{-1}(z_1)} \sup_{m_2 \in f^{-1}(z_2)} (B_1(m_1, m_2))$$

$$\cap \left(\sup_{m_1 \in f^{-1}(z_1)} \sup_{m_2 \in f^{-1}(z_2)} B_2(m_1, m_2) \right)$$
$$= f(B_1)(z_1, z_2)$$

Hence $f(B_1 \cap B_2) = f(B_1) \cap f(B_2)$.

Example 7 (continued to example 2) Let $S_1 = \{m_1, m_2, m_3, m_4\}$ and $S_2 = \{z_1, z_2, z_3\}$ f_2: $S_1 \rightarrow S_2$ is a mapping from S_1 to S_2 and defined as: $f_2(m_1) = f_2(m_2) = z_1$; $f_2(m_3) = z_2$; $f_2(m_4) = z_3$;

Fuzzy similarity relations for attributes reading (R), writing (W), and speaking (S) given in Table 1 are calculated using Definition 3 and shown in Table 2. ($R_R \cap R_W \cap R_S$) is given in Table 3.

Images of R, W, and S using 'f_2' are computed and presented in Table 4. $f_2(R_R) \cap f_2(R_W) \cap f_2(R_S)$ and $f_2(R_R \cap R_W \cap R_S)$ are given in Table 5.

We find that $f_2(R_R \cap R_W \cap R_S) = f_2(R_R) \cap f(R_W) \cap f(R_S)$, since mapping f_2 is a compatible mapping.

4 Homomorphism Between FSIS

This section focuses on the concept of homomorphism and discusses important properties of FSIS using homomorphism.

Definition 9 Let S_1 and S_2 are two universes, f: $S_1 \rightarrow S_2$ be its mapping from S_1 to S_2. Assume $\boldsymbol{R} = \{B_1, B_2, \ldots, B_n\}$ is collection of fuzzy relations on S_1; then $f(\boldsymbol{R}) = \{f(B_1), f(B_2), \ldots, f(B_n)\}$. Then $(S_1, \boldsymbol{R})$ is termed as FSRIS, and the corresponding $(V_1, f(\boldsymbol{R}))$ is its induced FSRIS [5].

Definition 9 Let $(S_1, \boldsymbol{R})$ be a FSRISs and 'f' is a function mapping. We define homomorphism using 'f' satisfying certain conditions as follows [5]:

(1) $\forall$ B_i, $B_j \in \boldsymbol{R}$, if 'f' is consistent with every B_i and B_j, then we say that 'f' is consistent homomorphism.
(2) $\forall$ $B_i \in \boldsymbol{R}$, *if is* compatible with each B_i, then we say that 'f' is compatible homomorphism.

Definition 10 Let $(S_1, \boldsymbol{R})$ be FSRISs and $P \subseteq \boldsymbol{R}$ satisfies the following:

(1) $\cap P = \cap R$
(2) $\forall B_i \in P, \cap P \subset \cap(P - \{B_i\})$.

Then we say P is a reduct of $\boldsymbol{R}$

Theorem 11 *Let $(S_1, \boldsymbol{R})$ be FSRISs and 'f' be a consistent homomorphism mapping from S_1 to S_2. $P \subseteq \boldsymbol{R}$ is a reduct of $\boldsymbol{R}$ only when f(P) is reduct of f($\boldsymbol{R}$) and vice versa.*

Table 2 Similarity relations for attribute R, W, and S

R_R	m_1	m_2	m_3	m_4	R_W	m_1	m_2	m_3	m_4	R_S	m_1	m_2	m_3	m_4
m_1	1	1	0.4	0.6	m_1	1	1	0.3	0.5	m_1	1	1	0.6	0.9
m_2	1	1	0.4	0.6	m_2	1	1	0.3	0.5	m_2	1	1	0.6	0.9
m_3	0.4	0.4	1	0.3	m_3	0.3	0.3	1	0.7	m_3	0.6	0.6	1	0.6
m_4	0.6	0.6	0.3	1	m_4	0.5	0.5	0.7	1	m_4	0.9	0.9	0.6	1

Table 3 Relation for ($R_R \cap R_W \cap R_S$)

($R_R \cap R_W \cap R_S$)	m_1	m_2	m_3	m_4
m_1	1	1	0.3	0.5
m_2	1	1	0.3	0.5
m_3	0.3	0.3	1	0.3
m_4	0.5	0.5	0.3	1

Table 4 Images of *R*, *W*, and *S* using f_2

f_2 (R_R)	z_1	z_2	z_3	f_2 (R_W)	z_1	z_2	z_3	f_2 (R_S)	z_1	z_2	z_3
z_1	1	0.4	0.6	z_1	1	0.3	0.5	z_1	1	0.6	0.9
z_2	0.4	1	0.3	z_2	0.3	1	0.7	z_2	0.6	1	0.6
z_3	0.6	0.3	1	z_3	0.5	0.7	1	z_3	0.9	0.6	1

Table 5 Relation for $f_2(R_R) \cap f(R_W) \cap f(R_S)$ and $f_2(R_R \cap R_W \cap R_S)$

$f_2(R_R \cap R_W \cap R_S)$	z_1	z_2	z_3	$f_2(R_R) \cap f_2(R_W) \cap f_2(R_S)$	z_1	z_2	z_3
z_1	1	0.3	0.5	z_1	1	0.3	0.5
z_2	0.3	1	0.3	z_2	0.3	1	0.3
z_3	0.5	0.3	1	z_3	0.5	0.3	1

Proof $\Rightarrow$ Let us suppose, *P* be reduct of ***R***. Therefore, $\cap P \neq \cap(P - \{B_i\})$.

Then, there must be $m_1, m_2 \in S_1$, such that $\cap P(m_1, m_2) < \cap\ (P - \{B_i\})(m_1, m_2)$, which implies,

$$
\begin{aligned}
& f(\cap(P - \{B_i\}))(f(m_1),\ f(m_2)) \\
& = \sup_{z_1 \in f^{-1} f(m_1))} \sup_{z_2 \in f^{-1} f(m_2))} \cap(P - \{B_i\})\ (z_1, z_2) \\
& > \sup_{z_1 \in f^{-1} f(m_1))} \sup_{z_2 \in f^{-1} f(m_2))} \cap P\ (z_1, z_2) \\
& = f(\cap P)\ (f(m_1), f(m_2)) = f(\cap R)(f(m_1), f(m_2))
\end{aligned}
$$

Now, $\cap P = \cap \boldsymbol{R}$. Hence, $f(\cap P) = f(\cap \boldsymbol{R})$
Using Theorem 6, $\cap f(P) = \cap f(\boldsymbol{R})$.
Assume, $f(P)$ is not reduct of $f(\boldsymbol{R})$, $\exists\ B_i \in P$ such that $\cap(f(P) - f(B_i)) = \cap f(P)$.
Since, $f(P) - f(R_i) = f(P - \{B_i\})$; therefore, $\cap f(P - \{B_i\}) = \cap f(P) = \cap f(\boldsymbol{R})$
Again, by Theorem 6, $f(\cap(P - B_i)) = f(\cap \boldsymbol{R})$
which is a conflict to the assumption that $f(P)$ is not reduct of $f(\boldsymbol{R})$.
$\therefore f(P)$ is not reduct of $f(\boldsymbol{R})$.

Example 12 (continued to example 7) Let $S_1 = \{m_1, m_2, m_3, m_4\}$ and $S_2 = \{z_1, z_2, z_3\}$

It is evident from Tables 2 and 4 that $\{R_R, R_W\}$ is reduct of '**R**', if and only if $\{f_2(R_R), f_2(R_W)\}$ is a reduct of $f_2(\boldsymbol{R})$.

5 Conclusion

This paper aims to discuss important properties related to communication between FSIS using FRS model. The definition of fuzzy relation mapping from the perspective of FSIS is discussed. The proposed approach proved that feature selection and other properties of the original FSIS and the corresponding image FSIS are assured in the case of consistent and compatible homomorphism.

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