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Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2

QTS-X/LT-XII, Varna, Bulgaria, June 2017

Springer

Springer Proceedings in Mathematics & Statistics

Volume 255

Springer Proceedings in Mathematics & Statistics

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Vladimir Dobrev
Editor

Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2

QTS-X/LT-XII, Varna, Bulgaria, June 2017

Springer

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ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-981-13-2178-8 ISBN 978-981-13-2179-5 (eBook)
https://doi.org/10.1007/978-981-13-2179-5

Library of Congress Control Number: 2018950773

Mathematics Subject Classification (2010): 83E30, 81T30, 83E50, 83F05, 81T40, 81T60, 81T13, 70S15, 81P40, 81P10, 81Q60, 83C47, 83C57, 83C20, 85A40, 81T28, 81R12

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Preface of Volume 2

This is the Second Volume of the Proceedings of the joint conference X. International Symposium "Quantum Theory and Symmetries" (QTS-10) and XII. International Workshop "Lie Theory and Its Applications in Physics" (LT-12), 19–25 June 2017, Varna, Bulgaria.

The first symposium of the QTS series was held in Goslar (Germany) in 1999, and then it was held in Cracow (2001), Cincinnati (2003), Varna (2005), Valladolid (2007), Lexington (2009), Prague (2011), Mexico City (2013), Yerevan (2015). The series started around the core concept that symmetries underlie all descriptions of quantum systems. It has since evolved to a symposium on the frontiers of theoretical and mathematical physics (for more details on this series, see here http://theo.inrne.bas.bg/~dobrev/QTS-homepage.htm).

The LT series covers the whole field of Lie Theory in its widest sense together with its applications in many facets of physics. As the interface between mathematics and physics, the workshop serves as a meeting place for mathematicians and theoretical and mathematical physicists. The first three workshops of the LT series were organised in Clausthal (1995, 1997, 1999), the fourth was part of the Second Symposium "Quantum Theory and Symmetries" in Cracow (2001), the fifth was organised in Varna (2003), the sixth was part of the Fourth Symposium "Quantum Theory and Symmetries" in Varna (2005), but has its own volume of Proceedings, and the seventh, eighth, ninth, tenth were organised in Varna (2007, 2009, 2011, 2013); see: http://theo.inrne.bas.bg/-dobrev/.

In the division of the material between the two volumes, we have tried to select for the first, respectively, second, volume more mathematics, respectively, physics, oriented papers. However, this division is relative since many papers could have been placed in either volume.

The scientific level was very high as can be judged by the speakers. The *plenary speakers* contributing to Volume 2 are: Benjamin Basso (ENS, Paris), Loriano Bonora (SISSA, Trieste), Martin Cederwall (Chalmers University of Technology, Gothenburg), Sumit R. Das (University of Kentucky, Lexington) joint paper with Antal Jevicki (Brown University, Providence), Evgeny Ivanov (JINR, Dubna), Ivan Kostov (CNRS, Saclay), Emil Nissimov (INRNE, BAS), Emery Sokatchev

(LAPTh, Annecy), Fumihiko Sugino (Institute for Basic Science, Seoul), Apostolos Vourdas (University of Bradford).

The topics covered the most modern trends in the field of the joint conferences: Symmetries in String Theories, Conformal Field Theory, Holography, Gravity Theories and Cosmology, Gauge Theories, Foundations of Quantum Theory, Nonrelativistic and Classical Theories.

The joint meeting of QTS-10 and LT-12 was organised by the Institute of Nuclear Research and Nuclear Energy of the Bulgarian Academy of Sciences (BAS) in June 2017 at the International House of Scientists "Frederic Joliot-Curie" (IHS) on the Black Sea Coast near Varna. The overall number of participants in the 2017 joint conference was 130, and they came from 33 countries. (The list is given at the end of the volume.)

The Organizing Committee was: V. K. Dobrev (Chairman), L. K. Anguelova, V. I. Doseva, V. G. Filev, A. Ch. Ganchev, K. K. Marinov, D. T. Nedanovski, T. V. Popov, D. R. Staicova, M. N. Stoilov, N. I. Stoilova, S. T. Stoimenov.

Acknowledgements

We express our gratitude to the

- Institute of Nuclear Research and Nuclear Energy
- Abdus Salam International Centre for Theoretical Physics
- International Association of Mathematical Physics

for financial help.

We thank the International Advisory Committee and the QTS Conference Board (see the web page) for the invaluable help.

We thank the Publisher, Springer Japan, represented by Ms. Chino Hasebe (Executive Editor in Mathematics, Statistics, Business, Economics, Computer Science) and Mr. Masayuki Nakamura (Editorial Department), for assistance in the publication.

Last, but not least, I thank the members of the Organizing Committee who, through their efforts, made the workshop run smoothly and efficiently.

Sofia, Bulgaria Vladimir Dobrev
May 2018

Contents

Part I Plenary Talks

From Hexagons to Feynman Integrals 3
Benjamin Basso

Higher Spins from One-Loop Effective Actions 17
L. Bonora, M. Cvitan, P. Dominis Prester, S. Giaccari and T. Štemberga

Algebraic Structures in Exceptional Geometry 31
Martin Cederwall

Duality in the Sachdev-Ye-Kitaev Model 43
Sumit R. Das, Animik Ghosh, Antal Jevicki and Kenta Suzuki

Implications of Hidden $\mathcal{N} = (0, 1)$ Super-Symmetry in $\mathcal{N} = (1, 1)$, 6*D* SYM Theory 63
Evgeny Ivanov

TBA and Tree Expansion 77
Ivan Kostov, Didina Serban and Dinh-Long Vu

Wheeler–DeWitt Quantization of Gravity Models of Unified Dark Energy and Dark Matter 99
Eduardo Guendelman, Emil Nissimov and Svetlana Pacheva

Wilson Loop Form Factors: A New Duality 115
Dmitry Chicherin, Paul Heslop, Gregory P. Korchemsky and Emery Sokatchev

Many-Body Localization in Large-*N* Conformal Mechanics 137
Fumihiko Sugino and Pramod Padmanabhan

Generalized Bases with a Resolution of the Identity: A Cooperative Game Theory Approach 149
A. Vourdas

Part II Holography and String Theory

On Non-slow Roll Inflationary Regimes 161
Lilia Anguelova, Peter Suranyi and L. C. Rohana Wijewardhana

Precision Test of Holographic Flavourdynamics 173
Yuhma Asano, Veselin G. Filev, Samuel Kováčik
and Denjoe O'Connor

$\mathcal{N} = 4$ Polygonal Wilson Loops: Fermions 181
Alfredo Bonini, Davide Fioravanti, Simone Piscaglia and Marco Rossi

Hidden Symmetry in String Theory 193
Yi Yang

Information Geometry of Strings on Plane Wave Background 205
H. Dimov, S. Mladenov, R. Rashkov and T. Vetsov

Part III Gravity and Cosmology

The Global Formulation of Generalized Einstein-Scalar-Maxwell Theories 217
C. I. Lazaroiu and C. S. Shahbazi

Two-Field Cosmological Models and the Uniformization Theorem 233
Elena Mirela Babalic and Calin Iuliu Lazaroiu

The Cohomological Structure of Generalized Killing Spinor Equations 243
Dario Rosa

Cosmological Solutions from Models with Unified Dark Energy and Dark Matter and with Inflaton Field 251
Denitsa Staicova and Michail Stoilov

Relation Between Dilatonic Pressure and Cosmological Pressure for Neutron Stars in Minimal Dilatonic Gravity 261
Kalin Marinov and Plamen Fiziev

Part IV Conformal and Gauge Theories

Conformal Ward–Takahashi Identity at Finite Temperature 271
Satoshi Ohya

Renormalization in Some 2D $\widehat{su}(2)$ Coset Models 283
Marian Stanishkov

Time-Dependent Free-Particle Salpeter Equation: Features of the Solutions 297
Amalia Torre, Ambra Lattanzi and Decio Levi

Leptons, Quarks, and Gauge Symmetries, from a Clifford Algebra 309
Ovidiu Cristinel Stoica

Part V Foundations of Quantum Theory

Infinitesimal Symmetries in Covariant Quantum Mechanics 319
Josef Janyška, Marco Modugno and Dirk Saller

The Quantum Detection Problem: A Survey 337
Sara Botelho-Andrade, Peter G. Casazza, Desai Cheng, John Haas and Tin T. Tran

Possible Alternative Mechanism to SUSY: Conservative Extensions of the Poincaré Group 353
András László

Noether's Theorem and Its Complement in Multi-Particle Systems 363
Walter Smilga

***SU*(2), Associated Laguerre Polynomials and Rigged Hilbert Spaces** 373
Enrico Celeghini, Manuel Gadella and Mariano A. del Olmo

Part VI Nonrelativistic and Classical Theories

Metage Symmetry Group of Non-barotropic Magnetohydrodynamics and the Conservation of Cross Helicity 387
Asher Yahalom

Three Quarks Confined by an Area-Dependent Potential in Two Dimensions 403
Igor Salom and V. Dmitrašinović

A Group Action Principle for Nambu Dynamics of Spin Degrees of Freedom 411
Stam Nicolis, Pascal Thibaudeau and Thomas Nussle

Newton–Cartan Trace Anomalies and Renormalization Group Flows 421
Roberto Auzzi

Evolution Operator Method and a Non-relativistic Particle in the Time-Dependent Homogeneous Field 431
E. I. Jafarov and S. M. Nagiyev

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Summary

Countries (33)	Number
Bulgaria	21
USA	11
France	9
Czech Republic	8
Brazil	7
Italy	7
S. Korea	6
Japan	5
Spain	5
Canada	5
Serbia	4
Belgium	4
UK	4
Switzerland	4
Romania	3
Russia	3
Germany	3
Israel	3
Mexico	2
Singapore	2
Poland	2
Armenia	1
Austria	1
Azerbaijan	1
China	1
Cyprus	1
Greece	1
Hungary	1
Norway	1
Slovenia	1
Sweden	1
Ukraine	1
Taiwan	1
Total	130

Part I
Plenary Talks

From Hexagons to Feynman Integrals

Benjamin Basso

Abstract I briefly describe some of the recent advances in the computation of structure constants of local operators in planar $\mathcal{N} = 4$ SYM theory using the so-called hexagon formalism. I then report on the application of this technique to the computation of a family of planar massless Feynman integrals.

Keywords Integrability · Structure constants · Feynman integrals

1 Introduction

Studies of anomalous dimensions in QCD [1–3] and, more recently, in supersymmetric gauge theories [4–8] have revealed an interesting connection between planar graphs and integrable spin chains. In some cases, this connection seems to pervade every corner of the theory, at weak and strong coupling, raising hopes that complicated interacting higher dimensional systems can be solved by means of the *Bethe ansatz*.

The best example of a higher dimensional theory that is being intensely investigated using integrability is the (4d) $\mathcal{N} = 4$ SYM theory. This theory outstands for its huge amount of symmetries, being exactly conformal and maximally supersymmetric. It is also a string theory in disguise, if seen through the lens of the AdS/CFT correspondence [9]. The correspondence and other pieces of evidence [8] lend support to the idea that the theory should be solvable, at least in the planar regime. This idea lies at the heart of many important all-order conjectures recently pushed forward, like the conjecture that the spectrum of scaling dimensions of the planar theory is encoded, at any value of the 't Hooft coupling constant $g^2 = \lambda/(16\pi^2)$, in a closed system of discrete Schrödinger-like equations, named Quantum Spectral Curve [7].

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_1

The most recent developments in this area aim at extending this "conjectural" technology to the computation of more complicated quantities, like the gluon scattering amplitudes [10], the higher-point functions [11–14] or some of the many conformal Feynman integrals that the theory can produce [15–19].

In this short note, I will lightly review some of these developments, focusing on the so-called hexagon approach for structure constants [12] and correlators [13, 14]. I will then illustrate through a specific example how the method can help computing complicated 4d massless integrals [20].

2 Integrable Spin Chain and Hexagon Form Factors

Integrability in planar $\mathcal{N} = 4$ SYM theory was primarily developed for finding the conformal single-trace operators of the theory and computing their scaling dimensions [8] or, equivalently, their two-point functions,

$$\langle \mathcal{O}_{\Delta_A}(x)\mathcal{O}^*_{\Delta_B}(0)\rangle = \delta_{A,B}x^{-2\Delta_A} \,. \tag{1}$$

In this framework, the local single-trace operators are identified with the states of a quantum spin chain and their scaling dimensions to the spin chain energies. The canonical example is given by composite operators made out of products of two complex scalar fields, ϕ_1 and ϕ_2. They map to states of a spin one-half periodic chain,

$$\mathcal{O} \sim \mathrm{tr}\, \phi_1\phi_2\phi_1 \dots \phi_2\phi_1 \qquad \leftrightarrow \qquad |\uparrow\downarrow\uparrow \cdots \downarrow\uparrow\rangle \,. \tag{2}$$

with the fields ϕ_1 and ϕ_2 mapping to up and down spins, respectively, and with the length L of the chain counting the total number of fields in the operator. In the free theory, operators with the same length L have the same canonical dimension $\Delta_0 = L$ and the spectrum is maximally degenerate. This state of affairs does not continue at loop level: the operators start mixing with each others and the associated eigenvalue problem is a serious task at generic length L. This is overlooking that the spin chain is secretly integrable, in the planar limit. At one loop, for instance, the chain happens to be identical to the Heisenberg magnet [4, 6], which is the prototypical example of an integrable quantum mechanical system.

There are operators that never renormalize, regardless of the strength of the interaction. These protected states form the multiplet of the ferromagnetic vacuum of the spin chain, with highest weight state

$$\mathrm{tr}\, \phi_1^L = |\uparrow \cdots \uparrow\rangle = |0\rangle \,. \tag{3}$$

More complicated operators, or excited states, describe linear superpositions of spin waves or magnons, which are down spins propagating and scattering on top of the up-spin background. Diagonalising the spin-chain Hamiltonian amounts to finding those linear combinations that are proper scattering eigenstates. They correspond to conformal primaries. The main idea, which goes back to Bethe, is that the integrability of the model translates into the factorization of the magnon S-matrix. Accordingly, the M-magnon S-matrix boils down to a product a 2-body S-matrices,

$$S_{12\ldots M} = \prod_{i<j}^{M} S(p_i, p_j)\,, \tag{4}$$

where $S(p_i, p_j)$ is the elastic S-matrix for two magnons carrying momenta p_i and p_j. This remarkable property allows one to immediately write down the quantization conditions on the magnons' momenta, the celebrated Bethe ansatz equations,

$$e^{ip_k L} \prod_{j\neq k}^{M} S(p_k, p_j) = 1\,, \tag{5}$$

which result from the periodicity conditions on the magnon wave function.

The energy spectrum is determined by adding up the individual energies of the magnons, $\Delta = L - M + \sum_{j=1}^{M} E(p_j)$, for those momenta that solve the above equations. In this form the original eigenvalue problem is reduced to finding the magnon S-matrix and energy [21]. The large amount of residual supersymmetries of the spin-chain vacuum, together with some extra requirements [22], has permitted to fix completely this data for the $\mathcal{N} = 4$ SYM spin chain [23–25]. The dispersion relation for a magnon, for instance, is concisely given to all loops by the simple expression

$$E^2 - 16g^2 \sin^2\left(\frac{p}{2}\right) = 1. \tag{6}$$

The S-matrix approach briefly described here is believed to take into account all the short-range spin-chain interactions. These ones are associated to diagrams that do not explicitely depend on the length L and survive in the asymptotic limit $L \to \infty$. A graph of this type is shown in the left panel of Fig. 1. For any finite length, however, a new type of interactions will appear at some point and must be added to the formalism. These interactions are associated to diagrams that wrap around the operator, like the one shown in the right panel of Fig. 1. They are exponentially suppressed with the length, being of order $\sim g^{2L}$ at weak coupling. Nicely, these finite-size corrections can be included in the form of so-called mirror magnons that wind around the chain [26, 27]. The latter mimic the effect of virtual particles propagating in the loops surrounding the operator. Unlike the spin waves associated to excitations of the operator, mirror magnons carry imaginary scaling dimensions, or energies $E \sim 2iu$, which must be integrated over. Heuristically, the imaginary energy acts as a momentum conjugated to displacement along the radial direction in Fig. 1 and the integration is needed to obtain a loop of arbitrary shape. Summing over a complete basis of mirror magnons should in principle accomodate for all the virtual particles wrapping around the operator. In this manner, finite-length corrections can be systematically included within the framework of the thermodynamical Bethe ansatz,

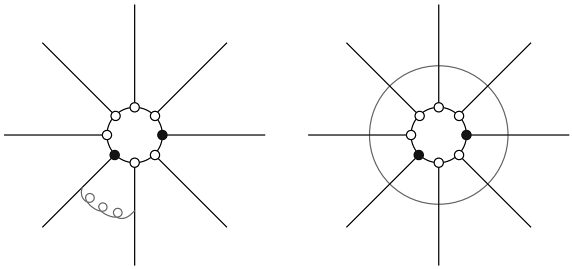

Fig. 1 Examples of planar graphs contributing to the mixing of local single-trace operators $\mathcal{O} \sim \mathrm{tr}\, \phi_1\phi_2\phi_1 \ldots \phi_1\phi_2$. The operators are represented by spin-chain states, with fields ϕ_1 (white nodes) standing for up spins and fields ϕ_2 (black nodes) for down spins. Typical quantum corrections to the operators, like the one-loop gluon exchange represented in the left panel, produce short-range interactions among the spins. The asymptotic S-matrix description is obtained by keeping the interactions that survive in the limit where the spin-chain length $L \to \infty$. The right panel provides an example of a wrapping graph, which is not captured by the S-matrix approach. It translates into a long-range interaction that is included in the formalism in the form of a virtual magnon propagating around the world. Its contribution to the scaling dimension of an operator of size $\sim L$ is of order $\sim g^{2L}$ at weak coupling

which upgrades the Bethe ansatz equations (5). The same end point can be more elegantly reached within the Quantum Spectral Curve framework [7], which treats equally all the contributions, regardless of their origins.

As mention in the introduction, in recent years, a lot of progress has been made at extending the spin-chain approach to the computation of more complicated observables. Among them are the structure constants characterizing the operator product algebra. Some of the new tools for computing planar correlators of single-trace operators by means of integrability [13, 14] stem from their study.

By definition, the structure constants measure the couplings among three conformal operators, or three-point functions,

$$C_{123} = \langle \mathcal{O}_{\Delta_1}(0)\mathcal{O}_{\Delta_2}(1)\mathcal{O}_{\Delta_3}(\infty)\rangle\,. \tag{7}$$

Like the scaling dimensions, these are, in general, complicated functions of the quantum numbers of the operators and of coupling constant g^2. Unlike the scaling dimensions, the structure constants are generically small in the planar limit, $N \to \infty$. They scale like $1/N$ for typical correlators, with $\Delta_{ij} = (\Delta_i + \Delta_j - \Delta_k) \neq 0$, when the single-trace operators are normalized as in Eq. (1). As such, computing the structure constants is a necessary step towards understanding the infinite tail of $1/N$ corrections. In the dual string theory, they are associated to the fundamental string vertex or pair-of-pants diagram.

At weak coupling, the computation of the structure constants follows from the Wick theorem and the fields in the operators are pairwise contracted as shown in Fig. 2. The analysis in the spin-chain framework was initiated in [11] where it was found to result in partial overlaps of the Bethe wave functions characterizing the three

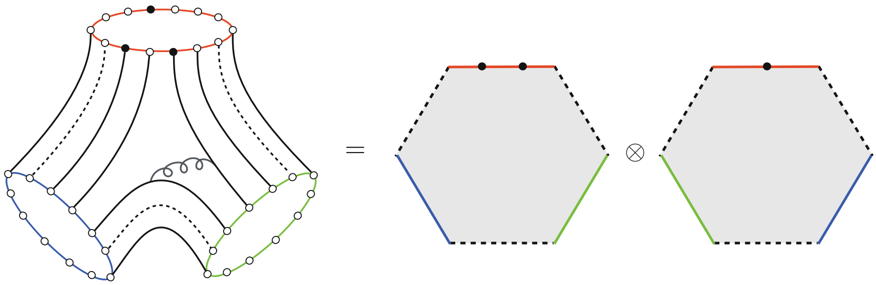

Fig. 2 Factorization of the pair-of-pants diagram into two hexagons. In the left panel we show a typical planar diagram contributing to the structure constant of three single-trace operators. Only the top operator is excited here and carries magnons (black dots). Straight lines stand for tree-level Wick contractions among the fundamental fields. At loop level one must add decorations, as shown in the left panel in the form of a gluon exchange between two neighbouring lines. In the hexagon approach one imagines opening the planar diagrams along three so-called mirror cuts, shown here as dashed lines. Each hexagon is made of three spin-chains edges and three mirror cuts

operators. At higher loops, one must decorate the Wick contractions with virtual gluons, etc. From the color viewpoint, this dressing turns the object into a three-punctured sphere, or equivalently a pair of pants. The hexagon approach [12] allows us to analyse the pair-of-pants diagram more directly, when all the characteristic lengths in the problem are large, roughly when $\Delta_{ij} \gg 1$. The main idea is that in this asymptotic regime the pair of pants factorizes into two hexagons, as shown in Fig. 2. Each hexagon is made of 3 edges taken from the spin chains and 3 mirror edges obtained by cutting along the seams of the pair of pants. The hexagons are easier to deal with than their parent pair of pants. Since all characteristic lengths are assumed to be large, the magnons can propagate freely on the hexagons. As such they are amenable to powerful bootstrap techniques developed for computing form factors in infinite volume.

The main information captured by the hexagon form factors are the amplitudes for creation or annihilation of magnons on the boundary of the hexagon. For illustration, in the simplest situation where only one edge is excited and carries a pair of magnons, with momenta p_1, p_2, the hexagon form factor is merely a function of these two momenta,

$$h(p_1, p_2) = \langle h || p_1, p_2 \rangle \otimes |0\rangle \otimes |0\rangle \, , \tag{8}$$

and, implicitely, of the coupling contant. Remarkably, in $\mathcal{N} = 4$ SYM, the two-point hexagon form factor can be determined at any value of the coupling through the integrable bootstrap, quite similarly to what happens for the magnon S-matrix. It is also the seed for the general form factor conjecture of [12] which, disregarding flavor indices, states that the M-magnon form factor can be expressed as a product of two-magnon form factors,

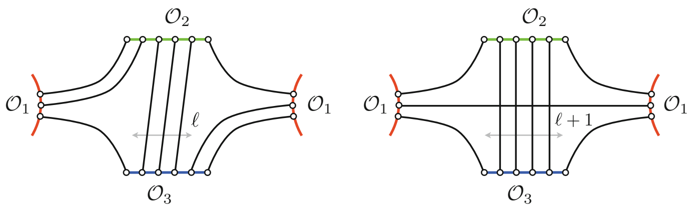

Fig. 3 The left panel shows a tree-level structure constant in the planar limit. The bridge length ℓ counts the number of Wick contractions between two operators, here between operators 2 and 3. The two hexagons decouple when all the bridge lengths are large. The right panel displays a loop correction to the same structure constant with a propagator stretching across the bridge. The interpretation of this graph in the hexagon framework is that a mirror magnon, produced on one hexagon and annihilated on the other, crosses the mirror cut connecting 2 and 3. The effect is of order $\sim g^{2\ell}$ at weak coupling and dies off at large separation $\ell \gg 1$

$$h(p_1, p_2, \dots, p_M) = \prod_{i<j}^{M} h(p_i, p_j)\,. \tag{9}$$

More generic configurations where the magnons are distributed along the various edges of the hexagon can be obtained by crossing symmetry. Structure constants are then determined to all loops in the asymptotic regime $\Delta_{ij} \gg 1$, by tensoring two form factors and summing over all the inequivalent ways of distributing the magnons on the two hexagons. Schematically,

$$C_{123} \propto \frac{1}{N} \times \sum_{\alpha} h(\alpha)h(\bar{\alpha})\,, \tag{10}$$

where the sum runs over the bipartite partitions of the set of momenta, $\alpha \cup \bar{\alpha} = \{p_1, \dots, p_M\}$.

The hexagon are seen to capture a certain half of the structure constant. Graphically, only the diagrams that fill the gaps between the tree-level contractions supporting the hexagon are expected to be encoded in the hexagon form factors. On finite-length pair of pants, one also finds long-range diagrams that are not contained inside one particular hexagon but instead overlap the two hexagons, see Fig. 3. They form the links between the hexagons and map to finite-size corrections. More technically, they come about when stitching the hexagons back together along the mirror cuts, by performing sums over complete basis of mirror magnons, like the ones discussed earlier for the finite-size corrections to the scaling dimensions. For example, the loop diagram shown in the right panel of Fig. 3 maps to a mirror magnon linking the hexagons along a seam of the pair of pants. Other diagrams, like the ones involving particles wrapping around an operator, can also be interpreted in this way. However, unlike the previous example, wrapping processes lead to technical problems which have not been completely solved yet [28].

As alluded to before, the hexagon approach can also be used to study higher-point functions [13, 14]. The underlying idea, which is reminiscent of the triangulation of 2d surfaces, is that any punctured sphere can be covered by means of hexagons, if enough of them are used and if they are stitched together in the appropriate manner. In the following, we illustrate the power of the hexagonalisation procedure on a simple planar four-point function.

3 Hexagonalisation and Fishnet Integrals

In $\mathcal{N} = 4$ SYM it is not possible in general to extract the information about a single diagram within the integrability framework. It is a virtue of the formalism to subsume into a single entity the many diagrams contributing to a given observable. The loop diagram shown in the right panel of Fig. 3, for instance, corrects the structure constant shown in the left panel. But many similar looking diagrams, featuring gluons, scalars and gluinos, also contribute at the same loop order. The hexagon approach does not a priori distinguish among them and only gives us access to their overall sum.

Nonetheless, it is sometimes possible to twist the sum such as to disentangle a particular graph. There are several ways of doing. One can choose very carefully the quantum numbers of the operators such that only one graph remains in the sum, at a certain loop order. Alternatively, one can deform the theory itself so that it only generates a particular set of diagrams from the onset. If the deformation preserves integrability then the technique can be used to compute graphs directly. This is what is happening in the so-called fishnet theory [15, 16]. The latter is obtained by twisting $\mathcal{N} = 4$ SYM through a procedure known as the γ-deformation [29, 30] and then letting the deformation parameter go to infinity. All the excitations of the theory are forcefully decoupled, if not for two scalar fields which interact in the deformed theory only by means of a quartic potential,

$$\mathcal{L}_{\text{int}} = (4\pi g)^2 \text{tr}\, \phi_1 \phi_2 \phi_1^* \phi_2^* \,. \tag{11}$$

As a vestige of the original gauge symmetry, the scalar fields are matrices in the adjoint representation of $U(N)$ and one still obtains a decent planar theory by taking N large at fixed g^2.

This procedure significantly reduces the number of planar graphs. In most cases only one graph remains, at a given loop order. Furthermore, all the graphs look locally the same and take the form of the fishnet mesh displayed in Fig. 4. Despite the massive cut in the number of graphs, the fishnet theory has a lot in common with $\mathcal{N} = 4$ SYM. It is conformal in the planar limit for any value of g^2, if completed with properly tuned double-trace interactions [31, 32], and it is "as integrable as" $\mathcal{N} = 4$ SYM in the planar limit. In fact, the integrability of the fishnet graphs has been known for many years and explored in the pioneering work of Zamolodchikov [33] (see also Isaev's quantum mechanical approach to Feynman diagrams [34]). The

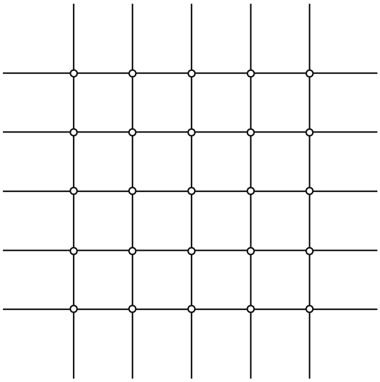

Fig. 4 The planar graphs of the fishnet theory all look locally like the fishnet mesh shown here. Every line represents a massless propagator $1/(x-y)^2$ and each node stands for a ϕ^4 interaction

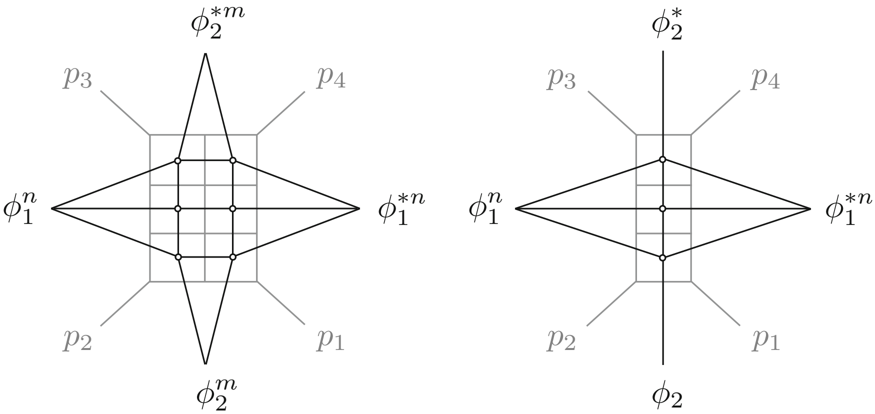

Fig. 5 Left panel: the Feynman graph contributing to the fishnet correlator. A beam of n massless particles is produced at x_1 on the left and annihilated at x_2 on the right. Similarly, a beam of m massless particles is created at x_3 at the bottom and annihilated at x_4 at the top. The particles in the beams interact locally at the white nodes, by means of a ϕ^4 potential. In grey we represent the dual color-ordered amplitude, with momentum conservation at every vertex. The duality transformation maps a propagator in coordinate space to a propagator in momentum space, $1/p^2 = 1/(x-y)^2$, where x and y are the spacetime points on the two sides of the momentum p. Note that the external momenta are off shell when the four external points are space-like separated from each other. Right panel: for $m = 1$ the Feynman integral reduces to the n-rung ladder integral [35]

only price to pay here is that the fishnet theory is not unitary. However, this is of no concern when the goal is to compute individual graphs.

The fishnet theory gives us access to an interesting class of conformal integrals which we can study using the techniques described earlier. Consider for instance the Feynman graph depicted in the left panel of Fig. 5. It is the sole graph contributing in the planar limit to the color-ordered correlator

$$G_{m,n}(\{x_i\}) = \langle \mathrm{tr}\, \phi_1(x_1)^n \phi_2(x_3)^m \phi_1^*(x_2)^n \phi_2^*(x_4)^n \rangle\,, \tag{12}$$

where the trace here embraces all the operators. For a generic configuration of the four external points, the resulting integral is both UV and IR finite. As such it is a function of the conformal cross ratios,

$$u = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2} = \frac{z\bar{z}}{(1-z)(1-\bar{z})}\,, \qquad v = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} = \frac{1}{(1-z)(1-\bar{z})}\,, \tag{13}$$

where $(z, \bar{z})$ are the holomorphic (or light-cone) coordinates of the 4-th point in the conformal frame $(x_1, x_3, x_2) = (0, 1, \infty)$. More precisely,

$$G_{m,n} \sim \frac{g^{2nm}}{x_{12}^{2n} x_{34}^{2m}} \times \Phi_{m,n}(z, \bar{z})\,, \tag{14}$$

after stripping out the weights of the fields and the obvious powers of the coupling. Here,

$$\Phi_{m,n}(z, \bar{z}) = \Phi_{n,m}(z, \bar{z}) = \Phi_{m,n}(1/z, 1/\bar{z}) = \Phi_{m,n}(\bar{z}, z)\,, \tag{15}$$

due to reflection and cyclic symmetry of the correlator. For $m = 1$ we obtain the ladder integrals, see Fig. 5. They have been computed long ago [35] and found to be given by

$$\Phi_{1,n} = \frac{(1-z)(1-\bar{z})}{z-\bar{z}} \times L_n(z, \bar{z})\,, \tag{16}$$

where $L_p(z, \bar{z})$ is an iterated integral (or pure function) of weight $2p$,

$$L_p(z, \bar{z}) = \sum_{j=0}^{p} \frac{(2p-j)!}{p!j!(p-j)!} (-\log z\bar{z})^j \left[\mathrm{Li}_{2p-j}(z) - \mathrm{Li}_{2p-j}(\bar{z})\right], \tag{17}$$

with $\mathrm{Li}_k(z) = \sum_{a=1}^{\infty} z^a/a^k$ the polylogarithm. This peculiar combination of logarithms and polylogarithms defines for any p a real analytical function for complex conjugated $z, \bar{z} \neq 0$; the function has a logarithmic singularity at $\rho = \sqrt{z\bar{z}} = 1$. For $p = 1$ this single-valued version of the dilogarithm,

$$L_1(z, \bar{z}) = 2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log(z\bar{z}) \log\left(\frac{1-z}{1-\bar{z}}\right), \tag{18}$$

is known as the Bloch–Wigner dilogarithm, or box integral.

The hexagon fomalism allows us to write down an integral representation for this two-parameter family of integrals which generalises the ladders. The hexagon factorization of the fishnet integral is shown graphically in Fig. 6 and consists in three components. (i) First we have the hexagon form factors. They account for the production and absorption of m mirror magnons on the bottom and top hexagons.

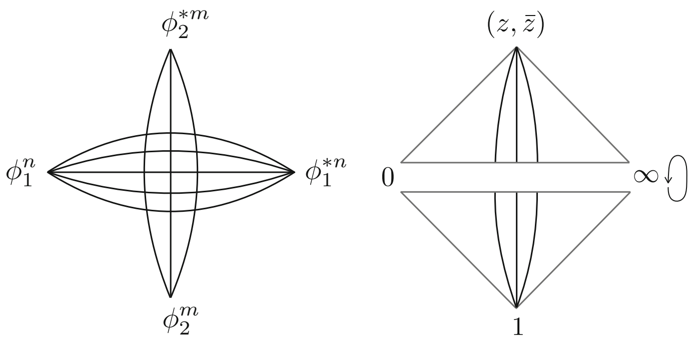

Fig. 6 The planar correlator can be covered with two hexagons. The vertical propagators map to m mirror magnons, on the mirror edges of the hexagons. (The spin chain edges are not depicted here; they sit at the tips of the triangles.) These magnons are created on the bottom hexagon and absorbed on the top one. The two hexagons are identical, if not for the fact that they stretch between different points. Their form factors are related by a conformal transformation [13] that preserves 0 and ∞ and maps 1 to $(z, \bar{z})$

A mirror magnon can be viewed here as describing the propagation of a scalar field ϕ_2. It carries two labels: a rapidity u, which is a momentum conjugated to shift along the direction $(0, \infty)$, and a positive integer a, which enumerates the Lorentz harmonics of the field. Alternatively, $s = a - 1$ counts the number of derivatives attached to the field, $\partial_{\alpha_1\dot{\alpha}_1} \ldots \partial_{\alpha_s\dot{\alpha}_s}\phi_2$, such that a magnon in the a-th mode forms a symmetric (traceless) representation of the Lorentz group, with spins $(\frac{s}{2}, \frac{s}{2})$ and dimension a^2. Once properly normalized, and disregarding Lorentz indices, these hexagon amplitudes result in

$$|h_{a_1,\ldots,a_M}(u_1, \ldots, u_M)|^2 = \prod_{i=1}^{m} \frac{a_i}{(u_i^2 + \frac{a_i}{4})^m} \times \prod_{i<j}^{m} \Delta_{a_i a_j}(u_i, u_j)\,, \tag{19}$$

where

$$\Delta_{ab}(u, v) = \left[(u-v)^2 + \frac{(a+b)^2}{4}\right] \times \left[(u-v)^2 + \frac{(a-b)^2}{4}\right]. \tag{20}$$

(ii) The next ingredient accounts for the propagation of the m mirror magnons through the n lines connecting 0 and ∞. Alternatively, the edge along which the two hexagons are stitched together has a thickness, given by the number of lines in the horizontal beam. The transport of the magnons through this stack results in the factor

$$\prod_{i=1}^{m} \left(\frac{g^2}{u_i^2 + \frac{a_i}{4}}\right)^n. \tag{21}$$

(iii) Finally, there is a factor that accounts for the 4pt geometry. In the four-point function the two hexagons do not end on the same 3 points; they only meet at 0 and ∞. Therefore, one must perform a conformal transformation, among those that fix 0 and ∞, to transform one hexagon into the other [13]. This transformation consists in a dilatation by $\rho = \sqrt{z\bar{z}}$ and a rotation by $e^{i\phi} = \sqrt{z/\bar{z}}$. This is how the information about the cross ratios enters into the construction. Under this transformation, a mirror magnon picks a factor $\rho^{-2iu}e^{ij\phi}$, where j is the total spin of the magnon. After tracing over the Lorentz multiplet, and using that only left-right symmetric states contribute to the 4p function, one finds that the dilatation-rotation brings, for every magnon, the factor [13]

$$(z\bar{z})^{-iu} \times \chi_a(\phi)\,, \tag{22}$$

where χ_a is the spin $\frac{1}{2}(a-1)$ character, $\chi_a(\phi) = \sin(a\phi)/\sin\phi$.

Putting these three ingredients together yields an integral representation for the fishnet correlator (14). After normalizing the correlator as in (14) and stripping off an overall rational factor,

$$\Phi_{m,n} = \left[\frac{(1-z)(1-\bar{z})}{z-\bar{z}}\right]^m \times I_{m,n}\,, \tag{23}$$

one obtains

$$I_{m,n} = \sum_{a_1,\ldots,a_m} \int \frac{du_1\ldots du_m}{(2\pi)^m m!} \prod_{i=1}^{m} \frac{a_i z^{-iu_i+a_i/2}\bar{z}^{-iu_i-a_i/2}}{(u_i^2+a_i^2/4)^{m+n}} \prod_{i<j}^{m} \Delta_{a_ia_j}(u_i,u_j)\,, \tag{24}$$

where the sum runs over $a_i = \pm 1, \pm 2, \ldots$ and the integral is taken over R^m. Though not obvious, the integral (24) is symmetric under exchange of m and n, up to an overall weight,

$$I_{m,n} = \left[\frac{z-\bar{z}}{(1-z)(1-\bar{z})}\right]^{m-n} \times I_{n,m}\,, \tag{25}$$

as required by $\Phi_{n,m} = \Phi_{m,n}$. The other relations in (15) are trivially satisfied.

For $m = 1$, the integral is seen [13] to reproduce the ladders (17). For higher values of m, one finds a surprising simplicity as well. Evaluating the matrix-model-like integral (24) explicitely for various values of m, n suggests that it can be cast into the form of a determinant of ladders for any m, n. The proposal put forward in [20] is that $I_{m,n}$ is a pure function of weight $2mn$, for $n+1 > m$, that can be written as

$$I_{m,n} = \frac{\det N}{\prod_{k=1}^{m}(n-m+2k-2)!(n-m+2k-1)!}\,, \tag{26}$$

where N is a $m \times m$ Hankel matrix with ij element[1]

[1] The overall factor in (26) can be absorbed in the columns of the matrix and the determinant written in the alternative form given in [20].

$$N_{ij} = (n-m+i+j-2)!(n-m+i+j-1)! \times L_{n-m+i+j-1}(z,\bar{z})\,. \tag{27}$$

For $n = m = 2$, it evaluates to

$$I_{2,2} = L_1 L_3 - \frac{1}{3} L_2^2\,, \tag{28}$$

a form that is equivalent to the result obtained in [36] by direct integration of the 2×2 fishnet Feynman integral. In the general case the determinant formula has not been proven yet, and thus stands as a conjecture. One verifies however that it meets non-trivial analyticity requirements. Among them is the fact that the integral should admit a dual interpretation as an off-shell color-ordered amplitude. The amplitude is obtained by dualising the propagators, as shown in Fig. 5. Amplitudes are subject to stringent constraints like the Steinmann relations [37]. In our case, they enforce that the integral should have a single discontinuity upon continuing $(1-z) \to (1-z)e^{i\pi}$, $(1-\bar{z}) \to (1-\bar{z})e^{i\pi}$. The ladder amplitude L_p fulfills this requirement. Upon the continuation it shifts by the amount

$$\text{disc}\, L_p = 2\pi i \frac{(-1)^p}{p!(p-1)!} \log{(z/\bar{z})} (\log z \log \bar{z})^{p-1}\,, \tag{29}$$

which itself has no discontinuity, $\text{disc}\,\text{disc}\, L_p = 0$. Only special combinations of products of ladders will obey the Steinmann relations. Under extra mild assumptions on these combinations [20], there is only one possibility at a given weight, the determinant (26). Under the continuation every column in N shifts by the same amount, which itself is free of discontinuity. Hence, $\det N$ has no higher discontinuity and the Steinmann relations are satisfied.

4 Conclusion

I have briefly described the use and utility of integrability for computing correlators in $\mathcal{N} = 4$ SYM and its siblings. There is of course a much wider range of observables that can be analyzed along these lines. For instance, a typical fishnet correlator, like the one shown in Fig. 4, with the external legs ending at arbitrary spacetime points, can be cut down into hexagons. There are actually many ways of doing, and many hexagons are needed. All the different cuttings should in principle return the same answer. However, the proliferation of hexagons with the number of external legs makes the resulting integrals rather intricate. These integrals are significantly more difficult than the ones considered here and it is currently unknown how to evaluate them efficiently.

Owing to its proximity to $\mathcal{N} = 4$ SYM and its elegant simplicity, the fishnet theory appears as an ideal laboratory for testing the integrability conjectures, the many excursions behind the spectral problem and the recent explorations of the non-planar regime [38–40]. It also provides a more transparent setting for understanding these conjectures from the first principles, that is from the graphs directly. In the long run, its study should enrich the dictionary between Feynman diagrams and integrable quantum mechanics [34] and bring new information about a broad family of conformal integrals.

Recent results obtained in the fishnet theory [31] and in ladder limits of $\mathcal{N} = 4$ SYM [41] hint at other ways of representing correlators within the integrability framework, which could bypass the limitations of the hexagon method. The Yangian invariance of fishnet correlators [17, 18] is also suggestive of additional structures that could complement the hexagon approach. It would be interesting to see if the combinations of all these methods can help computing more general conformal integrals and teach us something new about the functions that are needed to represent them.

Acknowledgements I would like to thank Lance Dixon, Volodya Kazakov and Grisha Korchemsky for enlightening discussions.

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Higher Spins from One-Loop Effective Actions

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Abstract In this contribution we review the method to obtain information about the classical dynamics of a higher spin field by minimally coupling the field via a conserved current to a simple free fermion, and by integrating out the latter. We consider here the two point correlators of two conserved currents, which allow us to determine the effective action to quadratic order. We show that this gives rise to the classical equation of motion of the Fronsdal type. We point out the importance of the contributions of the tadpole and seagull terms and the ambiguity related to the choice of the conserved currents.

1 Introduction

A widespread idea is that, in order to describe quantum matter and quantum gravity together in the framework of field theory, an infinite number of fields is needed. This is obvious in string and superstring theories, but has also been argued on a general

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_2

footing as a condition to avoid a causality violation, see [1]. What should be the theory with infinite many fields is of course *the* problem. Apart from superstring theories other proposals for consistent theories involving fields with infinite spins have been made, in particular, by Vasiliev [2] in the framework of 4d anti-De Sitter spacetime. Various consistent models have been shown to exist also in 3d.

However, despite such impressive results, many problems remain. Even before a comparison with experimental data can be approached, one is faced with a series of questions concerning various aspects of the would be theory. We have already mentioned the problem of causality, but even more basic is the problem of locality, which is naturally raised by the introduction of an infinite number of fields. For instance, string theory is non-local, but in a mild way does not spoil causality and unitarity. What should be the right amount of non-locality in a theory with infinite many fields is still a not understood and untackled problem. Beside locality, another basic problem in such kind of theories is their symmetries. The latter must be local and very large in order for them to reduce the enormous number of degrees of freedom. In trying to construct local higher spin theory free equations of motion, covariant under such gauge symmetries, one soon realizes that locality and covariance come to an apparent clash, [3], and in order to preserve both of them one is obliged to introduce auxiliary fields. It is a remarkable result that this is possible, at least at the lowest (linear) level, [4].

Notwithstanding the progress made so far, it is clear that analyzing the problem of theories with infinite many higher spin fields by trial and error is a daunting, if not hopeless, task. In this contribution we would like to review a proposal made, and carried out to some extent, in a series of papers, [5–7], which may help outlining a viable work program. The basic idea is to exploit the one-loop effective actions of elementary field theories coupled via conserved currents to external higher spin sources, in order to extract information about the (classical) dynamics of the latter. We will focus here, in particular, on a massive Dirac fermion model coupled to external sources, although an analogous treatment can be extended to a massive scalar, as was done in the above cited papers, and, no doubt, to other elementary fields. We will show that the effective action of this model contains the local quadratic action of all the low and higher spin fields and it is built out of the corresponding Fronsdal differential operators.

Already at this stage, i.e. even before coping with interactions, one can extract several derived results and mark some interesting points. The first is that all the local actions obtained in this way can be expressed in geometric form by means of tensors which are the generalizations of the Riemann one in 4d. The second is connected with the conserved currents to which one couples the external fields: they are not unique and their choice affect the form of the effective actions. This is important in particular in relation to the non-diagonal correlators. The third point is connected with the technique we use to compute the effective actions, that of Feynman diagrams: our basic diagram is the bubble diagram, with two interaction vertices and two fermion propagators. However we will consider also others, the tadpoles and seagull ones, which may give precious information on the covariance of the model.

The paper is organized as follows. In the next section we consider a few introductory examples in 3d and 4d. Section 3 outlines and reviews the analysis of 2pt current correlators and illustrates the results. Section 4 is devoted to a discussion of the Fronsdal-type equations and their appearance in our result. Section 5 contains some auxiliary material, such as the discussion about tadpole and seagull terms and non-diagonal correlators. Section 6 is devoted to a few concluding remarks.

2 A Few Examples in 3d and 4d

To justify our faith in the effective action method, let us consider a few low dimension and low spin examples. Let us start from the free Dirac field action

$$S_0 = \int d^d x \left[i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \right], \tag{1}$$

Out of it we can construct on shell conserved currents of any spin. For instance, if ψ belongs to the fundamental representation of a Lie algebra with generators T^a, the spin 1 current

$$J^a_\mu(x) = i\bar{\psi}\gamma_\mu T^a \psi \tag{2}$$

is conserved on shell. We can couple it to the a gauge field A^a_μ via the interaction term $S_1 = \int d^d x\, J^a_\mu A^{a\mu}$. Considering $S = S_0 + S_1$ one can define the interaction vertex between the gauge field and two fermions, as well as the fermion propagator. One can easily compute the two currents amplitude $J^{ab}_{\mu\nu}(x, y) = \langle 0|\mathcal{T} J^a_\mu(x) J^b_\nu(y)|0\rangle$ in any dimension; for instance, in 3d for large m, we have[1]

$$\tilde{J}^{ab}_{\mu\nu}(k) = \frac{i}{4\pi}\frac{1}{3|m|}\delta^{ab}(k_\mu k_\nu - k^2\eta_{\mu\nu}) \tag{3}$$

This term is local. By Fourier anti-transforming it and inserting it into the quadratic effective action $\int d^3x d^3y\, A^\mu_a(x) J^{ab}_{\mu\nu}(x, y) A^\nu_b(y)$, it takes the form

$$S \sim \frac{1}{|m|}\int d^3x \left(A^a_\mu \partial^\mu \partial^\nu A^a_\nu - A^a_\nu \Box A^{a\nu} \right) \tag{4}$$

which is the lowest term in the expansion of the YM action

$$S_{YM} = -\frac{1}{g_{YM}}\int d^3x\, \mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) \tag{5}$$

[1] This is the even parity part of the correlator. Since the mass term breaks the reflection invariance in 3d, there is also an odd part of the correlators, which we overlook here. It gives rise to the Chern–Simons action, see [5, 6].

where $g_{YM} \sim |m|$.

Similarly, we can couple ψ to the metric. If we represent $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$, then the flat space energy-momentum tensor

$$T_{\mu\nu} = \frac{i}{4}\bar{\psi}\left(\gamma_\mu \overleftrightarrow{\partial}_\nu + \gamma_\nu \overleftrightarrow{\partial}_\mu\right)\psi. \tag{6}$$

is conserved on shell, and can be linearly coupled to the metric fluctuation $h_{\mu\nu}$ via $S_1 = \int d^d x\, T^{\mu\nu} h_{\mu\nu}$. In the same way as above we can easily determine the vertex with one h and two fermion legs, and compute the two-point function of the em tensor. In 3d, in the IR limit, the 2pt e.m. tensor correlator behaves as

$$\begin{aligned}\langle T_{\mu\nu}(k) T_{\lambda\rho}(-k)\rangle^{IR} &= \frac{i|m|}{96\pi}\Big[\frac{1}{2}\left(\left(k_\mu k_\lambda \eta_{\nu\rho} + \lambda \leftrightarrow \rho\right) + \mu \leftrightarrow \nu\right) \\ -\left(k_\mu k_\nu \eta_{\lambda\rho} + k_\lambda k_\rho \eta_{\mu\nu}\right) &- \frac{k^2}{2}\left(\eta_{\mu\lambda}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\lambda}\right) + k^2 \eta_{\mu\nu}\eta_{\lambda\rho}\Big].\end{aligned} \tag{7}$$

This is a local expression multiplied by $|m|$. In fact Fourier anti-transforming it and integrating over spacetime after saturating it with $h^{\mu\nu}$ and $h^{\lambda\rho}$, it gives rise to the action

$$S \sim |m| \int d^3x \left(-2\partial_\mu h^{\mu\lambda}\partial_\nu h^\nu_\lambda - 2h\,\partial_\mu\partial_\nu h^{\mu\nu} - h^{\mu\nu}\Box h_{\mu\nu} + h\Box h\right), \tag{8}$$

which is the linearized Einstein–Hilbert action:

$$S_{EH} = \frac{1}{2\kappa}\int d^3x\,\sqrt{g}\,R \tag{9}$$

where $\kappa \sim \frac{1}{|m|}$.

As explained in the previous footnote, the odd parity two–point correlators give rise, in an analogous way, to the gauge and gravity Chern–Simons action in 3d. Analogous results can be obtained also in 4d, with some differences: the coupling constant of the EH and YM actions are momentum dependent and the EH appears in a non-local form (see below). Some of these results in 3d and 4d for spin-1 and -2 have been known for a long time, see for instance [8]. But the systematic way in which they appear creates the expectation that this method might lead to the same results in all dimensions.

Not only that. The fermion model (like the scalar one, see [9]) does not admit only spin one and two conserved currents, but in fact it has conserved currents of any spin. So the question arises as to whether something similar to spin 1 and 2 may be true also for higher spins: can one extract from the two-point correlators of such currents the quadratic action of the corresponding higher spin source fields, or, equivalently, their linearized equations of motion? The answer is yes for both questions. The relevant results have been obtained in [6, 7].

3 Two-Point Correlators and Effective Actions

As pointed out above the free fermion model (1) admits on shell symmetric conserved currents of any spin s, $J^{(s)}_{\mu_1\ldots\mu_s}$. Their form is not uniquely determined by conservation. The simplest choice is

$$J^{(s)}_{\mu\mu_1\ldots\mu_s} = i^{s-1}\frac{\partial}{\partial z^{(\mu_1}}\cdots\frac{\partial}{\partial z^{\mu_{s-1}}}\overline{\psi}\left(x+\frac{z}{2}\right)\gamma_{\mu_s)}\psi\left(x-\frac{z}{2}\right)\Big|_{z=0} \tag{10}$$

For instance

$$J^{(1)}_{\mu} = \overline{\psi}\gamma_\mu\psi \tag{11}$$

$$J^{(2)}_{\mu_1\mu_2} = i\left(\partial_{(\mu_2}\overline{\psi}\gamma_{\mu_1)}\psi - \overline{\psi}\gamma_{(\mu_1}\partial_{\mu_2)}\psi\right) \tag{12}$$

$$\ldots \tag{13}$$

Our method consists in linearly coupling such currents to external sources $a^{\mu_1\cdots\mu_s}$, in analogy with the gauge field and the metric fluctuation, via the action term $\int d^dx\, J^{(s)}_{\mu_1\mu_2\ldots\mu_s}a^{\mu_1\cdots\mu_s}$, extracting the Feynman vertices and proceeding to the calculation of the correlators. The complete effective action for this model is

$$\begin{aligned} W[a,s] = W[0] + \sum_{n=1}^{\infty}\frac{i^{n-1}}{n!}\int\prod_{i=1}^{n} d^dx_i a^{\mu_{11}\cdots\mu_{1s}}(x_1)\ldots a^{\mu_{n1}\cdots\mu_{ns}}(x_n) \\ \times\, \langle 0|\mathcal{T} J^{(s)}_{\mu_{11}\ldots\mu_{1s}}(x_1)\ldots J^{(s)}_{\mu_{n1}\ldots\mu_{ns}}(x_n)|0\rangle . \end{aligned} \tag{14}$$

In particular $a_\mu = A_\mu$, $a_{\mu\nu} = \frac{1}{4}h_{\mu\nu}$ and $J^{(2)}_{\mu\nu} = 2T_{\mu\nu}$. For the time being we will limit ourselves to the two-point correlators and the main object of investigation will be the bubble diagram with two external a legs and two internal fermion propagators. This will generate the quadratic part of the effective action.

The typical integral we meet in this calculation is the tensor integral

$$\tilde{J}_{\mu_1\ldots\mu_p}(d;\alpha,\beta;k,m) = \int\frac{d^dp}{(2\pi)^d}\frac{p_{\mu_1}\cdots p_{\mu_p}}{\left(p^2-m^2\right)^{\alpha}\left((p-k)^2-m^2\right)^{\beta}} \tag{15}$$

To evaluate it we follow the method of Davydychev et al. [10], which consists in reducing it to a combination of scalar integrals in different dimensions, and evaluating the latter with the dimensional regularization. The result is expressed in terms of generalized hypergeometric functions. In fact this is so in two distinct ways, by means of convergent series, one relevant to the IR, with $\frac{|k|}{2m} << 1$, and one to the UV, with $\frac{|k|}{2m} >> 1$. The two expressions are the analytic continuation of each other.

In [6, 7] this analysis was carried out for external fields with spin up to 5 and in dimensions up to 8. In fact in some cases it is possible to obtain compact expressions for the 2pt correlators of any spins and any dimensions (for instance, in the UV limit

$m \to 0$). The exact formulas are usually gigantic expressions, whose physical (and geometrical) content is not always simple to read out. It is easier to view it by using the IR and the UV series expansions alluded to above. The IR expressions are series of local terms (when inserted in the effective action (14)). It is often the case that some of these terms, corresponding to non-negative powers of m, are non-conserved (non-transverse with respect to k, the external momentum). Similar non-conserved terms exist also in the UV. However, since the IR non-conserved terms are local, we are allowed to subtract them from the effective action. We obtain in this way conserved expressions. Here we give one single example for spin 3 in 4d. In such a case the even power of order 0, logarithmic and 2,4,6 in m are not conserved, but

$$\begin{aligned} &\mathcal{O}_{UV}(m^0) - \mathcal{O}_{IR}(m^0) - \mathcal{O}_{IR}(\log(m)) \\ &= -\frac{2ik^6}{99225\pi^2}\left(-210\log\left(-\frac{k^2}{m^2}\right)+599\right)\left(n_1\cdot\pi^{(k)}\cdot n_2\right)^3 \\ &\quad + \frac{ik^6}{1587600\pi^2}\left(-3885\log\left(-\frac{k^2}{m^2}\right)+13339\right) \\ &\qquad \times(n_1\cdot\pi^{(k)}\cdot n_2)(n_1\cdot\pi^{(k)}\cdot n_1)(n_2\cdot\pi^{(k)}\cdot n_2) \end{aligned} \tag{16}$$

and

$$\mathcal{O}_{UV}(m^2) - \mathcal{O}_{IR}(m^2) = -\frac{4im^2k^4}{2025\pi^2}\left(15\log\left(-\frac{k^2}{m^2}\right)-16\right)\left(n_1\cdot\pi^{(k)}\cdot n_2\right)^3 \tag{17}$$
$$+\frac{im^2k^4}{16200\pi^2}\left(480\log\left(-\frac{k^2}{m^2}\right)-857\right)(n_1\cdot\pi^{(k)}\cdot n_2)(n_1\cdot\pi^{(k)}\cdot n_1)(n_2\cdot\pi^{(k)}\cdot n_2)$$

$$\mathcal{O}_{UV}(m^4) - \mathcal{O}_{IR}(m^4) = \frac{4im^4k^2}{27\pi^2}\left(n_1\cdot\pi^{(k)}\cdot n_2\right)^3 \tag{18}$$
$$-\frac{im^4k^2}{144\pi^2}\left(18\log\left(-\frac{k^2}{m^2}\right)-23\right)(n_1\cdot\pi^{(k)}\cdot n_2)(n_1\cdot\pi^{(k)}\cdot n_1)(n_2\cdot\pi^{(k)}\cdot n_2)$$

$$\mathcal{O}_{UV}(m^6) - \mathcal{O}_{IR}(m^6) = \frac{4im^6}{81\pi^2}\left(6\log\left(-\frac{k^2}{m^2}\right)-7\right)\left(n_1\cdot\pi^{(k)}\cdot n_2\right)^3 \tag{19}$$
$$+\frac{im^6}{162\pi^2}\left(69\log\left(-\frac{k^2}{m^2}\right)-70\right)(n_1\cdot\pi^{(k)}\cdot n_2)(n_1\cdot\pi^{(k)}\cdot n_1)(n_2\cdot\pi^{(k)}\cdot n_2),$$

where the above mentioned subtractions have already been carried out, are all conserved. Here we are using a compact notation where

$$\pi^{(k)}_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \tag{20}$$

and n_1, n_2 are two polarization vectors, a bookkeeping device to guarantee the correct Lorentz index symmetry. The final contribution to the effective action is obtained by differentiating with respect to n_1 and n_2, Fourier anti-transforming and inserting the result into (14).

Equation (18) is related to a non-local version of the spin 3 Fronsdal equation. Let us consider, for instance the piece $k^2 \left(n_1 \cdot \pi^{(k)} \cdot n_2\right)^3$. After stripping it of the polarization vectors, Fourier anti-transforming and inserting it into (14), the corresponding piece gives rise to the following equation of motion

$$\Box\varphi_{\mu\nu\lambda} - \partial_{\underline{\mu}}\partial\cdot\varphi_{\underline{\nu\lambda}} + \frac{1}{\Box}\partial_{\underline{\mu}}\partial_{\underline{\nu}}\partial\cdot\partial\cdot\varphi_{\underline{\lambda}} - \frac{1}{\Box^2}\partial_\mu\partial_\nu\partial_\lambda\partial\cdot\partial\cdot\partial\cdot\varphi = 0 \tag{21}$$

where the spin three field $a_{\mu\nu\lambda}$ has been called, as it is customary, $\varphi_{\mu\nu\lambda}$. In this equation a dot denotes index contraction and underlined indices mean the sum over the minimum number of terms necessary to completely symmetrize the expression in μ, ν and λ. Equation (21) is clearly non-local, but, as we shall see, it corresponds to a non-local form of the spin 3 Fronsdal equation. However, before carrying on, we need a bit of recalling.

4 The Local and Non-local Fronsdal Equations

Historically the first formulation of equations for the *unconstrained* free massless symmetric spin 3 field φ was given by Fronsdal [3]

$$\mathcal{F}_{\mu\nu\lambda} \equiv \Box\varphi_{\mu\nu\lambda} - \partial_{\underline{\mu}}\partial\cdot\varphi_{\underline{\nu\lambda}} + \partial_{\underline{\mu}}\partial_{\underline{\nu}}\varphi'_{\underline{\lambda}} = 0 \tag{22}$$

where a prime $'$ means that the tensor is traced over a pair of indices. From now on we shall use a more concise notation in which all indexes are suppressed, see [4]. In this new notation the Fronsdal equation for a spin s field takes the form

$$\mathcal{F} \equiv \Box\varphi - \partial\,\partial\cdot\varphi + \partial^2\varphi' = 0 \tag{23}$$

where φ represents a completely symmetric rank-s tensor field $\varphi \equiv \varphi_{\mu_1\cdots\mu_s}$. In the case of $s = 3$ Eq. (23) coincide with Eq. (22).

The Fronsdal equation (23) is invariant under local transformations that are parametrised by traceless completely symmetric rank-$(s-1)$ tensor fields $\Lambda \equiv \Lambda_{\mu_1\cdots\mu_{s-1}}$

$$\delta\varphi = \partial\Lambda \tag{24}$$

with $\Lambda' = 0$. While this gauge symmetry guarantees that the field propagates only free spin-s excitations, for $s \geq 3$ the gauge symmetry is constrained to trace-free parameters Λ. This unwelcome limitation may be avoided by sacrificing locality, at least in an intermediate stage, and recovering it by introducing additional auxiliary fields. In fact one can rewrite the Fronsdal equation in an unconstrained form by introducing a rank-$(s-3)$ compensator field α transforming on (unconstrained) gauge transformations (24) as $\delta\alpha = \Lambda'$, in the following way

$$\mathcal{F} = \partial^3 \alpha \tag{25}$$

This equation is invariant under the unconstrained gauge transformations (24) because the variation of α exactly cancels the variation of the Fronsdal tensor. Most important, the system φ, α can be cast in a (local) Lagrangian form. By the partial gauge fixing condition $\alpha = 0$ one obtains the original Fronsdal's equation (23).

There exists a generalization $\mathcal{F}^{(n)}$ of the Fronsdal differential operator, which is gauge invariant for n large enough. It is given in terms of the recursive formula [4, 11]

$$\mathcal{F}^{(n+1)} = \mathcal{F}^{(n)} + \frac{1}{(n+1)(2n+1)} \frac{\partial^2}{\Box} \mathcal{F}^{(n)\prime} - \frac{1}{n+1} \frac{\partial}{\Box} \partial \cdot \mathcal{F}^{(n)} \tag{26}$$

with $\mathcal{F}^{(0)} = \Box\varphi$. So, in particular,

$$\mathcal{F}^{(1)} \equiv \mathcal{F} = \Box\varphi - \partial\partial\cdot\varphi + \partial^2\varphi' \tag{27}$$

is the original Fronsdal operator.

The operators (26) are in general non-local and are non-divergenceless. Therefore they do not have the right form to represent our results in [6, 7], because the latter, like (16) and the following ones, can always be expressed as the product of a form factor times products of the projector (20), which are automatically conserved. However one can easily realize that, once one accepts the option of non-locality, there is large freedom in constructing linearized higher spin equations of motion, and the choice of $\mathcal{F}^{(n)}$ is far from unique.

The right object to make the connection with our results is the Einstein-like tensor

$$\mathcal{G}^{(n)} = \sum_{p=0}^{n} (-1)^p \frac{(n-p)!}{2^p n!} \eta^p \mathcal{F}^{(n)[p]} \tag{28}$$

where the superscript in square bracket denotes the number of time $\mathcal{F}^{(n)}$ has been traced, and η is the Minkowski metric. The association of φ with the spin s is as follows:

$$\begin{cases} s = 2n & s \ \text{even} \\ s = 2n-1 & s \ \text{odd} \end{cases} \tag{29}$$

The $\mathcal{G}^{(n)}$ tensor is divergenceless

$$\partial \cdot \mathcal{G}^{(n)} = 0 \tag{30}$$

The free (unconstrained) linearized equations of motion for φ are

$$\mathcal{G}^{(n)} = 0 \tag{31}$$

Once again, it can be shown that such an equation can be cast in local Lagrangian form, provided one introduces auxiliary fields (compensators). One can show that the Eq. (21) is precisely of the type (31).

On a general footing one can show that all the 2pt correlators of the two conserved currents can be expressed (in the usual concise notation) as follows

$$\sum_{l=0}^{\lfloor s/2 \rfloor} a_l \tilde{A}_l^{(s)}(k, n_1, n_2) \tag{32}$$

where the coefficients a_l are functions of k and m and

$$\tilde{A}_l^{(s)}(k, n_1, n_2) = (n_1 \cdot \pi^{(k)} \cdot n_2)^{s-2l} (n_1 \cdot \pi^{(k)} \cdot n_1)^l (n_2 \cdot \pi^{(k)} \cdot n_2)^l \tag{33}$$

On the other hand we can also show that

$$\begin{aligned} & k^2 (n_1 \cdot \pi^{(k)} \cdot n_2)^{s-2l} (n_1 \cdot \pi^{(k)} \cdot n_1)^l (n_2 \cdot \pi^{(k)} \cdot n_2)^l \\ & = \frac{1}{\binom{\lfloor s/2 \rfloor}{l}} \sum_{p=l}^{\lfloor s/2 \rfloor} \left(-\frac{1}{2}\right)^p \binom{p}{l} \frac{(2\lfloor s/2 \rfloor + D - 2p - 4)!!}{p!(2\lfloor s/2 \rfloor + D - 4)!!} \\ & \quad \times (n_1 \cdot \pi^{(k)} \cdot n_1)^p \, \tilde{\mathcal{G}}^{(n)[p]}(k, n_1, n_2) \end{aligned} \tag{34}$$

where $\tilde{\mathcal{G}}^{(n)}(k, n_1, n_2)$, the generalized Einstein symbol, is the Fourier transform of the differential operator that in $\mathcal{G}^{(n)}$ acts on φ. The latter is saturated with n_1 as far as the naked indices of $\mathcal{G}^{(n)}$ are concerned, while n_2 replace the symmetric indices of φ (for more details, see [7]).

In conclusion, any expression of the type (32), i.e. any conserved structure, can be expressed in terms of the generalized Einstein symbols $\tilde{\mathcal{G}}^{(n)}(k, n_1, n_2)$ and its traces. Thus any EA (or any eom) we obtain from our model, by integrating out matter, can be expressed in terms of the generalized Einstein tensor $\mathcal{G}^{(n)}$ and its traces preceded by a function of $\Box$ and the mass m of the model, with suitable multiples of the operator

$$\eta^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\Box}.$$

We can conclude that, although the quadratic effective action obtained by integrating out matter minimally coupled to a given higher spin source field is a highly non-local expression, its backbone is determined by the corresponding generalized Einstein differential operator.

5 More Results and Remarks

The main result of our work has been enunciated at the end of the previous section. However one can consider in addition several refinements and related problems.

5.1 Geometrization

To start with, the above results can be formulated in a more 'geometrical' way by introducing the Jacobi tensors $\mathcal{R}_{\mu_1\dots\mu_s\nu_1\dots\nu_s}$, [12]. The latter are one of the possible generalizations of the 4d Riemann tensor. They are defined by

$$\frac{1}{(s!)^2}(m^s\cdot\mathcal{R}^{(s)}\cdot n^s) = \sum_{l=0}^{s}\frac{(-1)^l}{s!(s-l)!l!}(m\cdot\partial)^{s-l}(n\cdot\partial)^l(m^l\cdot\varphi\cdot n^{s-l}) \qquad (35)$$

The tensors $\mathcal{R}^{(s)}$ are connected to the $\mathcal{F}^{(n)}$ as follows:

$$\mathcal{F}^{(n)} = \begin{cases} \frac{1}{\Box^{n-1}}\mathcal{R}^{(s)[n]} & s = 2n \\ \frac{1}{\Box^{n-1}}\partial\cdot\mathcal{R}^{(s)[n-1]} & s = 2n-1 \end{cases} \qquad (36)$$

where the traces in square brackets refer to the first set of indices. Using (28) one can replace the dependence on $\mathcal{G}^{(n)}$ in (34) with the dependence on $\mathcal{F}^{(n)}$, and the dependence on the latter by the dependence on the Jacobi tensors using (36). In this way we can express any EA or any eom in terms of $\mathcal{R}^{(s)}$ and traces (in the second set of indices) thereof.

For completeness it should be added that the Jacobi tensors are not the only possible generalizations of the 4d Riemann tensor.

5.2 Tadpoles and Seagulls

Above, in order to evaluate the two point correlators of conserved currents we computed only the bubble diagrams formed by two internal scalar or fermion lines and two vertices. In this way we found that several local terms (polynomials of the external momentum k) were not transverse. In such cases we recovered conservation by subtracting local counterterms from the EA. However it is in general not necessary to do so provided one takes into account not only the two-point bubble diagrams but also other diagrams such as tadpole and seagull ones. A tadpole diagram has one external a leg attached to an internal fermion loop, while seagull terms are similar but with two or more a legs attached to the same point of an internal fermion loop. For the quadratic action of the source fields both tadpole terms and seagull terms with two external legs may contribute. So far we have not taken them into account and, for higher spins, not knowing the full action, we are unable to compute the latter.

Let us see an example of gravity in any dimension. In this case we know the exact form of the action, therefore we can compute not only the tadpole terms but also the seagull contributions. When these contributions are taken into account the Ward identity for the 2pt correlator is not simply the transversality condition with respect to the external momentum k, but (in momentum space) takes the following form

$$k_\mu \tilde{T}^{\mu\mu\nu\nu}(k) = \left[-k^\nu \eta^{\mu\nu} + \frac{1}{2} k^\mu \eta^{\nu\nu}\right] \tilde{\Theta} \tag{37}$$

where $\tilde{\Theta}$ is a constant and represents the tadpole contribution, which, due to translation invariance, takes the form: $\tilde{\Theta}^{\mu\mu}(k) = \tilde{\Theta}\, \eta^{\mu\mu}$. Here repeated indices mean symmetrization, for instance ${}^{\mu\mu}$ stands for ${}^{(\mu_1\mu_2)}$.

From the explicit computation, the tadpole contribution turns out to be

$$\tilde{\Theta}^{\mu\mu}(k) = -2^{-2-d+\lfloor \frac{d}{2} \rfloor}\, i\, m^d \pi^{\frac{d}{2}} \Gamma\left(-\frac{d}{2}\right) \eta^{\mu\mu} \equiv \tilde{\Theta}\, \eta^{\mu\mu}, \tag{38}$$

and the seagull term

$$\tilde{T}^{\mu\mu\nu\nu}_{(s)}(k) = 2^{-3-d+\lfloor \frac{d}{2} \rfloor}\, i\, m^d \pi^{\frac{d}{2}} \Gamma\left(-\frac{d}{2}\right) (3\eta^{\mu\nu}\eta^{\mu\nu} - 2\eta^{\mu\mu}\eta^{\nu\nu})\,. \tag{39}$$

On the other hand the bubble diagram contributes two parts, the transverse part,

$$\begin{aligned}
&\tilde{T}^{\mu\mu\nu\nu}_{t}(k) = -\frac{1}{d(d+1)k^2} 2^{-2-d+\lfloor \frac{d}{2} \rfloor}\, i\, m^d \pi^{\frac{d}{2}} \Gamma\left(1-\frac{d}{2}\right) \\
&\left[\left(-8m^2 + (d+1)k^2 + {}_2F_1\left[1, -\frac{d}{2}, \frac{1}{2}, \frac{k^2}{4m^2}\right](8m^2 + (d-1)k^2)\right) \pi^{\mu\nu}\pi^{\mu\nu}\right. \\
&\left. + \left(-4m^2 + (d+1)k^2 + {}_2F_1\left[1, -\frac{d}{2}, \frac{1}{2}, \frac{k^2}{4m^2}\right](4m^2 - k^2)\right) \pi^{\mu\mu}\pi^{\nu\nu}\right]
\end{aligned} \tag{40}$$

and the non-transverse part

$$\tilde{T}^{\mu\mu\nu\nu}_{nt}(k) = -2^{-3-d+\lfloor \frac{d}{2} \rfloor}\, i\, m^d \pi^{\frac{d}{2}} \Gamma\left(-\frac{d}{2}\right) (\eta^{\mu\nu}\eta^{\mu\nu} - \eta^{\mu\mu}\eta^{\nu\nu})\,. \tag{41}$$

Taking formulas (38), (39), (40) and (41) and inserting them into (37) one can verify that the Ward identity is satisfied in any dimension d.

The result for spin 2 represents a useful suggestion for higher spins. As pointed out above, in the higher spin case we can easily compute the tadpoles, but we do not know the seagull terms, because in the elementary model $S_0 + S_1$ the external source appears only linearly. Imposing that the relevant Ward identity be satisfied may determine or, at least, considerably restrict the form of the interaction among the fermions and two external source fields.

5.3 Non-diagonal Correlators

So far we have been considering only 2pt correlators of a given current with itself, but there are also many non-vanishing 2pt of currents with different spins. Here we content ourselves with an example in 4d, the correlator of a spin 1 current with a spin 3 current. The transverse part is

$$\tilde{T}^{\rm t}_{\mu\nu\nu\nu} = k^4 \pi^{(k)}_{\nu\nu}\pi^{(k)}_{\mu\nu} \left(\frac{i}{5\pi^2}\left(\left(\frac{31}{180} - \frac{L_0}{12}\right) + \frac{2}{3}\frac{m^2}{k^2} - 4\frac{m^4}{k^4}\right) + \right.$$
$$\left. + \frac{iS}{5\pi^2}\left(-\frac{1}{6}\frac{1}{k} - \frac{1}{3}\frac{m^2}{k^3} + 4\frac{m^4}{k^5}\right)\right) \tag{42}$$

while the non-transverse part is

$$\tilde{T}^{\rm nt}_{\mu\nu\nu\nu} = \eta_{\nu\nu}\eta_{\mu\nu}\left(\tfrac{iL_2}{2\pi^2} m^4\right) \tag{43}$$

where $k = \sqrt{k^2}$ and

$$L_n = \frac{2}{\varepsilon} + \log\left(\frac{m^2}{4\pi}\right) + \gamma - \sum_{k=1}^{n}\frac{1}{k}, \qquad S = \sqrt{4m^2 - k^2}\,\csc^{-1}\left(\frac{2m}{k}\right)$$

This and analogous results mean that in the effective action there will be non-diagonal kinetic operators which entangle fields of different spins. This is a new factor of complexity along the way that hopefully will lead to a covariant classical higher spin action. On the other hand these results depend very much on the form of the currents we choose. Different choices lead to different results for the correlators and in some cases even to vanishing non-diagonal correlators. It is clearly important to determine the form of the currents that induces the simplest possible structure for the correlators. This research is under way.

6 Comments

In this paper we have reviewed the progress made in the effective action approach to higher spin theories. By computing, in a model of free Dirac fermions (or scalars) coupled to external (higher spin) fields, the 2pt correlators of conserved currents we have calculated the relevant one-loop effective action. This is quadratic in the fields and, to a large extent, non-local. It also contains a local part which can be interpreted as a classical quadratic action of the corresponding higher spin fields. Equivalently, it gives rise to the corresponding generalized (generally non-local, but covariant) Fronsdal equations. We have also reviewed the mechanism by which the non-locality can be reabsorbed by means of auxiliary fields, so as to transform the action to a local one. Returning to the effective action for a given higher spin field, we have also noticed that, although it is non-local, it can be broken down to pieces (preceded by a suitable form factor), each one of which is characterized by a differential operator which is a particular (generally non-local) version of the same Fronsdal (or Einstein) operator. Related to this we have shown how to carry out a complete geometrization of our results.

In summary, the one-loop effective action extracted from such simple models by integrating out matter, is strictly connected to the classical dynamics of the source fields. In other words, quantizing these simple models yields information about the dynamics of higher spin fields. Of course this is only a beginning. The real challenge now is to tackle the problem of interaction of higher spin fields. Carrying on our program, this means that we have to study the three point correlators of conserved currents. We are aware that it is not just a problem of evaluating an integral more complicated than (15). Before that a few preliminary problems must be investigated and understood. One such problem is the of role tadpole and seagull terms, which we have partly clarified above. Another important point is the choice of conserved currents that minimize the complexity of the calculations, as outlined above. An additional fundamental issue is the form of the gauge symmetry, which, in this paper, we have introduced only at the lowest level, see Eq. (24).

These are the visible problems within our present horizon, not forgetting that the final goal is a consistent theory with an infinite number of fields. This thought actually suggests another consideration. Any free field theory coupled to external sources via conserved currents can be treated in the same way as we have done with the scalar and fermion models, i.e. we can integrate out the fields in the original model and obtain a classical dynamics for the sources. It would seem that a theory is complete only when the field content of the original model coincides with the sources, otherwise when we quantize the initial model we excite always new dynamics. Perhaps this concept of completeness is what is in store for us beyond the present horizon.

Acknowledgements This research has been supported by the Croatian Science Foundation under the project No. 8946 and by the University of Rijeka under the research support No. 13.12.1.4.05. L.B., in particular, would like to thank the Croatian Science Foundation and Organizing Committee of the Varna Conference QTS10 & LT12 for their support on the occasion of the conference.

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Algebraic Structures in Exceptional Geometry

Martin Cederwall

Abstract Exceptional field theory (EFT) gives a geometric underpinning of the U-duality symmetries of M-theory. In this paper I give an overview of the surprisingly rich algebraic structures which naturally appear in the context of EFT. This includes Borcherds superalgebras, Cartan type superalgebras (tensor hierarchy algebras) and L_∞ algebras. This is the written version of a talk based mainly on Refs. [1–6].

Keywords Exceptional geometry · Superalgebras · Extended geometry

Duality symmetries in string theory/M-theory mix gravitational and non-gravitational fields. Manifestation of such symmetries calls for a generalisation of the concept of geometry. It has been proposed that the compactifying space (torus) is enlarged to accommodate momenta (representing momenta and brane windings) in modules of a duality group. This leads to *double geometry* [7–31] in the context of T-duality, and *exceptional geometry* [32–52] in the context of U-duality. These classes of models are special cases of *extended geometries*, and can be treated in a unified manner [4]. The duality group is in a certain sense present already in the uncompactified theory. It becomes "geometrised".

In the present paper, I will

- Describe the basics of extended geometry, with focus on the gauge transformations;
- Describe the appearance of Borcherds superalgebras and Cartan-type superalgebras (tensor hierarchy superalgebras);
- Indicate why L_∞ algebras provide a good framework for describing the gauge symmetries.
- Point out some questions and directions.

The focus will thus be on algebraic aspects, and less on geometric ones (Table 1).

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_3

Table 1 A list of U-duality groups

n	$E_{n(n)}$	R_1
3	$SL(3) \times SL(2)$	$(\mathbf{3}, \mathbf{2})$
4	$SL(5)$	**10**
5	$Spin(5,5)$	**16**
6	$E_{6(6)}$	**27**
7	$E_{7(7)}$	**56**
8	$E_{8(8)}$	**248**
9	$E_{9(9)}$	fund

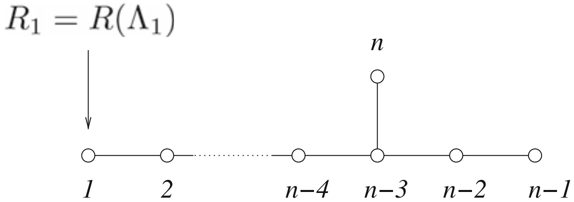

Fig. 1 The module R_1

Consider compactification from 11 to $11 - n$ dimensions on T^n. As is well known, fields and charges fall into modules of $E_{n(n)}$ (Fig. 1).

To be explicit, take $n = 7$ as an example. The gauge parameters ξ^M in **56** of E_7 decompose as:

$$\begin{array}{ccccccccc} \xi^m & & \lambda_{mn} & & \tilde{\lambda}_{mnpqr} & & \tilde{\xi}_{m,n_1\ldots n_7} & \leftarrow & \xi^M \\ 7 & + & 21 & + & 21 & + & 7 & = & 56 \end{array} \tag{1}$$

We recognise the parameters for diffeomorphisms, gauge transformations of the 3-form and dual 6-form and a parameter for "dual diffeomorphisms". The scalar fields are in the coset $E_{7(7)}/K(E_{7(7)}) = E_{7(7)}/(SU(8)/\mathbb{Z}_2)$. The dimension of coset is: $133 - 63 = 70$, and it is parameterized by

$$\begin{array}{ccccccc} g_{mn} & & C_{mnp} & & \tilde{C}_{mnpqrs} & \leftarrow & G_{MN} \\ 28 & + & 35 & + & 7 & = & 70 \end{array} \tag{2}$$

From the point of view of $N = 8$ supergravity in $D = 4$, this is the scalar field coset. Now it becomes a generalised metric. There are also mixed fields (generalised graviphotons): 1-forms in R_1, etc.

The situation for T-duality is simpler. Compactification from 10 to $10 - d$ dimensions gives the (continuous) T-duality group $O(d,d)$. The momenta are complemented with string windings to form the $2d$-dimensional module.

Note that the continuous duality group is not to be seen as a global symmetry. Discrete duality transformations in $O(d,d;\mathbb{Z})$ or $E_{n(n)}(\mathbb{Z})$ arise as symmetries in certain backgrounds, roughly as the mapping class group $SL(n;\mathbb{Z})$ arises as discrete isometries of a torus. The rôle of the continuous versions of the duality groups is analogous to that of $GL(n)$ in ordinary geometry (gravity).

One has to decide how tensors transform. The generic recipe is to mimic the Lie derivative for ordinary diffeomorphisms:

$$L_U V^m = U^n \partial_n V^m - \partial_n U^m V^n \ . \tag{3}$$

The first term is a transport term, and the second one a $\mathfrak{gl}$ transformation, with parameter in red.

In the case of U-duality, the role of GL is assumed by $E_{n(n)} \times \mathbb{R}^+$, and

$$\begin{aligned}\mathcal{L}_U V^M &= L_U V^M + Y^{MN}{}_{PQ}\partial_N U^P V^Q \\ &= U^N \partial_N V^M + Z^{MN}{}_{PQ}\partial_N U^P V^Q \ ,\end{aligned} \tag{4}$$

where $Z^{MN}{}_{PQ} = -\alpha_n P^M_{\text{adj}\,Q,}{}^N{}_P + \beta_n \delta^M_Q \delta^N_P = Y^{MN}{}_{PQ} - \delta^M_P \delta^N_Q$ projects on the adjoint of $E_{n(n)} \times \mathbb{R}$, so that the red factor becomes a parameter for an $\mathfrak{e}_n \oplus \mathbb{R}$ transformation.

The transformations form an "algebra" for $n \le 7$:

$$[\mathcal{L}_U, \mathcal{L}_V] W^M = \mathcal{L}_{[U,V]} W^M \ , \tag{5}$$

where the "Courant bracket" is $[U, V]^M = \frac{1}{2}(\mathcal{L}_U V^M - \mathcal{L}_V U^M)$, provided that the derivatives fulfil a "*section constraint*".

The section constraint ensures that fields locally depend only on an n-dimensional subspace of the coordinates, on which a $GL(n)$ subgroup acts. It reads $Y^{MN}{}_{PQ}\partial_M \ldots \partial_N = 0$, or

$$(\partial \otimes \partial)|_{\overline{R}_2} = 0 \ . \tag{6}$$

For $n \ge 8$ more local transformations, so called "ancillary transformations" [4] emerge, which are constrained local transformations in $\mathfrak{g}$ (Fig. 2 and Table 2).

The interpretation of the section condition is that the momenta locally are chosen so that they may span a linear subspace of cotangent space with maximal dimension, such that any pair of covectors p, p' in the subspace fulfil $(p \otimes p')|_{\overline{R}_2} = 0$.

The corresponding statement for double geometry is $\eta^{MN}\partial_M \otimes \partial_N = 0$, where η is the $O(d, d)$-invariant metric. The maximal linear subspace is a d-dimensional isotropic subspace, and it is determined by a pure spinor Λ. Once a Λ is chosen, the section condition can be written $\Gamma^M \Lambda \partial_M = 0$. An analogous linear construction can

Fig. 2 The module R_2

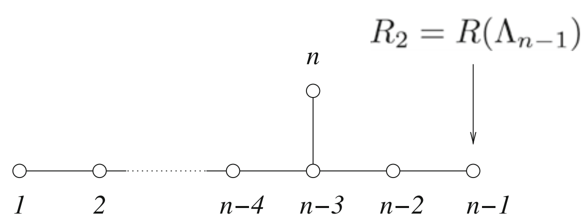

Table 2 A list of R_1 and R_2 for different E_n

n	R_1	R_2
3	$(\mathbf{3}, \mathbf{2})$	$(\overline{\mathbf{3}}, \mathbf{1})$
4	$\mathbf{10}$	$\overline{\mathbf{5}}$
5	$\mathbf{16}$	$\mathbf{10}$
6	$\mathbf{27}$	$\overline{\mathbf{27}}$
7	$\mathbf{56}$	$\mathbf{133}$
8	$\mathbf{248}$	$\mathbf{1} \oplus \mathbf{3875}$

be performed in the exceptional setting. The section condition in double geometry derives from the level matching condition in string theory. Locally, supergravity is recovered. Globally, non-geometric solutions are also obtained.

There is a universal form [1, 3, 4] of the generalised diffeomorphisms for any Kac–Moody algebra and choice of coordinate representation. Let the coordinate representation be $R(\lambda)$, for λ a fundamental weight dual to a simple root α (the construction can be made more general). Then

$$\sigma Y = -\eta_{AB} T^A \otimes T^B + (\lambda, \lambda) + \sigma - 1 \ , \tag{7}$$

where η is the Killing metric and σ the permutation operator, $\sigma(a \otimes b) = (b \otimes a)\sigma$.

This follows from the existence of a solution to the section constraint in the form of a linear space:

- Each momentum must be in the minimal orbit. Equivalently, $p \otimes p \in \overline{R(2\lambda)}$.
- Products of different momenta may contain $\overline{R(2\lambda)}$ and $\overline{R(2\lambda - \alpha)}$, where $R(2\lambda - \alpha)$ is the highest representation in the antisymmetric product. Expressing these conditions in terms of the quadratic Casimir gives the form of Y.

I will skip the detailed description of the generalised gravity. It effectively provides the local dynamics of gravity and 3-form, which are encoded by a vielbein $E_M{}^A$ in the coset $(E_{n(n)} \times \mathbb{R})/K(E_{n(n)})$ (Table 3).

Table 3 A list of compact subgroups

n	$E_{n(n)}$	$K(E_{n(n)})$
3	$SL(3) \times SL(2)$	$SO(3) \times SO(2)$
4	$SL(5)$	$SO(5)$
5	$Spin(5, 5)$	$(Spin(5) \times Spin(5))/\mathbb{Z}_2$
6	$E_{6(6)}$	$USp(8)/\mathbb{Z}_2$
7	$E_{7(7)}$	$SU(8)/\mathbb{Z}_2$
8	$E_{8(8)}$	$Spin(16)/\mathbb{Z}_2$
9	$E_{9(9)}$	$K(E_{9(9)})$

The T-duality case is described by a generalised metric in the coset $O(d,d)/(O(d) \times O(d))$, parametrised by the ordinary metric and B-field.

With some differences from ordinary geometry, one can go through the construction of connection, torsion, metric compatibility etc., and arrive at generalised Einstein's equations encoding the equations of motion for all fields. (This has been done for $n \leq 8$.)

For $n \geq 8$, the coset $E_{n(n)}/K(E_{n(n)})$ contains higher mixed tensors that do not carry independent physical degrees of freedom. They are removed by ancillary transformations that arise in the commutator between generalised diffeomorphisms [3, 4, 45, 48, 49].

One may introduce (local) supersymmetry. In the case of T-duality, the superspace is based on the fundamental representation of an orthosymplectic supergroup $OSp(d,d|2s)$. The exceptional cases are unexplored, but will be based on ∞-dimensional superalgebras [53].

The generalised diffeomorphisms do not satisfy a Jacobi identity. On general grounds, it can be shown that the "Jacobiator"

$$[[U, V, W] + \text{cycl} \neq 0\ , \tag{8}$$

but is proportional to $([U,V],W) + \text{cycl}$, where $(U,V) = \frac{1}{2}(\mathcal{L}_U V + \mathcal{L}_V U)$.

It is important to show that the Jacobiator in some sense is trivial. It turns out that $\mathcal{L}_{(U,V)} W = 0$ (for $n \leq 7$), and the interpretation is that it is a gauge transformation with a parameter representing reducibility (for $n \leq 6$). (The limits on n in the statements here are due to non-covariance of the derivative arising at some point in the tensor hierarchy, see below. I will not go into details.)

In double geometry, this reducibility is just the scalar reducibility of a gauge transformation: $\delta B_2 = d\lambda_1$, with the reducibility $\delta\lambda_1 = d\lambda'_0$.

In exceptional geometry, the reducibility turns out to be more complicated, leading to an infinite (but well defined) reducibility, containing the modules of tensor hierarchies, and providing a natural generalisation of forms (having connection-free covariant derivatives). One may ask why it does not reproduce only the finite reducibility of the gauge transformations of the 3-form and 6-form fields. These are of course contained in the tower of ghosts, but do not by themselves fill out modules of E_n. Insisting on having E_n modules at each level of reducibility means (given a solution to the section constraint) that the forms sooner or later need to be accompanied by some mixed tensors. Some examples of this phenomenon, including tables of the decompositions of some R_n's into $GL(n)$ representations are given in Appendix B of Ref. [43].

The reducibility continues, and there are ghosts at all levels > 0. The representations are those of a "tensor hierarchy", a sequence of representations R_n that for low n agrees with the representations of n-form gauge fields in the dimensionally reduced theory.

$$R_1 \overset{\partial}{\longleftarrow} R_2 \overset{\partial}{\longleftarrow} R_3 \overset{\partial}{\longleftarrow} \ldots \tag{9}$$

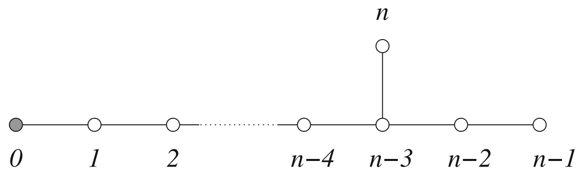

Fig. 3 Dynkinj diagram for $\mathscr{B}(E_n)$

Example, $n = 5$:

$$\mathbf{16} \xleftarrow{\partial} \mathbf{10} \xleftarrow{\partial} \overline{\mathbf{16}} \xleftarrow{\partial} \mathbf{45} \xleftarrow{\partial} \overline{\mathbf{144}} \xleftarrow{\partial} \ldots \tag{10}$$

$$16 - 10 + 16 - 45 + 144 - \ldots = 11 \tag{11}$$

(suitably regularised), which is the number of degrees of freedom of a pure spinor.

The representations $\{R_n\}_{n=1}^{\infty}$ agree with [54].

- The ghosts for a "pure spinor" constraint (a constraint implying an object lies in the minimal orbit);
- The positive levels of a Borcherds superalgebra $\mathcal{B}(E_n)$ (Fig. 3).

Indeed, the denominator appearing in the denominator formula for $\mathcal{B}(E_n)$ is identical to the partition function of a "pure spinor" [54].

$\mathcal{B}(D_n) \approx \mathfrak{osp}(n, n|2)$
$\mathcal{B}(A_n) \approx \mathfrak{sl}(n + 1|1)$

$$\cdots \xleftarrow{\partial} R_{-1} \xleftarrow{\partial} R_0 \xleftarrow{\partial} \underbrace{R_1 \xleftarrow{\partial} R_2 \xleftarrow{\partial} \ldots \xleftarrow{\partial} R_{8-n}}_{\text{covariant}} \xleftarrow{\partial} R_{9-n} \xleftarrow{\partial} R_{10-n} \xleftarrow{\partial} \cdots \tag{12}$$

The modules $R_1, \ldots, R_{8-n}$ behave like forms. The "exterior derivative" is connection-free (for a torsion-free connection), and there is a wedge product [43].

The modules show a symmetry: $R_{9-n} = \overline{R}_n$. There is another extension to negative levels that respects this symmetry, and seems more connected to geometry: tensor hierarchy algebras [2, 5].

In the classification of finite-dimensional superalgebras by Kac, there is a special class, "Cartan-type superalgebras". The Cartan-type superalgebra $W(n)$, which I prefer to call $W(A_{n-1})$, is asymmetric between positive and negative levels, and (therefore) not defined through generators corresponding to simple roots and Serre relations.

$W(A_{n-1})$ is the superalgebra of derivations on the superalgebra of (pointwise) forms in n dimensions.

Any operation $\omega \rightarrow \Omega \wedge \imath_V \omega$, where Ω is a form and V a vector, belongs to $W(A_{n-1})$. A basis is given by (Table 4)

Table 4 The level decomposition of $W(A_{n-1})$

level = 1	ι_a
0	$e^b \iota_a$
-1	$e^{b_1} e^{b_2} \iota_a$
-2	$e^{b_1} e^{b_2} e^{b_3} \iota_a$
...	...

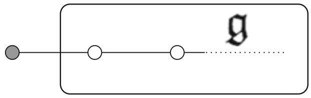

Fig. 4 Dynkin diagram for $\mathscr{B}(\mathfrak{g})$ and $W(\mathfrak{g})$

A subalgebra $S(A_{n-1})$ contains traceless tensors. The positive levels agree with $\mathcal{B}(A_{n-1}) \approx \mathfrak{sl}(n|1)$. Note that the representations of torsion and torsion Bianchi identity appear at levels -1 and -2.

In spite of the absence of a Cartan involution, there is a way to give a systematic Chevalley–Serre presentation of the superalgebra, based on the same Dynkin diagram as the Borcherds superalgebra [5] (Fig. 4).

The construction can be extended to $W(D_n)$, and, most interestingly, $W(E_n)$ (and the corresponding $S(\mathfrak{g})$). The statements about torsion and Bianchi identities remain true (but we still lack a good geometric argument).

Back to the Jacobi identity. Expressed in terms of a fermionic ghost in R_1,

$$[[c, c], c] \neq 0 \ . \tag{13}$$

How is this remedied? The most general formalism for gauge symmetries is the Batalin–Vilkovisky formalism, where everything is encoded in the master equation $(S, S) = 0$.

If transformations are field-independent, one may consider the ghost action consistently. An L_∞ algebra is a (super)algebraic structure which provides a perturbative solution to the master equation.

Let C denote *all* ghosts. Then the master equation states the nilpotency of a transformation

$$\delta C = (S, C) = \partial C + [C, C] + [C, C, C] + [C, C, C, C] + \ldots \tag{14}$$

The identities that need to be fulfilled are:

$$\begin{aligned} &\partial^2 C = 0 \ , \\ &\partial[C, C] + 2[\partial C, C] = 0 \ , \\ &\partial[C, C, C] + 2[[C, C], C] + 3[\partial C, C, C] \ , \\ &\ldots \end{aligned} \tag{15}$$

Assuming $\partial c = 0$, the non-vanishing of $[[c, c], c]$ can be compensated by the derivative of an element in R_2 (representing reducibility). One needs to introduce a 3-bracket

$$[c, c, c] \in R_2\ . \tag{16}$$

Then, there are more identities to check.

For double field theory, a 3-bracket is enough [55].

For exceptional field theory, there are signs, that one will in fact obtain arbitrarily high brackets [6]. There are also other issues concerning the non-covariance outside the "form window". I will not go into detail.

In conclusion, the area has rich connections to various areas of pure mathematics, some of which are under investigation:

- Group theory and representation theory
- Minimal orbits
- Superalgebras
- Cartan-type superalgebras
- Infinite-dimensional (affine, hyperbolic,...) Lie algebras
- Geometry and generalised geometry
- Automorphic forms
- L_∞ algebras
- ...

There are many open questions:

- Can the formalism be continued to $n > 9$? The transformations work for e.g. E_{10} [4], and there seems to be no reason (other than mathematical difficulties) that it stops there. Is there a connection to the "E_{10} proposal" [56] with emergent space?
- *Geometry from algebra?* What is the precise geometric relation between the tensor hierarchy algebra and the generalised diffeomorphisms?
- *Superspace/supergeometry?* And some formalism generalising that of pure spinor superfields, that manifests supersymmetry?
- *The section constraint:* Can it be lifted, or dynamically generated?
- What can be learnt about the full string theory/M-theory?
- . . . ?

Thank you for your attention.

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Duality in the Sachdev-Ye-Kitaev Model

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This work is dedicated to the memory of Joe Polchinski

Abstract In this article, starting from a review of basic aspects of the Sachdev-Ye-Kitaev (SYK) model in the large N limit, we discuss at non-linear-level the deduction of the zero mode effective Schwarzian action, featuring emergent finite reparametrization symmetry. We then discuss the question of identifying the bulk space-time of the SYK model. We explain the need for non-local (Radon-type) transformations on external legs of n-point Green's functions, leading to a dual theory with Euclidean AdS signature with additional leg-factors. We show that the SYK spectrum and the bi-local propagator can be obtained from a Horava-Witten type compactification of a three dimensional model.

Keywords large N expansion · Holographic correspondence · Collective field theory

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_4

1 Introduction

The Sachdev-Ye-Kitaev (SYK) model [14, 15, 25] has been recently attracting a lot of attention as a useful laboratory to understand the origins of the AdS/CFT duality. The model is notable for several reasons. It features an emergent conformal reparametrization invariance in the IR limit (i.e. the strong coupling limit $J|t| \gg 1$). The out-of-time-order correlators exhibit quantum chaos, with a maximal Lyapunov exponent characteristic of black holes, thus providing an example of the butterfly effect [15].

Once we deviate away from the critical IR point, the kinetic term sources the breaking of the conformal symmetry. In the following we first summarize the treatment of the dynamical symmetry mode and its coupling to matter developed in [11, 12]. The effective action describing the symmetry mode was originally suggested by Kitaev as given by a Schwarzian derivative [15] and confirmed at the quadratic level by Maldacena and Stanford [19]. We will explain the full non-linear derivations of the Schwarzian effective action of the symmetry mode with full implementation of reparametrization symmetry.

We then consider the important question regarding the duality of the SYK model and the identification of the dual spacetime. The Large N representation of the theory is based on bi-local composite variables which have in general been proposed as a vehicle for AdS holography [8]. For the present one dimensional theory this provides a two-dimensional representation: in terms of the center-of-mass and relative coordinates, one sees a Lorentzian AdS_2 or dS_2. The SYK eigenfunctions are derived exactly in the IR limit, which are seen to correspond to Lorentzian AdS_2 or dS_2 wave functions. This presents a conundrum, since we expect that the dual theory of the Euclidean SYK model should have Euclidean spacetime. We describe a resolution of this question. As we explain below, there is a need for a non-local space-time transformation leading to the Euclidean AdS_2 bulk space-time picture. This transformation is such that it brings Lorentzian wave functions into those of Euclidean AdS_2. At the same time, this transformation leads to additional factors which morally resemble the leg pole factors of the $c = 1$ matrix model necessary to relate the collective field to the usual tachyon field of the dual $2D$ string theory and reproduce the S-Matrix. We speculate that these factors incorporate the coupling of additional bulk states similar to the discrete states of $2D$ string theory. Finally we discuss the highly nontrivial matter spectrum of the model. We show that this can be realized as a Kaluza–Klein reduction of a three dimensional model where the additional dimension is an interval, similar to Horava-Witten compactification. Perhaps more significantly, the $3D$ propagator between points at the center of the interval exactly reproduces the SYK bilocal propagator.

2 Overview of SYK

In this section, we will give a brief review of the Large N formalism and results as developed in [11, 12]. In the IR limit this framework is used for solving for the spectrum, and study of correlation functions. We will describe the emergence of reparametrization symmetry and will give a details on the nonlinear derivation of the zero-mode effective action, given by the Schwarzian derivative.

The Sachdev-Ye-Kitaev model [14, 15] is a quantum mechanical many body system with all-to-all interactions on fermionic N sites ($N \gg 1$), represented by the Hamiltonian

$$H = \frac{1}{4!} \sum_{i,j,k,l=1}^{N} J_{ijkl}\, \chi_i\, \chi_j\, \chi_k\, \chi_l\,, \tag{1}$$

where χ_i are Majorana fermions, which satisfy $\{\chi_i, \chi_j\} = \delta_{ij}$. The coupling constant J_{ijkl} are random with a Gaussian distribution with width J. The generalization to analogous q-point interacting model is straightforward [15, 19]. After the disorder averaging for the random coupling J_{ijkl}, there is only one effective coupling J in the effective action. The model is usually treated by replica method. One does not expect a spin glass state in this model [25] so that we can restrict to the replica diagonal subspace. The Large N theory is simply represented through a (replica diagonal) bi-local collective field:

$$\Psi(t_1, t_2) \equiv \frac{1}{N} \sum_{i=1}^{N} \chi_i(t_1)\chi_i(t_2)\,, \tag{2}$$

where we have suppressed the replica index. The corresponding path-integral is

$$Z = \int \prod_{t_1,t_2} \mathcal{D}\Psi(t_1, t_2)\; \mu[\Psi]\, e^{-S_{\mathrm{col}}[\Psi]}\,, \tag{3}$$

where S_{col} is the collective action:

$$S_{\mathrm{col}}[\Psi] = \frac{N}{2} \int dt \left[\partial_t \Psi(t, t')\right]_{t'=t} + \frac{N}{2} \operatorname{Tr} \log \Psi - \frac{J^2 N}{2q} \int dt_1 dt_2\, \Psi^q(t_1, t_2)\,. \tag{4}$$

Here the trace term comes from a Jacobian factor due to the change of path-integral variable, and the trace is taken over the bi-local time. One also has an appropriate order $\mathcal{O}(N^0)$ measure $\mu[\Psi]$. This action being of order N gives a systematic $1/N$ expansion, while the measure $\mu[\Psi]$ begins to contribute at one-loop level (in $1/N$). There is another formulation with the two bi-local fields: the fundamental fermion propagator $G(t_{12})$ and the self energy $\Sigma(t_{12})$. This is equivalent to the above formulation after elimination of $\Sigma(t_{12})$. In this article, we focus on this Euclidean time SYK model.

In the above action, the first linear term represents a conformal breaking term, while the other terms respect conformal symmetry. In the IR limit with strong coupling $|t|J \gg 1$, the collective action is reduced to the critical action

$$S_{\rm c}[\Psi] = \frac{N}{2}\,{\rm Tr}\log\Psi - \frac{J^2N}{2q}\int dt_1dt_2\,\Psi^q(t_1,t_2)\,, \tag{5}$$

which exhibits an emergent conformal reparametrization symmetry

$$\Psi(t_1,t_2) \to \Psi_f(t_1,t_2) = \Big|f'(t_1)f'(t_2)\Big|^{\frac{1}{q}}\,\Psi(f(t_1),f(t_2))\,, \tag{6}$$

with an arbitrary function $f(t)$. This symmetry is responsible for the appearance of zero modes in the strict IR critical theory. This problem was addressed in [11] with analog of the quantization of extended systems with symmetry modes. The above symmetry mode representing time reparametrization can be elevated to a dynamical variable through the Faddeev–Popov method, leading to a Schwarzian action for this variable [12], which was originaly proposed by Kitaev:

$$S[f] = -\frac{N\alpha}{24\pi J}\int dt\left[\frac{f'''(t)}{f'(t)} - \frac{3}{2}\left(\frac{f''(t)}{f'(t)}\right)^2\right], \tag{7}$$

where the coefficient $\alpha = -12\pi B_1\gamma$ with B_1 representing the strength of the first order correction, established in numerical studies of the free energy by Maldacena and Stanford [19], who also verified the action in the linearized approximation, the full non-linear evaluation was give in [12]. Similar evaluation was also recently given by Kitaev and Suh [17].

In the rest of this section, we summarize the basic steps entering the evaluation of the non-linear Schwarzian action. One starts with the fact that the kinetic term in the action (4) breaks the conformal symmetry, and therefore the effective action (in the leading order of $1/J$) for the IR breaking is given by

$$S[f] = \frac{N}{2}\int dt_1\Big[\partial_1\Psi_{0,f}(t_1,t_2)\Big]_{t_2=t_1}\,, \tag{8}$$

where Ψ_0 is the saddle-point solution for the critical action (5) given by

$$\Psi_0(t_1,t_2) = b\,\frac{{\rm sgn}(t_{12})}{|Jt_{12}|^{\frac{2}{q}}}\,, \tag{9}$$

with $t_{ij} \equiv t_i - t_j$ and q-dependent numerical coefficient b. Then, $\Psi_{0,f}$ is the general reparametrized solution according to the transformation (6). If we use subleading corrections (in terms of $1/J$) for the saddle-point solution, instead of Ψ_0, we will obtain $1/J$ corrections for the zero-mode effective action. The first subleading correction is discussed in [17], but in this article we focus only on the leading contribution, which leads to the Schwarzian action.

For the $q = 2$ model, the induced action (8) can be directly evaluated [11]. We first expand the reparametrized critical solution in the $t_1 \to t_2$ limit as

$$\Psi_{0,f}(t_1, t_2) = -\frac{1}{\pi J}\left(\frac{1}{|t_{12}|} + \frac{|t_{12}|}{12}\,\mathrm{Sch}(f(t_2), t_2) + \cdots\right), \tag{10}$$

where $\mathrm{Sch}(f(t), t)$ is defined by the inside of the square bracket of Eq. (7). We define our regularization scheme to eliminate the first term. Substituting this expansion into Eq. (8), this leads to

$$S[f] = -\frac{N}{24\pi J}\int dt\,\mathrm{Sch}(f(t), t)\,. \tag{11}$$

The evaluation for general q involves a non-trivial regularization of the source term [12]. This is implemented by replacing the delta function source by a series in terms of powers:

$$\delta'(t_{12}) \Rightarrow Q_s(t_1, t_2) \equiv (s - \tfrac{1}{2})\,\frac{6qB_1\gamma J^2}{b}\,\frac{\mathrm{sgn}(t_{12})}{|Jt_{12}|^{2-\frac{2}{q}+2s}} + \mathcal{O}\big((s - \tfrac{1}{2})^2\big)\,. \tag{12}$$

The leading contribution is given by $s \to 1/2$ limit. Then, the indued action is given by

$$S[f] = -\frac{N}{2}\lim_{s\to\frac{1}{2}}\int dt_1 dt_2\,\Psi_{0,f}(t_1, t_2)\,Q_s(t_1, t_2)\,. \tag{13}$$

Let us explain this regularization scheme more. The original $\delta'(t_{12})$ source represents a UV source while the evaluation of the action started in the IR region. Consequently one is to represent the UV source as a series in terms of IR basis functions. The first term in the source expansion with $s = 1/2$ represents the leading correction to the critical conformal theory. Similar regularization and evaluation of the effective action is done by Kitaev and Suh [17]. One can indeed evaluate the integrals for any general reparametrization by expansion in series [12] or non-linearly as in [11]. After taking the $s \to 1/2$ limit, we find the Schwarzian action

$$S[f] = -\frac{N\alpha}{24\pi J}\int dt\left[\frac{f'''(t)}{f'(t)} - \frac{3}{2}\left(\frac{f''(t)}{f'(t)}\right)^2\right], \tag{14}$$

where $\alpha = -12\pi B_1\gamma$ with B_1 representing the strength of the first order correction.

The path-integral for this Schwarzian action is proven to be one-loop exact in [26]. Also the gravitational dual description of this Schwarzian action is archived in [9, 10, 20] based on the AdS_2 dilaton-gravity model of [2]. In the dual gravity theory the Schwarzian action describes the effective action of the boundary graviton in AdS_2.

3 Bulk Space-Time

Fluctuations around the critical IR saddle point background $\Psi_0(t_1, t_2)$ can be studied by expanding the bi-local field as [11]

$$\Psi(t_1, t_2) = \Psi_0(t_1, t_2) + \frac{1}{\sqrt{N}} \overline{\Psi}(t_1, t_2) , \tag{15}$$

where Ψ_0 is the IR large N saddle-point solution and $\overline{\Psi}$ is the fluctuation. At the quadratic level, we have a quadratic kernel $\mathcal{K}$. The diagonalization of this quadratic kernel is done by the eigenfunction $u_{\nu,\omega}$ and the eigenvalue $\widetilde{g}(\nu)$ as

$$\int dt_1' dt_2' \, \mathcal{K}(t_1, t_2; t_1', t_2') \, u_{\nu,\omega}(t_1', t_2') = \widetilde{g}(\nu) \, u_{\nu,\omega}(t_1, t_2) . \tag{16}$$

The quadratic kernel $\mathcal{K}$ can be diagonalized by using $SL(2, \mathcal{R})$ invariance. Consider the bi-local $SL(2, \mathcal{R})$ Casimir

$$\begin{aligned} C_{1+2} &= \big(\hat{D}_1 + \hat{D}_2\big)^2 - \frac{1}{2}\big(\hat{P}_1 + \hat{P}_2\big)\big(\hat{K}_1 + \hat{K}_2\big) - \frac{1}{2}\big(\hat{K}_1 + \hat{K}_2\big)\big(\hat{P}_1 + \hat{P}_2\big) \\ &= -(t_1 - t_2)^2 \, \partial_1 \partial_2 , \end{aligned} \tag{17}$$

with the $SL(2, \mathcal{R})$ generators $\hat{D} = -t\partial_t$, $\hat{P} = \partial_t$, and $\hat{K} = t^2\partial_t$. The eigenfunctions of the bi-local $SL(2, \mathcal{R})$ Casimir (17) are, due to the properties of the conformal block, given by the three-point function of the form

$$|t_{12}|^{2\Delta} \Big\langle \mathcal{O}_h(t_0) \, \mathcal{O}_\Delta(t_1) \, \mathcal{O}_\Delta(t_2) \Big\rangle = \frac{\mathrm{sgn}(t_{12})}{|t_{10}|^h |t_{20}|^h |t_{12}|^{-h}} . \tag{18}$$

The eigenvalues of the kernel, $\widetilde{g}(\nu)$, are given by the expression

$$\frac{1}{\widetilde{g}(\nu)} = -(q-1) \frac{\Gamma\left(\frac{3}{2} - \frac{1}{q}\right) \Gamma\left(1 - \frac{1}{q}\right) \Gamma\left(\frac{h}{2} + \frac{1}{q}\right) \Gamma\left(\frac{1}{2} + \frac{1}{q} - \frac{h}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{q}\right) \Gamma\left(\frac{1}{q}\right) \Gamma\left(\frac{3}{2} - \frac{1}{q} - \frac{h}{2}\right) \Gamma\left(1 - \frac{1}{q} + \frac{h}{2}\right)} \tag{19}$$

where we have defined

$$h \equiv \nu + 1/2 \tag{20}$$

This equation becomes simpler for $q = 4$,

$$\widetilde{g}^{q=4}(\nu) = -\frac{2\nu}{3} \cot\left(\frac{\pi\nu}{2}\right) . \tag{21}$$

The SYK quadratic kernel $\mathcal{K}$ is a function of the bi-local $SL(2, \mathcal{R})$ Casimir acting on fermions which have conformal dimension Δ. This means that the three-point function $\left\langle \mathcal{O}_h(t_0)\, \mathcal{O}_\Delta(t_1)\, \mathcal{O}_\Delta(t_2) \right\rangle$ is also the eigenfunction of the SYK quadratic kernel. For the investigation of dual gravity theory, it is more useful to Fourier transform from t_0 to ω by

$$\left\langle \widetilde{\mathcal{O}}_h(\omega)\, \mathcal{O}_\Delta(t_1)\, \mathcal{O}_\Delta(t_2) \right\rangle \equiv \int dt_0\, e^{i\omega t_0} \left\langle \mathcal{O}_h(t_0)\, \mathcal{O}_\Delta(t_1)\, \mathcal{O}_\Delta(t_2) \right\rangle$$
$$\propto \frac{\text{sgn}(t_{12})}{|t_{12}|^{2\Delta-\frac{1}{2}}}\, e^{i\omega(\frac{t_1+t_2}{2})} Z_\nu(|\tfrac{\omega t_{12}}{2}|)\,, \tag{22}$$

with $h = \nu + 1/2$ and

$$Z_\nu(x) \;=\; J_\nu(x) \,+\, \xi_\nu\, J_{-\nu}(x)\,, \qquad \xi_\nu \;=\; \frac{\tan(\pi\nu/2)+1}{\tan(\pi\nu/2)-1}\,. \tag{23}$$

The t_0 integral in the Fourier transform can be performed by decomposing the integration region into three pieces. The complete set of ν can be understood from the representation theory of the conformal group, as discussed recently in [16]. We have the discrete modes $\nu = 2n + 3/2$ with $(n = 0, 1, 2, \cdots)$ and the continuous modes $\nu = ir$ with $(0 < r < \infty)$. Adjusting the normalization, we define our eigenfunctions by

$$u_{\nu,\omega}(t,\hat{z}) \;\equiv\; \text{sgn}(\hat{z})\, \hat{z}^{\frac{1}{2}}\, e^{i\omega t}\, Z_\nu(|\omega\hat{z}|)\,. \tag{24}$$

which have normalization condition

$$\int_{-\infty}^{\infty} \frac{dt}{2\pi} \int_0^{\infty} \frac{d\hat{z}}{\hat{z}^2}\, u^*_{\nu,\omega}(t,\hat{z})\, u_{\nu',\omega'}(t,\hat{z}) \;=\; N_\nu\, \delta(\nu-\nu')\delta(\omega-\omega')\,, \tag{25}$$

with

$$N_\nu \;=\; \begin{cases} (2\nu)^{-1} & \text{for } \nu = 3/2 + 2n \\ 2\nu^{-1}\sin\pi\nu & \text{for } \nu = ir\,. \end{cases} \tag{26}$$

Here we used the change of the coordinates by

$$t \;\equiv\; \frac{t_1+t_2}{2}\,, \qquad \hat{z} \;\equiv\; \frac{t_1-t_2}{2}\,. \tag{27}$$

The bi-local $SL(2,\mathcal{R})$ Casimir can be seen to take the form of a Laplacian of Lorentzian two dimensional Anti de-Sitter or de-Sitter space (in this two dimensional case they are characterized by the same isometry group SO(2,1) or SO(1,2)). Under the canonical identification with AdS

$$ds^2 \;=\; \frac{-dt^2 + d\hat{z}^2}{\hat{z}^2}\,, \tag{28}$$

it equals

$$C_{1+2} = \hat{z}^2(-\partial_t^2 + \partial_{\hat{z}^2}) \,. \tag{29}$$

Consequently the SYK eigenfunctions should be compared with known AdS_2 or dS_2 basis wave functions. Note that the Bessel function Z_ν (23) are not the standard normalizable modes used in quantization of scalar fields in AdS_2: in particular they have rather different boundary conditions at the Poincare horizon. Another important property of this basis is that when viewed as a Schrodinger problem as in [24] it has a set of bound states, in addition to the scattering states. This will be discussed in detail below.

This leads one to try an identification with de-Sitter basis functions. In fact the bi-local SYK wave functions can be realized as a particular α-vacuum of Lorentzian dS_2 with a metric

$$ds^2 = \frac{-d\eta^2 + dt^2}{\eta^2} \,. \tag{30}$$

This can be obtained by the coordinate change (27) by replacing $\hat{z} \to \eta$. The Euclidean (Bunch–Davies [4]) wave function of a massive scalar field is given by

$$\phi_\omega^E(\eta)\, e^{i\omega t} \,, \tag{31}$$

with

$$\phi_\omega^E(\eta) = \eta^{\frac{1}{2}}\, H_\nu^{(2)}(|\omega|\eta) \,, \qquad \nu = \sqrt{\frac{1}{4} - m^2} \,, \tag{32}$$

where $H_\nu^{(2)}$ is the Hankel function of the second kind. Since the t-dependence is always like $e^{i\omega t}$, in the following we will focus only on the η dependence. The α-vacuum wave function is defined by Bogoliubov transformation from this Euclidean wave function [1, 22] as

$$\begin{aligned} \phi_\omega^\alpha(\eta) &\equiv N_\alpha\Big[\phi_\omega^E(\eta) + e^\alpha \phi_\omega^{E*}(\eta)\Big] \\ &= N_\alpha\, \eta^{\frac{1}{2}}\Big[H_\nu^{(2)}(|\omega|\eta) + e^\alpha H_\nu^{(1)}(|\omega|\eta)\Big] \,, \end{aligned} \tag{33}$$

where

$$N_\alpha = \frac{1}{\sqrt{1 - e^{\alpha+\alpha^*}}} \,, \tag{34}$$

and α is a complex parameter. Now let us consider a possibility of α-vacuum with

$$\alpha = i\pi\left(\nu + \frac{1}{2}\right) = i\pi h \,. \tag{35}$$

With this choice of α, using the definition of the Hankel functions

$$H_\nu^{(1)}(x) = \frac{J_{-\nu}(x) - e^{-i\pi\nu} J_\nu(x)}{i \sin(\pi\nu)}, \qquad H_\nu^{(2)}(x) = \frac{J_{-\nu}(x) - e^{i\pi\nu} J_\nu(x)}{-i \sin(\pi\nu)}, \tag{36}$$

one can rewrite the α-vacuum wave function as

$$\phi_\omega^\alpha(\eta) = \left(\frac{2\,\eta^{\frac{1}{2}}}{1+\xi_\nu\, e^{-i\pi\nu}}\right) Z_\nu(|\omega|\eta), \tag{37}$$

where Z_ν is defined in Eq.(23). After excluding the η-independent part of the wave function, we can write the η-dependent part as

$$\phi_\omega^\alpha(\eta) = \eta^{\frac{1}{2}}\, Z_\nu(|\omega|\eta). \tag{38}$$

This wave function agrees with the eigenfunction of the SYK quadratic kernel (24) after the identifications of $\eta = (t_1 - t_2)/2$ and $t = (t_1 + t_2)/2$.

Therefore, the SYK bi-local propagator has a natural interpretation as a two-point function in this Lorentzian dS_2 α-vacuum. Due to this observation, one might attempt to claim that the dual gravity theory of the SYK model is given by Lorentzian dS_2 space. However, there is a critical issue in this claim. Apart from the Lorentzian signature in this metric (30), we still have a discrepancy in the exponent of the partition function (3) with a factor of "i". Namely, if the dual gravity theory (higher spin gravity or string theory) is Lorentzian dS_2, it must have

$$Z = \int \mathcal{D}h_n\, \mathcal{D}\Phi_m\, \exp\left[i\Big(S_{\text{grav}}[h,\Phi] + S_{\text{matter}}[h,\Phi]\Big)\right], \tag{39}$$

where we collectively denote the graviton and other "higher spin" gauge fields by h_n and the dilaton and other matter fields by Φ_m. Hence the agreement of the SYK bi-local propagator

$$\mathcal{D}_{\text{SYK}}(t_1,t_2;t_1',t_2') = \Big\langle \overline{\Psi}(t_1,t_2)\overline{\Psi}(t_1',t_2')\Big\rangle = \sum_{m=0}^{\infty} G_{p_m}(t_1,t_2;t_1',t_2'), \tag{40}$$

with a dS_2 propagator

$$\mathcal{D}_{\text{dS}}(\eta,t;\eta',t') = \frac{1}{i}\sum_{m=0}^{\infty}\Big\langle \Phi_m(\eta,t)\Phi_m(\eta',t)\Big\rangle = \frac{1}{i}\sum_{m=0}^{\infty} G_m(\eta,t;\eta',t'), \tag{41}$$

is only up to the factor i. Namely, even if we have a complete agreement of G_{p_m} with G_m by identifying the coordinates by (27) (with a replacement of $\hat{z} \to \eta$), there is a problem with the signature (i.e. the discrepancy of the factor i). For higher point functions, the same i-problem proceeds due to the i factors coming from the propagator and each vertex.

To conclude, for the Euclidean SYK model under consideration, one needs a dual gravity theory to be in the hyperbolic plane H_2 (i.e. Euclidean AdS_2) for the matching of n-point functions.

To reach such an Euclidean bulk dual description, we employ a nonlocal map constructed so that it brings the SYK eigenfunctions (as given on bi-local space-time) to the standard eigenfunctions of the $EAdS_2$ Laplacian. The need for a non-local transform on external legs appears to be characteristic of collective theory (which as a rule contains a minimal set of physical degrees of freedom). For the Bi-local Vectorial/Higher Spin duality in higher dimension such maps were constructed in several papers (see for example [18]). The present $d = 1$ map is even simpler, it will be seen to take the form of the well known H^2 Radon transform (a related suggestion was made in [19]).[1]

Let us describe the simple method of construction, based on a canonical transformation, from the bi-local phase space;(t_1, p_1), (t_2, p_2) to $EAdS_2$ (τ, p_τ), (z, p_z) phase space.

For this,one equates the $SL(2, \mathcal{R})$ generators

$$\hat{J}_{1+2} = \hat{J}_{\text{EAdS}} . \tag{42}$$

The bi-local conformal generators are

$$\hat{D}_{1+2} = t_1\, p_1 + t_2\, p_2 , \qquad \hat{P}_{1+2} = -p_1 - p_2 , \qquad \hat{K}_{1+2} = -t_1^2\, p_1 - t_2^2\, p_2 , \tag{43}$$

and the $EAdS_2$ generators are given by

$$\hat{D}_{\text{EAdS}} = \tau\, p_\tau + z\, p_z , \qquad \hat{P}_{\text{EAdS}} = -p_\tau , \qquad \hat{K}_{\text{EAdS}} = (z^2 - \tau^2)\, p_\tau - 2\tau z\, p_z , \tag{44}$$

where we defined $p_x \equiv -\partial_x$, with $(x = t_1, t_2, \tau$ or $z)$. Equating the generators, we can determine the map is uniquely given by:

$$\tau = \frac{t_1\, p_1 - t_2\, p_2}{p_1 - p_2} , \quad p_\tau = p_1 + p_2 , \quad z^2 = -\left(\frac{t_1 - t_2}{p_1 - p_2}\right)^2 p_1 p_2 , \quad p_z^2 = -4 p_1 p_2 . \tag{45}$$

One can see that the canonical commutators are preserved under the transform (at least classically, i.e. in terms of the Poisson bracket). Namely, $[\tau, p_\tau] = [z, p_z] = 1$ and others vanish provided that $[t_i, p_j] = \delta_{ij}$, with $(i, j = 1, 2)$. Hence, we conclude the map is canonical transformation, which is also a point transformation in momentum space. For the kernel which implements this momentum space correspondence we can take

[1] The first appearance of Radon type transforms in identifying holographic space-time was seen in the $c = 1$ / $D = 2$ string correspondence. The transformation maps the eigenvalue density field of the $c = 1$ matrix model to the tachyon field in a 2D (black hole) space-time (see Eq.(90)). Related maps from the collective field or fermions to fields in a black hole background have been proposed which are also possibly related to Radon transforms. As a transformation to EAdS (from dS) this transform was introduced in [3].

$$\mathcal{R}(p_1, p_2; p_\tau, p_z) = \frac{\delta(p_\tau - (p_1 + p_2))}{\sqrt{p_z^2 + 4p_1 p_2}} . \tag{46}$$

Through Fourier transforming all momenta to corresponding coordinates, the associated coordinate space kernel is seen to be related to the well known Circular Radon transform (47) given by

$$\big[\mathcal{R}f\big](\eta, t) = 2\eta \int_{t-\eta}^{t+\eta} d\tau \int_0^\infty \frac{dz}{z}\, \delta\Big(\eta^2 - (\tau - t)^2 - z^2\Big)\, f(\tau, z) , \tag{47}$$

where the resulting function $[\mathcal{R}f](\eta, t)$ is understood as a function on the Lorentzian dS_2 (30).

In particular, we evaluate the transformation of (unit-normalized) $EAdS_2$ wave functions

$$\overline{\phi}_{EAdS_2}(\tau, z) = \alpha_\nu\, z^{\frac{1}{2}}\, e^{-i\omega\tau}\, K_\nu(|\omega| z) , \tag{48}$$

to obtain the relation:

$$\mathcal{R}\, \overline{\phi}^{(EAdS_2)}_{\omega,\nu}(\tau, z) = L(\nu)\, \overline{\psi}^{(dS_2)}_{\omega,\nu}(\eta, t) , \tag{49}$$

where $\overline{\phi}_{EAdS_2}$ and $\overline{\psi}_{dS_2}$ are the unit-normlized wave functions of Euclidean AdS_2 and Lorentzian dS_2, respectively, The extra factor respect to the unit-normlized wave functions are identified as a leg factor, defined by

$$L(\nu) \equiv (\text{Leg Factor}) = -2i\sqrt{\pi}\, \frac{\Gamma\left(\frac{1}{4} + \frac{\nu}{2}\right)}{\Gamma\left(\frac{3}{4} + \frac{\nu}{2}\right)} . \tag{50}$$

For the SYK case one will be employing then the inverse-Radon transform, which is expected to take us from the bi-local to $EAdS_2$ bulk space-time. The relevant expressions are

$$\mathcal{R}^{-1}\, \overline{\psi}^{(dS_2)}_{\omega,\nu}(\eta, t) = L^{-1}(\nu)\, \overline{\phi}^{(EAdS_2)}_{\omega,\nu}(\tau, z) . \tag{51}$$

for $\nu \neq 3/2 + 2n$, while for $\nu_n = 3/2 + 2n$ we have instead

$$\mathcal{R}^{-1}\, \overline{\psi}^{(dS_2)}_{\omega,\nu_n}(\eta, t) = \alpha'_{\nu_n} z^{1/2} e^{-ik\tau} I_{\nu_n}(|k| z) \tag{52}$$

where

$$\alpha'_{\nu_n} = \left(\frac{2\nu_n}{\pi}\right)^{\frac{1}{2}} \frac{\Gamma(\frac{3}{4} + \frac{\nu_n}{2})}{\Gamma\left(\frac{1}{4} + \frac{\nu_n}{2}\right)} \tag{53}$$

We start with bi-local space propagator given by

$$G(t_1, t_2; t_1', t_2') \;\propto\; J^{-1} \int_{-\infty}^{\infty} d\omega \sum_{\nu} \frac{u^*_{\nu,\omega}(t_1, t_2) u_{\nu,\omega}(t_1', t_2')}{N_\nu[\widetilde{g}(\nu) - 1]}\,, \tag{54}$$

where $u_{\nu,\omega}$ are the eigenfunctions defined in Eq.(24). Here the summation over ν is a short-hand notation denotes the discrete mode sum and the continuous mode sum. Next, the inverse Radon transform (51) and (52) are applied. This transforms the bi-local/dS wave functions into the EAdS wave functions with a result

$$G(\tau, z; \tau', z') \;=\; 2\pi J^{-1} \int_{-\infty}^{\infty} d\omega \Bigg\{ \sum_{n=0}^{\infty} \frac{4 \sin \pi \nu_n}{\widetilde{g}(\nu_n) - 1}\, |L^{-1}(\nu_n)|^2\, \overline{\phi}^{\,*}_{\omega,\nu_n}(\tau, z)\, \overline{\phi}_{\omega,\nu_n}(\tau', z') + \int_0^{\infty} dr\, |L^{-1}(\nu)|^2\, \frac{\overline{\phi}^{\,*}_{\omega,\nu}(\tau, z)\, \overline{\phi}_{\omega,\nu}(\tau', z')}{\widetilde{g}(\nu) - 1}\Bigg|_{\nu = ir} \Bigg\}. \tag{55}$$

Here we have *defined* $\overline{\phi}_{\omega,\nu_n}(\tau, z)$ by

$$\overline{\phi}_{\omega,\nu_n}(\tau, z) = \alpha'_{\nu_n} z^{1/2} e^{-ik\tau} I_{\nu_n}(|k|z) \tag{56}$$

Note that $\overline{\phi}_{\omega,\nu_n}(\tau, z)$ is not really a EAdS wave function.

One can transform this representation (after contour integration) to a more recognizable form (from the EAdS viewpoint):

$$\begin{aligned} &G(\tau, z; \tau', z') \\ &= \frac{|zz'|^{\frac{1}{2}}}{2\pi J} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega(\tau-\tau')} \Bigg\{ \sum_{m=0}^{\infty} \frac{\Gamma(\frac{3}{4} + \frac{p_m}{2})\Gamma(\frac{3}{4} - \frac{p_m}{2})}{\Gamma(\frac{1}{4} + \frac{p_m}{2})\Gamma(\frac{1}{4} - \frac{p_m}{2})} \frac{p_m}{\widetilde{g}'(p_m)}\, K_{p_m}(|\omega|z^>) I_{p_m}(|\omega|z^<) \\ &\quad + \sum_{n=0}^{\infty} \frac{\Gamma^2(\frac{3}{4} + \frac{\nu_n}{2})}{\Gamma^2(\frac{1}{4} + \frac{\nu_n}{2})} \left(\frac{\nu_n}{\widetilde{g}(\nu_n) - 1}\right) \Big[2 I_{\nu_n}(|\omega|z^>) - I_{-\nu_n}(|\omega|z^>)\Big] I_{\nu_n}(|\omega|z^<) \Bigg\}. \end{aligned} \tag{57}$$

Here we still have the zero mode ($p_0 = 3/2$) coming from $\Gamma(\frac{3}{4} - \frac{p_0}{2}) = \infty$ which is projected out through our Schwarzian mode gauge fixing. In this expression, the Bessel function part of the first contribution in the RHS is the standard form for EAdS propagator, while the extra factor coming from the leg-factors can be possibly understood as a contribution from the naively pure gauge degrees of freedom as in the $c = 1$ model, in which case the second contribution in RHS represents the contribution from these modes.

4 3D Realization and $c = 1$

To leading order in strong coupling, the spectrum of the theory is given by the equation

$$\widetilde{g}(\nu) = 1 \tag{58}$$

The solutions of this transcendental equation will be denoted by p_m. The bilocal propagator at strong coupling can be then written as

$$\mathcal{G}(t, \hat{z}; t', \hat{z}') \sim -\frac{1}{J}|\hat{z}\hat{z}'|^{\frac{1}{2}} \sum_m \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \int \frac{d\nu}{N_\nu} \frac{Z_\nu^*(|\omega\hat{z}|) Z_\nu(|\omega\hat{z}'|)}{\nu^2 - p_m^2} (2p_m) R(p_m) \tag{59}$$

where $R(p_m)$ denotes the residue at the pole $\nu = p_m$,

$$[R(\nu)]^{-1} = N_h \left[H_{-1+\frac{h}{2}+\frac{1}{q}} + H_{\frac{1}{2}-\frac{h}{2}-\frac{1}{q}} - H_{\frac{h}{2}-\frac{1}{q}} - H_{-\frac{1}{2}-\frac{h}{2}+\frac{1}{q}} \right] \tag{60}$$

where H_n denotes the Harmonic number, and

$$N_h = \frac{\left(\sin \pi h + \sin \frac{2\pi}{q}\right) \Gamma\left(\frac{2}{q}\right) \Gamma\left(2 - h - \frac{2}{q}\right) \Gamma\left(1 + h - \frac{2}{q}\right)}{\pi q \Gamma\left(3 - \frac{2}{q}\right)} \tag{61}$$

where, as before, we have defined $h = \nu + 1/2$.

This form of the propagator shows that the theory can be thought of an infinite number of fields living in AdS_2 or dS_2. However these fields cannot have conventional kinetic terms, as is clear from the nontrivial residue.

In [5, 7], we presented at 3D picture of the SYK theory, based on the fact that the non-trivial spectrum predicted by the model, which are solutions of $\widetilde{g}(p_m) = 1$ with ($m = 0, 1, 2, \ldots$) can be reproduced through Kaluza–Klein mechanism in one higher dimension.

This picture is more natural in the AdS_2 interpretation of the bilocal space. The action of the theory is

$$\frac{1}{2} \int dt dz dx \sqrt{-g} \left[-g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - V(x) \Phi^2 \right] \tag{62}$$

where the background metric which describes the bilocal theory of the strong coupling SYK model is

$$ds^2 = |x|^{\frac{4}{q}-1} \left[\frac{1}{\hat{z}^2}(-dt^2 + d\hat{z}^2) + \frac{dx^2}{4|x|(1-|x|)} \right] \tag{63}$$

and the direction x lies in the interval $-1 < x < 1$. The space-time is then *conformal* to $AdS_2 \times S^1/Z_2$. The potential which appears in (62) is given by

$$V(x) = \frac{1}{|x|^{\frac{4}{q}-1}} \left[4\left(\frac{1}{q} - \frac{1}{4}\right)^2 - \frac{1}{4} + \frac{2V}{J(x)}\left(1 - \frac{2}{q}\right)\delta(x) \right] \tag{64}$$

where V is a constant to be determined below and

$$J(x) = \frac{|x|^{\frac{2}{q}-1}}{2\sqrt{1-|x|}} \tag{65}$$

We will impose Dirichlet boundary conditions at $x = \pm 1$,

$$\Phi(t, z, \pm 1) = 0 \tag{66}$$

while the delta function discontinuity in the potential determines the discontinuity at $x = 0$ to be

$$\mathrm{Lim}_{\epsilon \to 0} \left[|x|^{2/q} \sqrt{1 - |x|} \partial_x \Phi \right]_{-\epsilon}^{\epsilon} = \left(1 - \frac{2}{q}\right) V \Phi(t, z, 0) \tag{67}$$

In the following we will be interested in fields which are even under $x \to -x$. For such fields (67) implies

$$\left[x^{2/q} \partial_x \Phi \right]_{x=0} = \left(1 - \frac{2}{q}\right) \frac{V}{2} \Phi(t, z, 0) \tag{68}$$

Once we impose this we can restrict to $0 < x < 1$ and forget about the delta function.

It turns out that the Kaluza–Klein spectrum of this model is in exact agreement with the SYK spectrum. Furthermore the bulk propagator of this theory between points $(t, \hat{z}, x = 0)$ and $(t', \hat{z}', x' = 0)$ exactly reproduce the SYK propagator. In the following we will indicate how this happens for the simplest case $q = 4$, following [7]. The treatment for general q is entirely analogous and given in [5].

For $q = 4$ the background is $AdS_2 \times S^1/Z_2$. In this case it is conveninent to use coordinates $-L \leq y \leq L$, in terms of which the metric is

$$ds^2 = \frac{1}{\hat{z}^2}[-dt^2 + d\hat{z}^2] + dy^2 \tag{69}$$

and the potential is simply a delta function at $y = 0$. This metric is that of the near-horizon geometry of an extremal BTZ black hole.

To obtain the spectrum we decompose the field

$$\Phi(t, \hat{z}, y) = \sum_k \int d\omega \int \frac{d\nu}{N_\nu} e^{-i\omega t} \hat{z}^{1/2} Z_\nu(|\omega z|) f_k(y)\, \xi(\omega, \nu, k) \tag{70}$$

where the function $f_k(y)$ denote the even parity eigenfunction of the operator $-\partial_y^2 + V\delta(y)$ satisfying Dirichlet conditions at the ends of the interval. This is a standard Schrodinger problem. The solutions are

$$f(y) = \begin{cases} B_k \sin(k(y-L)) & (0 < y < L) \\ -B_k \sin(k(y+L)) & (-L < y < 0) \end{cases} \tag{71}$$

where k satisfies the equation

$$-\frac{2}{V} k = \tan(kL) . \tag{72}$$

If we choose $L = \pi/2$ and $V = 3$ we precisely reproduce the spectum of the SYK model $\widetilde{g}^{q=4}(\nu) = 1$ given by (21). The normalization factor B_k is given by

$$B_k = \sqrt{\frac{2k}{2kL - \sin(2kL)}} . \tag{73}$$

We can now proceed and calculate the propagator using the above mode expansion. The result is

$$G^{(0)}(\hat{z}, t, y; \hat{z}', t', y') = |\hat{z}\hat{z}'|^{\frac{1}{2}} \sum_k f_k(y) f_k(y') \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \int \frac{d\nu}{N_\nu} \frac{Z_\nu^*(|\omega\hat{z}|)\, Z_\nu(|\omega\hat{z}'|)}{\nu^2 - k^2} , \tag{74}$$

Let us now evaluate this propagator with $y = y' = 0$.

$$G^{(0)}(t, \hat{z}, 0; t', \hat{z}', 0) = -|\hat{z}\hat{z}'|^{\frac{1}{2}} \sum_{k=0}^{\infty} C(k) \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \int \frac{d\nu}{N_\nu} \frac{Z_\nu^*(|\omega\hat{z}|)\, Z_\nu(|\omega\hat{z}'|)}{\nu^2 - p_m^2} , \tag{75}$$

where we have defined

$$C(k) \equiv f_k(0) f_k(0) = B_k^2 \frac{k^2}{k^2 + (3/2)^2} = \frac{2k^3}{[k^2 + (3/2)^2][\pi k - \sin(\pi k)]} . \tag{76}$$

On the other hand the residue factor in the SYK propagator (60) simplifies for $q = 4$ with the result

$$R^{q=4}(p_m) = \frac{3p_m^2}{[p_m^2 + (3/2)^2][\pi p_m - \sin(\pi p_m)]} \tag{77}$$

Therefore we have the relation

$$C(p_m) = \frac{2p_m}{3} R(p_m) , \tag{78}$$

Consequently the 3d propagator (75) becomes exactly equal to the SYK propagator with the index k simply renamed to p_m. Integration over ν yields the expression

$$G^{(0)}(t,z,0;t',z',0) = \frac{1}{3}|zz'|^{\frac{1}{2}} \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-t')}\, R(p_m)\, \frac{Z_{-p_m}(|\omega|z^>)J_{p_m}(|\omega|z^<)}{N_{p_m}}\,. \tag{79}$$

This strong coupling propagator is of course divergent because of the contribution of $p_m = 3/2$ where $Z_{-3/2}$ diverges. This is in fact precisely the zero mode of reparametrization invariance. At finite coupling we do not expect this diveregence since the kinetic term in the SYK action breaks the symmetry. In the 3d picture, the background (63) needs to be corrected. For $q = 4$ this modification is simple,

$$ds^2 = \frac{1}{\hat{z}^2}[-dt^2 + d\hat{z}^2] + \left[1 + \frac{a}{\hat{z}}\right] dy^2 \tag{80}$$

where $a \sim \frac{1}{J}$. One can now proceed to solve the eigenvalue equation for the third dimension perturbatively in a. This results in a shift of the eigenvalue $k = \nu = 3/2$ to

$$\nu = \frac{3}{2} + \frac{a|\omega|}{6\pi}\left(2 + q_0^2\right) + \mathcal{O}(a^2)\,. \tag{81}$$

where q_m denotes the expectation value of the operator $-\partial_y^2 - V(y)$ in the state whose wavefunction is $f_{p_m}(y)$ and $p_0 = 3/2$. The shift in eigenvalue is in agreement with the SYK result obtained by Maldacena and Stanford. Furthermore if we now use the corrected eigenvalue to compute the contribution of this lowest mode to the proagator we get

$$G^{(0)}_{\text{zero-mode}}(t,z,0;t',z',0) = -\frac{9\pi}{4a}\frac{B_0^2}{(2+q_0^2)}|zz'|^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{d\omega}{|\omega|}\, e^{-i\omega(t-t')} J_{\frac{3}{2}}(|\omega z|) J_{\frac{3}{2}}(|\omega z'|)\,. \tag{82}$$

which is also in agreement with the "enhanced" propagator in the SYK model.

The situation for general q is exactly analogous and described in detail in [5]. In this case the eigenfunctions of the operator in the third dimension are hypergeometric functions, and we have not been able to obtain analytic expressions for $C(k)$. However, a numerical evaluation of $C(k)$ shows that the ratio $\frac{C(k)}{2kR(k)}$ is *independent of* k, and depends only on q. This establishes the agreement of the propagators.

The limit of large q is interesting. If one naively performs a $1/q$ expansion of the right hand side of (19) one finds that the only solution to the eigenvalue equation is the zero mode $h = 2$ or $\nu = 3/2$. This is, however, incorrect. In fact, for any q there are an infinite number of solutions. However, as q increases the residues of the poles for all $p_m \neq 3/2$ go to zero,

$$R(p_m) \to \frac{1}{q}\frac{4(2m^2+m)}{(2m^2+m-1)^2} + O(1/q^2)\,, \quad p_m \neq 3/2$$
$$R(3/2) = \frac{2}{3} - \frac{1}{q}\left(\frac{5}{2}+\frac{\pi^2}{3}\right) + O(1/q^2) \tag{83}$$

This means that to leading order at large q only the $p_m = 3/2$ mode contributes to the propagator.

Now, we will point out a similarity between the 3D picture of the SYK model [5, 7] and the $c = 1$ Liouville theory (2D string theory).

Consider the three dimensional propagator (74). After the (inverse) Radon transform and the contour integral for the continuous mode sum, the propagator is reduced to

$$\begin{aligned} &G^{(0)}_{\omega;-\omega}(z,y;z',y') \\ &= \frac{|zz'|^{\frac{1}{2}}}{4\pi}\sum_k f_k(y)f_k(y')\left\{\frac{\Gamma\left(\frac{3}{4}+\frac{k}{2}\right)\Gamma\left(\frac{3}{4}-\frac{k}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{k}{2}\right)\Gamma\left(\frac{1}{4}-\frac{k}{2}\right)}K_k(|\omega|z^>)I_k(|\omega|z^<)\right. \\ &\left.\quad + 2\sum_{n=0}^{\infty}\frac{\Gamma^2\left(\frac{3}{4}+\frac{\nu_n}{2}\right)}{\Gamma^2\left(\frac{1}{4}+\frac{\nu_n}{2}\right)}\left(\frac{\nu_n}{\nu_n^2-k^2}\right)I_{\nu_n}(|\omega|z^<)\left[2I_{\nu_n}(|\omega|z^>)-I_{-\nu_n}(|\omega|z^>)\right]\right\}. \end{aligned} \tag{84}$$

On the other hand, for the $c = 1$ matrix model / 2D string duality, the Wilson loop operator is related to the matrix eigenvalue density field ϕ by

$$W(t,\ell) \equiv \mathrm{Tr}\left(e^{-\ell M(t)}\right) = \int_0^\infty dx\, e^{-\ell x}\,\phi(t,x)\,. \tag{85}$$

The corresponding propagator was found by Moore and Seiberg [21] as

$$\left\langle w(t,\varphi)w(t',\varphi')\right\rangle = \int_{-\infty}^{\infty} dE\int_0^\infty dp\,\frac{p}{\sinh\pi p}\,\frac{\phi^*_{E,p}(t,\varphi)\phi_{E,p}(t',\varphi')}{E^2-p^2}\,, \tag{86}$$

with $\ell = e^{-\varphi}$ and the normalized wave function

$$\phi_{E,p}(t,\varphi) = \sqrt{p\sinh\pi p}\,e^{-iEt}\,K_{ip}(\sqrt{\mu}e^{-\varphi})\,. \tag{87}$$

After evaluating the p-integral as a contour integral, we obtain the propagator as

$$\begin{aligned} \left\langle w(t,\varphi)w(t',\varphi')\right\rangle = -\pi\int_{-\infty}^{\infty} dE\, e^{-iE(t-t')}&\left\{\frac{\pi E}{2\sinh\pi E}K_{iE}(\sqrt{\mu}e^{-\varphi^<})\,I_{iE}(\sqrt{\mu}e^{-\varphi^>})\right. \\ &\left.+\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{E^2+n^2}K_n(\sqrt{\mu}e^{-\varphi^<})\,I_n(\sqrt{\mu}e^{-\varphi^>})\right\}. \end{aligned} \tag{88}$$

The point we want to make here is that this 3D picture is completely parallel to the $c = 1$ Liouville theory (2D string theory) [23]. Namely, if we make a change of coordinate by $z = e^{-\varphi}$, then the φ-direction becomes the Liouville direction, while the y-direction (at least in the leading order of $1/J$) can be understood as the $c = 1$ matter direction. In this comparison, the τ-direction serves as an extra direction. Finally, the ν appearing in the SYK model is realized as a momentum k along the y-direction in the 3D picture (84). Therefore, we have the following correspondence between the $c = 1$ non-critical string and the 3D picture of the SYK model.

$$\begin{aligned} ie^{-\varphi} \quad &\Leftrightarrow \quad z\,, \\ -it \quad &\Leftrightarrow \quad y\,, \\ ip \quad &\Leftrightarrow \quad \nu\,, \\ iE \quad &\Leftrightarrow \quad k\,, \\ \sqrt{\mu} \quad &\Leftrightarrow \quad |\omega|\,, \end{aligned} \tag{89}$$

where the LHS corresponds to quantities in 2D string theory while the RHS is the 3D picture of the SYK model.

There is also an appearance of Radon type transforms in the context of the $c = 1/D = 2$ string correspondence in identifying the holographic space-time. In [13] one had a transformation from the collective to a 2D (black hole) space-time of the form

$$T(u, v) = \int_{-\infty}^{\infty} dt \int_0^{\infty} dx\, \delta\left(\frac{ue^{-t} + ve^{t}}{2} - x^2\right) \gamma(i\partial_t)\, \phi(t, x)\,, \tag{90}$$

where $T(u, v)$ is the tachyon field in the Kruskal coordinates representing the target space-time and $\phi(t, x)$ is related to the eigenvalue density field. This map is identical to the Radon map. It appears that non-local map are in general required in going from collective to bulk space-time.

Acknowledgements S.R.D. and A.J. thank the organizers of QTS10 for organizing a very stimulating meeting. We acknowledge helpful discussions with Robert de Melo Koch, Alexei Kitaev, Juan Maldacena, Gautam Mandal, Cheng Peng and Edward Witten on the topics of this work. We also thank Junggi Yoon for collaborations. This work of A.J. and K.S. are supported by the Department of Energy under contract DE-SC0010010. The work of KS is also supported by the Galkin Fellowship Award at Brown University. The work of S.R.D and A.G is partially supported by the National Science Foundation grant NSF-PHY-1521045.

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Implications of Hidden $\mathscr{N} = (0, 1)$ Super-Symmetry in $\mathscr{N} = (1, 1)$, $6D$ SYM Theory

Evgeny Ivanov

Abstract Using the harmonic superfield description of $\mathscr{N} = (1, 1)$ SYM theory, the list of possible candidate counterterms with the canonical dimensions d = 6, 8 and 10 is derived from hidden $\mathscr{N} = (0, 1)$ supersymmetry. The $d = 6$ and $d = 8$ counterterms are at least on-shell vanishing, that means the one- and two-loop UV finiteness of $\mathscr{N} = (1, 1)$ SYM theory. The explicit quantum calculations in fact demonstrate a stronger property of its *off-shell* one-loop finiteness.

Keywords Supersymmetry · Superspace · Gauge fields

1 Motivations and Contents

For the last years, maximally extended supersymmetric gauge theories (with 16 supersymmetries) are a subject of intensive study. In diverse space-time dimensions, these theories are realized as

$$(a)\ \mathscr{N} = 4,\ 4D\ SYM \quad (b)\ \mathscr{N} = (1, 1),\ 6D\ SYM \quad (c)\ \mathscr{N} = (1, 0),\ 10D\ SYM\,,$$

where SYM stands for "Super Yang–Mills". Among them, $\mathscr{N} = 4,\ 4D$ SYM theory is most renowned. It is UV finite and, perhaps, completely integrable at the quantum level [1]. On the other hand, $\mathscr{N} = (1, 1),\ 6D$ SYM is not renormalizable by formal counting (the coupling constant is dimensionful) but it was also found to exhibit various unique properties. In particular, it enjoys the so-called "dual conformal symmetry" like its $4D$ counterpart [2]. It gives the effective theory descriptions of some particular low energy sectors of string theory, like D5-brane dynamics. The full low-energy effective action of D5-brane is expected to be a non-abelian Born-Infeld-type generalization of the microscopic $\mathscr{N} = (1, 1)$ SYM action [3].

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_5

The $\mathcal{N}=(1,1)$ SYM theory is anomaly free [4], as opposed to $\mathcal{N}=(1,0)$ SYM, which makes unambiguous the perturbative quantum calculations in this extended supersymmetric $6D$ theory. The $\mathcal{N}=(1,1)$ and $\mathcal{N}=(1,0)$ SYM theories are analogs of $\mathcal{N}=8, 4D$ supergravity and its lower $\mathcal{N}$ descendants, as well as of their higher-dimensional cousins, which all are also non-renormalizable by formal counting. So the study of the quantum properties of these $6D$ gauge theories can shed more light on the quantum structure of diverse extended supergravities.

Recent explicit quantum calculations in $\mathcal{N}=(1,1)$ SYM (treated as a low-energy limit of type II superstrings) revealed a lot of cancelations of the UV divergencies which could not be expected in advance. The theory is UV-finite up to 2 loops, while at 3 loops only a single-trace counterterm of canonical dim 10 is required. The allowed double-trace counterterms do not appear [5–7]. To explain these peculiar features, one seemingly needs new non-renormalization theorems. As usual, the maximally supersymmetric off-shell formulations are required to clarify these issues.

However, maximum what one can achieve in $6D$ is off-shell $\mathcal{N}=(1,0)$ supersymmetry.[1] The most natural off-shell formulation of $\mathcal{N}=(1,0)$ SYM is achieved in harmonic $\mathcal{N}=(1,0), 6D$ superspace (HSS) [9, 10] as a generalization of $\mathcal{N}=2, 4D$ HSS [11, 12]. In HSS, the $\mathcal{N}=(1,1)$ SYM theory action can be presented as a sum (schematically) [$\mathcal{N}=(1,1)$ SYM] = [$\mathcal{N}=(1,0)$ SYM + $6D$ hypermultiplet], with the second hidden on-shell $\mathcal{N}=(0,1)$ supersymmetry.[2]

In order to reveal the possible structure of the effective action and candidate counterterms for $\mathcal{N}=(1,1)$ SYM theory, one should learn how to construct higher-dimension $\mathcal{N}=(1,1)$ invariants in terms of $\mathcal{N}=(1,0)$ superfields. In the "brute-force" method one starts with the appropriate $\mathcal{N}=(1,0)$ SYM invariant and then completes it to $\mathcal{N}=(1,1)$ invariant by adding the proper hypermultiplet terms. It is very cumbersome technically, though the life is somewhat simplified by the fact that for finding all admissible superfield counterterms it is sufficient to stay *on shell*.[3]

In [13] there was developed a new approach to constructing higher-dimension $\mathcal{N}=(1,1)$ invariants, based on the concept of on-shell $\mathcal{N}=(1,1)$ harmonic superspace [14]. It provides a systematic way of setting up candidate counterterms for quantum $\mathcal{N}=(1,1)$ SYM theory, as well as possible finite contributions to its superfield effective action.

The hidden supersymmetry in itself tells us nothing about the precise coefficients with which the various $\mathcal{N}=(1,1)$ invariants constructed in one or another way can enter the effective action. One can reproduce them from the superfield perturbation theory. The first steps towards this goal were recently undertaken in [15–17].

All these issues will be addressed in my talk. It basically follows Refs. [13, 15–19].

[1]The maximal off-shell supersymmetry with 16 supercharges is attainable in the "pure spinor" superfield formalism [8], but here we limit our attention to the standard superspaces.

[2]We use the term "hidden supersymmetry" for some historical reasons. Perhaps, "non-manifest" would be more appropriate.

[3]This just means that the on-shell vanishing counterterms can be absorbed into the microscopic action by a field redefinition. No equations of motion are assumed for the involved (super)fields.

2 6*D* Superspace Techniques

2.1 Basic Superspaces

- Standard $\mathcal{N} = (1, 0), 6D$ superspace [20] is defined as the following set of coordinates:

$$z = (x^M, \theta^a_i)\,, \quad M = 0, \ldots, 5\,,\ a = 1, \ldots, 4\,,\ i = 1, 2\,, \tag{1}$$

  with the Grassmann pseudoreal θ^a_i variables.
- Harmonic $\mathcal{N} = (1, 0), 6D$ superspace [9, 10] is constructed by adding $SU(2)$ harmonics to (1):

$$Z := (z, u) = (x^M, \theta^a_i, u^{\pm i})\,, \quad u^-_i = (u^+_i)^*, u^{+i}u^-_i = 1\,,\ u^{\pm i} \in SU(2)_R/U(1)\,. \tag{2}$$

- *Analytic* $\mathcal{N} = (1, 0), 6D$ superspace has half the number of Grassmann coordinates as compared to (2):

$$\zeta := (x^M_{(\mathrm{an})}, \theta^{+a}, u^{\pm i}) \subset Z\,, \quad x^M_{(\mathrm{an})} = x^M + \frac{i}{2}\theta^a_k \gamma^M_{ab}\theta^b_l u^{+k}u^{-l}, \quad \theta^{\pm a} = \theta^a_i u^{\pm i}\,. \tag{3}$$

  It is still closed under the action of $\mathcal{N} = (1, 0), 6D$ supersymmetry.

The basic differential operators in the analytic basis of 6*D* HSS read:

$$\begin{aligned}
&D^+_a = \partial_{-a}\,, \quad D^-_a = -\partial_{+a} - 2i\theta^{-b}\partial_{ab}\,,\\
&D^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}} + \theta^{+a}\partial_{+a} - \theta^{-a}\partial_{-a}\\
&D^{++} = \partial^{++} + i\theta^{+a}\theta^{+b}\partial_{ab} + \theta^{+a}\partial_{-a}\,,\\
&D^{--} = \partial^{--} + i\theta^{-a}\theta^{-b}\partial_{ab} + \theta^{-a}\partial_{+a}\,,
\end{aligned} \tag{4}$$

where $\partial_{\pm a}\theta^{\pm b} = \delta^b_a$ and $\partial^{++} = u^{+i}\frac{\partial}{\partial u^{-i}}, \quad \partial^{--} = u^{-i}\frac{\partial}{\partial u^{+i}}$. They obey the following (anti)commutation relations

$$\begin{aligned}
&\{D^+_a, D^-_b\} = -2i\partial_{ab}\,, \quad [D^{++}, D^+_a] = 0\,,\ [D^{++}, D^-_a] = D^+_a\,,\\
&[D^{--}, D^+_a] = D^-_a\,, \quad [D^{--}, D^-_a] = 0\,,\\
&[D^{++}, D^{--}] = D^0\,, \quad [D^0, D^{\pm\pm}] = \pm 2D^{\pm\pm}\,.
\end{aligned} \tag{5}$$

2.2 Basic Superfields

- The basic geometric object of $\mathcal{N} = (1,0)$ SYM theory is the analytic gauge connection V^{++}:

$$\nabla^{++} = D^{++} + V^{++}\,, \quad \delta V^{++} = -\nabla^{++}\Lambda\,, \quad \Lambda = \Lambda(\zeta)\,.$$

- The second harmonic (non-analytic) connection V^{--} covariantizes the second harmonic derivative:

$$\nabla^{--} = D^{--} + V^{--}\,, \quad \delta V^{--} = -\nabla^{--}\Lambda\,.$$

It s related to V^{++} by the harmonic flatness condition

$$[\nabla^{++}, \nabla^{--}] = D^0 \;\Rightarrow\; D^{++}V^{--} - D^{--}V^{++} + [V^{++}, V^{--}] = 0$$
$$\Rightarrow\; V^{--} = V^{--}(V^{++}, u^{\pm})\,.$$

One can make use of the analytic gauge freedom to choose the Wess–Zumino gauge:

$$V^{++} \;=\; \theta^{+a}\theta^{+b}A_{ab} + 2(\theta^+)^3_a\lambda^{-a} - 3(\theta^+)^4\mathscr{D}^{--}\,.$$

Here A_{ab} is the gauge field, $\lambda^{-a} = \lambda^{ai}u^-_i$ is the gaugino and $\mathscr{D}^{--} = \mathscr{D}^{ik}u^-_i u^-_k$, where $\mathscr{D}^{ik} = \mathscr{D}^{ki}$ are the auxiliary fields. This is just the irreducible content of the $\mathcal{N} = (1,0)$ vector (gauge) multiplet.

- Using V^{--}, one can define the covariant spinor and vector derivatives

$$\nabla^-_a = [\nabla^{--}, D^+_a] = D^-_a + \mathscr{A}^-_a\,, \qquad \nabla_{ab} = \frac{1}{2i}[D^+_a, \nabla^-_b] = \partial_{ab} + \mathscr{A}_{ab}\,,$$
$$\mathscr{A}^-_a(V) = -D^+_a V^{--}\,, \quad \mathscr{A}_{ab}(V) = \frac{i}{2}D^+_a D^+_b V^{--}\,,$$
$$[\nabla^{++}, \nabla^-_a] = D^+_a\,, \quad [\nabla^{++}, D^+_a] = [\nabla^{--}, \nabla^-_a] = [\nabla^{\pm\pm}, \nabla_{ab}] = 0\,,$$

and the covariant superfield strengths

$$[D^+_a, \nabla_{bc}] \;=\; \frac{i}{2}\varepsilon_{abcd}W^{+d}\,, \qquad [\nabla^-_a, \nabla_{bc}] \;=\; \frac{i}{2}\varepsilon_{abcd}W^{-d}\,,$$
$$W^{+a} = -\frac{1}{6}\varepsilon^{abcd}D^+_b D^+_c D^+_d V^{--}\,, \quad W^{-a} := \nabla^{--}W^{+a}\,,$$
$$\nabla^{++}W^{+a} = \nabla^{--}W^{-a} \;=\; 0\,, \quad \nabla^{++}W^{-a} = W^{+a}\,,$$
$$D^+_b W^{+a} = \delta^a_b F^{++}\,, \quad F^{++} = \frac{1}{4}D^+_a W^{+a} = (D^+)^4 V^{--}\,,$$
$$\nabla^{++}F^{++} = 0\,, \quad D^+_a F^{++} = 0\,.$$

- The hypermultiplet superfield, like in the $\mathcal{N} = 2, 4D$ case, has an infinite number of auxiliary fields off shell:

$$q^{+A}(\zeta) = q^{iA}(x)u_i^+ - \theta^{+a}\psi_a^A(x) + \text{ An infinite tail of auxiliary fields}\,, \;\; \mathrm{A} = 1, 2\,.$$

These fields come from the expansion of $q^{+A}(\zeta)$ over harmonic variables.

2.3 $\mathcal{N} = (1, 0)$ *Superfield Actions*

The $\mathcal{N} = (1, 0)$ SYM action in $6D$ HSS was suggested by Boris Zupnik [10]:

$$S^{SYM} = \frac{1}{f^2}\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n}\mathrm{Tr}\int d^6x\, d^8\theta\, du_1 \dots du_n \frac{V^{++}(z, u_1)\dots V^{++}(z, u_n)}{(u_1^+u_2^+)\dots(u_n^+u_1^+)}\,,$$
$$\delta S^{SYM} = 0 \;\Rightarrow\; F^{++} = 0\,,$$

where $1/(u_1^+u_2^+), \dots$ are the harmonic distributions defined in [11, 12].

The hypermultiplet action (with q^{+A} in the adjoint representation of gauge group for simplicity) reads

$$S^q = -\frac{1}{2f^2}\mathrm{Tr}\int d\zeta^{(-4)}q^{+A}\nabla^{++}q_A^+\,, \quad \nabla^{++}q_A^+ = D^{++}q_A^+ + [V^{++}, q_A^+]\,,$$
$$\delta S^q = 0 \;\Rightarrow\; \nabla^{++}q^{+A} = 0\,.$$

The $\mathcal{N} = (1, 0)$ superfield form of the $\mathcal{N} = (1, 1)$ SYM action is a sum of the two actions just defined:

$$S^{(V+q)} = S^{SYM} + S^q = \frac{1}{f^2}\left(\int dZ\mathcal{L}^{\mathrm{SYM}} - \frac{1}{2}\mathrm{Tr}\int d\zeta^{(-4)}q^{+A}\nabla^{++}q_A^+\right),$$
$$\delta S^{(V+q)} = 0 \;\Rightarrow\; F^{++} + \frac{1}{2}[q^{+A}, q_A^+] = 0\,, \quad \nabla^{++}q^{+A} = 0\,.$$

It possesses an invariance under the second hidden $\mathcal{N} = (0, 1)$ supersymmetry:

$$\delta V^{++} = \varepsilon^{+A}q_A^+\,, \quad \delta q^{+A} = -(D^+)^4(\varepsilon_A^- V^{--})\,, \quad \varepsilon_A^\pm = \varepsilon_{aA}\theta^{\pm a}\,. \tag{6}$$

While $\mathcal{N} = (1, 0)$ supersymmetry closes *off shell* on the analytic harmonic superfields V^{++} and q^{+A}, the transformations (6) form, together with $\mathcal{N} = (1, 0)$ supersymmetry, a closed $\mathcal{N} = (1, 1)$ supersymmetry only *on shell*, i.e. with taking into account the equations of motion for V^{++} and q^{+A} (see details in [13]). The situation here is quite similar to what one observes in $\mathcal{N} = 2$ superfield formulation of

$\mathscr{N} = 4, 4D$ SYM theory: there, only $\mathscr{N} = 2$ supersymmetry is manifest and off-shell, whereas the rest of $\mathscr{N} = 4$ supersymmetry is realized by the transformations like (6), with the on-shell closure [12].

3 Higher-Dimension $\mathscr{N} = (1, 0)$ and $\mathscr{N} = (1, 1)$ Invariants

It is natural to ask how to construct the higher-dimension $\mathscr{N} = (1, 1)$ invariants from the $\mathscr{N} = (1, 0)$ gauge superfield strength W^{+a} and the hypermultiplet superfield q^{+A}. First attempts to solve this problem were based on direct calculations.

- <u>$d = 6$</u>: In the pure SYM case the invariant of this dimension was uniquely constructed as [18]

$$S^{(6)}_{SYM} = \frac{1}{2g^2}\mathrm{Tr}\int d\zeta^{(-4)}du\,\left(F^{++}\right)^2 \sim \mathrm{Tr}\int d^6x[(\nabla^M F_{ML})^2 + \ldots]\,.$$

  Does its off-shell completion to an off-shell $\mathscr{N} = (1, 1)$ invariant exist? The answer is NO, only an expression whose $\mathscr{N} = (0, 1)$ variation vanishes *on-shell* can be found. It is unique up to two real parameters

$$\mathscr{L}^{d=6} = \frac{c_0}{2g^2}\mathrm{Tr}\int dud\zeta^{(-4)}\left(F^{++} + \frac{1}{2}[q^{+A}, q^+_A]\right)\left(F^{++} + 2\beta[q^{+A}, q^+_A]\right).$$

  But it vanishes on-shell by itself! Thus the non-vanishing on-shell counterterms of the canonical dimension 6 are absent, and this proves the *one-loop finiteness* of $\mathscr{N} = (1, 1)$ SYM theory.
  Recently, <u>$d = 6$</u> counterterms were found by the explicit quantum calculations in $\mathscr{N} = (1, 0)$ superspace [15–17]. It was shown that they vanish *off shell*, without any need in the equations of motion, just because of vanishing of the corresponding numerical coefficients!
- <u>$d = 8$</u>: All $\mathscr{N} = (1, 0)$ superfield terms of such dimension in the pure $\mathscr{N} = (1, 0)$ SYM theory prove to vanish on the gauge fields mass shell, in accord with the old statement of Ref. [21]. Could adding the hypermultiplet terms somehow change this result?
  Our analysis showed that there exist NO $\mathscr{N} = (1, 0)$ supersymmetric off-shell invariants of the dimension 8 which would respect the on-shell $\mathscr{N} = (1, 1)$ invariance.
  This means that $\mathscr{N} = (1, 1)$ SYM theory is at least *on-shell* finite at two loops. It is still an open question whether it is *off- shell* finite, i.e. whether the coefficients of the candidate counterterms are vanishing, like in the one-loop approximation (now under examination).

Surprisingly, the $\underline{d = 8}$ superfield expression which is non-vanishing on shell and respects an on-shell $\mathscr{N} = (1, 1)$ supersymmetry can be constructed by *giving up* the requirement of *off-shell* $\mathscr{N} = (1, 0)$ supersymmetry.

An example of such an invariant in $\mathscr{N} = (1, 0)$ SYM is very simple

$$\tilde{S}_1^{(8)} \sim \mathrm{Tr} \int d\zeta^{(-4)}\, \varepsilon_{abcd} W^{+a} W^{+b} W^{+c} W^{+d} .$$

Since $D_a^+ W^{+b} = \delta_a^b F^{++}$, it vanishes on shell, when $F^{++} = 0$. Thus, W^{+a} is an analytic superfield, when disregarding the terms proportional to the equations of motion, and so the above action respects $\mathscr{N} = (1, 0)$ supersymmetry on shell. Also, a double-trace on-shell invariant exists. Both such on-shell invariants admit $\mathscr{N} = (1, 1)$ completions.

Though the nontrivial on-shell $\underline{d = 8}$ invariants exist, they cannot appear as counterterms for the $\mathscr{N} = (1, 1)$ SYM theory. The reason is that they do *not* possess the off-shell $\mathscr{N} = (1, 0)$ supersymmetry which the physically relevant counterterms should obey within the manifestly $\mathscr{N} = (1, 0)$ invariant supergraph techniques. The non-existence of such counterterms agrees with the component consideration of Ref. [22].

Apart from the fact that such $\underline{d = 8}$ terms cannot appear as *counterterms* in $\mathscr{N} = (1, 1)$ SYM theory, they can appear, e.g., as *finite quantum corrections* to the effective Wilsonian action. For the pure $\mathscr{N} = (1, 0)$ SYM theory this was substantiated in [23, 24].

It was of clear necessity to develop some simple and systematic way of constructing higher-order on-shell $\mathscr{N} = (1, 1)$ supersymmetric invariants. This became possible within the on-shell harmonic $\mathscr{N} = (1, 1)$ superspace.

4 $\mathscr{N} = (1, 1)$ Harmonic Superspace

The first step in constructing such a superspace is to promote $\mathscr{N} = (1, 0)$ superspace to $\mathscr{N} = (1, 1)$ one,

$$z = (x^{ab}, \theta_i^a) \;\Rightarrow\; \hat{z} = (x^{ab}, \theta_i^a, \hat{\theta}_a^A).$$

As a result, the double set of covariant spinor derivatives comes out,

$$\nabla_a^i = \frac{\partial}{\partial \theta_i^a} - i\theta^{bi}\partial_{ab} + \mathscr{A}_a^i\,, \; \hat{\nabla}^{aA} = \frac{\partial}{\partial \hat{\theta}_{Aa}} - i\hat{\theta}_b^A \partial^{ab} + \hat{\mathscr{A}}^{aA} .$$

The defining constraints of $\mathscr{N} = (1, 1)$ SYM in this extended superspace read as [20, 21]:

$$\{\nabla_a^{(i}, \nabla_b^{j)}\} = \{\hat{\nabla}^{a(A}, \hat{\nabla}^{bB)}\} = 0\,, \quad \{\nabla_a^i, \hat{\nabla}^{bA}\} = \delta_a^b \phi^{iA}$$
$$\Rightarrow \quad \nabla_a^{(i}\phi^{j)A} = \hat{\nabla}^{a(A}\phi^{B)i} = 0 \quad \text{(By Bianchis)}\,.$$

Next, we define $\mathscr{N} = (1,1)$ HSS with the double set of $SU(2)$ harmonics [14]:

$$Z = (x^{ab}, \theta_i^a, u_k^{\pm}) \;\Rightarrow\; \hat{Z} = (x^{ab}, \theta_i^a, \hat{\theta}_b^A, u_k^{\pm}, u_A^{\hat{\pm}})\,.$$

Then we pass to the analytic basis and choose the "hatted" spinor derivatives short, $\nabla^{\hat{+}a} = D^{\hat{+}a} = \frac{\partial}{\partial\theta_a^{\hat{-}}}$. The $\mathscr{N} = (1,1)$ SYM constraints are rewritten in $\mathscr{N} = (1,1)$ HSS as

$$\{\nabla_a^+, \nabla_b^+\} = 0\,, \quad \{D^{\hat{+}a}, D^{\hat{+}b}\} = 0\,, \quad \{\nabla_a^+, D^{\hat{+}b}\} = \delta_a^b \phi^{+\hat{+}}\,,$$
$$[\nabla^{\hat{+}\hat{+}}, \nabla_a^+] = 0\,,\; [\tilde{\nabla}^{++}, \nabla_a^+] = 0\,,\; [\nabla^{\hat{+}\hat{+}}, D^{a\hat{+}}] = 0\,,\; [\tilde{\nabla}^{++}, D^{a\hat{+}}] = 0\,,$$
$$[\tilde{\nabla}^{++}, \nabla^{\hat{+}\hat{+}}] = 0\,.$$

Here

$$\nabla_a^+ = D_a^+ + \mathscr{A}_a^+(\hat{Z})\,,\; \tilde{\nabla}^{++} = D^{++} + \tilde{V}^{++}(\hat{\zeta})\,, \quad \nabla^{\hat{+}\hat{+}} = D^{\hat{+}\hat{+}} + V^{\hat{+}\hat{+}}(\hat{\zeta})\,,$$
$$\hat{\zeta} = (x_{\text{an}}^{ab}, \theta^{\pm a}, \theta_c^{\hat{+}}, u_i^{\pm}, u_A^{\hat{\pm}})\,.$$

The starting point of solving these constraints is to fix, using the $\Lambda(\hat{\zeta})$ gauge freedom, the WZ gauge for the second harmonic connection $V^{\hat{+}\hat{+}}(\hat{\zeta})$

$$V^{\hat{+}\hat{+}} = i\theta_a^{\hat{+}}\theta_b^{\hat{+}}\hat{\mathscr{A}}^{ab} + \varepsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}\varphi_d^A u_A^{\hat{-}} + \varepsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}\theta_d^{\hat{+}}\mathscr{D}^{AB}u_A^{\hat{-}}u_B^{\hat{-}}\,,$$

with $\hat{\mathscr{A}}^{ab}$, φ_d^A and $\mathscr{D}^{(AB)}$ being some $\mathscr{N} = (1,0)$ harmonic superfields.

Then the above constraints are reduced to some harmonic equations which can be explicitly solved. The central point is the requirement that the vector $6D$ connections in the sectors of hatted and unhatted variables **are identical** to each other.

As the final result, we have found that the first harmonic connection V^{++} coincides precisely with the standard $\mathscr{N} = (1,0)$ one, $V^{++} = V^{++}(\zeta)$, while the dependence of all other $\mathscr{N} = (1,1)$ objects on the variables with "hat" is strictly fixed as

$$V^{\hat{+}\hat{+}} = i\theta_a^{\hat{+}}\theta_b^{\hat{+}}\mathscr{A}^{ab} - \frac{1}{3}\varepsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}D_d^+ q^{-\hat{-}} + \frac{1}{8}\varepsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}\theta_d^{\hat{+}}[q^{+\hat{-}}, q^{-\hat{-}}]$$
$$\phi^{+\hat{+}} = q^{+\hat{+}} - \theta_a^{\hat{+}}W^{+a} - i\theta_a^{\hat{+}}\theta_b^{\hat{+}}\nabla^{ab}q^{+\hat{-}} + \frac{1}{6}\varepsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}[D_d^+ q^{-\hat{-}}, q^{+\hat{-}}]$$
$$+ \frac{1}{24}\varepsilon^{abcd}\theta_a^{\hat{+}}\theta_b^{\hat{+}}\theta_c^{\hat{+}}\theta_d^{\hat{+}}[q^{+\hat{-}}, [q^{+\hat{-}}, q^{-\hat{-}}]]\,.$$

Here, $q^{+\hat{\pm}} = q^{+A}(\zeta)u_A^{\hat{\pm}}\,,\; q^{-\hat{\pm}} = q^{-A}(\zeta)u_A^{\hat{\pm}}$ and W^{+a}, $q^{\pm A}$ are just the $\mathscr{N} = (1,0)$ superfields used previously. In the process of solving the constraints, there appeared the analyticity conditions for q^{+A}, as well as the full set of the superfield equations of motion

$$\nabla^{++}q^{+A}=0\,,\quad F^{++}=\frac{1}{4}D^{+}_{a}W^{+a}=-\frac{1}{2}[q^{+A},q^{+}_{A}]\,.$$

The basic advantage of using the constrained $\mathcal{N}=(1,1)$ strengths $\phi^{\pm\hat{+}}$ for the purpose of constructing various invariants is their extremely simple transformation rules under the hidden $\mathcal{N}=(0,1)$ supersymmetry

$$\delta\phi^{\pm\hat{+}}=-\varepsilon^{\hat{+}}_{a}\frac{\partial}{\partial\theta^{\hat{+}}_{a}}\phi^{\pm\hat{+}}-2i\varepsilon^{\hat{-}}_{a}\theta^{\hat{+}}_{b}\partial^{ab}\phi^{\pm\hat{+}}-[\Lambda^{(comp)},\phi^{\pm\hat{+}}]\,,$$

where $\Lambda^{(comp)}$ is some composite gauge parameter which does not contribute under the trace.

5 Invariants in $\mathcal{N}=(1,1)$ Superspace

The single-trace on-shell $\underline{d=8}$ invariant admits a simple rewriting in $\mathcal{N}=(1,1)$ superspace

$$S_{(1,1)}=\int du d\zeta^{(-4)}\mathcal{L}^{+4}_{(1,1)}\,,\ \mathcal{L}^{+4}_{(1,1)}=-\mathrm{Tr}\,\frac{1}{4}\int d\hat{\zeta}^{(-4)}d\hat{u}\,(\phi^{+\hat{+}})^{4},\ d\hat{\zeta}^{(-4)}\sim(D^{\hat{-}})^{4}$$

$$\delta\mathcal{L}^{+4}_{(1,1)}=-2i\partial^{ab}\mathrm{Tr}\int d\hat{\zeta}^{(-4)}d\hat{u}\left[\varepsilon^{\hat{-}}_{a}\theta^{\hat{+}}_{b}\frac{1}{4}(\phi^{+\hat{+}})^{4}\right].$$

The double-trace $\underline{d=8}$ invariant can also be straightforwardly constructed.

Now it is easy to construct the single- and double-trace $\underline{d=10}$ invariants as candidates for the 3-loop counterterms

$$S^{(10)}_{1}=\mathrm{Tr}\int dZd\hat{\zeta}^{(-4)}d\hat{u}\,(\phi^{+\hat{+}})^{2}(\phi^{-\hat{+}})^{2},\quad \phi^{-\hat{+}}=\nabla^{--}\phi^{+\hat{+}}\,,$$
$$S^{(10)}_{2}=-\int dZd\hat{\zeta}^{(-4)}d\hat{u}\,\mathrm{Tr}\Big(\phi^{+\hat{+}}\phi^{-\hat{+}}\Big)\,\mathrm{Tr}\Big(\phi^{+\hat{+}}\phi^{-\hat{+}}\Big)\,.$$

These are $\mathcal{N}=(1,1)$ extensions of the $\mathcal{N}=(1,0)$ SYM invariants $\sim\varepsilon_{abcd}\mathrm{Tr}\left(W^{+a}W^{-b}W^{+c}W^{-d}\right)$ and $\sim\varepsilon_{abcd}\,\mathrm{Tr}\,(W^{+a}W^{-b})\,\mathrm{Tr}\,(W^{+c}W^{-d})$.

It is notable that the single-trace $\underline{d=10}$ invariant admits a representation as an integral over the *full* $\mathcal{N}=(1,1)$ superspace

$$S^{(10)}_{1}\sim\mathrm{Tr}\int dZd\hat{Z}\,\phi^{+\hat{+}}\phi^{-\hat{-}}\,,\quad \phi^{-\hat{-}}=\nabla^{\hat{-}\hat{-}}\phi^{-\hat{+}}\,.$$

On the other hand, the double-trace $\underline{d = 10}$ invariant *cannot* be written as the full integral and so it looks as being UV protected.

This could partly explain why in the perturbative calculations of the $\mathcal{N} = (1, 1)$ SYM amplitudes, the single-trace 3-loop divergence is only seen, while no double-trace structures at the same order were observed [6, 7]

However, this does not seem to be like the standard non-renormalization theorems because the quantum calculation of $\mathcal{N} = (1, 0)$ supergraphs should generate invariants in the off-shell $\mathcal{N} = (1, 0)$ superspace, not in the on-shell $\mathcal{N} = (1, 1)$ superspace. So some additional piece of reasoning is needed to explain the absence of the double-trace divergences.

6 Quantum $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ SYM

For calculating various $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ invariants, including counterterms, there was an urgent need to formulate self-consistent $\mathcal{N} = (1, 0)$ superfield perturbation techniques: superpropagators, background field method, etc. All that was recently given in a few papers by Buchbinder, Ivanov, Merzlikin and Stepanyantz, [15–17]. These methods were used to prove the one-loop **off-shell** finiteness of $\mathcal{N} = (1, 1)$ SYM theory formulated in terms of $\mathcal{N} = (1, 0)$ superfields.

The basic idea of the background field method is to split the relevant superfields into the sum of the "background" superfields V^{++}, Q^+ and the "quantum" ones v^{++}, q^+,

$$V^{++} \to V^{++} + f v^{++}, \qquad q^+ \to Q^+ + q^+ , \tag{7}$$

and then to expand the action in a power series in quantum fields.

In brief, in the background field approach the $\mathcal{N} = (1, 0)$, $6D$ SYM theory with hypermultiplets is described by the three quantum superfield ghosts: two fermionic Faddeev-Popov ghosts **b** and **c** together with the single bosonic Nielsen-Kallosh ghost φ, in addition to the quantum v^{++} and q^+ superfields. It was convenient to start with the model in which hypermultiplet belongs to an arbitrary representation R of gauge group, not just to the adjoint one.

After integrating, in the functional integral, over quantum superfields, the following representation for the one-loop quantum correction to the classical action was obtained

$$\Gamma^{(1)}[V^{++}, Q] = \frac{i}{2}\mathrm{Tr}\ln\left\{ \widehat{\Box}^{AB} - 2f^2 \widetilde{Q}^{+m}\left(T^A G_{(1,1)} T^B\right)_m{}^n Q^+_n \right\} - \frac{i}{2}\mathrm{Tr}\ln\widehat{\Box}$$
$$- i\,\mathrm{Tr}\ln(\nabla^{++})^2_{\mathrm{Adj}} + \frac{i}{2}\mathrm{Tr}\ln(\nabla^{++})^2_{\mathrm{Adj}} + i\,\mathrm{Tr}\ln\nabla^{++}_{\mathrm{R}} ,$$

where subscripts Adj and R mean that the corresponding operators are taken in the adjoint representation and the representation R of the hypermultiplet, and

$$\widehat{\Box}= \frac{1}{2}(D^{+})^{4}(\nabla^{--})^{2}$$

is the covariant Box operator.

The complete one-loop divergent part of the effective action reads

$$\begin{aligned}\Gamma^{(1)}_{div}[V^{++}, Q^{+}] = \ & \frac{C_2 - \frac{d_R}{d_G}C_2(R)}{3(4\pi)^3\varepsilon}\,\mathrm{Tr}\int d\zeta^{(-4)}du\,(F^{++})^2 \\ & -\frac{2if^2}{(4\pi)^3\varepsilon}\int d\zeta^{(-4)}du\,\widetilde{Q}^{+}(C_2 - C_2(R))F^{++}Q^{+}. \qquad (8)\end{aligned}$$

The coefficients of the $(F^{++})^2$ and $\widetilde{Q}^{+}F^{++}Q^{+}$ terms in the divergent part of one-loop effective action are proportional to the differences between the second order Casimir operator C_2 for adjoint representation of gauge group and the operators $T(R) = \frac{d_R}{d_G}C_2(R)$ and $C_2(R)$ for the hypermultiplet representation R, respectively.

Since $\mathscr{N}=(1,1)$, $6D$ supersymmetric Yang–Mills theory involves the hypermultiplet in **adjoint** representation of gauge group, with $d_R = d_G,\ C_2(R) = C_2$, the divergent part vanishes for this case. Hence, the $\mathscr{N}=(1,1)$ SYM theory is one-loop finite, and there is no need to use the equations of motion to prove this property.

For any other choice of the hypermultiplet irrep, (8) does not vanish even on shell, so in general the theory is divergent already at one loop. The pure $\mathscr{N}=(1,0)$ SYM corresponds to $C_2(R) = 0$ and the one-loop divergency is vanishing on the equation of motion $F^{++} = 0$, in accord with the old result by Howe and Stelle [21].

7 Summary and Outlook

Let me summarize the above presentation.

- Off-shell $\mathscr{N}=(1,0)$ and on-shell harmonic $\mathscr{N}=(1,1)$, $6D$ superspaces can be efficiently used to construct higher-dimensional invariants in the $\mathscr{N}=(1,0)$ and $\mathscr{N}=(1,1)$ SYM theories.
- $\mathscr{N}=(1,1)$ SYM constraints were solved in terms of harmonic $\mathscr{N}=(1,0)$ superfields. This allowed to explicitly construct the full set of the superfield dimension $d=8$ and $d=10$ invariants with $\mathscr{N}=(1,1)$ on-shell supersymmetry.
- All $\underline{d=6}$ $\mathscr{N}=(1,1)$ invariants are at least on-shell vanishing, proving the UV finiteness of $\mathscr{N}=(1,1)$ SYM at one loop.
- The off-shell $\underline{d=8}$ $\mathscr{N}=(1,1)$ invariants are absent. Assuming that the $\mathscr{N}=(1,0)$ supergraphs yield integrals over the full $\mathscr{N}=(1,0)$ harmonic superspace, this means the absence of two-loop counterterms.
- Two $\underline{d=10}$ invariants were explicitly constructed as integrals over the whole $\mathscr{N}=(1,0)$ harmonic superspace. The single-trace invariant can be rewritten as an integral over $\mathscr{N}=(1,1)$ superspace, while the double-trace one cannot. This

property combined with an additional reasoning could explain why the double-trace invariant is UV protected.
- The quantum techniques for $\mathcal{N} = (1, 0)$ SYM theory was worked out and used to show that $\mathcal{N} = (1, 1)$ SYM theory is one-loop finite off shell, without need in equations of motion.

7.1 Further Lines of Study

In conclusion, we outline some further possible lines of study:

(**a**) To construct the next $d \geq 12$ invariants in the $\mathcal{N} = (1, 1)$ SYM theory with the help of the on-shell $\mathcal{N} = (1, 1)$ harmonic superspace techniques (Buyukli & Ivanov, in preparation);

(**b**) To reproduce higher-dimensional invariants from the quantum superfield perturbation theory, to examine whether $\mathcal{N} = (1, 1)$ SYM theory is two-loop finite off shell (Buchbinder et al., in preparation)[4];

(**c**) To work out the quantum superfield perturbation theory directly in $\mathcal{N} = (1, 1)$ double-harmonized superspace;

(**d**) To apply the same methods for constructing the Born-Infeld action with manifest off-shell $\mathcal{N} = (1, 0)$ and hidden on-shell $\mathcal{N} = (0, 1)$ supersymmetries. To check the hypothesis that such an action could be identified with the full quantum effective action of $\mathcal{N} = (1, 1)$ SYM;

(**e**) Applications in supergravity?

Acknowledgements I thank the organizers of the X International Symposium "Quantum Theory and Symmetries" and XII International Workshop "Lie Theory and Its Applications in Physics" (and particularly Vladimir Dobrev) for the kind hospitality in Varna. I am indebted to my collaborators G. Bossard, I. Buchbinder, B. Merzlikin, A. Smilga, K. Stepanyantz and B. Zupnik. A partial support from the RSF Grant No. 16-12-10306, the RFBR grant No. 15-02-06670, and a grant of the Heisenberg - Landau program is acknowledged.

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[4]Some latest results along this line are published in [25, 26].

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TBA and Tree Expansion

Ivan Kostov, Didina Serban and Dinh-Long Vu

Abstract We propose an alternative, statistical, derivation of the Thermodynamic Bethe Ansatz based on the tree expansion of the Gaudin determinant. We illustrate the method on the simplest example of a theory with diagonal scattering and no bound states. We reproduce the expression for the free energy density and the finite size corrections to the energy of an excited state as well as the LeClair-Mussardo series for the one-point function for local operators.

Keywords Thermodynamic Bethe Ansatz · Integrable models · Matrix-Tree Theorem

1 Introduction

The finite size effects in 1+1 dimensional field theories come from the quantisation of the momenta of the physical particles, as well as from the virtual "mirror" particles winding around the space circle R [1]. When R is large, the exponentially small contribution from the mirror particles can be neglected and the spectrum is determined by the "asymptotic" Bethe–Yang equations, which take into account only the scattering processes between the physical particles. As it was first realised by Al. Zamolodchikov [2], for finite R a powerful technique for summing up the finite size corrections is given by the Thermodynamical Bethe Ansatz, or TBA [3]. If the the theory is Lorentz invariant, the finite size effects can be traded to finite

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_6

temperature effects. The main idea of the TBA is that the thermal trace is dominated by a saddle point for the density of states, which is obtained as the solution of some non-linear integral equations. By analytical continuation one can obtain the “exact Bethe equations” for the spectrum of the excited states in finite volume [4].

In the last decades much attention is been focused on combining the TBA and the form factor bootstrap in order to compute the correlation functions at finite volume/temperature. This is a problem of higher complexity and in spite of the considerable progress a systematic procedure is not yet available for the higher point functions. The main difficulty is to learn how to insert efficiently the resolution of the identity between the local operators in order to split the correlation function into simpler objects, the elementary form factors at infinite volume. In other words, the saddle point analysis of the TBA is not sufficient and has to be replaced by a more subtle, field-theoretical, consideration.

Another motivation for looking at the sum over the intermediate states is the recently proposed hexagon bootstrap program in the AdS/CFT integrable model [5] which can be applied for the computation of higher point correlation functions. The proposal prescribes to insert complete sets of mirror particles between the hexagon operators. Although these effects resemble the wrapping corrections in the spectral problem, no TBA methods have yet been developed to resum them.

In this paper, we address the problem of performing the sum over the mirror states in the simplest case of a theory with diagonal scattering and no bound states. Our proposal is close in spirit to some previous works [6, 8] where the excluded volume in the sum over the intermediate states is compensated by including into the sum non-physical solutions of the asymptotic Bethe–Yang equations. The new development is that we succeeded to perform explicitly the sum over the states using a graph expansion of the Gaudin determinant which gives the integration measure over the Bethe states in the mirror channel. This graph expansion leads to a Feynman-like diagram technique which allows us to write the free energy as a sum over tree Feynman diagrams.

In Sect. 2 we explain our method on the simplest example of a diagonal theory without bound states for which we compute the partition function on a cylinder with circumference R as the thermal trace in the mirror theory. In the rest of the text we consider two more examples, where we re-derive the formulas obtained previously by ingenuous application of the TBA. In Sect. 3 we compute the energy of an excited state in the physical channel. In Sect. 4 we derive the LeClair-Mussardo series for the one-point function. In all three examples we reduce the computation to a combinatorial problem involving the sum over tree graphs.

2 Integrable Quantum Field Theory on a Cylinder: The Partition Function

2.1 Physical and Mirror Channels

Consider an integrable 1+1 dimensional field theory with one single type of particle excitations above the vacuum. The dispersion relation between the momentum p and the energy E of the particle is parametrised by the rapidity variable u:

$$p = p(u), E = E(u). \tag{1}$$

We assume that there exists a transformation to the "mirror" theory in which the role of the time t and the space x are exchanged. The physical and the mirror channels are related by a "mirror" transformation $x = -i\tilde{t}$, $t = -i\tilde{x}$ and $E = i\tilde{p}$, $p = i\tilde{E}$. The mirror transformation can be encoded in a transformation $\gamma : u \to \tilde{u}$ of the rapidity parameter, so that

$$E(\tilde{u}) = i\tilde{p}(u), \quad p(\tilde{u}) = i\tilde{E}(u). \tag{2}$$

The square of the mirror transformation gives the crossing transformation $\gamma^2 : u \to \bar{u} = \gamma\tilde{u}$ which relates particles to anti-particles. If the theory is Lorentz invariant, then the mirror and the physical theories are identical. The diagonal S-matrix $S(u, v)$ is supposed to satisfy, besides the Yang-Baxter equations, unitarity $S(u, v)S(v, u) = 1$, crossing symmetry $S(u, v) = S(\bar{v}, \bar{u})$, and the condition $S(u, u) = -1$. We will not need to assume that the S-matrix is a function of the difference of the two rapidities.

If the theory is confined in a finite volume R with periodic boundary conditions, the eigenstates of the Hamiltonian can be constructed as superpositions of plane waves according to the Bethe Ansatz, with the spectrum of the rapidities determined by condition of periodicity. Each eigenstate from the N-particle sector is characterised by a set of rapidities $\mathbf{u} = \{u_1, \ldots, u_N\}$ and the energy of this state is equal to

$$E(\mathbf{u}) = \sum_{j=1}^{N} E(u_j). \tag{3}$$

When R is sufficiently large, the spectrum of the energies are determined by the Asymptotic Bethe Ansatz. The quantisation condition for the rapidities is expressed in terms of the total phase factor corresponding to a process in which one of the N particles winds once around the space circle,

$$\phi_j(u_1,\ldots,u_N) \equiv p(u_j)R + \frac{1}{i}\sum_{k(\neq j)}^{N} \log S(u_j,u_k) \qquad (j=1,\ldots,N). \tag{4}$$

For periodic boundary conditions the scattering phases can take integer values modulo 2π

$$\phi_j(u_1,\ldots,u_N) = 2\pi n_j \quad \text{with } n_j \text{ integer}, \quad j=1,\ldots,N\,. \tag{5}$$

In a system of units where the mass of the particle is equal to one, the asymptotic expression (37) for the scattering phases is true up to $o(e^{-R})$ terms. For finite R the Bethe–Yang equations (4)–(5) are deformed by the scattering with the virtual particles in the mirror channel which wrap the space circle [24]. One can study the finite volume effects using the TBA in the mirror channel. One can introduce an infrared cutoff in the mirror theory by considering the cylinder as the limit of a torus obtained as the product of the space circle with a time circle with asymptotically large circumference L. When L is large, one can construct a complete set of states in the mirror channel whose spectrum is given by the asymptotic Bethe–Yang equations. Then the partition function can be computed by taking the thermal trace in the mirror Hilbert space.

The standard TBA approach due to Yang and Yang [3] is to express the thermal trace as an integral over the density of one-particle rapidities, taking into account both the energy and the entropy of the states. The free energy is expressed as a functional of the rapidity density and the critical point of this functional gives both the thermal equilibrium state and the expression for the extensive piece $LF_0(R)$ of the free energy. In field-theoretical terms this translates to replace the sum over the intermediate states by a single "thermal state" characterised by the saddle point density. This approximation works well for evaluating the free energy and the one-point functions, where a single insertion of the identity is to be made, but it is not sufficient e.g. for the computation of the two-point functions.[1]

2.2 Thermal Partition Function

Below we will perform a direct summation in the mirror Hilbert space. Our method is exact up to corrections exponentially small in L and allows to control the whole $1/L$ expansion of the partition function. The simplest object to compute is the partition function on the torus, $Z(R,L)$, which can be evaluated as a thermal trace in the physical or in the mirror channels of the Euclidean theory,

$$\mathcal{Z}(L,R) = \operatorname*{Tr}_{\text{phys}}[e^{-LH_{\text{phys}}}] = \operatorname*{Tr}_{\text{mir}}[e^{-RH_{\text{mir}}}]. \tag{6}$$

[1]There however is a class of two-point functions for which a single insertion is sufficient [9].

Assuming that $R \ll L$, our goal is to evaluate the the free energy

$$\log \mathcal{Z}(L, R) = LF_0(R) + F_1(R) + \cdots \tag{7}$$

up to corrections exponentially small in L.

Let us stress that such an exponential accuracy is beyond the reach of the standard TBA approach which is essentially a collective field theory for the rapidity density and as such suffers from ambiguities beyond the first two terms of the expansion (7). The leading term in the TBA approach is determined by the saddle point of the integral over the densities, while the subleading term is produced by the gaussian fluctuations about the saddle point [10] and the normalisation of the wave function of the thermal state [11], with the two effects cancelling completely for periodic boundary conditions. Our approach does not suffer from the ambiguities of the collective theory and allows to obtain the whole series (7), which in the case of periodic boundary conditions consists of a single term $LF_0(R)$.

2.3 The Partition Function as a Sum over Mode Numbers

The quantisation condition in the mirror channel is given by the Bethe–Yang equations

$$\tilde{\phi}_j = 2\pi n_j \quad \text{with } n_j \text{ integer}, \quad j = 1, \ldots, M\ , \tag{8}$$

where $\tilde{\phi}_j$ is the total scattering phase for the jth mirror particle,

$$\tilde{\phi}_j(u_1, \ldots, u_M) \equiv \tilde{p}(u_j)L + \frac{1}{i}\sum_{k(\neq j)}^{M} \log \tilde{S}(u_j, u_k). \tag{9}$$

Here $\tilde{S}(u, v) = S(\tilde{u}, \tilde{v})$ denotes the S-matrix for the mirror particles. The states in the M-particle sector of the Hilbert space are labeled by M distinct mode numbers $n_1, \ldots, n_M$ and the identity operator in this sector can be decomposed as a sum of products of normalised states

$$\mathbb{I}_M = \sum_{n_1<\ldots<n_M} |n_1, \ldots, n_M\rangle\langle n_1, \ldots, n_M|. \tag{10}$$

If we denote by $\tilde{E}_M(n_1, \ldots, n_M)$ the eigenvalue of the Hamiltonian for the state $|n_1, \ldots, n_M\rangle$, the partition function (6) is given by the series

$$\mathcal{Z}(L,R)=\sum_{M=0}^{\infty}\sum_{n_1<n_2<\cdots<n_M} e^{-R\tilde{E}(n_1,\dots,n_M)}. \tag{11}$$

Our goal is to replace in the thermodynamical limit $L\to\infty$ the discrete sums by multiple integrals. For that we have first to get rid of the ordering of the quantum numbers. For that we insert a factor which kills the configurations with coinciding quantum numbers and take the sum over non-restricted integers,

$$\mathcal{Z}(L,R)=\sum_{M=0}^{\infty}\frac{1}{M!}\sum_{n_1,\dots,n_M}\prod_{j<k}^{M}\left(1-\delta_{n_j,n_k}\right)e^{-R\tilde{E}(n_1,\dots,n_M)}. \tag{12}$$

Expanding the product of Kronecker symbols, leads to a series

$$\mathcal{Z}(L,R)=1+\sum_{n}e^{-R\tilde{E}(n)}+\frac{1}{2!}\sum_{n_1,n_2}e^{-R\tilde{E}(n_1,n_2)}-\frac{1}{2}\sum_{n}e^{-R\tilde{E}(n,n)}+\cdots \tag{13}$$

which we are going to write as an exponential. The sum in (13) goes over all sequences $(n_1^{r_1},\dots,n_m^{r_m})$ of positive integers n_j with multiplicities r_j. For example, $(n^2)=(n,n)$. Each such sequence defines an (unphysical) Bethe state obtained by identifying some of the momenta of a Bethe state with $M=r_1+\cdots+r_m$ magnons. This state is a linear combination of plane waves with momenta $r_j\tilde{p}(u_j), j=1,\dots,m.$ and energy

$$\tilde{E}(n_1^{r_1},\ \dots\ ,\ n_m^{r_m})=r_1\tilde{E}(u_1)+\cdots+r_m\tilde{E}(u_m). \tag{14}$$

The relevance of such states has been already pointed out by Woynarovich [8] and by Dorey et al in [7].

The rapidities $u_1,\cdots,u_m$ are determined by the Bethe–Yang equations (8) with $M=r_1+\cdots+r_m$. The phase $\tilde{\phi}_j$ is acquired by the wave function if to one of the r_j particles with rapidity u_j winds once around the time circle,

$$\tilde{\phi}_j\equiv\tilde{p}(u_j)L+\frac{1}{i}\sum_{k(\neq j)}^{m}r_k\log\tilde{S}(u_j,u_k)+\pi(r_j-1)=2\pi n_j\quad(j=1,\dots,m). \tag{15}$$

The term $\pi(r_j-1)$ originates in the scattering of the probe particle with the r_j-1 particles with the same rapidity u_j.

The full series (13) has the form

$$\mathcal{Z}(L,R)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{n_1,\dots,n_m}\sum_{r_1,\dots,r_m}(-1)^{r_1+\dots+r_m}\,C_{r_1\dots r_m}\,e^{-R\tilde{E}(n_1^{r_1},\,\dots\,,n_m^{r_m})},\qquad(16)$$

where the coefficients $C_{r_1\dots r_m}$ are purely combinatorial. They can be fixed from the expansion of the thermal partition function when the quasiparticles are free fermions, $S(u_i,u_j)=-1$ and $\tilde{E}(n_1,\dots,n_M)=\tilde{E}(n_1)+\dots+\tilde{E}(n_M)$. In the occupation numbers representation, the partition function for free fermions can be written as an infinite product

$$\begin{aligned}\mathcal{Z}^{\text{free fermions}}&=\prod_{n\in\mathbb{Z}}\left(1+e^{-R\tilde{E}(n)}\right)=\exp\sum_{n\in\mathbb{Z}}\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r}e^{-rR\tilde{E}(n)}\\&=1+\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\sum_{n_1,\dots,n_m}\sum_{r_1,\dots,r_m}\frac{(-1)^{r_1+\dots+r_m}}{r_1\dots r_m}\prod_{j=1}^{m}e^{-Rr_jE(n_j)}.\end{aligned}\qquad(17)$$

Comparing with (16) we find for the combinatorial coefficients

$$C_{r_1\dots r_m}=\frac{1}{r_1\dots r_m}.\qquad(18)$$

In the case of free fermions, the multiplicities r_j have obvious meaning. The vacuum energy is a sum of all fermionic loops including those winding r times around the space circle. The weight of an r-winding loop consists of a Boltzmann factor e^{-rRE_n}, a sign $(-1)^r$ due to the Fermi statistics and a combinatorial factor $1/r$ counting for the Z_r cyclic symmetry. It is natural to interpret the multiplicities r_j as winding, or wrapping, numbers also in the case of non-trivial scattering, which we are going to do in the following.

2.4 From Mode Numbers to Rapidities

The discrete sum over the allowed values of the phases $\tilde{\phi}_j(u_1,r_1;\dots,u_m,r_m)$ for given wrapping numbers can be replaced, up to exponentially small in L terms, by an integral,

$$\sum_{n_1,\cdots,n_m}=\int\frac{d\tilde{\phi}_1}{2\pi}\cdots\frac{d\tilde{\phi}_m}{2\pi}.\qquad(19)$$

Since the energy takes a simple form as a function of the rapidities, Eq. (14), we are going to change the variables from scattering phases ϕ_j to rapidities u_j,

$$\mathcal{Z}(L,R) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{r_1,\dots,r_m} \frac{(-1)^{r_1+\dots+r_m}}{r_1\dots r_m} \int \frac{du_1}{2\pi}\dots\frac{du_m}{2\pi} \times \tilde{G}(u_1^{r_1},\dots,u_m^{r_m})\, e^{-r_1\tilde{E}(u_1)}\dots e^{-r_m\tilde{E}(u_m)}. \tag{20}$$

The change of variables brings a volume-dependent Jacobian (the Gaudin determinant)

$$\tilde{G} = \det_{m\times m} \tilde{G}_{kj}, \qquad \tilde{G}_{kj} = \frac{\partial}{\partial u_k}\tilde{\phi}_j(u_1^{r_1},\dots,u_m^{r_m}), \tag{21}$$

which gives the density of the particle states in the rapidity space. The explicit form of the Gaudin matrix $\tilde{G}_{jk}$ is

$$\tilde{G}_{kj} = \left(L\tilde{p}'(u_j) + \sum_{l=1}^{m} r_l K(u_j,u_l)\right)\delta_{jk} - r_k K(u_k,u_j), \tag{22}$$

where $K(u,v) = \frac{1}{i}\partial_u \log\tilde{S}(u,v)$.

2.5 Graph Expansion of the Gaudin Determinant

Let us denote for brevity

$$\tilde{p}'_j \equiv \tilde{p}'(u_j) \qquad K_{jk} \equiv K(u_j,u_k). \tag{23}$$

Inspecting the expansion of the Gaudin determinant for $m = 1, 2, 3$

$$\begin{aligned}
\tilde{G}(u^r) &= L\tilde{p}'\,,\\
\tilde{G}(u_1^{r_1},u_2^{r_2}) &= L^2\tilde{p}'_1\tilde{p}'_2 + L\tilde{p}'_1 r_1 K_{21} + L\tilde{p}'_2 r_2 K_{12},\\
\tilde{G}(u_1^{r_1};u_2^{2},u_3^{r_3}) &= L^3\tilde{p}'_1\tilde{p}'_2\tilde{p}'_3\\
&+ L^2\tilde{p}'_2\tilde{p}'_3 r_2 K_{12} + L^2\tilde{p}'_2\tilde{p}'_3 r_3 K_{13} + L^2\tilde{p}'_1\tilde{p}'_3 r_1 K_{21}\\
&+ L^2\tilde{p}'_1\tilde{p}'_3 r_3 K_{23} + L^2\tilde{p}'_1\tilde{p}'_2 r_1 K_{31} + L^2\tilde{p}'_1\tilde{p}'_2 r_2 K_{32}\\
&+ \tilde{p}'_3 L r_1 r_3 K_{13}K_{21} + \tilde{p}'_3 L r_2 r_3 K_{12}K_{23} + \tilde{p}'_3 L r_3^2 K_{13}K_{23}\\
&+ \tilde{p}'_2 L r_1 r_2 K_{12}K_{31} + \tilde{p}'_1 L r_1^2 K_{21}K_{31} + \tilde{p}'_1 L r_3 r_1 K_{23}K_{31}\\
&+ \tilde{p}'_1 L r_2 r_1 K_{21}K_{32} + \tilde{p}'_2 L r_2^2 K_{12}K_{32} + \tilde{p}'_2 L r_2 r_3 K_{13}K_{32},
\end{aligned} \tag{24}$$

we see that there are no cycles of the type $K_{12}K_{21}$ or $K_{12}K_{23}K_{31}$. We will see below that this property hold for general order m. To evaluate the Gaudin determinant for general state $\{u_1^{r_1}, \dots, u_m^{r_m}\}$, we will consider in the following a slightly modified Gaudin matrix, $\hat{G}_{kj} = \tilde{G}_{kj} r_j$. The determinants of the two matrices are simply related,

$$\tilde{G} = \frac{\det \hat{G}_{jk}}{\prod_{j=1}^{m} r_j}, \qquad \hat{G}_{kj} \equiv \tilde{G}_{kj} r_j. \tag{25}$$

The the modified Gaudin matrix has the advantage that it is a sum of a diagonal matrix $\hat{D}_j \delta_{jk}$ and a Laplacian matrix $\hat{K}_{kj}$ (a matrix with zero row sums):

$$\begin{aligned} &\hat{G}_{kj} = \hat{D}_k \, \delta_{kj} - \hat{K}_{kj} \\ &\text{with } \hat{D}_j = L r_j \tilde{p}'(u_j) \text{ and } \hat{K}_{k,j} = r_k r_j K(u_k, u_j) - \delta_{kj} \sum_{l=1}^{m} r_j r_l K(u_j, u_l) \end{aligned} \tag{26}$$

According to the *Matrix-Tree Theorem* (see e.g. [12, 13]), the determinant of the matrix $\hat{G}_{ij}$ can be expanded as a sum of graphs called *directed spanning forests*. A directed forest spanning the graph Γ is an oriented subgraph $\mathcal{F}$ fulfilling the following three conditions:

(i) $\mathcal{F}$ contains all vertices of Γ;
(ii) $\mathcal{F}$ does not contain cycles;
(iii) For any vertex of Γ there is at most one oriented edge of $\mathcal{F}$ ending at this vertex.

The vertices with no incoming lines are called *roots*. Any forest $\mathcal{F}$ is decomposed into connected components called *directed trees*. Each tree contains one and only one root. The Matrix-Tree Theorem states that the determinant of the matrix $\hat{G}$ is a sum of all directed forests $\mathcal{F}$ spanning the totally connected graph with vertices labeled by $j = 1, \dots, m$:

$$\det_{m\times m}\left(\hat{D}_j \delta_{jk} - \hat{K}_{jk}\right) = \sum_{\mathcal{F}} \prod_{v_i \in \text{roots}} \hat{D}_i \prod_{\ell_{jk} \in \mathcal{F}} \hat{K}_{kj}. \tag{27}$$

The weight of a forest $\mathcal{F}$ is a product of factors $\hat{D}_k$ associated with the roots and factors $\hat{K}_{kj}$ associated with the oriented edges $\ell_{jk} = \langle v_j \to v_k \rangle$ of the $\mathcal{F}$. The expansion in spanning forests for $m = 1, 2, 3$ is depicted in Fig. 1.

Applying the above graph expansion to the Jacobian, we write the partition function as

$$\begin{aligned} \mathcal{Z}(L, R) = &\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{r_1,\dots,r_m} \int \prod_{j=1}^{m} \frac{du_j}{2\pi} \frac{[-e^{-R\tilde{E}(u_j)}]^{r_j}}{r_j^2} \\ &\times \sum_{\mathcal{F}} \prod_{j\in \text{roots}} L r_j \tilde{p}'(u_j) \prod_{\ell_{ij} \in \mathcal{F}} r_i r_j K(u_j, u_i). \end{aligned} \tag{28}$$

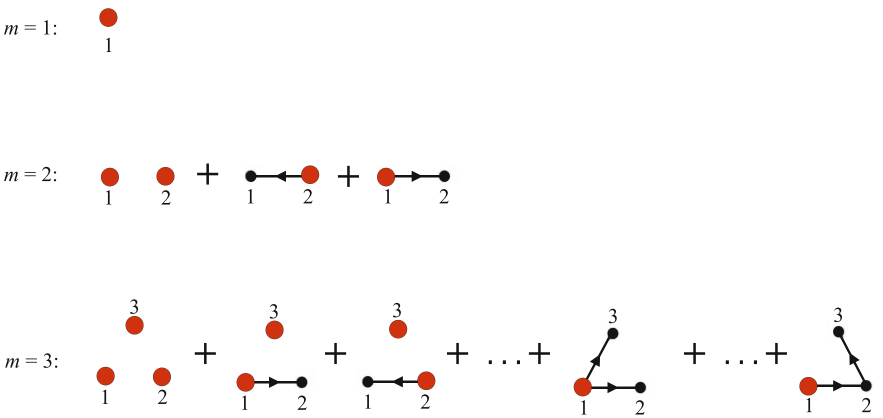
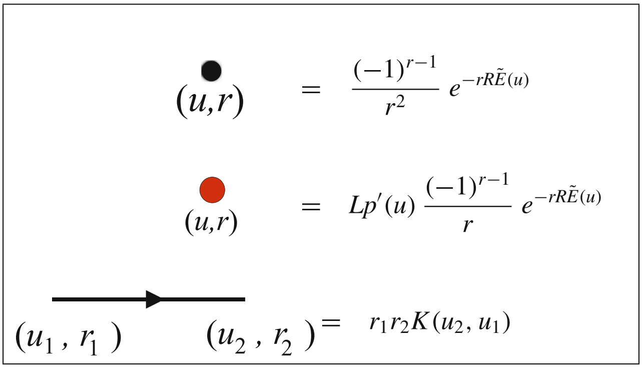


Fig. 1 The expansion of a determinant in directed spanning forests for $m = 1, 2, 3$. Ellipses mean sum over the permutations of the vertices of the preceding graph. Each vertex of a directed tree, except for the root, has exactly one incoming edge and an arbitrary number of outgoing edges. The root can have only outgoing edges

The next step is to invert the order of the sum over graphs and the integral/sum over the coordinates (u_j, r_j) assigned to the vertices. As a result we obtain a sum over the ensemble of abstract oriented tree graphs, with their symmetry factors, embedded in the space $\mathbb{R} \times \mathbb{N}$ where the coordinates u, r of the vertices take values. The embedding is free, in the sense that the sum over the positions of the vertices is taken without restriction. As a result, the sum over the embedded tree graphs is the exponential of the sum over connected ones. One can think of these graphs as tree level Feynman diagrams obtained by applying the following Feynman rules:



$$\bullet_{(u,r)} = \frac{(-1)^{r-1}}{r^2}\, e^{-rR\tilde{E}(u)}$$

$$\bullet_{(u,r)} = Lp'(u)\,\frac{(-1)^{r-1}}{r}\, e^{-rR\tilde{E}(u)} \tag{29}$$

$$(u_1, r_1) \longrightarrow (u_2, r_2) = r_1 r_2 K(u_2, u_1)$$

In this way we can write the free energy as

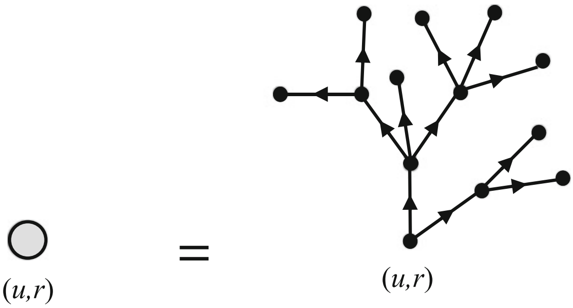


Fig. 2 The generating function $Y_r(u)$ of the directed trees with root at (u, r). The weight of each tree in the sum is a product of factors associated with its vertices and edges according to the Feynman rules (29)

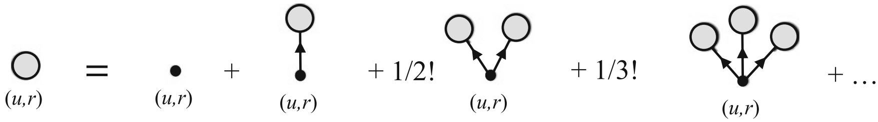


Fig. 3 The non-linear equation for the generating function $\tilde{Y}_r(u)$ of the trees with root at (u, r)

$$\log \mathcal{Z}(L, R) = L \int \frac{du}{2\pi} \tilde{p}'(u) \sum_{r=1}^{\infty} r \tilde{Y}_r(u), \tag{30}$$

where $\tilde{Y}_r(u)$ is the partition sum of all *connected* directed rooted trees with root at the point (u, r), Fig. 2.

Equation (30) gives the free energy up to e^{-L} terms, hence the subleading terms in the expansion (7) vanish. Of course this is true only for periodic boundary conditions.

2.6 Performing the Sum over Trees

As any partition sum of trees, $\tilde{Y}_r(u)$ satisfies a simple non-linear equation (a Schwinger–Dyson equation in the QFT language) depicted in Fig. 3,

$$\begin{aligned} \tilde{Y}_r(u) &= \frac{(-1)^{r-1}}{r^2} e^{-rR\tilde{E}(u)} \sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_s \int \frac{dv}{2\pi} rsK(v, u) \tilde{Y}_s(v) \right)^n \\ &= \frac{(-1)^{r-1}}{r^2} \left[e^{-R\tilde{E}(u)} \, e^{\sum_s \int \frac{dv}{2\pi} sK(v,u) \tilde{Y}_s(v)} \right]^r . \end{aligned} \tag{31}$$

In particular for $r = 1$

$$\tilde{Y}_1(u) = e^{-R\tilde{E}(u)}\, e^{\sum_s \int \frac{dv}{2\pi} sK(v,u)\tilde{Y}_s(v)}. \tag{32}$$

Substituting the rhs of (32) in the square brackets in the second line of Eq. (31), we express all Y_r in terms of Y_1,

$$\tilde{Y}_r(u) = \frac{(-1)^{r-1}}{r^2}[\tilde{Y}_1(u)]^r, \quad r = 1, 2, 3, \dots. \tag{33}$$

Now we can express the rhs of (30) and the exponent in on the rhs of (32) in terms of Y_1 only,

$$\sum_r r\tilde{Y}_r(v) = \log\Big[1 + \tilde{Y}_1(v)\Big]. \tag{34}$$

Now Eq. (32) becomes a closed equation for Y_1,

$$\tilde{Y}_1(u) = e^{-R\tilde{E}(u) + \int \frac{dv}{2\pi} K(v,u) \log\left[1+\tilde{Y}_1(v)\right]}, \tag{35}$$

which determines completely the free energy

$$\log \mathcal{Z}(L, R) = L \int \frac{du}{2\pi} \tilde{p}'(u) \log\Big[1 + \tilde{Y}_1(v)\Big] + o(e^{-L}). \tag{36}$$

In this way we reproduced, by summing up the tree expansion of the free energy, the TBA equation for the pseudoenergy $\epsilon(u) = -\frac{1}{L}\log \tilde{Y}_1(u)$. The expression (36) for the free energy is true in all orders in $1/L$. In particular, there is no $O(1)$ piece, in accord with the TBA based computation in [11].

3 The Energy of an Excited State

In this section we will apply the tree expansion to the case of an excited state $|\mathbf{u}\rangle$ in the physical channel characterised by a set of rapidities $\mathbf{u} = \{u_1, \dots, u_N\}$. We assume that the excited state is an eigenstate of the Hamiltonian with energy given by Eq. (3).

For large R the wrapping phenomena can be neglected and the rapidities $\mathbf{u}$ satisfy the asymptotic Bethe equations (8)–(4). In order to determine the exact energy and the exact values of the rapidities for finite R, we again introduce a cutoff L by compactifying the cylinnder into a torus obtained as the product of a space-like circle R-circle and a time-like L-circle, with a projector $|\mathbf{u}\rangle\langle\mathbf{u}|$ inserted in the physical channel. The phases of the mirror particles now contain an extra piece which comes from the scattering with the physical particles:

$$\tilde{\phi}_j(v_1,\ldots,v_M) \equiv \tilde{p}(v_j)L + \frac{1}{i}\sum_{k=1}^{N}\log S(\tilde{v}_j,u_k) + \frac{1}{i}\sum_{l(\neq j)}^{M}\log S(\tilde{v}_j,\tilde{v}_l), \qquad j=1,\ldots,M. \tag{37}$$

The computation of the partition function then follows strictly the argument of the previous section, with the only difference that the mirror energy is modified by the scattering with the physical particles. We have to replace

$$e^{-L\tilde{E}(v)} \;\to\; \tilde{Y}_1^{\circ}(v) \equiv e^{-L\tilde{E}(v)}\prod_{k=1}^{M} S(\tilde{v},u_k)\,. \tag{38}$$

Furthermore we have to add to the free energy the contribution from the physical particles that go directly to the opposite edge without scattering,

$$\log\mathcal{Z}(L,R,\mathbf{u}) = -L\sum_{j=1}^{N}E(u_j) + L\int\frac{du}{2\pi}\tilde{p}'(u)\log\left[1+\tilde{Y}_1(v)\right] + O(e^{-L}). \tag{39}$$

with the function $Y(u)$ satisfying non-linear integral equation which slightly generalises Eq. (35),

$$\tilde{Y}_1(v) = \tilde{Y}_1^{\circ}(v)\, e^{\int\frac{du}{2\pi}\log(1+\tilde{Y}_1(u))K(u,v)}. \tag{40}$$

The rapidities of the physical particles are no longer determined by the asymptotic Bethe–Yang equations but by the "exact Bethe equations" which take into account all virtual excitations in the mirror channel. The exact Bethe equations are formulated in terms of the function $\tilde{Y}_1$. In order to avoid confusion we introduce the Y-function in the physical channel, which is related to $\tilde{Y}$ by

$$\tilde{Y}_1(v) = Y_1(\tilde{v}). \tag{41}$$

The exact Bethe equations are obtained by the following requirement. Let $\mathcal{Z}_j(R,L)$ be the partition function with the jth physical particle winding once around the space circle before winding around the time circle. The configurations that contribute to $\mathcal{Z}(R,L)$ and $\mathcal{Z}_j(R,L)$ are depicted in Fig. 4a, b. In order to compute the partition function $\mathcal{Z}_j(R,L)$ we notice that the configurations in Fig. 4b can be simulated by pulling one of the mirror particles out of the thermal ensemble giving to its rapidity a physical value u_j. Indeed, since $S(u_j,u_j)=-1$, the partition function in presence of such extra mirror particle is $-\mathcal{Z}_j(R,L)$. In this way $\mathcal{Z}_j(R,L)$ is given by the sum

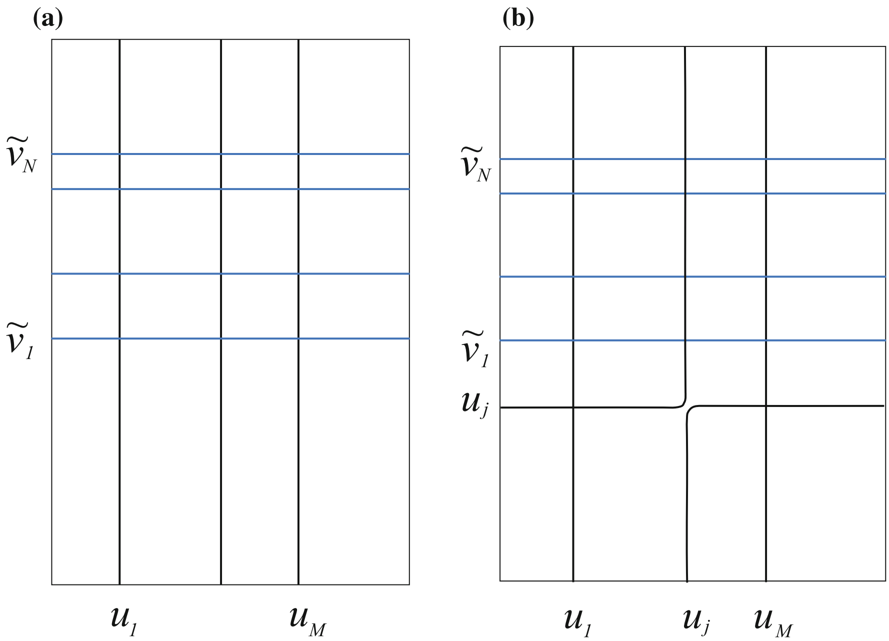


Fig. 4 The configurations that lead to the exact Bethe equation. The physical magnon winding once around the space circle has the same effect, up to a factor (-1), as a physical magnon going straight in presence of a mirror magnon with rapidity u_j

over all trees, with one extra tree having a root $\tilde{v} = u_j$ and $r = 1$. The generating function for such trees is $Y_1(u_j)$, while the contribution of the "vacuum" trees give the partition function: $\mathcal{Z}_j = -Y_1(u_j)\,\mathcal{Z}$. The periodicity in the space direction requires that $\mathcal{Z}_j = \mathcal{Z}$, which gives the exact Bethe–Yang equations

$$Y_1(u_j) = -1, \quad j = 1, \ldots, N. \tag{42}$$

4 One-Point Functions at Finite Volume/Temperature

In this section we will apply the tree expansion to compute the diagonal matrix elements of a local operator at finite volume R. The LeClair-Mussardo conjecture [14] gives an expression for the exact finite temperature one-point functions. In terms of infinite-volume diagonal connected form factors, and densities of mirror states determined by the TBA equation. The conjecture was proven for operators representing densities of conserved quantities in [15] and for general local operator in [16]. The proof of [16] concerns the formula about the diagonal form factors in asymptotically large volume conjectured by Pozsgay and Takacs [17], which is equivalent to the L-M formula. The Pozsgay-Takacs formula, which generalises a result by Saleur [15], gives an expansion of the diagonal matrix elements of a local operator in terms of the infinite-volume form factors with the same or lower number of particles.

4.1 The One-Point Function in Terms of Connected Diagonal Form Factors

In order to simplify the notations, in this section we assume that the physical Hilbert space is associated with the L-circle and the mirror Hilbert space is associated with the R-circle. In infinite volume, all matrix elements of a local operator $\mathcal{O}$ can be expressed, with the help of the crossing formula, in terms of the *elementary form factors*

$$F_n^{\mathcal{O}}(u_1, \ldots, u_n) = \langle 0|\mathcal{O}|u_1, \ldots, u_n\rangle_\infty. \tag{43}$$

The elementary form factors for local operators satisfy the Watson equations

$$F_n(u_1, \ldots, u_j, u_{j+1}, \ldots, u_n) = S(u_j, u_{j+1}) F_n(u_1, \ldots, u_{j+1}, u_j, \ldots, u_n) \tag{44}$$

and have kinematical singularities

$$F(v, u, u_1, \ldots, u_n) = \frac{i}{\bar{v} - u}\left(1 - \prod_{j=1}^{n} S(u, u_j)\right) F_n(u_1, \ldots, u_n) + \text{regular}, \tag{45}$$

where $\bar{v}$ is obtained from v by a crossing transformation. Here it is assumed that the infinite volume states are normalised as $\langle u|v\rangle = 2\pi\delta(u - v)$.

The diagonal limit of the form factors for local operators is ambiguous[2] and there are two prescriptions for evaluating the finite piece, the symmetric and the connected one [17]. The connected diagonal form factor $F^c_{2n}(u_1, \ldots, u_n)$ is obtained by performing the simultaneous limit $\varepsilon_1, \ldots, \varepsilon_n \to 0$ of the elementary form factor $F_{2n}(u_1, \ldots, u_{2n})$ defined by Eq. (43), with $u_{2n-j+1} = \bar{u}_j + i\varepsilon_j$. The limit is not uniform and depends on the prescription, which in this case is to retain only the ε-independent part:

$$F^c_{2n}(\bar{u}_n + i\varepsilon_n, \ldots, \bar{u}_1 + i\varepsilon_1, u_1, \ldots, u_n) = F^c_{2n}(u_1, \ldots, u_n) + \ \varepsilon\text{-dependent terms}. \tag{46}$$

[2]In the case of the non-local operators the situation is even worse: their diagonal limit diverges as L^M where M is the number of the particle pairs.

The Saleur-Pozsgay-Takacs formula [15, 17] relates the diagonal matrix elements in asymptotically large but finite volume L to the connected diagonal form-factors. The formula reads

$$\langle \mathbf{u}|\mathcal{O}|\mathbf{u}\rangle_L = \sum_{\alpha\cup\bar{\alpha}=\mathbf{u}} F^c_{2|\alpha|}(\alpha) \times \det_{j,k\in\bar{\alpha}} G_{jk} + O(e^{-L}), \tag{47}$$

where the sum goes over all partitions of the rapidities $\mathbf{u} = \{u_1, \ldots, u_n\}$ in to two complementary sets α and $\bar{\alpha}$, and $G_{jk} = \partial_{u_j}\phi_k$ is the Gaudin matrix for the n rapidities. It is assumed that $F^c_0 = 0$, so there is no term with $\alpha = \emptyset$. The formula is written for the normalisation with the Gaudin norm

$$\langle \mathbf{u}|\mathbf{u}\rangle = \det_{j,k\in\mathbf{u}} G_{jk}. \tag{48}$$

The determinants on the rhs are the minors of the Gaudin determinant obtain by deleting the lines and the columns that belong to the subset α. It is shown [11, 18] that the expansion (47) is equivalent to the Leclair and Mussardo series for the one-point function of a local operator [14]

$$\langle\mathcal{O}\rangle_R = \sum_{n=1}^{\infty}\frac{1}{n!}\int\prod_{j=1}^{n}\frac{du_j}{2\pi}f(u_j)\, F^c_{2n}(u_1,\ldots,u_n), \qquad f(u) = \frac{Y_1(u)}{1+Y_1(u)}. \tag{49}$$

Below we will derive the Leclair-Mussardo formula from the tree expansion method. In particular, we will reproduce the result obtained by Saleur [15] for the one-point function of a conserved charge. For that we will need the diagonal matrix elements also for the multi-wrapping states $|u_1^{r_1}, \ldots, u_m^{r_m}\rangle$. We will make a very natural conjecture about this action, which turns out to be compatible with the correct formula (49), namely

$$\langle u_m^{r_m}, \ldots, u_1^{r_1}|\mathcal{O}|u_1^{r_1}, \ldots, u_m^{r_m}\rangle_L = \sum_{\alpha\cup\bar{\alpha}=\{u_1,\ldots,u_m\}}\prod_{j\in\alpha} r_j\, F^c_{2|\alpha|}(\alpha)\times\det_{j,k\in\bar{\alpha}} G_{jk}. \tag{50}$$

The logic behind this conjecture is that the action of the operator on a multi wrapping particle is the same as if it were single wrapping particle. The only difference is that the r-wrapping particle appears r times in the same time slice, the operator acts on each copy, which brings an overall factor of r. We should mention here that a discussion about the "multi-diagonal" matrix elements was presented in [19].

4.2 LeClair-Mussardo Series from the Tree Expansion

Repeating the argument from the beginning of Sect. 2.4, we can perform the sum over the complete set of states in the thermal expectation value of the operator $\mathcal{O}$

$$\langle \mathcal{O} \rangle_R = \sum_{M=0}^{\infty} \sum_{n_1 < n_2 < \cdots < n_M} e^{-RE(n_1,\ldots,n_M)} \langle n_1, \ldots, n_M | \mathcal{O} | n_M, \ldots, n_1 \rangle \tag{51}$$

by inserting the expansion (47) in each term of the sum and proceeding as in Sect. 2.3. The expansion analogous to the formula (20) for the partition function is

$$\begin{aligned} \langle \mathcal{O} \rangle_R = & \frac{1}{\mathcal{Z}(L,R)} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{r_1,\ldots,r_m} \int \frac{du_1}{2\pi} \cdots \frac{du_m}{2\pi} \frac{e^{-Lr_1E(u_1) - Lr_mE(u_m)}}{r_1 \ldots r_m} \\ & \times \sum_{\alpha \cup \bar{\alpha} = \{u_1,\ldots,u_m\}} \prod_{j \in \alpha} r_j \, F^c_{2|\alpha|}(\alpha) \frac{\det_{j,k\in\bar{\alpha}} \hat{G}_{jk}}{\prod_{i\in\bar{\alpha}} r_i}, \end{aligned} \tag{52}$$

where the matrix $\hat{G}_{jk}$ is defined by Eq. (26) with $\tilde{p}$ replaced by p and the scattering kernel defined as $K(u, v) = \frac{1}{i}\partial_u \log S(u, v)$.

The next step is to apply the matrix-tree theorem for the diagonal minors of the Gaudin determinant in the last factor in the integrand in (52). A minor obtained by removing all edges and all columns from the subset $\alpha \subset \{1, \ldots, m\}$ of the matrix $\hat{G}_{jk}$ defined in Eq. (26) has the following expansion,

$$\det_{j,k\in\bar{\alpha}} \hat{G}_{jk} = \sum_{\mathcal{F}\in\mathcal{F}_{\alpha,\bar{\alpha}}} \prod_{\text{roots}\in\bar{\alpha}} \hat{D}_i \prod_{\ell_{jk}\in\mathcal{F}} \hat{K}_{kj}. \tag{53}$$

The spanning forests $\mathcal{F} \in \mathcal{F}_{\alpha,\bar{\alpha}}$ are subjected to conditions $(i) - (iii)$ of Sect. 2.5, with the additional restriction that all vertices belonging to α are roots. The weight of these roots is one. An example is given in Fig. 5.

The expansion (53) follows directly from the expansion (27) of the previous section which corresponds to the particular case $\alpha = \emptyset$, $\bar{\alpha} = \{u_1, \ldots, u_m\}$. Indeed, the rhs of (53) is obtained by retaining only the terms in the rhs of (27) that contain the factor $\prod_{j\in\alpha} \hat{D}_j$ and then dividing the sum by this factor.

Now we can proceed similarly to what we have done in the computation of the partition function, where rearranging of the order of summation allowed us to rewrite the sum as a series of tree Feynman diagrams. This time there will be two kinds of Feynman graphs: the "vacuum trees" and diagrams representing a vertex F^c_{2n} with n lines and a tree attached to each line. The weight of such tree is the same as the weight of the vacuum trees except for a factor of r^2 associated with the root. This factor becomes obvious if one writes the dependence of the integrand/summand of (52) on the wrapping numbers $r_1, \ldots, r_m$ as

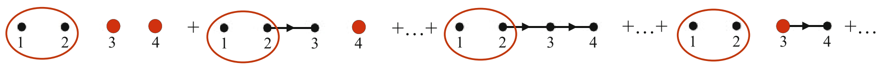


Fig. 5 The tree expansion for a principal minor of the Gaudin matrix $\det_{j,k\in\bar{\alpha}} \hat{G}_{jk}$ for $\alpha = \{1, 2\}$ and $\bar{\alpha} = \{3, 4\}$

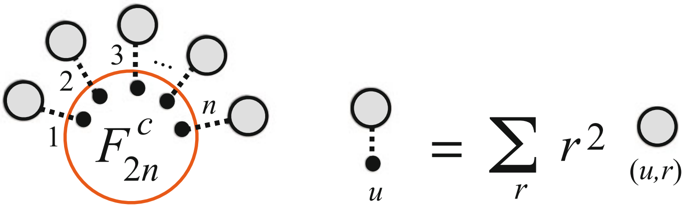


Fig. 6 The tree expansion for the thermal expectation value of a local operator

$$\frac{1}{r_1^2 \dots r_m^2} \prod_{j \in \alpha} r_j^2 .$$

The sum over the vacuum trees cancels with the partition function and the sum over the surviving terms has the same structure as (49), which is depicted in Fig. 6. The factor $f(u)$ is obtained as the sum of all trees with a root at the point u, with extra weight r^2 associated with the root:

$$\sum_r r^2 Y_r(u) = \sum_r (-1)^{r-1} [Y_1(u)]^r = \frac{Y_1(u)}{1 + Y_1(v)} = f(u). \tag{54}$$

The difference of the sum over trees in the factor $f(u)$ compared with the sum over vacuum trees (34) is that there is an extra factor r associated with the root reflecting the breaking of the Z_r symmetry of the corresponding wrapping process.

4.3 The Case of a Conserved Charge

The simplest local operator $\mathcal{O}$ is of the type of conserved charge, such as the energy or the momentum. Such operators act diagonally on multi-particle states with one-particle values $o(u)$. The matrix elements of the operator on a multi-particle state at zero temperature are

$$\mathcal{O} = L^{-1} \int dx \mathcal{O}(x), \qquad \frac{\langle u_n, \dots, u_1 | \mathcal{O} | u_1, \dots, u_n \rangle}{\langle u_n, \dots, u_1 | u_1, \dots, u_n \rangle} = \frac{1}{L} \sum_{j=1}^{n} o(u_j). \tag{55}$$

By direct computation one obtains [15]

$$F^c_{2n}(u_1, \dots, u_n) = p'(u_1) K(u_2, u_1) K(u_3, u_2) \dots K(u_n, u_{n-1})\, o(u_n) \\ + \text{permutations}, \tag{56}$$

to be substituted in the LeClair-Mussardo series (49).

This formula can be readily obtained from the tree expansion using only the definition (55). We start with the series for the partition function (28), with $\tilde{p}$ and $\tilde{E}$ replaced by p and E, and multiply each term by the eigenvalue of the operator $\mathcal{O}$, which acts on the states $|u_1^{r_1} \ldots, u_m^{r_m}\rangle$ as

$$\mathcal{O}|u_1^{r_1} \ldots, u_m^{r_m}\rangle = \frac{1}{L}\sum_j r_j o(u_j)\, |u_1^{r_1} \ldots, u_m^{r_m}\rangle. \tag{57}$$

After expanding the Gaudin norm in trees, one of the trees will acquire an extra factor $r_j o(u_j)$ associated with one of its vertices. The sum over the vacuum gives the partition function which is to be stripped off and one is left with the sum over connected trees with one marked point,

$$\langle\mathcal{O}\rangle_{L,R} = \int \frac{du_1}{2\pi}\int \frac{du_2}{2\pi}\sum_{r_1,r_2} L r_1 p'(u_1)\, Y(u_1, r_1; u_2, r_2)\frac{1}{L} r_2 o(u_2) \tag{58}$$

where $Y(u_1, r_1; u_2, r_2)$ is the partition function of all directed trees with root at (u_1, r_1) and a marked vertex at (u_2, r_2). Any such tree can be decomposed into a backbone consisting of the edges connecting the root and the marked point, and a collection of trees rooted at the vertices along the backbone. We will associate a factor K_{jk} with the edge ℓ_{kj} of the backbone, while the factors r_k and r_j will be absorbed into the weights of the trees rooted at the vertices k and j. In this way the trees rooted at the point j of the backbone contain a factor r_j^2 coming from the two adjacent edges. The sum of such trees gives the factor $f(u)$, Eq. (54). The net result is

$$\langle\mathcal{O}\rangle_R = \sum_{n=1}^{\infty}\int\prod_{j=1}^{n}\frac{du_j}{2\pi}\, p'(u_1)f(u_1)K(u_2, u_1)f(u_2)K(u_3, u_2)\ldots K(u_n, u_{n-1})f(u_n)o(u_n) \tag{59}$$

which is illustrated by Fig. 7

Another way to obtain the one-point function of a conserved charge is by replacing the energy $E(u)$ in the thermal factors with $E(u) - \alpha o(u)$. In this way the problem is

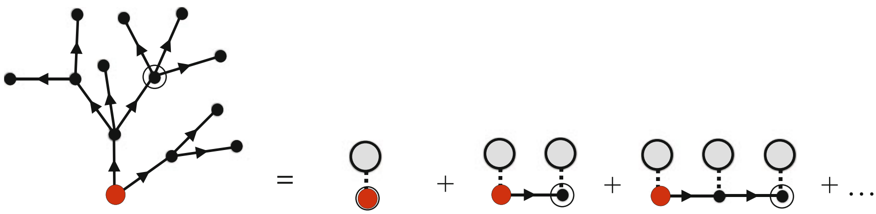


Fig. 7 The tree expansion for the thermal expectation value of a conserved charge. The circle symbolises the vertex where the one-particle operator $o(u)$ is inserted

reduced to the problem of the computation of the thermal partition function, but with slightly changed form of the energy. Since the computation of the partition function does not depend on the specific form of the energy, we can use the formulas of the previous section where $Y_1(u)$ is replaced by $Y_1(u, \alpha)$ determined by the non-linear integral equation

$$\log Y(u, \alpha) = -RE(u) + \alpha o(u) + \int \frac{dv}{2\pi} K(v, u) \log [1 + Y_1(v, \alpha)] \,. \qquad (60)$$

The one-point function is given by the derivative

$$\langle \mathcal{O} \rangle_R = \frac{\partial}{\partial \alpha} \int \frac{du}{2\pi} p'(u) \log(1 + Y_1(u, \alpha)) \Big|_{\alpha=0} = \int \frac{du}{2\pi} p'(u) f(u) \tilde{o}(u), \qquad (61)$$

with $\tilde{o}(u)$ satisfying a linear integral equation obtained by differentiating (35),

$$\tilde{o}(u) = o(u) + \int \frac{dv}{2\pi} K(v, u) f(v)\, \tilde{o}(v). \qquad (62)$$

This gives again the series (58).

5 Conclusion

We proposed a method for computing the finite volume (or finite temperature for the mirror theory) observables in (1+1)-dimensional field theories with factorised diagonal scattering and no bound states. The method is based on an exact treatment of the sum over a complete set of eigenstates of the Hamiltonian of the mirror theory using a graph expansion of the Gaudin measure using the Matrix-Tree Theorem. The free energy and the observables are expressed in terms of tree Feynman graphs. The vertices of such a graph correspond to virtual particles winding multiple times around the compact dimension and the oriented propagators correspond to scattering kernels. The method generalises trivially to the case of a theory with bound states. It is very natural to conjecture that the method can be generalised to theories with non-diagonal scattering.

The tree expansion derived here does not use relativistic invariance, hence the scattering matrix is not necessarily of difference form. Our principal motivation comes from AdS/CFT, where the world sheet (1+1)-dimensional field theory is not Lorentz invariant. We believe that after being generalised for a theory with non-diagonal scattering and bound states, our construction will help to give a renormalised formulation of the hexagon proposal of [5] for computation of correlation functions of trace operators.

Another exercise would be to re-derive the g-functions in the case of integrable boundaries [7, 20]. The exact g-function for diagonal scattering is known [11] but the extension to non-diagonal scattering is still out of reach. The method might be also relevant for the one-point functions in AdS/dCFT [21, 22].

Note Added

After the completion of this work we learned about the earlier papers by G. Kato and M. Wadati [23] where the expression for the free energy of the Lieb–Liniger model and the XXX Heisenberg ferromagnetic has been obtained by a direct combinatorial method which is essentially identical to the one we are proposing here. We thank Balázs Pozsgay for bringing these works to our knowledge.

Acknowledgements We thank Benjamin Basso for enlightening discussions, to Zoltan Bajnok for bringing to our attention ref. Reference [8]. This research was supported in part by the National Science Foundation under Grant No. NSF PHY11-25915.

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Wheeler–DeWitt Quantization of Gravity Models of Unified Dark Energy and Dark Matter

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Abstract First, we describe the construction of a new type of gravity-matter models based on the formalism of non-Riemannian space-time volume forms - alternative generally covariant integration measure densities (volume elements) defined in terms of auxiliary antisymmetric tensor gauge fields. Here gravity couples in a non-conventional way to two distinct scalar fields providing a unified Lagrangian action principle description of: (i) the evolution of both "early" and"late" Universe - by the "inflaton" scalar field; (ii) dark energy and dark matter as a unified manifestation of a single material entity - the "darkon" scalar field. A physically very interesting phenomenon occurs when including in addition interactions with the electro-weak model bosonic sector - we obtain a gravity-assisted dynamical generation of electroweak spontaneous gauge symmetry breaking in the post-inflationary "late" Universe, while the Higgs-like scalar remains massless in the "early" Universe. Next, we proceed to the Wheeler–DeWitt minisuperspace quantization of the above models. The "darkon" field plays here the role of cosmological"time". In particular, we show the absence of cosmological space-time singularities.

Keywords Dark energy · Dark matter · Non-Riemannian volume-forms Electroweak symmetry breaking · Wheeler-DeWitt quantization

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_7

1 Introduction

Among the most important paradigms at the interface of particle physics and cosmology [1–7] one should mention:

- (i) The nature of dark energy and dark matter – both "dark" species occupying around 70 and 25% of the matter content of the "late" (today's) Universe, respectively, continue to be the two most unexplained "mysteries" in cosmology and astrophysics (for a background, see [8–17]).
- (ii) The interplay between the cosmological dynamics and the evolution of the symmetry breaking patterns along the history of the Universe – specifically, for the present epoch's phase of slowly accelerating Universe (dark energy domination) see [8–14], and for a recent general account see [18, 19].

There exist a multitude of proposals for an adequate description of dark energy's and dark matter's dynamics within the framework of standard general relativity or its modern extensions, among them: "Chaplygin gas" models [20–22], "purely kinetic k-essence" models [23, 24], "mimetic" dark matter models [25–28].

Addressing issue (i) above, in Sect. 2 we will briefly review our own approach [29, 30] (for some earlier works, see also [31, 32]) to one of the principal challenge in modern cosmology to understand theoretically from first principles the nature of both "dark" species as a manifestation of the dynamics of a single entity of matter. In the simplest setting we achieve unified description of dark energy and dark matter based on a class of generalized non-canonical models of gravity interacting with a single scalar *"darkon"* field employing the method of non-Riemannian volume-forms on the pertinent spacetime manifold, i.e., non-Riemannian volume elements. Originally [33–35] this approach was proposed as introducing alternative generally covariant integration measure densities in terms of auxiliary "measure" scalar fields. Later [36–38] it was reformulated in a more consistent geometrical setting, namely, the non-Riemannian volume-forms are constructed in terms of auxiliary higher-rank antisymmetric tensor gauge fields, which were shown to be essentially pure-gauge degrees of freedom, i.e., *no* additional propagating field-theoretic (gravitational) degrees of freedom are introduced.

Next, addressing issue (ii) we extend [39, 40] the above non-canonical gravity-matter model by adding coupling to a second scalar *"inflaton"* field describing the universe's evolution in a unified way ("quintessence"), as well as coupling to the fields of the electroweak bosonic sector. In this way we obtain a *gravity-assisted* generation of electro-weak spontaneous gauge symmetry breaking in the post-inflationary "late" Universe, while the Higgs-like scalar remains massless in the "early" Universe [40, 41].

In Sect. 3 we perform Wheeler–DeWitt [42, 43] minisuperspace quantization of the above models. The "darkon" field plays the role of cosmological "time" in the pertinent Wheeler–DeWitt equation in the "early" universe. We show explicitly the absence of cosmological singularities in the wave function of the universe.

2 Quintessence, Unified Dark Energy and Dark Matter, and Gravity-Assisted Higgs Mechanism

2.1 *Hidden Noether Symmetry and Unification of Dark Energy and Dark Matter*

First, let us consider the following simple particular case of a non-conventional gravity-scalar-field action – a member of the general class of the non-Riemannian-volume-element-based gravity-matter theories [38, 39] (for simplicity we use units with the Newton constant $G_N = 1/16\pi$):

$$S = \int d^4x \sqrt{-g}\, R + \int d^4x \big(\sqrt{-g} + \Phi(C)\big) L(u, Y) \; . \tag{1}$$

Here R denotes the standard Riemannian scalar curvature for the pertinent Riemannian metric $g_{\mu\nu}$. In the second term in (1) – the scalar field Lagrangian is coupled *symmetrically* to two mutually independent spacetime volume-elements – the standard Riemannian $\sqrt{-g}$ and to an alternative non-Riemannian one:

$$\Phi(C) = \frac{1}{3!} \varepsilon^{\mu\nu\kappa\lambda} \partial_\mu C_{\nu\kappa\lambda} \; . \tag{2}$$

$L(u, Y)$ is general-coordinate invariant Lagrangian of a single scalar field $u(x)$, the simplest example being:

$$L(u, Y) = Y - V(u) \quad , \quad Y \equiv -\frac{1}{2} g^{\mu\nu} \partial_\mu u \partial_\nu u \; , \tag{3}$$

Crucial new property – we obtain *dynamical constraint* on $L(u, Y)$ as a result of the equations of motion w.r.t. $C_{\mu\nu\lambda}$:

$$\partial_\mu L(u, Y) = 0 \;\longrightarrow\; L(u, Y) = -2M_0 = \text{const} \; , \tag{4}$$

i.e., $Y = V(u) - 2M_0$. M_0 will play the role of dynamically generated cosmological constant.

A second crucial property – *hidden strongly nonlinear Noether symmetry* of scalar field action in (1) – is due to the presence of the non-Riemannian volume element $\Phi(C)$. The scalar field action is invariant (up to a total derivative) under the following nonlinear symmetry transformations:

$$\delta_\epsilon u = \epsilon \sqrt{Y} \quad , \quad \delta_\epsilon g_{\mu\nu} = 0 \quad , \quad \delta_\epsilon \mathcal{C}^\mu = -\epsilon \frac{1}{2\sqrt{Y}} g^{\mu\nu} \partial_\nu u \big(\Phi(C) + \sqrt{-g}\big) \; , \tag{5}$$

where $\mathcal{C}^\mu \equiv \frac{1}{3!} \varepsilon^{\mu\nu\kappa\lambda} C_{\nu\kappa\lambda}$.

Then, standard Noether procedure yields a conserved current:

$$\nabla_\mu J^\mu = 0 \quad , \quad J^\mu \equiv -\Big(1 + \frac{\Phi(C)}{\sqrt{-g}}\Big)\sqrt{2Y}\, g^{\mu\nu}\partial_\nu u \tag{6}$$

The energy-momentum tensor $T_{\mu\nu}$ and J^μ (6) can be cast into a relativistic hydrodynamical form (taking into account (4)):

$$T_{\mu\nu} = -2M_0 g_{\mu\nu} + \rho_0 u_\mu u_\nu \quad , \quad J^\mu = \rho_0 u^\mu \; , \tag{7}$$

where the pressure $p = -2M_0 = \text{const}$ and:

$$\rho_0 \equiv \Big(1 + \frac{\Phi(C)}{\sqrt{-g}}\Big) 2Y \quad , \quad u_\mu \equiv -\frac{\partial_\mu u}{\sqrt{2Y}} \quad , \quad u^\mu u_\mu = -1 \; . \tag{8}$$

The total energy density is $\rho = \rho_0 - p = 2M_0 + \Big(1 + \frac{\Phi(C)}{\sqrt{-g}}\Big) 2Y$.

Because of the constant pressure ($p = -2M_0$) $\nabla^\nu T_{\mu\nu} = 0$ implies *both* hidden Noether symmetry current $J^\mu = \rho_0 u^\mu$ conservation, as well as *geodesic fluid motion*:

$$\nabla_\mu\big(\rho_0 u^\mu\big) = 0 \quad , \quad u_\nu \nabla^\nu u_\mu = 0 \; . \tag{9}$$

Therefore, $T_{\mu\nu} = -2M_0 g_{\mu\nu} + \rho_0 u_\mu u_\nu$ represents an exact sum of two contributions of the two dark species:

$$p = p_{\rm DE} + p_{\rm DM} \quad , \quad \rho = \rho_{\rm DE} + \rho_{\rm DM} \tag{10}$$

$$p_{\rm DE} = -2M_0 \;\; , \;\; \rho_{\rm DE} = 2M_0 \quad ; \quad p_{\rm DM} = 0 \;\; , \;\; \rho_{\rm DM} = \rho_0 \; , \tag{11}$$

i.e., the dark matter component is a dust fluid flowing along geodesics. This is explicit unification of dark energy and dark matter originating from the dynamics of a single scalar field - the "darkon" u.

2.2 *Quintessential Inflation and Unified Dark Energy and Dark Matter*

We will now extend our previous gravity-"darkon" model to gravity coupled to both "inflaton" $\varphi(x)$ and "darkon" $u(x)$ scalar fields within the non-Riemannian volume-form formalism, as well as we will also add coupling to the bosonic sector of the electro-weak model:

$$S = \int d^4x\, \Phi(A)\Big[g^{\mu\nu} R_{\mu\nu}(\Gamma) + L_1(\varphi, X) + L_2(\sigma, \nabla\sigma; \varphi)\Big] + \\ \int d^4x\, \Phi(B)\Big[U(\varphi) + L_3(\mathcal{A}, \mathcal{B}) + \frac{\Phi(H)}{\sqrt{-g}}\Big] + \int d^4x \big(\sqrt{-g} + \Phi(C)\big) L(u, Y) \; . \tag{12}$$

Here the following notations are used:

- $\Phi(A) = \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu A_{\nu\kappa\lambda}$ and $\Phi(B) = \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu B_{\nu\kappa\lambda}$ – two new independent non-Riemannian volume-forms (non-Riemannian volume elements) apart from $\Phi(C)$;
- $\Phi(H) = \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu H_{\nu\kappa\lambda}$ is the dual field-strength of an additional auxiliary tensor gauge field $H_{\nu\kappa\lambda}$ crucial for the consistency of (12).
- Important – we use Palatini formalism: $R = g^{\mu\nu}R_{\mu\nu}(\Gamma)$; $g_{\mu\nu}$, $\Gamma^\lambda_{\mu\nu}$ – metric and affine connection are apriori independent.
- $\sigma \equiv (\sigma_a)$ is a complex $SU(2) \times U(1)$ iso-doublet Higgs-like scalar field with a Lagrangian:

$$L_2(\sigma, \nabla\sigma; \varphi) = -g^{\mu\nu}\big(\nabla_\mu\sigma_a\big)^*\nabla_\nu\sigma_a - V_0(\sigma)e^{\alpha\varphi} . \tag{13}$$

The gauge-covariant derivative acting on σ reads:

$$\nabla_\mu\sigma = \Big(\partial_\mu - \frac{i}{2}\tau_A\mathcal{A}^A_\mu - \frac{i}{2}\mathcal{B}_\mu\Big)\sigma , \tag{14}$$

with $\frac{1}{2}\tau_A$ (τ_A – Pauli matrices, $A = 1, 2, 3$) indicating the $SU(2)$ generators.

- The "bare" σ-field potential is of the same form as the standard Higgs potential:

$$V_0(\sigma) = \frac{\lambda}{4}\left((\sigma_a)^*\sigma_a - \mu^2\right)^2 . \tag{15}$$

- The $SU(2) \times U(1)$ gauge field action $L(\mathcal{A}, \mathcal{B})$ is of the standard Yang–Mills form (all $SU(2)$ indices $A, B, C = (1, 2, 3)$):

$$L_3(\mathcal{A}, \mathcal{B}) = -\frac{1}{4g^2}F^2(\mathcal{A}) - \frac{1}{4g'^2}F^2(\mathcal{B}) , \tag{16}$$

$$F^2(\mathcal{A}) \equiv F^A_{\mu\nu}(\mathcal{A})F^A_{\kappa\lambda}(\mathcal{A})g^{\mu\kappa}g^{\nu\lambda} \quad , \quad F^2(\mathcal{B}) \equiv F_{\mu\nu}(\mathcal{B})F_{\kappa\lambda}(\mathcal{B})g^{\mu\kappa}g^{\nu\lambda} ,$$
$$F^A_{\mu\nu}(\mathcal{A}) = \partial_\mu\mathcal{A}^A_\nu - \partial_\nu\mathcal{A}^A_\mu + \epsilon^{ABC}\mathcal{A}^B_\mu\mathcal{A}^C_\nu \quad , \quad F_{\mu\nu}(\mathcal{B}) = \partial_\mu\mathcal{B}_\nu - \partial_\nu\mathcal{B}_\mu .$$

$\mathcal{A}^A_\mu$ ($A = 1, 2, 3$) and $\mathcal{B}_\mu$ denote the corresponding $SU(2)$ and $U(1)$ electroweak gauge fields.

- The "inflaton" φ Lagrangian terms are given by:

$$L_1(\varphi, X) = X - V_1(\varphi) \quad , \quad X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi , \tag{17}$$

$$V_1(\varphi) = f_1 \exp\{\alpha\varphi\} \quad , \quad U(\varphi) \equiv f_2 \exp\{2\alpha\varphi\} , \tag{18}$$

where α, f_1, f_2 are dimensionful positive parameters.

- The form of the action (12) is fixed by the requirement of invariance under global Weyl-scale transformations:

$$g_{\mu\nu} \to \lambda g_{\mu\nu} \; , \; \Gamma^{\mu}_{\nu\lambda} \to \Gamma^{\mu}_{\nu\lambda} \; , \; \varphi \to \varphi - \frac{1}{\alpha} \ln \lambda \; ,$$
$$A_{\mu\nu\kappa} \to \lambda A_{\mu\nu\kappa} \; , \; B_{\mu\nu\kappa} \to \lambda^2 B_{\mu\nu\kappa} \; , \; H_{\mu\nu\kappa} \to H_{\mu\nu\kappa} \; , \tag{19}$$

and the electro-weak sector $(\sigma, \mathcal{A}, \mathcal{B})$ is inert w.r.t. (19).

Equations of motion w.r.t. affine connection $\Gamma^{\mu}_{\nu\lambda}$ yield a solution for the latter as a Levi-Civita connection:

$$\Gamma^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda}(\bar{g}) = \frac{1}{2}\bar{g}^{\mu\kappa}\left(\partial_\nu \bar{g}_{\lambda\kappa} + \partial_\lambda \bar{g}_{\nu\kappa} - \partial_\kappa \bar{g}_{\nu\lambda}\right) \; , \tag{20}$$

w.r.t. to the *Weyl-rescaled metric* $\bar{g}_{\mu\nu}$:

$$\bar{g}_{\mu\nu} = \chi_1 g_{\mu\nu} \quad , \quad \chi_1 \equiv \frac{\Phi_1(A)}{\sqrt{-g}} \; . \tag{21}$$

Transition from original metric $g_{\mu\nu}$ to $\bar{g}_{\mu\nu}$: *"Einstein-frame"*, where the gravity equations of motion are written in the standard form of Einstein's equations: $R_{\mu\nu}(\bar{g}) - \frac{1}{2}\bar{g}_{\mu\nu}R(\bar{g}) = \frac{1}{2}T^{\rm eff}_{\mu\nu}$ with an appropriate *effective* energy-momentum tensor given in terms of an Einstein-frame matter Lagrangian $L_{\rm eff}$ (see (25) below).

Solutions of the eqs. of motion of the action (12) w.r.t. auxiliary tensor gauge fields $A_{\mu\nu\lambda}$, $B_{\mu\nu\lambda}$ and $H_{\mu\nu\lambda}$ yield:

$$\frac{\Phi(B)}{\sqrt{-g}} \equiv \chi_2 = \text{const} \quad , \quad R + L_1(\varphi, X) + L_2(\sigma, \nabla\sigma; \varphi) = M_1 = \text{const} \; ,$$
$$U(\varphi) + L_3(\mathcal{A}, \mathcal{B}) + \frac{\Phi(H)}{\sqrt{-g}} = -M_2 = \text{const} \; . \tag{22}$$

Here M_1 and M_2 are arbitrary dimensionful and χ_2 arbitrary dimensionless integration constants, similar to M_0 (4).

Within the canonical Hamilton formalism we have shown [37, 38, 44] that M_0, $M_{1,2}$, χ_2 are the only remnant of the auxiliary gauge fields $C_{\mu\nu\lambda}$, $A_{\mu\nu\lambda}$, $B_{\mu\nu\lambda}$, $H_{\mu\nu\lambda}$ entering (12) – they have the meaning of conserved Dirac-constrained canonical momenta conjugated to some of the components of the latter.

We derive from (12) the physical *Einstein-frame* theory w.r.t. Weyl-rescaled Einstein-frame metric $\bar{g}_{\mu\nu}$ (21) and perform an additional "darkon" field redefinition $u \to \widetilde{u}$:

$$\frac{\partial \widetilde{u}}{\partial u} = \left(V_1(u) - 2M_0\right)^{-\frac{1}{2}} \quad ; \quad Y \to \widetilde{Y} = -\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu \widetilde{u}\partial_\nu \widetilde{u} \; . \tag{23}$$

The Einstein-frame action reads:

$$S = \int d^4x \sqrt{-\bar{g}}\Big[R(\bar{g}) + L_{\rm eff}\big(\varphi, \bar{X}, \widetilde{Y}; \sigma, \bar{X}_\sigma, \mathcal{A}, \mathcal{B}\big)\Big] \, , \tag{24}$$

where (now the kinetic terms are given in terms of the Einstein-frame metric (21), e.g. $\bar{X} = -\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$, etc.):

$$L_{\rm eff}\big(\varphi, \bar{X}, \widetilde{Y}; \sigma, \bar{X}_\sigma, \mathcal{A}, \mathcal{B}\big) = \bar{X} - \widetilde{Y}\Big(V_1(\varphi) + V_0(\sigma)e^{\alpha\varphi} + M_1\Big) + \widetilde{Y}^2\Big[\chi_2(U(\varphi) + M_2) - 2M_0\Big] + L[\sigma, \bar{X}_\sigma, \mathcal{A}, \mathcal{B}] \,, \qquad (25)$$

with $L[\sigma, \bar{X}_\sigma, \mathcal{A}, \mathcal{B}] \equiv -\bar{g}^{\mu\nu}\big(\nabla_\mu\sigma_a)^*\nabla_\nu\sigma_a - \frac{\chi_2}{4g^2}\bar{F}^2(\mathcal{A}) - \frac{\chi_2}{4g'^2}\bar{F}^2(\mathcal{B})$.

For static (spacetime independent) scalar field configurations we obtain from (25) the following Einstein-frame effective scalar "inflaton+Higgs" effective potential:

$$U_{\rm eff}\big(\varphi, \sigma\big) = \frac{\Big(V_1(\varphi) + V_0(\sigma)e^{\alpha\varphi} + M_1\Big)^2}{4\big[\chi_2(U(\varphi) + M_2) - 2M_0\big]} = \frac{\Big[\Big(f_1 + \frac{\lambda}{4}\big((\sigma_a)^*\sigma_a - \mu^2\big)^2\Big)e^{\alpha\varphi} + M_1\Big]^2}{4\big[\chi_2(f_2 e^{2\alpha\varphi} + M_2) - 2M_0\big]} \,. \qquad (26)$$

$U_{\rm eff}\big(\varphi, \sigma\big)$ has few remarkable properties. First, $U_{\rm eff}\big(\varphi, \sigma\big)$ possesses two infinitely large flat regions as function of φ (when σ is fixed):
(a) $(-)$ flat region for large negative values of the "inflaton" φ;
(b) $(+)$ flat region and large positive values of φ,
respectively, as depicted in Fig. 1.

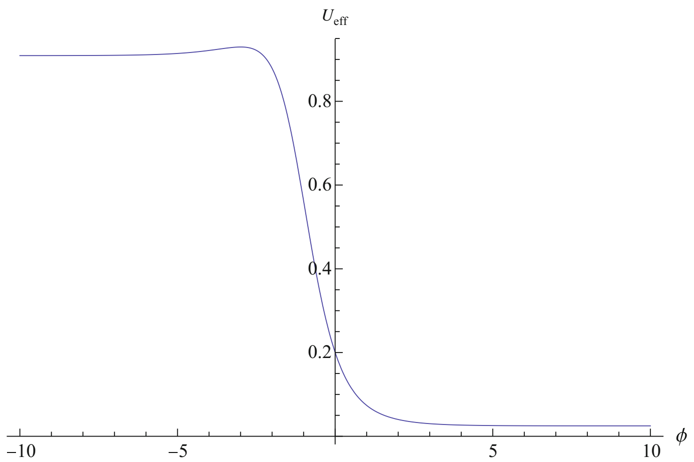

Fig. 1 Qualitative shape of the effective scalar potential $U_{\rm eff}$ (26) as function of φ at σ = fixed for $M_1 > 0$

- In the $(+)$ flat region (large positive "inflaton" values) (26) reduces to:

$$U_{\text{eff}}\big(\varphi,\sigma\big) \simeq U_{(+)}(\sigma) = \frac{\Big(\frac{\lambda}{4}\left((\sigma_a)^*\sigma_a - \mu^2\right)^2 + f_1\Big)^2}{4\chi_2 f_2} \;. \tag{27}$$

- Equation (27) yields as a lowest lying vacuum the Higgs one:

$$|\sigma| = \mu \;, \tag{28}$$

  i.e., we obtain the standard spontaneous breakdown of $SU(2)\times U(1)$ gauge symmetry.
- At the Higgs vacuum (28) we get from (27) a dynamically generated cosmological constant $\Lambda_{(+)}$:

$$U_{(+)}(\mu) \equiv 2\Lambda_{(+)} = \frac{f_1^2}{4\chi_2 f_2} \;. \tag{29}$$

- If we identify the integration constants in (26) with the fundamental constants of Nature – M_{Pl} (Planck mass) and M_{EW} (electro-weak mass scale) as $f_1 \sim M_{EW}^4$, $f_2 \sim M_{Pl}^4$, we are then naturally led to a very small vacuum energy density:

$$U_{(+)}(\mu) \sim M_{EW}^8/M_{Pl}^4 \sim 10^{-122} M_{Pl}^4 \;, \tag{30}$$

  which is the right order of magnitude for the present epoch's vacuum energy density according to [45]. Therefore, we can identify the $(+)$ flat region (large positive "inflaton" values) of U_{eff} (26) as describing the present "late" universe.

- In the $(-)$ flat region (large negative "inflaton" values) (26) reduces to:

$$U_{\text{eff}}\big(\varphi,\sigma\big) \simeq U_{(-)} \equiv \frac{M_1^2}{4(\chi_2 M_2 - 2M_0)} \;. \tag{31}$$

  If we take $M_1 \sim M_2 \sim 10^{-8} M_{Pl}^4$ and $M_0 \sim M_{EW}^4$, then the vacuum energy density $U_{(-)}$ (31) becomes $U_{(-)} \sim 10^{-8} M_{Pl}^4$, which conforms to the Planck Collaboration data [46, 47] for the energy scale of inflation (of order $10^{-2} M_{Pl}$). This allows to identify the $(-)$ flat region (large negative "inflaton" values) of the "inflaton+Higgs" effective potential (26) as describing the"early" universe, in particular, the inflationary epoch.
- In the $(-)$ flat region the effective potential (31) is σ-field idependent. Thus, the Higgs-like iso-doublet scalar field σ_a remains *massless* in the "early" (inflationary) Universe and accordingly there is *no electro-weak spontaneous symmetry breaking* there.

3 Wheeler–De Witt Minisuperspace Quantization

For simplicity here we will consider the unified dark energy/dark matter "quintessential" model (12) without the coupling to the bosonic electro-weak sector. The corresponding Einstein-frame action reads:

$$S = \int d^4x \sqrt{-\bar{g}} \Big[R(\bar{g}) + L_{\rm eff}\big(\varphi, \bar{X}, \widetilde{Y}\big) \Big] \,, \tag{32}$$

where (recall $\bar{X} = -\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$ and $\widetilde{Y} = -\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\widetilde{u}\partial_\nu\widetilde{u}$):

$$L_{\rm eff}\big(\varphi, \bar{X}, \widetilde{Y}\big) = \bar{X} - \widetilde{Y}\Big(V(\varphi) - M_1\Big) + \widetilde{Y}^2\Big[\chi_2(U(\varphi) + M_2) - 2M_0\Big] \,, \tag{33}$$

To study cosmological implications of (32) we perform a Friedmann–Lemaitre–Robertson–Walker (*FLRW*) reduction to the class of FLRW metrics:

$$ds^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = -N^2(t)dt^2 + a^2(t)d\mathbf{x}.d\mathbf{x} \tag{34}$$

and take the "inflaton" and"darkon" to be time-dependent only, i.e.:

$$\bar{X} = \frac{1}{2}\,\dot{\varphi}^2 \quad , \quad \widetilde{Y} = \frac{1}{2}w^2 \quad , \quad w \equiv \frac{d\widetilde{u}}{dt} \;. \tag{35}$$

The FLRW reduced action corresponding to (32) reads:

$$S_{\rm FLRW} = \int dt \Bigg\{ -\frac{1}{N} 6a\;\dot{a}^2 + Na^3 \Big[\frac{\dot{\varphi}^2}{2N^2} - (f_1 e^{\alpha\varphi} + M_1)\frac{w^2}{2N^2} \tag{36}$$

$$+ \Big(\chi_2(f_2 e^{2\alpha\varphi} + M_2) - 2M_0\Big)\frac{w^4}{4N^4}\Big]\Bigg\} \tag{37}$$

Calculating the canonically conjugated momenta $p_a, p_\varphi, p_{\widetilde{u}}$, we arrive at the canonical FLRW Hamiltonian:

$$\mathcal{H} = N\mathcal{H}_{WDW} = N\Bigg\{ -\frac{p_a^2}{24a} + \frac{p_\varphi^2}{a^3} + p_{\widetilde{u}} w \tag{38}$$

$$+ a^3\Big[(f_1 e^{\alpha\varphi} + M_1)\frac{w^2}{2} - \Big(\chi_2(f_2 e^{2\alpha\varphi} + M_2) - 2M_0\Big)\frac{w^4}{4}\Big]\Bigg\} \tag{39}$$

$\mathcal{H}$ turns out to be pure first-class constraint $\mathcal{H}_{WDW}$ a'la Dirac with the lapse N as Lagrange multiplier.

In (39) the "darkon" velocity w is determined as function of the canonical variables $(a, \varphi, p_{\widetilde{u}})$ being the real root (for all values of $(a, \varphi, p_{\widetilde{u}})$) of the cubic algebraic equation:

$$w^3 - 3A(\varphi)\, w - 2\frac{B(\varphi, p_{\widetilde{u}})}{a^3} = 0 \tag{40}$$

where the coefficients are given by:

$$A(\varphi) \equiv \frac{1}{3}\, \frac{(f_1 e^{\alpha\varphi} + M_1)}{\chi_2(f_2 e^{2\alpha\varphi} + M_2) - 2M_0}\,,$$
$$B(\varphi, p_{\widetilde{u}}) \equiv \frac{p_{\widetilde{u}}}{2}\, \frac{1}{\chi_2(f_2 e^{2\alpha\varphi} + M_2) - 2M_0}\,. \tag{41}$$

The solution of (40) for $w = w(a, \varphi, p_{\widetilde{u}})$ reads:

$$w = \mathrm{sign}(B(\varphi, p_{\widetilde{u}}))|A(\varphi)|^{1/2}|\xi|^{-1/6}\Big[\big(1+\sqrt{1-\xi}\big)^{1/3} + \big(1-\sqrt{1-\xi}\big)^{1/3}\Big] \tag{42}$$

where $\xi \equiv \xi(a, \varphi, p_{\widetilde{u}}) = \frac{A^3(\varphi)}{9B^2(\varphi, p_{\widetilde{u}})}\, a^6$.

Quantization of the Dirac-constrained canonical Hamilton (39) yields the Wheeler–DeWitt (WDW) equation for the wave function of the universe $\Psi = \Psi(a, \varphi; p_{\widetilde{u}})$:

$$\widehat{\mathcal{H}}_{WDW}\Psi(a, \varphi; p_{\widetilde{u}}) = 0\,, \tag{43}$$

where $\widehat{\mathcal{H}}_{WDW}$ is the quantum version of $\mathcal{H}_{WDW}$ in (39). We resolve the ordering ambiguity there by changing variables:

$$a \to \widetilde{a} = \frac{4}{\sqrt{3}} a^{3/2}\,, \tag{44}$$

and taking the special operator ordering:

$$\frac{p_a^2}{24a} \to \frac{1}{2}\frac{1}{\sqrt{12a}}\widehat{p}_a \frac{1}{\sqrt{12a}}\widehat{p}_a = -\frac{1}{2}\frac{\partial^2}{\partial \widetilde{a}^2}\,. \tag{45}$$

The WDW operator $\widehat{\mathcal{H}}_{WDW}$ becomes:

$$\widehat{\mathcal{H}}_{WDW} = \frac{1}{2}\frac{\partial^2}{\partial \widetilde{a}^2} + \frac{8}{3\widetilde{a}^2}\widehat{p}_\varphi^2 + \frac{3}{4} p_{\widetilde{u}} w + \frac{3}{64} w^2 \widetilde{a}^2 (f_1 e^{\alpha\varphi} + M_1)\,, \tag{46}$$

where $\widehat{p}_\varphi = -i\partial/\partial\varphi$ and $w = w(\widetilde{a}, \varphi, p_{\widetilde{u}})$ is the solution (42) of the cubic equation (40).

The final form of WDW equation reads:

$$\Big[\frac{1}{2}\Big(\frac{\partial}{\partial \widetilde{a}}\Big)^2 + \frac{8}{3\widetilde{a}^2}\widehat{p}_\varphi^2 + U(\widetilde{a}, \varphi, p_{\widetilde{u}})\Big]\Psi(\widetilde{a}, \varphi; p_{\widetilde{u}}) = 0\,, \tag{47}$$
$$U(\widetilde{a}, \varphi, p_{\widetilde{u}}) \equiv \frac{\widetilde{a}^2 (f_1 e^{\alpha\varphi} + M_1)^2}{64(\chi_2 f_2 e^{2\alpha\varphi} + \chi_2 M_2 - 2M_0)}\mathcal{F}\big(\xi(\widetilde{a}, \varphi, p_{\widetilde{u}})\big)) \tag{48}$$

with the following notations:

$$\xi(\widetilde{a},\varphi,p_{\widetilde{u}}) \equiv \frac{\widetilde{a}^4 (f_1 e^{\alpha\varphi} + M_1)^3}{192 p_{\widetilde{u}}^2 (\chi_2 f_2 e^{2\alpha\varphi} + \chi_2 M_2 - 2M_0)} , \tag{49}$$

$$\mathcal{F}(\xi) \equiv \xi^{-1/3}\Big[\big(1+\sqrt{1-\xi}\big)^{1/3} + \big(1-\sqrt{1-\xi}\big)^{1/3}\Big] \times \Big[2\xi^{-1/3} + \big(1+\sqrt{1-\xi}\big)^{1/3} + \big(1-\sqrt{1-\xi}\big)^{1/3}\Big] . \tag{50}$$

Analytic solutions of (47) can be found when the "inflaton" φ is either on the $(-)$ flat region (φ large negative – "early" universe) or on $(+)$ flat region (φ large positive – "late"/nowadays universe), cf. Fig. 1 above.

In the $(+)$ flat region of the "inflaton" φ ("late" universe) the WDW equation (47) reduces to the quantum mechanical Schrödinger equation:

$$\Big[\frac{1}{2}\frac{\partial^2}{\partial\widetilde{a}^2} + \mathcal{W}_{(+)}(\widetilde{a},p_\varphi)\Big]\Psi(\widetilde{a},p_\varphi) = 0 , \tag{51}$$

$$\mathcal{W}_{(+)}(\widetilde{a},p_\varphi) \equiv \frac{3f_1^2}{64\chi_2 f_2}\widetilde{a}^2 + \frac{8p_\varphi^2}{3}\widetilde{a}^{-2} \quad , \quad p_\varphi - \text{small} . \tag{52}$$

The solution of (51) reads (here $c_{1,2}$ are constants):

$$\Psi(\widetilde{a},p_\varphi) = \sqrt{\widetilde{a}}\Big[c_1 J_{\frac{1}{4}\sqrt{1-\gamma}}\Big(\frac{1}{2}\beta\widetilde{a}^2\Big) + c_2 J_{-\frac{1}{4}\sqrt{1-\gamma}}\Big(\frac{1}{2}\beta\widetilde{a}^2\Big)\Big] , \tag{53}$$

$$\beta \equiv \sqrt{\frac{3f_1^2}{32\chi_2 f_2}} \quad , \quad \gamma \equiv \frac{64}{3}p_\varphi^2 \quad (\gamma - \text{small}) , \tag{54}$$

$$\Psi(\widetilde{a},p_\varphi) \simeq \text{const}\ \widetilde{a}^{\frac{1}{2}(1-\sqrt{1-\gamma})} \quad \text{for} \quad \widetilde{a} \to 0 , \tag{55}$$

i.e., the wave function (53) vanishes at $\widetilde{a} = 0$.

Similarly, in the $(-)$ flat region of the "inflaton" φ ("early" universe) the WDW equation (47) reduces to the quantum mechanical Schrödinger equation:

$$\Big[\frac{1}{2}\frac{\partial^2}{\partial\widetilde{a}^2} + \mathcal{W}_{(-)}(\widetilde{a},p_\varphi,p_{\widetilde{u}})\Big]\Psi(\widetilde{a},p_\varphi,p_{\widetilde{u}}) = 0 , \tag{56}$$

$$\mathcal{W}_{(-)}(\widetilde{a},p_\varphi,p_{\widetilde{u}}) = \frac{3M_1^2}{64(\chi_2 M_2 - 2M_0)}\widetilde{a}^2 + \frac{8p_\varphi^2}{3}\widetilde{a}^{-2} + p_{\widetilde{u}}\sqrt{\frac{M_1}{\chi_2 M_2 - 2M_0}} + O\Big(\frac{p_{\widetilde{u}}^2}{\widetilde{a}^2}\Big) . \tag{57}$$

In (56) and (57) the canonical "darkon" momentum (times a constant) plays the role of energy eigenvalue $E \equiv p_{\widetilde{u}}\sqrt{\frac{M_1}{\chi_2 M_2 - 2M_0)}}$, meaning that the "darkon" field $\widetilde{u}$ plays the role of cosmological"time" in the "early" universe.

We can solve explicitly WDW equation (56) for small "darkon" momenta $p_{\widetilde{u}}$ ignoring the last term in (57):

$$\Psi(\widetilde{a}, p_\varphi, p_{\widetilde{u}}) = \text{const}\; \widetilde{a}^{\frac{1}{2}(1+\sqrt{1-\gamma})}\, e^{\frac{i}{2}\beta\widetilde{a}^2} \times U\Big(\frac{1}{4}(2+\sqrt{1-\gamma}) - i\frac{E}{2\beta}, \frac{1}{2}(2+\sqrt{1-\gamma}); -i\beta\widetilde{a}^2\Big)\,, \tag{58}$$

$$\beta \equiv \sqrt{\frac{3M_1^2}{32(\chi_2 M_2 - 2M_0}} \quad , \quad \gamma \equiv \frac{64}{3} p_\varphi^2 \quad , \quad E \equiv p_{\widetilde{u}} \sqrt{\frac{M_1}{\chi_2 M_2 - 2M_0}}\,, \tag{59}$$

where $U(\cdot, \cdot; z)$ denotes the confluent hypergeometric function of the second kind. Again as in (55) the wave function (58) vanishes at $\widetilde{a} = 0$:

$$\Psi(\widetilde{a}, p_\varphi, p_{\widetilde{u}}) \simeq \text{const}\; \widetilde{a}^{\frac{1}{2}(1-\sqrt{1-\gamma})} \quad \text{for} \quad \widetilde{a} \to 0\,, \tag{60}$$

In the inflationary "slow-roll" regime in the"early" Universe the "inflaton" canonical momentum p_φ is very small. Thus, ignoring also the second term in $\mathcal{W}_{(-)}$ (57) and Fourier-transforming (58) w.r.t. canonical "darkon" momentum $p_{\widetilde{u}}$ with E as in (59):

$$\Psi(\widetilde{a}, \tau) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} \Psi(\widetilde{a}, p_\varphi{=}0, p_{\widetilde{u}})\, e^{-iE\tau} \quad , \quad E \equiv p_{\widetilde{u}} \sqrt{\frac{M_1}{\chi_2 M_2 - 2M_0}}\,, \tag{61}$$

i.e., $\tau \sim \widetilde{u}$ being the "cosmological" time, the WDW equation (56) and (57) acquires the form of a time-dependent Schrödinger equation for the inverted harmonic oscillator:

$$i\frac{\partial}{\partial\tau}\Psi(\widetilde{a}, \tau) = \Big[-\frac{1}{2}\frac{\partial^2}{\partial\widetilde{a}^2} - \omega^2\widetilde{a}^2\Big]\Psi(\widetilde{a}, \tau) \tag{62}$$

with a negative "frequency" squared:

$$-\omega^2 \equiv -\frac{3M_1^2}{64(\chi_2 M_2 - 2M_0)} \equiv -\frac{3}{16} U_{(-)}\,, \tag{63}$$

where $U_{(-)}$ (31) is the vacuum energy density of the inflationary epoch.

The solution of equation (62) in the form of a normalized (on the semiaxis $\widetilde{a} \in (0, \infty)$) wave packet has already been found in [48]:

$$\Psi(\widetilde{a}, \tau) = \Big(\frac{2\omega}{\pi}\sin(2b)\Big)^{1/4} \big(\cos(b - i\omega\tau)\big)^{-1/2} \times \exp\Big\{-\frac{1}{2}\widetilde{a}^2\omega\tan(b - i\omega\tau)\Big\}\,, \tag{64}$$

where the parameter b describes the width of the wave packet. Calculating the average value of the FLRW scale factor $a = \frac{\sqrt{3}}{4}\widetilde{a}^{2/3}$ (cf. (44)) we obtain:

$$\langle\widetilde{a}\rangle \equiv \int_0^\infty d\widetilde{a}\,\widetilde{a}|\Psi(\widetilde{a},\tau)|^2 = \Big[\frac{\cos(2b)+\cosh(2\omega\tau)}{\pi\omega\sin(2b)}\Big]^{1/2}. \tag{65}$$

Thus, the quantum average of the FLRW scale factor does not exhibit any singularity ($\langle\widetilde{a}\rangle \to 0$) at any "time" τ.

4 Conclusions

Employing non-Riemannian spacetime volume-forms (non-Riemannian volume elements) in generalized gravity-matter theories allows for several interesting developments:

- Simple unified description of dark energy and dark matter as manifestation of the dynamics of a single non-canonical scalar field ("darkon").
- Construction of a new class of models of gravity interacting with a scalar "inflaton" φ, as well as with other phenomenologically relevant matter including Higgs-like scalar σ, which produce an effective full scalar potential of φ, σ with few remarkable properties.
- The "inflaton" effective potential (at fixed σ) possesses two infinitely large flat regions with vastly different energy scales for large negative and large positive values of φ. This allows for a unified description of both "early" universe inflation as well as of present "dark energy"-dominated epoch in universe's evolution.
- In the "early" universe the would-be Higgs field σ remains massless and decouples from the "inflaton" φ. The"early" universe evolution is described entirely in terms of the "inflaton" dynamics.
- In the post-inflationary epoch φ and σ exchange roles. The inflaton φ becomes massless and decoupled, whereas σ becomes a genuine Higgs field with a dynamically generated electro-weak-type symmetry breaking effective potential.
- A natural choice for the parameters involved conforms to quintessential cosmological and electro-weak phenomenologies.
- Minisuperspace Wheeler–DeWitt quantization reveals the role of the "darkon" scalar field as cosmological"time" in the "early" Universe. The quantum average of the FLRW scale factor does not exhibit any singularity in its "time" evolution.

Let us also note that applying the non-Riemannian volume-form formalism to minimal $N = 1$ supergravity we arrived at a novel mechanism for the supersymmetric Brout-Englert-Higgs effect, namely, the appearance of a dynamically generated cosmological constant triggering spontaneous supersymmetry breaking and mass generation for the gravitino [36, 44]. Applying the same non-Riemannian volume-form formalism to anti-de Sitter supergravity produces simultaneously a very large physical gravitino mass and a very small *positive* observable cosmological constant [36, 44] in accordance with modern cosmological scenarios for slowly expanding universe of the present epoch [8–14].

As a final comment let us mention some further extensions of the method of non-Riemannian volume elements – gravity models with dynamical spacetime [49] further developed into models of interacting diffusive unified dark energy and dark matter (see [50] and references therein).

Acknowledgements E.G., E.N. and S.P. gratefully acknowledge support of our collaboration through the academic exchange agreement between the Ben-Gurion University in Beer-Sheva, Israel, and the Bulgarian Academy of Sciences. E.N. and E.G. have received partial support from European COST actions MP-1405 and CA-16104, and from CA-15117 and CA-16104, respectively. E.N. and S.P. are also thankful to Bulgarian National Science Fund for support via research grant DN-18/1.

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Wilson Loop Form Factors: A New Duality

Dmitry Chicherin, Paul Heslop, Gregory P. Korchemsky and Emery Sokatchev

Abstract We find a new duality for form factors of lightlike Wilson loops in planar $\mathcal{N} = 4$ super-Yang-Mills theory. The duality maps a form factor involving an n-sided lightlike polygonal super-Wilson loop together with m external on-shell states, to the same type of object but with the edges of the Wilson loop and the external states swapping roles. This relation can essentially be seen graphically in Lorentz harmonic chiral (LHC) superspace where it is equivalent to planar graph duality. However there are some crucial subtleties with the cancellation of spurious poles due to the gauge fixing. They are resolved by finding the correct formulation of the Wilson loop and by careful analytic continuation from Minkowski to Euclidean space. We illustrate all of these subtleties explicitly in the simplest non-trivial NMHV-like case.

1 Introduction

The natural gauge invariant objects in any gauge theory include scattering amplitudes, Wilson loops, correlation functions and form factors of local operators. In the past years numerous studies have revealed interesting duality relations between the

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_8

first three objects in planar $\mathcal{N} = 4$ SYM theory. The simplest MHV gluon scattering amplitude $\mathrm{A}_n(p_1, \ldots, p_n)$ has been shown [1–3] to be dual to a Wilson loop $\mathcal{W}_n(x_1, \ldots, x_n)$ defined on a lightlike contour,

$$\mathrm{A}_n(p_1, \ldots, p_n) = \mathcal{W}_n(x_1, \ldots, x_n)\,, \tag{1}$$

upon the identification of the separation between the cusp points x_i of the contour with the particle momenta p_i in Minkowski space, $x_i - x_{i+1} = p_i$ for $i = 1, \ldots, n$ and $x_{n+1} \equiv x_1$. This duality has a natural supersymmetric extension [4–6] where the super-lightlike contour is built out of the on-shell supermomenta of the scattered particles. The correlation functions $G_n = \langle O(x_1) \ldots O(x_n)\rangle$ of local gauge invariant operators $O(x)$ are dual to the Wilson loops (and hence to the amplitudes) in the lightlike limit [7, 8], $\lim_{x^2_{i,i+1}\to 0} x^2_{12} \ldots x^2_{n1}\, G_n = \mathcal{W}_n$. This duality has a supersymmetric generalisation as well [9–11].

The fourth object is the form factor $\langle 0|O(x)|k_1, \ldots, k_m\rangle$ of a local operator $O(x)$ with an asymptotic $m-$particle state of on-shell momenta $k_j^2 = 0$ for $j = 1, \ldots, m$. It is a hybrid between correlation functions and scattering amplitudes because it lives simultaneously in coordinate and momentum spaces. Such form factors (and their supersymmetric extensions in $\mathcal{N} = 4$ SYM) have been actively studied in the recent years [12–18]. It is interesting to know if there are possible duality relations for them as well. This question has been addressed in [19] but for a more complicated object, the matrix element of a lightlike bosonic Wilson loop stretched between local operators along a single light-cone direction, with an on-shell state. It has been shown that this object is dual to itself upon swapping the coordinate and momentum data. It has also been conjectured there that the new duality may extend to a larger class of objects, namely the form factor $W_{n,m} = \langle 0|\mathcal{W}_n(x_1, \ldots, x_n)|k_1, \ldots, k_m\rangle$ of an $n-$gon lightlike (supersymmetric) Wilson loop with an $m-$particle state. Schematically, the suggested duality takes the form

$$W_{n,m}(\{x\}|\{k\}) = W_{m,n}(\{y\}|\{p\})\,, \tag{2}$$

where the kinematical data on both sides are related like in (1),

$$x_i - x_{i+1} = p_i\,, \qquad y_j - y_{j+1} = k_j\,, \tag{3}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ provided that the total momenta of the particles vanish, $\sum_{i=1}^n p_i = \sum_{j=1}^m k_j = 0$. This conjecture has been successfully tested in [20] in the simplest case of a Wilson loop with a state of helicity $(+1)$ gluons and in the Born approximation.

Building upon the observations in [19, 20], in this paper we study the general case of the form factor for a lightlike supersymmetric Wilson loop and we argue that it has a remarkable duality property in planar $\mathcal{N} = 4$ SYM. It extends the bosonic relation (2) and the identification of coordinates with momenta (3) to their supersymmetric analogs. The super-Wilson loop form factors are considered in the planar limit and in the lowest-order perturbative approximation (Born level). The introduction of Grassmann variables (θ_i on the Wilson loop contour and η_j for the on-shell states) allows us to probe the duality for more complicated configurations of particle helicities. By analogy with the amplitudes, we call the contributions at the lowest level in the Grassmann expansion MHV-like, at the next level NMHV-like, etc. At MHV level we confirm the result of [20]. The NMHV level is much more complicated, the form factor being a non-trivial rational function of the kinematical data. Yet, we show that the duality still works, in a rather simple and suggestive way, by just matching planar Feynman diagrams. This allows us to argue that it should hold for the complete supersymmetric object (at all Grassmann levels) and also beyond the Born approximation.

The key to understanding the duality is the appropriate superspace formulation of the Wilson loop and its form factor. In the conventional approach the chiral supersymmetric Wilson loop [4–6] is formulated in terms of constrained on-shell super-connections [21, 22], which makes the Feynman diagram technique highly inefficient. In this paper we prefer to use the Lorentz harmonic chiral (LHC) superspace approach [23]. It provides an off-shell formulation of the chiral $\mathcal{N}=4$ SYM theory in terms of unconstrained prepotentials, best suited for supersymmetric quantisation. LHC superspace is an alternative to the twistor formulation [24, 25], closer in spirit to traditional field theory (see also [26]). The main idea is to consider the interacting theory as a perturbation of the self-dual sector. The twistor formulation has been successfully used to justify the so-called MHV rules for the computation of the amplitude [27], to prove the duality between supersymmetric Wilson loops and amplitudes [5], to compute off-shell correlation functions of the $\mathcal{N}=4$ stress-tensor multiplet [28]. More recently, the LHC formalism was applied to finding the non-chiral completion of the correlators [29] and to the calculation of form factors of local operators [30]. In this paper, after explaining the kinematical setup in Sect. 2, we formulate the lightlike Wilson loop in LHC superspace in Sect. 3 and apply the Feynman rules of [30] to the computation of its form factors in Sect. 4. We find an important additional contribution to the Wilson loop, compared to the twistor formulation [5]. It is needed to make the Wilson loop gauge invariant.

The duality essentially works on a graph-to-graph basis. More precisely, we find two types of Feynman graphs corresponding to two different helicity configurations at NMHV-like level. These graphs are dual to each other after identifying the kinematical data as in (3) and redrawing the graph following a simple rule. In addition to these graphs there are sets of graphs whose role is to restore gauge invariance. We use a light-cone gauge whose parameter is the so-called reference spinor. A known problem of such gauges is the presence of spurious poles. Their elimination in the Feynman graphs (and hence the restoration of gauge invariance) is a somewhat subtle procedure.

2 Definitions and Summary of the Results

2.1 Generalised Form Factors of Wilson Loops

In this paper, we study a new object – the generalised form factor of the lightlike Wilson loop. In $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ it is defined as the matrix element of a lightlike $n-$gon supersymmetric Wilson loop $\mathcal{W}_n$ with the on-shell $m-$particle state $|1^{a_1} \dots m^{a_m}\rangle$:

$$\langle 0|\mathcal{W}_n|1^{a_1} \dots m^{a_m}\rangle = \frac{1}{N}\langle 0|\text{tr}\, P \exp\left[i \oint_{\mathcal{C}_n} \left(dx^\mu \mathcal{A}_\mu(x,\theta) + d\theta^{\alpha A}\mathcal{A}_{\alpha A}(x,\theta)\right)\right]|1^{a_1} \dots m^{a_m}\rangle\,, \tag{4}$$

where the integration goes over a closed contour $\mathcal{C}_n$ formed by n straight lightlike segments connecting the superspace points (x_i, θ_i). The bosonic and fermionic gauge connections, $\mathcal{A}_\mu$ and $\mathcal{A}_{\alpha A}$, have expansions in powers of θ's with coefficients given in terms of the gluon, gaugino and scalar fields. Their explicit expressions are shown below in (31).

In the planar limit, the form factor can be decomposed in the standard manner over the basis of single traces,

$$\langle 0|\mathcal{W}_n|1^{a_1} \dots m^{a_m}\rangle = \sum_{\sigma \in S_m/Z_m} \text{tr}(T^{a_{\sigma_1}} \dots T^{a_{\sigma_m}})F_{n,m}(\sigma_1, \dots, \sigma_m)\,, \tag{5}$$

where the sum runs over all permutations of the external particles $\sigma_1, \dots, \sigma_m$ modulo cyclic shifts. The matrix element (5) is a natural generalisation of lightlike Wilson loops $\langle 0|\mathcal{W}_n|0\rangle$ and scattering amplitudes $\mathrm{A}(1^{a_1} \dots m^{a_m})$. In fact, it gets a disconnected contribution given by their product. In what follows we discard it and consider only the connected contribution to (5).

The color-ordered form factors $F_{n,m}$ depend on two sets of variables. The first set consists of n coordinates in Minkowski space-time and their odd superpartners $(x_i^{\dot\alpha\alpha}, \theta_i^{\alpha A})$ specifying the position of the vertices of a lightlike $n-$gon,[1]

$$(x_i - x_{i+1})^2 = 0\,, \qquad (x_i - x_{i+1})^{\dot\alpha\alpha}\,(\theta^A_{i,\alpha} - \theta^A_{i+1,\alpha}) = 0 \tag{6}$$

for $i = 1, \dots, n$, with the cyclicity conditions $x_{n+1} = x_1$ and $\theta_{n+1} = \theta_1$. Here the first relation means that the Wilson loop is built from lightlike segments and the second relation is its superpartner.

The second set of variables consists of the on-shell momenta of m particles $(k_j^{\dot\alpha\alpha}, \eta_{jA})$

$$k_j^{\dot\alpha\alpha} = \tilde{k}_j^{\dot\alpha} k_j^\alpha \equiv |k_j]\langle k_j| \tag{7}$$

[1] We use two-component spinor notation for vectors, e.g., $x^{\dot\alpha\alpha} = (\sigma_\mu)^{\dot\alpha\alpha}x^\mu$. The Lorentz and R symmetry indices take values $\alpha = 1, 2$, $\dot\alpha = 1, 2$ and $A = 1, 2, 3, 4$, respectively.

with $k_j^2 = 0$ and $j = 1, \ldots, m$. Like the scattering amplitudes, the expansion of the on-shell state in powers of η_{jA} corresponds to particles with different helicity (gluons, gaugini and scalars). Each particle superstate carries one unit of helicity. It is then convenient to introduce the helicity-free function $W_{n,m}$ multiplying (5) by the so-called Parke-Taylor factor

$$W_{n,m} = \langle k_1 k_2 \rangle \langle k_2 k_3 \rangle \ldots \langle k_m k_1 \rangle \ F_{n,m}(1, \ldots, m) \,, \tag{8}$$

where $\langle k_i k_j \rangle = k_i^\alpha \epsilon_{\alpha\beta} k_j^\beta$. The scalar function $W_{n,m}$ defined in this way depends on the two sets of variables introduced above,

$$W_{n,m} = W_{n,m}(\{x, \theta\}; \{k, \eta\}) \,. \tag{9}$$

As follows from the definition (5), this function is invariant under cyclic shifts of the coordinates and momenta.

2.2 Dual Variables

To elucidate the interesting properties of $W_{n,m}$ we introduce the so-called dual superspace variables [31]. The coordinates of the Wilson loop (x_i, θ_i^A) have the dual momenta (p_i, ω_i^A) defined as

$$x_i - x_{i+1} = p_i \,, \qquad |\theta_i^A\rangle - |\theta_{i+1}^A\rangle = |p_i\rangle\, \omega_i^A \,, \tag{10}$$

where we do not display the Lorentz indices for simplicity. It follows from (6) that p_i are lightlike vectors, $p_i^2 = 0$, satisfying the condition $\sum_{i=1}^n p_i = 0$. Similarly, the odd variables ω_i^A satisfy the relation $\sum_{i=1}^n |p_i\rangle \omega_i^A = 0$ and solve the second condition in (6). Note that the properties of (p_i, ω_i^A) (with $i = 1, \ldots, n$) match those of the supermomenta of the on-shell states in the scattering amplitude A_n. This observation was crucial in establishing the duality between the lightlike Wilson loop $\mathcal{W}_n$ and the scattering amplitude A_n.

For the set of on-shell momenta (k_j, η_{jA}), the dual coordinates are defined as

$$k_j = y_j - y_{j+1} \,, \qquad |k_j\rangle\, \eta_{jA} = |\psi_{j,A}\rangle - |\psi_{j+1,A}\rangle \,. \tag{11}$$

Here the dual momenta $y_1, \ldots, y_{m+1}$ are consecutively lightlike separated, $(y_i - y_{i+1})^2 = 0$ and their superpartners satisfy $(y_j - y_{j+1})(|\psi_{j,A}\rangle - |\psi_{j+1,A}\rangle) = 0$. Note the striking similarity between relations (10) and (11). Namely, these relations can be mapped into each other by exchanging coordinates with dual momenta, $(x, \theta) \to (y, \psi)$, and momenta with dual coordinates, $(k, \eta) \to (p, \omega)$.

There is however an important difference between the two sets of dual coordinates. The dual vectors p_i define the edges of a closed $n-$gon and their sum equals zero.

The same is true for the sum of dual odd coordinates $|p_i\rangle\,\omega_i^A$,

$$\sum_{i=1}^{n} p_i = x_1 - x_{n+1} = 0\,, \qquad \sum_{i=1}^{n} |p_i\rangle\,\omega_i^A = |\theta_1^A\rangle - |\theta_{n+1}^A\rangle = 0\,, \tag{12}$$

so that the dual variables satisfy the periodicity conditions $x_i = x_{i+n}$ and $\theta_i^A = \theta_{i+n}^A$. For the dual momenta the analogous relations read

$$\sum_{j=1}^{m} k_j = y_1 - y_{m+1} = K\,, \qquad \sum_{j=1}^{m} |k_j\rangle\,\eta_{jA} = |\psi_{1,A}\rangle - |\psi_{m+1,A}\rangle = Q_A\,, \tag{13}$$

where K and Q are the total momentum and supercharge of the m particles in (5), respectively. In contrast with (12), K and Q can take arbitrary values and there are no reasons to impose the periodicity conditions $y_{m+1} = y_1$ and $\psi_{m+1,A} = \psi_{1,A}$. Indeed, the function (9) is well defined for arbitrary K and Q.

2.3 Duality Relation

Setting $K = Q_A = 0$ in (13) we restore the symmetry between (12) and (13). This allows us to treat the original variables and their dual counterparts on an equal footing. In this paper we argue that for $K = Q_A = 0$ the symmetry of $W_{n,m}$ is enhanced and yields an interesting duality relation for $W_{n,m}$ that we shall formulate in a moment. More precisely, we can use the dual variables to define, following (5), the matrix element of the lightlike Wilson loop $\langle 0|\mathcal{W}_m|1^{a_1}\ldots n^{a_n}\rangle$. Here the Wilson loop is evaluated along a closed lightlike $m-$gon with vertices located at (y_j, ψ_j) and the on-shell state consists of n particles with supermomenta (p_i, w_i^A). This matrix element has the same general form (5) and (8), with the corresponding scalar function $W_{m,n}$ given by

$$W_{m,n} = W_{m,n}(\{y,\psi\};\{p,\omega\})\,. \tag{14}$$

Applying relations (10) and (11) we can express it in terms of the original variables $\{x_i, \theta_i\}$ and $\{k_j, \eta_j\}$.

The duality relation that we propose states that the functions (9) and (14) coincide in planar $\mathcal{N} = 4$ SYM,

$$W_{n,m}(\{x,\theta\};\{k,\eta\}) = W_{m,n}(\{y,\psi\};\{p,\omega\})\,. \tag{15}$$

Using the definition of the dual variables we can rewrite the duality relation in other equivalent forms, e.g.

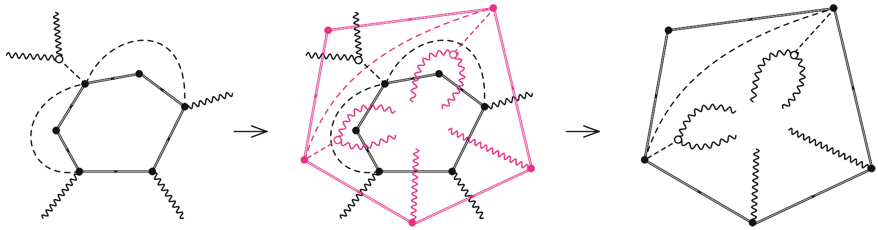

Fig. 1 Diagrammatic representation of the duality relation (16). The Wilson loop on the left is built out of lightlike vectors $p_1, \ldots, p_n$, the wavy lines denote on-shell particles with momenta $k_1, \ldots, k_m$ and the dash lines stand for free propagators. Black and white dots denote effective vertices. The dual Wilson loop form factor on the right has the lightlike vectors and momenta exchanged. The middle figure explains the duality by superimposing the two graphs

$$W_{n,m}(\{p, \omega\}; \{k, \eta\}) = W_{m,n}(\{k, \eta\}; \{p, \omega\}) . \tag{16}$$

This relation is represented diagrammatically in Fig. 1.

The duality relation (15) should hold for any values of n and m. As a simple illustration, we examine it for the lowest values of n and m. In the special cases $m = 0$ (or $n = 0$) we recover the well-known duality between the $n-$point superamplitude and the $n-$point super-Wilson loop. Since the $n-$gon Wilson loop is well defined for $n \geq 2$, we start with $n = 2, 3$. In this case, the cusp points x_i satisfying (6) have to lie on the same light-ray in Minkowski space-time. Then, the integration contour of the Wilson loop collapses to a backtracking path leading to $W_2 = W_3 = 1$. As a consequence, the matrix element on the left-hand side of (5) only receives disconnected contributions yielding the vanishing of $W_{n,m}(\{x, \theta\}; \{k, \eta\})$ for $n = 2, 3$. The duality relation (15) implies that the same should be true for $W_{m,n}(\{y, \psi\}; \{p, \omega\})$ for $n = 2, 3$. Indeed, the corresponding matrix element (5) involves an on-shell state with (real valued) lightlike momenta k_i that are necessarily aligned due to $\sum_i k_i = 0$. In this case $\langle k_i k_j \rangle = 0$ and it follows from (8) that $W_{m,n}$ vanishes, in agreement with (14).

2.4 Duality Relation at MHV Level

Let us now consider the duality relation for $n, m \geq 4$. In this case both sides of (15) are different from zero and are given by nontrivial functions of the kinematical variables and of the 't Hooft coupling constant. In what follows we shall restrict our consideration to the lowest order in the coupling (Born level). Expanding both sides of (15) in the Grassmann variables, we can get relations between the different components. By analogy with the scattering amplitudes, we shall refer to the terms of the expansion as MHV, NMHV, etc. Notice that since $W_{n,m}(\{x, \theta\}; \{k, \eta\})$ depends on two sets of Grassmann variables θ_i and η_j, we will have to deal with a double expansion of the form $\mathrm{N}^{\kappa}\mathrm{MHV} \times \mathrm{N}^{\sigma}\mathrm{MHV}$.

The lowest term of the expansion, MHV×MHV, corresponds to (15) with all Grassmann variables put to zero on both sides of the relation. Namely, for $\theta_i = 0$ the super Wilson loop $\mathcal{W}_n$ reduces to the bosonic lightlike Wilson loop and for $\eta_j = 0$ the on-shell state in (5) reduces to a gluon state of helicity $(+1)$. In this way, from (4) and (5) we obtain

$$F_{n,m}^{\mathrm{MHV}\times\mathrm{MHV}}(x, k) = \frac{1}{N}\langle 0|\mathrm{tr}\big(E_{1n}\dots E_{32}E_{21}\big)|k_1^+\dots k_m^+\rangle \,, \tag{17}$$

where $E_{i+1,i}$ denotes a bosonic Wilson line in the fundamental of $SU(N)$ evaluated along the lightlike segment $[x_i, x_{i+1}]$

$$E_{i+1,i} = P\exp\left(-i\int_0^1 dt\; p_i\cdot A(x_i - p_i t)\right)\,, \tag{18}$$

with $p_i = x_i - x_{i+1}$. Notice that the ordering of the $E-$factors inside the trace in (17) is opposite to that of the gluons in the on-shell state.

In the Born approximation, $\mathrm{A}_{n,m}^{\mathrm{MHV}\times\mathrm{MHV}}$ is given by the sum of tree Feynman diagrams in which the on-shell gluons are attached to the lightlike $n-$gon contour either directly or through $3-$ and $4-$gluon interaction vertices. The calculation can be simplified by introducing the notion of a "wedge", i.e. a cusped Wilson line built from two semi-infinite rays running along the lightlike vectors $-p_1$ and p_2 and joining at point x:

$$W_{p_2,p_1}(x) = P\left[\exp\left(i\int_0^\infty dt\; p_2\cdot A(x+p_2 t)\right)\exp\left(-i\int_{-\infty}^0 dt\; p_1\cdot A(x-p_1 t)\right)\right]. \tag{19}$$

In the product $W_{p_3,p_2}(x_3)W_{p_2,p_1}(x_2)$ with $p_2 = x_2 - x_3$, it is easy to see that the two semi-infinite rays running along p_2 partially cancel against each other giving rise to E_{32}. In this way, we can rewrite (17) as

$$F_{n,m}^{\mathrm{MHV}\times\mathrm{MHV}}(x, k) = \frac{1}{N}\langle 0|\mathrm{tr}\big[W_{p_n,p_{n-1}}(x_n)\dots W_{p_2,p_1}(x_2)W_{p_1,p_n}(x_1)\big]|k_1^+\dots k_m^+\rangle\,. \tag{20}$$

The advantage of this representation is that, in the Born approximation, the on-shell gluons can be emitted by one of the $W-$factors thus allowing us to express the matrix element on the right-hand side of (20) as the sum over all possible attachments of m gluons to n wedges

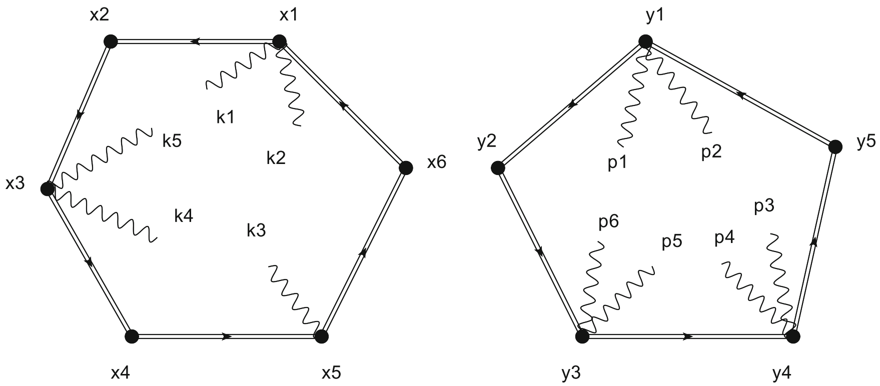

Fig. 2 Diagrammatic representation of the duality relation (28) for $n = 6$ and $m = 5$. Notice that the polygon vertices and the gluons are ordered in opposite directions. Black blobs with outgoing gluons denote wedge form factors (22). The lightlike edges of the Wilson loops are mapped to the momenta of the on-shell gluons, $k_i = y_i - y_{i+1}$ and $p_j = x_j - x_{j+1}$

$$F_{n,m}^{\mathrm{MHV}\times\mathrm{MHV}} = \sum_{\ell_1<\cdots<\ell_s} \sum_{1\leq i_s<\cdots<i_1\leq n} \langle 0|W_{p_{i_1},p_{i_1-1}}(x_{i_1})|k_{\ell_1}^+\dots k_{\ell_2-1}^+\rangle$$
$$\times \langle 0|W_{p_{i_2},p_{i_2-1}}(x_{i_2})|k_{\ell_2}^+\dots k_{\ell_3-1}^+\rangle \dots \langle 0|W_{p_{i_s},p_{i_s-1}}(x_{i_s})|k_{\ell_s}^+\dots k_{\ell_1-1}^+\rangle \,. \tag{21}$$

Here the first sum goes over all possible partitions of m gluons over s clusters (with $s \leq n$) and the second sum runs over all possible wedges $x_{i_1}, \dots, x_{i_s}$ to which these clusters are attached. The difference in the ordering of indices ℓ_k and i_k in (21) is due to the opposite ordering of the $E-$factors and gluons in (17). Relation (21) is represented diagrammatically in Fig. 2.

Relation (21) involves the so-called wedge form factor $\langle 0|W_{p_2,p_1}(x)|k_1^+\dots k_\ell^+\rangle$. Since the on-shell state contains only gluons of the same helicity, its calculation in the Born approximation can be performed in the self-dual sector of Yang-Mills theory [20, 26]

$$\langle 0|W_{p_2,p_1}(x)|k_1^+\dots k_\ell^+\rangle = F(p_2, k_1, \dots, k_\ell, p_1)\, e^{ix(k_1+\dots+k_\ell)} \,. \tag{22}$$

Here the dependence on x is fixed by Poincaré symmetry and the order of the arguments of the $F-$function matches the color ordering of the gluons. Its explicit expression reads (see Sect. 4.1 for more details)

$$F(p_2, k_1, \dots, k_\ell, p_1) = \frac{\langle p_2 p_1\rangle}{\langle p_2 k_1\rangle\langle k_1 k_2\rangle \dots \langle k_\ell p_1\rangle} \,. \tag{23}$$

Substituting (22) and (23) in (21) and matching the result with (8) we find

$$W_{n,m}^{\text{MHV}\times\text{MHV}}(x,k) = \sum e^{ix_{i_1}y_{\ell_1\ell_2}+ix_{i_2}y_{\ell_2,\ell_3}+\ldots+ix_{i_s}y_{\ell_s,\ell_1}} \\ \times \frac{\langle k_{\ell_1-1}k_{\ell_1}\rangle\langle p_{i_1}p_{i_1-1}\rangle\langle k_{\ell_2-1}k_{\ell_2}\rangle\langle p_{i_2}p_{i_2-1}\rangle\ldots\langle k_{\ell_s-1}k_{\ell_s}\rangle\langle p_{i_s}p_{i_s-1}\rangle}{\langle k_{\ell_1}p_{i_1}\rangle\langle p_{i_1-1}k_{\ell_2-1}\rangle\langle k_{\ell_2}p_{i_2}\rangle\langle p_{i_2-1}k_{\ell_3-1}\rangle\ldots\langle k_{\ell_s}p_{i_s}\rangle\langle p_{i_s-1}k_{\ell_1-1}\rangle}, \quad (24)$$

where the sum covers the same range as in (21). Here we used (11) to switch to dual momenta in the exponent, e.g. $y_{\ell_1\ell_2} = k_{\ell_1} + \cdots + k_{\ell_2-1}$. We recall that for vanishing total momentum $K = \sum_{i=1}^{m} k_i = 0$, the dual momenta satisfy the periodicity condition $y_{m+1} = y_1$. Using this property, we can rewrite the exponential factor in (24) in the equivalent form

$$e^{iy_{\ell_s}x_{i_s,i_{s-1}}+\cdots+iy_{\ell_2}x_{i_2i_1}+iy_{\ell_1}x_{i_1i_s}}. \quad (25)$$

We observe that it can be obtained from the original factor by swapping the variables

$$x_{i_1} \leftrightarrow y_{\ell_s}, \qquad x_{i_2} \leftrightarrow y_{\ell_{s-1}}, \quad \ldots, \quad x_{i_s} \leftrightarrow y_{\ell_1}. \quad (26)$$

Let us now examine the expression in the second line of (24). It depends on two sets of null vectors p_i and k_i defining the edges of the lightlike Wilson loop and the momenta of the on-shell gluons, respectively. It is straightforward to verify that it is invariant under the swapping of these vectors

$$k_{\ell_1} \leftrightarrow p_{i_s}, \qquad k_{\ell_2} \leftrightarrow p_{i_{s-1}}, \quad \ldots, \quad k_{\ell_s} \leftrightarrow p_{i_1}. \quad (27)$$

Putting together (26) and (27), we immediately conclude that the expression on the right-hand side of (24) is invariant under the exchange of the original variables (x, k) with their dual partners (y, p). This yields the duality relation

$$W_{n,m}^{\text{MHV}\times\text{MHV}}(x,k) = W_{m,n}^{\text{MHV}\times\text{MHV}}(y,p), \quad (28)$$

in agreement with [20].

2.5 *Duality Beyond MHV*

To test the duality relation (15) beyond MHV level, we have to take into account the dependence of the Wilson loop form factor (4) on the Grassmann variables θ_i^A and η_{jA}. The dependence on η comes from the expansion of the on-shell super-state in (4) over the states of particles (gluons, gaugino and scalars) with different helicity.

At the same time, the dependence of (4) on θ comes from the expansion of the supersymmetric n−gon Wilson loop

$$\mathcal{W}_n = \frac{1}{N}\text{tr}\big(\mathcal{E}_{1n}\ldots\mathcal{E}_{32}\,\mathcal{E}_{21}\big) \quad (29)$$

in powers of θ_i defining the position of vertices of the lightlike $n-$gon in (chiral) superspace. Here the supersymmetric Wilson line $\mathcal{E}_{i+1,i}$ is evaluated along the straight segment connecting the superspace points (x_i, θ_i) and (x_{i+1}, θ_{i+1})

$$\mathcal{E}_{i+1,i} = P \exp\left[-i \int_0^1 dt \left(\frac{1}{2} x_{i,i+1}^{\dot\alpha\alpha} \mathcal{A}_{\alpha\dot\alpha}(x(t), \theta(t)) + \theta_{i,i+1}^{\alpha A} \mathcal{A}_{\alpha A}(x(t), \theta(t))\right)\right], \tag{30}$$

where $x(t) = x_i - x_{i,i+1}\, t$ and $\theta(t) = \theta_i - \theta_{i,i+1}\, t$. The super-connections $\mathcal{A}$ are subject to the defining *on-shell* constraints of $\mathcal{N} = 4$ SYM [32]. One way of solving them is to fix the *non-supersymmetric* Wess–Zumino gauge and express the components of $\mathcal{A}$ in terms of the propagating gluon, gaugino and scalar fields [21, 22]

$$\begin{aligned}
\mathcal{A}_{\alpha\dot\alpha} &= A_{\alpha\dot\alpha} + i\theta_\alpha^A \bar\psi_{\dot\alpha A} + \frac{i}{2!}\theta_\alpha^A \theta^{\beta B} D_{\beta\dot\alpha}\bar\phi_{AB} - \frac{1}{3!}\epsilon_{ABCD}\theta_\alpha^A \theta^{\beta B}\theta^{\gamma C} D_{\beta\dot\alpha}\psi_\gamma^D + \cdots \\
\mathcal{A}_{\alpha A} &= \frac{i}{2}\bar\phi_{AB}\theta_\alpha^B - \frac{1}{3!!}\epsilon_{ABCD}\theta_\alpha^B \theta^{\gamma C}\psi_\gamma^D + \frac{i}{4!!}\epsilon_{ABCD}\theta_\alpha^B\theta^{\beta C}\theta^{\gamma D} F_{\beta\gamma} + \cdots,
\end{aligned} \tag{31}$$

where the dots denote higher-order terms in θ.

Before continuing let us examine the superspace structure we should expect this object to have arising from supersymmetry. The chiral supersymmetry of (9) yields the Ward identity

$$\left(\sum_{i=1}^n \frac{\partial}{\partial\theta_i^A} + \sum_{j=1}^m |k_j\rangle \eta_{j,A}\right) W_{n,m}(\{x,\theta\};\{k,\eta\}) = 0\,. \tag{32}$$

The duality relation is expected to hold if the total particle supercharge vanishes, $Q_A = \sum_{j=1}^m |k_j\rangle\, \eta_{jA} = 0$. Then (32) implies that $W_{n,m}$ can be an arbitrary function of $\theta_{ij}^A = \theta_i^A - \theta_j^A$ and η_{kA}. In virtue of the R symmetry, these variables must form $SU(4)$ invariants. The latter are of three different kinds: $\epsilon_{ABCD}\theta_{ii'}^A\theta_{jj'}^B\theta_{kk'}^C\theta_{ll'}^D$, $\eta_{ijkl}^4 = \epsilon^{ABCD}\eta_{iA}\eta_{jB}\eta_{kC}\eta_{lD}$ and $(\theta_{ij}\eta_k) = \theta_{ij}^A\, \eta_{kA}$. The dependence on these invariants simplifies further in the Born approximation.

To compute $W_{n,m}$ in the Born approximation, we substitute (29)–(31) into the definition (4) and retain the contribution at the lowest order in the coupling. Since the dependence on θ's comes from the expansion of the bosonic and fermionic connections in (31), the number of contributing diagrams and their complexity increases significantly as compared with the MHV case described in the previous subsection. Moreover, the use of the Wess–Zumino gauge (31) breaks manifest supersymmetry. This makes the conventional approach impractical.

In this paper we prefer the off-shell formulation of the chiral $\mathcal{N} = 4$ SYM theory in terms of *unconstrained* prepotentials in LHC superspace [23], better suited for supersymmetric quantisation. In Sect. 3 we formulate the lightlike Wilson loop in LHC superspace and apply the Feynman rules of [30] to the computation of its form factors.

In this new formulation, $W_{n,m}(\{x_i,\theta_i\};\{k_j,\eta_j\})$ is given by a sum of contributions having a similar structure to (21), with the important difference that the wedge form factors are replaced by their supersymmetric generalisations depending on the Grassmann variables θ_i^A and η_{jA}. This leads to the following general expression for $W_{n,m}$,

$$W_{n,m} = \sum e^{ix_{i_1}y_{\ell_1\ell_2}+ix_{i_2}y_{\ell_2,\ell_3}+\dots+ix_{i_s}y_{\ell_s,\ell_1}} \times e^{\langle\theta_{i_1}\psi_{\ell_1\ell_2}\rangle+\langle\theta_{i_2}\psi_{\ell_2\ell_3}\rangle+\dots+\langle\theta_{i_s}\psi_{\ell_s\ell_1}\rangle} \times \widehat{W}_{n,m}\,, \tag{33}$$

which should be compared with (24). Here we used shorthand notation $\langle\theta_{i_1}\psi_{\ell_1\ell_2}\rangle = \theta_{i_1}^{\alpha A}(\psi_{\ell_1,\alpha A}-\psi_{\ell_2,\alpha A})$ with the dual ψ−variables defined in (11). The sum in (33) has the same form as in (21) and runs over all possible partitions of m super particles over s clusters. Notice that the function $\widehat{W}_{n,m}$ depends on the choice of partition. The second exponent on the right-hand side of (33) is the supersymmetric completion of the first exponent depending on the bosonic variables.

Most importantly, as we show below by exploring the structure of the Feynman diagrams, the function $\widehat{W}_{n,m}$ does not depend on the mixed products of Grassmann variables $(\theta_{ij}\eta_k)$ in the Born approximation.[2] This allows us to expand $\widehat{W}_{n,m}$ in powers of the two remaining invariants leading to the following relation

$$\widehat{W}_{n,m} = W_{n,m}^{(0,0)} + \left(W_{n,m}^{(1,0)}+W_{n,m}^{(0,1)}\right) + \left(W_{n,m}^{(2,0)}+W_{n,m}^{(1,1)}+W_{n,m}^{(0,2)}\right) + \cdots\,, \tag{34}$$

where $W_{n,m}^{(\kappa,\sigma)}$ is a homogenous polynomial in θ's and η's of degree 4κ and 4σ, respectively. Schematically, $W_{n,m}^{(\kappa,\sigma)} \sim \theta^{4\kappa}\eta^{4\sigma}$. By analogy with the superamplitude, we refer to the terms on the right-hand side of (34) with $\kappa+\sigma=k$ as N^{k}MHV-like. The lowest term of the expansion, $W_{n,m}^{(0,0)}$, defines the MHV-like contribution $W_{n,m}^{\text{MHV}\times\text{MHV}}$ discussed in the previous subsection. Its explicit expression can be read from (24).

Substituting (33) and (34) into (15), we can formulate the duality relation in each sector,

$$W_{n,m}^{(\kappa,\sigma)}(\{x,\theta\};\{k,\eta\}) = W_{m,n}^{(\sigma,\kappa)}(\{y,\psi\};\{p,\omega\})\,. \tag{35}$$

The explicit expressions for $W_{n,m}^{(\kappa,\sigma)}$ for generic κ and σ are rather complicated even in the Born approximation. Nevertheless, as we show below, the duality relation (35) can be verified by matching into each other the diagrams contributing to both sides of (35).

[2]This does not follow from chiral supersymmetry (32) and it would be interesting to understand the symmetry leading to such a structure.

3 Lightlike Wilson Loop in LHC Superspace

As mentioned in the introduction, the conventional formulation (30) of the chiral supersymmetric Wilson loops, making use of constrained super-connections, is not convenient for quantum calculations. The LHC superspace approach, where the dynamical gauge prepotentials are unconstrained, is much more efficient. In this section we start by a brief summary of the LHC superspace description of $\mathcal{N}=4$ SYM. Then we present the explicit form of the Wilson loop in LHC superspace, in terms of the two unconstrained gauge prepotentials. Our formulation is similar to the twistor one of Mason and Skinner in [5] but differs from it on an essential point.

3.1 $\mathcal{N}=4$ Super-Yang-Mills in LHC Superspace

Here we recall some basic facts about $\mathcal{N}=4$ SYM in LHC superspace (for details see [23]). The theory is formulated in terms of two dynamical chiral superfields (prepotentials),

$$A^{++}(x,\theta^+,u)\,,\qquad A^+_{\dot\alpha}(x,\theta^+,u)\,. \tag{36}$$

Here $\theta^{+A}=\theta^A_\alpha u^{+\alpha}$ is a projection of the chiral Grassmann variable with a harmonic variable $u^{+\alpha}$. This commuting spinor variable together with its conjugate $u^{-\alpha}$ form a matrix of the chiral half $SU(2)_L$ of the Euclidean Lorentz group $SO(4)\sim SU(2)_L\times SU(2)_R$. The harmonic variables $u^\pm$ parametrise the coset space $S^2\sim SU(2)_L/U(1)$. The superfields (36) are interpreted as infinite harmonic expansions on the sphere, i.e. homogeneous series in the harmonic variables $u^\pm$ with fixed $U(1)$ charge. For example, in the expansion of $A^+_{\dot\alpha}(x,\theta^+,u)=A_{\alpha\dot\alpha}(x)u^{+\alpha}+A_{\alpha\beta\gamma\dot\alpha}(x)u^{+\alpha}u^{+\beta}u^{-\gamma}+\cdots+O(\theta)$ we find the ordinary gauge field $A_{\alpha\dot\alpha}(x)$ and an infinite set of auxiliary higher-spin fields $A_{\alpha\beta\gamma\dot\alpha}(x),\ldots$. Note the absence of the other projection $\theta^{-A}=\theta^A_\alpha u^{-\alpha}$ in (36). Such superfields are called chiral-analytic.

The prepotentials have the meaning of connections for two of the gauge covariant derivatives in the theory, namely

$$\nabla^{++}=\partial^{++}+A^{++}\,,\qquad \nabla^+_{\dot\alpha}=\partial^+_{\dot\alpha}+A^+_{\dot\alpha}\,. \tag{37}$$

Here $\partial^+_{\dot\alpha}=u^{+\alpha}\partial_{\alpha\dot\alpha}$ is a projection of the space-time derivative ∂_x while $\partial^{++}=u^{+\alpha}\partial/\partial u^{-\alpha}$ is one of the two covariant derivatives on S^2. These derivatives transform with a gauge parameter of the chiral-analytic type,

$$\nabla\ \to\ e^{\Lambda(x,\theta^+,u)}\,\nabla\,e^{-\Lambda(x,\theta^+,u)}\,. \tag{38}$$

The remaining gauge connections can be constructed from the prepotentials by solving the various super-curvature constraints. In particular, the projected spinor derivative $\partial_A^+ = u^{+\alpha}\partial/\partial\theta^{\alpha A}$ commutes with the gauge parameter $\Lambda(x, \theta^+, u)$, hence it needs no connection, $\nabla_A^+ = \partial_A^+$.

The action of the theory consists of two terms,

$$S_{\mathcal{N}=4\ SYM} = \int dud^4xd^4\theta^+\ L_{CS}(x, \theta^+, u) + \int d^4xd^8\theta\ L_Z(x, \theta)\,. \tag{39}$$

The first term in (39) is of the Chern–Simons type,

$$L_{\text{CS}}(x, \theta^+, u) = \text{tr}\left(A^{++}\partial^{+\dot\alpha}A^+_{\dot\alpha} - \frac{1}{2}A^{+\dot\alpha}\partial^{++}A^+_{\dot\alpha} + A^{++}A^{+\dot\alpha}A^+_{\dot\alpha}\right) \tag{40}$$

and it describes the self-dual sector of the theory [33]. The second term in (39) involves only the prepotential A^{++} in a non-polynomial way [24, 34],

$$L_{\text{Z}} = \text{tr}\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\int du_1\dots du_n\ \frac{A^{++}(x, \theta_1^+, u_1)\dots A^{++}(x, \theta_n^+, u_n)}{\langle u_1^+u_2^+\rangle\dots\langle u_n^+u_1^+\rangle}\,, \tag{41}$$

where $\theta_i^{+A} = \theta^{\alpha A}(u_i)_\alpha^+$ with $i = 1, \dots, n$ and $\langle u_i^+u_j^+\rangle = u_i^{+\alpha}\epsilon_{\alpha\beta}\,u_j^{+\beta}$. This Lagrangian is local in (x, θ) space but non-local in the harmonic space (each copy of A^{++} depends on its own harmonic variable). The gauge coupling constant g can be restored by redefining $A \to gA$ and $L \to g^{-2}L$.

In this paper we are dealing with form factors, so we need to define the supersymmetric on-shell states. A detailed discussion can be found in [30], here we only recall that the super-wave functions of the prepotentials A in the state with (super)momentum (k, η) have the form

$$\langle k, \eta|A^{++}(x, \theta^+, u)|0\rangle = \delta^2(k, u)e^{ikx+\langle k\theta\rangle\eta}\,, \qquad \langle k, \eta|A^+_{\dot\alpha}(x, \theta^+, u)|0\rangle = 0 \tag{42}$$

provided we quantise the theory in the light-cone gauge. The harmonic delta function $\delta^2(k, u)$ identifies the harmonic variable of the field with the chiral spinor momentum, $u_\alpha^+ = k_\alpha$. Notice that only the prepotential A^{++} has a non-trivial wave function, while $A^+_{\dot\alpha}$ does not appear in external states.

3.2 Chiral Wilson Loop in LHC Superspace

Now, the question arises how to reformulate the Wilson loop (29), (30) in terms of the prepotentials?

The chiral lightlike Wilson loop in LHC superspace takes the following form:

$$\mathcal{W}_n = \frac{1}{N}\mathrm{tr}\prod_{i=1}^{n} U(x_i, \theta_i; p_i, p_{i-1}) E_{i+1,i}\,. \tag{43}$$

Here the so-called bilocal bridge

$$U(x,\theta; p_2, p_1) = 1 + \sum_{n=1}^{\infty}(-1)^n \int du_1 \ldots du_n \frac{\langle p_2 p_1\rangle A^{++}(1)\ldots A^{++}(n)}{\langle p_2 u_1^+\rangle\langle u_1^+ u_2^+\rangle \ldots \langle u_n^+ p_1\rangle} \tag{44}$$

resembles the interaction Lagrangian (41). The bridges glue together adjacent Wilson line segments in (43),

$$E_{i+1,i} = P\exp\left\{-\frac{i}{2}\int_0^1 dt\; \tilde{p}_i^{\dot\alpha} A^+_{\dot\alpha}\Big(x_i - t\tilde{p}_i p_i, \langle p_i\theta_i\rangle, |p_i\rangle\Big)\right\}\,. \tag{45}$$

We remark that in the expression for the Wilson loop (43) the prepotential A^{++} appears only at the cusps of the Wilson loop contour via the bilocal bridge U (44), while the other prepotential $A^+_{\dot\alpha}$ contributes only through the edges of the contour.

We would like to emphasise that the definition of the Wilson loop (43) differs from the twistor formulation of Mason and Skinner [5] in that it contains the additional Wilson line segments $E_{i+1,i}$. We believe that the definition of the Wilson loop in [5] is not gauge invariant and hence it is incomplete. Still, the result of their calculation of the NMHV Wilson loop is correct. However, as we show in this paper, the Wilson line segments in (43) are indispensable for obtaining a gauge-invariant result for the Wilson loop form factor.

4 Diagrammatic Approach to the Duality

In this section we illustrate the duality (35) in the simplest MHV×MHV case. It corresponds to the first term on the right-hand side of (34) which has the lowest Grassmann degree ($\kappa = 0, \sigma = 0$). We apply the Feynman rules to the calculation of the Wilson loop form factor defined in (43), in the planar limit and to the lowest order in the coupling and rederive the result (24). This example illustrates both the graph duality and the simplicity of the LHC computation.

We end the section by a discussion of the general structure of the non-MHV diagrams.

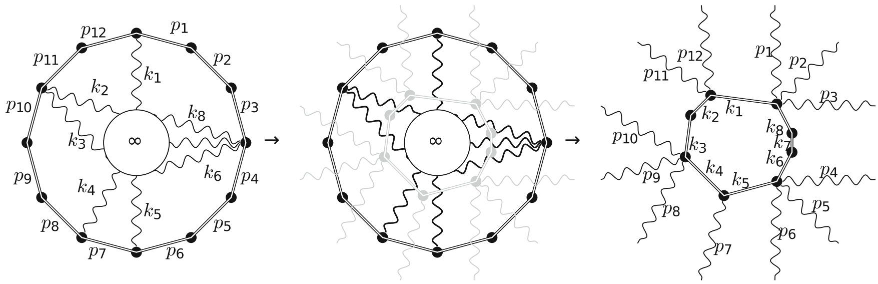

Fig. 3 The left figure represents a planar Born-level diagram for the Wilson loop form factor $W^{(0,0)}_{12,8}$. The external particles are coming from infinity which is chosen inside the Wilson loop contour. The right figure represents a diagram for $W^{(0,0)}_{8,12}$ where the variables specifying the Wilson loop contour and the external particles are swapped. Here infinity is chosen to lie outside the Wilson loop contour. In the middle figure the two diagrams are superimposed so that the planar graph duality is manifest

4.1 MHV Example

As follows from the definition of the Wilson loop (43)–(45), to the lowest degree in the Grassmann variables, the Born-level contribution only comes from diagrams without internal propagators and interaction vertices and with the prepotential A^{++} replaced by the wave function. Indeed, the propagators of the prepotentials A^{++} and $A^{+}_{\dot\alpha}$ are nilpotent (either $\sim \theta^4$ or $\sim \eta^4$) and increase the Grassmann degree. This leaves us with only one type of diagram illustrated in Fig. 3.

Here in the diagram on the left-hand side we draw all external legs inside the Wilson loop contour. These legs are ordered according to (5) and they end at a point that we call ‘infinity’. This graph contains $n = 12$ edges and $m = 8$ external particles and contributes to $F_{12,8}$. In the second diagram in Fig. 3 we show the planar dual graph faintly superimposed. For every face of the original graph we draw a vertex, then we join the vertices up by appropriate edges going through the boundaries of two faces as described above. This results in the third diagram which we recognise as a valid MHV$\times$MHV diagram contributing to $F_{8,12}$ with all external legs outside the Wilson loop.

Let us compute the graph expressions using the simple Feynman rules. First consider the left diagram in Fig. 3. There are five non-trivial cusps of the Wilson loop emitting particles. We obtain the following contribution to $F_{12,8}$

$$
\begin{aligned}
F_{12,8} = {} & \frac{e^{ik_1x_1+Q_1\theta_1}\langle p_{12}p_1\rangle}{\langle p_1k_1\rangle\langle k_1p_{12}\rangle} \times \frac{e^{i(k_2+k_3)x_{11}+(Q_2+Q_3)\theta_{11}}\langle p_{10}p_{11}\rangle}{\langle p_{11}k_2\rangle\langle k_2k_3\rangle\langle k_3p_{10}\rangle} \\
& \times \frac{e^{ik_4x_8+Q_4\theta_8}\langle p_7p_8\rangle}{\langle p_8k_4\rangle\langle k_4p_7\rangle} \times \frac{e^{ik_5x_7+Q_5\theta_7}\langle p_6p_7\rangle}{\langle p_7k_5\rangle\langle k_5p_6\rangle} \times \frac{e^{i(k_6+k_7+k_8)x_4+(Q_6+Q_7+Q_8)\theta_4}\langle p_3p_4\rangle}{\langle p_4k_6\rangle\langle k_6k_7\rangle\langle k_7k_8\rangle\langle k_8p_3\rangle}\,.
\end{aligned}
\tag{46}
$$

where $Q_i\theta_j \equiv \eta_{iA}\langle k_i\theta_j^A\rangle$. The dependence on the Grassmann variables follows the similar bosonic variables exponents. Substituting $F_{12,8}$ into (8) and (33) we obtain the corresponding contribution to $\widehat{W}_{12,8}$

$$\widehat{W}_{12,8} = \frac{\langle k_8k_1\rangle\langle k_1k_2\rangle\langle k_3k_4\rangle\langle k_4k_5\rangle\langle k_5k_6\rangle \times \langle p_{12}p_1\rangle\langle p_{10}p_{11}\rangle\langle p_7p_8\rangle\langle p_6p_7\rangle\langle p_3p_4\rangle}{\langle p_1k_1\rangle\langle k_1p_{12}\rangle\langle p_{11}k_2\rangle\langle k_3p_{10}\rangle\langle p_8k_4\rangle\langle k_4p_7\rangle\langle p_7k_5\rangle\langle k_5p_6\rangle\langle p_4k_6\rangle\langle k_8p_3\rangle} \,. \tag{47}$$

Let us now look at the right diagram in Fig. 3. It depends on the variables (y_j, ψ_j) defining the Wilson loop contour and the variables (p_i, ω_i) specifying the external particles. Using the effective Feynman rules, we obtain the following contribution to $F_{8,12}$:

$$\begin{aligned} F_{8,12} = {} & \frac{e^{i(p_1+p_2+p_3)y_1+(\tilde{Q}_1+\tilde{Q}_2+\tilde{Q}_3)\psi_1}\langle k_8k_1\rangle}{\langle k_8p_3\rangle\langle p_3p_2\rangle\langle p_2p_1\rangle\langle p_1k_1\rangle} \times \frac{e^{i(p_{11}+p_{12})y_2+(\tilde{Q}_{11}+\tilde{Q}_{12})\psi_2}\langle k_1k_2\rangle}{\langle k_1p_{12}\rangle\langle p_{12}p_{11}\rangle\langle p_{11}k_2\rangle} \\ & \times \frac{e^{i(p_8+p_9+p_{10})y_4+(\tilde{Q}_8+\tilde{Q}_9+\tilde{Q}_{10})\psi_4}\langle k_3k_4\rangle}{\langle k_3p_{10}\rangle\langle p_{10}p_9\rangle\langle p_9p_8\rangle\langle p_8k_4\rangle} \times \frac{e^{ip_7y_5+\tilde{Q}_7\psi_5}\langle k_4k_5\rangle}{\langle k_4p_7\rangle\langle p_7k_5\rangle} \\ & \times \frac{e^{i(p_4+p_5+p_6)y_6+(\tilde{Q}_4+\tilde{Q}_5+\tilde{Q}_6)\psi_6}\langle k_5k_6\rangle}{\langle k_5p_6\rangle\langle p_6p_5\rangle\langle p_5p_4\rangle\langle p_4k_6\rangle} \,, \end{aligned} \tag{48}$$

where $\tilde{Q}_i\psi_j \equiv \omega_i^A\langle p_i\psi_{jA}\rangle$. Substituting this expression into (8) and (33) we find that its contribution to $\widehat{W}_{8,12}$ is precisely equal to (47),

$$\widehat{W}_{8,12} = \widehat{W}_{12,8} \,. \tag{49}$$

This example illustrates the general diagrammatic proof of the duality in the MHV case: there are mixed $\langle k_i p_j\rangle$ brackets, common to both the graph and its dual. Then the missing $\langle k_ik_j\rangle$ brackets in the denominator on one side become explicit numerator terms from the Wilson loop vertices on the other, and vice versa for the $\langle p_ip_j\rangle$ brackets.

The exponential factors can be seen to agree in general, also diagrammatically. Using $k_j = y_j - y_{j+1}$ we find that there is an exponent $e^{ix_iy_j}$ in the left diagram if and only if the face y_j has a corner x_i. In the dual picture faces and vertices are swapped, but the result is unchanged. The Grassmann exponents follow the same pattern.

5 Concluding Remarks

In this paper we have given the proof of the new duality for Wilson loop form factors at the first non-trivial NMHV-like level and in the Born approximation. Can we go beyond?

Consider the general duality (35) in the Born approximation. In this case the cusp diagrams involve several propagators (see Fig. 1) and the corresponding edge diagrams also have a more complicated structure. In particular we need diagrams involving higher-order edge terms in the expansion of the Wilson lines $E_{i+1,i}$, Eq. (45). Also, we encounter diagrams of the mixed type, with cups-to-cusp and cusp-to-edge propagators. Nevertheless the mechanism of spurious pole cancellation is expected to be essentially the same.

We can start with the cusp diagrams for which the duality is evident since it is a duality of planar graphs. These diagrams provide the physical poles corresponding to vanishing invariant masses, $(k_i + \cdots + k_{j-1})^2 = y_{ij}^2 = 0$, or to the distance between two distant points of the Wilson loop contour becoming lightlike, $x_{ij}^2 = 0$. However they contain various complex spurious poles. These poles are removed by adding the appropriate mixed and edge diagrams. For each spurious pole there are correction terms obtained by sliding an external leg along a propagator. The mechanism is expected to work iteratively, first removing the poles of the pure cusp diagrams, then of the mixed, etc.

We can also think of the duality beyond the Born approximation. The loop corrections to the vacuum expectation value of the Wilson loop create UV-divergences. At loop level the scattering amplitude suffers from IR-divergences. Since the Wilson loop form factor is a hybrid observable interpolating between the two, its perturbative corrections are both UV- and IR-divergent.[3] So one needs to introduce a regularisation which can handle both types of divergences. Instead, we can consider the duality for the four-dimensional loop integrands corresponding to Lagrangian insertions into the Born-level object.

In the planar limit the loop integrands are unambiguously well-defined rational functions. So it is natural to expect that the duality works for them similarly to the Born approximation. Indeed, using the effective Feynman rules together with the Euclidean Fourier integration rules[4] one can see that the cusp diagrams are dual to each other as loop integrands. The corresponding edge diagrams play an auxiliary role cancelling spurious complex poles. The duality (15) is again translated into a planar graph duality. In Fig. 4 we give an example of the duality in the MHV$\times$MHV sector in the one-loop approximation. There the Wilson loop contour is purely bosonic and the scattered particles are $(+1)$ helicity gluons (this is equivalent to explicitly performing the integration over the superspace variable related to point y_0 which the effective rules naturally give us). In the left diagram we introduce the region momenta $y_0, y_1, \ldots, y_6$ associated with faces and represent the momentum space integral as an integration over y_0. Multiplying it by the Parke-Taylor prefactor we obtain the contribution to $W_{4,6}^{(0,0)}$,

[3]Notice that the divergent part of the Wilson loop form factor automatically satisfies the duality relation (15). Namely, the IR divergencies of $W_{n,m}$ match the UV divergences of $W_{m,n}$ and vice versa.

[4]We cannot fully justify applicability of the Fourier transform in Euclidean space until we have checked for integrand level cancellation of spurious poles, but we assume this here.

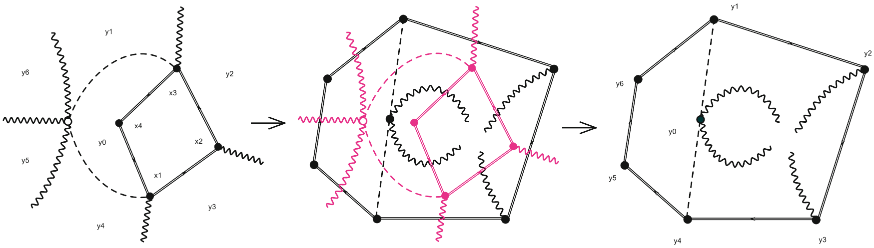

Fig. 4 Diagrammatic representation of the duality relation $W_{4,6}^{(0,0)} \leftrightarrow W_{6,4}^{(0,0)}$ in the one-loop approximation

$$\int d^4 y_0 \frac{e^{ix_{23}y_2} e^{ix_{12}y_3} e^{iy_0 x_{31}} [\xi|y_{10}y_{04}|\xi]^3}{y_{10}^2 y_{04}^2 [\xi|y_{04}|k_4\rangle\langle k_6|y_{10}|\xi]\langle p_3|y_{10}|\xi][\xi|y_{10}|k_1\rangle\langle k_3|y_{04}|\xi][\xi|y_{04}|p_4\rangle} \times \frac{\langle p_4 p_1\rangle\langle p_1 p_2\rangle\langle p_2 p_3\rangle\langle k_6 k_1\rangle\langle k_1 k_2\rangle\langle k_2 k_3\rangle\langle k_3 k_4\rangle}{\langle k_1 p_2\rangle\langle p_2 k_2\rangle\langle k_2 p_1\rangle\langle p_1 k_3\rangle} . \quad (50)$$

In the right diagram we use the Euclidean Fourier transform to write it down immediately in coordinate space and integrate over position y_0 of the interaction vertex. Its contribution to $W_{6,4}^{(0,0)}$ coincides with (50). So we see the duality at the level of the integrand.

There are several directions for further investigations. It is well known that the Born-level amplitudes have a remarkable dual superconformal symmetry which, combined with the native superconformal symmetry, results in a Yangian structure [31, 35–37]. As a result, the form of the amplitude is completely determined by this powerful symmetry and the requirement of absence of spurious poles. In this context we may ask the question if the new duality found in this paper could be a manifestation of some hidden symmetry? The first step in this direction should be to elucidate the role of conformal symmetry. It is supposed to simultaneously act on the Wilson loop component of the form factor as a local symmetry, and on its amplitude component as a non-local symmetry. This issue is under investigation.

It would also be interesting to understand how to properly regularise the loop correction integrals so that the duality still holds at loop level. Another challenging problem is to find a strong coupling or AdS/CFT analog of this duality.

Acknowledgements We profited from numerous discussions with Simon Caron-Huot and Ömer Gürdoğan. G.K. would like to thank Ömer Gürdoğan for collaboration at the early stage of this project. We acknowledge partial support by the French National Agency for Research (ANR) under contract StrongInt (BLANC-SIMI-4-2011). The work of D.C. has been partially supported by the RFBR grant 14-01-00341. The work of P.H. has been partially supported by an STFC Consolidated Grant ST/L000407/1. P.H. would also like to thank the CNRS for financial support and LAPTh for hospitality where part of this work was done.

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Many-Body Localization in Large-*N* Conformal Mechanics

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Abstract In quantum statistical mechanics, closed many-body systems that do not exhibit thermalization after an arbitrarily long time in spite of the presence of interactions are called as many-body localized systems, and recently have been vigorously investigated. After a brief review of this topic, we consider a many-body interacting quantum system in one dimension, which has conformal symmetry and integrability. We exactly solve the system and discuss its thermal or non-thermal behavior.

Keywords Thermalization · Many-body localization · Conformal quantum mechanics · Integrable models · Hopf algebra

1 Introduction

In quantum statistical physics, it is still a big challenge to formulate and understand how systems out of thermal equilibrium settle down to systems in thermal equilibrium, although innumerable attempts has been done toward its understanding for over a century. Recently, by investigating closed quantum many-body systems and their time evolution for a sufficiently long time, two qualitaitvely different phases have been found in the thermodynamic limit, which are referred to as *thermalization/delocalization* and *localization*. First, we start with a brief review of these phases.[1]

[1]For review articles, see [1, 10, 12] for example.

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_9

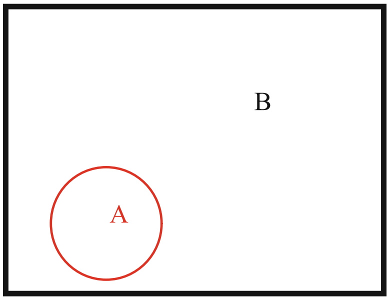


Fig. 1 The closed system S is the inside of the box. The subregion A is a region bounded by the red circle, and $B = S - A$ is the rest

1.1 Thermalization

Let us consider a closed quantum system S, for which the Hamiltonian H is defined. The density matrix of the system ρ evolves with the time t as

$$\rho(t) = e^{-iHt}\rho(0)\, e^{iHt}. \tag{1}$$

Suppose the same system is put in thermal equilibrium at temperature β^{-1}. Its density matrix is expressed as

$$\rho^{(\mathrm{eq})}(\beta) = \frac{1}{Z(\beta)} e^{-\beta H} \qquad \text{with} \qquad Z(\beta) = \mathrm{Tr}\, e^{-\beta H}. \tag{2}$$

Next, we pick any small subregion A in S in real space, and regard $B = S - A$ as a reservior (Fig. 1). The reduced density matrix of A for (1) and (2) is obtained from ρ by tracing out the states belonging to the Hilbert space of the subsystem B:

$$\rho_A(t) = \mathrm{Tr}_B\, \rho(t), \tag{3}$$

and

$$\rho_A^{(\mathrm{eq})}(\beta) = \mathrm{Tr}_B\, \rho^{(\mathrm{eq})}(\beta), \tag{4}$$

respectively. Then, we define thermalization as follows.

Definition 1 If

$$\rho_A(t) \to \rho_A^{(\mathrm{eq})}(\beta) \tag{5}$$

as sending t and the volume of S to infinity with the volume of A being fixed, and if it holds for any choice of the subsystem A, the system S thermalizes for the temperature β^{-1}.

Note that since in a closed system the density matrix of the total system $\rho(t)$ undergoes unitary time-evolution, $\rho(t)$ does not evolve to $\rho^{(\mathrm{eq})}(\beta)$ in general. This brings us to the Eigenstate Thermalization Hypothesis.

1.2 Eigenstate Thermalization Hypothesis

Suppose the initial state $\rho(0)$ is a pure state for an energy eigenstate of the energy E_n:

$$\rho(0) = |E_n\rangle \langle E_n| \qquad \text{with} \qquad H\,|E_n\rangle = E_n\,|E_n\rangle\,. \tag{6}$$

Then, ρ is time-independent: $\rho(t) = \rho(0)$, which leads to $\rho_A(t) = \rho_A(0)$ for any A from (3). In this case, noting Definition 1, we could expect that all the energy eigenstates are thermalized, which is called as the Eigenstate Thermalization Hypothesis (ETH) [8, 13–15].

If ETH holds, the temperature at the thermal equilibrium, denoted by β_n^{-1}, is determined by

$$E_n = \langle H\rangle_{\beta_n} \equiv \frac{1}{Z(\beta_n)}\,\mathrm{Tr}\left(H\,e^{-\beta_n H}\right). \tag{7}$$

The entanglement entropy of the subsystem A:

$$S_A = -\mathrm{Tr}_A\,(\rho_A \ln \rho_A) \tag{8}$$

coincides with the equilibrium thermal entropy of A. In particular, S_A is an extensive quantity, proportional to the volume of A.

However, ETH is a hypothesis, and not true for one class of systems. Such systems are called as localized systems.

1.3 Localized Systems

A simple example of single-particle localization is given by the one-dimensional Hamiltonian:

$$H = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V_p(x) + V_q(x), \tag{9}$$

where $V_p(x)$ is a periodic potential, and $V_q(x)$ is a random noise. If the noise is absent ($V_q(x) = 0$), the wave function of the particle is oscillating due to the Bloch wave, and delocalized. However, when the noise is turned on, the wave function becomes localized as

$$\psi(x) \sim e^{-\mu_q x} \qquad \text{as} \qquad |x| \to \infty \tag{10}$$

with a strictly positive constant μ_q. This phenomenon is well-known as the Anderson localization [2, 5]

Next, we turn to many-body localization (MBL), which takes place in the presence of many-body interactions and for highly excited states. A typical example is given by a one-dimensional quantum spin-1/2 chain, whose Hamiltonian takes the form

$$H = \sum_i h_i \sigma_i^z + J \sum_{<i,j>} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j. \tag{11}$$

Here, $i, j \in \{1, 2, \ldots\}$ denote the sites of the system, h_i are random magnetic fields at the site i distributed over the range $[-W, W]$, and the second term represents the nearest neighbor interactions of the Pauli spins.

At $J = 0$, the eigenstates of (11) are product states of the σ^z eigenstates: $\left|\sigma_1^z\right\rangle \otimes \left|\sigma_2^z\right\rangle \otimes \cdots$ with $|\sigma^z\rangle = |\uparrow\rangle$ or $|\downarrow\rangle$. Each spin variable is completely decoupled and undergoes independent time evolution. This system is fully localized, and essentially the same as the above single-particle localization. There are strictly local integrals of motions (LIOM) σ_i^z $(i = 1, 2 \ldots)$, whose supports are on single sites.

When turning on J but $J \ll W$, the localization property somehow remains. This case is called as MBL. There are also LIOM, but they satisfy milder locality condition with exponentially decaying tails in large distances (called as quasi LIOM). Such quasi LIOM are constructed, and DC spin transport and energy transport are shown to be absent perturbatively and nonperturbaively with respect to the coupling J [4, 9].

On increasing J, the localization ceases and ETH starts to hold eventually. Interestingly, there will be a transition between MBL (localized) and delocalized phases around $J \sim W$, which is a new type of phase transition between thermal equilibrium and out-of-equilibrium. It is expected that the localization is an intriguing phenomenon that protects the system from thermal decoherence and can be useful to construct devices for quantum computations.

However, analyses for MBL have been performed mainly for quantum spin systems. Extension to other quantum systems should be important to find new aspects and understand universal properties for localizations. In the rest of this contribution, we construct an integrable model of many-body conformal quantum mechanics by using its coalgebra structure, and analyze its thermal or localization properties.

2 Many-Body Interacting Model by Using Coproducts

The conformal group in one dimension, $SL(2, \mathbf{R})$, is generated by the Lie algebra generators L_0, $L_\pm$ satisfying

$$\left[L_0, L_\pm\right] = \pm L_\pm, \qquad [L_+, L_-] = -2L_0 \tag{12}$$

with the quadrartic Casimir

$$C = L_0^2 - L_0 - L_+ L_-. \tag{13}$$

This is realized in one-dimensional quantum mechanical system [7] as

$$L_0 = \frac{1}{4}\left(p^2 + \frac{g}{x^2} + x^2\right), \tag{14}$$

$$L_\pm = \frac{1}{4}\left(-p^2 - \frac{g}{x^2} + x^2\right) \mp i\frac{1}{4}(xp + px) \tag{15}$$

with $[x,\ p] = i$ and $C = -\frac{3}{16} + \frac{1}{4}g$. L_0 plays a role of the Hamiltonian. For simplicity, we will consider the case of $g = 0$, in which the system reduces to a harmonic oscillator.

2.1 Coproducts

In treating N-body systems, it is convenient to introduce coproducts denoted by $\Delta^{(k)}$ $(k = 2, 3, \ldots, N)$. Let $L_{a,\,i}$ $(a = 0,\ \pm)$ be the L_a-operator for particle i (or at site i). $\Delta^{(2)}(L_a)$ acts on two-particle states, which is defined by

$$\Delta^{(2)}(L_a) = L_a \otimes 1 + 1 \otimes L_a = L_{a,\,1} + L_{a,\,2}. \tag{16}$$

Also, $\Delta^{(2)}(1) = 1 \otimes 1$. Then, $\Delta^{(3)}(L_a)$ acting on three-particle states is given as

$$\begin{aligned} \Delta^{(3)}(L_a) &= (\mathbf{1} \otimes \Delta^{(2)}) \circ \Delta^{(2)}(L_a) \\ &= (\mathbf{1} \otimes \Delta^{(2)}) \circ (L_a \otimes 1 + 1 \otimes L_a) \\ &= L_a \otimes \Delta^{(2)}(1) + 1 \otimes \Delta^{(2)}(L_a) \\ &= L_a \otimes 1 \otimes 1 + 1 \otimes (L_a \otimes 1 + 1 \otimes L_a) \\ &= L_{a,\,1} + L_{a,\,2} + L_{a,\,3}, \end{aligned} \tag{17}$$

In general, $\Delta^{(k)}(L_a)$ is inductively given as

$$\begin{aligned} \Delta^{(k)}(L_a) &= (\overbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}^{k-2} \otimes \Delta^{(2)}) \circ \Delta^{(k-1)}(L_a) \\ &= L_{a,\,1} + \cdots + L_{a,\,k}. \end{aligned} \tag{18}$$

Note that the coproducts act as homomorphism and preserve the algebra (12):

$$[\Delta^{(k)}(L_0),\ \Delta^{(k)}(L_\pm)] = \pm\Delta^{(k)}(L_\pm), \tag{19}$$

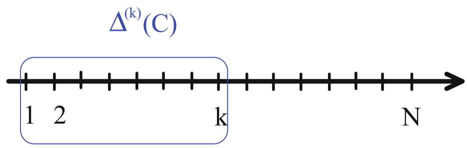
$$[\Delta^{(k)}(L_+),\ \Delta^{(k)}(L_-)] = -2\Delta^{(k)}(L_0) \tag{20}$$

with the quadratic Casimir

$$\Delta^{(k)}(C) = \left(\Delta^{(k)}(L_0)\right)^2 - \Delta^{(k)}(L_0) - \Delta^{(k)}(L_+)\Delta^{(k)}(L_-). \tag{21}$$

We can see that $\Delta^{(k')}(C)$ commutes with $\Delta^{(k)}(L_a)$ for $k' \le k$.

Fig. 2 The operator $\Delta^{(k)}(C)$ has support on sites $\{1,\ 2,\ \ldots,\ k\}$



2.2 Hamiltonian

We consider the Hamiltonian for N-particle interacting conformal system as

$$H_N = \Delta^{(N)}(L_0) + \sum_{k=2}^{N} \alpha_k \Delta^{(k)}(C), \tag{22}$$

where the first term describes N free harmonic oscillators, and the rest are interactions with coupling constants α_k. $\Delta^{(k)}(C)$ is an interaction with support on sites 1 to k as depicted in Fig. 2. The construction of (22) is based on the idea in [3, 11]. Eventually, we send N to infinity.

Since $\Delta^{(N)}(L_0)$ and $\Delta^{(k)}(C)$ $(k = 2, \ldots, N)$ mutually commute, they give N conserved quantities. This implies that the system is integrable. However, they are not local in general, and it is nontrivial whether the system exhibits MBL. If we choose the coupling constants behaving as

$$\alpha_k \sim e^{-k/\xi} \qquad \text{with } \xi \text{ some positive number}, \tag{23}$$

all the interactions become quasi local and the above conserved quantities can be regarded as quasi LIOM.

In terms of the position and momentum variables, (22) is expressed as

$$H_N = \sum_{i=1}^{N} \frac{1}{4}\left(p_i^2 + x_i^2\right) + \sum_{k=2}^{N} \alpha_k \left\{ \frac{1}{4} \sum_{1 \le i < j \le k} M_{ij}^2 + \frac{k(k-4)}{16} \right\} \tag{24}$$

with $M_{ij} \equiv x_i p_j - x_j p_i$ being an analog of angular momentum operators.

3 Eigenstates and Eigenvalues

In order to exactly solve the system (22), we first consider the lowest weight states (level 0 states) satisfying

$$L_{-,i}\,|s\rangle_N = 0 \qquad \text{for} \qquad i = 1, \ldots, N. \tag{25}$$

Here, the subscript 'N' in the state vector is used to denote the N-particle state. The conditions are solved as

$$|s\rangle_N = \left|r_0^{(1)}, \ldots, r_0^{(N)}\right\rangle \equiv \left|r_0^{(1)}\right\rangle \otimes \cdots \otimes \left|r_0^{(N)}\right\rangle \tag{26}$$

with $\left|r_0^{(i)}\right\rangle$ being the eigenstate of $L_{0,\,i}$ with the weight $1/4$ or $3/4$:

$$L_{0,\,i}\left|r_0^{(i)}\right\rangle = r_0^{(i)}\left|r_0^{(i)}\right\rangle \qquad \left(r_0^{(i)} = \frac{1}{4}, \frac{3}{4}\right). \tag{27}$$

The weights $1/4$ and $3/4$ correspond to the ground state energy and the first excited energy of the harmonic oscillator, respectively. The energy eigenvalue is given by

$$E_0 = R_N + \sum_{k=2}^{N} \alpha_k R_k \left(R_k - 1\right), \qquad R_k \equiv r_0^{(1)} + \cdots + r_0^{(k)} \tag{28}$$

Any state vector in the Fock space can be obtained by successively acting $L_{+,\,i}$ operators on the level 0 states. From the $SL(2,\ \mathbf{R})$ algebra (12), the states containing $n\ L_{+,\,i}$ operators increase the weight by n, and correspond to $2n$-th excited states of the harmonic oscillator. The Fock space is decomposed as

$$\mathcal{F} = \bigoplus_{r_0^{(1)},\ldots,r_0^{(N)}} \mathcal{F}_{(r_0^{(1)},\ldots,r_0^{(N)})} \tag{29}$$

with

$$\mathcal{F}_{(r_0^{(1)},\ldots,r_0^{(N)})} \equiv \left\{L_{+,\,1}^{k_1} \ldots L_{+,\,N}^{k_N}\,|s\rangle_N\ ;\ k_1, \ldots, k_N = 0, 1, 2, \ldots\right\}. \tag{30}$$

$L_{+,\,1}^{k_1} \ldots L_{+,\,N}^{k_N}\,|s\rangle_N$ is the eigenstate of $\Delta^{(N)}(L_0)$ with the eigenvalue $k_1 + \cdots + k_n + R_N$, and called as a level $k_1 + \cdots + k_n$ state.

3.1 Level 1 States

We find the following N states of level 1:

$$\begin{aligned}
|v_1\rangle_N &= \Delta^{(N)}(L_+)\,|s\rangle_N\,, \\
\left|v_{1,\,(1,1)}\right\rangle_N &= F_1\left(L_{+,\,1},\ L_{+,\,2}\right)|s\rangle_N\,, \\
\left|v_{1,\,(1,2)}\right\rangle_N &= F_1\left(\Delta^{(2)}(L_+),\ L_{+,\,3}\right)|s\rangle_N\,, \\
&\ \ \vdots \\
\left|v_{1,\,(1,\,N-1)}\right\rangle_N &= F_1\left(\Delta^{(N-1)}(L_+),\ L_{+,\,N}\right)|s\rangle_N\,,
\end{aligned} \tag{31}$$

where $F_1\left(\varDelta^{(n)}(L_+),\ L_{+,\,n+1}\right)$ is a linear function of $\varDelta^{(n)}(L_+)$ and $L_{+,\,n+1}$ given by

$$F_1\left(\varDelta^{(n)}(L_+),\ L_{+,\,n+1}\right) = -\frac{r_0^{(n+1)}}{R_n}\varDelta^{(n)}(L_+) + L_{+,\,n+1} \tag{32}$$

for $n = 1, \ldots, N-1$, and hereafter $\varDelta^{(1)}(L_+)$ is regarded as $L_{+,\,1}$. Notice that

$$\begin{aligned} &\varDelta^{(m)}(L_-)\left|v_{1,\,(1,n)}\right\rangle_N = 0, \\ &F_1\left(\varDelta^{(m)}(L_-),\ L_{-,\,m+1}\right)\left|v_{1,\,(1,n)}\right\rangle_N = 0 \end{aligned} \tag{33}$$

hold for $m > n$, which leads to the orthogonality of the states (31).

The energy eigenvalues are obtained as

$$E_1 = R_N + 1 + \sum_{k=2}^{N} \alpha_k R_k (R_k - 1) \tag{34}$$

for $|v_1\rangle_N$, and

$$E_{1,\,(1,n)} = R_N + 1 + \sum_{k=2}^{n} \alpha_k R_k (R_k - 1) + \sum_{k=n+1}^{N} \alpha_k (R_k + 1) R_k \tag{35}$$

for $\left|v_{1,\,(1,n)}\right\rangle_N$.

3.2 Level p States

General level p states are obtained as

$$\begin{aligned} \left|v_p\right\rangle_N &= \left(\varDelta^{(N)}(L_+)\right)^p |s\rangle_N\,, \\ \left|v_{p,\,(m_1,n_1),\,\ldots,(m_q,n_q)}\right\rangle_N &= \left(\varDelta^{(N)}(L_+)\right)^{p-m_1-\cdots-m_q} \\ &\quad \times F_{m_1}\left(\varDelta^{(n_1)}(L_+),\ L_{+,\,n_1+1}\right)_{+m_2+\cdots+m_q} \\ &\quad \times F_{m_2}\left(\varDelta^{(n_2)}(L_+),\ L_{+,\,n_2+1}\right)_{+m_3+\cdots+m_q} \\ &\quad \times \cdots \\ &\quad \times F_{m_{q-1}}\left(\varDelta^{(n_{q-1})}(L_+),\ L_{+,\,n_{q-1}+1}\right)_{+m_q} \\ &\quad \times F_{m_q}\left(\varDelta^{(n_q)}(L_+),\ L_{+,\,n_q+1}\right)|s\rangle_N\,, \end{aligned} \tag{36}$$

where q runs from 1 to p, and $m_1, \ldots m_q \in \{1, \ldots, p\}$ satisfy $\sum_{i=1}^{q} m_i \leq p$. The integers n_i should be taken as $N-1 \geq n_1 > n_2 > \cdots > n_q \geq 1$. $F_m\left(\Delta^{(n)}(L_+), L_{+,n+1}\right)$ is a degree-m homogeneous polynomial of $\Delta^{(n)}(L_+)$ and $L_{+,n+1}$, whose explicit form is

$$F_m\left(\Delta^{(n)}(L_+),\ L_{+,n+1}\right) = c_0^{(m)}\left(\Delta^{(n)}(L_+)\right)^m + c_1^{(m)}\left(\Delta^{(n)}(L_+)\right)^{m-1} L_{+,n+1} + \cdots + c_{p-1}^{(m)}\Delta^{(n)}(L_+)\left(L_{+,n+1}\right)^{m-1} + \left(L_{+,n+1}\right)^m \tag{37}$$

with the coefficients

$$c_k^{(m)} \equiv (-1)^{m-k}\binom{m}{k}\frac{\Gamma\left(2r_0^{(n+1)}+m\right)}{\Gamma\left(2r_0^{(n+1)}+k\right)}\frac{\Gamma\left(2R_n\right)}{\Gamma\left(2R_n+m-k\right)}. \tag{38}$$

Note that (37) is independent of the couplings α_k's. $F_m\left(\Delta^{(n)}(L_+),\ L_{+,n+1}\right)_{+\ell}$ denotes (37) with every R_n appearing in (38) replaced by $R_n+\ell$. The states in (36) consist of mutually orthogonal $\binom{p+N-1}{p}$ states. All of the states have no dependence on the couplings, which comes from the Hamiltonian (22) consists of the mutually commuting operators.

The norms of the states are computed as

$$\left\|\left|v_p\right\rangle_N\right\|^2 = p!\frac{\Gamma\left(2R_N+p\right)}{\Gamma\left(2R_N\right)}, \tag{39}$$

$$\begin{aligned}\left\|\left|v_{p,(m_1,n_1),\ldots,(m_q,n_q)}\right\rangle_N\right\|^2 &= (p-M_1)!\frac{\Gamma\left(2R_N+M_1+p\right)}{\Gamma\left(2R_N+2M_1\right)} \\ &\times\prod_{a=1}^{q}\left[m_a!\frac{\Gamma\left(2r_0^{(n_a+1)}+m_a\right)}{\Gamma\left(2r_0^{(n_a+1)}\right)}\frac{\Gamma\left(2R_{n_a}+2M_{a+1}\right)}{\Gamma\left(2R_{n_a}+2M_{a+1}+m_a\right)}\right. \\ &\left.\times\frac{\Gamma\left(2R_{n_a+1}+2M_a-1\right)}{\Gamma\left(2R_{n_a+1}+2M_{a+1}+m_a-1\right)}\right]\end{aligned} \tag{40}$$

with

$$M_a \equiv \sum_{k=a}^{q} m_k. \tag{41}$$

The energy eigenvalues are

$$E_p = R_N + p + \sum_{k=2}^{N}\alpha_k R_k(R_k-1) \tag{42}$$

for $\left|v_p\right\rangle_N$, and

$$\begin{aligned} E_{p,\,(m_1,n_1),\dots,(m_q,n_q)} &= R_N + p + \sum_{k=2}^{n_q} \alpha_k R_k (R_k - 1) \\ &\quad + \sum_{\ell=2}^{q} \sum_{k=n_\ell+1}^{n_{\ell-1}} \alpha_k \left(R_k + M_\ell\right)\left(R_k + M_\ell - 1\right) \\ &\quad + \sum_{k=n_1+1}^{N} \alpha_k \left(R_k + M_1\right)\left(R_k + M_1 - 1\right) \end{aligned} \tag{43}$$

for $\left|v_{p,\,(m_1,n_1),\,\dots,(m_q,n_q)}\right\rangle_N$.

We can see that all the level p states are degenerate for the free case, while the degeneracy is completely resolved by turning on the couplings α_k. Note for the choice (23), the level splitting between states with different m_j's is of the order $O\left(e^{N/\xi}\right)$, which yields continuous spectrum at large N. This seems a situation in which thermalization takes place. On the other hand, there are quasi local LIOM that support MBL as we have seen in Sect. 2. Thus, it is interesting to see which property of ETH and MBL is realized in this case.

4 Entanglement Entropy

Let us start with the density matrix for the pure state:

$$\rho = \frac{1}{\left\| \left|v_{p,\,(m_1,n_1),\dots,(m_q,n_q)}\right\rangle_N \right\|^2} \left|v_{p,\,(m_1,n_1),\dots,(m_q,n_q)}\right\rangle_N \left\langle v_{p,\,(m_1,n_1),\dots,(m_q,n_q)}\right| . \tag{44}$$

We divide the total system $S = \{1, 2, \dots, N\}$ into a small subsystem $A = \{N - \nu + 1, \dots, N\}$ with $\nu \ll N$ and the rest $B = \{1, 2, \dots, N - \nu\}$. For simplicity, we consider the case of $n_1 \leq N - \nu - 1$, in which all the F_m operators in (36) act only on B. For such pure states, the reduced density matrix ρ_A takes a diagonal form with each diagonal entry taking a simple form:

$$\lambda_{A,\,\tilde{n}} \equiv \binom{p - M_1}{\tilde{n}} \frac{B\left(2R_{N-\nu} + 2M_1 + \tilde{n},\, 2\bar{R}_\nu + p - M_1 - \tilde{n}\right)}{B\left(2R_{N-\nu} + 2M_1,\, 2\bar{R}_\nu\right)}, \tag{45}$$

where

$$\bar{R}_\nu \equiv \sum_{i=N-\nu+1}^{N} r_0^{(i)}, \tag{46}$$

and $\tilde{n}$ runs from 0 to $p - M_1$.

We find the large-N behavior of the entanglement entropy

$$S_A = - \sum_{\tilde{n}=0}^{p-M_1} \lambda_{A,\tilde{n}} \ln \lambda_{A,\tilde{n}} \tag{47}$$

in the following two cases:

- For $p - M_1 \ll R_N + M_1$ (case 1),

$$S_A \sim \bar{R}_\nu \frac{p - M_1}{R_N + M_1} \ln (R_N + M_1) . \tag{48}$$

  Since $\bar{R}_\nu$ grows with ν (the volume of A), this result exhibits the volume-law like behavior although the multiplicative factor $\frac{p-M_1}{R_N+M_1} \ln (R_N + M_1)$ is tiny for the case.
- For $p - M_1 \gg R_N + M_1$ (case 2),

$$S_A \sim \ln(p - M_1). \tag{49}$$

  This result is independent of ν, and exhibits the area law, which supports the localization phase.

In the case 1, the energy is relatively lower, but the result (48) seems to support thermal like phase. On the other hand, in the case 2, the energy is relatively higher, and the result (49) suggests localization. Interestingly, because the states (44) do not depend on the couplings α_k, the above results hold for any choice of α_k. In particular, the result means that there are some highly excited states which exhibit the area law behavior (49) even in the presence of nonlocal interactions. It is also interesting to analyze the case in which $p - M_1$ is comparable to $R_N + M_1$ (the intermediate region of the cases 1 and 2), and to see how the volume-law like behavior changes to the area law.

5 Discussion

In this contribution, first we have briefly reviewed topics on quantum thermalization and localization. Second, we have constructed an integrable model with many-body interactions by using coproducts, and obtained the exact spectrum of the model. Third, by computing the entanglement entropy, we have found a localization property in highly excited states in spite of nonlocal interactions. We guess that this captures a new aspect of localization, which has not been seen yet.

Since the entanglement entropy does not depend on the couplings, it will be interesting to analyze other quantities that are sensitive to the couplings. Actually, we introduced a deformation breaking the integrablity, and computed how the entanglement entropy of the level 1 states changes with the time t. For general couplings for which interactions are nonlocal, the entanglement entropy initially grows as t^2, but saturates at some value soon after and keeps oscillating. On the other hand, for the choice (23), the entanglement entropy keeps growing as t^2, and never reaches the point that is saturated in the nonlocal case. We can see that the exponential decreasing couplings crucially slow down the spreading of the entanglement. We are also considering to measure transport properties by computing connected two point correlation functions.

The $SL(2, \mathbf{R})$ conformal symmetry plays a crucial role to construct the Hamiltonian (22) and thus to make the energy eigenstates independent of the couplings. Investigating this model from the viewpoint of AdS/CFT correspondence [6] will also be intriguing.

Acknowledgements We would like to thank Catherine Meusburger for discussing the construction of integrable models by using coproducts. F. S. would also like to thank the organizers and Vladimir Dobrev for invitation to the workshop.

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Generalized Bases with a Resolution of the Identity: A Cooperative Game Theory Approach

A. Vourdas

Abstract A quantum system with d-dimensional Hilbert space H_d, is considered. A dresssing mechanism inspired by Shapley's methodology in cooperative game theory, is used to convert a total set of $n \geq d$ states (for which we have no resolution of the identity), into a 'generalized basis' of n mixed states with a resolution of the identity. Results based on these generalized bases are sensitive to physical changes and robust in the presence of noise. An arbitrary vector is expanded in these generalized bases, in terms of n component vectors. The concept of location index of a Hermitian operator, is introduced. Hermitian operators are studied using the concepts of comonotonic operators and comonotonicity intervals.

Keywords Generalized bases · Resolution of the identity · Shapley formalism in quantum context

1 Introduction

Coherent states, POVMs (positive operator valued measures) and frames and wavelets (e.g., [1–3]), lead to generalized bases. They are advantageous in comparison to orthonormal bases because calculations that use them are robust in the presence of noise, due to redundancy. An arbitrary state can be expanded in terms of coherent states or POVMs, because they form a resolution of the identity. In frames we have no exact resolution of the identity, but the frame operator is upper and lower bounded by the identity times a constant.

In recent work [4, 5], we proposed a novel approach in this general area. We started from a pre-basis, i.e., a total set of $n \geq d$ states (which might not form a

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_10

resolution of the identity). Using a formalism inspired by Shapley's methodology in cooperative game theory [6–9], we renormalized them into n density matrices $R(i)$, that resolve the identity. They can be used as a generalized basis, which is robust in the presence of noise and yet sensitive to physical changes. This is because noise is uniformly distributed in the whole phase space, while a physical change is associated with local changes of certain quantities. In the present paper we review this work. The aim is to give a physical presentation of these ideas, without the 'distraction' from technical proofs based on combinatorics.

In Sect. 2 we give briefly the basics of cooperative game theory. It considers a number of players and divides ('resolves') the 'total worth' of the game into the various players. Their individual contribution is 'dressed' (renormalized) with their contribution to various coalitions. This terminology (taken from quantum field theory) refers to the fact that we start with some initial quantity ('bare' quantity), and after we add corrections to it, we get the 'dressed or renormalized' quantity. In analogy to the cooperative game theory methodology, we consider in Sect. 3, $n \geq d$ vectors $|i\rangle$ in H_d, and we renormalize them into mixed states $R(i)$ that resolve the identity.

In Sect. 4, we show how to use these mixed states as a generalized basis. A vector in H_d is represented by n component vectors.

In Sect. 5, we represent Hermitian operators θ with n coefficients, and introduce the concept of location index. For Hermitian operators $\theta(\lambda)$ that depend on a coupling parameter λ, we introduce the concept of comonotonicity intervals of the coupling parameter λ, within which the location index remains constant. Crossing points from one comonotonicity interval to another, indicate a possible drastic change in the system.

We conclude in Sect. 6, with a discussion of our results.

2 Cooperative Game Theory: The Whole is Greater than the Sum of Its Parts

Given a set Ω of players, a coalition is a subset $A \subseteq \Omega$. Von Neumann and Morgenstern introduced the characteristic function which is a real valued function that assigns a value $v(A)$ to each subset of players $A \subseteq \Omega$. For the empty set $v(\emptyset) = 0$. The total worth of the whole game is $v(\Omega)$.

The characteristic function takes $2^{|\Omega|}$ values, because there are $2^{|\Omega|}$ subsets of Ω. If $A = \{i_1, \ldots, i_k\}$, then in general

$$v(A) \neq v(i_1) + \cdots + v(i_k). \quad (1)$$

This formalizes the expression 'the whole is greater than the sum of its parts' or 'one plus one makes three'. The coalition provides an added value, which can be positive or negative.

The Shapley methodology divides the total worth $v(\Omega)$ to the various players, taking into account their contribution to the various coalitions. We call this methodology of dividing the total worth, Shapley's resolution (i.e., division) of the total worth $v(\Omega)$, to the various players. This introduces physics terminology, and provides a link with the use of a similar methodology for a resolution of the identity in a quantum context, later.

Möbius transform has been introduced by Rota in combinatorics [10, 11]. It generalizes the 'inclusion-exclusion' principle in set theory, for the cardinality of the union of overlapping sets. Rota generalized this to partially ordered structures.

The Möbius transform of the characteristic function $v(A)$ and its inverse, are defined as:

$$\begin{aligned} \mathfrak{m}(B) &= \sum_{A \subseteq B} (-1)^{|A|-|B|} v(A); \qquad A, B \subseteq \Omega \\ v(A) &= \sum_{B \subseteq A} \mathfrak{m}(B). \end{aligned} \tag{2}$$

It quantifies the added value in the various coalitions. For example:

$$\begin{aligned} &\mathfrak{m}(i_1) = v(i_1); \quad \mathfrak{m}(i_1, i_2) = v(i_1, i_2) - v(i_1) - v(i_2) \\ &\mathfrak{m}(i_1, i_2, i_3) = v(i_1, i_2, i_3) - v(i_1, i_2) - v(i_1, i_3) - v(i_2, i_3) + v(i_1) + v(i_2) + v(i_3). \end{aligned} \tag{3}$$

In the special case that there is no added value in the coalitions, i.e.,

$$v(A) = v(i_1) + \cdots + v(i_k), \tag{4}$$

for all subsets A of Ω, then all the $\mathfrak{m}(B)$ with $|B| \geq 2$, are zero.

Shapley divided equally the added value of a coalition to all members of the coalition. The Shapley value for the player i is given by

$$\begin{aligned} S(i) &= \sum_{A \ni i} \frac{\mathfrak{m}(A)}{|A|} = v(i) + \frac{1}{2} \sum_{j} \mathfrak{m}(i, j) + \frac{1}{3} \sum_{j,k} \mathfrak{m}(i, j, k) + \ldots \\ \sum S(i) &= v(\Omega). \end{aligned} \tag{5}$$

The player i gets the worth of his individual contribution $v(i)$, half of the worth of the added value in the coalitions (i, j) (for all j), one third of the worth of the added value in the coalitions (i, j, k) (for all j, k), etc.

If there is no added value in any coalition, i.e., for every coalition $A = \{i_1, \ldots, i_k\}$ the $v(A) = v(i_1) + \cdots + v(i_k)$, then $S(i) = v(i)$.

Example 1 Workers $1, 2, 3$ working individually, produce $1, 1, 2$ items of the same product (per hour), correspondingly. The collaboration of $(1, 2)$ produces 4 items, the collaboration of $(1, 3)$ produces 5 items, and the collaboration of $(2, 3)$ produces 3 items. The collaboration of $(1, 2, 3)$ produces 7 items. In this case

$$\begin{aligned}&v(\emptyset) = 0; \quad v(1) = 1; \quad v(2) = 1; \quad v(3) = 2\\&v(1,2) = 4; \quad v(1,3) = 5; \quad v(2,3) = 3; \quad v(1,2,3) = 7.\end{aligned} \tag{6}$$

The Möbius transform of the characteristic function is

$$\begin{aligned}&\mathfrak{m}(1) = 1; \quad \mathfrak{m}(2) = 1; \quad \mathfrak{m}(3) = 2\\&\mathfrak{m}(1,2) = 2; \quad \mathfrak{m}(1,3) = 2; \quad \mathfrak{m}(2,3) = 0; \quad \mathfrak{m}(1,2,3) = -1,\end{aligned} \tag{7}$$

and it gives

$$\begin{aligned}&S(1) = \mathfrak{m}(1) + \frac{1}{2}[\mathfrak{m}(1,2) + \mathfrak{m}(1,3)] + \frac{1}{3}\mathfrak{m}(1,2,3) = \frac{8}{3}\\&S(2) = \frac{5}{3}; \quad S(3) = \frac{8}{3}\\&S(1) + S(2) + S(3) = v(1,2,3) = 7.\end{aligned} \tag{8}$$

3 Generalized Bases of Mixed States with a Resolution of the Identity

We consider a quantum system with positions and momenta in $\mathbb{Z}(d)$ (the integers modulo d). The associated Hilbert space H_d is d-dimensional.

In H_d we consider a 'pre-basis', i.e., a set

$$\Sigma = \{|i\rangle \mid i \in \Omega\}; \quad \Omega = \{1, \ldots, n\}; \quad n \geq d, \tag{9}$$

of n states $|i\rangle$, which are not necessarily an orthonormal basis. Ω is a set of labels for these states. Any d of these states are assumed to be linearly independent. Then any $r \geq d$ of these states, are a total set (i.e., there is no state which is orthogonal to all of them). In general, we have no resolution of the identity in terms of these n states.

Let $H(A)$ be the subspace of H_d spanned by the states $|i_1\rangle, \ldots |i_r\rangle$ where $A = \{i_1, \ldots, i_r\} \subseteq \Omega$. If $r < d$ then $H(A)$ is an r-dimensional subspace of H. If $r \geq d$, then $H(A) = H_d$. We call $\Pi(A)$ the projector to the subspace $H(A)$. If the cardinality of A is $|A| \geq d$ then $\Pi(A) = \mathbf{1}$. In general

$$\Pi(A) \neq \Pi(i_1) + \cdots + \Pi(i_r). \tag{10}$$

We note the analogy between this and Eq. (1). More generally, this section is inspired by the Shapley formalism in the previous section, with

$$v(A) \ \rightarrow \ \Pi(A). \tag{11}$$

In order to emphasize this analogy, we use a similar notation for the corresponding quantities in the two theories.

We define Möbius transforms in the present context as

$$\mathfrak{M}(B) = \sum_{A \subseteq B} (-1)^{|A|-|B|} \Pi(A); \quad A, B \subseteq \Omega. \tag{12}$$

The inverse Möbius transform is

$$\Pi(A) = \sum_{B \subseteq A} \mathfrak{M}(B). \tag{13}$$

Some examples are:

$$\begin{aligned} \mathfrak{M}(1) &= \Pi(1); \quad \mathfrak{M}(1,2) = \Pi(1,2) - \Pi(1) - \Pi(2) \\ \mathfrak{M}(1,2,3) &= \Pi(1,2,3) - \Pi(1,2) - \Pi(1,3) - \Pi(2,3) \\ &+ \Pi(1) + \Pi(2) + \Pi(3), \end{aligned} \tag{14}$$

and then

$$\begin{aligned} \Pi(1,2) &= \mathfrak{M}(1,2) + \mathfrak{M}(1) + \mathfrak{M}(2) \\ \Pi(1,2,3) &= \mathfrak{M}(1,2,3) + \mathfrak{M}(1,2) + \mathfrak{M}(1,3) + \mathfrak{M}(2,3) \\ &+ \mathfrak{M}(1) + \mathfrak{M}(2) + \mathfrak{M}(3). \end{aligned} \tag{15}$$

The $\mathfrak{M}(B)$ are related to commutators that involve the projectors $\Pi(A)$. For example:

$$[\Pi(i), \Pi(j)] = \mathfrak{M}(i,j)[\Pi(i) - \Pi(j)] \tag{16}$$

and

$$\begin{aligned} [[\Pi(i), \Pi(k)], \Pi(j)] &= \Pi(j)\mathfrak{M}(i,j,k)[\Pi(i) - \Pi(k)] \\ &+ [\Pi(i) - \Pi(k)]\mathfrak{M}(i,j,k)\Pi(j). \end{aligned} \tag{17}$$

So the use of Möbius operators is intimatelly related to non-commutativity.

We introduce the analogue of the Shapley values in Eq. (5) for the projectors:

$$\begin{aligned} &\mathscr{S}(i) = \sum_{A \ni i} \frac{\mathfrak{M}(A)}{|A|} = \Pi(i) + \frac{1}{2} \sum_{j} \mathfrak{M}(i,j) + \frac{1}{3} \sum_{j,k} \mathfrak{M}(i,j,k) + \ldots \\ &\sum_{i=1}^{n} \mathscr{S}(i) = \Pi(\Omega) = \mathbf{1}. \end{aligned} \tag{18}$$

It has been proved in [4, 5] that they are positive semi-definite operators, and that they all have the same trace which is $\frac{d}{n}$. The proof involves another approach to Shapley values which involves combinatorics, and which is omitted here.

Therefore we introduce the density matrices $R(i)$

$$R(i) = \frac{n}{d}\mathscr{S}(i); \quad \frac{d}{n}\sum_{i=1}^{n} R(i) = \mathbf{1}. \tag{19}$$

They satisfy the above resolution of the identity, and they can be used as a generalized basis of mixed states.

Example 2 In the special case that Σ is an orthonormal set of d states, the inequality in Eq. (10) becomes equality, and all the $\mathfrak{M}(B)$ with $|B| \geq 2$, are zero. In this case $R(i) = \Pi(i)$.

Example 3 In H_2 we consider the total set of states:

$$|0\rangle; \quad \frac{1}{\sqrt{5}}(2|0\rangle + |1\rangle); \quad \frac{1}{\sqrt{10}}(|0\rangle + 3i|1\rangle). \tag{20}$$

In this case $n = 3$, and using Eq. (3) we get

$$\begin{aligned}
\mathfrak{M}(1,2) &= \begin{pmatrix} -0.8 & -0.4 \\ -0.4 & 0.8 \end{pmatrix}; \quad \mathfrak{M}(1,3) = \begin{pmatrix} -0.1 & 0.3i \\ -0.3i & 0.1 \end{pmatrix} \\
\mathfrak{M}(2,3) &= \begin{pmatrix} 0.1 & -0.4 + 0.3i \\ -0.4 - 0.3i & -0.1 \end{pmatrix} \\
\mathfrak{M}(1,2,3) &= \begin{pmatrix} -0.1 & 0.4 - 0.3i \\ 0.4 + 0.3i & -0.9 \end{pmatrix}.
\end{aligned} \tag{21}$$

Therefore

$$\begin{aligned}
\Pi(1) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow R(1) = \begin{pmatrix} 0.775 & -0.100 + 0.075i \\ -0.100 - 0.075i & 0.225 \end{pmatrix}; \\
\Pi(2) &= \begin{pmatrix} 0.8 & 0.4 \\ 0.4 & 0.2 \end{pmatrix} \rightarrow R(2) = \begin{pmatrix} 0.625 & 0.200 + 0.075i \\ 0.200 - 0.075i & 0.375 \end{pmatrix}; \\
\Pi(3) &= \frac{1}{2}\begin{pmatrix} 0.1 & -0.3i \\ 0.3i & 0.9 \end{pmatrix} \rightarrow R(3) = \begin{pmatrix} 0.10 & -0.10 - 0.15i \\ -0.10 + 0.15i & 0.90 \end{pmatrix}.
\end{aligned} \tag{22}$$

The resolution of the identity is

$$\frac{2}{3}[R(1) + R(2) + R(3)] = \mathbf{1}. \tag{23}$$

4 Representation of Vectors in the Generalized Basis

An arbitrary normalized vector in H_d can now be expanded in terms of $n \geq d$ component vectors, as

$$|V\rangle = \sum_{i=1}^{n} |V(i)\rangle; \quad |V(i)\rangle = \frac{d}{n} R(i)|V\rangle. \tag{24}$$

Example 4 In H_2 we consider the vector

$$|V\rangle = \frac{1}{\sqrt{13}} \begin{pmatrix} 2i \\ 3 \end{pmatrix}. \tag{25}$$

Using the matrices $R(1)$, $R(2)$, $R(3)$, in Eq. (22), and the resolution of the identity in Eq. (23) we expand this vector as

$$|V\rangle = \begin{pmatrix} -0.055 + 0.328i \\ 0.152 - 0.037i \end{pmatrix} + \begin{pmatrix} 0.110 + 0.272i \\ 0.235 + 0.074i \end{pmatrix} + \begin{pmatrix} -0.055 - 0.046i \\ 0.443 - 0.037i \end{pmatrix}. \tag{26}$$

There is redundancy in this approach, which is precisely the merit for using it. In [4, 5] we gave numerical examples, where in spite of noise in the components, the overall error is small.

5 Representation of Hermitian Operators

Let $\theta(\lambda)$ be a Hermitian operator, e.g. a Hamiltonian that depends on a coupling parameter λ. Also let $s_\theta(i|\lambda)$ be the n coefficients

$$s_\theta(i|\lambda) = \frac{d}{n} \mathrm{Tr}[\theta(\lambda) R(i)]; \quad \sum_{i=1}^{n} s_\theta(i|\lambda) = \mathrm{Tr}(\theta). \tag{27}$$

In [4] we have considered a pre-basis that consists of coherent states, and in this case the $s_\theta(i|\lambda)$ is the Q-function. In general, we can regard $s_\theta(i|\lambda)$ as a generalized Q-function.

5.1 Location Indices, Comonotonic Operators and Comonotonicity Intervals

We order the $s_\theta(1|\lambda), \ldots, s_\theta(n|\lambda)$ as

$$s_\theta(i_1|\lambda) \geq s_\theta(i_2|\lambda) \geq \cdots \geq s_\theta(i_n|\lambda). \tag{28}$$

The location index of $\theta(\lambda)$, with respect to $\{R(i)\}$, is the n-tuple

$$\mathscr{L}[\theta(\lambda)] = (i_1, \ldots, i_n). \tag{29}$$

The $\mathscr{L}[\theta(\lambda)]$ indicates the position of $\theta(\lambda)$ with respect to the generalized basis $\{R(i)\}$. $\theta(\lambda)$ is more close to $R(i_1)$ (because $s_\theta(i_1|\lambda)$ is the largest), less close to $R(i_2)$, even less close to $R(i_3)$, etc. In this sense the location index is a kind of 'postcode within the Hilbert space'. The location index is unique if there are no equalities in Eq. (28).

We consider the set $\Theta = \{\theta(\lambda) \mid \lambda \in [a, b]\}$. From $[a, b]$ we will exclude values of λ for which some of the n values $s_\theta(i|\lambda)$ (with fixed λ and $i = 1, \ldots, n$) are equal to each other. In this way we get an interval $I \subseteq [a, b]$, and the

$$\widetilde{\Theta} = \{\theta(\lambda) \mid \lambda \in I\} \subseteq \Theta. \tag{30}$$

By definition, if $\theta(\lambda) \in \widetilde{\Theta}$, there are no equalities in the corresponding Eq. (28).

Within the set $\widetilde{\Theta}$, we say that $\theta(\lambda_1)$ and $\theta(\lambda_2)$ are comonotonic or cohabitant, and denote it as $\theta(\lambda_1) \sim \theta(\lambda_2)$, if they have the same location index, i.e., if $\mathscr{L}[\theta(\lambda_1)] = \mathscr{L}[\theta(\lambda_2)]$. $\sim$ is an equivalence relation within $\widetilde{\Theta}$ (but not within Θ because transitivity does not hold). Then $\widetilde{\Theta}$ is partitioned into equivalence classes, each of which contains operators which are comonotonic to each other.

If all $\theta(\lambda)$ with $\lambda \in (c_1, c_2) \subseteq I$ are comonotonic to each other, the $I_1 = (c_1, c_2)$ is called comonotonicity interval. In other words, all $\theta(\lambda)$ within a comononicity interval have the same location index. The points in the set $[a, b] \setminus I$ are crossing points from one comonotonicity region to another.

Example 5 In the Hilbert space H_2 we consider the Hermitian operator

$$\theta(\lambda) = \begin{pmatrix} 1+\lambda & \lambda i \\ -\lambda i & 2 \end{pmatrix}. \tag{31}$$

Using the matrices $R(1)$, $R(2)$, $R(3)$, in Eq. (22), we get

$$\begin{aligned} s_\theta(1|\lambda) &= \frac{2}{3}\mathrm{Tr}[\theta(\lambda)R(1)] = 0.816 + 0.616\lambda \\ s_\theta(2|\lambda) &= \frac{2}{3}\mathrm{Tr}[\theta(\lambda)R(2)] = 0.916 + 0.516\lambda \\ s_\theta(3|\lambda) &= \frac{2}{3}\mathrm{Tr}[\theta(\lambda)R(3)] = 1.266 - 0.134\lambda \end{aligned} \tag{32}$$

From this we find that

$$\begin{aligned}
\lambda \leq 0.538 &\rightarrow s_\theta(1|\lambda) \leq s_\theta(2|\lambda) \leq s_\theta(3|\lambda)\\
0.538 \leq \lambda \leq 0.6 &\rightarrow s_\theta(1|\lambda) \leq s_\theta(3|\lambda) \leq s_\theta(2|\lambda)\\
0.6 \leq \lambda \leq 1 &\rightarrow s_\theta(3|\lambda) \leq s_\theta(1|\lambda) \leq s_\theta(2|\lambda)\\
1 \leq \lambda &\rightarrow s_\theta(3|\lambda) \leq s_\theta(2|\lambda) \leq s_\theta(1|\lambda).
\end{aligned} \tag{33}$$

Therefore we have the following comonotonicity intervals, and the corresponding location indices:

$$\begin{aligned}
(-\infty, 0.538) &\rightarrow \mathscr{L} = (3, 2, 1)\\
(0.538, 0.6) &\rightarrow \mathscr{L} = (2, 3, 1)\\
(0.6, 1) &\rightarrow \mathscr{L} = (2, 1, 3)\\
(1, \infty) &\rightarrow \mathscr{L} = (1, 2, 3).
\end{aligned} \tag{34}$$

The points $\lambda = 0.538$, $\lambda = 0.6$, $\lambda = 1$, are crossing points.

We conjecture that comonotonic operators are physically similar operators. As λ varies within a comonotonicity interval, we get mild physical changes in the system. The crossing points from one comonotonicity interval to another, might be related with drastic physical changes in the system. In [4, 5] we gave examples, which support this conjecture.

6 Discussion

Our approach extends the area of coherent states, POVMs and frames and wavelets, in a new direction. It starts from a pre-basis (i.e., a total set of $n \geq d$ vectors in H_d), and leads to n mixed states $R(i)$ that resolve the identity (Eq. (19)). Then an arbitrary vector can be written as a sum of n component vectors, as in Eq. (24).

A Hermitian operator can be represented with n numbers given in Eq. (27). Based on an ordering of these numbers, we have defined the location index of a Hermitian operator. Comonotonic (or cohabitant) operators have the same location index, and in this sense they 'live' in the same part of the Hilbert space.

For Hermitian operators $\theta(\lambda)$ that depend on a coupling parameter λ, we have defined comonotonicity regions, so that all $\theta(\lambda)$ within a given comonotonicity region are comonotonic to each other. Then we conjecture that comonotonic operators are physically similar. As λ varies within a comonotonicity interval, we do not get any drastic physical changes in the system. Drastic physical changes might occur at the crossing points, from one comonotonicity interval to another.

Our work has used cooperative game theory in the context of quantum systems with discrete variables that take a finite number of values. There is work on cooperative game theory with a continuum of players [12], which could be used to extend these ideas into quantum systems with continuous variables.

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Part II
Holography and String Theory

On Non-slow Roll Inflationary Regimes

Lilia Anguelova, Peter Suranyi and L. C. Rohana Wijewardhana

Abstract We summarize our work on constant roll inflationary models. It was understood recently that constant roll inflation, in a regime beyond the slow roll approximation, *can* give models that are in agreement with the observational constraints. We describe a new class of constant roll inflationary models and investigate the behavior of scalar perturbations in them. We also comment on other non-slow roll regimes of inflation.

Keywords Cosmological inflation · Alpha-attractors · Noether symmetry

1 Introduction

It has long been a standard lore that, to agree with the observational constraints, an inflationary model has to be in the so called slow-roll regime. This is an approximation that allows an easy solution of the coupled equations of motion. The background metric is (near-)de Sitter and the spectrum of scalar perturbations turns out to be (nearly-)scale invariant, as required for consistency with the data from current cosmological observations. However, it is known since [1] that a scale invariant spectrum can also be obtained from a non-slow roll inflationary expansion. Although, the

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_11

ultra-slow roll regime investigated in [1] is unstable (i.e. very short-lived) and thus cannot provide a full-fledged inflationary model by itself.

Despite the stability issue, ultra-slow roll inflation has received considerable attention during the last several years in relation to the observed low-l anomaly of the CMB [2]. It was also understood recently how to construct a class of ultra-slow roll composite inflation models in the context of the gauge/gravity duality [3–5]. Much more importantly, [6] showed that a certain generalization of ultra-slow roll, called constant roll, can give a long-lasting/stable expansion in addition to producing a scale invariant spectrum of scalar perturbations. Therefore, constant roll inflation is an observationally viable alternative to the standard slow roll one.

In view of the great, and continually growing, precision of present day cosmological observations, it is undoubtedly worth investigating in more depth the full set of viable inflationary regimes. In [7] we performed a systematic study of the constant roll condition and found a new class of solutions of this type. These solutions are stable under scalar perturbations and have a corner of their parameter space, in which one obtains a nearly scale invariant spectrum of scalar perturbations. Here we discuss their properties and comment on broader non-slow roll regimes.

2 Constant Roll Inflation

Within the standard field theoretic description, inflation is obtained as a solution of the equations of motion following from the action

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2} + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] , \tag{1}$$

upon using the metric ansatz

$$ds_4^2 = -dt^2 + a^2(t)\, d\mathbf{x}^2 \tag{2}$$

with $a(t)$ being the scale factor.

The condition for inflationary solutions is $\ddot{a}(t) > 0$. In principle, such solutions may or may not satisfy the slow roll approximation, which can be defined in terms of the Hubble parameter $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$ as [8, 9]:

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} \ll 1 \quad \text{and} \quad \eta \equiv -\frac{\ddot{H}}{2H\dot{H}} \ll 1 \quad . \tag{3}$$

The standard lore for decades has been that conditions (3) are necessary in order to obtain a long-lasting (i.e. stable) inflationary expansion, which produces a scale invariant spectrum of scalar perturbations. In other words, it is usually assumed that (3) is needed for consistency with the observational data.

However, it is also well-known that the ultra-slow roll regime of [1, 10], defined by the conditions

$$\varepsilon \ll 1 \qquad \text{and} \qquad \eta = 3 \qquad , \tag{4}$$

similarly gives a scale invariant spectrum, i.e. with $n_s = 1$. This regime, though, is unstable and can last only a few e-folds. Recently [6] showed that a generalization of the η-condition, given by

$$\eta = const \equiv c \tag{5}$$

and called constant roll regime, can lead to a long-lasting inflationary expansion, while preserving the $n_s = 1$ result, for some values of $c \neq 3$. The considerations of [6] were based on the definition of the η-parameter as $\eta = -\frac{\ddot{\phi}}{H\dot{\phi}}$, which is equivalent to the one in (3) upon using the field equations. A more straightforward and systematic analysis can be performed by studying instead the condition

$$-\frac{\ddot{H}}{2H\dot{H}} = c \ \ , \tag{6}$$

following from the η-definition in (3) together with (5).

Investigating Eq. (6), the work [7] reproduced the solutions of [6] and, in addition, found a new class of constant roll solutions. The Hubble parameter, scale factor and inflaton of the new solutions have the following form:

$$\begin{aligned} H(t) &= \frac{N}{c}\cot(Nt) \ \ , \\ a(t) &= C_a \sin^{1/c}(Nt) \ \ , \\ \phi(t) &= \pm\sqrt{\frac{2}{c}}\ln\left[\cot\left(\frac{Nt}{2}\right)\right] + C_\phi \ \ , \end{aligned} \tag{7}$$

where N, C_a and C_ϕ are integration constants. Also, the parameter c has to satisfy

$$c > 0 \ \ , \tag{8}$$

to ensure $\dot{H} < 0$ (and thus a real inflaton ϕ), while the combination Nt has to be in the finite interval

$$Nt \in \left[0, \frac{\pi}{2}\right] \ \ , \tag{9}$$

to have $H(t) > 0$ during the entire inflationary period. Clearly, by taking

$$N \ll 1 \ \ , \tag{10}$$

one can have as large a t-interval as desired.

Finally, the scalar potential is given by

$$V(\phi) = \frac{N^2}{2c^2}\left[(3-c)\cosh\Big(\sqrt{2c}\,(\phi+\phi_0)\Big) - (3+c)\right] \ , \tag{11}$$

where the constant $\phi_0 \equiv -C_\phi$. We will see shortly that $V(\phi)$ is positive-definite within the entire inflationary parameter space of these constant roll models.

3 Parameter Space of the New Solutions

The class of solutions (7), with parameter space as in (8)–(9), was obtained only by studying the defining equation for constant roll, namely Eq. (6), and imposing the requirements for a positive Hubble parameter and a real inflaton. However, we still need to consider the condition for inflation $\ddot{a}(t) > 0$. Now we will discuss the additional constraints on the parameter space of the new solutions that follow from this condition.

First, however, let us make an important observation. Note that the Nt-interval in Eq. (9) can be shortened by a rescaling of the integration constant N [7]. Indeed, introducing the constant $\hat{N} = \frac{2}{\pi}\theta_* N$ with some fixed $\theta_* < \frac{\pi}{2}$, we can see that $Nt \in [0, \frac{\pi}{2}]$ becomes $\hat{N}t \in [0, \theta_*]$. So the freedom to redefine the integration constant N implies that we are free to restrict the Nt-interval to a convenient subinterval. Clearly, this does not affect the above statement that the t-interval can be as large as desired, since the rescaled integration constant $\hat{N}$ is, obviously, just as arbitrary as N. However, it will be useful, at some point later on, to restrict the Nt interval to $[0, \frac{\pi}{4}]$.

Now let us turn to investigating the condition $\ddot{a} > 0$. From (7), we find:

$$\ddot{a}(t) = \frac{N^2}{c^2}\frac{a(t)}{\sin^2(Nt)}\left[\cos^2(Nt) - c\right] \ . \tag{12}$$

Therefore, to ensure $\ddot{a} > 0$, one needs to satisfy the inequality

$$\cos^2(Nt) > c \ . \tag{13}$$

To be able to do that, we must have $c < 1$. Together with (8), this implies that:

$$0 < c < 1 \ . \tag{14}$$

Then we can solve (13), finding:

$$Nt \in \left[0,\ \arccos(\sqrt{\mathrm{c}})\right) \ . \tag{15}$$

Note that (15) guarantees the positive-definiteness of the inflaton potential (11); see [7].

Finally, let us discuss what are the conditions for the acceleration in the new class of models to be increasing or decreasing. Computing the time-derivative of (12), we have:

$$\dddot{a} = \frac{N^2}{c^2} \frac{aH}{\sin^2(Nt)} \left[\cos^2(Nt) - 3c + 2c^2\right] \ . \tag{16}$$

Hence, the condition $\dddot{a} > 0$ is equivalent to

$$\cos^2(Nt) > 3c - 2c^2 \ . \tag{17}$$

Note that, when $\frac{1}{2} < c < 1$, one always has $3c - 2c^2 > 1$. So, in that case, $\ddot{a}(t)$ is always decreasing with time. On the other hand, when $c < \frac{1}{2}$, one can solve the condition for increasing acceleration (17), obtaining:

$$Nt < \arccos\left(\sqrt{3c - 2c^2}\right) \ . \tag{18}$$

In conclusion, to have any period of increasing acceleration (like in the familiar de Sitter case), one has to have $c < \frac{1}{2}$.

4 Stability Under Scalar Perturbations

Let us now discuss the scalar perturbations in the new class of models (7) with parameter space (9) and (14). We will denote the perturbations of the inflaton and the spatial part of the metric as $\delta\phi$ and δg_{ij} respectively, where $i, j = 1, 2, 3$. It is convenient to work in comoving gauge, where $\delta\phi = 0$ and $\delta g_{ij} = a^2 \left[(1 - 2\zeta)\delta_{ij} + h_{ij}\right]$ with h_{ij} being the tensor perturbations; see [11] for instance. As is well-known, the perturbation ζ inside δg_{ij} is the only independent scalar degree of freedom.

Upon Fourier transforming $\zeta(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \zeta_k(t)\, e^{i\mathbf{k}.\mathbf{x}}$, one can introduce the mode function $v_k \equiv \sqrt{2}\, z\zeta_k$ with $z^2 \equiv -a^2 \frac{\dot{H}}{H^2}$. In terms of v_k , the evolution equation for the perturbations is the Mukhanov–Sasaki equation [12, 13]:

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0 \ , \tag{19}$$

where $k \equiv |\mathbf{k}|$ and $' \equiv \partial_\tau$ with τ being conformal time defined as usual via $dt^2 = a^2 d\tau^2$. Note also that the z''/z term in (19) can be rewritten *exactly* (as opposed to in the slow-roll approximation) as [6, 14]:

$$\frac{\tilde{z}''}{\tilde{z}} = a^2 H^2 \left(2 - \epsilon_1 + \frac{3}{2}\epsilon_2 + \frac{1}{4}\epsilon_2^2 - \frac{1}{2}\epsilon_1\epsilon_2 + \frac{1}{2}\epsilon_2\epsilon_3\right) \ , \tag{20}$$

where ϵ_i are the following series of slow roll parameters:

$$\epsilon_1 \equiv -\frac{\dot{H}}{H^2} \qquad \text{and} \qquad \epsilon_{i+1} \equiv \frac{\dot{\epsilon}_i}{H\epsilon_i} \quad . \tag{21}$$

To investigate the issue of stability of the new models under scalar perturbations, we will consider the super-Hubble limit of the evolution Eq. (19), where $k^2 \ll \tilde{z}''/\tilde{z}$. Clearly, in that case, (19) simplifies to:

$$v_k'' - \frac{\tilde{z}''}{\tilde{z}} v_k = 0 \ . \tag{22}$$

It was already observed in [6] that the general solution of (22) gives the following form for $\zeta_k = \frac{\sqrt{2}}{2}\frac{v_k}{\tilde{z}}$:

$$\zeta_k = A_k + B_k \int \frac{dt}{a^3 \epsilon_1} \quad , \tag{23}$$

where $A_k, B_k = const$ and $\tau = \tau(t)$ is any function. Using (21) and absorbing a minus sign in the arbitrary integration constant B_k, we can conveniently rewrite (23) as:

$$\zeta_k = A_k + B_k \int \frac{H^2}{a^3 \dot{H}} dt \quad . \tag{24}$$

The goal now will be to investigate the behavior of this integral at late times. If it turns out that ζ_k decreases (or stays constant), then the corresponding model would be stable. On the other hand, if it were to increase with time, this would indicate instability.

Substituting the Hubble parameter H and scale factor a from (7), we find:

$$\int \frac{H^2}{a^3 \dot{H}} dt = \frac{1}{3\,c\,N(C_a)^3} \cos^3(Nt)\ {}_2F_1\left(\frac{3}{2}, \frac{c+3}{2c}, \frac{5}{2}; \cos^2(Nt)\right) \ . \tag{25}$$

Note that the parameter c here and the parameter α in [6] are related to each other via $c = 3 + \alpha$. Using this, one can immediately verify that the indices of the hypergeometric function in (25) coincide precisely with those in Eq. (47) of [6]. In fact, if we denote $x \equiv \cos^2(Nt)$, we have in (25) exactly the same function (up to an overall numerical constant) as in Eq. (47) of [6], namely

$$f(x) \equiv x^{\frac{3}{2}}\ {}_2F_1\left(\frac{3}{2}, \frac{c+3}{2c}, \frac{5}{2}; x\right) \ . \tag{26}$$

However, there is a crucial difference due to the fact that Ref. [6] needed to investigate that function in the limit $x \to \infty$, whereas for us $x \in [c, 1]$ because of (15).

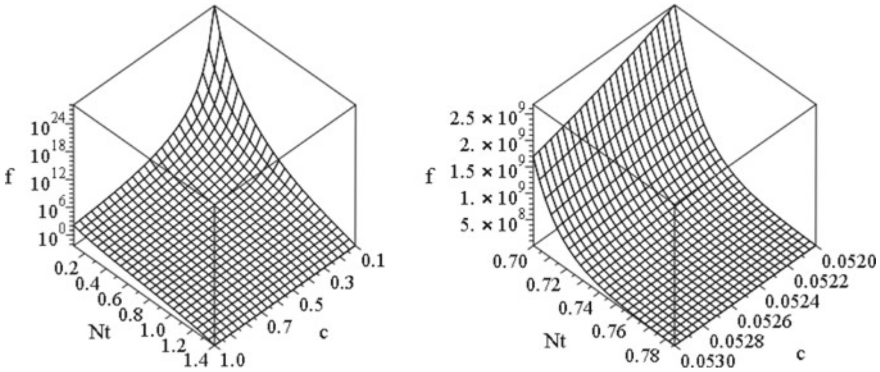

Fig. 1 Plot of $f(x)$ in Eq. (26), with $x = \cos^2(Nt)$, as a function of both Nt and c. On the left side, $f(Nt, c)$ is plotted for $Nt \in (0, \frac{\pi}{2})$ and $c \in (0, 1)$. On the right side, we have plotted a representative slice for the intervals $Nt \in [\,0.7\,,\ \frac{\pi}{4}\,]$ and $c \in [\,0.053\,,\ 0.054\,]$, which will be useful in the next section

In [7] the function (26) was considered in the full parameter ranges, given by $0 < Nt < \frac{\pi}{2}$ and $0 < c < 1$, and it was shown that it is always decreasing with time regardless of the values of the constants c and N. This behavior is illustrated on Fig. 1.

5 Scalar Spectral Index

In order to determine the scalar spectral index n_s, we need to investigate the Mukhanov–Sasaki equation (19) in a regime when the terms with k^2 and $\frac{z''}{z}$ are comparable. We will impose the usual initial condition:

$$v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \qquad \text{for} \qquad \tau \to -\infty \quad . \tag{27}$$

To make further progress, we need the explicit relation $\tau = \tau(t)$. Using $a(t)$ in (7), one finds:

$$\tau = \int \frac{dt}{a} = -\frac{1}{C_a N}\ \cos(Nt)\ {}_2F_1\!\left(\frac{1}{2}\,,\ \frac{c+1}{2c}\,,\ \frac{3}{2}\,;\ \cos^2(Nt)\right)\ +\ const\ \ . \tag{28}$$

The integration constant here can easily be chosen such that the range of τ is [7]:

$$\tau \in (-\infty\,,\ 0\,] \tag{29}$$

for t varying in the entire interval

$$t \in \left[0,\ \frac{1}{N}\arccos(\sqrt{c})\right) \ , \tag{30}$$

according to the inflationary condition (15).

As discussed in Sect. 3 though, we can restrict to any subinterval of (30) as part of the freedom to redefine the integration constant N. It will turn out below to be particularly useful to consider the subinterval

$$t \in \left[0,\ \frac{\pi}{4N}\right] \tag{31}$$

when $c < \frac{1}{2}$. In this case, the integration constant guaranteeing (29) is such that:

$$\tau = -\frac{1}{C_a N}\left[\cos(Nt)\ {}_2F_1\left(\frac{1}{2},\ \frac{c+1}{2c},\ \frac{3}{2};\ \cos^2(Nt)\right) - \frac{\sqrt{2}}{2}\ {}_2F_1\left(\frac{1}{2},\ \frac{c+1}{2c},\ \frac{3}{2};\ \frac{1}{2}\right)\right] \ . \tag{32}$$

Solving Eq. (19) in full generality is rather complicated because the potential term z''/z depends on the background. In principle, one needs to use numerical methods [11]. However, one can find an analytical estimate, compatible with the observational constraint $n_s \approx 1$, in the approximation

$$c \ll 1 \ . \tag{33}$$

In this limit, we are free to choose the interval (31) as our inflationary period. And, furthermore, during that entire period the slow roll parameters ϵ_i in (21) are almost constant [7]. More concretely, we have:

$$\epsilon_1 \approx 2c \quad , \quad \epsilon_2 \approx 2c \quad , \quad \epsilon_3 \approx 4c \quad . \tag{34}$$

Hence, the ϵ_i-expression in (20) acquires the form:

$$\left(2 - \epsilon_1 + \frac{3}{2}\epsilon_2 + \frac{1}{4}\epsilon_2^2 - \frac{1}{2}\epsilon_1\epsilon_2 + \frac{1}{2}\epsilon_2\epsilon_3\right) \approx 2 + c + 3c^2 = \nu^2 - \frac{1}{4} \quad , \tag{35}$$

where for convenience we have introduced the notation

$$\nu^2 \equiv \frac{9}{4} + c + 3c^2 \quad . \tag{36}$$

In addition, in the approximation (33), one can verify from (7) and (32) that [7]:

$$aH \approx -\frac{1}{\tau} \quad , \tag{37}$$

similarly to inflation in pure de Sitter space. Now, making use of (35) and (37) inside (19), one can easily obtain the spectral index n_s by following the standard computation [6, 7]. The result is:

$$n_s = 4 - 2\sqrt{\frac{9}{4} + c + 3c^2} \quad . \tag{38}$$

To find the values of c that lead to agreement with the observational constraint $n_s \approx 0.96$, we need to solve the quadratic equation that follows from imposing it on Eq. (38). It turns out that only one of the two roots lies within the parameter space of our class of models, namely within (14). That solution is:

$$c \approx 0.0522 \quad . \tag{39}$$

As explained in [7], this result is consistent with the approximations made in deriving it. More precisely, for this value of the parameter c, the approximations (34) and (37) hold to a very good degree of accuracy. Hence, we have found a corner of the parameter space of the new constant roll models, in which they are compatible with the present day observational data.

6 Other Non-slow Roll Regimes

The constant roll regime studied here can be viewed as a generalization of ultra-slow roll inflation, that was first considered in [10]. Other non-slow roll inflationary regimes have also been investigated during the last couple of decades. See, for example [2, 15], for different cases of 'fast roll' inflation, depending on which (and how many) of the slow roll parameters in (21) are actually large during the inflationary period. Usually, such stages of expansion are expected to be rather short-lived. So they are viewed as useful only for setting certain initial conditions for a subsequent stage of regular slow-roll inflation. A transient non-slow roll stage preceding slow roll is, in fact, considered to be important in explaining the observed low multipole moment anomaly in the CMB [2].

However, in view of the recent realization [6, 7], that a constant roll inflationary expansion can last long enough to produce a full-fledged inflationary model (compatible with $n_s \approx 1$ in a part of its parameter space), it makes sense to ask whether it is possible to find other stable non-slow roll regimes. In particular, it would be interesting to investigate whether there is a suitable generalization of fast roll inflation, conceptually similar to how constant roll generalizes ultra-slow roll. We hope to come back to this question in the future.

Finally, in view of the fact that, at present, the Universe has a (small) positive cosmological constant, it is worth exploring models that can have more than one inflationary stage. This would enable the development of a unified description, that can account for *both* inflation in the Early Universe *and* accelerated expansion in the present day. Important progress in that direction was achieved in [16]. The early inflationary period in their considerations was with constant rate of roll. It would be interesting to explore how our new constant-roll solutions fit in this framework and, in particular, whether they can lead to some specific observational features in this context.

Acknowledgements L.A. has received partial support from the Bulgarian NSF grant DN 08/3 and the bilateral grant STC/Bulgaria-France 01/6.

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Precision Test of Holographic Flavourdynamics

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Abstract We study the Berkooz–Douglas matrix model using holography, lattice simulation and high temperature perturbative expansion. In particular we calculate the mass susceptibility of the theory. Our results show excellent agreement between lattice simulations and holography at low and intermediate temperatures $T \leq \lambda^{1/3}$. We also report a surprisingly good agreement between holography and perturbative high temperature expansion at $T \sim \lambda^{1/3}$.

1 Introduction

Among the most profound developments of modern physics is the quantum description of reality. Quantum field theory (QFT) is our main tool to describe physics on a diverse range of scales, from the standard model of interactions to the theory of superconductivity. Yet there are regimes, when QFTs are strongly coupled and perturbation theory breaks down. These regimes are prevalent in Nature from confinement, chiral

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_12

symmetry breaking and quark matter in particle physics to high temperature superconductors, strange metals and graphene in condensed matter phenomena. This calls for novel nonperturbative tools to study QFTs at strong coupling. A promising such tool is the AdS/CFT correspondence [1, 2], also referred to as the holographic correspondence, which is a duality between a strongly coupled quantum field theory and a higher dimensional weakly interacting gravitational system. There is overwhelming evidence that the supersymmetric regime of the correspondence is correct, yet the most relevant phenomenological applications of the duality, when supersymmetry is broken are poorly tested, making the nature of these studies somewhat speculative.

Testing the AdS/CFT correspondence requires an alternative nonperturbative approach and for a four dimensional gauge theory lattice simulations on a computer seem a natural approach. Unfortunately, although the subject of active research, the lattice formulation of four dimensional Supersymmetric Yang–Mills (SYM) theory is still problematic. When faced with such difficulties, a useful approach is to study simplified versions of the correspondence. Recently progress in this direction has been made by studying a $0 + 1$ dimensional version of the correspondence, between the supersymmetric BFFS matrix model [3] and its dual type IIA supergravity background [4–10].

In this report we are interested in generalisation of the AdS/CFT correspondence [11] including matter in the fundamental representation of the gauge group. The idea of Ref. [11] is to introduce a probe D7–brane to the $\text{AdS}_5 \times S^5$ supergravity background. The corresponding dual field theory has $\mathcal{N} = 2$ supercharges and is the $\mathcal{N} = 4$ SYM theory coupled to an $\mathcal{N} = 2$ fundamental hypermultiplet, which is the effective low energy theory of the D3/D7 brane intersection. This holographic set-up has received a great deal of attention and has led to numerous theoretical and phenomenological applications. In particular at the finite temperature regime of the theory features a first order meson melting phase transition [12–17].

In Ref. [18] the lattice formulation of the Berkooz–Douglas matrix model [19] (see also [20]) was studied. The main result of Ref. [18] is a numerical calculation of the fundamental condensate of the theory using computer simulations. Comparison with holographic calculations show remarkable agreement in the deconfined phase of the theory. The most plausible explanation for that agreement is that in the deconfined phase the α' corrections due to the high curvature of the background are cancelled in the calculation of the condensate, since it involves a derivative of the free energy with respect to the bare mass of the theory. It is then natural to propose that this agreement should be even better if one considers the mass susceptibility of the condensate, which is a second derivative of the free energy with respect to the bare mass. In fact the mass susceptibility can be evaluated at zero bare mass, when analytic result for the susceptibility can be obtained from holography. In Ref. [21] a perturbative approach (at high temperature) was used to calculate the condensate susceptibility, remarkably in Ref. [22] it was shown that at intermediate temperature ($T \sim \lambda^{1/3}$) the holographic calculations agree with the perturbative high temperature expansion of Ref. [21]. Furthermore, computer simulations of the lattice discretisation developed in Ref. [18] show agreement at lower temperatures. Reference [22] also considered an alternative lattice formulation providing an independent check of the numerical results.

The goal of this report is to discuss the results of Refs. [18, 21, 22]. The structure of the paper is as follows:

In Sect. 2 we describe the general properties of the Dp/Dq brane intersections T-dual to the D3/D7 system. We comment on the universal properties of the Dp/Dq system and the difficulties in simulating supersymmetric theories in higher than $1 + 0$ dimensions. Section 3 describes the holographic calculation of the condensate susceptibility performed in Ref. [22]. In Sect. 4 we compare the holographic results with the results from field theory using both perturbative high temperature expansion [21] and lattice simulations [22]. Finally, Sect. 5 contains a brief conclusion.

2 The Dp/Dq Brane Holographic Set-Up

The Dp/Dq holographic set-up is inspired by the Dp/Dq brane intersection T-dual to the D3/D7 one. In this set-up a probe Dq brane is introduced to the near horizon limit of the supergravity background describing a Dp-brane. In the dual field theory this corresponds to adding $\mathcal{N} = 2$ fundamental hypermultiplets in the Lagrangian of the $\mathcal{N} = 4$ four-dimensional SYM theory. The probe approximation corresponds to a quenched approximation in the dual field theory when fundamental loops are ignored in correlation functions involving only adjoint fields. In other words the dynamics of the adjoint degrees of freedom is not affected by the presence of the fundamental fields. Note that this is not the same as the quenched approximation in lattice gauge theory since the fermionic determinant is not suppressed when fundamental fields are present in the correlators (as in the case of the fundamental condensate).

In this set-up the asymptotic separation of the Dq–brane corresponds to the bare mass of the fundamental hypermultiplet and the bending of the probe Dq-brane at infinity encodes the fundamental condensate. Furthermore, the spectrum of the semi-classical fluctuations of the probe corresponds to the meson spectrum in the dual field theory. This allows one to use semi-classical calculations in supergravity to obtain non-perturbative quantum results for the dual field theory. In particular one can explore the phase structure of the dual theory at finite temperature and in the presence of various other control parameters. It turns out that the thermal properties of the Dp/Dq system exhibit some universal features [14] in particular the pattern of the first order meson melting phase transition depends only on the dimension of the internal cycle wrapped by the probe Dq-brane in the transverse to the Dp–brane subspace. One can show that the D3/D7 system is in the same class of universality as the D0/D4 system, which is dual to the Berkooz–Douglas matrix model and can be relatively easily simulated on a computer. Therefore, by performing a precision test of the holographic correspondence in the D0/D4 system we indirectly test properties of the D3/D7 system.

3 Holographic Calculation of the Condensate Susceptibility

At low temperature the BD model is proposed to be dual to the D0/D4 holographic set-up.[1] The most understood case that we will focus on is the so called quenched approximation, when the flavour D4–branes are in the probe approximation [11]. In the near horizon limit the D0-brane supergravity background is given by:

$$ds^2 = -H^{-\frac{1}{2}}\, f\, dt^2 + H^{\frac{1}{2}} \left(\frac{du^2}{f} + u^2\, d\Omega_8^2 \right) ,$$
$$e^{\Phi} = H^{\frac{3}{4}} , \qquad C_0 = H^{-1} , \tag{1}$$

where $H = (L/u)^7$ and $f(u) = 1 - (u_0/u)^7$. Here u_0 is the radius of the horizon related to the Hawking temperature via $T = 7/(4\,\pi\, L)\,(u_0/L)^{5/2}$ and the length scale L is given by $L^7 = 15/2\,(2\pi\alpha')^5\,\lambda$, with λ the 't Hooft coupling.

To introduce matter in the fundamental representation we consider the addition of N_f D4–probe branes. In the probe approximation $N_f \ll N$, their dynamics is governed by the Dirac-Born-Infeld action:

$$S_{\mathrm{DBI}} = -\frac{N_f}{(2\pi)^4\,\alpha'^{5/2}\,g_s} \int d^4\xi\, e^{-\Phi}\,\sqrt{-\det||G_{\alpha,\beta} + (2\pi\alpha')F_{\alpha,\beta}||}\,, \tag{2}$$

where $G_{\alpha,\beta}$ is the induced metric and $F_{\alpha,\beta}$ is the $U(1)$ gauge field of the D4–brane, which we will set to zero. Parametrising the unit S^8 in the metric (1) as:

$$d\Omega_8^2 = d\theta^2 + \cos^2\theta\, d\Omega_3^2 + \sin^2\theta\, d\Omega_4^2 \tag{3}$$

and taking a D4–brane embedding extended along: $t,\ u,\ \Omega_3$ with a non-trivial profile $\theta(u)$, we obtain (after Wick rotation):

$$S^E_{\mathrm{DBI}} = \frac{N_f\,\beta}{8\,\pi^2\,\alpha'^{5/2}\,g_s} \int du\, u^3 \cos^3\theta(u)\,\sqrt{1 + u^2\, f(u)\,\theta'(u)^2}\,. \tag{4}$$

The embedding extremising the action (4) can be obtained by solving numerically the corresponding non-linear equation of motion. The AdS/CFT dictionary then relates the behaviour of the solution at large radial distance u to the bare mass and condensate of the theory via [11, 14]:

$$\sin\theta = \frac{\tilde{m}}{\tilde{u}} + \frac{\tilde{c}}{\tilde{u}^3} + \dots\,, \tag{5}$$

[1]The D0/D4 set-up belongs to a large class of Dp/Dp+4 -brane intersections exhibiting universal properties such as the presence of a meson melting phase transition. For more details look at Refs. [12–15, 17] as well as Ref. [16] for an extensive review.

where $\tilde{u} = u/u_0$ and the parameters $\tilde{m}$ and $\tilde{c}$ are proportional to the bare mass and condensate of the theory. Therefore, the mass susceptibility of the condensate at zero bare mass $\langle \mathcal{C}^m \rangle$ is proportional to:

$$\langle \mathcal{C}^m \rangle \propto -\left(\frac{d\tilde{c}}{d\tilde{m}}\right)\Big|_{\tilde{m}=0} = \frac{7\pi}{2}\frac{\csc(\pi/7)\,\Gamma(3/7)\,\Gamma(5/7)}{\Gamma(1/7)^2\,\Gamma(2/7)\,\Gamma(4/7)}\ . \tag{6}$$

The last expression was obtained by using that small $\tilde{m}$ implies small θ, and hence the equation of motion for θ can be linearised and solved analytically. Combining equation (6) with the exact expressions for the mass and condensate in terms of $\tilde{m}$ and $\tilde{c}$ [14, 18]:

$$\begin{aligned} m = m_q/\lambda^{1/3} &= \frac{u_0\,\tilde{m}}{2\pi\alpha'} = \left(\frac{120\,\pi^2}{49}\right)^{1/5}\left(\frac{T}{\lambda^{1/3}}\right)^{2/5}\tilde{m}\ , \\ \langle \mathcal{O}_m \rangle &= -\frac{N_f\,u_0^3}{2\,\pi\,g_s\,\alpha'^{3/2}}\,\tilde{c} = \left(\frac{2^4\,15^3\,\pi^6}{7^6}\right)^{1/5} N_f\,N_c\left(\frac{T}{\lambda^{1/3}}\right)^{6/5}(-2\,\tilde{c})\ , \end{aligned} \tag{7}$$

we obtain:

$$\begin{aligned} \langle \mathcal{C}^m \rangle &= 14^{1/5}15^{2/5}\pi^{9/5}\left(\frac{\csc(\pi/7)\,\Gamma(3/7)\,\Gamma(5/7)}{\Gamma(1/7)^2\,\Gamma(2/7)\,\Gamma(4/7)}\right)N_f\,N_c\left(\frac{T}{\lambda^{1/3}}\right)^{4/5} \\ &\approx 1.136\,N_f\,N_c\left(\frac{T}{\lambda^{1/3}}\right)^{4/5}\ . \end{aligned} \tag{8}$$

Equation (8) is the holographic prediction for the mass susceptibility of the fundamental condensate, which in the next section we compare to the field theory results obtained by high temperature expansion and lattice simulations.

4 Field Theory Comparison

In Fig. 1 we present a comparison between the analytic expression (8) obtained from holography and our field theory results. The red curve represents the holographic prediction (8), while the black dashed curve corresponds to the high temperature expansion curve:

$$\langle \mathcal{C}^m \rangle = 14.08\left(\frac{T}{\lambda^{1/3}}\right)^{1/2} - 3.02\left(\frac{T}{\lambda^{1/3}}\right)^{-1} + O(T^{-\frac{5}{2}})\ , \tag{9}$$

obtained in Ref. [21]. The blue bars represent the results of lattice simulations using the lattice discretisation in Ref. [18]. The red bars correspond to independent lattice simulations based on a different lattice discretisation.

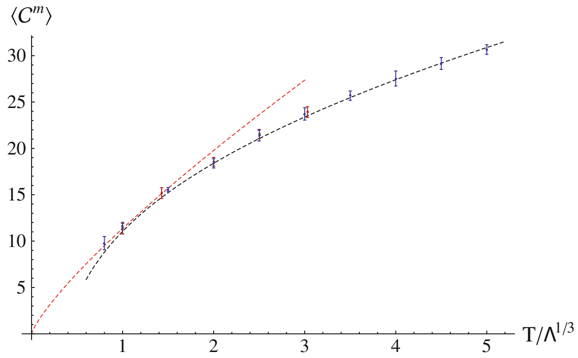

Fig. 1 The red curve represents the holographic prediction (8), while the black dashed curve corresponds to the high temperature expansion curve (9). The blue bars represent the results of lattice simulations using the lattice discretisation in Ref. [18]. The red bars correspond to an independent lattice simulation based on an alternative formulation

Overall, one can observe excellent agreement between the lattice simulation and the high T curve even for temperatures as low as $T = \lambda^{1/3}$. One can also observe excellent agreement with holographic predictions at temperatures $T \sim \lambda^{1/3}$. Remarkably, even the high temperature curve is very close to the holographic curve in this regime. As mentioned earlier this suggests that the α' corrections to the mass susceptibility are indeed very small.

5 Conclusion

In this paper we reported on a recent study of the Berkooz–Douglas matrix model using both holography and field theory approaches. We focus on the study of the mass susceptibility of the condensate, for which we derive an analytic expression from holography. Since the curvature of the D0 supergravity background grows with the radial distance, significant α' corrections are expected at large and intermediate temperature (radious of the black hole). Naively one would expect that the holographic result for the susceptibility should be valid only at low temperature ($T < \lambda^{1/3}$). However, as argued in Ref. [18], in the deconfined phase of the theory the derivatives of the free energy with respect to the bare mass should largely cancel the curvature α' corrections. Therefore, one can expect a good agreement even at intermediate temperature ($T \sim \lambda^{1/3}$). Remarkably, this is exactly what we observe in Sect. 4, where not only lattice simulation agree with the theoretical curve (8), but also the high temperature expansion curve (9) is very close to the holographic one at $T \sim \lambda$. Overall our results provide a solid evidence for the validity of the holographic description of the Berkooz–Douglas model, even when supersymmetry is broken by a finite temperature.

Acknowledgements V. F. would like to thank the organisers of 10th International Symposium "Quantum Theory and Symmetries" for the opportunity to present this work.

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$\mathcal{N} = 4$ Polygonal Wilson Loops: Fermions

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Abstract The contributions of scalars and fermions to the null polygonal bosonic Wilson loops/gluon MHV scattering amplitudes in $\mathcal{N} = 4$ SYM are considered. We first examine the re-summation of scalars at strong coupling. Then, we disentangle the form of the fermion contribution and show its strong coupling expansion. In particular, we derive the leading order with the appearance of a fermion-anti-fermion bound state first and then effective multiple bound states thereof. This reproduces the string minimal area result and also applies to the Nekrasov instanton partition function $\mathcal{Z}$ of the $\mathcal{N} = 2$ theories. Especially, in the latter case the method appears to be suitable for a systematic expansion.

Keywords AdS/CFT Correspondence · N=4 SYM scattering amplitudes · Form factors and scattering matrix for integrable theories · N=2 SYM partition functions

1 Introduction and Summary

$\mathcal{N} = 4$ Super Yang–Mills (SYM) in the planar limit, with 't Hooft coupling $\lambda = 16\pi^2 g^2$, appears at one side (of one example) of the AdS/CFT correspondence [33] and, interestingly, shows remarkable connections with $1 + 1$ dimensional integrable

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_13

models [12]. Even if integrability was discovered in the study of anomalous dimensions of local operators, recently techniques borrowed from integrable systems have been used for exact computations of other quantities in the same theory, e.g. the expectation values of null polygonal (bosonic) Wilson loops (Wls). These Wls are dual to (MHV) gluon scattering amplitudes [2, 18, 24, 30, 31], which makes them even more interesting, and can be efficiently studied by the all order expansion of the collinear limit of two consecutive edges: their value takes on the form of a (sort of non-local) Operator Product Expansion (OPE) [4, 7]. In fact, this is the same as the insertion of the identity (operator) as an infinite series of basis states in the space of the integrable quantum GKP string, namely the Form Factor series which sums over the flux-tube excitations: gluons and their bound states, fermions, anti-fermions and scalars.

The validity of the integrable OPE series has been successfully checked, by explicit computations, both in the weak and in the strong coupling regime [5, 6, 8, 11, 13–16, 19, 25–29, 32]. In this letter we shall focus on the latter, whose leading contributions are of the same order and come from two sectors. The first – due to the non-perturbative string dynamics on S^5 –, is computable by considering the scalar excitations [10, 20, 21]; the second one – caused by the classical string minimal action in AdS_5 [1, 2]–, comes from gluons, their bound states and fermions. As for the scalar series contribution, W_s, it is resolutive considering the series for $\ln W_s$: in this manner, each term is proven to be proportional to $\sqrt{\lambda}$. Then, because of the fermion-anti-fermion short range potential (15), they contribute at leading order not as single particles but through a bound state $f\bar{f}$ [9, 19, 32] which arises only at infinite coupling. Now, the (effective) sum runs on these (free) particles, named 'mesons' ($SU(4)$ singlets). Moreover, it has the same mathematical structure of the Nekrasov instanton partition function $\mathcal{Z}$ of the $\mathcal{N}=2$ theories with $\epsilon_2 \sim 1/g$ [34]. In fact, there is a short range potential (13) between two mesons which our method uses to produce a systematic expansion at small $\epsilon_2 \sim 1/g$. The leading of the latter is given by a simplified sum on mesons and their multiple bound states which gives rise to the dilogarithm of the Yang–Yang potential, proportional to $\sqrt{\lambda} \propto g \sim 1/\epsilon_2$, for the Thermodynamic Bethe Ansatz (TBA). Actually, we have conjectured this kind of TBA contribution in [19, 32] on the ground of the scattering theory. In this way we can make a parallel with gluon (stable) bound states and reproduce precisely (the middle node of) the TBA governing the string classical minimal action/area (= free energy) [1, 4]. In Sect. 2 we briefly describe the contribution of scalars. In Sect. 3, that of fermions: first, we work out the contribution of n couples $f\bar{f}$ as that of n mesons; then, the sum on (free) mesons (analogues of the instantons in $\mathcal{N}=2$ partition functions) is expanded at small $\epsilon_2 \sim 1/g$. At leading order it becomes the sum on multiple meson bound states which originates the TBA.

2 Non-perturbative Scalars in the Wilson Loop

The pentagon OPE approach [7] allows us to represent the Wl as a superposition of pentagonal transitions (squared form factors) and propagations. If we go to the non-perturbative strong coupling regime, scalars decouple themselves to give rise to a relativistic $O(6)$ non-linear σ-model [3]. Therefore, we can single out their contribution W_s to the hexagonal Wl OPE

$$W_s = \sum_{n=0}^{\infty} W_s^{(2n)}\,, \quad W_s^{(2n)} = \frac{1}{(2n)!}\int \prod_{i=1}^{2n} \frac{d\theta_i}{2\pi}\, G^{(2n)}(\theta_1, \ldots, \theta_{2n})\, e^{-z\sum\limits_{i=1}^{2n}\cosh\theta_i}\,, \quad (1)$$

where only even numbers $2n$ of scalars (with rapidities θ_i) are considered, for the Wl/MHV needs to be chargeless under $SU(4)$; the parameter $z = m_{gap}\sqrt{\tau^2+\sigma^2}$ encloses the dependence on two conformal ratios σ, τ and is proportional to the dynamically generated mass $m_{gap}(\lambda)$. Each function $G^{(2n)}$ factorizes $G^{(2n)} = \Pi^{(2n)}_{dyn}\Pi^{(2n)}_{mat}$ into a dynamical factor $\Pi^{(2n)}_{dyn}$, expressed as a product over two-particle functions, and a coupling-independent matrix part[1] $\Pi^{(2n)}_{mat}$, encoding the internal $SO(6)$ structure of scalars [10]. A dramatic improvement occurs when, rather than computing the scalar contribution (1), we consider its logarithm

$$\mathcal{F}_s = \ln W_s = \sum_{n=1}^{\infty} \mathcal{F}_s^{(2n)} = \sum_{n=1}^{\infty} \frac{1}{(2n)!}\int \prod_{i=1}^{2n} \frac{d\theta_i}{2\pi} g^{(2n)}(\theta_1, \ldots, \theta_{2n}) e^{-z\sum_{i=1}^{2n}\cosh\theta_i} \quad (2)$$

by passing from the functions $G^{(2n)}$ to their 'connected' counterparts $g^{(2n)}$, under a customary procedure. The crucial point concerns the *asymptotic factorization* of the Gs: that is to say, when one shifts $2k$ rapidities by a large amount $\Lambda \to \infty$, while holding fixed the remaining $2n - 2k$, $G^{(2n)}$ splits as

$$\begin{aligned} &G^{(2n)}(\theta_1 + \Lambda, \ldots, \theta_{2k} + \Lambda, \theta_{2k+1}, \ldots, \theta_{2n}) \xrightarrow{\Lambda\to\infty} \\ &G^{(2k)}(\theta_1, \ldots, \theta_{2k})\, G^{(2n-2k)}(\theta_{2k+1}, \ldots, \theta_{2n}) + O(\Lambda^{-2})\,. \end{aligned} \quad (3)$$

This remarkable property crucially affects the connected functions, as

$$\lim_{\Lambda\to\infty} g^{(2n)}(\theta_1 + \Lambda, \ldots, \theta_m + \Lambda, \theta_{m+1}, \ldots, \theta_{2n}) \simeq \frac{1}{\Lambda^2} \to 0\,, \qquad \text{for } m < 2n\,, \quad (4)$$

ensuring eventually their integrability. Clearly, the property (4) defines the conformal limit at small z for the logarithm of the Wilson loop, since, jointly to the relativistic behaviour of the $G^{(2n)}$ (hence the $g^{(2n)}$), it allows us to integrate out one rapidity for each $\mathcal{F}_s^{(2n)}$ in (2), giving

$$\mathcal{F}_s^{(2n)} = \frac{2}{(2\pi)^n (2n)!}\int \prod_{i=1}^{2n-1} d\theta_i\, g^{(2n)}(\theta_1, \ldots, \theta_{2n-1}) K_0(z\xi)\,, \quad (5)$$

[1]This factor exhibits an interesting resemblance with the $\mathcal{N} = 2$ instanton partition function: in fact, a Young tableaux approach was developed in [21] to compute $\Pi^{(2n)}_{mat}$.

for some known function of the rapidities $\xi(\theta_1, \ldots, \theta_{2n-1})$ [20]. Now, we can expand (inside) for small argument the Bessel function $K_0(z\xi) = -\ln z - \ln \xi + O(1)$ (whilst we could not before with the $G^{(2n)}$). Straightforwardly we can work this out for the leading term and obtain

$$\ln W_s \simeq A \ln(1/z) \simeq -A \ln m_{gap} \simeq A \frac{\sqrt{\lambda}}{4}, \tag{6}$$

where the coefficient A is given by a series $A = \sum_{n=1}^{\infty} A^{(2n)}$ over the multi-particle contributions, numerically very convenient as it is rapidly converging [10, 17]. For the sub-leading terms we need a further step as the weak power decay (4) compels us to restrict the integral at the region $|z\xi| < 1$ and carefully estimate how the rest behaves at small z: this is ultimately a consequence of the asymptotic freedom of the $O(6)$ σ-model and gives rise to the peculiar logarithmic behaviour of the two point 2D CFT correlation function [20, 21]. This procedure can be generalized to higher number of edges and still gives [22] a leading term of the form (6), competing with the minimal area term as conjectured in [10].

3 Fermion Contribution to the Wilson Loop

We now focus on the contribution to the hexagonal Wilson Loop due to the fermionic sector only: the singlet condition requires $N_f = N_{\bar{f}} \bmod 4$, but in the strong coupling limit only states with $N_f = N_{\bar{f}}$ contribute at the leading order. Anew, the pentagonal OPE writes as a form-factor series

$$W_f = \sum_{n=0}^{\infty} W_f^{(n)} \tag{7}$$

in terms of the contribution of n fermion-anti-fermion couples:

$$W_f^{(n)} = \frac{1}{n!n!} \int_C \prod_{k=1}^{n} \left[\frac{du_k}{2\pi} \frac{dv_k}{2\pi} \, \mu_f(u_k)\mu_f(v_k) \, e^{-\tau E_f(u_k) + i\sigma p_f(u_k)} \cdot \right. \\ \left. \cdot e^{-\tau E_f(v_k) + i\sigma p_f(v_k)} \right] \Pi_{dyn}^{(n)}(\{u_i\}, \{v_j\}) \, \Pi_{mat}^{(n)}(\{u_i\}, \{v_j\}) \, . \tag{8}$$

The open integration contour C, restricted to the small fermion sheet, is described in detail in [9, 19]. The dynamical quantities are parametrised through the set of fermion $\{u_k\}$ and anti-fermion rapidities $\{v_k\}$: energy and momentum of a particle correspond respectively to $E_f(u)$ and $p_f(u)$ and couple in the propagation phase to the cross ratios τ and σ, determining the conformal geometry of the polygon. Analogously to scalars, the multiparticle pentagonal transitions factorize into the product of a dynamical and a (coupling independent) matrix part [10]. The dynamical part in turn is factorized in terms of two particles amplitudes

$$\Pi^{(n)}_{dyn}(\{u_i\},\{v_j\}) = \prod_{i<j}^{n} \frac{1}{P(u_i|u_j)P(u_j|u_i)} \frac{1}{P(v_i|v_j)P(v_j|v_i)} \prod_{i,j=1}^{n} \frac{1}{\bar{P}(u_i|v_j)\bar{P}(v_j|u_i)} \tag{9}$$

where P stands for the transition between particles of the same type ($i.e.$ fermion-fermion or anti-fermion-anti-fermion) and $\bar{P}$ for the transition between a fermion and an anti-fermion. The function $P(u|v)$ is endowed with a single pole for coinciding rapidities $v = u$, whose residue determines the measure $\mu_f(u)$ [7]: $\text{Res}_{v=u} P(u|v) = i/\mu_f(u)$. The factor $\Pi^{(n)}_{mat}$, encoding the $SU(4)$ matrix structure, has an integral representation [10] in terms of the auxiliary variables $a,\ b,\ c$, corresponding to the nodes of the $SU(4)$ Dynkin diagram. In a system composed of n couples $f\bar{f}$ with rapidities $u_i,\ v_j$, in a $SU(4)$ singlet, the matrix factor reads

$$\begin{aligned}\Pi^{(n)}_{mat}(\{u_i\},\{v_j\}) &= \frac{1}{(n!)^3}\int \prod_{k=1}^{n}\left(\frac{da_k db_k dc_k}{(2\pi)^3}\right)\cdot \\ &\cdot \frac{\prod\limits_{i<j}^{n} g(a_i-a_j)g(b_i-b_j)g(c_i-c_j)}{\prod\limits_{i,j}^{n} f(a_i-b_j)f(c_i-b_j)\prod\limits_{i,j}^{n} f(u_i-a_j)f(v_i-c_j)},\end{aligned} \tag{10}$$

where the integrations are performed on the whole real axis and $f(u) = u^2 + \frac{1}{4},\quad g(u) = u^2(u^2+1)$. Similarly to the scalars above [21], the multiple integrals (10) can be evaluated by a Young tableaux method [22] and assume, eventually, the polar structure

$$\Pi^{(n)}_{mat}(\{u_i\},\{v_j\}) = \frac{P^{(n)}(u_1,\ldots,u_n,v_1,\ldots,v_n)}{\prod\limits_{i<j}^{n}[(u_i-u_j)^2+1]\prod\limits_{i<j}^{n}[(v_i-v_j)^2+1]\prod\limits_{i,j=1}^{n}[(u_i-v_j)^2+4]}. \tag{11}$$

$P^{(n)}(u_1,\ldots,u_n,v_1,\ldots,v_n)$ is a degree $2n(n-1)$ polynomial in the $u_i,\ v_j$.

3.1 Emergence of a Bound State

As we will present in this sub-section, the polar structure of the $SU(4)$ matrix factor (11) and the properties of the polynomials $P^{(n)}$ play a crucial role to unravel how, in the perturbative strong coupling regime (i.e. $\lambda \to \infty$ with the ratios $\bar{u}_i = u_i/2g$, $\bar{v}_i = v_i/2g$ finite), the sum on the fermionic sector can be performed as if there is an effective particle, named 'meson', coalescence of a fermion and an anti-fermion. In turn coalescences of many mesons will be summed up (in the next sub-section) to obtain effectively the right strong coupling limit of the series, in place of the sum over fermions. In this way, we complete the work of [19], where only two couples $f\bar{f}$ were analyzed ($n = 2$) (cf. also $n = 1$ [9]). Actually, already [32] conjectured the possibility of substituting the original sum over fermions and anti-fermions with the sum over mesons and their multiple bound states, supposed on the basis of the analytic structure (particle content) of the S-matrix. In details, on the ground of the Bethe Ansatz equations, the meson does not show up in the spectrum at finite coupling, as it lies outside the physical sheet [9, 32]; on the contrary, it comes into existence at infinitely large values of the coupling and starts contributing to the OPE differently from (unbounded) fermions and anti-fermions, whose contribution is subdominant. The multi-meson bound states share the same destiny [19, 32]. To ease our task, we re-cast (8) in the form (we could have privileged the v_j)

$$W_f^{(n)} = \frac{1}{n!}\int_C \prod_{i=1}^{n} \frac{du_i}{2\pi} I_n(u_1, \ldots, u_n) \prod_{i<j}^{n} p(u_{ij}) \,, \tag{12}$$

by highlighting a factor accounting for poles and zeroes in the u_i rapidities,

$$p(u_{ij}) = \frac{u_{ij}^2}{u_{ij}^2 + 1} \,, \quad u_{ij} = u_i - u_j \,, \tag{13}$$

the (meson-meson) short range potential, and enclosing the integrals on the anti-fermionic rapidities v_j inside the functions

$$I_n(u_1, \ldots, u_n) \equiv \frac{1}{n!}\int_C \prod_{i=1}^{n} \frac{dv_i}{2\pi} R_n(\{u_i\}, \{v_j\}) P^{(n)}(\{u_i\}, \{v_j\}) \prod_{i,j=1}^{n} h(u_i - v_j) \prod_{i<j}^{n} p(v_{ij}) \,, \tag{14}$$

where we defined the fermion-anti-fermion short range potential [9]

$$h(u_i - v_j) = \frac{1}{(u_i - v_j)^2 + 4} \,. \tag{15}$$

R_n is a regular function, with no poles nor zeroes in the rapidities u_i, v_j and related to the dynamical factor (9) by

$$R_n(\{u_i\}, \{v_j\}) \prod_{i<j}^{n} u_{ij}^2 v_{ij}^2 \equiv \Pi_{dyn}^{(n)}(\{u_i\}, \{v_j\}) \prod_{i=1}^{n} \hat{\mu}_f(u_i)\hat{\mu}_f(v_i) \,, \tag{16}$$

where the measure and the propagation phase are combined into $\hat{\mu}_f(u) = \mu_f(u)e^{-\tau E_f(u)+i\sigma p_f(u)}$. The strong coupling limit of (14) can be evaluated by integrating the rapidities v_i by closing the contour C for taking the residues and obtaining the result I_n^{closed}. Because of the properties of $P^{(n)}$ [22], only the poles in the fermion-anti-fermion short range potential (15) $v_i = u_j - 2i$ survive and provide a contribution to

$$I_n^{closed}(u_1, \ldots, u_n) = (-1)^n R_n(u_1, \ldots, u_n, u_1 - 2i, \ldots, u_n - 2i)\,, \tag{17}$$

which means that fermion and anti-fermion pair up to form a complex two-strings with spacing $2i$. A comparison with (9), (16) suggests to interpret this two-string (appearing in the OPE) as a bound state particle, the meson, whose energy and momentum are given additively

$$E_M(u) \equiv E_f(u+i) + E_f(u-i), \quad p_M(u) \equiv p_f(u+i) + p_f(u-i)\,, \tag{18}$$

along with the pentagon transition amplitude built up in the form

$$P^{MM}(u|v) = -(u-v)(u-v+i)P(u+i|v+i)P(u-i|v-i)| \\ \bar{P}(u-i|v+i)\bar{P}(u+i|v-i)\,.$$

From this, we can introduce the regular function (no poles, no zeroes)

$$P_{reg}^{MM}(u|v) = P^{MM}(u|v)\frac{u-v}{u-v+i}\,, \tag{19}$$

for later use and, from $\text{Res}_{\,v=u}\, P^{MM}(u|v) = i/\mu_M(u)$, the (hatted) measure

$$\hat{\mu}_M(u) = \mu_M(u)e^{-\tau E_M(u)+i\sigma p_M(u)} = -\frac{\hat{\mu}_f(u+i)\hat{\mu}_f(u-i)}{\bar{P}(u+i|u-i)\bar{P}(u-i|u+i)}\,, \tag{20}$$

which both allow us to recast (17) in a form with only reference to mesons

$$I_n^{closed}(u_1, \ldots, u_n) = \frac{\prod_{i=1}^{n} \hat{\mu}_M(u_i - i)}{\prod_{i<j}^{n} P_{reg}^{MM}(u_i - i|u_j - i)P_{reg}^{MM}(u_j - i|u_i - i)}\,. \tag{21}$$

Upon plugging this strong coupling limit into (12), we can efficiently reformulate the fermionic contribution (7) in terms of (free) mesons only:

$$W_f \simeq W^{(M)} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_C \prod_{i=1}^{n} \frac{du_i}{2\pi} \hat{\mu}_M(u_i - i) \cdot \tag{22}$$
$$\cdot \prod_{i<j}^{n} \frac{1}{P_{reg}^{MM}(u_i - i|u_j - i) P_{reg}^{MM}(u_j - i|u_i - i)} \prod_{i<j}^{n} p(u_{ij}) \,.$$

Evidently, this expression gives the exact strong coupling limit, though the next orders need a careful reconsideration of the above procedure.

3.2 Mesons Bound States, TBA and Beyond

Now, we shall show that in $W^{(M)}$ (22), thanks to the short range potential (13), the sum on mesons may be traded, at leading order, for one on 'TBA effective bound states' (no new nodes for them): this issue reveals a general feature beneath the appearance of a TBA integral equation and a possible physical interpretation of ordinary TBA. Actually, we will develop here a method to go also beyond the leading TBA order, as in and beyond [23], in principle at all orders. In fact, formula (22) for $W^{(M)}$ shares its form with the instanton partition function $\mathcal{Z}$ of $\mathcal{N} = 2$ theories, and from this perspective the large coupling $g \sim 1/\epsilon_2$ for $W^{(M)}$ corresponds to the so-called Nekrasov–Shatashvili limit of $\mathcal{Z}$, where the omega background ϵ_2 approaches zero [34]. Our approach relies on the introduction of a quantum gaussian field $X(u)$

$$e^{\langle X(u_i) X(u_j) \rangle} \equiv \frac{1}{P_{reg}^{MM}(u_i - i|u_j - i) P_{reg}^{MM}(u_j - i|u_i - i)} \,, \tag{23}$$

so that, upon a Hubbard–Stratonovich transformation, we can rewrite the Wl [32]

$$W^{(M)} = \left\langle \sum_{n=0}^{\infty} \frac{1}{n!} \int_C \prod_{i=1}^{n} \frac{du_i}{2\pi} \hat{\mu}_M(u_i - i) e^{X(u_i)} \prod_{i<j}^{n} p(u_{ij}) \right\rangle \,, \tag{24}$$

where the expectation value involves a gaussian path integral over the field $X(u)$ (cf. an analogous development for $\mathcal{Z}$ of $\mathcal{N} = 2$ theories [23]). Above we have neglected the diagonal terms $u_i = u_j$ of the Gaussian identity as they are sub-leading. The short range potential (13) part can be recast into a determinant form by means of the Cauchy identity

$$\prod_{i<j}^{n} p(u_{ij}) = \prod_{i<j}^{n} \frac{u_{ij}^2}{u_{ij}^2 + 1} = \frac{1}{i^n} \det \left(\frac{1}{u_i - u_j - i} \right) \,. \tag{25}$$

Thus, we are encouraged to define the matrix

$$M(u_i, u_j) \equiv \frac{\left[\hat{\mu}_M(u_i - i)e^{X(u_i)}\hat{\mu}_M(u_j - i)e^{X(u_j)}\right]^{1/2}}{u_i - u_j - i}, \tag{26}$$

so to obtain the following determinant

$$W^{(M)} = \left\langle \sum_{n=0}^{\infty} \frac{1}{n!} \int_C \prod_{i=1}^{n} \frac{du_i}{2\pi i} \det_{ij} M(u_i, u_j) \right\rangle . \tag{27}$$

In conclusion, this entails the average of a Fredholm determinant

$$W^{(M)} = \langle \det(1 + M) \rangle = \left\langle \exp\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} TrM^n\right]\right\rangle , \tag{28}$$

as expanded in the peculiar power traces

$$TrM^n \equiv \int_C \prod_{i=1}^{n} \frac{du_i}{2\pi i} \hat{\mu}_M(u_i - i)e^{X(u_i)} \prod_{i=1}^{n} \frac{1}{u_i - u_{i+1} - i}, \quad u_{n+1} \equiv u_1 . \tag{29}$$

This holds in the same manner for the instanton partition function $\mathcal{Z}$ of $\mathcal{N} = 2$ theories. Now, we need to compute the expansion for large $g \sim 1/\epsilon_2$ of the traces (29). At leading order, we can again close the contour C for $n - 1$ rapidities and compute the residues for $u_i - u_{i+1} = i$, obtaining

$$TrM^n \simeq \frac{(-1)^{n-1}}{n} \int_C \frac{du}{2\pi} \hat{\mu}_M^n(u - i)e^{nX(u)} \simeq \frac{(-1)^{n-1}}{n} \int_C \frac{du}{2\pi} \hat{\mu}_M^n(u)e^{nX(u)} \tag{30}$$

where the imaginary shifts $\sim 1/g \sim \epsilon_2$ in $\bar{u} = u/(2g)$ have been neglected: this is indeed the contribution of a n-meson bound state (like for gluons [32]). Notice that in $\mathcal{N} = 2$ theories all the integration contours are closed ab initio [34], so that the traces (29) can be, in principle, computed at all orders more easily [22]. Instead, for Wls the corrections at next orders have many origins and the computation of the one-loop contribution is much more involved than in [23], but here we give a path [22]. Within the bound state approximation (30), we can re-sum the Wilson loop (28) ($\mathcal{N} = 2$ too [34]) into a simple path integral

$$W^{(M)} \simeq \left\langle \exp\left[-\int_C \frac{du}{2\pi} \mu_M(u) Li_2\left[-e^{-\tau E_M(u) + i\sigma p_M(u)} e^{X(u)}\right]\right]\right\rangle , \tag{31}$$

upon use of (20) (further simplification $\mu_M(u) \simeq -1$). In details, the last gaussian path integral (31) can be re-interpreted as the partition function with an effective action, Yang–Yang potential, with dilogarithm potential and coincides with the conjecture of [32] for the middle node of the A_3 TBA [1]: the stationary point of the Yang–Yang potential gives the TBA equations. In fact, the other two nodes TBA contributions to the effective action can be obtained by summing up the contribution of the two (components of the) gluons, which genuinely form bound states (and then the dilogarithm potential [32]). Of course, the saddle point TBA equations are indeed the leading order since the effective action is proportional to g; moreover, they coincide with those arising, in a fully different manner, by minimizing the string area/action. The whole procedure of this section in two steps, – emergence of meson and effectiveness of its bound states –, extends to all the other polygons thus opening the way to the treatment of [19].

4 Conclusions and Perspectives

For scalars and fermions we compute the coupling independent parts of the OPE series as some random partitions on Young tableaux. This allows us to disentangle their respective two contributions (of the same order) at large coupling. At infinite coupling, fermion-anti-fermion pairs have been thought of as mesons which, by virtue of the short range potential (13), form bound states namely generate the $1/n$ factor (in the traces (30)) which yields the typical TBA (di)logarithm form. Importantly, the method is amenable to give a systematic expansion also for the partition function $\mathcal{Z}$ of $\mathcal{N}=2$ gauge theories, with instanton positions u_i (and their bound states at leading order) [22].

Acknowledgements Supporting grants: GAST (from INFN), EC Network Gatis and JSPS Postdoctoral Fellows 16F16735 for S.P.

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Hidden Symmetry in String Theory

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Abstract We explore the hidden symmetry in string theory by studying string scattering amplitudes. We calculate 4-point open string scattering amplitudes with three tachyons and a massive higher spin string state. The result can be expressed as Type D Lauricella functions which are generalization of Gaussian Hypergeometric functions. In various high energy limits, the string amplitudes reduced to the expected results that we obtained previously. We find that the symmetry of the string amplitudes at general energy is associated with $SL(3+K, C)$ algebra.

Keywords Hidden symmetry · String amplitudes · Lauricella functions

1 Introduction

Quantum Field Theory (QFT) is a powerful theory in modern physics. Based on QFT, Standard Model of particle physics successfully describes our microcosmic world. All important predictions by Standard Model have been observed in various experiments under rather precise level. However, to solve the UV divergence problem in QFT, the key technical procedure, i.e. renormalization, is complicated and has not been fully understood. More seriously, the renormalization procedure does not work for gravity at all. It means that it is impossible to construct a consistent quantum gravity theory by using the conventional method of QFT. Most of people believe that the divergence in QFT comes from the topological structure of the interaction among point-like particles, and it cannot be cured without modifying its topological structure. In string theory, one extends a point-like particle to a small piece of string. This simple extension dramatically changes the topological structure of the interaction in the theory. The new "Feynman diagram" now is a smooth world-sheet instead of a world-line with singularity at the interacting points.

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_14

To understand the problem of the UV divergence in QFT clearly, let us review the high energy behavior in QFT by a simple power counting. In the high energy hard limit, the tree amplitude by interchanging a spin-J particle behaves as $A_{tree}^{(J)} \sim E^{-2(1-J)}$, so that the one-loop amplitude behaves as

$$A_{1-loop}^{(J)} \sim \int d^4 p \frac{\left(A_{tree}^{(J)}\right)^2}{\left(p^2\right)^2} \sim \int E^{-4(2-J)}\, d^4 E, \tag{1}$$

which is finite for a scalar particle ($J = 0$) and renormalizable for a vector particle ($J = 1$), but is nonrenormalizable for a particle with $J \geq 2$, including graviton ($J = 2$). Nevertheless, if we sum over all the tree amplitudes by interchanging states with different spins, the final amplitude will be

$$A_{tree} = \sum_J A_{tree}^{(J)} \sim \sum_J a_J E^{-2(1-J)}, \tag{2}$$

which could have the enough soft behavior so that the loop amplitudes would be finite, if the following two conditions are satisfied simultaneously:

1. there are infinite intermedian higher spin states,
2. the coefficients a_J's are precisely related to each other in a certain way.

In string theory, string scattering amplitudes are exponentially fall-off in the high energy hard limit, which promises that string theory is a finite theory without the problem of UV divergence. We believe that the reason why the high energy behavior of string theory is so soft is that string theory satisfies the above two conditions.

The first condition is trivially satisfied in string theory because a string has infinite oscillation modes which correspond to infinite higher spin states, i.e. the Regge spectrum. The second condition is highly nontrivial. We conjecture that it corresponds to a huge symmetry in string theory, which is complicated and not apparent so that we usually call it hidden symmetries. A useful way to investigate the hidden symmetry is to study the symmetry among the string scattering amplitudes. Gross has conjectured that the string scattering amplitudes are linearly related to each other in the high energy, fixed scattering angle limit [1–3]. Using the three different methods, including the zero norm states Ward identities [4–6], the Virasoro algebra and the direct calculation of the scattering amplitudes, we have proved the Gross conjecture and obtained all of the linear ratios among the string amplitudes [7–17]. We also extend our study to the high energy, small angle limit, i.e. Regge scattering, and studied the recurrence relations among the scattering amplitudes [18–22]. Recently, we calculated the four-point string amplitudes at arbitrary energy and found that the amplitudes is associated with $SL(K+3,\mathbb{C})$ algebra [23, 24]. In the Sect. 2, we will calculate the four-point string amplitudes. The relations among the amplitudes in various high energy limits will be studied in Sect. 3. In Sect. 4, we will show that how to get the $SL(K+3,\mathbb{C})$ algebra from string amplitudes. We conclude our result in Sect. 5.

2 Four-Point String Amplitudes

To study the symmetry of string scattering amplitudes, we consider four-point open bosonic string scattering amplitudes with three tachyons and an arbitrary massive higher spin string state of the form,

$$\left|r_n^T, r_m^P, r_l^L\right\rangle = \prod_{n>0}\left(\alpha_{-n}^T\right)^{r_n^T}\prod_{m>0}\left(\alpha_{-m}^P\right)^{r_m^P}\prod_{l>0}\left(\alpha_{-l}^L\right)^{r_l^L}|0,k\rangle, \tag{3}$$

where $M_1^2 = M_3^2 = M_4^2 = -2$ are three tachyons and $M_2^2 \equiv M^2 = 2(N-1)$ is the higher spin string state with the mass level $N = \sum_{n,m,l>0}\left(nr_n^T + mr_m^P + lr_l^L\right)$, as showed in the Fig. 1.

In the center-of-mass frame, the four-momentum can be expressed as

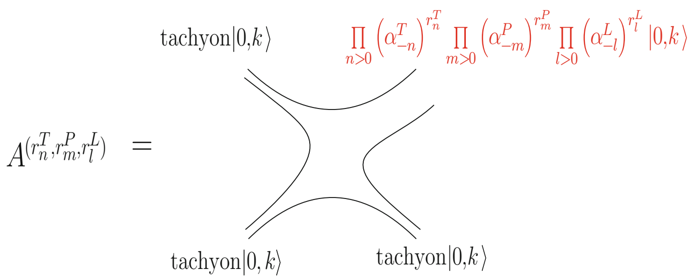

Fig. 1 The four-point open bosonic string scattering amplitudes with three tachyons and an arbitrary massive higher spin string state

$$k_1 = \left(\sqrt{M_1^2 + |\mathbf{k_1}|^2}, -|\mathbf{k_1}|, 0\right), \tag{4}$$

$$k_2 = \left(\sqrt{M_2^2 + |\mathbf{k_1}|^2}, +|\mathbf{k_1}|, 0\right), \tag{5}$$

$$k_3 = \left(-\sqrt{M_3^2 + |\mathbf{k_3}|^2}, -|\mathbf{k_3}|\cos\phi, -|\mathbf{k_3}|\sin\phi\right), \tag{6}$$

$$k_4 = \left(-\sqrt{M_4^2 + |\mathbf{k_3}|^2}, +|\mathbf{k_3}|\cos\phi, +|\mathbf{k_3}|\sin\phi\right), \tag{7}$$

where $\mathbf{k}_i$ is the three dimensional momentum vector and ϕ is the scattering angle. The Mandelstam variables are defined as usual as

$$s = -\left(k_1 + k_2\right)^2, t = -\left(k_2 + k_3\right)^2, u = -\left(k_1 + k_3\right)^2, \tag{8}$$

with $s + t + u = \sum M_i^2$.

On the two dimensional scattering plane, there are only three independent polarizations which we can choose to be

$$e^T = (0, 0, 1), \tag{9}$$

$$e^L = \frac{1}{M_2}\left(|\mathbf{k_1}|, \sqrt{M_2 + |\mathbf{k_1}|^2}, 0\right), \tag{10}$$

$$e^P = \frac{1}{M_2}\left(\sqrt{M_2 + |\mathbf{k_1}|^2}, |\mathbf{k_1}|, 0\right). \tag{11}$$

Note that the string amplitude with polarizations orthogonal to the scattering plane vanish in our setup. For later use, we also define

$$k_i^X \equiv e^X \cdot k_i \text{ for } X = (T, P, L)\,. \tag{12}$$

The simplest four-point string amplitude is scattered by four tachyons with $M_i^2 = -2$, i.e. the Veneziano amplitudes. In (s, t) channel, the four-tachyon scattering amplitude can be easily calculated,

$$A_{st}^{(\text{4-tachyon})} = \left\langle e^{ik_1\cdot X(x_1)} e^{ik_2\cdot X(x_2)} e^{ik_3\cdot X(x_3)} e^{ik_4\cdot X(x_4)}\right\rangle = B\left(-\frac{s}{2} - 1, -\frac{t}{2} - 1\right), \tag{13}$$

where

$$B\left(-\frac{s}{2} - 1, -\frac{t}{2} - 1\right) = \frac{\Gamma\left(-\frac{s}{2} - 1\right)\Gamma\left(-\frac{t}{2} - 1\right)}{\Gamma\left(\frac{u}{2} + 2\right)}, \tag{14}$$

is the Beta function. It is easy to verify that the Veneziano amplitude behaves as exponentially fall-off $A_{st}^{\text{(4-tachyon)}} \sim e^{-E}$ in the high energy limit ($s \sim E^2 \to \infty, t \to \infty$ with s/t fixed). This property holds for all four-point scattering amplitudes in string theory as we will show later.

The (s, t) and (t, u) channels string scattering amplitudes of states in Eq. (3) can be calculated to be [23, 24]

$$A_{st}^{(r_n^T, r_m^P, r_l^L)} = B\left(-\frac{t}{2}-1, -\frac{s}{2}-1\right) F_D^{(K)}\left(-\frac{t}{2}-1; R_n^T, R_m^P, R_l^L; \frac{u}{2}+2-N; \tilde{Z}_n^T, \tilde{Z}_m^P, \tilde{Z}_l^L\right)$$
$$\cdot \prod_{n=1} \left[-(n-1)!k_3^T\right]^{r_n^T} \prod_{m=1} \left[-(m-1)!k_3^P\right]^{r_m^P} \prod_{l=1} \left[-(l-1)!k_3^L\right]^{r_l^L}, \tag{15}$$
$$A_{tu}^{(r_n^T, r_m^P, r_l^L)} = B\left(-\frac{t}{2}-1, -\frac{u}{2}-1\right) F_D^{(K)}\left(-\frac{t}{2}-1; R_n^T, R_m^P, R_l^L; \frac{s}{2}+2-N; Z_n^T, Z_m^P, Z_l^L\right)$$
$$\cdot \prod_{n=1} \left[-(n-1)!k_3^T\right]^{r_n^T} \prod_{m=1} \left[-(m-1)!k_3^P\right]^{r_m^P} \prod_{l=1} \left[-(l-1)!k_3^L\right]^{r_l^L}, \tag{16}$$

where we have defined

$$R_k^X \equiv \left\{-r_1^X\right\}^1, \ldots, \left\{-r_k^X\right\}^k \text{ with } \{a\}^n = \underbrace{a, a, \ldots, a}_{n}, \tag{17}$$

$$Z_k^X \equiv \left[z_1^X\right], \ldots, \left[z_k^X\right] \text{ with } \left[z_k^X\right] = z_{k0}^X, \ldots, z_{k(k-1)}^X, \tag{18}$$

$$z_{kk'}^X = \left|\frac{k_1^X}{k_3^X}\right|^{\frac{1}{k}} e^{\frac{2\pi i k'}{k}} \text{ and } \tilde{z}_{kk'}^X \equiv 1 - z_{kk'}^X, k' = 0, \ldots, k-1, \tag{19}$$

and the integer K is defined to be

$$K = \sum_{\substack{j=1 \\ \{\text{for all } r_j^T \neq 0\}}}^{n} j + \sum_{\substack{j=1 \\ \{\text{for all } r_j^P \neq 0\}}}^{m} j + \sum_{\substack{j=1 \\ \{\text{for all } r_j^L \neq 0\}}}^{l} j, \tag{20}$$

which is usually different from the mass level N.

In Eqs. (15) and (16), $F_D^{(K)}$ is the D-type Lauricella function, which is one of the four extensions of the Gauss hypergeometric function to K variables and is defined as

$$F_D^{(K)}(a; b_1, \ldots, b_K; c; x_1, \ldots, x_K)$$
$$= \sum_{k_i} \frac{(a)_{k_1+\cdots+k_K}}{(c)_{k_1+\cdots+k_K}} \frac{(b_1)_{k_1}\cdots(b_n)_{k_K}}{k_1!\cdots k_K!} x_1^{k_1}\cdots x_K^{k_K}$$
$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 dt\, t^{a-1}(1-t)^{c-a-1} \cdot (1-x_1 t)^{-b_1}(1-x_2 t)^{-b_2} \ldots (1-x_K t)^{-b_K}, \tag{21}$$

where the integral representation of the Lauricella function $F_D^{(K)}$ in the last line was discovered by Appell and Kampe de Feriet (1926) [25].

By using the identity of the Lauricella functions for $b_i \in Z^-$

$$F_D^{(K)}(a; b_1, \ldots, b_K; c; x_1, \ldots, x_K)$$
$$= \frac{\Gamma(c)\,\Gamma\left(c - a - \sum b_i\right)}{\Gamma(c-a)\,\Gamma\left(c - \sum b_i\right)} F_D^{(K)}\left(a; b_1, \ldots, b_K; 1 + a + \sum b_i - c; 1 - x_1, \ldots, 1 - x_K\right), \quad (22)$$

we can express the (s, t) channel amplitude (15) in the following form

$$A_{st}^{(r_n^T, r_m^P, r_l^L)} = B\left(-\frac{t}{2} - 1, -\frac{s}{2} - 1 + N\right) F_D^{(K)}$$
$$\left(-\frac{t}{2} - 1; R_n^T, R_m^P, R_l^L; \frac{s}{2} + 2 - N; Z_n^T, Z_m^P, Z_l^L\right)$$
$$\cdot \prod_{n=1} \left[-(n-1)!k_3^T\right]^{r_n^T} \cdot \prod_{m=1} \left[-(m-1)!k_3^P\right]^{r_m^P} \prod_{l=1} \left[-(l-1)!k_3^L\right]^{r_l^L}. \quad (23)$$

Now it is easy to see the string BCJ relation from the Eqs. (23) and (16),

$$\frac{A_{st}^{(r_n^T, r_m^P, r_l^L)}}{A_{tu}^{(r_n^T, r_m^P, r_l^L)}} = \frac{(-)^N \Gamma\left(-\frac{s}{2} - 1\right) \Gamma\left(\frac{s}{2} + 2\right)}{\Gamma\left(\frac{u}{2} + 2 - N\right) \Gamma\left(-\frac{u}{2} - 1 + N\right)} = \frac{\sin\left(\frac{\pi u}{2}\right)}{\sin\left(\frac{\pi s}{2}\right)} = \frac{\sin(\pi k_2 \cdot k_4)}{\sin(\pi k_1 \cdot k_2)}, \quad (24)$$

which was proved by monodromy of integration of string amplitudes [26, 27] and explicitly proved recently in [28].

3 Symmetry in High Energy Limits

To study the relations among the string scattering amplitudes, we consider two different high energy limits: hard scattering limit and Regge scattering limit. We will briefly describe the results in the following. Readers can find the detail in a current review paper [29].

3.1 Linear Relations in Hard Limit

Hard scattering limit is the fixed angle scattering with $s \sim E^2 \to \infty$ and $\frac{t}{s} \sim \sin^2 \frac{\phi}{2} =$ constant. The linear relations of string amplitudes in the hard scattering limit were conjectured by Gross [1–3] and proved in [7–11, 13]. In the hard scattering limit $e^P = e^L$ [7, 8], we can only consider the polarization e^L. The relevant kinematics are

$$k_1^T = 0, k_3^T \simeq -E \sin\phi, \tag{25}$$

$$k_1^L \simeq -\frac{2p^2}{M_2} \simeq -\frac{2E^2}{M_2}, \tag{26}$$

$$k_3^L \simeq \frac{2E^2}{M_2} \sin^2 \frac{\phi}{2}, \tag{27}$$

with $\tilde{z}_{kk'}^T = 1,\ \tilde{z}_{kk'}^L = 1 - \left(-\frac{s}{t}\right)^{1/k} e^{\frac{i2\pi k'}{k}} \sim O(1)$.

In the hard limit, the (s, t) channel string amplitude in Eq. (15) reduces to

$$A_{st}^{(r_n^T, r_l^L)} = B\left(-\frac{t}{2} - 1, -\frac{s}{2} - 1\right) F_D^{(K)}\left(-\frac{t}{2} - 1; R_n^T, R_l^L; \frac{u}{2} + 2 - N; (1)_n, \tilde{Z}_l^L\right)$$
$$\cdot \prod_{n=1} [(n-1)! E \sin\phi]^{r_n^T} \prod_{l=1}\left[-(l-1)!\frac{2E^2}{M_2}\sin^2\frac{\phi}{2}\right]^{r_l^L}. \tag{28}$$

Next, we propose the following identity

$$\sum_{k_r=0}^{r_1^L} \frac{\left(-\frac{t}{2}-1\right)_{k_r}}{\left(\frac{u}{2}+2-N\right)_{k_r}} \frac{\left(-r_1^L\right)_{k_r}}{k_r!}\left(1+\frac{s}{t}\right)^{k_r}$$
$$= 0 \cdot \left(\frac{tu}{s}\right)^0 + 0 \cdot \left(\frac{tu}{s}\right)^{-1} + \cdots + 0 \cdot \left(\frac{tu}{s}\right)^{-\left[\frac{r_1^L+1}{2}\right]-1} + C_{r_l^L}\left(\frac{tu}{s}\right)^{-\left[\frac{r_1^L+1}{2}\right]}$$
$$+ O\left\{\left(\frac{tu}{s}\right)^{-\left[\frac{r_1^L+1}{2}\right]+1}\right\}, \tag{29}$$

where $C_{r_1^L}$ is independent of the energy E and depends on r_1^L and possibly the scattering angle ϕ. For $r_1^L = 2m$ being an even number, we can show that $C_{r_1^L} = \frac{(2m)!}{m!}$ which is independent of the scattering angle ϕ. We have verified Eq. (29) for $r_1^L = 0, 1, 2, \ldots, 10$ explicitly.

It is noted that, taking the Regge limit ($s \to \infty$ with t fixed) and setting $r_1^L = 2m$, Eq. (29) reduces to the Stirling number identity,

$$\sum_{k_r=0}^{2m} \frac{\left(-\frac{t}{2}-1\right)_{k_r}}{\left(-\frac{s}{2}\right)_{k_r}} \frac{(-2m)_{k_r}}{k_r!} \left(\frac{s}{t}\right)^{k_r} \simeq \sum_{k_r=0}^{2m} (-2m)_{k_r} \left(-\frac{t}{2}-1\right)_{k_r} \frac{(-2/t)^{k_r}}{k_r!}$$
$$= 0 \cdot (-t)^0 + 0 \cdot (-t)^{-1} + \cdots + 0 \cdot (-t)^{-m+1} + \frac{(2m)!}{m!} (-t)^{-m} + O\left\{\left(\frac{1}{t}\right)^{m+1}\right\}, \tag{30}$$

which was proposed in [30] and proved in [31].

Finally, the leading order string amplitudes in the hard scattering limit can be calculated to be

$$\begin{aligned} A_{st}^{(r_1^T,2m,r_2^L)} &\simeq B\left(-\frac{s}{2}-1, -\frac{t}{2}-1\right) \cdot F_D^{(4)}\left(-\frac{t}{2}-1; R_1^T, R_2^L; \frac{u}{2}+2-N; 1, Z_2^L\right) \\ &\cdot [E\sin\phi]^{r_1^T} \cdot \left[-\frac{2E^2}{M_2}\sin^2\frac{\phi}{2}\right]^{2m} \cdot \left[-\frac{2E^2}{M_2}\sin^2\frac{\phi}{2}\right]^{r_2^L} \\ &= B\left(-\frac{s}{2}-1, -\frac{t}{2}-1\right)(E\sin\phi)^N \cdot \left(r_1^L-1\right)!!\left(-\frac{1}{M_2}\right)^{2m+r_2^L}\left(\frac{1}{2}\right)^{m+r_2^L} \\ &= \left(r_1^L-1\right)!!\left(-\frac{1}{M_2}\right)^{2m+r_2^L}\left(\frac{1}{2}\right)^{m+r_2^L} \cdot A^{(N,0,0)}, \end{aligned} \tag{31}$$

which reproduces the ratios

$$\frac{A_{st}^{(r_1^T,2m,r_2^L)}}{A_{st}^{(N,0,0)}} = (2m-1)!!\left(-\frac{1}{M_2}\right)^{2m+r_2^L}\left(\frac{1}{2}\right)^{m+r_2^L}, \tag{32}$$

which is consistent with the previous result [7–11, 13].

3.2 Recurrence Relations in Regge Limit

Regge scattering limit is the small angle scattering with $s \sim E^2 \to \infty$ and $t \sim E^2 \sin^2 \frac{\phi}{2} =$ constant. The recurrence relations of string amplitudes in the Regge scattering limit have been studied in [30, 32, 33]. The relevant kinematics in Regge limit are

$$k_1^T = 0, k_3^T \simeq -\sqrt{-t}, \tag{33}$$

$$k_1^P \simeq -\frac{s}{2M_2}, k_3^P \simeq -\frac{\tilde{t}}{2M_2} = -\frac{t-M_2^2-M_3^2}{2M_2}, \tag{34}$$

$$k_1^L \simeq -\frac{s}{2M_2}, k_3^L \simeq -\frac{\tilde{t}'}{2M_2} = -\frac{t+M_2^2-M_3^2}{2M_2}, \tag{35}$$

with $\tilde{z}^T_{kk'} = 1$, $\tilde{z}^P_{kk'} = 1 - \left(-\frac{s}{\tilde{t}}\right)^{1/k} e^{\frac{i2\pi k'}{k}} \sim s^{1/k}$ and $\tilde{z}^L_{kk'} = 1 - \left(-\frac{s}{\tilde{t}'}\right)^{1/k} e^{\frac{i2\pi k'}{k}} \sim s^{1/k}$.

In the Regge limit, the (s, t) channel string amplitude in Eq. (15) reduces to

$$A_{st}^{(r_n^T, r_m^P, r_l^L)} \simeq B\left(-\frac{t}{2} - 1, -\frac{s}{2} - 1\right) F_1\left(-\frac{t}{2} - 1; -q_1, -r_1; -\frac{s}{2}; \frac{s}{\tilde{t}}, \frac{s}{\tilde{t}'}\right) \cdot \prod_{n=1} \left[(n-1)!\sqrt{-t}\right]^{r_n^T} \cdot \prod_{m=1} \left[(m-1)!\frac{\tilde{t}}{2M_2}\right]^{r_m^P} \prod_{l=1} \left[(l-1)!\frac{\tilde{t}'}{2M_2}\right]^{r_l^L}, \tag{36}$$

where F_1 is the Appell function. Equation (36) agrees with the result obtained in [33].

The string amplitudes in the Regge limit are much more complicated than that in the hard limit and do not have linear relations. However, there are a series of recurrence relations for the Appell functions F_1,

$$(a - b_1 - b_2) F_1 - aF_1(a+1) + b_1 F_1(b_1+1) + b_2 F_1(b_2+1) = 0, \tag{37}$$

$$cF_1 - (c-a) F_1(c+1) - aF_1(a+1; c+1) = 0, \tag{38}$$

$$cF_1 + c(x-1) F_1(b_1+1) - (c-a) xF_1(b_1+1; c+1) = 0, \tag{39}$$

$$cF_1 + c(y-1) F_1(b_2+1) - (c-a) yF_1(b_2+1; c+1) = 0. \tag{40}$$

Using the above recurrence relations, we can obtain a lot of recurrence relations among the string amplitudes in Eq. (36). One can show that by solving the recurrence relations, all the string amplitudes at certain mass level can be expressed in term of a single amplitude [33].

4 Symmetry of Four-Point Amplitudes at General Energy

Let us recall the (s, t) channels string scattering amplitudes with three tachyons and a massive higher spin string state in Eq. (15),

$$A_{st}^{(r_n^T, r_m^P, r_l^L)} = B\left(-\frac{t}{2} - 1, -\frac{s}{2} - 1\right) F_D^{(K)}\left(-\frac{t}{2} - 1; R_n^T, R_m^P, R_l^L; \frac{u}{2} + 2 - N; \tilde{Z}_n^T, \tilde{Z}_m^P, \tilde{Z}_l^L\right) \cdot \prod_{n=1} \left[-(n-1)!k_3^T\right]^{r_n^T} \cdot \prod_{m=1} \left[-(m-1)!k_3^P\right]^{r_m^P} \prod_{l=1} \left[-(l-1)!k_3^L\right]^{r_l^L}. \tag{41}$$

To explore the symmetry of the above amplitudes, we need to understand their mathematical structure in a deeper way. To do it, we follow the mathematical construction in [34] and define the generating functions associated with the D-type Lauricella function $F_D^{(K)}$ as

$$f_c^{a,b_j}\left(s,u_j,t,x_j\right)\equiv B\left(a,c-a\right)F_D^{(K)}\left(a;b_j;c;x_j\right)s^a u_1^{b_1}\cdots u_K^{b_K}t^c,\ j=1,\ldots,K. \tag{42}$$

Now the (s,t) channels string scattering amplitudes can be expressed in term of the generating functions as

$$A_{st}^{\left(r_n^T,r_m^P,r_l^L\right)}\sim f_{\frac{u}{2}+2-N}^{-\frac{t}{2}-1,R_j^X}\left(1,k_3^X,1,\tilde{Z}_j^X\right),\ X=T,P,L. \tag{43}$$

We next define the operators,

$$\begin{aligned}
E^{\alpha}&=s\left(\sum_j x_j\partial_{x_j}+s\partial_s\right),\ E^{\alpha\gamma}=st\left(\sum_j\left(1-x_j\right)\partial_{x_j}-s\partial_s\right),\\
E^{\beta_k}&=u_k\left(x_k\partial_{x_k}+u_k\partial_{u_k}\right),\ E^{\beta_k\gamma}=u_kt\left(-\left(1-x_k\right)\partial_{x_k}+u_k\partial_{u_k}\right),\\
E^{\gamma}&=t\left(\sum_j\left(1-x_j\right)\partial_{x_j}+t\partial_t-s\partial_s-\sum_j u_j\partial_{u_j}\right),\ E^{\alpha\beta_k\gamma}=su_kt\partial_{x_k},\\
J_{\alpha}&=s\partial_s,\ J_{\beta_k}=u_k\partial_{u_k}-\frac{1}{2}t\partial_t+\frac{1}{2}\sum_{j\neq k}u_j\partial_{u_j},\ J_{\gamma}=t\partial_t-\frac{1}{2}\left(s\partial_s+\sum_j u_j\partial_{u_j}+1\right),
\end{aligned} \tag{44}$$

which acting on the generating function gives,

$$\begin{aligned}
E^{\alpha}f_c^{a,b_j}&=\left(c-a-1\right)f_c^{a+1,b_j},\ E^{\alpha\gamma}f_c^{a,b_j}=\left(\sum_j b_j-1\right)f_{c+1}^{a+1,b_j},\\
E^{\beta_k}f_c^{a,b_j}&=b_kf_c^{a,b_k+1},\qquad E^{\beta_k\gamma}f_c^{a,b_j}=b_kf_{c+1}^{a,b_k+1},\\
E^{\gamma}f_c^{a,b_j}&=\left(c-\sum_j b_j\right)f_{c+1}^{a,b_j},\ E^{\alpha\beta_k\gamma}f_c^{a,b_j}=b_kf_{c+1}^{a+1,b_k+1},\\
J_{\alpha}f_c^{a,b_j}&=\left(a-\frac{c}{2}\right)f_c^{a,b_j},\qquad J_{\beta_k}f_c^{a,b_j}=\left(b_k-\frac{c}{2}+\frac{1}{2}\sum_{j\neq k}b_j\right)f_c^{a,b_j},\\
J_{\gamma}f_c^{a,b_j}&=\left[c-\frac{1}{2}\left(a+\sum_j b_j+1\right)\right]f_c^{a,b_j}.
\end{aligned} \tag{45}$$

Finally, by defining a set of new operators $\mathcal{E}_{ij}$ in the following way,

$$\begin{aligned}
E^{\alpha}&=\mathcal{E}_{12},\qquad E^{\alpha\gamma}=\mathcal{E}_{32},\qquad E^{\gamma}=\mathcal{E}_{31},\\
E^{\beta_k\gamma}&=-\mathcal{E}_{k+3,1},\quad E^{\alpha\beta_k\gamma}=-\mathcal{E}_{k+3,2},\quad E^{\beta_k}=\mathcal{E}_{k+3,3},\\
J_{\alpha}&=\frac{1}{2}\left(\mathcal{E}_{11}-\mathcal{E}_{22}\right),\ J_{\gamma}=\frac{1}{2}\left(\mathcal{E}_{33}-\mathcal{E}_{11}\right),\ J_{\beta_k}=\frac{1}{2}\left(\mathcal{E}_{k+3,k+3}-\mathcal{E}_{33}\right).
\end{aligned} \tag{46}$$

the algebra satisfied by the new operators becomes

$$\left[\mathcal{E}_{ij}, \mathcal{E}_{kl}\right] = \delta_{jk} E_{il} - \delta_{li} E_{kj}, \tag{47}$$

which can be identified as $sl\,(K+3, \mathbb{C})$ algebra.

5 Conclusion

In this article, I briefly reviewed the hidden symmetry in string theory by studying the string scattering amplitudes. The four-point bosonic open string scattering amplitude with three tachyons and an arbitrary massive higher spin string state in both (s, t) and (t, u) channels have been explicitly calculated and expressed in term of D-type Lauricella function in Eqs. (15) and (16). The string BCJ relation can be verified easily. We also considered two high energy limits. In hard limit, the hidden symmetry reduces to the linear relations among the string amplitude. In Regge limit, the hidden symmetry exhibit to be the recurrence relations of the string amplitudes. At general energy, we mathematically showed that the hidden symmetry is associated to $sl\,(K+3, \mathbb{C})$ algebra. To explore its physical picture in more details is important to understand the hidden symmetry in string theory in the future.

Acknowledgements The main results in this talk are based on a series of works collaborated with S.H. Lai and Jen-Chi Lee. I would like to thank the organizers of The 24th International Conference on Integrable Systems and Quantum symmetries (ISQS24) for inviting me to present this work. This work is supported by the Ministry of Science and Technology (MoST), Taiwan.

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Information Geometry of Strings on Plane Wave Background

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Abstract In this report we consider the information-theoretic approach to closed bosonic strings in homogeneous plane wave background. We derive the extended renormalized entanglement entropy of the string and the corresponding Fisher metric on its statistical manifold. Our investigations are conducted exclusively within the framework of Thermo Field Dynamics. At the end of the report we discuss a procedure for reconstructing probability density functions from a given Fisher information metric.

Keywords String theory · Information geometry · Fisher information metric
Phase transitions

1 Introduction

It is common knowledge that different descriptions of probabilistic phenomena in nature can be conveniently accommodated within the framework of Statistical Physics. However, in the recent years, a new approach gains popularity, called Information Geometry (IG) [1, 2]. It has the potential to encompass all statistically based occurrences not only in Physics, but also in other sciences [2, 3], thus paving the way for new and unexpected discoveries. With its variety of powerful analytic tools,

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_15

namely Fisher information metric (FIM), relative entropies, mutual information, etc., IG allows one to find connections between phenomena and processes, initially regarded as distinct and unrelated.

The traditional way of gaining knowledge on a physical system is through observations and measurements that faithfully describe the different states of the system. The accumulated data is then fed to some phenomenological model, which allows for further refinement of the ideas and new physical predictions. From this point of view IG inserts additional layer of knowledge by considering the nature of the information that is conveyed by the data. This is regardless of the particular effect that is under observation and thus providing one unified information-theoretic viewpoint.

Analysing physical systems on the basis of information flow upon phenomenological modelling already amounts to some very interesting and intriguing results. For example, in String Theory, in the context of the holographic duality between gravitational and gauge theories, the concept of emergent space-time became relevant [4, 5], due to the Ryu–Takayanagi conjecture [6, 7]. In Black Hole (BH) physics an alternative geometric approach to BH thermodynamics appears as thermodynamic limit of the Fisher information metric [8, 9]. In Condensed Matter Physics FIM and its algebraic invariants have proven to contain information about the phase structure of the system [10]. The latter are only some of many applications of IG in modern Physics (more applications can be found in [2, 3]).

This paper is organized as follows. In Sect. 2 we shortly discuss the basic concepts in Thermo Field Dynamics (TFD). In Sect. 3 we apply TFD to closed bosonic strings on $D = 10$ regular homogeneous plane wave background in order to derive the extended entanglement entropy (EEE) and the Fisher metric of the system. In Sect. 4 we focus on a method of reconstructing the probability density functions (PDFs) from given FIM. Finally, in Sect. 5 we give a short overview of our findings.

2 Basics of Thermo Field Dynamics

The essential quantity in statistical mechanics in thermal equilibrium is the statistical average of a quantity A, say over the grand canonical ensemble at temperature T, given by

$$\langle A \rangle = \frac{\mathrm{Tr}[A\, e^{-\beta H}]}{Z(\beta)}\,, \tag{1}$$

where H is the Hamiltonian of the system, $Z(\beta) = \mathrm{Tr} e^{-\beta H}$ is the partition function, and $\beta = k_B T^{-1}$ is the inversed temperature. In 1955 Matsubara [11] observed that the statistical average $\langle A \rangle$ has properties similar to the vacuum expectation value of A in Quantum Field Theory. The latter observation lead him to construct a field theory in which the vacuum expectation value coincides with the statistical average, i.e.

$$\langle A\rangle = \frac{\mathrm{Tr}[A\,e^{-\beta H}]}{Z(\beta)} \equiv \langle 0(\beta)|A|0(\beta)\rangle. \tag{2}$$

Here $|0(\beta)\rangle$ is a temperature dependent vacuum state in a new space to be constructed. In general one can define a suitable thermal state $|0(\beta)\rangle$, which satisfy

$$\langle 0(\beta)|F|0(\beta)\rangle = Z^{-1}(\beta)\sum\nolimits_n \langle n|F|n\rangle\, e^{-\beta E_n}, \tag{3}$$

for arbitrary dynamical variable F, where

$$H\,|n\rangle = E_n\,|n\rangle\,, \quad \langle n|m\rangle = \delta_{nm}. \tag{4}$$

If now one expands the thermal state $|0(\beta)\rangle$ in terms of the energy eigenstates $|n\rangle$,

$$|0(\beta)\rangle = \sum\nolimits_n |n\rangle\, f_n(\beta), \tag{5}$$

and inserts Eq. (5) back in Eq. (3), one finds the following condition [11]:

$$f_n^*(\beta)\, f_m(\beta) = Z^{-1}(\beta)\, e^{-\beta E_n}\,\delta_{mn}. \tag{6}$$

Equation (6) is not possible if f_n are mere c-numbers, but one has to consider them as state vectors in some specific Hilbert space, in which Eq. (6) is an orthogonality condition. The new Hilbert space is called tilde space $\tilde{\mathcal{H}}$. It introduces a fictitious system, which is of exactly the same structure as the physical one under consideration,

$$\tilde{H}\,|\tilde{n}\rangle = E_n\,|\tilde{n}\rangle\,, \quad \langle\tilde{n}|\tilde{m}\rangle = \delta_{nm}. \tag{7}$$

Therefore the equilibrium thermal vacuum state $|0(\beta)\rangle$ is a state vector in the double Hilbert space, $\mathcal{H} = \widehat{\mathcal{H}} \bigotimes \tilde{\mathcal{H}}$, and is given by

$$|0(\beta)\rangle = \frac{1}{Z(\beta)}\sum\nolimits_n e^{-\beta E_n}\,|n,\tilde{n}\rangle. \tag{8}$$

One can think of particle states in this TFD construction as follow. The one particle state is build up from the thermal equilibrium state $|0(\beta)\rangle$ by adding one particle without tilde or by eliminating one particle with tilde.[1]

As we have shown Thermo Field Dynamics requires a statistical state defined in a double Hilbert space, which is a direct product of the original space and an isomorphic copy of it. Although the state in (8) is in the energy eigenbasis, it turns out that it is independent from the chosen representation. This result is known as the general representation theorem [12] and makes TFD very useful for treating quantum states directly via density matrix approach.

[1]For more information on TFD formalism see [11].

In TFD one defines an extended density matrix, $\hat{\rho} = |\Psi\rangle \langle\Psi|$, where $|\Psi\rangle$ is a general TFD state of the form given in Eq. (8). Now, one can use the extended density operator to calculate the extended entanglement entropy via the standard expression

$$S_A = -Tr_{\{A\}} \left(\hat{\rho}_A \, \ln \hat{\rho}_A\right) , \tag{9}$$

where the trace is over the degrees of freedom of the suitably taken subsystem A, while $\hat{\rho}_A = Tr_{\{B\}}\hat{\rho}$ is the reduced density operator for the same subsystem. In this case the whole quantum system M is partitioned such as $M = A \bigcup B$. By definition the Fisher information metric is now given as the Hessian of the entanglement entropy [13, 14]:

$$g_{\mu\nu}(\theta) = \epsilon \, \frac{\partial^2 S_A(\theta)}{\partial\theta^\mu \, \partial\theta^\nu} , \tag{10}$$

where the relative sign, $\epsilon = \pm 1$, is chosen such as the metric components be positive defined, as thermodynamic stability requires. The set of parameters θ^μ span the parameter space, i. e. the statistical manifold of the system. The Fisher metric (10) naturally defines a Riemannian metric on this manifold.

In what follows we will calculate the EEE and the FIM for closed bosonic string in a simple homogeneous plane wave background within the framework of TFD.

3 The Fisher Information Metric for Closed Strings in Plane Wave Geometry

Consider a closed relativistic string in non-singular $2 + d$ dimensional homogeneous plane wave background with metric of the form

$$ds^2 = 2\, du\, dv + k_{ij}\, x^i\, x^j\, du^2 + 2\, f_{ij}\, x^i\, dx^j\, du + dx^i\, dx^j\, . \tag{11}$$

Here k_{ij} and f_{ij} are constant, and the B-field is given by $B_{iu} = -h_{ij}\, x^j$. In Ref. [15] the authors showed that the string Hamiltonian,

$$H = \frac{1}{2\,\pi} \int_0^\pi d\sigma \left[\delta_{ij} \left(\dot{X}^i\, \dot{X}^j + X^{i\,\prime}\, X^{j\,\prime} - k_i\, X^i\, X^j\right) - 2\, h_i\, j\, X^i\, X^{j\,\prime}\right] , \tag{12}$$

can be written as a sum of n-level harmonic oscillator Hamiltonians:

$$H = \sum_{n=0}^{\infty} H^{(n)}\, . \tag{13}$$

The zero-mode part Hamiltonian assumes the form

$$H^0 = \sum_{j=1}^{d} \text{sign}\left(C_j\right)\, \Omega_j \left(\mathcal{N}_j + \frac{1}{2}\right), \tag{14}$$

with frequencies

$$\Omega_j = \frac{\sum_{i=1}^{d} (\omega_j^2 - k_i)\, m_{ii}(\omega_j)}{2\,\omega_j \prod_{k \neq j} (\omega_j^2 - \omega_k^2)}. \tag{15}$$

Likewise, the Hamiltonians for higher modes of the string are given by

$$H^{(n)} = \sum_{J=1}^{2d} \text{sign}\left(C_J^{(n)}\right)\, \Omega_J^{(n)} \left(\mathcal{N}_J^{(n)} + \frac{1}{2}\right), \quad n > 0, \tag{16}$$

where the frequency $\Omega_J^{(n)}$ is a sum of two terms – one coming from the plane wave metric, and the other coming from the B-field:

$$\Omega_J^{(n)} = 2\,\omega_J^{(n)}\, C_J^{(n)}\, m_{11}\left(\omega_J^{(n)}\right) \sum_{i,j} \left(\omega_J^{(n)}\, \delta_{ij} + i\,(-1)^{i+j}\, f_{ij}\right) m_{ij}\left(\omega_J^{(n)}\right). \tag{17}$$

Now we can apply the TFD technique to obtain the equilibrium extended entanglement entropy on every energy level of the string spectrum. However, for simplicity, we will consider only the $n = 0$ Hamiltonian of the string from Eq. (14). Assume the following two subsystems:

$$\{\mathcal{N}_j\}_{j=1}^{d} = \{\mathcal{N}_\mu\}_{\mu=1}^{p} \bigcup \{\mathcal{N}_k\}_{k=p+1}^{d} = A \bigcup B, \quad p \le d-1, \quad 2 \le d \le 9, \tag{18}$$

Once the Hamiltonian is fixed it is straightforward to find the corresponding partition function,

$$Z = Tr_{\{AB\}}\left(e^{-\beta \hat{H}}\right) = \prod_{i=1}^{N} \frac{e^{-\beta E_0}}{1 - e^{-\beta E_i}} = \prod_{i=1}^{N} \frac{e^{-K_0}}{1 - e^{-K_i}}, \tag{19}$$

where we have introduced the inverse scaled temperatures $K_0 = \beta$, $E_0 = \beta \sum_{j=1}^{d} sign(C_j)\, \Omega_j/2$ and $K_i = \beta$, $E_i = \beta\, sign(C_i)\, \Omega_i$, $i = 1, \ldots, N$. In this case the TFD statistical state assumes the form

$$|\Psi\rangle = \sum_{\{n_i\}=0}^{\infty} \sqrt{(\hat{\rho}_{eq})_{ii}}\; |\{n_i\}\rangle\; |\{\tilde{n}_i\}\rangle, \tag{20}$$

where $\hat{\rho}_{eq} = \frac{e^{-\beta \hat{H}}}{Z}$ is the ordinary density matrix in equilibrium. After some lengthy computations the EEE for the closed bosonic string in equilibrium on the regular pp-wave background (11) is found explicitly as

$$S_A\left(t_\mu\right) = \left(\prod_{\mu=1}^{p} \frac{t_\mu + 1}{t_\mu - 1}\right) \sum_{\mu=1}^{p} \left(\frac{2\, t_\mu \, \ln t_\mu}{t_\mu - 1} - \ln\left(t_\mu^2 - 1\right)\right), \tag{21}$$

where $t_\mu = e^{K_\mu/2}$, $\mu = 0,\ 1,\ 2, \ldots$. The EEE is divergent at $t_\mu = 1$, suggesting a critical phase transition point. This point corresponds to $K_\mu = 0$, which puts the system at very high temperatures ($T \to \infty$). The Fisher metric follows immediately from Eqs. (21) and (10),

$$\begin{aligned} g_{\mu\nu} &= 4\left(\prod_{\sigma=1}^{p} \frac{\delta_{\mu\sigma}\, \delta_{\nu\sigma}}{(t_\sigma - 1)^3}\right) \sum_{\sigma=1}^{p} \left[\frac{2\, t_\sigma \, \ln t_\sigma}{t_\sigma - 1} - \ln\left(t_\sigma^2 - 1\right)\right] \\ &+ 4\left(\prod_{\sigma=1}^{p} \frac{\delta_{\nu\sigma}}{(t_\sigma - 1)^2}\right) \sum_{\sigma=1}^{p} \left[\left(\frac{1}{1 + t_\sigma} + \frac{\ln t_\sigma}{(t_\sigma - 1)^2}\right) \delta_{\mu\sigma}\right] \\ &+ 4\left(\prod_{\sigma=1}^{p} \frac{\delta_{\mu\sigma}}{(t_\sigma - 1)^2}\right) \sum_{\sigma=1}^{p} \left[\left(\frac{1}{1 + t_\sigma} + \frac{\ln t_\sigma}{(t_\sigma - 1)^2}\right) \delta_{\nu\sigma}\right] \\ + 4\left(\prod_{\sigma=1}^{p} \frac{t_\sigma + 1}{t_\sigma - 1}\right) &\sum_{\sigma=1}^{p} \left[\left(\frac{1}{2t(1 + t_\sigma)^2} + \frac{2 - 4t_\sigma + t_\sigma^2}{2(1 + t_\sigma)^2} t_\sigma + \ln t_\sigma\right) \frac{\delta_{\mu\sigma}\delta_{\nu\sigma}}{(t_\sigma - 1)^3}\right]. \end{aligned} \tag{22}$$

From information-theoretic point of view FIM represents a continuous setting even if the underlying features of the system are discrete. This allows one to take advantage of the powerful framework of differential geometry to treat probabilistic structures as geometrical ones.

For two-dimensional statistical manifold one needs only to investigate the properties of the Fisher scalar curvature to obtain information about the critical behaviour of the system. The latter is well-known result in differential geometry, where in $2d$ space the components of the Riemann tensor are just proportional to the scalar curvature, e.g. there is only one degree of freedom. The latter implies that any critical points, corresponding to second order phase transitions, are encoded in the singularities of the Ricci scalar R_{FIM}.

Furthermore, by studying the properties of R_{FIM}, one can say something about the type of interactions between the constituents of the system. For example, if $R_{FIM} = 0$ the system is non-interacting. While, on the other hand, a maximum positive information curvature, $max|R_{FIM} > 0|$, corresponds to maximal repulsive interactions, the maximum of the absolute value of the negative curvature, $max|R_{FIM} < 0|$, generates maximal attractive force. The critical behaviour of the closed bosonic string in $D = 5 + 2$ dimensional regular plane wave background was studied in Ref. [16].

In the higher dimensional case the problem is not that simple and only the invariants of the Fisher metric may not be enough to encompass all critical phase phenomena. Here, an additional information analysis may has to be invoked.

4 Reconstruction of Probability Density Functions from Fisher Metric

In this section we are going to give an idea of how to reconstruct a family of probability density functions from a given Fisher information metric.

By definition the Fisher information metric can be straightforwardly calculated once a probability distribution has been chosen. A set of distributions $P(\mathbf{x}, \boldsymbol{\theta})$, parametrized by $\boldsymbol{\theta}$, forms a statistical manifold. The Riemannian metric on such manifold is the FIM, defined by the following expectation value

$$g_{\mu\nu}(\boldsymbol{\theta}) = \int\limits_X \mathcal{D}_X \, P(\mathbf{x}, \boldsymbol{\theta}) \, \frac{\partial \ln P(\mathbf{x}, \boldsymbol{\theta})}{\partial \theta^\mu} \, \frac{\partial \ln P(\mathbf{x}, \boldsymbol{\theta})}{\partial \theta^\nu} \, . \tag{23}$$

Here $\mathbf{x} \in X$ is a point from the random sample space X. It turns out that the only Riemannian metric is the Fisher metric, which is invariant under coordinate transformations of $\boldsymbol{\theta}$ and also under one-to-one transformations of the random variable $\mathbf{x}$ [1]. Therefore the natural question arising is how to revert Eq. (23) with respect to $P(\mathbf{x}, \boldsymbol{\theta})$? Also it is relevant to define under what conditions such operation is possible?

For this purpose let us consider a family of normalized Gaussian PDFs in the form:

$$P(x; \theta) = \frac{1}{\sqrt{(2\,\pi)^n}} \, e^{-\frac{1}{2} \sum\limits_{i=1}^{n} \left(x^i - h^i(\theta)\right)^2} \, . \tag{24}$$

The Nash embedding theorem tells us that there is an $n \in N$ such that a p-dimensional manifold $(\mathcal{M}^p, g)$ may be C^1 isometrically embedded in n- dimensional Euclidean space (E^n, δ). Thus, it tells us that there exists an h such that the metric g on the manifold is the pullback $g = h * \delta$.

The CMS (Clingman–Murugan–Shock) method, proposed in [17], is focused on the computation of the transition functions $h^i(\theta)$, $i = 1, \ldots, n$, from E^n to the statistical manifold $\mathcal{M}$. The CMS ansatz for FIM is given by

$$g_{ab} = (\partial_a h^i) \, (\partial_b h^j) \, \delta_{ij} \, . \tag{25}$$

The latter expression gives a set of non-linear first-order partial differential equations for $h^i(\theta)$. A simple example is the metric on the unit 2-sphere,

$$ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2 \, . \tag{26}$$

where an easy check shows that the set of transition functions

$$h = (h^1(\theta, \varphi), \; h^2(\theta, \varphi), \; h^3(\theta, \varphi)) = (\sin\theta \, \cos\varphi, \; \sin\theta \, \sin\varphi, \; cos\theta) \tag{27}$$

satisfy Eq. (25) and thus the 2-sphere can be embedded in 3-dimensional Euclidean space. These are only the set of spherical coordinates.

The Gaussian PDF ansatz (24) is not the only possibility. There are an infinite number of different PDFs giving the same FIM. In other words, one can choose an infinite-to-one families of probability density functions to be parametrized by h^i, which originate from the same Riemannian metric tensor. For example, the hyperbolic secant PDF is a valid case:

$$P(x;\theta) = \frac{1}{\pi^n} \prod_{i=1}^{n} \operatorname{sech}(x^i - h^i \sqrt{2})\,. \tag{28}$$

Now let us get back to the string system considered in this report. When $p = 2$ the statistical manifold $\mathcal{M}^2$ of the string is $2d$ with components of the Fisher metric given by

$$g_{t_1 t_1} = \frac{4\,(t_2+1)}{(t_2-1)\,(t_1-1)^3} \left(\ln \frac{t_1^{\frac{3(t_1+1)}{t_1-1}}\, t_2^{\frac{2t_2}{t_2-1}}}{\left(t_1^2-1\right)\left(t_2^2-1\right)} - \frac{1+6\,t_1+3\,t_1^2}{2\,t_1\,(t_1+1)} \right), \tag{29}$$

$$g_{t_2 t_2} = \frac{4\,(t_1+1)}{(t_1-1)\,(t_2-1)^3} \left(\ln \frac{t_1^{\frac{2t_1}{t_1-1}}\, t_2^{\frac{3(t_2+1)}{t_2-1}}}{\left(t_1^2-1\right)\left(t_2^2-1\right)} - \frac{1+6\,t_2+3\,t_2^2}{2\,t_2\,(t_2+1)} \right), \tag{30}$$

$$g_{t_1 t_2} = g_{t_2 t_1} = \frac{4}{(t_1-1)^2\,(t_2-1)^2} \left(\ln \frac{t_1^{\frac{1+3t_1}{t_1-1}}\, t_1^{\frac{1+3t_2}{t_2-1}}}{\left(t_1^2-1\right)\left(t_2^2-1\right)} - 2 \right). \tag{31}$$

The Riemannian immersion of this manifold in 3-dimensional Euclidean space E^3 is defined by three transition functions $h = (h^1(t_1,t_2),\ h^2(t_1,t_2), h^3(t_1,t_2))$, satisfying the following equations:

$$g_{t_1 t_1}(t_1,t_2) = \left(\frac{\partial h^1}{\partial t_1}\right)^2 + \left(\frac{\partial h^2}{\partial t_1}\right)^2 + \left(\frac{\partial h^3}{\partial t_1}\right)^2, \tag{32}$$

$$g_{t_2 t_2}(t_1,t_2) = \left(\frac{\partial h^1}{\partial t_2}\right)^2 + \left(\frac{\partial h^2}{\partial t_2}\right)^2 + \left(\frac{\partial h^3}{\partial t_2}\right)^2, \tag{33}$$

$$g_{t_1 t_2}(t_1,t_2) = \left(\frac{\partial h^1}{\partial t_1}\right)\left(\frac{\partial h^1}{\partial t_2}\right) + \left(\frac{\partial h^2}{\partial t_1}\right)\left(\frac{\partial h^2}{\partial t_2}\right) + \left(\frac{\partial h^3}{\partial t_1}\right)\left(\frac{\partial h^3}{\partial t_2}\right). \tag{34}$$

Solutions to this non-linear system of first-order PDEs are not easy to obtain. However, we intend to address the problem in a future work, where additional PDF reconstruction techniques will be presented.

5 Conclusion

In this paper, using TFD techniques, we derived explicit expression for the extended renormalized entanglement entropy of a system of closed bosonic strings, vibrating in curved D-dimensional plane wave background. The Hessian of the EEE enabled us to obtain positive defined analytical expressions for the components of the Fisher information metric, which locally measures distances on the parameter space of the given string system.

Our investigation showed that the EEE and FIM are divergent at $t_\mu = 1$, suggesting a critical phase transition point. This point corresponds to $K_\mu = 0$, which puts the system at very high temperatures ($T \to \infty$). In [16] the authors showed that, for $D = 5 + 2$ dimensional pp-wave space-time, the statistical manifold of the string is 2-dimensional and the corresponding Fisher scalar curvature is regular at $T \to \infty$. The latter excludes critical behaviour at high temperature for the string system with $2d$ parameter space. For higher space-time dimensions ($>5 + 2$) and higher parameter space dimensions (>2) additional analysis is required.

Finally, a specific technique for reconstructing PDFs from a given Fisher information metric was presented and supplied with examples. The method leads to a system of non-linear first-order PDEs, which can be solved relatively easy for simple components of the Fisher metric. For complicated metric coefficients the PDEs are also complicated and analytical results for the transition functions are hard to obtain. In this case additional constraints and techniques need to be invoked. Furthermore, by using only one seed metric, the presented reconstruction method generates an infinite number of possible PDFs, parametrized by the set of transition functions it produces.

Acknowledgements This work was partially supported by the Bulgarian NSF grants № DN18/1 and № DM18/1. The support by the Sofia University grants No. 8010-104 and No. 8010-105/19.04.2018 is also gratefully acknowledged.

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Part III
Gravity and Cosmology

The Global Formulation of Generalized Einstein-Scalar-Maxwell Theories

C. I. Lazaroiu and C. S. Shahbazi

Abstract We summarize the global geometric formulation of Einstein-Scalar-Maxwell theories *twisted* by flat symplectic vector bundle which encodes the duality structure of the theory. We describe the scalar-electromagnetic symmetry group of such models, which consists of flat unbased symplectic automorphisms of the flat symplectic vector bundle lifting those isometries of the scalar manifold which preserve the scalar potential. The Dirac quantization condition for such models involves a local system of integral symplectic spaces, giving rise to a bundle of polarized Abelian varieties equipped with a symplectic flat connection, which is defined over the scalar manifold of the theory. Generalized Einstein-Scalar-Maxwell models arise as the bosonic sector of the effective theory of string/M-theory compactifications to four-dimensions, and they are characterized by having non-trivial solutions of "U-fold" type.

Keywords Cosmology · Two-field models · Alpha-attractors · Mathematical physics · Uniformization · Hyperbolic geometry

1 Introduction

Supergravity theories [1, 2] are classical theories of gravity coupled to matter, formulated using systems of "fields" defined on a manifold M of appropriate dimension and subject to certain partial differential equations known as the equations of motion. An

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_16

unambiguous formulation of such theories on non-contractible spacetimes requires that one specifies the global nature of the fields and of the differential operators arising in these equations. Currently, the literature gives only *local* descriptions[1] of these objects. The *globalization problem* asks for globally-unambiguous mathematical definitions of such theories which reduce locally to the known description. The solution of this problem is non-unique since, on a non-contractible space-time, there can be many global definitions of "fields" subject to globally-defined differential equations which reduce to a given local description.

Since supergravity theories contain spinors, their general global formulation involves subtle questions in spin geometry (see [3–5]). In this note, we simplify the globalization problem by ignoring the spinor fields and the supersymmetry conditions, thus considering only the so-called *universal bosonic sector*. The latter arises in any supergravity theory, though it is subject to increasingly stringent supplementary constraints (not discussed in this paper) as the number of supersymmetries present in the theory increases. In addition, we focus exclusively on the case when M is a four-manifold.

In four dimensions, the universal bosonic sector is the so-called Einstein-Scalar-Maxwell (ESM) theory defined on a four-manifold M, which involves gravity (modeled globally by a Lorentzian metric on M), a finite number of real scalar fields (modeled globally by a smooth map from M to a manifold $\mathcal{M}$ of arbitrary dimension) and a finite number of Abelian gauge fields, whose field strengths can be modeled *locally* as 2-forms defined on M. While the local form of ESM theories is well-known, their precise global formulation on arbitrary spacetimes was systematically addressed only recently [6]. It turns out that the naive globalization of the local formulation fails to capture the classical limit of certain string theory backgrounds known as "U-folds" and hence is insufficient for the application of such models to string theory. The geometric description of the classical limit of U-fold backgrounds [7] requires that one globalizes ESM models by including a "twist" of the Abelian gauge field sector through the (pull-back of) a flat symplectic vector bundle defined on $\mathcal{M}$. This produces so-called *generalized ESM models*, which are *locally* indistinguishable from the naive globalization but have non-trivial global behavior. The naive globalization corresponds to using a *trivial* flat symplectic vector bundle on $\mathcal{M}$.

The global mathematical formulation of generalized ESM theories given in [6] is summarized in this note; this construction can be extended further as shown in [8]. We follow the notations and conventions of op. cit. In particular, all manifolds considered are smooth, paracompact and connected and all bundles and maps considered are smooth. In this note, a Lorentzian metric is a smooth metric of signature (3, 1) defined on a four-manifold.

[1]Descriptions which are valid only if one restricts all fields to contractible open subsets of M.

2 Generalized Einstein-Scalar-Maxwell Theories

We start be defining certain mathematical objects which arise in the global formulation.

2.1 Scalar Structures and Related Notions

Definition 2.1 A *scalar structure* is a triplet $\Sigma = (\mathcal{M}, \mathcal{G}, \Phi)$, where $(\mathcal{M}, \mathcal{G})$ is a Riemannian manifold (called the *scalar manifold*) and $\Phi \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ is a smooth real-valued function defined on $\mathcal{M}$ (called the *scalar potential*).

Let $\Sigma = (\mathcal{M}, \mathcal{G}, \Phi)$ be a scalar structure and M be a (generally non-compact) oriented four-manifold.

Definition 2.2 The *modified density* of a smooth map $\varphi \in \mathcal{C}^\infty(M, \mathcal{M})$ relative to a Lorentzian metric $g \in \mathrm{Met}_{3,1}(M)$ and to the scalar structure Σ is the real-valued map defined on M through:

$$e_\Sigma(g, \varphi) \stackrel{\text{def.}}{=} \frac{1}{2}\mathrm{Tr}_g \varphi^*(\mathcal{G}) + \Phi^\varphi \in \mathcal{C}^\infty(M, \mathbb{R}) \quad , \tag{1}$$

where $\Phi^\varphi \stackrel{\text{def.}}{=} \Phi \circ \varphi$ and Tr_g denotes trace taken with respect to g.

Definition 2.3 The *modified tension field* of a smooth map $\varphi \in \mathcal{C}^\infty(M, \mathcal{M})$ relative $g \in \mathrm{Met}_{3,1}(M)$ and to Σ is the section of the pulled-back bundle $(T\mathcal{M})^\varphi$ defined through:

$$\theta_\Sigma(g, \varphi) \stackrel{\text{def.}}{=} \theta_\mathcal{G}(g, \varphi) - (\mathrm{grad}_\mathcal{G}\Phi)^\varphi \in \Gamma(M, (T\mathcal{M})^\varphi)\,. \tag{2}$$

Here $\mathrm{grad}_\mathcal{G}\Phi \in \mathcal{X}(\mathcal{M})$ is the gradient vector field of Φ with respect to $\mathcal{G}$ and $\theta_\mathcal{G}(g, \varphi)$ is the tension field of φ relative to g and $\mathcal{G}$ [9]:

$$\theta_\mathcal{G}(g, \varphi) \stackrel{\text{def.}}{=} \mathrm{Tr}_g \nabla \widetilde{\mathrm{d}\varphi} \in \Omega^0(M, (T\mathcal{M})^\varphi)\,, \tag{3}$$

where $\widetilde{\mathrm{d}\varphi} \in \Omega^1(M, (T\mathcal{M})^\varphi)$ is the $(T\mathcal{M})^\varphi$-valued one-form associated to the differential $\mathrm{d}\varphi\colon TM \to T\mathcal{M}$ and ∇ is the connection induced on $(T\mathcal{M})^\varphi$ by the Levi-Civita connections of g and $\mathcal{G}$.

2.2 Duality Structures

Let N be a manifold.

Definition 2.4 A *duality structure* on N is a flat symplectic vector bundle $\Delta = (\mathcal{S}, D, \omega)$ defined over N, where ω denotes the symplectic pairing on the vector bundle $\mathcal{S}$ and D denotes the ω-compatible flat connection on $\mathcal{S}$.

Definition 2.5 Let $\Delta_i = (\mathcal{S}_i, D_i, \omega_i)$ with $i = 1, 2$ be two duality structures defined on N. A *morphism of duality structures* from Δ_1 to Δ_2 is a based morphism of vector bundles $f \in \mathrm{Hom}(\mathcal{S}_1, \mathcal{S}_2)$ such that $\omega_2(f \otimes f) = \omega_1$ and such that $D_2 \circ f = (\mathrm{id}_{\Omega^1(N)} \otimes f) \circ D_1$.

Duality structures on N form a category denoted $\mathrm{DS}(N)$. Let $\Delta = (\mathcal{S}, D, \omega)$ be a duality structure defined on N such that $\mathrm{rk}\,\mathcal{S} = 2n$. Let Symp be the category of finite-dimensional symplectic vector spaces over $\mathbb{R}$. Let $\mathrm{Symp}^\times$ denote the unit groupoid of this category and $\Pi_1(N)$ denote the fundamental groupoid of N. Let T^Δ_γ denote the parallel transport of D along a path $\gamma : [0, 1] \to N$.

Definition 2.6 The *parallel transport functor* of Δ is the functor $T_\Delta : \Pi_1(N) \to \mathrm{Symp}^\times$ which associates to any point $x \in N$ the symplectic vector space $T_\Delta(x) = (\mathcal{S}_x, \omega_x)$ and to any homotopy class $c \in \Pi_1(N)$ with fixed initial point x and fixed final point y the invertible symplectic morphism $T_\Delta(c) = T^\Delta_\gamma : (\mathcal{S}_x, \omega_x) \xrightarrow{\sim} (\mathcal{S}_y, \omega_y)$, where $\gamma \in \mathcal{P}(N)$ is any path which represents the class c.

Notice that T_Δ can be viewed as a $\mathrm{Symp}^\times$-valued local system defined on N. The map sending Δ to T_Δ is an equivalence between $\mathrm{DS}(N)$ and the functor category $[\Pi_1(N), \mathrm{Symp}^\times]$. This implies that duality structures on N are classified up to isomorphism by the symplectic character variety:

$$C_{\pi_1(N)}(\mathrm{Sp}(2n, \mathbb{R})) \overset{\text{def.}}{=} \mathrm{Hom}(\pi_1(N), \mathrm{Sp}(2n, \mathbb{R}))/\mathrm{Sp}(2n, \mathbb{R}) \ . \tag{4}$$

Definition 2.7 A *duality frame* of Δ is a D-flat symplectic frame $\mathcal{E} \overset{\text{def.}}{=} (e_1, \ldots e_n, f_1, \ldots, f_n)$ of $(\mathcal{S}, \omega)$ defined on an open subset $\mathcal{U} \subset N$.

Definition 2.8 The duality structure Δ is called *trivial* if it is trivial as a flat symplectic vector bundle.

Remark 1 A duality structure is trivial iff it admits a globally-defined duality frame. If N is simply connected, then any duality structure on N is trivial.

2.3 Electromagnetic Structures

Let N be a manifold.

Definition 2.9 An *electromagnetic structure* defined on N is a quadruplet $\Xi \overset{\text{def.}}{=} (\mathcal{S}, D, J, \omega)$, where $(\mathcal{S}, D, \omega)$ is a duality structure defined on N and J is a taming of $(\mathcal{S}, \omega)$.

Remark 2 Notice that we do not require J to be compatible with D. Together with ω, J defines an Euclidean scalar product Q on $\mathcal{S}$ given by $Q(\cdot,\cdot) \stackrel{\text{def.}}{=} \omega(J\cdot,\cdot)$.

Definition 2.10 Let $\Xi_1 = (\mathcal{S}_1, D_1, J_1, \omega_1)$ and $\Xi_2 = (\mathcal{S}_1, D_1, J_1, \omega_1)$ be two electromagnetic structures defined on N. A *morphism of electromagnetic structures* from Ξ_1 to Ξ_2 is a morphism of duality structures $f : (\mathcal{S}_1, D_1, \omega_1) \to (\mathcal{S}_2, D_2, \omega_2)$ such that $J_2 \circ f = f \circ J_1$.

Electromagnetic structures defined on N form a category denoted $\mathrm{ES}(N)$ which fibers over $\mathrm{DS}(N)$; the fiber at a duality structure $\Delta = (\mathcal{S}, D, \omega)$ can be identified with the set $\mathfrak{J}_+(\mathcal{S},\omega)$ of tamings of $(\mathcal{S},\omega)$, which is a contractible topological space. The set of isomorphism classes of $\mathrm{ES}(N)$ fibers over the disjoint union $\sqcup_{n\geq 0} C_{\pi_1(N)}(\mathrm{Sp}(2n,\mathbb{R}))$. Let $\Xi = (\mathcal{S}, D, J, \omega)$ be an electromagnetic structure defined on N and $h = Q + \mathbf{i}\omega$ be the Hermitian scalar product defined by ω and J on $\mathcal{S}$.

Definition 2.11 The *fundamental form* of Ξ is the $End(\mathcal{S})$-valued one-form on N defined through:

$$\Theta_\Xi \stackrel{\text{def.}}{=} D^{\mathrm{ad}}(J) \stackrel{\text{def.}}{=} D \circ J - J \circ D \in \Omega^1(N, End(\mathcal{S})) \ .$$

The electromagnetic structure Ξ is called *unitary* if $\Theta_\Xi = 0$, i.e. if J is parallel with respect to D.

If Ξ is unitary, then D is a unitary connection on the Hermitian vector bundle $(\mathcal{S}, J, h)$. In this case, we have $\mathrm{Hol}_D^x \subset \mathrm{U}(\mathcal{S}_x, J_x, h_x)$ for all $x \in N$. We have a full sub-category of $\mathrm{ES}(N)$ whose objects are the unitary electromagnetic structures. This is equivalent with the category of Hermitian vector bundles which are endowed with a flat $\mathbb{C}$-linear Hermitian connection. In particular, isomorphism classes of *unitary* electromagnetic structures are in bijection with the points of the character variety:

$$C_{\pi_1(N)}(\mathrm{U}(n)) \stackrel{\text{def.}}{=} \mathrm{Hom}(\pi_1(N), \mathrm{U}(n))/\mathrm{U}(n) \ ,$$

where $\mathrm{U}(n)$ acts by conjugation.

2.4 Scalar-Duality and Scalar-Electromagnetic Structures

Definition 2.12 A *scalar-duality structure* is an ordered system (Σ, Ξ), where $\Sigma = (\mathcal{M}, \mathcal{G}, \Phi)$ is a scalar structure and $\Xi = (\mathcal{S}, D, \omega)$ is a duality structure defined on $\mathcal{M}$. A *scalar-electromagnetic structure* is an ordered system $\mathcal{D} = (\Sigma, \Xi)$, where $\Sigma = (\mathcal{M}, \mathcal{G}, \Phi)$ is a scalar structure and $\Xi = (\mathcal{S}, D, J, \omega)$ is an electromagnetic structure defined on $\mathcal{M}$. In this case, the system $\mathcal{D}_0 \stackrel{\text{def.}}{=} (\Sigma, \Xi_0)$ is called the underlying scalar-duality structure, where $\Xi_0 \stackrel{\text{def.}}{=} (\mathcal{S}, D, \omega)$ is the duality structure underlying Ξ.

Let $\mathcal{D}$ be a scalar-electromagnetic structure as in the definition.

Definition 2.13 The *fundamental field* of $\mathcal{D}$ is defined through:

$$\Psi_{\mathcal{D}} \stackrel{\text{def.}}{=} (\sharp_{\mathcal{G}} \otimes \mathrm{id}_{End(\mathcal{S})})(\Theta_{\Xi}) \in \Gamma(\mathcal{M}, T\mathcal{M} \otimes End(\mathcal{S}))\,.$$

2.5 Pulled-Back Electromagnetic Structures

Let $\mathcal{D} = (\Sigma, \Xi)$ be a scalar-electromagnetic structure with underlying scalar structure $\Sigma = (\mathcal{M}, \mathcal{G}, \Phi)$ and underlying electromagnetic structure $\Xi = (\mathcal{S}, D, J, \omega)$. Let M be a four-manifold and $\varphi \in \mathcal{C}^\infty(M, \mathcal{M})$ be a smooth map from M to $\mathcal{M}$.

Definition 2.14 The φ-pullback of the electromagnetic structure Ξ defined on $\mathcal{M}$ is the electromagnetic structure $\Xi^\varphi \stackrel{\text{def.}}{=} (\mathcal{S}^\varphi, D^\varphi, J^\varphi, \omega^\varphi)$ defined on M.

The Hodge operator $*_g : \wedge T^*M \to \wedge T^*M$ of (M, g) induces the endomorphism $\circledast \stackrel{\text{def.}}{=} \circledast_g \stackrel{\text{def.}}{=} *_g \otimes \mathrm{id}_{\mathcal{S}^\varphi}$ of the bundle $\wedge_M(\mathcal{S}^\varphi) \stackrel{\text{def.}}{=} \wedge T^*M \otimes \mathcal{S}^\varphi$.

Definition 2.15 The *twisted Hodge operator* of Ξ^φ is the bundle endomorphism $\star := \star_{g,J^\varphi} \in \mathrm{End}(M, \wedge T^*M \otimes \mathcal{S}^\varphi)$ defined through $\star_{g,J^\varphi} \stackrel{\text{def.}}{=} *_g \otimes J^\varphi = \circledast_g \circ J^\varphi = J^\varphi \circ \circledast_g$.

Let $\alpha \stackrel{\text{def.}}{=} \oplus_{k=0}^4 (-1)^k \mathrm{id}_{\wedge^k T^*M}$ be the *main automorphism* of $\wedge T^*M$. We have:

$$\star^2 = \alpha \otimes \mathrm{id}_{\mathcal{S}^\varphi}\ \ . \tag{5}$$

The operator $\star_{g,J^\varphi}$ preserves the sub-bundle $\wedge^2_M(\mathcal{S}^\varphi) = \wedge^2 T^*M \otimes \mathcal{S}^\varphi$, on which it squares to plus the identity. Accordingly, we have a direct sum decomposition $\wedge^2 T^*M \otimes \mathcal{S}^\varphi = (\wedge^2 T^*M \otimes \mathcal{S}^\varphi)^+ \oplus (\wedge^2 T^*M \otimes \mathcal{S}^\varphi)^-$, where $(\wedge^2 T^*M \otimes \mathcal{S}^\varphi)^\pm$ are the sub-bundles of eigenvectors of $\star$ corresponding to the eigenvalues ± 1.

Definition 2.16 An $\mathcal{S}^\varphi$-valued two-form $\eta \in \Omega^2(M, \mathcal{S}^\varphi)$ defined on M is called *positively polarized* with respect to g and J^φ if it is a section of the vector bundle $(\wedge^2 T^*M \otimes \mathcal{S}^\varphi)^+$, which amounts to the requirement that it satisfies the *positive polarization condition*:

$$\star_{g,J^\varphi}\, \eta = \eta \ \text{ i.e. } \ *_g\, \eta = -J^\varphi \eta\ \ . \tag{6}$$

For any open subset U of M, let $g_U \stackrel{\text{def.}}{=} g|_U$, $\varphi_U \stackrel{\text{def.}}{=} \varphi|_U$ and let $\Omega^{\Xi,g,\varphi}$ be the sheaf of smooth sections of the bundle $(\wedge^2 T^*M \otimes \mathcal{S}^\varphi)^+$. Globally-defined and positively-polarized $\mathcal{S}^\varphi$-valued forms are the global sections of this sheaf. Notice that $\eta \in \Omega^2(M, \mathcal{S}^\varphi)$ is positively polarized iff $\star\eta$ is.

2.6 The Mathematical Formulation of Generalized ESM Theories

Let M be a four-manifold and $\mathcal{D} = (\Sigma, \Xi)$ be a scalar-electromagnetic structure with underlying scalar structure $\Sigma = (\mathcal{M}, \mathcal{G}, \Phi)$ and underlying electromagnetic structure $\Xi = (\mathcal{S}, D, J, \omega)$. The φ-pullback Q^φ of the Euclidean scalar product Q induced by ω and J on $\mathcal{S}$ is a Euclidean scalar product on $\mathcal{S}^\varphi$. Let $\oslash_g : \otimes^4 T^*M \to \otimes^2 T^*M$ be the bundle morphism given by g-contraction of the two middle indices. This is uniquely determined by the condition:

$$(\omega_1 \otimes \omega_2) \oslash (\omega_3 \otimes \omega_4) = (\omega_2, \omega_3)_g \omega_1 \otimes \omega_4 \ \ \forall \omega_1, \omega_2, \omega_3, \omega_4 \ \ \forall \omega \in \Omega^1(\mathcal{M}) \ \ ,$$

where $(\ ,\ )_g$ is the pseudo-Euclidean metric induced by g on $\wedge T^*M$. Viewing $\wedge^2 T^*M$ as the sub-bundle of antisymmetric 2-tensors inside $\otimes^2 T^*M$, this restricts to a morphism of vector bundles $\oslash_g : \wedge^2 T^*M \otimes \wedge^2 T^*M \to \otimes^2 T^*M$, which we call the *inner g-contraction of 2-forms*.

Definition 2.17 The *twisted inner contraction* of $\mathcal{S}^\varphi$-valued 2-forms is the unique morphism of vector bundles $\boldsymbol{\oslash} := \boldsymbol{\oslash}_{g,J,\omega,\varphi} : \wedge^2_M(\mathcal{S}^\varphi) \times_M \wedge^2_M(\mathcal{S}^\varphi) \to \otimes^2(T^*M)$ which satisfies:

$$(\rho_1 \otimes \xi_1) \boldsymbol{\oslash} (\rho_2 \otimes \xi_2) = Q^\varphi(\xi_1, \xi_2) \rho_1 \oslash_g \rho_2$$

for all $\rho_1, \rho_2 \in \Omega^2(M)$ and all $\xi_2, \xi_2 \in \Gamma(M, \mathcal{S}^\varphi)$.

Let $\Psi \stackrel{\text{def.}}{=} \Psi_{\mathcal{D}} \in \Gamma(\mathcal{M}, T\mathcal{M} \otimes End(\mathcal{S}))$ be the fundamental field of $\mathcal{D}$ and let $\Psi^\varphi \in \Gamma(M, (T\mathcal{M})^\varphi \otimes End(\mathcal{S}^\varphi))$ be its pullback through φ. Let $(\ ,\ )$ be the pseudo-Euclidean scalar product induced by g and Q^φ on the vector bundle $\wedge_M(\mathcal{S}^\varphi) \stackrel{\text{def.}}{=} \wedge T^*M \otimes \mathcal{S}^\varphi$. For any vector bundle $\mathcal{T}$ defined on M, we extend this trivially to a $\mathcal{T}$-valued pairing (denoted by the same symbol) between the bundles $\mathcal{T} \otimes \wedge_M(\mathcal{S}^\varphi)$ and $\wedge_M(\mathcal{S}^\varphi)$. Similarly, we trivially extend the twisted wedge product $\boldsymbol{\wedge}_\omega$ defined in Appendix C of Ref. [6] to a $\mathcal{T} \otimes \wedge T^*M$-valued pairing (denoted by the same symbol) between the bundles $\mathcal{T} \otimes \wedge_M(\mathcal{S}^\varphi)$ and $\wedge_M(\mathcal{S}^\varphi)$.

Definition 2.18 The *sheaf of ESM configurations* $\mathrm{Conf}_{\mathcal{D}}$ determined by $\mathcal{D}$ is the sheaf of sets defined on M through:

$$\mathrm{Conf}_{\mathcal{D}}(U) \stackrel{\text{def.}}{=} \{(g, \varphi, \mathcal{V}) | g \in \mathrm{Met}_{3,1}(U), \varphi \in \mathcal{C}^\infty(U, \mathcal{M}), \mathcal{V} \in \Omega^{\Xi, g, \varphi}(U)\}$$

for all open subsets $U \subset M$, with the obvious restriction maps. An element $(g, \varphi, \mathcal{D}) \in \mathrm{Conf}_{\mathcal{D}}(U)$ is called a *local ESM configuration* of type $\mathcal{D}$ defined on U. The *set of global configurations* of type $\mathcal{D}$ is the set $\mathrm{Conf}_{\mathcal{D}}(M)$ of global sections of this sheaf. An element $(g, \varphi, \mathcal{D}) \in \mathrm{Conf}_{\mathcal{D}}(M)$ is called a *global ESM configuration* of type $\mathcal{D}$.

Definition 2.19 The *generalized ESM theory* associated to $\mathcal{D}$ is defined by the following set of partial differential equations on M with unknowns $(g, \varphi, \mathcal{V}) \in \mathrm{Conf}_{\mathcal{D}}(M)$:

1. The *Einstein equation*:
$$\mathrm{G}(g) = \kappa\, \mathrm{T}(g, \varphi, \mathcal{V}) \ , \tag{7}$$
with *energy-momentum tensor* $\mathrm{T}_{\mathcal{D}}$ given by:
$$\mathrm{T}_{\mathcal{D}}(g, \varphi, \mathcal{V}) \stackrel{\text{def.}}{=} g\, e_{\Sigma}(g, \varphi) + 2\, \mathcal{V} \oslash \mathcal{V} - \varphi^*(\mathcal{G})\,. \tag{8}$$

2. The *scalar equations*:
$$\theta_{\Sigma}(g, \varphi) - \frac{1}{2}(*\mathcal{V}, \Psi^{\varphi}\mathcal{V}) = 0 \ . \tag{9}$$

3. The *twisted electromagnetic equations*:
$$\mathrm{d}_{D^{\varphi}}\mathcal{V} = 0 \ , \tag{10}$$
where $\mathrm{d}_{D^{\varphi}} : \Omega^k(M, \mathcal{S}^{\varphi}) \to \Omega^{k+1}(M, \mathcal{S}^{\varphi})$ is the de Rham differential of M twisted by the pulled-back flat connection D^{φ}.

The *sheaf of ESM solutions* $\mathrm{Sol}_{\mathcal{D}}$ of type $\mathcal{D}$ is the sheaf of sets whose sections on an open subset $U \subset M$ is the set of all local solutions defined on U. A *global solution* is a global section of $\mathrm{Sol}_{\mathcal{D}}$.

Remark 3 It is shown in [6] that a generalized ESM model is *locally* indistinguishable from an ordinary ESM model, in the sense that the global partial differential equations (7)–(10) reduce locally to those found in the literature (see [1][2]) upon choosing a local flat symplectic frame of the duality structure $\Delta = (\mathcal{S}, D, \omega)$. Global solutions of generalized ESM theories afford a geometric description of a certain type of classical U-folds, thereby realizing the proposal of [7].

2.7 Sheaves of Scalar-Electromagnetic Configurations and Solutions

Let M and $\mathcal{D}$ be as above and fix a metric $g \in \mathrm{Met}_{3,1}(M)$.

Definition 2.20 The *sheaf of local scalar-electromagnetic configurations* $\mathrm{Conf}^{g}_{\mathcal{D}}$ relative to g is the sheaf of sets defined on M whose set of sections on an open subset $U \subset M$ is defined through:

[2]Notice however that we use different conventions.

$$\mathrm{Conf}^g_{\mathcal{D}}(U) \stackrel{\text{def.}}{=} \{(\varphi, \mathcal{V}) | \varphi \in \mathcal{C}^\infty(U, \mathcal{M}), \mathcal{V} \in \Omega^{\Xi, g, \varphi}(U)\} \ .$$

The *set of global scalar-electromagnetic configurations* relative to g is the set $\mathrm{Conf}^g_{\mathcal{D}}(M)$.

Definition 2.21 The *sheaf of local scalar-electromagnetic solutions* relative to g is the sheaf whose set of sections $\mathrm{Sol}^g_{\mathcal{D}}(U)$ on an open $U \subset M$ consists of all solutions of (9) and (10) defined on U. The *global scalar-electromagnetic solutions* relative to g are the elements of $\mathrm{Sol}^g_{\mathcal{D}}(M)$.

Since it will be of use later, we define:

$$\mathrm{Conf}^g_{\mathcal{D}_0}(M) \stackrel{\text{def.}}{=} \cup_{J \in \mathfrak{J}_+(\mathcal{S}, \omega)} \mathrm{Conf}^g_{(\mathcal{D}_0, J)}(M) \ , \tag{11}$$

where $\mathcal{D}_0$ is a scalar-duality structure.

2.8 Electromagnetic Field Strengths

Definition 2.22 An *electromagnetic field strength* on M with respect to $\mathcal{D}$ and relative to $g \in \mathrm{Met}_{3,1}(M)$ and to the map $\varphi \in \mathcal{C}^\infty(M, \mathcal{M})$ is an $\mathcal{S}^\varphi$-valued 2-form $\mathcal{V} \in \Omega^2(M, \mathcal{S}^\varphi)$ which satisfies the following two conditions:

1. $\mathcal{V}$ is positively polarized with respect to J^φ, i.e. we have $\star_{g, J^\varphi} \mathcal{V} = \mathcal{V}$.
2. $\mathcal{V}$ is d_{D^φ}-closed, i.e.:

$$\mathrm{d}_{D^\varphi} \mathcal{V} = 0 \ . \tag{12}$$

The second condition is called *the electromagnetic equation*.

For any open subset U of M, let:

$$\Omega^{\Xi, g, \varphi}_{\mathrm{cl}}(U) \stackrel{\text{def.}}{=} \{\mathcal{V} \in \Omega^{\Xi, g, \varphi}(U) | \mathrm{d}_{D^\varphi} \mathcal{V} = 0\} \tag{13}$$

denote the set of electromagnetic field strengths defined on U, which is an (infinite-dimensional) subspace of the $\mathbb{R}$-vector space $\Omega^{\Xi, g, \varphi}(U)$. This defines a *sheaf of electromagnetic field strengths* $\Omega^{\Xi, g, \varphi}_{\mathrm{cl}}$ relative to φ and g, which is a locally-constant sheaf of $\mathbb{R}$-vector spaces defined on M.

3 Scalar-Electromagnetic Dualities and Symmetries

Let $\Delta = (\mathcal{S}, D, \omega)$ be a duality structure on $\mathcal{M}$ and J be a taming of $(\mathcal{S}, \omega)$. Let $\Xi = (\mathcal{S}, D, J, \omega)$ be the corresponding electromagnetic structure with underlying duality structure $\Delta = (\mathcal{S}, D, \omega)$.

Definition 3.1 An *unbased* automorphism $f \in \mathrm{Aut}^{\mathrm{ub}}(\mathcal{S})$ is called:

1. A *symmetry of the duality structure* Δ, if f is symplectic with respect to ω and covariantly constant with respect to D.
2. A *symmetry of the electromagnetic structure* Ξ, if f is complex with respect to J and is a symmetry of the duality structure Δ.

Let $\mathrm{Aut}^{\mathrm{ub}}(\Delta) = \mathrm{Aut}^{\mathrm{ub}}(\mathcal{S}, D, \omega)$ and $\mathrm{Aut}^{\mathrm{ub}}(\Xi) = \mathrm{Aut}^{\mathrm{ub}}(\mathcal{S}, D, J, \omega)$ denote the groups of symmetries of Δ and Ξ. We have:

$$\mathrm{Aut}^{\mathrm{ub}}(\Xi) = \mathrm{Aut}^{\mathrm{ub}}(\Delta) \cap \mathrm{Aut}(\mathcal{S}, J) = \mathrm{Aut}^{\mathrm{ub}}(\mathcal{S}, D) \cap \mathrm{Aut}^{\mathrm{ub}}(\mathcal{S}, J, \omega)$$
$$\mathrm{Aut}^{\mathrm{ub}}(\Delta) = \mathrm{Aut}^{\mathrm{ub}}(\mathcal{S}, \omega) \cap \mathrm{Aut}^{\mathrm{ub}}(\mathcal{S}, D) \enspace .$$

Given a symplectic automorphism $f \in \mathrm{Aut}^{\mathrm{ub}}(\mathcal{S}, \omega)$, the endomorphism $\mathbf{Ad}(f)(J)$ is again a taming of $(\mathcal{S}, \omega)$, where $\mathbf{Ad}(f)$ denotes the adjoint action of f on ordinary sections of the bundle $End(\mathcal{S})$ (see [6]). For any electromagnetic structure $\Xi = (\mathcal{S}, D, J, \omega)$ refining Δ, the quadruplet:

$$\Xi_f \stackrel{\text{def.}}{=} (\mathcal{S}, D, \mathbf{Ad}(f)(J), \omega) \tag{14}$$

is again an electromagnetic structure refining Δ. This defines a left action of the group $\mathrm{Aut}^{\mathrm{ub}}(\mathcal{S}, \omega)$ on the set $\mathrm{ES}_\Delta(\mathcal{M})$ of all electromagnetic structures whose underlying duality structure equals Δ.

Let M be a four-manifold and $\mathcal{D} = (\Sigma, \Xi)$ be a scalar-electromagnetic structure with underlying scalar structure $\Sigma = (\mathcal{M}, \mathcal{G}, \Phi)$ and electromagnetic structure $\Xi = (\mathcal{S}, \mathcal{D}, J, \omega)$. Let $\mathcal{D}_0 = (\Sigma, \Delta)$ be the scalar-duality structure underlying $\mathcal{D}$, where $\Delta = (\mathcal{S}, D, \omega)$. Let $g \in \mathrm{Met}_{3,1}(M)$ and:

$$\mathrm{Aut}(\Sigma) \stackrel{\text{def.}}{=} \{\psi \in \mathrm{Iso}(\mathcal{M}, \mathcal{G}) | \Phi \circ \psi = \Phi\} \enspace ,$$

where $\mathrm{Iso}(\mathcal{M}, \mathcal{G})$ denote the isometry group of $(\mathcal{M}, \mathcal{G})$.

Definition 3.2 The *scalar-electromagnetic duality group* of $\mathcal{D}_0$ is the following subgroup of $\mathrm{Aut}^{\mathrm{ub}}(\Delta)$:

$$\mathrm{Aut}(\mathcal{D}_0) \stackrel{\text{def.}}{=} \{f \in \mathrm{Aut}^{\mathrm{ub}}(\Delta) | f_0 \in \mathrm{Aut}(\Sigma)\} \enspace ,$$

an element of which is called a *scalar-electromagnetic duality*. The *duality action* is the action of $\mathrm{Aut}(\mathcal{D}_0)$ on $\mathrm{Conf}^g_{\mathcal{D}_0}(M)$ given by:

$$f \diamond (\varphi, \mathcal{V}) \stackrel{\text{def.}}{=} (f_0 \circ \varphi, \hat{f}^\varphi(\mathcal{V})) \enspace , \quad \forall f \in \mathrm{Aut}(\mathcal{D}_0) \enspace ,$$

where $f_0 \in \mathrm{Diff}(\mathcal{M})$ is the projection of f to $\mathcal{M}$ and $\hat{f} : \mathcal{S} \to \mathcal{S}^{f_0}$ is the based isomorphism of vector bundles induced by f.

Theorem 3.3 ([6]) *For any* $f \in \mathrm{Aut}(\mathcal{D}_0)$*, we have:*

$$f \diamond \mathrm{Sol}^g_{\mathcal{D}}(M) = \mathrm{Sol}^g_{\mathcal{D}_f}(M) \ , \tag{15}$$

where $\mathcal{D}_f \stackrel{\text{def.}}{=} (\Sigma, \Xi_f)$.

Definition 3.4 The *scalar-electromagnetic symmetry group* of $\mathcal{D}$ is the subgroup of $\mathrm{Aut}(\mathcal{D}_0)$ given by:

$$\mathrm{Aut}(\mathcal{D}) \stackrel{\text{def.}}{=} \{f \in \mathrm{Aut}(\mathcal{D}_0) | \mathbf{Ad}(f)(J) = J\} = \{f \in \mathrm{Aut}^{\mathrm{ub}}(\Xi) | f_0 \in \mathrm{Aut}(\Sigma)\} \, .$$

An element of this group is called a *scalar-electromagnetic symmetry*.

Corollary 3.5 *For all* $f \in \mathrm{Aut}(\mathcal{D})$*, we have* $f \diamond \mathrm{Sol}^g_{\mathcal{D}}(M) = \mathrm{Sol}^g_{\mathcal{D}}(M)$*. Thus* $\mathrm{Aut}(\mathcal{D})$ *consists of symmetries of the scalar-electromagnetic equations* (9) *and* (10)*, for any fixed Lorentzian metric* $g \in \mathrm{Met}_{3,1}(M)$.

We have short exact sequences:

$$\begin{aligned} &1 \to \mathrm{Aut}(\Delta) \hookrightarrow \mathrm{Aut}(\mathcal{D}_0) \longrightarrow \mathrm{Aut}^{\Delta}(\Sigma) \to 1 \\ &1 \to \mathrm{Aut}(\Xi) \hookrightarrow \mathrm{Aut}(\mathcal{D}) \longrightarrow \mathrm{Aut}^{\Xi}(\Sigma) \to 1 \ , \end{aligned}$$

where $\mathrm{Aut}(\Delta) \stackrel{\text{def.}}{=} \mathrm{Hom}_{\mathrm{DS}(N)^\times}(\Delta, \Delta)$ and $\mathrm{Aut}(\Xi) \stackrel{\text{def.}}{=} \mathrm{Hom}_{\mathrm{ES}(N)^\times}(\Xi, \Xi)$ consist of *based* automorphisms of Δ and Ξ, while $\mathrm{Aut}^{\Delta}(\Sigma)$ and $\mathrm{Aut}^{\Xi}(\Sigma)$ consist of those automorphisms of Σ which admit lifts to scalar-electromagnetic duality transformations and scalar-electromagnetic symmetries, respectively. Let Hol^p_D be the holonomy group of D at a point $p \in \mathcal{M}$. Then we can identify $\mathrm{Aut}(\Delta)$ with the commutant of Hol^p_D inside the group $\mathrm{Sp}(\mathcal{S}_p, \omega_p) \simeq \mathrm{Sp}(2n, \mathbb{R})$.

4 Twisted Dirac Quantization

Let N be a manifold.

4.1 Integral Duality Structures and Integral Electromagnetic Structures

Definition 4.1 Let $\Delta = (\mathcal{S}, D, \omega)$ be a duality structure of rank $2n$ defined on N. A *Dirac system* for Δ is a fiber sub-bundle $\Lambda \subset \mathcal{S}$ which satisfies the following conditions:

1. For any $x \in X$, the triple $(\mathcal{S}_x, \omega_x, \Lambda_x)$ is an integral symplectic space, i.e. Λ_x is a full lattice in $\mathcal{S}_x$ and $\omega_x(\Lambda_x, \Lambda_x) \subset \mathbb{Z}$.

2. Λ is invariant under the parallel transport of D, i.e. the following condition is satisfied for any path $\gamma \in \mathcal{P}(N)$:

$$T_{\gamma}^{\Delta}(\Lambda_{\gamma(0)}) = \Lambda_{\gamma(1)} \quad . \tag{16}$$

For every $x \in N$, the lattice $\Lambda_x \subset \mathcal{S}_x$ is called the *Dirac lattice defined by Λ at the point x*.

Definition 4.2 An *integral duality structure* defined on N is a pair $\boldsymbol{\Delta} \stackrel{\text{def.}}{=} (\Delta, \Lambda)$, where Δ is is a duality structure defined on N and Λ is a Dirac system for Δ.

Relation (16) implies that the type $\mathbf{t}$ (the ordered list of elementary divisors) of the integral symplectic space $(\mathcal{S}_x, \omega_x, \Lambda_x)$ does not depend on the point $x \in N$. This quantity is denoted $\mathbf{t}(\boldsymbol{\Delta})$ and called the *type* of $\boldsymbol{\Delta}$.

Definition 4.3 Let $\boldsymbol{\Delta} = (\Delta_1, \Lambda_1)$ and $\boldsymbol{\Delta}_2 = (\Delta_2, \Lambda_2)$ be two integral duality structures defined on N. An morphism of of integral duality structures from $\boldsymbol{\Delta}_1$ to $\boldsymbol{\Delta}_2$ is a morphism of duality structures $f : \Delta_1 \to \Delta_2$ such that $f(\Lambda_1) \subset \Lambda_2$.

Remark 4 The set of isomorphism classes of integral duality structures of type $\mathbf{t}$ defined on N is in bijection with the character variety:

$$C_{\pi_1(N)}(\mathrm{Sp}_{\mathbf{t}}(2n, \mathbb{Z})) = \mathrm{Hom}(\pi_1(N), \mathrm{Sp}_{\mathbf{t}}(2n, \mathbb{Z}))/\mathrm{Sp}_{\mathbf{t}}(2n, \mathbb{Z}) \quad ,$$

where $\mathrm{Sp}_{\mathbf{t}}(2n, \mathbb{Z})$ is the modified Siegel modular group of type $\mathbf{t}$.

Let $\boldsymbol{\Delta} \stackrel{\text{def.}}{=} (\mathcal{S}, D, \omega, \Lambda)$ be an integral duality structure of rank $2n$ and type $\mathbf{t}$, defined on N. For any $x \in N$, the integral symplectic space $(\mathcal{S}_x, \omega_x, \Lambda_x)$ defines a symplectic torus $X_s(\mathcal{S}_x, \omega_x, \Lambda_x)$. These tori fit into a fiber bundle $\mathcal{X}_s(\boldsymbol{\Delta})$ endowed with a complete flat Ehresmann connection $\mathcal{H}_{\boldsymbol{\Delta}}$ induced by D. The Ehresmann transport of this connection preserves the group structure and symplectic form of the fibers; in particular, the holonomy group of $\mathcal{H}_{\boldsymbol{\Delta}}$ is contained in $\mathrm{Sp}_{\mathbf{t}}(2n, \mathbb{Z})$.

Definition 4.4 The pair $(\mathcal{X}_s(\boldsymbol{\Delta}), \mathcal{H}_{\boldsymbol{\Delta}})$ is called the *flat bundle of symplectic tori* defined by the integral duality structure $\boldsymbol{\Delta}$.

Definition 4.5 An *integral electromagnetic structure* defined on N is a pair $\boldsymbol{\Xi} = (\Xi, \Lambda)$, where $\Xi = (\Delta, J)$ is an electromagnetic structure and Λ is a Dirac system for the underlying duality structure $\Delta = (\mathcal{S}, D, \omega)$. The type of $\boldsymbol{\Delta} = (\mathcal{S}, D, \omega, \Lambda)$ is called the *type* $\mathbf{t}(\boldsymbol{\Xi})$ of $\boldsymbol{\Xi}$:

$$\mathbf{t}(\boldsymbol{\Xi}) \stackrel{\text{def.}}{=} \mathbf{t}(\boldsymbol{\Delta}) \quad .$$

Let $\boldsymbol{\Xi} = (\mathcal{S}, D, J, \omega, \Lambda)$ be an integral electromagnetic structure of rank $2n$ and type $\mathbf{t}$, with underlying duality structure $\Delta = (\mathcal{S}, D, \omega)$. For every $x \in N$, the fiber

$(\mathcal{S}_x, J_x, \omega_x, \Lambda_x)$ is an integral tamed symplectic space which defines a polarized Abelian variety $X_h(\mathcal{S}_x, J_x, \omega_x, \Lambda_x)$ of type $\mathbf{t}$, whose underlying symplectic torus is given by $X_s(\mathcal{S}_x, \omega_x, \Lambda_x)$. These Abelian varieties fit into a smooth fiber bundle $\boldsymbol{\mathcal{X}}_h(\boldsymbol{\Xi})$. As above, the connection D induces a complete integrable Ehresmann connection $\mathcal{H}_{\boldsymbol{\Xi}} \stackrel{\text{def.}}{=} \mathcal{H}_{\boldsymbol{\Delta}}$ on this bundle, whose transport reserves the Abelian group structure and symplectic form of the fibers but not their complex structure.

Definition 4.6 The pair $(\mathcal{X}_h(\boldsymbol{\Xi}), \mathcal{H}_{\boldsymbol{\Xi}})$ is called the *bundle of polarized Abelian varieties* defined by the integral electromagnetic structure $\boldsymbol{\Xi}$.

4.2 The Twisted Dirac Quantization Condition

Let (M, g) be a Lorentzian four-manifold and $(\mathcal{M}, \mathcal{G})$ be a Riemannian manifold. Let $\varphi \in \mathcal{C}^\infty(M, \mathcal{M})$. Let $\boldsymbol{\Xi} = (\Xi, \Lambda)$ be an integral electromagnetic structure defined on $\mathcal{M}$, with underlying electromagnetic structure $\Xi = (\mathcal{S}, D, J, \omega)$ and underlying duality structure $\Delta = (\mathcal{S}, D, \omega)$. Then $\boldsymbol{\Xi}^\varphi \stackrel{\text{def.}}{=} (\Xi^\varphi, \Lambda^\varphi)$ is an integral electromagnetic structure on M, where Λ^φ is the φ-pullback of the fiber sub-bundle $\Lambda \subset \mathcal{S}$; this refines the duality structure $\Delta^\varphi = (\mathcal{S}^\varphi, D^\varphi, \omega^\varphi)$. Let $\boldsymbol{\Delta}^\varphi \stackrel{\text{def.}}{=} (\Delta^\varphi, \Lambda^\varphi)$. Let Symp_0 denote the category of finite-dimensional integral symplectic vector spaces. Let $H^\bullet(M, \boldsymbol{\Delta}^\varphi)$ denote the total twisted singular cohomology group of M with coefficients in the $\mathrm{Symp}_0^\times$-valued local system $T_{\boldsymbol{\Delta}^\varphi}$ and let $H^\bullet(M, \Delta^\varphi)$ denote the total twisted singular cohomology space of M with coefficients in the $\mathrm{Symp}^\times$-valued local system T_{Δ^φ}. The latter can be identified with the total cohomology space $H^\bullet_{\mathrm{d}_{D^\varphi}}(M, \mathcal{S}^\varphi)$ of the twisted de Rham complex $(\Omega^\bullet(M, \mathcal{S}^\varphi), \mathrm{d}_{D^\varphi})$. Since $\mathcal{S}^\varphi = \Lambda^\varphi \otimes_{\mathbb{Z}} \mathbb{R}$, the coefficient sequence gives a map $j_* : H^\bullet(M, \boldsymbol{\Delta}^\varphi) \to H^\bullet(M, \Delta^\varphi)$, whose image $H^\bullet_{\Lambda^\varphi}(M, \Delta^\varphi) \stackrel{\text{def.}}{=} j_*(H^\bullet(M, \boldsymbol{\Delta}^\varphi))$ is a graded subgroup of the graded additive group $H^\bullet(M, \Delta^\varphi)$.

Definition 4.7 An electromagnetic field $\mathcal{V} \in \Omega^2(M, \mathcal{S}^\varphi)$ is called Λ^φ*-integral* if its D^φ-twisted cohomology class $[\mathcal{V}] \in H^2_{\mathrm{d}_{D^\varphi}}(M, \mathcal{S}^\varphi) \equiv H^2(M, \Delta^\varphi)$ belongs to $H^2_{\Lambda^\varphi}(M, \Delta^\varphi)$:

$$[\mathcal{V}] \in H^2_{\Lambda^\varphi}(M, \Delta^\varphi) = j_*(H^2(M, \boldsymbol{\Delta}^\varphi)) \ . \tag{17}$$

Condition (17) is called the *twisted Dirac quantization condition* defined by the Dirac structure Λ. This condition constrains semiclassical Abelian gauge field configurations; a mathematical model for such configurations can be given using a certain version of twisted differential cohomology.

4.3 Integral Scalar-Electromagnetic Duality and Symmetry Groups

Definition 4.8 An *integral scalar-duality structure* is a pair $\boldsymbol{\mathcal{D}}_0 \stackrel{\text{def.}}{=} (\mathcal{D}_0, \Lambda)$, where $\mathcal{D}_0 = (\Sigma, \Delta)$ is a scalar-duality structure and Λ is a Dirac system for Δ. An *integral scalar-electromagnetic structure* is a pair $\boldsymbol{\mathcal{D}} = (\mathcal{D}, \Lambda)$, where $\mathcal{D} = (\Sigma, \Xi)$ is a scalar-electromagnetic structure and Λ is a Dirac system for the underlying duality structure of the electromagnetic structure Ξ.

Let $\boldsymbol{\mathcal{D}} = (\mathcal{D}, \Lambda)$ be an integral scalar-electromagnetic structure with underlying scalar-electromagnetic structure $\mathcal{D} = (\Sigma, \Xi)$, where $\Sigma = (\mathcal{M}, \mathcal{G}, \Phi)$ and $\Xi \stackrel{\text{def.}}{=} (\mathcal{S}, D, J, \omega)$. Let $\Delta = (\mathcal{S}, D, \omega)$ be the underlying duality structure and let $\boldsymbol{\Delta} = (\Delta, \Lambda)$ and $\boldsymbol{\Xi} = (\Xi, \Lambda)$ be the underlying integral duality structure and integral electromagnetic structure. Let $\mathcal{D}_0 = (\Sigma, \Delta)$ be the underlying scalar-duality structure and $\boldsymbol{\mathcal{D}}_0 = (\mathcal{D}_0, \Lambda)$ be the underlying integral scalar-duality structure.

Definition 4.9 The *integral scalar-electromagnetic duality group* defined by the integral scalar-duality structure $\boldsymbol{\mathcal{D}}_0$ is the following subgroup of the scalar-electromagnetic duality group $\mathrm{Aut}(\mathcal{D}_0)$:

$$\mathrm{Aut}(\boldsymbol{\mathcal{D}}_0) \stackrel{\text{def.}}{=} \{f \in \mathrm{Aut}(\mathcal{D}_0) | f(\Lambda) = \Lambda\} \subset \mathrm{Aut}(\mathcal{D}_0) \ ,$$

elements of which are called *integral scalar-electromagnetic dualities*. The *integral scalar-electromagnetic symmetry group* of $\boldsymbol{\mathcal{D}}$ is the following subgroup of the scalar-electromagnetic symmetry group $\mathrm{Aut}(\mathcal{D})$:

$$\mathrm{Aut}(\boldsymbol{\mathcal{D}}) \stackrel{\text{def.}}{=} \{f \in \mathrm{Aut}(\mathcal{D}) | f(\Lambda) = \Lambda\} \subset \mathrm{Aut}(\mathcal{D}) \ ,$$

elements of which are called *integral scalar-electromagnetic symmetries*.

Notice that $\mathrm{Aut}(\boldsymbol{\mathcal{D}})$ is a subgroup of $\mathrm{Aut}(\boldsymbol{\mathcal{D}}_0)$. These groups are highly sensitive to global topological data.

Acknowledgements The authors thank Tomas Ortin for discussions and correspondence. The work of C. I. L. was supported by grant IBS-R003-S1. The work of C.S.S. was supported by the ERC Starting Grant 259133 – Observable String.

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Two-Field Cosmological Models and the Uniformization Theorem

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Abstract We propose a class of two-field cosmological models derived from gravity coupled to non-linear sigma models whose target space is a non-compact and geometrically-finite hyperbolic surface, which provide a wide generalization of so-called α-attractor models and can be studied using uniformization theory. We illustrate cosmological dynamics in such models for the case of the hyperbolic triply-punctured sphere.

Keywords Non-linear sigma models · Cosmology · Inflation · Hyperbolic geometry

1 Introduction

Inflation in the early universe can be described reasonably well by so-called cosmological α-attractor models [1–3], which provide a good fit to current observational results. The observational predictions of these models are to a large extent determined by the geometry of the scalar manifold rather than by the scalar potential.

The best studied α-attractor models contain a single scalar field, being obtained by radial truncation of two-field models based on the hyperbolic disk [1]. The latter arise from cosmological solutions of 4-dimensional gravity coupled to a non-linear sigma model whose scalar manifold is the open unit disk endowed with its unique complete metric $\mathcal{G}$ of constant negative Gaussian curvature $K(\mathcal{G}) = -\frac{1}{3\alpha}$, where α is a positive

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parameter. As shown in [1], the "universal" behavior of such models in the radial one-field truncation close to the conformal boundary of the disk is a consequence of the hyperbolic character of $\mathcal{G}$, when the scalar potential is "well-behaved" near the conformal boundary. It is thus natural to consider *two-field* α-attractor models in which the hyperbolic disk is replaced by an arbitrary hyperbolic surface Σ which is geometrically finite in the sense that its fundamental group is finitely-generated.

Definition 1 ([4]) A *generalized two-field α-attractor model* is defined by a triplet $(\Sigma, \mathcal{G}, V)$, where $(\Sigma, \mathcal{G})$ is a complete geometrically-finite hyperbolic surface and $V : \Sigma \to \mathbb{R}$ is a smooth potential function, while $K(\mathcal{G}) = -\frac{1}{3\alpha}$ with $\alpha > 0$.

This class of models is very rich. Since in general Σ has non-trivial topology, a complete understanding requires going beyond one field truncations. Instead, one can use the theoretical and numerical methods of [5, 6] and certain other approximation techniques [4].

2 Cosmological Models with Two Real Scalar Fields Minimally Coupled to Gravity

Let us recall the general description of cosmological models with two real scalar fields minimally coupled to gravity, allowing for scalar manifolds of non-trivial topology, in a global and coordinate-free description.

2.1 Einstein–Scalar Theories with 2-Dimensional Scalar Manifolds

Let $(\Sigma, \mathcal{G})$ be any oriented, connected, complete and possibly non-compact two-dimensional Riemannian manifold without boundary called the *scalar manifold* and $V : \Sigma \to \mathbb{R}$ be a smooth function called the *scalar potential*. We require completeness of the metric $\mathcal{G}$ in order to avoid problems with conservation of energy. For applications to cosmology, it is important to allow (Σ, G) to be non-compact and of possibly infinite area.

Any triplet $(\Sigma, \mathcal{G}, V)$ as above allows one to define an Einstein–Scalar theory on any four-dimensional oriented manifold X which admits Lorentzian metrics. This theory includes 4-dimensional gravity (described by a Lorentzian metric g defined on X) and a smooth map $\varphi : X \to \Sigma$ (which locally describes two real scalar fields), with action:

$$S[g, \varphi] = \int_X \mathcal{L}(g, \varphi)\mathrm{vol}_g \ , \tag{1}$$

where vol_g is the volume form of (X, g) and $\mathcal{L}(g, \varphi)$ is the Lagrange density:

$$\mathcal{L}(g, \varphi) = \frac{M^2}{2}\mathrm{R}(g) - \frac{1}{2}\mathrm{Tr}_g\varphi^*(\mathcal{G}) - V \circ \varphi \ . \qquad (2)$$

Here $\mathrm{R}(g)$ is the scalar curvature of g and M is the reduced Planck mass. The quantity $\varphi^*(\mathcal{G})$ is the pull-back through φ of the metric $\mathcal{G}$, while $\mathrm{Tr}_g\varphi^*(\mathcal{G})$ denotes the trace of the tensor field of type $(1, 1)$ obtained by raising one of the indices of $\varphi^*(\mathcal{G})$ using the metric g. The coordinate-free formulation (2) allows one to define such a theory globally for any topology of Σ and X. The last two terms in the Lagrange density (2) describe the non-linear sigma model with source (X, g), target space $(\Sigma, \mathcal{G})$ and scalar potential V.

2.2 Cosmological Models Defined by $(\Sigma, \mathcal{G}, V)$

By definition, a *cosmological model* defined by $(\Sigma, \mathcal{G}, V)$ is a solution of the equations of motion of the theory (1)–(2) when (X, g) is a FLRW universe and φ depends only on the cosmological time. We assume for simplicity that the spatial section is flat and simply connected. Hence the cosmological models of interest are defined by the following conditions:

1. X is diffeomorphic with $\mathbb{R}^4$, with global coordinates (t, x^1, x^2, x^3).
2. The squared line element of g has the form:

$$\mathrm{d}s_g^2 = -\mathrm{d}t^2 + a(t)^2\sum_{i=1}^{3}(\mathrm{d}x^i)^2 \quad , \quad \text{with } \ \mathrm{a(t)} > 0 \ .$$

3. φ depends only on t.
4. $(a(t), \varphi(t))$ are such that (g, φ) is a solution of the equations of motion derived from (1).

With these assumptions, the cosmological equations of motion are:

$$\nabla_t\dot{\varphi} + 3H\dot{\varphi} + (\mathrm{grad}_{\mathcal{G}}V) \circ \varphi = 0 \ , \ \ \frac{1}{3}\dot{H} + H^2 - \frac{V \circ \varphi}{3M^2} = 0 \ , \ \ \dot{H} + \frac{\dot{\sigma}^2}{2M^2} = 0 \ ,$$

where $\dot{} \overset{\mathrm{def.}}{=} \frac{\mathrm{d}}{\mathrm{d}t}$, $\nabla_t \overset{\mathrm{def.}}{=} \nabla_{\dot{\varphi}(t)}$ is the covariant derivative with respect to $\dot{\varphi}(t)$, σ is the proper length parameter on the curve $\varphi(t)$, while $H \overset{\mathrm{def.}}{=} \frac{\dot{a}}{a}$ denotes the Hubble parameter. The *inflationary regions* of a trajectory $\varphi(t)$ are defined as the time intervals for which the scale factor $a(t)$ is a convex and increasing function of t ($\dot{a} > 0, \ \ddot{a} > 0$) and are given by the condition:

$$H(t) < H_c(\varphi(t)) \ ,$$

where $H_c(p) \stackrel{\text{def.}}{=} \frac{1}{M}\sqrt{\frac{V(p)}{2}}$ is the *critical Hubble parameter* at a point $p \in \Sigma$. Approximations useful for studying such models are discussed in [4].

2.3 Two-Field Generalized α-Attractor Models

By definition, a *hyperbolic surface* is a connected, oriented, borderless and complete Riemannian two-manifold (Σ, G) of constant Gaussian curvature equal to -1. A *two-field generalized* α*-attractor model* is a two-field cosmological model defined by a triple $(\Sigma, \mathcal{G}, V)$ as above, where $\mathcal{G} = 3\alpha G$ with α a positive parameter and (Σ, G) is a hyperbolic surface.

3 Uniformization of Hyperbolic Surfaces

An isometric model of the Poincaré disk is provided by the Poincaré half-plane, defined as the upper half-plane $\mathbb{H} \stackrel{\text{def.}}{=} \{\tau \in \mathbb{C} | \Im\tau > 0\}$ endowed with its unique complete hyperbolic metric $\mathrm{d}s^2_{\mathbb{H}} = \lambda^2_{\mathbb{H}}(\tau, \bar{\tau})\mathrm{d}\tau^2$, where $\lambda_{\mathbb{H}}(\tau, \bar{\tau}) = \frac{1}{\Im\tau}$. The orientation-preserving isometries of $\mathbb{H}$ form the projective special linear group $\mathrm{PSL}(2, \mathbb{R})$. An element $A \in \mathrm{PSL}(2, \mathbb{R})$ is called *elliptic* if $|\mathrm{tr}(A)| < 2$. By definition, a *surface group* is a discrete subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ which contains no elliptic elements. Our analysis of generalized α-attractor models is based on the uniformization theorem [7]:

Theorem 1 *For any hyperbolic surface* (Σ, G) *there is a surface group* Γ *and a holomorphic covering map (uniformization map)* $\pi_{\mathbb{H}} : \mathbb{H} \longrightarrow \Sigma$ *defined on the Poincaré half-plane* $\mathbb{H}$ *such that* Σ *is isometric to the quotient* $\mathbb{H}/\Gamma$.

In this theorem, holomorphicity of $\pi_{\mathbb{H}}$ is understood with respect to the complex structure J induced on Σ by the conformal class of G.

3.1 Lifted Trajectories and Tilings

Consider the generalized α-attractor model defined by a hyperbolic surface (Σ, G) at a fixed positive value of the parameter α. To study the cosmological trajectories $\varphi : \mathfrak{I} \longrightarrow \Sigma$ (where $\mathfrak{I} \subset \mathbb{R}$ is an interval), it is convenient to first study their lifts $\widetilde{\varphi} : \mathfrak{I} \longrightarrow \mathbb{H}$ to the hyperbolic plane and then project them back to Σ as $\varphi = \pi_{\mathbb{H}} \circ \widetilde{\varphi}$. The projection from $\mathbb{H}$ to Σ can be determined if we know the tiling of $\mathbb{H}$ determined by a fundamental polygon of Γ. There is no fully general stopping algorithm known for computing fundamental polygons of surface groups. However, a general algorithm [8] is known when Γ is an arithmetic surface group such that $\mathbb{H}/\Gamma$ has finite hyperbolic area. The connection to uniformization theory shows that the study of generalized

α-attractor models requires sophisticated results from uniformization theory. When Σ has finite hyperbolic area, this is closely connected to the theory of modular forms and hence to number theory.

3.2 The End and Conformal Compactifications

A non-compact hyperbolic surface (Σ, G) has two natural compactifications, namely the *end compactification* [9] of Freudenthal and Kerekjarto–Stoilow (which depends only on the topology of Σ) and the *conformal compactification* (which depends on the conformal class of the hyperbolic metric G). In the geometrically-finite case, these two compactifications can be described as follows:

- Since $\pi_1(\Sigma)$ is finitely-generated, Σ is homeomorphic with $\widehat{\Sigma} \setminus \{p_1, \ldots, p_n\}$, where $\widehat{\Sigma}$ is a closed oriented surface and $p_1, \ldots, p_n$ are distinct points on $\widehat{\Sigma}$. The compact surface $\widehat{\Sigma}$ can be identified with the *end compactification* of Σ, while the points $p_1, \ldots, p_n$ can be identified with the *ends* of Σ.
- As shown by Maskit, the *conformal compactification* $\bar{\Sigma}$ of Σ (with respect to the complex structure J) can be identified with the topological closure of Σ inside a closed Riemann surface which is obtained from Σ by adding a finite number n_c of points and a finite number n_f of disks, where $n_f + n_c = n$. The topological boundary $\partial_\infty \Sigma = \bar{\Sigma} \setminus \Sigma$ consists of n_c isolated points and n_f disjoint simple closed curves and is called the *conformal boundary* of Σ. Contracting each of the n_f curves to a point recovers the end compactification, the n_c isolated points and the n_f contraction points recovering the ends of Σ. Accordingly, the ends of Σ divide into n_c *cusp ends* (those corresponding to points in the conformal compactification) and n_f *flaring ends* (those corresponding to simple closed curves in the conformal compactification).

On the neighborhoods of each end, the hyperbolic metric can be brought to one of four explicitly known forms (namely for the “cusp”, “funnel”, “horn” or “plane” end), thus providing the *isometric classification of ends*.

3.3 Well-Behaved Scalar Potentials

Let $\widehat{\Sigma}$ be the end compactification of Σ. A scalar potential $V : \Sigma \rightarrow \mathbb{R}$ is called *well-behaved* at an end $p \in \widehat{\Sigma} \setminus \Sigma$ if there exists a smooth function $\widehat{V}_p : \Sigma \sqcup \{p\} \rightarrow \mathbb{R}$ such that $V = \widehat{V}_p|_\Sigma$. The potential V is called *globally well-behaved* if it is well-behaved at each end of Σ, i.e. if there exists a globally-defined smooth function $\widehat{V} : \widehat{\Sigma} \rightarrow \mathbb{R}$ such that $V = \widehat{V}|_\Sigma$.

3.4 Behavior Near the Ends

The cosmological equations of motion in semi-geodesic coordinates (r, θ) on an appropriate vicinity of an end $p \in \widehat{\Sigma} \setminus \Sigma$ reduce to [4]:

$$\ddot{r} - 3\epsilon_p \alpha \left(\frac{C_p}{4\pi}\right)^2 e^{2\epsilon_p r}\dot{\theta}^2 + 3H\dot{r} + \frac{1}{3\alpha}\partial_r V = 0 \ ,$$
$$\ddot{\theta} + 2\epsilon_p \dot{r}\dot{\theta} + 3H\dot{\theta} + \frac{1}{3\alpha}\left(\frac{4\pi}{C_p}\right)^2 e^{-2\epsilon_p r}\partial_\theta V = 0 \ ,$$

where C_p and ϵ_p are known constants depending on the type of end. Since θ is periodic, a generic trajectory will spiral around the ends for any well-behaved scalar potential. Using an argument similar to that of [3], we showed in [4] that generalized α-attractor models have the same kind of "universal" behavior as the disk models of [1] in the *naive* one field truncation near each end obtained by fixing θ. The cosmological behavior away from the ends is much more subtle than that of ordinary α-attractors; some of its qualitative features were discussed in [4]. Various examples are discussed in [10, 11].

4 Examples of Trajectories for the Hyperbolic Triply Punctured Sphere

Consider those generalized α-attractor models for which the scalar manifold Σ is the triply-punctured Riemann sphere (a.k.a. the modular curve) $Y(2) \stackrel{\text{def.}}{=} \mathbb{CP}^1 \setminus \{p_1, p_2, p_3\}$. This is diffeomorphic with the doubly-punctured complex plane endowed with its complete hyperbolic metric $\mathrm{d}s^2 = \rho(\zeta, \bar{\zeta})^2 \mathrm{d}\zeta^2$, where:

$$\rho(\zeta, \bar{\zeta}) = \frac{\pi}{8|\zeta(1-\zeta)|}\frac{1}{\mathrm{Re}[\mathcal{K}(\zeta)\mathcal{K}(1-\bar{\zeta})]} \ , \ \mathcal{K}(\zeta) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-\zeta t^2)}} \ .$$

Each of the three punctures p_i corresponds to a cusp end, so the end compactification is $\widehat{\Sigma} = \mathbb{CP}^1 \simeq \mathrm{S}^2$. The surface $Y(2)$ is uniformized by the principal congruence subgroup of level 2, $\Gamma(2) = \left\{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{Z}) \mid a, d = \text{odd}\ , \ b, c = \text{even}\right\}$, with uniformization map given by the elliptic modular lambda function $\pi_{\mathbb{H}}(\tau) \equiv \lambda(\tau) = \frac{\wp_\tau(\frac{1+\tau}{2}) - \wp_\tau(\frac{\tau}{2})}{\wp_\tau(\frac{1}{2}) - \wp_\tau(\frac{\tau}{2})}$, where $\wp$ is the Weierstrass elliptic function of modulus τ. A fundamental polygon for the action of $\Gamma(2)$ on $\mathbb{H}$ is given by the hyperbolic quadrilateral $\mathfrak{D}_{\mathbb{H}} = \{\tau \in \mathbb{H} | -1 < \mathrm{Re}\tau < 0, |\tau + \frac{1}{2}| > \frac{1}{2}\} \cup \{\tau \in \mathbb{H} | 0 \le \mathrm{Re}\tau < 1, |\tau - \frac{1}{2}| > \frac{1}{2}\}$.

Consider the following two globally well-behaved scalar potentials:

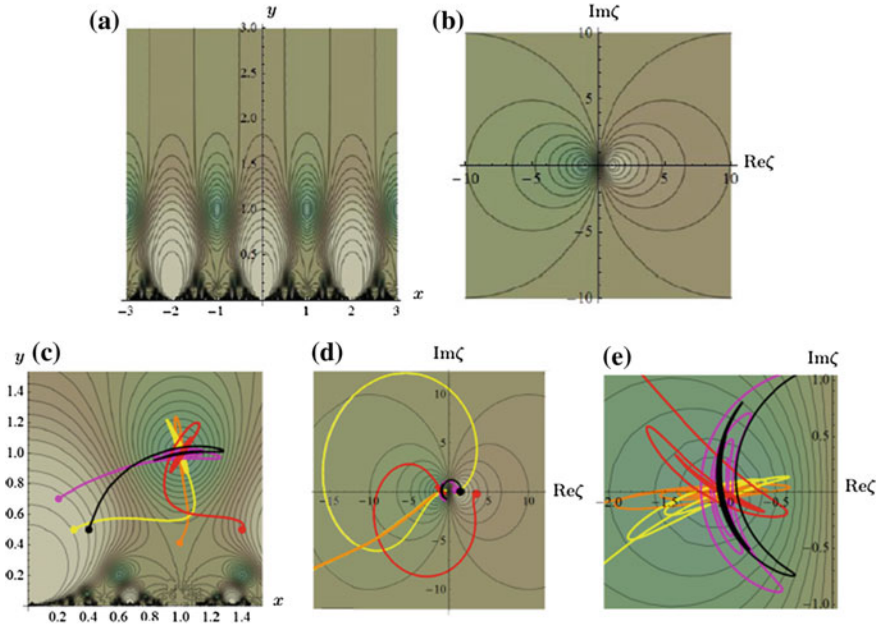

Fig. 1 **a** Level plot of the lifted potential $\widetilde{V}_0$ on $\mathbb{H}$. **b** Level plot of V_0 on $Y(2)$. **c** Lifted trajectories on $\mathbb{H}$, with initial conditions given in Table 1. **d** Projected trajectories on $Y(2)$, where the orange trajectory is too long to fit into the plot at the scale shown. **e** Detail of the spiral ends of the trajectories on $Y(2)$. The beginning points of the trajectories are indicated by fat dots. In all figures, dark green indicates minima of the potential while light brown indicates maxima

$$\widehat{V}_0(\psi,\theta) \stackrel{\text{def.}}{=} M_0(1+\sin\psi\cos\theta) \ ,$$
$$\widehat{V}_+(\psi) \stackrel{\text{def.}}{=} M_0\cos^2\frac{\psi}{2} \ ,$$

where $M_0 \stackrel{\text{def.}}{=} M\sqrt{2/3}$ and ψ,θ are spherical coordinates on the end compactification $\widehat{\Sigma} = \mathrm{S}^2$. Fixing $\alpha = \frac{1}{3}$ and choosing the initial conditions τ_0 and $\widetilde{v}_0 \stackrel{\text{def.}}{=} \dot{\tilde{\varphi}}(t_0)$ given in Table 1, we compute [10] the lifted trajectories on the Poincaré half-plane with coordinate $\tau = x + \mathbf{i}y$ and then project them to $Y(2)$ (see Figs. 1 and 2). The potentials $\widehat{V}_0$ and respectively $\widehat{V}_+$ correspond to $\widetilde{V}_0$ and $\widetilde{V}_+$ on $\mathbb{H}$ and to V_0 and V_+ on $Y(2)$.

For the potential $\widetilde{V}_0$, we find that the red, magenta, yellow and orange trajectories start in inflationary regime (see Fig. 3), but computations show they have small number of e-folds (less than 5); on the other hand, the black trajectory is not inflationary. For potential $\widetilde{V}_+$, we find that the red, yellow and orange trajectories (see Fig. 4) start in inflationary regime, while the magenta and black trajectories are not inflationary. The orange trajectory has 50 e-folds and using very small variations of its initial conditions given in Table 1 we can easily find other trajectories with 50–60 e-folds;

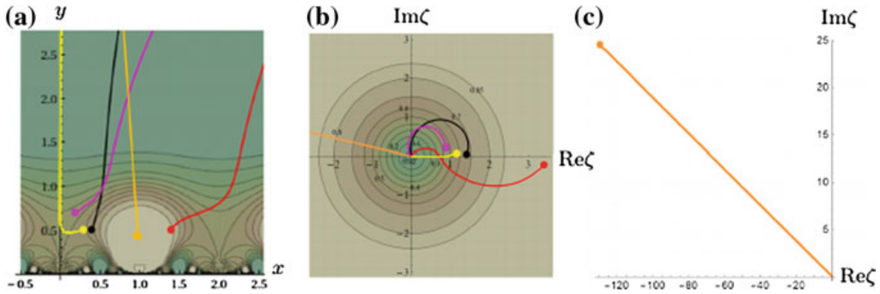

Fig. 2 **a** Level plots of the lifted potential $\widetilde{V}_+$ on $\mathbb{H}$ and the lifted trajectories with initial conditions given in Table 1. **b** Level plots of V_+ on $Y(2)$ and the corresponding projected trajectories. **c** The full orange trajectory projected on $Y(2)$

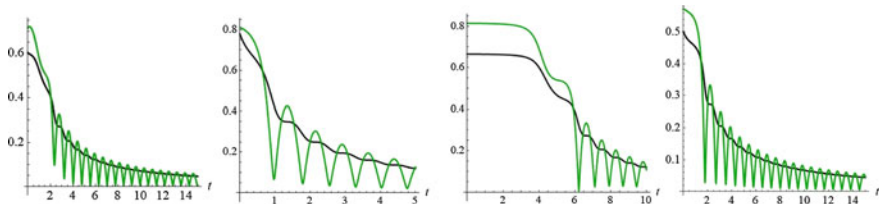

Fig. 3 Plot of $H(t)/\sqrt{M_0}$ (black) and $H_c(t)/\sqrt{M_0}$ (green) for the red, magenta, yellow and orange trajectories for the potential $\widetilde{V}_0$

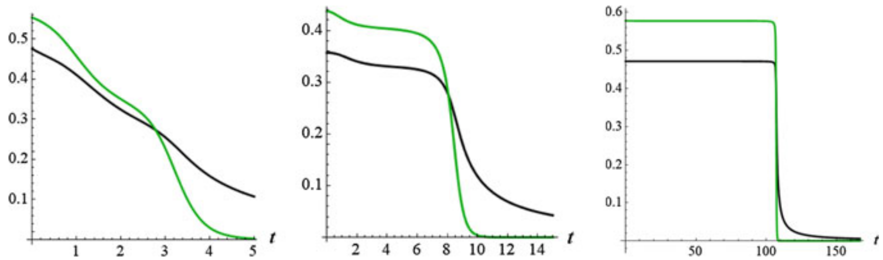

Fig. 4 Plot of $H(t)/\sqrt{M_0}$ (black) and $H_c(t)/\sqrt{M_0}$ (green) for the red, yellow and orange trajectories in the potential $\widetilde{V}_+$. The red and yellow trajectory have small number of e-folds (less than 2), but the orange trajectory has 50 e-folds

this shows that generalized α attractors with $\Sigma = Y(2)$ can produce phenomenologically realistic predictions. The number of e-folds is given by $N = \int_0^{t_I} H(t)\mathrm{d}t$, where t_I is the inflationary period (the duration of the first inflationary regime).

Table 1 Initial conditions on the Poincaré half-plane

Trajectory	τ_0	$\widetilde{v}_0$
Black	$0.4 + 0.5\mathbf{i}$	$0.3 + \mathbf{i}$
Red	$1.4 + 0.5\mathbf{i}$	$0.1 + 0.2\mathbf{i}$
Magenta	$0.2 + 0.7\mathbf{i}$	$0.7 + 0.5\mathbf{i}$
Yellow	$0.3 + 0.5\mathbf{i}$	0
Orange	$0.99 + 0.415\mathbf{i}$	0

5 Conclusions

We proposed [4, 10, 11] a wide generalization of two-field α-attractor models obtained by promoting the scalar manifold from the Poincaré disk to an arbitrary geometrically finite non-compact hyperbolic surface and a procedure for studying such models through uniformization techniques. Our models are parameterized by a constant $\alpha > 0$, by the choice of a surface group Γ and of a smooth scalar potential V. They have the same universal behavior as ordinary α-attractors in the naive one-field truncation near each end, provided that the scalar potential is well-behaved near that end.

Acknowledgements This work was supported by grant IBS-R003-S1.

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The Cohomological Structure of Generalized Killing Spinor Equations

Dario Rosa

Abstract We review the topological structure, sitting in any supergravity theory, which has been recently discovered in [7]. We describe how such a structure allows for a cohomological reformulation of the generalized Killing spinor equations which characterize bosonic supergravity solutions with unbroken supersymmetry.

Keywords Supersymmetric localization · Topological gravity · Topological YM Supergravity

1 Introduction

Localization has been a powerful tool to obtain exact results for supersymetric quantum field theories (SQFT) on curved spaces.[1] To put a SQFT on a curved background preserving supersymmetry is a non-trivial task. A general strategy to address this problem[2] is the following: one couples the SQFT under study to classical *off-shell* supergravity. Putting to zero the supersymmetry variations of the fermionic fields of supergravity one gets equations involving the bosonic supergravity fields. These equations, named *generalized Killing spinor equations*, can be solved only for specific configurations of the supergravity background fields. We will refer to the space of these configurations as the *localization locus*.

In [1, 6] the generalized Killing spinor equations for certain extended supergravity in two and three dimensions have been rewritten in a cohomological form. These cohomological equations were shown to be equivalent to the equations obtained setting to zero the BRST variations of the fermionic fields of topological gravity coupled to a given topological Yang–Mills system. A conceptual explanation of this

[1] See [8] for an extensive overview.

[2] First considered in [3], using superspace formalism, and more recently re-discovered, using component formalism, starting from [4].

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_18

equivalence has been furnished in [7]. In this contribution we will review this equivalence. The main technical tool we will use is the BRST formulation of supergravity, to which now we turn.

2 The BRST Formulation of Supergravity

In the BRST formalism one introduces ghost fields, of ghost number $+1$, associated to each of the local symmetries. In supergravity, the *bosonic* local symmetries include diffeomorphisms and YM gauge symmetries; among the latter there are always local Lorentz transformations, plus additional local YM gauge symmetries which depend on the particular supergravity one is considering (a typical example is provided by the R-symmetry). We denote with ξ^μ the *anticommuting* vector ghost field associated to diffeomorphisms, and with c the *anticommuting* scalar ghost field associated with the YM gauge symmetries, c is valued in the adjoint representation of the total YM gauge algebra. The fermionic local symmetries are the N local supersymmetries; for them one introduces *commuting* spinorial Majorana[3] ghosts ζ^i, with $i = 1, \ldots, N$.

The spinorial ghosts ζ^i, the vierbein $e^a \equiv e^a_\mu dx^\mu$ and the diffeomorphisms ghost ξ^μ constitute the *universal* sector of supergravity, in the sense that their BRST variations are the same for any supergravity theory

$$\begin{aligned} s\,\zeta^i &= \iota_\gamma(\psi^i) + \text{diffeos} + \text{gauge transfs}\,, \\ s\,e^a &= \sum_i \bar{\psi}^i\, \Gamma^a\, \zeta^i + \text{diffeos} + \text{local Lorentz}\,, \\ s\,\xi^\mu &= -\frac{1}{2}\,\mathcal{L}_\xi \xi^\mu - \frac{1}{2}\sum_i \bar{\zeta}^i\,\Gamma^a \zeta^i\, e^\mu_a = -\frac{1}{2}\,\mathcal{L}_\xi \xi^\mu + \gamma^\mu\,, \end{aligned} \tag{1}$$

where s is the nilpotent BRST operator, $\psi^i \equiv \psi^i_\mu dx^\mu$ are the Majorana gravitinos, $\mathcal{L}_\xi$ denotes the Lie derivative along the vector ξ^μ and the vector γ^μ is the following bilinear[4]

$$\gamma^\mu \equiv -\frac{1}{2}\sum_i \bar{\zeta}^i\,\Gamma^a \zeta^i\, e^\mu_a\,, \tag{2}$$

with e^μ_a the inverse of the vierbein. It was observed in [2], that the universal BRST variations (1) imply the following universal BRST variation for the vector bilinear γ^μ

$$s\,\gamma^\mu = -\mathcal{L}_\xi\,\gamma^\mu\,. \tag{3}$$

[3] We will refer to Majorana spinors for simplicity. The discussion can be extended, when N is even, to Dirac spinors.

[4] We will denote with γ^μ the vectorial bilinear (2) and with Γ^a the Dirac matrices.

In [7] it has been recognized that the universal BRST variations (1) and (3) precisely match the BRST variations of topological gravity, once one identifies the vector bilinear γ^μ with the superghost of topological gravity. Indeed, the BRST variations of topological gravity read

$$\begin{aligned} s\, g_{\mu\nu} &= -\mathcal{L}_\xi g_{\mu\nu} + \psi_{\mu\nu}\,, \qquad & s\, \psi_{\mu\nu} &= -\mathcal{L}_\xi \psi_{\mu\nu} + \mathcal{L}_\gamma g_{\mu\nu}\,, \\ s\, \xi^\mu &= -\frac{1}{2}\mathcal{L}_\xi \xi^\mu + \gamma^\mu\,, \qquad & s\, \gamma^\mu &= -\mathcal{L}_\xi \gamma^\mu\,, \end{aligned} \tag{4}$$

where $g_{\mu\nu}$ is the space-time metric, $\psi_{\mu\nu}$ is the topological gravitino and γ^μ is the topological gravity superghost. We have thus obtained that the universal sector of supergravity exactly coincides with topological gravity. We want now bring to light the full topological structure sitting inside any supergravity theory.

3 The Full Topological Structure of Supergravity

Beyond the ghost fields of ghost number $+1$ introduced in the previous section, any supergravity theory includes also fields of ghost number 0. In the rest of this section we will call both the fields of ghost number 0 and the *commuting* supergravity ghosts ζ^i as the *matter* fields and we will denote them with M.

The supergravity BRST variations of the matter fields read

$$s\, M = -\mathcal{L}_\xi\, M - \delta_c\, M + \hat{M}(M)\,, \tag{5}$$

where δ_c is a gauge transformation with the ghost field c and $\hat{M}(M)$ denotes a *composite* of the matter fields M only. The expressions $\hat{M}(M)$, except for the universal supergravity fields discussed in Sect. 2, are the *non-universal* parts of the supergravity BRST transformations; they are theory-dependent functionals of the matter fields. As an example, from (1) we find that for the universal fields ζ^i, we have

$$\hat{\zeta}^i = \iota_\gamma\, \psi^i\,. \tag{6}$$

The BRST variations of the anticommuting ghost fields take a slightly different structure

$$s\, \xi^\mu = -\frac{1}{2}\,\mathcal{L}_\xi\, \xi^\mu + \gamma^\mu\,, \qquad s\, c = -c^2 - \mathcal{L}_\xi c + \hat{c}\,, \tag{7}$$

where $\gamma^\mu \equiv \hat{\xi}^\mu$ is the vector bilinear (2) and $\hat{c}$ are functions, of ghost number 2, of the matter fields. The fields $\hat{c}$ are theory-dependent.

Imposing the nilpotency of the BRST operator s on the matter fields M, one obtains the BRST rules for the composite $\hat{M}$ to be

$$s\,\hat{M} = -\mathcal{L}_\xi \hat{M} - \delta_c \hat{M} + \mathcal{L}_\gamma M + \delta_{\hat{c}} M\ . \tag{8}$$

The Eqs. (5) and (8) make convenient to define another operator S, obtained by subtracting from s both diffeomorphisms and YM transformations

$$S\,M \equiv s\,M + \mathcal{L}_\xi M + \delta_c M\ , \qquad S\,M = \hat{M}(M)\ . \tag{9}$$

By applying S on the composites $\hat{M}$ it follows

$$\frac{\partial\,\hat{M}}{\partial\,M}(M)\hat{M}(M) = S\,\hat{M} = S^2\,M = \mathcal{L}_\gamma\,M + \delta_{\hat{c}}\,M\ , \tag{10}$$

which defines a set of differential conditions that must be satisfied by $\hat{M}(M)$. Moreover, by computing $S^2\hat{M}$ one gets

$$S^2\hat{M} = \mathcal{L}_\gamma\,\hat{M} + \delta_{\hat{c}}\,\hat{M} + \delta_{S\,\hat{c}}\,M\ , \tag{11}$$

where the relation $S\,\gamma^\mu = 0$, which follows from (3), has been used. On the other hand, since the fields $\hat{M}(M)$ are composite, and since the operator S acts as a derivative, it must be

$$S^2\hat{M} = \mathcal{L}_\gamma\,\hat{M} + \delta_{\hat{c}}\,\hat{M}\ . \tag{12}$$

By comparing (11) and (12) one obtains that the composite $\hat{c}$ must satisfy the condition

$$S\,\hat{c} = 0\ . \tag{13}$$

Hence, a supergravity theory is specified by the composites $\hat{M}$ and $\hat{c}$, plus the universal composite γ^μ that has been discussed in the previous section. On them one has to impose the constraints

$$\begin{aligned} &S\,\hat{c} = 0\ ,\\ &\frac{\partial\,\hat{M}}{\partial\,M}(M)\hat{M}(M) = \mathcal{L}_\gamma\,M + \delta_{\hat{c}}\,M\ . \end{aligned} \tag{14}$$

When the constraints (14) are imposed, the operator S satisfies the algebra

$$S^2 = \mathcal{L}_\gamma + \delta_{\hat{c}}\ . \tag{15}$$

It can be shown (see [7] for the details) that the composite $\hat{c}$ takes the general form

$$\hat{c} = \iota_\gamma(A) + \phi\ , \tag{16}$$

where A is the gauge field associated to the local YM symmetry and ϕ is a scalar composite of the matter fields, bilinear in the supersymmetry ghosts ζ^i and valued in the adjoint of the YM Lie algebra. Its explicit form is *theory-dependent*.

The consistency condition $S\hat{c} = 0$ gets translated into the equation

$$S\,\phi = \iota_\gamma(S\,A) = \iota_\gamma(\hat{A})\ . \tag{17}$$

The composite $S\,A = \hat{A}$ is the topological gaugino, usually denoted with λ. Together, the fields ϕ and λ sit into a multiplet valued in the adjoint of the gauge algebra and whose BRST transformations are

$$\begin{aligned} &S\,A = \lambda\ ,\\ &S\,\lambda = \iota_\gamma\,(F) - D\,\phi\ ,\\ &S\,\phi = \iota_\gamma\,(\lambda)\ , \end{aligned} \tag{18}$$

where F is the field strength associated to the local YM symmetry.

The transformations (18) are exactly the BRST variations of topological YM coupled to topological gravity, first derived in this form in [5, 6]. This topological multiplet represents the universal topological sector sitting inside any supergravity theory.

Summarizing, the supergravity BRST algebra takes the universal form

$$S^2 = \mathcal{L}_\gamma + \delta_{\iota_\gamma(A)+\phi}\ , \tag{19}$$

and it is characterized by the two topological fields γ^μ and ϕ. The vector γ^μ has a universal form and it is identified with the superghost of topological gravity. The scalar ϕ has a theory-dependent form and it is identified with the superghost of topological YM coupled to topological gravity. We have thus identified the full topological content sitting inside any supergravity theory: the supergravity BRST algebra is characterized, universally, by two composite fields having clear topological roots.

4 The Cohomological Equations of Localization

As mentioned in the Introduction, the localization locus of a given supergravity theory is obtained by setting to zero the supersymmetry variations of the fermionic supergravity fields. The resulting spinorial equations defining the localization locus are typically involved, and it is hard to extract their gauge invariant content.

In the previous sections it has been shown that a topological sector sits inside any supergravity theory. In particular, the composite topological fermions $\psi_{\mu\nu}$ and λ_μ have been constructed. Hence, on the localization locus the following equations must hold

$$S\,\psi_{\mu\nu} = \mathcal{L}_\gamma g_{\mu\nu} = 0\,, \qquad S\,\lambda = D\,\phi - \iota_\gamma(F) = 0\,, \tag{20}$$

since both $\psi_{\mu\nu}$ and λ_μ are composites containing the fermionic supergravity fields. The first equation in (20) states that the vector bilinear γ^μ has to be an isometry of the spacetime metric $g_{\mu\nu}$. This equation is indeed well-known in the supergravity literature.

On the other hand, the second equation is novel and it has not been studied extensively in both supergravity and topological field theory literature.[5] This equation, when the YM gauge symmetry is non-abelian, is not gauge invariant: its gauge invariant content is captured by considering the following generalized Chern classes

$$c_n(F+\phi) \equiv \mathrm{Tr}\,(F+\phi)^n\;. \tag{21}$$

Indeed, the generalized Chern classes c_n satisfy the equations

$$\mathcal{D}_\gamma\, c_n \equiv (d - \iota_\gamma)\, c_n = 0\,, \tag{22}$$

which states that the c_n's, on the localization locus, are closed under the coboundary operator

$$\mathcal{D}_\gamma \equiv (d-\iota_\gamma)\,, \qquad \mathcal{D}_\gamma^2 = 0\,, \tag{23}$$

associated to the de Rham cohomology of forms on space-time, *equivariant* with respect to the action of the Killing vector γ^μ. In the following, forms closed under the operator $\mathcal{D}_\gamma$ will be called $\gamma - equivariant$.

It should be stressed that the Eq. (20) are *universal*, in the sense that they have to be satisfied, with a specific ϕ which is theory-dependent, on the localization locus of any supergravity theory.

It should be also stressed that the Eq. (20) in general do *not* completely specify the localization locus. Indeed they are obtained by setting to zero the supergravity BRST variations of specific (fermionic) supergravity bilinears, and there might be inequivalent bosonic supergravity backgrounds that give rise to c_n's which are different representatives of the same γ-equivariant classes. As a matter of facts, the γ-equivariant classes c_n parametrize different *branches* of the localization locus. On each of these branches, a moduli space of inequivalent solutions of the generalized Killing spinor equations can be usually found.

In the following, other independent and gauge invariant composite fermions, which can be defined for specific supergravities only, will be introduced. Setting to zero their BRST variations one obtains additional cohomological equations which must be satisfied on the localization locus. These equations allow for a finer classification of the localization locus, i.e. they allow to characterize the moduli space sitting inside each of the branches defined by the c_n's.

---

[5] The author has been informed that this same equation is currently under investigation in a slightly different context [9].

To see how to extract these additional equations, one observes that the crucial property of ϕ, which made possible to construct the topological multiplet $F + \lambda + \phi$ satisfying on the localization locus the second equation in (20), is that its BRST variation is

$$S\,\phi = \iota_\gamma(\lambda)\ . \tag{24}$$

We note that also the supersymmetry ghosts ζ^i have a BRST variation of the same kind:

$$S\,\zeta^i = \iota_\gamma(\psi^i)\ . \tag{25}$$

Hence, scalar and gauge invariant ghost bilinears which are *independent* of extra bosonic fields automatically give rise to other topological multiplets whose BRST take the form (18) and so, putting to zero the BRST variations of the corresponding fermions, one gets additional cohomological equations which are satisfied on the localization locus.

To provide an example, we wil consider the case of $N = (2, 2)$ supergravity in two dimensions. In $N = (2, 2)$ 2d supergravity, it is convenient to combine the two Majorana spinors ζ^i, $i = 1,\ 2$ into a single Dirac spinor ζ, on which the R-symmetry gauge group $U(1)$ acts as a phase multiplication. One can then construct the two scalar bilinears[6]

$$\varphi_1 \equiv \bar{\zeta}\zeta\ , \qquad \varphi_2 \equiv \bar{\zeta}\,\Gamma_3\,\zeta, \tag{26}$$

which are gauge invariant. Therefore, their BRST variations read

$$S\,\phi_i = \iota(\lambda_i)\ , \qquad i = 1,\ 2\ , \tag{27}$$

where

$$\lambda_1 \equiv \bar{\psi}\zeta + \bar{\zeta}\psi\ , \qquad \lambda_2 \equiv \bar{\psi}\,\gamma_3\,\zeta + \bar{\zeta}\Gamma_3\psi\ . \tag{28}$$

As consequence, the BRST algebra (19) tells that the generalized forms

$$\mathcal{H}_i \equiv \phi_i + \lambda_i + \hat{H}_i^{(2)}\ , \tag{29}$$

satisfy

$$(S + d - \iota_\gamma)\,\mathcal{H}_i = 0\ . \tag{30}$$

The 2-forms $\hat{H}_i^{(2)}$ write

[6] Barred spinors are defined in the usual way: $\bar{\zeta} \equiv \zeta^\dagger\,\Gamma_0$.

$$\hat{H}_1^{(2)} = \bar{\psi}\,\psi + H_1^{(2)} \qquad \hat{H}_2^{(2)} = \bar{\psi}\,\Gamma_3\,\psi + H_2^{(2)}\,, \tag{31}$$

where $H_i^{(2)}$, with $i = 1, 2$, are the graphiphoton field strengths. Note that these 2-forms are non universal: they depend indeed on the auxiliary fields of $N = (2, 2)$ supergravity and their explicit form can be found in [7]. From (30) one deduces that on the localization locus the following cohomological equations hold

$$d\,\varphi_i - i_\gamma(H_i^{(2)}) = 0\,. \tag{32}$$

It has been shown[7] in [1] that the Eq. (32), together with the universal equations (20), fully characterize the localization locus of $N = (2, 2)$ supergravity: the localization locus splits in three branches which are parametrized by the integer values of the flux of the R-symmetry field strength; on each branch the Eq. (32) give rise to a moduli space of inequivalent supersymmetric supergravity backgrounds. This moduli space is parametrized by two real moduli.

Acknowledgements I am particularly grateful to C. Imbimbo for a long collaboration on this subject over the years. I also thank J. Bae and J. Winding for discussions and collaboration. I finally thank V. Dobrev and all the organizers of the "X. International Symposium QUANTUM THEORY AND SYMMETRIES" for the invitation to this very interesting workshop.

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Cosmological Solutions from Models with Unified Dark Energy and Dark Matter and with Inflaton Field

Denitsa Staicova and Michail Stoilov

Abstract Recently, few cosmological models with additional non-Riemannian volume form(s) have been proposed. In this article we use Supernovae type Ia experimental data to test one of these models which provides a unified description of both dark energy via dynamically generated cosmological constant and dark matter as a "dust" fluid due to a hidden nonlinear Noether symmetry. It turns out that the model allows various scenarios of the future Universe evolution and in the same time perfectly fits contemporary observational data. Further, we investigate the influence of an additional inflaton field with a step like potential. With its help we can reproduce the Universe inflation epoch, matter dominated epoch and present accelerating expansion in a seamless way. Interesting feature is that inflaton undergoes a finite change during its evolution. It can be speculated that the inflaton asymptotic value is connected to the vacuum expectation value of the Higgs field.

Keywords Cosmology · Dark matter · Dark energy · Two-measures model

1 The Two-Measures Model

The application of the two-measures model [1–4] to cosmology has been pioneered in series of articles by Guendelman, Nissimov and Pacheva [5–12]. In those articles, it has been described a model which is able to describe both dark matter and dark energy in the Universe and also early inflation. This is achieved by the introduction of two scalar fields – a darkon and an inflaton – in a scalar Lagrangian coupled both to the standard Riemannian volume-form (the square root of the metric determinant) and to another non-Riemannian volume form (given in terms of auxiliary

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_19

maximal-rank antisymmetric tensor gauge field). The effect of the additional measure in the theory is felt only through the ratio of the two measures – a constraint determined by an algebraic equation. The equations of motion of such a theory generate dynamically a cosmological constant and a dark matter dust fluid term and also inflation-inducing terms.

In this article, we will discuss our recent numerical investigations of this model in the case of both darkon-only universe and of darkon-inflaton universe. In the first case, we were able to successfully fit the model with the Supernova Type 1 data and to limit its parameter space to observationally acceptable values. We also showed that in this case it is possible for the Universe to undergo a phase transition. In the second case, we were able to reproduce the stages of the Universe expansion – early inflation, matter domination and late inflation under certain choices for the parameters. We also observed some novel features, like the matter-dominated early epoch and a non-zero scalar field in the late Universe. In both cases, we have numerically confirmed that the two-measures model can be a viable cosmological model.

2 The Darkon Model in FLRW Metric

The action of the two-measures darkon model in the f(R) gravity (Guendelman, Nissimov and Pacheva [6, 9]) has the following form:

$$S_{darkon} = \int d^4x\sqrt{-g}(R(g,\Gamma) - \alpha R^2(g,\Gamma)) + \int d^4x(\sqrt{-g} + \Phi(C))L(u,X)$$

where $\Phi(C) = \frac{1}{3}\epsilon^{\mu\nu\kappa\lambda}\partial_\mu C_{\nu\kappa\lambda}$ is the non-Riemannian measure and $L(u,X) = -\frac{1}{2}g^{\mu\nu}\partial_\mu u\partial_\nu u - V(u)$ is the matter Lagrangian of the darkon scalar field u.

If one applies the equations of motion obtained from this action to the Friedman–Lemaître–Robertson–Walker metric with $k = 0$:

$$ds^2 = -dt^2 + a(t)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right]. \quad (1)$$

one obtains from the Friedman equations ($G_{00} = T_{00}$) the following relations for the energy density:

$$\rho = \frac{1}{8\alpha}\dot{u}^2 + \frac{3}{4}\frac{p_u}{a(t)^3}\dot{u} - \frac{1}{4\alpha} \quad (2)$$

$$p_u = a(t)^3\left[-\frac{1}{2\alpha}\dot{u} + \left(\frac{1}{4\alpha} - 2M_0\right)\dot{u}^3\right] \quad (3)$$

where $p_u = const$.

Following our work in [13], we rewrite the last cubic equation for $\dot{u}$ (Eq. (3)), as $y^3 + 3\mathbf{a}y + 2\mathbf{b} = 0$ with $\mathbf{a} = -\frac{2}{3-24\alpha M_0}$ and $\mathbf{b} = -\frac{2\alpha p_u}{a(t)^3(1-8\alpha M_0)}$, $y = \dot{u}$.

The solutions are:

$$y_1 = -\frac{\mathbf{a}}{(-\mathbf{b} + \sqrt{\mathbf{a}^3 + \mathbf{b}^2})^{1/3}} + (-\mathbf{b} + \sqrt{\mathbf{a}^3 + \mathbf{b}^2})^{1/3}$$

$$y_2 = \frac{\mathbf{a}}{(\mathbf{b} - \sqrt{\mathbf{a}^3 + \mathbf{b}^2})^{1/3}} - (\mathbf{b} - \sqrt{\mathbf{a}^3 + \mathbf{b}^2})^{1/3}$$

$$y_3 = \frac{y_2 - i\sqrt{3}y_1}{2}$$

Since no real smooth solution exists in the whole [**a**, **b**] plane, we define the following piecewise functions, **real** in the whole plane [**a**, **b**]:

$$y_b = \begin{cases} y_1 & \text{for } (a, b) \in \{a \geq 0\} \cup \{a < 0 \cap b < 0\} \\ y_2 & \text{for } (a, b) \in \{a < 0 \cap b > 0\} \end{cases} \qquad y_s = \begin{cases} y_1 & \text{for } b > 0 \\ y_2 & \text{for } b < 0. \end{cases}$$

We obtain the final form of the Friedman equation after rescaling time by $2|\alpha|/3 = 1$ and absorbing α into Hubble constant ($\bar{\rho} = 4|\alpha|\rho$):

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \bar{\rho} = \left(\frac{1}{2}y^2 + \frac{\mathbf{b}}{\mathbf{a}}y - 1\right) \tag{4}$$

The asymptotics, corresponding to the dark energy term in the late universe, is:

$$\bar{\rho} \xrightarrow[a(t)\to\infty]{} \begin{cases} 1 & \text{for } \mathbf{a} > 0 \\ -\frac{3}{2}\mathbf{a} - 1 & \text{for } \mathbf{a} < 0 \end{cases}$$

We use as independent real solutions y_b (our basic solution) and y_s and integrate numerically Eq. (4) to find the evolution of the universe.

Phase transition: From the numerical integration we have seen that it is possible to obtain both Universes with or without phase transition. The possibility for such transition comes from the fact that in the [**a**, **b**] plane exist sectors where two of our solutions have positive energy density ρ.

Explicitly, let's denote $\bar{\bar{\rho}}$ the density corresponding to solution y_s ($\bar{\rho}$ corresponds to y_b). At the moment $t_1 = 0$, it will be negative, i.e. $\bar{\bar{\rho}}(t_1) < 0$. For certain moment t_p, however, it will change sign: $\bar{\bar{\rho}}(a(t_p)) = 0$. Therefore, for any moment $t > t_p$ we have two "states" of the Universe $\bar{\rho}$ and $\bar{\bar{\rho}}$ such as:

$$0 \leq \bar{\bar{\rho}} < \bar{\rho} \quad \text{for} \quad t \geq t_p. \tag{5}$$

This opens the possibility the Universe to undergo *"phase transition"* or *"quenching"* to the lower state.

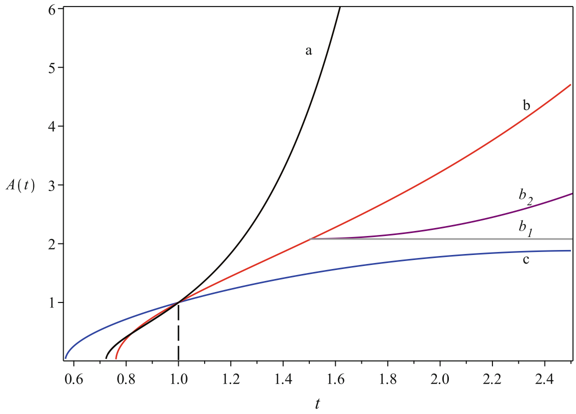

Fig. 1 Graphics of the $a(t)$ evolution for: $\mathbf{a} = -5.987, \mathbf{b} = \frac{-2.932}{a^3}$ (a) $\mathbf{a} = -1, \mathbf{b} = -\frac{2}{a^3}$ (b), $t_p = 1.5074, a_s(t_p) = 2.0825$ (b_2), $\mathbf{a} = -.5, \mathbf{b} = -\frac{0.5}{a^3}$ (c)

The moment of the phase transition is crucial for the further evolution, since if $t = t_p$ the evolution stops ($\bar{\bar{\rho}} = 0$), if $t = t_p + \delta t$, we observe phase transition of the first kind. An illustration of this process can be seen on Fig. 1, where the transition happens between lines b, b_1 and b_2.

The Supernova Fit: Using the freely available data of Supernovae Type 1a [14], we were able to fit the distance modulus d_m[1] as a function of z using using energy density from the two-measures model:

$$d_m = 5\log_{10}\left((1+z)\int_0^z dx \frac{a(x)}{\dot{a}(x)}\right) = \\ \text{const} + 5\log_{10}\left((1+z)\int_0^z dx \frac{1}{\sqrt{\rho(x)}}\right) \quad (6)$$

Using a symplectic fit, we were able to prove that our model is able to reproduce the observational data. The details on the fit can be seen in [13], here we will emphasize only that the precision of the fit ($\chi^2 \sim 562$ for $\mathbf{a} < -2/3$, $\chi^2 \sim 578$ for $\mathbf{a} > 1$.) is similar to the one of the standard model ($\chi^2 = 562$).

The best fit of SN data using the proposed model is not unique. We find one parametric family of solutions producing the same $d_m(z)$ function. An approximate formula for the dependency $\mathbf{b}(\mathbf{a})$ can be obtained using the LeastSquares algorithm in Maple and it gives:

$$b_\pm = \sum_0^4 \pm c_i a^i + O(a^5), \text{ with coefficients}$$

$$c_i = [0.337906, 0.376679, -0.0251697, 0.00148545, 0.11272710^{-3}]$$

[1] $d_m = 5\log_{10}\left(\frac{d}{10}\right)$, where d is the distance in parsecs.

On Fig. 1, curve (d) represents one such evolution with parameters **a** and **b** corresponding to the observational data.

3 The Two-Measures Theory – Including the Inflaton

In order to produce inflation in the model, one needs to include a new scalar field – the inflaton ϕ. Following Guendelman, Nissimov and Pacheva [6, 12] (where in S_{darkon} $\alpha = 0$)), the action, featuring two non-Riemannian measures $\Phi_1(A)$ and $\Phi_2(B)$, becomes:

$$S = S_{darkon} + \int d^4x \Phi_1(A)(R + L^{(1)}) + \int d^4x \Phi_2(B) \left(L^{(2)} + \frac{\Phi(H)}{\sqrt{-g}} \right)$$

where:

$$L^{(1)} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi), \;\; V(\phi) = f_1 e^{-\alpha\phi} \tag{7}$$

$$L^{(2)} = -\frac{b}{2} e^{-\alpha\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + U(\phi), \;\; U(\phi) = f_2 e^{-2\alpha\phi} \tag{8}$$

From the equations of motion we have:

$$p = -2M_0 = const, \frac{\Phi_2(B)}{\sqrt{-g}} = \chi_2 = const$$

$$R + L^{(1)} = -M_1 = const, L^{(2)} + \frac{\Phi(H)}{\sqrt{-g}} = -M_2 = const$$

$$U_{eff}(\phi) = \frac{(f_1 e^{-\alpha\phi} + M_1)^2}{4\chi_2(f_2 e^{-2\alpha\phi} + M_2) - 8M_0} \text{with } U_- = \frac{f_1^2}{4\chi_2 f_2}, U_+ = \frac{M_1^2}{4\chi_2 M_2 - 8M_0}$$

An important condition following from the requirement that the vacuum energy density of the early Universe U_- should be much higher than that of the late Universe U_+ gives:

$$\frac{f_1^2}{\chi_2 f_2} >> \frac{M_1^2}{\chi_2 M_2 - 2M_0} \tag{9}$$

This ensures that the effective potential has the form of two infinite plateaus connected with a steep slope.

Additionally, one can postulate:

$$|M_1| \sim M_{EW}^4, M_2 \sim M_{Pl}^4, f_1 \sim f_2 \sim 10^{-8} M_{Pl}^4,$$

so that one can connect the theory with the electroweak and the Planck scales.

The system of equations that need to be solved numerically in order to obtain the evolution of the Universe is the following:

$$v^3 + 3av + 2b = 0 \text{ for} \tag{10}$$

$$a = \frac{-1}{3}\frac{V(\phi) + M_1 - \frac{1}{2}\chi_2 b e^{-\alpha\phi}\dot{\phi}^2}{\chi_2(U(\phi) + M_2) - 2M_0}, b = \frac{-p_u}{2a(t)^3(\chi_2(U(\phi) + M_2) - 2M_0)} \tag{11}$$

$$\dot{a}(t) = \sqrt{\frac{\rho}{6}}a(t),\ \rho = \frac{1}{2}\dot{\phi}^2(1 + \frac{3}{4}\chi_2 b e^{-\alpha\phi}v^2) + \frac{v^2}{4}(V + M_1) + \frac{3p_u v}{4a(t)^3} \tag{12}$$

$$\ddot{a}(t) = -\frac{1}{12}(\rho + 3p)a(t),\ p = \frac{1}{2}\dot{\phi}^2(1 + \frac{1}{4}\chi_2 b e^{-\alpha\phi}v^2) - \frac{v^2}{4}(V + M_1) + \frac{p_u v}{4a(t)^3} \tag{13}$$

$$\frac{d}{dt}\left(a(t)^3\dot{\phi}(1 + \frac{\chi_2}{2}be^{-\alpha\phi}v^2)\right) + a(t)^3(\alpha\frac{\dot{\phi}^2}{4}\chi_2 b e^{-\alpha\phi}v^2 + \frac{1}{2}V_\phi v^2 - \chi_2 U_\phi\frac{v^4}{4}) = 0 \tag{14}$$

Here Eq. (13) is optional and it offers an independent way to evaluate $\ddot{a}(t)$. This differential system is of first order with respect to $a(t)$ and of second order with respect to $\phi(t)$. Once again, we first solve the cubic equation by choosing a base solution and then we use it, to integrate the differential system with the implemented in Maple Fehlberg fourth-fifth order Runge-Kutta method with degree four interpolant.

Study the [a, b] plane Unlike the previous case, where **a** was a constant, here it depends on $\phi, \dot{\phi}$. Because of this, the trajectories in the [**a**, **b**] plane which the Universe will describe in its evolution won't be straight lines like in the darkon case, but curves. For example, on Fig. 2 we have plotted the trajectory for one set of parameters (dots). It starts at $b \to -\infty$ and ends at $b \to 0$. In its evolution, it crosses the $a^3 + b^2 = 0$ line (solid line). On the plot, one can see also the trajectories for the darkon case plotted with dashed lines. In order to work in the sector III, where both solutions y_b and y_s are valid we have chosen the parameters in such a way that $b = \frac{-p_u}{2a(t)^3(\chi_2(U+M_2)-2M_0)} < 0$.

Numerical integration and equation of state For the numerical integration, we impose as initial conditions:

$$a(0) = 10^{-12}, \phi(0) = \phi_0, \dot{\phi}(0) = 0.$$

Additionally, we impose the gauge condition: $a(1) = 1$.

This problem has 12 parameters of the system. Our numerical experiments show that the system is extremely sensitive to them and small changes can lead to either eternally exponentially expanding Universe or collapsing without inflation Universe. In order for an evolution to reproduce the known past of the Universe the second derivative of the scale factor $\ddot{a}(t)$ has to change sign at least 2 times: to be positive during early inflation $\ddot{a}_i(t) > 0$, to be negative during matter domination period ($\ddot{a}_{MD}(t) < 0$, and to be positive during the late (current) expansion $\ddot{a}_{LE}(t) > 0$.

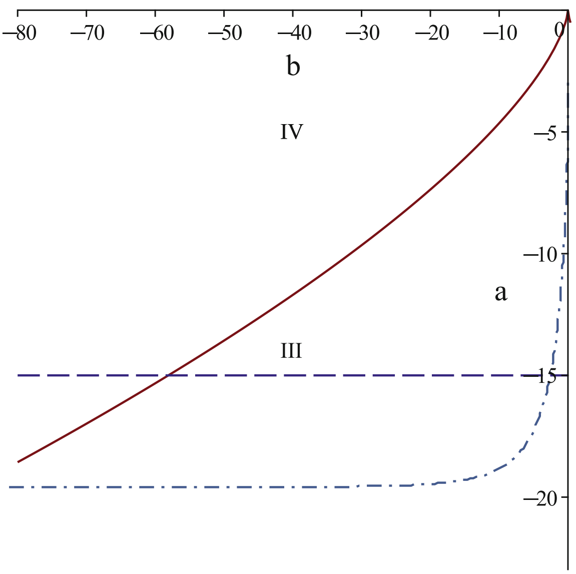

Fig. 2 The evolution of the solutions in the [**a**, **b**] plane. The dashed line correspond to the darkon case, the dash-dot line – to the inflation case, the solid lines – the lines of validity of the solutions

Our numerical investigations show that such "physical" cases are indeed possible, for example Fig. 3, but require careful fine-tuning of the parameters.[2] One can see the different epochs by plotting the equation of state $w = p/\rho$ (see Fig. 3b). The times in which they kick in correspond to the change of sign of $\ddot{a}(t)$.

A notable result from our work is that it is not numerically possible to start from the left plateau and to obtain a "physical" evolution. Instead, the evolution explodes to eternal inflation. It is not possible also to finish on the right plateau, because the evolution of the scalar field stops before reaching it (there is a friction term). This illustrated on Fig. 3a, where one can see the effective potential in this case.

A very important feature of the model is that it starts with a pre-inflation mater domination epoch with exponentially high energy density, which quickly cools to enter in the early inflation stage (see Fig. 3b). Another important feature is the fact that the scalar field does not reach zero in the late Universe as expected by the theory. This is also due to the friction term which stops its evolution ($\dot{\phi} = 0$) before it can reach zero (Fig. 4).

In our numerical simulations we have discovered some interesting features of the model, which is not in accorance with the asymptotic found in [6]. One needs to keep in mind, however, that due to the numerical complexity of the problem and its big parameter-space, we have not yet reached the theoretically predicted values of somevadjust of the parameters discussed in there. While the main requirement of the

[2]The plots are for parameters $M_0 = -0.01$, $M_1 = 0.1$, $M_2 = 4$, $\alpha = 0.7$, $b_0 = 1 \times 10^{-5}$, $p_u = 0.15$, $\chi_2 = 3.3 \times 10^{-4}$, $f_1 = 3 \times 10^{-5}$, $f_2 = 1 \times 10^{-8}$, integrated for $t = 0.4$.

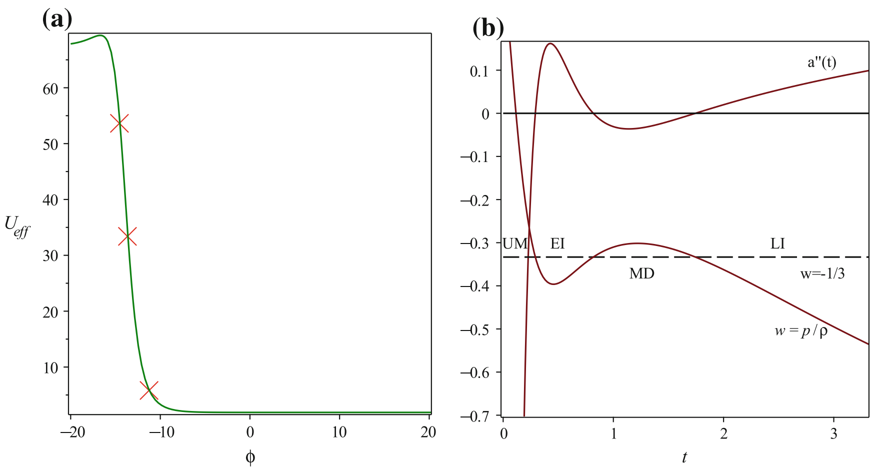

Fig. 3 **a** The effective potential, where the crosses signify the moments: $t = 0$, $t = 1$ and $t = 4$. **b** Plot of $\ddot{a}(t)$ and the equation of state $w = p/\rho$. One can see the different epochs – ultra-relativistic matter domination (UM), the early inflation (EI), the matter domination (MD) and the late inflation (LI)

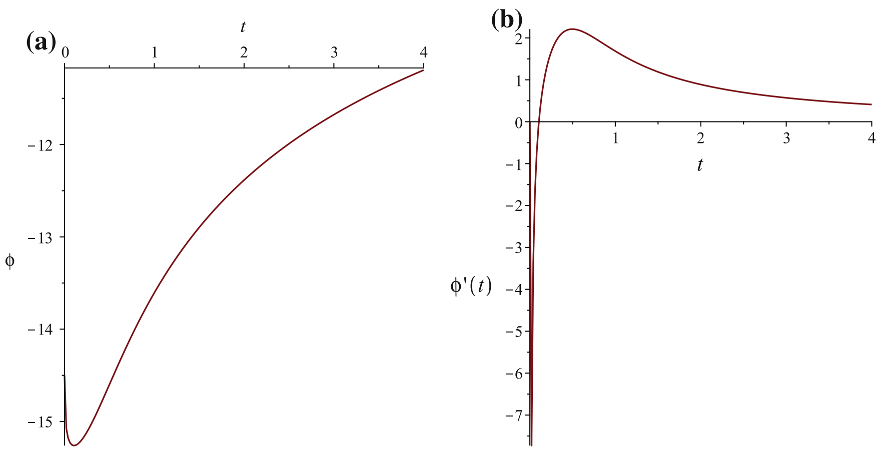

Fig. 4 **a** The inflaton scalar field $\phi(t)$. **b** Its derivative $\dot{\phi}(t)$

model Eq. (9) is satisfied, the values of the other parameters, which can be found in the Table 1, should be extended to a more physically realistic domain.

Another way to define "physicality" of the problem, is the problem of the time scales. The observationally expected values for the time at which $\ddot{a}(t)$ changes its sign comes from the time when different epochs start. Theoretically, matter domination is considered to start at $a_{MD}(t) \sim 3 \times 10^{-4}$ and the accelerated expansion – at $a_{AE}(t) \gtrsim 0.6$. Our current best result is $a_{MD} = 0.2$, $a_{AE} = 1.2$. It is yet to be seen whether the

Table 1 Comparison between the theoretically stipulated and the numerical values of some of the parameters

Parameter	Theory	Numerics
M_1	$\sim M_{EW}^4 = 4.10^{-60}$	$1/15 =$ 6.67×10^{-2}
M_2	$\sim M_{Pl}^4 = 4$	4
f_1	$\sim 10^{-8}$	2×10^{-5}
f_2	$\sim 10^{-8}$	10^{-8}
α	10^{-20}–0.2	0.64

observational values can be reached trough fine-tuning of the parameters. Because of the complexity of the problem, this fine-tuning needs to be done step by step and cannot be automatized for the moment.

Finally, due to the extreme predicted ratio $U_+/U_- \sim 10^{120}$, reaching the theoretically predicted values of the parameters may be computationally impossible, due to possible increase in the required precision for the numerical integration of the system. A fuller investigation of the parameter space of the problem will be presented in future works.

4 Conclusions

In our numerical work on the application of the two-measures model to cosmology we have confirmed that this model can be considered as an alternative of the standard model of dark matter and dark energy. Through numerical integration of the Friedman equations in the K-essence theory in the darkon and the inflaton case, along with detailed study of the plane [**a**, **b**], we have obtained interesting numerical results.

It was shown that in the darkon model we can obtain both a Universe with and without phase transition and those models were fit to the data of Supernovae Type 1a. In the case of inflaton model, we have performed first steps in the study of the the parameter space of the model and we have found solutions for which one can obtain the two inflationary epochs and one matter dominated epoch. It was shown that the inflation experiences friction, due to which inflation stops before reaching the U_+ part of the potential. This was unexpected result which is to be further investigated, because it also means that there should be a non-zero scalar field surviving to the modern epoch.

Acknowledgements It is a pleasure to thank E. Nissimov and S. Pacheva for discussions.The work is supported by BAS contract DFNP – 49/21.04.2016, by Bulgarian NSF grant DN-18/1/10.12.2017 and by Bulgarian NSF Grant 08-17.

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Relation Between Dilatonic Pressure and Cosmological Pressure for Neutron Stars in Minimal Dilatonic Gravity

Kalin Marinov and Plamen Fiziev

Abstract The minimal dilatonic gravity (MDG) is a proper generalization of the Einstein general relativity (GR), which uses one gravitation-dilaton field Φ, an offers a simultaneous explanation of the effects of dark matter and dark energy. We present an in depth research of the dark matter and dark energy effects in the interior of the non-rotating neutron star models in MDG. We use different realistic equations of state, which are in good agreement with the latest observational data. The equations describing relativistic static spherically symmetric stars are solved numerically for the different equations of state and we present results for the center and the edge of the stars.

Keywords Extended gravity · Neutron star · Gravitational dilation · Dark matter
Dark energy

1 Introduction

The model of minimal dilatonic gravity (MDG) is an alternative model of gravitation. First it was introduced by O'Hanlon [21]. Later, the possible relation of the O'Hanlon model with cosmology and astrophysics was explored and the name MDG appeared [7–16], where the cosmological constant Λ was also used. MDG is a proper, simple modification of GR, based on the following action of the gravi-dilaton sector:

$$S_{g,\Phi} = -\frac{c}{2k}\int d^4x\sqrt{|g|}(\Phi R + 2\Lambda U(\Phi)), \tag{1}$$

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_20

where $k = 8\pi G/c^2$ is the Einstein constant, G is the Newton gravitational constant, Λ is the cosmological constant, and $\Phi \in (0, \infty)$ is the dilaton field. The values of Φ must be positive because change of the sign would lead to a change of the sign of the gravitational factor G/Φ, which would lead to antigravity. We rule out the possibility of antigravity, since it is unphysical. The value $\Phi = \infty$ must also be excluded, because the gravity is eliminated. $\Phi = 0$ is also unacceptable since it leads to an infinite gravitational factor, and the Cauchy problem is not well posed.

The scalar field Φ is introduced in order to have a variable gravitational factor $G(\Phi) = G/\Phi$, instead of gravitational constant G. Φ does not enter in the action of the matter S_{matter}, because it has no interaction with ordinary matter. Due to its specific physical meaning it has unusual properties. The function $U(\Phi)$ defines the cosmological potential. It is introduced in order to have a variable cosmological factor instead of the cosmological constant Λ. $U(\Phi)$ must be a singe valued function of the dilaton field due to astrophysical reasons. The MDG without cosmological term corresponds to the Brans–Dicke theory with identically vanishing parameter w. If we set $\Phi = 1$ and $U(\Phi) = 1$, we are back into GR.

A special class of potential are introduced in [9]. They are called withholding potentials and they confined the dynamical values of the dilaton Φ in the physical domain. It is also shown that MDG model is only locally equivalent to the $f(R)$ theories and leads to different physical consequences.

A lot of $f(R)$ functions could be found in the literature. For example [2, 18, 25, 26]. More extensive information about $f(R)$ theories can be found in [3–5, 19, 20].

2 Basic Equations and Boundary Conditions

The field equation of MDG with matter fields can be found in [10, 12, 15, 16]. The inner domain $r \in [0, r^\star]$, where $r^\star$ is the radius of the star, the structure is described by the following system of four first order differential equation, which represent a generalization of the Tolman–Oppenheimer–Volkoff equations:

$$\begin{aligned}
\frac{\mathrm{d}m}{\mathrm{d}r} &= \frac{4\pi r^2 \epsilon_{eff}}{\Phi} \\
\frac{\mathrm{d}p}{\mathrm{d}r} &= -\frac{(p+\epsilon)}{r\big(\Delta - 2\pi r^3 p_\Phi/\Phi\big)}\left(\frac{4\pi r^3 p_{eff}}{\Phi} + m\right) \\
\frac{\mathrm{d}\Phi}{\mathrm{d}r} &= -\frac{4\pi r^2 p_\Phi}{\Delta} \\
\frac{\mathrm{d}p_\Phi}{\mathrm{d}r} &= -\frac{p_\Phi}{\Delta r}\left(3r - 7m - \frac{2}{3}\Lambda r^3 + \frac{4\pi r^3 \epsilon_{eff}}{\Phi}\right) - \frac{2\epsilon_\Phi}{r}.
\end{aligned} \tag{2}$$

Here we have four unknown functions, $m = m(r)$, $p = p(r)$, $\Phi = \Phi(r)$ and $p_\Phi = p_\Phi(r)$, the mass, the pressure, the dilaton and the dilaton pressure. The following indications are used in the system:

$$\Delta = r - 2m - \frac{\Lambda r^3}{3},$$
$$\epsilon_{eff} = \epsilon + \epsilon_\Lambda + \epsilon_\Phi, \quad p_{eff} = p + p_\Lambda + p_\Phi,$$
$$\epsilon_\Lambda = \frac{\Lambda}{8\pi}\big(U(\Phi) - \Phi\big), \quad p_\Lambda = -\frac{\Lambda}{8\pi}\left(U(\Phi) - \frac{\Phi}{3}\right),$$
$$\epsilon_\Phi = p - \frac{1}{3}\epsilon + \frac{\Lambda}{8\pi}V'(\Phi) + \frac{p_\Phi\left(\frac{4\pi r^3}{\Phi}p_{eff} + m\right)}{2\left(\Delta - \frac{2\pi r^3 p_\Phi}{\Phi}\right)}.$$

In the above equation ϵ_Λ and p_Λ are the cosmological energy density and cosmological pressure, ϵ_Φ and p_Φ are the dilaton energy density and dilaton pressure. p_Λ and p_Φ correspond to the effects of dark energy and dark matter respectively and the need for dark matter and dark energy is firmly established [24]. We combine cosmological, dilaton and matter energy density in a new variable ϵ_{eff}. We do the same thing for the cosmological, dilaton and matter pressure in the variable p_{eff}.

3 Numerical Results

In the current research we use three equations of state BSk19, BSk20, BSk21 [17, 22, 23]. Those equations are compatible with the latest results for the maximum mass of neutron stars [1, 6]. We use the simplest withholding dilaton potential in the form [8, 9, 14]

$$U(\Phi) = \Phi^2 + \frac{3}{16d^2}(\Phi - 1/\Phi)^2, \tag{3}$$

where we use the dimensionless Compton length $d = \lambda_\Phi\sqrt{\Lambda}$, and λ_Φ is the dilaton Compton length. More information fo the numerical procedure can be found in [10, 15, 16].

On Figs. 1, 2 and 3 are shown the results dilaton pressure and the cosmological pressure in the center of the star. For all three equations of state the cosmological pressure in the center is always negative and the dilaton pressure in the center of the star can be positive or negative depending on the initial conditions. Both p_Λ and p_Φ contribute to the effective pressure p_{eff}, which lead to masses of the stars very different from the ones obtained in general relativity [12, 15, 16].

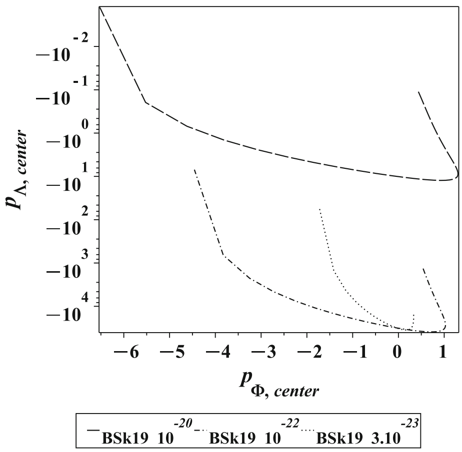


Fig. 1 Here is shown the relation between the cosmological pressure p_Λ in the center of the star and the dilaton pressure p_Φ in the center of the star, for BSk19 equation of state. On the figure $p_{\Lambda,center}$ is in 10^{29} dyne.cm^2 and $p_{\Phi,center}$ is in 10^{35} dyne.cm^2

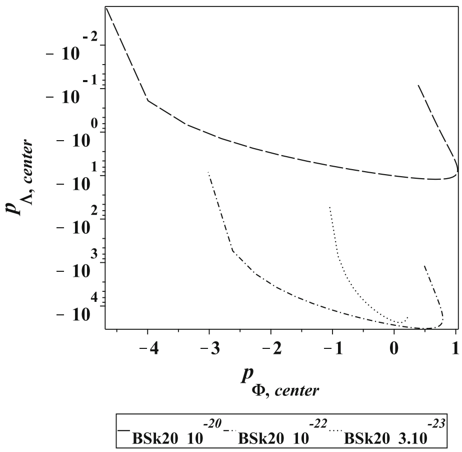


Fig. 2 Here is shown the relation between the cosmological pressure p_Λ in the center of the star and the dilaton pressure p_Φ in the center of the star, for BSk20 equation of state. On the figure $p_{\Lambda,center}$ is in 10^{29} dyne.cm^2 and $p_{\Phi,center}$ is in 10^{35} dyne.cm^2

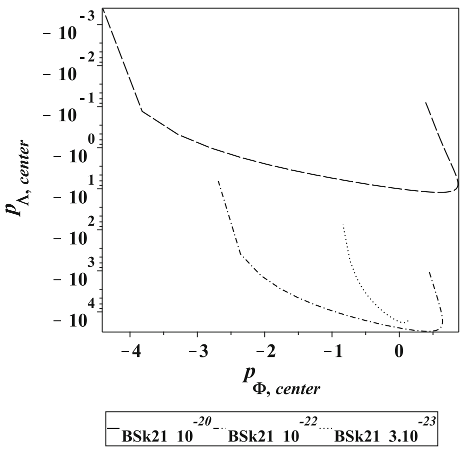


Fig. 3 Here is shown the relation between the cosmological pressure p_Λ in the center of the star and the dilaton pressure p_Φ in the center of the star, for BSk21 equation of state. On the figure $p_{\Lambda,center}$ is in 10^{29} dyne.cm^2 and $p_{\Phi,center}$ is in 10^{35} dyne.cm^2

Fig. 4 Here is shown the relation between the cosmological pressure p_Λ on the edge of the star and the dilaton pressure p_Φ on the edge of the star, for BSk19 equation of state. On the figure $p_{\Lambda,center}$ is in 10^{29} dyne.cm^2 and $p_{\Phi,center}$ is in 10^{35} dyne.cm^2

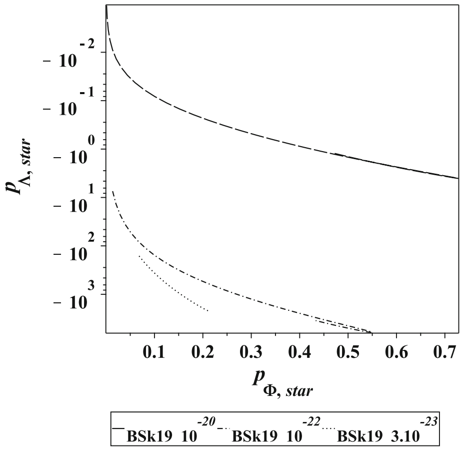
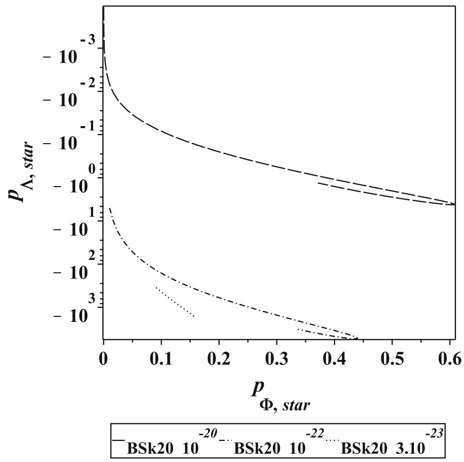


Fig. 5 Here is shown the relation between the cosmological pressure p_Λ on the edge of the star and the dilaton pressure p_Φ on the edge of the star, for BSk20 equation of state. On the figure $p_{\Lambda,center}$ is in 10^{29} dyne.cm^2 and $p_{\Phi,center}$ is in 10^{35} dyne.cm^2



On Figs. 4, 5 and 6 are shown the results dilaton pressure and the cosmological pressure on the edge of the star. For all three equations of state the cosmological pressure on the edge is negative or positive depending on the initial conditions and the dilaton pressure on the edge of the star is always positive. This leads to existence of a sphere of dilaton around the star, called dilasphere. The dilasphere contributes to the mass of the object significantly and it is the main reason behind the high neutron star masses in the minimal dilatonic gravity [12, 15, 16].

In the center of the star p_Φ is of the same order for all initial conditions and Compton lengths, but p_Λ vary greatly with the Compton length. The same observation is true and for the edge of the star and for all used equations of state.

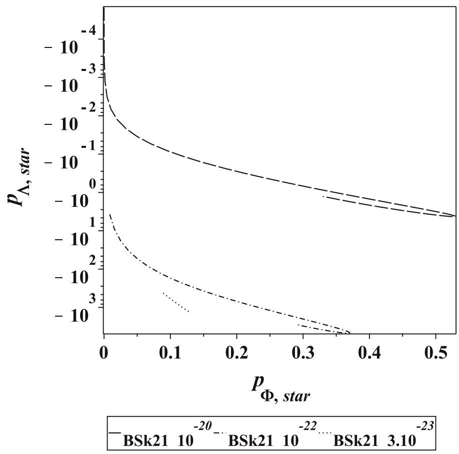


Fig. 6 Here is shown the relation between the cosmological pressure p_Λ on the edge of the star and the dilaton pressure p_Φ on the edge of the star, for BSk21 equation of state. On the figure $p_{\Lambda,center}$ is in 10^{29} dyne.cm^2 and $p_{\Phi,center}$ is in 10^{35} dyne.cm^2

Acknowledgements Kalin Marinov wants to express gratitude to the Program for career development of young scientist, Bulgarian academy of sciences project DFNP-17-168/03.06.2017. Plamen Fiziev is deeply indebted to the Directorate of the Laboratory of Theoretical Physics, JINR, Dubna, for the good working conditions and support. This research is supported in part by the Foundation for Theoretical and Computational Physics and Astrophysics and by Bulgarian Nuclear Regulatory Agency Grants for 2014, 2015, 2016, 2017.

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Part IV
Conformal and Gauge Theories

Conformal Ward–Takahashi Identity at Finite Temperature

Satoshi Ohya

Abstract We study conformal Ward–Takahashi identities for two-point functions in $d(\geq 3)$-dimensional finite-temperature conformal field theory. We first show that the conformal Ward–Takahashi identities can be translated into the intertwining relations of conformal algebra $\mathfrak{so}(2, d)$. We then show that, at finite temperature, the intertwining relations can be translated into the recurrence relations for two-point functions in complex momentum space. By solving these recurrence relations, we find the momentum-space two-point functions that satisfy the Kubo–Martin–Schwinger thermal equilibrium condition.

Keywords Conformal field theory · Finite temperature · Two-point function

1 Introduction

It is widely believed that conformal symmetry is always broken at finite temperature. This comes from the naive argument that finite-temperature field theory necessarily contains one particular scale—the temperature—and hence must break scale and conformal invariance. Contrary to this popular belief, however, finite temperature and conformal invariance can in fact be compatible with each other: If conformal field theory (CFT) is thermalized via the *Unruh effect*, conformal symmetry remains intact even at finite temperature. The purpose of this paper is to report our recent work on this subject [13] and to see how the conformal symmetry determines finite-temperature two-point functions in momentum space. The key is the *intertwining relations* of conformal algebra $\mathfrak{so}(2, d)$ [6, 8, 12, 16], which follow from the conformal Ward–Takahashi identities for two-point functions. We shall show that, at finite temperature, the intertwining relations are recast into the recurrence relations in complex momentum space. These recurrence relations can be regarded as the

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_21

conformal Ward–Takahashi identities at finite temperature, from which we can deduce the possible forms of momentum-space two-point functions.

The rest of the paper is organized as follows: In Sect. 2 we first introduce the intertwining operator, which is defined as an integral transform whose kernel is the two-point function. We then discuss that the conformal Ward–Takahashi identities are rewritten as the intertwining relations. In Sect. 3 we introduce the $d(\geq 3)$-dimensional Rindler wedge, light-cone, and diamond, all of which are subspaces of Minkowski spacetime and conformal to $\mathbb{H}^1 \times \mathbb{H}^{d-1}$. These subspaces are the whole universes of our finite-temperature CFT and possess the global timelike conformal Killing vectors associated with the subgroup $SO(1,1) \subset SO(2,d)$. In Sect. 4 we study the intertwining relations in the basis in which the $SO(1,1)$ generator becomes diagonal. We shall see that in this basis the intertwining relations reduce to the recurrence relations for momentum-space two-point functions. We also give two minimal solutions that correspond to the positive- and negative-frequency two-point Wightman functions and satisfy the Kubo–Martin–Schwinger (KMS) thermal equilibrium condition.

Throughout the paper we work with the metric signature $(-,+,\dots,+)$.

2 From Conformal Ward–Takahashi Identities to Intertwining Relations

To begin with, let us consider a scalar primary operator $\mathcal{O}_\Delta(x)$ of scaling dimension Δ. Let $g \in SO(2,d)$ be an element of the conformal group and $x \mapsto x_g$ be the associated conformal transformation. Then the scalar primary operator transforms as follows:

$$U(g)\mathcal{O}_\Delta(x)U^{-1}(g) = \left|\frac{\partial x_g}{\partial x}\right|^{\Delta/d} \mathcal{O}_\Delta(x_g), \tag{1}$$

where U is a unitary representation of the conformal group and $|\partial x_g/\partial x|$ stands for the Jacobian of the conformal transformation.

Let us next consider a two-point function $G_\Delta(x,y)$ of $\mathcal{O}_\Delta$. For example, one may consider this to be the positive- or negative-frequency two-point Wightman functions, $\langle 0|\mathcal{O}_\Delta(x)\mathcal{O}_\Delta^\dagger(y)|0\rangle$ or $\langle 0|\mathcal{O}_\Delta^\dagger(y)\mathcal{O}_\Delta(x)|0\rangle$, where $|0\rangle$ stands for the conformally-invariant vacuum state that satisfies $U(g)|0\rangle = |0\rangle$ for any $g \in SO(2,d)$. Then $G_\Delta(x,y)$ satisfies the following identity:

$$G_\Delta(x,y) = \left|\frac{\partial x_g}{\partial x}\right|^{\Delta/d} \left|\frac{\partial y_g}{\partial y}\right|^{\Delta/d} G_\Delta(x_g, y_g). \tag{2}$$

As is well-known, this identity—the finite form of conformal Ward–Takahashi identity—fully determines the possible forms of two-point functions. For example, up to the $i\epsilon$ prescription the Wightman functions must be of the form $G_\Delta(x, y) \propto [(x-y)^2]^{-\Delta}$.

Now, let us consider another scalar primary operator $\mathcal{O}_{d-\Delta}(x)$ of scaling dimension $d-\Delta$. Once $\mathcal{O}_{d-\Delta}(x)$ and $G_\Delta(x, y)$ are given, we can define an operator G_Δ through the following integral transform:

$$G_\Delta : \mathcal{O}_{d-\Delta}(x) \mapsto (G_\Delta \mathcal{O}_{d-\Delta})(x) := \int d^d y \, G_\Delta(x, y) \mathcal{O}_{d-\Delta}(y). \tag{3}$$

It is easy to check that thus defined operator $(G_\Delta \mathcal{O}_{d-\Delta})(x)$ satisfies the transformation law (1) and hence is a primary operator of scaling dimension Δ. Conversely, one can start from $\mathcal{O}_\Delta(x)$ and $G_{d-\Delta}(x, y)$ and then define an operator $G_{d-\Delta}$ through the integral $(G_{d-\Delta}\mathcal{O}_\Delta)(x) := \int d^d y \, G_{d-\Delta}(x, y)\mathcal{O}_\Delta(y)$. In this case $(G_{d-\Delta}\mathcal{O}_\Delta)(x)$ becomes a primary operator of scaling dimension $d-\Delta$. In short, G_α is a map from one primary operator to another, where $\alpha \in \{\Delta, d-\Delta\}$. In the literature [7] $(G_\alpha \mathcal{O}_{d-\alpha})(x)$ is called the shadow operator of $\mathcal{O}_{d-\alpha}(x)$.

Let us now turn to the infinitesimal conformal invariance. If $g \in SO(2, d)$ is infinitesimally close to the identity element, (1) is recast into the following commutation relations:

$$[J^{ab}, \mathcal{O}_\Delta(x)] = -J^{ab}_\Delta(x, \partial_x)\mathcal{O}_\Delta(x). \tag{4}$$

Likewise, (2) becomes the following identities (the infinitesimal form of conformal Ward–Takahashi identities):

$$\left(J^{ab}_\Delta(x, \partial_x) + J^{ab}_\Delta(y, \partial_y)\right) G_\Delta(x, y) = 0. \tag{5}$$

Here $J^{ab} = -J^{ba}$ $(a, b = 0, 1, \ldots, d+1)$ are the generators of $SO(2, d)$ and satisfy the following commutation relations of the Lie algebra $\mathfrak{so}(2, d)$:

$$[J^{ab}, J^{cd}] = i(\eta^{ac}J^{bd} - \eta^{ad}J^{bc} - \eta^{bc}J^{ad} + \eta^{bd}J^{ac}), \tag{6}$$

where $\eta_{ab} = \eta^{ab} = \mathrm{diag}(-1, +1, \ldots, +1, -1)$. On the other hand, $J^{ab}_\Delta(x, \partial_x)$ are the following differential representations of J^{ab}:

$$J^{ab}_\Delta(x, \partial_x) = i\left(k^{\mu ab}(x)\partial_\mu + \frac{\Delta}{d}(\partial_\mu k^{\mu ab})(x)\right), \tag{7}$$

where $k^{\mu ab}(x) = -k^{\mu ba}(x)$ are the conformal Killing vectors given by

$$k^{\mu\nu\lambda}(x) = \eta^{\mu\nu}x^\lambda - \eta^{\mu\lambda}x^\nu, \qquad k^{\mu\nu d}(x) = \frac{\ell^2 - x\cdot x}{2\ell}\eta^{\mu\nu} + \frac{x^\mu x^\nu}{\ell}, \tag{8}$$

$$k^{\mu\nu, d+1}(x) = \frac{\ell^2 + x\cdot x}{2\ell}\eta^{\mu\nu} - \frac{x^\mu x^\nu}{\ell}, \qquad k^{\mu d, d+1}(x) = -x^\mu. \tag{9}$$

Here $\ell > 0$ is an arbitrary reference length scale which needs to be introduced to adjust the length dimensions of the equations. Note that these vectors satisfy the conformal Killing equations $\partial_\mu k_\nu{}^{ab} + \partial_\nu k_\mu{}^{ab} = \frac{2}{d}\eta_{\mu\nu}\partial_\rho k^{\rho ab}$.

Now, let $G_\Delta(x, y)$ satisfy the infinitesimal conformal Ward–Takahashi identities (5). Then, upon integration by parts one can prove the following identities:

$$\int d^d y\, J_\Delta^{ab}(x, \partial_x) G_\Delta(x, y) \mathcal{O}_{d-\Delta}(y) = \int d^d y\, G_\Delta(x, y) J_{d-\Delta}^{ab}(y, \partial_y) \mathcal{O}_{d-\Delta}(y), \quad (10)$$

or, more compactly,

$$(J_\Delta^{ab} G_\Delta \mathcal{O}_{d-\Delta})(x) = (G_\Delta J_{d-\Delta}^{ab} \mathcal{O}_{d-\Delta})(x), \quad (11)$$

where $(J_\alpha^{ab}\mathcal{O}_\alpha)(x) := J_\alpha^{ab}(x, \partial_x)\mathcal{O}_\alpha(x)$, $\alpha \in \{\Delta, d-\Delta\}$. Since this holds for arbitrary $\mathcal{O}_{d-\Delta}$ we get the following operator identities:

$$J_\Delta^{ab} G_\Delta = G_\Delta J_{d-\Delta}^{ab}. \quad (12)$$

These are the intertwining relations, and in this respect G_Δ is called the intertwining operator. As is evident from the above discussions the intertwining relations are essentially the same as the conformal Ward–Takahashi identities. There is, however, a big advantage of using (12): The operator identities (12) are basis independent and hence easy to manipulate in an algebraic language. In the rest of the paper we shall apply the intertwining relations to a certain (improper) basis for a representation space of conformal algebra. In other words, we shall apply (12) to a mode function $f_{\alpha,p}(x)$ in terms of which the operator $\mathcal{O}_\alpha(x)$ is expanded as $\mathcal{O}_\alpha(x) = \int \frac{d^d p}{(2\pi)^d}\, \tilde{\mathcal{O}}_\alpha(p) f_{\alpha,p}(x)$. In zero-temperature CFT such mode function is just the plane wave $e^{ip\cdot x}$. In this case the intertwining relations just result in the well-known momentum-space conformal Ward–Takahashi identities at zero temperature. In finite-temperature CFT thermalized via the Unruh effect, on the other hand, $f_{\alpha,p}(x)$ becomes a quite nontrivial function. In a more algebraic language, $f_{\alpha,p}(x)$ is chosen to be an eigenfunction for the generator of one-parameter subgroup $SO(1,1) \subset SO(2,d)$. Before going to study the intertwining relations in the $SO(1,1)$ diagonal basis, let us first recall the significance of $SO(1,1)$ for finite-temperature CFT.

3 Timelike Conformal Killing Vectors Associated with the Subgroup $SO(1,1) \subset SO(2,d)$

Let us start with the KMS condition [9]. The KMS condition is a thermal equilibrium condition for quantum systems and expressed as an analytic condition for positive- and negative-frequency two-point Wightman functions $G^+(t)$ and $G^-(t)$. It demands that (i) $G^+(t)$ ($G^-(t)$) should be an analytic function on the strip $-\beta < \operatorname{Im} t < 0$

($0 < \mathrm{Im}\, t < \beta$); and (ii) $G^+(t)$ and $G^-(t)$ should satisfy the following boundary conditions on the strips:

$$G^+(t) = G^-(t + i\beta) \;\;\&\;\; G^-(t) = G^+(t - i\beta), \quad \forall t \in \mathbb{R}, \tag{13}$$

where $\beta = 1/T$ is the inverse temperature. (For the moment we will suppress the spatial coordinates.) The advantage of using the KMS condition is that these analytic conditions remain valid even after the thermodynamic limit. (Note that the extensive property of the free energy $F = -(1/\beta) \log \mathrm{Tr}\, \mathrm{e}^{-\beta H}$ would render the density matrix $\rho = \mathrm{e}^{-\beta(H-F)}$ ill-defined in the thermodynamic limit.) For a full account of the KMS condition we refer to [9, 10].

Now, let us take a closer look at the boundary conditions (13). These conditions are best understood in statistical mechanics for finite degrees of freedom in a finite box. Let $\mathcal{O}(t) = \mathrm{e}^{iHt}\mathcal{O}(0)\mathrm{e}^{-iHt}$ be a Heisenberg operator. Then we have

$$\begin{aligned}\langle \mathcal{O}(t)\mathcal{O}^\dagger(t')\rangle &= \frac{1}{Z}\mathrm{Tr}\left(\mathrm{e}^{-\beta H}\mathcal{O}(t)\mathcal{O}^\dagger(t')\right) = \frac{1}{Z}\mathrm{Tr}\left(\mathrm{e}^{-\beta H}\mathcal{O}(t)\mathrm{e}^{\beta H}\mathrm{e}^{-\beta H}\mathcal{O}^\dagger(t')\right)\\ &= \frac{1}{Z}\mathrm{Tr}\left(\mathrm{e}^{-\beta H}\mathcal{O}^\dagger(t')\mathrm{e}^{-\beta H}\mathcal{O}(t)\mathrm{e}^{\beta H}\right)\\ &= \frac{1}{Z}\mathrm{Tr}\left(\mathrm{e}^{-\beta H}\mathcal{O}^\dagger(t')\mathcal{O}(t+i\beta)\right) = \langle \mathcal{O}^\dagger(t')\mathcal{O}(t+i\beta)\rangle,\end{aligned} \tag{14}$$

where $Z = \mathrm{Tr}\, \mathrm{e}^{-\beta H}$ is the partition function. The second line follows from the cyclic property of trace and the last line the identity $\mathrm{e}^{izH}\mathcal{O}(t)\mathrm{e}^{-izH} = \mathcal{O}(t+z)$ with $z = i\beta$. Setting $t' = 0$ we get the condition $G^+(t) = G^-(t + i\beta)$. Likewise, one can prove $G^-(t) = G^+(t - i\beta)$ in a similar manner.

The above discussion is based on the expectation value with respect to the density matrix $\rho = \mathrm{e}^{-\beta H}/Z$. However, the boundary conditions (13) themselves can be formulated without recourse to the density matrix. Suppose that there exist a state $|\Omega\rangle$ and an antiunitary operator J such that the following identity holds:

$$J\mathrm{e}^{-\frac{\beta}{2}H}\mathcal{O}(t)|\Omega\rangle = \mathcal{O}^\dagger(t)|\Omega\rangle, \tag{15}$$

where $\mathcal{O}(t)$ is an arbitrary Heisenberg operator and H is assumed to satisfy $H|\Omega\rangle = 0$. Once we have the identity (15), we can prove that the Wightman functions with respect to the state $|\Omega\rangle$ satisfy (13). Indeed, by using the inner product notation $(*, *)$ we have (see also Chapter 5 of [15])

$$\begin{aligned}\langle \Omega|\mathcal{O}(t)\mathcal{O}^\dagger(t')|\Omega\rangle &= (|\Omega\rangle, \mathcal{O}(t)\mathcal{O}^\dagger(t')|\Omega\rangle) = (\mathcal{O}^\dagger(t)|\Omega\rangle, \mathcal{O}^\dagger(t')|\Omega\rangle)\\ &= (J\mathrm{e}^{-\frac{\beta}{2}H}\mathcal{O}(t)|\Omega\rangle, J\mathrm{e}^{-\frac{\beta}{2}H}\mathcal{O}(t')|\Omega\rangle)\\ &= (\mathrm{e}^{-\frac{\beta}{2}H}\mathcal{O}(t')|\Omega\rangle, \mathrm{e}^{-\frac{\beta}{2}H}\mathcal{O}(t)|\Omega\rangle)\\ &= (|\Omega\rangle, \mathcal{O}^\dagger(t')\mathrm{e}^{-\beta H}\mathcal{O}(t)\mathrm{e}^{\beta H}\mathrm{e}^{-\beta H}|\Omega\rangle)\end{aligned}$$

$$= (|\Omega\rangle, \mathcal{O}^\dagger(t')\mathcal{O}(t+i\beta)|\Omega\rangle)$$
$$= \langle\Omega|\mathcal{O}^\dagger(t')\mathcal{O}(t+i\beta)|\Omega\rangle, \tag{16}$$

where the second line follows from the assumption (15), the third line the antiunitarity of J (i.e., $(J|\Psi\rangle, J|\Phi\rangle) = \overline{(|\Psi\rangle, |\Phi\rangle)} = (|\Phi\rangle, |\Psi\rangle)$), and the fifth line the relations $e^{-\beta H}\mathcal{O}(t)e^{\beta H} = \mathcal{O}(t+i\beta)$ and $e^{-\beta H}|\Omega\rangle = |\Omega\rangle$. Setting $t' = 0$ we get the condition $G^+(t) = G^-(t+i\beta)$. Likewise, one can prove $G^-(t) = G^+(t-i\beta)$. These mean that, if (15) holds, the Wightman functions with respect to the state $|\Omega\rangle$ are nothing but the thermal Wightman functions at temperature $T = 1/\beta$ (except the question of analyticity on the strips).

The above discussion, though simplified, captures the essence of the interplay between the KMS condition and the Bisognano–Wichmann theorem [1, 2]. In the mid-1970s Bisognano and Wichmann showed that there exists the identity (15) in generic Poincaré-invariant quantum field theories. There, the state $|\Omega\rangle$ is the vacuum state $|0\rangle$ for inertial observers, J is the CPT conjugate (with a partial reflection), and $\frac{\beta}{2\pi}H$ is the generator of Lorentz boost. The temporal coordinate t is proportional to the dimensionless Lorentz boost parameter θ and identified as $\theta = (2\pi/\beta)t$. Physically speaking, t is identical to the proper time for uniformly accelerating observers and the proportional coefficient $2\pi/\beta$ is identical to the proper acceleration a, from which we can deduce the Unruh temperature $T = a/(2\pi)$. This is the physical content of Bisognano–Wichmann theorem, which provides a nonperturbative proof for the thermality of Wightman functions with respect to the vacuum [14].

Now we have come to the point. From a group theoretical viewpoint the most important thing in the Bisognano–Wichmann theorem is that the time-translation generator H is given by the Lorentz boost generator—the generator of one-parameter subgroup $SO(1,1)$ of the Poincaré group $ISO(1,d-1)$. In Poincaré-invariant quantum field theories the Lorentz boost is the only way to realize the group $SO(1,1)$ as a coordinate transformation. However, there emerge several options if the theory enjoys conformal invariance. Typical examples are the following [13]:

$$SO(1,1): x^\mu \mapsto x^\mu(\theta) = \Lambda^\mu{}_\nu x^\nu, \tag{17}$$
$$SO(1,1): x^\mu \mapsto x^\mu(\theta) = e^{-\theta}x^\mu, \tag{18}$$
$$SO(1,1): x^\mu \mapsto x^\mu(\theta) = e^{-\varphi}\frac{x^\mu - b^\mu(x\cdot x)}{1-2(b\cdot x)+(b\cdot b)(x\cdot x)} + a^\mu, \tag{19}$$

where $\Lambda = \begin{pmatrix} \cosh\theta & \sinh\theta & \\ \sinh\theta & \cosh\theta & \\ & & \mathbf{1}_{d-2} \end{pmatrix}$, $\varphi = 2\log\cosh\frac{\theta}{2}$, $b^\mu = (\frac{1}{\ell}\tanh\frac{\theta}{2}, 0, \ldots, 0)$, and $a^\mu = (\ell\tanh\frac{\theta}{2}, 0, \ldots, 0)$. Note that (17) is a Lorentz boost on the (x^0, x^1)-plane, (18) is a dilatation, and (19) is a special conformal transformation followed by a dilatation followed by a translation. Note also that these transformations are the solutions to the following flow equations generated by the conformal Killing vectors $k^{\mu 10}$, $k^{\mu d,d+1}$, and $k^{\mu d0} = -k^{\mu 0d}$:

$$\dot{x}^\mu(\theta) = \eta^{\mu 1} x^0(\theta) - \eta^{\mu 0} x^1(\theta), \tag{20}$$

$$\dot{x}^\mu(\theta) = -x^\mu(\theta), \tag{21}$$

$$\dot{x}^\mu(\theta) = -\frac{\ell^2 - x(\theta)\cdot x(\theta)}{2\ell}\eta^{\mu 0} - \frac{x^\mu(\theta) x^0(\theta)}{\ell}, \tag{22}$$

where dot stands for the derivative with respect to θ.

Now we wish to identify the parameter θ with the temporal coordinate t (up to the factor $2\pi/\beta$). To justify this, the above conformal Killing vectors must be *timelike*; that is, $\dot{x}(\theta)\cdot\dot{x}(\theta) < 0$. It is a straightforward exercise to classify their timelike domains. The results are as follows (see also Fig. 1):

- **Rindler wedge**. The Killing vector (20) becomes timelike in the following domains:

$$W_\pm = \{x^\mu : \pm x^1 > |x^0|\}, \tag{23}$$

  which are nothing but the right and left Rindler wedges. The coordinate systems in which (17) yields the time-translation are given by

$$x^0 = \pm\ell\frac{\sinh(t/\ell)}{H^0 + H^1}, \quad x^1 = \pm\ell\frac{\cosh(t/\ell)}{H^0 + H^1}, \quad x^i = \ell\frac{H^i}{H^0 + H^1}, \tag{24}$$

  where $H^\mu = (H^0, H^1, \ldots, H^{d-1})$ describes the upper half of two-sheeted hyperbolic space $\mathbb{H}^{d-1}$ and is subject to the conditions $H\cdot H \equiv -(H^0)^2 + (H^1)^2 + \cdots + (H^{d-1})^2 = -1$ and $H^0 \geq 1$. The induced metrics on $W_\pm$ are

$$ds^2_{W_\pm} = \frac{-dt^2 + \ell^2 dH\cdot dH}{(H^0 + H^1)^2}. \tag{25}$$

- **Light-cone**. The conformal Killing vector (21) becomes timelike in the following domains:

$$V_\pm = \{x^\mu : \pm x^0 > |\mathbf{x}|\}, \tag{26}$$

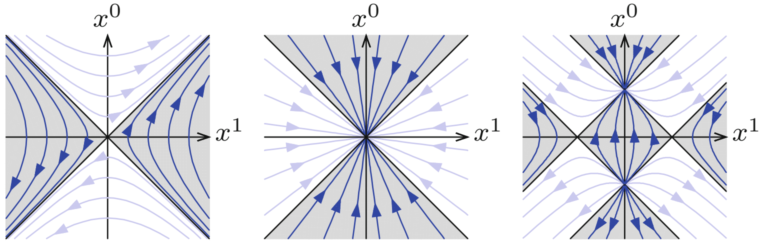

Fig. 1 Timelike domains of the $SO(1,1)$ conformal Killing vectors (20)–(22) in the (x^0, x^1)-plane. Temporal flows are identified with thick blue curves

which are nothing but the future and past light-cones. The coordinate systems in which (18) yields the time-translation are given by

$$x^\mu = \pm \ell e^{-t/\ell} H^\mu, \tag{27}$$

where H^μ is the same as above. The induced metrics on $V_\pm$ are

$$ds^2_{V_\pm} = e^{-2t/\ell}(-dt^2 + \ell^2 dH \cdot dH). \tag{28}$$

- **Diamond**. The conformal Killing vector (22) becomes timelike in the following domain[1]:

$$D = \{x^\mu : |\mathbf{x}| + |x^0| < \ell\}, \tag{29}$$

which is nothing but the diamond (or double cone). The coordinate system in which (19) yields the time-translation is given by

$$x^0 = \ell \frac{\sinh(t/\ell)}{\cosh(t/\ell) + H^0}, \quad x^i = \ell \frac{H^i}{\cosh(t/\ell) + H^0}, \tag{30}$$

where H^μ is the same as above. The induced metric on D is

$$ds^2_D = \frac{-dt^2 + \ell^2 dH \cdot dH}{(\cosh(t/\ell) + H^0)^2}. \tag{31}$$

Now it is obvious that these subspaces of the flat Minkowski spacetime $\mathbb{R}^{1,d-1}$ are all conformal to $\mathbb{H}^1 \times \mathbb{H}^{d-1} \ni (t, H^\mu)$. Hence the correlation functions on $\mathbb{H}^1 \times \mathbb{H}^{d-1}$ with respect to the inertial vacuum $|0\rangle$ are just given by conformal transformations of those in the Cartesian coordinate system. For example, the positive- and negative-frequency two-point Wightman functions $\langle 0|\mathcal{O}_\Delta(t, H)\mathcal{O}^\dagger_\Delta(t', H')|0\rangle$ and $\langle 0|\mathcal{O}^\dagger_\Delta(t', H')\mathcal{O}_\Delta(t, H)|0\rangle$ are given by

$$\left[\frac{2\pi^2 T^2}{-\cosh(2\pi T(t - t' \mp i\epsilon)) - H \cdot H'}\right]^\Delta, \tag{32}$$

where $T = 1/(2\pi\ell)$. It can be shown that these Wightman functions indeed satisfy the KMS condition and hence give the thermal correlation functions on $\mathbb{H}^1 \times \mathbb{H}^{d-1}$ at temperature T [13]. We note that there also exist theorems [3, 11] which generalize the Bisognano–Wichmann theorem and consider the conformal Killing vectors (21) and (22) and their timelike domains.

So far we have considered correlation functions in position space. For practical applications, however, we often need to know correlation functions in momentum

[1] In fact, as depicted in Fig. 1 the conformal Killing vector (22) becomes timelike also in the domains $K = \{x^\mu : |\mathbf{x}| - |x^0| > \ell\}$ and $\mathcal{V}_\pm = \{x^\mu : \pm x^0 > |\mathbf{x}| + \ell\}$.

space. A standard approach to momentum-space correlators is the Fourier transform of position-space correlators. However, the Fourier transform of correlation functions is generally hard to carry out. In fact, the Fourier transform of (32) is, though not impossible, quite complicated and requires a lot of integration techniques. Hence it would be desirable to develop a method which bypasses Fourier integrals and directly leads to momentum-space expressions.[2] In the rest of the paper we will see that the intertwining relations do the job: The operator identities (12) enable us to deduce momentum-space two-point functions in a purely Lie-algebraic fashion.

4 Intertwining Relations in the $SO(1, 1)$ Basis

Let us finally move on to the intertwining relations in the $SO(1, 1)$ diagonal basis—the conformal Ward–Takahashi identities at finite temperature. We emphasize that this section is rather sketchy. For more details we refer to our paper [13]. In what follows we shall set $2\pi T = 1/\ell = 1$ for simplicity. The temperature dependence is easily restored by dimensional analysis.

To begin with, let us consider the quadratic Casimir operator of the Lie algebra $\mathfrak{so}(2, d)$, which is given by

$$C_2[\mathfrak{so}(2, d)] = \frac{1}{2} J_{ab} J^{ab}. \tag{33}$$

We wish to identify the $SO(1, 1)$ generator as the time-translation generator H. In group theoretical language, this means that we need to work with the basis where the following subgroup becomes diagonal:

$$SO(1, 1) \times SO(1, d-1) \subset SO(2, d). \tag{34}$$

Correspondingly, the quadratic Casimir operator is decomposed as follows:

$$C_2[\mathfrak{so}(2, d)] = -H(H \pm id) - \eta_{ab} E^{\mp a} E^{\pm b} + C_2[\mathfrak{so}(1, d-1)], \tag{35}$$

where $E^{\pm a}$ are certain linear combinations of J^{ab} and $C_2[\mathfrak{so}(1, d-1)]$ is the quadratic Casimir operator of the subalgebra $\mathfrak{so}(1, d-1)$. For example, in the case of Rindler wedge we have $H = J^{10}$, $E^{\pm a} = J^{0a} \pm J^{1a}$, and $C_2[\mathfrak{so}(1, d-1)] = (1/2) J_{ab} J^{ab}$, where a and b run through 2 to $d+1$.

[2] In zero-temperature CFT such a method would simply fall into the study of conformal Ward–Takahashi identities in momentum space. In Euclidean signature this approach was thoroughly discussed in [4] (see also [5]).

Now let $|\Delta, \omega, k; \sigma\rangle$ be a simultaneous eigenstate of $C_2[\mathfrak{so}(2,d)]$, H, and $C_2[\mathfrak{so}(1,d-1)]$ that satisfies the following eigenvalue equations:

$$C_2[\mathfrak{so}(2,d)]|\Delta, \omega, j; \sigma\rangle = \Delta(\Delta - d)|\Delta, \omega, j; \sigma\rangle, \tag{36}$$

$$H|\Delta, \omega, j; \sigma\rangle = \omega|\Delta, \omega, j; \sigma\rangle, \tag{37}$$

$$C_2[\mathfrak{so}(1,d-1)]|\Delta, \omega, j; \sigma\rangle = j(j-d+2)|\Delta, \omega, j; \sigma\rangle, \tag{38}$$

where σ stands for eigenvalues of other simultaneously commuting generators which are irrelevant in the following discussion. Below we shall focus on the case $j(j-d+2) < -(d-2)^2/4$ and parameterize j as follows:

$$j = \frac{d-2}{2} + ik, \quad k \in (0, \infty). \tag{39}$$

In other words, we shall focus on the principal series representation of $\mathfrak{so}(1,d-1)$. Note that $j(j-d+2) = -k^2 - (d-2)^2/4$ is real though j is complex. Physically, k plays the role of the modulus of spatial momentum. From now on we shall write the eigenstate as $|\Delta, \omega, k; \sigma\rangle$.

Now there are two important things for the following discussion. The first is that the eigenvalue $\Delta(\Delta - d)$ is invariant under the exchange $\Delta \to d - \Delta$, which means that the vectors $|\Delta, \omega, k; \sigma\rangle$ and $|d-\Delta, \omega, k; \sigma\rangle$ share the same eigenvalue of $C_2[\mathfrak{so}(2,d)]$. These two vectors are mapped to each other by the intertwining operators and satisfy the following relations:

$$G_\alpha|d-\alpha, \omega, k; \sigma\rangle = \tilde{G}_\alpha(\omega, k)|\alpha, \omega, k; \sigma\rangle, \quad \alpha \in \{\Delta, d-\Delta\}, \tag{40}$$

where the proportional coefficients $\tilde{G}_\alpha(\omega, k)$ are the momentum-space two-point functions. From now on J_α^{ab}, H_α, $E_\alpha^{\pm a}$, etc. denote the $SO(2,d)$ generators that act on the representation space spanned by the vectors $\{|\alpha, \omega, k; \sigma\rangle\}$. For example, their differential representations are given in (7).

The second important thing is the set of generators $E_\alpha^{\pm a}$. One can show that there exist certain linear combinations $E_\alpha^\pm$ of these generators that satisfy the following ladder equations:

$$\begin{aligned} E_\alpha^\pm|\alpha, \omega, k; \sigma\rangle &= A^\pm\left[\alpha - \tfrac{d-2}{2} \mp i(\omega \pm k)\right]|\alpha, \omega \pm i, k+i; \sigma\rangle \\ &\quad + B^\pm\left[\alpha - \tfrac{d-2}{2} \mp i(\omega \mp k)\right]|\alpha, \omega \pm i, k-i; \sigma\rangle. \end{aligned} \tag{41}$$

For example, in the case of Rindler wedge they are given by $E_\alpha^\pm = E_\alpha^{\pm d} + E_\alpha^{\pm(d+1)}$. Note that $A^\pm$ and $B^\pm$ are α-independent irrelevant factors.

Now we have almost done. Let us finally consider the intertwining relations $E_\Delta^\pm G_\Delta = G_\Delta E_{d-\Delta}^\pm$. Applying these to the state $|d-\Delta, \omega, k; \sigma\rangle$ we get

$$E_\Delta^\pm G_\Delta|d-\Delta, \omega, k; \sigma\rangle = G_\Delta E_{d-\Delta}^\pm|d-\Delta, \omega, k; \sigma\rangle. \tag{42}$$

It follows from (40) and (41) that the identities (42) result in the following nontrivial functional equations in complex momentum space:

$$\tilde{G}_\Delta(\omega \pm i, k \pm i) = \frac{\Delta - \frac{d-2}{2} \mp i(\omega + k)}{\tilde{\Delta} - \frac{d-2}{2} \mp i(\omega + k)} \tilde{G}_\Delta(\omega, k), \tag{43}$$

$$\tilde{G}_\Delta(\omega \pm i, k \mp i) = \frac{\Delta - \frac{d-2}{2} \mp i(\omega - k)}{\tilde{\Delta} - \frac{d-2}{2} \mp i(\omega - k)} \tilde{G}_\Delta(\omega, k), \tag{44}$$

where $\tilde{\Delta} = d - \Delta$ is the scaling dimension of the shadow operator. Since these are kind of recurrence relations, we can guess the solution by iteration. "Minimal" solutions to the recurrence relations are as follows:

$$\tilde{G}^{\pm}_\Delta(\omega, k) \propto \mathrm{e}^{\pm\pi\omega} \left| \Gamma\left(\frac{\Delta - \frac{d-2}{2} + i(\omega+k)}{2} \right) \right|^2 \left| \Gamma\left(\frac{\Delta - \frac{d-2}{2} + i(\omega-k)}{2} \right) \right|^2, \tag{45}$$

which can be interpreted as the positive- and negative-frequency Wightman functions. Indeed, these satisfy the KMS condition in momentum space, $\tilde{G}^+_\Delta(\omega, k) = \mathrm{e}^{2\pi\omega} \tilde{G}^-_\Delta(\omega, k)$. One can also check that the solutions (45) exactly coincide with the Fourier transform of (32) [13]. Note that T can be restored by the replacements $\omega \to \omega/(2\pi T)$ and $k \to k/(2\pi T)$.

To summarize, we have seen that the intertwining relations, which are just the conformal Ward–Takahashi identities in disguise, result in the recurrence relations (43) and (44) when applied to the $SO(1, 1)$ diagonal basis. These are the conformal Ward–Takahashi identities at finite temperature and give us nontrivial constraints on momentum-space two-point functions. Though may need a bit of experience, one can deduce the momentum-space correlators from these constraints without recourse to the notoriously complicated Fourier transform. We think this is a big step toward the understanding of real-time momentum-space correlators in $d (\geq 3)$-dimensional finite-temperature CFT, because these have not been studied in the literature. In fact, for $d \geq 3$ and at nonzero temperature, even the momentum-space two-point functions of scalar primary operators have been unknown. It would be quite interesting to generalize our approach to thermal spinning correlators.

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Renormalization in Some 2D $\widehat{su}(2)$ Coset Models

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Abstract We consider a RG flow in certain 2D coset models perturbed by the least relevant field. In the case of the symmetric su(2) coset model we show, up to second order of the perturbation theory, that there exists a nontrivial IR fixed point. We obtain the structure constants and the four-point functions of certain fields by deriving specific recursive relations. This allows us to compute the anomalous dimensions and the mixing coefficients of these fields in the UV and IR theories. In the case of another su(2) coset model, describing the N = 2 superconformal theories, we show that there does not exists a nontrivial IR fixed point up to second order.

Keywords RG flow · Mixing of fields · Anomalous dimensions · Coset models

1 Introduction

In the first part of this paper we consider the symmetric $\widehat{su}(2)$ coset model $M(k, l)$ [1] perturbed by the least relevant operator. It is known [2] that there exists an infrared fixed point of the renormalization group flow of this theory which coincides with the model $M(k-l, l)$. Here we are interested in the mixing of certain fields under the corresponding RG flow. It is known that the mixing coefficients coincide for $l = 1$ (Virasoro) and $l = 2$ (superconformal) theories (particular cases of $M(k, l)$) up to the second order of the perturbation theory [3]. We will show that this is the case in the general theory, i.e. they do not depend on l and are finite up to the second order. For that purpose one needs in addition to the structure constants also the corresponding four-point functions which are not known exactly. We find it convenient, following [2], to use the construction presented in [4]. Namely, we define the perturbing field and the other fields in consideration recursively as a product of lower level fields. Then the corresponding structure constants and four-point functions at some level l,

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_22

governing the perturbation expansion, can be obtained recursively from those of the lower levels by certain projected tensor product.

In the second part of the paper we discuss the renormalization group properties of the $N = 2$ superconformal minimal models. It is known that these models are connected to another $\hat{su}(2)$ based coset theories. The latter determine the so called parafermionic construction [5, 6]. It is very useful for the calculation of the 4-point functions and the structure constants of the 2D OPE algebra. The reason for that is in the relation of the parafermionic models with the $su(2)$ Wess–Zumino–Witten (WZW) models [7]. We compute the β function up to second order in the perturbation theory and show that it doesn't possesses a non-trivial fixed point. We argue that this is true also in higher orders.

2 Symmetric $\hat{su}(2)$ Coset Models

In this Section we present the general $\hat{su}(2)$ coset model perturbed by the least relevant field. We obtain the β function and show that it has a non-trivial fixed point up to second order in the perturbation theory. We also construct certain fields and find their anomalous dimensions and the corresponding mixing matrix.

2.1 The Theory

Consider a two-dimensional CFT $M(k, l)$ based on the coset:

$$\frac{\widehat{su}(2)_k \times \widehat{su}(2)_l}{\widehat{su}(2)_{k+l}}, \tag{1}$$

where k and l are integers, we assume $k > l$. It is written in terms of $\widehat{su}(2)_k$ WZNW models with current J^a, k is the level. The latter are CFT's with a stress tensor expressed through the currents by the Sugawara construction, the central charge is $c_k = \frac{3k}{k+2}$. The energy momentum tensor of the coset is then $T = T_k + T_l - T_{k+l}$ and:

$$c = \frac{3kl(k + l + 4)}{(k + 2)(l + 2)(k + l + 2)} = \frac{3l}{l + 2}\left(1 - \frac{2(l + 2)}{(k + 2)(k + l + 2)}\right).$$

The dimensions of the primary fields $\phi_{m,n}(l, p)$ of the "minimal models" (m, n are integers) are computed in [8]:

$$\Delta_{m,n}(l,p) = \frac{((p+l)m - pn)^2 - l^2}{4lp(p+l)} + \frac{s(l-s)}{2l(l+2)},$$
$$= |m-n|(mod(l)), \qquad 0 \le s \le l,$$
$$1 \le m \le p-1, \qquad 1 \le n \le p+l-1 \tag{2}$$

where we introduced $\mathbf{p} = \mathbf{k} + \mathbf{2}$ (note that we inverted k and l in the definition of the fields).

In this paper we will use a description of the theory $M(k,l)$ presented in [4]. It was shown there that this theory is not independent but can be built out of products of theories of lower levels. Schematically this can be written as a recursion:

$$M(1, l-1) \times M(k,l) = \mathbf{P}(M(k,1) \times M(k+1, l-1)) \tag{3}$$

where $\mathbf{P}$ in the RHS is a specific projection. It allows the multiplication of fields of the same internal indices and describes primary and descendent fields.

In the following we will be interested in the CFT $M(k,l)$ perturbed by the least relevant field. The theory is described by the Lagrangian:

$$\mathcal{L}(x) = \mathcal{L}_0(x) + \lambda\tilde{\phi}(x)$$

where $\mathcal{L}_0(x)$ describes the theory $M(k,l)$ itself. We define the field $\tilde{\phi} = \tilde{\phi}_{1,3}$ in terms of lower level fields:

$$\tilde{\phi}_{1,3}(l,p) = a(l,p)\phi_{1,1}(1,p)\tilde{\phi}_{1,3}(l-1,p+1) + b(l,p)\phi_{1,3}(1,p)\phi_{3,3}(l-1,p+1). \tag{4}$$

Here the field $\phi_{3,3}(l,p)$ is just a primary field form (2). The dimension of the field (4) is:

$$\Delta = \Delta_{1,3} + \frac{l}{l+2} = 1 - \frac{2}{p+l} = 1 - \epsilon. \tag{5}$$

In this paper we consider the case $p \to \infty$ and assume that $\epsilon = \frac{2}{p+l} \ll 1$ is a small parameter. The coefficients $a(l,p)$ and $b(l,p)$ as well as the structure constants of the fields involved in the construction (4) can be found by demanding the closure of the fusion rules [2].

The mixing of the fields along the RG flow is connected to the two-point function. Up to the second order of the perturbation theory it is given by:

$$< \phi_1(x)\phi_2(0) > = < \phi_1(x)\phi_2(0) >_0 - \lambda \int < \phi_1(x)\phi_2(0)\tilde{\phi}(y) >_0 d^2y +$$
$$+ \frac{\lambda^2}{2} \int < \phi_1(x)\phi_2(0)\tilde{\phi}(x_1)\tilde{\phi}(x_2) >_0 d^2x_1 d^2x_2 + \ldots \tag{6}$$

where ϕ_1, ϕ_2 can be arbitrary fields of dimensions Δ_1, Δ_2. The first order corrections are expressed through the structure constants. Let us focus here on the second order.

One can use the conformal transformation properties of the fields to bring the double integral to the form:

$$\int < \phi_1(x)\phi_2(0)\tilde{\phi}(x_1)\tilde{\phi}(x_2) >_0 d^2x_1 d^2x_2 =$$
$$= (x\bar{x})^{2-\Delta_1-\Delta_2-2\Delta} \int I(x_1) < \tilde{\phi}(x_1)\phi_1(1)\phi_2(0)\tilde{\phi}(\infty) >_0 d^2x_1 \tag{7}$$

where:

$$I(x) = \int |y|^{2(a-1)}|1-y|^{2(b-1)}|x-y|^{2c} d^2y \tag{8}$$

and $a = 2\epsilon + \Delta_2 - \Delta_1, b = 2\epsilon + \Delta_1 - \Delta_2, c = -2\epsilon$. It is well known that the integral for $I(x)$ can be expressed in terms of hypergeometric functions whose behaviour around the points 0, 1 and ∞ is well known. It is clear that the integral (7) is singular. We follow the regularization procedure proposed in [10]. It was proposed there to cut discs in the two-dimensional surface of radius r ($\frac{1}{r}$) around singular points 0, 1 (∞) with $0 \ll r_0 \ll r < 1$, where r_0 is the ultraviolet cut-off. The additional parameter r is not physical and should not appear in the final result. The region outside these discs, where the integration is well-defined, is called Ω_{r,r_0}. Near the singular points one can use the OPE. The final result is a sum of all these contributions. It turns out however that we count twice two lens-like regions around the point 1 so we have to subtract those integrals.

Let us consider the correlation function that enters the integral (7). The basic ingredients for the computation of the four-point correlation functions are the conformal blocks. According to the construction (3) any field $\phi_{m,n}(l, p)$ (or its descendent) can be expressed recursively as a product of lower level fields. Therefore the corresponding conformal blocks will be a product of lower level conformal blocks. Due to the RHS of (3) only certain products of conformal blocks will survive the projection **P**.

Let us consider for example the correlation function of the perturbing field itself. The corresponding conformal blocks are linear combinations of products of conformal blocks at levels 1 and $l-1$. In view of the construction (4) there are in general 16 terms. Some of them are absent because of the fusion rules in each intermediate channel. Here there are three channels: identity $\phi_{1,1}$, the field $\tilde{\phi}_{1,3}$ itself and the descendent field $\tilde{\phi}_{1,5}$ which is defined in a way similar to that of $\tilde{\phi}_{1,3}$. We compute the conformal blocks up to a sufficiently high level and make a guess (remind that we need the result in the leading order in $\epsilon \to 0$). As a result, we obtain the following 2D correlation function:

$$< \tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}(1)\tilde{\phi}(\infty) >= \tag{9}$$
$$= \left| \frac{1}{x^2(1-x)^2} \left[1 - 2x + \left(\frac{5}{3} + \frac{4}{3l}\right)x^2 - \left(\frac{2}{3} + \frac{4}{3l}\right)x^3 + \frac{1}{3}x^4 \right] \right|^2 +$$

$$+\frac{16}{3l^2}\left|\frac{1}{x(1-x)^2}\left[1-\frac{3}{2}x+\frac{l+1}{2}x^2-\frac{l}{4}x^3\right]\right|^2+$$
$$+\frac{5}{9}\left(\frac{2(l-1)}{l}\right)^2\left|\frac{1}{(1-x)^2}\left[1-x+\frac{l}{2(l-1)}x^2\right]\right|^2.$$

One can check that this function is crossing symmetric and has a correct behaviour near the singular points.

We now use this function for the computation of the β-function up to the second order. For that purpose we have to compute the integral in (7). The integration over the safe region far from the singularities yields ($I(x) \sim \frac{\pi}{\epsilon}$):

$$\begin{aligned}&\int_{\Omega_{r,r_0}} I(x) < \tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}(1)\tilde{\phi}(\infty) > d^2x = \\ &= \frac{(29l^2-128l)\pi^2}{24\epsilon l^2}+\frac{2\pi^2}{\epsilon r^2}+\frac{\pi^2}{2\epsilon r_0^2}-\frac{64\pi^2\log r}{3\epsilon l^2}-\frac{32\pi^2\log 2r_0}{3\epsilon l^2}\end{aligned} \tag{10}$$

and we omitted the terms of order r or r_0/r.

We have to subtract the integrals over the lens-like regions since they would be accounted twice. Here is the result of that integration:

$$\frac{\pi^2}{\epsilon}\left(-\frac{1}{r^2}+\frac{1}{2r_0^2}+\frac{1}{24}\left(29+\frac{64}{l}\right)+\frac{32}{3l^2}\log\frac{r}{2r_0}\right).$$

Next we have to compute the integrals near the singular points 0, 1 and ∞. For that purpose we can use the OPE of the fields and take the appropriate limit of $I(x)$. Near the point 0 the relevant OPE is (by definition (4):

$$\tilde{\phi}(x)\tilde{\phi}(0) = (x\bar{x})^{-2\Delta}(1+\ldots)+C_{(1,3)(1,3)}^{(1,3)}(x\bar{x})^{-\Delta}(\tilde{\phi}(0)+\ldots).$$

The structure constant was computed in [2]. The value of $I(x)$ near 0 is given in [10] and finally we obtain:

$$\int_{D_{r,0}\backslash D_{r_0,0}} I(x) < \tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}(1)\tilde{\phi}(\infty) > d^2x = -\frac{\pi^2}{r^2\epsilon}+\frac{32\pi^2}{3l^2\epsilon^2}-\frac{32\pi^2}{l\epsilon}+\frac{32}{3l^2}\frac{\pi^2\log r}{\epsilon} \tag{11}$$

where the region of integration $D_{r,0}\backslash D_{r_0,0}$ is a ring with internal and external radiuses r_0 and r respectively. Since the integral near 1 gives obviously the same result, we just need to add the above result twice. To compute the integral near infinity, we use a relation

$$< \phi_1(x)\phi_2(0)\phi_3(1)\phi_4(\infty) >= (x\bar{x})^{-2\Delta_1} < \phi_1(1/x)\phi_4(0)\phi_3(1)\phi_2(\infty) >$$

and $I(x) \sim \frac{\pi}{\epsilon}(x\bar{x})^{-2\epsilon}$. This gives

$$\int_{D_{r,\infty}\setminus D_{r_0,\infty}} I(x) < \tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}(1)\tilde{\phi}(\infty) > d^2x = -\frac{\pi^2}{r^2\epsilon} + \frac{16\pi^2}{3l^2\epsilon^2} - 16\frac{\pi^2}{l\epsilon} + \frac{32\pi^2 \log r}{3l^2\epsilon}$$

where now $D_{r,\infty}\setminus D_{r_0,\infty}$ is a ring between $\frac{1}{r}$ and $\frac{1}{r_0}$.

Putting altogether, we obtain the finite part of the integral:

$$\frac{80\pi^2}{3l^2\epsilon^2} - \frac{88\pi^2}{l\epsilon}. \tag{12}$$

We want to remind also that we follow the renormalization scheme proposed in [10]. Therefore we already omitted the terms proportional to $r_0^{4\epsilon-2}$ which could be canceled by an appropriate counterterm in the action.

Taking into account also the first order term, we get the final result (up to the second order) for the two-point function of the perturbing field:

$$\begin{aligned} G(x,\lambda) &= < \tilde{\phi}(x)\tilde{\phi}(0) > \\ &= (x\bar{x})^{-2+2\epsilon}\left[1 - \lambda\frac{4\pi}{\sqrt{3}}\left(\frac{2}{l\epsilon} - 3\right)(x\bar{x})^{\epsilon} + \frac{\lambda^2}{2}\left(\frac{80\pi^2}{3l^2\epsilon^2} - \frac{88\pi^2}{l\epsilon}\right)(x\bar{x})^{2\epsilon} + \dots\right]. \end{aligned} \tag{13}$$

We now introduce a renormalized coupling constant g and a renormalized field $\tilde{\phi}^g = \partial_g\mathcal{L}$ analogously to $\tilde{\phi} = \partial_\lambda\mathcal{L}$. It is normalized by $< \tilde{\phi}^g(1)\tilde{\phi}^g(0) >= 1$. In this renormalization scheme the β-function is given by [9, 10]:

$$\beta(g) = \epsilon\lambda\frac{\partial g}{\partial\lambda} = \epsilon\lambda\sqrt{G(1,\lambda)}$$

One can invert this and compute the bare coupling constant and the β-function in terms of g:

$$\lambda = g + g^2\frac{\pi}{\sqrt{3}}\left(\frac{2}{l\epsilon} - 3\right) + g^3\frac{\pi^2}{3}\left(\frac{4}{l^2\epsilon^2} - \frac{10}{l\epsilon}\right) + \mathcal{O}(g^4), \tag{14}$$

$$\beta(g) = \epsilon g - g^2\frac{\pi}{\sqrt{3}}\left(\frac{2}{l} - 3\epsilon\right) - \frac{4\pi^2}{3l}g^3 + \mathcal{O}(g^4). \tag{15}$$

A nontrivial IR fixed point occurs at the zero of the β-function:

$$g^* = \frac{l\sqrt{3}}{2\pi}\epsilon\left(1 + \frac{l}{2}\epsilon\right). \tag{16}$$

It corresponds to the IR CFT $M(k-l,l)$ as can be seen from the central charge difference:

$$c^* - c = -\frac{4(l+2)}{l}\pi^2\int_0^{g^*}\beta(g)dg = -l\left(1+\frac{l}{2}\right)\epsilon^3 - \frac{3l^2}{4}(l+2)\epsilon^4 + \mathcal{O}(\epsilon^5).$$

The anomalous dimension of the perturbing field becomes

$$\Delta^* = 1 - \partial_g \beta(g)|_{g^*} = 1 + \epsilon + l\epsilon^2 + \mathcal{O}(\epsilon^3)$$

which matches with that of the field $\phi_{3,1}(l, p-l)$ of $M(k-l, l)$ (defined precisely below).

2.2 Mixing of the Fields

Let us define recursively the descendant fields $\tilde{\phi}_{n,n\pm 2}$:

$$\tilde{\phi}_{n,n+2}(l,p) = x\phi_{n,n}(1,p)\tilde{\phi}_{n,n+2}(l-1,p+1) + y\phi_{n,n+2}(1,p)\phi_{n+2,n+2}(l-1,p+1),$$
$$\tilde{\phi}_{n,n-2}(l,p) = \tilde{x}\phi_{n,n}(1,p)\tilde{\phi}_{n,n-2}(l-1,p+1) + \tilde{y}\phi_{n,n-2}(1,p)\phi_{n-2,n-2}(l-1,p+1)$$

(where x, $\tilde{x}$ and y, $\tilde{y}$ are at (l, p)) and the derivative $\partial\phi_{n,n}$ of the primary field

$$\phi_{n,n}(l,p) = \phi_{n,n}(1,p)\phi_{n,n}(l-1,p+1). \tag{17}$$

They have dimensions close to 1

$$\tilde{\Delta}_{n,n\pm 2} = 1 + \frac{n^2-1}{4p} - \frac{(2\pm n)^2-1}{4(p+l)} = 1 - \frac{1\pm n}{2}\epsilon + O(\epsilon^2),$$
$$1 + \Delta_{n,n} = 1 + \frac{n^2-1}{4p} - \frac{n^2-1}{4(p+l)} = 1 + \frac{(n^2-1)l}{16}\epsilon^2 + O(\epsilon^3). \tag{18}$$

This suggests that they mix along the RG-trajectory. To ensure this we ask that their fusion rules with the perturbing field are closed. This requirement defines the coefficients and the corresponding structure constants [11].

We want to compute the matrix of anomalous dimensions and the corresponding mixing matrix of the fields defined above. For that purpose we compute their two-point functions up to second order and the corresponding integrals (7). The first order integrals are proportional to the structure constants. For the second order calculation we need the corresponding four point functions. They are obtained in a way similar to that of the perturbing field $\tilde{\phi}(z)$ itself. The explicit form of the four-point functions we need: $< \tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n+2}(\infty) >$, $< \tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n-2}(\infty) >$ and $< \tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}_{n,n}(1)\tilde{\phi}_{n,n+2}(\infty) >$ can be found in [12].

Let us describe briefly the renormalization scheme. We introduce renormalized fields ϕ^g_α which are expressed through the bare ones by:

$$\phi^g_\alpha = B_{\alpha\beta}(\lambda)\phi_\beta \tag{19}$$

(here ϕ could be a primary or a descendent field). The two-point functions of the renormalized fields

$$G^g_{\alpha\beta}(x) = < \phi^g_\alpha(x)\phi^g_\beta(0) >, \quad G^g_{\alpha\beta}(1) = \delta_{\alpha\beta} \tag{20}$$

satisfy the Callan–Symanzik equation:

$$(x\partial_x - \beta(g)\partial_g)G^g_{\alpha\beta} + \sum_{\rho=1}^{2}(\Gamma_{\alpha\rho}G^g_{\rho\beta} + \Gamma_{\beta\rho}G^g_{\alpha\rho}) = 0.$$

The matrix of anomalous dimensions Γ that appears above is given by

$$\Gamma = B\hat{\Delta}B^{-1} - \epsilon\lambda B\partial_\lambda B^{-1} \tag{21}$$

where $\hat{\Delta} = diag(\Delta_1, \Delta_2)$ is a diagonal matrix of the bare dimensions. The matrix B, as defined in (19), is computed from the matrix of the bare two-point functions we computed, using the normalization condition (20) and requiring the matrix Γ to be symmetric.

Let us combine the fields in consideration in a vector with components:

$$\phi_1 = \tilde{\phi}_{n,n+2}, \quad \phi_2 = (2\Delta_{n,n}(2\Delta_{n,n}+1))^{-1}\partial\bar{\partial}\phi_{n,n}, \quad \phi_3 = \tilde{\phi}_{n,n-2}.$$

The field ϕ_2 is normalized so that its bare two-point function is 1.

We can write the matrix of the bare two-point functions $G_{\alpha,\beta}(x,\lambda) = < \phi_\alpha(x)$ $\phi_\beta(0) >$ up to the second order in the perturbation expansion as:

$$G_{\alpha,\beta}(x,\lambda) = (x\bar{x})^{-\Delta_\alpha-\Delta_\beta}\left[\delta_{\alpha,\beta} - \lambda C^{(1)}_{\alpha,\beta}(x\bar{x})^\epsilon + \frac{\lambda^2}{2}C^{(2)}_{\alpha,\beta}(x\bar{x})^{2\epsilon} + \dots\right]. \tag{22}$$

As we already mentioned, the two-point functions in the first order are proportional to the structure constants [9]. The second order contribution is a result of the double integration in (7) of the four-point functions mentioned above. This integration goes along the same lines as in the case of the perturbing field.

Using the entries $C^{(1)}$ and $C^{(2)}$ thus obtained we can apply the renormalization procedure and obtain the matrix of anomalous dimensions (21). The bare coupling constant λ is expressed through g by (14) and the bare dimensions, up to order ϵ^2. Evaluating this matrix at the fixed point (16), we get:

$$\begin{aligned}
\Gamma^{g^*}_{1,1} &= 1 + \frac{(20-4n^2)\epsilon}{8(n+1)} + \frac{l(39-n-7n^2+n^3)\epsilon^2}{16(n+1)}, \\
\Gamma^{g^*}_{1,2} = \Gamma^{g^*}_{2,1} &= \frac{(n-1)\sqrt{\frac{n+2}{n}}\epsilon(1+l\epsilon)}{n+1}, \\
\Gamma^{g^*}_{1,3} = \Gamma^{g^*}_{3,1} &= 0,
\end{aligned}$$

$$\Gamma_{2,2}^{g^*} = 1 + \frac{4\epsilon}{n^2-1} + \frac{l(65-2n^2+n^4)\epsilon^2}{16(n^2-1)},$$
$$\Gamma_{2,3}^{g^*} = \Gamma_{3,2}^{g^*} = \frac{\sqrt{\frac{n-2}{n}}(n+1)\epsilon(1+l\epsilon)}{n-1},$$
$$\Gamma_{3,3}^{g^*} = 1 + \frac{(n^2-5)\epsilon}{2(n-1)} + \frac{l(-39-n+7n^2+n^3)\epsilon^2}{16(n-1)}$$

Its eigenvalues are (up to order ϵ^2):

$$\Delta_1^{g^*} = 1 + \frac{1+n}{2}\epsilon + \frac{l(7+8n+n^2)}{16}\epsilon^2,$$
$$\Delta_2^{g^*} = 1 + \frac{l(n^2-1)}{16}\epsilon^2,$$
$$\Delta_3^{g^*} = 1 + \frac{1-n}{2}\epsilon + \frac{l(7-8n+n^2)}{16}\epsilon^2.$$

This result coincides with the dimensions $\tilde{\Delta}_{n+2,n}(l, p-l)$, $\Delta_{n,n}(l, p-l)+1$ and $\tilde{\Delta}_{n-2,n}(l, p-l)$ of the model $M(k-l, l)$ up to this order. The corresponding normalized eigenvectors should be identified with the fields of $M(k-l, l)$:

$$\tilde{\phi}_{n+2,n}(l, p-l) = \frac{2}{n(n+1)}\phi_1^{g^*} + \frac{2\sqrt{\frac{n+2}{n}}}{n+1}\phi_2^{g^*} + \frac{\sqrt{n^2-4}}{n}\phi_3^{g^*},$$
$$\phi_2(l, p-l) = -\frac{2\sqrt{\frac{n+2}{n}}}{n+1}\phi_1^{g^*} - \frac{n^2-5}{n^2+1}\phi_2^{g^*} + \frac{2\sqrt{\frac{n-2}{n}}}{n-1}\phi_3^{g^*},$$
$$\tilde{\phi}_{n-2,n}(l, p-l) = \frac{\sqrt{n^2-4}}{n}\phi_1^{g^*} - \frac{2\sqrt{\frac{n-2}{n}}}{n-1}\phi_2^{g^*} + \frac{2}{n(n-1)}\phi_3^{g^*}.$$

We used as before the notation $\tilde{\phi}$ for the descendent field defined as in the UV theory and:

$$\phi_2(l, p-l) = \frac{1}{2\Delta_{n,n}^{p-l}(2\Delta_{n,n}^{p-l}+1)}\partial\bar{\partial}\phi_{n,n}(l, p-l)$$

is the normalized derivative of the corresponding primary field. We notice that these eigenvectors are finite as $\epsilon \to 0$ with exactly the same entries as in $l = 1$ [10] and $l = 2$ [3] minimal models. This is one of the main results of this paper.

3 $N = 2$ Superconformal Models

The $N = 2$ superconformal theories are invariant under the corresponding algebra generated by the stress-energy tensor $T(z)$, the supercurrents $G^{(\pm)}(z)$ and the $U(1)$ current $J(z)$. We shall be interested here in the simplest minimal models of this theory, labeled by an integer p, containing a finite number of fields. It is well known that the latter are connected to a coset $\frac{\hat{su}(2)\times u(1)}{u(1)}$. The fields of the $N = 2$ theories belong to different sectors, depending on the boundary conditions of the supercurrents. Here we will be interested in the fields of the Neveu–Schwartz (NS) sector only.

As it is clear from the coset construction, the $N = 2$ superconformal minimal models admit a representation in terms of the D_{2p} parafermionic (PF) theories. It is based on the observation [5, 6] of the fact that the generators of the $N = 2$ supersymmetric theory could be expressed in terms of the PF currents and a free scalar field.

The primary fields in the $N = 2$ theories are constructed from the lowest fields of the PF theory and exponentials of the free scalar field φ. For the NS sector we have:

$$N^l_m(z) = \phi^l_m(z)\exp\left(i\frac{m}{\sqrt{2p(p+2)}}\varphi(z)\right),$$
$$l = 0, 1, \ldots, p \quad m = -l, -l+2, \ldots, l, \tag{23}$$

where ϕ^l_m is the lowest dimensional fields of the parafermionic theory.

The $U(1)$ charge of this field is:

$$q^l_m = \frac{m}{2(p+2)} \tag{24}$$

and its dimension is simply the sum of the dimensions of the two ingredients:

$$\Delta^l_m = d^l_m + \frac{m^2}{2p(p+2)} = \frac{l(l+2)}{4(p+2)} - \frac{m^2}{4(p+2)}. \tag{25}$$

The product with the supercurrents defines the second component of the field N^l_m:

$$(N^l_m)^{II\pm} \sim \phi^{p-l}_{m\pm(p+2)} e^{i\left(m\pm(p+2)/\sqrt{2p(p+2)}\right)\varphi} \tag{26}$$

Investigating the FR's in the NS sector one must keep attention that they have more complicated structure due to the fact that there exist three different 3-point functions of the NS superfields - one even and two odd ones. The meaning of the odd FR's in terms of component fields is that in the product of two first components of given superfields the second component of the RHS superfield appears. Taking all this into account we obtain the following FR's in the NS sector:

$$N^{l_1}_{m_1} N^{l_2}_{m_2} = \sum_{l=|l_1-l_2|}^{L} [\Psi^l_m],$$
$$L = \min(l_1 + l_2, 2p - l_1 - l_2) \tag{27}$$

where:

$$\Psi^l_m = (N^l_{m_1+m_2})^{even}, \quad |m_1 + m_2| \le l,$$
$$\Psi^l_m = (N^{p-l}_{m_1+m_2\pm(p+2)})^{odd}, \quad |m_1 + m_2| > l.$$

In this Section we would like to discuss the renormalization group properties of the $N = 2$ minimal models. In other words we would like to describe the RG flow of these models perturbed by the least relevant field. In the case of $N = 2$ minimal models the latter is constructed from the chiral and antichiral fields $N^p_{\pm p}$ of dimension $\Delta = 1/2 - 1/(p+2)$ and $U(1)$ charge $q = \pm\Delta$. The suitable perturbation term, neutral and of dimension close to one, is therefore constructed out of the second components of such chiral fields. Explicitly we consider:

$$\mathcal{L} = \mathcal{L}_0 + \int d^2z \Phi(z) \tag{28}$$

where $\mathcal{L}_0$ represents the minimal model itself and the field $\Phi(z)$ is a combination of the second components:

$$\Phi = (N^p_p)^{II} + (N^p_{-p})^{II} \equiv \phi_+ + \phi_-. \tag{29}$$

It is neutral and has a dimension $\Delta = 1 - 1/(p+2) = 1 - \epsilon$. Similarly to what we did in the previous Section, we consider the case $p \to \infty$ and assume $\epsilon = 1/(p+2)$ to be a small parameter. Also, according to our parafermionic construction, we can express the perturbing field in terms of the PF currents and exponents of the scalar field as follows:

$$(N^p_p)^{II} = \sqrt{\frac{2p}{p+2}} \psi_1{}^{\dagger} e^{-i\frac{2}{\sqrt{2p(p+2)}}} \equiv \phi_+,$$
$$(N^p_{-p})^{II} = \sqrt{\frac{2p}{p+2}} \psi_1 e^{i\frac{2}{\sqrt{2p(p+2)}}} \equiv \phi_-$$

where $\psi_1^{(\dagger)}$ are the simplest parafermionic currents.

Our purpose now is to compute the beta-function of this theory and to check for an eventual fixed point. For that we need to compute the two-point function of the perturbing field up to a second order. The expansion was already written in (6). As in the case of the symmetric coset, we need the 3- and 4-point functions of the perturbing field. We note that, due to the FR's computed above, the 3-point function of the field $\Phi(z)$, and therefore the first term in (6), is identically zero. So we are left with the computation of the second order term only. This computation goes along the same lines as above. We need to compute the 4-point function of $\Phi(z)$ up to zeroth order in ϵ and to integrate it in the safe region Ω_{r,r_0} far from the singularities. Near the singular points 0, 1 and ∞ we use the OPE's that we computed above.

The 4-point function of the perturbing field $\Phi(z)$ is expressed through the corresponding functions of the parafermionic fields which are known [5] and the trivial power-like contribution of the exponents. The final result is (up to zeroth order in ϵ):

$$< \Phi(x)\Phi(0)\Phi(1)\Phi(\infty) >= C|1 + \frac{1}{x^2} + \frac{1}{(1-x)^2}|^2 \tag{30}$$

where C is some structure constant. We will not need its explicit expression here. The integration of this function over the safe region gives:

$$\frac{2\pi^2}{\epsilon}\left(\frac{31}{16} + \frac{1}{r^2} + \frac{1}{4r_0^2}\right). \tag{31}$$

From this we have to subtract the contribution of the lens-like region:

$$\frac{\pi^2}{\epsilon}\left(\frac{31}{16} - \frac{1}{r^2} + \frac{1}{2r_0^2}\right). \tag{32}$$

At the end, we add the result of the integration near the singular points:

$$2\left(-\frac{\pi^2}{r^2\epsilon}\right) + \frac{2\pi^2}{\epsilon}\left(-\frac{1}{2r^2} + \frac{1}{2r_0^2}\right) \tag{33}$$

corresponding to the integrals around 0 (and 1) and ∞ respectively. Summing all the contributions we get finally as a result:

$$\frac{\pi^2}{\epsilon r_0^2}. \tag{34}$$

Two comments are in order. First, this result contains only the cut-off parameter and could be cancelled by adding an appropriate counterterm in the action. Second, the finite contribution is identically zero. This means that there is no contribution to the beta-function neither in the first nor in the second order. One can speculate that this is the case also in higher orders. This result leads us to the conclusion that there do not exits a nontrivial fixed point of the beta-function close to the UV one. If such a fixed point exists it should be due to some non-perturbative effects.

Acknowledgements This work is supported in part by the NSFB grant DFNI T02/6 and the bilateral grant STC/Bulgaria-France 01/6.

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Time-Dependent Free-Particle Salpeter Equation: Features of the Solutions

Amalia Torre, Ambra Lattanzi and Decio Levi

Abstract An analysis of the spinless (1 + 1)D free-particle Salpeter equation is presented. Future investigations are suggested.

Keywords Relativistic wave equation · Schrödinger equation · Wigner distribution function

1 Introduction

The formulation of relativistic quantum mechanics had a rather articulated and in some ways controversial path, as witnessed by the presence of three relativistic wave equations: the Salpeter equation, the Klein–Gordon equation and the Dirac equation, each having its own advantages and disadvantages.

The spinless Salpeter equation is the relativistic version of the Schrödinger equation [5, 7–9, 12, 13, 15]. The latter is a well defined partial differential equation characterized by the Laplacian operator. In contrast, the Salpeter equation presents a pseudo-differential operator since the relativistic kinetic energy operator is the square-root of a Laplacian-related operator, that is inherently non-local [2].

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_23

In 1926, Weyl proposed such a square-root operator for the formulation of relativistic quantum mechanics but he could not find a clear interpretation of it although he came very close to the concept of pseudo differential operator as it is today formulated. The square root operator was abandoned and new methods were proposed leading to the Klein–Gordon and Dirac equations.

Recently, the Salpeter equation has been the object of renewed interest in both the eigenvalues/eigenstates problem [4, 6, 10] and the time-evolution issue [5, 7–9, 15]. The spinless Salpeter equation represents a well-defined standard approximation to the Bethe-Salpeter formalism [10, 13], and offers definite advantages with respect to the Klein–Gordon and Dirac equations. It is first-order in the time-derivative, and preserves the scalar nature of the wave function. It possesses solutions of positive energies only, and supports the probabilistic interpretation of its solutions, whose norm is a conserved quantity [7, 8]. Also, it has a well-defined classical relativistic counterpart.

Subject of this paper is the initial condition problem for the spinless time-dependent free-particle Salpeter equation in one space-dimension, as explicitly formulated in Sect. 2. An accurate analysis of such an issue has been presented in [9] and further developed in [15]. Analyzing specific solutions of the equation for assigned initial conditions on the basis of simple concepts and methods, like those of symmetries, fundamental solution, Fourier transform, asymptotic analysis and stationary phase method, peculiar features of the evolution governed by the Salpeter equation have been fixed. The "degree of localization" of the initial condition has been shown to mark the border between the relativistic and non relativistic behavior. A quantitative definition of it has been achieved for some initial conditions.

We will recall from [9, 15] the results, that are functional to the present analysis. Thus, Sects. 3 and 4 are concerned with the fundamental solution and the evolution of a Gaussian input, respectively.

Inspired by the well-known formal analogy between the $(1+1)$D Schrödinger equation for a free-particle and the 2D paraxial wave equation for free propagation, the possibility of establishing a plain correspondence between the evolution under the Salpeter equation and the optical free propagation has been thoroughly examined in [15]. Indeed, the analogy between the solution of the Salpeter equation in the asymptotic limit and the Huygens-Fresnel integral for free propagation has been established, when interpreting the former in the light of the pseudo-Euclidean metric pertaining to the Minkowski spacetime. As a consequence, a long-time evolution rule (paralleling the "Fraunhofer diffraction rule') for the quantum relativistic evolution has been formulated.

It is on theses lines that we move here. In Sect. 5, we examine the evolution of Bessel functions, related to a new class of paraxial optical beams [11]; also, they represent a band-limited input for the equation. Then we inspect the joint coordinate-momentum representation of the wave function, as it is provided by the Wigner distribution function [16], which, as is well known, finds applications also in optics and in signal theory [14]. As a preliminary analysis, we compare the Wigner charts of the Gaussian input evolving under the Salpeter equation and the Schrödinger equation. The latter means paraxial propagation of the wave field in the optics language. Concluding notes (Sect. 6) close the paper.

2 Salpeter Equation: Coordinate and Momentum Representations

The initial condition problem for the spinless $(1 + 1)$D free-particle Salpeter equation can be formulated in both the coordinate and momentum representations. In both cases one deals with an evolution equation for the position wave function $\psi(x, t)$ of the particle,

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi(x,t) &= \sqrt{m^2c^4 - c^2\hbar^2\frac{\partial^2}{\partial x^2}}\,\psi(x,t), \\ \psi(x,0) &= \psi_0(x), \end{aligned} \tag{1}$$

and for the momentum wave function $\widetilde{\psi}(p, t)$,

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\widetilde{\psi}(p,t) &= \sqrt{m^2c^4 + c^2p^2}\,\widetilde{\psi}(p,t), \\ \widetilde{\psi}(p,0) &= \widetilde{\psi}_0(p). \end{aligned} \tag{2}$$

Equation (2) is easily solved while (1) requires a more thorough analysis. Indeed, the Salpeter equation is usually approached in the momentum-space representation. However, as proved in [9, 15], a parallel analysis in both representations allows for a full-view account of the features of the solutions.

2.1 Solution: General Expressions

We introduce the dimensionless variables

$$\xi = \frac{x}{\lambda\!\!\!^{-}_c}, \quad \tau = \frac{ct}{\lambda\!\!\!^{-}_c}, \quad \kappa = \frac{p/\hbar}{\kappa_c}, \quad \lambda\!\!\!^{-}_c = \frac{\hbar}{mc}; \tag{3}$$

the Compton wavelength $\lambda\!\!\!^{-}_c$ and the corresponding wavenumber $\kappa_c = \lambda\!\!\!^{-}_c{}^{-1}$ are the scale factors for the space and spatial frequency variables.

On the basis of the Fourier relation linking $\psi(\xi, \tau)$ and $\widetilde{\psi}(\kappa, \tau)$, from the solution of Eq. (2), which in terms of (3) writes as

$$\widetilde{\psi}(\kappa, \tau) = e^{-i\tau\sqrt{1+\kappa^2}}\widetilde{\psi}_0(\kappa), \tag{4}$$

two expressions for the solution of (1) can be elaborated. The momentum wave-function based expression amounts to an integration over the momentum domain, being the (inverse) Fourier transform of $\widetilde{\psi}(\kappa, \tau)$:

$$\psi(\xi, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\tau\sqrt{1+\kappa^2}} e^{i\xi\kappa} \widetilde{\psi}_0(\kappa) d\kappa. \tag{5}$$

The position wave-function based expression amounts to an integration over the space domain, being the convolution of the initial condition with the *fundamental solution* $S(\xi, \tau)$:

$$\psi(\xi, \tau) = \int_{-\infty}^{+\infty} S(\xi - \xi', \tau)\psi_0(\xi')d\xi'. \tag{6}$$

We highlight the main results of the study presented in [9, 15], concerning in particular the fundamental solution and the evolution of a Gaussian input.

The numerical analysis is based on (5); for the interpretation of the results we resort to the features of the fundamental solution. Unless otherwise specified, all the displayed plots have been obtained using Mathcad 15 infinite limit routine with absolute error tolerance set to 10^{-6} (10^{-9} for Fig. 6).

3 Extremely Localized Input: Fundamental Solution

The fundamental solution $S(\xi, \tau)$ is the wave function evolving from the δ-function. It is amenable for a closed-form expression. Thus,

$$\psi_0(\xi) = \delta(\xi) \tag{7}$$

evolves in [1]

$$S(\xi, \tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\tau\sqrt{1+\kappa^2}} e^{i\kappa\xi} d\kappa = \frac{\tau}{\pi} \frac{K_1(i\sqrt{\tau^2 - \xi^2})}{\sqrt{\tau^2 - \xi^2}}, \tag{8}$$

where K_1 is the McDonald function of first order.

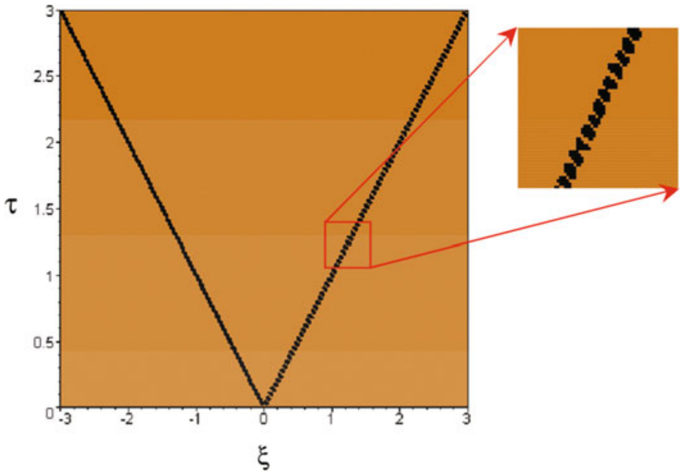

Fig. 1 (ξ, τ)-contourplot of the probability density $|S(\xi, \tau)|^2$. The plots have been obtained by Maple 8 setting to 20 the number of digits in the evaluation of (8)

The space and time variables enter expression (8) through the pseudo-Euclidean norm $s^2 \equiv \tau^2 - \xi^2$ as it is defined in the $(1+1)$D Minkowski space-time in accord with the inherent metric of signature $(1, -1)$. Being a solution of an evolution equation, $S(\xi, \tau)$ should encode causality and hence its support in the (ξ, τ)-plane should be inside the time-like region delimited by the light cone. This is confirmed by an accurate study based on the properties of the McDonald function K_1 and a numerical and asymptotic analysis [15].

3.1 Numerical Analysis: V-Like Shape of $|S(\xi, \tau)|^2$

In fact, as shown in Fig. 1, the probability density $|S(\xi, \tau)|^2$ covers almost exclusively the time-like region in the Minkowski plane. Here, it displays hyperbolic level curves, whose main characteristic is convexity. Convexity of the level curves appears to be a peculiar feature of the relativistic evolution. The change of the curvature of the level curves from convex to concave signalizes the passage from the relativistic to the non-relativistic regime.

4 Gaussian Input

The solution to the initial condition problem (1) for the Gaussian input

$$\psi_0^{(G)}(\xi) = \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{\xi^2}{2w^2}}, \qquad w > 0 \tag{9}$$

is provided by the integral transform

$$\psi_{\text{Salp}}^{(G)}(\xi, \tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\tau\sqrt{1+\kappa^2}} e^{i\kappa\xi} e^{-\frac{w^2\kappa^2}{2}} d\kappa. \tag{10}$$

The parameter w fixes the widths of $\psi_0^{(G)}(\xi)$ in both the ξ- and κ-domains. As "width" of a wave function, we address, as usual, the variance of the probability density, so that for (9) one has $\sigma_\xi = \frac{w}{\sqrt{2}}$ and $\sigma_\kappa = \frac{1}{\sqrt{2}w}$.

The numerical and asymptotic analysis of (10), the latter based on the stationary-phase method, reveals that the relevant probability density is mostly contained in the triangular region subtended by the lines $\xi = \pm\tau$ (Fig. 2), reproducing more or less faithfully the peculiar V-shape of $|S(\xi, \tau)|^2$ as far as the width parameter w is such that

$$0 \leq w \leq 1.225. \tag{11}$$

Increasing w, $|\psi_{\text{Salp}}^{(G)}(\xi, \tau)|^2$ tends to round the steepness at the edges and to change the level-curves curvature from convex to concave, approaching the non relativistic behavior. The plots in Fig. 3 show the behavior of the probability densities associated with $\psi_{\text{Salp}}^{(G)}(\xi, \tau)$ and with the solution of the Schrödinger equation $\psi_{\text{Schr}}^{(G)}(\xi, \tau)$, which is the well-known Gaussian wave packet for a free particle:

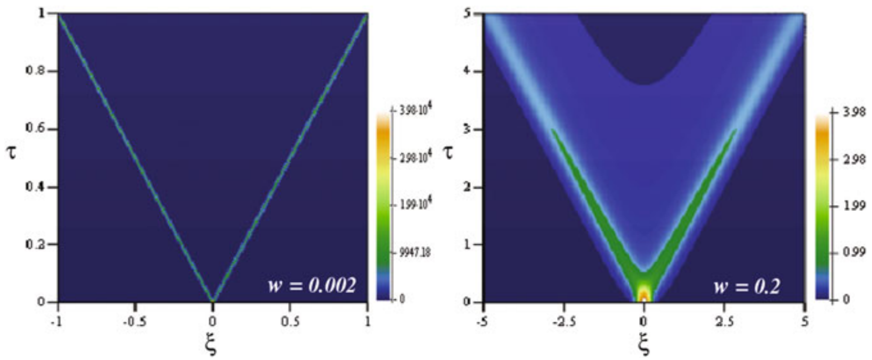

Fig. 2 (ξ, τ)-contourplots of $|\psi_{\text{Salp}}^{(G)}(\xi, \tau)|^2$ for values of w well inside the range (11)

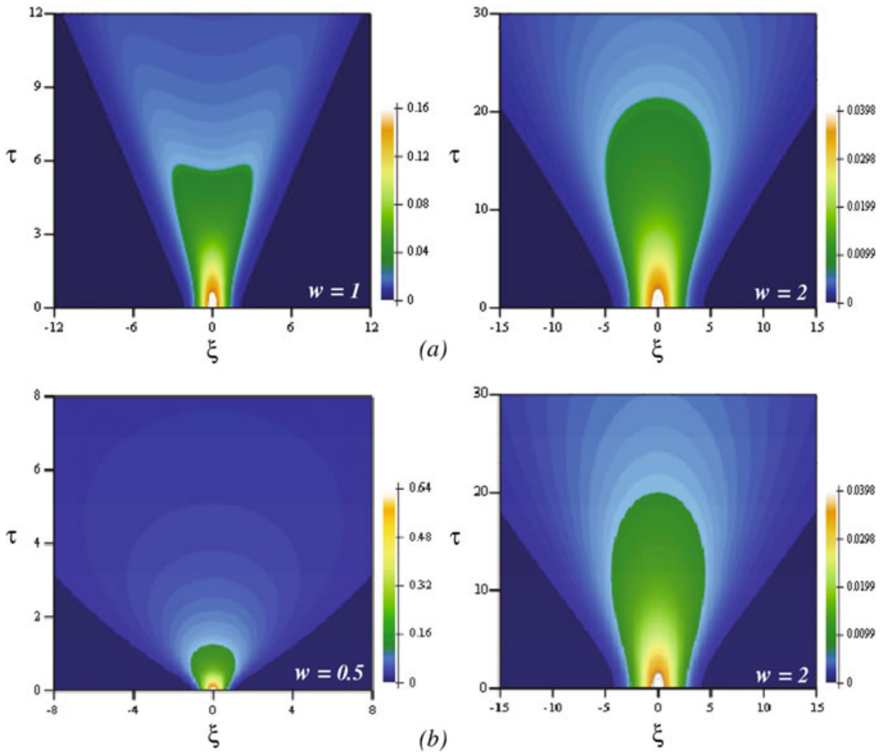

Fig. 3 (ξ, τ)-contourplots of *(a)* $|\psi^{(G)}_{\text{Salp}}(\xi, \tau)|^2$ and *(b)* $|\psi^{(G)}_{\text{Schr}}(\xi, \tau)|^2$ for some values of w

$$\psi^{(G)}_{\text{Schr}}(\xi, \tau) = \frac{1}{\sqrt{2\pi w^2 \mu(\tau, w)}} e^{-\frac{\xi^2}{2w^2\mu(\tau,w)}}, \quad \mu(\tau, w) = 1 + i\frac{\tau}{w^2}. \tag{12}$$

Note that $\psi^{(G)}_0(\xi)$ is normalized to the δ-function as $\lim_{w\to 0}\psi^{(G)}_0(\xi) = \delta(\xi)$.

5 Salpeter Equation and Optics

All the initial conditions examined in [9, 15] have relevance in optics as well. This grounds on the formal analogy between the $(1 + 1)$D Schrödinger equation for a free-particle and the 2D paraxial wave equation for free-propagation so that solutions of the former are also solutions of the latter, and hence can be interpreted as optical beams with definite and properly exploited features. For instance, the Gaussian wave packet (12) corresponds to the optical Gaussian beam produced by a laser system.

Further inspired by this analogy we explore the behavior of the Bessel functions and the Gaussian-apodized Bessel functions, which represent a new class of paraxial optical beams [11], exhibiting a discrete-like diffraction pattern, similar to that observed in periodic evanescently coupled waveguide lattices endowed with coupling interactions up to second order.

Then, we examine the Wigner distribution function of the Gaussian input, comparing its evolution under the Schrödinger and Salpeter equations.

5.1 Bessel Input: Band-Limited Initial Condition

The Bessel function is interesting because it provides an example of a band-limited input. We consider the initial wave function

$$\psi_0^{(J_n)}(\xi) = J_n(\alpha\xi),$$

where the parameter α fixes the width of the function in the κ-domain, since the Fourier transform is [3]

$$\widetilde{\psi}_0^{(J_n)}(\kappa) = \begin{cases} \frac{(-i)^n}{\alpha}\sqrt{\frac{2}{\pi(1-(\kappa/\alpha)^2)}}\,T_n(\kappa/\alpha), & |\kappa/\alpha| \le 1, \\ 0, & |\kappa/\alpha| > 1, \end{cases} \tag{13}$$

where T_n denotes the Chebyshev polynomial of first kind and degree n.

The analysis of the evolution of $\psi_0^{(J_n)}(\xi)$ reveals interesting aspects. The relativistic and nonrelativistic evolutions do not differ dramatically from each other, but in a form that highlights the basic features of the former. In both cases, one observes a spot-structure of the probability density (Fig. 4). Is it due to a sort of interference of the contributions emanating from the various lobes of the Bessel function? In the case of the Salpeter solution, the spot-structure persists also when, with increasing α, that amounts to increasing the width of the input in the κ-domain (and hence to approaching the δ -function in the ξ-domain) and the frequency of the Bessel oscillations, the probability density tends to mimic $|S(\xi, \tau)|^2$. This yields a certain "granularity" of the "light-lines", the same displayed by $|S(\xi, \tau)|^2$ (Fig. 1) as well as by the probability density associated with other inputs when they are quite similar to the δ-function (Fig. 2). Can such a "granularity" be considered as merely due to a numerical effect?

The question is also stimulated by the fact that a similar spot-structure appears in the contour plots of the probability density conveyed by a uniform band-limited input (spatially meaning a sinc function). However, it disappears when, increasing α, the probability density tends to be like $|S(\xi, \tau)|^2$.

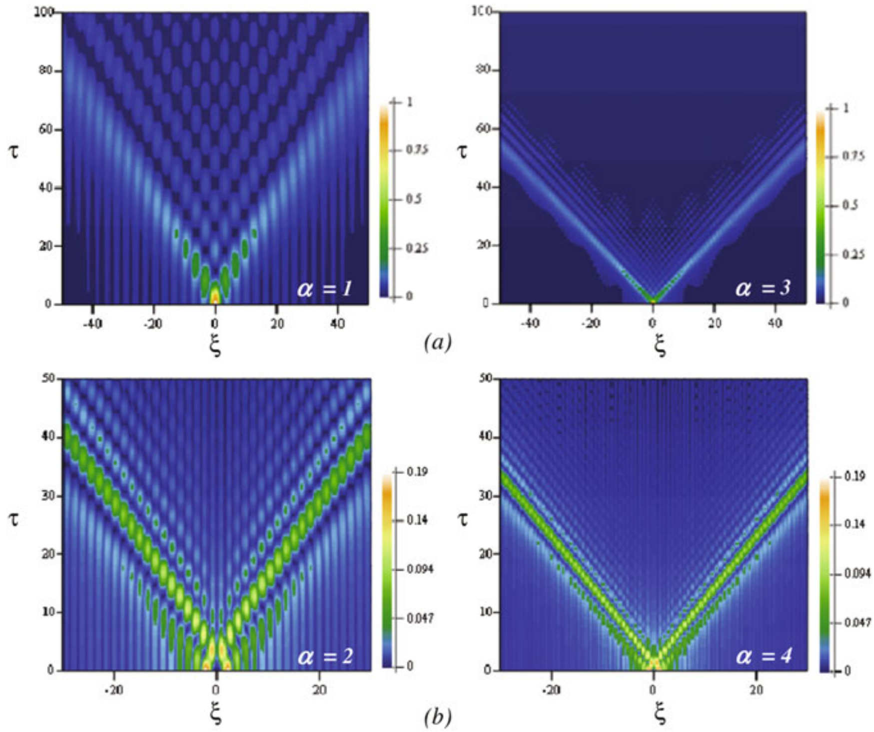

Fig. 4 (ξ, τ)-contourplots of *(a)* $\left|\psi_{\mathrm{Salp}}^{(J_0)}(\xi,\tau)\right|^2$ for $\alpha = 1$ and $\alpha = 3$, and *(b)* $\left|\psi_{\mathrm{Salp}}^{(J_3)}(\xi,\tau)\right|^2$ for $\alpha = 2$ and $\alpha = 4$

5.2 Coordinate-Momentum Representation: Wigner Distribution Function

The Wigner distribution function is the simplest tool for the phase-space representation in quantum mechanics [16]. Defined as

$$\mathcal{W}(\xi,\kappa,\tau) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \psi^*(\xi-\xi',\tau)\psi(\xi+\xi',\tau)e^{-2i\kappa\xi'}d\xi'$$
$$= \frac{1}{\pi}\int_{-\infty}^{+\infty} \widetilde{\psi}^*(\kappa-\kappa',\tau)\widetilde{\psi}(\kappa+\kappa',\tau)e^{2i\xi\kappa'}d\kappa', \qquad (14)$$

equivalently through the position and momentum wave functions, it is introduced in signal analysis and optics through an expression formally identical, with properly involved "conjugate variables" and "system state descriptor".

The Wigner distribution function represents the wave-optical tool closest to the geometric-optical concept of light ray, due to its localization properties and dynamical

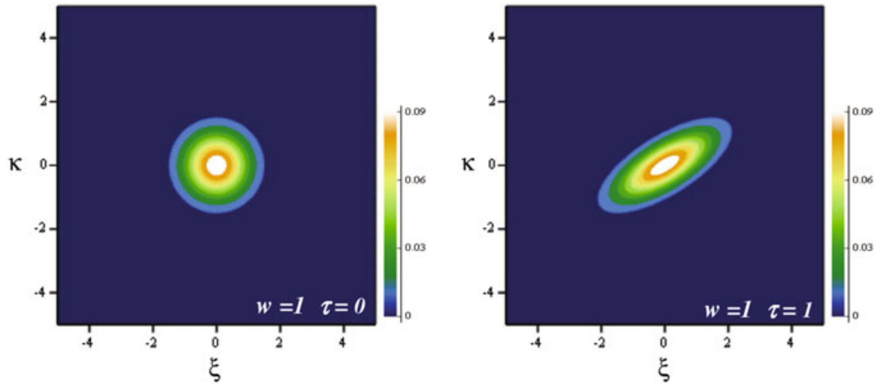

Fig. 5 Wigner chart of $\psi_0^{(G)}(\xi)$ and of the relevant time-evolved wave function for $w = 1$ at $\tau = 1$

behavior, which under paraxial propagation through real optical systems is ruled by the same transfer law of ray optics. In fact, under free propagation of the wave field, and equivalently under the Schrödinger evolution of the wave function, it changes as

$$\mathcal{W}(\xi, \kappa, \tau) = \mathcal{W}_0(\xi - \kappa\tau, \kappa), \tag{15}$$

where $\mathcal{W}_0(\xi, \kappa)$ is the Wigner function at the initial time $\tau = 0$, which for $\psi_0^{(G)}(\xi)$ writes as

$$\mathcal{W}_0^{(G)}(\xi, \kappa) = \frac{1}{2\pi^{3/2} w} e^{-\frac{\xi^2}{w^2} - w^2\kappa^2}. \tag{16}$$

Relation (15) amounts to a q-shear of the Wigner chart (Fig. 5).

Using expression (4) in (14), for $\psi_{\text{Salp}}^{(G)}(\xi, \tau)$ we obtain

$$\mathcal{W}_{\text{Salp}}^{(G)}(\xi, \kappa, \tau) = \frac{e^{-w^2\kappa^2}}{\pi^2} \int_0^{+\infty} e^{-w^2 y^2} \cos[2\xi y + \tau(\sqrt{1 + (\kappa - y)^2} - \sqrt{1 + (\kappa + y)^2})] dy. \tag{17}$$

Figure 6 shows the Wigner chart of $\psi_{\text{Salp}}^{(G)}(\xi, \tau)$ for $w = 1$ at $\tau = 0.5$ and $\tau = 1$.

$\mathcal{W}_{\text{Salp}}^{(G)}$ is real as it should be, and is nonnegative everywhere through the (ξ, κ)-plane. This is a prerogative of the Gaussian function only; it seems then to be preserved by the relativistic evolution as well. Evidently, the structure of $\mathcal{W}_{\text{Salp}}^{(G)}(\xi, \kappa, \tau)$, as it appears in Fig. 6, deserves further investigations.

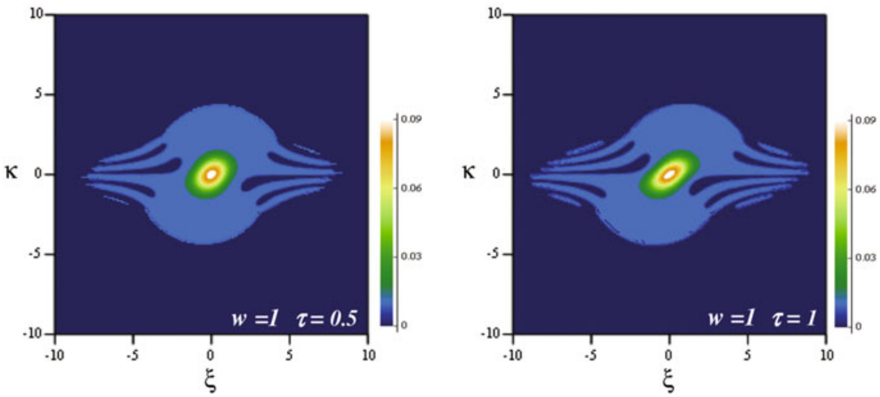

Fig. 6 Wigner chart of $\psi^{(G)}_{\text{Salp}}(\xi, \tau)$ for $w = 1$ at $\tau = 0.5$ and $\tau = 1$. The relativistic evolution seems to manifest through a q-shear as well

6 Concluding Notes

An extension of the study of the spinless $(1 + 1)$D free-particle Salpeter equation, reported in [9, 15], has been presented, comprising the evolution of Bessel inputs and the coordinate-momentum representation, the latter addressed through the Wigner distribution function. From the inherent results, hints for further investigations are drawn.

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Leptons, Quarks, and Gauge Symmetries, from a Clifford Algebra

Ovidiu Cristinel Stoica

Abstract Spinors having the discrete properties of the leptons and quarks in a family of the Standard Model, with the proper symmetries, are obtained using the left ideals of a Clifford algebra. This algebra is the complex Clifford algebra $\mathbb{C}\ell_6$ obtained from the exterior algebra of a complex three-dimensional vector space and its dual, this giving the ideal decomposition and representing the electric charges, the quark colors, and the proper $\mathrm{SU}(3)$ symmetries. The Lorentz and Dirac algebras appear as subalgebras, their left actions on the ideals representing therefore the leptons and the quarks. Because the representation of the Dirac algebra on the minimal left ideals of $\mathbb{C}\ell_6$ is reducible, the weak symmetry emerges as well, with the isospins, hypercharges, and chirality. The electroweak symmetry is broken geometrically, without resulting in additional exchange bosons or other fermions. The bare Weinberg angle θ_W predicted by this model is given by $\sin^2\theta_W = 0.25$. The mass-related parameters and the three families of leptons and quarks are not yet obtained in this model.

Keywords Beyond the Standard Model · Clifford algebras · Grand Unified theory · Gauge theory

1 Introduction

At first sight, the number of types of particles is immense. But they are all known to be composed out of a small number of elementary particles. There are three known families of fundamental fermions – the *leptons* and *quarks*, and their antiparticles. They interact via *exchange bosons* - *gluons*, the $W^{\pm}$ and Z particles, and the *photon*. The *Standard Model* of particle physics (SM) is the model we currently have which describes these particles, their properties, and their interactions. To give mass to the particles, it relies on spontaneous symmetry breaking, which requires a scalar particle

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_24

– the *Higgs boson*. The SM is completed with the data about neutrinos, coming from the discovery that they oscillate.

The electric charges, colors, and isospins of leptons and quarks show that there is a pattern, which is not explained by the SM. It is the purpose of this article to provide a possible explanation of this pattern. In fact, the model proposed here gives a generic family of leptons and quarks, together with their electric charges, colors, weak isospins, hypercharges, chiralities, as well as the gauge symmetries at the origin of the electromagnetic, weak, and color forces. They will be shown to emerge out of a simple Clifford algebra. Except for the Weinberg angle, this model currently doesn't say anything about the masses and other parameters of the SM, and about the number of families.

Other proposals to explain these patterns are known. The most popular are based on finding a simple Lie group, and obtaining the SM symmetries by symmetry breaking, and the leptons and quarks as representations of this group (SU(5) [1] and Spin(10) [2, 3]). They do not explain the group and the chosen representation, and come with additional exchange bosons and proton decay, which are not confirmed experimentally. Models based on Clifford algebras and other algebras are also known [4–9]. I compare these and other models with this one in [10].

2 The Standard Model Algebra

The electric charges of leptons and quarks are all multiple of $1/3$ of the electron's charge, and the quarks have colors, while the leptons are singlets with respect to the color $SU(3)_c$. This suggests the following. Let us represent electromagnetism by the action of the symmetry group $U(1)_{em}$ on a complex 1-dimensional vector space χ_{em}, so that this action preserves a Hermitian inner product h_{em} on χ_{em}. The tensor products of the form $\bigotimes^k \chi_{em}$ will therefore represent an electric charge k times larger than that of χ_{em} (which we take to be $1/3$), and the tensor products of the form $\bigotimes^{-k} \chi_{em} := \bigotimes^k \chi_{em}^\dagger$ the opposite charges. But why is k limited between -3 and 3? To see this, consider the color as given by the group $SU(3)_c$, of symmetries of a complex 3-dimensional vector space χ endowed with a Hermitian inner product $\mathfrak{h}$. Also consider the action of $U(1)_{em}$ on χ by multiplication with $e^{\frac{i}{3}\varphi}$. The exterior spaces $\bigwedge^k \chi$ and $\bigwedge^{-k} \chi := \bigwedge^k \chi^\dagger$ will have the right colors and charges to represent the internal spaces of leptons, quarks, and their antiparticles.

To see this, let us pick an orthonormal basis of χ, and its dual basis in $\chi^\dagger$,

$$\begin{cases} (\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3) \\ (\mathfrak{q}^\dagger_1, \mathfrak{q}^\dagger_2, \mathfrak{q}^\dagger_3). \end{cases} \tag{1}$$

Consider the following basis for the exterior algebra of χ, $\bigwedge^\bullet \chi$,

$$(1, \mathfrak{q}_{23}, \mathfrak{q}_{31}, \mathfrak{q}_{12}, \mathfrak{q}_{321}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3), \tag{2}$$

where $\mathfrak{q}_{j_1 \ldots j_k} := \mathfrak{q}_{j_1} \ldots \mathfrak{q}_{j_k}$. The inner product $\mathfrak{h}$ extends on the exterior algebra $\bigwedge^{\bullet}\chi$ to a positive definite Hermitian inner product for which the basis (2) is orthonormal.

The internal charge and color spaces for leptons and quarks are therefore $\circ \mapsto \bigwedge^0\chi$, $\mathbf{d} \mapsto \bigwedge^1\chi^\dagger$, $\mathbf{u} \mapsto \bigwedge^2\chi$, and $\mathbf{e}^- \mapsto \bigwedge^3\chi^\dagger$. Then, the spaces $\bigwedge^k\chi$, $-3 \leq k \leq 3$, are the correct representations of $U(1)_{\text{em}}$ and $SU(3)_{\text{c}}$.

The complex 6-dimensional vector space $\chi^\dagger \oplus \chi$ is endowed with the *inner product* given by the contraction between 1-forms in $\chi^\dagger$ with vectors in χ,

$$\langle u_1^\dagger + u_2, u_3^\dagger + u_4 \rangle := \frac{1}{2}\left(u_1^\dagger(u_4) + u_3^\dagger(u_2)\right) \in \mathbb{C}, \tag{3}$$

where $u_1^\dagger, u_3^\dagger \in \chi^\dagger$ and $u_2, u_4 \in \chi$.

In the following, the *Standard Model Algebra* (SMA) is the Clifford algebra of the space $\chi^\dagger \oplus \chi$ with the inner product (3),

$$\mathcal{A}_{\text{SM}} := \mathbb{C}\ell(\chi^\dagger \oplus \chi) \cong \mathbb{C}\ell_6, \tag{4}$$

together with the *Witt decomposition* $\chi^\dagger \oplus \chi$, and the inner product $\mathfrak{h}$. It is isomorphic with the matrix algebra $\mathbb{C}(8)$.

The elements of the bases (1) satisfy the anticommutation relations

$$\{\mathfrak{q}_j, \mathfrak{q}_k\} = 0, \{\mathfrak{q}_j^\dagger, \mathfrak{q}_k^\dagger\} = 0, \{\mathfrak{q}_j, \mathfrak{q}_k^\dagger\} = \delta_{jk} \tag{5}$$

for $j, k \in \{1, 2, 3\}$.

The matrix representation of the Standard Model Algebra has in a basis that will be described later the form from Fig. 1.

The minimal left ideals of $\mathbb{C}\ell(\chi^\dagger \oplus \chi)$, which in the matrix representation are columns, will turn out to be reducible representations of the Dirac algebra, and will thus represent pairs of spinors whose left chiral components are parts of the same weak doublet. These ideals are determined by the decomposition $\chi^\dagger \oplus \chi$. Each column contains two 4-spinors associated to two different flavors. The symmetry group $SU(3)_{\text{c}}$ acts by linearly combining the columns according to the representations $\mathbf{1}_c$, $\mathbf{3}_c$, $\bar{\mathbf{1}}_c$, and $\bar{\mathbf{3}}_c$. To each ideal corresponds an electric charge, multiple of $\frac{1}{3}$, representing in part the charge of the upper particle. The colors are determined for each ideal according to the representation of $SU(3)_{\text{c}}$. The *Dirac algebra* and the *Lorentz group* preserving a metric with signature $(+, -, -, -)$ have reducible representations on each column, and they permute the rows of each ideal. When decomposed into irreducible representations, they split each column into two 4-spinors, each of them being decomposed into left and right Weyl spinors. The weak symmetry group $SU(2)_{\text{L}}$ acts by permuting the rows according to the representations $\mathbf{1}_w$ and $\mathbf{2}_w$. Therefore, the leptons, quarks, and gauge symmetries of the SM are reproduced.

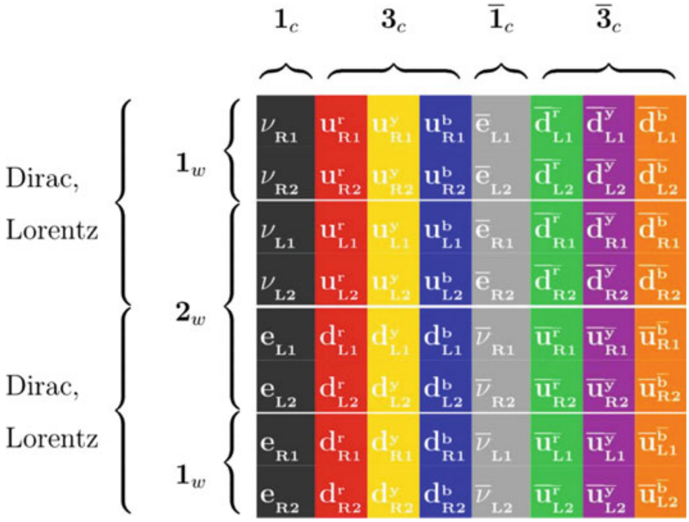

Fig. 1 Matrix representation of the leptons and quarks. The Dirac algebra, the Lorentz group, and the weak symmetry group, act by multiplication at left, permuting the rows. The color symmetry group acts at right, and permutes the columns

3 Representing Leptons and Quarks

Leptons and quarks are spinor fields. Since the Dirac algebra is represented in physics as a matrix algebra, for a physicists the spinors are column matrices. From mathematical point of view, spinors are elements of minimal left ideals, and they appear as column vectors only in the matrix representation. To obtain a minimal left ideal, we need a primitive idempotent element of $\mathcal{A}_{\text{SM}}$. As a matrix, this is a projector on a 1-dimensional space, thus giving us the column vector. Let us define $\mathfrak{q} := \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3$ and $\mathfrak{q}^\dagger := \mathfrak{q}^\dagger_3\mathfrak{q}^\dagger_2\mathfrak{q}^\dagger_1$, and the idempotents $\mathfrak{p} := \mathfrak{q}\mathfrak{q}^\dagger$ and $\mathfrak{p}' := \mathfrak{q}^\dagger\mathfrak{q}$, $(\mathfrak{p})^2 = \mathfrak{p}$ and $(\mathfrak{p}')^2 = \mathfrak{p}'$.

The representation of the algebra $\mathcal{A}_{\text{SM}}$ on its ideal $\bigwedge^\bullet \chi^\dagger \mathfrak{p}$ is obtained by taking the Clifford product between $u^\dagger + v \in \chi^\dagger \oplus \chi$ and $\omega\mathfrak{p} \in \bigwedge^\bullet \chi^\dagger \mathfrak{p}$ as

$$(u^\dagger + v)\omega\mathfrak{p} = (u^\dagger \wedge \omega)\mathfrak{p} + (i_v\omega)\mathfrak{p} \in \bigwedge{}^\bullet \chi^\dagger \mathfrak{p}, \tag{6}$$

where $i_v\omega$ is the *interior product*, defined for any $\omega \in \bigwedge^k \chi^\dagger$ by

$$(i_v\omega)\,(u_1, \ldots, u_k) = \begin{cases} \omega(v, u_1, \ldots, u_{k-1}), & \text{for } k \in \{1, 2, 3\}, \text{ and} \\ 0 & \text{for } k = 0. \end{cases} \tag{7}$$

Then, the vectors $\mathfrak{q}_j$ and $\mathfrak{q}^\dagger_j$ act as *ladder operators* on $\bigwedge^\bullet \chi^\dagger \mathfrak{p}$:

$$\begin{cases} \mathfrak{q}^{\dagger}_{j}(\omega\mathfrak{p}) = (\mathfrak{q}^{\dagger}_{j} \wedge \omega)\mathfrak{p}, \\ \mathfrak{q}_{j}(\omega\mathfrak{p}) = (i_{\mathfrak{q}_j}\omega)\mathfrak{p}. \end{cases} \tag{8}$$

A basis of the ideal $\bigwedge^{\bullet}\chi^{\dagger}\mathfrak{p}$ is

$$(1\ \mathfrak{p}, \mathfrak{q}^{\dagger}_{23}\ \mathfrak{p}, \mathfrak{q}^{\dagger}_{31}\ \mathfrak{p}, \mathfrak{q}^{\dagger}_{12}\ \mathfrak{p}, \mathfrak{q}^{\dagger}_{321}\ \mathfrak{p}, \mathfrak{q}^{\dagger}_{1}\ \mathfrak{p}, \mathfrak{q}^{\dagger}_{2}\ \mathfrak{p}, \mathfrak{q}^{\dagger}_{3}\ \mathfrak{p}). \tag{9}$$

The basis (9) is written in terms of the idempotent element $\mathfrak{p}$. It is equal to the basis

$$(\mathfrak{q}\ \mathfrak{q}^{\dagger}, -\mathfrak{q}_1\ \mathfrak{q}^{\dagger}, -\mathfrak{q}_2\ \mathfrak{q}^{\dagger}, -\mathfrak{q}_3\ \mathfrak{q}^{\dagger}, 1\ \mathfrak{q}^{\dagger}, \mathfrak{q}_{23}\ \mathfrak{q}^{\dagger}, \mathfrak{q}_{31}\ \mathfrak{q}^{\dagger}, \mathfrak{q}_{12}\ \mathfrak{q}^{\dagger}) \tag{10}$$

written in terms of the nilpotent $\mathfrak{q}^{\dagger}$, which determines the same ideal as $\mathfrak{p}$.

It is convenient to use the *Pauli matrices* $\sigma_1, \sigma_2, \sigma_3$, and the matrices $\sigma_+ = \frac{1}{2}(\sigma_1 + i\sigma_2)$, $\sigma_- = \frac{1}{2}(\sigma_1 - i\sigma_2)$, $\sigma_3^+ = \frac{1}{2}(1 + \sigma_3) = \sigma_+\sigma_-$, and $\sigma_3^- = \frac{1}{2}(1 - \sigma_3) = \sigma_-\sigma_+$.

Then, in the representation (9) of $\mathcal{A}_{\mathrm{SM}}$ on its ideal $\bigwedge^{\bullet}\chi^{\dagger}\mathfrak{q}$,

$$\mathfrak{q}_1 = \begin{pmatrix} 0 & 0 & i\sigma_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i\sigma_2 & 0 & 0 \end{pmatrix}, \mathfrak{q}_2 = \begin{pmatrix} 0 & 0 & 0 & \sigma_3^+ \\ 0 & 0 & -\sigma_3^+ & 0 \\ 0 & -\sigma_3^- & 0 & 0 \\ \sigma_3^- & 0 & 0 & 0 \end{pmatrix}, \mathfrak{q}_3 = \begin{pmatrix} 0 & 0 & 0 & \sigma_+ \\ 0 & 0 & -\sigma_- & 0 \\ 0 & \sigma_- & 0 & 0 \\ -\sigma_+ & 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

The elements $\mathfrak{q}^{\dagger}_1$, $\mathfrak{q}^{\dagger}_2$, and $\mathfrak{q}^{\dagger}_3$ are represented by the corresponding adjoint matrices. From the matrices $\mathfrak{q}_1$, $\mathfrak{q}_2$, $\mathfrak{q}_3$, $\mathfrak{q}^{\dagger}_1$, $\mathfrak{q}^{\dagger}_2$, and $\mathfrak{q}^{\dagger}_3$, we can calculate the matrix form of any element of interest in the algebra $\mathcal{A}_{\mathrm{SM}}$.

From the Witt decomposition $\mathcal{A}^1_{\mathrm{SM}} = \chi^{\dagger} \oplus \chi$ we obtain a natural decomposition of $\mathcal{A}_{\mathrm{SM}}$ as a direct sum of left ideals

$$\mathcal{A}_{\mathrm{SM}} = \bigoplus_{k=0}^{3} \left(\bigwedge^{\bullet}\chi^{\dagger}\right)\mathfrak{p}\bigwedge^{k}\chi, \tag{12}$$

and the ideals have internal degrees of freedom in $\bigwedge^{k}\chi$, corresponding to the charges and colors of leptons and quarks, as explained in Sect. 2.

The *Dirac algebra* is the complexified of the Clifford algebra of the Minkowski spacetime, $\mathrm{D}_{\mathbb{C}} := C\ell_{1,3} \otimes \mathbb{C} \cong \mathbb{C}\ell_4$. To represent the Dirac algebra on the minimal left ideals of the algebra $\mathcal{A}_{\mathrm{SM}}$, we use the following representation

$$\Gamma^{\mu} = \begin{pmatrix} \widetilde{\gamma}^{\mu} & 0 \\ 0 & \gamma^{\mu} \end{pmatrix}, \text{ where } \gamma^0 = \widetilde{\gamma}^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \gamma^j = -\widetilde{\gamma}^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \tag{13}$$

where γ^{μ} is the chiral (Weyl) representation. The representation (13) is reducible, and the minimal left ideals decompose into two 4-dimensional spaces which give irreducible representations of the Dirac algebra. Each of these spaces decomposes as the direct sum of two 2-dimensional chiral subspaces.

4 The Gauge Symmetries

We construct an orthonormal basis of $\chi^\dagger \oplus \chi$, $(\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \tilde{\mathfrak{e}}_1, \tilde{\mathfrak{e}}_2, \tilde{\mathfrak{e}}_3)$,

$$\begin{cases} \mathfrak{e}_j = \mathfrak{q}_j + \mathfrak{q}^\dagger{}_j \\ \tilde{\mathfrak{e}}_j = i\left(\mathfrak{q}^\dagger{}_j - \mathfrak{q}_j\right) \end{cases} \quad \begin{cases} \mathfrak{q}_j = \frac{1}{2}\left(\mathfrak{e}_j + i\tilde{\mathfrak{e}}_j\right) \\ \mathfrak{q}^\dagger{}_j = \frac{1}{2}\left(\mathfrak{e}_j - i\tilde{\mathfrak{e}}_j\right) \end{cases} \tag{14}$$

where $j \in \{1, 2, 3\}$. Then, $\mathfrak{e}_j^2 = \tilde{\mathfrak{e}}_j^2 = 1$.

The weak symmetry group $SU(2)_L$ is generated by the elements

$$\begin{cases} \widetilde{T}_1 := \mathfrak{u}_u \mathfrak{u}'_d - \mathfrak{u}'_u \mathfrak{u}_d \\ \widetilde{T}_2 := \mathfrak{u}_u \mathfrak{u}_d + \mathfrak{u}'_u \mathfrak{u}'_d \\ \widetilde{T}_3 := \mathfrak{u}_u \mathfrak{u}'_u - \mathfrak{u}_d \mathfrak{u}'_d \end{cases}, \tag{15}$$

where $\mathfrak{u}_u = -i\mathfrak{e}_3\tilde{\mathfrak{e}}_1$, $\mathfrak{u}'_u = \tilde{\mathfrak{e}}_1\tilde{\mathfrak{e}}_2\tilde{\mathfrak{e}}_3\mathfrak{e}_2$, $\mathfrak{u}_d = i\tilde{\mathfrak{e}}_1\tilde{\mathfrak{e}}_2\tilde{\mathfrak{e}}_3$, and $\mathfrak{u}'_d = i\mathfrak{e}_1\mathfrak{e}_2\mathfrak{e}_3$.

From (11) follows that the elements (15) have the following matrix form:

$$\widetilde{T}_1 = 2i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1_2 & 0 \\ 0 & 1_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \widetilde{T}_2 = 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1_2 & 0 \\ 0 & 1_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \widetilde{T}_3 = 2i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1_2 & 0 & 0 \\ 0 & 0 & 1_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{16}$$

We define $\mathfrak{u}_\circ := i\mathfrak{e}_3\tilde{\mathfrak{e}}_2$ and $\mathfrak{u}'_\circ := \tilde{\mathfrak{e}}_1\tilde{\mathfrak{e}}_2\tilde{\mathfrak{e}}_3\mathfrak{e}_1$, and

$$\begin{cases} \omega_j = \frac{1}{2}\left(\mathfrak{u}_j + i\mathfrak{u}'_j\right) \\ \omega^\dagger{}_j = \frac{1}{2}\left(\mathfrak{u}_j - i\mathfrak{u}'_j\right), \end{cases} \tag{17}$$

where $j \in \{u, d, \circ\}$.

The elements ω_j and $\omega^\dagger{}_j$, $j \in \{u, d\}$, split the ideal $\mathcal{A}_{SM}\mathfrak{p}$ into subspaces which correspond to the singlets and doublets of the weak symmetry:

$$\begin{cases} \text{Right-handed up singlet space: } \mathbb{W}_{0R} := 1\,\mathrm{span}_{\mathbb{C}}\left(\mathfrak{p}, \omega^\dagger{}_\circ\mathfrak{p}\right), \\ \text{Left-handed up doublet space: } \mathbb{W}_{0L} := \omega^\dagger{}_u\mathbb{W}_{0R}, \\ \text{Right-handed down singlet space: } \mathbb{W}_{1R} := \omega^\dagger{}_u\omega^\dagger{}_d\mathbb{W}_{0R}, \\ \text{Left-handed down doublet space: } \mathbb{W}_{1L} := \omega^\dagger{}_d\mathbb{W}_{0R}. \end{cases} \tag{18}$$

As proven in [10], the bivectors in Eq. (15) are spinorial generators of the $SU(2)_L$ group, by the adjoint action

$$e^{-i\varphi T_j}a = e^{\frac{\varphi}{2}\widetilde{T}_j}ae^{-\frac{\varphi}{2}\widetilde{T}_j} = e^{-i\varphi\sigma_j\otimes\frac{1-\gamma^5}{2}}a \tag{19}$$

for any $a \in \mathbb{W}_{0\mathrm{L}} \oplus \mathbb{W}_{1\mathrm{L}}$ and $j \in \{1, 2, 3\}$. Therefore, the ideals give the correct representations of the symmetry group $\mathrm{SU}(2)_{\mathrm{L}}$.

In [10] it is also shown that for the algebra $\mathcal{A}_{\mathrm{SM}}$, the Weinberg angle is $\theta_{W,\mathcal{A}_{\mathrm{SM}}} = \frac{\pi}{6}$, given by

$$\sin^2 \theta_{W,\mathcal{A}_{\mathrm{SM}}} = 0.25. \tag{20}$$

Different experiments gave different values for $\sin^2 \theta_W$, ranging between $\sim$0.223 and $\sim$0.24 [11, 12]. So the value predicted by this model is a bit larger, but not as large as that of 0.375 predicted by the SU(5), Spin(10), and other GUTs, but the same as other models like [8, 13]. For a correct comparison we should consider the running of the coupling constants due to higher order perturbative corrections. The Higgs scalar field is, internally, a vector $\phi \in \mathbb{W}_w$. The direction of the vector ϕ in $\mathbb{W}_w$ is the element $\omega_{\mathrm{u}}^{\dagger}$ [10]. While in this model the symmetry breaking appears to be due to geometry, the Higgs field is still responsible for the masses of particles.

As for the $\mathrm{SU}(3)_{\mathrm{c}}$, it is generated by the adjoint action of

$$\begin{array}{l}
\widetilde{\lambda}_1 = \mathfrak{e}_1\tilde{\mathfrak{e}}_2 - \tilde{\mathfrak{e}}_1\mathfrak{e}_2,\ \widetilde{\lambda}_2 = \mathfrak{e}_1\mathfrak{e}_2 + \tilde{\mathfrak{e}}_1\tilde{\mathfrak{e}}_2,\ \widetilde{\lambda}_3 = \mathfrak{e}_1\tilde{\mathfrak{e}}_1 - \mathfrak{e}_2\tilde{\mathfrak{e}}_2,\\
\widetilde{\lambda}_4 = \mathfrak{e}_1\tilde{\mathfrak{e}}_3 - \tilde{\mathfrak{e}}_1\mathfrak{e}_3,\ \widetilde{\lambda}_5 = \mathfrak{e}_1\mathfrak{e}_3 + \tilde{\mathfrak{e}}_1\tilde{\mathfrak{e}}_3,\\
\widetilde{\lambda}_6 = \mathfrak{e}_2\tilde{\mathfrak{e}}_3 - \tilde{\mathfrak{e}}_2\mathfrak{e}_3,\ \widetilde{\lambda}_7 = \mathfrak{e}_2\mathfrak{e}_3 + \tilde{\mathfrak{e}}_2\tilde{\mathfrak{e}}_3,\ \widetilde{\lambda}_8 = \frac{1}{\sqrt{3}}(\mathfrak{e}_1\tilde{\mathfrak{e}}_1 + \mathfrak{e}_2\tilde{\mathfrak{e}}_2 - 2\mathfrak{e}_3\tilde{\mathfrak{e}}_3).
\end{array} \tag{21}$$

The standard Gell-Mann matrices are defined by $\lambda_j = i\widetilde{\lambda}_j$. Then,

$$e^{-i\varphi\lambda_j}a = e^{\frac{\varphi}{2}\widetilde{\lambda}_j}ae^{-\frac{\varphi}{2}\widetilde{\lambda}_j}, \tag{22}$$

for the $\mathrm{SU}(3)_{\mathrm{c}}$ representation **3**. The spinorial action generated by (21) is equivalent to the right multiplication with the matrix $O_1 \oplus e^{i\varphi\lambda_j} \oplus O_1 \oplus e^{-i\varphi\lambda_j}$, according to the representations $\mathbf{1}_c$, $\mathbf{3}_c$, $\overline{\mathbf{1}}_c$, and $\overline{\mathbf{3}}_c$ [6, 10, 14, 15].

Since the $\mathrm{U}(1)_{\mathrm{em}}$ gauge transformation only multiplies the vectors in χ by a phase factor $e^{i\varphi}$, the generator is the identity of $\mathrm{End}_{\mathbb{C}}(\chi)$ [7, 10]

$$Q = \mathfrak{e}_1\tilde{\mathfrak{e}}_1 + \mathfrak{e}_2\tilde{\mathfrak{e}}_2 + \mathfrak{e}_3\tilde{\mathfrak{e}}_3. \tag{23}$$

The action of the group $\mathrm{U}(1)_{\mathrm{em}}$ being adjoint, it is consistent with the exterior product, and it transforms both $\mathfrak{p}\bigwedge^k\chi$ and $\omega_{\mathrm{d}}^{\dagger}\mathfrak{p}$. The way it transforms gives charges that are multiple of $\frac{1}{3}$, and they depend not only on the minimal left ideal, but also each ideal represents two particles with charges differing by -1.

5 Open Problems

The algebra $\mathcal{A}_{\mathrm{SM}}$ proposes an explanation for the leptons and quarks, their discrete properties, and their gauge symmetries. But it does not explain why there are three families, the masses, the quark and lepton mixing matrices, the mechanism responsible for the neutrino masses. Maybe there is another algebra which explains all these, or maybe it is this one, supplemented with additional features. It is not clear at this point what happens in perturbative regime. Another open problem is the connection with general relativity, and quantum gravity. Does this algebra arise from a geometric structure intimately related with the spacetime geometry?

Acknowledgements I wish to thank Prof. Ivan Todorov and Igor Salom for inspiring discussions about this model.

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Part V
Foundations of Quantum Theory

Infinitesimal Symmetries in Covariant Quantum Mechanics

Josef Janyška, Marco Modugno and Dirk Saller

Abstract We discuss the Lie algebras of infinitesimal symmetries of the main covariant geometric objects of covariant quantum mechanics: the time form, the hermitian metric, the upper quantum connection, the quantum lagrangian. Indeed, these infinitesimal symmetries are generated, in a covariant way, by the Lie algebra of time preserving conserved special phase functions. Actually, this Lie algebra of special phase functions generates also the Lie algebra of infinitesimal symmetries of the main classical objects: the time form and the cosymplectic 2-form. A natural output of the classification of the quantum symmetries is a covariant approach to quantum operators and to quantum currents associated with special phase functions.

Keywords Covariant classical mechanics · Covariant quantum mechanics
Quantum symmetries

2010 MSC: 81Q99 · 81S10 · 83C00 · 70H40 · 70G45 · 58A20.

1 Introduction

Several covariant formulations of Classical and Quantum Mechanics in a curved spacetime with absolute time have been proposed by different authors (see, for instance, [2–18, 28–32, 37–40] and citations therein).

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_25

In particular, “Covariant Quantum Mechanics” is an approach to Quantum Mechanics in a curved spacetime fibred over absolute time, equipped with a riemannian metric on its fibres, and aimed at implementing several features of General Relativity in this riemannian framework. This formulation started some years ago [20] and has been further developed by several papers (see, for instance, [19, 21–23, 25, 26, 34–36, 41] and citations therein).

The infinitesimal symmetries of Covariant Classical Mechanics have been discussed in [26, 34–36]. In the present paper, we discuss the infinitesimal symmetries of the fundamental objects of Covariant Quantum Mechanics: the time form dt , the η-hermitian metric $\mathfrak{h}_\eta$ and the upper quantum connection $\text{Ч}^\uparrow$, which is the source of all other quantum objects. We find that the Lie algebra of the infinitesimal symmetries of these objects is isomorphic, in a covariant way, to the Lie algebra of time preserving conserved special phase functions [35]. Moreover, we find that the Lie algebra of infinitesimal symmetries of the quantum lagrangian L and of the time form dt coincides with the Lie algebra of the above fundamental quantum objects and also with the Lie algebra of the fundamental classical objects: the time form dt and the cosymplectic 2-form Ω . Hence, the results of this paper underline the meaning of the Lie algebra of special phase functions and its distinguished subalgebras within this approach to Classical and Quantum Mechanics. This again confirms the covariant approach, which was crucial for the discovery of special phase functions.

We deal with units of measurement on the same footing of coordinates, gauges and observers. So, in order to make our theory explicitly independent of “*units of measurement*”, we use the notion of “*spaces of scales*” [25, 27].

We consider the following *basic spaces of scales*: (1) the space $\mathbb{T}$ of *time intervals*, (2) the space $\mathbb{L}$ of *lengths*, (3) the space $\mathbb{M}$ of *masses*. Then, other *space of scales* are obtained by tensor products of rational powers of the above basic spaces.

We consider the *Planck constant* $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$ as a “*universal scale*”. Moreover, we will consider a *mass* $m \in \mathbb{M}$ and *charge* $q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$. We denote a *time unit of measurement* and its dual, respectively, by $u_0 \in \mathbb{T}$ and $u^0 \in \mathbb{T}^* \simeq \mathbb{T}^{-1}$.

2 Sketch of the Classical Background

The classical background of Covariant Quantum Mechanics is provided by a suitable formulation of Classical Mechanics (for a short account of it, see, for instance, [25], where the reader can find further details).

In the present model, we postulate *time* as an oriented 1-dimensional affine space $\boldsymbol{T}$, associated with the vector space $\mathbb{T} \otimes \mathbb{R}$, and *spacetime* as an oriented 4-dimensional manifold $\boldsymbol{E}$ equipped with a *time fibring* $t : \boldsymbol{E} \to \boldsymbol{T}$.

The time fibring yields the distinguished *time form* $dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$.

We shall refer to *spacetime charts* $(x^\lambda) \equiv (x^0, x^i)$, defined as charts of the manifold $\boldsymbol{E}$, which are adapted to the time fibring, the affine structure of $\boldsymbol{T}$ and the orientation of $\boldsymbol{E}$ and $\boldsymbol{T}$. Every spacetime chart (x^λ) yields a *time scale* $u_0 \in \mathbb{T}$. The associated bases of vector fields and forms are denoted by $(\partial_\lambda) \equiv (\partial_0, \partial_i)$ and

$(d^\lambda) \equiv (d^0, d^i)$. Accordingly, we obtain the linear fibred charts of the tangent bundle $\boldsymbol{TE} \to \boldsymbol{E}$ by $(x^\lambda, \dot{x}^\lambda)$.

We denote by $\boldsymbol{VE} \subset \boldsymbol{TE}$ the 3-dimensional *vertical subbundle* annihilated by dt and by $H^*\boldsymbol{E} \subset T^*\boldsymbol{E}$ the 1-dimensional *horizontal subbundle* generated by dt . The vertical projection $T^*\boldsymbol{E} \to V^*\boldsymbol{E}$ is denoted by the restriction symbol $\vee$.

The *classical motions* are the sections $s : \boldsymbol{T} \to \boldsymbol{E}$.

The *classical phase space* is the 7-dimensional 1st jet space of motions $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$, equipped with the fibred charts (x^λ, x^i_0) .

The phase space is naturally equipped with the *contact map* and the *complementary contact map* $\text{д} : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{TE}$ and $\theta : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E}$, with coordinate expressions $\text{д} = u^0 \otimes (\partial_0 + x^i_0\, \partial_i)$ and $\theta = (d^i - x^i_0\, d^0) \otimes \partial_i$.

The *classical observers* are the sections $o : \boldsymbol{E} \to J_1\boldsymbol{E}$.

An observer o is characterised by the "observed" *contact map* and *complementary contact map* $\text{д}[o] := \text{д} \circ o : \boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{TE}$ and $\theta[o] := \theta \circ o : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E}$, with coordinate expressions $\text{д}[o] = u^0 \otimes (\partial_0 + o^i_0\, \partial_i)$ and $\theta[o] = (d^i - o^i_0\, d^0) \otimes \partial_i$.

Then, we postulate the *galileian metric* to be a spacelike riemannian metric $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})$. With reference to a particle of mass $m \in \mathbb{M}$, and to the Planck constant $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$, the *rescaled galileian metric* is $G := \frac{m}{\hbar}\, g : \boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})$.

We have the coordinate expressions $g = g_{ij}\, \check{d}^i \otimes \check{d}^j$ and $G = G^0_{ij}\, u_0 \otimes \check{d}^i \otimes \check{d}^j$, with $g_{ij} \in \text{map}(\boldsymbol{E}, \mathbb{L}^2 \otimes \mathbb{R})$ and $G^0_{ij} \in \text{map}(\boldsymbol{E}, \mathbb{R})$.

The spacelike metric g and the spacetime orientation yield the scaled *spacelike volume form* $\eta : \boldsymbol{E} \to \mathbb{L}^3 \otimes \wedge^3 V^*\boldsymbol{E}$, with coordinate expression $\eta = \sqrt{|g|}\, \check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3$.

Then, we obtain the scaled *spacetime volume form* $\upsilon := dt \wedge \eta : \boldsymbol{E} \to \mathbb{T} \otimes \wedge^4 T^*\boldsymbol{E}$, with coordinate expression $\upsilon = \upsilon^0\, u_0 = \sqrt{|g|}\, u_0 \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3$.

Given an observer o , we define the *observed kinetic energy*, the *observed kinetic momentum* and the *observed Poincaré–Cartan form* to be, respectively, the sections

$$\begin{aligned} \mathscr{K}[G, o] &:= \tfrac{1}{2}\, G\, (\nabla[o], \nabla[o]) && \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E})\,, \\ \mathscr{Q}[G, o] &:= \theta[o] \lrcorner (G^\flat\, \nabla[o]) && \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})\,, \\ \Theta[G, o] &:= -\mathscr{K}[G, o] + \mathscr{Q}[G, o] && \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})\,, \end{aligned}$$

with coordinate expressions

$$\begin{aligned} \mathscr{K}[G, o] &= \tfrac{1}{2}\, G^0_{ij} (x^i_0 - o^i_0)\, (x^j_0 - o^j_0)\, d^0\,, \\ \mathscr{Q}[G, o] &= (-\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + \tfrac{1}{2}\, G^0_{ij}\, o^i_0\, o^j_0)\, d^0 + G^0_{ij}\, (x^j_0 - o^j_0)\, d^i\,, \\ \Theta[G, o] &= G^0_{ij} (x^j_0 - o^j_0)\, (d^i - o^i_0\, d^0)\,. \end{aligned}$$

We define a *galileian spacetime connection* to be a spacetime connection K , which is linear, torsion free and which fulfills the conditions $\nabla dt = 0$, $\nabla g = 0$ and $R_{i\mu j\nu} = R_{j\nu i\mu}$, where R is the curvature tensor of K . Its coordinate expression is of the type

$$
\begin{aligned}
K &= d^\lambda \otimes (\partial_\lambda + K_\lambda{}^i{}_\mu\, \dot{x}^\mu\, \dot{\partial}_i) \\
&= d^\lambda \otimes \partial_\lambda - \tfrac{1}{2}\, G^{ij}_0 \big(\partial_0 G^0_{hj}\, (\dot{x}^h\, d^0 + \dot{x}^0\, d^h) + (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\, \dot{x}^k\, d^h\big) \otimes \dot{\partial}_i \\
&\quad - G^{ij}_0 \big(\Phi_{0j}\, \dot{x}^0\, d^0 + \tfrac{1}{2}\, \Phi_{hj}\, (\dot{x}^h\, d^0 + \dot{x}^0\, d^h)\big) \otimes \dot{\partial}_i\,,
\end{aligned}
$$

where $\Phi \equiv \Phi[K, G, o] = \Phi_{\lambda\mu}\, d^\lambda \wedge d^\mu : \boldsymbol{E} \to \wedge^2 T^*\boldsymbol{E}$ is a closed spacetime 2-form, which depends on K, on G and on the observer o associated with the chosen spacetime chart (x^λ), by the condition $o^i_0 = 0$.

Further, we postulate, as *gravitational and electromagnetic fields*, a galileian spacetime connection and a closed scaled spacetime 2-form [33]

$$
K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E} \qquad \text{and} \qquad F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \wedge^2 T^*\boldsymbol{E}\,.
$$

With reference to a particle of mass m and charge q, we couple $K^\natural$ and F into the *joined galileian spacetime connection* $K \equiv K^\natural + K^\mathfrak{e} := K^\natural - \frac{1}{2}\frac{q}{\hbar}\,(dt \otimes \widehat{F} + \widehat{F} \otimes dt)$, where $\widehat{F} := G^{\sharp 2}(F) : \boldsymbol{E} \to (\mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes (T^*\boldsymbol{E} \otimes V\boldsymbol{E})$.

From now on, we shall refer to the joined spacetime connection K.

The joined observed spacetime 2-form $\Phi \equiv \Phi[K, G, o]$ splits as $\Phi = \Phi^\natural + \frac{1}{2}\frac{q}{\hbar}\, F$.

We consider as *law of motion* for a particle, with mass m and charge q, effected by the gravitational and electromagnetic fields, the equation $\nabla[K]ds = 0$.

We define a *phase connection* to be a connection $\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TJ_1\boldsymbol{E}$ of the affine bundle $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$.

There is a bijection between time preserving, linear spacetime connections K and affine phase connections Γ with coordinate expression $K_\lambda{}^i{}_\mu \mapsto \Gamma_{\lambda}{}^{i}{}_{0\mu}^{0}$ [21].

Each affine phase connection Γ yields the "quadratic" *dynamical phase connection*, the *dynamical phase 2-form*, the *dynamical phase 2-vector*

$$
\begin{aligned}
\gamma &\equiv \gamma[\Gamma] \quad := \text{Д} \lrcorner\, \Gamma : \boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}\,, \\
\Omega &\equiv \Omega[\Gamma, G] := G \lrcorner \big(\nu[\Gamma] \wedge \theta\big) : J_1\boldsymbol{E} \to \wedge^2 T^* J_1\boldsymbol{E}\,, \\
\Lambda &\equiv \Lambda[\Gamma, G] := \bar{G} \lrcorner\, (\check{\Gamma} \wedge \nu) : J_1\boldsymbol{E} \to \wedge^2 V J_1\boldsymbol{E}\,.
\end{aligned}
$$

Therefore, the joined spacetime connection K yields the distinguished affine phase connection, dynamical phase connection, dynamical phase 2-form, dynamical phase 2-vector $\Gamma \equiv \Gamma[K]$, $\gamma \equiv \gamma[K]$, $\Omega \equiv \Omega[K, G]$, $\Lambda \equiv \Lambda[K, G]$.

We have the coordinate expressions

$$
\begin{aligned}
\Gamma[K] &= d^\lambda \otimes \partial_\lambda - G^{ij}_0 \big(\Phi_{0j} + \tfrac{1}{2}\, (\partial_0 G^0_{hj} + \Phi_{hj})\, x^h_0)\big)\, d^0 \otimes \partial^0_i \\
&\quad - G^{ij}_0\, \tfrac{1}{2} \big((\partial_0 G^0_{kj} + \Phi_{kj}) + (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\, x^h_0)\big)\, d^k \otimes \partial^0_i\,, \\
\gamma[K] &= u^0 \otimes (\partial_0 + x^i_0\, \partial_i) \\
&\quad - G^{ij}_0 \big((\partial_h G^0_{jk} - \tfrac{1}{2}\, \partial_j G^0_{hk})\, x^h_0\, x^k_0 + \partial_0 G^0_{hj}\, x^h_0 + (\Phi_{hj}\, x^h_0 + \Phi_{0j})\big)\, u^0 \otimes \partial^0_i\,,
\end{aligned}
$$

$$\Omega[K, G] = (\partial_0 G^0_{hj}\, x^h_0 + \tfrac{1}{2}\, \partial_j G^0_{hk}\, x^h_0\, x^k_0)\, d^0 \wedge d^j + (\partial_i G^0_{jh}\, x^h_0)\, d^i \wedge d^j$$
$$+ G^0_{hj}\, x^h_0\, d^0 \wedge d^j_0 - G^0_{ij}\, d^i \wedge d^j_0 + \tfrac{1}{2}\, \Phi_{\lambda\mu}\, d^\lambda \wedge d^\mu\,,$$
$$\Lambda[K, G] = G^{ij}_0\, \partial_i \wedge \partial^0_j + G^{ih}_0\, G^{jk}_0 \left(\partial_h G^0_{kr}\, x^r_0 + \tfrac{1}{2}\, \Phi_{hk}\right) \partial^0_i \wedge \partial^0_j\,.$$

We can prove that $\Omega[K, G]$ turns out to be closed if and only if K is galileian.

Hence, the pair (dt, Ω) turns out to be a scaled *cosymplectic structure* of the phase space [24]. In other words, $dt \wedge \Omega \wedge \Omega \wedge \Omega : J_1\boldsymbol{E} \to \mathbb{T} \otimes \wedge^7 T^* J_1\boldsymbol{E}$ is a scaled volume form of the phase space and $d\Omega = 0$.

The cosymplectic 2-form Ω admits an *"upper" horizontal potential* of the type $A^\uparrow : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$, according to the equation $\Omega = dA^\uparrow$. Clearly, the horizontal potential $A^\uparrow$ is locally defined up to a gauge of the type $df : \boldsymbol{E} \to T^*\boldsymbol{E}$, with $f \in \text{map}(\boldsymbol{E}, \mathbb{R})$.

For each observer o, we have $\Phi[K, G, o] = 2\, o^* \Omega[K, G]$. Hence, the observed potential $A[K, G, o]$ of $\Phi[K, G, o]$ turns out to be given (up to a gauge) by the equality $A[K, G, o] = o^* A^\uparrow$.

The classical law of motion for a motion s effected by the gravitational and electromagnetic fields is expressed equivalently by the equations $\nabla[K]ds = 0$, or $dj_1 s = \gamma[K] \circ j_1 s$.

The *classical lagrangian*, the *classical momentum*, the *observed classical hamiltonian* and the *observed classical momentum* are, respectively, the horizontal and vertical components and the observed horizontal and vertical components of $A^\uparrow$

$$\mathscr{L} \equiv \mathscr{L}[A^\uparrow] := д \lrcorner A^\uparrow \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E})\,,$$
$$\mathscr{P} \equiv \mathscr{P}[A^\uparrow] := \theta \lrcorner A^\uparrow \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})\,,$$
$$\mathscr{H}[A^\uparrow, o] := -\, д[o] \lrcorner A^\uparrow = \mathscr{K}[G, o] - A[G, o] \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E})\,,$$
$$\mathscr{P}[A^\uparrow, o] := \theta[o] \lrcorner A^\uparrow = \mathscr{Q}[G, o] + A[G, o] \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})\,.$$

We have the coordinate expressions

$$\mathscr{L}[A^\uparrow] = (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + A_j\, x^j_0 + A_0)\, d^0\,, \qquad \mathscr{P}[A^\uparrow] = (G^0_{ij}\, x^j_0 + A_i)\, (d^i - x^i_0\, d^0)\,,$$
$$\mathscr{H}[A^\uparrow, o] = (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0\,, \qquad \mathscr{P}[A^\uparrow, o] = (G^0_{ij}\, x^j_0 + A_i)\, (d^i - o^i_0\, d^0)\,.$$

3 Setting of the Quantum Theory

Next, we sketch the starting setting of Covariant Quantum Mechanics (for a short account of it, see, for instance, [25], where the reader can find further details).

We postulate the *quantum bundle* to be a 1-dimensional complex vector bundle over spacetime $\pi : \boldsymbol{Q} \to \boldsymbol{E}$, equipped with a scaled *η-hermitian quantum metric* $\mathfrak{h}_\eta : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \wedge^3 V^*\boldsymbol{E} \otimes \mathbb{C}$.

We shall refer to normalised scaled *quantum bases* $\mathfrak{b} : \boldsymbol{E} \to \mathbb{L}^{3/2} \otimes \boldsymbol{Q}$, which fulfill the condition $\mathfrak{h}_\eta(\mathfrak{b}, \mathfrak{b}) = \eta$. Accordingly, we shall refer to scaled linear fibred charts (x^λ, z), where the scaled complex function $z : \boldsymbol{Q} \to \mathbb{L}^{-3/2} \otimes \mathbb{C}$, fulfills the condition $z(\mathfrak{b}) = 1$, and to the associated real fibred charts $(x^\lambda,\ w^1, w^2)$, given by $z = w^1 + \mathfrak{i}\, w^2$.

The quantum states are represented by the *quantum sections* $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$. We shall write $\Psi = \psi\, \mathfrak{b}$, with $\psi \equiv |\psi| \exp(\mathfrak{i}\, \varphi) \in \mathrm{map}(\boldsymbol{E}, \mathbb{L}^{-3/2} \otimes \mathbb{C})$.

We define the *upper quantum bundle* to be the 1-dimensional complex vector bundle $\pi^\uparrow : \boldsymbol{Q}^\uparrow \to J_1\boldsymbol{E}$ over the phase space, given by the pullback $\boldsymbol{Q}^\uparrow := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q}$.

The η-hermitian quantum metric $\mathfrak{h}$ yields, by pullback, the η-hermitian upper quantum metric $\mathfrak{h}^\uparrow$.

We say that a complex linear connection $\text{Ч}^\uparrow : \boldsymbol{Q}^\uparrow \underset{J_1\boldsymbol{E}}{\times} T J_1\boldsymbol{E} \to T\boldsymbol{Q}^\uparrow$ is *reducible* if it factorises through a system of quantum connections $\text{Ч}[o] : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{Q}$. Indeed, $\text{Ч}^\uparrow$ turns out to be reducible if and only if, in coordinates, $\text{Ч}^{\uparrow}{}^0_i = 0$.

We postulate the *galileian upper quantum connection* $\text{Ч}^\uparrow : \boldsymbol{Q}^\uparrow \to T^*J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^\uparrow$ to be a connection of the upper quantum bundle, which is hermitian and reducible and whose curvature fulfills the condition $R[\text{Ч}^\uparrow] = -2\,\mathfrak{i}\,\Omega \otimes \mathbb{I}^\uparrow$, where $\mathbb{I}^\uparrow : \boldsymbol{Q}^\uparrow \to \boldsymbol{Q}^\uparrow$ is the Liouville vector field of $\boldsymbol{Q}^\uparrow$ (see also [32]). The closure of Ω turns out to be a necessary integrability condition for the local existence of $\text{Ч}^\uparrow$, because of the Bianchi identity. The integer cohomology class of Ω turns out to be a necessary integrability condition for the global existence of $\text{Ч}^\uparrow$ [41]. The upper quantum connections $\text{Ч}^\uparrow$ are defined locally up to a gauge of the type $\mathfrak{i}\, df \otimes \mathbb{I}^\uparrow$, where $f : \boldsymbol{E} \to \mathbb{R}$.

With reference to a quantum basis $\mathfrak{b}$, the coordinate expression of an upper quantum connection $\text{Ч}^\uparrow$ is locally of the type

$$\begin{aligned}
\text{Ч}^\uparrow &= \chi^\uparrow[\mathfrak{b}] + \mathfrak{i}\, A^\uparrow[\mathfrak{b}] \otimes \mathbb{I}^\uparrow \\
&= \chi^\uparrow[\mathfrak{b}] + \mathfrak{i}\,\big(\Theta[o] + A[\mathfrak{b}, o]\big) \otimes \mathbb{I}^\uparrow \\
&= \chi^\uparrow[\mathfrak{b}] + \mathfrak{i}\,\big(-\mathscr{K}[o] + \mathscr{Q}[o] + A[\mathfrak{b}, o]\big) \otimes \mathbb{I}^\uparrow \\
&= \chi^\uparrow[\mathfrak{b}] + \mathfrak{i}\,\big(-\mathscr{H}[\mathfrak{b}, o] + \mathscr{P}[\mathfrak{b}, o]\big) \otimes \mathbb{I}^\uparrow \\
&= d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + \mathfrak{i}\,\big(-(\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0 + (G^0_{ij}\, x^j_0 + A_i)\, d^i\big) \otimes \mathbb{I}^\uparrow,
\end{aligned}$$

where $\chi^\uparrow[\mathfrak{b}] : \boldsymbol{Q}^\uparrow \to T^*J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^\uparrow$ is the flat hermitian upper quantum connection induced by the quantum basis $\mathfrak{b}$.

Thus, the *upper quantum potential* $A^\uparrow[\mathfrak{b}]$ appearing in the above expression of $\text{Ч}^\uparrow$ is just a potential of Ω and a potential of K, that have been discussed previously.

We suppose the cohomology class of Ω to be integer and postulate a *galileian upper quantum connection* $\text{Ч}^\uparrow$, as source of all further quantum developments.

We observe that the quantum bases $\mathfrak{b}$ allow us to parametrise the upper quantum potentials $A^\uparrow$, hence the *observed quantum potentials* $A[\mathfrak{b}, o]$.

With reference to two quantum bases $\mathfrak{b}$ and $\acute{\mathfrak{b}} = \exp(\mathrm{i}\,\vartheta)\,\mathfrak{b}$ and two observers o and $\acute{o} = o + v$, with $v \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes V\boldsymbol{E})$, we have the transition rules

$$A^{\uparrow}[\acute{\mathfrak{b}}] = A^{\uparrow}[\mathfrak{b}] - d\vartheta \qquad \text{and} \qquad A[\acute{\mathfrak{b}}, \acute{o}] = A[\mathfrak{b}, o] - d\vartheta + \theta[o] \lrcorner G^{\flat}(v) - \tfrac{1}{2}\, G(v, v)\,.$$

From the quantum connection $\text{Ч}^{\uparrow}$ we derive, by a covariant procedure, the *kinetic quantum momentum*, the *probability current*, the *Schrödinger operator*, the *quantum lagrangian* and the *quantum Poincaré–Cartan form*

$$\begin{aligned}
&\mathrm{Q}(\Psi) := \text{д} \otimes \Psi - \mathrm{i}\, G^{\sharp} \nabla^{\uparrow} \Psi : \boldsymbol{E} \to \mathbb{T}^* \otimes (T\boldsymbol{E} \otimes \boldsymbol{Q})\,,\\
&\mathrm{J}(\Psi) := \text{д} \otimes \|\Psi\|^2 - \mathrm{re}\, \mathfrak{h}(\Psi,\, \mathrm{i}\, G^{\sharp} \nabla^{\uparrow} \Psi) : \boldsymbol{E} \to \mathbb{L}^{-3} \otimes (\mathbb{T}^* \otimes T\boldsymbol{E})\,,\\
&\mathrm{S}(\Psi) := \tfrac{1}{2}\Big(\text{д} \lrcorner \nabla^{\uparrow}\Psi + \delta^{\uparrow}\big(\mathrm{Q}(\Psi)\big)\Big) : \boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{Q}\,,\\
&\mathrm{L}(\Psi) := -dt \wedge \big(\mathrm{im}\, \mathfrak{h}_\eta(\Psi,\, \text{д} \lrcorner \nabla^{\uparrow}\Psi) + \tfrac{1}{2}\, (\bar{G} \otimes \mathfrak{h}_\eta)(\check{\nabla}^{\uparrow}\Psi,\, \check{\nabla}^{\uparrow}\Psi) : \boldsymbol{E} \to \wedge^4 T^*\boldsymbol{E}\,,\\
&\Theta[\mathrm{L}] := \mathrm{L} + \vartheta \mathbin{\bar{\wedge}} V_{\boldsymbol{Q}}\mathrm{L} : J_1\boldsymbol{Q} \to \wedge^4 T^*\boldsymbol{Q}\,,
\end{aligned}$$

with coordinate expressions

$$\begin{aligned}
\mathrm{Q}[\Psi] &= \big(\psi\, \partial_0 - \mathrm{i}\, G^{ij}_0\, (\partial_j \psi - \mathrm{i}\, A_j[\mathfrak{b}, o]\, \psi)\, \partial_i\big) \otimes u^0 \otimes \mathfrak{b}\,,\\
\mathrm{J}(\Psi) &= \big(|\psi|^2\, \partial_0 + (\mathrm{i}\, \tfrac{1}{2}\, G^{ij}_0\, (\psi\, \partial_j \bar{\psi} - \bar{\psi}\, \partial_j \psi) - A^i_0[\mathfrak{b}, o]\, |\psi|^2)\, \partial_i\big) \otimes u^0\,,\\
\mathrm{S}(\Psi) &= \Big((\partial_0 \psi - \tfrac{1}{2}\, \mathrm{i}\, G^{ij}_0\, \partial_{ij}\psi) - \mathrm{i}\, (A_0 - \tfrac{1}{2}\, A_i\, A^i_0)\, \psi\\
&\quad - \left(\left(A^j_0 + \tfrac{1}{2}\, \mathrm{i}\, \frac{\partial_i (G^{ij}_0\, \sqrt{|g|})}{\sqrt{|g|}}\right) \partial_j \psi\right) + \tfrac{1}{2}\left(\left(\frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i (A^i_0\, \sqrt{|g|})}{\sqrt{|g|}}\right) \psi\right) u_0 \otimes \mathfrak{b}\,,\\
\mathrm{L}(\Psi) &= \tfrac{1}{2}\,\big(-G^{ij}_0\, \partial_i \bar{\psi}\, \partial_j \psi + \mathrm{i}\, A^{\lambda}_0\, (\bar{\psi}\, \partial_\lambda \psi - \psi\, \partial_\lambda \bar{\psi}) + 2\, (A_0 - \tfrac{1}{2}\, A_i\, A^i_0)\big)\, \upsilon^0\,,\\
\Theta[\mathrm{L}] &= \tfrac{1}{2}\, \mathrm{i}\, (\bar{z}\, dz - z\, d\bar{z}) \wedge \upsilon^0_0 - \tfrac{1}{2}\, \big(G^{ij}_0\, (\bar{z}_i\, dz + z_i\, d\bar{z}) + \mathrm{i}\, A^i_0\, (\bar{z}\, dz - z\, d\bar{z})\big) \otimes \upsilon^0_j\\
&\quad + \big(\tfrac{1}{2}\, G^{ij}_0\, \bar{z}_i\, z_j + (A_0 - \tfrac{1}{2}\, A_i\, A^i_0) \bar{z}\, z\big)\, \upsilon^0\,,
\end{aligned}$$

where $\upsilon_\lambda := i_{\partial_\lambda} \upsilon$, $A^0_0 := 1$ and $A^i_0 := G^{ij}_0\, A_j$.

In the particular case of a flat spacetime and an inertial observer, S turns out to be the standard Schrödinger operator.

4 Lie Algebra of Special Phase Functions

Definition 1 ([20, 23]) A *special phase function* (s.p.f.) is defined to be a phase function $f \in \mathrm{map}(J_1\boldsymbol{E}, \mathbb{R})$, such that its 2nd fibre derivative with respect to the affine bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$ is of the type $D^2 f = f'' \otimes G$, with $f'' \in \mathrm{map}(\boldsymbol{E}, \mathbb{T} \otimes \mathbb{R})$.

In coordinates, a special phase function is characterised by an expression of the type $f = f^0 \frac{1}{2} G^0_{ij} x^i_0 x^j_0 + f^i G^0_{ij} x^j_0 + \breve{f}$, with $f^0, f^i, \breve{f} \in \text{map}(\boldsymbol{E}, \mathbb{R})$. Accordingly, we have $f'' = f^0 u_0$.

We denote the subsheaf of s.p.f. by $\text{spe}\,(J_1\boldsymbol{E}, \mathbb{R}) \subset \text{map}(J_1\boldsymbol{E}, \mathbb{R})$. □

We have the following distinguished subsheaves of $\text{spe}\,(J_1\boldsymbol{E}, \mathbb{R})$

subsheaf of projectable s.p.f. $:= \text{pro spe}\,(J_1\boldsymbol{E}, \mathbb{R}) := \{f \mid \partial_j f^0 = 0\}$.
subsheaf of time preserving s.p.f. $:= \text{tim spe}\,(J_1\boldsymbol{E}, \mathbb{R}) := \{f \mid \partial_\lambda f^0 = 0\}$,
subsheaf of affine s.p.f. $:= \text{aff spe}\,(J_1\boldsymbol{E}, \mathbb{R}) := \{f \mid f^0 = 0\}$,
subsheaf of spacetime s.p.f. $:= \text{map}(\boldsymbol{E}, \mathbb{R}) := \{f \mid f^\lambda = 0\}$.

Example 1 We have the distinguished special phase functions

$$x^\lambda, \quad A^{\uparrow}{}_i[\mathfrak{b}, o] = \mathscr{P}_i[\mathfrak{b}, o] = G^0_{ij} x^j_0 + A_i, \quad -A^{\uparrow}{}_0[\mathfrak{b}, o] = \mathscr{H}_0[\mathfrak{b}, o] = \tfrac{1}{2} G^0_{ij} x^i_0 x^j_0 - A_0.$$
□

Proposition 1 *For each* $f \in \text{spe}\,(J_1\boldsymbol{E}, \mathbb{R})$, *we obtain, in a covariant way, the spacetime vector field, called its* tangent lift, $X[f] = f'' \lrcorner\, д - G^\sharp(Df) \in \sec(\boldsymbol{E}, T\boldsymbol{E})$, *with coordinate expression* $X[f] = f^0 \partial_0 - f^i \partial_i$.

For instance, we have: $X[\mathscr{P}_i] = -\partial_i, \quad X[\mathscr{H}_0] = \partial_0, \quad X[\mathscr{L}_0] = \partial_0 - A^i_0 \partial_i$. □

Proposition 2 *With reference to an observer* o *and to a quantum basis* $\mathfrak{b}$, *we can split each* $f \in \text{spe}\,(J_1\boldsymbol{E}, \mathbb{R})$, *respectively, as*

$$f = -X[f] \lrcorner\, \Theta[o] + \breve{f}[o] = (f^0 \mathscr{K}_0 + f^i \mathscr{Q}_i) + \breve{f},$$
$$f = -X[f] \lrcorner\, A^{\uparrow}[\mathfrak{b}] + \hat{f}[\mathfrak{b}] = (f^0 \mathscr{H}_0 + f^i \mathscr{P}_i) + \hat{f},$$

where $\breve{f}[o] = \breve{f}$ *and* $\hat{f}[\mathfrak{b}] = \breve{f} + A_0 f^0 - A_i f^i$. □

Thus, each $f \in \text{spe}\,(J_1\boldsymbol{E}, \mathbb{R})$ is characterised:

- with reference to an observer o, by its observer and gauge independent tangent lift $X[f]$ and observer dependent and gauge independent spacetime function $\breve{f}[o]$,
- with reference to a quantum basis $\mathfrak{b}$, by its observer and gauge independent tangent lift $X[f]$ and gauge dependent and observer independent spacetime function $\hat{f}[\mathfrak{b}]$.

Proposition 3 *We have two distinguished phase lifts of a special phase function* f:

- *the* holonomic phase lift, *which involves only the time fibring of spacetime,*
- *the* hamiltonian phase lift, *which involves the cosymplectic structure of the phase space (here* r_1 *is the natural fibred morphism* $r_1 : J_1T\boldsymbol{E} \to TJ_1\boldsymbol{E}$*),*

$$X^{\uparrow}{}_{\text{hol}}[f] := r_1 \circ J_1X[f], \qquad X^{\uparrow}{}_{\text{ham}}[f] := \gamma(f'') + \Lambda^\sharp(df),$$

with coordinate expressions

$$X^{\uparrow}{}_{\mathrm{hol}}\,[f] = f^0\,\partial_0 - f^i\,\partial_i - (\partial_0 f^i + \partial_j f^i\,x_0^j + \partial_0 f^0\,x_0^i + \partial_j f^0\,x_0^j\,x_0^i)\,\partial_i^0\,,$$
$$X^{\uparrow}{}_{\mathrm{ham}}\,[f] = f^0\,\partial_0 - f^i\,\partial_i + G_0^{ij}\Big(-f^0\,(\partial_0\mathscr{P}_j - \partial_j A_0) + f^h\,(\partial_h\mathscr{P}_j - \partial_j A_h) + \partial_j f^0\,\mathscr{K}_0 + \partial_j f^h\,\mathscr{Q}_h + \partial_j \breve{f}\Big)\,\partial_i^0\,.$$
□

Theorem 1 *The equality* $[\![f, \acute{f}]\!] := \Lambda(df, d\acute{f}) + \gamma(f'').\acute{f} - \gamma(\acute{f}'').f$ *equips the sheaf of special phase functions with an* $\mathbb{R}$*-lie bracket, called* special phase Lie bracket. *This bracket can also be expressed by the following equalities*

$$[\![f, \acute{f}]\!] = -\big[X[f], X[\acute{f}]\big] \lrcorner\, \Theta[o] + X[f].\breve{\acute{f}} - X[\acute{f}].\breve{f} + \Phi[o]\big(X[f], X[\acute{f}]\big)\,,$$
$$[\![f, \acute{f}]\!] = -\big[X[f], X[\acute{f}]\big] \lrcorner\, A^{\uparrow}[\mathfrak{b}] + X[f].\hat{\acute{f}} - X[\acute{f}].\hat{f}\,,$$
$$[\![f, \acute{f}]\!] = X^{\uparrow}[f].\acute{f} - X^{\uparrow}[\acute{f}].f + 2\,\Omega\big(X^{\uparrow}[f], X^{\uparrow}[\acute{f}]\big)\,,$$

where $X^{\uparrow}[f] \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ *is any phase prolongation (in particular, the holonomic lift and the hamiltonian lift) of the tangent lift* $X[f] \in \sec(\boldsymbol{E}, T\boldsymbol{E})$.

In coordinates, we have the following expression

$$[\![f, \acute{f}]\!]^{\lambda} = X[f]^{\mu}\,\partial_{\mu}X[\acute{f}]^{\lambda} - X[\acute{f}]^{\mu}\,\partial_{\mu}X[f]^{\lambda},$$
$$[\![f, \breve{\acute{f}}]\!] = X[f]^{\mu}\,\partial_{\mu}\breve{\acute{f}} - X[\acute{f}]^{\mu}\,\partial_{\mu}\breve{f} + X[f]^{\lambda}\,X[\acute{f}]^{\mu}\,(\partial_{\lambda}A_{\mu} - \partial_{\mu}A_{\lambda}).$$

The projectable, time preserving and affine subsheaves of special phase functions turn out to be closed with respect to the special phase Lie bracket.

The holonomic lift and the hamiltonian lift of special phase functions turn out to be Lie algebra homomorphisms. □

For each $f \in \mathrm{pro\ spe}\,(J_1\boldsymbol{E}, \mathbb{R})$, we set $\mathrm{div}_{\eta}\, f := \mathrm{div}_{\eta}\, X[f]$.

Indeed, we have $\mathrm{div}_{\eta}\, [\![f, \acute{f}]\!] = X[f].\,\mathrm{div}_{\eta}\,\acute{f} - X[\acute{f}].\,\mathrm{div}_{\eta}\, f$.

The subsheaves $\mathrm{uni}_{\eta}\,\mathrm{spe}\,(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{duni}_{\eta}\,\mathrm{spe}\,(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{pro\ spe}\,(J_1\boldsymbol{E}, \mathbb{R})$, of projectable special phase functions with vanishing divergence and with constant divergence, respectively, are closed with respect to the special Lie bracket.

Definition 2 A special phase function f is said to be *holonomic* if its holonomic phase lift and hamiltonian phase lift coincide: $X^{\uparrow}{}_{\mathrm{ham}}\,[f] = X^{\uparrow}{}_{\mathrm{hol}}\,[f]$. □

Proposition 4 *A special phase function* f *turns out to be holonomic if and only if it fulfills the following conditions, in coordinates,*

$$\partial_i f^0 = 0\,,$$
$$\partial_0 f^0\,G_{ij}^0 - (f^0\,\partial_0 - f^h\,\partial_h)\,G_{ij}^0 + \partial_j f^h\,G_{ih}^0 + \partial_i f^h\,G_{jh}^0 = 0\,,$$
$$\partial_i \breve{f} + \partial_0 f^h\,G_{ih}^0 - f^0\,(\partial_0 A_i - \partial_i A_0) + f^h\,(\partial_h A_i - \partial_i A_h) = 0\,.$$

The subsheaf of holonomic *special phase functions* hol spe $(J_1\boldsymbol{E}, \mathbb{R}) \subset$ spe $(J_1\boldsymbol{E}, \mathbb{R})$ *is closed with respect to the special phase Lie bracket.* □

A special phase function f is said to be *conserved* if it is constant along the classical motions solutions of the law of motion, i.e. if $\gamma.f = 0$. We denote the subsheaf of conserved special phase functions by cns spe $(J_1\boldsymbol{E}, \mathbb{R}) \subset$ spe $(J_1\boldsymbol{E}, \mathbb{R})$.

Lemma 1 *For each* $X^{\uparrow} \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ *and* $f \in \mathrm{map}(J_1\boldsymbol{E}, \mathbb{R})$, *the following implication holds*

$$i_{X^{\uparrow}}\,\Omega = df \qquad \Rightarrow \qquad X^{\uparrow} = \gamma\big(dt(X^{\uparrow})\big) + \Lambda^{\sharp}(df) \quad \textit{and} \quad \gamma.f = 0\,.$$

Proof The proof can be achieved from the identities $\Lambda\big(i_{X^{\uparrow}}\Omega\big) = X^{\uparrow} - \gamma(X^{\uparrow})$ and $i_{\gamma}\Omega = 0$. QED

Theorem 2 *For each* $f \in$ spe $(J_1\boldsymbol{E}, \mathbb{R})$, *the following conditions are equivalent:*

$$(1)\ \ 0 = \gamma.f\,, \qquad (2)\ \ df = i_{X^{\uparrow}_{\mathrm{ham}}[f]}\,\Omega\,, \qquad (3)\ \ df = i_{X^{\uparrow}_{\mathrm{hol}}[f]}\,\Omega\,,$$

$$(4)\qquad \begin{cases} 0 = \partial_i f^0\,,\\ 0 = \partial_0 f^0\, G^0_{hk} - f^0\,\partial_0 G^0_{hk} + f^i\,\partial_i G^0_{hk} + \partial_h f^i\, G^0_{ik} + \partial_k f^i\, G^0_{ih}\,,\\ 0 = \partial_h \breve{f} - f^0\,(\partial_0 A_h - \partial_h A_0) + f^i\,(\partial_i A_h - \partial_h A_i) + \partial_0 f^i\, G^0_{ih}\,,\\ 0 = \partial_0 \breve{f} - f^i\,(\partial_0 A_i - \partial_i A_0)\,. \end{cases}$$

Indeed, if the above equivalent conditions are fulfilled, then $X^{\uparrow}_{\mathrm{ham}}[f] = X^{\uparrow}_{\mathrm{hol}}[f]$, *i.e.,* cns spe $(J_1\boldsymbol{E}, \mathbb{R}) \subset$ hol spe $(J_1\boldsymbol{E}, \mathbb{R})$.

Proof The proof can be achieved from the above Lemma 1 and from the coordinate expression of the condition for a special phase function to be conserved. QED

The time preserving conserved special phase functions constitute a further Lie subalgebra tim cns spe $(J_1\boldsymbol{E}, \mathbb{R}) \subset$ cns spe $(J_1\boldsymbol{E}, \mathbb{R}) \subset$ spe $(J_1\boldsymbol{E}, \mathbb{R})$.

For each $f \in$ tim cns spe $(J_1\boldsymbol{E}, \mathbb{R})$, we have $L_{X[f]}G = 0$, hence $\mathrm{div}_{\eta} f = 0$.

5 Quantum Symmetries

A vector field $Y \in \sec(\boldsymbol{Q}, T\boldsymbol{Q})$ is said to be *real linear* if it is projectable on $\boldsymbol{E}$ and a real linear morphism over its spacetime projection $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$, i.e. if it is of the type $Y = X^{\lambda}\,\partial_{\lambda} + (Y^1_1\, w^1 + Y^1_2\, w^2)\,\partial w_1 + (Y^2_1\, w^1 + Y^2_2\, w^2)\,\partial w_2$, with $X^{\lambda},\ Y^1_1,\ Y^1_2,\ Y^2_1,\ Y^2_2 \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$.

A vector field $Y \in \sec(\boldsymbol{Q}, T\boldsymbol{Q})$ is said to be *complex linear* if it is real linear and a complex linear morphism over its spacetime projection $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$, i.e. if it is of the type $Y = X^{\lambda}\,\partial_{\lambda} + Y^1_1\,(w^1\,\partial w_1 + w^2\,\partial w_2) + Y^1_2\,(w^2\,\partial w_1 - w^1\,\partial w_2)$, with $X^{\lambda},\ Y^1_1,\ Y^1_2 \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$.

The sheaves $\text{lin}_{\,\mathbb{R}}\,\text{pro}_{\boldsymbol{E}}(\boldsymbol{Q}, T\boldsymbol{Q})$ and $\text{lin}_{\,\mathbb{C}}\,\text{pro}_{\boldsymbol{E}}(\boldsymbol{Q}, T\boldsymbol{Q})$ of $\mathbb{R}$-linear and $\mathbb{C}$-linear quantum vector fields turn out to be closed with respect to the Lie bracket of vector fields.

Lemma 2 *If* $f \in \text{spe}\,(J_1\boldsymbol{E}, \mathbb{R})$*, then:*

- *for each observer* o, *the vector field* $Y[f, o] := X[f] \lrcorner \text{Ч}[o] + \mathfrak{i}\,\breve{f}[o]\,\mathbb{I} \in \text{sec}\,(\boldsymbol{Q}, T\boldsymbol{Q})$ *turns out to be gauge independent;*
- *for each basis* $\mathfrak{b}$, *the vector field* $Y[f, \mathfrak{b}] := X[f] \lrcorner \chi[\mathfrak{b}] + \mathfrak{i}\,\hat{f}[\mathfrak{b}]\,\mathbb{I} \in \text{sec}(\boldsymbol{Q}, T\boldsymbol{Q})$ *turns out to be observer independent.*

Moreover, we have $X[f] \lrcorner \text{Ч}[o] + \mathfrak{i}\,\breve{f}[o]\,\mathbb{I} = X[f] \lrcorner \chi[\mathfrak{b}] + \mathfrak{i}\,\hat{f}[\mathfrak{b}]\,\mathbb{I}$.

Proof The proof follows from the transition rules of the quantum potential [25] and of the components of the special phase function, which fit very well. QED

Definition 3 We define the *η-hermitian quantum vector fields* to be the infinitesimal symmetries of the η-quantum metric $\mathfrak{h}_\eta$, i.e. the vector fields

$$Y_\eta \in \text{lin}_{\,\mathbb{R}}\,\text{pro}_{\boldsymbol{E},\boldsymbol{T}}(\boldsymbol{Q}, T\boldsymbol{Q}),$$

such that $L_{Y_\eta}\mathfrak{h}_\eta = 0$. We denote the Lie algebra subsheaf of η-hermitian quantum vector fields by $\text{her}_{\,\eta}(\boldsymbol{Q}, T\boldsymbol{Q}) \subset \text{sec}(\boldsymbol{Q}, T\boldsymbol{Q})$. □

Theorem 3 ([23]) *The η-hermitian quantum vector fields are of the type*

$$\begin{aligned} Y_\eta = Y_\eta[f] &= X[f] \lrcorner \chi[\mathfrak{b}] + \big(\mathfrak{i}\,\hat{f}[\mathfrak{b}] - \tfrac{1}{2}\,\text{div}_\eta X[f]\big)\,\mathbb{I} \\ &= X[f] \lrcorner \text{Ч}[o] + \big(\mathfrak{i}\,\breve{f}[o] - \tfrac{1}{2}\,\text{div}_\eta X[f]\big)\,\mathbb{I} \\ &= f^0\,\partial_0 - f^i\,\partial_i + \big(\mathfrak{i}\,(\breve{f} + A_0\,f^0 - A_i\,f^i) - \tfrac{1}{2}\,\text{div}_\eta f\big)\,\mathbb{I} \\ &= f^0\,\partial_0 - f^i\,\partial_i + \big(\mathfrak{i}\,\hat{f} - \tfrac{1}{2}\,\text{div}_\eta f\big)\,\mathbb{I}, \end{aligned}$$

with $f \in \text{pro spe}\,(J_1\boldsymbol{E}, \mathbb{R})$. *Indeed, the map* $Y_\eta : \text{pro spe}\,(J_1\boldsymbol{E}, \mathbb{R}) \to \text{her}_{\,\eta}(\boldsymbol{Q}, T\boldsymbol{Q})$ *turns out to be an $\mathbb{R}$-Lie algebra isomorphisms with respect to the special phase Lie bracket and the Lie bracket of vector fields.*

Proof The proof can be achieved by comparing the splitting of Y_η into its horizontal and vertical components with respect to the observed quantum connection $\text{Ч}[o]$ (or with respect to the flat quantum connection $\chi[\mathfrak{b}]$) and the splittings of a special phase function f into its spacetime lift $X[f]$ and its observed spacetime component $f[o]$ (or its gauge components $f[\mathfrak{b}]$) (Proposition 2). QED

Example 2 We have the following distinguished η-hermitian quantum vector fields

$$Y_\eta[x^\lambda] = \mathfrak{i}\,x^\lambda\,\mathbb{I}, \qquad Y_\eta[A^\uparrow{}_\lambda] = -\partial_\lambda + \tfrac{1}{2}\,\frac{\partial_\lambda\sqrt{|g|}}{\sqrt{|g|}}\,\mathbb{I}.$$

□

Definition 4 We define the *η-hermitian upper quantum vector fields* to be the infinitesimal symmetries of the η-hermitian upper quantum metric $\mathfrak{h}^{\uparrow}{}_{\eta}$, i.e. the vector fields $Y^{\uparrow}{}_{\eta} \in \text{lin}_{\mathbb{R}}\, \text{pro}_{J_1\boldsymbol{E},\boldsymbol{E},\boldsymbol{T}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$, such that $L_{Y^{\uparrow}{}_{\eta}} \mathfrak{h}^{\uparrow}{}_{\eta} = 0$.

We denote the Lie algebra subsheaf of η-hermitian upper quantum vector fields by $\text{her}^{\uparrow}{}_{\eta}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) \subset \sec(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$. □

Proposition 5 *The η-hermitian upper quantum vector fields are of the type*

$$Y^{\uparrow}{}_{\eta} = Y^{\uparrow}{}_{\eta}[X^{\uparrow}, f] := \text{Ч}^{\uparrow}(X^{\uparrow}) + (\mathfrak{i}\, f - \tfrac{1}{2}\, \text{div}_{\eta} X)\, \mathbb{I}^{\uparrow},$$

with $(X^{\uparrow},\ f) \in \text{pro}_{\boldsymbol{E},\boldsymbol{T}}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \times \text{map}(J_1\boldsymbol{E}, \mathbb{R})$, *where* $X \in \text{pro}_{\boldsymbol{T}}(\boldsymbol{E}, T\boldsymbol{E})$ *is the spacetime projection of* $X^{\uparrow}$, *i.e., in coordinates, of the type*

$$\begin{aligned} Y^{\uparrow}{}_{\eta} &= X^{\lambda}\, \partial_{\lambda} + X^{i}_{0}\, \partial^{0}_{i} + (f + A^{\uparrow}{}_{\lambda}\, X^{\lambda})\, (w^{1}\, \partial w_{2} - w^{2}\, \partial w_{1}) \\ &\quad - \tfrac{1}{2}\, \text{div}_{\eta} f\, (w^{1}\, \partial w_{1} + w^{2}\, \partial w_{2}), \end{aligned}$$

where $X^{0} \in \text{map}(\boldsymbol{T}, \mathbb{R})$, $\quad X^{i} \in \text{map}(\boldsymbol{E}, \mathbb{R})$, $\quad X^{i}_{0},\ f \in \text{map}(J_1\boldsymbol{E}, \mathbb{R})$.

Proof The proof can be achieved by splitting $Y^{\uparrow}{}_{\eta}$ into its horizontal and vertical components with respect to the upper quantum connection $\text{Ч}^{\uparrow}$. QED

Proposition 6 *The subsheaf of η-hermitian upper quantum vector fields*

$$\text{her}^{\uparrow}{}_{\eta}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) \subset \text{lin}_{\mathbb{R}}\, \text{pro}_{J_1\boldsymbol{E},\boldsymbol{E},\boldsymbol{T}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$$

turns out to be closed with respect to the Lie bracket of vector fields. Indeed, the map

$$Y^{\uparrow}{}_{\eta} : \text{pro}_{\boldsymbol{E},\boldsymbol{T}}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \times \text{map}(J_1\boldsymbol{E}, \mathbb{R}) \to \text{her}^{\uparrow}{}_{\eta}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) : (X^{\uparrow}, f) \mapsto Y^{\uparrow}{}_{\eta}[X^{\uparrow}, f]$$

turns out to be an $\mathbb{R}$-Lie algebra isomorphism with respect to the Lie bracket of phase pairs

$$\Big[(X^{\uparrow}, f),\, (\acute{X}^{\uparrow}, \acute{f})\Big]_{2\Omega} = \Big([X^{\uparrow}, \acute{X}^{\uparrow}],\ X^{\uparrow}.\acute{f} - \acute{X}^{\uparrow}.f + 2\,\Omega(X^{\uparrow}, \acute{X}^{\uparrow})\Big)$$

and the Lie bracket of vector fields. □

Theorem 4 *An η-hermitian upper quantum vector field $Y^{\uparrow}{}_{\eta}[X^{\uparrow}, f]$ is projectable on $\boldsymbol{Q}$ if and only if* $f \in \text{pro spe}\,(J_1\boldsymbol{E}, \mathbb{R})$ *and $X^{\uparrow}$ is any phase prolongation of the tangent lift* $X[f] \in \text{pro sec}(\boldsymbol{E}, T\boldsymbol{E})$.

Proof The proof can be achieved from the coordinate expression of $L_{Y^{\uparrow}{}_{\eta}} \mathfrak{h}^{\uparrow}{}_{\eta}$ and the splittings of the special phase functions (Proposition 2). QED

Corollary 1 *For each* $f \in \text{pro spe}\,(J_1\boldsymbol{E}, \mathbb{R})$*, we have two distinguished $\mathbb{R}$-Lie algebra isomorphisms (Proposition 3)*

$$Y^{\uparrow}{}_{\eta\,\mathrm{hol}} : \mathrm{pro\ spe}\,(J_1\boldsymbol{E}, \mathbb{R}) \to \mathrm{her}\,{}^{\uparrow}{}_{\eta}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) : f \mapsto Y^{\uparrow}{}_{\eta}[X^{\uparrow}{}_{\mathrm{hol}}\,, f]\,,$$
$$Y^{\uparrow}{}_{\eta\,\mathrm{ham}} : \mathrm{pro\ spe}\,(J_1\boldsymbol{E}, \mathbb{R}) \to \mathrm{her}\,{}^{\uparrow}{}_{\eta}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) : f \mapsto Y^{\uparrow}{}_{\eta}[X^{\uparrow}{}_{\mathrm{ham}}\,, f]\,. \quad \square$$

Example 3 We have the following distinguished infinitesimal symmetries of the η-hermitian upper quantum metric

$$Y^{\uparrow}{}_{\eta\,\mathrm{hol}}\,[x^{\lambda}] = \mathrm{i}\,x^{\lambda}\,\mathbb{I}^{\uparrow}\,, \qquad Y^{\uparrow}{}_{\eta\,\mathrm{hol}}\,[A^{\uparrow}{}_{\lambda}] = -\partial_{\lambda} + \tfrac{1}{2}\,\tfrac{\partial_{\lambda}\sqrt{|g|}}{\sqrt{|g|}}\,\mathbb{I}^{\uparrow}\,,$$
$$Y^{\uparrow}{}_{\eta\,\mathrm{ham}}\,[x^{\lambda}] = \delta^{\lambda}_{j}\,G^{ij}_{0}\,\partial^{0}_{i} + \mathrm{i}\,x^{\lambda}\,\mathbb{I}^{\uparrow}\,,$$
$$Y^{\uparrow}{}_{\eta\,\mathrm{ham}}\,[A^{\uparrow}{}_{\lambda}] = -\partial_{\lambda} + G^{ih}_{0}\,\partial_{\lambda}\mathscr{P}_{h}\,\partial^{0}_{i} + \tfrac{1}{2}\,\tfrac{\partial_{\lambda}\sqrt{|g|}}{\sqrt{|g|}}\,\mathbb{I}^{\uparrow}\,.$$

□

Definition 5 We define the *infinitesimal symmetries* of the upper quantum connection $Ч^{\uparrow}$ to be the upper quantum vector fields $Y^{\uparrow} \in \mathrm{lin}_{\mathbb{R}}\,\mathrm{pro}_{J_1\boldsymbol{E}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$, such that $L_{Y^{\uparrow}}Ч^{\uparrow} = 0$. □

Proposition 7 *The infinitesimal symmetries of* $Ч^{\uparrow}$ *are of the type*

$$Y^{\uparrow} = Ч^{\uparrow}(X^{\uparrow}) + \check{Y}^{\uparrow}\,,$$

where $X^{\uparrow} \in \sec(J_1\boldsymbol{E},\, T J_1\boldsymbol{E})$ *and* $\check{Y}^{\uparrow} \in \mathrm{lin}_{\mathbb{R}}\,\mathrm{pro}_{J_1\boldsymbol{E}}(\boldsymbol{Q}^{\uparrow},\, V_{J_1\boldsymbol{E}}\boldsymbol{Q}^{\uparrow})$ *fulfill the following two equivalent conditions:*
(1) $L_{\check{Y}^{\uparrow}}Ч^{\uparrow} = -\mathrm{i}\,(i_{X^{\uparrow}}\Omega) \otimes \mathbb{I}^{\uparrow}$, (2) $\nabla^{\uparrow}\check{Y}^{\uparrow} = \mathrm{i}\,(i_{X^{\uparrow}}\Omega) \otimes \mathbb{I}^{\uparrow}$.

Indeed, the sheaf $\mathrm{cnc}^{\uparrow}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$ *of infinitesimal symmetries of* $Ч^{\uparrow}$ *turns out to be closed with respect to the Lie bracket of vector fields.*

Proof The proof can be achieved by means of our postulate $R[Ч^{\uparrow}] = -2\,\mathrm{i}\,\Omega \otimes \mathbb{I}^{\uparrow}$. QED

Proposition 8 *The infinitesimal symmetries* $Y^{\uparrow}{}_{\eta} \in \mathrm{lin}_{\mathbb{R}}\,\mathrm{pro}_{J_1\boldsymbol{E},\boldsymbol{E},\boldsymbol{T}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$ *of* $\mathfrak{h}^{\uparrow}{}_{\eta}$ *and* $Ч^{\uparrow}$ *are of the type* $Y^{\uparrow}{}_{\eta} = Y^{\uparrow}{}_{\eta}[f] := Ч^{\uparrow}\big(X^{\uparrow}[f]\big) + (\mathrm{i}\,f - \tfrac{1}{2}\,\mathrm{div}_{\eta}\,f)\,\mathbb{I}^{\uparrow}$, *with* $f \in \mathrm{duni}_{\eta}\,\mathrm{cns\ spe}\,(J_1\boldsymbol{E},\, \mathbb{R})$ *and* $X^{\uparrow}[f] = X^{\uparrow}{}_{\mathrm{hol}}\,[f] = X^{\uparrow}{}_{\mathrm{ham}}\,[f]$, *i.e. of the type*

$$Y^{\uparrow}{}_{\eta} = f^{0}\,\partial_{0} - f^{i}\,\partial_{i} + X^{j}_{0}\,\partial^{0}_{j}$$
$$\qquad + (\check{f} + A_{0}\,f^{0} - A_{i}\,f^{i})\,(w^{1}\,\partial w_{2} - w^{2}\,\partial w_{1}) - \tfrac{1}{2}\,\mathrm{div}_{\eta}\,f\,(w^{1}\,\partial w_{1} + w^{2}\,\partial w_{2})\,,$$

where the spacetime functions $f^{0} \in \mathrm{map}(\boldsymbol{T}, \mathbb{R})$, f^{i}, $\check{f} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$ *fulfill the conditions*

$$
\begin{aligned}
0 &= \partial_i f^0 \,, \\
0 &= \partial_0 f^0\, G^0_{hk} - f^0\, \partial_0 G^0_{hk} + f^i\, \partial_i G^0_{hk} + \partial_h f^i\, G^0_{ik} + \partial_k f^i\, G^0_{ih} \,, \\
0 &= \partial_h \breve{f} - f^0\, (\partial_0 A_h - \partial_h A_0) + f^i\, (\partial_i A_h - \partial_h A_i) + \partial_0 f^i\, G^0_{ih} \,, \\
0 &= \partial_0 \breve{f} - f^i\, (\partial_0 A_i - \partial_i A_0) \,, \\
0 &= d\big(f^0\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i (f^i\, \sqrt{|g|})}{\sqrt{|g|}}\big)
\end{aligned}
$$

and

$$
\begin{aligned}
X^{\uparrow}[f] &= f^0\, \partial_0 - f^i\, \partial_i - (\partial_0 f^i + \partial_j f^i\, x^j_0 + \partial_0 f^0\, x^i_0 + \partial_j f^0\, x^j_0\, x^i_0)\, \partial^0_i \,, \\
&= f^0\, \partial_0 - f^i\, \partial_i + G^{ij}_0 \Big(\partial_j \breve{f} + \partial_j f^0\, \tfrac{1}{2}\, G^0_{hk}\, x^h_0\, x^k_0 + \partial_j f^h\, G^0_{hk}\, x^k_0 \\
&\quad - f^0\, \big(\partial_0 G^0_{hj}\, x^h_0 + (\partial_0 A_j - \partial_j A_0)\big) + f^h\, \big(\partial_h G^0_{jk}\, x^k_0 - (\partial_j A_h - \partial_h A_j)\big)\Big)\, \partial^0_i \,,
\end{aligned}
$$

$$
\operatorname{div}_\eta f = f^0\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i (f^i\, \sqrt{|g|})}{\sqrt{|g|}} \,.
$$

Indeed, the upper quantum vector field $Y^{\uparrow}{}_\eta[f]$ *turns out to be projectable on the* η*-hermitian quantum vector field* $Y_\eta[f] \in \text{her}\,{}_\eta(\boldsymbol{Q}, T\,\boldsymbol{Q})$.

Moreover, the map $Y^{\uparrow}{}_\eta : \text{duni}\,{}_\eta\, \text{cns spe}\, (J_1\boldsymbol{E}, \mathbb{R}) \to \text{cns her}\,{}^{\uparrow}{}_\eta(\boldsymbol{Q}^{\uparrow}, T\,\boldsymbol{Q}^{\uparrow}) : f \mapsto Y^{\uparrow}{}_\eta[f]$ *turns out to be an* $\mathbb{R}$*-Lie algebra isomorphism with respect to the special phase Lie bracket and the Lie bracket of vector fields.*

Furthermore, the map $\text{pro}_{\,\boldsymbol{Q}} : \text{cnc her}\,{}^{\uparrow}{}_\eta(\boldsymbol{Q}^{\uparrow}, T\,\boldsymbol{Q}^{\uparrow}) \to \text{her}\,{}_\eta(\boldsymbol{Q}, T\,\boldsymbol{Q}) : Y^{\uparrow}{}_\eta[f] \mapsto Y_\eta[f]$ *turns out to be an* $\mathbb{R}$*-Lie algebra morphism with respect to the Lie bracket of vector fields.*

Proof The proof can be achieved from the coordinate expression of $L_{Y^{\uparrow}} \text{Ч}^{\uparrow}$ and Theorem 2. QED

Theorem 5 *The infinitesimal symmetries* $Y^{\uparrow}{}_\eta \in \text{lin}_{\,\mathbb{R}}\, \text{pro}_{J_1\boldsymbol{E},\boldsymbol{E},\boldsymbol{T}}(\boldsymbol{Q}^{\uparrow}, T\,\boldsymbol{Q}^{\uparrow})$ *of* dt, $\mathfrak{h}^{\uparrow}{}_\eta$ *and* $\text{Ч}^{\uparrow}$ *are of the type* $Y^{\uparrow} = Y^{\uparrow}[f] := \text{Ч}^{\uparrow}\big(X^{\uparrow}[f]\big) + \mathfrak{i}\, f\, \mathbb{I}^{\uparrow}$, *with* $f \in \text{cns tim spe}\, (J_1\boldsymbol{E}, \mathbb{R})$ *and* $X^{\uparrow}[f] = X^{\uparrow}{}_{\text{hol}}\,[f] = X^{\uparrow}{}_{\text{ham}}\,[f]$, *i.e. of the type*

$$
Y^{\uparrow}{}_\eta = f^0\, \partial_0 - f^i\, \partial_i + X^j_0\, \partial^0_j + (\breve{f} + A_0\, f^0 - A_i\, f^i)\, (w^1\, \partial w_2 - w^2\, \partial w_1) \,,
$$

where the spacetime functions $f^0 \in \mathbb{R}\,,\ f^i,\ \breve{f} \in \text{map}(\boldsymbol{E}, \mathbb{R})$ *fulfill the conditions*

$$
\begin{aligned}
0 &= \partial_0 f^0\, G^0_{hk} - f^0\, \partial_0 G^0_{hk} + f^i\, \partial_i G^0_{hk} + \partial_h f^i\, G^0_{ik} + \partial_k f^i\, G^0_{ih} \,, \\
0 &= \partial_h \breve{f} - f^0\, (\partial_0 A_h - \partial_h A_0) + f^i\, (\partial_i A_h - \partial_h A_i) + \partial_0 f^i\, G^0_{ih} \,, \\
0 &= \partial_0 \breve{f} - f^i\, (\partial_0 A_i - \partial_i A_0) \,,
\end{aligned}
$$

$$X^{\uparrow}[f] = f^0\,\partial_0 - f^i\,\partial_i - (\partial_0 f^i + \partial_j f^i\,x_0^j + \partial_0 f^0\,x_0^i + \partial_j f^0\,x_0^j\,x_0^i)\,\partial_i^0\,,$$

$$= f^0\,\partial_0 - f^i\,\partial_i + G_0^{ij}\Big(\partial_j \breve{f} + \partial_j f^0\,\tfrac{1}{2}\,G^0_{hk}\,x_0^h\,x_0^k + \partial_j f^h\,G^0_{hk}\,x_0^k$$
$$- f^0\big(\partial_0 G^0_{hj}\,x_0^h + (\partial_0 A_j - \partial_j A_0)\big) + f^h\big(\partial_h G^0_{jk}\,x_0^k - (\partial_j A_h - \partial_h A_j)\big)\Big)\,\partial_i^0\,.$$

Indeed, the upper quantum vector field $Y^{\uparrow}{}_{\eta}[f]$ *turns out to be projectable on the hermitian quantum vector field* $Y_\eta[f] \in \text{her}\,_\eta(\boldsymbol{Q}, T\boldsymbol{Q})$.

Moreover, the map $Y^{\uparrow}{}_{\eta} : \text{cns tim spe}\,(J_1\boldsymbol{E}, \mathbb{R}) \to \text{cns her}\,^{\uparrow}{}_{\eta}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) : f \mapsto Y^{\uparrow}{}_{\eta}[f]$ *turns out to be an* $\mathbb{R}$*-Lie algebra isomorphism with respect to the special phase Lie bracket and the Lie bracket of vector fields.*

Furthermore, the map $\text{pro}\,_{\boldsymbol{Q}} : \text{cnc her}\,^{\uparrow}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) \to \text{her}\,_\eta(\boldsymbol{Q}, T\boldsymbol{Q}) : Y^{\uparrow}{}_{\eta}[f] \mapsto Y_\eta[f]$ *turns out to be an* $\mathbb{R}$*-Lie algebra morphism with respect to the Lie bracket of vector fields.*

Proof The proof follows from Proposition 8. QED

Definition 6 We define the *infinitesimal symmetries of the quantum lagrangian* to be the $\mathbb{R}$-linear quantum vector fields $Y \in \text{lin}\,_{\mathbb{R}}\,\text{pro}_{\boldsymbol{E}}(\boldsymbol{Q}, T\boldsymbol{Q})$, such that $L_{Y_1}\mathrm{L} = 0$, where $Y_1 := r_1 \circ J_1 Y \in \text{lin}\,_{\mathbb{R}}\,\text{pro}_{\boldsymbol{E},\mathrm{Q}}(J_1\boldsymbol{Q}, T J_1\boldsymbol{Q})$, is the 1st holonomic prolongation of Y, with coordinate expression

$$Y_1 = X^\lambda\,\partial_\lambda + Y^{\mathrm{a}}_{\mathrm{b}}\,w^{\mathrm{b}}\,\partial w_{\mathrm{a}} + (\partial_\mu Y^{\mathrm{a}}_{\mathrm{b}}\,w^{\mathrm{b}} + Y^{\mathrm{a}}_{\mathrm{b}}\,w^{\mathrm{b}}_\mu - \partial_\mu X^\nu\,w^{\mathrm{a}}_\nu)\,\partial w^\mu_{\mathrm{a}}\,.$$

□

Proposition 9 *The infinitesimal symmetries* Y *of* L *are characterised, in coordinates, by the following conditions*

$$Y_1^1 = Y_2^2\,, \qquad Y_2^1 = -Y_1^2\,, \qquad \partial_j Y_1^1 = 0\,,$$
$$0 = X^\lambda\,\partial_\lambda(A_0 - A_j\,A_0^j) - (\partial_0 - A_0^j\,\partial_j)\,Y_1^2 + (A_0 - A_i\,A_0^i)\,(2\,Y_1^1 + \mathrm{div}_\upsilon\,X)\,,$$
$$0 = -(\partial_0 X^0 - A_0^j\,\partial_j)\,X^0 + (2\,Y_1^1 + \mathrm{div}_\upsilon\,X)\,,$$
$$0 = (\partial_0 - A_0^j\,\partial_j)\,X^i + X^\lambda\,\partial_\lambda A_0^i - G_0^{ij}\,\partial_j Y_1^2 + A_0^i\,(2\,Y_1^1 + \mathrm{div}_\upsilon\,X)\,,$$
$$0 = X^\lambda\,\partial_\lambda G_0^{ij} - G_0^{hj}\,\partial_h X^i - G_0^{ih}\,\partial_h X^j + G_0^{ij}\,(2\,Y_1^1 + \mathrm{div}_\upsilon\,X)\,.$$

□

Theorem 6 *The infinitesimal symmetries of* L *and* dt *are the* η*-hermitian quantum vector fields generated by time preserving conserved special phase functions*

$$Y_\eta = Y_\eta[f]\,, \qquad \text{with} \qquad f \in \text{tim cns spe}\,(J_1\boldsymbol{E}, \mathbb{R})\,.$$

Thus, they are of the type $Y_\eta = f^0\,\partial_0 - f^i\,\partial_i + \mathfrak{i}\,(\breve{f} + A_0\,f^0 - A_i\,f^i)\,\mathbb{I}$, *where the functions* $f^0 \in \mathbb{R}$, f^i, $\breve{f} \in \operatorname{map}(\boldsymbol{E}, \mathbb{R})$ *fulfill the conditions*

$$\begin{aligned}
0 &= f^0\,\partial_0 G^0_{hk} - f^i\,\partial_i G^0_{hk} + \partial_h f^i\,G^0_{ik} + \partial_k f^i\,G^0_{ih}\,,\\
0 &= \partial_h \breve{f} - f^0\,(\partial_0 A_h - \partial_h A_0) + f^i\,(\partial_i A_h - \partial_h A_i) + \partial_0 f^i\,G^0_{ih}\,,\\
0 &= \partial_0 \breve{f} - f^i\,(\partial_0 A_i - \partial_i A_0)\,.
\end{aligned}$$

Proof The proof follows from the coordinate expressions of $L_{Y_1}\mathrm{L}$ and $Y_\eta[f]$. QED

Corollary 2 *For each* $f \in \operatorname{tim\ spe}(J_1\boldsymbol{E}, \mathbb{R})$, *we have the equivalences:*

$$L_{Y_{\eta\,1}[f]}\mathrm{L} = 0 \quad\Leftrightarrow\quad L_{Y_{\eta\,1}[f]}\Theta[\mathrm{L}] = 0 \quad\Leftrightarrow\quad f \in \operatorname{tim\ cns\ spe}(J_1\boldsymbol{E}, \mathbb{R}).$$

Proof The 1st equivalence follows from a general result of variational calculus [42]. QED

It is remarkable that the $\mathbb{R}$-Lie algebra of infinitesimal symmetries of $(\Omega\,,\ dt)$ (see [34, 36]), of $(\mathfrak{h}^\uparrow\,,\ \text{Ч}^\uparrow\,,\ dt)$ and of $(\mathrm{L}\,,\ dt)$ (see [35]) be generated by the same Lie subsheaf of special phase functions $\operatorname{tim\ cns\ spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$.

The above classifications of quantum infinitesimal symmetries can be used as the source of further developments.

In particular, the classification of η-hermitian quantum vector fields yields, in a covariant way, the *quantum operators* associated with projectable special phase functions $\mathrm{O}[f] = \mathfrak{i}\,\big(Y_\eta[f] - f'' \lrcorner\, \mathrm{S}\big) : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \boldsymbol{Q})$, with coordinate expression

$$\mathrm{O}[f](\Psi) = \left(\left(\breve{f} - A_i\,f^i - \mathfrak{i}\left(f^i\,\partial_i + \tfrac{1}{2}\,\frac{\partial_i(f^i\,\sqrt{|g|})}{\sqrt{|g|}}\right) - \tfrac{1}{2}\,f^0\,\Delta_0\right)\psi\right)\mathfrak{b}\,.$$

For instance,

$$\begin{gathered}
\mathrm{O}[x^\lambda](\Psi) = x^\lambda\,\psi\,\mathfrak{b}\,,\quad \mathrm{O}[\mathscr{P}_j](\Psi) = -\mathfrak{i}\left(\partial_j\psi + \tfrac{1}{2}\,\tfrac{\partial_j\sqrt{|g|}}{\sqrt{|g|}}\,\psi\right)\mathfrak{b}\,,\\
\mathrm{O}[\mathscr{H}_0](\Psi) = -\big(\tfrac{1}{2}\,\Delta_0\,\psi + A_0\,\psi\big)\,\mathfrak{b}\,.
\end{gathered}$$

Moreover, we obtain, in a covariant way, for each $f \in \operatorname{pro\ spe}(J_1\boldsymbol{E}, \mathbb{R})$ and $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, the *quantum current*

$$\mathrm{j}_\eta[f](\Psi) := -\,(j_1\Psi)^*(i_{Y_{\eta\,1}[f]}\,\Theta[\mathrm{L}]) \in \sec(\boldsymbol{E}, \wedge^3 T^*\boldsymbol{E}).$$

For instance, the quantum current associated to the special phase function $f = 1$ turns out to be just the probability current.

These objects will be the subject of another paper.

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The Quantum Detection Problem: A Survey

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Abstract We will look at the development of the quantum detection problem and its equivalent forms. We conclude with a complete solution to the POVM version of the problem, recast in terms of frame theory, as in [7].

1 Introduction and Preliminaries

At the beginning of the 20th century, quantum theory arose to deal with results of physical measurements which could not be explained [20]. Retrieving data from quantum systems is carried out according to *quantum measurement theory* [43]. A measurement is performed by a *quantum instrument*. The goal is to precisely determine a quantum state which is necessary for quantum information processing devices such as quantum teleporters and quantum computers [23, 41].

A drawback of quantum theory is that the predictions are *probabilistic*. Quantum theory tries to predict the probability of observing outcomes from a sequence of measurements of the system in an unknown state. This process is called *quantum state tomography* [45]. The outcome statistics are described by a *positive operator-valued measure* (POVM) [9, 16, 35, 40, 46].

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_26

Important Notation. Throughout the paper we will let $\{e_i\}_{i=1}^n$ be the canonical orthonormal basis of $\mathbb{R}^n$ or $\mathbb{C}^n$ and $\{e_i\}_{i=1}^\infty$ will denote the canonical orthonormal basis of real or complex ℓ_2. Also, ι will be used to denote the complex unit.

For a vector x_k in $\mathbb{R}^n$ or $\mathbb{C}^n$, we denote its coordinates as

$$x_k = (x_{k1}, x_{k2}, \ldots, x_{kn}).$$

Similarly, for x_k belonging to ℓ_2, we write

$$x_k = (x_{k1}, x_{k2}, \ldots, x_{ki}, \ldots).$$

To explain exactly what the quantum detection is, we need to introduce the basics of quantum detection. Let $L^\infty(\mathbb{H})$ be the space of bounded linear operators on a finite or infinite dimensional (real or complex) Hilbert space $\mathbb{H}$. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathbb{H}$. For an operator $T \in L_0(\mathbb{H})$, the finite rank operators on $\mathbb{H}$, the *trace* of T is given by: $\text{tr}(T) = \sum_{i\in I}\langle Te_i, e_i\rangle$, which is finite and independent of the orthonormal basis. The trace induces a scalar product by $\langle T, S\rangle_{HS} = \text{tr}(TS^*)$. The closure of $L_0(\mathbb{H})$ with respect to this scalar product, denoted $L^2(\mathbb{H})$ is the space of the Hilbert-Schmidt operators on $\mathbb{H}$. For any $T \in L^\infty(\mathbb{H})$ we denote by $|T| = \sqrt{TT^*}$, the positive square root of TT^*. We say that T is a *trace class operator* if $\text{tr}(|T|) < \infty$. The set of all trace class operators is denoted by $L^1(\mathbb{H})$ and forms a Banach space under the *trace norm* $\|T\|_1 = \text{tr}(|T|)$.

Let

$$Sym(\mathbb{H}) = \{T : T \in L^\infty(\mathbb{H}),\ T = T^*\},$$

denote the real Banach space of self-adjoint operators on $\mathbb{H}$ and let

$$Sym^+(\mathbb{H}) = \{T = T^* \geq 0\},$$

denote the real cone of positive self-adjoint operators on $\mathbb{H}$. The main objects to analyze these operators are the *positive operator-valued measures*.

Let X denote a set of outcomes (e.g. this could be a finite or infinite subset of $\mathbb{Z}^d$ or $\mathbb{R}^d$). Let β denote a sigma algebra of subsets of X.

Definition 1 A **positive operator-valued measure** (**POVM**) is a function $\Pi : \beta \to Sym^+(\mathbb{H})$ satisfying:

1. $\Pi(\emptyset) = 0$ (the zero operator).
2. For every disjoint family $\{U_i\}_{i\in I} \subset \beta$, $x, y \in \mathbb{H}$ we have

$$\langle \Pi\left(\cup_{i\in I} U_i\right) x, y\rangle = \sum_{i\in I}\langle \Pi(U_i)x, y\rangle.$$

3. $\Pi(X) = I$ (the identity operator).

A **quantum system** is defined as a von Neumann algebra $\mathcal{A}$ of operators acting on $\mathbb{H}$. The set of **states** on $\mathbb{H}$ is

$$\mathcal{S}(\mathbb{H}) = \{T \in L^1(\mathbb{H}),\ T = T^* \geq 0,\ \text{tr}(T) = 1\},$$

and it represents the reservoir of **quantum states** for any quantum system. The set of **quantum states** $\mathcal{S}(\mathcal{A})$ associated to a quantum system $\mathcal{A}$ is obtained by identifying states that differ by a null state with respect to $\mathcal{A}$. Thus, the set of quantum states are in one-to-one correspondance with the linear functionals on $\mathcal{A}$ of the form:

$$\rho : \mathcal{A} \to \mathbb{C},\ \text{for some } S \in \mathcal{S}(\mathbb{H}),\ \rho(T) = \text{tr}(TS),\quad \text{for every } T \in \mathcal{A}.$$

Given a quantum state ρ, the **quantum measurement** performed by the POVM Π is the map $p : \beta \to \mathbb{R}$ defined by $p(U) = \rho(\Pi(U)) = \text{tr}(\Pi(U)T)$, where $T \in \mathcal{S}(\mathbb{H})$ is in the equivalence class associated to ρ.

Let $L(\beta, \mathbb{R})$ denote the set of bounded functions defined on β. Given a POVM Π associated to a von Neumann algebra $\mathcal{A}$, the **quantum detection problem** asks if there is a unique quantum state $\rho \in \mathcal{S}(\mathcal{A})$ compatible with the set of quantum measurements performed by the POVM Π? Specifically, the quantum detection problem asks two questions.

Quantum State Separability Problem: Is the following map injective?

$$\mathbb{M} : \mathcal{S}(\mathcal{A}) \to L(\beta, \mathbb{R}),\quad \mathbb{M}(\rho)(U) = \rho(\Pi(U))?$$

A POVM is **informationally complete** (IC-POVM) if every quantum state is uniquely determined by its measurement statistics [8, 9, 18, 19, 24, 28, 48, 52]. I.e. Does it give quantum state separability?

Quantum State Estimation Problem: Assume $\mathbb{M}$ is injective. Then, given a map $p \in L(\beta, \mathbb{R})$, determine if p is in the range of $\mathbb{M}$, and hence is of the form $p = \mathbb{M}(\rho)$ for some unique $\rho \in \mathcal{S}(\mathcal{A})$. If not, find a quantum state ρ that best approximates p in some sense (e.g. robustness to noise).

POVMs have a natural, and very valuable, subset which comes from Hilbert space *frame theory* [12, 14, 15]. For a background on frame POVMs we recommend [3, 22, 27, 42].

Definition 2 A family of vectors $\{x_k\}_{k \in I}$ is a **frame** for a real or complex, finite or infinite dimensional Hilbert space $\mathbb{H}$ if there are constants $0 < A \leq B < \infty$ satisfying:

$$A\|x\|^2 \leq \sum_{k \in I} |\langle x, x_k \rangle|^2 \leq B\|x\|^2,\ \text{for all } x \in \mathbb{H}.$$

We have

1. A, B are the **lower and upper frame bounds** of the frame.
2. If $A = B = \lambda$ this is a λ-**tight frame**. If $\lambda = 1$ this is a **Parseval frame**.
3. If we only assume we have $0 < B < \infty$, this is called a **B-Bessel sequence**. Note that $\|x_k\|^2 \leq B$, for all $k \in I$.
4. The frame is **bounded** if there is a $C > 0$ so that $\|x_k\| \geq C$ for all $k \in I$.

5. The frame is **unit norm** if $\|x_k\| = 1$, for all $k \in I$.
6. The frame is **equiangular** if it is unit norm and there is some $\alpha > 0$ so that

$$|\langle x_j, x_k\rangle| = \alpha, \text{ for all } j \neq k.$$

We define the **analysis operator** of the frame as $T : \mathbb{H} \to \ell_2(I)$ by

$$T(x) = (\langle x, x_1\rangle, \langle x, x_2\rangle, \dots) = \sum_{k\in I} \langle x, x_k\rangle e_k.$$

The **synthesis operator** T^* is given by:

$$T^*\left(\{a_k\}_{k\in I}\right) = \sum_{k\in I} a_k x_k.$$

The **frame operator** is $S = T^*T$. This is a positive, self-adjoint, invertible operator on $\mathbb{H}$ satisfying:

$$S(x) = \sum_{k\in I} \langle x, x_k\rangle x_k.$$

That is,

$$S = \sum_{k\in I} x_k x_k^*.$$

It is known that for any frame $\{x_k\}_{k\in I}$, $\{S^{-1/2}x_k\}_{k\in I}$ is a Parseval frame. It is also known that a frame is Parseval if and only if its frame operator is the identity operator. Frames are spanning sets of vectors which allow stable expansion and reconstruction of vectors. But, in contrast to orthonormal bases, frame vectors need not be linearly independent. This gives design flexibility not available to orthonormal bases. Parseval frames also have the advantage that they give immediate reconstruction. These objects are as *close as possible* to being orthonormal bases for quantum states. Parseval frames have wide application to quantum measurements and encryption schemes [14, 21, 22, 46, 47].

If $\{x_k\}_{k\in I}$ is a Parseval frame for a Hilbert space $\mathbb{H}$, it naturally induces a POVM Π on $X = I$ with $\beta = 2^I$ (the power set of I):

$$\Pi(U) = \sum_{k\in U} x_k x_k^*, \text{ where } x_k^* : \mathbb{H} \to \mathbb{C},\ x_k^*(x) = \langle x, x_k\rangle,$$

with strong convergence for any $U \subset I$. So from now on we will refer to a positive operator-valued measure as a Parseval frame given by rank-1 operators $\{\pi_k\}_{k=1}^m$.

Parseval frames have broad application to engineering problems such as A/D conversion [4, 5, 26], multiple description coding [25, 50], wireless communication [30], matched filtering in the quantum setting [29], and detection of radar and laser signals [29, 37–39], and applications to astronomy [29, 51]. Parseval frames have also been shown to be optimal for linear quantum state tomography and measurement based quantum cloning [49].

Given a state $T \in \mathcal{S}(\mathbb{H})$ (i.e. a unit-trace, trace class, positive, self-adjoint operator on $\mathbb{H}$), the frame induced quantum measurement is given by the function

$$p : \beta \to \mathbb{R}, \quad p(U) = \sum_{k\in U} \mathrm{tr}(T x_k x_k^*) = \sum_{k\in U} \langle T x_k, x_k \rangle.$$

For the von Neumann algebra $\mathcal{A} = L^\infty(\mathbb{H})$, the quantum states coincide with the convex set of states $\mathcal{S}(\mathbb{H})$. In this case, the state separability problem and the state estimation problem ask:

Quantum State Separability Problem: Is there a Parseval frame $\mathcal{X} = \{x_k\}_{k\in I}$ so that the map $\mathbb{M} : \mathcal{S}(\mathbb{H}) \to L(\beta, \mathbb{R})$ defined by $\mathbb{M}(T)(U) = \sum_{k\in U} \langle T x_k, x_k \rangle$ for $U \subset I$ is injective? That is, given $T, S \in S(\mathbb{H})$, if

$$\langle T x_k, x_k \rangle = \langle S x_k, x_k \rangle, \text{ for all } k,$$

then $T = S$.

Quantum State Estimation Problem: Given a separable Parseval frame $\{x_k\}_{k\in I}$ and a function $p : \beta \to \mathbb{R}$, is there any $T \in \mathcal{S}(\mathbb{H})$ so that $\mathbb{M}(T) = p$? If not, find a quantum state T that best approximates p. In the finite case, this means, given separable Parseval frame $\{x_k\}_{k=1}^m$ on $\mathbb{H}^n$ and a measurement vector $a = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$, can we find a positive self-adjoint trace one operator T so that

$$\langle T x_k, x_k \rangle = a_k, \text{ for all } k?$$

And if not, find the T that best approximates a solution.

2 The Frame Optimization Problem

The goal for the frame optimization problem is to construct a tight frame which minimizes the error in quantum detection [3]. In quantum physics this has the interpretation of the probability of a detection error. Solutions to this problem can also be viewed as a generalization of classical matched filtering solutions. Thus, this is a generalization of fundamental detection techniques in radar [3].

Frame Optimization Problem: Let $\mathbb{H}$ be an n-dimensional Hilbert space. Given a sequence $\{x_k\}_{k=1}^m \subset \mathbb{H}$ of unit normed vectors and a sequence $\{w_k\}_{k=1}^m \subset \mathbb{R}$ of positive weights that sums to one, the frame optimization problem is to construct a Parseval frame $\{e_k\}_{k=1}^m$ that minimizes the quantity:

$$P_e(\{e_k\}_{k=1}^m) = 1 - \sum_{k=1}^m w_k |\langle x_k, e_k\rangle|^2.$$

Kennedy et al. [54] gave necessary and sufficient conditions on a POVM so that it minimizes P_e. Helstrom [29] solved the problem completely for the case in which the quantum system is limited to be in one of two possible states. The solution appears in [3] and relies on the solution for Newton's equation of motion that minimizes energy [3].

3 The Solution to the Finite Dimensional Quantum Detection Problem

In this section we look at the solution to the finite dimensional cases of the quantum detection problems and simple methods for constructing unlimited numbers of solutions.

3.1 The Solution to the Quantum State Separability Problem

As we have seen, the solution to the state separability problem is the informationally complete positive operator-valued measures (IC-POVMs). This problem was first solved by Scott [49] (See also [6]).

Theorem 1 *A positive operator-valued measure $\{\pi_k\}_{k=1}^m$ is informationally complete if and only if the real span of $\{\pi_k\}_{k=1}^m$ spans the space of Hermitian operators.*

It follows that

Corollary 1 *If a POVM $\{\pi_k\}_{k=1}^m$ on a n-dimensional complex Hilbert space is informationally complete, then $m \geq n^2$ (and $m \geq \frac{n(n+1)}{2}$ in the real case).*

The spanning property above is equivalent to $\{\pi_k\}_{k=1}^m$ being a frame for the Hermitian operators which is an inner product space with the Hilbert-Schmidt inner product. In this case, the *pseudo-inverse* of the Gram matrix gives reconstruction. The Gram matrix for a family of vectors $\{x_k\}_{k=1}^m$ is given by:

$$G = \left(\langle x_j, x_k\rangle\right)_{j,k=1}^m.$$

If $\mathbb{H}$, $\mathbb{K}$ are Hilbert spaces and $U : \mathbb{K} \to \mathbb{H}$ is a bounded operator with closed range R, then there exists a bounded operator (the **pseudo-inverse** of U) $U^\dagger : \mathbb{H} \to \mathbb{K}$ satisfying

$$UU^\dagger x = x, \text{ for all } x \in R.$$

In [7], a different approach is taken for solving the separability problem. This approach has the advantage that it generalizes to infinite dimensions and it gives a simple method for constructing unlimited numbers of solutions. First we need a definition.

Definition 3 To a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we associate a vector $\tilde{x}$ in $\mathbb{R}^{\frac{n(n+1)}{2}}$ by:

$$\tilde{x} = (x_1x_1, x_1x_2, \ldots, x_1x_n; x_2x_2, x_2x_3, \ldots, x_2x_n; \ldots; x_{n-1}x_{n-1}, x_{n-1}x_n; x_nx_n),$$

and for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$, we define

$$\tilde{x} = (|x_1|^2, \mathrm{Re}(\bar{x}_1x_2), \mathrm{Im}(\bar{x}_1x_2), \ldots, \mathrm{Re}(\bar{x}_1x_n), \mathrm{Im}(\bar{x}_1x_n);$$

$$|x_2|^2, \mathrm{Re}(\bar{x}_2x_3), \mathrm{Im}(\bar{x}_2x_3), \ldots, \mathrm{Re}(\bar{x}_2x_n), \mathrm{Im}(\bar{x}_2x_n); \ldots;$$

$$|x|^2_{n-1}, \mathrm{Re}(\bar{x}_{n-1}x_n), \mathrm{Im}(\bar{x}_{n-1}x_n); |x_n|^2) \in \mathbb{R}^{n^2}.$$

The solution to the separability problem then becomes [7]:

Theorem 2 *Let $\mathcal{X} = \{x_k\}_{k=1}^m$ be a frame for $\mathbb{R}^n$ or $\mathbb{C}^n$. The following are equivalent:*

1. *$\mathcal{X}$ gives separability.*
2. *We have that $\{\tilde{x}_k\}_{k=1}^m$ spans $\mathbb{R}^{\frac{(n(n+1)}{2}}$ in the real case and $\mathbb{R}^{n^2}$ in the complex case.*

The class of (IC-POVMs) are the same as the class of *weighted 2-designs* in complex projective space [17]. Spherical t-designs were first extended to projective space by Neumaier [44], and were extensively studied in [1, 2, 31–34].

Definition 4 A sequence of rank-one projections $\{\pi_k\}_{k=1}^m$ with weights $\{w_k\}_{k=1}^m$ on a n-dimensional Hilbert space is a **weighted projective** 2**-design** if

$$\sum_{k=1}^m w_j\pi_j\pi_j^* = \frac{2}{n(n+1)}\Pi_{sym},$$

where Π_{sym} is the projection onto the symmetric subspace of $\mathbb{H} \otimes \mathbb{H}$. That is,

$$\Pi_{sym} = \frac{1}{2}\sum_{j,k=1}^n (E_{j,j} \otimes E_{k,k} + E_{j,k} \otimes E_{k,j}),$$

where $E_{j,k} = e_je_k^*$.

Scott [49] showed the connection with POVMs.

Theorem 3 *Given a POVM $\{A_k\}_{k=1}^m$ on a n-dimensional Hilbert space $\mathbb{H}$ and its operator valued canonical dual $\{B_k\}_{k=1}^m$, we have*

$$\sum_{k=1}^{m} tr[\frac{A_k}{n}]tr[B_k^2] \geq \frac{1}{n} + \frac{(n^2-1)(n+1)}{n}.$$

Moreover, equality holds if and only if $\{A_k\}_{k=1}^m$ *is a rank-one POVM and* $w_k = \frac{tr[A_k]}{n}$, $\pi_k = \frac{A_k}{tr[A_k]}$ *forms a weighted projective 2-design.*

3.2 Examples of (IC-POVMs)

An optimal class of (IC-POVMs) are the maximal classes of *mutually unbiased bases* [49].

Definition 5 Two orthonormal bases for an n-dimensional Hilbert space $\mathbb{H}$, $\{x_k\}_{k=1}^n$ and $\{y_k\}_{k=1}^n$ is a **pair of mutually unbiased bases** (MUBs) if

$$|\langle x_k, y_j \rangle|^2 = \frac{1}{n}, \text{ for all } j, k = 1, 2, \dots, n.$$

If $\mathcal{B}$ is a family of pairwise mutually unbiased bases for $\mathbb{C}^n$ then $|\mathcal{B}| \leq n+1$ [17]. It is rare that this bound can be met. Cameron and Seidel [10] showed that maximal sets of MUBs exist for $\mathbb{C}^p$ if p is a prime number. Later this was extended to prime powers [11, 53]. In general, little is known about the existence of maximal numbers of MUBs. For example, the answer is not known even for $\mathbb{C}^6$ although numerical evidence suggests it can have only 3 MUBs [36].

Another class that solves the quantum state separability problem are the (SIC-POVMs) - *Symmetric Informationally Complete Positive Operator-Valued Measures*). For one, these are minimal in the sense that they contain n^2 elements in an n-dimensional Hilbert space which we know is the required minimal number. These objects correspond to the maximal equiangular tight frames. Very few examples of this type exist. **Zauner's Conjecture** [55, 56] asserts that there should exist a SIC-POVM in $\mathbb{C}^n$ for all $n = 1, 2, \dots$. Actually, Zauner conjectured that these exist in the *Weyl-Heisenberg groups*.

In [7], many concrete constructions of IC-POVMs are given. To help with this, they show several simplifying assumptions for the constructions. First, since applying an invertible operator does not change a solution to the quantum separability problem, one does not need a Parseval frame here since given a frame, we can turn it into a Parseval frame by applying $S^{-1/2}$, where S is the frame operator of the frame. They then show that we do not really have to assume our operators are positive, or trace one since this can be made up for later. Their construction becomes in the real case (a similar construction works in the complex case):

Theorem 4 *Let* $\{x_k\}_{k=1}^n$ *be a linearly independent set in* $\mathbb{R}^n$ *such that the first coordinates of these vectors are non-zero. Now choose* $(n-1)$ *linearly independent vectors* $\{x_k\}_{k=n+1}^{2n-1}$ *in* $\mathbb{R}^n$ *such that each vector is zero in the first coordinate and is non-zero* *in the second coordinate. Continuing this procedure we get a frame* $\{x_k\}_{k=1}^{\frac{n(n+1)}{2}}$ *which gives separability.*

In [7] there is also a direct construction of large classes of Parseval frames giving separability.

Theorem 5 *Let* $\{\lambda_{ij}\}_{i=1,}^{n}{}_{j=i}^{n}$ *be non-negative numbers satisfying:*

1. $\lambda_{ij} = 0$ *if and only if* $j < i$.
2. *For each* $j = 1, 2, \ldots, n$ *we have* $\sum_{i=1}^{n} \lambda_{ij} = 1$.

Let $\mathcal{E} = \{e_j\}_{j=1}^{n}$ *be the canonical basis of* $\mathbb{R}^n$*. Let* $\{x_k\}_{k=1}^{\frac{n(n+1)}{2}}$ *be vectors in* $\mathbb{R}^n$ *which satisfy:*

1. $\{x_k\}_{k=1}^{n}$ *is a linearly independent set with* $x_{k1} \neq 0$ *for all* $k = 1, \ldots, n$ *and it has frame operator* S_1 *with eigenvectors* $\mathcal{E}$ *and respective eigenvalues* $\{\lambda_{1j}\}_{j=1}^{n}$ *(See [13].)*
2. $\{x_k\}_{k=n+1}^{2n-1}$ *is a linearly independent set with* $x_{k1} = 0$*, for all* k*,* $x_{k2} \neq 0$ *for all* k*, and it has frame operator* S_2 *with eigenvectors* $\mathcal{E}$ *and respective eigenvalues* $\{\lambda_{2j}\}_{j=1}^{n}$.
3. *continue.*

Then the vectors $\{x_k\}_{k=1}^{\frac{n(n+1)}{2}}$ *form a Parseval frame for* $\mathbb{R}^n$ *which gives separability.*

It is also shown in [7] that there are an unlimited number of solutions to the separability problem.

Theorem 6 *The family of all* m*-element frames on* $\mathbb{H}^n$ *that give injectivity in the frame quantum detection problem is open and dense in the space of all* m*-element frames on* $\mathbb{H}^n$.

We also have:

Corollary 2 *The set of all* m*-element Parseval frames which give injectivity is dense in the set of all* m*-element Parseval frames.*

3.3 The Solution to the Quantum State Estimation Problem

For the real state estimation problem we have [7].

Theorem 7 *Let* $\mathcal{X} = \{x_k\}_{k=1}^{\frac{n(n+1)}{2}} \subset \mathbb{R}^n$ *be an separable Parseval frame. Then the state estimation problem has a unique solution for all choices of vectors* $a = (a_1, a_2, \ldots, a_{\frac{n(n+1)}{2}})$.

There is a similar theorem for the complex case. There is a technical problem with the state estimation problem. If the measurement vector is larger than the dimension of the Hilbert space, this problem is rarely solvable. For example, if we are given $\{x_k\}_{k=1}^m$ with $m > n(n+1)/2$ in the real case or $m > n^2$ in the complex case with $x_1 = x_2$ and a measurement vector $a = (a_1, a_2, \ldots, a_m)$ with $a_1 \neq a_2$, then we clearly cannot solve the state estimation problem. However, in these cases we can find the *best approximation* to a solution.

We consider the real case. Note that there always exists a subset $I \subset \{1, 2, \ldots, m\}$ of size $\frac{n(n+1)}{2}$, and a self-adjoint operator T so that $\langle Tx_k, x_k\rangle = a_k$, for all $k \in I$. Therefore, if the state estimation problem is not solvable, it is natural to find such T so that the distance to the measurement vector a:

$$\sum_{k=1}^{m} |\langle Tx_k, x_k\rangle - a_k|^2$$

is minimum.

To do this, let $\mathcal{S}$ be the set of all bases of $\mathbb{R}^{\frac{n(n+1)}{2}}$ that are subsets of $\{\tilde{x}_k\}_{k=1}^m$. This set is obviously finite. Since each element $\{\tilde{x}_k\}_{k\in I}$ in $\mathcal{S}$ determines a unique self-adjoint operator T satisfying $\langle Tx_k, x_k\rangle = a_k$, for all $k \in I$, we can find the quantum state T that gives the best approximation to the measurement vector a by choosing the set which minimizes the distance above.

4 The Solution to the Infinite Dimensional Quantum Detection Problem

The infinite dimensional case of the quantum detection problem was solved in [7]. Here there is a list of technical problems which we do not have in the finite dimensional case:

1. In the finite dimensional cases we often show that our POVM vectors span by showing we have *enough* to span the operator space and they are independent. This does not work in the infinite dimensional case.
2. There are unending problems with it convergence of the necessary series.
3. Trace class operators are much harder to construct with additional properties than self-adjoint operators.

4.1 The Solution to the Quantum State Separability Problem

In infinite dimensions the quantum state separability problem becomes:

Quantum State Separability Problem: For what frames $\{x_k\}_{k=1}^\infty$ in real or complex infinite dimensional Hilbert space $\mathbb{H}$ do we have the property: Whenever T, S are

Hilbert-Schmidt positive self-adjoint operators on $\mathbb{H}$ and $\langle Tx_k, x_k\rangle = \langle Sx_k, x_k\rangle$, for all $k = 1, 2, \ldots$, then $T = S$.

Remark 1 If in the problem we switch to the operator $L = T - S$, then the problem asks if L is a Hilbert-Schmidt, self-adjoint operator, and $\langle Lx_k, x_k\rangle = 0$, for all $k = 1, 2, \ldots$, then $L = 0$.

We need a definition.

Definition 6 We define a subspace of the real space ℓ_1 as follows:

$$W := \left\{ (\lambda_1, \lambda_2, \ldots) \in \ell_1 : \sum_{j=1}^{\infty} \lambda_j = 0 \right\}.$$

The first solution to the problem is difficult to check except in certain special circumstances.

Theorem 8 *Let $\mathcal{X} = \{x_k\}_{k=1}^{\infty}$ be a frame for an infinite dimensional real or complex Hilbert space $\mathbb{H}$. The following are equivalent:*

1. *If T is a trace class self-adjoint operator of trace zero such that*

$$\langle Tx_k, x_k\rangle = 0, \text{ for all } k,$$

then $T = 0$.

2. *For every $\lambda = (\lambda_1, \lambda_2, \ldots) \in W$ and for every orthonormal basis $\{e_j\}_{j=1}^{\infty}$ for $\mathbb{H}$, if $\sum_{j=1}^{\infty} \lambda_j |\langle x_k, e_j\rangle|^2 = 0$ for all k then $\lambda = 0$.*

The next solution has the advantage that the needed Parseval frames can be constructed.

Definition 7 Denote by $\tilde{\mathbb{H}}$ the direct sum of the real Hilbert spaces ℓ_2:

$$\tilde{\mathbb{H}} = \left(\sum_{i=1}^{\infty} \oplus \ell_2 \right)_{\ell_2}.$$

A vector in this direct sum will be written in the form:

$$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \ldots),$$

and we have

$$\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^{\infty} \langle \mathbf{x}_i, \mathbf{y}_i\rangle.$$

We also need another definition:

Definition 8 For $x = (x_1, x_2, \ldots) \in \ell_2$, we define

$$\tilde{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \ldots) \in \tilde{\mathbb{H}},$$

where

$$\mathbf{x}_1 = (x_1x_1, x_1x_2, \ldots);\ \mathbf{x}_2 = (x_2x_2, x_2x_3, \ldots); \ldots; \mathbf{x}_n = (x_nx_n, x_nx_{n+1}, \ldots); \ldots$$

One first has to show that these vectors are actually in $\tilde{\mathbb{H}}$. Now we will give the solution to the infinite dimensional state separability problem in the real case.

Theorem 9 *Let $\mathcal{X} = \{x_k\}_{k=1}^{\infty}$ be a frame in the real Hilbert space ℓ_2. The following are equivalent:*

1. *$\mathcal{X}$ is separable.*
2. *$\overline{span}\{\tilde{x}_k\}_{k=1}^{\infty} = \tilde{\mathbb{H}}$.*

The complex case is similar with slight adjustments.

Definition 9 For $x = (x_1, x_2, \ldots) \in \ell_2$, we define

$$\tilde{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \ldots),$$

where

$$\mathbf{x}_1 = (|x_1|^2, \mathrm{Re}(\bar{x}_1x_2), \mathrm{Im}(\bar{x}_1x_2), \mathrm{Re}(\bar{x}_1x_3), \mathrm{Im}(\bar{x}_1x_3), \ldots);$$

$$\mathbf{x}_2 = (|x_2|^2, \mathrm{Re}(\bar{x}_2x_3), \mathrm{Im}(\bar{x}_2x_3), \mathrm{Re}(\bar{x}_2x_4), \mathrm{Im}(\bar{x}_2x_4), \ldots); \ldots;$$

$$\mathbf{x}_n = (|x_n|^2, \mathrm{Re}(\bar{x}_nx_{n+1}), \mathrm{Im}(\bar{x}_nx_{n+1}), \mathrm{Re}(\bar{x}_nx_{n+2}), \mathrm{Im}(\bar{x}_nx_{n+2}), \ldots); \ldots.$$

The solution to the complex separability problem is now:

Theorem 10 *Let $\mathcal{X} = \{x_k\}_{k=1}^{\infty}$ be a frame in the complex Hilbert space ℓ_2. The following are equivalent:*

1. *$\mathcal{X}$ gives separability.*
2. *$\overline{span}\{\tilde{x}_k\}_{k=1}^{\infty} = \tilde{\mathbb{H}}$.*

4.2 Constructing Solutions to the Quantum State Separability Problem

It is difficult, but possible to give a concrete construction of the solutions to the separability problem.

Theorem 11 *Let $\{e_i\}_{i=1}^{\infty}$ be the canonical basis for the real Hilbert space ℓ_2 and let $a_i \neq 0$ for $i = 1, 2, \ldots$ be such that $\sum_{i=1}^{\infty} a_i^2 < \infty$. Define*

$$x_k = a_k(e_1 + e_{k+1}), \text{ for } k = 1, 2, \ldots.$$

Let L be the right shift operator on ℓ_2. Then the family

$$\{e_i\}_{i=1}^{\infty} \cup \left\{ \frac{1}{2^i} L^i x_k \right\}_{i=0,k=1}^{\infty,\ \infty}$$

is a frame for ℓ_2 which gives state separability.

The complex case is an adjustment of this.

It is further shown in [7] that the solutions to the state separability problem are neither open nor dense in the family of frames. It is also shown that the solutions $\{x_k\}_{k=1}^{\infty}$ can never have the property that $\{\tilde{x}_k\}_{k=1}^{\infty}$ is a frame for $\tilde{\mathbb{H}}$.

4.3 The Solution to the State Estimation Problem

The infinite dimensional state estimation problem has an unlimited number of problems since the problems for the finite dimensional case can appear infinitely often here. The following result points out a major problem with state estimation in infinite dimensions.

Theorem 12 *There is no separable frame $\mathcal{X} = \{x_k\}_{k=1}^{\infty}$ in the real or complex space ℓ_2 so that for every $a = \{a_k\}_{k=1}^{\infty} \in \ell_2$, there is a Hilbert-Schmidt operator T so that*

$$\langle T x_k, x_k \rangle = a_k, \text{ for all } k = 1, 2, \ldots.$$

However, if we drop the requirement that the vectors form a frame for ℓ_2, then the problem is solvable.

A case where we always have solutions will now be addressed. For the solution of the state estimation problem we will need the notion of a separated sequence in ℓ_2.

Definition 10 A family of vectors $\{x_i\}_{i=1}^{\infty}$ in ℓ_2 is **separated** if for every $j \in \mathbb{N}$,

$$x_j \notin \overline{\text{span}}\{x_i\}_{i \neq j}.$$

It is δ-**separated** if the projection P_j onto $\overline{\text{span}}\{x_i\}_{i \neq j}$ satisfies

$$\|(I - P_j)x_j\| \geq \delta.$$

Theorem 13 *Let $\mathcal{X} = \{x_k\}_{k=1}^{\infty}$ be a frame for the real or complex space ℓ_2. The following are equivalent:*

1. *For every real vector $a = (a_1, a_2, \ldots) \in \ell_1$, there is a Hilbert-Schmidt self-adjoint operator T so that*

$$\langle Tx_k, x_k \rangle = a_k, \text{ for all } k = 1, 2, \ldots.$$

2. *The sequence $\{\tilde{x}_k\}_{k=1}^{\infty}$ is δ-separated.*

Acknowledgements The authors were supported by NSF DMS 1609760; NSF ATD 1321779; and ARO W911NF-16-1-0008.

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Possible Alternative Mechanism to SUSY: Conservative Extensions of the Poincaré Group

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Abstract A group theoretical mechanism is outlined, which can indecomposable extend the Poincaré group by the compact internal (gauge) symmetries at the price of allowing some nilpotent (or, more precisely: solvable) internal symmetries in addition. Due to the presence of this nilpotent part, the prohibitive argument of the well known Coleman-Mandula, McGlinn no-go theorems do not go through. In contrast to SUSY or extended SUSY, in our construction the symmetries extending the Poincaré group will be all internal, i.e. they do not act on the spacetime, merely on some internal degrees of freedom — hence the name: *conservative* extensions of the Poincaré group. Using the Levi decomposition and O'Raifeartaigh theorem, the general structure of all possible conservative extensions of the Poincaré group is outlined, and a concrete example group is presented with U(1) being the compact gauge group component. It is argued that such nilpotent internal symmetries may be inapparent symmetries of some more fundamental field variables, and therefore do not carry an ab initio contradiction with the present experimental understanding in particle physics. The construction is compared to (extended) SUSY, since SUSY is somewhat analogous to the proposed mechanism. It is pointed out, however, that the proposed mechanism is less irregular in comparison to SUSY, in certain aspects. The only exoticity needed in comparison to a traditional gauge theory setting is that the full group of internal symmetries is not purely compact, but is a semi-direct product of a nilpotent and of a compact part.

Keywords GUT · Unification · Poincaré group · Gauge group · O'Raifeartaigh theorem · Levi decomposition theorem

1 Introduction

In Lagrangian field theories it is well understood that larger amount of symmetries of the Lagrangian gives less room for variants of the theory. In particular, the

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_27

larger amount of direct-indecomposable (unified) symmetries reduce the number of possible free coupling parameters. This phenomenon motivated the search for unified symmetries in field theory, meaning that a plausible direct-indecomposable symmetry group was being searched for, which contained the known symmetry groups as subgroups. When it comes to building relativistic field theories to be applied in particle physics, the known symmetry groups are the Poincaré group and the compact internal (gauge) symmetries of the Standard Model, commuting with eachother. Therefore, a rather plausible idea was to try to find a direct-indecomposable symmetry group, which contains Poincaré symmetries and compact internal symmetries, indecomposably. In 1964 it was realized by McGlinn [1] that whenever the compact internal symmetries are semi-simple, i.e. purely non-abelian, this is group theoretically impossible. This motivated the work of O'Raifeartaigh in 1965 [2] to try to understand all possible group extensions of the Poincaré symmetries. The pertinent O'Raifeartaigh theorem made it clear that the Lie group theoretical possibilities for a direct-indecomposable extension of the Poincaré group is rather limited. Historically, at the time of the publication of O'Raifeartaigh theorem, no constructive examples for the potentially allowed direct-indecomposable Poincaré group extensions were known. For instance supersymmetry (SUSY) was not known at the time, and the conformal Poincaré group, being a direct-indecomposable extension of the Poincaré group, was not in the physics folklore. Therefore, the potentially allowed Poincaré group extensions by means of the O'Raifeartaigh theorem were talked away by a littlebit handwaving physics arguments. Not much later, in 1967 the famous Coleman-Mandula theorem [3] was published, stating that given some plausible assumptions, a unification of the Poincaré group with purely compact internal symmetries is not possible in the framework of quantum field theory. These attempts were historically reviewed in [4]. A few years later, the famous paper of Wess and Zumino was published [5], implicitly providing an example Lie group (the super-Poincaré group) which is an indecomposable extension of the Poincaré group, and thus providing an explicit example for one of the cases of O'Raifeartaigh theorem, allowing a direct-indecomposable extension of the Poincaré group. Motivated by this, Haag, Lopuszański and Sohnius [6] generalized the Coleman-Mandula theorem also allowing for super-Poincaré transformations. Since that work, the so called super-Lie algebra view of those transformations is the most popular in the literature, making it less obvious to see the underlying ordinary Lie group structure of the super-Poincaré transformations, and their relations to O'Raifeartaigh theorem. In the recent years it was re-understood that there do exist also other direct-indecomposable extensions of the Poincaré group. A rather well-understood example is the conformal Poincaré group, being isomorphic to $SO(2, 4)$, but also others have been found [7–10], some of which can lead to field theories which may not be *ab initio* pathological. They bypass the Coleman-Mandula theorem by weakening some of its assumptions, for instance allowing for symmetry breaking.

In this paper a newly found direct-indecomposable Poincaré group extension [11, 12] is discussed, which contains a Poincaré component, a compact internal group component, and a nilpotent internal group component. From the Lie group theoretical point of view, it resembles to the (extended) super-Poincaré group, since in its Levi decomposition its radical is a nilpotent Lie algebra. However, in contrast to SUSY, this group respects vector bundle structure of fields, i.e. all the non-Poincaré symmetries act spacetime pointwise on some internal degrees of freedom. This implies that symmetry breaking is not necessary in order to make this new symmetry concept to harmonize with a gauge-theory-like setting, where vector bundle structure of fundamental fields is essential to preserve. Hence, we call these constructions *conservative* extensions of the Poincaré group.

The outline of the paper is as follows. In Sect. 2 the general structure of Lie groups is recalled in the light of Levi decomposition theorem. In Sect. 3 the O'Raifeartaigh classification theorem on Poincaré group extensions is recalled. In Sect. 4 the structure of conservative extensions of the Poincaré group is outlined. In Sect. 5 the Lie algebra of the concrete conservative Poincaré group extension defined in [11, 12] is presented.

2 General Structure of Lie Groups: Levi Decomposition

In every finite dimensional real Lie algebra, one has the Killing form, being a real valued bilinear form defined by the formula $x \cdot y := \mathrm{Tr}\left(\mathrm{ad}_x\,\mathrm{ad}_y\right)$ for two elements x, y of the Lie algebra. The Levi decomposition theorem [13, 14] states that the structure of a generic real finite dimensional connected and simply connected Lie group is as follows:

$$\underbrace{E}_{\text{Lie group}} = \underbrace{R}_{\substack{\text{degenerate directions of Killing form} \\ \text{(called to be the } radical\text{)}}} \rtimes \underbrace{\left(L_1 \times \cdots \times L_n\right)}_{\substack{\text{non}-\text{degenerate directions of Killing form} \\ \text{(called to be the } Levi\ factor\text{)}}} \tag{1}$$

A subgroup spanned by the non-degenerate directions of the Killing form is called the *Levi factor* or *semisimple part*. It falls apart to direct product of subgroups which contain no proper normal subgroups, and are called the *simple components*. The normal (invariant) subgroup spanned by the degenerate directions of the Killing form is called the *radical* or *solvable part*. The radical R can also be equivalently characterized by the property that the Lie algebra r of R has terminating derived series. Namely, with the definition $r^0 := r$, $r^k := \left[r^{k-1}, r^{k-1}\right]$, there exists a finite k such that $r^k = \{0\}$. A special case is when R is said to be *nilpotent*: in this case there exists a finite k such that for all $x_1, \ldots, x_k \in r$ one has $\mathrm{ad}_{x_1} \ldots \mathrm{ad}_{x_k} = 0$. The extreme case is when R is said to be *abelian*: in this case for all $x \in r$ one has $\mathrm{ad}_x = 0$.

Whenever also non-simply connected or non-connected Lie groups are considered, their generic structure can be slightly more complex:

$$\underbrace{E}_{\text{Lie group}} = \left(\left(\underbrace{R}_{\text{radical}} \rtimes \underbrace{\left(L_1 \times \cdots \times L_n\right)}_{\text{Levi factor}}\right) / \underbrace{\mathcal{I}}_{\text{discrete}}\right) \rtimes \underbrace{\mathcal{J}}_{\text{discrete}} \tag{2}$$

where $\mathcal{I}$ is some discrete normal subgroup of $R \rtimes (L_1 \times \cdots \times L_n)$ and $\mathcal{J}$ is some discrete subgroup of the outer automorphisms of the quotient group $(R \rtimes (L_1 \times \cdots \times L_n))\,/\mathcal{I}$. It is not complicated to see that whenever a Lie group is injectively embedded into another, then its Lie algebra must be injectively embedded into the Lie algebra of the other. Thus, for studying necessary condition for injective embedding of Lie groups, one first needs to study the injective embeddings of Lie algebras, or equivalently, of connected and simply connected Lie groups. From now on, by Lie groups we shall always mean connected and simply connected ones, i.e. the universal covering groups.

Levi decomposition theorem can be illustrated with the Poincaré group:

$$\underbrace{\mathcal{P}}_{\text{Poincaré group}} = \underbrace{\mathcal{T}}_{\text{translations (radical)}} \rtimes \underbrace{\mathcal{L}}_{\text{Lorentz group (Levi factor)}} \tag{3}$$

3 A Classification of Poincaré Group Extensions

A classification scheme of Poincaré group extensions was outlined by O'Raifeartaigh [2], using the Levi decomposition theorem. It is based on the simple observation that when injectively embedding a finite dimensional real Lie algebra into another, then the Levi factor of the smaller Lie algebra cannot intersect with the radical of the larger one. This implies the following disjoint possibilities for a connected and simply connected extension $E = R \rtimes (L_1 \times \cdots \times L_n)$ of the Poincaré symmetries $\mathcal{P} = \mathcal{T} \rtimes \mathcal{L}$.

A One has $E = \mathcal{P} \times \{\text{some other Lie group}\}$, i.e. no unification occurs.
B One has not A and $\mathcal{T} \subset R$ and $\mathcal{L} \subset L_1$, meaning that the translations $\mathcal{T}$ are injected into the radical R and the homogeneous Lorentz group $\mathcal{L}$ is injected into one of the simple components L_1 of E.
C One has $(\mathcal{T} \rtimes \mathcal{L}) \subset L_1$, i.e. the entire Poincaré group is injected into one of the simple components L_1 of E.

Examples for case B are detailed in [12], namely the super-Poincaré group or the extended super-Poincaré group [5, 15, 16], as well as the extensions of the Poincaré group proposed by us [12]. Example for case C is the conformal Poincaré group, being isomorphic to SO(2, 4). However, also more complicated examples are being constructed [7–9] in the literature.

Knowing O'Raifeartaigh theorem, the argument of Coleman-Mandula theorem in case of a finite dimensional Poincaré group extension can be greatly simplified. First, Coleman-Mandula assumes implicitly that symmetry breaking is not present, which excludes case C. Secondly, it implicitly assumes that one has a positive definite invariant scalar product on the non-Poincaré directions of the Lie algebra, which excludes case B (along with SUSY, for instance). In case of SUSY or our Poincaré group extensions, the pertinent invariant scalar product is merely positive *semi*definite, which provides a backdoor to the otherwise prohibitive argument.

4 Conservative Extensions of the Poincaré Group

As outlined in [12], the super-Poincaré group or extended super-Poincaré group cannot be considered as a vector bundle automorphism group with the spacetime being the base manifold. This implies that in a supersymmetric model a heavy symmetry breaking needs to be introduced in order to recover a gauge-theory-like setting, so characteristic to the Standard Model. Also in [12] the question is asked: what are those finite dimensional direct-indecomposable extensions E of the Poincaré group $\mathcal{P} = \mathcal{T} \rtimes \mathcal{L}$, which respect the vector bundle structure of fundamental fields as well as the Lorentz metric of the spacetime? Technically, this means that one has $E = \mathcal{T} \rtimes \{\text{some pointwise acting symmetries}\}$ with a surjective homomorphism $\{\text{some pointwise acting symmetries}\} \to \mathcal{L}$ onto the Lorentz group. The answer [12] is a simple consequence of the Levi decomposition / O'Raifeartaigh theorem and of the definition of semidirect product:

$$E = \Big(\underbrace{\mathcal{T}}_{\text{translations}} \times \underbrace{\mathcal{N}}_{\substack{\text{solvable}\\ \text{internal symmetries}}} \Big) \rtimes \Big(\underbrace{\mathcal{G}_1 \times \cdots \times \mathcal{G}_m}_{\substack{\text{semisimple}\\ \text{internal symmetries}}} \times \underbrace{\mathcal{L}}_{\substack{\text{Lorentz}\\ \text{symmetries}}} \Big) \tag{4}$$

must hold, where the semisimple internal symmetries $\mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ commute with the translations $\mathcal{T}$, the Lorentz symmetries $\mathcal{L}$ have the canonical adjoint action on the translations $\mathcal{T}$, but the invariant subgroup of solvable internal symmetries $\mathcal{N}$ does not commute with the Lorentz symmetries nor with the semisimple internal symmetries. If one requires in addition that there exists a positive *semi*definite invariant bilinear form on the Lie algebra of the non-Poincaré symmetries, then it also follows that $\mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ is compact. (Such a requirement is motivated by the positive energy condition for gauge fields.) With this requirement, the full internal symmetry group of such a Poincaré group extension shall have the structure $\{\text{solvable}\} \rtimes \{\text{compact}\}$. These kind of Poincaré group extensions we named *conservative* extensions, and are seen to have a number of rather favorable properties [12]: they are direct-indecomposable, preserve causal structure of the spacetime, preserve vector bundle structure of fundamental fields, obey positive energy condition etc. Ideally, one could look for such a setting in which case the group of compact internal symmetries is identical to the Standard Model gauge group $\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)$.

It is not difficult to see that conservative extensions of the Poincaré group do exist, i.e. that our definition is not empty. Take, for instance, the complexified Schrödinger Lie group, which is isomorphic to $\mathrm{H}_3(\mathbb{C}) \rtimes \mathrm{SL}(2, \mathbb{C})$. Here $H_3(\mathbb{C})$ denotes the complexified Heisenberg Lie group with three generators, being the lowest dimensional complex non-abelian nilpotent Lie group. Clearly, from this there exists a homomorphism onto $\mathrm{SL}(2, \mathbb{C})$ and therefore also onto the homogeneous Lorentz group $\mathcal{L}$, which acts canonically on the group of spacetime translations $\mathcal{T}$ in its adjoint representation. With these subgroup actions, the group $(\mathcal{T} \times \mathrm{H}_3(\mathbb{C})) \rtimes \mathcal{L}$ is uniquely well-defined and is direct-indecomposable. (Note that from the Lie algebra point of view, one has $\mathrm{SL}(2, \mathbb{C}) \equiv \mathcal{L}$). This provides the simplest conservative extension of the Poincaré group, and the non-Poincaré symmetries span a nilpotent Lie group $\mathrm{H}_3(\mathbb{C})$, being part of the radical.

An other example is constructed in [11, 12], which is expected to be more interesting for physics. It contains a Poincaré component, a compact internal group component ($\mathrm{U}(1)$ in the example), and unavoidably a nilpotent internal group component. In particular, it has the group structure $(\mathcal{T} \times N) \rtimes (\mathrm{U}(1) \times \mathcal{L})$, where N is a 20 dimensional real nilpotent Lie group, the Lorentz group $\mathcal{L}$ acts with the canonical adjoint action on the translations $\mathcal{T}$, and both the compact $\mathrm{U}(1)$ component and the Lorentz group component $\mathcal{L}$ has non-vanishing adjoint action on N, which provides the direct-indecomposability. Clearly, it is essential in the construction that the radical $\mathcal{T}$ of the Poincaré group is extended by N, without which such a direct-indecomposability is not possible according to O'Raifeartaigh theorem. Also note, that the construction resembles to (extended) super-Poincaré group as outlined in [12], with the important difference that in case of the (extended) super-Poincaré group the translations are direct-indecomposably part of the nilpotent symmetries, called to be the group of supertranslations, forming a direct-indecomposable two-step nilpotent Lie group. In case of our construction, however, the translations are direct-decomposable from other symmetries within the radical, which makes it a conservative extension of the Poincaré group, in contrast to (extended) SUSY. It is also an important piece of information that the concrete conservative extension of the Poincaré group proposed in [11, 12] can be shown to have faithful unitary representations on some separable complex Hilbert space.

An important feature of the conservative extensions of the Poincaré group $\mathcal{P}$ is that there exists a homomorphism:

$$\underbrace{\underbrace{\mathcal{N}}_{\substack{\text{solvable}\\ \text{internal symmetries}}} \rtimes \left(\underbrace{\mathcal{G}_1 \times \cdots \times \mathcal{G}_m}_{\substack{\text{compact}\\ \text{internal symmetries}}} \times \underbrace{\mathcal{P}}_{\substack{\text{Poincaré}\\ \text{symmetries}}} \right)}_{\substack{\text{direct-indecomposable conservative extension of the Poincaré group,}\\ \text{acting on fundamental field degrees of freedom}}}$$

$$\longrightarrow \underbrace{\underbrace{\mathcal{G}_1 \times \cdots \times \mathcal{G}_m}_{\substack{\text{compact}\\ \text{internal symmetries}}} \times \underbrace{\mathcal{P}}_{\substack{\text{Poincaré}\\ \text{symmetries}}}}_{\substack{\text{observed direct-decomposable symmetries,}\\ \text{acting on some derived field quantities}\\ \text{which are function of fundamental degrees of freedom}}} \tag{5}$$

and potentially can explain a Standard Model-like gauge theory setting from a direct-indecomposable fundamental symmetry, without a breaking of it.

5 Commutation Relations of the Concrete Example

In this section the commutation relations of the generators of the Lie algebra of our concrete example group [11, 12] is outlined. The pertinent direct-indecomposable conservative extension of the Poincaré group is the automorphism group of some finite dimensional unital associative algebra valued classical fields over the four dimensional spacetime. Similar algebra valued field construction was tried by Anco and Wald in the end of '80s [17], but they could not achieve the goal of direct-indecomposability due to the too simple structure of the algebra of fields which they applied.

In the followings S shall denote a complex two-dimensional vector space ("spinor space"), and S^*, $\bar{S}$, $\bar{S}^*$ shall denote its dual, complex conjugate, complex conjugate dual vector space, respectively. Let us consider the complex unital associative algebra $\Lambda(\bar{S}^*) \otimes \Lambda(S^*)$, where $\Lambda()$ denotes exterior algebra formation. Observe that this algebra also has an antilinear involution defined by the complex conjugation, which is compatible with the algebraic product in the sense that $\overline{x\,y} = \bar{x}\,\bar{y}$ holds for any two algebra elements x, y. We shall call a finite dimensional complex unital associative algebra A together with an antilinear involution $(\cdot)^+$ obeying $(x\,y)^+ = x^+\,y^+$ a *spin algebra* whenever the pair $\big(A,\ (\cdot)^+\big)$ is isomorphic to $\Big(\Lambda(\bar{S}^*) \otimes \Lambda(S^*),\ \overline{(\cdot)}\Big)$. The antilinear involution $(\cdot)^+$ (or, $\overline{(\cdot)}$) shall be referred to as *charge conjugation*. Thus, a spin algebra A is (not naturally) isomorphic to the concrete spin algebra $\Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ with spinorial realization. In the followings, we shall often use a representation $A \cong \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ so that the simple formalism of traditional two-spinor calculus can be used.

For the sake of simplicity, we shall give our construction in the flat spacetime limit. Let $\mathcal{M}$ denote a four real dimensional affine space, modeling a (flat) spacetime manifold, and let T be its underlying vector space ("tangent space"). Take the trivial vector bundle $A(\mathcal{M}) := \mathcal{M} \times A$. Our direct-indecomposable conservative Poincaré group extension containing also $\mathrm{U}(1)$ shall be nothing but the automorphism group of the algebra of the sections of the $A(\mathcal{M})$, i.e. of the spin algebra valued fields [11, 12]. In the followings Penrose abstract indices shall be used for the spacetime degrees of freedom and for the spinor degrees of freedom, as usual in the General Relativity literature [18, 19]. The symbol ∇_a shall denote the affine covariant derivation of the affine space $\mathcal{M}$. Also, given a point o ("origin") of $\mathcal{M}$, the symbol X_o shall denote the vectorization map against o, which is the vector field $X_o :\ \mathcal{M} \to T,\ x \mapsto (x-o)$. Let in the spinorial representation $\sigma_a^{AA'}$ denote the usual Infeld-Van der Waerden symbol, also called Pauli injection, or soldering form. It is some preferred injective linear map $T \to \mathrm{Re}\big(\bar{S} \otimes S\big)$, and is shown in [11, 12] to be $\mathrm{Aut}(A)$-invariant. Its inverse map is denoted by $\sigma^a_{AA'}$. Let $\omega_{[A'B'][CD]}$ be a positive maximal form from A. Then, it is well-known that $g(\sigma, \omega)_{ab} := \sigma_a^{AA'} \sigma_b^{BB'} \omega_{[A'B'][AB]}$ is a Lorentz signature metric on T, and its inverse metric is denoted by $g(\sigma, \omega)^{ab}$. The symbol $\Sigma(\sigma)_a{}^b{}_B{}^A :=$ $\mathrm{i}\,\big(\sigma_a^{AC'} \sigma^b_{BC'} - g(\sigma, \omega)^{cb} g(\sigma, \omega)_{da} \sigma_c^{AC'} \sigma^d_{BC'}\big)$ is called the spin tensor in the literature, and can be considered as the generators of the $\mathrm{SL}(2, \mathbb{C})$ group, as it is well-known. It can uniquely act on the full mixed tensor algebra of S, S^*, $\bar{S}$, $\bar{S}^*$ by requiring vanishing action on scalars, commutativity with duality form, realness of $\mathrm{i}\Sigma(\sigma)_a{}^b$, and Leibniz rule over tensor product. Given a concrete spinorial representation $A \equiv \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$, thus the spin tensor can be uniquely extended to A as an algebra derivation valued tensor $\Sigma(\sigma)_a{}^b$, and it shall have vanishing action on scalars, shall obey Leibniz rule against algebra multiplication of A, and shall have realness of $\mathrm{i}\Sigma(\sigma)_a{}^b$ against the charge conjugation within A. The spin tensor $\Sigma(\sigma)_a{}^b$, however, is not invariant to the full action of $\mathrm{Aut}(A)$: the nilpotent normal subgroup within $\mathrm{Aut}(A)$ which do not preserve the subspaces $\Lambda_{\bar{p}q} := \wedge^p \bar{S}^* \otimes \wedge^q S^*$ of pure p, q-forms do not preserve $\Sigma(\sigma)_a{}^b$. That is, the definition of $\Sigma(\sigma)_a{}^b$ is relative to a concrete spinorial representation $A \equiv \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$, which is also not preserved by the pertinent nilpotent normal subgroup.

Introduce the differential operators $J_o{}^{ab} := \left(X_o{}^a \mathrm{i}\nabla^b - X_o{}^b \mathrm{i}\nabla^a\right) + \frac{1}{2}\Sigma^{ab}$ and $P_a := \mathrm{i}\nabla_a$ over the sections of the spin algebra bundle $A(\mathcal{M})$, i.e. over the spin algebra valued fields. They are called the o-angular momentum and momentum operators, respectively, and are known to provide a faithful representation of the Poincaré Lie algebra in the Lie algebra of differential operators of the sections of $A(\mathcal{M})$. Given a concrete spinorial representation $A \equiv \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$, for each complex number c introduce the unique algebra derivation operator which acts as $\zeta_c(\bar{\xi}_{A'}) := c\,\bar{\xi}_{A'}$ for all $\bar{\xi}_{A'} \in \Lambda_{\bar{1}0} \equiv \wedge^1 \bar{S}^* \otimes \wedge^0 S^*$. By construction, the map $\mathrm{i}\varphi \mapsto \zeta_{\mathrm{i}\varphi}\quad (\varphi \in \mathbb{R})$ provides a faithful representation of the Lie algebra of the $\mathrm{U}(1)$ group on the algebra derivations of the spin algebra A, and thus on the algebra derivations of the spin algebra valued fields. Similarly to the spin tensor Σ^{ab}, the definition of the operator ζ depends on a concrete chosen spinorial representation $A \equiv \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$. By construction, the operators P_a, $J_{o\,ab}$, ζ provide a faithful representation of the Lie algebra of $\mathcal{P} \times \mathrm{U}(1)$.

The direct-indecomposable unification of $\mathcal{P}$ and of $\mathrm{U}(1)$ shall happen because $\mathrm{Aut}(A)$ has a nilpotent normal subgroup on which both $\mathcal{P}$ and $\mathrm{U}(1)$ has nonvanishing adjoint action. The generators of this nilpotent normal subgroup shall be detailed as follows. Take a concrete chosen spinorial representation $A \equiv \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$. Take any element $\beta \in \mathrm{Re}\left(\Lambda_{\bar{1}2} \otimes \Lambda^*_{\bar{1}0} \oplus \Lambda_{\bar{2}1} \otimes \Lambda^*_{\bar{0}1}\right) \subset \mathrm{Re}\,(\mathrm{Lin}(A))$. Such an element, in the spinorial notation, can be represented as $\left(\beta_{B'[CD]}{}^{A'},\ \bar{\beta}_{B[C'D']}{}^{A}\right)$, uniquely determined by the spinor tensor $\beta_{B'[CD]}{}^{A'}$. Such an element β defines a $\Lambda_{\bar{1}0} \to \Lambda_{\bar{1}2}$ linear operator via the formula $\bar{\xi}_{A'} \mapsto \beta_{B'[CD]}{}^{A'}\bar{\xi}_{A'}$. Direct verification shows that this can be uniquely extended as an algebra derivation operator ν_β of A, via requiring vanishing on scalars $\Lambda_{\bar{0}0}$, realness, and Leibniz rule. Also, for all elements $a \in \mathrm{Re}\,(A)$, the linear map $\mathrm{ad}_a : A \to A$ is an algebra derivation of A, called inner derivation. They can be uniquely parameterized by real elements not in the center of A, i.e. with elements $a \in \mathrm{Re}\left(\Lambda_{\bar{1}0} \oplus \Lambda_{\bar{0}1} \oplus \Lambda_{\bar{1}1} \oplus \Lambda_{\bar{2}1} \oplus \Lambda_{\bar{1}2}\right)$.

Let $\beta, \beta' \in \mathrm{Re}\left(\Lambda_{\bar{1}2} \otimes \Lambda^*_{\bar{1}0} \oplus \Lambda_{\bar{2}1} \otimes \Lambda^*_{\bar{0}1}\right) \subset \mathrm{Re}\,(\mathrm{Lin}(A))$ and take the elements $a, a' \in \mathrm{Re}\left(\Lambda_{\bar{1}0} \oplus \Lambda_{\bar{0}1} \oplus \Lambda_{\bar{1}1} \oplus \Lambda_{\bar{2}1} \oplus \Lambda_{\bar{1}2}\right)$ and $\varphi, \varphi' \in \mathbb{R}$, regarded as constant fields over the spacetime manifold $\mathcal{M}$. Then the relations

$$
\begin{aligned}
[\mathrm{ad}_a,\ \mathrm{ad}_{a'}] &= \mathrm{ad}_{[a,a']},\\
\left[\mathrm{ad}_a,\ \nu_{\beta'}\right] &= -\,\mathrm{ad}_{\nu_{\beta'}(a)},\\
\left[\mathrm{ad}_a,\ \zeta_{\mathrm{i}\varphi'}\right] &= -\,\mathrm{ad}_{\zeta_{\mathrm{i}\varphi'}(a)},\\
\left[\mathrm{ad}_a,\ J_{o\,cd}\right] &= -\,\mathrm{ad}_{J_{o\,cd}(a)},\\
[\mathrm{ad}_a,\ P_c] &= 0,\\
\left[\nu_\beta,\ \nu_{\beta'}\right] &= 0,\\
\left[\nu_\beta,\ \zeta_{\mathrm{i}\varphi'}\right] &= -\nu_{[\zeta_{\mathrm{i}\varphi'},\beta]},\\
\left[\nu_\beta,\ J_{o\,cd}\right] &= -\nu_{[J_{o\,cd},\beta]},\\
\left[\nu_\beta,\ P_c\right] &= 0,\\
\left[\zeta_{\mathrm{i}\varphi},\ \zeta_{\mathrm{i}\varphi'}\right] &= 0,\\
\left[\zeta_{\mathrm{i}\varphi},\ J_{o\,cd}\right] &= 0,\\
\left[\zeta_{\mathrm{i}\varphi},\ P_c\right] &= 0,\\
\left[J_{o\,cd},\ J_{o\,ef}\right] &= \mathrm{i}\,g_{de}\,J_{o\,cf} - \mathrm{i}\,g_{ce}\,J_{o\,df} + \mathrm{i}\,g_{cf}\,J_{o\,de} - \mathrm{i}\,g_{df}\,J_{o\,ce},\\
\left[J_{o\,cd},\ P_e\right] &= \mathrm{i}\,g_{de}\,P_c - \mathrm{i}\,g_{ce}\,P_d,\\
[P_c,\ P_d] &= 0
\end{aligned}
\tag{6}
$$

are seen to hold, where the operators ad_a, ν_β, $\zeta_{\mathrm{i}\varphi}$, $J_{o\,cd}$, P_e are regarded as acting on the smooth sections of $A(\mathcal{M})$, i.e. on spin algebra valued fields. These operators are algebra derivation valued on the algebra of smooth sections of $A(\mathcal{M})$, where in a concrete spinor representation $A \equiv \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$, these fields can be regarded as a 9-tuple of spinor tensor fields

$$
\Big(\varphi,\ \xi_{(+)\,A'},\ \xi_{(-)\,A'},\ \epsilon_{(+)\,[B'C']},\ v_{DD'},\ \epsilon_{(-)\,[BC]},\ \chi_{(+)\,[B'C']A},\ \chi_{(-)\,A'[BC]},\ \omega_{[A'B'][CD]}\Big) \tag{7}
$$

in the usual spinor index notation. The symmetry generators in Eq. (6) respect the vector bundle structure of $A(\mathcal{M})$, the spin algebra structure of the fibers of $A(\mathcal{M})$, as well as the soldering form $\sigma_a^{AA'}$ viewed as a $T^* \otimes \mathrm{Re}\left(\Lambda^*_{\bar{1}1}\right)$ valued constant field over the affine space $\mathcal{M}$. They also happen to preserve the constant maximal forms $\omega_{[A'B'][CD]}$, i.e. constant sections of value in $\Lambda_{\bar{2}2} \equiv \wedge^2 \bar{S}^* \otimes \wedge^2 S^*$. If an additional generator, i.e. the operator $\rho \mapsto \zeta_\rho \quad (\rho \in \mathbb{R})$ is also included among the Lie algebra generators of Eq. (6), then also the generators of the constant Weyl (conformal) rescalings of the flat spacetime metric g_{ab} is included in the Lie algebra, and in that case the maximal forms are not preserved, but acted on with the Weyl rescalings.

Acknowledgements The author would like to thank to the organizers of QTS10. Special thanks to prof. Evgeny Ivanov for discussions on a simplified example group for extended Poincaré symmetries, based on the complexified Schrödinger group. This work was supported in part by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. Financial coverage for the participation of the author on QTS10 was provided in part via the Momentum ('Lendület') program of the Hungarian Academy of Sciences under grant number LP2013-60, special thanks to Dezső Varga for that. Moreover, significant part of the conference participation was covered by the travel grant of the Wigner Research Centre for Physics of the Hungarian Academy of Sciences, which is also greatly acknowledged.

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Noether's Theorem and Its Complement in Multi-Particle Systems

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Abstract Noether's theorem has gained outstanding importance in theoretical particle physics, because it leads to basic conservation laws, such as the conservation of momentum and of angular momentum. Closely related to this theorem, but unnoticed so far, is a complementary law, which requires the (virtual) exchange of momentum between the particles of an isolated multi-particle system. This exchange of momentum determines an interaction. For a two-particle system defined by an irreducible representation of the Poincaré group, this interaction is identified as the electromagnetic interaction. This sheds new light on the particle interactions described by the Standard Model. It resolves long-standing questions about the value of the electromagnetic coupling constant, and about divergent integrals in quantum electrodynamics.

Keywords Noether's theorem · Multi-particle systems · Poincaré group
Momentum entanglement · Electromagnetic interaction · Fine-structure constant

1 Introduction

Since quantum electrodynamics (QED) was cast in its present form in 1949–50, we have been faced with two problems. The first problem concerns the mathematical inconsistencies of the perturbation algorithm, which become apparent in divergent integrals as soon as higher-order approximations to the perturbation series are considered. Although we have learned to remove these divergences by a mathematical trick called renormalization, this trick has neither made the mathematics consistent nor contributed to a better understanding of QED. The second problem is that QED cannot determine the value of the electromagnetic coupling constant, which indicates that QED in its current form is not a closed theory of the electromagnetic interaction but merely its phenomenological description.

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_28

The central question to be answered concerns the phenomenon of interaction itself. The Feynman rules in momentum space [1] assert that for the construction of the S matrix, a factor $(2\pi)^4\,\delta^{(4)}(p - p' + k)$ has to be inserted at each vertex, where p and p' are the momenta of the fermion lines and k is that of a photon line, and one is to integrate over the momenta of all internal lines. The integration leads to momentum-entangled structures, describing a 'virtual exchange of momentum' between the particles while conserving the total momentum. A short analysis of S-matrix elements, calculated by Feynman rules, shows that it is this exchange of momentum that is de facto responsible for the interaction described by QED, irrespective of the physical or mathematical cause of the entanglement.

The Standard Model suggests that the reason for entanglement is the exchange of virtual gauge particles, their existence being postulated by the principle of gauge invariance. Considering, however, that the structure of not only single- but also two-particle states is largely determined by their symmetry group, that is, the Poincaré group, we can expect group theory to provide a different, strictly mathematical answer to the question: What physical or mathematical conditions can force two single particles into a momentum-entangled two-particle state?

In answering this question, the following group theoretical approach will provide unexpected insights into the quantum mechanics of multi-particle systems and, as a by-product, give answers to the issues raised about divergences and coupling constants.

2 The Complement to Noether's Theorem

Let me start with Noether's theorem, which, on a very basic level, links continuous symmetry groups with conservation laws. In quantum mechanics, this linkage is especially close, because the generators of unitary symmetry transformations are, at the same time, self-adjoint operators that represent observables. In the case of translation symmetry, the generators of the translations represent the (conserved) momentum; in the case of rotational symmetry, the generators of the rotations represent the (conserved) angular momentum. In the Heisenberg picture, the proof of Noether's theorem is extremely simple: the invariance of the Hamiltonian H with respect to unitary symmetry transformations means that it commutes with the generator X of the symmetry operations. According to the Heisenberg equation

$$\frac{dX}{dt} = i\,[H, X]\ , \tag{1}$$

the self-adjoint operator X, now understood as the representation of an observable, is therefore conserved in time.

Consider an isolated multi-particle system that is composed of independent ("free") particles. According to the axioms of quantum mechanics the state space of this system is the direct product of the state spaces of the particles; the total momentum P is given by the sum of the individual particle momenta P_j:

$$P = \sum_{j=1}^{n} P_j \ . \tag{2}$$

Being an isolated system, its Hamiltonian H is invariant under translations of the whole system. Therefore, according to Noether's theorem, the total momentum P is conserved in time:

$$\frac{dP}{dt} = i\,[H, P] = 0 \ . \tag{3}$$

This does not imply that the system is also invariant under translations of a single particle. If it is not invariant, the Heisenberg equation (1), applied to the generator P_j of such a transformation, states that the momentum of this particle is not conserved in time:

$$\frac{dP_j}{dt} = i\left[H, P_j\right] \neq 0 \ . \tag{4}$$

In general terms, this can be formulated as follows:

Theorem 1 (Complement to Noether's theorem) *If the Hamiltonian is not invariant with respect to a continuous unitary transformation, then the generator of the transformation is not conserved in time.*

Proof The proof follows analogously to the proof of Noether's theorem from the Heisenberg equation (1).

Although the mathematical basis of this theorem is the same as that of Noether's original theorem, its physical implications are quite different. Because in the above-mentioned configuration the total momentum is conserved in time, a possible change of the momentum of particle j must be compensated for by an opposite change of momentum of (at least) one other particle k:

$$\frac{dP_j}{dt} = i\left[H, P_j\right] \neq 0 \ , \tag{5}$$

$$\frac{dP_k}{dt} = i\,[H, P_k] \neq 0 \ . \tag{6}$$

With Eqs. (2) and (3) we have

$$\frac{dP_j}{dt} + \frac{dP_k}{dt} = \frac{dP}{dt} = 0 \tag{7}$$

and, therefore,

$$\frac{dP_j}{dt} = -\,\frac{dP_k}{dt} \ . \tag{8}$$

Relation (8) can be interpreted as (modelled by) an exchange of momentum between particles j and k. Hence, instead of a conservation law, the complement of Noether's theorem causes an interaction law:

Proposition 1 (Interaction law) *If the Hamiltonian of an isolated multi-particle system composed of independent particles is not invariant under translations of the individual particles, the particles are correlated by exchange of momentum.*

Proof The proof follows from Eqs. (2)–(8).

Here and in the following, the term 'exchange of momentum' is to be understood in the sense of Eq. (8).

Two questions have to be answered: Can we find a realistic multi-particle system where the condition for the application of this law is fulfilled, namely, the existence of a Hamiltonian that is not invariant under translations of the individual particles? What are the physical consequences of this law? The following Sect. 3 will show how this condition is actually met for a simple two-particle system. Section 4 will illustrate the practical consequences of this law.

3 Two-Particle States and Interaction

According to the axioms of quantum mechanics in combination with Poincaré invariance, two independent particles with 4-momenta p_1 and p_2 are described by a product representation of the Poincaré group. A product representation can be reduced to the direct sum of irreducible representations.

The irreducible representations of the Poincaré group are characterized by fixed eigenvalues of two Casimir operators [2]

$$P = p^\mu p_\mu \ \text{ and } \ W = -w^\mu w_\mu \ , \ \text{ with } \ w_\sigma = \frac{1}{2}\epsilon_{\sigma\mu\nu\lambda} M^{\mu\nu} p^\lambda \ . \tag{9}$$

Here, p^μ and $M^{\mu\nu}$ are the operators of 4-momentum and angular momentum.

The state space H_I of an irreducible representation is a subspace of the state space H_P of the corresponding product representation. In H_I there exists a basis of eigenstates $|\mathbf{p}, m\rangle$ of the total 3-momentum $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$ and of a component of $M^{\mu\nu}$ with eigenvalue m [2]. The translations of a single particle, generated by the operators of the individual particle momenta, are well-defined unitary transformations within H_P, but, in contrast to H_P, they are (in general) not symmetry transformations of H_I. In other words, they lead out of H_I. This follows from the commutation relations between p^μ and $M^{\mu\nu}$ [2]

$$\left[p^\sigma, M^{\mu\nu}\right] = i\left(g^{\mu\sigma} p^\nu - g^{\nu\sigma} p^\mu\right) , \tag{10}$$

which are equal to 0 only if $\sigma \neq \mu, \nu$. If the total momentum $\mathbf{p}$ points in the direction σ, then $M^{\mu\nu}$ commutes with $\mathbf{p}$, but does not commute with $\mathbf{p}_1$ and $\mathbf{p}_2$, unless $\mathbf{p}_1$

and $\mathbf{p}_2$ are parallel or anti-parallel to $\mathbf{p}$. This means the basis states are, in general, not eigenstates of the individual particle momenta: in consequence, they are not invariant with respect to translations of a single particle. Hence, with the exception of the parallel/anti-parallel cases, the following lemma applies.

Lemma 1 *Two-particle eigenstates of total momentum and orbital angular momentum are not invariant with respect to translations of the individual one-particle states.*

One can say that based on the commutation relations of the Poincaré group, the conservation of total and angular momentum breaks the translation invariance of the individual particles.

Lemma 1 determines the general structure of the basis states $|\mathbf{p}, m\rangle$: They are a momentum-entangled superposition of product states $|\mathbf{p_1}, \mathbf{p_2}\rangle$ with the same total momentum $\mathbf{p}$

$$|\mathbf{p}, m\rangle = \int_{\Omega} d^3\mathbf{p}_1 d^3\mathbf{p}_2 \; c(\mathbf{p}, m, \mathbf{p}_1, \mathbf{p}_2) \; |\mathbf{p}_1, \mathbf{p}_2\rangle \quad . \tag{11}$$

The coefficients $c(\mathbf{p}, m, \mathbf{p}_1, \mathbf{p}_2)$ are the analogues of the Clebsch–Gordan coefficients, as known from the coupling of angular momenta. The domain of integration Ω is a finite subspace of the two-particle mass shell. The product states are (in general) momentum-entangled, because otherwise the basis states would be eigenstates also of the individual particle momenta, which (in general) is excluded by Lemma 1.

The product states $|\mathbf{p_1}, \mathbf{p_2}\rangle$ are normalized with respect to the parameter space $\mathbb{R}^3 \times \mathbb{R}^3$ of the product representation:

$$\langle \mathbf{p}_1, \mathbf{p}_2 | \mathbf{p}'_1, \mathbf{p}'_2 \rangle = \delta(\mathbf{p}_1 - \mathbf{p}'_1)\, \delta(\mathbf{p}_2 - \mathbf{p}'_2) \; . \tag{12}$$

In an irreducible two-particle representation, the constancy of the first Casimir operator P restricts the parameter space to the two-particle mass shell

$$(p_1^0 + p_2^0)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 = m_{tot}^2 \; . \tag{13}$$

The constancy of the second Casimir operator W gives the mass shell the topological structure of a fibre space with circle fibres over the mass hyperboloid of the total momentum [3]. The circle fibres correspond to rotations generated by the component of angular momentum that commutes with the total momentum. The restriction of the parameter space requires that the two-particle states (11) are re-normalized by a factor $\omega = V(\Omega)^{-\frac{1}{2}}$, where $V(\Omega)$ is the (finite) volume of the domain of integration Ω. Together, $d^3\mathbf{p}_1 d^3\mathbf{p}_2$ and ω form an infinitesimal volume element that ensures the correct normalization.

For expository reasons, I will not include ω in the two-particle states, but write $\omega\,|\mathbf{p}, m\rangle$ for the normalized states.

Given this basis, the Hamiltonian of the two-particle system can be written in the form

$$H = \omega^2 \sum_m \int d^3\mathbf{p} \; \left|\mathbf{p}, m\right\rangle h_{pm} \left\langle \mathbf{p}, m\right| \; . \tag{14}$$

Note that H cannot be made ‘diagonal’ with respect to the product states $|\mathbf{p}_1, \mathbf{p}_2\rangle$,

$$H \neq \int d^3\mathbf{p}_1 \, d^3\mathbf{p}_2 \; \left|\mathbf{p}_1, \mathbf{p}_2\right\rangle h_{p_1 p_2} \left\langle \mathbf{p}_1, \mathbf{p}_2\right| \; , \tag{15}$$

because the product states do not belong to the state space of the considered irreducible representation.

Since the basis states are eigenstates of the Hamiltonian (14), they do not change in time – except for a phase factor. They describe a stable configuration with conserved total and angular momenta: in other words, they describe an isolated system. On the other side, the basis states and, therefore, the Hamiltonian, are not invariant under translations of the individual particles. Since these translations are well-defined unitary transformations of the underlying product state space, Proposition 1 can be applied, leading, together with Lemma 1, to the following statement.

Corollary 1 *In two-particle systems described by an irreducible representation of the Poincaré group, the particles exchange (virtual) quanta of momentum.*

Note that the ‘exchange of momentum’ does not necessarily imply a dynamic process: in the first place, it refers to the static entangled structure of two-particle states, which, however, bears the potential for a dynamic process. Relating to such structures, Feynman used the phrase ‘exchange of virtual quanta’ [4].

The (virtual or real) exchange of momentum between two particles defines an interaction; the similarity to the electromagnetic interaction as described in the Introduction is obvious. However, this interaction mechanism is not based on a purposely constructed model, as in the case of the Standard Model, but is rooted in the principles of quantum mechanics. The particles interact directly and necessarily, without any mediating particles or fields, thanks to the entangled structure of the two-particle states [5].

Alternatively, Lemma 1 and Corollary 1 can be obtained directly from the fact that an eigenstate of angular momentum must have a rotational symmetry, which means that it must be a superposition of product states $|\mathbf{p}_1, \mathbf{p}_2\rangle$ such that along with any pure product state, the rotated versions of this state also contribute to the eigenstate. This necessarily gives the eigenstate a momentum-entangled structure.

4 Illustration: A Scattering Experiment

The following thought experiment, describing a scattering process, will illustrate the interaction mechanism.

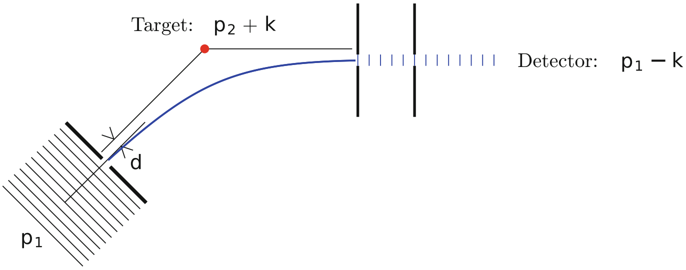

Fig. 1 Geometry of the scattering experiment

Figure 1 shows an incoming plane wave of particles, some apertures, a target, and a detector.

Between the first and second aperture, an incoming particle and a particle of the target form a two-particle state. The apertures of the collimator in front of the detector select an outgoing plane wave. The total momentum $\mathbf{p}$ is equal to the incoming momentum $\mathbf{p}_1$. In the semi-classical view, $\mathbf{p}_1$, together with the perpendicular distance $\mathbf{d}$ between the beam and the target, define an angular momentum $\mathbf{m} = \mathbf{d} \times \mathbf{p}_1$. Therefore the experimental setup can be considered a filter that selects intermediate eigenstates of angular momentum such as $\omega\,|\mathbf{p}, m\rangle$.

Note the similarities with the diffraction of a plane wave at a pin hole: both here and there, the basic scattering mechanism becomes visible when essential parts of the incoming plane wave are blocked by an aperture. Here, the momentum-entangled two-particle state is left, there, the spherical elementary wave.

The scattering amplitude from the incoming product state $|\mathbf{p}_1, \mathbf{p}_2\rangle$ to the outgoing product state $|\mathbf{p}_1 - \mathbf{k}, \mathbf{p}_2 + \mathbf{k}\rangle$ is given by

$$S(\mathbf{k}) = \omega^2 \langle \mathbf{p}_1, \mathbf{p}_2 | \mathbf{p}, m \rangle \langle \mathbf{p}, m | \mathbf{p}_1 - \mathbf{k}, \mathbf{p}_2 + \mathbf{k} \rangle \ . \tag{16}$$

Since the intermediate state $|\mathbf{p}, m\rangle$ is momentum-entangled, it connects incoming and outgoing states also for non-zero values of $\mathbf{k}$. There is, in fact, an interaction by an exchange of momentum.

In the thought experiment, only the value of the Casimir operator P is determined by the momentum of the incoming plane wave. The second Casimir operator W is determined by the geometry of the setup.

In Eq. (16), the square of the normalization factor ω of the intermediate two-particle state acts like a coupling constant between the incoming and the outgoing states.

The numerical value of ω^2 is determined by the geometry of the two-particle mass shell (13). It has been calculated [3, 5] with the result:

$$\omega^2 = \frac{9}{16\pi^3} \left(\frac{\pi}{120} \right)^{1/4} = 1/137.03608245 \ . \tag{17}$$

This value matches the value of the fine-structure constant α, the square of the electromagnetic coupling constant, with the empirical CODATA value [6] of

$$\alpha = 1/137.035999139 \ . \tag{18}$$

This agreement provides strong evidence that the exchange of momentum within the states of an irreducible two-particle representation manifests itself as the electromagnetic interaction. This can be formulated as

Corollary 2 (Conjecture) *The electromagnetic interaction is a model-independent property of irreducible two-particle representations of the Poincaré group.*

5 Conclusions

Noether's theorem in connection with its complement provides a new and unbiased view on the electromagnetic interaction; this view is mathematically well founded on the principles of quantum mechanics, is independent of any model, and is physically supported by the matching of the calculated and empirical values of the fine-structure constant α. It simply says: In an isolated multi-particle system with well-defined and conserved total and angular momenta, the translation invariance of the individual particles is broken; therefore, the corresponding multi-particle states cannot be plain product states, but must be momentum-entangled; this is synonymous with a virtual exchange of momentum between the individual particle states.

In contrast to the standard formulation of QED, the group theoretical approach uniquely determines the electromagnetic coupling constant, which is identified as the normalization factor for two-particle states of an irreducible two-particle representation of the Poincaré group.

Noether's theorem and its complement basically confirm the interaction mechanism of the Standard Model, which is an interaction by exchange of momentum. However, in the Standard Model, the exchange of momentum is modelled by the exchange of virtual gauge bosons; this approach does not take note of the special topology of the fibered two-particle mass shell: In a correctly defined two-particle state, the exchange of momentum is controlled by a one-dimensional and bounded parameter on a circle fibre. In contrast, gauge particles come along with three independent components of momentum. The Feynman rules prescribe integration over these three (unbounded) parameters, rather than — as would be correct — over a single parameter on a circle fibre. An inspection of the integrals of the standard perturbation algorithm (cf., e.g. [7]) clearly shows that it is the excessive number and ranges of integration variables that are responsible for the well-known divergences.

The crucial insight of the foregoing analysis is that the structure of the Poincaré group, via its irreducible representations, completely determines not only the basic properties of single particles, but also their (electromagnetic) interaction. There is no need for phenomenological interaction terms or additional physical principles, such as gauge invariance, which the developers of the Standard Model considered an indispensable prerequisite of interaction. As far as the electromagnetic interaction is concerned, Poincaré invariance alone provides the rules for the complete description of multi-particle configurations. This presents us with the challenge to comprehend also the weak, strong, and, finally, gravitational interactions, as inherent structural properties of multi-particle states – instead of merely modelling them.

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$SU(2)$, Associated Laguerre Polynomials and Rigged Hilbert Spaces

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Abstract We present a family of unitary irreducible representations of $SU(2)$ realized in the plane, in terms of the associated Laguerre polynomials. These functions are similar to the spherical harmonics defined on the sphere. Relations with a space of square integrable functions defined on the plane, $L^2(\mathbb{R}^2)$, are analyzed. We have also enlarged this study using rigged Hilbert spaces that allow to work with discrete and continuous bases like is the case here.

Keywords Lie group representations · Special functions · Rigged Hilbert spaces

1 Introduction

The representations of a Lie algebra are usually considered as ancillary to the algebra and developed starting from the algebra, i.e. from the generators and their commutation relations. The universal enveloping algebra (UEA) is constructed and a complete set of commuting observables selected, choosing between the invariant operators of the algebra and of a chain of its subalgebras. The common eigenvectors of this complete set of operators are a basis of a vector space where the Lie algebra generators are realized as operators.

We propose here an alternative construction that allows to add to the representations obtained following the reported recipe, new ones not achievable following the

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_29

previous approach. Starting from a concrete vector space of functions with discrete labels and continuous variables, we consider the recurrence relations that allow to connect functions with different values of the labels. These recurrence relations are not operators but allow us to introduce, for each label and for each continuous variable, an operator that reads its value. In this way, recurrence relations are rewritten in terms of rising and lowering operators built by means of the above defined operators. These rising and lowering operators are often genuine generators of the Lie algebras considered by Miller [1] and the procedure gives simply the representations of the algebras in a well defined function space [2, 3]. However it can happen that the commutators, besides the values required by the algebra, have additional contributions. The essential point of this paper is that these additional contributions (as exhibited here) can be proportional to the null identity that defines the starting vector space. As this identity is zero on the whole representation, the Lie algebra is well defined and a new representation in a space of functions has been found.

We do not discuss here the general approach, but we limit ourselves to a simple example where all aspects are better understandable. We start thus from the associated Laguerre polynomials (ALP) and, following the proposed construction, we realize the algebra $su(2)$ in terms of the appropriate rising and lowering operators. The ALP support in reality a larger algebra [4] but we prefer to consider here only the subalgebra $su(2)$. The reasons for this choice are twofold: first in this way the technicalities are reduced at the minimum and second it has been very nice for us to discover that not all representations of a so elementary group like $SU(2)$ where known.

As discussed in [5–7] the presence of operators with spectrum of different cardinality implies that, as considered for the first time in Lie algebras in [8], the space of the group representation is not a Hilbert space but a rigged Hilbert space (RHS) [9]. Thus, we introduce the above setting within the context of RHS since the RHS is the perfect framework where discrete and continuous bases coexist. In addition, the same RHS serves as a support for a representation on it of a Lie algebra as continuous operators as well as for its UEA. Therefore, the connection between discrete and continuous bases and Lie algebras with RHS is well established.

2 Associated Laguerre Polynomials

The ALP [10], $L_n^{(\alpha)}(x)$, depend from a real continuous variable $x \in [0, \infty)$ and from two other real labels (n, α) : $n = 0, 1, 2, \dots$ and α (usually assumed as a fixed parameter) continuous and > -1 . They reduce to the Laguerre polynomials for $\alpha = 0$ and are defined by the second order differential equation

$$\left[x \frac{d^2}{dx^2} + (1 + \alpha - x)\frac{d}{dx} + n\right] L_n^{(\alpha)}(x) = 0 \, . \tag{1}$$

From the many recurrence relations that can be found in literature [10, 11], we consider the following ones, all first order differential recurrence relations:

$$\begin{aligned}
&\left[x\frac{d}{dx}+(n+1+a-x)\right]L_n^{(\alpha)}(x)=(n+1)L_{n+1}^{(\alpha)}(x)\,,\\
&\left[-x\frac{d}{dx}+n\right]L_n^{(\alpha)}(x)=(n+\alpha)L_{n-1}^{(\alpha)}(x)\,,\\
&\left[-\frac{d}{dx}+1\right]L_n^{(\alpha)}(x)=L_n^{(\alpha+1)}(x)\,,\\
&\left[x\frac{d}{dx}+\alpha\right]L_n^{(\alpha)}(x)=(n+\alpha)L_n^{(\alpha-1)}(x)\,.
\end{aligned} \tag{2}$$

Starting from $L_n^{(\alpha)}(x)$, by means of repeated applications of equations (2), $L_{n+k}^{(\alpha+h)}(x)$ –with h and k arbitrary integers– can be obtained through a differential relation of higher order. But, by means of Eq. (1), every differential relation of order two or higher can be rewritten as a differential relation of order one. In particular we can obtain

$$\begin{aligned}
&\left[\frac{d}{dx}+\frac{n}{\alpha+1}\right]L_n^{(\alpha)}(x)\;=\;-\frac{\alpha}{\alpha+1}\;L_{n-1}^{(\alpha+2)}(x),\\
&\left[x(\alpha-1)\frac{d}{dx}-x\left(n+3\;\frac{\alpha}{2}\right)+\alpha(\alpha-1)\right]L_n^{(\alpha)}(x)\\
&\qquad\qquad=(j+\alpha)(\alpha+1)\;L_{n+1}^{(\alpha-2)}(x),
\end{aligned} \tag{3}$$

that are the recurrence relations we employ in this paper.

The ALP $L_n^{(\alpha)}(x)$ are –for $\alpha > -1$ and fixed– orthogonal in n with respect the weight measure $d\mu(x) = x^\alpha\, e^{-x}\, dx$ [10]:

$$\begin{aligned}
&\int_0^\infty dx\; x^\alpha\; e^{-x}\; L_n^{(\alpha)}(x)\; L_{n'}^{(\alpha)}(x)\;=\;\frac{\Gamma(n+\alpha+1)}{n!}\;\delta_{nn'}\;,\\
&\sum_{n=0}^{\infty} x^\alpha\, e^{-x}\, L_n^{(\alpha)}(x)L_n^{(\alpha)}(x')=\delta(x-x')\,.
\end{aligned} \tag{4}$$

The parameter α can be extended to arbitrary complex values [10] and, in particular, for α integer and such that $0 \le |\alpha| \le n$, we have the relation

$$L_n^{(-\alpha)}(x)=(-x)^\alpha\,\frac{(n-\alpha)!}{n!}\,L_{n-\alpha}^{(\alpha)}(x)\,. \tag{5}$$

Here we assume consistently that $n \in \mathbb{N}$, $\alpha \in \mathbb{Z}$ and $n-\alpha \in \mathbb{N}$, and we also consider α as a label, like n, and not a parameter fixed at the beginning. Following the approach of [2], we introduce now a set of alternative variables and include the weight measure inside the functions, in such a way to obtain the bases we are used in quantum mechanics. We define indeed $j := n + \alpha/2$ and $m := -\alpha/2$ that are such that $j \in \mathbb{N}/2\,,\ \ j-m \in \mathbb{N}$ and $|m| \le j$. Note that they look like the parameters j and m used in $SU(2)$. Now we write

$$\mathcal{L}_j^m(x) := \sqrt{\frac{(j+m)!}{(j-m)!}}\; x^{-m}\, e^{-x/2}\, L_{j+m}^{(-2m)}(x)$$

so that, from Eq. (5), $\mathcal{L}_j^m(x)$ is symmetric/antisymmetric in the exchange $m \leftrightarrow -m$ since $\mathcal{L}_j^m(x) = (-1)^{2j}\, \mathcal{L}_j^{-m}(x)$. From Eq. (4), we see that the $\mathcal{L}_j^m(x)$ verify, for m fixed, the following orthonormality and completeness relations

$$\int_0^\infty \mathcal{L}_j^m(x)\; \mathcal{L}_{j'}^m(x)\; dx \;=\; \delta_{jj'}\,, \qquad \sum_{j=|m|}^{\infty} \mathcal{L}_j^m(x)\; \mathcal{L}_j^m(x') \;=\; \delta(x-x')\,, \tag{6}$$

and are thus, for any fixed value of m, an orthonormal basis of $L^2(\mathbb{R}^+)$.

Note that, in the algebraic description of the spherical harmonics, the functions $T_j^m(x) = \sqrt{\frac{(j-m)!}{(j+m)!}}\, P_j^m(x)$, related to the associated Legendre functions $P_l^m(x)$ and introduced in [2], satisfy $T_j^m(x) = (-1)^m\, T_j^{-m}(x)$ which is a relation similar to those verified by the $\mathcal{L}_j^m(x)$. Moreover the $T_j^m(x)$, like the $\mathcal{L}_j^m(x)$ on the half-line, are orthogonal –for fixed m– in the interval $(-1,+1) \subset \mathbb{R}$ and a basis for $L^2[-1,1]$.

3 $SU(2)$ Representations in the Plane

Following now Ref. [2], we define four operators X, D_x, J and M such that

$$\begin{aligned} X\, \mathcal{L}_j^m(x) &= x\, \mathcal{L}_j^m(x), & D_x\, \mathcal{L}_j^m(x) &= \mathcal{L}_j^m(x)'\,, \\ J\, \mathcal{L}_j^m(x) &= j\, \mathcal{L}_j^m(x), & M\, \mathcal{L}_j^m(x) &= m\, \mathcal{L}_j^m(x), \end{aligned} \tag{7}$$

and we can rewrite Eq. (1) in terms of the $\mathcal{L}_j^m(x)$ and in operatorial form as

$$E\, \mathcal{L}_j^m(x) \;\equiv\; \left[X\, D_x^2 + D_x - \frac{1}{X} M^2 - \frac{X}{4} + J + \frac{1}{2}\right] \mathcal{L}_j^m(x) = 0\,. \tag{8}$$

Thus, the identity $E \equiv 0$ defines $L^2(\mathbb{R}^+)$.

The relations (3) can now be rewritten on terms of the $\mathcal{L}_j^m(x)$ as

$$\begin{aligned} K_+\, \mathcal{L}_j^m(x) &= \sqrt{(j-m)(j+m+1)}\;\; \mathcal{L}_j^{m+1}(x)\,, \\ K_-\, \mathcal{L}_j^m(x) &= \sqrt{(j+m)(j-m+1)}\;\; \mathcal{L}_j^{m-1}(x)\,, \end{aligned} \tag{9}$$

where

$$
\begin{aligned}
K_+ &= -2D_x\left(M+\frac{1}{2}\right)+\frac{2}{X}M\left(M+\frac{1}{2}\right)-\left(J+\frac{1}{2}\right),\\
K_- &= 2D_x\left(M-\frac{1}{2}\right)+\frac{2}{X}M\left(M-\frac{1}{2}\right)-\left(J+\frac{1}{2}\right).
\end{aligned}
\tag{10}
$$

Since, from Eq. (9), we have $[K_+, K_-]\ \mathcal{L}_j^m(x) = 2\,m\ \mathcal{L}_j^m(x)$ and defining $K_3 := M$ (i.e. $K_3\ \mathcal{L}_j^m(x) = m\ \mathcal{L}_j^m(x)$) we get the relations

$$
[K_+, K_-]\ \mathcal{L}_j^m(x) = 2\,K_3\ \mathcal{L}_j^m(x), \quad [K_3, K_\pm]\ \mathcal{L}_j^m(x) = \pm\,K_\pm\ \mathcal{L}_j^m(x)\,, \tag{11}
$$

that display the fact that, for fixed j, under the action of $K_\pm$ and K_3, the $\mathcal{L}_j^m(x)$ supports the irreducible representation of dimension $2j+1$ of $su(2)$.

However, while as exhibited by (6) the space $\{\mathcal{L}_j^m(x)\}$ has an inner product for m fixed and $j \geq |m|$ (thus supporting a set of UIR of $SU(1,1)$ [4]), the representation (11) of $SU(2)$ is not faithful, since $\mathcal{L}_j^m(x) = (-1)^{2j}\ \mathcal{L}_j^{-m}(x)$, and not unitary. The definition of a scalar product is indeed one of the problems we have in the connection of hypergeometric functions and Lie algebras. Hence, we have two problems: the $\mathcal{L}_j^m(x)$ are not orthonormal for j fixed and functions with opposite m are not independent (as it happens also with the $P_j^m(x)$). Following the same approach of the spherical harmonics to construct the inner product space for j fixed and $|m| \leq j$ we, thus, introduce a new real variable ϕ ($-\pi < \phi \leq \pi$) and the new objects

$$
\mathcal{Z}_j^m(r,\phi) := e^{i\,m\,\phi}\,\mathcal{L}_j^m(r^2),
$$

that verify $\mathcal{Z}_j^m(r,\phi+2\pi) = (-1)^{2j}\ \mathcal{Z}_j^m(r,\phi)$. Under the change of variable $x \to r^2$ equation (8) becomes for $\mathcal{Z}_j^m(r,0)$

$$
\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{4m^2}{r}-r^2+4\left(j+\frac{1}{2}\right)\right]\ \mathcal{Z}_j^m(r,0)=0. \tag{12}
$$

The functions $\mathcal{Z}_j^m(r,\phi)$ are the analogous on the plane of the spherical harmonics $Y_{lm}(\theta,\phi)$ on the sphere. The orthonormality and completeness of the $\mathcal{Z}_j^m(r,\phi)$ is similar to that of $Y_j^m(\theta,\phi)$

$$
\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} d\phi \int_0^\infty r\,dr\ \ \mathcal{Z}_j^m(r,\phi)^*\ \mathcal{Z}_{j'}^{m'}(r,\phi) &= \delta_{j,j'}\ \ \delta_{m,m'}\,,\\
\sum_{j,m}\ \mathcal{Z}_j^m(r,\phi)^*\ \mathcal{Z}_j^m(r',\phi') &= \frac{\pi}{r}\ \delta(r-r')\ \delta(\phi-\phi')\,.
\end{aligned}
\tag{13}
$$

This means that $\{\mathcal{Z}_j^m(r,\phi)\}$ is a basis of the space $L^2(\mathbb{R}^2)$ with measure $d\mu(r,\phi) = r\,dr\,d\phi/\pi$ like $\{Y_j^m(\theta,\phi)\}$ is a basis of $L^2(S^2)$ with $d\Omega$.

Now we consider an abstract Hilbert space $\mathcal{H}$ supporting the $2j+1$ dimensional IR of $su(2)$ spanned by the eigenvectors of J and M (see Eq. (7))

$$J\,|j,m\rangle = j\,|j,m\rangle\,, \quad M\,|j,m\rangle = m\,|j,m\rangle\,, \qquad 2j \in \mathbb{N}, \quad |m| \leq j\,.$$

These vectors $|j,m\rangle$ constitute a basis of $\mathcal{H}$ verifying the properties of orthogonality and completeness

$$\langle j,m|j',m'\rangle = \delta_{j,j'}\,\delta_{m,m'}\,, \qquad \sum_{j=0}^{\infty}\sum_{m=-j}^{j} |j,m\rangle\langle j,m| = I.$$

Any $|f\rangle \in \mathcal{H}$ may be written as $|f\rangle = \sum_{j=0}^{\infty}\sum_{m=-j}^{j} f_{j,m}\,|j,m\rangle$ if and only if

$$\sum_{j=0}^{\infty}\sum_{m=-j}^{j} |f_{j,m}|^2 < \infty\,, \qquad f_{l,m} = \langle l,m|f\rangle\,. \tag{14}$$

A canonical injection $S : \mathcal{H} \to L^2(\mathbb{R}^2)$ can be defined by $|j,m\rangle \to \mathcal{Z}_j^m(r,\phi)$ and extended by linearity and continuity to the whole $\mathcal{H}$. One can easily check that S is unitary. For any $|f\rangle \in \mathcal{H}$ we have the following expression

$$S|f\rangle = \sum_{j=0}^{\infty}\sum_{m=-j}^{j} f_{j,m}\,S\,|j,m\rangle = \sum_{j=0}^{\infty}\sum_{m=-j}^{j} f_{j,m}\,\mathcal{Z}_j^m(r,\phi)\,.$$

We now introduce a continuous basis, $\{|r,\phi\rangle\}$, depending on the values of the variables r and ϕ with the help of the discrete basis $\{|j,m\rangle\}$ by

$$\langle r,\phi|j,m\rangle := \mathcal{Z}_j^m(r,\phi)\,. \tag{15}$$

In reality, because of the different cardinality of r and j, we are not dealing with a Hilbert space but with a rigged Hilbert space (see next Section). The $\mathcal{Z}_j^m(r,\phi)$ can be seen as the transformation matrices from the irreducible representation states $\{|j,m\rangle\}$ to the localized states in the plane $\{|r,\phi\rangle\}$, like $Y_j^m(\theta,\phi) = \langle j,m|\theta,\phi\rangle$ are the corresponding ones to the localized states $\{|\theta,\phi\rangle\}$ in the sphere [7, 12]. Indeed

$$|j,m\rangle = \frac{1}{\pi}\int_{\mathbb{R}^2} |r,\phi\rangle\mathcal{Z}_j^m(r,\phi) r\,dr\,d\phi, \quad |j,m\rangle = \int_{S^2} |\theta,\phi\rangle\sqrt{j+1/2}\,Y_j^m(\theta,\phi)\,d\Omega.$$

We continue with the analogy and, from $K_\pm$ and K_3 (10), we define

$$J_\pm := e^{\pm i\phi}\,K_\pm, \qquad J_3 := K_3, \tag{16}$$

with act on the $\mathcal{Z}_j^m(r, \phi)$ as

$$\begin{aligned} J_+ \;\; \mathcal{Z}_j^m(r,\phi) &= \sqrt{(j-m)(j+m+1)}\;\; \mathcal{Z}_j^{m+1}(r,\phi), \\ J_- \;\; \mathcal{Z}_j^m(r,\phi) &= \sqrt{(j+m)(j-m+1)}\;\; \mathcal{Z}_j^{m-1}(r,\phi), \\ J_3 \;\; \mathcal{Z}_j^m(r,\phi) &= m\; \mathcal{Z}_j^m(r,\phi)\,. \end{aligned} \tag{17}$$

The functions $\mathcal{Z}_j^m(r, \phi)$ with j fixed and $|m| \leq j$, are orthonormal and determine the representation of dimension $2j + 1$ of $su(2)$ as it happens for the $Y_j^m(\theta, \phi)$. However there is a essential difference between the operators $\{J_\pm, J_3\}$ that act on the sphere S^2 that are true generators of $su(2)$ and the $\{J_\pm, J_3\}$ of (16), defined in $\mathbb{R}^2$, that do not close a Lie algebra. Indeed, when we calculate the commutator $[J_+, J_-]$ in terms of the differential operators defined in the Eqs. (10) and (16), we obtain $[J_+, J_-] = 2\, J_3 + \dfrac{8}{R^2}\, J_3\, E$, and only when $E \equiv 0$, i.e. only in the unitary space $L^2(\mathbb{R}^2)$, the $su(2)$ algebra is recovered. On the other hand, E is related to the $su(2)$ Casimir $\mathcal{C}$

$$E \;=\; -\frac{R^2}{4J_3^2+1}\;[\mathcal{C} - J(J+1)] \;\equiv\; -\frac{R^2}{4J_3^2+1}\left[J_3^2 + \frac{1}{2}\,\{J_+, J_-\}\; - J(J+1)\right],$$

so equation $E = 0$ is equivalent to the $su(2)$ Casimir condition $\mathcal{C} - J(J + 1) = 0$, that entails the usual Lie algebra in each $su(2)$ representation space.

4 Rigged Hilbert Space Formulation

A RHS (or Gelf'and triplet) is a triplet of spaces $\Phi \subset \mathcal{H} \subset \Phi^\times$, where $\mathcal{H}$ is an infinite dimensional separable Hilbert space, Φ is a dense subspace of $\mathcal{H}$ endowed with its own topology, and $\Phi^\times$ is the dual (or the antidual) space of Φ [9, 13, 14]. The topology considered on Φ is finer (contains more open sets) than the topology that Φ has as subspace of $\mathcal{H}$, and $\Phi^\times$ is equipped with a topology compatible with the dual pair $(\Phi, \Phi^\times)$ [15], usually the weak topology. The topology of Φ [16, 17] allows that all sequences which converge on Φ, also converge on $\mathcal{H}$ but the converse is not true. The difference between topologies gives rise that $\Phi^\times$ is bigger than $\mathcal{H}$, which is self-dual.

Here, any $F \in \Phi^\times$ is a continuous linear mapping from Φ into $\mathbb{C}$.

An essential property is that if A is a densely defined operator on $\mathcal{H}$, such that Φ be a subspace of its domain and that $A\varphi \in \Phi$ for all $\varphi \in \Phi$, we say that Φ reduces A or that Φ is invariant under the action of A, (i.e., $A\Phi \subset \Phi$). Then A may be extended unambiguously to $\Phi^\times$ by the duality formula

$$\langle A^\times\, F|\varphi\rangle := \langle F|A\varphi\rangle\,, \qquad \forall\, \varphi \in \Phi\,, \;\; \forall\, F \in \Phi^\times\,. \tag{18}$$

Moreover if A is continuous on Φ, then $A^\times$ is continuous on $\Phi^\times$.

The topology on Φ is given by an infinite countable set of norms $\{||-||_{n=1}^{\infty}\}$. A linear operator A on Φ is continuous if and only if for each norm $||-||_n$ there is a $K_n > 0$ and a finite sequence of norms $||-||_{p_1}, ||-||_{p_2}, \dots, ||-||_{p_r}$ such that for any $\varphi \in \Phi$, one has [18]

$$||A\varphi||_n \leq K_n \left(||\varphi||_{p_1} + ||\varphi||_{p_2} + \cdots + ||\varphi||_{p_r}\right) . \tag{19}$$

Now let us go to define and use the RHS $\mathfrak{G} \subset \mathcal{H} \subset \mathfrak{G}^{\times}$ where discrete and continuous bases coexist and the meaningful operators are well defined and continuous. Since we have a representation in terms of the $\mathcal{Z}_j^m(r,\phi)$, it would be more convenient to start with an equivalent RHS $\mathfrak{D} \subset L^2(\mathbb{R}^2) \subset \mathfrak{D}^{\times}$, such as $\mathfrak{D}$ is a test functions space with $f(r,\phi) \in L^2(\mathbb{R}^2)$, which therefore admit the span

$$f(r,\phi) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} f_{j,m}\, \mathcal{Z}_j^m(r,\phi) , \tag{20}$$

where the series converges in the sense of the norm in $L^2(\mathbb{R}^2)$. A necessary and sufficient condition for it is $\sum_{j=0}^{\infty} \sum_{m=-j}^{j} |f_{j,m}|^2 < \infty$. Thus, from (20), we define $\mathfrak{D}$ as the space of functions $f(r,\phi)$ in $L^2(\mathbb{R}^2)$ such that

$$||f(r,\phi)||_n^2 := \sum_{j=0}^{\infty} \sum_{m=-j}^{j} (j + |m| + 1)^{2n}\, |f_{j,m}|^2 < \infty , \qquad n = 0, 1, 2, \dots . \tag{21}$$

Obviously, all the finite linear combinations of the $\mathcal{Z}_j^m(r,\phi)$ are in $\mathfrak{D}$, hence $\mathfrak{D}$ is dense in $L^2(\mathbb{R}^2)$. Thus, the family of norms $||-||_n$ on $\mathfrak{D}$ (21) gives a topology such that $\mathfrak{D}$ is a Frèchet space (metrizable and complete). Since for $n = 0$ we have the Hilbert space norm, the canonical injection from $\mathfrak{D}$ into $L^2(\mathbb{R}^2)$ is continuous.

Because j goes from 0 to ∞, the operators $J_\pm$, J_3 are all unbounded and, therefore, their respective domains are densely defined on $L^2(\mathbb{R}^2)$, but not on the whole $L^2(\mathbb{R}^2)$. We can prove that all these operators are defined on the whole $\mathfrak{D}$ and are continuous with the topology on $\mathfrak{D}$. The proof is simple and it is essentially the same for all operators. As an example, let us give the proof for J_+. For any function f in $\mathfrak{D}$, we have $J_+ f$, i.e.,

$$J_+ \sum_{j=0}^{\infty} \sum_{m=-j}^{j} f_{j,m}\, \mathcal{Z}_j^m(r,\phi) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} f_{j,m} \sqrt{(j-m)(j+m+1)}\, \mathcal{Z}_j^{m+1}(r,\phi) .$$

To show that $J_+ f \in \mathfrak{D}$ we have to prove that for any $n \in \mathbb{N}$, it satisfies (21). So taking into account the shift on the index m (17) we have

$$\sum_{j=0}^{\infty}\sum_{m=-j}^{j} |f_{j,m}|^2\,(j-m)(j+m+1)\,(j+1+|m|+1)^{2n}\,. \tag{22}$$

The following two inequalities are straightforward:

$$(j-m)(j+m+1) \le (j+|m|+1)^2\,, \qquad (j+1+|m|+1)^{2n} \le 2^{2n}\,(j+|m|+1)^{2n}\,.$$

Using these inequalities we see that (22) is bounded by

$$2^{2n}\sum_{j=0}^{\infty}\sum_{m=-j}^{j} |f_{j,m}|^2\,(j+1+|m|+1)^{2n+2}\,, \tag{23}$$

which converges after (21). Hence, $J_+ f \in \mathfrak{D}$. In order to show the continuity of J_+ on $\mathfrak{D}$, we use (19). Thus, applying J_+ to any $f(r,\phi) \in \mathfrak{D}$ we get

$$||J_+ f(r,\phi)||_n^2 \le 2^{2n}\,||f(r,\phi)||_{n+1}^2 \Longrightarrow ||J_+ f(r,\phi)||_n \le 2^n\,||f(r,\phi)||_{n+1}\,,$$

which satisfies (19) for all $n = 0, 1, 2, \ldots$. Hence, the continuity of J_+ on $\mathfrak{D}$ has been proved. By means of the duality formula, we extend J_+ to a weakly continuous operator on $\mathfrak{D}^\times$. Same properties can be proved for J_- and J_3.

Now we are able to define the abstract RHS $\mathfrak{G} \subset \mathcal{H} \subset \mathfrak{G}^\times$ using the unitary mapping $S : \mathcal{H} \to L^2(\mathbb{R}^2)$ introduced in the previous section. Thus, we define $\mathfrak{G} := S^{-1}\mathfrak{D}$. Hence the topology on $\mathfrak{G}$ is the transported topology from $\mathfrak{D}$ by S, so that if $f \in \mathfrak{G}$, the semi-norms are

$$||f||_n^2 = \sum_{j=0}^{\infty}\sum_{m=-j}^{j} (j+|m|+1)^{2n}\,|f_{j,m}|^2 < \infty\,, \qquad n = 0, 1, 2, \ldots\,.$$

The topology on $\mathfrak{G}$ uniquely defines $\mathfrak{G}^\times$. Moreover there exists a one-to-one continuous mapping from $\mathfrak{G}$ onto $\mathfrak{D}$ with continuous inverse. It is given by an extension, $\widetilde{S}$, of S defined via the duality formula $\langle \widetilde{S} f | \widetilde{S} F\rangle = \langle f | F\rangle$, with $f \in \mathfrak{G}$ and $F \in \mathfrak{G}^\times$.

On the other hand, if an operator O satisfies $O\mathfrak{D} \subset \mathfrak{D}$ with continuity, the same property works for $\widehat{O} = S^{-1} O S$ on $\mathfrak{G}$.

5 Conclusions

Starting from the recurrence relations (3) we obtained the operators $\{J_\pm, J_3\}$ (16). Their general linear algebra is not a Lie algebra. However its representation on $L^2(\mathbb{R}^2)$, characterized by the eigenvalue zero of the operator E, is isomorphic to the regular representation $\{|j, m\rangle\}$ of $su(2)$ and it has therefore a stronger symmetry than the general linear operator structure itself.

We are used in Lie algebra theory to representations that preserve the symmetry of the algebra and to algebras that have the same symmetry of the space where the representation is defined. This is exactly what happens with the spherical harmonics, that are solution of Laplace equation and, thus, have the same intrinsic symmetry of the group $SU(2)$ of which they are representation bases. However, here the situation is different since we represent $SU(2)$ in the plane $\mathbb{R}^2$ which geometry preserves only the subgroup $SO(2)$ of $SU(2)$. Indeed $\{J_\pm, J_3\}$ (16) are defined for arbitrary E, but they generate $su(2)$ only under the assumption $E \equiv 0$, i.e. when we restrict ourselves to functions f verifying the Casimir condition $\mathcal{C} f = J(J+1) f$,, i.e. that belong to $L^2(\mathbb{R}^2)$.

Reversing the connection, the representations of a Lie algebra have been related not only to the Lie algebra itself but also to a set of operators that do not close a Lie algebra in an universal way but reduce to a Lie algebra only when applied to well defined vector spaces.

This paper offers a method to introduce representations of Lie groups in spaces that are not symmetric under the group action and in situations where the general linear group of operators is not a Lie group in a universal way.

We have also constructed two RHS ($\mathfrak{G} \subset \mathcal{H} \subset \mathfrak{G}^\times$ and $\mathfrak{D} \subset L^2(\mathbb{R}^2) \subset \mathfrak{D}^\times$) supporting two UIR of $SU(2)$, the first one is related with the discrete basis $\{|j, m\rangle\}$ and the other RHS with the continuous one $\{|r, \phi\rangle\}$. Both are related by the unitary map $S : |j, m\rangle \to \mathcal{Z}_j^m(r, \phi)$ that also transports the topologies of the first RHS and other properties to the second RHS.

Acknowledgements Partial financial support is acknowledged to the Junta de Castilla y León and FEDER (Project VA057U16) and MINECO of Spain (Project MTM2014-57129-C2-1-P).

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Part VI
Nonrelativistic and Classical Theories

Metage Symmetry Group of Non-barotropic Magnetohydrodynamics and the Conservation of Cross Helicity

Asher Yahalom

Abstract Standard cross helicity is not conserved in non-barotropic magnetohydrodynamics (MHD) (as opposed to barotropic or incompressible MHD). It was shown that a new kind of cross helicity which is conserved in the non barotropic case can be introduced. The non barotropic cross helicity reduces to the standard cross helicity under barotropic assumptions. Here we show that the new cross helicity can be deduced from a variational principle using the Noether's theorem. The symmetry group associated with the new cross helicity is related to translation in a labelling of the flow elements connected to the magnetic field lines known as magnetic metage.

Keywords Symmetry group · Magnetohydrodynamics · Topological conservation laws · Metage · Cross helicity

1 Introduction

The theorem of Noether dictates that for every continuous symmetry group of an Action the system must possess a conservation law. For example time translation symmetry results in the conservation of energy, while spatial translation symmetry results in the conservation of linear momentum and rotation symmetry in the conservation of angular momentum to list some well known examples. But sometimes the conservation law is discovered without reference to the Noether theorem by using the equations of the system. In that case one is tempted to inquire what is the hidden symmetry associated with this conservation law and what is the simplest way to represent it.

The concept of metage as a label for fluid elements along a vortex line in ideal fluids was first introduced by Lynden-Bell and Katz [1]. A translation group of this label was found to be connected to the conservation of Moffat's [2] helicity by Yahalom [3]. The concept of metage was later generalized by Yahalom and Lynden-Bell [4] for barotropic MHD, but now as a label for fluid elements along magnetic

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_30

field lines which are comoving with the flow in the case of ideal MHD. Yahalom and Lynden-Bell [4] have also shown that the translation group of the magnetic metage is connected to Woltjer [5, 6] conservation of cross helicity for barotropic MHD. Recently the concept of metage was generalized also for non barotropic MHD in which magnetic field lines lie on entropy surfaces [7]. This will be generalized in this paper by dropping the entropy condition on magnetic field lines.

Cross Helicity was first described by Woltjer [5, 6] and is give by:

$$H_C \equiv \int \mathbf{B} \cdot \mathbf{v} d^3 x, \tag{1}$$

in which $\mathbf{B}$ is the magnetic field, $\mathbf{v}$ is the velocity field and the integral is taken over the entire flow domain. H_C is conserved for barotropic or incompressible MHD and is given a topological interpretation in terms of the knottiness of magnetic and flow field lines. A generalization of barotropic fluid dynamics conserved quantities including helicity to non barotropic flows including topological constants of motion is given by Mobbs [13]. However, Mobbs did not discuss the MHD case.

Both conservation laws for the helicity in the fluid dynamics case and the barotropic MHD case were shown to originate from a relabelling symmetry through the Noether theorem [3, 4, 8, 9]. Webb et al. [10] have generalized the idea of relabelling symmetry to non-barotropic MHD and derived their generalized cross helicity conservation law by using Noether's theorem but without using the simple representation which is connected with the metage variable. The conservation law deduction involves a divergence symmetry of the action. These conservation laws were written as Eulerian conservation laws of the form $D_t + \nabla \cdot \mathbf{F} = 0$ where D is the conserved density and F is the conserved flux. Webb et al. [11] discuss the cross helicity conservation law for non-barotropic MHD in a multi-symplectic formulation of MHD. Webb et al. [10, 11] emphasize that the generalized cross helicity conservation law, in MHD and the generalized helicity conservation law in non-barotropic fluids are non-local in the sense that they depend on the auxiliary nonlocal variable σ, which depends on the Lagrangian time integral of the temperature $T(x, t)$. Notice that a potential vorticity conservation equation for non-barotropic MHD is derived by Webb, G. M. and Mace, R.L. [12] by using Noether's second theorem.

It should be mentioned that Mobbs [13] derived a helicity conservation law for ideal, non-barotropic fluid dynamics, which is of the same form as the cross helicity conservation law for non-barotropic MHD, except that the magnetic field induction is replaced by the generalized fluid helicity $\mathbf{\Omega} = \nabla \times (\mathbf{v} - \sigma \nabla s)$. Webb et al. [10, 11] also derive the Eulerian, differential form of Mobbs [13] conservation law (although they did not reference Mobbs [13]). Webb and Anco [14] show how Mobbs conservation law arises in multi-symplectic, Lagrangian fluid mechanics.

Variational principles for magnetohydrodynamics were introduced by previous authors both in Lagrangian and Eulerian form. Sturrock [15] has discussed in his book a Lagrangian variational formalism for magnetohydrodynamics. Vladimirov and Moffatt [16] in a series of papers have discussed an Eulerian variational principle for incompressible magnetohydrodynamics. However, their variational principle contained three more functions in addition to the seven variables which appear in the standard equations of incompressible magnetohydrodynamics which are the magnetic field **B** the velocity field **v** and the pressure P. Kats [17] has generalized Moffatt's work for compressible non barotropic flows but without reducing the number of functions and the computational load. Sakurai [18] has introduced a two function Eulerian variational principle for force-free magnetohydrodynamics and used it as a basis of a numerical scheme, his method is discussed in a book by Sturrock [15]. Yahalom and Lynden-Bell [4] combined the Lagrangian of Sturrock [15] with the Lagrangian of Sakurai [18] to obtain an **Eulerian** Lagrangian principle for barotropic magnetohydrodynamics which depends on only six functions. The variational derivative of this Lagrangian produced all the equations needed to describe barotropic magnetohydrodynamics without any additional constraints. The equations obtained resembled the equations of Frenkel, Levich and Stilman [30] (see also [19]). Yahalom [32] have shown that for the barotropic case four functions will suffice. Moreover, it was shown that the cuts of some of those functions [20] are topological local conserved quantities.

Previous work was concerned only with barotropic magnetohydrodynamics. Variational principles of non barotropic magnetohydrodynamics can be found in the work of Bekenstein and Oron [21] in terms of 15 functions and V.A. Kats [17] in terms of 20 functions. Morrison [22] has suggested a Hamiltonian approach but this also depends on 8 canonical variables (see table 2 [22]). The variational principle introduced in [23, 24] show that only five functions will suffice to describe non barotropic MHD in the case that we enforce a Sakurai [18] representation for the magnetic field (see also [29] for the stationary case).

The plan of this paper is as follows: First we introduce the basic quantities and equations of non-barotropic MHD. Then we describe the concept of magnetic metage for non-barotropic MHD. This is followed by a description of a Lagrangian variational principle for non-barotropic MHD. Finally we derive a non-barotropic cross helicity conservation law for non-barotropic MHD using Noether's theorem.

2 Basic Equations

Consider the equations of non-barotropic MHD [15, 23]:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{4}$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla p(\rho, s) + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi}, \tag{5}$$

$$\frac{ds}{dt} = 0. \tag{6}$$

In the above the following notations are utilized: $\frac{\partial}{\partial t}$ is the temporal derivative, $\frac{d}{dt}$ is the temporal material derivative and ∇ has its standard meaning in vector calculus. ρ is the fluid density and s is the specific entropy. Finally $p(\rho, s)$ is the pressure which depends on the density and entropy (the non-barotropic case). Equation (2) describes the fact that the magnetic field lines are moving with the fluid elements ("frozen" magnetic field lines), Eq. (3) describes the fact that the magnetic field is solenoidal, Eq. (4) describes the conservation of mass and Eq. (5) is the Euler equation for a fluid in which both pressure and Lorentz magnetic forces apply. Equation (6) describes the fact that heat is not created (zero viscosity, zero resistivity) in ideal non-barotropic MHD and is not conducted, thus only convection occurs. The number of independent variables for which one needs to solve is eight ($\mathbf{v}$, $\mathbf{B}$, ρ, s) and the number of equations (2), (4), (5), (6) is also eight. Notice that Eq. (3) is a condition on the initial $\mathbf{B}$ field and is satisfied automatically for any other time due to Eq. (2).

In non-barotropic MHD one can calculate the temporal derivative of the cross helicity (1) using the above equations and obtain:

$$\frac{dH_C}{dt} = \int T \nabla s \cdot \mathbf{B} d^3x, \tag{7}$$

in which T is the temperature. Hence, generally speaking cross helicity is not conserved.

3 Load and Metage

The following section follows closely a similar section in [4, 31]. Consider a thin tube surrounding a magnetic field line as described in Fig. 1,

the magnetic flux contained within the tube is:

$$\Delta\Phi = \int \mathbf{B} \cdot d\mathbf{S} \tag{8}$$

and the mass contained with the tube is:

$$\Delta M = \int \rho d\mathbf{l} \cdot d\mathbf{S}, \tag{9}$$

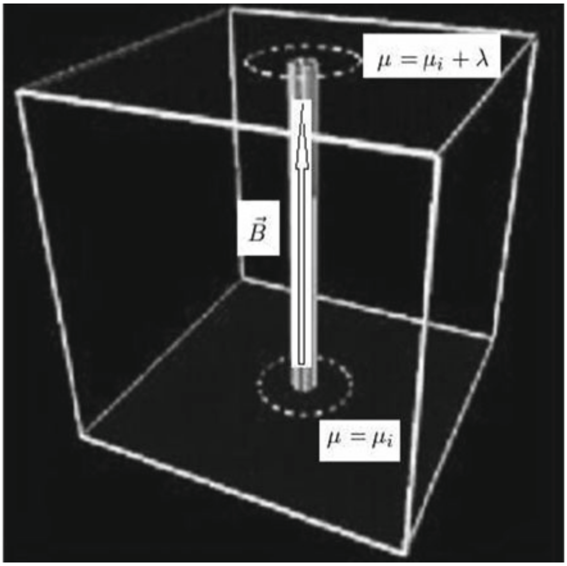

Fig. 1 A thin tube surrounding a magnetic field line

in which dl is a length element along the tube. Since the magnetic field lines move with the flow by virtue of Eqs. (2) and (4) both the quantities $\Delta\Phi$ and ΔM are conserved and since the tube is thin we may define the conserved magnetic load:

$$\lambda = \frac{\Delta M}{\Delta\Phi} = \oint \frac{\rho}{B} dl, \tag{10}$$

in which the above integral is performed along the field line. Obviously the parts of the line which go out of the flow to regions in which $\rho = 0$ have a null contribution to the integral. Notice that λ is a **single valued** function that can be measured in principle. Since λ is conserved it satisfies the equation:

$$\frac{d\lambda}{dt} = 0. \tag{11}$$

By construction surfaces of constant magnetic load move with the flow and contain magnetic field lines. Hence the gradient to such surfaces must be orthogonal to the field line:

$$\boldsymbol{\nabla}\lambda \cdot \mathbf{B} = 0. \tag{12}$$

Now consider an arbitrary comoving point on the magnetic field line and denote it by i, and consider an additional comoving point on the magnetic field line and denote it by r. The integral:

$$\mu(r) = \int_i^r \frac{\rho}{B} dl + \mu(i), \tag{13}$$

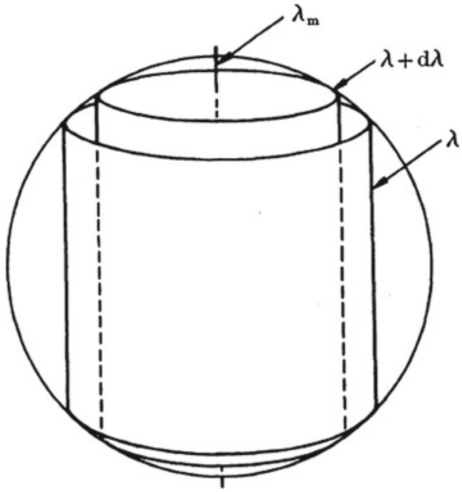

Fig. 2 Surfaces of constant load

is also a conserved quantity which we may denote following Lynden-Bell and Katz [1] as the magnetic metage. $\mu(i)$ is an arbitrary number which can be chosen differently for each magnetic line. By construction:

$$\frac{d\mu}{dt} = 0. \tag{14}$$

Also it is easy to see that by differentiating along the magnetic field line we obtain:

$$\nabla\mu \cdot \mathbf{B} = \rho. \tag{15}$$

Notice that μ will be generally a **non single valued** function, we will show later in this paper that symmetry to translations in μ; will generate through the Noether theorem the conservation of the magnetic cross helicity.

At this point we have two comoving coordinates of flow, namely λ, μ obviously in a three dimensional flow we also have a third coordinate. However, before defining the third coordinate we will find it useful to work not directly with λ but with a function of λ. Now consider the magnetic flux within a surface of constant load $\Phi(\lambda)$ as described in Fig. 2 (the figure was given by Lynden-Bell and Katz [1]). The magnetic flux is a conserved quantity and depends only on the load λ of the surrounding surface. Now we define the quantity:

$$\chi = \frac{\Phi(\lambda)}{2\pi}. \tag{16}$$

Obviously χ satisfies the equations:

$$\frac{d\chi}{dt} = 0, \qquad \mathbf{B} \cdot \nabla \chi = 0. \tag{17}$$

Let us now define an additional comoving coordinate η^* since $\nabla \mu$ is not orthogonal to the $\mathbf{B}$ lines we can choose $\nabla \eta^*$ to be orthogonal to the $\mathbf{B}$ lines and not be in the direction of the $\nabla \chi$ lines, that is we choose η^* not to depend only on χ. Since both $\nabla \eta^*$ and $\nabla \chi$ are orthogonal to $\mathbf{B}$, $\mathbf{B}$ must take the form:

$$\mathbf{B} = A \nabla \chi \times \nabla \eta^*. \tag{18}$$

However, using Eq. (3) we have:

$$\nabla \cdot \mathbf{B} = \nabla A \cdot (\nabla \chi \times \nabla \eta^*) = 0. \tag{19}$$

Which implies that A is a function of χ, η^*. Now we can define a new comoving function η such that:

$$\eta = \int_0^{\eta^*} A(\chi, \eta^{'*}) d\eta^{'*}, \qquad \frac{d\eta}{dt} = 0. \tag{20}$$

In terms of this function we obtain the Sakurai (Euler potentials) presentation:

$$\mathbf{B} = \nabla \chi \times \nabla \eta. \tag{21}$$

And the density is now given by the Jacobian:

$$\rho = \nabla \mu \cdot (\nabla \chi \times \nabla \eta) = \frac{\partial(\chi, \eta, \mu)}{\partial(x, y, z)}. \tag{22}$$

It can easily be shown using the fact that the labels are comoving that the above forms of $\mathbf{B}$ and ρ satisfy Eqs. (2), (3) and (4) automatically.

Notice however, that η is defined in a non unique way since one can redefine η for example by performing the following transformation: $\eta \to \eta + f(\chi)$ in which $f(\chi)$ is an arbitrary function. The comoving coordinates χ, η serve as labels of the magnetic field lines. Moreover the magnetic flux can be calculated as:

$$\Phi = \int \mathbf{B} \cdot d\mathbf{S} = \int d\chi d\eta. \tag{23}$$

In the case that the surface integral is performed inside a load contour we obtain:

$$\Phi(\lambda) = \int_\lambda d\chi d\eta = \chi \int_\lambda d\eta = \begin{cases} \chi[\eta] \\ \chi(\eta_{max} - \eta_{min}) \end{cases} \tag{24}$$

There are two cases involved; in one case the load surfaces are topological cylinders; in this case η is not single valued and hence we obtain the upper value for $\Phi(\lambda)$. In a second case the load surfaces are topological spheres; in this case η is single valued and has minimal η_{min} and maximal η_{max} values. Hence the lower value of $\Phi(\lambda)$ is obtained. For example in some cases η is identical to twice the latitude angle θ. In those cases $\eta_{min} = 0$ (value at the "north pole") and $\eta_{max} = 2\pi$ (value at the "south pole").

Comparing the above equation with Eq. (16) we derive that η can be either **single valued** or **not single valued** and that its discontinuity across its cut in the non single valued case is $[\eta] = 2\pi$.

The triplet χ, η, μ will suffice to label any fluid element in three dimensions. But for a non-barotropic flow there is also another label s which is comoving according to Eq. (6). The question then arises of the relation of this label to the previous three. As one needs to make a choice regarding the preferred set of labels it seems that the physical ones are χ, η, s in which we use the surfaces on which the magnetic fields lie and the entropy, each label has an obvious physical interpretation. In this case we must look at μ as a function of χ, η, s. If the magnetic field lines lie on entropy surface then μ regains its status as an independent label. The density can now be written as:

$$\rho = \frac{\partial \mu}{\partial s} \frac{\partial(\chi, \eta, s)}{\partial(x, y, z)}. \tag{25}$$

Now as μ can be defined for each magnetic field line separately according to Eq. (13) it is obvious that such a choice exist in which μ is a function of s only. One may also think of the entropy s as a functions χ, η, μ. However, if one change μ in this case this generally entails a change in s and the symmetry described in Eq. (13) is lost.

4 Lagrangian variational principle of MHD

A Lagrangian variational principle for barotropic MHD has been discussed by a number of authors (see for example [4, 15]), we repeat the derivation with the necessary modifications which are required for the non-barotropic case. Consider the action:

$$A \equiv \int \mathcal{L} d^3x dt,$$
$$\mathcal{L} \equiv \rho \left(\frac{1}{2} \mathbf{v}^2 - \varepsilon(\rho, s) \right) - \frac{\mathbf{B}^2}{8\pi}, \tag{26}$$

In the above ε is the specific internal energy (internal energy per unit of mass). The reader is reminded of the following thermodynamic relations which will become useful later:

$$
\begin{aligned}
d\varepsilon &= Tds - Pd\frac{1}{\rho} = Tds + \frac{P}{\rho^2}d\rho \\
\frac{\partial \varepsilon}{\partial s} &= T, \qquad \frac{\partial \varepsilon}{\partial \rho} = \frac{P}{\rho^2} \\
w &= \varepsilon + \frac{P}{\rho} = \varepsilon + \frac{\partial \varepsilon}{\partial \rho}\rho = \frac{\partial(\rho\varepsilon)}{\partial \rho} \\
dw &= d\varepsilon + d\left(\frac{P}{\rho}\right) = Tds + \frac{1}{\rho}dP
\end{aligned} \tag{27}
$$

in the above T is the temperature and w is the specific enthalpy. A variation in any quantity F for a fixed position $\mathbf{r}$ is denoted as δF hence:

$$
\begin{aligned}
\delta A &= \int \delta \mathcal{L} d^3x dt, \\
\delta \mathcal{L} &= \delta\rho\left(\frac{1}{2}\mathbf{v}^2 - w(\rho)\right) - \rho T \delta s + \rho \mathbf{v} \cdot \delta \mathbf{v} - \frac{\mathbf{B} \cdot \delta \mathbf{B}}{4\pi},
\end{aligned} \tag{28}
$$

A change in a position of a fluid element located at a position $\mathbf{r}$ at time t is given by $\boldsymbol{\xi}(\mathbf{r}, t)$. A change involving both a local variation coupled with a change of element position of the quantity F is given by:

$$
\Delta F = \delta F + (\boldsymbol{\xi} \cdot \nabla)F, \tag{29}
$$

hence

$$
\Delta \mathbf{v} = \delta \mathbf{v} + (\boldsymbol{\xi} \cdot \nabla)\mathbf{v}. \tag{30}
$$

However, since:

$$
\Delta \mathbf{v} = \Delta \frac{d\mathbf{r}}{dt} = \frac{d\Delta \mathbf{r}}{dt} = \frac{d\boldsymbol{\xi}}{dt}. \tag{31}
$$

We obtain:

$$
\delta \mathbf{v} = \frac{d\boldsymbol{\xi}}{dt} - (\boldsymbol{\xi} \cdot \nabla)\mathbf{v} = \frac{\partial \boldsymbol{\xi}}{\partial t} + (\mathbf{v} \cdot \nabla)\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla)\mathbf{v}. \tag{32}
$$

For any of the labels χ, η, μ a change in a specific spatial location is only possible by the displacement of the fluid element to a new position. However, if one takes into account both the spatial change in value and change due to the displacement then obviously the total change is zero as each fluid element retains its labels. Hence:

$$
\begin{aligned}
\Delta \chi &= \delta \chi + (\boldsymbol{\xi} \cdot \nabla)\chi = 0 \Rightarrow \delta \chi = -(\boldsymbol{\xi} \cdot \nabla)\chi, \\
\Delta \eta &= \delta \eta + (\boldsymbol{\xi} \cdot \nabla)\eta = 0 \Rightarrow \delta \eta = -(\boldsymbol{\xi} \cdot \nabla)\eta, \\
\Delta \mu &= \delta \mu + (\boldsymbol{\xi} \cdot \nabla)\mu = 0 \Rightarrow \delta \mu = -(\boldsymbol{\xi} \cdot \nabla)\mu,
\end{aligned} \tag{33}
$$

Now since s is a comoving quantity depending only on the fluid element labels we have:

$$\Delta s = 0 \Rightarrow \delta s = -(\boldsymbol{\xi} \cdot \boldsymbol{\nabla})s. \tag{34}$$

Using Eqs. (22) and (33) we obtain a mass conserving variation of ρ:

$$\delta\rho = -\boldsymbol{\nabla} \cdot (\rho\boldsymbol{\xi}) \tag{35}$$

Using Eqs. (21) and (33) a magnetic flux conserving variation takes the form:

$$\delta\mathbf{B} = \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \mathbf{B}). \tag{36}$$

Introducing the result of Eqs. (32), (34), (35), (36) into (28) and integrating by parts we arrive at the result:

$$\begin{aligned}\delta A &= \int d^3x \rho \mathbf{v} \cdot \boldsymbol{\xi}|_{t_0}^{t_1} \\ &+ \int dt \left\{ \oint d\mathbf{S} \cdot \left[-\rho\boldsymbol{\xi}\left(\frac{1}{2}\mathbf{v}^2 - w(\rho)\right) + \rho\mathbf{v}(\mathbf{v}\cdot\boldsymbol{\xi}) + \frac{1}{4\pi}\mathbf{B} \times (\boldsymbol{\xi} \times \mathbf{B}) \right] \right. \\ &+ \left. \int d^3x \boldsymbol{\xi} \cdot \left[-\rho\boldsymbol{\nabla} w + \rho T \boldsymbol{\nabla} s - \frac{\partial(\rho\mathbf{v})}{\partial t} - \frac{\partial(\rho\mathbf{v}v_k)}{\partial x_k} - \frac{1}{4\pi}\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B}) \right] \right\},\end{aligned} \tag{37}$$

in which a summation convention is assumed. Taking into account the continuity equation (4) we obtain:

$$\begin{aligned}\delta A &= \int d^3x \rho \mathbf{v} \cdot \boldsymbol{\xi}|_{t_0}^{t_1} \\ &+ \int dt \left\{ \oint d\mathbf{S} \cdot \left[-\rho\boldsymbol{\xi}\left(\frac{1}{2}\mathbf{v}^2 - w(\rho)\right) + \rho\mathbf{v}(\mathbf{v}\cdot\boldsymbol{\xi}) + \frac{1}{4\pi}\mathbf{B} \times (\boldsymbol{\xi} \times \mathbf{B}) \right] \right. \\ &+ \left. \int d^3x \boldsymbol{\xi} \cdot \left[-\boldsymbol{\nabla} P - \rho\frac{\partial\mathbf{v}}{\partial t} - \rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} - \frac{1}{4\pi}\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B}) \right] \right\},\end{aligned} \tag{38}$$

hence we see that if $\delta A = 0$ for a $\boldsymbol{\xi}$ vanishing at the initial and final times and on the surface of the domain but otherwise arbitrary then Euler's equation (5) is satisfied (taking into account that $\boldsymbol{\nabla} w - T\boldsymbol{\nabla} s = \frac{\boldsymbol{\nabla} p}{\rho}$).

The Lagrangian density given in Eq. (26) does not admit a μ modification symmetry since we assume that the entropy $s = s(\chi, \eta, \mu)$ is a given function of the labels. This problem can be overcome by taking s as a dynamical variable and enforcing its conservation by using a Lagrange multiplier. In this approach the variational principle takes the form:

$$A \equiv \int \mathcal{L} d^3x dt,$$
$$\mathcal{L} \equiv \rho \left(\frac{1}{2}\mathbf{v}^2 - \varepsilon(\rho, s) \right) - \frac{\mathbf{B}^2}{8\pi} - \rho\sigma \frac{ds}{dt}, \tag{39}$$

A variation with respect to the Lagrange multiplier σ will obviously result in Eq. (6). A variation with respect to s will result in:

$$\delta_s A = \int d^3x dt \delta s \left[\frac{\partial(\rho\sigma)}{\partial t} + \nabla \cdot (\rho\sigma\mathbf{v}) - \rho T \right]$$
$$+ \int dt \oint d\mathbf{S} \cdot \rho\sigma\mathbf{v}\delta s - \int d^3x \rho\sigma\delta s|_{t_0}^{t_1}, \tag{40}$$

Taking into account the continuity equation (4) we obtain for locations in which the density ρ is not null the result:

$$\frac{d\sigma}{dt} = T, \tag{41}$$

provided that $\delta_s A$ vanished for an arbitrary δs. Now let us turn our attention to the variation with respect to the fluid element displacement which takes the form:

$$\delta A_{\boldsymbol{\xi}} = \int \delta\mathcal{L}_{\boldsymbol{\xi}} d^3x dt,$$
$$\delta\mathcal{L}_{\boldsymbol{\xi}} = \delta\rho \left(\frac{1}{2}\mathbf{v}^2 - w(\rho) \right) - \rho\sigma\delta\mathbf{v} \cdot \nabla s + \rho\mathbf{v} \cdot \delta\mathbf{v} - \frac{\mathbf{B} \cdot \delta\mathbf{B}}{4\pi}, \tag{42}$$

As most of the terms were calculated previously we will only calculate the term $-\rho\sigma\delta\mathbf{v} \cdot \nabla s$ which is equal to:

$$-\rho\sigma\delta\mathbf{v} \cdot \nabla s = \boldsymbol{\xi} \cdot \rho T \nabla s - \frac{\partial(\rho\sigma\nabla s \cdot \boldsymbol{\xi})}{\partial t} - \nabla \cdot (\rho\sigma(\nabla s \cdot \boldsymbol{\xi})\mathbf{v}). \tag{43}$$

The above result was obtained using Eqs. (32), (6) and (41). Hence the variation of the action with respect to a displacement of the fluid elements is:

$$\delta_{\boldsymbol{\xi}} A = \int d^3x \rho(\mathbf{v} - \sigma\nabla s) \cdot \boldsymbol{\xi}|_{t_0}^{t_1}$$
$$+ \int dt \left\{ \oint d\mathbf{S} \cdot \left[-\rho\boldsymbol{\xi} \left(\frac{1}{2}\mathbf{v}^2 - w(\rho) \right) + \rho\mathbf{v}\left((\mathbf{v} - \sigma\nabla s) \cdot \boldsymbol{\xi}\right) + \frac{1}{4\pi}\mathbf{B} \times (\boldsymbol{\xi} \times \mathbf{B}) \right] \right.$$
$$\left. + \int d^3x \boldsymbol{\xi} \cdot \left[-\rho\nabla w + \rho T \nabla s - \frac{\partial(\rho\mathbf{v})}{\partial t} - \frac{\partial(\rho\mathbf{v}v_k)}{\partial x_k} - \frac{1}{4\pi}\mathbf{B} \times (\nabla \times \mathbf{B}) \right] \right\}, \tag{44}$$

in which a summation convention is assumed. Taking into account the continuity equation (4) and the thermodynamic identities given in Eq. (27) we obtain:

$$
\begin{aligned}
&\delta_{\boldsymbol{\xi}} A = \int d^3 x \rho (\mathbf{v} - \sigma \nabla s) \cdot \boldsymbol{\xi}|_{t_0}^{t_1} \\
&+\int dt \left\{ \oint d\mathbf{S} \cdot \left[-\rho \boldsymbol{\xi} \left(\frac{1}{2}\mathbf{v}^2 - w(\rho) \right) + \rho \mathbf{v}((\mathbf{v} - \sigma \nabla s) \cdot \boldsymbol{\xi}) + \frac{1}{4\pi} \mathbf{B} \times (\boldsymbol{\xi} \times \mathbf{B}) \right] \right. \\
&\left. +\int d^3 x \boldsymbol{\xi} \cdot \left[-\nabla P - \rho \frac{\partial \mathbf{v}}{\partial t} - \rho (\mathbf{v} \cdot \nabla) \mathbf{v} - \frac{1}{4\pi} \mathbf{B} \times (\nabla \times \mathbf{B}) \right] \right\}, \qquad (45)
\end{aligned}
$$

Hence we obtain the correct dynamical equations for an arbitrary $\boldsymbol{\xi}$. Now suppose that the equations and boundary conditions hold. Then:

$$
\delta_{\boldsymbol{\xi}} A = \int d^3 x \rho (\mathbf{v} - \sigma \nabla s) \cdot \boldsymbol{\xi}|_{t_0}^{t_1} \qquad (46)
$$

If in addition $\boldsymbol{\xi}$ is a small symmetry displacement such that $\delta_{\boldsymbol{\xi}} A = 0$ we obtained a conserved Noether current:

$$
\delta J = \int d^3 x \rho (\mathbf{v} - \sigma \nabla s) \cdot \boldsymbol{\xi} \qquad (47)
$$

5 Non Barotropic Cross Helicity Conservation via the Noether Theorem

It is obvious that the choice of fluid labels is quite arbitrary. However, when enforcing the χ, η, μ coordinate system such that:

$$
\rho = \frac{\partial(\chi, \eta, \mu)}{\partial(x, y, z)}. \qquad (48)
$$

The choice is restricted to $\tilde{\chi}, \tilde{\eta}, \tilde{\mu}$:

$$
\frac{\partial(\tilde{\chi}, \tilde{\eta}, \tilde{\mu})}{\partial(\chi, \eta, \mu)} = 1. \qquad (49)
$$

Moreover the Euler potential magnetic field representation:

$$
\mathbf{B} = \nabla \chi \times \nabla \eta, \qquad (50)
$$

reduces the choice further to:

$$
\frac{\partial(\tilde{\chi}, \tilde{\eta})}{\partial(\chi, \eta)} = 1. \qquad (51)
$$

In what follows we consider the transformation (see also Eq. (13)):

$$\tilde{\chi} = \chi, \tilde{\eta} = \eta, \tilde{\mu} = \mu + a(\chi, \eta) \tag{52}$$

Hence a is a label displacement which may be different for each magnetic field line, as the field line is closed one need not worry about edge difficulties. This transformation satisfies trivially the conditions (49), (51). If $a = \delta\mu$ is small we can use Eq. (33) to calculate the associated fluid element displacement with this relabelling.

$$\boldsymbol{\xi} = -\frac{\partial \mathbf{r}}{\partial \mu}\delta\mu = -\delta\mu\frac{\mathbf{B}}{\rho}. \tag{53}$$

Inserting this expression into the boundary term in Eq. (45) will result in:

$$\delta A_B = \int dt \oint d\mathbf{S} \cdot \left[\mathbf{B}\left(\frac{1}{2}\mathbf{v}^2 - w(\rho)\right) - \mathbf{v}((\mathbf{v} - \sigma\boldsymbol{\nabla}s)\cdot\mathbf{B})\right]\delta\mu = 0, \tag{54}$$

which is the condition for magnetic cross helicity conservation. Inserting Eq. (53) into (47) we obtain the conservation law:

$$\delta J = \int d^3x\rho(\mathbf{v} - \sigma\boldsymbol{\nabla}s)\cdot\boldsymbol{\xi} = -\int d^3x\delta\mu(\mathbf{v} - \sigma\boldsymbol{\nabla}s)\cdot\mathbf{B} \tag{55}$$

In the simplest case we may take $\delta\mu$ to be a small constant, hence:

$$\delta J = -\delta\mu\int d^3x(\mathbf{v} - \sigma\boldsymbol{\nabla}s)\cdot\mathbf{B} = -\delta\mu H_{CNB} \tag{56}$$

Where H_{CNB} is the non barotropic global cross helicity [11, 25, 26] defined as:

$$H_{CNB} \equiv \int d^3x(\mathbf{v} - \sigma\boldsymbol{\nabla}s)\cdot\mathbf{B} = \int d^3x\mathbf{v}_t\cdot\mathbf{B} \tag{57}$$

in which $\mathbf{v}_t \equiv \mathbf{v} - \sigma\boldsymbol{\nabla}s$ is the topological velocity field. We thus obtain the conservation of non-barotropic cross helicity using the Noether theorem and the symmetry group of metage translations. Of course one can perform a different translation on each magnetic field line, in this case one obtains:

$$\delta J = -\int d^3x\delta\mu\mathbf{v}_t\cdot\mathbf{B} = -\int d\chi d\eta\delta\mu\oint_{\chi,\eta} d\mu\rho^{-1}\mathbf{v}_t\cdot\mathbf{B} \tag{58}$$

Now since $\delta\mu$ is an arbitrary (small) function of χ, η it follows that:

$$I = \oint_{\chi,\eta} d\mu\rho^{-1}\mathbf{v}_t\cdot\mathbf{B} \tag{59}$$

is a conserved quantity for each magnetic field line. Along a magnetic field line the following equations hold:

$$d\mu = \nabla\mu \cdot d\mathbf{r} = \nabla\mu \cdot \hat{B} dr = \frac{\rho}{B} dr \tag{60}$$

in the above $\hat{B}$ is an unit vector in the magnetic field direction an Eq. (15) is used. Inserting Eq. (60) into (59) we obtain:

$$I = \oint_{\chi,\eta} dr \mathbf{v}_t \cdot \hat{B} = \oint_{\chi,\eta} d\mathbf{r} \cdot \mathbf{v}_t. \tag{61}$$

which is just the circulation of the topological velocity along the magnetic field lines. This quantity can be written in terms of the generalized Clebsch representation of the velocity [23]:

$$\mathbf{v} = \nabla\nu + \alpha\nabla\chi + \beta\nabla\eta + \sigma\nabla s. \tag{62}$$

as:

$$I = \oint_{\chi,\eta} d\mathbf{r} \cdot \mathbf{v}_t = \oint_{\chi,\eta} d\mathbf{r} \cdot \nabla\nu = [\nu]. \tag{63}$$

$[\nu]$ is the discontinuity of ν. This was shown to be equal to the amount of non barotropic cross helicity per unit of magnetic flux [25, 26].

$$I = [\nu] = \frac{dH_{CNB}}{d\Phi}. \tag{64}$$

6 Conclusion

We have shown the connection of the translation symmetry group of labels which is a subgroup of the relabelling group to both the global non barotropic cross helicity conservation law and the conservation law of circulations of topological velocity along magnetic field lines. The latter were shown to be equivalent to the amount of non barotropic cross helicity per unit of magnetic flux [25, 26]. Possible applications of the generalized cross helicity conservation law (both local and global) may arise in solar MHD, where rotation, and the baroclinic instability can give rise to magnetic tornadoes, in which vorticity of the fluid is generated in part by the baroclinic term $\nabla T \times \nabla s$ (see for example [12] eqn. (4.54) and also [27]). Other possible applications for MHD constraints of the current constants of motion are described in [26]. The importance of constants of motion for stability analysis is also discussed in [28].

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Three Quarks Confined by an Area-Dependent Potential in Two Dimensions

Igor Salom and V. Dmitrašinović

Abstract We study the low-lying parts of the spectrum of three-quark states with definite permutation symmetry bound by an area-dependent three-quark potential. Such potentials generally confine three quarks in non-collinear configurations, but classically allow for free (unbound) collinear motion. We use our previous work to evaluate the low-lying parts of the spectrum in a non-adiabatic approximation. We show that the eigen-energies are positive and discrete, i.e., that the system is quantum-mechanically confined in spite of the classically allowed free collinear motion.

Keywords Potential models · Baryons · Y-junction string

1 Introduction

In a recent series of papers we have developed an algebraic theory of quantum mechanical three-body bound states in two [1–3] and in three dimensions [4–6]. This theory is based on the O(4) and O(6) symmetries, respectively, of the relativistic kinetic energy and the corresponding O(4) and O(6) hyperspherical harmonics. One expands the three-body potential and the wave functions in these hyperspherical harmonics and then uses the O(n) algebra to simplify the Schrödinger equation.

If the three-body potential is homogenous, then, under certain conditions on the expansion coefficients v of the three-body potential, allow for an energy spectrum that depends essentially only on the said coefficients. This fact leads to a well-known theorem [7–12] about energy-level ordering in the lower shells of the spectrum. Most three-body confining potentials, such as the Δ- and Y-string ones, satisfy these

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_31

conditions, and consequenty their states have the "universal" ordering properties, so that their low-lying spectra look alike.

In continuation of our previous work on the quantum mechanics of three-particle bound states, here we present an example of a potential that is homogenous and generally confines classically, except under (very) special circumstances, and yet does not satisfy the aforementioned conditions. Consequently its energy spectrum is not readily calculable using our previous (adiabatic) formulae/results and does not have the "universal" properties. We present the results of a non-adiabatic calculation for low-lying parts of the spectrum in two dimensions.[1] We show that the ordering of states is significantly distorted, as compared with the conventional one, but the energy spectrum remains discrete and positive, i.e., it corresponds to a quantum-mechanically confined system.

2 An Area-Dependent Potential

We define the model potential as an harmonic oscillator perturbed by an "area term", with the coupling strength v_b,

$$V_{\mathrm{HY}} = \frac{k}{2}\left(v_a(\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2) + v_b|\boldsymbol{\rho} \times \boldsymbol{\lambda}|\right). \tag{1}$$

This potential is homogenous with homogeneity coefficient $\alpha = 2$. It can be viewed as harmonic ($\alpha = 2$) generalization of the Y-string potential, which is homogenous with $\alpha = 1$, so we may call it the "harmonic Y-string".

In the limit $v_a > 0$, $v_b = 0$ this potential turns into the standard harmonic oscillator, with the well-known discrete, equidistant energy spectrum. In the limit $v_a = 0$, $v_b > 0$ the potential is still harmonic in the sense that it is proportional to the square of the hyper-radius R^2, but it depends only on the area of the triangle $|\boldsymbol{\rho} \times \boldsymbol{\lambda}|$. Manifestly, this area vanishes for all collinear quark configurations, i.e. whenever vector $\boldsymbol{\rho}$ is parallel with the vector $\boldsymbol{\lambda}$, thus making such collinear classical motions free, i.e., unconfined.

An interesting question is what happens to this one unconfined mode of classical motion?[2] In other words, can such a "deformation" of the harmonic oscillator potential change the discrete nature of the original harmonic oscillator energy spectrum? In order to try and answer that question we shall solve the full (i.e. non-adiabatic) Schrödinger equation, Ref. [3]. The three-body potential Eq. (1) can be expanded in terms of $L = 0$ $SO(4)$ hyper-spherical harmonics $\mathcal{Y}^J_{0M}(\alpha, \phi, \Phi)$, Ref. [3]

$$V_{3-\mathrm{body}}(\alpha, \phi) = \sqrt{\frac{\pi}{2}} \sum_{J,M}^{\infty} v^{3-\mathrm{body}}_{JM} \mathcal{Y}^J_{0M}(\alpha, \phi, \Phi) \tag{2}$$

[1]The three-dimensional calculation will be shown elsewhere.

[2]which has measure zero as compared with the set of all three-body configurations - "shape space".

which is equivalent to an expansion in $SO(3)$ hyper-spherical harmonics,

$$V_{3-\text{body}}(\alpha,\phi) = \sum_{J,M}^{\infty} v_{JM}^{3-\text{body}} Y_{JM}(\alpha,\phi) \tag{3}$$

that are functions of the two hyper-angles (α, ϕ). The fact that the potential does not depend on the angle ϕ implies that only $SO(3)$ hyper-spherical harmonics with $M = 0$ enter the expansion, which is equivalent to an expansion in Legendre polynomials of the variable $x = \cos\alpha$, see the Appendix.

In the following we shall keep only the first two terms in the Legendre expansion of the potential, Eq. (10) and then use it to solve the Schrodinger equation numerically for an arbitrarily large ratio of strengths of the area- and harmonic potentials $\frac{v_b}{v_a} \to \infty$, which corresponds to the $v_a \to 0$ limit, while keeping v_b finite.

In that limit, the ratio of the two "harmonic Y string" effective potential coefficients $\lim_{k\to 0}\left(v_{20}^{HY}/v_{00}^{HY}\right) = \frac{\sqrt{5}}{4}$ remains finite, however, as can be seen in the Appendix, and thus ensures that there remains an effective harmonic oscillator component in the effective potential and thus preserves confinement.

3 Low-Lying Energy Spectrum

The Hilbert space of this problem naturally separates into the even- and odd-parity parts, that are fully disconnected from each other. Moreover, other conserved quantities, ("good quantum numbers"), such as the total angular momentum L and the permutation symmetry multiplets, also provide other "super-selection rules" that further split the Hilbert space into smaller subsets. One particularly interesting Hilbert sub-space is the $L = 0$ space: this is where the deconfined ("continuum") states ought to appear, provided that they exist at all. This is because collinear motion implies vanishing angular momentum, but not *vice versa*.

Following Sect. IV.A in Ref. [3] we may use $m_1 = \frac{1}{2}\left(l_\rho + l_\lambda\right) = \frac{L}{2} = 0$ and $m_2 = G_3 = 0$ as the definition of the invariant sub-space. This condition means that these are the $[SU(6), L^P] = [56, 0^+]$ and $[SU(6), L^P] = [20, 0^+]$ states (in the spectroscopy notation), the former appears first in the $K = 0$ band and the latter in the $K = 2$ band. They re-appear at even K's, with increasing multiplicity.

We look at the strongly perturbed spectrum of the first 21 even-K states (K = 0, 2, 4, 6, 8, 10) sub-space satisfying the $m_1 = m_2 = 0$, i.e., $L = 0 = G_3$ condition. For convenience we (re)define the Hamiltonian as

$$H = H_0 + \mathrm{C}_{pot}\frac{R^2}{\sqrt{2}\pi}\left(\mathcal{Y}_{00}^{J=0} + \frac{2}{\sqrt{5}}\mathcal{Y}_{00}^{J=2}\right) \tag{4}$$

where H_0 is the harmonic oscillator Hamiltonian, with eigenvalues that are multiples of $\mathrm{C}_0 = \hbar\omega$ and C_{pot} is the coefficient multiplying the area term, i.e., $\mathrm{C}_{pot} \simeq v_b$.

Equation (4) contains the $\mathcal{Y}_{00}^{J=0}$ term which is the part of the area-term and due to the presence of this term in expansion there is no appearance of artificial negative eigenvalues.

Thus, we need the Hamiltonian matrix for the 21-dimensional even-K state (K = 0, 2, 4, ..., 10) sub-space of the full Hilbert space satisfying the condition $m_1 = m_2 = 0$. We must diagonalize the corresponding 21 × 21 Hamiltonian matrix; below we show the upper-left-hand corner 6 × 6 submatrix, corresponding to K = 0, 2, 4 states, of the full 21 × 21 matrix

$$\begin{pmatrix} \frac{C_{pot}}{2\lambda\pi^2}+2C_0 & 0 & \frac{\sqrt{\frac{3}{2}}C_{pot}}{5\lambda\pi^2} & 0 & 0 & 0 \\ 0 & \frac{9C_{pot}}{10\lambda\pi^2}+4C_0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{\frac{3}{2}}\mathrm{Cpot}}{5\lambda\pi^2} & 0 & \frac{11C_{pot}}{14\lambda\pi^2}+6C_0 & \frac{\sqrt{3}C_{pot}}{5\lambda\pi^2} & 0 & \frac{C_{pot}}{5\lambda\sqrt{2}\pi^2} \\ 0 & 0 & \frac{\sqrt{3}C_{pot}}{5\lambda\pi^2} & \frac{C_{pot}}{2\lambda\pi^2}+4C_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{9C_{pot}}{10\lambda\pi^2}+6C_0 & 0 \\ 0 & 0 & \frac{C_{pot}R^2}{5\sqrt{2}\pi^2} & 0 & 0 & \frac{C_{pot}}{2\lambda\pi^2}+6C_0 \end{pmatrix} \quad (5)$$

where $\lambda = \frac{m\omega}{\hbar}$ and $\omega = \sqrt{\frac{k}{m}}$. In the vanishing "area interaction" coupling constant limit $\frac{C_{pot}}{C_0} \propto \frac{v_b}{\omega} \to 0$ we recover the usual harmonic oscillator spectrum together with its characteristic degeneracy, see Fig. 1top. As we increase the ratio of the "area interaction" coupling constant and the harmonic one, first to unity, $\frac{C_{pot}}{C_0} \propto \frac{v_b}{v_a} \to 1$ Fig. 1middle, and then to seven $\frac{C_{pot}}{C_0} \propto \frac{v_b}{v_a} \to 7$, Fig. 1bottom, one can see that the states are shifted, at first a little, and then much more into a more-or-less smooth distribution of states, with no degeneracies, or manifest accumulation points.

Another interesting limit is $\frac{v_b}{v_a} \to \infty$, i.e., $v_a \to 0$, when this Hamiltonian does not confine all three-body configurations: the collinear classical motion is free in this potential. What this means in the quantized case is not yet clear: naively one might expect to see (at least) one continuum in the spectrum, corresponding to the unconfined ("free") collinear motion.

The lowest-lying such continuum ought to correspond to states with vanishing total angular momentum $L = 0$ and high values of K, as the collinear motion implies: 1) vanishing total angular momentum $L = 0$; 2) one (hyper)-angle in the triangle always being precisely equal to $\Phi = \pi$. The second requirement leads to the vanishing of the (hyper)-angular uncertainty $\Delta\Phi = 0$, which, in turn demands, an infinite uncertainty in the corresponding (hyper)angular momentum $\Delta K \to \infty$. That can be fulfilled only by states with vanishing total and very large/infinite/ values of the hyper-angular momentum K. In other words, one might expect the (binding) energy of some high hyper-angular momentum K states to decrease and ultimately to vanish in the infinite angular momentum limit. If there are sufficiently many such states, they may form something that resembles a continuum.

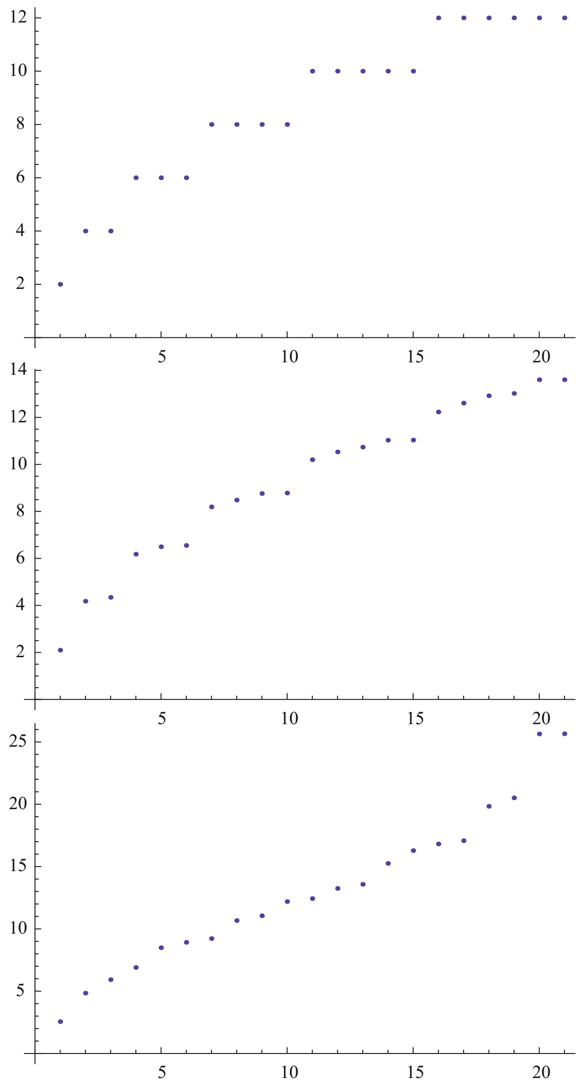

Fig. 1 The spectrum of the first six even-K bands (K = 0, 2, 4, 6, 8, 10) of the three-body harmonic oscillator perturbed by the area-dependent three-body potential with coupling constant $\frac{C_{pot}}{C_0}$ equal to 0, 1 and 7. This is a 21-dimensional sub-space of the full Hilbert space consisting of states satisfying the conditions $m_1 = m_2 = 0$, or $L = 0 = G_3$, equivalent to $[SU(6), L^P] = [20, 0^+]$ in the spectroscopy notation. Note the rearrangement of the levels until the K-shells become practically indiscernible

In order to check this limit numerically we increase the "area interaction" coupling constant ratio $\frac{v_b}{v_0}$ to e.g. $\frac{C_{pot}}{C_0}(\propto \frac{v_b}{v_a}) \to 10$, 100, and 1000, and show the results in Fig. 2.

There one sees a spectrum consisting of discrete, positive energy eigen-values. Of course, one cannot expect to find a "true" continuum with a finite number of states N, but one might see some hints thereof, if the number of states N and the off-diagonal matrix elements are large enough: our results shown in Fig. 2 do not give even a hint of such a continuum at $N = 19$ and $\frac{C_{pot}}{C_0} = 1000$.

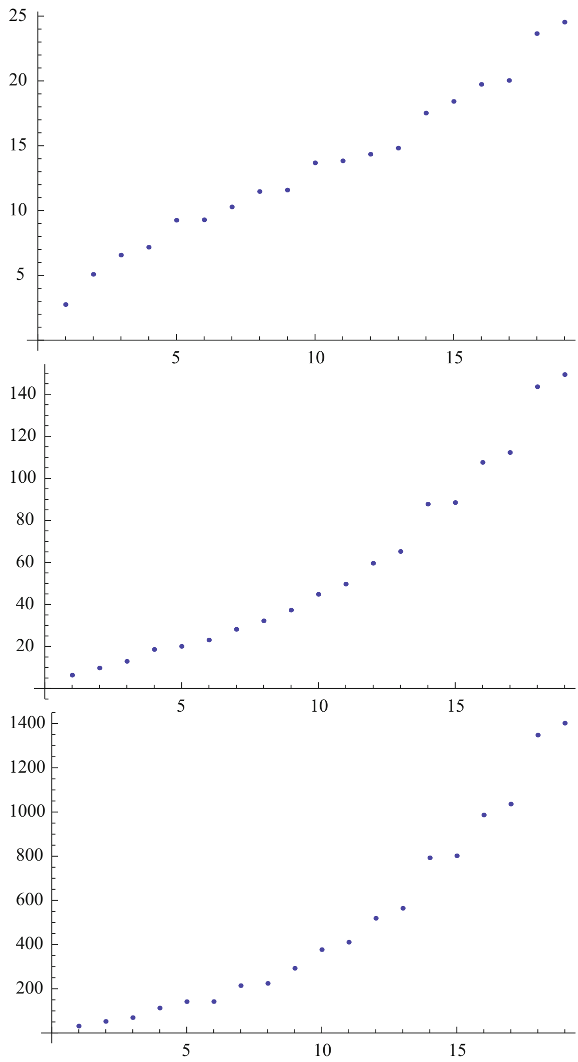

Fig. 2 The spectrum of the first six even-K bands (K = 2, 4, 6, 8, 10, 12) of the three-body harmonic oscillator perturbed by the area-dependent three-body potential with coupling constant $\frac{C_{pot}}{C_0}$ equal to 10, 100 and 1000. This is a 21-dimensional sub-space of the full Hilbert space consisting of states satisfying the conditions $m_1 = m_2 = 0$, or $L = 0 = G_3$, equivalent to $[SU(6), L^P] = [20, 0^+]$ in the spectroscopy notation. We show only $N = 19$ levels, as the last two seem to be adversely affected by the boundary. Note that the pattern of the levels is essentially unchanged, only the scale on the ordinate is different

4 Summary

In this paper we have reported our calculation of three-quark energy spectrum in a three-body potential that depends only on the area of the triangle subtended by the three quarks. The spectrum shows no signs of deconfinement in spite of classically allowed unbound one-dimensional motion.

Acknowledgements This work was financed by the Serbian Ministry of Science and Technological Development under grant numbers OI 171031, OI 171037 and III 41011.

Appendix

Equation (1) can be re-written as a function of (the absolute value of) only one O(3) (hyper-)spherical harmonic in the shape (hyper-)space: the $|Y_{10}(\alpha, \phi)|$:

$$\frac{2}{R^2}|\boldsymbol{\rho} \times \boldsymbol{\lambda}| = |\cos\alpha| = \sqrt{\frac{4\pi}{3}}|Y_{10}(\alpha, \phi)|. \tag{6}$$

Now, the absolute value of $|Y_{10}(\alpha, \phi)|$ can be expressed as $\sqrt{Y_{10}^{*}(\alpha, \phi)Y_{10}(\alpha, \phi)}$ and the O(3) Clebsch–Gordan expansion can be applied to $Y_{10}^{*}(\alpha, \phi)Y_{10}(\alpha, \phi)$, which contains only the (obviously even) values of $L = 0, 2$, as in Eq. (A12) of Ref. [3].

$$\frac{2}{R^2}|\boldsymbol{\rho} \times \boldsymbol{\lambda}| = \sqrt{\frac{1}{3}}\sqrt{1 + \frac{2}{\sqrt{5}}\frac{Y_{20}(\alpha, \phi)}{Y_{00}(\alpha, \phi)}}. \tag{7}$$

The square root can be expanded in a Taylor-like series, the first two terms of which coincide with the expansion in Legendre polynomials, or O(3) spherical harmonics, and for $L = 0$, even in O(4) hyper-spherical harmonics

$$\frac{2}{R^2}|\boldsymbol{\rho} \times \boldsymbol{\lambda}| = \sqrt{\frac{1}{3}}\left(1 + \frac{1}{\sqrt{5}}\frac{Y_{20}(\alpha, \phi)}{Y_{00}(\alpha, \phi)} + \cdots\right). \tag{8}$$

Manifestly the Legendre polynomial expansion, Eq. (8) is limited to even-order $J = 0, 2, 4, \ldots$ terms only,

$$\begin{aligned} V_{\mathrm{HY}}(R, \alpha, \phi) &= \frac{k}{2}\left(v_a(\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2) + v_b|\boldsymbol{\rho} \times \boldsymbol{\lambda}|\right). \qquad (9)\\ &= \frac{k}{2}R^2\left(v_a + v_b\frac{1}{2}\sqrt{\frac{1}{3}}\left(1 + \frac{1}{\sqrt{5}}\frac{Y_{20}(\alpha, \phi)}{Y_{00}(\alpha, \phi)} + \cdots\right)\right)\\ &= \frac{k}{2}R^2\frac{v_0^{HY}}{\sqrt{4\pi}}\left(1 + \frac{v_2^{HY}}{v_0^{HY}}\sqrt{4\pi}Y_{20}(\alpha, \phi) + \cdots\right). \qquad (10) \end{aligned}$$

Note, however, that $v_b/v_a \neq v_2^{HY}/v_0^{HY}$. In particular the additive constant in the expansion Eq. (8) is important, as it ensures the (overall) positivity of this potential and leads to the change of “effective couplings”

$$v_{00}^{HY} = \sqrt{4\pi}\left(v_a + v_b\frac{1}{2}\sqrt{\frac{1}{3}}\right),$$

and

$$v_2^{HY} = v_b \frac{1}{2}\sqrt{\frac{4\pi}{15}}.$$

These two equations in turn lead to

$$\frac{v_{20}^{HY}}{v_{00}^{HY}} = \frac{v_b \frac{1}{2}\sqrt{\frac{4\pi}{15}}}{\sqrt{4\pi}\left(v_a + v_b \frac{1}{2}\sqrt{\frac{1}{3}}\right)} = \frac{v_b}{2\sqrt{15}\left(v_a + v_b \frac{1}{2}\sqrt{\frac{1}{3}}\right)},$$

and in particular in the $v_a \to 0$ limit, this ratio for the HY potential equals that of the pure area potential:

$$\lim_{v_a \to 0}\left(\frac{v_{20}^{HY}}{v_{00}^{HY}}\right) = \frac{1}{\sqrt{5}}.$$

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A Group Action Principle for Nambu Dynamics of Spin Degrees of Freedom

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Abstract We describe a formulation of the group action principle, for linear Nambu flows, that explicitly takes into account all the defining properties of Nambu mechanics and illustrate its relevance by showing how it can be used to describe the off–shell states and superpositions thereof that define the transition amplitudes for the quantization of Larmor precession of a magnetic moment. It highlights the relation between the fluctuations of the longitudinal and transverse components of the magnetization. This formulation has been shown to be consistent with the approach that has been developed in the framework of the non commutative geometry of the 3–torus. In this way the latter can be used as a consistent discretization of the former.

Keywords Nambu mechanics · Magnetization dynamics · Finite dimensional phase spaces

1 Introduction

Nambu mechanics is the generalization of Hamiltonian mechanics to phase spaces of arbitrary dimension [1]. The reason it is useful to consider such spaces at all is to describe the dynamics of extended objects [2] (which was Nambu's original motivation [3]). It represents, in fact, the generalization of the area preserving diffeomorphisms of Hamiltonian mechanics to the corresponding group(s) of transfor-

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_32

mations that preserve the volume in spaces of odd dimension, too [4]. In this regard it appears much less “exotic” and, indeed, has found many applications in classical fluid mechanics, where the study of incompressible flows is the natural context [5, 6].

While it became the subject of interest in the effort to understand the dynamics of multiple M2–branes [7], it was quickly realized that there were conceptual issues that remain to be clarified for the description of their quantum effects. A workaround that used only generalized Poisson brackets, i.e. a purely Hamiltonian formulation in a non-flat metric, was found to be sufficient for many cases of practical interest [8]; however a deeper understanding of the properties of the models proposed for M2–branes must solve the problem of the consistent definition of the quantization of the Nambu bracket [9–11], which is a generalization of the Poisson bracket with more than 2 elements. To this end it is useful to understand the properties of simpler quantum systems, that can be described with the framework of Nambu mechanics. Such examples are provided by magnetic systems [12], where the three components of the magnetization, are, naturally, identified with the canonical variables of Nambu mechanics.

In this contribution we shall show that a recently proposed quantization scheme [13] can be applied to describe the quantum dynamics of Larmor precession of a magnetic moment in an external field and can describe the off–shell states in a way that provides insights that are much harder to grasp using the traditional Hamiltonian formalism.

What has been lacking, indeed, is a consistent group action principle, that leads to the definition of consistent unitary, linear, evolution operators that, acting on the space of states, describe consistent superpositions of states and their transition probabilities. While the proposal was set forth in Ref. [13], it does require fleshing out for concrete applications, such as the one discussed here.

Therefore, in the following section, we shall review the salient properties of classical linear Nambu flows in the continuum, focusing on the volume preserving diffeomorphisms; we shall, then, show how the consistent quantization of these can be understood in terms of the properties of the non–commutative 3–torus. We shall then construct the unitary evolution operators on it, implementing a regularization in terms of finite dimensional matrices and show that the size of the matrices has a physical basis. We conclude with a discussion of further avenues of inquiry, in particular, regarding consistent coupling to baths.

2 Linear Nambu Flows for Classical and Quantum Magnets

In 3-dimensional Nambu mechanics, linear Nambu flows are defined by the time evolution equations for each component of the vector of dynamical variables

$$\frac{dx^I}{dt} = \left\{x^I, H_1, H_2\right\} = \mathsf{M}^{IJ} x^J(t), \tag{1}$$

with M a constant antisymmetric matrix and the Nambu 3-bracket is defined as

$$\{f, g, h\} = \varepsilon^{IJK} \partial_I f \partial_J g \partial_K h, \tag{2}$$

where f, g, h are functions of $\boldsymbol{x}$, $\partial_I \equiv \partial/\partial x^I$ and ε^{IJK} is the fully anti-symmetric Levi-Civita pseudo-tensor of rank 3.

The solution of equation (1) can be written as

$$\boldsymbol{x}(t) = e^{\mathsf{M}t}\boldsymbol{x}(0) \equiv \mathsf{A}(t)\boldsymbol{x}(0) \tag{3}$$

Since M is traceless, $\mathsf{A}(t) \equiv e^{\mathsf{M}t}$, the classical, one–step evolution operator, can be shown to be an element of the group $SL(3, \mathbb{R})$.

Linear systems can be defined by two conserved quantities, $H_1 = \boldsymbol{a} \cdot \boldsymbol{x}$ with $\boldsymbol{a}$ a constant vector and $H_2 = (1/2)\boldsymbol{x}^{\mathrm{T}}\mathsf{B}\boldsymbol{x}$, with B a constant symmetrical matrix. In the framework of Nambu mechanics, these systems define the simplest systems to consider [13] and are formal analogues of the harmonic oscillator from Hamiltonian mechanics.

Such systems are not only toy models, but can be considered as prototypes for modeling dynamics of magnets, dominated by the exchange interaction. For instance, if an anti-ferromagnetic material is defined by a magnetic crystalline cell that can be mapped on two sublattices with spins $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$, then for each spin in its first neighbor cell, without any further interaction, we have the following equations of motion:

$$\frac{ds_1^I}{dt} = \varepsilon^{IJK} J_{12} s_2^J s_1^K \tag{4a}$$

$$\frac{ds_2^I}{dt} = \varepsilon^{IJK} J_{12} s_1^J s_2^K \tag{4b}$$

The average ferromagnetic magnetization vector $\boldsymbol{M} \equiv \frac{1}{2}(\boldsymbol{s}_1 + \boldsymbol{s}_2)$ and the anti-ferromagnetic Néel vector $\boldsymbol{m} \equiv \frac{1}{2}(\boldsymbol{s}_1 - \boldsymbol{s}_2)$ can be defined, and for these vectors we immediately have

$$\frac{dM^I}{dt} = 0 \tag{5a}$$

$$\frac{dm^I}{dt} = 2J_{12}\varepsilon^{IJK} M^J m^K \tag{5b}$$

Because $\boldsymbol{M}$ is then a constant of motion, Eq. (5b) describes a linear Nambu flow for the antiferromagnetic vector $\boldsymbol{m}$, associated with a traceless matrix M introduced in Eq. (1).

As in Hamiltonian mechanics, this flow is on the phase space of the system and describes, through Liouville's theorem, a flow for the probability density $\rho(\boldsymbol{x}, t)$ therein:

$$\frac{\partial \rho(\boldsymbol{x}, t)}{\partial t} = \{\rho(\boldsymbol{x}, t), H_1, H_2\} \tag{6}$$

Of particular interest are the moments of this probability density and their evolution in time, since they can be related to observable quantities.

For linear Nambu flows this equation takes the form

$$\frac{\partial \rho(\boldsymbol{x},t)}{\partial t} = \varepsilon^{IJK}\omega_J B_{KL}\partial_I \rho(\boldsymbol{x},t) x^L = \det\left(\nabla\rho(\boldsymbol{x},t), \boldsymbol{\omega}, \nabla H_2\right) \tag{7}$$

and describes, also, the classical dynamics in phase space, i.e. the properties of the solutions of the classical equations of motion.

The problem here is that, in Hamiltonian mechanics, it is known that Poisson brackets are the classical limits of commutators [14]. In Nambu mechanics what is the quantum structure that preserves all its properties, whose classical limit would be the Nambu bracket, is not known [11].

In Ref. [13] the off–shell states and consistent evolution operators for classical and quantum, linear Nambu flows were constructed.

Let us review the idea of the construction. It is based on introducing an infrared cutoff, by compactifying the phase space on a 3–torus, $\mathbb{T}^3$; and on introducing a short–distance ("ultraviolet") cutoff, by considering only points with rational coordinates and common denominator, N.

In this way the differential equations become linear recurrences on the finite field, $\mathbb{Z}_N$

$$\boldsymbol{x}_{n+1} = \mathsf{A}\boldsymbol{x}_n \bmod N \tag{8}$$

which can display quite complex, indeed, deterministic chaotic, behavior as N varies. Already, at the classical level, this means that the system possesses a finite number of states and how these are visited during the evolution is of interest. If there is periodic behavior, the period, $T(N)$, satisfies $\mathsf{A}^{T(N)} \equiv I \bmod N$; and the quantum counterpart, $U(\mathsf{A})$ shares this property, by construction, since $U(\mathsf{A}^{T(N)}) = [U(\mathsf{A})]^{T(N)}$. What happens as $N \to \infty$ is a quite delicate issue, that has been investigated in the context of quantum chaos [15], but there is, still, much to be clarified.

The integer N controls in this way the fluctuations at both ends, infrared and ultraviolet and $2\pi/N$ plays, indeed, the role of Planck's constant for describing the quantum fluctuations [13].

The construction of the quantum evolution operator, $U(\mathsf{A})$, on the 3–torus proceeds, in fact, in complete analogy to the Hamiltonian quantization of toroidal phase spaces. The idea will be to construct a unitary operator, $U(\mathsf{A})$, that realizes a consistent quantization, of the classical evolution operator, $\mathsf{A} \in SL(3,\mathbb{R})$, in the sense that it satisfies the correspondence principle–which means that it realizes the metaplectic representation –and provides a faithful representation, in the sense that, for any classical evolution operators, A and B, we have the composition rule that

$$U(\mathsf{A}\circ\mathsf{B}) = U(\mathsf{A})\circ U(\mathsf{B}) \tag{9}$$

This property is necessary to ensure that time evolution is well–defined, that it depends only on the endpoints in phase space and not on the parametrization of the path(s).

For a finite-dimensional representation N of the operator A, as sketched in Ref. [13], the construction of an $N \times N$ matrix $U(\mathsf{A})$ with these properties is realized by showing that it can be mapped exactly to the construction of the corresponding unitary operator of a Hamiltonian system.

This proceeds as follows: first, the linear Nambu flow has the property that $\boldsymbol{\omega}$ is an eigenvector of the one–step evolution operator, A, with eigenvalue 1:

$$\mathsf{A}\boldsymbol{\omega} = \boldsymbol{\omega} \Leftrightarrow \boldsymbol{\omega}^T = \boldsymbol{\omega}^T \mathsf{A} \tag{10}$$

The convention of multiplication from the right was used (i.e for the dual vectors) in Ref. [13]; in the present contribution we use the, perhaps, more familiar convention of multiplication from the left. Because one deals with finite dimensional ordinary and possibly complex valued vectors, it does not really matter, up to transposition and conjugate. The property (10) is also true for the any collinear vector $\lambda\boldsymbol{\omega}$ that form an infinite collection of fixed vector with 1 as eigenvalue.

The property that the vector of the linear Hamiltonian is left invariant by the flow implies, in turn, for Eq. (8), that

$$[\boldsymbol{x} \times \boldsymbol{\omega}]_{n+1} = \mathsf{A}[\boldsymbol{x} \times \boldsymbol{\omega}]_n \tag{11}$$

for any time step n.

This expression can now be shown to be equivalent to

$$[\boldsymbol{x} \times \boldsymbol{\omega}]_{n+1} = \begin{pmatrix} \widetilde{\mathsf{A}} & \boldsymbol{0} \\ \widetilde{\boldsymbol{\omega}} & 0 \end{pmatrix} [\boldsymbol{x} \times \boldsymbol{\omega}]_n \tag{12}$$

which provides a definition of the 2×2 evolution operator $\widetilde{\mathsf{A}}$. It has been shown in Ref. [16] that one can construct a basis using the initial magnetization state $\boldsymbol{x}(0)$, the precession vector $\boldsymbol{\omega}$ and their vector product $\boldsymbol{x}(0) \times \boldsymbol{\omega}$, and it is possible to decompose the time evolution of the solution on this basis as

$$\boldsymbol{x}(t) = A(t)\boldsymbol{x}(0) + B(t)\frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|} + C(t)\boldsymbol{x}(0) \times \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|} \tag{13}$$

As long as the precession vector $\boldsymbol{\omega}$ is constant, one can always choose a reference frame such that said vector is aligned with the $\boldsymbol{z}$-axis. If we now consider the time evolution of $\boldsymbol{x}(t) \times \boldsymbol{\omega}$ one can see that this vector remains in the $(\boldsymbol{x}, \boldsymbol{y})$ plane. As such the last component remains null over time. Hence one can restrain Eqs. (11)–(12).

This can be shown to be symplectic, therefore the corresponding quantum evolution operator, $U(\widetilde{\mathsf{A}})$, can be constructed by known techniques. It is this operator that we shall define as the unitary evolution operator of the quantum Nambu evolution. In subsequent sections we shall show that our construction passes a non–trivial test, by checking that it provides results that are consistent with those obtained by the canonical quantization of the Larmor precession.

The off–shell states are, therefore, those that are defined by the action of operators mod N, whereas the on–shell states are those that do not require the mod N operation. However, quantum effects are, also, described by superpositions of pure states. We shall show the relevance of such superpositions in the following section.

3 Computing Transition Probabilities la Nambu

In this section we shall show how to use the unitary evolution operator, $U(\mathsf{A})$ to compute transition probabilities for Larmor precession, from any initial to any final state of the magnetic moment.

Our starting point is the identification of the Larmor precession equation as a linear Nambu flow, following the notation of Ref. [13]

$$\frac{dx^I}{dt} = \epsilon^{IJK} a_J B_{KL} x^L \Leftrightarrow \frac{ds^I}{dt} = \epsilon^{IJK} \omega_J s_K \equiv \mathsf{M}^{IK} s_K \tag{14}$$

where $H_1 \equiv \boldsymbol{a} \cdot \boldsymbol{x} = \boldsymbol{\omega} \cdot \boldsymbol{s}$ and $H_2 \equiv (1/2)(\boldsymbol{x}, \mathsf{B}\boldsymbol{x})$, with $\mathsf{B} = \mathbb{1}$.

For Larmor precession around an external field described by $\boldsymbol{\omega}$ we, thus, have

$$\mathsf{M} = \boldsymbol{\omega} \times . = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \tag{15}$$

If Larmor precession happens around a fixed vector $\boldsymbol{\omega}$, we can always choose our reference frame such that only the component along the z-axis is non-zero $\boldsymbol{\omega} = (0, 0, \omega_3)$.

Its exponential, $\mathsf{A} = \exp(\mathsf{M})$ is the one–step evolution operator. This acts on a finite set of states, labeled by the integers mod N [13], which means that the matrix $\mathsf{A} \in SL(3, \mathbb{Z}_N)$, which, also, has integer entries, mod N, has the form

$$A = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{16}$$

where a and b are integers mod N, which satisfy $a^2 + b^2 \equiv 1 \bmod N$. We may, hence, work out the form of the "reduced" evolution operator $\tilde{A} \in SL(2, \mathbb{Z}_N)$

$$\tilde{A} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{17}$$

If the three–component states are labeled by the vector $\boldsymbol{s}$, then the "reduced" states are labeled by the vector $\tilde{\boldsymbol{s}}$

$$\tilde{\boldsymbol{s}} = (\omega_3 s_1 - \omega_1 s_3, \omega_3 s_2 - \omega_2 s_3) \tag{18}$$

Now by choosing the unit of time appropriately, we may set $\omega_3 \equiv 1 \bmod N$.

The previous expression thus becomes

$$\tilde{\boldsymbol{s}} = (s_1, s_2) \bmod N \tag{19}$$

This means that all the interesting dynamics happens on a plane, orthogonal to the magnetization vector $\boldsymbol{\omega}$ and as such, once we choose an initial state with fixed value for s_3 we have to satisfy $s_1^2 + s_2^2 = (1 - s_3^2) \bmod N$.

What is interesting in this expression is that, if $1 - s_3^2$ is a quadratic residue mod N, this expression can be reduced to $\sigma_1^2 + \sigma_2^2 \equiv 1 \bmod N$. If not, we must work in the quadratic extension of the number field. In this way the transverse fluctuations, described by s_1 and s_2 are related to the longitudinal fluctuations, described by s_3. We now have a way to count all the states which are accessible from any one initial state and as such we can label them.

All that remains to be computed, therefore, is the quantum evolution operator $U(\tilde{\mathsf{A}})$, whose classical limit would be $\widetilde{\mathsf{A}}$.

According to reference [13],

$$\left[U(\widetilde{\mathsf{A}})\right]_{k,l} = \frac{(2b|N)}{\sqrt{N}} \Omega_N^{\frac{ak^2 - 2kl + dl^2}{2b}} \tag{20}$$

where $\Omega_N = e^{2\pi \mathrm{i}/N}$ is the Nth root of unity, and $(2b|N)$ is the Jacobi symbol, for $2b$ and N, equal to 1 if $2b$ is a quadratic residue mod N, -1 if not and 0 if $2b \equiv 0 \bmod N$.

While these expressions were originally derived for N prime, it has been shown that the matrices factorize over the prime factorization of N [17].

Both the time evolution of the quantum states $|\tilde{\boldsymbol{s}}\rangle$ and the transition probabilities between them are given by the evolution operator as

$$|\tilde{\boldsymbol{s}}\rangle_n \equiv U(\widetilde{\mathsf{A}}^n)|\tilde{\boldsymbol{s}}\rangle_0 \tag{21}$$

$$P_n(\tilde{\boldsymbol{s}}', \tilde{\boldsymbol{s}}) = |\langle \tilde{\boldsymbol{s}}'|U(\widetilde{\mathsf{A}}^n)|\tilde{\boldsymbol{s}}\rangle|^2 \tag{22}$$

Let us illustrate this abstract framework with a specific example by taking $N = 5$, $\omega_3 = 1$, and the initial magnetization state to be normal to the external field described by $\boldsymbol{\omega}$ (i.e in the $(\boldsymbol{x}, \boldsymbol{y})$ plane, so that $s_3 \equiv 0$). This means all the accessible states are those for which

$$s_1^2 + s_2^2 \equiv 1 \bmod 5 \tag{23}$$

To count and label them we have

$$|1\rangle = (1, 0) \quad |2\rangle = (0, 1) \quad |3\rangle = (4, 0) \quad |4\rangle = (0, 4)$$

We note that these are, also, "classical" states. Quantum effects are described by their superpositions, that don't have a classical analog.

Furthermore, the only three, non–trivial, evolution operators, $\widetilde{\mathsf{A}}$, satisfying the constraint $a^2 + b^2 \equiv 1 \bmod 5$ are

$$\tilde{A}_1 = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}, \tilde{A}_2 = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} \text{ and } \tilde{A}_3 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = (\tilde{A}_1)^2$$

These matrices describe rotations by $\pm 90^{\mathrm{o}}$ in phase space–the Fourier transform. This means that the quantum evolution operator, $U(\widetilde{\mathsf{A}})$ is, in fact, the Discrete Fourier Transform, over five states. This, apparently, is one more state than necessary, since $\widetilde{\mathsf{A}}^4 = \mathbb{1}_{2\times 2} \Leftrightarrow U(\widetilde{\mathsf{A}})^4 = U(\widetilde{\mathsf{A}}^4) = U(\mathbb{1}_{2\times 2}) = \mathbb{1}_{5\times 5}$. This, of course, means that the states are degenerate–their degeneracies were studied in detail by Mehta [18].

However, any state $|\tilde{\boldsymbol{s}}\rangle$, can be expanded in the basis of the eigenstates of $U(\widetilde{\mathsf{A}})$

$$|\tilde{\boldsymbol{s}}\rangle = \sum_{k=0}^{N-1} c_k |\psi_k\rangle \tag{24}$$

These states are superpositions of the states of definite magnetization. This, however, means that there are only $N - 1$ independent, relative phases, since the evolution operator is unitary. Therefore there are only $N - 1 = 4$, in the case at hand, "non–trivial" states. So, let us label the additional state as $|0\rangle$. To define a more convenient way of dealing with the superposition of states, we will use the following notation

$$|\alpha, \beta, \gamma, \delta, \epsilon\rangle \equiv \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle + \delta|3\rangle + \epsilon|4\rangle \tag{25}$$

To compute transition probabilities depending on time (the integer n playing the role of a discrete time evolution here, hence a "kicked"-precession), if we start with an initial state $|0, 1, 0, 0, 0\rangle \equiv |1\rangle$, in the basis of the position operator, after one time-step, the next state is given by

$$|\tilde{\boldsymbol{s}}\rangle_1 = U(\tilde{A})|\tilde{\boldsymbol{s}}\rangle_0 \tag{26}$$

The evolution operator $U(\widetilde{\mathsf{A}}_1)$, describing the Discrete Fourier Transform, applied to the initial state $|1\rangle$

$$U(\tilde{A}_1)|1\rangle = \frac{1}{\sqrt{5}}\left[-1|0\rangle + e^{\frac{3\imath\pi}{5}}|1\rangle + e^{\frac{4\imath\pi}{5}}|2\rangle] - e^{\frac{1\imath\pi}{5}}|3\rangle - e^{\frac{2\imath\pi}{5}}|4\rangle\right] \tag{27}$$

where we have highlighted the relative phases, of the other pure states wrt the state $|0\rangle$.

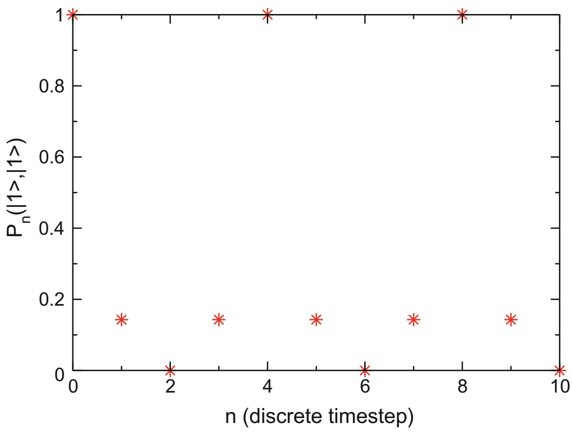

Fig. 1 Transition probability $P_n(|1\rangle, |1\rangle) = |\langle 1|U(\tilde{A}_1^n)|1\rangle|^2$ as function of the time-step $0 \leq n \leq 10$ highlighting the periodicity

The time evolution of the transition probability between the same initial and final pure state, say $|1\rangle$, as a function of the discrete time-step n

$$P_n(|1\rangle, |1\rangle) = |\langle 1|U(\tilde{A}_1)^n|1\rangle|^2 \tag{28}$$

computes the Nambu path integral and should display the appropriate periodicity, viz.

$$U(\tilde{A}_1)^{T(N)} = \mathbb{1}_{5\times 5}$$

Results are displayed in Fig. 1.

4 Conclusion and Outlook

In this contribution we have proposed a group action principle for linear Nambu flows, that is consistent with the properties of classical Nambu mechanics, as well as the correspondence principle of quantum mechanics and can, thus, be considered as a consistent quantization of linear Nambu flows. Our formalism provides an explicit prescription for the space of states, both "on–shell" and "off–shell" and linear superpositions, that are the hallmark of non–classical behavior. Such flows are relevant for describing the Larmor precession of the magnetization of nanamagnets and, thus, their quantization is relevant for describing its quantum fluctuations. We have applied our framework to the calculation of transition probabilities and computing the time evolution of a simple model for quantum Larmor precession mod 5, that is relevant for a spin 2 nanomagnet.

The semi–classical limit can be obtained, as N, therefore the number of spin states becomes large, as might be expected and is quite subtle.

When Gilbert damping is taken into account, the equations of motion become non–linear, but can, still be solved; their solutions can be interpreted as describing instantons. Interestingly the damped linear Nambu flow admits a continuous evolution solution [16] that it would be of practical interest to set in the presented framework.

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Newton–Cartan Trace Anomalies and Renormalization Group Flows

Roberto Auzzi

Abstract I will discuss trace anomalies for non-relativistic Schrödinger theories in $2+1$ dimensions coupled to a Newton–Cartan gravity background, which is used as a source of the energy-momentum tensor. The motivation is to identify candidates for a possible non-relativistic version of the a-theorem for theories with RG flows interpolating between an UV and an IR Schroedinger-invariant non-relativistic conformal fixed points. I will first discuss the general structure of the anomaly, which is determined by the Wess–Zumino consistency condition. Then I will present an explicit calculation for the anomaly in the case of a free scalar and of a free fermion, using heat kernel. There is a type A anomaly which is proportional to 1/m, where m is the mass of the particle. In analogy with the relativistic case, the irreversibility properties of the renormalization group can also be investigated by studying the Wess–Zumino consistency conditions for the trace anomaly of the theory in a Newton Cartan background with space-time dependent couplings.

Keywords Nonrelativistic field theories · Anomalies · Renormalization group flow

1 Introduction

A renormalization group (RG) flow is a trajectory in the space of theories, induced by a change of scale. In the case of (unitary) relativistic theories in even dimension, the trace anomaly gives us an useful measure of the degrees of freedom at a conformal fixed point and a powerful universal constraint on the possible infrared dynamics which can emerge from an ultraviolet theory, valid also in the regime of strong coupling. In 2 dimensions this comes from Zamolodchikov c theorem [1]. In 4 dimensions this comes from the a-theorem, which was first conjectured in [2]; a

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_33

perturbative proof was given by [3–5] with the local renormalization group (RG) equations. A proof using dispersion relations was given in [6, 7].

The local RG equations [3–5] can be derived imposing the Wess–Zumino (WZ) consistency conditions for the trace anomaly [8, 9] of the theory in a gravity background with spacetime-dependent couplings. It is a useful tool to study relativistic RG flows nearby conformal fixed points in various dimensions [10–15] and it can be applied to the supersymmetric case [12, 16, 17]. For reviews see [18, 19].

It would be interesting, for condensed matter applications, to extend similar results to the case of non-relativistic theories. Non-relativistic fixed points allow for different scaling in time and in space, which can be parameterized by the dynamical exponent z:

$$x^i \to e^{\sigma} x^i \ , \qquad t \to e^{z\sigma} t \ . \tag{1}$$

The details of the trace anomaly depend also on the symmetry content of the theories, in particular if we require or not boost invariance. The case of theories without boost invariance (Lifshitz) were studied by several authors, e.g. [20–25]. It turns out that, in all the cases that have been studied so far, the scheme-independent trace anomalies at the fixed point have vanishing Weyl variation (type-B anomalies [26]).

Here we will consider the case of non-relativistic theories with galilean boost invariance, and we will focus for concreteness with the case of $2+1$ dimensions and $z = 2$. The natural background to which the non-relativistic theory is coupled is provided by the Newton–Cartan (NC) gravity. Two different perspectives concerning trace anomalies can be taken:

- The case where causality is not required on the gravity background; this setting was first studied by [27]. Note that causality is not a fundamental ingredient here, because the gravity background is not dynamical and it is just used as a source to define operators in the energy-momentum tensor multiplet. In principle the anomaly has a very rich structure, because an infinite number of terms can be written by dimensional analysis. All these terms live in separated sectors, which are labelled by the integer N_n; the sector in not changed by Weyl variations. In particular, in the simplest sector $N_n = 0$ the anomaly structure is identical to the trace anomaly of relativistic theories in 4 dimensions:

$$\mathcal{A} = T^i_{\ i} - 2T^0_{\ 0} = \sigma \left(-a\, E_4 + c\, W^2 + bR^2 \right) + \mathcal{A}_{\rm ct} + \dots \ . \tag{2}$$

  with $b = 0$ due to Wess–Zumino consistency conditions. A natural type-A candidate for a monotonicity theorem is the coefficient of the E_4 term. In Eq. (2) the curvatures are computed in term of an auxiliary relativistic 4 dimensional space, using the null-reduction trick (see Sect. 2); $\mathcal{A}_{\rm ct}$ indicates terms which can be removed by local counterterms, such as $D^2 R$. The dots in Eq. (2) correspond to possible extra terms from other sectors with $N_n \neq 0$.
- The case in which causality is imposed on the background metric (and consequently the Frobenius condition is imposed on the NC geometry) has a much simpler anomaly structure. The number of terms allowed by dimensional analysis

is finite [28] and the only scheme-independent anomaly turns out to be of type B [24, 28]. The structure of the anomaly is:

$$\mathcal{A} = 2T_0^0 + T_i^i = \mathcal{A} = b\sigma J^2 + \mathcal{A}_{\text{ct}} \,, \tag{3}$$

where J^2 is an anomaly term with zero Weyl variation, see [28] for the definition.

The anomaly with the Frobenius condition does not have any interesting candidates for an a-theorem; on the other hand the one without Frobenius condition has an interesting analog of the four dimensional a coefficient, which is a type A anomaly and is constrained by Wess–Zumino consistency condition in a non-trivial way. An analysis using the local RG approach, analog to the one performed in the relativistic case by [3–5], was performed in [29]; the outcome is that the same consistency conditions which give the perturbative a-theorem for four-dimensional relativistic theories can be written. On the other hand, we did not manage to prove that some anomaly coefficients which are positive-definite in the relativistic case are still positive definite, and so we were not able to extend Osborn's proof to the non-relativistic case.

In Sect. 2 I will review some basic elements of the Newton–Cartan gravity background, which here is used as a source to define the energy-tensor multiplet. In Sect. 3 I will discuss the calculation of the anomaly for free scalars and fermions, recently done in [30] and [31]. I conclude in Sect. 4.

2 Newton–Cartan Gravity Background

A Newton–Cartan (NC) gravity background in $d+1$ spacetime dimensions is defined by a 1-form n_μ (which corresponds to the local time direction), by a positive-definite symmetric tensor $h^{\mu\nu}$ with rank d (which corresponds to the space metric) for which n_μ is a zero eigenvector

$$n_\mu h^{\mu\alpha} = 0 \,, \tag{4}$$

and by a background gauge field A_μ for the particle number symmetry. A vector field v^μ, with property $n_\mu v^\mu = 1$, is also introduced; once v^μ is fixed, it is possible to uniquely define a degenerate rank d symmetric tensor $h_{\mu\nu}$, which corresponds to the metric along the spatial directions, which satisfies:

$$h^{\mu\alpha} h_{\alpha\nu} = \delta^\mu_\nu - v^\mu n_\nu = P^\mu_\nu \,, \qquad h_{\mu\alpha} v^\alpha = 0 \,, \tag{5}$$

where P^μ_ν is the projector onto the spatial directions. The NC geometry was originally introduced as a tool to write newtonian gravity in a diffeomorphism-invariant way; for a review see [32]. Recently it was realized in [33–36] that it is a very useful tool for condensed matter physics, because it is a very convenient way to parameterize the sources of the non-relativistic energy-momentum tensor. Useful recent references on NC geometry include [37–43].

The local version of the Galilean boost symmetry, which is usually called Milne boost, acts in a rather non-trivial way on the NC gravity fields. This makes the classification of geometrical invariants complicated. For this reason it is convenient to use an extra-dimensional null reduction (x^-, x^μ) from a relativistic parent space [44]:

$$G_{MN} = \begin{pmatrix} 0 & n_\mu \\ n_\nu & n_\mu A_\nu + n_\nu A_\mu + h_{\mu\nu} \end{pmatrix} ,$$
$$G^{MN} = \begin{pmatrix} A^2 - 2v \cdot A & v^\mu - h^{\mu\sigma} A_\sigma \\ v^\nu - h^{\nu\sigma} A_\sigma & h^{\mu\nu} \end{pmatrix} . \tag{6}$$

Diffeomorphism-invariant scalars in $d + 2$ dimensions are automatically Milne boos invariants in the non-relativistic $d + 1$ dimensional theory. We denote by D_A the covariant derivative defined by the Levi-Civita connection from the metric in Eq. (6), and by R_{ABCD}, R_{AB}, R the corresponding Riemann, Ricci and scalar curvatures. It is important to stress that, even if we are often using this extra-dimensional trick, we will compute the anomaly of the non-relativistic theory in $d + 1$ spacetime dimensions, and not of the $d + 2$ dimensional relativistic parent theory.

The anomaly then can be written [27] in term of the well-known four dimensional relativistic expression, see Eq. (2). Here there is a subtlety: extra type A terms in the anomaly may be present, because dimensional analysis allows for an infinite number of terms, divided in sectors which are not linked by Weyl transformations. An analysis of one of these sectors was done in [29]; no extra terms in the anomaly was found.

The null-reduction trick is also useful to write the action for non relativistic matter. For example, the action of the non-relativistic scalar comes form the null reduction of the relativistic one. Indeed, from the relativistic action

$$S = \frac{1}{4\pi} \int d^{d+2}x \sqrt{-\det G_{AB}} \left(-G^{MN} \partial_M \Phi^\dagger \partial_N \Phi - \xi R \Phi^\dagger \Phi \right) , \tag{7}$$

using the following ansatz along the null direction x^-:

$$\Phi(x^-, x^\mu) = \phi(x^\mu) e^{imx^-} . \tag{8}$$

and considering the reduction on a circle x^- with radius 4π, we get the non relativistic action of a scalar coupled to background NC geometry:

$$\int d^3x \sqrt{g} \left\{ imv^\mu \left(\phi^\dagger D_\mu \phi - D_\mu \phi^\dagger \phi \right) - h^{\mu\nu} D_\mu \phi^\dagger D_\nu \phi - \xi R \phi^\dagger \phi \right\} . \tag{9}$$

Here the covariant derivative includes just the gauge part:

$$D_\mu \phi = \partial_\mu \phi - imA_\mu \phi . \tag{10}$$

A similar null-reduction trick can be used to write the covariant non-relativistic action for a fermion (see e.g. [31] for details).

3 Calculation of the Anomaly for Scalars and Fermions

Here I shall review the calculation of the trace anomaly for the free non relativistic scalar and fermion recently done in [30] and [31] by the heat kernel method. We denote by W the vacuum functional of our theory:

$$e^{iW} = \int \mathcal{D}\phi^\dagger \, \mathcal{D}\phi \, e^{iS_D[\phi^\dagger,\, \phi]} \tag{11}$$

where S_D is the classical action specified by a differential operator D. In the bosonic case, the path integral is evaluated in terms of the functional determinant of the operator D as

$$iW = -\log\det(D)\,. \tag{12}$$

We can compute the anomaly in perturbation theory from the flat background, which corresponds to a differential operator $D = \triangle$:

$$\triangle = \left(-2im\partial_t + \partial_i^2\right) = \left(-2m\sqrt{-\partial_t^2} + \partial_i^2\right)\,. \tag{13}$$

The free flat-space heat kernel (which was introduced in [45] for the purpose to study entanglement entropy) is:

$$K_\triangle(s) = \langle xt|e^{s\triangle}|x't'\rangle = \frac{1}{2\pi}\,\frac{ms}{m^2s^2 + \frac{(t-t')^2}{4}}\,\frac{1}{(4\pi s)^{d/2}}\exp\left(-\frac{(x-x')^2}{4s}\right)\,. \tag{14}$$

The diagonal matrix elements in the coordinate basis of the curved-space heat kernel can be expanded in powers of s as:

$$\tilde{K}_{\hat\triangle}(s) = \langle xt|e^{s\hat\triangle}|xt\rangle_g = \frac{1}{s^{d/2+1}}\left(a_0(\hat\triangle) + a_2(\hat\triangle)s + a_4(\hat\triangle)s^2 + \ldots\right)\,. \tag{15}$$

This defines the De Witt–Seeley–Gilkey coefficients $a_{2k}(\hat\triangle)$ of the problem. In non-relativistic $2+1$ dimensional theories, the trace anomaly is proportional to the a_4 coefficient [30].

One can then evaluate the heat kernel in curved spacetime as a perturbative expansion from the flat case. The techniques can be borrowed from the relativistic case, as explained in the textbook [46]. A simple choice of background, that was used in [30], is:

$$n_\mu = \left(\frac{1}{1-\eta(x^i)}, 0, 0 \right) , \qquad v^\mu = \left(1-\eta(x^i), 0, 0\right) , \qquad h_{ij} = \delta_{ij} , \tag{16}$$

with $A_\mu = 0$[1]. We refer to [30] for the technical details of the calculation.

The result for the anomaly coefficients in Eq. (2) in the scalar case is:

$$a = \frac{1}{8m\pi^2} \frac{1}{360} , \quad c = \frac{1}{8m\pi^2} \frac{3}{360} , \quad b = \frac{1}{8m\pi^2} \frac{1}{2} \left(\xi - \frac{1}{6} \right)^2 . \tag{17}$$

The the case of a fermionic spin doublet was studied in [31] with similar methods. The result for the anomaly coefficients is:

$$a = \frac{1}{8m\pi^2} \frac{1}{360} \frac{11}{2} , \quad c = \frac{1}{8m\pi^2} \frac{9}{360} , \quad b = 0 . \tag{18}$$

Up to an overall $1/m$ factor, the anomaly coefficients in Eqs. (17), (18) are the same as the ones in the relativistic case in 4 dimensions, respectively for a free scalar and a free fermion. It would be interesting to understand the origin of this numerical coincidence.

The results in Eq. (17) have been recently reproduced with Fujikawa's method in [47]. On the other hand, we do not agree with the result found in [49].

4 Conclusions

It is tempting to conjecture that an analog of the a-theorem may hold for the E_4 coefficient of Schrödinger-invariant theories in 2 + 1 dimensions; indeed this is the simplest type A anomaly candidate which naturally occurs if we couple the theory to a NC background geometry. In the simple example where both the elementary (UV) and the composite (IR) degrees of freedom would be free scalars and fermions with spin 1/2, it would give the following non-trivial constraint:

$$a_{\rm UV} \propto \sum_{\rm scalars}^{\rm UV} \frac{1}{m} + \frac{11}{2} \sum_{\rm fermions}^{\rm UV} \frac{1}{m} \geq \sum_{\rm scalars}^{\rm IR} \frac{1}{m} + \frac{11}{2} \sum_{\rm fermions}^{\rm IR} \frac{1}{m} \propto a_{\rm IR} . \tag{19}$$

In Galilean-invariant theories the mass is a conserved quantity and the mass of a bound state is equal to the sum of the masses of the elementary constituents. Contrarily to the relativistic case, no bound-state contribution to the mass is present. As proposed in [30], the $1/m$ dependence is non-trivially consistent with the physical intuition that bound states form in the infrared: as energy is added, bound states tend to be broken and degrees of freedom cease to be frozen.

[1]The backgrounds with non-vanishing A_μ has some interesting subtleties: the anomaly contains some terms with are not $U(1)$ gauge-invariant, see Refs. [47, 48].

Several interesting issues call for further investigation:

- In the relativistic case, the anomaly coefficients are directly related to the the correlation functions of the energy-momentum evaluated at non-coincident points. For example, in four dimension c can be extracted from the two points functions and a from the three points functions of energy-momentum tensor. In the non-relativistic case, these correlators have support just at coincident points. It would be interesting to understand if the anomaly coefficients can be related to correlation function of energy momentum tensor also in the non-relativistic case.
- Another interesting direction is to attempt a perturbative proof of the non-relativistic a-theorem using Osborn's local renormalization group approach [29]. There is still a missing ingredient in the proof: one should show that some anomaly coefficients, whose relativistic analog turn out to be proportional to the Zamolodchikov metric, are positive.
- It might be interesting to study the relation between the anomaly and the dilaton effective action; in the relativistic case, this leads to a non-perturbative proof of the a-theorem [6]. Non-relativistic dilaton was recently studied in [50].
- The anomaly coefficients for anyons coupled to NC backgrounds should be computed. This might have condensed matter applications for the quantum Hall effect.
- For relativistic theories in four dimension at a fixed point, the trace anomaly coefficients and superconformal R-charges are related [51]: this is a powerful exact result which can be used to determine anomalous dimension of chiral operators in strongly coupled theories. It would be interesting to understand if a similar relation exists also in the non-relativistic case. The supersymmetric local RG formalism, investigated in [16], might be a convenient framework to investigate these issues. Newton–Cartan supergravity was studied in [52].

Acknowledgements I'm grateful to Stefano Baiguera, Francesco Filippini and Giuseppe Nardelli for collaboration on [28–31, 48] and comments on the manuscript. I thank the organisers of the Symposium QTS10 and Workshop LT12 for the nice and friendly atmosphere and for their warm hospitality in Varna.

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Evolution Operator Method and a Non-relativistic Particle in the Time-Dependent Homogeneous Field

E. I. Jafarov and S. M. Nagiyev

Abstract A unitary equivalence between the free non-relativistic particle problem and a particle in the time-dependent homogeneous field is established. Coherent states of a non-relativistic free particle and particle with a mass $M(t)$ in the time-dependent homogeneous field are constructed by the use of the evolution operator method. Main properties of the constructed coherent states, including time-dependent evolution of the corresponding probability density are discussed in detail. Explicit expressions of the oscillator-like wavefunctions for the systems under consideration are obtained, too.

Keywords Unitary equivalence · Evolution operator method · Time-dependent homogeneous field

1 Introduction

Explicit non-stationary solutions of quantum systems with time-dependent mass are of great interest among physicists due to number of applications of such solutions in various fields of modern physics [1–7]. It is well known that an attempt to solve the non-stationary Schrödinger equation or its relativistic generalizations does not lead always to the explicit solutions. Then, one needs to clarify, what kind of approaches can be used to obtain time-dependent explicit solution of the differential or finite-difference equation under consideration in terms of certain orthogonal polynomials or other special functions. Application of the evolution operator method is one of such powerful methods. Beauty of this method is that through knowledge of some initial state $\psi_0(x)$ of quantum dynamical system under consideration, one can find

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_34

time evolution of it by computing explicit form of state $\psi(x,t)$ at some other time t as action of $\hat{U}(x,t)$ evolution operator to initial state $\psi_0(x)$ as follows

$$\psi(x,t) = \hat{U}(x,t)\,\psi_0(x)\,. \tag{1}$$

Main goal of present work is study of main properties of a non-relativistic free particle and particle with a mass $M(t)$ in the time-dependent homogeneous field by the use of the evolution operator method. We construct coherent states of these dynamical systems and discuss in detail main properties of them, including time-dependent evolution of the corresponding probability density. Thereafter, explicit expressions of the oscillator-like wavefunctions for the systems under consideration are obtained, too. A unitary equivalence between the free non-relativistic particle problem with a time-dependent mass and a particle in the time-dependent homogeneous field is established.

Our paper is structured as follows: in Sect. 2, we discuss time evolution of a free non-relativistic particle and a particle in a homogeneous field. We show, how unitary equivalence between the free non-relativistic particle problem with a time-dependent mass and a particle in the time-dependent homogeneous field easily can be established. Further, in Sect. 3, we construct generalized coherent states for a free non-relativistic particle problem with a time-dependent mass and extend explicit expressions to the case of the non-relativistic particle in a time-dependent homogeneous field. Section 4 is devoted to the unitary equivalence for oscillator-like solutions of a free non-relativistic particle with time-dependent mass and a particle in a time-dependent field. By the use of evolution operator method, we compute explicitly oscillator-like wavefunctions of both problems under consideration. In Sect. 5, we briefly discuss obtained results.

2 Time Evolution of a Free Non-relativistic Particle and a Particle in a Homogeneous Field

Our starting point for study of time evolution of a free non-relativistic particle and a particle in a time-dependent homogeneous field is the Schrödinger equation with the time-dependent mass that has the following form for both problems under consideration:

$$\hat{S}_F(x,t)\,\psi_F(x,t) = 0, \quad \hat{S}_F(x,t) = i\hbar\partial_t + \frac{\hbar^2}{2M(t)}\partial_x^2, \tag{2a}$$

$$\hat{S}_L(x,t)\,\psi_L(x,t) = 0, \quad \hat{S}_L(x,t) = i\hbar\partial_t + \frac{\hbar^2}{2M(t)}\partial_x^2 + F(t)x, \tag{2b}$$

where, $M(t)$ and $F(t)$ are time-dependent mass and external field, respectively.

Then, evolution operators corresponding to Eqs. (2a) and (2b) have the following form:

$$U_F(x,t) = e^{i\hbar S_2(t)\partial_x^2}, \tag{3a}$$

$$U_L(x,t) = e^{ix\delta(t)/\hbar} e^{-\frac{i}{\hbar}\int\limits_0^t \frac{1}{2M(t')}\left[-i\hbar\partial_x + \delta(t')\right]^2 dt'}. \tag{3b}$$

By the use of (3a), one can rewrite (3b) in terms of the evolution operator of a free particle with time-dependent mass as follows:

$$U_L(x,t) = U_l(x,t)U_F(x,t), \tag{4}$$

where

$$U_l(x,t) = e^{i\varphi_0(x,t)} e^{-S_1(t)\partial_x}, \tag{5}$$

and

$$\delta(t) = \int\limits_0^t F(t')dt', \quad \varphi_0(x,t) = \frac{1}{\hbar}\left[x\delta(t) - S_0(t)\right], \tag{6}$$

$$S_0(t) = \int\limits_0^t \frac{\delta^2(t')}{2M(t')}dt', \quad S_1(t) = \int\limits_0^t \frac{\delta(t')}{M(t')}dt', \quad S_2(t) = \int\limits_0^t \frac{dt'}{2M(t')}.$$

In case, when $M(t) = m = const$, our notations become simpler as follows:

$$S_0(t) = \tfrac{\delta_2(t)}{2m}, \quad S_1(t) = \tfrac{\delta_1(t)}{2m}, \quad S_2(t) = \tfrac{t}{2m},$$
$$\delta_1(t) = \int\limits_0^t \delta(t')dt', \quad \delta_2(t) = \int\limits_0^t \delta^2(t')dt'.$$

Now, based on definition (1), one can write that

$$\psi_F(x,t) = U_F(x,t)\psi_0(x), \tag{7a}$$

$$\psi_L(x,t) = U_L(x,t)\psi_0(x). \tag{7b}$$

It is obvious that by choosing different values of initial stationary state $\psi_0(x)$, we will come to different expressions for time evolution of the non-stationary state $\psi_F(x,t)$ or $\psi_L(x,t)$. Also, one easily observes that $U_l(x,t)$ defined in (5) is an operator that transforms explicit solution of the Schrödinger equation for a free particle with time-dependent mass to the solution of a particle in a time-dependent external field, whereas, $U_l^{-1}(x,t)$ plays the role of the operator performing inverse transform of the solution of a particle in a time-dependent external field to the solution for a free particle with time-dependent mass. Symbolic expressions for both transforms are the following:

$$\psi_L(x,t) = U_l(x,t)\psi_F(x,t), \tag{8a}$$
$$\psi_F(x,t) = U_l^{-1}(x,t)\psi_L(x,t). \tag{8b}$$

In the momentum representation, the Schrödinger equation corresponding to both problems under consideration is written down as follows:

$$\hat{S}_F(p,t)\,\phi_F(p,t) = 0, \quad \hat{S}_F(p,t) = i\hbar\partial_t - \frac{p^2}{2M(t)}, \tag{9a}$$
$$\hat{S}_L(p,t)\,\phi_L(p,t) = 0, \quad \hat{S}_L(p,t) = i\hbar\partial_t - \frac{p^2}{2M(t)} + i\hbar F(t)\partial_p. \tag{9b}$$

Then, evolution operators (3a) and (3b) are also simpler than in the position representation and first one is multiplication operator, whereas, second one is simply shift operator:

$$U_F(p,t) = e^{-iS_2(t)p^2/\hbar}, \tag{10a}$$
$$U_L(p,t) = e^{-\frac{i}{\hbar}\int\limits_0^t \frac{1}{2M(t')}\left[p-\delta(t)+\delta(t')\right]^2 dt'} e^{-\delta(t)\partial_p} = U_l(p,t)U_F(p,t), \tag{10b}$$

with the following notations:

$$U_l(p,t) = e^{-i\varphi_1(p,t)}e^{-\delta(t)\partial_p}, \quad \varphi_1(p,t) = \frac{1}{\hbar}\left\{S_0(t) + S_1(t)\left[p-\delta(t)\right]\right\}.$$

3 Generalized Coherent States

In order to construct generalized coherent states [9], first of all, we introduce two creation and annihilation operators

$$a^+ = \frac{1}{\sqrt{2\hbar}}\left(\lambda_1^*\hat{x} - i\lambda_2^*\hat{p}\right), \quad a^- = \frac{1}{\sqrt{2\hbar}}\left(\lambda_1\hat{x} + i\lambda_2\hat{p}\right). \tag{11}$$

Here, λ_1 and λ_2 are complex parameters. It follows from commutation relation $\left[a^-, a^+\right] = 1$ that the condition

$$\mathrm{Re}\left(\lambda_1^*\lambda_2\right) = 1$$

should be satisfied. Then, using (11), it is easy to show that operators of the integrals of motion of a free non-relativistic particle have the following forms:

$$\hat{A}_F^{+}(t) = U_F(x,t)a^{+}U_F^{-1}(x,t) = \frac{1}{\sqrt{2\hbar}}\left[\lambda_1^{*}\hat{x} - i\varepsilon^{*}(t)\hat{p}\right], \tag{12a}$$

$$\hat{A}_F^{-}(t) = U_F(x,t)a^{-}U_F^{-1}(x,t) = \frac{1}{\sqrt{2\hbar}}\left[\lambda_1\hat{x} + i\varepsilon(t)\hat{p}\right], \tag{12b}$$

where, $\varepsilon(t) = \lambda_2 + 2i\lambda_1 S_2(t)$. From (12a) and (12b), it is clear that

$$\hat{x} = \sqrt{\frac{\hbar}{2}}\left[\varepsilon^{*}(t)\hat{A}_F^{-}(t) + \varepsilon(t)\hat{A}_F^{+}(t)\right], \tag{13}$$

$$\hat{p} = i\sqrt{\frac{\hbar}{2}}\left[\lambda_1\hat{A}_F^{+}(t) - \lambda_1^{*}\hat{A}_F^{-}(t)\right]. \tag{14}$$

Now, we can introduce generalized coherent states $|z,t\rangle_F$ of a free non-relativistic particle with time-dependent mass as eigenstate of operator $A_F^{-}(t)$ as follows:

$$\hat{A}_F^{-}(t)\,|z,t\rangle_F = z\,|z,t\rangle_F\,, \tag{15}$$

with eigenvalue z being a complex number. We compute explicit form of the coherent states of a free non-relativistic particle by the use of evolution operator method and it has the following explicit form:

$$\psi_z^F(x,t) = U_F(x,t)\psi_z(x) = N_0\exp\left\{-\frac{\lambda_1\left[x - \bar{x}_F(t)\right]^2}{2\hbar\varepsilon(t)} - \frac{i}{\hbar}p_0\left[x - p_0 S_2(t)\right]\right\}, \tag{16}$$

where, $\psi_z^F(x,t) \equiv \langle x\,|z,t\rangle_F$, $\psi_z(x) \equiv \langle x\,|z\rangle$ and the normalization factor being equal to $N_0 = (\pi\hbar)^{-\frac{1}{4}}\left[\varepsilon(t)\right]^{-\frac{1}{2}}$ is found from the condition $\int\limits_{-\infty}^{\infty}\left|\psi_z^F(x,t)\right|^2 dx = 1$. Also, here

$$\bar{x}_F(t) \equiv {}_F\langle z,t|\,\hat{x}\,|z,t\rangle_F = x_0 + 2p_0 S_2(t)\,, \quad \bar{p}_F(t) \equiv {}_F\langle z,t|\,\hat{p}\,|z,t\rangle_F = p_0,$$
$$z = {}_F\langle z,t|\,\hat{A}_F^{-}(t)\,|z,t\rangle_F = \tfrac{1}{\sqrt{2\hbar}}\left[\lambda_1 x_0 + i\lambda_2 p_0\right], \quad a\,|z\rangle = z\,|z\rangle\,,$$
$$x_0 \equiv \langle z|\,\hat{x}\,|z\rangle = \sqrt{\tfrac{\hbar}{2}}\left(\lambda_2^{*}z + \lambda_2 z^{*}\right), \quad p_0 \equiv \langle z|\,\hat{p}\,|z\rangle = i\sqrt{\tfrac{\hbar}{2}}\left(\lambda_1 z^{*} - \lambda_1^{*}z\right).$$

Now, as a next step one can easily compute probability density of position generated by coherent states (16) as follows:

$$\rho_z^F(x,t) = \left|\psi_z^F(x,t)\right|^2 = \frac{1}{\sqrt{\pi\hbar}\,|\varepsilon(t)|}\exp\left\{-\frac{\left[x - \bar{x}_F(t)\right]^2}{\hbar\,|\varepsilon(t)|^2}\right\}. \tag{17}$$

Explicit expression of the coherent states can be computed also by the use of Glauber method [8], which is based on action of shift operator $D_F(z,t) = \exp\left[z\hat{A}_F^+(t) - z^*\hat{A}_F^-(t)\right]$ to the ground state $\psi_0^F(x,t)$:

$$\tilde{\psi}_z^F(x,t) = D_F(z,t)\,\psi_0^F(x,t) = \exp\left(-\frac{1}{2}|z|^2\right)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\psi_n^F(x,t). \tag{18}$$

Here

$$\psi_n^F(x,t) = \frac{\left[\hat{A}_F^+(t)\right]^n}{\sqrt{n!}}\psi_0^F(z,t) = N_n\left(\frac{\varepsilon^*}{\varepsilon}\right)^{n/2} H_n\left(\frac{x}{\sqrt{\hbar}\,|\varepsilon|}\right)e^{-\frac{\lambda_1 x^2}{2\hbar\varepsilon}}, \tag{19}$$

with $H_n(x)$ being Hermite polynomials and $N_n = N_0/\sqrt{2^n n!}$.

One observes that the difference between two explicit expressions (16) and (18) is the phase factor, i.e. $\psi_z^F(x,t) = \exp(ip_0x_0/2\hbar)\,\tilde{\psi}_z^F(x,t)$.

We can do similar computations and find explicit expression of the coherent states of a free non-relativistic particle in the momentum representation, too. In this case, explicit expression will have the following form:

$$\varphi_z^F(p,t) = N_0'\exp\left\{-\frac{\varepsilon(t)}{2\hbar\lambda_1}(p-p_0)^2 - \frac{i\overline{x}_F(t)}{\hbar}(p-p_0) - \frac{ip_0^2}{\hbar}S_2(t)\right\}, \tag{20}$$

where, $N_0' = (\pi\hbar)^{-1/4}\lambda_1^{-1/2}$.

Probability density of momentum computed by the use of coherent states (20) will lead to the expression

$$\rho_z^F(p,t) = \left|\varphi_z^F(p,t)\right|^2 = \left|N_0'\right|^2\exp\left\{-\frac{(p-p_0)^2}{\hbar|\lambda_1|^2}\right\}. \tag{21}$$

Here, one observes that (21) does not depend on time t.

Now, having complete information about the coherent states of a free non-relativistic particle with the time-dependent mass, it is possible to generalize obtained results to the case of the non-relativistic particle in a time-dependent external field. It is possible to show that integrals of motion of position and momentum operators $\hat{x}$ and $\hat{p}$ should be defined through integrals of motion of a free non-relativistic particle (12a) and (12b) as follows:

$$\hat{A}_L^-(t) = U_l\hat{A}_F^-(t)\,U_l^{-1} = \frac{1}{\sqrt{2\hbar}}\left[\lambda_1\hat{x}_1(t) + i\varepsilon(t)\,\hat{p}_1(t)\right], \tag{22a}$$

$$\hat{A}_L^+(t) = U_l\hat{A}_F^+(t)\,U_l^{-1} = \frac{1}{\sqrt{2\hbar}}\left[\lambda_1^*\hat{x}_1(t) - i\varepsilon^*(t)\,\hat{p}_1(t)\right], \tag{22b}$$

where, $\hat{x}_1(t) = \hat{x} - S_1(t)$ and $\hat{p}_1(t) = \hat{p} - \delta(t)$. Hence, one easily finds that

$$\hat{x} = S_1(t) + \sqrt{\frac{\hbar}{2}}\left[\varepsilon^*(t)\hat{A}_L^-(t) + \varepsilon(t)\hat{A}_L^+(t)\right], \ \hat{p} = \delta(t) + i\sqrt{\frac{\hbar}{2}}\left[\lambda_1\hat{A}_L^+(t) - \lambda_1^*\hat{A}_L^-(t)\right]. \quad (23)$$

We introduce generalized coherent states of the non-relativistic particle in a time-dependent external field as eigenfunction of operator $\hat{A}_L^-$ (22a):

$$\hat{A}_L^-(t)\,|z,t\rangle_L = z\,|z,t\rangle_L\,. \quad (24)$$

Taking into account a unitary equivalence between the free particle problem and a particle in the time-dependent homogeneous field, explicit form of the coherent state in position representation $\psi_z^L(x,t)$ is computed as an action of operator $U_l(x,t)$ (5) to coherent state of the free non-relativistic particle $\psi_z^F(x,t)$ (16) as follows:

$$\psi_z^L(x,t) = U_l(t)\,\psi_z^F(x,t) = N_0\exp\left\{-\frac{\lambda_1[x-\bar{x}_L(t)]^2}{2\hbar\varepsilon(t)} + \frac{i}{\hbar}p_0[x_1(t) - p_0S_2(t)] + i\varphi_0(x,t)\right\}, \quad (25)$$

where, $x_1(t) = x - S_1(t)$ and

$$\bar{x}_L(t) \equiv {}_L\langle z,t|\,\hat{x}\,|z,t\rangle_L = \bar{x}_F(t) + S_1(t)\,,$$
$$\bar{p}_L(t) \equiv {}_L\langle z,t|\,\hat{p}\,|z,t\rangle_L = p_0 + \delta(t)\,.$$

Like the case of free non-relativistic particle with time-dependent mass, generalized coherent states of a non-relativistic particle in a time-dependent external field in position representation can be computed explicitly by the use of Glauber method through the action of shift operator $D_L(z,t) = U_lD_F(z,t)\,U_l^{-1} = \exp\left[z\hat{A}_L^+(t) - z^*\hat{A}_L^-(t)\right]$ to the ground state $\psi_0^L(x,t)$ [8]:

$$\tilde{\psi}_z^L(x,t) = D_L(z,t)\,\psi_0^L(x,t) = \exp\left(-\frac{1}{2}|z|^2\right)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\psi_n^L(x,t), \quad (26)$$

where

$$\psi_n^L(x,t) = \frac{\left[\hat{A}_L^+(t)\right]^n}{\sqrt{n!}}\psi_0^L(x,t) = N_n\left(\frac{\varepsilon^*}{\varepsilon}\right)^{n/2}H_n\left(\frac{x_1(t)}{\sqrt{\hbar}\,|\varepsilon|}\right)e^{-\frac{\lambda_1x_1^2(t)}{2\hbar\varepsilon}+i\varphi_0(x,t)}.$$

Probability density of the position corresponding to coherent state (25) equals to

$$\rho_z^L(x,t) = \left|\psi_z^L(x,t)\right|^2 = \frac{1}{\sqrt{\pi\hbar}\,|\varepsilon(t)|}\exp\left\{-\frac{[x-\bar{x}_L(t)]^2}{\hbar\,|\varepsilon(t)|^2}\right\}. \quad (27)$$

4 Unitary Equivalence for Oscillator-Like Solutions of a Free Non-relativistic Particle with Time-Dependent Mass and a Particle in a Time-Dependent Field

By the use of the evolution operator method, one can obtain various solutions of a free non-relativistic particle with time-dependent mass and then easily show that evolution operator easily transforms obtained solutions to similar oscillatory solutions for a non-relativistic particle in a time-dependent homogeneous field. We choose the following function as our initial wavefunction [10]:

$$\psi_n(x) = \frac{\left(a^+\right)^n}{\sqrt{n!}}\psi_0(x) = c_n\left(\frac{\lambda_2^*}{\lambda_2}\right)^{n/2} H_n\left(\frac{x}{\sqrt{\hbar}\,|\lambda_2|}\right)e^{-\frac{\lambda_1 x^2}{2\hbar\lambda_2}}, \tag{28}$$

where

$$c_n = c_0/\sqrt{2^n n!}, \quad c_0 = (\pi\hbar)^{-1/4}\,\lambda_2^{-1/2}, \quad n = 0, 1, 2, \ldots.$$

Then, simple straightforward computations through the action of the evolution operator of a free non-relativistic particle with the time-dependent mass $U_F(x, t)$ (3a) to (28) allow us to obtain the following explicit expression of the oscillator-like non-stationary wavefunctions of a problem under consideration:

$$\psi_n^F(x,t) = U_F(x,t)\psi_n(x) = N_n\left(\frac{\varepsilon^*}{\varepsilon}\right)^{n/2} H_n\left(\frac{x}{\sqrt{\hbar}|\varepsilon(t)|}\right)e^{-\frac{\lambda_1 x^2}{2\hbar\varepsilon(t)}}, \tag{29}$$

which one completely overlaps with (19). The probability density of position corresponding to (29) also can be computed easily as follows:

$$\rho_n^F(x,t) = \left|\psi_n^F(x,t)\right|^2 = |N_n|^2\, H_n^2\left(\frac{x}{\sqrt{\hbar}|\varepsilon(t)|}\right)e^{-\frac{x^2}{\hbar|\varepsilon(t)|}}. \tag{30}$$

Now, we are able to extend these results to the case of a non-relativistic particle in a time-dependent homogeneous field. In this case, it is sufficient to act by $U_l(x, t)$ to the oscillator-like wavefunctions of a free non-relativistic particle with the time-dependent mass (29):

$$\psi_n^L(x,t) = U_l(x,t)\psi_n^F(x,t) = N_n\left(\frac{\varepsilon^*}{\varepsilon}\right)^{n/2} H_n\left(\frac{x_1(t)}{\sqrt{\hbar}|\varepsilon(t)|}\right)e^{i\varphi_0(x,t)-\frac{\lambda_1 x_1^2(t)}{2\hbar\varepsilon(t)}}. \tag{31}$$

Further, the probability density of position corresponding to (31) also can be computed easily as follows:

$$\rho_n^L(x,t) = \left|\psi_n^L(x,t)\right|^2 = |N_n|^2\, H_n^2\left(\frac{x_1(t)}{\sqrt{\hbar}|\varepsilon(t)|}\right)e^{-\frac{x_1^2(t)}{\hbar|\varepsilon(t)|}}. \tag{32}$$

5 Conclusions

In this paper, a unitary equivalence between the free non-relativistic particle problem and a particle in the time-dependent homogeneous field is established. We were able to compute explicitly generalized coherent states of a non-relativistic free particle and particle with a mass $M(t)$ in the time-dependent homogeneous field $F(t)$. For our computations, we used evolution operator method and simply extended our results for a free non-relativistic particle with a time-dependent mass to the case of a non-relativistic particle in a time-dependent external field. Some basic properties of the generalized coherent states for these problems are discussed, too. Then, we also computed explicitly oscillator-like wavefunctions for both systems under consideration.

Evolution operator method is extremely fruitful being applied to non-relativistic quantum (both stationary and non-stationary) systems. This is explained by the following reasons:

- Evolution operator greatly simplifies the procedures for obtaining solutions of the non-stationary Schrödinger equation for a free particle and a particle in a time-dependent homogeneous field;
- Explicit expressions of the evolution operators for a free particle with a time-dependent mass and a particle in the time-dependent homogeneous field makes it possible to establish a unitary equivalence between these systems;
- Evolution operator method is the simplest direct method for finding the solutions with given property (for example, quadratically integrable and non-integrable). In fact, it is sufficient for this that the initial wave function possesses this property.

Acknowledgements This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan - Grants Nr EIF-KETPL-2-2015-1(25)-56/01/1 & EIF-KETPL-2-2015-1(25)-56/02/1.

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