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Performance and Safety Management
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Operations Research in Development Sector

Springer

Asset Analytics

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Operations Research in Development Sector

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Preface

The papers in this book have been selected from among those presented at the 48th convention of the Operational Research Society of India (ORSI) organised at Bhubaneswar, Odisha, India, during 17–19 December 2015. The theme of the conference was Operations Research in Development Sector. More than 100 abstracts were received for presentation at the conference. Out of these, over 80 papers were presented at the conference. A total of 8 papers have been selected for this book after a rigorous peer review. As in most Operational Research (OR) conventions, large numbers of papers are technique related and few with some applications. Though the convention theme was focussed on the development sector, hardly any paper could qualify for the same.

The relevance of Operational Research for Development has been recognised from the 1950s. The use of Operational Research for Planning was used for national planning in India by Prof. Mahalanobis in the mid-1950s. Recognising its importance, the International Federation of Operational Research Societies (IFORS) organised the First International Conference on Operational Research for Development (ICORD) in 1992 at Ahmedabad, India. Now, ICORD is organised every year in a workshop format.

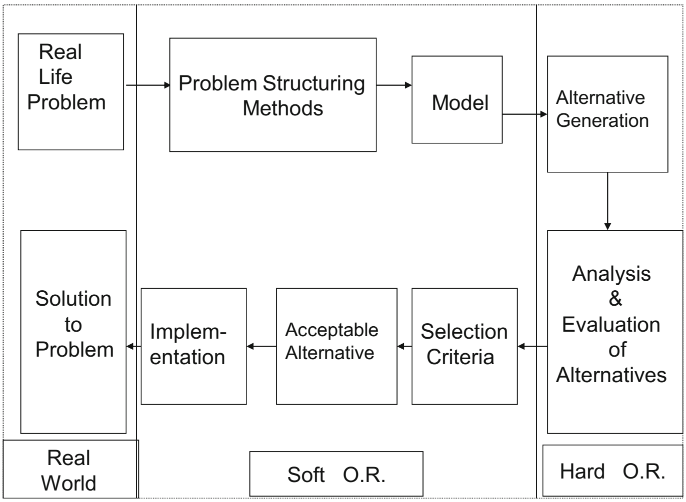
Operational Research originated with the focus on problem-solving. One of the essential requirements for effective problem-solving is to have enough clarity on the problem itself. In addition, in a problem-solving environment, there are many stakeholders. Every stakeholder has their own perspective while dealing with a problem, which hinders the problem-solving process. It helps to have a common understanding of the problem endorsed by all stakeholders. This thought led to the development of Soft OR. While Hard OR deals with various techniques and OR tools, Soft OR deals with the stakeholders' perspectives in getting a common understanding of the problem facilitating the solution process. Any problem can be thought to have the following characteristics—number of actors/stakeholders, diversity of perspectives, conflicting interests/objectives, uncertainty and such others.

Additionally, problems in the development domain are substantially messy; that is., clarity of the problems and objectives/outcomes to be pursued are substantially ambiguous, and the relationship between various elements is complex.

A schematic problem-solving model is presented below showing the role of Soft and Hard ORs. As presented in the diagram of the model, Operational Research is concerned with the real-life problems and to find an implementable solution. Usually, a Hard OR approach is taken to address the problem with (in general) sophisticated OR tools and techniques. After analysing and evaluating various alternatives, the most desirable one (a so-called optimal solution) is recommended for implementation. Typically, that marks the end of the initiative. The stakeholders, who are to implement the recommended solution, do not show enough enthusiasm in implementation. This is largely due to the fact that the concerns of the stakeholders are seldom taken into consideration. This often results in difficulty in implementation, or in some cases, the recommendation does not have much of the relevance in addressing the issues. This had prompted Prof. Ackoff[1] to come up with Future of OR is Past in the 1970s. Subsequently, a series of new initiatives of problem-structuring methods were used to get a better understanding of the problem. These methods are the essence of Soft OR. Soft OR aims at achieving a common and in-depth understanding of the problem by all stakeholders. This, in turn, facilitates problem-solving.

As in the diagram, in an OR problem-solving situation, both Soft OR and Hard OR have roles. The extent to which each of these will be used in a particular problematic situation depends on the nature of the problem. In the development-related issues, Soft OR has a significant importance due to complexity, multiple stakeholders and messy nature. This necessitates a different framework to look at OR in the development sector. These frameworks and approaches are deliberated rather less. This results in relatively less or hardly any coverage of OR in the development sector in most conferences on OR.

SCHEMETIC PROBLEM SOLVING MODEL



[1] "The Future of Operational Research is Past" Russell L. Ackoff; The Journal of the Operational Research Society, Vol. 30, No. 2, (Feb. 1979) pp 93–104.

In the present volume, eight papers are included in the constituent chapters. These have been selected from 84 papers presented at the conference. The chapters have been grouped into three categories: theory and methodology; inventory and stochastic processes; and application in industry, government, agriculture and engineering.

In the category of theory and methodology, three chapters are in the area of linear programming and its variation by Pradhan and Biswal, Nayak and Mishra, and Dash and Acharya. Pradhan and Biswal deal with a linear programming problem with additional constraints (conditions) relating to randomness, while Nayak and Mishra deal with some duality aspects in non-convex programming problem.

In the category of inventory and stochastic processes, Kumar deals with inventory issues influenced by trade credits while introducing new products. Sahoo and Dash deal with the effect of type 1 fuzzy set in chance-constrained single-period inventory model. Kozlovskaya et al. deal with an interesting situation of re-manufacturing and parts dismantling on the EOQ model.

In the final section of application in industry, government, agriculture and engineering, Jain and Dharmaraja carry out a mathematical review of the problematic area of crop insurance in India. Singh et al. address the adoption issues in e-governance initiatives in the Income Tax Department of India.

On the whole, the book provides a glimpse of the areas and the type of work the operational researchers are engaged in at the moment. It will also provide inputs to the present-day operational researchers the status and way ahead for further work on contemporary issues.

We would like to thank Prof. Manojranjan Nayak, Founder of Siksha 'O' Anusandhan (SOA), for his continuous support for this endeavour. For the preparation of the manuscript and correspondence, we thank Mitali Madhusmita Nayak (SOA) and Rajesh Gudepu and Srinivas Chary (ICFAI Foundation of Higher Education, Hyderabad, India). Lastly, we thank Springer for the publication of this volume.

Bhubaneswar, India
November 2017

Arabinda Tripathy
Rabi Narayan Subudhi
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Jyotiranjan Nayak is Associate Professor of mathematics at the Faculty of Science and Technology, IFHE, Hyderabad, India. Prior to this, he was Associate Professor at ITER, Siksha ‘O’ Anusandhan University, Bhubaneswar, India, and has more than 27 years of teaching experience. He obtained his Ph.D. degree from Utkal University. His research areas include non-convex programming, fuzzy optimisation, optimal control and topology optimisation. He has published ten papers in leading national and international journals. He has taught engineering mathematics and optimisation techniques at undergraduate and postgraduate levels. He is also former Associate Editor of the *Journal of Orissa Mathematical Society*. Currently, he is Vice-President of the Operational Research Society of India, Bhubaneswar Chapter.

Trade Credit Induced Inventory Model for New Products

Alok Kumar

1 Introduction

The decision-making is more effective when the effects of several parameters are taken into account together. Usually, in inventory management the optimal decision is made by developing mathematical models and by performing sensitivity analysis of these models with respect to different parameters. The different parameter values depict different scenarios of the market and different characteristics of the social system. When the analysis is made by taking these different parameters together, it brings towards effective decision-making because of concentration of integrated effect of different features and scenarios of the market and the social system. The paper develops an inventory model for new products by considering the effect of deterioration, inflation, time value of money and permissible delay in payments. There are several authors who have developed the inventory models by considering these effects. Some of them have been briefly described as follows. As we know that deterioration is defined in terms of obsolescence, spoilage, decay, damage, evaporation, pilferage, loss of utility of items, etc. Here, the items considered in this model are measured to be deteriorated in the sense that after certain time interval its value is forced to be lowered because of introduction of more advanced technology products into the market. Ghare and Schrader [10] first formulated a mathematical model over a finite-planning horizon having a constant rate of deterioration and constant rate of demand where, the deteriorated inventory has been categorized as three types as direct spoilage, physical depletion and deterioration. The Ghare and Schrader's model was extended by Covert and Philip [7] by using variable rate of deterioration and derived an EOQ model with Weibull distribution deterioration. Dave and Patel [8] developed an inventory model over a finite-planning horizon using linearly changing demand rate and constant deterioration. The up-to-date survey of published literature

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for the deteriorating inventory models has been given by Goyal and Giri [12] in their review articles. Now, the effect of inflation on the procurement policy of the items is also a major concern. Buzacott [4] and Misra [15] were the pioneers to develop EOQ models using the effect of inflation. Aggarwal [1] discussed purchase-inventory decision models for inflationary conditions. An inventory model was developed by Bose et al. [3] by considering the effect of inflation on deteriorating items under time discounting. Wee and Law [21] discussed an inventory model with finite replenishment rate where the effect of deterioration and time value of money are considered. Also, credit period offered by supplier to retailer brings changes in the schedule of optimal ordering policies, therefore its effect on the optimal inventory policies must be given due care. The conditions of permissible delay in payments in an EOQ model was first considered by Goyal [11]. A probabilistic inventory model under conditions of permissible delay in payments was discussed by Shah and Shah [17]. Huang and Chung [13] discussed the effect of cash discount and payment delay from the retailer's point of view. An inventory model for deteriorating items under inflation, in which the supplier provides the purchaser a permissible delay of payments if the purchaser orders a large quantity was discussed by Chang [5]. Chang et al. [6] studied an integrated vendor–buyer inventory system, where the demand rate is assumed to be a decreasing function of the retail price and the trade credit is linked to the order quantity. Manna et al. [14] developed an inventory model using discounted cash flows approach to analyze the optimal ordering policies in presence of inflation and trade credit. Ouyang et al. [16] discussed an integrated inventory model with capacity constraint and a order-size dependent permissible delay payment period. The inventory models discussed above unfortunately said nothing about the optimal inventory policy for the items which are newly introduced into the market. The need of the hours is to develop optimal inventory policy for new products by incorporating the concepts of deterioration, inflation, trade credit, etc. In marketing literature, there are several authors who have discussed the adoption behavior of new products. Some of them are discussed as follows. Fourt and Woodlock [9] discussed pure innovative model. Bass [2] captures both the innovative and imitative aspects of product adoption. Sharif and Ramanathan [18] studied some modified binomial innovation diffusion model that incorporate dynamic potential adopter populations. A meta-analysis of applications of diffusion models was discussed by Sultan et al. [19]. Talukdar et al. [20] have studied the diffusion of new product across products and countries. The objective of the paper is to discuss the optimal inventory policy for new products by developing economic order quantity model where demand follows innovation diffusion process. The effect of deterioration, inflation, time value of money and trade credit has also been incorporated in the model. The paper is arranged as follows. After introduction the paper describes the assumptions and notations which are useful to develop the mathematical model. Based upon the assumptions and notations, the mathematical model is formulated subsequently. Next, a solution procedure in the form of algorithm is developed to solve the model numerically. Subsequently, the model describes the sensitivity analysis with respect to the different parameters which are essential to understand the nature of the model. Later, some graphical representations were shown to examine the convexity of the cost functions. Afterward, observations with managerial implications are well described which are based upon the results obtained previously. Finally, the paper is concluded.

2 Assumptions and Notations

2.1 Assumptions

The model is based upon a number of assumptions which are stated below. These assumptions are based upon few references such as Bass [2], Buzacott [4], Fourt and Woodlock [9], Goyal [11] and Wee and Law [21].

- Inflation and Time value of money build up continuously with time and with constant rates.
- Demand rate follows innovation diffusion process and is governed by the innovation effect only.
- The replenishment rate is instantaneous.
- Shortages are not allowed.
- Lead time is zero.
- The coefficient of innovation is constant over the planning period.
- There is only one product bought per new adopter.
- The supplier offers credit period to the retailer.
- The potential market size is constant throughout the cycle time.
- The rate of deterioration is constant.

2.2 Notations

The following notations are used to develop the mathematical model

i	Inflation rate
r	Discount rate representing time value of money
R	Discount rate of net inflation
A	Ordering cost per order
C	Cost per unit item
I	Inventory carrying charge per unit per unit time
IC	Inventory carrying cost per unit per unit time
T	Cycle time
Q	Order quantity
θ	Rate of deterioration
$I(t)$	Inventory level at time t
$\lambda(t)$	Demand rate at time t
I_c	Interest charged per \$ per unit time

I_e	Interest earned per \$ per unit time
M	the retailer's trade credit period offered by the supplier to the retailer in years
d	Deterioration cost per unit per unit time
P	Selling price per unit item
p	Coefficient of innovation
$\overline{N}$	Potential market size
$PV_1(T)$	The total present worth of the cost of the system per unit time when $M \le T$
$PV_2(T)$	The total present worth of the cost of the system per unit time when $T \le M$

3 Model Formulation

In this section a mathematical model is developed based upon the assumptions and notations as discussed above. The model developed here is based upon the assumptions that the demand of the product follows innovation diffusion process and is governed by innovation effect only. Here, the mathematical formulation of demand is based upon the Fourt and Woodlock [9] and has the following functional form:

$$\lambda(t) = \frac{dN(t)}{dt} = p\left[\overline{N} - N(t)\right] \quad (1)$$

The inventory is depleted partly due to demand and partly due to deterioration over the time period $(0, T)$. The on-hand inventory level at any instant of time t is described as follows. Let $I(t)$ be the on-hand inventory level at any instant of time t, $(0 \le t \le T)$. The instantaneous state of $I(t)$ at any instant of time t in the interval $(0, T)$ is described by the differential equation as follows:

$$\frac{dI(t)}{dt} + \theta I(t) = -\lambda(t) \quad 0 \le t \le T \quad (2)$$

$$I(0) = Q, \quad I(T) = 0 \quad (3)$$

Using (1) and (2) and with initial conditions as mentioned in (3) we have

$$I(t) = \frac{p\overline{N}}{(\theta - p)} e^{-\theta t}\left[e^{(\theta-p)T} - e^{(\theta-p)t}\right] \quad 0 \le t \le T \quad (4)$$

and

$$I(0) = Q = \frac{p\overline{N}}{(\theta - p)}\left[e^{(\theta-p)T} - 1\right] \quad (5)$$

Let $D(T)$ be the number of units deteriorated during a cycle time T. Then it is given by

$$D(T) = Q - \int_0^T \lambda(t)\mathrm{d}t \tag{6}$$

$$\Rightarrow D(T) = \frac{p\overline{N}}{(\theta - p)}\left[\mathrm{e}^{(\theta-p)T} - 1\right] - \overline{N}\left[1 - \mathrm{e}^{-pT}\right] \tag{7}$$

Present worth of the ordering cost per unit time is given by

$$\mathrm{PV_{OC}} = \frac{A}{T} \tag{8}$$

Present worth of the material cost per unit time is given by

$$\mathrm{PV_{MC}} = \frac{CQ}{T} = \frac{Cp\overline{N}}{T(\theta - p)}\left[\mathrm{e}^{(\theta-p)T} - 1\right] \tag{9}$$

Present worth of the holding cost per unit time for carrying inventory over the period is given by

$$\mathrm{PV_{HC}} = \frac{\mathrm{IC}}{T}\int_0^T I(t)\mathrm{e}^{-Rt}\mathrm{d}t \tag{10}$$

$$\Rightarrow \mathrm{PV_{HC}} = \frac{\mathrm{IC}p\overline{N}}{(\theta - p)(R+\theta)T}\mathrm{e}^{(\theta-p)T}\left(1 - \mathrm{e}^{-(R+\theta)T}\right) - \frac{\mathrm{IC}p\overline{N}}{T(R+p)(\theta - p)}\left(1 - \mathrm{e}^{(R+p)T}\right) \tag{11}$$

Present worth of the deterioration cost per unit time is given by

$$\mathrm{PV_{DC}} = \frac{\mathrm{d}}{T}\int_0^T \theta I(t)\mathrm{e}^{-Rt}\mathrm{d}t \tag{12}$$

$$\Rightarrow \mathrm{PV_{DC}} = \frac{\mathrm{d}\theta p\overline{N}}{(\theta - p)(R+\theta)T}\mathrm{e}^{(\theta-p)T}\left(1 - \mathrm{e}^{-(R+\theta)T}\right) - \frac{\mathrm{d}\theta p\overline{N}}{T(R+p)(\theta - p)}\left(1 - \mathrm{e}^{(R+p)T}\right) \tag{13}$$

The model incorporates the trade credit period where supplier offers credit period to the retailer. Therefore, in this regard two cases exist as discussed below

Case-(1) $M \leq T$

Present worth of the Interest Charged per unit time is given by

$$\mathrm{PV_{IC}} = \frac{I_c C \mathrm{e}^{-RM}}{T} \int_M^T I(t)\mathrm{e}^{-Rt}\mathrm{d}t \tag{14}$$

$$\begin{aligned} \Rightarrow \mathrm{PV_{IC}} = {} & \frac{\mathrm{CI_c} p\overline{N}\mathrm{e}^{-RM}}{(\theta - p)(R+\theta)T}\mathrm{e}^{(\theta-p)T}\left[\mathrm{e}^{-(R+\theta)M} - \mathrm{e}^{-(R+\theta)T}\right] \\ & - \frac{\mathrm{CI_c} p\overline{N}\mathrm{e}^{-RM}}{T(R+p)(\theta-p)}\left[\mathrm{e}^{-(R+p)M} - \mathrm{e}^{-(R+p)T}\right] \end{aligned} \tag{15}$$

Present worth of the Interest earned per unit time up to the permissible time period M is given by

$$\mathrm{PV_{IE}} = \frac{I_e P}{T} \int_0^M t\lambda(t)\mathrm{e}^{-Rt}\mathrm{d}t \tag{16}$$

$$\Rightarrow \mathrm{PV_{IE}} = \frac{\mathrm{PI_e} p\overline{N}}{(R+p)^2 T}\left[1 - \mathrm{e}^{-(R+p)M}\right] - \frac{\mathrm{PI_e} p\overline{N}M}{T(R+p)}\mathrm{e}^{-(R+p)M} \tag{17}$$

Total present worth per unit time $\mathrm{PV_1}(T)$ is given by

$$\mathrm{PV_1}(T) = \mathrm{PV_{OC}} + \mathrm{PV_{MC}} + \mathrm{PV_{HC}} + \mathrm{PV_{DC}} + \mathrm{PV_{IC}} - \mathrm{PV_{IE}} \tag{18}$$

$$\begin{aligned} \Rightarrow \mathrm{PV_1}(T) = {} & \frac{A}{T} + \frac{Cp\overline{N}}{T(\theta-p)}\left[\mathrm{e}^{(\theta-p)T} - 1\right] \\ & - \frac{\mathrm{PI_e} p\overline{N}}{(R+p)^2 T}\left[1 - \mathrm{e}^{-(R+p)M}\right] + \frac{\mathrm{PI_e} p\overline{N}M}{T(R+p)}\mathrm{e}^{-(R+p)M} \\ & + \frac{\mathrm{IC} p\overline{N}}{(\theta-p)(R+\theta)T}\mathrm{e}^{(\theta-p)T}\left(1 - \mathrm{e}^{-(R+\theta)T}\right) \\ & - \frac{\mathrm{IC} p\overline{N}}{T(R+p)(\theta-p)}\left(1 - \mathrm{e}^{(R+p)T}\right) \\ & + \frac{\mathrm{d}\theta p\overline{N}}{(\theta-p)(R+\theta)T}\mathrm{e}^{(\theta-p)T}\left(1 - \mathrm{e}^{-(R+\theta)T}\right) \\ & - \frac{\mathrm{d}\theta p\overline{N}}{T(R+p)(\theta-p)}\left(1 - \mathrm{e}^{(R+p)T}\right) \\ & + \frac{\mathrm{CI_c} p\overline{N}\mathrm{e}^{-RM}}{(\theta-p)(R+\theta)T}\mathrm{e}^{(\theta-p)T}\left[\mathrm{e}^{-(R+\theta)M} - \mathrm{e}^{-(R+\theta)T}\right] \\ & - \frac{\mathrm{CI_c} p\overline{N}\mathrm{e}^{-RM}}{T(R+p)(\theta-p)}\left[\mathrm{e}^{-(R+p)M} - \mathrm{e}^{-(R+p)T}\right] \end{aligned} \tag{19}$$

For optimum total present worth, the necessary criterion is

$$\frac{\mathrm{dPV_1}(T)}{\mathrm{d}T} = 0 \tag{20}$$

Now, for $\mathrm{PV}_1(T)$ to be convex

$$\frac{\mathrm{d}^2\mathrm{PV}_1(T)}{\mathrm{d}T^2} > 0 \tag{21}$$

The solution of the equation $\frac{\mathrm{dPV}_1(T)}{\mathrm{d}T} = 0$ say $T = T_1$ gives the optimum value of T provided it satisfies the condition $\frac{\mathrm{d}^2\mathrm{PV}_1(T)}{\mathrm{d}T^2} > 0$. Since the above cost Eq. (20) is highly nonlinear, the problem has been solved numerically for given parameter values. The solution gives the optimum value T^* of the replenishment cycle time $T = T_1$. Once T^* is known the value of optimum order quantity Q^* and the optimum cost $\mathrm{PV}_1(T^*)$ can easily be obtained from the Eqs. (5) and (19) respectively. The numerical solution for the given base value has been obtained by using software package Excel-Solver.

Case-(2) $T \leq M$

Present worth of the Interest earned per unit time up to the cycle length T is given by

$$\mathrm{PV}_{\mathrm{IE}_1} = \frac{I_\mathrm{e} P}{T} \int_0^T t\lambda(t)\mathrm{e}^{-Rt}\mathrm{d}t \tag{22}$$

$$\Rightarrow \mathrm{PV}_{\mathrm{IE}_1} = \frac{\mathrm{PI_e} p\overline{N}}{(R+p)^2 T}\left[1 - \mathrm{e}^{-(R+p)T}\right] - \frac{\mathrm{PI_e} p\overline{N}}{(R+p)}\mathrm{e}^{-(R+p)T} \tag{23}$$

Present worth of the Interest earned per unit time during $(M - T)$ period is given by

$$\mathrm{PV}_{\mathrm{IE}_2} = \frac{I_\mathrm{e} P\mathrm{e}^{-RT}}{T} \int_T^M \left[\int_0^T \lambda(t)\mathrm{d}t\right]\mathrm{e}^{-Rt}\mathrm{d}t \tag{24}$$

$$\Rightarrow \mathrm{PV}_{\mathrm{IE}_2} = \frac{I_\mathrm{e} P\mathrm{e}^{-RT}\overline{N}\left(1 - \mathrm{e}^{-pT}\right)}{RT}\left(\mathrm{e}^{-RT} - \mathrm{e}^{-RM}\right) \tag{25}$$

Total present worth per unit time $\mathrm{PV}_2(T)$ is given by

$$\mathrm{PV}_2(T) = \mathrm{PV}_{\mathrm{OC}} + \mathrm{PV}_{\mathrm{MC}} + \mathrm{PV}_{\mathrm{HC}} + \mathrm{PV}_{\mathrm{DC}} - \mathrm{PV}_{\mathrm{IE}_1} - \mathrm{PV}_{\mathrm{IE}_2} \tag{26}$$

$$\begin{aligned}\Rightarrow \mathrm{PV}_2(T) = {} & \frac{A}{T} + \frac{Cp\overline{N}}{T(\theta-p)}\left[\mathrm{e}^{(\theta-p)T} - 1\right] \\ & - \frac{\mathrm{PI_e}p\overline{N}}{(R+p)^2T}\left[1-\mathrm{e}^{-(R+p)T}\right] + \frac{\mathrm{PI_e}p\overline{N}}{(R+p)}\mathrm{e}^{-(R+p)T} \\ & + \frac{\mathrm{IC}p\overline{N}}{(\theta-p)(R+\theta)T}\mathrm{e}^{(\theta-p)T}\left(1-\mathrm{e}^{-(R+\theta)T}\right) \\ & - \frac{\mathrm{IC}p\overline{N}}{T(R+p)(\theta-p)}\left(1-\mathrm{e}^{(R+p)T}\right) \\ & + \frac{\mathrm{d}\theta p\overline{N}}{(\theta-p)(R+\theta)T}\mathrm{e}^{(\theta-p)T}\left(1-\mathrm{e}^{-(R+\theta)T}\right) \\ & - \frac{\mathrm{d}\theta p\overline{N}}{T(R+p)(\theta-p)}\left(1-\mathrm{e}^{(R+p)T}\right) \\ & - \frac{I_\mathrm{e}P\mathrm{e}^{-RT}\overline{N}\left(1-\mathrm{e}^{-pT}\right)}{RT}\left(\mathrm{e}^{-RT}-\mathrm{e}^{-RM}\right)\end{aligned} \quad (27)$$

For optimum total present worth, the necessary criterion is

$$\frac{\mathrm{dPV}_2(T)}{\mathrm{d}T} = 0 \quad (28)$$

Now, for $\mathrm{PV}_2(T)$ to be convex

$$\frac{\mathrm{d}^2\mathrm{PV}_2(T)}{\mathrm{d}T^2} > 0 \quad (29)$$

The solution of the equation $\frac{\mathrm{dPV}_2(T)}{\mathrm{d}T} = 0$ say $T = T_2$ gives the optimum value of T provided it satisfies the condition $\frac{\mathrm{d}^2\mathrm{PV}_2(T)}{\mathrm{d}T^2} > 0$. Since the above cost Eq. (28) is highly nonlinear, the problem has been solved numerically for given parameter values. The solution gives the optimum value T^* of the replenishment cycle time $T = T_2$. Once T^* is known the value of optimum order quantity Q^* and the optimum cost $\mathrm{PV}_2(T^*)$ can easily be obtained from the Eqs. (5) and (27) respectively. The numerical solution for the given base value has been obtained by using software package Excel-Solver.

Also, at $T = M$, the present worth functions $\mathrm{PV}_1(T)$ and $\mathrm{PV}_2(T)$ are identical, denoted as $\mathrm{PV}(M)$ and has been expressed as follows:

$$
\begin{aligned}
\mathrm{PV}(M) = {} & \frac{A}{M} + \frac{Cp\overline{N}}{M(\theta - p)}\left[\mathrm{e}^{(\theta-p)M} - 1\right] \\
& - \frac{\mathrm{PI_e}p\overline{N}}{(R+p)^2 M}\left[1 - \mathrm{e}^{-(R+p)M}\right] + \frac{\mathrm{PI_e}p\overline{N}}{(R+p)}\mathrm{e}^{-(R+p)M} \\
& + \frac{\mathrm{IC}p\overline{N}}{(\theta - p)(R+\theta)M}\mathrm{e}^{(\theta-p)M}\left(1 - \mathrm{e}^{-(R+\theta)M}\right) \\
& - \frac{\mathrm{IC}p\overline{N}}{M(R+p)(\theta - p)}\left(1 - \mathrm{e}^{(R+p)M}\right) \\
& + \frac{\mathrm{d}\theta p\overline{N}}{(\theta - p)(R+\theta)M}\mathrm{e}^{(\theta-p)M}\left(1 - \mathrm{e}^{-(R+\theta)M}\right) \\
& - \frac{\mathrm{d}\theta p\overline{N}}{M(R+p)(\theta - p)}\left(1 - \mathrm{e}^{(R+p)M}\right)
\end{aligned} \tag{30}
$$

For optimum total present worth, the necessary criterion is

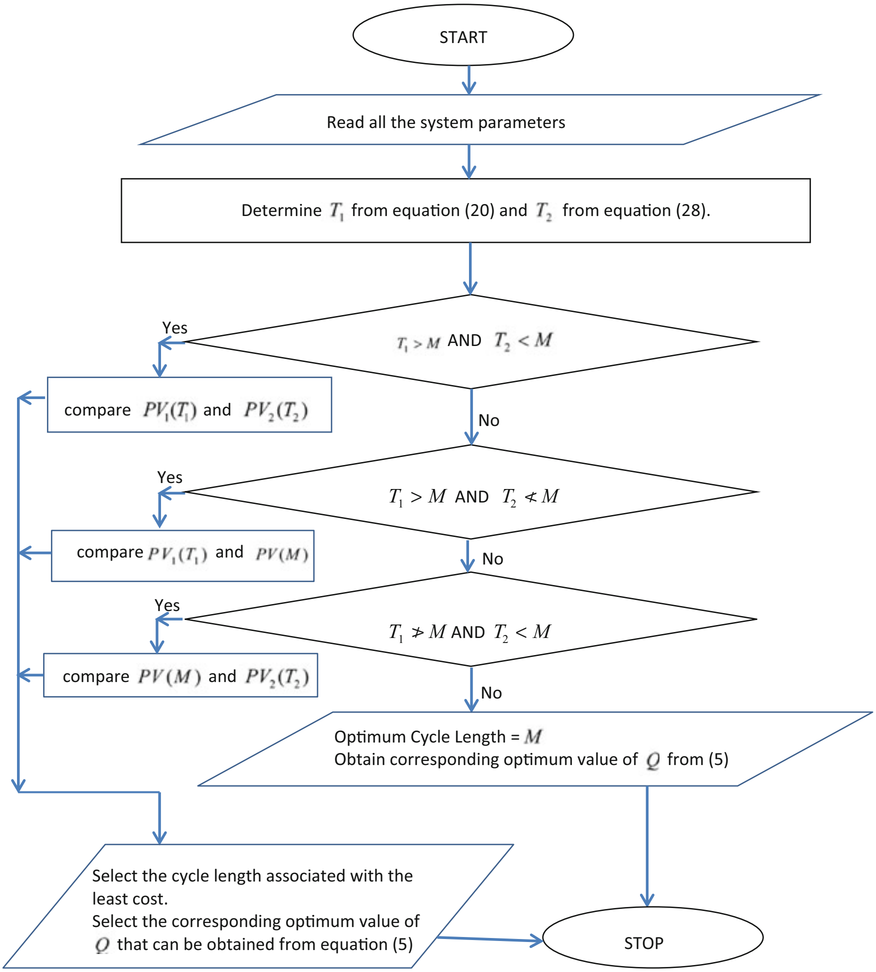
$$
\frac{\mathrm{dPV}(M)}{\mathrm{d}M} = 0 \tag{31}
$$

Now, for PV(M) to be convex

$$
\frac{\mathrm{d}^2\mathrm{PV}(M)}{\mathrm{d}M^2} > 0 \tag{32}
$$

The solution of the equation $\frac{\mathrm{dPV}(M)}{\mathrm{d}M} = 0$ say $M = M_1$ gives the optimum value of M provided it satisfies the condition $\frac{\mathrm{d}^2\mathrm{PV}(M)}{\mathrm{d}M^2} > 0$. Since the above cost Eq. (31) is highly nonlinear, the problem has been solved numerically for given parameter values. The solution gives the optimum value M^* of the replenishment cycle time $M = M_1$. Once M^* is known the value of optimum order quantity Q^* and the optimum cost PV(M^*) can easily be obtained from the Eqs. (5) and (30) respectively. The numerical solution for the given base value has been obtained by using software package Excel-Solver.

4 Solution Procedure



5 Numerical Example

To validate the model numerically and to understand the nature of the parameters, the sensitivity analysis with respect to some parameters has been performed in different numerical tables. Some base values are given as follows:

$$A = \$1100/\text{order},\ C = \$200/\text{unit},\ P = \$600,$$
$$I = 0.25,\ \bar{N} = 1000,\ I_c = 0.20,\ I_e = 0.17$$
$$r = 0.15,\ \theta = 0.15,\ d = 0.65$$

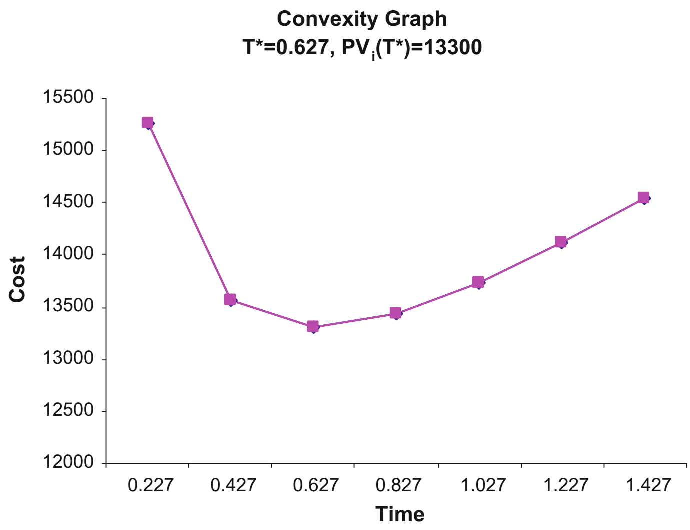


Fig. 1 Convexity Graph for function (present worth)

6 Graphical Representations

It has been observed that the total present worth of the cost functions are highly nonlinear where it becomes difficult to prove the convex nature of these cost functions analytically. Therefore, convexity of the cost functions is shown here graphically for some optimal values as shown in Figs. 1 and 2.

7 Observations with Managerial Implications

The patterns observed over the optimal solutions with respect to the parameters such as coefficient of innovation (p), credit period offered by the supplier to the retailer (M), deterioration rate (θ), inflation rate (i) and discount rate representing time value of money (r) in the numerical example section are elaborated here. The managerial implications of these patterns are also discussed in this section. The points elaborated here are as follows:

i. It has been observed from Table 1 that as p increases keeping other parameter values constant then the optimal cycle length decreases whereas the optimum cost and the optimum order quantity increases. Here, the manager should follow the pattern as mentioned above as investment on promotion increases.
ii. Table 2 suggests that on increasing M keeping other parameter values constant, the manager should follow the policy of keeping fewer inventories for less time period to obtain the optimum cost.

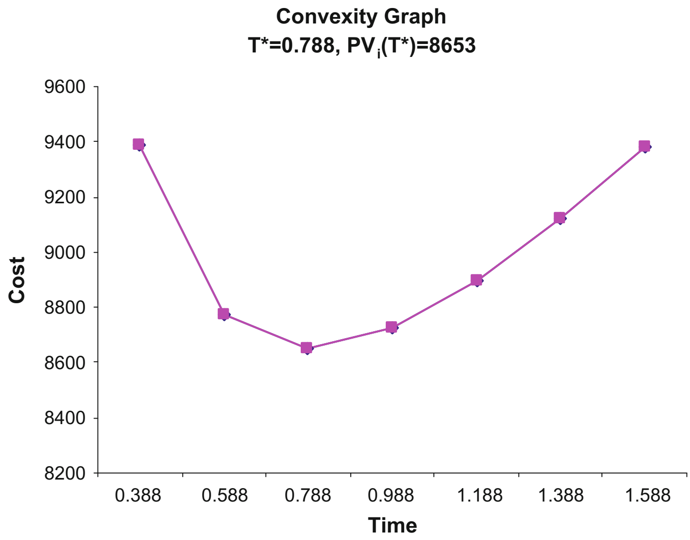


Fig. 2 Convexity Graph for the function (present worth)

Table 1 Sensitivity analysis on coefficient of innovation 'p' For M = 0.0821 (30/365), $r = 0.15, i = 0.1, \theta = 0.15$

p	T^*	$\text{PV}_i(T^*)$	$Q(T^*)$
0.02	0.950	6213	20.22
0.03	0.788	8653	24.82
0.04	0.693	11,005	28.80
0.05	0.627	13,300	32.40
0.06	0.580	15,552	35.74

Table 2 Sensitivity analysis on credit period 'M' For $p =$ 0.05, $r = 0.15, i = 0.1, \theta = 0.15$

M	T^*	$\text{PV}_i(T^*)$	$Q(T^*)$
(15/365)	0.629	13,397	32.48
(30/365)	0.627	13,300	32.40
(60/365)	0.620	13,084	31.99
(90/365)	0.606	12,837	31.24
(120/365)	0.585	12,557	30.13

iii. In Table 4 it is observed that as θ increases keeping other parameter values constant the optimum cycle length and the optimum order quantity decreases whereas the optimum cost increases. This is consistent with the reality that as the rate of deterioration is more keep fewer inventories for less time period to obtain the optimum cost.

iv. Tables 3 and 5 describes the pattern with respect to r and i respectively. The inventory manager should follow the identical pattern if condition remains identical and feasible to the situation arises.

Table 3 Sensitivity analysis on 'r' For $M = 0.0821$, $p = 0.05, i = 0.1, \theta = 0.15$

r	T^*	$PV_i(T^*)$	$Q(T^*)$
0.11	0.623	13,313	32.16
0.12	0.624	13,310	32.22
0.13	0.625	13,306	32.28
0.14	0.626	13,303	32.34
0.15	0.627	13,300	32.40

Table 4 Sensitivity analysis on 'θ' For $M = 0.0821$, $p = 0.05, i = 0.1, r = 0.15$

θ	T^*	$PV_i(T^*)$	$Q(T^*)$
0.11	0.658	13,156	33.58
0.12	0.650	13,192	33.27
0.13	0.642	13,229	32.97
0.14	0.635	13,264	32.68
0.15	0.627	13,300	32.40

Table 5 Sensitivity analysis on 'i' For $M = 0.0821$, $p = 0.05, \theta = 0.15, r = 0.20$

i	T^*	$PV_i(T^*)$	$Q(T^*)$
0.11	0.632	13,287	32.64
0.12	0.631	13,290	32.58
0.13	0.630	13,294	32.52
0.14	0.629	13,297	32.46
0.15	0.627	13,300	32.40

8 Conclusion

The effective inventory management is a significant task of every inventory manger and this is applicable to every organization. In this era of dynamic environment the proper scheduling and managing of inventory is crucial to obtain the optimum result. Here, the paper develops an inventory model and discusses the optimum schedules which are consistent with the reality and are beneficial to the managers who are involved in this kind of identical activities. The model is based upon the demand function which is time dependent governed by innovation diffusion process and considers the innovation effect only. The effect of credit period offered by the supplier to the retailer, deterioration, inflation, and discount representing time value of money has been incorporated in the model. A solution procedure in the form of algorithm has been developed which helps in obtaining the optimum results numerically. The results have been well discussed in the observations section followed by some managerial insights.

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Solving a Chance-Constrained Single-Period Inventory Model with Type-1 Fuzzy Set

Anuradha Sahoo and J. K. Dash

1 Introduction

The single-period inventory model (SPIM) has a wide application such as fashion products business, service industries, fresh food business like cheese, milk, ice-cream, and yogurt, etc. Similar use of SPIM can be found in sporting, style goods, newspaper, magazines and Christmas trees, etc. In inventory management the SPIM is the most well-made studied model. In this SPIM with various cost factors and random demand, an optimal order quantity is determined.

However, in real-life situations, for profit maximization or cost minimization, market demand estimation is always a principal factor. Due to so many reasons demand estimation is so difficult in inventory management. So the theory of fuzzy set has been used to this inventory problem, where demand is uncertain. The problem involving both fuzziness and randomness for which fuzzy chance constrained programme is perhaps a best mathematical tool.

There are several studies on SPIM fuzzy problem such as Zhou et al. [1] formulated the stochastic demand model. Wang et al. [2] developed single item and multi-item SPIMS with uncertain random demands.

Sarkar and Mahapatra [3] investigated a periodic review fuzzy inventory model. Sen and Malakar [4] studied inventory model with shortage by considering the associated costs involved as fuzzy numbers. Dash and Sahoo [5] presented a multi-item newsboy problem with demand is uncertain and purchasing cost is a fuzzy number.

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Nayebi et al. [6] discussed the fuzzy chance constrained multi-Objective Programming inventory problems. Dey and Chakraborty [7] has been developed a SPIM in the mixed fuzzy random environment.

Chen and Ho [8] discussed the single order newsboy problem. They used the ranking method. Ding [9] extended the analysis of the classic newsboy problem when market demand is an uncertain random variable.

Okyay et al. [10] discussed an extension of SPIM problem. Chen and Chen [11] developed a newsboy problem with a reservation policy to meet marketing requirements.

Ziukov [12] formulated a suitable inventory model which is one of the major concerns for an industry. Messaoudi et al. [13] presented a goal programming model with fuzzy chance constraint.

Wu et al. [14] studied the news vendor problem to maximize the profit with risk approach. Ding and Gao [15] derived an optimal policy to estimate demands for multi-item newsboy problem. Ershi [16] developed an effect of loss-averse preference and product substitutability on the loss-averse.

Kampempe and Luhandjula [17] approached Chance-constrained for multi-objective stochastic linear programming problems ranging from portfolio selection to water resource management. Aisbett and Rickard [18] discussed the membership functions (MF) in which type 1 and type 2 fuzzy logic systems has been used as a method of de-fuzzification and type-reduction.

Trksen [19] reviewed and presented the essentials fuzzy system models with type 1 fuzzy system problems and Nie and Tan [20] evaluated the modeling capability of a type 1 and an interval type 2 to determine the accuracy of the centroid.

Rajati and Mendel [21] discussed an effective methodology which involves implicit assignments of linguistic truth, probability, and possibility. Wu [22] explained the interval type 2 and type 1 fuzzy logic. Zheng and Liu [23] investigated and constructed a single- period supply chain problem where demand is a fuzzy random variable in both non-cooperation and cooperation situations.

Chen and Ho [24] formulated a multi-item newsboy model which is a nonlinear programming problem. Grubbstrm [25] involved point-in and point-out investment problem in stochastic version.

In this paper randomness and fuzziness are considered under one umbrella in term of fuzzy random variable (FRV). Different from previous studies, the important feature of the present work is to consider a chance constrained SPIM in the presence of FRVs in the objective function and in the constraint whose parameters are independent log-normally distributed. More over theorems have established, to convert the chance constrained SPIM into a deterministic one. Our objective to obtain the maximum profit for both overstock and understock situation, where all the costs, i.e., purchasing cost, selling price, and salvage value are type-1 fuzzy number.

The rest part of the paper is planned as follows. Section 2 recollects some primary ideas and findings of uncertain theory. An analytical model is formulated for the chance constrained SPIM under fuzziness and randomness in Sect. 3. In Sect. 4, we explore a transformation technique for objective function and chance constrained to change the problem to an equivalent deterministic problem. Numerical examples with sensitivity analysis are also provided for illustration purpose in Sect. 5. At last, some conclusions are listed in Sect. 6.

2 Preliminaries

From literature study, here we present some basic concepts regarding the uncertainty theory.

Definition 2.1 Fuzzy set [5] A fuzzy set $\tilde{A}$ is a set of ordered pairs in an Universal set X defined as

$$\tilde{A} = \left\{\left(x, \mu\tilde{A}(x)\right): x \in X\right\}$$

Where the mapping $\mu\tilde{A}(x)$: $X \to [0, 1]$ for each $x \in X$ is known as the grade of membership or membership function.

Definition 2.2 Fuzzy number [26] A fuzzy set $\tilde{A}$ defined over an Universal set X is a fuzzy number (FN) if

1. There exist at least one $x \in X$ such that $\mu\tilde{A}(x) = 1$
2. The membership function $\mu\tilde{A}(x)$ is piecewise continuous.

Definition 2.3 Triangular fuzzy number [5] The triangular fuzzy number is a FN denoted as $\tilde{A} = (A_1, A_2, A_3)$ and is interpreted by its membership function as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-A_1}{A_2-A_1}, \text{ if } A_1 \le x \le A_2 \\ \frac{A_3-x}{A_3-A_2}, \text{ if } A_2 \le x \le A_3 \\ 0, \quad \text{Otherwise} \end{cases}$$

Definition 2.4 Type-1 fuzzy set [26] The concept of an ordinary fuzzy set is called a type-1 fuzzy set. A type-1 fuzzy set has a grade of membership that is crisp. In the definition of ordinary fuzzy sets, the range of membership function is [0, 1] and the interval [0, 1] forms a linear ordered set. A is a type-1 fuzzy set and the membership grade i.e. the degree of membership of $x \in X$ in A is $\mu_A(x)$, which is a crisp number in X.

Definition 2.5 Type-1 fuzzy number [26] A type-1 FN is an generalization and an extension of a regular real number which does not refer to one single value but rather to a connected set of possible values, where each possible value has its own weight between 0 and 1. This weight is called the membership function. A type-1 FN is a quantity whose value is imprecise and the calculations with type-1 FNs allow the incorporation of uncertainty on parameters, rather than exact as is the case with ordinary numbers. A type-1 FN is a type-1 fuzzy set on the real line that satisfies the conditions of normality and convexity.

Definition 2.6 α-cut of a fuzzy number [5] The set $\{x|\mu_A(x) \geq \alpha\}$ where $\alpha \in (0, 1)$ is the α-cut of a FN $\tilde{A}$. It is identified as $\tilde{A}[\alpha]$.

Definition 2.7 Partial order relation of two fuzzy numbers [27] Let $\tilde{A} = (A_1, A_2, A_3)$ is a FN with α-cut $\tilde{A}[\alpha] = [A_*, A^*]$ and $\tilde{B} = (B_1, B_2, B_3)$ is another FN with $\tilde{B}[\alpha] = [B_*, B^*]$. Then $\tilde{A} \leq \tilde{B}$ iff $A^* \leq B_*$ and $\tilde{A} \leq \tilde{B}$ iff $A_* \leq B^*$.

Definition 2.8 Fuzzy random variable [7] FRV is a random variable whose parameters are FNs. A FRV $\tilde{X}$ with mean $\tilde{\mu}_x$ and variance $\tilde{\sigma}_x^2$ is identified by $\tilde{X}(\tilde{\mu}, \tilde{\sigma}^2)$.

Definition 2.9 Fuzzy log-normal distribution [27] A random variable x is log normally distributed if $\ln x$ is normally distributed. Let $N(\ln x, \mu, \sigma^2)$ denote the crisp log-normal random variable with mean μ, variance σ^2 and with the density function $f(\ln x, \mu, \sigma^2)$ where

$$f(\ln x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(\ln x - \mu)^2}{2\sigma^2}}, \quad \sigma > 0$$

Definition 2.10 Mean and variance of log-normal distribution [27] Let X be a normally distributed random variable with mean μ and variance σ^2. If $Y = e^X$, then Y follows log normal distribution. The mean and variance of Y is calculated by the relation

$$E(Y) = \mathrm{e}^{\mu + \frac{1}{2}\sigma^2}$$
$$\mathrm{Var}(Y) = \mathrm{e}^{2\mu + \sigma^2}\left(\mathrm{e}^{\sigma^2} - 1\right)$$

Definition 2.11 Fuzzy chance constrained programming problem (FCCP) [6, 17] A general chance constraint programming problem CCP is of the following form

$$\text{(CCP):} \quad \text{Optimize } \sum_{j=1}^{k} c_j x_j$$
$$\text{Subject to } P\left(\sum a_{ij} x_j \leq b_i\right) \geq \gamma_i$$
$$x_j \geq 0 \quad i = 1, 2, \ldots, m, \;\; j = 1, 2, \ldots, k \text{ and } 0 \leq \gamma_i \leq 1$$

where at least one of a_{ij}, c_j and b_i is treated as random variable.

So FCCP is a CCP given as above form where at least one of a_{ij}, c_j and b_i is considered as FRV.

3 Mathematical Formulation

Notations

In this paper we used the following symbols for the proposed problem:

Q_j quantity of order for product j ($j = 1, 2, 3, \ldots, k$).
$\tilde{D}_j$ demand as a FRV for product j.
$\tilde{C}_j$ fuzzy purchasing cost for 1 unit of product j.
$\tilde{P}_j$ fuzzy selling price for 1 unit of product j.
$\tilde{S}_j$ fuzzy salvage value for 1 unit of product j.
a_j area required for each unit of product j.
$\tilde{Z}_j$ profit function for product j.
$\tilde{Z}$ total profit function for all products.
$\tilde{W}_j$ storage space, a fuzzy random variable

Assumptions [5] We imposed the following assumptions to develop the model:

- For all products, $\tilde{D}_j$ are independent FRVs whose mean $\tilde{\mu}_D$ are type-1 fuzzy numbers.
- For all j, $\tilde{W}_j$ are independent fuzzy random variables whose mean $\tilde{\mu}_W$ and variance are type-1 fuzzy numbers.
- The time period is precise.
- In the starting of time period, decision maker's order quantity Q_j is optimal for all products.
- The left over items are salvaged in the season end.
- Zero penalty cost for the lost sales.

Single-Period Inventory Fuzzy Probabilistic Model (SPIFPM)

Here two situations explore to discuss about the profit function $\tilde{Z}_j$ related with each order quantity Q_j and fuzzy demand $\tilde{D}_j$ i.e. overstock situation ($\tilde{D}_j \leq Q_j$) and understock situation ($\tilde{D}_j \geq Q_j$). So the profit is as follows:

$$Z_j\left(Q_j, \tilde{D}_j\right) = \begin{cases} \tilde{\bar{Z}}_j\left(Q_j, \tilde{D}_j\right) = \left(\tilde{P}_j - \tilde{S}_j\right)\tilde{D}_j - \left(\tilde{C}_j - \tilde{S}_j\right)Q_j & \tilde{D}_j \leq Q_j \\ \tilde{\underline{Z}}_j\left(Q_j, \tilde{D}_j\right) = \left(\tilde{P}_j - \tilde{C}_j\right)Q_j, & \tilde{D}_j \geq Q_j \end{cases},$$

where $\bar{Z}_j$ and $\underline{Z}_j$ are profit functions for each product in overstock and understock situations respectively. Due to fuzziness of demand $\tilde{D}_j$, we get either a overstock gain $\tilde{\bar{Z}}_j\left(Q_j, \tilde{D}_j\right)$ or a understock gain $\tilde{\underline{Z}}_j\left(Q_j, \tilde{D}_j\right)$. Hence, total expected profit $\tilde{Z}_j\left(Q_j, \tilde{D}_j\right)$ corresponding to both overstock and understock situation is given as follows:

$$\tilde{Z}_j\left(Q_j, \tilde{D}_j\right) = \tilde{\bar{Z}}_j\left(Q_j, \tilde{D}_j\right) \cup \tilde{\underline{Z}}_j\left(Q_j, \tilde{D}_j\right)$$

Thus a multi-item SPIFPM subject to a storage space chance constraint with a confidence level γ can be stated as follows:

$$\text{(SPIFPM): Maximize } \tilde{Z}(Q_1, Q_2, \ldots, Q_k) = \sum_{j=1}^{k} \tilde{Z}_j\left(Q_j, \tilde{D}_j\right)$$

$$\text{subject to} \quad \tilde{P}\left(\sum_{j=1}^{k} a_j Q_j \leq \tilde{W}\right) \succeq \tilde{\gamma}$$

$$Q_j \geq 0 \quad \text{for } j = 1, 2, \ldots, k.$$

In the above model, we consider demand $\tilde{D}_j$ as a FRV present in the objective function and the total storage space $\tilde{W}$ normally distributed are independent FRVs present in the constraint log-normally distributed.

4 Transformation Technique

Crisp Equivalent of the Fuzzy Objective Function

In SPIM, the defined resultant profit function is fuzzy, since the customer demand for each product is imprecise. Now our objective to convert the fuzzy model to crisp model. The resultant profit function under overstock and understock situation is given as follows:

$$\begin{aligned}
\tilde{Z}\left(Q_j, \tilde{D}_j\right) &= \tilde{\bar{Z}}_j\left(Q_j, \tilde{D}_j\right) \cup \tilde{\underline{Z}}_j\left(Q_j, \tilde{D}_j\right) \\
\tilde{Z}\left(Q_j, D_j, \alpha\right) &= \bar{Z}_j\left(Q_j, D_j, \alpha\right) \cup \underline{Z}_j\left(Q_j, D_j, \alpha\right) \\
Z\left(Q_j, D_j, \alpha\right) &= \left[\bar{Z}_L\left(Q_j, D_j, \alpha\right), \underline{Z}_R\left(Q_j, D_j, \alpha\right)\right] \\
Z\left(Q_j, D_j, \alpha\right) &= \left[\min\left\{\bar{Z}_L\left(Q_j, D_j, \alpha\right)\right\}, \max\left\{\underline{Z}_R\left(Q_j, D_j, \alpha\right), \underline{Z}_R\left(Q_j, D_j, \alpha\right)\right\}\right]
\end{aligned} \tag{4.1}$$

Theorem 4.1 *Let [Z_{min}, Z_{max}] is the support of the fuzzy profit function $\tilde{Z}\left(Q_j, D_j\right)$, then*

$$Z_{min} = \sum_{j=1}^{k} \left(\underline{P}_j - \underline{S}_j\right)\underline{D}_j - \left(\underline{C}_j - \underline{S}_j\right)Q_j \quad \textit{and} \quad Z_{max} = \sum_{j=1}^{k} \left(\bar{P}_j - \bar{C}_j\right)Q_j$$

Proof To compute the left and right endpoints of fuzzy profit function $\tilde{Z}\left(Q_j, D_j\right)$ when $\alpha = 0$ in Eq. (4.1), we get

$$Z_L\big(Q_j, D_j, \alpha = 0\big) = \min\big\{\bar{Z}_L\big(Q_j, D_j, \alpha\big), \underline{Z}_L\big(Q_j, D_j, \alpha = 0\big)\big\}$$
$$Z_R\big(Q_j, D_j, \alpha = 0\big) = \max\big\{\bar{Z}_R\big(Q_j, D_j, \alpha\big), \underline{Z}_R\big(Q_j, D_j, \alpha = 0\big)\big\}$$

Now we have

$$Z_{\min} = \min\left\{\sum_{j=1}^{k}\Big(\underline{P}_j - \underline{S}_j\Big)\underline{D}_j - \Big(\underline{C}_j - \underline{S}_j\Big)Q_j, \sum_{j=1}^{k}\Big(\underline{P}_j - \underline{C}_j\Big)Q_j\right\}$$
$$Z_{\max} = \max\left\{\sum_{j=1}^{k}(\bar{P}_j - \bar{S}_j)\bar{D}_j - (\bar{C}_j - \bar{S}_j)Q_j, \sum_{j=1}^{k}(\bar{P}_j - C_{-j})Q_j\right\}$$

Since $\sum_{j-1}^{k} \underline{D}_j \leq Q_j$ and for each item j, the profit function associated with each order quantity Q_j and demand D_j, we have the following relation

$$\sum_{j=1}^{k}\Big(\underline{P}_j - \underline{S}_j\Big)\underline{D}_j - \Big(\underline{C}_j - \underline{S}_j\Big)Q_j \leq \sum_{j=1}^{k}(\bar{P}_j - \bar{C}_j)Q_j$$

Therefore, we have

$$Z_{\min} = \sum_{j=1}^{k}\Big(\underline{P}_j - \underline{S}_j\Big)\underline{D}_j - \Big(\underline{C}_j - \underline{S}_j\Big)Q_j$$
$$Z_{\max} = \sum_{j=1}^{k}(\bar{P}_j - \bar{C}_j)Q_j$$

Hence the fuzzy objective function of the model can be transformed to a crisp model as follows:

$$\text{Maximize } 0.5\left[\sum_{j=1}^{k}\Big(\underline{P}_j - \underline{S}_j\Big)\underline{D}_j - \Big(\underline{C}_j - \underline{S}_j\Big)Q_j + \sum_{j=1}^{k}(\bar{P}_j - \bar{C}_j)Q_j\right]$$

Crisp Equivalent of the Fuzzy Chance Constraint

To transform the fuzzy chance constraint to a crisp form following results are proved.

Theorem 4.2 *The deterministic form of the constraint*

$$\tilde{P}\left(\sum_{j=1}^{k}\alpha_j Q_j \leq \tilde{W}\right) \succeq \tilde{\gamma}$$

where $\tilde{W}$ is independent a FRV distributed log-normally is given by

$$\ln\left(\sum_{j=1}^{k} a_j Q_j\right) \leq \ln(\mu_{W*}) - 0.5\ln\left(1+\frac{\sigma_{W*}^2}{\mu_{W*}^2}\right) + F^{-1}\left(1-\gamma^*\right)\sqrt{\ln\left(1+\frac{\sigma_{W*}^2}{\mu_{W*}^2}\right)}$$

Here mean and variances of $\tilde{W}$ are $\tilde{\mu}_W$ and $\tilde{\sigma}_W^2$ respectively which follows fuzzy log-normal distribution.

Proof It is assumed that $\tilde{W}$ is independent FRV distributed log-normally. If W follows log-normal distribution, then $\ln W$ follows normal distribution with mean μ and variance σ^2.

$$E(\ln W) = \mu, \text{Var}(\ln W) = \sigma^2$$

So $\ln \tilde{W}$ is a fuzzy normal random variable, whose mean and variance are $\tilde{\mu}$ and $\tilde{\sigma}^2$ respectively. The α-cuts of $\tilde{W}$, $\tilde{\mu}$ and $\tilde{\sigma}^2$ are $\tilde{W}[\alpha] = [W_*, W^*]$, $\tilde{\mu}[\alpha] = [\mu_*, \mu^*]$ and $\tilde{\sigma}^2[\alpha] = \left[\sigma_*^2, \sigma^{*2}\right]$ respectively. Now let us consider the i-th constraint

$$\tilde{P}\left(\sum_{j=1}^{k} a_j Q_j \leq \tilde{W}\right) \succeq \tilde{\gamma}$$

where $\tilde{\gamma}$ is a fuzzy number and whose α-cuts is $\tilde{\gamma}[\alpha] = \left[\gamma_*, \gamma^*\right]$. The α-cut of the probabilistic constraint is expressed as:

$$\begin{aligned}
&\widetilde{P}\left(\sum_{j=1}^{k} a_j Q_j \leq \widetilde{W}\right)[\alpha] \\
&= \widetilde{P}\big(U_i \leq \widetilde{W}\big)[\alpha], \quad \text{where } U_i = \sum_{j=1}^{k} a_j Q_j \\
&= \left\{\widetilde{P}(U_i \leq W) | W \in \widetilde{W}[\alpha]\right\} \\
&= \left\{P(\ln U_i \leq \ln W) | W \in \widetilde{W}[\alpha]\right\} \\
&= \left\{1 - P(\ln W \leq \ln U_i) | W \in \widetilde{W}[\alpha]\right\} \\
&= \left\{1 - P\left(\frac{\ln W - \mu}{\sigma} \leq \frac{\ln U_i - \mu}{\sigma}\right) | W \in \widetilde{W}[\alpha], \mu \in \widetilde{\mu}[\alpha], \sigma^2 \in \widetilde{\sigma}^2[\alpha]\right\} \\
&= \left\{1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln U_i - \mu}{\sigma}} \exp\left(\frac{-z^2}{2} \mathrm{d}z\right) | W \in \widetilde{W}[\alpha], \mu \in \widetilde{\mu}[\alpha], \sigma^2 \in \widetilde{\sigma}^2[\alpha]\right\}
\end{aligned}$$

where

$$z = \frac{\ln W - \mu}{\sigma}$$

$$= \left\{1 - F\left(\frac{\ln U_i - \mu}{\sigma}\right) | \mu \in \tilde{\mu}[\alpha], \sigma^2 \in \tilde{\sigma}^2[\alpha]\right\}$$

where F is the cumulative distribution function of $N(0, 1)$ distribution. Now let

$$\min\left\{1 - F\left(\frac{\ln U_i - \mu}{\sigma}\right)\right\} = \left\{1 - F\left(\frac{\ln U_i - \mu_*}{\sigma_*}\right)\right\}$$
$$\max\left\{1 - F\left(\frac{\ln U_i - \mu}{\sigma}\right)\right\} = \left\{1 - F\left(\frac{\ln U_i - \mu^*}{\sigma^*}\right)\right\}$$
$$\tilde{P}\left(\sum_{j=1}^{k} a_j Q_j \le \tilde{W}\right)[\alpha] = \left[1 - F\left(\frac{\ln U_i - \mu_*}{\sigma_*}\right), 1 - F\left(\frac{\ln U_i - \mu^*}{\sigma^*}\right)\right]$$

Using fuzzy inequality, the α-cut of the fuzzy constraint is expressed as:

$$\tilde{P}\left(\sum_{j=1}^{k} a_j Q_j \le \tilde{W}\right)[\alpha] \succ \tilde{\gamma}[\alpha]$$
$$\Rightarrow 1 - F\left(\frac{\ln U_i - \mu_*}{\sigma_*}\right) \ge \tilde{\gamma}$$
$$\Rightarrow F\left(\frac{\ln U_i - \mu_*}{\sigma_*}\right) \le 1 - \tilde{\gamma}$$
$$\Rightarrow \frac{\ln U_i - \mu^*}{\sigma^*} \le F^{-1}(1 - \tilde{\gamma})$$
$$\Rightarrow \ln U_i - \mu_* \le F^{-1}(1 - \tilde{\gamma})\sigma_*$$
$$\Rightarrow \ln U_i \le F^{-1}(1 - \tilde{\gamma})\sigma_* + \mu_*$$
$$\Rightarrow U_i \le \exp(F^{-1}(1 - \tilde{\gamma})\sigma_* + \mu_*)$$

Hence the deterministic equivalent of the fuzzy probabilistic constraint is expressed as:

$$\sum_{j=1}^{k} a_j Q_j \le \exp\left(F^{-1}\left(1 - \gamma^*\right)\sigma_* + \mu_*\right)$$

Let W follows log-normal distribution with mean and variance μ_W and respectively.

Then the mean and variance of ln W follows normal distribution with mean and variance μ and σ^2 respectively. The relation between the mean and variances of log-normal and normal distributions are given by

$$\mu = \ln(\mu_W) - \tfrac{1}{2}\ln\left(1 + \tfrac{\sigma_W^2}{\mu_W^2}\right)$$
$$\sigma^2 = \ln\left(1 + \tfrac{\sigma_W^2}{\mu_W^2}\right)$$

So the above equation is equivalent to

$$\sum_{j=i}^{k} a_j Q_j \le \exp\left(F^{-1}(1-\tilde{\gamma})\sqrt{\ln\left(1 + \frac{\sigma_{W^*}^2}{\mu_{W^*}^2}\right)} + \ln\left(\mu_{W_*}\right) - \frac{1}{2}\ln\left(1 + \frac{\sigma_{W*}^2}{\mu_{W*}^2}\right)\right)$$

$$\Rightarrow \sum_{j=i}^{k} a_j Q_j \le \exp\left(F^{-1}(1-\tilde{\gamma})\sqrt{\ln\left(1 + \frac{\sigma_{W^*}^2}{\mu_{W^*}^2}\right)} + \ln\left(\frac{\mu_{W_*}}{\sqrt{\left(1 + \frac{\sigma_{W_*}^2}{\mu_{W_*}^2}\right)}}\right)\right)$$

$$\Rightarrow \sum_{j=i}^{k} a_j Q_j \le +\ln\left(\frac{\mu_{W_*}}{\sqrt{\left(1 + \frac{\sigma_{W*}^2}{\mu_{W*}^2}\right)}}\right)\exp\left(F^{-1}(1-\tilde{\gamma})\sqrt{\ln\left(1 + \frac{\sigma_{W^*}^2}{\mu_{W^*}^2}\right)}\right)$$

The deterministic equivalent of the SPIFPM problem is expressed as:

$$\text{SPICM : Maximize } 0.5\left[\sum_{j=1}^{k}\left(\underline{P}_j - \underline{S}_j\right)\underline{D}_j - 0\left(\underline{C}_j - \underline{S}_j\right)\underline{Q}_j + \sum_{j=1}^{k}\left(\bar{P}_j - \bar{C}_j\right)Q_j\right]$$

Subject to

$$\ln\left(\sum_{j=1}^{k} a_j Q_j\right) \le \ln\left(\mu_{W_*}\right) - 0.5\ln\left(1 + \frac{\sigma_{W_*}^2}{\mu_{W_*}^2}\right) + F^{-1}(1-\tilde{\gamma})\sqrt{\ln\left(1 + \frac{\sigma_{W_*}^2}{\mu_{W_*}^2}\right)}$$
$$Q_j \ge 0, \quad \text{for } j = 1, 2, \ldots, k$$

where $\tilde{\gamma}$ is a fuzzy number and $a_j \in R$ for all j and $\tilde{\mu}_W$ and $\tilde{\sigma}_W^2$ are mean and variances of $\tilde{W}$, which follows fuzzy log-normal distribution.

5 Numerical Example

Let us consider a SPIFPM to explain the effectiveness of the above approach with some data presented in a table.

$$\text{(SPIFPM)}: \text{Maximize}\left[\sum_{j=1}^{2}\left(\tilde{P}_j - \tilde{S}_j\right)\tilde{D}_j - \left(\tilde{C}_j - \tilde{S}_j\right)Q_j\right] \cup \left[\sum_{j=1}^{2}(P_j - C_j)Q_j\right]$$

$$\text{Subject to} \quad \tilde{P}\left(5Q_1 \oplus 4Q_2 \le \tilde{W}\right) \succeq \widetilde{0.4}$$

$$Q_1 \ge 0, \ Q_2 \ge 0$$

Here $\tilde{D}_1$, $\tilde{D}_2$ are independent FRVs with mean $\tilde{\mu}_{D_1}$, $\tilde{\mu}_{D_2}$ are as fuzzy numbers. Also $\tilde{W}$ is an independent log-normal distributed FRVs with mean $\tilde{\mu}_W$ and variance $\tilde{\sigma}_W^2$ as triangular fuzzy numbers and $\widetilde{0.4}$ are also triangular fuzzy numbers as, $\tilde{\mu}_W$ = (100/150/200), $\tilde{\sigma}_W^2$ (8/10/13), $\widetilde{0.4}$ = (0.3/0.4/0.5), where the other data for multi-items is given as follows:

Product name	P_j	C_j	S_j	a_j	$\tilde{\mu}_{D_j}$
Product 1	(12, 14, 16)	(9, 10, 11)	(7, 8, 9)	5	(40, 50, 60)
Product 2	(17, 20, 22)	(10, 15, 19)	(12, 13, 14)	4	(65, 70, 80)

The α-cuts of $\tilde{W}$, $\tilde{\mu}_W$ and $\widetilde{0.4}$ can be calculated as follows:

$$\tilde{\mu}_W[\alpha] = [100 + 50\alpha, 200 - 50\alpha]$$
$$\tilde{\sigma}_W^2[\alpha] = [8 + 2\alpha, 13 - 3\alpha]$$
$$\widetilde{0.4}[\alpha] = [0.3 + 0.1\alpha, 0.5 - 0.1\alpha]$$

Using the above theorem the corresponding single-period inventory crisp model (SPICM) is given as follows:

$$\text{(SPICM)}: \text{Maximize } 1.5Q_1 + 0.5Q_2 + 287.5$$
$$\text{Subject to} \quad \ln(5Q_1 + 4Q_2) \le 4.909013539$$
$$Q_1 \ge 0, \ Q_2 \ge 0$$

Using any optimization software its local solution can be determined. Here we have solved this by using LINGO and its solution is $Q_1 = 27.10115$, $Q_2 = 0$ and the corresponding objective value is 328.1517.

6 Conclusions

Initially some theorems proposed to modify the fuzzy problem into an equivalent crisp problem. Then the resultant problem is solved by using the LINGO software. Using this LINGO software we obtained its optimal solutions as $Q_1 = 27.10115$,

$Q_2 = 0$ and the corresponding objective value is 328.1517. Each value of α gives a corresponding solution. This work can be extended by considering other distributions namely uniform, exponential etc. Also type 2 fuzzy set can be considered instead of type 1, which is the future scope of the above work. Also this concept can be extended to other mathematical model.

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Multi-objective Multi-choice Random Linear Programming Problem

Avik Pradhan and M. P. Biswal

1 Introduction

In real life, the process of selecting a particular course of action from all the available alternatives is called decision-making problem. In almost all such decision making problems, the decision maker wants to attain more than one objective or goal in selecting the course of action while satisfying the constraints dictated by environment, processes, and resources. Mathematically, these problems can be represented as:

$$\max : [z_1(x), z_2(x), \ldots z_K(x)] \tag{1.1}$$

$$\text{s.t} \quad x \in X = \left\{x \big/ g_i(x) \leq 0\,,\ i = 1, 2, \ldots, m\right\}, \tag{1.2}$$

where x is an n-dimensional decision vector. In the literature, this problem is referred as multi-objective decision making (MODM) problem. Because of incommensurability and the conflicting nature of the multiple criterions, the problem becomes complex. A large variety of techniques have been published to find solution of a MODM problem. According to the information needed, Hwang and Masud [1] have systematically classified the MODM techniques into four classes [2]:

(a) no articulation of preference information;
(b) A priori articulation of preference information;
(c) Progressive articulation of preference information or interactive methods; and
(d) A posterior articulation of preference information or no dominated solution generation methods.

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In most of the solution methods such as global criterion method, bounded objective function method, goal-programming method, and fuzzy approaches for multi-objective programming problem, we construct an equivalent single objective problem with a scaled cost function. One of the deficiencies of such approaches is that it bases upon the concept that the chosen alternative should have only the shortest distance from the positive ideal solution (PIS). To avoid this deficiencies the technique for order preference by similarity to ideal solution or TOPSIS method was developed. The basic concept of this method is that the chosen alternative should have the shortest distance from the positive ideal solution (PIS) and the farthest from the negative ideal solution (NIS). Using this method, the decision maker can obtain a compromise solution of a MODM problem. This method transfers K-objectives (criteria) into two objectives (the shortest distance from the PIS and the longest distance from the NIS). These objectives are commensurable and most of the time conflicting. Then, this equivalent bi-objective problem can be solved by using the fuzzy set theoretic approach with the max-min operator [2–4]. It was Hwang and Yoon [5] who developed TOPSIS to solve a multiple attribute decision-making problem. Lia et al. [2] extended the concept of TOPSIS to develop a methodology for solving multiple objective decision-making (MODM) problems. A further extensions of TOPSIS for large-scale multi-objective nonlinear programming problems with block angular structure was presented by Abo-Sinna et al. in [3, 4]. Baky and Abo-Sinna [6, 7] extended the TOPSIS approach to solve Bi-Level MODM Problems and multi-level nonlinear multi-objective problems in their respective papers.

The values of the parameters of these MODM problems are not deterministic or particular in real situation. Hence, we need to develop new models to tackle these kinds of situations. In this study, we develop the mathematical model of a MODM problem where the parameters of the problem are multi-choice random parameters. Also, we develop the methodology to solve the problem. Multi-choice random parameters are that parameter which requires choosing a value from a set of alternative values and the alternatives are all random variables. In the following paragraph, we briefly describe about the multi-choice programming and stochastic programming.

In several decision-making problems, we need to choose one value from a set of values for a parameter to take the optimal decision. This kind of problem can be transformed into a mathematical programming model and the model is called multi-choice programming problem (MCPP). In MCPP, a multiple number of choices can be assigned by the decision maker for a parameter. Chang [8] proposed a new idea to solve multi-objective programming problem called multi-choice goal programming where for each objective a multiple number of aspiration levels or goals are set by the decision maker. In addition, to tackle these multi-choice aspiration levels a binary variable for each goal are introduced. In his other paper, Chang [9] used some continuous variable to tackle the situation instead of using binary variable. Liao [10] proposed a model called multi-segment goal-programming problem, in which the coefficients of the objective function take multiple aspiration level. Chang's binary variable method has been used to tackle the multi-choice parameters. Biswal and Acharya [11] proposed multi-choice linear programming problems (MCLPP) in which the right hand side parameters of the constraints of linear programming problem take multiple aspiration level. They proposed a transformation technique to obtain an equivalent mathematical model. In their other paper [12], they have introduced interpolating polynomial for each multi-choice type parameter to transform the model. However, multi-choice programming problem can be considered as a combinatorial optimization problem. Since in actual decision-making situations, we must often make a decision on the basis of uncertain data, so it makes proper sense to take the alternative choices as random variable in a MCLPP. Stochastic programming is used for such decision-making problems where randomness is involved. Stochastic programming has been developed in various ways, a bibliographic review of Stochastic programming can be found in [13, 14]. Among these approaches for stochastic programming, there are two very popular approaches namely,

(i) Chance constraints programming or probabilistic programming proposed by Charnes and Cooper [15,16]
(ii) Two-stage programming or Recourse programming proposed by Dantzig [17].

In optimization problems, to handle some random parameters, we use both these methods. In chance constrained programming (CCP) technique, constraints of the problem can be violated up to a given level of probability. These satisfactory levels of probability are fixed by the decision maker. The two-stage programming technique also converts the stochastic problem into a deterministic problem. But the two-stage programming does not allow any constraint to be violated. Several chance-constraint programming (CCP) technique are based on the probability of the occurrence of the constraint containing a random variable. In this technique, probabilistic constraints of the problem can be violated up to a given significance level of probability. These significance levels of probability are xed by the decision maker. After using chance-constraint programming technique on an optimization problem having a random parameter, the decision maker obtains a deterministic model. The two-stage programming technique also converts the stochastic programming problem into a deterministic model. But in this technique, the constraint violation is not allowed. Plenty of literatures can be found in the field of stochastic programming [18–24].

In this paper, we concentrate to find a proper methodology to solve the multi-objective multi-choice linear programming problem with random variable as the alternative values. Since most common distribution in nature is the normal distribution, we assume that all the alternative random variables of a multi-choice parameter are independent normal distributions with known mean and variance.

The organization of the paper is as follows: after giving a brief introduction about the problem in Sect. 1, in Sect. 2 we have presented the mathematical Model of Multi-objective Multi-choice random Linear Programming Problem. The deterministic form of the proposed model is established in Sect. 3. In Sect. 4, TOPSIS Method to solve Multi-objective Multi-choice random Linear Programming Problem is presented. In Sect. 5, an example in given to illustrate the model and methodology developed throughout this paper. Then in the very next Section, some conclusions are presented.

2 Mathematical Model of Multi-objective Multi-choice Random Linear Programming Problem

Definition 2.1 If a parameter of an optimization problem requires to choose a value from a set of alternative values and the alternatives are all random variables, then the parameter is said to be a multi-choice random parameter.

The mathematical model of a multi-objective linear programming problem with multi-choice random parameter can be stated as

$$\max:\ z_k = \sum_{j=1}^{n} c_{ij} x_j, \quad k = 1, 2, \ldots, K \tag{2.1}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i,\ i = 1, 2, \ldots, m \tag{2.2}$$

$$x_j \ge 0,\ j = 1, 2, \ldots, n, \tag{2.3}$$

where $X = (x_1, x_2, \ldots, x_n)$ is n-dimensional decision vector and it is deterministic. We consider the problem when c_{kj}; a_{ij}; b_i are multi-choice random variables. Then seven sub cases are arising which are as follows:

i. Only a_{ij} are multi-choice random variables
ii. Only b_i are multi-choice random variables
iii. a_{ij} and b_i are multi-choice random variables.
iv. Only c_{kj} are multi-choice random variables (it can be define as "Multi-Segment random linear programming problem").
v. a_{ij} and c_{kj} are multi-choice random variables.
vi. c_{kj} and b_i are multi-choice random variables.
vii. a_{ij}, b_i and c_{kj} are multi-choice random variables.

We will discuss about the deterministic form of the problem for only first four cases and rest of the cases are follows by combining these cases.

3 Deterministic Model Formulation

In this paper, we consider that all the random variables present in the problem follow independent normal distribution with known mean and variance. We established the deterministic form of the various cases of the model.

Case-I: Only a_{ij} are multi-choice random parameters

Let each alternative a_{ij}^{s}, $(s = 1, 2, \ldots, p_{ij},\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n)$ are distributed normally with known mean and variance. Let $a_{ij}^{s} \approx N(\mu_{ij}^{(s)}, \sigma_{ij}^{(s)})$ i.e., a_{ij}^{s} is normally distributed random variables with mean $\mu_{ij}^{(s)}$

Table 1 Data table for multi-choice parameter a_{ij}

w_{ij}	0	1	2	…	p_{ij} 1
f_{aij} (w_{ij})	$a_{ij}^{(1)}$	$a_{ij}^{(2)}$	$a_{ij}^{(3)}$	…	$a_{ij}^{(p_{ij})}$

and variance $(\sigma_{ij}^{(s)})^2$. We considered that the random variables are independently distributed, hence the covariance between any two random variables will be zero. To tackle the multi-choice parameter a_{ij}, we use interpolating polynomial approach introduced by Biswal et al. [12]. To formulate interpolating polynomials, we have to introduce an integer variable w_{ij} corresponding to multi-choice parameter a_{ij}; (i = 1,2,…, m; j = 1,2,…,n). Each integer variable takes exactly p_{ij} number of nodal points. Each node corresponds to exactly one functional value of a multi-choice parameter. Here the functional value of each node is a random variable. Details of it are given by the Table 1. Using Lagrange's formula, we formulate the interpolating polynomial corresponds to a_{ij} as:

$$\begin{aligned} f_{a_{ij}}(w_{ij}, a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}, \ldots, a_{ij}^{(p_{ij})}) = {} & \frac{(w_{ij}-1)(w_{ij}-2)\ldots(w_{ij}-p_{ij}+1)}{(-1)^{(p_{ij}-1)}(p_{ij}-1)!} a_{ij}^{(1)} \\ & + \frac{w_{ij}(w_{ij}-2)(w_{ij}-3)\ldots(w_{ij}-p_{ij}+1)}{(-1)^{(p_{ij}-2)}1!\,(p_{ij}-2)!} a_{ij}^{(2)} + \ldots \\ & + \frac{w_{ij}(w_{ij}-1)(w_{ij}-2)\ldots(w_{ij}-p_{ij}+2)}{(p_{ij}-1)!} a_{ij}^{(p_{ij})} \end{aligned} \tag{3.1}$$

Since the problem (2.1–2.3) has random variables as the alternative values of the multi-choice parameter a_{ij}, we can't able to apply the solution methods as usual manner for mathematical programming problem directly. So, we apply chance constrained programming to find the equivalent model which can be solved by usual methods. In the chance constraints programming method, the original m constraints are interpreted as:

$$\Pr\left(\sum \left\{a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(p_{ij})}\right\} x_j \le b_i\right) \ge 1-\gamma_i, \quad i = 1, 2, \ldots, m \tag{3.2}$$

where Pr means probability, and $0 < \gamma_i < 1, i = 1, 2, \ldots, m.(1-\gamma_i)$ is the given probability of the extents to which the i-th constraint violations are admitted. The inequalities (3.2) are called chance constraints, it means that the i-th constraint may be violated, but at most γ_i proportion of the time.

Let us define $M_{ij} = x_j f_{a_{ij}}(w_{ij}, a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, .a_{ij}^{(p_{ij})})$. since $a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, .a_{ij}^{(p_{ij})}$ are independent normally distributed random variable and M_{ij} is a linear combination of these random variables, M_{ij} will be normally distributed random variable. The mean and variance of (M_{ij}) are defined as,

$$E(M_{ij}) = \frac{(w_{ij}-1)(w_{ij}-2)\ldots(w_{ij}-p_{ij}+1)}{(-1)^{(p_{ij}-1)}(p_{ij}-1)!} x_j \mu_{ij}^{(1)}$$

$$+\frac{w_{ij}(w_{ij}-2)(w_{ij}-3)\dots(w_{ij}-p_{ij}+1)}{(-1)^{(p_{ij}-2)}1!\,(p_{ij}-2)!}x_j\mu_{ij}^{(2)}+\dots$$
$$+\frac{w_{ij}(w_{ij}-1)(w_{ij}-2)\dots(w_{ij}-p_{ij}+2)}{(p_{ij}-1)!}x_j\mu_{ij}^{(p_{ij})}$$

$$V(M_{ij})=\left[\frac{(w_{ij}-1)(w_{ij}-2)\dots(w_{ij}-p_{ij}+1)}{(-1)^{(p_{ij}-1)}(p_{ij}-1)!}x_j\right]^2(\sigma_{ij}^{(1)})^2$$
$$+\left[\frac{w_{ij}(w_{ij}-2)(w_{ij}-3)\dots(w_{ij}-p_{ij}+1)}{(-1)^{(p_{ij}-2)}1!\,(p_{ij}-2)!}x_j\right]^2(\sigma_{ij}^{(2)})^2$$
$$+\left[\frac{w_{ij}(w_{ij}-1)(w_{ij}-2)\dots(w_{ij}-p_{ij}+2)}{(p_{ij}-1)!}x_j\right]^2(\sigma_{ij}^{(p_{ij})})^2$$

Now from the Eq. (3.2) we have,

$$\Pr\left(\sum_{j=1}^{n}M_{ij}\le b_i\right)\ge 1-\gamma_i,\quad i=1,2,\dots,m \tag{3.3}$$

$$\Rightarrow \Pr\left(\frac{\sum_{j=1}^{n}M_{ij}\le b_i-\sum_{j=1}^{n}E(M_{ij})}{\sqrt{V(\sum_{j=1}^{n}M_{ij})}}\le\frac{b_i-\sum_{j=1}^{n}E(M_{ij})}{\sqrt{V(\sum_{j=1}^{n}M_{ij})}}\right)\ge 1-\gamma_i \tag{3.4}$$

M_{ij} are all independent normally distributed random variable, hence their sum will be a normally distributed random variable. Therefore, $\frac{\sum_{j=1}^{n}M_{ij}\le b_i-\sum_{j=1}^{n}E(M_{ij})}{\sqrt{V(\sum_{j=1}^{n}M_{ij})}}$ follows standard normal distribution N (0,1). The cumulative density function of the standard normal random variable evaluated at z is given by $\Phi(z)$. Hence from (3.4) we have,

$$\Phi\left(\frac{b_i-\sum_{j=1}^{n}E(M_{ij})}{\sqrt{V\left(\sum_{j=1}^{n}M_{ij}\right)}}\right)\ge\Phi(\eta_{\gamma_i}) \tag{3.5}$$

$$\Rightarrow\frac{b_i-\sum_{j=1}^{n}E(M_{ij})}{\sqrt{V(\sum_{j=1}^{n}M_{ij})}}\ge\eta_{\gamma_i} \tag{3.6}$$

$$\Rightarrow\sum_{j=1}^{n}E(M_{ij})+\eta_{\gamma_i}\sqrt{V\left(\sum_{j=1}^{n}M_{ij}\right)}\le b_i,\ \ i=1,2,\dots,m, \tag{3.7}$$

where η_{γ_i} denotes the value of the standard normal variable such that $\Phi(\eta_{\gamma_i})=1-\gamma_i$. The inequalities given by (3.7) are the equivalent deterministic form of the probabilistic form of the constraints 2.2. Hence the solution of the original problem can be found by solving the following equivalent deterministic model:

Table 2 Data table for multi-choice parameter b_i

v_i	0	1	2	…	$r_i - 1$
$f_{b_i}(v_i)$	$b_i^{(1)}$	$b_i^{(2)}$	$b_i^{(3)}$		$b_i^{(r_i)}$

$$\max: \ z_k = \sum_{j=1}^{n} c_{ij} x_j, \quad k = 1, 2, \ldots, K \tag{3.8}$$

$$\text{subject to} \ \sum_{j=1}^{n} E(M_{ij}) + \eta_{\gamma_i} \sqrt{V\left(\sum_{j=1}^{n} M_{ij}\right)} \le b_i, \quad i = 1, 2, \ldots, m \tag{3.9}$$

$$x_j \ge 0, \ \ j = 1, 2, \ldots, n \tag{3.10}$$

$$\gamma_i \in (0, 1), \ \ i = 1, 2, \ldots, n \tag{3.11}$$

$$0 \le w_{ij} \le p_{ij} - 1, \ \ w_{ij} \in N_0, \ \ i = 1, 2, \ldots, m, \ \ j = 1, 2, \ldots, n \tag{3.12}$$

Case-II: only b_i are multi-choice random parameter.

Let each alternative choices of a multi-choice random parameter are independent normally distributed random variables with known mean and variance. Let t-th alternative choice of i-th parameter b_i follows $N(\mu_i^{(t)}, \sigma_i^{(t)})$, i.e., mean and variance of $b_i^{(t)}$ are given by $\mu_i^{(t)}$ and $(\sigma_i^{(t)})^2$ respectively. To formulate the interpolating polynomial corresponding to multi-choice parameter b_i, we introduce an integer variable v_i. As we discussed in the previous case, here also v_i takes values from 0 to $r_i - 1$. Details are given by Table 2. Hence we formulate the Lagrange's interpolating polynomial as:

$$\begin{aligned} f_{b_i}(v_i; b_i^{(1)}, b_i^{(2)}, \ldots, b_i^{(r_i)}) = & \frac{(v_i - 1)(v_i - 2)\ldots(v_i - r_i + 1)}{(-1)^{(r_i - 1)}(r_i - 1)!} b_i^{(1)} \\ & + \frac{v_i(v_i - 2)(v_i - 3)\ldots(v_i - r_i + 1)}{(-1)^{(r_i - 2)} 1!\,(r_i - 2)!} b_i^{(2)} + \ldots \\ & + \frac{v_i(v_i - 1)(v_i - 2)\ldots(v_i - r_i + 2)}{(r_i - 1)!} b_i^{(r_i)}; \quad i = 1, 2, \ldots, m. \end{aligned} \tag{3.13}$$

Let us define $N_i = f_{b_i}(v_i; b_i^{(1)}, b_i^{(2)}, \ldots, b_i^{(r_i)})$ since $b_i^{(1)}, b_i^{(2)}, \ldots, b_i^{(r_i)}$ are independent normal variable and N_i is a linear function of them, then N_i is also normally distributed random variable. The mean and variance of N_i are given by

$$\begin{aligned} E(N_i) = & \frac{(v_i - 1)(v_i - 2)\ldots(v_i - r_i + 1)}{(-1)^{(r_i - 1)}(r_i - 1)!} \mu_i^{(1)} \\ & + \frac{v_i(v_i - 2)(v_i - 3)\ldots(v_i - r_i + 1)}{(-1)^{(r_i - 2)} 1!\,(r_i - 2)!} \mu_i^{(2)} + \ldots \\ & + \frac{v_i(v_i - 1)(v_i - 2)\ldots(v_i - r_i + 2)}{(r_i - 1)!} \mu_i^{(r_i)} \end{aligned}$$

$$V(N_i) = \left[\frac{(v_i-1)(v_i-2)\dots(v_i-r_i+1)}{(-1)^{(r_i-1)}(r_i-1)!}\right]^2(\sigma_i^{(1)})^2$$
$$+\left[\frac{v_i(v_i-2)(v_i-3)\dots(v_i-r_i+1)}{(-1)^{(r_i-2)}1!\,(r_i-2)!}\right]^2(\sigma_i^{(2)})^2$$
$$+\left[\frac{v_i(v_i-1)(v_i-2)\dots(v_i-r_i+2)}{(r_i-1)!}\right]^2(\sigma_i^{(r_i)})^2$$

respectively. Since random variables are involved in the constraints, we consider the constraints as chance constraint and we can restate them as,

$$\Pr\left(\sum_{j=1}^{n} a_{ij}x_j \le \left\{b_i^{(1)}, b_i^{(1)}, \dots, b_i^{(1)}\right\}\right) \ge 1-\gamma_i \tag{3.14}$$

$$\Rightarrow \Pr\left(\frac{N_i - E(N_i)}{\sqrt{V(N_i)}} \ge \frac{\sum_{j=1}^{n} a_{ij}x_j - E(N_i)}{\sqrt{V(N_i)}}\right) \ge 1-\gamma_i \tag{3.15}$$

$$\Rightarrow 1-\Pr\left(\frac{N_i - E(N_i)}{\sqrt{V(N_i)}} \le \frac{\sum_{j=1}^{n} a_{ij}x_j - E(N_i)}{\sqrt{V(N_i)}}\right) \ge 1-\gamma_i \text{ (say}\xi_i) \tag{3.16}$$

$$\Rightarrow \Pr\left(\frac{N_i - E(N_i)}{\sqrt{V(N_i)}} \le \frac{\sum_{j=1}^{n} a_{ij}x_j - E(N_i)}{\sqrt{V(N_i)}}\right) \le 1-\xi_i \tag{3.17}$$

Here $\frac{N_i - E(N_i)}{\sqrt{V(N_i)}}$ is a standard normal random variable. Let η_{ξ_i} be the representation of the value of the standard normal variable at which $\Phi(\eta_{\xi_i}) = 1-\xi_i$, then the above constraint can be rewritten as:

$$\Phi\left(\frac{\sum_{j=1}^{n} a_{ij}x_j - E(N_i)}{\sqrt{V(N_i)}}\right) \le \Phi(\eta_{\xi_i}) \tag{3.18}$$

$$\Rightarrow \frac{\sum_{j=1}^{n} a_{ij}x_j - E(N_i)}{\sqrt{V(N_i)}} \le \eta_{\xi_i} \tag{3.19}$$

$$\Rightarrow \sum_{j=1}^{n} a_{ij}x_j \le \eta_{\xi_i}\sqrt{V(N_i)} + E(N_i), \quad i = 1, 2, \dots, m \tag{3.20}$$

Thus, the equivalent deterministic form of the probabilistic problem (2.1–2.3) is given by the following problem:

$$\max: \; z_k = \sum_{j=1}^{n} c_{ij}x_j, \quad k = 1, 2, \dots, K \tag{3.21}$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij}x_j \le \eta_{\xi_i}\sqrt{V(N_i)} + E(N_i), \; i = 1, 2, \dots, m \tag{3.22}$$

$$x_j \geq 0, \;\; j = 1, 2, \ldots, n \tag{3.23}$$

$$\xi_i \in (0, 1), \;\; i = 1, 2, \ldots, m \tag{3.24}$$

$$0 \leq v_i \leq r_i - 1, \;\; v_i \in N_0, \;\; i = 1, 2, \ldots, m, \;\; j = 1, 2, \ldots, n \tag{3.25}$$

Case-III: a_{ij} and b_i are multi-choice random parameter.

$$\Pr\left(\sum_{j=1}^{n} \left\{a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(p_{ij})}\right\} x_j \leq \left\{b_i^{(1)}, b_i^{(1)}, \ldots, b_i^{(1)}\right\}\right) \geq 1 - \gamma_i, \;\; i = 1, 2, \ldots, m \tag{3.26}$$

Since the parameters a_{ij} and b_i are multi-choice random, the constraint (2.2) can be restated as a chance constraint. It can represent as,

Let us consider alternative choices $a_{ij}^{(s)}$ of the multi-choice random parameter a_{ij} follows $N(\mu_{ij}^{(s)}, \sigma_{ij}^{(s)})$ and the random variables are independent. Similarly, alternative choices $b_i^{(t)}$ of the multi-choice random parameter b_i follows $N(\mu_{ij}^{(t)}, \sigma_{ij}^{(t)})$. We use the interpolating polynomials 3.1 and 3.13 to tackle these multi-choice parameter respectively. Now, let us denote a random variable $A_i = \sum_{j=1}^{n} M_{ij} - N_i$, where $M_{ij}, \; N_i$ are defined in the discussion of previous two cases. Since all $M_{ij}, \; N_i$ are independent normally distributed random variables and A_i are linear combination of them, A_i follows normal distribution. The mean and variance of A_i are given by $E(A_i) = \sum_{j=1}^{n} E(M_{ij}) - E(N_i)$ and $V(A_i) = \sum_{j=1}^{n} V(M_{ij}) - V(N_i)$ respectively. Now the chance constraints can be written as,

$$\Pr(A_i \leq 0) \geq 1 - \gamma_i \tag{3.27}$$

$$\Pr\left(\frac{A_i - E(A_i)}{\sqrt{V(A_i)}} \leq -\frac{E(A_i)}{\sqrt{V(A_i)}}\right) \geq 1 - \gamma_i \tag{3.28}$$

where $\frac{A_i - E(A_i)}{\sqrt{V(A_i)}}$ follows standard normal distribution. Let η_{γ_i} be the value of the standard normal variable where $\Phi(\eta_{\gamma_i}) = 1 - \gamma_i$. Therefore, the above inequality can be written as,

$$\Phi\left(-\frac{E(A_i)}{\sqrt{V(A_i)}}\right) \geq \Phi(\eta_{\gamma_I}) \tag{3.29}$$

$$\Rightarrow -\frac{E(A_i)}{\sqrt{V(A_i)}} \geq \eta_{\gamma_I} \tag{3.30}$$

$$\Rightarrow \sum_{j=1}^{n} E(M_{ij}) + \eta_{\gamma_I} \sqrt{\sum_{j=1}^{n} V(M_{ij}) + V(N_i)} \geq E(N_i). \tag{3.31}$$

Thus, equivalent deterministic model of the problem for this case is given by,

$$\max : \; z_k = \sum_{j=1}^{n} c_{ij} x_j, \quad k = 1, 2, \ldots, K \tag{3.32}$$

Table 3 Data table for multi-choice parameter c_{qj}

u_{kk}	0	1	2	…	$r_i - 1$
$f_{c_{kj}}(u_{kj})$	$c_{kj}^{(1)}$	$c_{kj}^{(2)}$	$c_{kj}^{(3)}$	…	$c_{kj}^{(q_{kj})}$

$$\text{subject to } \sum_{j=1}^{n} E(M_{ij}) + \eta_{\xi_i}\sqrt{\sum V(M_{ij}) + V(N_i)} \geq E(N_i), \; i = 1, 2, \ldots, m \tag{3.33}$$

$$x_j \geq 0\,, j = 1, 2, \ldots, n \tag{3.34}$$

$$\gamma_i \in (0, 1), \; i = 1, 2, \ldots, m \tag{3.35}$$

$$0 \leq w_{ij} \leq p_{ij} - 1, \; w_{ij} \in N_0, \; i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n \tag{3.36}$$

$$0 \leq v_i \leq r_i - 1, \; v_i \in N_0, i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n \tag{3.37}$$

Case-IV: $\mathrm{c_{kj}}$ are multi-choice random parameters.

This case deals with the situation when only the cost coefficients of the problem i.e., c_{kj} are multi-choice random parameters. According to the definition, the alternative choices for each cost coefficients are random variable. Let c_{kj} ($k = 1,2,\ldots,K; j = 1,2,\ldots,n$) are independent and the alternative choices follows independent normal distribution. Let $c_{kj}^{l} = N(\mu_{kj}^{(l)}, \sigma_{kj}^{(l)})$ ($l = 1, 2, \ldots, q_{kj}$). To tackle the multi-choice parameter c_{kj}, we introduce an integer variable u_{kj} which takes q_{kj} number of values. We formulate a Lagrange interpolating polynomial $f_{c_{kj}}(u_{kj})$ which passes through all the q_{kj} number of points given by Table 3.

The interpolating polynomial for the multi-choice parameter c_{kj} is given by

$$\begin{aligned} f_{c_{kj}}(u_{kj}, c_{kj}^{(1)}, c_{kj}^{(2)}, c_{kj}^{(3)}, \ldots, c_{kj}^{(q_{kj})}) = {} & \frac{(u_{kj}-1)(u_{kj}-2)\ldots(u_{kj}-p_{ij}+1)}{(-1)^{(q_{kj}-1)}(q_{kj}-1)!} c_{kj}^{(1)} \\ & + \frac{u_{kj}(u_{kj}-2)(u_{kj}-3)\ldots(u_{kj}-q_{kj}+1)}{(-1)^{(q_{kij}-2)} 1!\,(q_{kj}-2)!} c_{kj}^{(2)} + \ldots \\ & + \frac{u_{kj}(u_{kj}-1)(u_{kj}-2)\ldots(u_{kj}-q_{kj}+2)}{(q_{kj}-1)!} c_{kj}^{(q_{kj})} \end{aligned} \tag{3.38}$$

Let us define $\mathrm{C}_{kj} = \mathrm{x}_j\, f_{c_{kj}}(u_{kj}, c_{kj}^{(1)}, c_{kj}^{(2)}, c_{kj}^{(3)}, \ldots, c_{kj}^{(q_{kj})})$, then C_{kj} is a linear function of independent normally distributed random variable. Thus, C_{kj} is also a random variable with mean and variance given by,

$$E(C_{kj}) = x_j \left[\begin{array}{c} \frac{(u_{kj}-1)(u_{kj}-2)\ldots(u_{kj}-q_{kj}+1)}{(-1)^{(q_{kj}-1)}(q_{kj}-1)!}{}_j\,\mu_{kj}^{(1)} \\ + \frac{u_{kj}(u_{kj}-2)(u_{kj}-3)\ldots(u_{kj}-q_{kj}+1)}{(-1)^{(q_{kj}-2)} 1!(q_{kj}-2)!} \mu_{kj}^{(2)} + \ldots \\ + \frac{u_{kj}(u_{kj}-1)(u_{kj}-2)\ldots(u_{kj}-p_{ij}+2)}{(q_{kj}-1)!} \mu_{kj}^{(q_{kj})} \end{array} \right]$$

$$V(M_{ij}) = x_j^2 \begin{pmatrix} \left[\frac{(u_{kj}-1)(u_{kj}-2)\ldots(u_{kj}-q_{kj}+1)}{(-1)^{(q_{kj}-1)}(q_{kj}-1)!}\right]^2 (\sigma_{kj}^{(1)})^2 \\ +\left[\frac{u_{kj}(u_{kj}-2)(u_{kj}-3)\ldots(u_{kj}-q_{kj}+1)}{(-1)^{(q_{kj}-2)}1!(q_{kj}-2)!}\right]^2 (\sigma_{kj}^{(2)})^2 \\ +\left[\frac{u_{kj}(u_{kj}-1)(u_{kj}-2)\ldots(u_{kj}-q_{kj}+2)}{(q_{kj}-1)!}\right]^2 (\sigma_{kj}^{(q_{kj})})^2 \end{pmatrix}$$

In this case, to establish the equivalent deterministic model of (2.1–2.3) we have to find the deterministic form of the objective function. Depending on the aim of the decision maker, we have considered four types of decision rules to optimize objective functions with random variables:

(i) The maximum expected value model (called 'E'-model)
(ii) The minimum variance model namely variance model (called 'V'-model)
(iii) The maximum probability model also called as dependent-chance model (called 'P'-model)
(iv) The fractile criterion model [25, 26].

If the decision maker wants to maximize the expected value of the objective functions, he/she can apply 'E'-model to solve the problem. Substituting the random variables by their expected values, we obtain the 'E'-model as:

$$\max: \ z_k = \sum_{j=1}^{n} E(C_{kj}), \quad k = 1, 2, \ldots, K \tag{3.39}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij}x_j \le b_i, \ i = 1, 2, \ldots, m \tag{3.40}$$

$$x_j \ge 0, \ j = 1, 2, \ldots, n \tag{3.41}$$

$$0 \le u_{kj} \le q_{kj} - 1, u_j \in N_0, \ k = 1, 2, \ldots, K; \ i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n \tag{3.42}$$

In the 'V'-model, the variances of the objective functions are minimized subjective to the fact that the expected value of the objective functions should be higher than a target value. The model can be given as

$$\min: \ z_k = \sum_{j=1}^{n} V(C_{kj}), \quad k = 1, 2, \ldots, K \tag{3.43}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij}x_j \le b_i \ , \ i = 1, 2, \ldots, m \tag{3.44}$$

$$\sum_{j=1}^{n} E(C_{kj}) \ge T_k, \ k = 1, 2, \ldots, K; \ j = 1, 2, \ldots, n \tag{3.45}$$

$$x_j \ge 0, \ j = 1, 2, \ldots, n \tag{3.46}$$

$$0 \le u_{kj} \le q_{kj} - 1,\ u_j \in N_0,\ k = 1, 2, \ldots, K;\ i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n, \tag{3.47}$$

where T_k is the fixed expected value of the objective function z_k by the decision maker. In order to deal with the situations where the decision maker of the problem wants to maximize the probability that the objective function with random variable is greater than or equal to a certain permissible level, we consider probability maximization model. Considering the minimum permissible level of the objective function as T_k, we establish the objective function of the model as:

$$\max \quad p_k(X, U) = \left(\Pr \sum_{j=1}^{n} C_{kj} x_j \ge T_k \right), \quad k = 1, 2, \ldots, K \tag{3.48}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i,\ i = 1, 2, \ldots, m \tag{3.49}$$

$$\sum_{j=1}^{n} E(C_{kj}) \ge T_k,\ k = 1, 2, \ldots, K;\ j = 1, 2, \ldots, n \tag{3.50}$$

$$x_j \ge 0,\ j = 1, 2, \ldots, n \tag{3.51}$$

$$0 \le u_{kj} \le q_{kj} - 1,\ u_j \in N_0, k = 1, 2, \ldots, K;\ i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n \tag{3.52}$$

Since c_{kj} are multi-choice random parameter, we use interpolating polynomial and chance constraint programming approach to find the equivalent deterministic form. Therefore,

$$\Pr\left(\sum_{j=1}^{n} c_{kj} x_j \ge T_k \right) = \Pr\left(\sum_{j=1}^{n} c_{kj} \ge T_k \right),\ i = 1, 2, \ldots, m \tag{3.53}$$

$$= \Pr\left(\frac{\sum_{j=1}^{n} C_{kj} - E\left(\sum_{j=1}^{n} C_{kj}\right)}{\sqrt{V\left(\sum_{j=1}^{n} C_{kj}\right)}} \ge \frac{T_k - E\left(\sum_{j=1}^{n} C_{kj}\right)}{\sqrt{V\left(\sum_{j=1}^{n} C_{kj}\right)}} \right) \tag{3.54}$$

$$= 1 - \Phi\left(\frac{T_k - E\left(\sum_{j=1}^{n} C_{kj}\right)}{\sqrt{V\left(\sum_{j=1}^{n} C_{kj}\right)}} \right) \tag{3.55}$$

hence we have the equivalent problem as

$$\max\ p_k(X,U) = 1 - \Phi\left(\frac{T_k - E\left(\sum_{j=1}^{n} C_{kj}\right)}{\sqrt{V\left(\sum_{j=1}^{n} C_{kj}\right)}}\right), \quad k = 1,2,\ldots,K \tag{3.56}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij}x_j \le b_i,\ i = 1,2,\ldots,m \tag{3.57}$$

$$\sum_{j=1}^{n} E(C_{kj}) \ge T_k, k = 1,2,\ldots,K;\ j = 1,2,\ldots,n \tag{3.58}$$

$$x_j \ge 0,\ j = 1,2,\ldots,n \tag{3.59}$$

$$0 \le u_{kj} \le q_{kj} - 1,\ u_j \in N_0,\ k = 1,2,\ldots,K;\ i = 1,2,\ldots,m;\ j = 1,2,\ldots,n \tag{3.60}$$

In the fractile criterion model, the decision maker tries to maximize the value of the objective function such that the probability of obtaining such value is greater than or equal to some given thresholds under chance constrained conditions. Let us denote that maximal value for the k-th objective function as T_k and the given threshold be λ_k. Then we have the equivalent model as:

$$\max:\ T_k, \quad k = 1,2,\ldots,K \tag{3.61}$$

$$\text{subject to} \sum_{j=1}^{n} E\left(\sum_{j=1}^{n} C_{kj}\right) + \eta_{1-\lambda_k}\sqrt{\sum_{j=1}^{n} V(M_{ij}) + V(N_i) \ge E(N_i)}, \tag{3.62}$$
$$i = 1,2,\ldots,m$$

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i, \quad i = 1,2,\ldots,m \tag{3.63}$$

$$x_j \ge 0,\ j = 1,2,\ldots,n \tag{3.64}$$

$$0 \le u_j \le k_j - 1, \tag{3.65}$$

$$u_j \in N_0,\ i = 1,2,\ldots,m, \quad j = 1,2,\ldots,n \tag{3.66}$$

Similarly, the equivalent model for the rest of the cases can be established by combining the described cases. The solution of these equivalent models will be the solution of the original problem. To find the solution of these equivalent multi-objective nonlinear mixed integer problems, we use TOPSIS algorithm. In the next section, we will describe the method to solve the equivalent model.

4 TOPSIS Method for Multi-objective Multi-choice Random Linear Programming Problem

In this section, we describe the TOPSIS method to find the non dominated solution of the multi-objective nonlinear mixed integer problems. To explain the method, we consider Case VII of the problem (2.1–2.3). The equivalent model ('E'-model) of the considered problem is given by

$$\max:\ Z_k = \sum_{j=1}^{n} E(C_{kj}),\quad k = 1, 2, \ldots, K \tag{4.1}$$

$$\text{subject to}\ \sum_{j=1}^{n} E(M_{ij}) + \eta_{\gamma_i}\sqrt{\sum_{j=1}^{n} V(M_{ij}) + V(N_i)} \geq E(N_i),\quad i = 1, 2, \ldots, m \tag{4.2}$$

$$x_j \geq 0,\ \ j = 1, 2, \ldots, n \tag{4.3}$$

$$\gamma_i \in (0, 1),\ \ i = 1, 2, \ldots, m \tag{4.4}$$

$$0 \leq u_j \leq k_j - 1 \tag{4.6}$$

$$0 \leq w_{ij} \leq p_{kj} - 1,\ \ w_{ij} \in N_0,\ \ i = 1, 2, \ldots, m,\ \ j = 1, 2, \ldots, n \tag{4.7}$$

$$0 \leq v_i \leq r_i - 1,\ \ v_i \in N_0,\ \ i = 1, 2, \ldots, m,\ \ j = 1, 2, \ldots, n \tag{4.8}$$

all the notations used in this equivalent model are already described in the previous Section. Before describing the TOPSIS method, we will recall some basis distance measures concepts for the problem (4.1–4.7).

4.1 Basic Concepts of Distance Measure

To solve a multi-objective decision making problem, compromise programming approach was developed. The method reduces the non dominated solution set of the problem. The solution 'closest' to the ideal solution are called compromise solution. The 'closeness' is defined by some distance measure.

Consider the objective vector $Z = (Z_1, Z_2, \ldots, Z_K)$ of the problem (4.1–4.7). The vector of the ideal solution (PIS) and anti-ideal solution (NIS) of the objective vector in the K objective space is given by $Z^* = (Z_1^*, Z_2^*, \ldots, Z_K^*)$ and $Z' = (Z_1', Z_2', \ldots, Z_K')$ respectively. Where $Z_k^* = \max Z_k$ and $Z_k' = \min Z_k$ over the feasible region of the problem. Here, we use 'L_p-metric' to define the 'closeness'. L_p-metric defines the distance between two points Z^* and Z in K dimensional space as:

$$d_p = \left\{ \sum_{k=1}^{K} \lambda_k^p (Z_k^* - Z_k)^p \right\}^{\frac{1}{p}}, \quad p = 1, 2, \ldots, \infty, \tag{4.8}$$

where λ_k; $k = 1,2,\ldots, K$ represent the relative importance (weights) of objectives. If the objective functions Z_k, $k = 1,2,\ldots, K$ are not expressed in commensurable units, a scaling function for each and every objective functions can be used, usually, this dimensionless is the interval [0, 1]. In this case, the following metric can be used:

$$d_p = \left\{ \sum_{k=1}^{K} \lambda_k^p \left(\frac{Z_k^* - Z_k}{Z_k^* - Z_k'} \right)^p \right\}^{\frac{1}{p}}, \quad p = 1, 2, \ldots, \infty \tag{4.9}$$

Considering the positive ideal solution (PIS) vector ($Z^* = (Z_1^*, Z_2^*, \ldots, Z_K^*)$) as the reference point, many methods such as global criterion method, goal-programming method, fuzzy programming method, and interactive method have been developed to find compromise solution of the multi-objective problems. In these methods the distance measure (4.8) and (4.9) are used. Then we need to optimize the following auxiliary objective over the feasible region [2]:

$$\min d_p = \left\{ \sum_{k=1}^{K} \lambda_k^p \left(\frac{Z_k' - Z_k^*}{Z_k^* - Z_k'} \right)^p \right\}^{\frac{1}{p}}, \quad p = 1, 2, \ldots, \infty \tag{4.10}$$

The value of d_p greatly depends on the parameter p, as p increase, the distance d_p decrease, i.e., $d_1 \geq d_2 \geq \ldots \geq d_\infty$ and greater emphasis is given to the largest deviation in forming the total. p is a 'balancing factor' between the group utility and maximal individual regret. In special cases, $p = 1$ implies an equal importance (weights) for all these deviations, while $p = 2$ implies these deviations are weighted proportionately with the largest deviation having the largest weight, for p_∞, the largest deviation completely dominates the distance determination [2].

4.2 TOPSIS Approach for Equivalent Deterministic Model

We use TOPSIS method to find the compromise solution of the equivalent deterministic problem (4.1–4.7). The TOPSIS model for the considered problem can be formulated as:

$$\min : \; d_p^{\text{PIS}}(X, U, W, V) \tag{4.11}$$

$$\max : \; d_p^{\text{NIS}}(X, U, W, V) \tag{4.12}$$

$$\text{subject to } \sum_{j=1}^{n} E(M_{ij}) + \eta_{\gamma_i} \sqrt{\sum_{j=1}^{n} V(M_{ij}) + V(N_i)} \geq E(N_i),\ \ i = 1, 2, \ldots, m \tag{4.13}$$

$$x_j \geq 0,\ \ j = 1, 2, \ldots, n \tag{4.14}$$

$$\gamma_i \in (0, 1),\ \ i = 1, 2, \ldots, m \tag{4.15}$$

$$0 \leq u_j \leq k_j - 1 \tag{4.16}$$

$$0 \leq w_{ij} \leq p_{kj} - 1,\ \ w_{ij} \in N_0,\ \ i = 1, 2, \ldots, m,\ \ j = 1, 2, \ldots, n \tag{4.17}$$

$$0 \leq v_i \leq r_i - 1, v_i \in N_0,\ \ i = 1, 2, \ldots, m,\ \ j = 1, 2, \ldots, n \tag{4.18}$$

where

$$X = (x_1, x_2, \ldots, x_n), U = (u_1, u_2, \ldots, u_n),\ \ W = (w_{11}, w_{12}, \ldots, w_{nm}),\ \ V = (v_1, v_2, \ldots, v_n),$$

$$d_p^{\text{PIS}}(X, U, W, V) = \left\{ \sum_{k=1}^{K} \lambda_k^p \left(\frac{Z_k^* - Z_k}{Z_k^* - Z_k'} \right)^p \right\}^{\frac{1}{p}},$$

$$d_p^{\text{NIS}}(X, U, W, V) = \left\{ \sum_{k=1}^{K} \lambda_k^p \left(\frac{Z_k - Z_k'}{Z_k^* - Z_k'} \right)^p \right\}^{\frac{1}{p}} \tag{4.19}$$

where $Z_k^* = \max Z_k$ and $Z_k' = \min Z_k$ over the feasible region of the problem are the PIS and NIS of the k-th objective function. Hence the TOPSIS model is a bi-objective problem. By using the TOPSIS approach, we reduce multi-objective model into a bi-objective model.

$$(d_p^{\text{PIS}})^* = \min:\ d_p^{\text{PIS}}(X, U, W, V) \text{ and the solution is } (X^p, U^p, W^p, V^p) \tag{4.20}$$

$$(d_p^{\text{PIS}})' = \max:\ d_p^{\text{PIS}}(X, U, W, V) \tag{4.21}$$

$$(d_p^{\text{NIS}})^* = \max:\ d_p^{\text{NIS}}(X, U, W, V) \text{ and the solution is } (X^n, U^n, W^n, V^n) \tag{4.22}$$

$$(d_p^{\text{NIS}})' = \min:\ d_p^{\text{NIS}}(X, U, W, V) \tag{4.23}$$

Then we can formulate the membership function for the objectives d_p^{PIS} and d_p^{NIS} as:

$$\mu_1(X, U, W, V) = \begin{cases} 1, & d_p^{\text{PIS}} \leq (d_p^{\text{PIS}})^* \\ \frac{(d_p^{\text{PIS}})' - d_p^{\text{PIS}}}{(d_p^{\text{PIS}})' - (d_p^{\text{PIS}})^*}, & (d_p^{\text{PIS}})^* < (d_p^{\text{PIS}}) < (d_p^{\text{PIS}})' \\ 0, & d_p^{\text{PIS}} \geq (d_p^{\text{PIS}})' \end{cases} \tag{4.24}$$

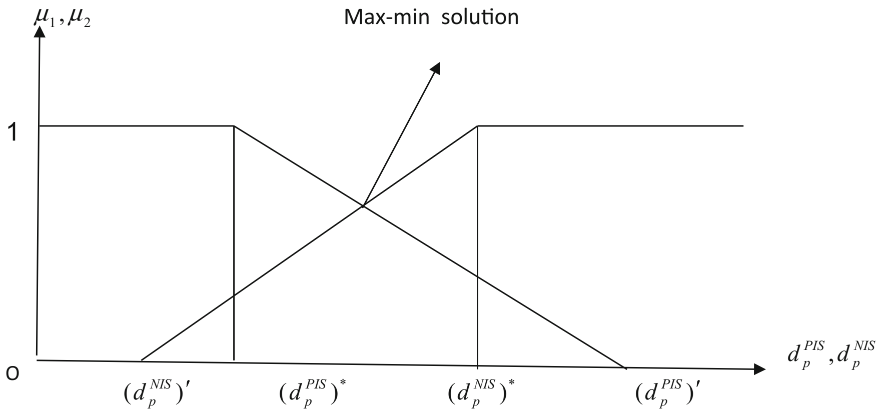

Fig. 1 Membership function of μ_1 and μ_2

$$\mu_2(X, U, W, V) = \begin{cases} 1, & d_p^{\text{NIS}} \geq (d_p^{\text{NIS}})^* \\ \frac{d_p^{\text{NIS}} - (d_p^{\text{NIS}})'}{(d_p^{\text{NIS}})' - (d_p^{\text{NIS}})^*}, & (d_p^{\text{NIS}})' < (d_p^{\text{NIS}}) < (d_p^{\text{NIS}})^* \\ 0, & d_p^{\text{NIS}} \leq (d_p^{\text{NIS}})' \end{cases} \tag{4.25}$$

These membership functions are described in Fig. 1.

To solve the bi-objective programming problem, we apply the max-min decision model proposed by Bellman and Zadeh [27] and extended by Zimmermann [28, 29]. We can obtain the optimal solution of the problem (4.11–4.18) by solving the following problem:

$$\mu(X, U, W, V) = \max\{\min(\mu_1(X, U, W, V), \mu_2(X, U, W, V))\} \tag{4.26}$$

If $\tau = \min(\mu_1(X, U, W, V), \mu_2(X, U, W, V))$ then the problem is eqivalent to the following

Tchebycheff model [2]:

$$\max : \tau \tag{4.27}$$

$$\text{subject to} \quad \mu_1(X, U, W, V) \geq \tau \tag{4.28}$$

$$\mu_2(X, U, W, V) \geq \tau \tag{4.29}$$

$$\sum_{j=1}^{n} E(M_{ij}) + \eta_{\gamma_i} \sqrt{\sum_{j=1}^{n} V(M_{ij}) + V(N_i)} \geq E(N_i), \quad i = 1, 2, \ldots, m \tag{4.30}$$

$$x_j \geq 0, \quad j = 1, 2, \ldots, n \tag{4.31}$$

$$\gamma_i \in (0, 1), \quad i = 1, 2, \ldots, m \tag{4.32}$$

Table 4 Values of multi-choice random parameters c_{1j}

c_{11}	{N (53, 5), N (60, 9), N (57, 8)}
c_{12}	{N (37, 4), N (40, 6), N (42, 7)}
c_{13}	{N (45,11), N (47,12), N (42,10), N (50,14)}
c_{13}	N (20, 5)

$$0 \leq u_j \leq k_j - 1 \tag{4.33}$$

$$0 \leq w_{ij} \leq p_{kj} - 1, \ \ w_{ij} \in N_0, \ \ i = 1, 2, \ldots, m, \ \ j = 1, 2, \ldots, n \tag{4.34}$$

$$0 \leq v_i \leq r_i - 1, \ \ v_i \in N_0, i = 1, 2, \ldots, m, \ \ j = 1, 2, \ldots, n, \tag{4.35}$$

where τ represent the satisfactory level for both the objectives, the shortest distance from the PIS and the farthest distance from the NIS. It is well known that if the optimal solution of (4.27–4.35) is the vector $(\tau^*, \ X^*, \ U^*, \ W^*, \ V^*)$, then $(X^*, \ U^*, \ W^*, \ V^*)$ is the optimal solution of model (4.11–4.18) and a satisfactory solution of the problem (4.1–4.7) [2].

5 Numerical Examples

In this section, we present an example to illustrate the methodology to solve a multi-objective multi-choice random linear programming problem. We consider the following example:

$$\text{Max}: Z_1 = c_{11}x_1 + c_{12}x_2 + c_{13}x_3 + c_{14}x_4 \tag{5.1}$$

$$Z_2 = c_{21}x_1 + c_{22}x_2 + c_{23}x_3 + c_{24}x_4 \tag{5.2}$$

$$Z_3 = c_{31}x_1 + c_{32}x_2 + c_{33}x_3 + c_{34}x_4 \tag{5.3}$$

subject to

$$x_1 + x_2 + x_3 + x_4 \geq b_1 \tag{5.4}$$

$$a_{11}x_1 + 2x_2 + x_3 + a_{24}x_4 \leq b_2 \tag{5.5}$$

$$x_1 + a_{32}x_2 + a_{33}x_3 + x_4 \leq b_1, \tag{5.6}$$

where $c_{qj}, \ \ q = 1, 2, 3, \ \ j = 1, 2, 3, 4$ are all multi-choice random parameters and their values are given by the Tables 4, 5, and 6, respectively.

Also, $a_{21} \sim$ {N(3, 1), N(4, 2)}, $a_{24} \sim$ N(2, 4), $a_{32} \sim$ {N (2; 1); N (3; 2); N (4; 4)}, and $a_{33} \sim$ {N (3; 2); N (5; 3)};

$b_1 \sim$ {N (300; 10); N (350; 17); N (325; 15)};

$b_2 \sim$ {1050; 1070; 1075; 1078; 1085},

$b_3 \sim$ {N (700; 20); N (720; 25); N (750; 30); N (760; 31)}.

Table 5 Values of multi-choice random parameters c_{2j}

c_{21}	{N (28,6), N (30,8), N (33,9), N (35,12)}
c_{22}	N (52,14)
c_{23}	{N (20,7), N (27,9)}
c_{24}	{N (20,5), N (48,14), N (55,16)}

Table 6 Values of multi-choice random parameters c_{3j}

c_{31}	N (17, 7)
c_{32}	{N(40,12), N(45,14), N(48,15)}
c_{34}	N (30, 15)
c_{33}	{N(29,11), N(33,14), N (35,15), N (38,16)}

Using our methodology, we obtain the equivalent deterministic form ('E'-model) of the multi-objective multi-choice random linear programming problem as:

$$\max : Z_1 = \left(53 + 12u_{11} - 5u_{11}^2\right)x_1 + \left(37 + 3.5u_{12} - 0.5u_{12}^2\right)x_2 + \left(45 + \frac{73}{6}u_{13} - \frac{27}{2}u_{13}^2 + \frac{10}{3}u_{13}^3\right)x_3 + 20x_4 \quad (5.7)$$

$$\max : Z_2 = \left(28 + \frac{5}{6}u_{21} - \frac{3}{2}u_{21}^2 - \frac{1}{3}u_{21}^3\right)x_1 + 52x_2 + (20 + 7 * u_{23})x_3 + \left(50 - \frac{13}{2}u_{24} + \frac{9}{2}u_{24}^2\right)x_4 \quad (5.8)$$

$$\max : Z_3 = 17x_1 + \left(40 + 6u_{32} - u_{32}^2\right)x_2 + 30x_3 + \left(29 + 6u_{34} - \frac{5}{2}u_{34}^2\right)x_4 \quad (5.9)$$

subject to

$$x_1 + x_2 + x_3 + x_4 \geq \left(300 + 175v_1 - \frac{75}{2}v_1^2\right) + 1.645\sqrt{10 - 30v_1 + \frac{417}{4}v_1^2 - \frac{181}{2}v_1^3 + \frac{93}{4}v_1^4} \quad (5.10)$$

$$(3 + w_{21})x_1 + 2x_2 + x_3 + 2x_4 + 1.81\sqrt{(1 - 2w_{21} + 3w_{21}^2)x_1^2 + 4x_4^2} \leq \left(1050 + \frac{403}{12}v_2 - \frac{413}{24}v_2^2 + \frac{47}{12}v_2^3 - \frac{7}{24}v_2^4\right) \quad (5.11)$$

Table 7 PIS pay-off table

	Z_1	Z_2	Z_3	x_1	x_2	x_3	x_4
max Z_1	17088.38	11242.456	7921.486	185.4382	52.27116	75.33402	0
max Z_2	11140.485	15458.67	8361.23.00	44.04432	148.8488	0	112.3088
max Z_3	11924.642	12148.176	12196.9	33.69139	12.01391	140.288	119.2087

PIS:$Z^* = (17088.38, 15458.67, 12196.9)$

Table 8 NIS pay-off table

	Z_1	Z_2	Z_3	x_1	x_2	x_3	x_4
min Z_1	9589.7970	13425.80	9841.520	20.1725	108.0003	44.73002	132.2991
min Z_2	14571.82	8158.4570	6605.744	194.8159	0	92.67569	17.71058
min Z_3	14571.82	8158.457	6605.7440	194.8157	0	92.67569	17.71055

NIS:$Z' = (9589.797, 8158.457, 6605.744)$

$$x_1 + (2 + w_{32})x_2 + (3 + 2w_{33})x_3 + x_4 + 2.054\left[\begin{array}{l}\left(1 - 3w_{32} + \frac{49}{4}w_{32}^2 - \frac{23}{2}w_{32}^3 + \frac{13}{4}w_{32}^4\right)x_2^2 \\ +\left(2 - 4w_{33} + 5w_{33}^2\right)x_3^2 + \left(20 - \frac{220}{3}v_3 + \frac{2419}{6}v_3^2\right. \\ \left. - \frac{1936}{3}v_3^3 + \frac{1319}{3}v_3^4 - \frac{403}{3}v_3^5 + \frac{91}{6}v_3^6\right)\end{array}\right]^{\frac{1}{2}} \leq (700 + 5v_3 + 20v_3 - 5v_3) \tag{5.12}$$

$$0 \leq u_{11} \leq 2;\; 0 \leq u_{12} \leq 2; 0 \leq u_{13} \leq 3 \tag{5.13}$$

$$0 \leq u_{21} \leq 3;\; 0 \leq u_{22} \leq 1; 0 \leq u_{23} \leq 2 \tag{5.14}$$

$$0 \leq u_{31} \leq 2;\; 0 \leq u_{32} \leq 3 \tag{5.15}$$

$$0 \leq w_{21} \leq 1;\; 0 \leq w_{32} \leq 2; 0 \leq w_{33} \leq 1 \tag{5.16}$$

$$0 \leq v_1 \leq 2;\; 0 \leq v_2 \leq 4; 0 \leq v_3 \leq 3 \tag{5.17}$$

As the transformed model (5.7–5.20) is the equivalent deterministic model of the problem (5.1–5.6), the solution of the transformed model will be the solution of the original problem. To solve the transformed problem, we first obtain the Positive Ideal Solution (PIS) and the Negative Ideal solution (NIS) of the problem. The PIS and NIS of the problem is given by the Tables 7 and 8 respectively.

Now, we obtain the distance function as follows:

$$d_p^{\mathrm{PIS}} = \left(w_1^p\left[\frac{17088.38 - Z_1}{7498.583}\right]^p + w_2^p\left[\frac{15458.67 - Z_2}{7300.213}\right]^p + w_3^p\left[\frac{12196.9 - Z_3}{5591.156}\right]^p\right)^{\frac{1}{p}} \tag{5.18}$$

$$d_p^{\mathrm{NIS}} = \left(w_1^p\left[\frac{Z_1 - 9589.797}{7498.583}\right]^p + w_2^p\left[\frac{Z_2 - 8158.457}{7300.213}\right]^p + w_3^p\left[\frac{Z_3 - 6605.744}{5591.156}\right]^p\right)^{\frac{1}{p}} \tag{5.19}$$

Table 9 Pay-off table

	$d_2^{\text{P IS}}$	d_2^{NIS}	Z_1	Z_2	Z_3
min d_2^{PIS}	0.9032757		15935.53198	9522.1697	6605.74355
max d^{NIS}		0.4704698	11124.5503831	15454.6525871	12164.742369

$d^* = (0{:}9032757;\ 0{:}4704698)$ and $d^0 = (1{:}96759;\ 0{:}06802568)$

Table 10 Optimal solution

Method	d_2^{PIS}	d_2^{NIS}	z_1	z_2	z_3	x_1	x_2	x_3	x_4
TOPSIS	1.29066	0.32399	16168.9	13397.6	9885.8	161.213	132.36	16.67	7.68

Table 11 Optimal values of the parameters

C_1	(N (60, 9), N (42, 7), N (47, 12), N (20, 5))
C_2	(N (35, 12), N (52, 14), N (27, 9), N (55, 16))
C_3	(N (17, 7), N (48, 15), N (30, 15), N (38, 16))
A_1	(1, 1, 1, 1)
A_2	(N (3, 1), 2, 1, N (2, 4))
A_3	(1, N (2, 1), N (3, 2), 1)
B	(N (300, 10), 1085, N (760, 31))

Where the coefficient matrix is given by $A = [A_1 A_2 A_3]^{\text{T}}$

Hence, we obtain a bi-objective problem with objective of maximization of 5.22 and minimization of 5.21 over the feasible region of the problem. Solution of this problem will lead us to the solution of the original problem. In order to find the numerical solution, we consider that all the objectives of the problem are of equal importance and $p = 2$. The payoff matrix of the newly formed bi-objective problem is given by the Table 9.

Hence, by using max-min operator we obtain the final model as (with $p = 2$, $w_1 = w_2 = w_3 = 1/3$):

$$\text{Max} : \lambda$$

Subject to

$$d_p^{\text{PIS}} - 0.9032757 \geq \lambda(1.96759 - 0.9032757) \tag{5.20}$$

$$0.4704698 - d_p^{\text{NIS}} \geq \lambda(0.4704698 - 0.06802568) \tag{5.21}$$

$$\text{with constraints}\,(5.10) - (5.20) \tag{5.22}$$

Using LINGO 11.0 [30] software, we obtain the optimal solution of the problem. The optimal solution of the problem is given by Table 10. Optimal values of the multi-choice random parameters are given by Table 11.

6 Conclusions

This paper deals with the problem called multi-objective multi-choice linear programming problem. The mathematical model as well as the methodology to solve multi-objective multi-choice linear programming problem are developed in this study. Considering the advantage of the TOPSIS method to solve a MODM problem, a hybrid method of chance constraint programming and TOPSIS is developed to solve the proposed problem. The present method can be serve as a useful decision making tool for a decision maker to find optimal solution with best alternative for a multi-choice parameter. The proposed hybrid method can be use to solve a bi-level as well as multi-level multi-choice random linear programming problem with some necessary changes.

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Symmetric Duality and Complementarity in Non-Convex Programming

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1 Introduction

The analysis of convex and non-convex programming are vividly studied and presented in Bazaraa et al. [1] Rockafellar [13], Mishra [11], Mangasarian [7]. Further details of symmetric duality and complementarity are available in Ekeland [5], Zalmai [14], Gao [6], Mangasarian [8, 9], Mishra, Nanda and Acharya [10]. Nayak [12] has obtained related results with emphasis to non-convex functions. In this presentation we have considered a pair of symmetric dual non-convex programs using complementarity defined over arbitrary cones. Our results are motivated by Craven and Mond [2, 3] and Danzig and Cottle [4]. These results were obtained by and Craven and Mond [2, 3] under the stronger assumptions on the cones and the functions (convex/concave) and the non-negative orthant as the cone. Craven and Mond [2, 3] generalized this to any arbitrary cone. With an additional feasibility condition on the weaker functions (strong pseudo-convex/strong pseudo-concave) the duality theorems are presented here.

These notations and terminologies will be used throughout this paper.

Let $T \subset X = R^m$ and $U \subset Y = R^n$ be convex cones with nonempty interiors; let the function $K : X \times Y \to R$ be twice continuously differentiable on R^{n+m}. Let T^* be the polar of T, that is

$$T^* = \left\{w : (\forall v \in T) : w^t v \geq 0, \text{ where } w^t \text{ represents the transpose of } w\right\}$$

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U^* is defined similarly. $K_x(x_0, y_0)$ denotes the partial derivative with respect to x at the point (x_0, y_0). $K_y(x_0, y_0)$ is defined similarly. $K_{xx}(x_0, y_0)$ denotes the matrix (Hessian) of second partial derivative with respect to x evaluated at (x_0, y_0), $K_y(x_0, y_0)$ is defined similarly.

We say that K is strong pseudo-convex/strong pseudo-concave on $T \times U$ iff $K(x, y)$ is strong pseudo-convex with respect to a positive scalar function ψ_1 on T for each give $y \in U$ and $K(x,.)$ is strong pseudo-concave with respect to a positive scalar function ψ_2 for each given $x \in T$.

Consider the minimization problem:

$$\text{Min}\{f(x) : -g(x) \in S\}, \tag{1}$$

where f is a real differentiable function, g is a differentiable function and S is a convex cone with nonempty interior. It is shown in Craven and Mond [2] that a necessary condition for (1) to attain a local minima at $x = a$ is that there exists $r \in R_+$ and $v \in S^*$, the polar of S, such that r, v are not both zero, and

$$rf'(a) + vg'(a) = 0, \quad rf'(a) + vg'(a) = 0.$$

The problem (1.1) is regular at the point a if $r > 0$; hence without loss of generality, we assume that $r = 1$.

Let us consider the following pair of nonlinear programs:

$$(P_0) \quad \text{Min}\left\{f(x, y) = K(x, y) - K_y(x, y)y : (x, y) \in T \times U, -K_y(x, y)^t \in U^*\right\}$$
$$(D_0) \quad \text{Max}\left\{g(u, v) = K(u, v) - K_u(u, v)u : (u, v) \in T \times U, -K_u(u, v)^t \in T^*\right\}$$

We represent the following theorem of Craven and Mond [2] which will be used in the sequel:

Proposition 1.1 *(Craven and Mond [2]) Let $V \subset R^n$ be a closed convex cone (not necessarily polyhedral) whose dual cone is V^*. Let $\psi : R^n \to R^n$ be a differentiable function, with derivative $\psi'(x)$.*

Consider the minimization problem

$$\underset{x \in R^n}{Min} \left\{x'\psi(x) \in V, x \in V^*\right\} \tag{2}$$

Let us assume that V has nonempty interior. Let (1.2) attain a local minimum at $x = a$. Either

(i) let $\psi'(a)$ be positive definite,

or

(ii) let $\psi'(a)$ be positive semi definite. Let (2) be regular at a. Then $a^t \cdot \psi(a) = 0$.

2 Main Results

Theorem 2.1 *Let K be a strong pseudo-convex/strong pseudo-concave on* $T \times U$ *with respect to scalar-valued function* $\psi_1 \geq 1$ *and* $\psi_2 \geq 1$ *respectively.*

Let the program

$$\underset{x,\, y}{Min}\left\{K_x\big(x,\ y\big)x - K_y(x, y)y{:}(x, y) \in T \times U, K_u(u, v)^t \in T^*, -K_v(u, v)^t \in U^*\big)\right\} \tag{3}$$

attain a minimum at (x_0, y_0). *At this point, either (3) be regular and both* $K_{xx}, -K_{yy}$ *be positive semi definite or both* $K_{xx}, -K_{yy}$ *be positive definite, then* (P_0) *and* (D_0) *are dual to each other.*

Let (x, y) and (u, v) be feasible for (P_0) *and* (D_0) *respectively.*

Since K is strong pseudo-convex/strong pseudo-concave with respect to ψ_1 *and* ψ_2 *respectively, the following inequalities hold:*

$$\psi_1(x, y)(K(x, v) - K(u, v)) \geq (x - u)^t K_u(u, v)$$

or

$$K(x, v) - K(u, v) \geq \frac{(x - u)^t K_u(u, v)}{\psi_1(u, x)} \tag{4}$$

and

$$\psi(y, v)(K(x, v) - K(x, y)) \leq (v - y)^t K_y(x, y)$$

or

$$K(x, v) - K(x, y) \leq \frac{(v - y)^t K_y(x, y)}{\psi_2(y, v)} \tag{5}$$

By multiplying -1 in (5) and adding it to (4) we get

$$\begin{aligned} K(u, v) + \frac{x^t K_u(u, x)}{\psi_1(u, v)} - K(x, y) - \frac{(v - y)^t K_y(x, y)}{\psi_2(y, v)} \\ = K(u, v) + \frac{x^t K_u(u, v)}{\psi_1(u, v)} - \frac{u^t K_u(u, v)}{\psi_1(u, v)} \\ - K(x, y) - \frac{v^t K(x, y)}{\psi_2(y, v)} + \frac{y^t K_y(x, y)}{\psi_2(y, v)} \leq 0 \end{aligned} \tag{6}$$

Since $u \in T$ and $K_u(u, v)^t \in T^*$, we have $K_u(u, v)u \geq 0$ and

$$\frac{-u^t K_u(u,v)}{\psi_1(u,x)} \geq -u^t K_u(u,v) \text{ as } \psi_1(u,x) \geq 1 \tag{7}$$

Similarly $y \in U$ and $-K(x,y)^t \in U^* \Rightarrow K_y(x,y)y \geq 0$, so we have

$$\frac{y^t K_y(x,y)}{\psi_2(y,v)} \geq y^t K_y(x,y) \text{ as } \psi_2(y,v) \geq 1 \tag{8}$$

$$\frac{x^t K_u(u,v)}{\psi_1(u,v)} \geq 0 \text{ as } + x^t K_u(u,v) \geq 0 \text{ and } \psi_1(u,v) \geq 1 \tag{9}$$

$$\frac{-v^t K_y(x,y)}{\psi_2(y,v)} \geq 0 \text{ as } -v^t K_y(x,y) \geq 0 \text{ and } \psi_2(y,v) \geq 1 \tag{10}$$

Using (7–10) in (6) we get

$$K(u,v) - u^t K_u(u,v) - K(x,y) + y^t K_y(x,y) = K(u,v) - u^t K_u(u,v)$$
$$- \left(K(x,y) - y^t K_y(x,y)\right) = g(u,v) - f(x,y) \leq 0$$

which implies that $f(x,y) \geq g(u,v)$, which proves weak duality theorem.

It remains to prove that

$$K(x_0,y_0) - K_y(x_0 y_0)y_0 = K(x_0,y_0) - K_x(x_0,y_0)x_0 \tag{11}$$

This is equivalent to requiring that $w_0^t \phi(w_0) = 0$, where

$$\phi(w_0) = \begin{bmatrix} K_x^t(w_0) \\ -K_y^t(w_0) \end{bmatrix} \text{ and } w_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

Now $w_0^t \psi'(w_0) w_0 = x_0^t K_{xx} x_0 - y_0^t K_{yy} y_0$. Since it is assumed that K_{xx} and $-K_{yy}$ are both positive semi definite, so $w_0^t \psi'(w_0) w_0 \geq 0$, which implies that ψ' is positive semi definite.

In case of both K_{xx} and $-K_{yy}$ are positive definite, then ψ' is positive definite. Since the system (3) is regular in both the cases therefore, $w_0^t \psi'(w_0) w_0 = 0$, by the Proposition 1.1.

Theorem 2.2 *Let K be strong pseudo-convex/strong pseudo-concave on $T \times U$. Let (x, y) and (u, v) be feasible for (P_0) and (D_0) respectively with $x - u \in T$ and $v - y \in U$. Let the program (3) be regular and both K_{xx} and $-K_{yy}$ be positive semi definite or both K_{xx} and $-K_{yy}$ be positive, then $(P_0)$and (D_0) are each dual to the other.*

Proof According to hypothesis $x - u \in T$. So $K_u(u,v)(x-u) \geq 0$ by the definition of polar; i.e., $(x-u)^t K_u(u,v) \geq 0$. As $K(.,y)$ is pseudo-convex on T for each given $y \in U$ and $(x-u)^t K_u(u,v) \geq 0$ so we have

$$K(x,v) \geq K(u,v) \tag{12}$$

Since $v - y \in U$ so $(v - y)^t K_y(x, y) \leq 0$. As $K(x,.)$ is pseudo—concave on U for each given $x \in T$ and $(v - y)^t K_y(x, y) \leq 0$, so we have

$$K(x, v) \leq K(x, y) \tag{13}$$

From (12) and (13) we get

$$K(x, y) \geq K(x, v) \geq K(u, v)$$

i.e.,

$$K(x, y) \geq K(u, v)$$

or

$$K(x, y) - y^t K_y(x, y) \geq K(u, v) - u^t K_u(u, v)$$

$$\left[\text{As} - y^t K_y(x, y) \geq 0 \text{ and } u^t K_u(u, v) \geq 0\right]$$

i.e.,

$$f(x, y) \geq g(u, v)$$

which proves the weak-duality. We next prove that

$$K(x_0, y_0) - y_0^t K_y(x_0 y_0) = K(x_0, y_0) - x_0^t K_x(x_0, y_0) \tag{14}$$

This is equivalent to show that $w_0^t \phi(w_0) = 0$

Now $w_0^t \psi'(w_0) w_0 = x_0^t K_{xx} x_0 - y_0^t K_{yy} y_0$. Since it is assumed that K_{xx} and $-K_{yy}$ are both positive semi definite, so $w_0^t \psi^{|}(w_0) w_0 \geq 0$, which implies that ψ' is positive semi-definite.

In case of both K_{xx} and $-K_{yy}$ are positive definite, so ψ' is positive definite. Since the system (3) is regular in both the cases hence $w_0^t \psi'(w_0) w_0 = 0$, by the Proposition 1.1.

3 Special Cases

Let us consider the following pair of nonlinear programming problems of Craven and Mond [2]

$$(P_1): \quad \underset{x}{\text{Min}} \left\{ f(x); g(x) \in S, x \in T^* \right\}$$

$$(D_1): \quad \underset{x,u}{\text{Max}}\,(\psi(x,u)) = f(x) - u'g(x) - \phi(u,x)x : u \in S^*, \phi(u,x)^t \in T$$

where $\phi(u,x) = f'(x) - u'g'(x)$. Here f and g are differentiable functions, S and T are closed convex cones (not necessarily polyhedral), with S having nonempty interior.

By combining (P_1) and (D_1), we get a single minimization problem of the complementarity type of the form (2) namely:

$$\underset{u,x}{\text{Min}}\left\{u^t g(x) + \phi(u,x)x : \phi(u,x)^t \in T, g(x) \in S, (u,x) \in S^* \times T^*\right\} \tag{15}$$

which is equivalent to:

$$\underset{w}{\text{Min}}\left\{w^t \psi(w) : w \in V^*, \psi(w) \in V\right\}$$

where $w = \begin{bmatrix} u \\ x \end{bmatrix}$, $\psi(w) = \begin{bmatrix} g(x) \\ \phi(u,x)^t \end{bmatrix}$, $V = \begin{bmatrix} S \\ T \end{bmatrix}$

Definition 3.1 f is S—strong pseudo-convex on U if

$$\psi_1(x,y)(f(y) - f(x)) - (y-x)^t \nabla f(x) \in S$$

for any arbitrary positive scalar function ψ_1, for each $x, y \in U$.

Lemma 3.1 *If f is strong pseudo-convex and $-g$ is S—strong pseudo-convex with respect to the same scalar function ψ_1; then $f - u^t g$ is strong pseudo-convex with respect to the same scalar function ψ_1.*

Proof Since f is strong pseudo-convex with respect to a positive scalar function ψ_1, so we have

$$\psi_1(x,y)(f(y) - f(x)) \geq (y-x)^t \nabla f(x). \tag{16}$$

Since $-g$ is S—strong pseudo-convex with respect to ψ_1 then we have

$$\psi_1(x,y)(-g(y) + g(x)) + (y-x)^t \nabla f(x) \in S$$

So

$$u^t\left[\psi_1(x,y)(-g(y) + g(x)) + (y-x)^t \nabla f(x)\right] \geq 0 \text{ as } u \in S^* \tag{17}$$

By adding (16) to (17) we get

$$\psi_1'(x, y)\big[\big(f - u^t g\big)(y) - \big(f - u^t g\big)(x)\big] \geq (y - x)^t \nabla\big(f - u^t g\big)(x)$$

which implies that $(f - u^t g)$ is strong pseudo-convex with respect to $\psi_1(x, y)$.

Theorem 3.1 *Let f and g be twice differentiable, let f be strong pseudo-convex with respect to a positive scalar function $\psi_1 \geq 1$ and $-g$ is S—strong pseudo-convex with respect to the same scalar function ψ_1. Let (15) attain a minimum at (u_0, x_0) and at this point assume either the system (15) is regular and $Q = \big(f - u^t g\big)_{xx}$ is positive semi definite or Q is positive definite then (P_1) and (D_1) are dual to each other.*

Proof Since f is strong pseudo-convex and $-g$ is S—strong pseudo-convex with respect to the same scalar function ψ_1, so the Lemma 3.1, it follows that $f - u^t g$ is strong pseudo-convex with respect to the same scalar function ψ_1. Hence

$$\psi_1(x, y)\big[\big(f - u^t g\big)(y) - \big(f - u^t g\big)(x)\big] \geq (y - x)^t \phi(u, x) \tag{19}$$

Let y_0 be feasible for (P_1) and (u_0, x_0) be feasible for (D_1), so from (18)
We have

$$\psi_1(x_0, y_0)\big[\big(f - u_0^t g\big)(y_0) - \big(f - u_0^t g_0\big)(x_0)\big] \geq (y_0 - x_0)^t \phi(u_0, x_0)$$

or

$$\begin{aligned}
&\big[\big(f - u_0^t g\big)(y_0) - \big(f - u_0^t g_0\big)(x_0)\big] \geq \frac{(y_0 - x_0)^t \phi(u_0, x_0)}{\psi_1(x_0, y_0)} \\
&= \frac{y_0 \phi(x, x_0)}{\psi_1(x_0, y_0)} - \frac{x_0^t \phi(u_0, x_0)}{\psi_1(x_0, y_0)} \geq \frac{-x_0^t \phi(u_0, x_0)}{\psi_1(x_0, y_0)} \\
&\text{as } y_0^t \phi(u_0, x_0) \geq 0 \text{ and } \geq -x_0^t \phi(u_0, x_0) \\
&\text{as } -x_0^t \phi(u_0, x_0) \geq 0 \text{ and } \psi_1(x_0, y_0) \geq 1
\end{aligned}$$

or

$$f(y_0) \geq u_0^t g(y_0) + f(x_0) - u_0^t g(x) - x_0^t \phi(u_0, x_0) \geq 0$$
$$f(y_0) \geq u_0^t g(y_0) + f(x_0) - u_0^t g(x) - x_0^t \phi(u_0, x_0) \geq 0$$

So the weak duality follows from above. It remains to prove that

$$f(x_0) = f(x_0) - u_0^t g(x) - x_0^t \phi(u_0, x_0),$$

where x_0 is feasible for (P_1) and (x_0, u_0) is feasible for (D_1). This is equivalent to show that

$$w_0^t \psi'(w_0) = 0,$$

where $w_0 = \begin{bmatrix} u_0 \\ x_0 \end{bmatrix}$ $\psi(w_0) = \left(\begin{bmatrix} g(x_0) \\ \phi(u_0, x_0)^t \end{bmatrix} \right)$

It is assumed that Q is positive semi definite if $w_t^1 = \left(u_1^t, x_1^t\right)$, then $w_t^1 \psi'(w_0) w_1 = x_1^t Q x_1$ which implies that $\psi'(w)$ is positive semi-definite. If Q is positive definite then also $\psi'(w_0)$. By hypothesis the system (15) is regular, in both the cases $w_0^t \psi'(w_0) = 0$ by the Proposition 1.1.

Theorem 3.2 *Let f and g be twice differentiable, let $f - u^t g$ be pseudo-convex on T^*. If $x_2$is feasible for (P_1) and (x_1, u_1) is feasible for (D_1) with $x_2 - x_1 \in T^*$ and $u_1 \in S^*$. Let (15) is regular and $Q = \left(f - u^t g\right)_{xx}$ is positive semi definite or Q is positive definite then (P_1) and (D_1) are dual to each other.*

Proof Since (x_1, u_1) is feasible for (D_1), we have $x_2 - x_1 \in T^*$. By hypothesis $x_2 - x_1 \in T^*$ therefore, $(x_2 - x_1)^t \phi(u_1, x_1) \geq 0$

Since $\left(f - u^t g\right)$ is pseudo-convex on T^* and $(x_2 - x_1)^t \left(f' - u_1^t g'\right)(x_1) \geq 0$, then

$$\left(f - u^t g\right)(x_2) \geq \left(f - u_1^t g\right)(x_1)$$

or

$$f(x_2) \geq f(x_1) - u_1^t g(x_1) + u_1^t g(x_2) \tag{18}$$

Since $u_1^t g(x_2) \geq 0$ and $x_1^t \left(f'(x_1) u_1^t g(x_1)\right) \geq 0$
i.e., $-x_1^t \left(f'(x_1) u_1^t g(x_1)\right) \geq 0$
so from (19)

$$f(x_2) \geq f(x_1) - u_1^t g(x_1) - x_1^t \left(f'(x_1) u_1' g'(x_1)\right)$$

i.e.,

$$f(x_2) \geq \psi(x_1, u_1)$$

which proves the weak duality. Hence if x_0 is feasible for (D_1) then $f(x_0) \geq \psi(x_0, u_0)$.

It remains to prove that

$$f(x_0) = f(x_0) - u_0^t g(x) - x_0^t \phi(u_0, x_0).$$

This is equivalent to show that $w_0^t \psi(w_0) = 0$

It is assumed that Q is positive semi-definite.

If $w_1^t = \left(u_1^t, x_1^t\right)$, then $w_t^1 \psi'(w_0) w_1 = x_1^t Q x_1$ which implies that $\psi'(w_0)$ is positive semi definite. If Q is positive definite then also $\psi'(w_0)$. Since by hypothesis the system (15) is regular, in both the cases $w_0^t \psi(w_0) = 0$.

4 Conclusion

In this paper, we have presented symmetric duality theorems in nonlinear programming problems using complementarily for weaker strong pseudo-convex/strong pseudo-concave functions. For pseudo-convex/pseudo-concave functions we have an additional feasibility condition. A special case of the general symmetric dual programs is discussed under the same weaker assumptions on the functions.

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An EOQ Inventory Model with Remanufacturing and Dismantling for Parts

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1 Introduction

In recent years reverse logistics is drawing increasing attention from both science community and manufacturing industries. There is increasing understanding that attentive management can bring both environmental protection and lower costs: environmental and economic examinations have led to manufacturers taking cores back at the end of their life circle. As a result reverse logistics is now considered as a basis for generating real economic value, as well as support of environmental concerns.

Rogers and Tibben-Lembke [27] determined Reverse logistics as the process of planning, implementing, and controlling the efficient, cost effective flow of raw materials, in-process inventory, finished goods and related information from the point of consumption to the point of origin for the purpose of recapturing value or proper disposal. Combination of forward and reverse supply chain contributed to emergence of the idea of a closed-loop supply chain. The whole chain can be developed in such a manner to service both forward and reverse channels.

One of the last most comprehensive studies, which is devoted to quantitative models for inventory and production planning in closed-loop supply chain was published by Akcaly and Cetinkaya [1].

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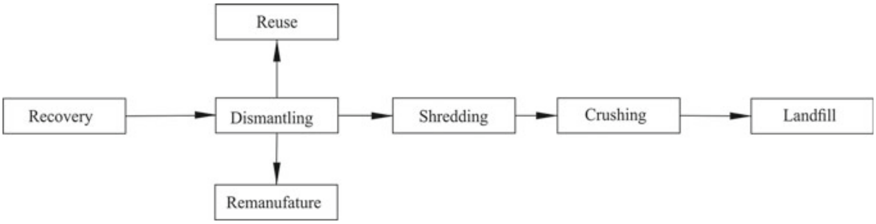

Fig. 1 End-of-life vehicles processing [10]

As an illustration of closed-loop supply chain Souza [32] examines Cummins, the producer of diesel engines located in Indiana. Forward channels contain new engines and engine parts, and reverse channels contain used products, and remanufactured products. For a diesel engine or part, remanufacturing includes six stages: full disassembly, cleaning of each part, taking a decision for each part: remanufacture it or recycle it for materials recovery, refurbishing parts to restore them to the new state, reassembly, and testing. Remanufactured products and parts are sold at a 35% discount comparing to new products. When one purchases a Cummins product, he gets a discount if he returns their old product. Cores are delivered from dealers to Cummins' warehouse for used products. At the warehouse, customers receive credit for returning the core, and products are then delivered to one of two plants: engine remanufacturing (first plant), or module remanufacturing (second plant). Remanufactured engines are delivered from first plant to the main distribution center, for distribution to the dealers. Remanufactured parts are delivered from second plant to another distribution center, or to the first plant, depending on forecasts and existing demand. Used parts not suited for remanufacturing are sold to recyclers.

The following example addresses the recovery activity of IBM. IBM organized remanufacturing facility, where returned servers and storage systems are remanufactured or dismantled. Remanufactured products are sold at a reduced price, dismantled parts are used to satisfy internal or external demand for spare parts [9, 12].

The growth in the automotive industry causes an annual increase of the number of end-of-life (ELV) vehicles. ELV processing is the same all over the world (Ferrão and Amaral [10]) (Fig. 1). For example, there are many companies that are engaged in the dismantling of cars, bicycles and other vehicles for spare parts. The damaged car, which is bought at the auction, is then delivered to the warehouse of car parts. Here the damaged vehicle is disassembled into parts, then the vehicle parts are packed for shipment and then shipped. Priority activity of these companies is purchase of the damaged cars and their cutting, dismantle of details for cars. Usually the prices of damaged cars offered for sale at insurance auctions are much lower than the market prices. Note that many parts in damaged cars remain in good condition. Buying such a vehicle, the buyer receives high-quality spare parts at a low price.

Among all countries, the dismantling activity in the United States is the most mature. The US has been studying and developing recovery approached for ELV for more than 30 years. The US has formed the world's largest system of decommissioning and restoration of ELV parts, which is becoming the main source of profit in the ELV recycling industry. Automotive parts recovery has become the largest industry recovery in the US [34].

Thus, in developed countries these products are prepared with remanufacturing policies and strategies in place. However, in developing countries, remanufacturing is still in the initial stages and is not a virtual application. Amelia et al. [2] cited the significant difference between EOL vehicle management strategies in developing and developed countries, noting that China, India, and Brazil are struggling with remanufacturing implementation. The paper [35] analyzes internal barriers met by automotive parts remanufacturers if China and evaluates causal barriers by a proposed model framework [35]. The paper [11] provides a brief road map and insights into future research for remanufacturing specifically in an Indian context.

Thus policies and strategies for recovery EOL vehicles exist in developed countries, but in developing countries the recovery process is still at an early stage. Amelie et al. [2] mentioned a significant difference between EOL vehicle management strategies in developing and developed countries. The paper [35] explores the internal barriers faced in the repair industry of automotive parts in China, using a special model. The paper [11] provides an analysis of future research for recovery, especially in the Indian context.

In this paper we consider the EOQ model with the recovery and dismantling of parts. According to [1] inventory models are divided into two main classes: deterministic and stochastic by modeling the demand and return. The subject of this work is a deterministic inventory model based on EOQ.

The economic order quantity model (EOQ model), which was formulated by Ford W. Harris in 1913, began the start for many models of reverse logistics because of its simplicity. The study [3] is the most detailed review of the EOQ problems.

Shady was the first to apply the EOQ model to reverse logistics, he formulated a model where production and remanufacturing processes occur instantly, and found a solution explicitly [31]. In this work the optimal policy was found, which is that within each cycle several batches of the same size are recovered and one new batch is produced.

The work [31] was expanded by Nahmias and Rivera [22], and further by Mabini et al. [21] to the multi-product case. Koh et al. [15] considered a model similar to Shady [31] with some differences.

Teunter [33] summarized the results of Schrady. He explored the various structures of the recovery cycle. He considered different types of policies, varying the number of batches of production and recovery, and concluded that the policy would never be optimal if both numbers of batches simultaneously more than one.

Choi et al. [4] summarized the Teunter policy by taking an ordered sequence of production and recovery batches as a variable. With sensitivity analysis, they found that only 0.2% of the tested problems have an optimal solution in which both number of batches are more than one. Liu et al. [20] generated and solved 60,000 problems and found that only 0.19% of them have an optimal solution in which both number of batches are more than one. Konstantaras and Papachristos [17] also received the solution in explicit form.

Richter is the author of some works, where he considered the EOQ model in terms of waste management. Richter [23] proposed an EOQ model that is different from Shrady. Richter [23] assumed a system of two workshops: the first workshop supplies products to the second workshop, the first workshop can both produce new products and recover the old ones, which are supplied from the second workshop in accordance with some disposal rate. Richter [25] in his work found the optimal policy of inventory management, taking the disposal rate as a variable. Result of this work is that the optimal policy has an extreme property: either to reuse all the elements without recycling, or to dispose of all the elements and produce new products. The research of this model was continued in the works [5, 24–26].

In papers [6, 7] Dobos and Richter investigated a production/recycling system with constant demand that is satisfied by non-instantaneous production and recycling. The result of this paper is that it is optimal either to produce or to recycle all bought back items. In paper [8] Dobos and Richter extended their previous work by considering the quality of returned items.

Al-Saadani and Jaber [29] felt that such a clean waste disposal policy was not technologically possible and suggested that a demand function should depend on purchase price and the level of quality.

Ahmed M. A. El-Saadani, Mohammed th. Jabber, Maurice bonnet [30] suggested that it is impossible to restore the goods an infinite number of times, because the material loses part of its mass each time. In [30], a model is proposed in which the product is restored a finite number of times.

Some authors have investigated the above models by canceling other initial assumptions, one option is to allow back orders when some customers receive compensation for waiting orders, or receive a discount, resulting in additional costs of the supplier. Constantaras and Scouri [18] consider two models investigating shortage case. Both models use the quantity of production and recovery batches as variables. For both models, sufficient optimal policies are found depending on the initial parameters.

Saadany and Jabber in their work [14] expanded Richter's work [23], suggesting that the demand for the new products is different from the demand for the restored products. The model takes into account the costs associated with lost sales when demand for new products is lost during recovery cycles and vice versa. Konstantaras and Papachristos [16] extended the work of Richter [23], allowing to schedule backorder. The work [13] expanded the work of Jabber and El Saadani [14] for cases of full and partial re-ordering, when buyers perceive the restored goods as lower quality goods than new ones.

The work of Konstantaras, Score and Jabber [19], which is an extension of the work [15], also considers the process of inspection and sorting. The study assumes that the recovered and new products are sold in different markets, primary and secondary.

In the work of Saadany and Jabber [28], it was proved that ignoring the first time interval in the Richter's model leads to excessive evaluation of holding costs. Also, the work [28] takes into account the costs of switching (for example, production losses, deterioration, additional labor). In process of the transition from production of one product to another on the same facility, the producer can incur additional costs, which are considered as switching costs.

The main feature of the work [28] is the accounting of switching costs. Our paper generalizes the approach of Saadany and Jabber [28] for a class of models and illustrates this approach by example. The model considers three types of costs. First, the EOQ not related cost, which is independent on the number of batches and the batch size (for, example production cost), the EOQ related cost, which depends on the dynamics of stocks, the size and number of batches (holding costs), and the EOQ related to the cost, which depends on the number of batches and production planning (the switching cost).

We study an EOQ inventory model with remanufacturing and dismantling for parts. We consider a firm that can manufacture new products and recover the value of a used product through remanufacturing with dismantling for parts. The firm provides product, which consists of two parts. Each part is manufactured or remanufactured separately and placed in inventory, then two parts are assembled. Also each part can be directly reused, if it is in good technical condition.

To determine the optimal production and remanufacturing policy means to find the optimal numbers of production and remanufacturing lots of each part for the minimization of the total cost. The optimal policy is found depending on the parameters of the model. For the solution of the problem a theorem was proved that provides solution to the certain class of deterministic inventory models with constant demand and return. The theorem can be used for the complete solution of some of above mentioned models.

This paper is organized as follows: the first section contains introduction, in the second section the formulation and analysis of an EOQ inventory model with remanufacturing and dismantling for parts is considered, in the third section the theorem is derived which gives closed-form expressions for the optimal policy parameters and can be used for other various remanufacturing problems, the fourth section contains the numerical example, the last section of the paper is conclusion.

2 Model Formulation

We consider a firm that receives recoverable product from the market. The firm can manufacture new products and recover the value of a used product or return through remanufacturing with dismantling for parts. The firm provides product at a constant demand rate of d items per time unit. Product consists of two parts, denoted as part 1 and part 2. Each part is manufactured separately and placed in inventory (SS1—serviceable stock inventory for part 1, SS2—serviceable stock inventory for part 2), then two parts are assembled with the cost c_A and are sold in a market. Products are returned to the firm according the rate β, other products are immediately disposed of at the rate $\alpha = 1 - \beta$. The dismantling operation costs c_D. Returned product is dismantled for parts, any part is inspected whether it is usable or not, and then is placed in inventory (RS1—inventory for returned stock of part 1,

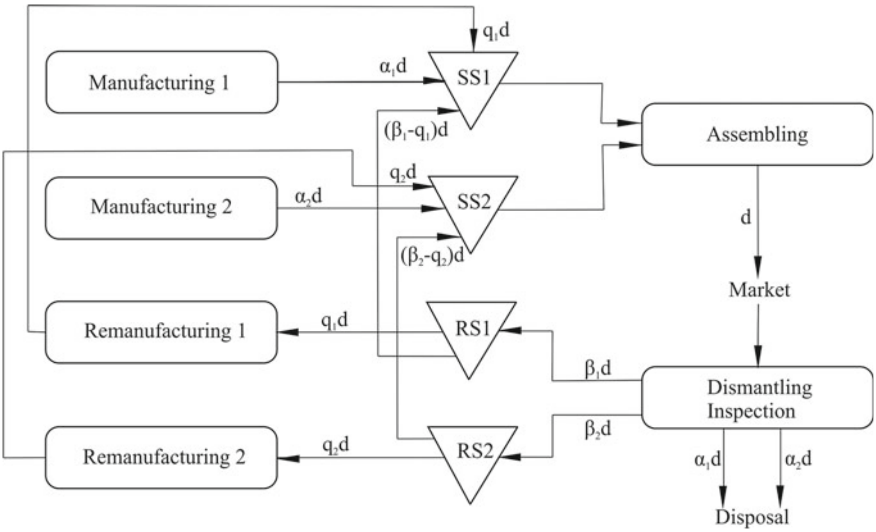

Fig. 2 The integrated closed-loop supply chain inventory system

RS2—inventory for returned stock of part 2). Part 1 is not usable at the rate q_1 and should be remanufactured, the rest $\beta_1 - q_1$ are as good as new and directly reused, part 2 isn't usable at the rate q_2. Figure 1 represents the integrated closed-loop supply chain inventory system. The sequence of production activities is the following: in any time cycle $[0, T]$ demand for part 1 and part 2 is satisfied firstly through usable parts, then through remanufacturing of used parts and at last manufacturing of new parts. All activities in the model are supposed to be instantaneous and lot-for-lot. The production activities of each part are evaluated on separate production lines (Fig. 2).

Assumptions

This paper assumes:

(1) production and recovery are instantaneous,
(2) remanufactured items are as good as new,
(3) demand is known, constant and independent,
(4) lead time is zero,
(5) the product consists of two parts
(6) no shortages are allowed,
(7) unlimited storage, and
(8) infinite planning horizon.

Notations:

c_M^1	part 1 unit manufacturing cost
c_M^2	part 2 unit manufacturing cost
c_R^1	part 1 unit remanufacturing cost

c_R^2	part 2 unit remanufacturing cost
c_D	dismantling operation cost
c_A	assembling operation cost
d	constant demand rate
$[0, T]$	time cycle interval
β	percentage of returned items
$\alpha = 1 - \beta$	disposal rate
q_1	percentage of not usable returned parts of type 1
q_2	percentage of not usable returned parts of type 2
Q_1^n	manufacturing lot size for part 1
Q_2^n	manufacturing lot size for part 2
Q_1^m	lot size for directly reused part 1
Q_2^m	lot size for directly reused part 2
Q_3^m	remanufacturing lot size for part 1
Q_4^m	remanufacturing lot size for part 2
H_1	holding cost for SS1 per item per time unit
H_2	holding cost for SS2 per item per time unit
h_1	holding cost for RS1 per item per time unit
h_2	holding cost for RS2 per item per time unit
R_1	fixed inspection cost for lot of usable part 1
R_2	fixed inspection cost for lot of usable part 2
R_3	fixed inspection cost and remanufacturing setup cost for part 1
R_4	fixed inspection cost and remanufacturing setup cost for part 2
S_1	manufacturing setup cost for part 1
S_2	manufacturing setup cost for part 2
P_1	the total switching costs for part 1, which include machine startup, when remanufacturing is started and machine adjustment, when remanufacturing is switched to manufacturing
P_2	the total switching costs for part 2, which include machine startup, when remanufacturing is started and machine adjustment, when remanufacturing is switched to manufacturing

In this model setup and switching cost are differentiated. The setup cost incurred every time a manufacturing or remanufacturing of next lot is started, and the switching costs are incurred when the activity is changed, for example, remanufacturing is changed to remanufacturing and on the contrary. If a machine is shut down in the middle of a production run of one item, a setup cost is incurred when production is resumed, but no switching costs are incurred. Switching costs are defined as the costs incurred whenever two consecutive jobs do not share the same features. Switching costs may include cleaning cost, machine adjustment/fine tuning cost, changing tools, changing product family, changing production supplies, equipment startup/shutdown.

This paper assumes demand is supplied by $\mathrm{d}T$ of part 1 and $\mathrm{d}T$ of part 2 per time interval $[0, T]$, which are assembled together. The quantity of $\mathrm{d}T$ of first parts are accomplished through $\alpha_1 \mathrm{d}T$ of newly manufactured items in n_1 lots of size Q_1^n,

$(\beta_1 - q_1)\mathrm{d}T$ of directly reused items in m_1 lots of size Q_1^m and $q_1\mathrm{d}T$ of remanufactured items in m_3 lots of size Q_3^m. Similarly $\mathrm{d}T$ of second parts are accomplished through $\alpha_2\mathrm{d}T$ of newly manufactured items in n_2 lots of size Q_2^n, $(\beta_2 - q_2)\mathrm{d}T$ of directly reused items in m_2 lots of size Q_2^m and $q_2\mathrm{d}T$ of remanufactured items in m_4 lots of size Q_4^m. The following system of equations is fulfilled:

$$\begin{aligned} n_1 Q_1^n &= \alpha_1 \mathrm{d}T \\ n_2 Q_2^n &= \alpha_2 \mathrm{d}T \\ m_1 Q_1^m &= (\beta_1 - q_1)\mathrm{d}T \\ m_2 Q_2^m &= (\beta_2 - q_2)\mathrm{d}T \\ m_3 Q_3^m &= q_1 \mathrm{d}T \\ m_4 Q_4^m &= q_2 \mathrm{d}T \\ \alpha_1 &+ q_1 + (\beta_1 - q_1) = 1 \\ \alpha_1 &+ q_2 + (\beta_1 - q_2) = 1 \end{aligned} \tag{1}$$

We divide all costs of the firm over $[0, T]$ into three groups:

(1) EOQ non related cost, which does not depend on the numbers of lots and lot sizes at all, i.e., manufacturing cost, remanufacturing cost, assemble and dismantle cost. It is assumed that EOQ non related costs in the model are proportional to the quantity or product $\mathrm{d}T$, i.e. T.
(2) EOQ related cost, which depends on dynamics of the inventories, lot sizes and numbers of lots, i.e., holding cost.
(3) EOQ related cost, which depends on the numbers of lots and production scheduling, i.e., setup cost, switching cost [28].

Let us denote the total cost of the firm over $[0, T]$ by

$$TC = TC(T, m_1, m_2, m_3, m_4, n_1, n_2, Q_1^m, Q_2^m, Q_3^m, Q_4^m, Q_1^n, Q_2^n), \tag{2}$$

Using (1) the lot sizes can be defined by the following formulas:

$$\begin{aligned} Q_1^n &= \frac{\alpha_1 \mathrm{d}T}{n_1} \\ Q_2^n &= \frac{\alpha_2 \mathrm{d}T}{n_2} \\ Q_1^m &= \frac{(\beta_1 - q_1)\mathrm{d}T}{m_1} \\ Q_2^m &= \frac{(\beta_2 - q_2)\mathrm{d}T}{m_2} \\ Q_3^m &= \frac{q_1 \mathrm{d}T}{m_3} \\ Q_4^m &= \frac{q_2 \mathrm{d}T}{m_4} \end{aligned} \tag{3}$$

Taking into account (3), the variables Q_i^m, Q_j^n, $i = 1\ldots4$, $j = 1,2$ can be excluded from (2):

$$TC = TC(T, m_1, m_2, m_3, m_4, n_1, n_2), \tag{4}$$

The total cost per $[0, T]$ is the sum of manufacturing cost, $\alpha_1 c_M^1 \mathrm{d}T$ of part 1, $\alpha_2 c_M^2 \mathrm{d}T$ of part 2, remanufacturing cost, $q_1 C_R^1 \mathrm{d}T$ of part 1, $q_2 C_R^2 \mathrm{d}T$ of part 2, dismantle and assemble cost, $(c_D + c_A)\mathrm{d}T$, fixed setup, switching, inspection cost of part 1, $G_1(m_1, m_3, n_1)$, and part 2 $G_2(m_2, m_4, n_2)$, holding cost of part 1, $H_1(m_1, m_3, n_1)$, and part 2 $H_2(m_2, m_4, n_2)$, and is given as

$$TC(T, m_1, m_2, m_3, m_4, n_1, n_2) = ((\alpha_1 c_M^1 + \alpha_2 c_M^2) + q_1 C_R^1 + q_2 C_R^2 + (c_D + c_A))\mathrm{d}T$$
$$+ G_1(m_1, m_3, n_1) + G_2(m_2, m_4, n_2) + H_1(T, m_1, m_3, n_1) + H_2(T, m_2, m_4, n_2)$$

Eoq related costs of type (3), i.e., setup costs and switching costs equals the sum of the total switching cost P_1, which include machine startup, when remanufacturing is started, and machine adjustment, when remanufacturing is switched to manufacturing, fixed inspection cost for m_1 lots of usable part 1, $R_1 m_1$, fixed inspection cost and remanufacturing setup cost for m_3 lots of part 1, $R_3 m_3$, and manufacturing setup cost for n_1 lots of part 1, $S_1 n_1$ (the setup and switching cost expression for the second part is obtained by the similar way):

$$G_1(m_1, m_3, n_1) = P_1 + R_1 m_1 + R_3 m_3 + S_1 n_1$$
$$G_2(m_2, m_4, n_2) = P_2 + R_2 m_2 + R_4 m_4 + S_2 n_2.$$

The behavior of RS and SS inventories for part 1 and part 2 is also similar. The behavior of RS1 and SS1 inventories is represented on Fig. 3.

The holding costs function for first part inventories RS1, SS1 are given by $H_1(T, m_1, m_3, n_1)$, for the second part inventories RS2, SS2 by $H_2(T, m_2, m_4, n_2)$:

$$H_1(T, m_1, m_3, n_1) = T^2 H_1(m_1, m_3, n_1)$$
$$H_2(T, m_2, m_4, n_2) = T^2 H_2(m_2, m_4, n_2),$$

$$H_1(m_1, m_3, n_1) = \frac{d}{2}\Bigg[h_1\alpha\beta + \frac{1}{n_1}H_1\alpha^2 + \frac{1}{m_1}(H_1 - h_1)(\beta - q_1)^2$$
$$+ \frac{1}{m_3}(H_1 q_1^2 + h_1\beta q_1 + h_1 q_1(\beta - q_1))\Bigg]$$
$$H_2(m_2, m_4, n_2) = \frac{d}{2}\Bigg[h_2\alpha\beta + \frac{1}{n_2}H_2\alpha^2 + \frac{1}{m_2}(H_2 - h_2)(\beta - q_2)^2$$
$$+ \frac{1}{m_4}(H_2 q_2^2 + h_2\beta q_2 + h_2 q_2(\beta - q_2))\Bigg].$$

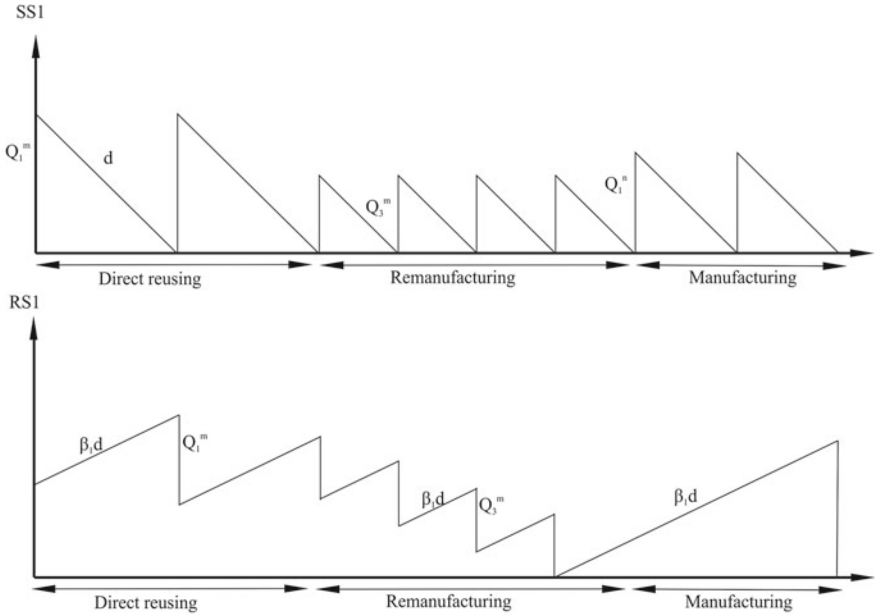

Fig. 3 The behavior of serviceable stock and recoverable stock inventories for part 1

Denote the total setup cost and total holding cost by

$$G(m_1, m_2, m_3, m_4, n_1, n_2) = G_1(m_1, m_3, n_1) + G_2(m_2, m_4, n_2)$$
$$H(m_1, m_2, m_3, m_4, n_1, n_2) = H_1(m_1, m_3, n_1) + H_2(m_2, m_4, n_2)$$

The unit time cost function is obtained by dividing by T the total cost function

$$ATC(T, m_1, m_2, m_3, m_4, n_1, n_2) = ((\alpha_1 c_M^1 + \alpha_2 c_M^2) + q_1 C_R^1 + q_2 C_R^2 + (c_D + c_A))d + \frac{G(m_1, m_2, m_3, m_4, n_1, n_2)}{T} + T(H(m_1, m_2, m_3, m_4, n_1, n_2)). \quad (5)$$

It can be obtained from (5) by differentiating that the length of the optimal time cycle is equal to

$$T = \sqrt{\frac{G(m_1, m_2, m_3, m_4, n_1, n_2)}{H(m_1, m_2, m_3, m_4, n_1, n_2)}}$$

and the corresponding cost equals

$$ATC(T, m_1, m_2, m_3, m_4, n_1, n_2) = ((\alpha_1 c_M^1 + \alpha_2 c_M^2) + q_1 C_R^1 + q_2 C_R^2 + (c_D + c_A))d + 2\sqrt{G(m_1, m_2, m_3, m_4, n_1, n_2) \cdot H(m_1, m_2, m_3, m_4, n_1, n_2)}. \quad (6)$$

Let the radicand of the root (6) be denoted by

$$L(m_1, m_2, m_3, m_4, n_1, n_2) = G(m_1, m_2, m_3, m_4, n_1, n_2) \cdot H(m_1, m_2, m_3, m_4, n_1, n_2), \tag{7}$$

where

$$G(m_1, m_2, m_3, m_4, n_1, n_2) = P_1 + P_2 + \sum_{j=1}^{4} R_j m_j + \sum_{i=1}^{2} S_i n_i$$
$$H(m_1, m_2, m_3, m_4, n_1, n_2) = h_1 + \sum_{j=1}^{4} \frac{h_2^j}{m_j} + \sum_{i=1}^{2} \frac{h_3^i}{n_i}.$$

The coefficients h_i^j, $i = 1, 2, 3$, $j = 1 \ldots 4$ equals:

$$\begin{aligned}
h_1 &= \frac{d}{2}(\alpha_1 \beta_1 h_1 + \alpha_2 \beta_2 h_2) \\
h_2^1 &= \frac{d}{2}(H_1 - h_1)(\beta_1 - q_1)^2 \\
h_2^2 &= \frac{d}{2}(H_2 - h_2)(\beta_2 - q_2)^2 \\
h_2^3 &= \frac{d}{2}(H_1 q_1^2 + h_1 \beta_1 q_1 + h_1 q_1(\beta_1 - q_1)) \\
h_2^4 &= \frac{d}{2}(H_2 q_2^2 + h_2 \beta_2 q_2 + h_2 q_2(\beta_2 - q_2)) \\
h_3^1 &= \frac{d}{2} H_1 \alpha_1^2 \\
h_3^2 &= \frac{d}{2} H_2 \alpha_2^2.
\end{aligned} \tag{8}$$

To determine the optimal policy means to find the optimal numbers $m_1, m_2, m_3, m_4, n_1, n_2$ for the minimum total unit time cost (6). The problem of determining the optimal numbers of lots takes the form of a nonlinear integer optimization problem

$$\begin{aligned}
&\min_{(m_1, m_2, m_3, m_4, n_1, n_2)} ATC(T, m_1, m_2, m_3, m_4, n_1, n_2) \\
&= ((\alpha_1 c_M^1 + \alpha_2 c_M^2) + q_1 C_R^1 + q_2 C_R^2 + (c_D + c_A))d + 2\sqrt{L(m_1, m_2, m_3, m_4, n_1, n_2)}, \\
&m_j, n_i \in \{1, 2, \ldots\},
\end{aligned} \tag{9}$$

where $L(m_1, m_2, m_3, m_4, n_1, n_2)$ is defined by (7).

3 Solution of the Model

Instead of solving the problem (9) the function $L(m, n)$ can be minimized subject to $m_j \geq 1, n_i \geq 1$, i.e., the following two-dimensional nonlinear integer optimization problem is relevant

$$\min_{(m,n)} L(m,n) = \min_{(m,n)} \left(P + \sum_{j=1}^{l} R_j m_j + \sum_{i=1}^{k} S_i n_i \right) \cdot \left(h_1 + \sum_{j=1}^{l} \frac{h_2^j}{m_j} + \sum_{i=1}^{k} \frac{h_3^i}{n_i} \right),$$
$$m = (m_1, m_2, \ldots, m_l), n = (n_1, n_2, \ldots, n_k)$$
$$m_j, n_i \in \{1, 2, \ldots\} \quad (10)$$

For the solution of the problem (10), consider the following two-dimensional nonlinear integer optimization problem

$$\min_{(x_1,x_2,\ldots,x_n)} K(x_1, x_2, \ldots, x_n) = \min_{(x_1,x_2,\ldots,x_n)} \left(b_0 + \sum_{i=1}^{i} b_i x_i \right) \cdot \left(a_0 + \sum_{i=1}^{n} \frac{a_i}{x_i} \right), \quad (11)$$
$$x_i \in \{1, 2, \ldots\}, \; i = 1, 2, \ldots n.$$

First, let us consider the following continuous auxiliary problem:

$$\min_{(x_1,x_2,\ldots,x_n)} K(x_1, x_2, \ldots, x_n) = \min_{(x_1,x_2,\ldots,x_n)} \left(b_0 + \sum_{i=1}^{i} b_i x_i \right) \cdot \left(a_0 + \sum_{i=1}^{n} \frac{a_i}{x_i} \right), \quad (12)$$
$$x_i \geq 1, \; i = 1, 2, \ldots, n.$$

By analysing the first partial derivatives, we can prove the following lemma:

Lemma 1 *If $x_i > 0, i = 1, 2, \ldots, n$, there are n curves of local minima (12) with respect to x_j:*

$$X_j(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) = \sqrt{\frac{a_j \left(b_0 + \sum_{i=1, i \neq j}^{n} b_i x_i \right)}{b_j \left(a_0 + \sum_{i=1, i \neq j}^{n} \frac{a_i}{x_i} \right)}}, \quad (13)$$

and the point of the local minimum

$$x_j^* = \sqrt{\frac{a_j b_0}{a_0 b_j}}, \quad i = 1, 2, \ldots, n. \quad (14)$$

Let us denote the radicands of the expressions (14) by

$$A_i = \frac{a_i b_0}{a_0 b_i}, \quad i = 1, 2, \ldots, n. \quad (15)$$

Without loss of generality, it is supposed that $A_1 < A_2 < \ldots < A_n$.
We denote

$$B_i(j) = \frac{a_i\left(b_0 + \sum_{k=1}^{j} b_k\right)}{b_i\left(a_0 + \sum_{k=1}^{j} a_k\right)}, \quad i = 1, 2, \ldots, n. \tag{16}$$

Then the optimal solution for the continuous problem (15) is provided by the following theorem.

Theorem 1 *The optimal solution to the problem (12) has the following structure depending on the value of the parameters* A_i, $B_i(j)$:

1. *If* $A_i \geq 1, i = 1, 2, \ldots, n$, *then* $x_i = \sqrt{A_i}, \quad i = 1, 2, \ldots, n$.
2. *If* $A_1 < 1$, *then consider* $B_2(1), B_3(2), \ldots\ldots, B_j(j-1), \ldots, B_n(n-1)$; *if* $B_j(j-1) < 1$ *and* $B_{j+1}(j) \geq 1$ *then* $x_i = 1, \quad i = 1, \ldots, j, \quad x_i = \sqrt{B_i(j)}, \quad i = j+1, \ldots, n$.
3. *If* $B_n(n-1) < 1$, *then* $x_i = 1, \quad i = 1, \ldots, n$.

This theorem gives the solution for problem (11) for any initial parameters. Also theorem can be used for solution more complicated models. The next section contains numerical illustrations.

4 Numerical Examples

Example 1 This section presents numerical example to illustrate the behavior of the model.

Let the parameters of the model be fixed

$$d = 10,000, H_1 = 15, H_2 = 12, h_1 = 10, h_2 = 7,$$
$$R_1 = 30, R_2 = 35, R_3 = 40, R_4 = 45, S_1 = 50, S_2 = 60, P_1 + P_2 = 100,$$
$$\alpha_2 = 0.5, \beta_2 = 0.5, q_1 = 0.2, q_2 = 0.3.$$

We consider different values of parameter $\alpha_1 \in [0.05, 0.75]$ with the step 0.05 and the function $L(\cdot)$ (7) as the function of α_1. If α_1 changes in [0.05, 0.75], then $\beta_1 = 1 - \alpha_1$ changes in [0.25, 0.95] and $\beta_1 - q_1$ changes in [0.05, 0.75]. So the interval [0.05, 0.75] covers all the range of possible values of α_1 with the step 0.05. In Fig. 4 the total average cost $L(\alpha_1)$ depending on depending on the disposal rate of part 1 α_1 is represented. It is obvious from Fig. 4, that the more disposal rate α_1, the more total average cost $L(\alpha_1)$. It is logic consequence of the fact that the more percentage of final product is accomplished through manufacturing, which is more expensive alternative of remanufacturing. Table 1 represents values of $L(\alpha_1)$ at different α_1 with the step 0,05, optimal lot numbers $(m_1^*, m_2^*, m_3^*, m_4^*, n_1^*, n_2^*)$, and optimal lot sizes $(Q_1^m, Q_2^m, Q_3^m, Q_4^m, Q_1^n, Q_2^n)$.

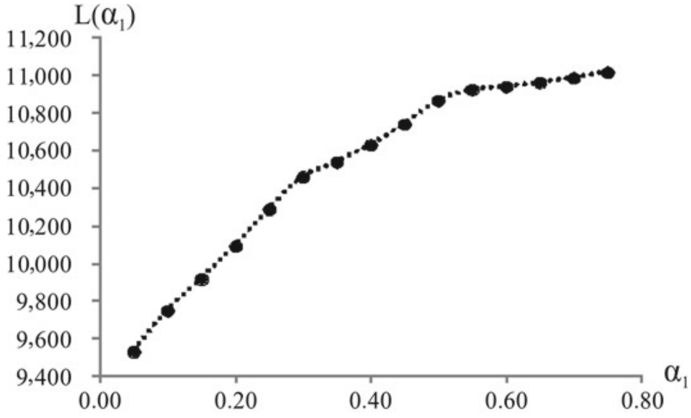

Fig. 4 Total average total cost $L(\alpha_1)$ depending on the disposal rate of part 1 α_1

Table 1 The results of example 1

α_1	$L(\alpha_1)$	$(m_1^*, m_2^*, m_3^*, m_4^*, n_1^*, n_2^*)$	$(Q_1^m, Q_2^m, Q_3^m, Q_4^m, Q_1^n, Q_2^n)$	T
0.05	9,531	(3, 1, 3, 2, 1, 2)	(317, 254, 85, 190, 63, 317)	0.127
0.1	9,750	(2, 1, 3, 2, 1, 2)	(413, 236, 79, 177, 118, 295)	0.118
0.15	9,918	(2, 1, 2, 2, 1, 2)	(351, 216, 108, 162, 162, 270)	0.108
0.2	10,095	(2, 1, 2, 2, 1, 2)	(318, 212, 106, 159, 212, 265)	0.106
0.25	10,288	(2, 1, 2, 2, 1, 2)	(286, 208, 104, 156, 260, 260)	0.104
0.3	10,461	(1, 1, 1, 1, 1, 1)	(344, 138, 138, 206, 206, 344)	0.069
0.35	10,538	(1, 1, 1, 1, 1, 1)	(307, 137, 137, 205, 239, 342)	0.068
0.4	10,632	(1, 1, 1, 1, 1, 1)	(271, 135, 135, 203, 271, 339)	0.068
0.45	10,741	(1, 1, 1, 1, 1, 1)	(235, 134, 134, 201, 302, 335)	0.067
0.5	10,866	(1, 1, 1, 1, 1, 1)	(199, 133, 133, 199, 331, 331)	0.066
0.55	10,925	(1, 1, 1, 1, 2, 1)	(188, 150, 150, 225, 206, 375)	0.075
0.6	10,941	(1, 1, 1, 1, 2, 1)	(150, 150, 150, 225, 225, 375)	0.075
0.65	10,962	(1, 1, 1, 1, 2, 1)	(112, 150, 150, 224, 243, 374)	0.075
0.7	10,988	(1, 1, 1, 1, 2, 1)	(75, 149, 149, 224, 261, 373)	0.075
0.75	11,018	(1, 1, 1, 1, 2, 1)	(37, 149, 149, 223, 279, 372)	0.074

Example 2 Let the parameters of the model be fixed

$$d = 10,000,\ H_1 = 20,\ H_2 = 20,\ h_1 = 7,\ h_2 = 19,$$
$$R_1 = 30,\ R_2 = 35,\ R_3 = 40,\ R_4 = 100,\ S_1 = 40,\ S_2 = 60,\ P_1 + P_2 = 100,$$
$$\alpha_1 = 0.5,\ \beta_1 = 0.5,\ \alpha_2 = 0.1,\ \beta_2 = 0.9,\ q_2 = 0.8.$$

We consider different values of parameter $q_1 \in [0.02, 0.48]$ and the function $L(\cdot)$ (7) as the function of q_1. The question is, how the change of the percentage of remanufactured items changes the total cost. In Fig. 5 the total average cost $L(q_1)$

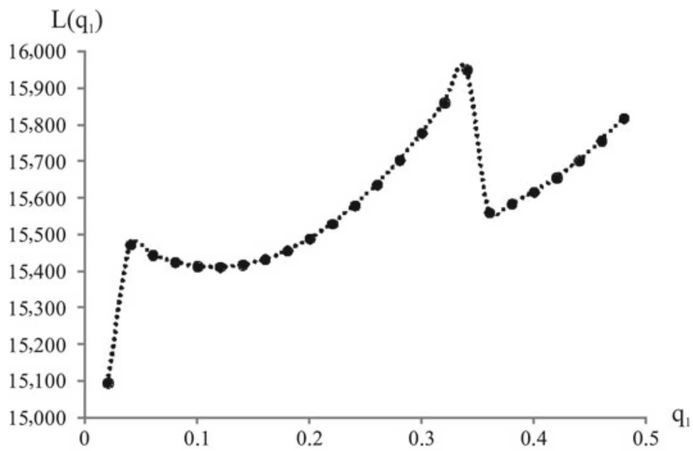

Fig. 5 Total average cost $L(q_1)$ depending on the percentage of remanufactured items q_1

depending on the percentage of remanufactured items q_1 is represented. Table 2 represents values of $L(q_1)$ at different q_1 with the step 0,02, optimal lot numbers $(m_1^*, m_2^*, m_3^*, m_4^*, n_1^*, n_2^*)$, and optimal lot sizes $(Q_1^m, Q_2^m, Q_3^m, Q_4^m, Q_1^n, Q_2^n)$.

If q_1 changes in [0.02, 0.48], then $\beta_1 - q_1$ also changes in [0.02, 0.48]. So the interval [0.02, 0.48] covers all the range of possible values of q_1. We can see from Fig. 5, that the dependency is rather complex. For instances, the "fall" at $q_1 = 0.36$ can be observed: the optimal policy (1, 1, 1, 3, 2, 1), which was invariable from $q_1 = 0.04$ to 0.36, was changed to (1, 1, 3, 4, 3, 1). The total cost increased at the values of q_1 from 0.04 to 0.36. However the policy (1, 1, 3, 4, 3, 1) remained invariable from $q_1 = 0.36$ to 0,48 and the total cost also increased. The minimum value of total cost 15,094 was reached at $q_1 = 0.02$ under the policy (3, 1, 1, 4, 3, 1).

5 Summary and Conclusions

Recently, a lot of attention has been paid to inventory management for the case where the production process involves the recovery of the used products and the production of new products. Most studies, however, have been concerned with finding the optimal policy by means of numerical optimization. In this paper, the main thing is that the parameters of optimal policy are obtained in an explicit form. We have considered a company that can produce new products and recover the value of the used product by remanufacturing and dismantling of parts. The cost structure consists of setup costs, switching costs, holding costs, and non-EOQ costs. The theorem in Sect. 3 presents simple closed-form formulas for approximating optimal policy parameters while minimizing costs.

Table 2 The results of example 2

q_1	$L(q_1)$	$(m_1^*, m_2^*, m_3^*, m_4^*, n_1^*, n_2^*)$	$(Q_1^m, Q_2^m, Q_3^m, Q_4^m, Q_1^n, Q_2^n)$	T
0.02	**15,094**	**(3, 1, 1, 4, 3, 1)**	**(179, 112, 22, 224, 187, 112)**	**0.112**
0.04	**15,471**	**(1, 1, 1, 3, 2, 1)**	**(384, 83, 33, 222, 208, 83)**	**0.083**
0.06	15,443	(1, 1, 1, 3, 2, 1)	(368, 84, 50, 223, 209, 84)	0.084
0.08	15,423	(1, 1, 1, 3, 2, 1)	(351, 84, 67, 223, 209, 84)	0.084
0.1	15,412	(1, 1, 1, 3, 2, 1)	(335, 84, 84, 223, 209, 84)	0.084
0.12	**15,410**	**(1, 1, 1, 3, 2, 1)**	**(318, 84, 100, 223, 209, 84)**	**0.084**
0.14	15,416	(1, 1, 1, 3, 2, 1)	(301, 84, 117, 223, 209, 84)	0.084
0.16	15,431	(1, 1, 1, 3, 2, 1)	(284, 84, 134, 223, 209, 84)	0.084
0.18	15,455	(1, 1, 1, 3, 2, 1)	(267, 83, 150, 223, 209, 83)	0.083
0.2	15,487	(1, 1, 1, 3, 2, 1)	(250, 83, 167, 222, 208, 83)	0.083
0.22	15,528	(1, 1, 1, 3, 2, 1)	(233, 83, 183, 222, 208, 83)	0.083
0.24	15,578	(1, 1, 1, 3, 2, 1)	(215, 83, 199, 221, 207, 83)	0.083
0.26	15,635	(1, 1, 1, 3, 2, 1)	(198, 83, 215, 220, 206, 83)	0.083
0.28	15,702	(1, 1, 1, 3, 2, 1)	(181, 82, 230, 219, 205, 82)	0.082
0.3	15,776	(1, 1, 1, 3, 2, 1)	(164, 82, 245, 218, 204, 82)	0.082
0.32	15,859	(1, 1, 1, 3, 2, 1)	(146, 81, 260, 217, 203, 81)	0.081
0.34	**15,949**	**(1, 1, 1, 3, 2, 1)**	**(129, 81, 275, 216, 202, 81)**	**0.081**
0.36	**15,559**	**(1, 1, 3, 4, 3, 1)**	**(156, 111, 133, 222, 185, 111)**	**0.111**
0.38	15,583	(1, 1, 3, 4, 3, 1)	(133, 111, 141, 222, 185, 111)	0.111
0.4	15,615	(1, 1, 3, 4, 3, 1)	(111, 111, 148, 222, 185, 111)	0.111
0.42	15,654	(1, 1, 3, 4, 3, 1)	(88, 111, 155, 221, 184, 111)	0.111
0.44	15,701	(1, 1, 3, 4, 3, 1)	(66, 110, 162, 220, 184, 110)	0.110
0.46	15,755	(1, 1, 3, 4, 3, 1)	(44, 110, 168, 220, 183, 110)	0.110
0.48	**15,817**	**(1, 1, 3, 4, 3, 1)**	**(22, 109, 175, 219, 182, 109)**	**0.109**

Bold flags the points, in which the function $L(q_1)$ changes direction of increase

The numerical examples were considered, which demonstrated the alteration of the optimal policy under different values of the initial parameter ceteris paribus. It also demonstrates that the dependency of total average cost from other parameters of the model can be rather complex, which is the consequence of integer optimal numbers of optimal policy.

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Computation of Multi-choice Multi-objective Fuzzy Probabilistic Transportation Problem

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1 Introduction

In a transportation problem (TP), a product is to be transported from different sources to different destinations such that the total transportation cost is minimum. In real-life applications, all the parameters of the transportation problem are not generally defined precisely such as the availability of source amount and requirement of demand amount are not certainly. To handle such type of uncertainty, supply, and demand parameters are considered as random variables rather than the deterministic one. These random variables are assumed to follow a given probability distribution or its probability distribution may be estimated. Such type of (TP) is known as Stochastic Transportation Problem (STP).

If more than one objective is to be optimized in an (STP), then the problem is called Multi-Objective Stochastic Transportation (MOST) problem, which is a special case of multi-objective decision-making problem.

Instead of single choice, if there may be several choices involved associated with the transportation parameters like cost, supply, or demand, then the decision maker is confused to select the proper choice for these parameters. In this circumstance, the study of transportation problem creates a new direction which is called Multi-Choice

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Multi-Objective Transportation (MCMOT) problem. In this special (MCMOT) problem, the demand is multi-choice and the supply constraint is stochastic in nature. In which the parameter supply is fuzzy Laplace distribution random variable. Here both fuzziness and randomness are present. So this problem is known as Multi-Choice Multi-Objective Fuzzy Probabilistic Transportation (MCMOFPT) problem.

Multiple Choice Programming Problem (MCPP) is originated by Healy [13]. (MCPP) is a mixed binary programming where all binary variables constitute a number of mutually exclusive choices where only one variable is to be selected. When we transform an (MCPP) into a standard mathematical programming problem involving more number of choices for a given parameter: We face the following major difficulties (i) selecting binary variables, (ii) selecting bounds for binary codes, (iii) restricting binary codes using auxiliary constraints.

(TP) is first developed and proposed by Hitchcock [14]. Charnes et al. [10] presented an efficient method for solving transportation problem using the simplex algorithm. Greig [12] have derived different solution procedure for transportation problem when prices and quantities are given as crisp numbers by using the table methods such as the northwest corner method, the shortcut method and Russel's approximation method.

Stochastic programming problem is first formulated by Dantzig and Madansky [11], who suggested a two-stage programming technique for its solutions.

Later Charnes and Cooper [9] developed the chance constrained programming technique in which the chance constraints are converted into equivalent deterministic nonlinear constraints.

The concept of fuzzy random variable is first introduced by Zadeh [23] and then introduced by Kwakernaak [15] to depict the simultaneous occurrence of randomness and fuzziness. Acharya et al. [1] developed solution of multi-objective fuzzy probabilistic programming problem where parameters are independent log normally

Bit et al. [5] presented the multi-objective transportation problem using the fuzzy programming technique on probabilistic constraints.

Biswal and Acharya [3] presented a solution of multi-choice linear programming problems by interpolating polynomials.

Multi-choice stochastic Transportation Problem (MCSTP) has been extensively studied by many researchers. Roy et al. [21] presented an equivalent deterministic model of (MCSTP) by assuming that both availability a_i and demand b_j as random variables following exponential distribution. Biswal and Samal [4] obtained an equivalent deterministic model of (MCSTP) in which they considered that both a_i and b_j follow Cauchy distribution. Mahapatra [16] also gave equivalent deterministic model of (MCSTP) involving Weibull distribution. Mahapatra et al. [17] considered the (MCSTP) involving Extreme value distribution. Quddoos et al. [19] discussed the (MCSTP) involving general form of distribution. Acharya et al. [2] have proposed the solution methodology of a multi-objective transportation problem where fuzziness and randomness occur under one roof. Ranarahu et al. [20] have discussed an efficient method for solving a multi-objective bilevel programming problem where the uncertain parameters are present in second level in form of fuzzy random variables.

In this paper, we have consider a multi-choice multi-objective fuzzy probabilistic transportation problem, where some parameters are multi-choice and other are fuzzy random variables. Fuzzy random variables are considered to follow fuzzy Laplace random variables. It is also assume that are independent. The solution procedure of the proposed model is discussed. Finally numerical example is presented in support of the methodology.

The organization of the paper is as follows: following the Introduction, basic preliminaries are presented in Sect. 2. The mathematical programming model of (MCMOFPT) problems is presented in Sect. 3. In order to solve (MCMOFPT) problem methodology is presented in Sect. 4. In order to demonstrate the proposed method a numerical example is included in Sect. 5. Finally, conclusions are presented in Sect. 6.

2 Basic Preliminaries

Definition 2.1 [8] A Fuzzy Number (FN) $\widetilde{A}$ is a convex normalized fuzzy set $\widetilde{A}$ of the real line R, with membership function μ_A: $R \to [0; 1]$, satisfying the following conditions.

1. There exists exactly one interval $I \in R$ such that $\mu_A(x) = 1; x \in I$
2. The membership function μ_A is piecewise continuous.

Definition 2.2 [6] α-cut of the FN $\widetilde{A}$ is the set $\{x|\mu_A(x) \geq \alpha\}$ for $0<\alpha<1$ and denoted $\widetilde{A}[\alpha]$.

Definition 2.3 [8] A FN $\widetilde{A}$ is said to be positive if it's membership function $\mu_A(x) = 0; \forall x \leq 0$. It may be stated as follow: Let $\widetilde{A}[\alpha] = [A_*; A^*]$ be the α-cut of the fuzzy number $\widetilde{A}$ for $0<\alpha<1$. $\widetilde{A}$ is said to be positive if $A_* > 0$.

Definition 2.4 [7] A fuzzy random variable is a random variable whose parameter is a FN. Let $\widetilde{X}$ be continuous random variable with fuzzy parameter $\tilde{\theta}$ and $\tilde{P}$ as fuzzy probability, then $\widetilde{X}$ is said to be a continuous fuzzy random variable with density function $f(x, \tilde{\theta})$. $\tilde{P}(\tilde{X} \leq x) = \tilde{\beta}$, where $0 \leq \tilde{\beta} \leq 1$; $\tilde{\beta} = (\beta^{(m)}; \beta^{(p)}; \beta^{(o)})$, $\beta^{(p)} \geq 0$ and $\beta^{(o)} \leq 1$.

Definition 2.5 [7] Let $E = [c; d]$ be an event. Then the probability of the event E of continuous fuzzy random variable $\widetilde{X}$ is a fuzzy number whose α-cut is

$$\widetilde{P}\big[c \leq \widetilde{X} \leq d\big] = \left[\min : \left\{ \int_a^b f(x, \theta)\mathrm{d}x | \theta \in \tilde{\theta}[\alpha], \int_{-\infty}^{\infty} f(x, \theta)\mathrm{d}x = 1 \right\}, \right.$$

$$\left. \max : \left\{ \int_a^b f(x, \theta)\mathrm{d}x | \theta \in \tilde{\theta}[\alpha], \int_{-\infty}^{\infty} f(x, \theta)\mathrm{d}x = 1 \right\} \right]$$

$$= \left[\beta_*[\alpha], \beta^*[\alpha]\right]$$

Definition 2.6 [18] Let $\widetilde{A} = \left(A^{(m)}; A^{(p)}; A^{(o)}\right)$ and $\widetilde{B} = \left(B^{(m)}; B^{(p)}; B^{(o)}\right)$ be two FNs with α-cut $\widetilde{A}[\alpha] = [A_*; A^*]$ and $\widetilde{B}[\alpha] = [B_*; B^*]$ respectively. Then $\widetilde{A} \lesssim \widetilde{B}$ iff $A^* \leq B_*$.

Fuzzy Laplace Distribution Let X be a Laplace distributed random variable having parameter α and β. Where α is the location parameter and β is the scale parameter of the random variable. Let the probability density function of the random variable X is

$$f(x, \alpha, \beta) = \frac{1}{2\beta} e^{\frac{-|(x-\alpha)|}{\beta}}, \quad -\infty \leq x \leq \infty, \quad \beta > 0$$

If the parameter α or β or both are positive FN then the Laplace random variable X will be fuzzy Laplace distributed random variable denoted by $\widetilde{X}$.

Now the fuzzy probability of the fuzzy Laplace random variable $\widetilde{X}$ on the interval $[c, d]$, is a FN whose α-cut is

$$\widetilde{P}(c \leq \widetilde{X} \leq d)[\alpha] = \int_a^b \frac{1}{2\beta} e^{\frac{-|(x-\alpha)|}{\beta}} \mathrm{d}x, \quad \alpha \in \tilde{\alpha}[\alpha], \quad \beta \in \tilde{\beta}[\alpha]$$

If both α and β are fuzzy numbers.

Fuzzy Exponential Distribution A FN $\widetilde{\boldsymbol{A}} = (\boldsymbol{A}^1, \boldsymbol{A}^2, \boldsymbol{A}^3, \boldsymbol{A}^4)$ is said to be exponential if its membership function satisfies the following:

$$\mu_{\tilde{A}}(x) = \begin{cases} \exp\left[\frac{-(A^2-x)}{A^2-A^1}\right], & \text{if } A^1 \leq x \leq A^2 \\ 1, & \text{if } A^2 \leq x \leq A^3 \\ \exp\left[\frac{-(x-A^3)}{A^4-A^3}\right], & \text{if } A^3 \leq x \leq A^4 \end{cases}$$

α-cut of a exponential fuzzy number $\tilde{A} = (A^1, A^2, A^3, A^4)$ is defined as

$$\widetilde{A}[\alpha] = \left[A^2 + (A^2 - A^1)\log\alpha, A^3 - (A^4 - A^3)\log\alpha\right]$$

3 Multi-choice Multi-objective Fuzzy Probabilistic Transportation Problem

The mathematical model of the **multi-objective transportation problem** is presented as follows:

$$\min: \quad Z_k = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^k x_{ij}, \quad k = 1, 2, 3, \ldots, K$$

$$\text{subject to } \sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \ldots, n$$

$$x_{ij} \geq 0, \quad \forall i, j$$

Here Z_k represents minimum values of k-th objective function and it is assumed that supply $a_i > 0$, demand $b_j > 0$ and transportation cost $c_{ij}^k > 0$ and

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j \quad \text{(for unbalanced Transportation Problem)}$$

Multi-choice multi-objective probabilistic transportation (MCMOPT) problem is presented as

$$\min: \quad Z_k = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^k x_{ij}, \quad k = 1, 2, 3, \ldots, K$$

$$\text{subject to } P\left(\sum_{j=1}^{n} x_{ij} \leq a_i\right) \geq \gamma_i, \quad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} \geq b_j \in \left\{b_j^{(1)}, b_j^{(2)}, \ldots, b_j^{(k_j)}\right\}, \quad j = 1, 2, \ldots, n$$

$$x_{ij} \geq 0, \quad \forall i, j$$

,

where $0 < \gamma_i < 1, \forall i$. a_i is random variable which is supply. Right hand side of the j-th constraint b_j is demand which is multi-choice in nature that is exactly one element is to be selected from the set $\left\{b_j^{(1)}, b_j^{(2)} \ldots b_j^{(k_j)}\right\}$ for j-th constraint.

Multi-choice multi-objective fuzzy probabilistic transportation (MCMOFPT) problem is presented as

$$\min: Z_k = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^k x_{ij}, \quad k = 1, 2, 3, \ldots, K \tag{1}$$

$$\text{subject to } \quad \tilde{P}\left(\sum_{j=1}^{n} x_{ij} \leq \tilde{a}_i\right) \geq \widetilde{\gamma}_i, \quad i = 1, 2, \ldots, m \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} \geq b_j \in \left\{b_j^{(1)}, b_j^{(2)}, \ldots, b_j^{(k_j)}\right\}, \quad j = 1, 2, \ldots, n, \tag{3}$$

$$x_{ij} \geq 0, \quad \forall i, j \tag{4}$$

where, $\tilde{\gamma}_i$ are fuzzy numbers, $\tilde{a}_i$ is Laplace distributed fuzzy random variable with location and scale parameter are $\tilde{\alpha}_{a_i}$ and $\tilde{\beta}_{a_i}$ respectively. The right hand side of j-th constraint has a set of k_j number of goals, out of which only one goal is to be selected so as to minimize the objective function.

4 Transformation Techniques

The mathematical models presented in Sect. 3 are difficult to solve directly. Therefore an equivalent crisp model is necessary in order to obtain the compromise solution for the proposed mathematical models. In this section an attempt has been made to find the crisp equivalent models for the (MCMOFPT) problem in the form of theorem.

Theorem 4.1 *If $\tilde{a}_i, i = 1, 2, \ldots, m$ are Laplace distributed fuzzy random variable then*

$$\tilde{P}\left(\sum_{j=1}^{n} x_{ij} \leq \tilde{a}_i\right) \gtrsim \tilde{\gamma}_i$$

is equivalent to

$$\sum_{j=1}^{n} x_{ij} \leq \alpha_{a_{i*}} - \beta^*_{a_i} \ln\left(2\gamma_i^*\right),$$

where, $\tilde{\gamma}_i$ are fuzzy numbers, $\tilde{a}_i$ is Laplace distributed fuzzy random variable with location and scale parameter are $\tilde{\alpha}_{a_i}$ and $\tilde{\beta}_{a_i}$ respectively.

Proof It is assumed that $\tilde{a}_i, i = 1, 2, 3 \ldots m$ are independent Laplace distributed fuzzy random variable. If a_i follows Laplace distribution with location parameter α_{a_i} and scale parameter β_{a_i}. Then $\tilde{a}_i$ *is a f*uzzy Laplace random variable, whose location and scale parameter are $\tilde{\alpha}_{a_i}$ and $\tilde{\beta}_{a_i}$ respectively. The α-cuts of $\tilde{a}_i$, $\tilde{\alpha}_{a_i}$ and $\tilde{\beta}_{a_i}$ are $\tilde{a}_i[\alpha] = [a_{i_*}, a_i^*]$, $\tilde{\alpha}_{a_i}[\alpha] = \left[\alpha_{a_{i*}}, \alpha^*_{a_i}\right]$ and $\tilde{\beta}_{a_i}[\alpha] = \left[\beta_{a_{i*}}, \beta^*_{a_i}\right]$ respectively.

Now let us consider the i-th constraint

$$\tilde{P}\left(\sum_{j=1}^{n} x_{ij} \leq \tilde{a}_i\right) \gtrsim \tilde{\gamma}_i, \quad i = 1, 2, 3 \ldots m, \tag{5}$$

where $\tilde{\gamma}_i$ are fuzzy numbers and whose α-cuts are $\tilde{\gamma}_i[\alpha] = [\gamma_{i_*}, \gamma_{i^*}]$.

The α-cut of the probabilistic constraint is expressed as

$$\tilde{P}\left(\sum_{j=1}^{n} x_{ij} \le \tilde{a}_i\right)[\alpha] = \widetilde{P}(A_i \le \tilde{a}_i)[\alpha],$$

where

$$\begin{aligned}
A_i &= \sum_{j=1}^{n} x_{ij} \\
&= \{P(A_i \le a_i) | a_i \in \tilde{\boldsymbol{a}}_i[\alpha]\} \\
&= \{1 - P(a_i \le A_i) | a_i \in \tilde{a}_i[\alpha]\} \\
&= \left\{1 - \int_{-\infty}^{A_i} f(a_i)\mathrm{d}a_i | a_i \in \tilde{a}_i[\alpha], \alpha_{a_i} \in \tilde{\alpha}_{a_i}[\alpha], \beta_{a_i} \in \tilde{\beta}_{a_i}[\alpha]\right\} \\
&= \left\{1 - \left[\int_{-\infty}^{\alpha_{a_i}} f(a_i)\mathrm{d}a_i + \int_{\alpha_{a_i}}^{A_i} f(a_i)\mathrm{d}a_i\right] | a_i \in \tilde{a}_i[\alpha], \alpha_{a_i} \in \tilde{\alpha}_{a_i}[\alpha], \beta_{a_i} \in \tilde{\beta}_{a_i}[\alpha]\right\} \\
&= \left\{1 - \left[\int_{-\infty}^{\alpha_{a_i}} \frac{1}{2\beta_{a_i}} \mathrm{e}^{\frac{(a_i - \alpha_{a_i})}{\beta_{a_i}}} \mathrm{d}a_i + \int_{\alpha_{a_i}}^{Ai} \frac{1}{2\beta_{a_i}} \mathrm{e}^{-\frac{(a_i - \alpha_{a_i})}{\beta_{a_i}}} \mathrm{d}a_i\right] | a_i \in \tilde{a}_i[\alpha], \alpha_{a_i} \in \tilde{\alpha}_{a_i}[\alpha], \beta_{a_i} \in \tilde{\beta}_{a_i}[\alpha]\right\} \\
&= \left[\frac{1}{2}\mathrm{e}^{\frac{(\alpha_{a_i} - A_i)}{\beta_{a_i}}}\right] | a_i \in \tilde{a}_i[\alpha], \alpha_{a_i} \in \tilde{\alpha}_{a_i}[\alpha], \beta_{a_i} \in \tilde{\beta}_{a_i}[\alpha] \\
&= [k_{i^*}, k_{i^*}].
\end{aligned}$$

where

k_{i^*} is the optimal objective values of the optimization problem $\min\left[\frac{1}{2}\mathrm{e}^{\frac{(\alpha_{a_i} - A_i)}{\beta_{a_i}}}\right]$

k_{i^*} is the optimal objective values of the optimization problem $\max\left[\frac{1}{2}\mathrm{e}^{\frac{(\alpha_{a_i} - A_i)}{\beta_{a_i}}}\right]$

Let

$$\min\left[\frac{1}{2}\mathrm{e}^{\frac{(\alpha_{a_i} - A_i)}{\beta_{a_i}}}\right] = \frac{1}{2}\mathrm{e}^{\frac{(\alpha_{a_{i_*}} - A_i)}{\beta^*_{a_i}}} = k_{i^*}$$

$$\max\left[\frac{1}{2}\mathrm{e}^{\frac{(\alpha_{a_i} - A_i)}{\beta_{a_i}}}\right] = \frac{1}{2}\mathrm{e}^{\frac{(\alpha^*_{a_i} - A_i)}{\beta_{a_{i_*}}}} = k_{i^*}$$

$$\widetilde{P}\left(\sum_{j=1}^{n} x_{ij} \le \tilde{a}_i\right)[\alpha] = \left[\frac{1}{2}\mathrm{e}^{\frac{\left(\alpha_{a_{i_*}} - A_i\right)}{\beta^*_{a_i}}}, \frac{1}{2}\mathrm{e}^{\frac{\left(\alpha^*_{a_i} - A_i\right)}{\beta_{a_{i_*}}}}\right]$$

Using fuzzy inequality, the α-cut of the fuzzy constraint is expressed as

$$\widetilde{P}\left(\sum_{j=1}^{n} x_{ij} \le \tilde{a}_i\right)[\alpha] \succsim \tilde{\gamma}_i[\alpha]$$

$$\Rightarrow \frac{1}{2}\mathrm{e}^{\frac{\left(\alpha_{a_{i*}}-A_i\right)}{\beta^*_{a_i}}} \geq \gamma_{i^*}$$

$$\Rightarrow \mathrm{e}^{\frac{\left(\alpha_{a_{i*}}-A_i\right)}{\beta^*_{a_i}}} \geq 2\gamma_{i^*}$$

$$\Rightarrow \frac{\left(\alpha_{a_{i*}}-A_i\right)}{\beta^*_{a_i}} \geq \ln 2\gamma_{i^*}$$

$$\Rightarrow \left(\alpha_{a_{i*}}-A_i\right) \geq \beta^*_{a_i}\ln(2\gamma_{i^*})$$

$$\Rightarrow A_i \leq \alpha_{a_{i*}} - \beta^*_{a_i}\ln(2\gamma_{i^*})$$

Hence the deterministic equivalent of the fuzzy probabilistic constraint (5) is expressed as

$$\sum_{j=1}^{n} x_{ij} \leq \alpha_{a_{i*}} - \beta^*_{a_i}\ln\left(2\gamma_i^*\right),$$

Hence the theorem.

The deterministic equivalent of the (MCMOFPT) problem (1)–(4) is expressed as

$$\min: Z_k = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^k x_{ij}, \quad k = 1, 2, 3, \ldots, K \tag{6}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} \leq \alpha_{a_{i*}} - \beta^*_{a_i}\ln\left(2\gamma_i^*\right), \quad i = 1, 2, 3 \ldots m \tag{7}$$

$$\sum_{i=1}^{m} x_{ij} \geq b_j \in \left\{b_j^{(1)}, b_j^{(2)} \ldots b_j^{(k_j)}\right\}, \quad j = 1, 2, 3 \ldots n \tag{8}$$

$$x_{ij} \geq 0, \quad \forall i, j, \tag{9}$$

where, $\widetilde{\gamma}_i$ are fuzzy numbers, $\forall_i \widetilde{a}_i$ is Laplace distributed fuzzy random variable with location and scale parameter are $\widetilde{\alpha}_{a_i}$ and $\widetilde{\beta}_{a_i}$ respectively. The right-hand side of j-th constraint has a set of k_j number of goals, out of which only one goal is to be selected so as to minimize the objective function.

It is difficult to solve multi-choice multi-objective transportation problem directly. Interpolating polynomials are formulated for all the multi-choice parameters of the problem. We take the help of Newtons forward difference interpolating polynomials to tackle the multi-choice parameter.

Table 1 Data points

$i = z^{(j)}$	0	1	2	…	$k_j - 1$
$f(z^{(j)}) = b_j$	$b_j^{(1)}$	$b_j^{(2)}$	$b_j^{(3)}$	…	$b_j^{(k_j)}$

4.1 Model Under Newtons Forward Difference Interpolating Polynomial

Interpolating polynomials are formulated for the multi-choice parameters taking some integral values for the nodal points.

Let $0, 1, 2, \ldots, (k_j - 1)$ be k_j number of node points where $\left\{b_j^{(1)}, b_j^{(2)} \ldots b_j^{(k_j)}\right\}$ be the associated functional values of the interpolating polynomial of k_j different node points. We derive a polynomial $P_{k_j-1}(z^{(j)})$ of degree $(k_j - 1)$ which interpolates the given data:

$$P_{(k_j-1)}(i) = b_j^{(i+1)}, \quad i = 0, 1, 2 \ldots (k_j - 1), \quad j = 1, 2, 3 \ldots n.$$

The given data points in Table 1 correspond to the sequence: $0, 1, 2, \ldots, (k_j - 1)$. Here we consider the initial point as 0 and the step length size as 1. These simple differences can be forward differences Δf_j. We will present the forward differences and then the interpolating polynomial based on the forward differences. The *p*th order forward difference $\Delta^p f_j$ can be calculated using the following formula:

$$\begin{aligned} \Delta f_j &= f_{j+1} - f_j \\ \Delta^2 f_j &= \Delta f_{j+1} - \Delta f_j \\ \Delta^p f_j &= \Delta^{p-1} f_{j+1} - \Delta^{p-1} f_j \end{aligned}$$

To find the forward differences, we use the forward difference Table 2.

We formulate an interpolating polynomial of the multi-choice parameters by Newtons forward difference formula as

$$\begin{aligned} P_{(k_j-1)}(z^j) = b_j^{(1)} + z^{(j)}\Delta f_0 + \frac{z^{(j)}\left(z^{(j)} - 1\right)}{2!}\Delta^2 f_0 \\ + \cdots + \frac{z^{(j)}\left(z^{(j)} - 1\right)\ldots\left(z^{(j)} - k_j + 2\right)}{(k_j - 1)!}\Delta^{k_j-1} f_0, \quad j = 1, 2, 3 \ldots n \end{aligned}$$

Now the (MCMOFPT) problem (6)–(9) can be presented as

$$\min : Z_k = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^k x_{ij}, \quad k = 1, 2, 3, \ldots, K \tag{10}$$

Table 2 Newtons forward difference table

$z^{(j)}$	$f(z^{(j)})$	First order forward difference	Second order forward difference	…	$k_j - 1$th order forward difference
0	$f(0)$	Δf_0			
1	$f(1)$	Δf_1	$\Delta^2 f_0$		
2	$f(2)$	Δf_2	$\Delta^2 f_1$		
3	$f(3)$		$\Delta^2 f_3$		
…	…	…	…	…	$\Delta^{k_j-1} f_0$
$k_j - 3$	$f(k_j - 3)$	Δf_{k_j-3}			
$k_j - 2$	$f(k_j - 2)$	Δf_{k_j-2}	$\Delta^2 f_{k_j-3}$		
$k_j - 1$	$f(k_j - 1)$				

$$\text{Subject to } \sum_{j=1}^{n} x_{ij} \le \alpha_{a_{i*}} - \beta^*_{a_i} \ln\left(2\gamma_i^*\right), \quad i = 1, 2, 3 \ldots m \tag{11}$$

$$\sum_{i=1}^{m} x_{ij} \ge b_j^{(1)} z^{(j)} \Delta f_0 + \frac{z^{(j)}\left(z^{(j)} - 1\right)}{2!} \Delta^2 f_0 + \cdots + \frac{z^{(j)}\left(z^{(j)} - 1\right) \ldots \left(z^{(j)} - k_j + 2\right)}{(k_j - 1)!} \Delta^{k_j-1} f_0, \quad j = 1, 2, 3 \ldots n \tag{12}$$

$$x_{ij} \ge 0, \quad \forall i, j, \tag{13}$$

where, $\widetilde{\gamma}_i$ are fuzzy numbers $\forall i$, $\widetilde{a}_i$ is Laplace distributed fuzzy random variable with location and scale parameter are $\widetilde{\alpha}_{a_i}$ and $\widetilde{\beta}_{a_i}$ respectively.

4.2 Weighting Method of Multi-objective Optimization

The weighting method is the method to find the optimal solution of multi-objective function if the objective space is convex.

If $f_1(x), f_2(x), \ldots f_n(x)$ are n objective functions for any vector $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ then we can define weighting method for their optimal solution as defined below.

Let $W = \left\{w : w \in R^n, w_j > 0, \sum_{j=1}^{n} w_j = 1\right\}$ to be the set of non-negative weights.

The weighted objective function for the multiple objective function defined above can be defined as $M(w)$, where

$$M(w) = \min \sum_{j=1}^{n} w_j f_j(x), \quad x \in X.$$

where $w_j \in [0, 1]$ is the weight of the j-th objective function such that their sum is one.

5 Numerical Example

A numerical example for a minimization type problem is presented to illustrate the solution procedure. Consider the following:

$$\begin{aligned} \min: Z_1 &= 20x_{11} + 13x_{12} + x_{13} + 2x_{21} + 3x_{22} + 2x_{23} \\ \min: Z_2 &= 10x_{11} + 2x_{12} + 5x_{13} + 9x_{21} + 19x_{22} + 5x_{23} \end{aligned}$$

$$\widetilde{P}\left(\sum_{j=1}^{3} x_{1j} \le \tilde{a}_1\right) \gtrsim \tilde{\gamma}_1$$

$$\widetilde{P}\left(\sum_{j=1}^{3} x_{2j} \le \tilde{a}_2\right) \gtrsim \tilde{\gamma}_2$$

$$\sum_{i=1}^{2} x_{i1} \ge \{5, 12, 21\}$$

$$\sum_{i=1}^{2} x_{i2} \ge \{4, 6\}$$

$$\sum_{i=1}^{2} x_{i3} \ge \{6\}$$

$$x_{ij} \ge 0, \quad i = 1, 2 \text{ and } j = 1, 2, 3,$$

where, $\widetilde{\gamma}_i$ are fuzzy numbers, $\forall i$, $\tilde{a}_i$ is Laplace distributed fuzzy random variable with location and scale parameter are $\tilde{\alpha}_{a_i}$ and $\tilde{\beta}_{a_i}$ respectively.

Where $\tilde{a}_1$ and $\tilde{a}_2$ are independent Laplace distributed fuzzy random variables with location parameters $\tilde{\alpha}_{a_1}$ and $\tilde{\alpha}_{a_2}$ and scale parameters $\tilde{\beta}_{a_1}$ and $\tilde{\beta}_{a_2}$ are exponential fuzzy numbers and $\tilde{\gamma}_1 = \widetilde{0.5}$ and $\widetilde{\gamma}_2 = \widetilde{0.6}$ are also exponential fuzzy numbers as $\tilde{\alpha}_{a_1} = \langle 5/6/7/8 \rangle$, $\tilde{\alpha}_{a_2} = \langle 13/14/15/16 \rangle$, $\tilde{\beta}_{a_1} = \langle 2/3/4/5 \rangle$, $\tilde{\beta}_{a_2} = \langle 4/6/8/9 \rangle$, $\widetilde{0.5} = \langle 0.1/0.2/0.4/0.6 \rangle$ and $\widetilde{0.6} = \langle 0.3/0.4/0.5/0.7 \rangle$ (All these fuzzy numbers are assumed to be exponential).

Solution α-cuts of $\tilde{\alpha}_{a_1}, \tilde{\alpha}_{a_2}, \tilde{\beta}_{a_1}, \tilde{\beta}_{a_2}, \widetilde{0.5}$ and $\widetilde{0.6}$ can be calculated. The α-cuts are

$$\begin{aligned}
\tilde{\alpha}_{a_1}[\alpha] &= [6+\ln\alpha, 7-\ln\alpha] = \left[\alpha_{a_{1*}}, \alpha^*_{a_1}\right] \\
\tilde{\alpha}_{a_2}[\alpha] &= [14+\ln\alpha, 15-\ln\alpha] = \left[\alpha_{a_{2*}}, \alpha^*_{a_2}\right] \\
\tilde{\beta}_{a_1}[\alpha] &= [3+\ln\alpha, 4-\ln\alpha] = \left[\beta_{a_{1*}}, \beta^*_{a_1}\right] \\
\tilde{\beta}_{a_2}[\alpha] &= [6+2\ln\alpha, 8-\ln\alpha] = \left[\beta_{a_{2*}}, \beta^*_{a_2}\right] \\
\widetilde{0.5}[\alpha] &= [0.2+0.1\ln\alpha, 0.4-0.2\ln\alpha] = \left[\gamma_{1_*}, \gamma^*_1\right] \\
\widetilde{0.6}[\alpha] &= [0.4+0.1\ln\alpha, 0.5-0.2\alpha] = \left[\gamma_{2_*}, \gamma^*_2\right]
\end{aligned}$$

Using the relation (11) and (12) the deterministic equivalent model is

$$\begin{aligned}
\text{min}: \quad & Z_1 = 20x_{11} + 13x_{12} + x_{13} + 2x_{21} + 3x_{22} + 2x_{23} \\
\text{min}: \quad & Z_2 = 10x_{11} + 2x_{12} + 5x_{13} + 9x_{21} + 19x_{22} + 5x_{23} \\
\text{subject to } & \sum_{j=1}^{3} x_{1j} \le (6+\ln\alpha) - (4-\ln\alpha)\ln\{2(0.4-0.2\ln\alpha)\} \\
& \sum_{j=1}^{3} x_{2j} \le (14+\ln\alpha) - (8-\ln\alpha)\ln\{2(0.5-0.2\ln\alpha)\} \\
& \sum_{i=1}^{2} x_{i1} \ge 5 + 6k_1 + k_1^2 \\
& \sum_{i=1}^{2} x_{i2} \ge 4 + 2k_2 \\
& \sum_{i=1}^{2} x_{i3} \ge 6 \\
& x_{ij} \ge 0, \quad i = 1, 2, \quad j = 1, 2, 3, \quad k_1 = 0, 1, 2, \quad k_2 = 0, 1
\end{aligned}$$

For $\alpha = 0.5$ the above deterministic equivalent becomes

$$\begin{aligned}
\text{min}: \quad & Z_1 = 20x_{11} + 13x_{12} + x_{13} + 2x_{21} + 3x_{22} + 2x_{23} \\
\text{min}: \quad & Z_2 = 10x_{11} + 2x_{12} + 5x_{13} + 9x_{21} + 19x_{22} + 5x_{23} \\
\text{subject to } & x_{11} + x_{12} + x_{13} \le 4.95759006 \\
& x_{21} + x_{22} + x_{23} \le 11.1794982 \\
& x_{11} + x_{21} \ge 5 + 6k_1 + k_1^2 \\
& x_{12} + x_{22} \ge 4 + 2k_2 \\
& x_{13} + x_{23} \ge 6 \\
& x_{ij} \ge 0, \quad i = 1, 2, \; j = 1, 2, 3, \; k_1 = 0, 1, \; k_2 = 0, 1
\end{aligned}$$

The above deterministic multi-objective fuzzy nonlinear programming problem is solved using weighted mean method. Using the weights the above objective function can be reduced to the new objective function as

$$\begin{aligned}
\text{min}: \quad & Z = w_1(20x_{11} + 13x_{12} + x_{13} + 2x_{21} + 3x_{22} + 2x_{23}) \\
& \quad + w_2(10x_{11} + 2x_{12} + 5x_{13} + 9x_{21} + 19x_{22} + 5x_{23}) \\
\text{subject to } & x_{11} + x_{12} + x_{13} \leq 4.95759006 \\
& x_{21} + x_{22} + x_{23} \leq 11.1794982 \\
& x_{11} + x_{21} \geq 5 + 6k_1 + k_1^2 \\
& x_{12} + x_{22} \geq 4 + 2k_2 \\
& x_{13} + x_{23} \geq 6 \\
& w_1 + w_2 = 1 \\
& x_{ij} \geq 0, \quad i = 1, 2, \; j = 1, 2, 3, \; k_1 = 0, 1, \; k_2 = 0, 1, \; w_1 > 0, \; w_2 > 0
\end{aligned}$$

The crisp problem is solved using the LINGO [22] software.

By considering different values of w_1 and w_2, the corresponding objectives are given in Table 3.

6 Conclusion

In this paper we presented a solution procedure for (MCMOFPT) problem. The parameter which follows fuzzy random variable is independent and place distributed in fuzzy sense. The fuzzy stochastic models have been converted into equivalent deterministic models in two steps. In first step using the concept of α-cut the fuzziness has been removed and in second step using chance constraint method randomness has been removed. In the third step multi-choice is handled by Interpolating Polynomial approach. In this paper in order to deal with multi-choice parameters Newton's forward interpolating polynomials has been used however other interpolating polynomial approaches such as Lagrange interpolating polynomial, Newtons divided difference interpolating polynomial can be used for same purpose. Finally using weighting method, a single objective problem is developed. Depending on different situation, other fuzzy random variables namely fuzzy log-normal random variable, fuzzy Cauchy random variable, fuzzy exponential random variable, etc. can be taken. However, other multi-objective techniques can be used, namely: goal programming, Fuzzy Programming method, ϵ-constraint method, etc.

Table 3 Solution for different values of w_1 and w_2

α	w_1	w_2	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	k_1	k_2	Z	Z_1	Z_2
0.5	0.1	0.9	0.0	4.0	0.9575901	5	0.0	5.042410	0.0	0.0	82.00424	73.04241	83.00
0.5	0.2	0.8	0.0	4.0	0.9575901	5	0.0	5.042410	0.0	0.0	81.00848	73.04241	83.00
0.5	0.3	0.7	0.0	4.0	0.9575901	5	0.0	5.042410	0.0	0.0	80.01272	73.04241	83.00
0.5	0.4	0.6	0.0	4.0	0.9575901	5	0.0	5.042410	0.0	0.0	79.01696	73.04241	83.00
0.5	0.5	0.5	0.0	4.0	0.9575901	5	0.0	5.042410	0.0	0.0	78.02120	73.04241	83.00
0.5	0.6	0.4	0.0	4.0	0.9575901	5	0.0	5.042410	0.0	0.0	77.02545	73.04241	83.00
0.5	0.7	0.3	0.0	0.0	4.9575900	5	4.0	1.042410	0.0	0.0	41.23817	29.04241	151.00
0.5	0.8	0.2	0.0	0.0	4.9575900	5	4.0	1.042410	0.0	0.0	41.23817	29.04241	151.00
0.5	0.9	0.1	0.0	0.0	4.9575900	5	4.0	1.042410	0.0	0.0	41.23817	29.04241	151.00

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Crop Insurance in India: A Mathematical Review

Vidyottama Jain and S. Dharmaraja

1 Introduction

Agriculture contributes to 17% of the India's Gross Domestic Product (GDP) and any modification has an extensive impact on the Indian economy as a whole. Agriculture production and farm incomes in India are majorly suffered by natural disasters such as droughts, foods, cyclones, storms, landslides and earthquakes. It is also frequently affected by man-made disasters such as fire, sale of spurious seeds, fertilizers and price crashes etc. It is obvious that all these circumstances are beyond the control of the farmers.

Farmers, being typically risk-averse, attempt to stabilize their incomes by resorting to a variety of measures, e.g., diversification, inter-cropping, and share-cropping arrangements. In recent times, governments of developing countries, concerned with the welfare of the farming community, have either encouraged or directly participated in setting up of risk-sharing institutions. Crop insurance is a basic tool by which farmers can stabilize their financial gain through farms and their investment against fatal results of losses due to natural hazards or low market prices. There are numerous crop insurance products that farmers could use to reduce such risks.

Researchers and policymakers have a huge interest in the area of management of such risks in agriculture through the different insurance plans. Though, it has been considered in theory that crop insurance is an efficient risk-sharing mechanism, but practically crop insurance has been a costly way to transfer the risk from farmers to governments and/or other insurers.

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In this paper, detailed study over crop insurance is arranged into three sections for convenience of analysis. Section 1 is introduction followed by a brief report over all crop insurance schemes implemented by the Government of India till date in Sect. 2. Section 3 represents the literature over mathematical techniques for crop insurance all over the world. Further, observations over crop insurance in India are presented in Sect. 4. Section 5 contains conclusion with the proposed future work.

2 Crop Insurance in India: Earlier Attempts and Schemes

In India, the agricultural sector occupies a vital position in the overall economy of the country. Consequently, growth of Indian economy is inextricably connected to Indian agricultural growth and vice versa. Therefore, policy makers in India have also been concerned about the risk and uncertainty involved in agriculture.

For the economic stability of the country, various crop insurance schemes have been implemented in India since independence. In 1947, in the Central Legislature, Rajendra Prasad, Minister of Food and Agriculture, gave an assurance that the government would examine the feasibility of introducing crop and cattle insurance in the country. To pursue this matter, one special officer studied this issue and formulated experimental schemes of crop and cattle insurance for operation in selected areas. He proposed two pilot schemes which were not accepted by any of the states. Later, Government of Punjab submitted a proposal to introduce crop insurance as a part of the Third Five-Year Plan of the state and requested the government of India for financial assistance. Since, under the Constitution, the central legislature alone is competent to enact necessary legislation for the purpose, the government of India decided in October 1965 to have a Crop Insurance Bill and a Model Scheme of Crop Insurance formulated so that the states, if needed, can introduce crop insurance in their areas.

After getting the state governments' view over the Draft Bill and the Model Scheme, the government of India referred the same to an expert committee for its full examination with respect to its economic, administrative, financial and actuarial implications. In accordance, a committee headed by the late Dharam Narain was set up to examine the pros and cons of the scheme. After a thorough scrutiny, the Dharam Narain Committee recommended against the introduction of the crop insurance scheme in the country.

In 1976, Dandekar [1], objecting to this recommendation, pointed out that many of the difficulties visualized by the Dharam Narain Committee could be overcome by adopting the 'area approach' to crop insurance. The very first crop insurance scheme, started by the government of India in 1985, was the Comprehensive Crop Insurance Scheme (CCIS) [2]. It was suggested by Prof. Dandekar and was based on area approach.

Though the scheme was based on the area approach, it had a difference with Dandekar's suggestions in some crucial features. It covered all natural risks excluding nuclear and war risks. This scheme kept uniform the premium and the indemnity rate for notified crop without considering their actual yield. When the average output of a given area was below the normal output, Indemnities were paid to all insured farmers.

CCIS coverage was limited to only some particular crops and also this scheme was intended only for the loanee farmers. Therefore, a new crop insurance scheme, Rashtriya Krishi Bima Yojana (RKBY) under the National Agricultural Insurance Scheme (NAIS), was launched by Prime Minister A. B. Vajpayee on June 23, 1999. This scheme was the first ever scheme for farmers for providing comprehensive risk insurance against loss or damage to crops.

This scheme is still in existence and covers all risks (e.g., re and lightning, storm, cyclone, food, drought, and pests) except war, nuclear risks, and malicious damage. One of the salient features of CCIS is that this scheme is available to both loanee and non-loanee farmers. Also, it is designed on both 'area approach', for widespread calamities, and 'individual approach' for localized calamities. The premium in the case of for the small and marginal farmers, premium is subsidized by 50% and is shared equally by the government of India and the concerned state/ union territory.

During the last 5 years, the amount of claims through NAIS is much higher than what the premium is, which indicates a magnitude of loss from the operation of the scheme. Overall, NAIS is considered to be an improvement over the CCIS, but it has simply replaced one flawed scheme with another slightly less flawed one. One of the major issue related to NAIS is reduction of insurance unit to Village Panchayat which causes difficulty to decide the insurance premium due to lack of past records of land surveys, ownerships, tenancy and yields at individual farm level. Other issues are unawareness of the farmers regarding NAIS, mandatory to loanee farmers, adverse selection in the case of non-loanee farmers, the area approach etc. Consequently, even after 23 years of existence of NAIS, less than one-fifth of the farmers are insured in the country, with only a few notable exceptions like Rajasthan, where about 50% of the farmers/holdings are insured. This scheme was improved as Modified National Agriculture Insurance Scheme (MNAIS). Further, Agriculture Insurance Company (AIC) was established to implement the available schemes with the help of state governments and the designated banks [3]. In MNAIS, loanee farmers are insured under compulsory category and non-loanee farmers are insured under voluntary category. There are some qualitative improvements in MNAIS, e.g., reduction of insurance unit size, on-account payment of claims, individual farm assessment of hailstorm and landslide losses, etc.

It was expected that MNAIS will resolve at least some of the key shortcomings of NAIS. However, it has its own issues. Since the Insurance unit for major crops is village/village panchayat, it has increased the work load required for crop cutting experiments (CCEs). This is one of the major reasons of the failure of this scheme. Due to the human interference, there is a huge pressure on the primary workers (those who conduct CCEs) to show lower yields, affect the actuarial soundness of the programmer.

Recently, Government of India launched Prime Minister's Crop Insurance Scheme. This scheme is prepared on the grounds that it will provide insurance coverage to all farmers, enhanced coverage in terms of sum insured, a good amount of subsidization, adoption of latest technologies and above all cooperation of more insurance companies [3]. However like in the past, the path is not going to be easy even for this new scheme. As this scheme has a high level of in built subsidy to be borne by Central and State Govt. It requires a full participation and commitment from the State Govt. too which looks difficult because of political and financial reasons. In addition, the success of the scheme is heavily based on its acceptance by the non-loanee farmers. The distribution setup of the insurance companies empaneled for the scheme is not up to the desired level and it requires huge effort in infrastructure building. There is a strong chance of the disagreement towards the claim settlement process for the indemnity between Govt. agencies and farmers. Some of the reasons could be corruption and lack of the technical aspects. In this scheme, relying only on the latest technology can create data error. Also, there has not been any solution provided for the division of the indemnity between the landowner and the cultivator, therefore cultivator will incur the loss and the land owner will get compensation (officially) in this scheme too. All these schemes have not really proved a significant risk mitigation tool for the farmers in many regions. Given this scenario in India with respect to crop insurance, the research article addresses the question, ‘Can we have a mathematical model that can throw some insight on schemes that can mitigate the risk for the farmers?’.

3 Mathematical Techniques for Crop Insurance

Crop insurance scheme and its mathematical models are extensively studied and analyzed by the academicians/policy makers. Studies over crop insurance schemes are focused on various issues particularly related to the failure of the performance of the crop insurance programs as required. The major reason for its failure is moral hazard, adverse selection, and systemic risks [4–9]. In this section, some mathematical model of crop insurance schemes, applied in all over the world, are considered and discussed in brief.

The significant role of crop insurance products has to indemnify adversely affected risk-averse individuals. The perspective of insurance is the distribution of the risk over a large number of individuals. For the success of the insurance program, it is required that the insurer should have adequate information about the nature of the risks being insured. It has been proven to be extremely difficult for farm-level yield insurance. To provide a sustainable insurance, an insurer should be able to properly classify risk. Farmers, who are favorably classified, will opt for the insurance. This phenomenon, known as adverse selection, initiates a cycle of losses [9, 10]. Moral hazard occurs when producers, after purchasing insurance, alter their production practices in a manner that increases their chances of collecting an indemnity [11]. Systemic risk is considered as the possibility by which an event can trigger a collapse in the agricultural industry.

Ahsan et al. [10] proposed a theoretical framework of agricultural insurance which fails if the risk-information is imperfect. It has been shown that output will be increased if the crop is insured. Hence, in less developed countries, crop insurance is beneficial to increase food supplies. For this purpose, it is concluded that agricultural insurance could be viable in case of public subsidies. Their conclusions were derived from a simplified model of production with one output and one input.

In order to combat adverse selection and moral hazard, Halcrow [12] analyzed and promoted area level yield crop insurance. In this risk-sharing mechanism between farmers and insurers, the indemnity schedule is based on the aggregate yield of a surrounding area. Later, Miranda [13] resurrected these mechanisms by examining the relative effects of area yields and individual yields on the variance of net farm yield. In this sequence, Mahul [14] presented a more general framework where the risk-averse farmer is assumed to maximize the expected utility of net yield. Mahul [14], Miranda [13] and Ramaswami [15] considered a linear additive model (LAM) which decomposes the variations in individual yield into the variations in aggregate area yield. In their proposed model, the parameter is the slope coefficient in that linear relation. Mahul [14] designed crop insurance policy optimally in which the indemnity is based upon the aggregate yield of a surrounding area. Miranda [13] proved empirically that individual-yield crop insurance offers insurance against both systemic and non-systemic risks but it has deductible. He also showed that area-yield crop insurance insures only against systemic risk and without deductible. When the deductible in the individual-yield insurance is large enough, area-yield insurance would reduce risk more effectively than individual-yield insurance [15].

Ramaswami et al. [15] have derived the LAM from aggregation of micro production functions (structural model) and derive the precise conditions under which the LAM is valid. They have shown that the linear additive model can be obtained when systemic and individual risks are additive in individual yields, and the law of large numbers holds in case of the aggregation. The additivity property of systemic and individual risks is a necessary condition. By using structural model of systemic and non-systemic risks, it has been shown that the level of aggregation affects the risk reduction of producers. In addition, they have found that dropping the large numbers restriction alone does not alter the linear relationship between individual and area yield but it results that the variations in individual crop insurance is now decomposed into two correlated risk components. As a result, it is not valid to estimate the beta parameter by ordinary least square technique of estimation.

When a linear relationship exists between farm-level yield and area-level yield, and the orthogonal component of farm-level yield is independent of area level yield, the principal result proves that the optimal area level yield crop insurance contract depends on the individual beta coefficient which measures the sensitivity of farm yield to area yield. Depending on the positive (negative) value of beta coefficient, indemnity is paid whenever the realized area level yield is lower (higher) than a critical yield. The optimal contract contains a disappearing deductible if the beta coefficient is higher than unity. An insurance model, designed by Duncan and Myers [16], discusses how catastrophic risk affects the nature and existence of crop insurance market equilibrium.

In recent years, there are numerous crop insurance options available to farmers. In addition to traditional crop insurance products, farmers have other options too, e.g., crop revenue insurance products, developed group insurance products, expanded the range of crops covered by insurance, etc. There are various other schemes such as new approaches for using crop yield histories in determining insurance premiums, implemented significant premium subsidies etc. to increase farmer participation towards crop insurance. Sherrick [17] concluded that such improvements in insurance motivated farmers to avail crop insurance product. In this sequel, Kaylen et al. [7] developed a model of farmer's decision-making under risk incorporating insurance contracts. This model can predict input–output responses to different insurance contracts. It helps to farmers to accept different contracts and the level of expected indemnities. This model identifies and evaluates the effects of different insurance policies concerning price, yield, price and yield, or revenue.

Kaylen et al. [7] considered that optimal input and output choices with risk can be defined by maximization of the expected utility of profit. The constant risk-aversion preference model $u\left(\prod\right) = -\exp^{\left(-r\prod\right)}$ is used where $u(.)$ is the utility function, denotes profit and r is considered as a risk-aversion parameter.

Consideration of constant risk-aversion model makes easy to calculate the farmer's benefit of an insurance program. This benefit can be measured as the maximum amount which the farmer would be willing to pay (WTP) for the insurance. Henceforth, it is obvious that the farmer will buy a particular insurance policy at a given premium if and only if his WTP for the insurance more than that premium value. The evaluation of his WTP can be calculated by solving the following integer programming problem:

$$\max_{x_{ij},\, A_j} \int \dots \int u\left(\prod\right) g(p, \theta) \mathrm{d}p \mathrm{d}\theta$$

subject to,

$$\begin{aligned} \sum A_j &\le \bar{A} \\ \sum_j \sum_i h_{ijk} x_{ij} + s_{jk} A_j &\le b_k, \quad (k = 1, 2, \dots, K) \\ x_{ij} &\ge 0 \\ A_j &\ge 0, \end{aligned} \tag{1}$$

where j denotes the crop, p denotes a vector of costs and denotes a vector of physical-biological variables. Also, A_j is the allocation of acreage for crop j in acres; x_{ij} is the per acre quantity of input i used in the production of crop j; $g(p, \theta)$ is the joint distribution of the random vectors p and the integration is over the domain of each random variable. The first constraint rules out purchasing or renting land, the next set of inequalities represents K technological or institutional constraints among the choice variables [7]. This model solution includes the optimal values for the choice variables: per acre input usages and the number of acres planted to the different crops. The model also returns the maximum amount the farmer would be willing to pay for the insurance policy.

On the similar lines, Liu et al. [8] introduced a mathematical framework for farmers to decide about buying crop insurance products to reduce climate and price risks. In this approach, Liu et al. [8] studied the impact of the accuracy of the El Nino-Southern Oscillation (ENSO)-phase forecasts and uncertain prices on crop insurance decisions. In their work [8], the aim is to minimize the expected loss subject to a risk-aversion constraint represented using Conditional Value-at-Risk (CVaR [18]). The number of scenarios is equal to the number of possible yields and market prices (historical data). The amount of land allotted to every planting date and the chosen crop insurance products are the decision variables [8].

The main objective is either to minimize the expected losses or to maximize the expected revenue, as the cost per crop is composed of the cost of its production, cost of the premium, and its operational cost. The total revenue is calculated by the revenue from selling of the actual yield and from the indemnity, if any received [8]. Here, Y_k^s is the total yield of crop k in scenario s; that is,

$$Y_k^s = \sum_{d_k} X_{d_k} y_{d_k}^s, \tag{2}$$

where for planting date d_k, $y_{d_k}^s$ is yield of crop k per hectare (ha) in scenario s and X_{d_k} is number of hectares of land for crop k.

Let $Z_{i,k}^s$ be the difference between the insured yield and the true yield:

$$Z_{i,k}^s = \sum_{d_k} X_{d_k} \left(y_{d_k}^* - y_{d_k}^s \right). \tag{3}$$

thus the indemnity yield is $(Z_{i,k}^s)^+ = \max(0, Z_{i,k}^s)$.

The total loss can be calculated as cost of the production plus cost of the premium minus indemnity gain minus market gain. Thus, the loss function

$$f(\tilde{x}, \tilde{\xi}) = \sum_{k=1}^{K} \left\{ C_k q_k - Y_k^s P_k^s + \sum \lambda_{i,k} \left[R_{i,k} q_k - \left(Z_{i,k}^i \right)^+ P_k^* \right] \right\}, \tag{4}$$

where C_k is the production cost of crop k per ha, P_k^s is the market price of crop k per kg at scenario s, $\lambda_{i,k}$ is the selection of insurance policy i for crop k, $\tilde{x} = \{X_{d_k}, \lambda_{i,k}\}$ is the decision vector and $\tilde{\xi} = \{Y_k^s, P_k^s\}$ is the random vector.

It is important that the expected loss exceeding VaR must be controlled and should be less than a certain threshold value v. This is modeled using CVaR as follows:

$$\mathrm{CVaR}_\alpha[f(\tilde{x}, \tilde{\xi})] \le v,$$

where $\alpha = \Pr\left[f(\tilde{x}, \tilde{\xi}) \le \mathrm{VaR} \right]$ is the confidence level and Pr[] is the probability function. In other words, CVaR is derived by taking a weighted average between the value-at-risk and losses exceeding the value-at-risk.

Therefore, the complete model formulation presented by Liu et al. [8] is as follows:

$$
\begin{aligned}
&\min\ E[f(\tilde{x},\tilde{\xi})] \\
&\text{such that} \\
&f(\tilde{x},\tilde{\xi}) = \sum_{k=1}^{K} \left\{ C_k q_k - Y_k^s P_k^s + \sum \lambda_{i,k} \left[R_{i,k} q_k - \left(Z_{i,k}^i\right)^+ P_k^* \right] \right\} \\
&Y_k^s = \sum_{d_k} X_{d_k} y_{d_k}^s \\
&Z_{i,k}^s = \sum_{d_k} X_{d_k} \left(y_{d_k}^* - y_{d_k}^s \right), \\
&\sum_{d_k} X_{d_k} = q_k \quad \text{and} \quad X_{d_k} \geq 0 \quad \text{for} \quad k = 1, 2, \ldots, K, \\
&\sum_i \lambda_{i,k} = 1 \quad \text{for} \quad k = 1, 2 \ldots, K, \qquad (6)
\end{aligned}
$$

where $\lambda_{i,k}$ are binary numbers, and CVaRα $[f(\tilde{x},\tilde{\xi})] \leq v$.

Therefore, it can be observed that there are two additional constraints associated with the decision variables in this model. First, Liu et al. [8] considered that the farm could grow q_k ha of crop k. Because every crop had different planting dates, the total available area is the sum of the area allocated to these planting dates. Also, they assumed that the farmer will buy one type of insurance policy for one crop. Binary variables $\lambda_{i,k}$ are used to represent this condition

$$\sum_i \lambda_{i,k} = 1 \quad \text{for} \quad k = 1, 2 \ldots, K.$$

Wang [19] analyzed the behavior of insured farmer by using Von-Neuman Morgenstern Utility Model. This model also investigated the factors affecting their behavior and tried to reveal the effective ways to increase the insurance rate.

4 Crop Insurance in India: Review and Observations

As of now, the crop insurance schemes in India have lots of issues related to its operation, governance and financial sustainability. Though different schemes are launched at times in India, yet agriculture insurance has served very limited purpose. To obtain the suitable mathematical model for crop insurance in Indian scenario, various research articles in this context are presented.

Venkatesh [20] proposed the weather insurance as a recovery to the loss of crop. It is accepted in countries like US, UK, and Canada. Further, Pal and Mondal [21] studied the different approaches and current challenges for the crop insurance in India. Aiming at the steady financial gain of the farmers, they advocated peril-indexed insurance and options as a risk management technique. In India, ICICI Lombard pioneered this insurance as a weather risk mitigation tool. To favor this concept, it [22] recommended use of indexed-based contracts, e.g., rain fall contracts. It simple means that if the rainfall in that area would be below a pre fixed level, farmers would be compensated with varying levels of payment depending upon the level of rainfall.

Raju and Chand [2, 23] presented the problems and prospects of agriculture insurance in the country in a working paper of National Centre for Agricultural Economics and Policy Research (Indian Council of Agricultural Research). They also empirically investigated the perceptions of the farmers in Andhra Pradesh regarding the Agricultural insurance. While working over the same, their findings were that the insured farmers mentioned crop insurance as the biggest financial security for their crop. In addition, they mentioned that farmers wanted fast and smooth settlement of their claims. At the same time, they found that the non-loanee farmers are not aware of such insurance.

Vyas [24] presented a overview of NAIS in India and suggested changes to make it more effective. On the basis of a detailed analysis of data for 11 different crop seasons. They investigated fields and had discussions with knowledgeable and experienced persons from the different government bodies and with the farmer representatives. Mahajan and Bobade [25] studied about the growth of NAIS and examined the important features and performance of NAIS. As per their findings, the major issue with the failure of NAIS is lack of awareness of farmers about it. Also, it has a major shortcoming in the view of insurer that the claims are higher than the received premium.

Kalavakondaa and Mahul [26] critically inspected the role of the crop insurance scheme in Karnataka. They proposed a structure to design a crop insurance scheme. They also presented some new concepts to improve the existing crop insurance scheme and to explore alternatives to the current product, based on an area-yield approach.

Nair [27] suggested that to build an efficient crop insurance mechanism, both multi-peril yield-based NAIS and weather-based insurance must be taken into consideration together. Later, Nair [27] focused on the recent developments over the weather insurance products and evaluated the performance of the weather-based crop insurance scheme in India in his research article.

Recently, Soni and Trivedi [28] presented a clear understanding about the existing scenario of crop insurance in India by considering Gujarat as a special reference. This study empirically investigated the level of farmer's awareness for crop insurance products in Anand district. Further, they examined the perception of insured and non-insured farmers.

Hence, it may be concluded that majority of the Indian studies were taken up at macro level in the context of crop insurance. Most of these were conceptual and theoretical. The main priority of these studies was to focus over the expansion and the growth of farmer's crop insurance with different government insurance schemes. There are a very few studies which are empirically presenting the perception of the farmers towards crop insurance. Hence, authors have made an attempt to address the mathematical models for better penetration of crop insurance.

5 Conclusion and Future Work

In developing countries like India, people earn their livelihood from agricultural sector as compared to any other economic sectors altogether. It is observed that agriculture production and farm incomes in India are frequently affected by natural disasters such as droughts, foods, cyclones, storms, landslides, and earthquakes. Agricultural products are vulnerable to such disasters compounded by the outbreak of epidemics and man-made disasters such as resale of spurious seeds, fertilizers and pesticides, price crashes, etc.

It is very unfortunate that crop insurance in India has not made much head-way despite the requirement of protecting farmers from agriculture variability has been a seamless concern of agriculture policy. Through this literature review, authors have observed the following issues:

Mathematical model of a crop insurance scheme covering all/certain specified risks.
Mathematical study of the factors affecting farmer's behavior on participating in crop insurance
The design of crop insurance scheme is based on data available, area level/ individual level approach and type of crops etc. However, in Indian scenario, modeling of a workable crop insurance scheme is difficult due to the lack of previous years' yield data, small size of farmers' land, low value of the crops and the high premium of the crop insurance.
It is observed that the second issue is more based on farmer's perception. Therefore, behavioral game theory/fuzzy set theory could be a useful armamentarium in this quest.

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Adoption of e-Government Services: A Case Study on e-Filing System of Income Tax Department of India

Harjit Singh, Arpan Kumar Kar and P. Vigneswara Ilavarasan

1 Introduction

The aim of e-Government is to provide better services for citizens by taking use of Information Communication Technology and to evaluate the success of an e-Service, its adoption is a standard criterion. Governments worldwide are developing multiple capabilities to deliver services to the global Citizens. With the intelligent use of ICT and especially the Internet, Indian governments have the unique opportunity to take advantage of the indisputable advantages that these technologies can offer, in order to achieve better and more functional government. ICT have a valuable potential to help Indian central and state governments deliver good governance to their constituents.

Acceptance is termed as agreeing to receive something new or the act of receiving it whereas adoption is an act of taking something new or different on as your own and process of beginning to use that. Technology adoption can be an appropriate and effective usage of technology. Technology adoption is important because it is the vehicle that allows most people to participate in a rapidly changing world where technology has become central to our lives. Factors responsible for and influencing adoption of e-Services could be different for developing countries than developed countries, as the priorities and ecosystem for execution is entirely different. Strategy and approach for diffusion of e-Services to maximum of the citizens with minimum resources in developing countries like India would also be different than the practise in developed countries. However, the governments for the developing countries can retrofit and build on the learnings of the countries who have already succeeded in e-Government implementations.

This paper consists of seven sections. First section provides the background, definitions and context of different terms used in this paper. Second section covers the

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findings from the literature reviewed for this study with focus on the e-Government adoption relationship with established IS adoption models with focus on studies in the taxation domain worldwide. Third section tells about the research gaps, objective and scope of this study based on the literature reviewed. Fourth section gives information about the research methodology used and the scope of the information gathered in this study. Fifth section covered the details of the case study of the e-Filing system of the Income Tax Department of India. Sixth section provides the findings and recommendations including limitation of this research paper the future research based on the findings of this case study. The last section contains the list to the research and studies referenced in this paper.

2 Literature Review

e-Government research touches many other research domains, such as Information Systems (IS), Public Administration, Management and Political Science. In literature review, it is observed that in every e-Government study or research some of the literature cited is definitely from Information Systems or e-Business concepts modified in some way to fit into a public administration perspective. Literature and ideas representation in e-Government research from economics and sociology is not much, despite the fact that these domains have contributed significantly to informatics and governance research [18].

2.1 e-Government to e-Services

In literature many different definitions of e-Government are available [25]. The definitions by different agencies/organizations/sources may differ among them, however, there is a commonality across these definitions as "an electronic Government contains, using of information and communication technology, especially the Internet, to improve the delivery of government services to its citizens, businesses, and other government agencies". E-Government enables the citizens/users to interact with and/or receive services from the central government, state or local governments on 24×7 bases. E-governance is generally considered as a wider concept than e-government. E-governance [25] is the use of ICT by government for improving information and service delivery to citizen and encouraging them to participate in the decision-making process in making government more accountable, transparent and effective. E-governance is generally considered as a wider concept than e-government, since it can bring about a change in the way citizens relate to governments and to each other. E-Service is an umbrella term for services being provided on the web like application hosted by services providers on web or e-commerce transactions services for handling online orders. There are different definitions of e-Services in literature. However, most reflects three components [26], service

Table 1 Overview of the prominent models of adoption of information systems

Model	Study	Key constructs
Technology acceptance model (TAM)	Davis [8]	Perceived usefulness (PU), perceived ease of use (PEU), attitude towards use (A), behavioral intention of use (BI) and actual use (usage)
TAM2	Venkatesh and Davis [36]	Extension of TAM, by including cognitive instrumental processes and social influence processes and excluding attitude towards use (A)
Theory of reasoned action (TRA)	Fishbein and Ajzen [14]	Behavioral beliefs, attitude towards behavior, normative beliefs, subjective norm, behavioral intention and actual behavior
Theory of planned behavior (TPB)	Ajzen [1]	Extension of TRA, by including a relationship of control beliefs, perceived behavioral control and behavioral intention
Unified theory of acceptance and use of technology (UTAUT)	Venkatesh et al. [37]	Performance expectancy (PE), effort expectancy (EE), social influence (SI) and facilitating conditions (FC)
Diffusion of innovations (DOI) theory	Rogers [28]	Relative advantage, compatibility, complexity, 'triability' and 'observability'

provider—government agency, service receiver—citizens and businesses and the channel of service delivery—Internet. The e-Filing system [12] is not about simply making it possible to transmit information electronically. It is about a change in the basic infrastructure used by citizens and the government to deal with one another. The electronic filing (e-Filing) is the electronic submission of information that is required by law using the designated e-Filing system.

2.2 Adoption of ICT

There are a number of proven theories and models relating to Adoption of IS available in literature, which are still in some form being referred for research/studies in IS domain. The prominent models (Table 1) of adoption of IS are as follows:

- Technology Acceptance Model [36]—TAM is one of the most frequently employed models for research into new information technology acceptance. It suggests that when users are presented with a new technology, two beliefs, the

perceived usability and utility and the perceived ease of application, determine attitudes to adopt new technologies.
- The Theory of Reasoned Action [14]—TRA says, "Behavior is determined by the Behavior Intension to emit the behavior. Attitudinal factor and normative factor determine the Behavior Intension."
- The Theory of Planned Behavior [1]—TPB says, "Human behavior is guided by beliefs about the likely consequences (Behavioral beliefs), beliefs about the normative expectations of other people (Normative beliefs) and beliefs about the presence of the factors that may impact the performance of the behavior (Control beliefs)."
- The Diffusion of Innovations (DOI) theory [28]—explain how an innovation diffuses through a society, it has five components as, relative advantage, compatibility, complexity, "triability" and "observability".
- The Unified Theory of Acceptance and Use of Technology [37]—UTAUT is an integration of predictability capabilities from different existing models of technology acceptance. It says, "Performance expectancy, effort expectancy, and social influence predict behavioral intention towards the acceptance of information technology."

In the existing body of knowledge [3] it was observed that theories and models generally cited are the theory in the research area of acceptance and adoption of technology. Most of these theories focus at a time on any of the individual theory as TAM, TRA, TPB, or UTAUT. The variables of success in IS in adoption are System Quality, Information Quality, Use, User Satisfaction, Individual Impact, and Organizational Impact [10] and inclusion of Service Quality as an additional aspect of IS success [11] assigning different weights to System Quality, Information Quality, and Service Quality depending on the context and scope of the project. Perceived ease of use was an important determinant factor of perceived usefulness [35]. Experience and voluntariness are the controlling factors of subjective norm [36] and external variables affect perceived usefulness. However, UTAUT, which touches impact of intervention, has four key constructs (i.e., performance expectancy, effort expectancy, social influence, and facilitating conditions) that influence behavioral intention to use a technology and/or technology use [37]. Most of the studies on technology adoption used modified versions of the TAM [33] rather than original model and the results were influenced by other variables that were introduced when using any of the modified versions of the TAM models. Interventions impact both determinants of technology adoption, perceived ease of use and perceived usefulness [34].

2.3 e-Government and Adoption

The successful implementation of e-Governments not only depends on the strong support and commitment from the governments, but also depends on whether the citizens would like to accept and adopt the e-Government services [2]. ICT investments are helpful [22] in improving the performance and productivity (throughput). The acceptance and adoption by user and ensuring the optimum use of ICT systems is a challenge in any organizations. In an e-Government initiative, [27] we need to understand and represent the relationships among different stakeholder of the initiative. The development and integration of a strong back-office to support is an essential factor in achieving the best of the frontend portal of an e-Government initiative. A weak or no backend support leads to failure of any e-Government initiative [15, 21, 32]. While planning an e-Government initiative, we must have a change management package as built-in [16]. When the roadmap/blueprint is being drawn for any e-Government initiative in any organization, there are no readymade and common or “one size fits all” e-Governance solutions available [17]. Every country, at each level of government within that country, has a distinctive combination of conditions, priorities and resources. The role of intermediaries [39] particularly for developing countries is vital for success of e-Government initiatives, as they implement their own infrastructure to bridge the digital divide and technology gap. There are also significant differences while measuring the “use of e-Government services” when accessed through intermediaries or there is a direct online access, by the citizens. It is empirically validated [38] in the perspective of UTAUT that performance expectancy, effort expectancy, and social influence have a significant positive influence on behavioral intention to use information kiosks and intermediaries. These kiosks operators and intermediaries are partners of the Government to support e-Services. We should not consider the e-Government a one-step process or a single project [19], instead, we should conceptualize it as evolutionary and progressive, which involves multiple stages or phases of execution. We find that clear vision and goal definition, with excellent leadership is essential for adoption of e-Government initiative. The measures to evaluate the success [29] of an e-Government project are the adoption of that e-Service.

The challenges and problems associated with the implementation and adoption of ICT systems have led scholars and practitioners to seek to understand and manage the processes and phenomena related to the field [22]. It was found that trust, financial security, and information quality are adoption barriers, whilst time and money are potential adoption benefits. For successful e-Government it requires the engagement of all stakeholders, and it is expected that development of strategies should align stakeholder interests, so that participation in e-Government for all stakeholder can be self-governing. It was found in a study [7] that most of the e-Government initiatives in developing countries abandoned very soon after implementation or major Goals were not attained. This is major problem, as developing countries have a limited number of resources at their disposal and cannot afford to waste large amount of money on such projects.

2.4 e-Governance in India

The National e-Governance Plan (NeGP) was launched by government of India on May 16, 2006 with focus on building the right governance and institutional mechanisms, setting up of the core infrastructure and framing policies along with the implementation of a large number of e-Governance projects called Mission Mode Projects at the levels of Union government, state governments as well as integrated service levels to develop a citizen-centric and business-centric environment for governance. The NeGP [9] list now comprises of 31 mission mode projects (MMPs). One of the important provisions of the NeGP is to encourage Public Private Partnership into various key projects. The focus of e-Government in India is to make all Government services accessible to the common citizen and use it as a tool to further economic development and good governance.

In Indian government sector, there is a strategic shift being observed [4], Instead of acquiring hardware and software, the governments in India (central and state) now have started buying ICT services. Most of these ICT procurements are as Managed Services projects under Public Private Partnership (PPP) model, which allows the governments to focus on critical value-adding business and activities (which only government can do as best way) and transferring the technology-related requirements to IT professionals (IT partner), most of the time a private organization, taking advantage of the matured Indian IT services industry. Successful implementation of PPP is not easy to implement in e-Governance sector in India. It requires [30], the adoption and intelligent use of key best practices and lessons learnt (from mistakes) from earlier PPP implementation experiences in e-Government and this is especially useful for developing countries, where there is already a scarcity of resources in terms of finances and technical skills.

2.5 e-Governance and Taxation

In order to reach taxpayers more effectively [31], the organization leveraged on its relationship with trade groups, bodies, NGOs and corporates to educate taxpayers on tax matters. Also, different type of incentives given to eFilers and providing easy internet access to taxpayers (kiosks and community clubs) helped in promoting e-Filing of tax returns. The automation of taxation department [23] led to improved efficiency by saving time and cost. Savings from automation is resulting from greater productivity of staff. The error and fraud detections significantly improve by automation. “E-Tax” is intended to radically improve tax administration efficiency [5] in both back-office tax record management and front-line tax consultation. The impact of e-Government requires mobilizing internal resources particularly the people to implement e-Government initiatives. Both compatibility (COMP) and PIIT strongly [24] influence BI. The direct significant effect of PEOU on BI supports a previous research finding that PEOU directly affects BI. Limitation of the study is the participants. However, this study used a convenience sample of graduating engineering and management students, who in reality have never filed any Income Tax return (e-filed or manual). Based on the analysis [6] of the factors surrounding e-Government adoption issues and identifying which ones should take priority, the findings of previous studies advocating that government website managers must work towards increasing and improving citizen awareness and adoption and usage rates were confirmed.

3 Research Gap, Contribution, and Scope

There are research citations available on Information Systems field on the adoption and acceptance of technology. Most of these researches and studies are in generic e-Governance perspectives than the specific ones and considering cases from developed countries. There is hardly any study to focus on adoption of use of e-Services of Government from developing countries, specifically from India. The finding of this study will contribute to the knowledge body in this gap. It focuses on efficiency and effectiveness of e-Services system. The finding of this study will be helpful in providing the direction to the planners of these services in similar context.

The objective of this study is to explicate the best practices and lessons learnt in the context of the technology adoption models, with a specific focus on use of the electronic services in India. Specifically, how the adoption of e-Services can be promoted through different stages of adoption is of greater focus. By exploring a qualitative case study which provides a greater overview of the stages descriptively, we attempt to explore greater breadth of incidents instead of focusing exclusively on validation of the findings or existing models through empirical evidences.

Scope of this exploratory research was to conduct a focused study work guided by availability of resources and time on a citizen focused e-Government initiative. Government of India has done huge investment for implementation of ICT systems (e-Government projects), which are impacting the citizens of India. It is significant to study the e-Government systems with a point of view of efficiency, effectiveness and adoption of these systems by users. As scope, we covered the case study e-Services on citizen portals of Income Tax Department (e-Filing System). Income Tax Department has initiated one of the major e-Governance initiatives of India. Under this initiative, the department offers e-Services through web-based, single-window, 24×7 Managed Services models. E-Filing system of the department is flagship e-Services initiative for the citizens.

4 Research Methodology

Case Study based qualitative research methods have been used in this research. It was decided that we collect information from those stakeholders who are close to the day-to-day activities of the systems. The chances are that we get information from important stakeholders while discussing the ICT systems, which the person (government official) may otherwise not be comfortable in sharing, especially through email or otherwise. The interview is also a fine method with which to identify incidents that are critical for (dis)satisfaction with the service.

Triangulation approach was adopted for information and findings, by using different data-sources and data types. We collected information from multiple sources (Interviews, System database, reports/documents available in public domains and project documents) with aim to corroborate the same finding. Multiple sources of evidences essentially provide multiple measures of the same phenomenon. We covered interviews with personal worked or working on key positions the project, to collect the data from both views, Service Provider and Sponsor (civil servants). When these stakeholders were interviewed, the focus was on their specific knowledge for the system. Interviews were semi-structured. We prepared a list of questions for each type of stakeholder interview but also on case-to-case bases deviated from the format to track interesting issues that come up during the conversation and shifted focus to other potentially relevant topics that were not previously included in the interview scheme. In second part, we did a focused discussion with different interest groups and intermediaries (Law firms, Chartered Accountants and Third party e-Filing utility providers) on their expectations and experience for the system. In parallel, we collected data from documents available in public domain and project documents (Functional Requirement and System Design documents) and from system databases extracted the relevant information of portal usage on different parameters, including portal data of stakeholders' grievance and feedback. Based on these as inputs, we did the analysis using critical factor analysis and human factor analysis. The outcome of this analysis becomes bases for findings and recommendations of this study.

5 Case Study: e-Filing System of Income Tax Department

The task force constituted by Government of India [13] on taxation recommended that tax departments should concentrate on its core functions with focus on assessment and enforcement duties, rather than logistics and support services. To achieve this, one the recommendation stressed on the use of information technology to provide better services to taxpayers and enhance tax efficiency. The approval was also given to the Scheme of e-filing of Income Tax Returns. This study covered in-depth analysis of the progress across all phases of implementation of e-Filing system of Income Tax department.

5.1 Initial Phase

As per the recommendations of the task force, in the budget speech in Feb 2006, Finance Minister announced e-Filing of Income Tax returns effective Assessment Year (AY) 2006–07. The time given to department was not enough to implement a comprehensive e-Filing system. So, similar to the concept/model generally being adopted by different Government departments in India for execution of ICT systems, Income Tax Department also followed the same approach of owning and managing the system under a multi-vendor execution model with department handling the management activities and specialist individual vendors for networks of the department (LANs and WANs), main hardware (Servers) had been supplied by a multinational hardware vendor and Application was developed by an Indian software services company and was being maintaining on annual basses by the same. Project Management and day-to-day system operations were being handled by technical staff of the department.

Department launched its website with URL as www.incometaxindiaefiling.gov.in for e-Filing of Income Tax Returns. The implementation of this system was full of challenges for all stakeholders, especially for the department. The taxpayers were not ready for this system and process. A feedback based study for done to find the reasons for reluctance of users to optimally use this system. The reasons were prioritized and actions were taken within the constraints of the system. No training or awareness was imparted to citizen to use the system. Usability (ease of use) of the system was considered as poor. Reliability and availability of system was also an issue. These are the important attributes for potential adopters to be motivated to connect to the system for usage [20]. However, actions to improve on these issues were taken by the department and within short time the system stabilized. The confidence of all stakeholders increased on this system. The total number of returns filed electronically increased from around 0.36 million in FY 2006–07 to around 5.7 million in FY 2009–10, which was more than the capacity for the implemented system. This was further expected to grow. The reason for this is considered as the usage of online systems increased tremendously due to widespread awareness of ICT and penetration of Internet, as well as the faster processing of electronically filed ITRs with respect to processing of paper returns (the returns filed in the jurisdictional offices of the department). This is because of inbuilt benefits of e-Filing and dedicated facility at CPC Bangalore for electronically filed IT Returns only in bulk mode. As an incentive, citizens who filed ITR electronically started receiving their tax refunds faster, hassle-free, including in most of the cases, directly into their bank accounts.

5.2 Transition Phase

Although every vendor was expert in its own area, however there are many other constraints/challenges in this type of execution model especially the coordination and communication among different vendor partners. Many times very small problem or issue caught into blame game among them. The problem gets worsen if the organization internally is not competent to handle technology. Many system environment components (like servers, network equipment and storage) had been declared obsolete by respective Original Equipment Manufacturers and needed a large investment for the system. The department decided to scrap the existing e-Filing system all together and development of fresh new ICT system with new thoughts and with long term benefits in consideration including more freedom/flexibility in terms of technology progress and adoption, functional scope and focusing on the lessons learnt from execution of the previous system.

5.3 Transformational Phase

Department finally took a conscious call to go for the new afresh e-Filing system with an innovative concept of partnership with IT service provider (Managed Services model). The new e-Filing System has been made live on November 9, 2012 with many new facilities to taxpayers and extension of electronic filing of other statutory forms and reports (other than ITR forms) as prescribed in the Income Tax Rules. This portal is a single-window interface to different category of users like Individual taxpayers, Business, Corporate, Tax Professionals, Electronic Return Intermarries (ERIs), internal departmental users and External Agencies (Banks and other departments) for information through role bases scoured access control system. There is a dedicated Call Centre service for all stakeholders of this system. Provision is there for online feedback and grievance from users and subsequently after analysis, tracking to closure. A few functionalize have been added and some are modified, based on the analysis of the online feedback received through portal. With new system in place and realizing the potential of e-Filing, department wants that most of the taxpayers who file paper tax returns to be encouraged to shift to e-Filing of IT returns. This e-Filing has an interface with both the Central Processing Centre (CPC) which is a dedicated facility of the department for a jurisdiction free bulk processing for Income Tax Returns (ITRs) and with the Core ITD Application system for information flow for Assessing Officers and other department officials. There is service level agreement with all service providers to ensure the timely completion of end-to-end activities. The information of the different status of the ITRs at CPC flows back to e-Filing system for showing the same status information to the taxpayer. Due to these interfaces and flow of information across systems, PAN validation facility and view of Tax Credits (26AS statement) are available online on new e-Filing System. The intimation of the final result of ITR processing is also sent through email as well as through SMS to taxpayer. Figure 1 shows the process of e-Filing the Income Tax Return by taxpayer.

For the FY 2014–15, more than 34 million Tax Returns were files as e-Returns. More than 43 million are registered users of this system (Retrieved from: https://incometaxindiaefiling.gov.in/). Department also experienced a sudden increase in

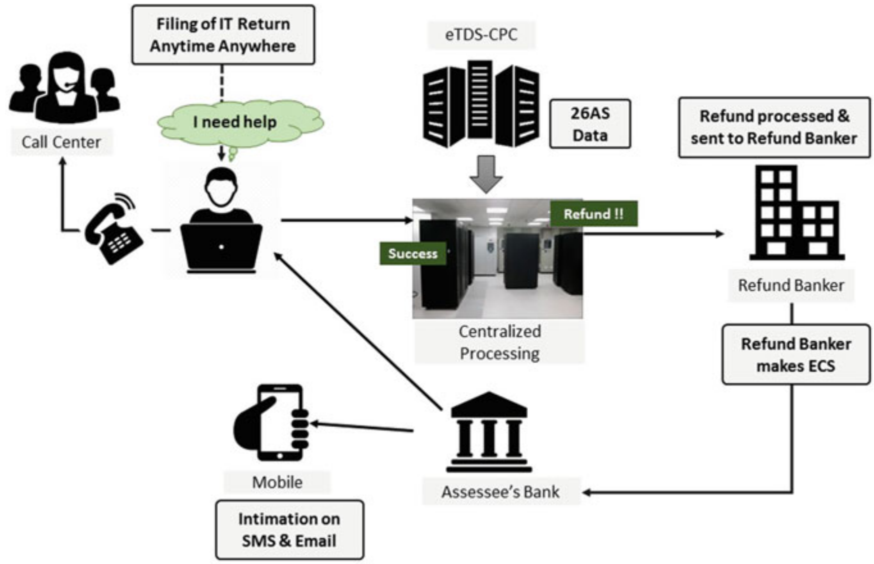

Fig. 1 Experience of taxpayers for e-Filing of tax return

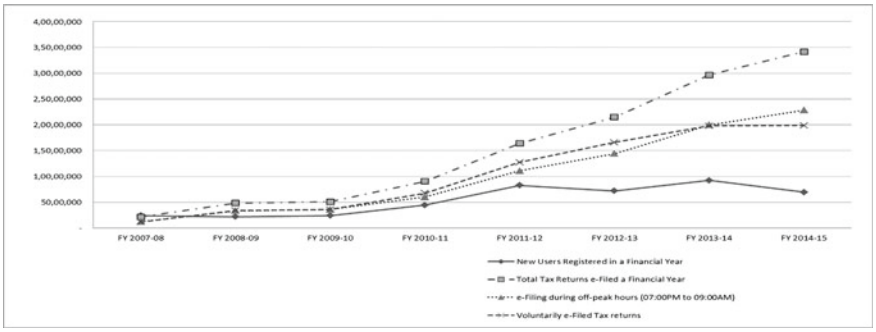

Fig. 2 Growth of e-Filing of income tax returns (*Source* Departmental Reports)

volumes of voluntary filing of Income Tax Returns electronically. Figure 2 shows the summary of growth trends of different activities of e-Filing system of Income Tax Department. Attitudinal factors as usefulness of the system and image enhancements of the user become important for continued usage of the ICT system for users [20]. To ensure that the users take maximum benefit of this system, there has been a further push by the department to focus on awareness and education of taxpayers, steadily inclusion of more taxpayers' categories as compulsory e-Filing of Income Tax Returns, involvement of Intermediaries and Tax Return Preparers (TRPs) to encourage the e-Filing of ITRs and include more business services of the department to e-Services through this system.

Since beginning, the ITR-V process established by the department has a big issue for both, the taxpayer and the department. Effective AY 2009–10, the ITR-V is to be sent by post to Central Process Centre (CPC) of the department in Bangalore. Many times the return filer (e-Filed return) claims, that the ITR-V was sent well within due date (120 days of filing the return on System). The facts in the department tells different story, either ITR-V has not been received or has been received after the due date. Even after this new system, this issue has not been tackled. Now, with advance in the technology and implementation in the e-Filing with best available technology, department implemented certain innovative solutions like One Time Electronic Signatures and "Aadhaar" based authentication and in these cases, the taxpayer is not required to send of signed ITR-V to department (Bengaluru). This further helped in faster processing of tax returns and hence faster refunds reaching faster to the taxpayers' accounts.

After the success and acceptability of this System, coupled with growing expectations of the citizens it has become a single window of interaction with department for all external stakeholders, including the taxpayers who are filing their ITRs in paper format, are also being given many services through this System. Income Tax Department (ITD) has recently introduced a new initiative as "e-Sahyog", in line with the government's commitment under Digital India to work in an e-Environment and to e-Enable public services for the benefit of the citizens. This has been done with a view to reduce compliance cost, to increase transparency and to reduce the need for the taxpayer to physically appear before tax authorities. With focus now to increase the further spread of e-Services, the department has already started the development and will launch a facility to interact with e-Filing system through different mobile platforms.

6 Discussion and Conclusion

The e-Government projects always require a lot of investment of resources, so the failure of the e-Government projects means great amount of loss on the citizens' money. The e-Government adoption model can identify the main influencing factors affecting adoption, which can help the successful implementation of e-Government projects. The analysis of these models shows that the two constructs ease of use and usefulness under different names are part of each model. Observing the adoption progress of the e-Filing system of the department and focusing on awareness by the department, taxpayers were being encouraged for opting using the e-Filing process of Income Tax Returns every year. With addition of many other e-Services to the citizens though this System, this e-Filing System of the Income Tax Department is now an e-Interaction System of the department (one stop shop—Single window). For end-to-end activities, e-Filing of Tax Return till receiving the refund in the bank account, taxpayer is not required to visit the income tax office. Most of the exceptions are also being handled through System or through emails ("e-Sahyog"). In context of developing country, awareness, affordability and accessibility, are the core factors which impact the adoption and acceptance of any e-Service or Digital Service, irrespective of the maturity stage or level. Important factors have clearly emerged during the study, which explain the significance of the roles of intermediaries, kiosks, back-office, other interfacing ICT systems (upstream ICT systems) and internal users of the organization for success of an e-Service. For organization like Income Tax Department of India, transparency could be a factor for adoption of e-Services by the taxpayers. However, each of these need to be further studied and can be other paths to further researches.

Although the initial adoption of usage of the system is important, the sustained use of the system by the same users is more important. The increase in efficiency, especially the response of the system to its users, led to further participation and hence to adoption of e-Service. There is clear evidence of the importance of citizens' participation for improving the performances of a policy-making process in adoption. For citizens, service maturity levels and trust in government are important factor for adoption of an e-Service. For e-Service, it is important to know the reasons why citizens are hesitant or resistant to use e-Services and evaluates different actions governments could take to increase e-Services usage. It requires the engagement of all stakeholders from an early stage of the project, and that a prerequisite to that engagement is a shared understanding of the interests, perspectives, value dimensions, and benefits sought from e-Government by the various stakeholders. Being citizen-centric systems, the reliability and availability has to be given priority in design this system. Social networking and other trends in the internet are also to be exploited for training and awareness of citizens. The criteria to judge the success of an e-Government project is the adoption. This is also on the same lines what is mentioned in UTAUT. The implementation of these e-Services should be in sustained manner with effectiveness (adoption) of each stage being measures before shifting to next stage of maturity of e-Services. However, plan of sustained release of e-Services should be different across identified countries based on the setup and conditions. For further spread of e-Services, provide the e-Services also through different mobile platforms. This study also confirmed that Managed Services model through Public Private Partnership (PPP) for e-Governance projects in India is successful. The corresponding author is a practitioner and in past more than one decade, he is closely associated with ICT implementation for Government of India. The observations from his experience up to an extent have also matches the findings of this study. Considering the penetration and diffusion of mobile technology in India, it is observed that these e-Services are being made available on mobile platforms (m-Services). Citizens prospective as a user of this System, preferably using quantitative technique would be further research step.

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