Metage Symmetry Group of Non-barotropic Magnetohydrodynamics and the Conservation of Cross Helicity

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Abstract Standard cross helicity is not conserved in non-barotropic magnetohydrodynamics (MHD) (as opposed to barotropic or incompressible MHD). It was shown that a new kind of cross helicity which is conserved in the non barotropic case can be introduced. The non barotropic cross helicity reduces to the standard cross helicity under barotropic assumptions. Here we show that the new cross helicity can be deduced from a variational principle using the Noether's theorem. The symmetry group associated with the new cross helicity is related to translation in a labelling of the flow elements connected to the magnetic field lines known as magnetic metage.

Keywords Symmetry group · Magnetohydrodynamics · Topological conservation laws · Metage · Cross helicity

1 Introduction

The theorem of Noether dictates that for every continuous symmetry group of an Action the system must possess a conservation law. For example time translation symmetry results in the conservation of energy, while spatial translation symmetry results in the conservation of linear momentum and rotation symmetry in the conservation of angular momentum to list some well known examples. But sometimes the conservation law is discovered without reference to the Noether theorem by using the equations of the system. In that case one is tempted to inquire what is the hidden symmetry associated with this conservation law and what is the simplest way to represent it.

The concept of metage as a label for fluid elements along a vortex line in ideal fluids was first introduced by Lynden-Bell and Katz [1]. A translation group of this label was found to be connected to the conservation of Moffat's [2] helicity by Yahalom [3]. The concept of metage was later generalized by Yahalom and Lynden-Bell [4] for barotropic MHD, but now as a label for fluid elements along magnetic

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field lines which are comoving with the flow in the case of ideal MHD. Yahalom and Lynden-Bell [4] have also shown that the translation group of the magnetic metage is connected to Woltjer [5, 6] conservation of cross helicity for barotropic MHD. Recently the concept of metage was generalized also for non barotropic MHD in which magnetic field lines lie on entropy surfaces [7]. This will be generalized in this paper by dropping the entropy condition on magnetic field lines.

Cross Helicity was first described by Woltjer [5, 6] and is give by:

$$H_C \equiv \int \mathbf{B} \cdot \mathbf{v} d^3 x, \tag{1}$$

in which $\mathbf{B}$ is the magnetic field, $\mathbf{v}$ is the velocity field and the integral is taken over the entire flow domain. H_C is conserved for barotropic or incompressible MHD and is given a topological interpretation in terms of the knottiness of magnetic and flow field lines. A generalization of barotropic fluid dynamics conserved quantities including helicity to non barotropic flows including topological constants of motion is given by Mobbs [13]. However, Mobbs did not discuss the MHD case.

Both conservation laws for the helicity in the fluid dynamics case and the barotropic MHD case were shown to originate from a relabelling symmetry through the Noether theorem [3, 4, 8, 9]. Webb et al. [10] have generalized the idea of relabelling symmetry to non-barotropic MHD and derived their generalized cross helicity conservation law by using Noether's theorem but without using the simple representation which is connected with the metage variable. The conservation law deduction involves a divergence symmetry of the action. These conservation laws were written as Eulerian conservation laws of the form $D_t + \mathbf{\nabla} \cdot \mathbf{F} = 0$ where D is the conserved density and F is the conserved flux. Webb et al. [11] discuss the cross helicity conservation law for non-barotropic MHD in a multi-symplectic formulation of MHD. Webb et al. [10, 11] emphasize that the generalized cross helicity conservation law, in MHD and the generalized helicity conservation law in non-barotropic fluids are non-local in the sense that they depend on the auxiliary nonlocal variable σ, which depends on the Lagrangian time integral of the temperature $T(x, t)$. Notice that a potential vorticity conservation equation for non-barotropic MHD is derived by Webb, G. M. and Mace, R.L. [12] by using Noether's second theorem.

It should be mentioned that Mobbs [13] derived a helicity conservation law for ideal, non-barotropic fluid dynamics, which is of the same form as the cross helicity conservation law for non-barotropic MHD, except that the magnetic field induction is replaced by the generalized fluid helicity $\mathbf{\Omega} = \mathbf{\nabla} \times (\mathbf{v} - \sigma \mathbf{\nabla} s)$. Webb et al. [10, 11] also derive the Eulerian, differential form of Mobbs [13] conservation law (although they did not reference Mobbs [13]). Webb and Anco [14] show how Mobbs conservation law arises in multi-symplectic, Lagrangian fluid mechanics.

Variational principles for magnetohydrodynamics were introduced by previous authors both in Lagrangian and Eulerian form. Sturrock [15] has discussed in his book a Lagrangian variational formalism for magnetohydrodynamics. Vladimirov

and Moffatt [16] in a series of papers have discussed an Eulerian variational principle for incompressible magnetohydrodynamics. However, their variational principle contained three more functions in addition to the seven variables which appear in the standard equations of incompressible magnetohydrodynamics which are the magnetic field **B** the velocity field **v** and the pressure P. Kats [17] has generalized Moffatt's work for compressible non barotropic flows but without reducing the number of functions and the computational load. Sakurai [18] has introduced a two function Eulerian variational principle for force-free magnetohydrodynamics and used it as a basis of a numerical scheme, his method is discussed in a book by Sturrock [15]. Yahalom and Lynden-Bell [4] combined the Lagrangian of Sturrock [15] with the Lagrangian of Sakurai [18] to obtain an **Eulerian** Lagrangian principle for barotropic magnetohydrodynamics which depends on only six functions. The variational derivative of this Lagrangian produced all the equations needed to describe barotropic magnetohydrodynamics without any additional constraints. The equations obtained resembled the equations of Frenkel, Levich and Stilman [30] (see also [19]). Yahalom [32] have shown that for the barotropic case four functions will suffice. Moreover, it was shown that the cuts of some of those functions [20] are topological local conserved quantities.

Previous work was concerned only with barotropic magnetohydrodynamics. Variational principles of non barotropic magnetohydrodynamics can be found in the work of Bekenstein and Oron [21] in terms of 15 functions and V.A. Kats [17] in terms of 20 functions. Morrison [22] has suggested a Hamiltonian approach but this also depends on 8 canonical variables (see table 2 [22]). The variational principle introduced in [23, 24] show that only five functions will suffice to describe non barotropic MHD in the case that we enforce a Sakurai [18] representation for the magnetic field (see also [29] for the stationary case).

The plan of this paper is as follows: First we introduce the basic quantities and equations of non-barotropic MHD. Then we describe the concept of magnetic metage for non-barotropic MHD. This is followed by a description of a Lagrangian variational principle for non-barotropic MHD. Finally we derive a non-barotropic cross helicity conservation law for non-barotropic MHD using Noether's theorem.

2 Basic Equations

Consider the equations of non-barotropic MHD [15, 23]:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{4}$$

$$\rho\frac{d\mathbf{v}}{dt} = \rho\left(\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p(\rho, s) + \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi}, \tag{5}$$

$$\frac{ds}{dt} = 0. \tag{6}$$

In the above the following notations are utilized: $\frac{\partial}{\partial t}$ is the temporal derivative, $\frac{d}{dt}$ is the temporal material derivative and ∇ has its standard meaning in vector calculus. ρ is the fluid density and s is the specific entropy. Finally $p(\rho, s)$ is the pressure which depends on the density and entropy (the non-barotropic case). Equation (2) describes the fact that the magnetic field lines are moving with the fluid elements ("frozen" magnetic field lines), Eq. (3) describes the fact that the magnetic field is solenoidal, Eq. (4) describes the conservation of mass and Eq. (5) is the Euler equation for a fluid in which both pressure and Lorentz magnetic forces apply. Equation (6) describes the fact that heat is not created (zero viscosity, zero resistivity) in ideal non-barotropic MHD and is not conducted, thus only convection occurs. The number of independent variables for which one needs to solve is eight ($\mathbf{v}$, $\mathbf{B}$, ρ, s) and the number of equations (2), (4), (5), (6) is also eight. Notice that Eq. (3) is a condition on the initial $\mathbf{B}$ field and is satisfied automatically for any other time due to Eq. (2).

In non-barotropic MHD one can calculate the temporal derivative of the cross helicity (1) using the above equations and obtain:

$$\frac{dH_C}{dt} = \int T\nabla s \cdot \mathbf{B}d^3x, \tag{7}$$

in which T is the temperature. Hence, generally speaking cross helicity is not conserved.

3   Load and Metage

The following section follows closely a similar section in [4, 31]. Consider a thin tube surrounding a magnetic field line as described in Fig. 1,
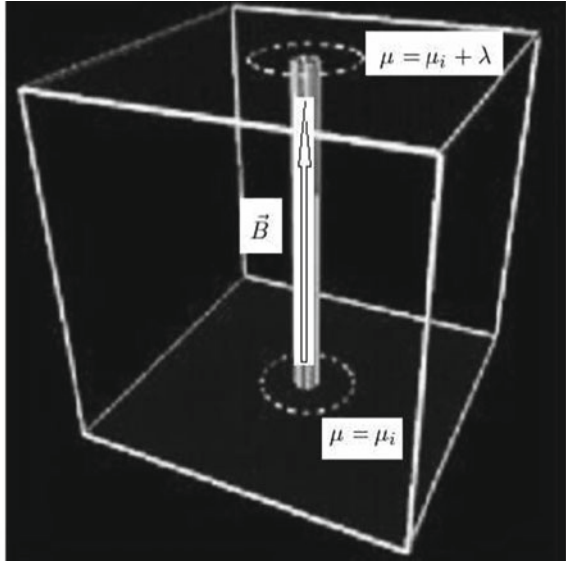
the magnetic flux contained within the tube is:

$$\Delta\Phi = \int \mathbf{B} \cdot d\mathbf{S} \tag{8}$$

and the mass contained with the tube is:

$$\Delta M = \int \rho d\mathbf{l} \cdot d\mathbf{S}, \tag{9}$$

in which dl is a length element along the tube. Since the magnetic field lines move
with the flow by virtue of Eqs. (2) and (4) both the quantities $\Delta\Phi$ and ΔM are
conserved and since the tube is thin we may define the conserved magnetic load:

$$\lambda = \frac{\Delta M}{\Delta \Phi} = \oint \frac{\rho}{B} dl, \tag{10}$$

in which the above integral is performed along the field line. Obviously the parts of
the line which go out of the flow to regions in which $\rho = 0$ have a null contribution
to the integral. Notice that λ is a **single valued** function that can be measured in
principle. Since λ is conserved it satisfies the equation:
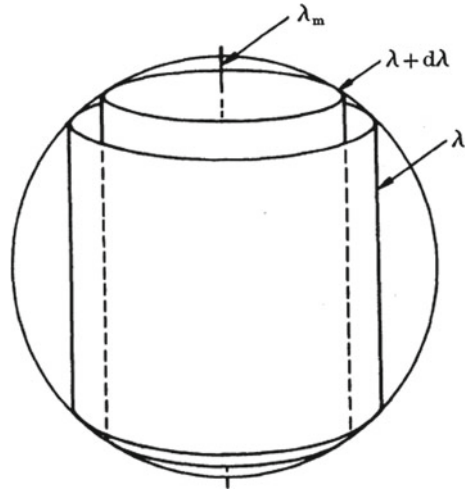
$$\frac{d\lambda}{dt} = 0. \tag{11}$$

By construction surfaces of constant magnetic load move with the flow and contain
magnetic field lines. Hence the gradient to such surfaces must be orthogonal to the
field line:

$$\nabla \lambda \cdot \mathbf{B} = 0. \tag{12}$$

Now consider an arbitrary comoving point on the magnetic field line and denote it by
i, and consider an additional comoving point on the magnetic field line and denote
it by r. The integral:

$$\mu(r) = \int_i^r \frac{\rho}{B} dl + \mu(i), \tag{13}$$

Fig. 2 Surfaces of constant
load

is also a conserved quantity which we may denote following Lynden-Bell and Katz [1]
as the magnetic metage. $\mu(i)$ is an arbitrary number which can be chosen differently
for each magnetic line. By construction:

$$\frac{d\mu}{dt} = 0. \tag{14}$$

Also it is easy to see that by differentiating along the magnetic field line we obtain:

$$\nabla\mu \cdot \mathbf{B} = \rho. \tag{15}$$

Notice that μ will be generally a **non single valued** function, we will show later
in this paper that symmetry to translations in μ; will generate through the Noether
theorem the conservation of the magnetic cross helicity.

At this point we have two comoving coordinates of flow, namely λ, μ obviously in
a three dimensional flow we also have a third coordinate. However, before defining
the third coordinate we will find it useful to work not directly with λ but with a
function of λ. Now consider the magnetic flux within a surface of constant load
$\Phi(\lambda)$ as described in Fig. 2 (the figure was given by Lynden-Bell and Katz [1]).
The magnetic flux is a conserved quantity and depends only on the load λ of the
surrounding surface. Now we define the quantity:

$$\chi = \frac{\Phi(\lambda)}{2\pi}. \tag{16}$$

Obviously χ satisfies the equations:

$$\frac{d\chi}{dt} = 0, \quad \mathbf{B} \cdot \nabla \chi = 0. \tag{17}$$

Let us now define an additional comoving coordinate η^* since $\nabla \mu$ is not orthogonal to the $\mathbf{B}$ lines we can choose $\nabla \eta^*$ to be orthogonal to the $\mathbf{B}$ lines and not be in the direction of the $\nabla \chi$ lines, that is we choose η^* not to depend only on χ. Since both $\nabla \eta^*$ and $\nabla \chi$ are orthogonal to $\mathbf{B}$, $\mathbf{B}$ must take the form:

$$\mathbf{B} = A \nabla \chi \times \nabla \eta^*. \tag{18}$$

However, using Eq. (3) we have:

$$\nabla \cdot \mathbf{B} = \nabla A \cdot (\nabla \chi \times \nabla \eta^*) = 0. \tag{19}$$

Which implies that A is a function of χ, η^*. Now we can define a new comoving function η such that:

$$\eta = \int_0^{\eta^*} A(\chi, \eta'^*) d\eta'^*, \quad \frac{d\eta}{dt} = 0. \tag{20}$$

In terms of this function we obtain the Sakurai (Euler potentials) presentation:

$$\mathbf{B} = \nabla \chi \times \nabla \eta. \tag{21}$$

And the density is now given by the Jacobian:

$$\rho = \nabla \mu \cdot (\nabla \chi \times \nabla \eta) = \frac{\partial(\chi, \eta, \mu)}{\partial(x, y, z)}. \tag{22}$$

It can easily be shown using the fact that the labels are comoving that the above forms of $\mathbf{B}$ and ρ satisfy Eqs. (2), (3) and (4) automatically.

Notice however, that η is defined in a non unique way since one can redefine η for example by performing the following transformation: $\eta \to \eta + f(\chi)$ in which $f(\chi)$ is an arbitrary function. The comoving coordinates χ, η serve as labels of the magnetic field lines. Moreover the magnetic flux can be calculated as:

$$\Phi = \int \mathbf{B} \cdot d\mathbf{S} = \int d\chi d\eta. \tag{23}$$

In the case that the surface integral is performed inside a load contour we obtain:

$$\Phi(\lambda) = \int_\lambda d\chi d\eta = \chi \int_\lambda d\eta = \begin{cases} \chi[\eta] \\ \chi(\eta_{max} - \eta_{min}) \end{cases} \tag{24}$$

There are two cases involved; in one case the load surfaces are topological cylinders; in this case η is not single valued and hence we obtain the upper value for $\Phi(\lambda)$. In a second case the load surfaces are topological spheres; in this case η is single valued and has minimal η_{min} and maximal η_{max} values. Hence the lower value of $\Phi(\lambda)$ is obtained. For example in some cases η is identical to twice the latitude angle θ. In those cases $\eta_{min} = 0$ (value at the "north pole") and $\eta_{max} = 2\pi$ (value at the "south pole").

Comparing the above equation with Eq. (16) we derive that η can be either **single valued** or **not single valued** and that its discontinuity across its cut in the non single valued case is $[\eta] = 2\pi$.

The triplet χ, η, μ will suffice to label any fluid element in three dimensions. But for a non-barotropic flow there is also another label s which is comoving according to Eq. (6). The question then arises of the relation of this label to the previous three. As one needs to make a choice regarding the preferred set of labels it seems that the physical ones are χ, η, s in which we use the surfaces on which the magnetic fields lie and the entropy, each label has an obvious physical interpretation. In this case we must look at μ as a function of χ, η, s. If the magnetic field lines lie on entropy surface then μ regains its status as an independent label. The density can now be written as:

$$\rho = \frac{\partial \mu}{\partial s} \frac{\partial(\chi, \eta, s)}{\partial(x, y, z)}. \tag{25}$$

Now as μ can be defined for each magnetic field line separately according to Eq. (13) it is obvious that such a choice exist in which μ is a function of s only. One may also think of the entropy s as a functions χ, η, μ. However, if one change μ in this case this generally entails a change in s and the symmetry described in Eq. (13) is lost.

4   Lagrangian variational principle of MHD

A Lagrangian variational principle for barotropic MHD has been discussed by a number of authors (see for example [4, 15]), we repeat the derivation with the necessary modifications which are required for the non-barotropic case. Consider the action:

$$A \equiv \int \mathcal{L} d^3 x dt,$$

$$\mathcal{L} \equiv \rho \left(\frac{1}{2} \mathbf{v}^2 - \varepsilon(\rho, s) \right) - \frac{\mathbf{B}^2}{8\pi}, \tag{26}$$

In the above ε is the specific internal energy (internal energy per unit of mass). The reader is reminded of the following thermodynamic relations which will become useful later:

$$d\varepsilon = T ds - P d\frac{1}{\rho} = T ds + \frac{P}{\rho^2} d\rho$$

$$\frac{\partial \varepsilon}{\partial s} = T, \qquad \frac{\partial \varepsilon}{\partial \rho} = \frac{P}{\rho^2}$$

$$w = \varepsilon + \frac{P}{\rho} = \varepsilon + \frac{\partial \varepsilon}{\partial \rho} \rho = \frac{\partial(\rho \varepsilon)}{\partial \rho}$$

$$dw = d\varepsilon + d\left(\frac{P}{\rho}\right) = T ds + \frac{1}{\rho} dP \tag{27}$$

in the above T is the temperature and w is the specific enthalpy. A variation in any quantity F for a fixed position $\mathbf{r}$ is denoted as δF hence:

$$\delta A = \int \delta \mathcal{L} d^3 x dt,$$

$$\delta \mathcal{L} = \delta \rho \left(\frac{1}{2} \mathbf{v}^2 - w(\rho)\right) - \rho T \delta s + \rho \mathbf{v} \cdot \delta \mathbf{v} - \frac{\mathbf{B} \cdot \delta \mathbf{B}}{4\pi}, \tag{28}$$

A change in a position of a fluid element located at a position $\mathbf{r}$ at time t is given by $\boldsymbol{\xi}(\mathbf{r}, t)$. A change involving both a local variation coupled with a change of element position of the quantity F is given by:

$$\Delta F = \delta F + (\boldsymbol{\xi} \cdot \nabla) F, \tag{29}$$

hence

$$\Delta \mathbf{v} = \delta \mathbf{v} + (\boldsymbol{\xi} \cdot \nabla) \mathbf{v}. \tag{30}$$

However, since:

$$\Delta \mathbf{v} = \Delta \frac{d\mathbf{r}}{dt} = \frac{d \Delta \mathbf{r}}{dt} = \frac{d\boldsymbol{\xi}}{dt}. \tag{31}$$

We obtain:

$$\delta \mathbf{v} = \frac{d\boldsymbol{\xi}}{dt} - (\boldsymbol{\xi} \cdot \nabla) \mathbf{v} = \frac{\partial \boldsymbol{\xi}}{\partial t} + (\mathbf{v} \cdot \nabla) \boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla) \mathbf{v}. \tag{32}$$

For any of the labels χ, η, μ a change in a specific spatial location is only possible by the displacement of the fluid element to a new position. However, if one takes into account both the spatial change in value and change due to the displacement then obviously the total change is zero as each fluid element retains its labels. Hence:

$$\Delta \chi = \delta \chi + (\boldsymbol{\xi} \cdot \nabla) \chi = 0 \Rightarrow \delta \chi = -(\boldsymbol{\xi} \cdot \nabla) \chi,$$
$$\Delta \eta = \delta \eta + (\boldsymbol{\xi} \cdot \nabla) \eta = 0 \Rightarrow \delta \eta = -(\boldsymbol{\xi} \cdot \nabla) \eta,$$
$$\Delta \mu = \delta \mu + (\boldsymbol{\xi} \cdot \nabla) \mu = 0 \Rightarrow \delta \mu = -(\boldsymbol{\xi} \cdot \nabla) \mu, \tag{33}$$

Now since s is a comoving quantity depending only on the fluid element labels we have:

$$\Delta s = 0 \Rightarrow \delta s = -(\boldsymbol{\xi} \cdot \nabla)s. \tag{34}$$

Using Eqs. (22) and (33) we obtain a mass conserving variation of ρ:

$$\delta\rho = -\nabla \cdot (\rho\boldsymbol{\xi}) \tag{35}$$

Using Eqs. (21) and (33) a magnetic flux conserving variation takes the form:

$$\delta\mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}). \tag{36}$$

Introducing the result of Eqs. (32), (34), (35), (36) into (28) and integrating by parts we arrive at the result:

$$
\delta A = \int d^3x \rho \mathbf{v} \cdot \boldsymbol{\xi}|_{t_0}^{t_1}
$$
$$
+ \int dt \left\{ \oint d\mathbf{S} \cdot \left[-\rho\boldsymbol{\xi} \left(\frac{1}{2}\mathbf{v}^2 - w(\rho) \right) + \rho\mathbf{v}(\mathbf{v} \cdot \boldsymbol{\xi}) + \frac{1}{4\pi}\mathbf{B} \times (\boldsymbol{\xi} \times \mathbf{B}) \right] \right.
$$
$$
+ \int d^3x \boldsymbol{\xi} \cdot \left[-\rho\nabla w + \rho T \nabla s - \frac{\partial(\rho\mathbf{v})}{\partial t} - \frac{\partial(\rho\mathbf{v}v_k)}{\partial x_k} - \frac{1}{4\pi}\mathbf{B} \times (\nabla \times \mathbf{B}) \right] \right\}, \tag{37}
$$

in which a summation convention is assumed. Taking into account the continuity equation (4) we obtain:

$$
\delta A = \int d^3x \rho \mathbf{v} \cdot \boldsymbol{\xi}|_{t_0}^{t_1}
$$
$$
+ \int dt \left\{ \oint d\mathbf{S} \cdot \left[-\rho\boldsymbol{\xi} \left(\frac{1}{2}\mathbf{v}^2 - w(\rho) \right) + \rho\mathbf{v}(\mathbf{v} \cdot \boldsymbol{\xi}) + \frac{1}{4\pi}\mathbf{B} \times (\boldsymbol{\xi} \times \mathbf{B}) \right] \right.
$$
$$
+ \int d^3x \boldsymbol{\xi} \cdot \left[-\nabla P - \rho\frac{\partial\mathbf{v}}{\partial t} - \rho(\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{1}{4\pi}\mathbf{B} \times (\nabla \times \mathbf{B}) \right] \right\}, \tag{38}
$$

hence we see that if $\delta A = 0$ for a $\boldsymbol{\xi}$ vanishing at the initial and final times and on the surface of the domain but otherwise arbitrary then Euler's equation (5) is satisfied (taking into account that $\nabla w - T\nabla s = \frac{\nabla p}{\rho}$).

The Lagrangian density given in Eq. (26) does not admit a μ modification symmetry since we assume that the entropy $s = s(\chi, \eta, \mu)$ is a given function of the labels. This problem can be overcome by taking s as a dynamical variable and enforcing its conservation by using a Lagrange multiplier. In this approach the variational principle takes the form:

$$A \equiv \int \mathcal{L} d^3 x dt,$$

$$\mathcal{L} \equiv \rho \left(\frac{1}{2} \mathbf{v}^2 - \varepsilon(\rho, s) \right) - \frac{\mathbf{B}^2}{8\pi} - \rho \sigma \frac{ds}{dt}, \tag{39}$$

A variation with respect to the Lagrange multiplier σ will obviously result in Eq. (6). A variation with respect to s will result in:

$$\delta_s A = \int d^3 x dt \delta s \left[\frac{\partial(\rho\sigma)}{\partial t} + \nabla \cdot (\rho\sigma\mathbf{v}) - \rho T \right]$$

$$+ \int dt \oint d\mathbf{S} \cdot \rho\sigma\mathbf{v}\delta s - \int d^3 x \rho\sigma\delta s |_{t_0}^{t_1}, \tag{40}$$

Taking into account the continuity equation (4) we obtain for locations in which the density ρ is not null the result:

$$\frac{d\sigma}{dt} = T, \tag{41}$$

provided that $\delta_s A$ vanished for an arbitrary δs. Now let us turn our attention to the variation with respect to the fluid element displacement which takes the form:

$$\delta A_\xi = \int \delta \mathcal{L}_\xi d^3 x dt,$$

$$\delta \mathcal{L}_\xi = \delta\rho \left(\frac{1}{2} \mathbf{v}^2 - w(\rho) \right) - \rho\sigma\delta\mathbf{v} \cdot \nabla s + \rho\mathbf{v} \cdot \delta\mathbf{v} - \frac{\mathbf{B} \cdot \delta\mathbf{B}}{4\pi}, \tag{42}$$

As most of the terms were calculated previously we will only calculate the term $-\rho\sigma\delta\mathbf{v} \cdot \nabla s$ which is equal to:

$$- \rho\sigma\delta\mathbf{v} \cdot \nabla s = \boldsymbol{\xi} \cdot \rho T \nabla s - \frac{\partial(\rho\sigma\nabla s \cdot \boldsymbol{\xi})}{\partial t} - \nabla \cdot (\rho\sigma(\nabla s \cdot \boldsymbol{\xi})\mathbf{v}). \tag{43}$$

The above result was obtained using Eqs. (32), (6) and (41). Hence the variation of the action with respect to a displacement of the fluid elements is:

$$\delta_\xi A = \int d^3 x \rho(\mathbf{v} - \sigma\nabla s) \cdot \boldsymbol{\xi}|_{t_0}^{t_1}$$

$$+ \int dt \left\{ \oint d\mathbf{S} \cdot \left[-\rho\boldsymbol{\xi} \left(\frac{1}{2} \mathbf{v}^2 - w(\rho) \right) + \rho\mathbf{v} ((\mathbf{v} - \sigma\nabla s) \cdot \boldsymbol{\xi}) + \frac{1}{4\pi} \mathbf{B} \times (\boldsymbol{\xi} \times \mathbf{B}) \right] \right.$$

$$\left. + \int d^3 x \boldsymbol{\xi} \cdot \left[-\rho\nabla w + \rho T \nabla s - \frac{\partial(\rho\mathbf{v})}{\partial t} - \frac{\partial(\rho\mathbf{v}v_k)}{\partial x_k} - \frac{1}{4\pi} \mathbf{B} \times (\nabla \times \mathbf{B}) \right] \right\}, \tag{44}$$

in which a summation convention is assumed. Taking into account the continuity equation (4) and the thermodynamic identities given in Eq. (27) we obtain:

$$\delta_\xi A = \int d^3x \rho(\mathbf{v} - \sigma\nabla s) \cdot \boldsymbol{\xi}|_{t_0}^{t_1}$$

$$+ \int dt \left\{ \oint d\mathbf{S} \cdot \left[-\rho\boldsymbol{\xi}\left(\frac{1}{2}\mathbf{v}^2 - w(\rho)\right) + \rho\mathbf{v}((\mathbf{v} - \sigma\nabla s) \cdot \boldsymbol{\xi}) + \frac{1}{4\pi}\mathbf{B} \times (\boldsymbol{\xi} \times \mathbf{B}) \right] \right.$$

$$+ \int d^3x \boldsymbol{\xi} \cdot \left[-\nabla P - \rho\frac{\partial\mathbf{v}}{\partial t} - \rho(\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{1}{4\pi}\mathbf{B} \times (\nabla \times \mathbf{B}) \right] \right\}, \tag{45}$$

Hence we obtain the correct dynamical equations for an arbitrary $\boldsymbol{\xi}$. Now suppose that the equations and boundary conditions hold. Then:

$$\delta_\xi A = \int d^3x \rho(\mathbf{v} - \sigma\nabla s) \cdot \boldsymbol{\xi}|_{t_0}^{t_1} \tag{46}$$

If in addition $\boldsymbol{\xi}$ is a small symmetry displacement such that $\delta_\xi A = 0$ we obtained a conserved Noether current:

$$\delta J = \int d^3x \rho(\mathbf{v} - \sigma\nabla s) \cdot \boldsymbol{\xi} \tag{47}$$

5 Non Barotropic Cross Helicity Conservation via the Noether Theorem

It is obvious that the choice of fluid labels is quite arbitrary. However, when enforcing the χ, η, μ coordinate system such that:

$$\rho = \frac{\partial(\chi, \eta, \mu)}{\partial(x, y, z)}. \tag{48}$$

The choice is restricted to $\tilde{\chi}, \tilde{\eta}, \tilde{\mu}$:

$$\frac{\partial(\tilde{\chi}, \tilde{\eta}, \tilde{\mu})}{\partial(\chi, \eta, \mu)} = 1. \tag{49}$$

Moreover the Euler potential magnetic field representation:

$$\mathbf{B} = \nabla\chi \times \nabla\eta, \tag{50}$$

reduces the choice further to:

$$\frac{\partial(\tilde{\chi}, \tilde{\eta})}{\partial(\chi, \eta)} = 1. \tag{51}$$

In what follows we consider the transformation (see also Eq. (13)):

$$\tilde{\chi} = \chi, \ \tilde{\eta} = \eta, \ \tilde{\mu} = \mu + a(\chi, \eta) \tag{52}$$

Hence a is a label displacement which may be different for each magnetic field line, as the field line is closed one need not worry about edge difficulties. This transformation satisfies trivially the conditions (49), (51). If $a = \delta\mu$ is small we can use Eq. (33) to calculate the associated fluid element displacement with this relabelling.

$$\boldsymbol{\xi} = -\frac{\partial \mathbf{r}}{\partial \mu} \delta\mu = -\delta\mu \frac{\mathbf{B}}{\rho}. \tag{53}$$

Inserting this expression into the boundary term in Eq. (45) will result in:

$$\delta A_B = \int dt \oint d\mathbf{S} \cdot \left[\mathbf{B} \left(\frac{1}{2} \mathbf{v}^2 - w(\rho) \right) - \mathbf{v}((\mathbf{v} - \sigma\nabla s) \cdot \mathbf{B}) \right] \delta\mu = 0, \tag{54}$$

which is the condition for magnetic cross helicity conservation. Inserting Eq. (53) into (47) we obtain the conservation law:

$$\delta J = \int d^3x \rho(\mathbf{v} - \sigma\nabla s) \cdot \boldsymbol{\xi} = -\int d^3x \delta\mu(\mathbf{v} - \sigma\nabla s) \cdot \mathbf{B} \tag{55}$$

In the simplest case we may take $\delta\mu$ to be a small constant, hence:

$$\delta J = -\delta\mu \int d^3x (\mathbf{v} - \sigma\nabla s) \cdot \mathbf{B} = -\delta\mu H_{CNB} \tag{56}$$

Where H_{CNB} is the non barotropic global cross helicity [11, 25, 26] defined as:

$$H_{CNB} \equiv \int d^3x (\mathbf{v} - \sigma\nabla s) \cdot \mathbf{B} = \int d^3x \mathbf{v}_t \cdot \mathbf{B} \tag{57}$$

in which $\mathbf{v}_t \equiv \mathbf{v} - \sigma\nabla s$ is the topological velocity field. We thus obtain the conservation of non-barotropic cross helicity using the Noether theorem and the symmetry group of metage translations. Of course one can perform a different translation on each magnetic field line, in this case one obtains:

$$\delta J = -\int d^3x \delta\mu \mathbf{v}_t \cdot \mathbf{B} = -\int d\chi d\eta \delta\mu \oint_{\chi,\eta} d\mu \rho^{-1} \mathbf{v}_t \cdot \mathbf{B} \tag{58}$$

Now since $\delta\mu$ is an arbitrary (small) function of χ, η it follows that:

$$I = \oint_{\chi,\eta} d\mu \rho^{-1} \mathbf{v}_t \cdot \mathbf{B} \tag{59}$$

is a conserved quantity for each magnetic field line. Along a magnetic field line the following equations hold:

$$d\mu = \boldsymbol{\nabla}\mu \cdot d\mathbf{r} = \boldsymbol{\nabla}\mu \cdot \hat{B}dr = \frac{\rho}{B}dr \tag{60}$$

in the above $\hat{B}$ is an unit vector in the magnetic field direction an Eq. (15) is used. Inserting Eq. (60) into (59) we obtain:

$$I = \oint_{\chi,\eta} dr\mathbf{v}_t \cdot \hat{B} = \oint_{\chi,\eta} d\mathbf{r} \cdot \mathbf{v}_t. \tag{61}$$

which is just the circulation of the topological velocity along the magnetic field lines. This quantity can be written in terms of the generalized Clebsch representation of the velocity [23]:

$$\mathbf{v} = \boldsymbol{\nabla}\nu + \alpha\boldsymbol{\nabla}\chi + \beta\boldsymbol{\nabla}\eta + \sigma\boldsymbol{\nabla}s. \tag{62}$$

as:

$$I = \oint_{\chi,\eta} d\mathbf{r} \cdot \mathbf{v}_t = \oint_{\chi,\eta} d\mathbf{r} \cdot \boldsymbol{\nabla}\nu = [\nu]. \tag{63}$$

$[\nu]$ is the discontinuity of ν. This was shown to be equal to the amount of non barotropic cross helicity per unit of magnetic flux [25, 26].

$$I = [\nu] = \frac{dH_{CNB}}{d\Phi}. \tag{64}$$

6 Conclusion

We have shown the connection of the translation symmetry group of labels which is a subgroup of the relabelling group to both the global non barotropic cross helicity conservation law and the conservation law of circulations of topological velocity along magnetic field lines. The latter were shown to be equivalent to the amount of non barotropic cross helicity per unit of magnetic flux [25, 26]. Possible applications of the generalized cross helicity conservation law (both local and global) may arise in solar MHD, where rotation, and the baroclinic instability can give rise to magnetic tornadoes, in which vorticity of the fluid is generated in part by the baroclinic term $\boldsymbol{\nabla}T \times \boldsymbol{\nabla}s$ (see for example [12] eqn. (4.54) and also [27]). Other possible applications for MHD constraints of the current constants of motion are described in [26]. The importance of constants of motion for stability analysis is also discussed in [28].

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