Chapter 21
A Fuzzy Random Continuous (Q, r, L) Inventory Model Involving Controllable Back-order Rate and Variable Lead-Time with Imprecise Chance Constraint

Debjani Chakraborty, Sushil Kumar Bhuiya and Debdas Ghosh

Abstract In this article, we analyze a fuzzy random continuous review inventory system with the mixture of back-orders and lost sales, where the annual demand is treated as a fuzzy random variable. The study under consideration assumes that the lead-time is a control variable and the lead-time crashing cost is being introduced as a negative exponential function of the lead-time. In a realistic situation, the back-order rate is dependent on the lead-time. Significantly large lead-times might lead to stock-out periods being longer. As a result, many customers may not be prepared to wait for back-orders. Instead of constant back-order rate, we introduce the back-order rate as a decision variable, which is a function of the lead-time throughout the amount of shortage. Moreover, a budgetary constraint is imposed on the model in the form of an imprecise chance constraint to capture the possible way of measuring the imprecisely defined uncertain information of the budget constraint. We develop a methodology to determine the optimum order quantity, reorder point, lead-time, and back-order rate such that the total cost is minimized in the fuzzy sense. Finally, a numerical example is presented to illustrate the proposed methodology.

Keywords Inventory · Imprecise chance constraint · Fuzzy random variable
Possibilistic mean value

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1 Introduction

Inventory control plays a significant role in every production house. The continuous review inventory model is one of the most important and useful problems in industrial applications. In the continuous review inventory system, the occurrence of the shortage is a major concern. In most of the real-life situations, when such a condition arises, back-orders and lost sales happen simultaneously. Thus, the inventory model, which constitutes both back-order and lost sale cases, is more efficient than the ones based on the individual cases. Montgomery et al. [23] first introduced the inventory model with a mixture of back-orders and lost sales. After the pioneering work of [23], numerous related studies have been developed considerably in the problem of mixture of back-orders and lost sales (see, among others [16, 22, 31]).

In the earlier literature dealing with inventory systems, the lead-time is commonly considered as a prescribed constant or a stochastic variable. Hence, the lead-time becomes uncontrollable [26]. But, production management philosophies like just in time (JIT) show that there are advantages and benefits associated with the effort of control of the lead-time. By shortening lead-time [35], we can decrease the safety stock, minimize the loss due to stock-out, improve the service level to the customer, and increase the competitive capability in business. Liao and Shyu [21] first introduced the problem of lead-time reduction in a continuous review inventory model, where the order quantity was predetermined, and the lead-time was assumed to be a decision variable. Thereafter, several researchers (see, among others [2, 14, 20, 22, 25, 28–30]) have studied lead-time reduction in different types of inventory system.

In addition to lead-time, another key aspect of the inventory system is back-orders. Most of the earlier work in the field of inventory control, it is assumed that the back-order rate is constant. However, in a realistic situation, the back-order rate is dependent on the lead-time. Bigger lead-times might lead to stock-out periods being longer; and as a result, many customers may not be willing for back-orders. This phenomenon reveals that under the longer length of the lead-time, the period of shortage becomes longer. It signifies that the proportion of customers that can wait goes down; as a result, back-order rate decreases. The interdependence between the back-order rate and the lead-time has been proposed by Ouyang and Chuang [28]. They have considered the back-order rate to be dependent on the length of the lead-time through the amount of shortage and that the back-order rate is a control variable. After the work of [28], researchers have been attracted on the problem of controlling back-order rate, and they have extended the inventory control in various directions (see, among others [20, 22, 33]).

On the other hand, most of the real-life business situation, the decision maker has to work under limited budget. According to Hadley and Whitin [15], the most significant real-world constraint is the budgetary restriction on the amount of capital that can be contributed to procure the items of inventory. Keeping this in mind, many inventory models (see, among others [1, 18, 24]) have been developed under budgetary constraint in stochastic environment.

During the mid-1980s, researchers have noticed that the fuzziness is also an intrinsic property of key parameters of the inventory system, particularly when given or obtained data is undetectable, insufficient or partially ignorant. After that, fuzzy set theory has been extensively employed in the problem of inventory system for capturing the uncertainties in the non-probabilistic sense. Park [32] introduced the fuzzy mathematics in the inventory system by developing economic order quantity (EOQ) model in which trapezoidal fuzzy numbers were represented the ordering and holding costs. Gen et al. [13] developed a continuous review inventory model where the values of the parameters of inventory system are considered to be triangular fuzzy numbers. Ouyang and Yao [27] extended min-max distribution-free procedure in the fuzzy environment by developing a continuous review inventory model with variable lead-time in which the annual demand was assumed as the triangular fuzzy number. Tütüncü et al. [36] and Vijayan and Kumaran [37] studied the continuous review inventory model by fuzzifying the cost parameters into fuzzy numbers. Tütüncü et al. obtain the solution using a simulation-based analysis, while an iterative algorithm was used to derive the optimal solution by Vijayan and Kumaran. Recently, Shekarian et al. [34] presented a comprehensive review of the most relevant works of fuzzy inventory model.

It can be noticed that the models, primarily the ones as mentioned above, capture the uncertainty of the parameters of inventory system by characterizing the corresponding variable as either fuzzy or random variable. In a real-life inventory system, fuzziness and randomness often co-occur. Kwakernaak [19] first described the fuzziness and randomness of an event simultaneously. Dutta et al. [12] first incorporated the mixture of fuzziness and randomness into annual demand and developed a single periodic review inventory model. Chang et al. [5] and Dutta and Chakraborty [11] analyzed and extended the continuous review inventory model into fuzzy random circumstances. Chang et al. [5] treated the lead time as the fuzzy random variable and annual expected demand as the fuzzy number. On the other hand, Dutta and Chakraborty [11] considered both the lead-time and annual demand as discrete fuzzy random variables. Dey and Chakraborty [10] considered the annual demand as a fuzzy random variable for developing a periodic review inventory model. Dey and Chakraborty [9] proposed a methodology for constructing a fuzzy random data set from the partially known information. This method is applied on the fuzzy random periodic review model developed by Dey and Chakraborty [10]. Moreover, Dey and Chakraborty [8] also extended the model [10] by incorporating negative exponential crashing cost and lead-time as a variable. Kumar and Goswami [17] extended the min-max distribution-free approach in fuzzy random environments by developing a continuous review production–inventory system. Now, with increased complexity of inventory problem domain, it is hard to define budgetary constraint with proper certainty and precision. Chance-constrained programming [6] can be providing a procedure to construct the constraints in the presence of randomness. However, the imprecision and randomness may appear combined in the information of the restriction. Keeping the issue of vagueness in mind, Chakraborty [4] redefined the chance constraint as the imprecise chance constraint in which the probability of satisfying the imprecise constraint is considered to be vague in nature and to be imprecisely greater than or equal to a specified probability. Recently, Dey et al. [7] incorporated imprecise chance constraint into a fuzzy random continuous review inventory model with a mixture of back-orders and lost sales.

An analysis of the literature reveals that there are some studies of the continuous review inventory system that consider both the fuzziness and randomness simultaneously. But, existing research does not assemble the controllable lead-time and back-order rate in the mixed fuzzy random framework. Here, our intention is to address this research gap of the continuous review inventory model under fuzzy random environment.

Thus, in this paper, we consider a fuzzy random continuous review (Q, r, L) inventory model inclusive of back-orders and lost sales by including the annual demand as the fuzzy random variable. The lead-time is taken as a decision variable, and the crashing cost is being introduced by the negative exponential function of the lead-time. The back-order rate is also a decision variable, which is a function of the lead-time through an amount of shortages. A budgetary constraint has been considered on the model in the form of an imprecise chance constraint. A methodology has been developed to determine the optimal values of the decision-making variable such that the annual cost of the inventory model is minimized in the fuzzy sense. Finally, a numerical example is provided to illustrate the proposed methodology.

The rest of paper is organized as follows: Sect. 2 presents some basic concepts of fuzzy set theory. In Sect. 3, development of proposed methodology is discussed. We present a numerical example to illustrate the methodology in Sect. 4. Paper has been summarized in Sect. 5.

2 Preliminaries

In this section, we review some basic concepts of the fuzzy set theory in which will be used in this paper.

Definition 1 (*Triangular fuzzy number* [38]). A normalized triangular fuzzy number $\tilde{A} = (a, b, c)$ is a fuzzy subset of the real line $\mathbb{R}$, whose membership function $\mu_{\tilde{A}}(x)$ satisfies the following conditions:

(i) $\mu_{\tilde{A}}(x)$ is a continuous function from $\mathbb{R}$ to the closed interval [0, 1],
(ii) $\mu_{\tilde{A}}(x) = \frac{x-a}{b-a}$ is strictly increasing function on $[a, b]$,
(iii) $\mu_{\tilde{A}}(x) = 1$ for $x = b$,
(iv) $\mu_{\tilde{A}}(x) = \frac{c-x}{c-b}$ is strictly decreasing function on $[b, c]$,
(v) $\mu_{\tilde{A}}(x) = 0$ elsewhere.

Without any loss of generality, all fuzzy quantities are assumed as triangular fuzzy numbers throughout this paper.

Definition 2 (*α-cut of fuzzy set* [38]). Let $\tilde{A}$ be a fuzzy set. The α-cut of the fuzzy set $\tilde{A}$, denoted by $\tilde{A}_\alpha = [A_\alpha^-, A_\alpha^+]$, is defined as follows:

$$\tilde{A}_\alpha = \begin{cases} \{x \in \mathbb{R} : \mu_{\tilde{A}}(x) \geq \alpha\} & \text{if } \alpha \in (0, 1] \\ cl\{x \in \mathbb{R} : \mu_{\tilde{A}}(x) > 0\} & \text{if } \alpha = 0. \end{cases} \tag{1}$$

Definition 3 (*Fuzzy random variable* [19]). Let $(\Omega, \mathcal{B}, \mathcal{P})$ be a probability space and $F(\mathbb{R})$ be the set all all fuzzy numbers, then a mapping $\tilde{\chi} : \Omega \to F(\mathbb{R})$ is said to be a fuzzy random variable (or FRV for short) if for all $\alpha \in [0, 1]$, the two real-valued mappings $\chi_\alpha^- : \Omega \to \mathbb{R}$ and $\chi_\alpha^+ : \Omega \to \mathbb{R}$ are real-valued random variable.

Definition 4 (*Expectation of fuzzy random variable* [19]). If $\tilde{X}$ is a fuzzy random variable, then the fuzzy expectation of $\tilde{X}$ is a unique fuzzy number. It is defined by

$$E(\tilde{X}) = \int \tilde{X} \, \mathrm{d}P = \left\{ \left(\int X_\alpha^- \mathrm{d}P, \int X_\alpha^+ \mathrm{d}P \right) : 0 \leq \alpha \leq 1 \right\}, \tag{2}$$

where the α-cut of fuzzy random variable is $[\tilde{X}]_\alpha = [X_\alpha^-, X_\alpha^+]$ for all $\alpha \in [0, 1]$.

Definition 5 (*Possibilistic mean value of a fuzzy number* [3]). Let $\tilde{A}$ be a fuzzy number with α-cut $\tilde{A}_\alpha = [A_\alpha^-, A_\alpha^+]$, and therefore, the possibilistic mean value of $\tilde{A}$ is denoted by $\overline{M}(\tilde{A})$ and defined as

$$\overline{M}(\tilde{A}) = \int_0^1 \alpha (A_\alpha^- + A_\alpha^+) d\alpha. \tag{3}$$

3 Methodology

3.1 Model and Assumptions

The inventory position is reviewed continuously in the (Q, r) continuous review inventory system. When the stock position falls to the reordering point r, an order quantity Q is placed to order. In inventory system, a state is said to be the stock-out state if inventory level falls to zero, at any point in time. Considering the simultaneous occurrence of back-orders and lost sales in real-world scenario, the effect of both are included in the model. The following notations have been used to develop the model:

Notations

P fixed ordering cost per order,
h holding cost per unit per year,
π stock-out cost per unit stock-out,

π_0 marginal profit per unit,
Q order quantity,
r reorder point,
β fraction of demand back-ordered during the stock-out period, $(0 \leq \beta \leq 1)$,
L lead-time (in years),
$R(L)$ lead-time crashing cost,
$\tilde{D}(\omega)$ annual demand ($\omega \in \Omega$ where $(\Omega, \mathcal{B}, \mathcal{P})$ is a probability space),
$\tilde{D}_L(\omega)$ lead-time demand ($\omega \in \Omega$),
x^+ $\max\{0, x\}$.

In continuous review inventory system, the safety stock or buffer stock is defined as the difference between reorder point r and the expected lead-time demand. Now, for all practical purposes, none of the manufacturer wants to have a negative safety stock. Therefore, nonnegative safety stock criterion is imposed on the model. To maintain the safety stock at nonnegative level, $r \geq \overline{M}(\tilde{D}_L)$ has been considered, where $\overline{M}(\tilde{D}_L)$ denotes the expected lead-time demand in possibilistic sense and defined by

$$\overline{M}(\tilde{D}_L) = \int_0^1 \alpha \left[D^-_{L,\alpha} + D^+_{L,\alpha} \right] \mathrm{d}\alpha. \tag{4}$$

In order to incorporate fuzziness and randomness simultaneously [11], the annual demand is assumed to be a discrete fuzzy random variable $\tilde{D}(\omega)$ ($\omega \in \Omega$ where $(\Omega, \mathcal{B}, \mathcal{P})$). Let us suppose that the annual customer demand $\tilde{D}(\omega)$ is of the form $\{(\tilde{D}_1, p_1), (\tilde{D}_2, p_2), \ldots, (\tilde{D}_n, p_n)\}$, where each of $\tilde{D}_i$'s are triangular fuzzy numbers of the form $(\underline{D}_i, D_i, \overline{D}_i)$ with corresponding probabilities p_i's, $i = 1, 2, \ldots, n$. Moreover, the lead-time demand is reflected by any fluctuation of the annual demand. Thus instead of independent parameter, the lead-time demand is assumed to be connected to the annual demand through the length of the lead-time in the following form [11]:

$$\tilde{D}_L(\omega) = \tilde{D}(\omega) \times L. \tag{5}$$

Since annual demand $\tilde{D}(\omega)$ is a fuzzy random variable of the form $\tilde{D}_i = (\underline{D}_i, D_i, \overline{D}_i)$ with associated probability p_i, $i = 1, 2, \ldots, n$, the lead-time demand is also fuzzy random variable. Thus, the lead-time demand is of the form $\tilde{D}_{L,i} = (\underline{D}_{L,i}, D_{L,i}, \overline{D}_{L,i})$ with associated probability p_i, $i = 1, 2, \ldots, n$. Hence, the expected lead-time demand can be expressed in triangular form. The triangular form of expected lead-time demand is given by $E(\tilde{D}_L(\omega)) = \tilde{D}_L = (\underline{D}_L, D_L, \overline{D}_L)$. The annual demand $\tilde{D}(\omega)$ and the lead-time demand $\tilde{D}_L(\omega)$ can be represented by its α-cut as $[\tilde{D}(\omega)]_\alpha = [D^-_\alpha(\omega), D^+_\alpha(\omega)]$ and $[\tilde{D}_L(\omega)]_\alpha = [D^-_{L,\alpha}(\omega), D^+_{L,\alpha}(\omega)]$ where $\alpha \in [0, 1]$. The α-cut representation of the expected lead-time demand is defined as follows:

$$D^-_{L,\alpha}(\omega) = D^-_{L,\alpha}(\omega) \times L \quad \text{and} \quad D^+_{L,\alpha}(\omega) = D^+_{L,\alpha}(\omega) \times L \tag{6}$$

$$\Rightarrow \begin{cases} E\left(D^{-}_{L,\alpha}(\omega)\right) = \sum_{i=1}^{n} D^{-}_{i,\alpha} p_i \times L \\ E\left(D^{+}_{L,\alpha}(\omega)\right) = \sum_{i=1}^{n} D^{+}_{i,\alpha} p_i \times L \end{cases} \tag{7}$$

In this study, we consider the lead-time is a decision variable and the lead-time crashing cost is assumed to be a negative exponential function [8] of the lead-time, which is given by

$$\text{Crashing cost } R(L) = \varepsilon e^{-\delta L} \tag{8}$$

where we can estimate the parameters ε, δ by some of known values of the lead-time crashing cost for a few values of L.

A function of fuzzy random variable is itself a fuzzy random variable; therefore, total cost function is also a fuzzy random variable. Thus, the fuzzy total cost function is given by

$$\tilde{C}(Q, r, L) = h\left[\frac{Q}{2} + r - \tilde{D}(\omega)L\right] + \left[h(1-\beta) + \{\pi + \pi_0(1-\beta)\}\frac{\tilde{D}(\omega)}{Q}\right]\overline{M}(\tilde{D}_L - r)^{+} \\ + \frac{\tilde{D}(\omega)}{Q}(R(L) + P) \tag{9}$$

where $\overline{M}(\tilde{D}_L - r)^{+}$ denote the expected shortage at each cycle in possibilistic sense and defined by

$$\overline{M}\left(\tilde{D}_L - r\right)^{+} = \int_0^1 \alpha\left[\left((\tilde{D}_L - r)^{+}\right)^{-}_{\alpha} + \left((\tilde{D}_L - r)^{+}\right)^{+}_{\alpha}\right]\mathrm{d}\alpha. \tag{10}$$

As mentioned earlier, in a realistic situation, the back-order rate is dependent on the lead-time. Significantly large lead-times might lead to stock-out periods longer, and as a result, many customers may not be willing for back-orders. This phenomenon reveals that with the longer length of lead-time, the time of shortage gets longer and with the increase of shortage the proportion of customers that can wait goes down resulting in the overall decrease of back-order rate. Therefore, we consider the back-orders rate, β, which is a decision variable instead of constant. During the stock-out period, the back-order rate β is a function of the lead-time through the amount of shortage $\overline{M}(\tilde{D}_L - r)^{+}$. The larger expected shortage quantity implies, the smaller back-order rate. Thus, we consider β as $\beta = \frac{1}{1 + \alpha\overline{M}(\tilde{D}_L - r)^{+}}$, where α the back-order parameter ($0 \leq \alpha < \infty$).

Hence, the fuzzy total cost function can be written as

$$\tilde{C}(Q,r,L) = \left[\left\{h + \pi_0\left(\frac{\tilde{D}(\omega)}{Q}\right)\right\}\frac{\alpha(\overline{M}(\tilde{D}_L - r)^+)^2}{1+\alpha\overline{M}(\tilde{D}_L - r)^+} + \pi\left(\frac{\tilde{D}(\omega)}{Q}\right)\overline{M}(\tilde{D}_L - r)^+\right] + h\left[\frac{Q}{2} + r - \tilde{D}(\omega)L\right] + \frac{\tilde{D}(\omega)}{Q}(P + \varepsilon e^{-\delta L}) \tag{11}$$

In real-life situation, decision maker has to work under limited budget. Let us consider that the cost of each item and the total available budget are c and C, respectively. Then since the order quantities are Q when an order is placed, the following inequality required to hold:

$$cQ \leq C \tag{12}$$

The information about the cost of each unit of the item and total budget available is estimated from past data. Let $\hat{c} \sim N(m^c, \sigma^c)$ and $\hat{C} \sim N(m^C, \sigma^C)$ be normally distributed and independent random variables of the cost of each unit of the item and the total available budget, respectively. Further, the fulfillment of the budget constraint is an individual, organizational decision. Again the decision maker allows some relaxation of the restriction; i.e., both sides of the constraint may be tied with the vague relationship '$\lessapprox$' which is the fuzzified version of '$\leq$.' As explained earlier, the decision maker may be more confident to select the probability level in linguistic terms. Thus, instead of crisp probability, a fuzzy probability measure, say around $p \in [0, 1]$ will be attached such that the constraint is satisfied with no less than this imprecise probability level. Because of this, the budgetary constraint (12) may be written in the form of the imprecise chance constraint as [4]

$$\widetilde{\text{Prob}}\left(\hat{c}Q \lessapprox \hat{C}\right) \gtrapprox p. \tag{13}$$

The goal of the decision maker is to determine the optimal order quantity, reorder point, lead-time, and back-order rate in order to minimize the total cost in fuzzy sense. Since the total cost function is a fuzzy random variable thus the expectation of total cost function is a unique fuzzy number. Let $\overline{M}(\tilde{C}(Q,r,L)(\omega))$ or simply $\overline{M}(Q,r,L)$ be the defuzzified representation of the expectation of the total cost. So mathematically, the problem can be formulated in the following optimization form:

$$(\text{P}_3)\quad \begin{cases} \min\limits_{Q,r,L} \overline{M}(Q,r,L) \\ \text{such that } \widetilde{\text{Prob}}\left(\hat{c}Q \lessapprox \hat{C}\right) \gtrapprox p \\ Q, r, L \geq 0; \end{cases}$$

where the value of $\overline{M}(Q, r, L)$ is need to be determined. Therefore, the following steps are required to find for obtaining the optimal solution of decision-making variables:

(i) The expected lead-time demand and the exact expression for expected shortage $\overline{M}(\tilde{D}_L - r)^+$ for a given $r \in [\underline{D}_L, \overline{D}_L]$ in possibilistic sense;
(ii) The expected value of the total cost function, which are a fuzzy random variable and the defuzzified representation of this fuzzy random variable;
(iii) The crisp equivalent form of the imprecise chance constraint;
(iv) The optimal values of order quantity Q^*, reorder point r^*, lead-time L^*, and back-order rate β^* in order to minimize the total cost.

3.2 Determination of the Expected Shortage

The expected lead-time demand is $\tilde{D}_L = (\underline{D}_L, D_L, \overline{D}_L)$. Now, in order to maintain the nonnegative safety or buffer stock, the lower bound of reorder point r is $\overline{M}(\tilde{D}_L)$. When the expected lead-time demand $\tilde{D}_L$ in each cycle is greater than r, then there is a shortage of amount $(\tilde{D}_L - r)$. Since the lead-time demand $\tilde{D}_L$ is a triangular fuzzy number, the upper bound of the reorder point r is $\overline{D}_L$ due to the nonnegative safety stock condition. Thus to determine the expected amount of shortage in each cycle, two situations will arise depending upon the position of $r \in [\underline{D}_L, \overline{D}_L]$ subject to condition that the safety or buffer stock is nonnegative.

Situation 1. For r lying between $\underline{D}_L$ and D_L, we have the α-level set of the lead-time demand as [11]

$$(\tilde{D}_L)_\alpha = \begin{cases} [r, D^+_{L,\alpha}], & \alpha \leq L(r) \\ [D^-_{L,\alpha}, D^+_{L,\alpha}], & \alpha > L(r) \end{cases}$$

which implies

$$\left((\tilde{D}_L - r)^+\right)_\alpha = \begin{cases} [0, D^+_{L,\alpha} - r], & \alpha \leq L(r) \\ [D^-_{L,\alpha} - r, D^+_{L,\alpha} - r], & \alpha > L(r) \end{cases} \tag{14}$$

Therefore, the possibilistic mean is obtained as follows:

$$\begin{aligned} \overline{M}\left(\tilde{D}_L - r\right)^+ &= \int_0^1 \alpha\left[\left((\tilde{D}_L - r)^+\right)^-_\alpha + \left((\tilde{D}_L - r)^+\right)^+_\alpha\right]\mathrm{d}\alpha \\ &= \int_0^1 \alpha D^+_{L,\alpha}\mathrm{d}\alpha + \int_{L(r)}^1 \alpha D^-_{L,\alpha}\mathrm{d}\alpha - (1 - 0.5L^2(r)) \end{aligned} \tag{15}$$

Situation 2. For r lying between D_L and $\overline{D}_L$, we have the α-level set of the lead-time demand as [11]

$$(\tilde{D}_L)_\alpha = \begin{cases} [r, D^+_{L,\alpha}], & \alpha \leq R(r) \\ \phi, & \alpha > R(r) \end{cases}$$

which implies

$$\left((\tilde{D}_L - r)^+\right)_\alpha = \begin{cases} [0, D^+_{L,\alpha} - r], & \alpha \leq R(r) \\ \phi, & \alpha > R(r) \end{cases} \tag{16}$$

Therefore, the possibilistic mean is obtained as follows:

$$\begin{aligned} \overline{M}\left(\tilde{D}_L - r\right)^+ &= \int_0^1 \alpha\left[\left((\tilde{D}_L - r)^+\right)^-_\alpha + \left((\tilde{D}_L - r)^+\right)^+_\alpha\right]\mathrm{d}\alpha \\ &= \int_0^{R(r)} \alpha D^+_{L,\alpha}\mathrm{d}\alpha - 0.5rR^2(r) \end{aligned} \tag{17}$$

3.3 *Defuzzification of the Fuzzy Expected Total Cost Function Using Possibilistic Mean Value*

We have obtained the total cost function in (11), which is given by

$$\begin{aligned} \tilde{C}(Q, r, L) = h\left[\frac{Q}{2} + r - \tilde{D}(\omega)L\right] + \left[\left\{h + \pi_0\left(\frac{\tilde{D}(\omega)}{Q}\right)\right\} \frac{\alpha(\overline{M}(\tilde{D}_L - r)^+)^2}{1 + \alpha\overline{M}(\tilde{D}_L - r)^+} \right. \\ \left. + \pi\left(\frac{\tilde{D}(\omega)}{Q}\right)\overline{M}(\tilde{D}_L - r)^+\right] + \frac{\tilde{D}(\omega)}{Q}(P + \varepsilon e^{-\delta L}) \end{aligned} \tag{18}$$

where $\overline{M}(\tilde{D}_L - r)^+$ is given by Eqs. (15) or (17) according to the position of the target inventory level r in the interval $[\underline{D}_L, \overline{D}_L]$. For computational purpose, we defuzzified the fuzzy expected total cost function using its possibilistic mean value. Let $E(\tilde{C}(\omega))$ be the fuzzy expectation of the total cost function. Then, the possibilistic mean value of the fuzzy expected total cost function is given by

$$\overline{M}(Q, r, L) = \int_0^1 \alpha\left(E(C^-_\alpha(\omega)) + E(C^+_\alpha(\omega))\right)\mathrm{d}\alpha \tag{19}$$

Now, the α-level set of $E(\tilde{C}(\omega))$ is then given by

$EC_\alpha(\omega) = E(C_\alpha(\omega) = [E(C_\alpha^-(\omega), E(C_\alpha^+(\omega)], \alpha \in [0,1], \omega \in (\Omega, \mathcal{B}, \mathcal{P})$, where

$$E(C_\alpha^-(\omega)) = \sum_{i=1}^{n}\left[\frac{D_\alpha^-(\omega)}{Q}\left\{P + \pi\overline{M}(\tilde{D}_L - r)^+ + \pi_0\frac{\alpha(\overline{M}(\tilde{D}_L - r)^+)^2}{1+\alpha\overline{M}(\tilde{D}_L - r)^+} + \varepsilon e^{-\delta L}\right\}\right.$$
$$= \sum_{i=1}^{n}\left[\frac{\{\underline{D}_i + \alpha(D_i - \underline{D}_i)\}}{Q}\left\{P + \pi\overline{M}(\tilde{D}_L - r)^+ + \pi_0\frac{\alpha(\overline{M}(\tilde{D}_L - r)^+)^2}{1+\alpha\overline{M}(\tilde{D}_L - r)^+} + \varepsilon e^{-\delta L}\right\}\right.$$
$$\left. + h\left\{\frac{Q}{2} + r - \{\overline{D}_i - \alpha(\overline{D}_i - D_i)\}L + \frac{\alpha(\overline{M}(\tilde{D}_L - r)^+)^2}{1+\alpha\overline{M}(\tilde{D}_L - r)^+}\right\}\right]p_i \tag{20}$$

and

$$E(C_\alpha^+(\omega)) = \sum_{i=1}^{n}\left[\frac{D_\alpha^+(\omega)}{Q}\left\{P + \pi\overline{M}(\tilde{D}_L - r)^+ + \pi_0\frac{\alpha(\overline{M}(\tilde{D}_L - r)^+)^2}{1+\alpha\overline{M}(\tilde{D}_L - r)^+} + \varepsilon e^{-\delta L}\right\}\right.$$
$$\left. + h\left\{\frac{Q}{2} + r - D_\alpha^-(\omega)L + \frac{\alpha(\overline{M}(\tilde{D}_L - r)^+)^2}{1+\alpha\overline{M}(\tilde{D}_L - r)^+}\right\}\right]p_i$$
$$= \sum_{i=1}^{n}\left[\frac{\{\overline{D}_i - \alpha(\overline{D}_i - D_i)\}}{Q}\left\{P + \pi\overline{M}(\tilde{D}_L - r)^+ + \pi_0\frac{\alpha(\overline{M}(\tilde{D}_L - r)^+)^2}{1+\alpha\overline{M}(\tilde{D}_L - r)^+} + \varepsilon e^{-\delta L}\right\}\right.$$
$$\left. + h\left\{\frac{Q}{2} + r - \{\underline{D}_i + \alpha(D_i - \underline{D}_i)\}L + \frac{\alpha(\overline{M}(\tilde{D}_L - r)^+)^2}{1+\alpha\overline{M}(\tilde{D}_L - r)^+}\right\}\right]p_i \tag{21}$$

Substituting the values of Eqs. (20) and (21) in (19), we find the possibilistic mean value of the fuzzy expected total cost function $\overline{M}(Q, r, L)$, which is given by

$$\overline{M}(Q, r, L) = \frac{1}{Q}\left[P + \pi\overline{M}(\tilde{D}_L - r)^+ + \pi_0\frac{\alpha(\overline{M}(\tilde{D}_L - r)^+)^2}{1+\alpha\overline{M}(\tilde{D}_L - r)^+} + \varepsilon e^{-\delta L}\right]$$
$$\left\{\frac{1}{6}\sum_{i=1}^{n}(\underline{D}_i + \overline{D}_i)p_i + \frac{2}{3}\sum_{i=1}^{n}D_i p_i\right\}$$
$$+ h\left[\frac{Q}{2} + r - \left\{\frac{1}{6}\sum_{i=1}^{n}(\underline{D}_i + \overline{D}_i)p_i + \frac{2}{3}\sum_{i=1}^{n}D_i p_i\right\}L\right.$$
$$\left. + \frac{\alpha(\overline{M}(\tilde{D}_L - r)^+)^2}{1+\alpha\overline{M}(\tilde{D}_L - r)^+}\right] \tag{22}$$

3.4 Crisp Equivalent Form of the Imprecise Chance Constraint

The imprecise chance constraint is as follows:

$\widetilde{\text{Prob}}\left(\hat{c}Q \lessapprox \hat{C}\right) \gtrapprox p$, where $\hat{c} \sim N(m^c, \sigma^c)$ and $\hat{C} \sim N(m^C, \sigma^C)$ are normally distributed and independent random variables of the cost of each unit of item and the total available budget, respectively. Since this constraint cannot be dealt with this form, hence the imprecise chance constraint is transformed to its crisp equivalent form using the concept which is mentioned in [4].

Suppose $\hat{Z} = \hat{c}Q - \hat{C}$. Then, $\hat{Z}$ follows the normal distribution with mean m^Z and standard deviation σ^Z where $m^Z = m^c Q - m^C$ and $\sigma^Z = \left[(\sigma^c)^2 Q^2 + (\sigma^C)^2\right]^{\frac{1}{2}}$. Resorting the fuzzy ordering between the left- and right-hand sides of '$\lessapprox$' in the parenthesis (), $\hat{Z}$ is then replaced by its standard normal variable $\left(\frac{\hat{Z}-m^Z}{\sigma^Z}\right)$ as follows

$$\widetilde{\text{Prob}}\left(\frac{\hat{Z}-m^Z}{\sigma^Z} \lessapprox \frac{-m^Z}{\sigma^Z}\right) \gtrapprox p. \tag{23}$$

Now, for a fuzzy event $(Z \lessapprox z)$, the following proposition, as proved by [4], holds:

$$F(z) \leq \widetilde{\text{Prob}}(Z \lessapprox z) \leq F(z + \Delta z) \tag{24}$$

where Δz is the extent of softness permitted and fixed by decision maker. Therefore using the result of (24) in (23), we have

$$\widetilde{\text{Prob}}\left(\frac{\hat{Z}-m^Z}{\sigma^Z} \lessapprox \frac{-m^Z}{\sigma^Z}\right) \leq \phi\left(\frac{-m^{Z'}}{\sigma^{Z'}}\right) \tag{25}$$

where $\hat{Z}' = \hat{c}Q - (\hat{C} + \Delta C) \leq \hat{Z}$ and $\phi(.)$ is the distribution function of standard normal variable. Here, ΔC (non-random) is the range of tolerance allowed and fixed by the decision maker for the fuzzy events $\hat{c}Q \lessapprox \hat{C}$. Hence, we get the following fuzzy relation

$$\phi\left(\frac{-m^{Z'}}{\sigma^{Z'}}\right) \gtrapprox p. \tag{26}$$

Assuming the following linear membership function of the above fuzzy relation with Δp assumed to be range of tolerance permitted,

$$\mu_{\phi(\cdot)}(p) = \begin{cases} 1 & \text{if } \phi(\cdot) > p \\ \frac{\phi(\cdot) - (p - \Delta p)}{\Delta p} & \text{if } p - \Delta p \leq \phi(\cdot) \leq p \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

Hence, the crisp equivalent form of the imprecise chance constraint is given as

$$m^{Z'} + \sigma^{Z'} \phi^{-1}(p - \Delta p) \leq 0. \tag{28}$$

3.5 Optimal Solution

Our main goal is to find the optimal solution. In order to find the optimal order quantity, reorder point, lead-time, and back-order rate for decision making, the following steps are required to execute.

Step (i): Input the values of $P, h, \pi, \pi_0, \hat{c}, \hat{C}, p, \alpha, \epsilon, \delta$.
Step (ii): Calculate the possibilistic mean value of the fuzzy expected shortage using either (15) or (17) with the condition $0 \leq L(r) \leq 1$ and $0 \leq R(r) \leq 1$, respectively.
Step (iii): Determine the safety stock criteria, i.e., $r - \overline{M}(D_L) \geq 0$.
Step (iv): Calculate the possibilistic mean value of the fuzzy expected total cost from (22).
Step (v): Find the crisp equivalent form of imprecise chance constraint using (28).
Step (vi): Use the Lingo, Lindo, or Mathematica to solve the following minimization problem

$$(\mathrm{P}_3) \quad \begin{cases} \min\limits_{Q,r,L} \overline{M}(Q, r, L) \\ \text{such that } m^{Z'} + \sigma^{Z'} \phi^{-1}(p - \Delta p) \leq 0 \\ r \geq \overline{M}(\tilde{D}_L) \\ r \leq \overline{D}_L \\ Q, r, L \geq 0. \end{cases}$$

4 Numerical Example

A Leather Good's company in a city, say X Leather private limited, sells a particular type of handbags. The cost of placing an order is assumed to be Rs. 200. The holding cost is Rs. 20 per item per year. The fixed penalty cost for the shortage is Rs. 50, and the cost of lost sales including marginal profit is Rs. 100. Suppose it is estimated that the expense of each handbag is normally distributed with mean Rs. 375 and standard deviation Rs. 5. The total budget available to the private limited is also normally distributed with mean Rs. 30,000 and standard deviation Rs. 75. The lead-time reduction cost is a negative exponential function of the lead-time, i.e., $R(L) = \varepsilon e^{-\delta L}$ with $\varepsilon = 156$ and $\delta = 114$. Now, the manager of X private limited is quite

Table 1 Demand information

Demand	Probability
(825, 1130, 1270)	.25
(775, 977, 1275)	.22
(1120, 1325, 1450)	.27
(1240, 1352, 1560)	.26

satisfied if the budgetary constraint attains to the probability of 'around 0.8'. The information about annual demand is given in Table 1.

Thence, the problem is to determine the optimal order quantity Q^*, reorder point r^*, lead-time L^*, and back-order rate β^* in such a way that the expected annual inventory cost incurred is minimum. From the above problem, we have the order cost $P = 200$, the inventory holding cost $h = 20$, the fixed shortage cost $\pi = 50$, the marginal profit $\pi_0 = 100$, the lead-time reduction cost $R(L) = 156e^{-114L}$, the cost of each handbag $\hat{c} \sim N(375, 5)$, the total budget $\hat{C} \sim N(30000, 75)$ and the probability $p =' $ around $0.8'$. Thus, the expected lead-time demand and possibilistic mean value of lead-time are given by

$$\tilde{D}_L = (1001.55, 1206.71, 1373.1)L \text{ and,} \tag{29}$$

$$\overline{M}(\tilde{D}_L) = \left\{ \frac{1}{6} \sum_{i=1}^{n} (\underline{D}_i + \overline{D}_i) p_i + \frac{2}{3} \sum_{i=1}^{n} D_i p_i \right\} L = 1200.248L. \tag{30}$$

Then, the defuzzified fuzzy expected total cost function is obtained as

$$\overline{M}(Q, r, L) = 20\left[\frac{Q}{2} + r - 1200.248L + \frac{\alpha(\overline{M}(\tilde{D}_L - r)^+)^2}{1 + \alpha\overline{M}(\tilde{D}_L - r)^+} \right] + \frac{1200.248}{Q}\left[200 + 50\overline{M}(\tilde{D}_L - r)^+ + 100\frac{\alpha(\overline{M}(\tilde{D}_L - r)^+)^2}{1 + \alpha\overline{M}(\tilde{D}_L - r)^+} + 156e^{-114L} \right] \tag{31}$$

Thus, mathematically, we need to solve the following optimization problem for determining the optimal solutions:

$$(\mathrm{P}_4) \begin{cases} \min\limits_{Q,r,L} \overline{M}(Q, r, L) \\ \text{such that } 140607.29Q^2 - 22575000Q + 906006016 \geq 0 \\ r \geq 1200.248L \\ r \leq 1373.1L \\ \dfrac{r - 1001.55L}{205.16L} \leq 1 \\ Q, r, L \geq 0. \end{cases}$$

Table 2 Optimal solutions of optimization problem (P_4) for different values of α

α	Q^*	r^*	β^*	L^*(in yearr)	$R(L)$	Total cost
0.0	79.36030	26.52812	1.0000000	0.02198384	12.00000	4467.313
0.5	79.36030	20.88508	0.8064689	0.01730796	20.85363	4641.311
1.0	79.36030	18.99215	0.6961684	0.01573879	24.93852	4730.848
10	79.36030	15.20582	0.2225064	0.01260105	35.66305	5039.961
∞	79.36030	14.84971	0.0000000	0.01230594	36.88325	5158.792

where $\overline{M}(\tilde{D}_L - r)^+ = (1200.248L - r) + (r - 1001.55L)^2 \left\{ \frac{1.18791\times10^{-5}r}{L^2} - \frac{.01187542}{L} \right\}$ $- \frac{7.91942\times10^{-6}}{L^2}(r - 1001.55L)^3$, $\Delta p = .01$ and $\Delta C = 100$. For the different values of α, the optimal solutions are presented in Table 2. Through numerical solutions, we have seen that as the back-order parameter α increases, the back-order rate decreases, and with the decreases of back-order rates, the total cost increases. It is also observed that the lead-time crashing cost increases as the length of the lead-time declines.

5 Conclusions

In this paper, we have proposed a fuzzy random continuous review inventory system with a mixture of back-orders and lost sales. The model is developed under the consideration that the order quantity, reorder point, back-order rate, and lead-time are the decision variables. We have considered the negative exponential function of lead-time and introduced a function of lead-time through an amount of shortages for controlling the lead-time and back-order rate, respectively, in the fuzzy random framework. We have considered the annual demand as a fuzzy random variable to capture the fuzziness and randomness simultaneously. To quantify the imprecise information, a budgetary constraint has been imposed on the model in the form of an imprecise chance constraint. We developed a methodology for obtaining the optimum decision-making variables in such a way that the total annual cost is minimized in the fuzzy sense. A numerical example has illustrated the proposed methodology. In future research on this model, it would be interesting to deal with imprecise probabilities. On the other hand, a possible extension of this model can be achieved by inclusion of the service-level constraint.

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