Chapter 14
Half-Normal Distribution: Ordinary Differential Equations

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Abstract In this chapter, homogenous ordinary differential equations (ODES) of different orders were obtained for the probability density function, quantile function, survival function inverse survival function, hazard function and reversed hazard functions of half-normal distribution. This is possible since the aforementioned probability functions are differentiable. Differentiation and modified product rule were used to obtain the required ordinary differential equations, whose solutions are the respective probability functions. The different conditions necessary for the existence of the ODEs were obtained and it is almost in consistent with the support that defined the various probability functions considered. The parameters that defined each distribution greatly affect the nature of the ODEs obtained. This method provides new ways of classifying and approximating other probability distributions apart from half-normal distribution considered in this chapter. In addition, the result of the quantile function can be compared with quantile approximation using the quantile mechanics.

Keywords Differential calculus · Half-normal distribution · Hazard function
Inverse survival function · Quantile function · Quantile mechanics
Reversed hazard function · Survival function

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14.1 Introduction

Calculus in general and differential calculus in particular is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. The method of maximum likelihood is an example.

Differential equations often arise from the understanding and modeling of real life problems or some observed physical phenomena. Approximations of probability functions are one of the major areas of application of calculus and ordinary differential equations in mathematical statistics. The approximations are helpful in the recovery of the probability functions of complex distributions [1–4] especially in quantile approximations.

Apart from mode estimation, parameter estimation and approximation, probability density function (PDF) of probability distributions can be expressed as ODE whose solution is the PDF. Some of which are available. They include: beta distribution [5], Lomax distribution [6], beta prime distribution [7], Laplace distribution [8] and raised cosine distribution [9].

The aim of this research is to develop homogenous ordinary differential equations for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of half-normal distribution. This will also help to provide the answers as to whether there are discrepancies between the support of the distribution and the necessary conditions for the existence of the ODEs. Similar results for other distributions have been proposed, see [10–23] for details.

Half-normal distribution is a normal distribution with a mean set at zero and parameterized to the domain of positive real numbers and zero being the lower bound. Pewsey [24, 25] worked on the improved statistical inference for the distribution while [26] proposed unbiased estimators for the parameters of the distribution, which according to them, performs better than the traditional maximum likelihood. Some generalizations are available for the distribution such as: the extended generalized half-normal distribution [27], beta generalized half-normal distribution [28], generalized half-normal distribution [29, 30], discrete half-normal distribution [31], an extension of the half-normal distribution called the slashed half-normal distribution [32], Kumaraswamy generalized half-normal distribution [33], beta generalized half-normal geometric distribution [34], gamma half-normal distribution [35] and alpha half-Normal Slash distribution [36]. Also available are epsilon half-normal distribution [37]. The distribution is a sub-model of exponentiated generalized gamma distribution proposed by [38] and generalized half-t distribution by [39]. Also available is the odd log-logistic generalized half-normal lifetime distribution [40].

In addition, the distribution was generalized with the Airy model to obtain the M-Wright distribution [41]. Details of the new method of generating the distribution is given in [42] and its application to quality control were highlighted by [43]. Lang [44] used the distribution to model wage gap between immigrants and natives in Germany.

The distribution was among those used by Schoenberg et al. [45] in modeling of the distribution of the sizes of wildfire.

Differential calculus was used to obtain the results.

14.2 Probability Density Function

The probability density function of the half-normal distribution is given by;

$$f(x) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}} \tag{14.1}$$

To obtain the first order ordinary differential equation for the probability density function of the half-normal distribution, differentiate Eq. (14.1);

$$f'(x) = -\frac{x\sqrt{2}}{\sigma^3\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}} \tag{14.2}$$

The condition necessary for the existence of the equation is $\sigma > 0$.

Simplify Eq. (14.2) using Eq. (14.1);

$$f'(x) = -\left(\frac{x}{\sigma^2}\right)\frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}} = -\frac{x}{\sigma^2} f(x) \tag{14.3}$$

The first order ordinary differential for the probability density function of the half-normal distribution is given as;

$$\sigma^2 f'(x) + x f(x) = 0 \tag{14.4}$$

with initial value condition; $f(1) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{1}{2\sigma^2}}$.

To obtain the second order ordinary differential equation for the probability density function of the half-normal distribution, differentiate Eq. (14.2);

$$f''(x) = \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}} \left\{ \frac{x^2}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} - e^{-\frac{x^2}{2\sigma^2}} \right\} \tag{14.5}$$

The condition necessary for the existence of the equation is $\sigma > 0$.

Two differential equations can be obtained from the simplification of Eq. (14.5). They are presented as Eqs. (14.6) and (14.7).

$$\sigma^4 f''(x) + (\sigma^2 - x^2) f(x) = 0 \tag{14.6}$$

$$\sigma^2 f''(x) + x f'(x) + f(x) = 0 \tag{14.7}$$

To obtain the third order ordinary differential equation for the probability density function of the half-normal distribution, differentiate Eq. (14.5);

$$f'''(x) = \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\left\{-\frac{x^3}{\sigma^4}\mathrm{e}^{-\frac{x^2}{2\sigma^2}} + \frac{3x}{\sigma^2}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\right\} \tag{14.8}$$

The condition necessary for the existence of the equation is $\sigma > 0$.

Six differential equations can be obtained from the simplification of equations. They are listed as Eqs. (14.9–14.14).

$$\sigma^6 f'''(x) - (3\sigma^2 x - x^3)f(x) = 0 \tag{14.9}$$

$$\sigma^4 f'''(x) + (3\sigma^2 - x^2)f'(x) = 0 \tag{14.10}$$

$$\sigma^4 f'''(x) - x^2 f'(x) - 3xf(x) = 0 \tag{14.11}$$

$$\sigma^4 f'''(x) + \sigma^2 x f''(x) - 2xf(x) = 0 \tag{14.12}$$

$$\sigma^2 f'''(x) + xf''(x) + 2f'(x) = 0 \tag{14.13}$$

$$\sigma^4 f'''(x) + \sigma^2 x f''(x) + \sigma^2 f'(x) - xf(x) = 0 \tag{14.14}$$

$$f'(1) = -\frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\mathrm{e}^{-\frac{1}{2\sigma^2}} \tag{14.15}$$

To obtain the fourth order ordinary differential equation for the probability density function of the half-normal distribution, differentiate Eq. (14.8);

$$f^{\prime v}(x) = \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\left\{\frac{x^4}{\sigma^6}\mathrm{e}^{-\frac{x^2}{2\sigma^2}} - \frac{6x^2}{\sigma^4}\mathrm{e}^{-\frac{x^2}{2\sigma^2}} + \frac{3}{\sigma^2}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\right\} \tag{14.16}$$

The condition necessary for the existence of the equation is $\sigma > 0$.

Twelve differential equations can be obtained from the simplification of Eq. (14.16);

$$\sigma^8 f^{\prime v}(x) - (x^4 - 6\sigma^2 x^2 + 3\sigma^4)f(x) = 0 \tag{14.17}$$

$$x\sigma^6 f^{\prime v}(x) + (x^4 - 6\sigma^2 x^2 + 3\sigma^4)f'(x) = 0 \tag{14.18}$$

$$\sigma^6 f^{\prime v}(x) + (x^3 - 6\sigma^2 x)f'(x) - 3\sigma^4 f(x) = 0 \tag{14.19}$$

$$\sigma^6 f^{\prime v}(x) - (\sigma^2 x^2 - 3\sigma^4)f''(x) + 2x^2 f(x) = 0 \tag{14.20}$$

$$\sigma^4 f^{\prime v}(x) - (x^2 - 3\sigma^2)f''(x) - 2xf'(x) = 0 \tag{14.21}$$

$$\sigma^6 f^{\prime v}(x) + x\sigma^4 f'''(x) - 3(\sigma^2 - x^2)f(x) = 0 \tag{14.22}$$

$$x\sigma^4 f^{\prime v}(x) + x^2\sigma^2 f'''(x) + 3(\sigma^2 - x^2)f'(x) = 0 \tag{14.23}$$

$$\sigma^2 f^{\prime v}(x) + xf'''(x) + 3f''(x) = 0 \tag{14.24}$$

$$\sigma^4 f^{\prime v}(x) - (x^2 - 2\sigma^2)f''(x) - 3xf'(x) - f(x) = 0 \tag{14.25}$$

$$\sigma^4 f^{\prime v}(x) + \sigma^2 x f'''(x) - 3xf'(x) - 3f(x) = 0 \tag{14.26}$$

$$\sigma^6 f'^v(x) + \sigma^4 x f'''(x) + 2\sigma^4 f''(x) + (x^2 - \sigma^2) f(x) = 0 \tag{14.27}$$

$$x\sigma^4 f'^v(x) + \sigma^2 x^2 f'''(x) + 2\sigma^2 x f''(x) - (x^2 - \sigma^2) f'(x) = 0 \tag{14.28}$$

14.3 Quantile Function

The Quantile function of the half-normal distribution is given by;

$$Q(p) = \sigma\sqrt{2} erf^{-1}(p) \tag{14.29}$$

To obtain the first order ordinary differential equation for the Quantile function of the half-normal distribution, differentiate Eq. (14.29);

$$Q'(p) = \frac{\sigma\sqrt{2\pi}}{2} e^{[erf^{-1}(p)]^2} \tag{14.30}$$

The condition necessary for the existence of the equation is

$$\sigma > 0, 0 \leq p < 1.$$

Simplify Eq. (14.30) using Eq. (14.29), however Eq. (14.29) becomes;

$$\frac{Q(p)}{\sigma\sqrt{2}} = erf^{-1}(p) \tag{14.31}$$

$$Q'(p) = \frac{\sigma\sqrt{2\pi}}{2} e^{\frac{Q^2(p)}{2\sigma^2}} \tag{14.32}$$

$$\ln Q'(p) = \ln\left(\frac{\sigma\sqrt{2\pi}}{2}\right) + \frac{Q^2(p)}{2\sigma^2} \tag{14.33}$$

$$2\sigma^2 \ln Q'(p) - Q^2(p) - 2\sigma^2 g = 0 \tag{14.34}$$

where $g = \ln\left(\frac{\sigma\sqrt{2\pi}}{2}\right)$ with initial value condition; $Q(0) = 0$.

To obtain the second order ordinary differential equation for the Quantile function of the half-normal distribution, differentiate Eq. (14.30);

$$Q''(p) = \frac{\sigma\pi\sqrt{2}}{2} erf^{-1}(p)\left(e^{[erf^{-1}(p)]^2}\right)^2 \tag{14.35}$$

The condition necessary for the existence of the equation is

$$\sigma > 0, 0 \leq p < 1.$$

Substitute Eq. (14.29) into Eq. (14.35);

$$Q''(p) = \frac{\pi}{2} Q(p)\left(e^{[erf^{-1}(p)]^2}\right)^2 \tag{14.36}$$

The following equations obtained from the simplification of Eq. (14.30) are needed to simplify Eq. (14.36);

$$Q'^2(p) = \frac{\pi\sigma^2}{2}\left(e^{[erf^{-1}(p)]^2}\right)^2 \tag{14.37}$$

$$\frac{Q'^2(p)}{\sigma^2} = \frac{\pi}{2}\left(e^{[erf^{-1}(p)]^2}\right)^2 \tag{14.38}$$

Substitute Eq. (14.38) into Eq. (14.36);

$$Q''(p) = Q(p)\frac{Q'^2(p)}{\sigma^2} \tag{14.39}$$

$$\sigma^2 Q''(p) - Q(p)Q'^2(p) = 0 \tag{14.40}$$

14.4 Survival Function

The survival function of the half-normal distribution is given by;

$$S(t) = 1 - erf\left(\frac{t}{\sigma\sqrt{2}}\right) \tag{14.41}$$

To obtain the first order ordinary differential equation for the Survival function of the half-normal distribution, differentiate Eq. (14.41);

$$S'(t) = -\frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{t^2}{2\sigma^2}} = -f(t) \tag{14.42}$$

The condition necessary for the existence of the equation is $\sigma > 0$.

The second and third order ordinary differential equations for the Survival function of the half-normal distribution can also be obtained using Eq. (14.42);

$$\sigma^2 S''(t) + tS'(t) = 0 \tag{14.43}$$

$$S'(0) = -\frac{\sqrt{2}}{\sigma\sqrt{\pi}} \tag{14.44}$$

$$\sigma^4 S'''(t) + (\sigma^2 - t^2)S'(t) = 0 \tag{14.45}$$

$$\sigma^2 S'''(t) + \sigma t S''(t) + S(t) = 0 \tag{14.46}$$

14.5 Inverse Survival Function

The inverse survival function of the half-normal distribution is given by;

$$Q(p) = \sigma\sqrt{2}erf^{-1}(1-p) \tag{14.47}$$

To obtain the first order ordinary differential equation for the Quantile function of the half-normal distribution, differentiate Eq. (14.47);

$$Q'(p) = -\frac{\sigma\sqrt{2\pi}}{2}e^{[erf^{-1}(1-p)]^2} \tag{14.48}$$

The condition necessary for the existence of the equation is

$$\sigma > 0, 0 \leq p < 1.$$

Simplify Eq. (14.48) using Eq. (14.47), however Eq. (14.47) becomes;

$$\frac{Q(p)}{\sigma\sqrt{2}} = erf^{-1}(1-p) \tag{14.49}$$

$$Q'(p) = -\frac{\sigma\sqrt{2\pi}}{2}e^{\frac{Q^2(p)}{2\sigma^2}} \tag{14.50}$$

$$\ln\ Q'(p) = -\ln\left(\frac{\sigma\sqrt{2\pi}}{2}\right) - \frac{Q^2(p)}{2\sigma^2} \tag{14.51}$$

$$2\sigma^2 \ln\ Q'(p) + Q^2(p) + 2\sigma^2 g = 0 \tag{14.52}$$

where $g = \ln\left(\frac{\sigma\sqrt{2\pi}}{2}\right)$ and with initial value condition; $Q(0) = 0$.

14.6 Hazard Function

The hazard function of the half-normal distribution is given by;

$$h(t) = \left(\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{t^2}{2\sigma^2}}\right)\left(1 - erf\left(\frac{t}{\sigma\sqrt{2}}\right)\right)^{-1} \tag{14.53}$$

To obtain the first order ordinary differential equation for the hazard function of the half-normal distribution, differentiate Eq. (14.53);

$$h'(t) = -\left\{ \frac{t}{\sigma^2} + \frac{\left(\frac{\sqrt{2}}{\sigma\sqrt{\pi}} \mathrm{e}^{-\frac{t^2}{2\sigma^2}} \right)}{\left(1 - erf\left(\frac{t}{\sigma\sqrt{2}} \right) \right)} \right\} h(t) \tag{14.54}$$

The condition necessary for the existence of the equation is $\sigma > 0$.

$$h'(t) = -\left\{ \frac{t}{\sigma^2} + h(t) \right\} h(t) \tag{14.55}$$

$$\sigma^2 h'(t) = -(t + \sigma^2 h(t))h(t) \tag{14.56}$$

$$\sigma^2 h'(t) + \sigma^2 h^2(t) + th(t) = 0 \tag{14.57}$$

with initial value condition: $h(0) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}}$.

Higher order ordinary differential equations can also be obtained such as;

$$\sigma^2 h''(t) + (t + 2\sigma^2 h(t))h'(t) + h(t) = 0 \tag{14.58}$$

$$\sigma^2 h'''(t) + (t + 2\sigma^2 h(t))h''(t) + 2\sigma^2 h'^2(t) + h'(t) + h(t) = 0 \tag{14.59}$$

14.7 Reversed Hazard Function

he reversed hazard function of the half-normal distribution is given by;

$$j(t) = \frac{\frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{\frac{t^2}{2\sigma^2}}}{erf\left(\frac{t}{\sigma\sqrt{2}} \right)} = \left(\frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{t^2}{2\sigma^2}} \right) \left(erf\left(\frac{t}{\sigma\sqrt{2}} \right) \right)^{-1} \tag{14.60}$$

To obtain the first order ordinary differential equation for the reversed hazard function of the half-normal distribution, differentiate Eq. (14.60);

$$j'(t) = -\left\{ \frac{t}{\sigma^2} + \frac{\left(\frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{t^2}{2\sigma^2}} \right)}{\left(erf\left(\frac{t}{\sigma\sqrt{2}} \right) \right)} \right\} j(t) \tag{61}$$

The condition necessary for the existence of the equation is $\sigma > 0$.

$$j'(t) = -\left\{ \frac{t}{\sigma^2} + j(t) \right\} j(t) \tag{62}$$

$$\sigma^2 j'(t) + \sigma^2 j^2(t) + tj(t) = 0 \tag{63}$$

with initial value condition; $j(1) = \frac{\frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{1}{2\sigma^2}}}{erf\left(\frac{1}{\sigma\sqrt{2}}\right)}$

14.8 Conclusion

Ordinary differential equations (ODEs) has been obtained for the probability density function (PDF), quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of the half-normal distribution.

This differential calculus, modified product rule and efficient algebraic simplifications were used to derive the various classes of the ODEs. The parameter and the supports that characterize the half-normal distribution determine the nature, existence, orientation and uniqueness of the ODEs. The results are in agreement with those available in scientific literature. Furthermore several methods can be used to obtain desirable solutions to the ODEs. This method of characterizing distributions cannot be applied to distributions whose PDF or CDF are either not differentiable or the domain of the support of the distribution contains singular points. Furthermore, the differential equations can be used to model phenomena described by the distribution.

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