On a Hyperconvex Manifold Without Non-constant Bounded Holomorphic Functions

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Abstract An example is given of a hyperconvex manifold without non-constant bounded holomorphic functions, which is realized as a domain with real-analytic Levi-flat boundary in a projective surface.

Keywords Hyperconvexity · Liouville property · Levi-flat · Grauert tube
Ergodicity

1 Introduction

In geometric complex analysis, hyperbolicity and parabolicity of non-compact complex manifolds are key properties governing behavior of holomorphic functions. Stoll [24] introduced the notion of *parabolic manifold* to investigate value distribution of holomorphic functions in several variables. We recall this notion using the formulation of Aytuna and Sadullaev [6]:

Definition 1.1 A complex manifold X is said to be *parabolic* if X does not admit non-constant bounded plurisubharmonic function. We say that X is *S-parabolic* if it possesses a plurisubharmonic exhaustion φ that satisfies the homogeneous complex Monge–Ampère equation $(i\partial\overline{\partial}\varphi)^n = 0$ on $X \setminus K$ for some compact subset $K \subset X$.

S-parabolic manifolds are parabolic, and their model case is $\mathbb{C}^n$ equipped with the exhaustion $\log \|z\|$. We refer the reader to Aytuna and Sadullaev [6] for the detail.

On the other hand, it would also be of interest to investigate non-compact complex manifolds that are not parabolic in the sense above but enjoy some weaker parabolicity. Myrberg [21] gave such an example in one dimensional setting, namely, an open Riemann surface of infinite genus that has smooth boundary component, hence, not

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parabolic, but on which all the bounded holomorphic functions are constant. This celebrated example was the driving force toward the classification theory of Riemann surfaces (cf. Heins [14]).

In several complex variables, this sort of intermediate parabolicity actually appears too. See Aytuna and Sadullaev [6] for an example of unbounded pseudoconvex domain in $\mathbb{C}^n$ containing countably many copies of $\mathbb{C}$ and having plurisubharmonic defining function but no bounded holomorphic function except for constant functions. The purpose of this article is to remark another kind of example of non-parabolic Stein manifold without non-constant bounded holomorphic function, which the author hopes to be useful for further study.

Theorem *There exists a hyperconvex manifold that does not possess any non-constant bounded holomorphic function and is realized as a domain with real-analytic Levi-flat boundary.*

Here hyperconvexity is defined as

Definition 1.2 A complex manifold X is said to be *hyperconvex* if it admits strictly plurisubharmonic bounded exhaustion.

Recall that a function on a topological space X, $\varphi\colon X \to [-\infty, c)$, is said to be *bounded exhaustion* if all the sublevel sets $\{x \in X \mid \varphi(x) < b\}$, $b < c$, are relatively compact in X. For example, any C^2-smoothly bounded pseudoconvex domains in Stein manifolds is hyperconvex (Diederich and Fornæss [9]. Later the required smoothness was relaxed to C^1 by Kerzman and Rosay [17], then to Lipschitz boundary by Demailly [7]). Clearly, a hyperconvex manifold is not parabolic, but Theorem states that it can satisfy the Liouville property.

Now we explain the construction of the manifold claimed in Theorem. Let Σ be a compact Riemann surface of genus ≥ 2 and fix its uniformization $\Sigma = \mathbb{D}/\Gamma$ by a Fuchsian group Γ acting on the unit disk $\mathbb{D}$. We make Γ act on the bidisk $\mathbb{D} \times \mathbb{D}$ diagonally but with conjugated complex structure for second factor, namely, for each $\gamma \in \Gamma$ and $(z, w) \in \mathbb{D} \times \mathbb{D}$, we let

$$\gamma \cdot (z, w) := (\gamma z, \overline{\gamma \overline{w}}).$$

We shall show that the quotient space $X := \mathbb{D} \times \mathbb{D}/\Gamma$ enjoys the desired property.

This example has two origins. One is the work by Diederich and Ohsawa [10], where holomorphic $\mathbb{D}$-bundles over compact Kähler manifolds are shown to be weakly 1-complete. Such a holomorphic $\mathbb{D}$-bundle is canonically embedded in the associated holomorphic $\mathbb{CP}^1$-bundle as a pseudoconvex domain with real-analytic Levi-flat boundary. In our case, the first and the second projection endow X structures of $\mathbb{D}$-bundle over Σ and $\overline{\Sigma}$, the quotient of $\mathbb{D}$ by the conjugated action of Γ, respectively. Hence, X has two realization as domains in ruled surfaces $Y := \mathbb{D} \times \mathbb{CP}^1/\Gamma$ and $Y' := \mathbb{CP}^1 \times \mathbb{D}/\Gamma$, where the action of Γ is the same as above thanks to the fact $\mathrm{Aut}(\mathbb{D}) \subset \mathrm{Aut}(\mathbb{CP}^1)$. The Levi-flat boundaries of X in Y and Y' are denoted by $M = \mathbb{D} \times \partial\mathbb{D}/\Gamma$ and $M' = \partial\mathbb{D} \times \mathbb{D}/\Gamma$ respectively. In summary, we have two natural ways to realize X in larger complex manifolds Y and Y' and the real-analytic boundaries M and M' are inequivalent CR manifolds in general (Mitsumatsu [20]). For further background on $\mathbb{D}$-bundles, we refer the reader to a recent study by Deng and Fornæss [8].

Another origin is the Grauert tube of maximal radius in the sense of Guillemin and Stenzel [13] and Lempert and Szőke [19]. Since the conjugated diagonal set $\{(z, \overline{z}) \mid z \in \mathbb{D}\} \subset \mathbb{D} \times \mathbb{D}$ is preserved under the action of Γ, its quotient S is totally-real submanifold of real dimension two and isomorphic to Σ as real-analytic manifold. Namely, X is a complexification of Σ. Not only that, we can find a plurisubharmonic *bounded* exhaustion that satisfies the homogeneous complex Monge–Ampère equation on $X \setminus S$.

In Sect. 2, we first confirm that our X coincides with the Grauert tube of Σ, then show the hyperconvexity of X. In Sect. 3, after explaining that the Liouville property of X is actually a corollary of Hopf's ergodicity theorem, we shall give another proof for the Liouville property using the plurisubharmonic bounded exhaustion. In Sect. 4, some open questions are posed.

2 Grauert Tube and Its Hyperconvexity

First we recall the notion of Grauert tube in the sense of Guillemin–Stenzel and Lempert–Szőke.

Fact 2.1 (Guillemin and Stenzel [13], Lempert and Szőke [19]) *Let (M, g) be a compact real-analytic Riemannian manifold of dimension n. Denote by $\rho\colon TM \to \mathbb{R}_{\geq 0}$ the length function, and we identify M with the zero section of TM. Then, there exists $R \in (0, \infty]$ and unique complex structure on $X := \{v \in TM \mid \rho(v) < R\}$ such that*

(1) ρ enjoys the homogeneous complex Monge–Ampère equation $(i\partial\overline{\partial}\rho)^n = 0$ on $X \setminus M$;
(2) ρ^2 is strictly plurisubharmonic on X;
(3) $i\partial\overline{\partial}(\rho^2)$ agrees with g on TM.

This X above is called *the Grauert tube of M of radius R*. Since our Σ is endowed with the hyperbolic metric of constant Gaussian curvature -1, whose fundamental form is

$$g(z) = \frac{2i dz \wedge d\overline{z}}{(1 - |z|^2)^2},$$

Lempert and Szőke [19, Theorem 4.3] yields an upper bound of the radius R of the Grauert tube of Σ, $R \leq \pi/2$.

Proposition 2.2 *The complex manifold X defined in Sect. 1 is biholomorphic to the Grauert tube of Σ of radius $\pi/2$, which is maximum possible, whose length function agrees with*

$$\rho(z,w) := \arccos\sqrt{\delta} \quad \textit{where} \quad \delta(z,w) := 1 - \left|\frac{w-\overline{z}}{1-zw}\right|^2.$$

Proof First note that $\delta\colon \mathbb{D}\times\mathbb{D} \to (0,1]$ is invariant under the action of Γ and induces a real-analytic function on X. Hence, $\rho\colon X \to [0,\pi/2)$ is well-defined bounded exhaustion and $\rho^{-1}(0) = S = \{(z,\overline{z}) \mid z \in \mathbb{D}\}/\Gamma$, which we identified with Σ. Moreover, ρ^2 is C^∞-smooth function on X since

$$\rho(z,w) = \arcsin\left|\frac{w-\overline{z}}{1-zw}\right|.$$

In view of Lempert and Szőke [19, Theorem 3.1], it suffices to confirm that ρ satisfies the three conditions in Fact 2.1. From direct computation, we have

$$\begin{aligned} i\partial\overline{\partial}(-\log\delta) &= \frac{idz\wedge d\overline{z}}{(1-|z|^2)^2} + \frac{idw\wedge d\overline{w}}{(1-|w|^2)^2},\\ \frac{i\partial(-\log\delta)\wedge\overline{\partial}(-\log\delta)}{1-\delta} &= \frac{idz\wedge d\overline{z}}{(1-|z|^2)^2} + \frac{idw\wedge d\overline{w}}{(1-|w|^2)^2} + \frac{i\varepsilon dz\wedge d\overline{w} + i\overline{\varepsilon}dw\wedge d\overline{z}}{(1-|z|^2)(1-|w|^2)} \end{aligned}$$

on $X\backslash S$, where $\varepsilon = -(w-\overline{z})(\overline{w}-z)^{-1}$. Hence, it follows that

$$\begin{aligned} \overline{\partial}\rho &= \frac{1}{2}\sqrt{\frac{\delta}{1-\delta}}\,\overline{\partial}(-\log\delta),\\ i\partial\overline{\partial}\rho &= \frac{1}{2}\sqrt{\frac{\delta}{1-\delta}}\left(i\partial\overline{\partial}(-\log\delta) - \frac{1}{2}\frac{i\partial(-\log\delta)\wedge\overline{\partial}(-\log\delta)}{1-\delta}\right)\\ &= \frac{1}{4}\sqrt{\frac{\delta}{1-\delta}}\left(\frac{idz\wedge d\overline{z}}{(1-|z|^2)^2} + \frac{idw\wedge d\overline{w}}{(1-|w|^2)^2} - \frac{i\varepsilon dz\wedge d\overline{w} + i\overline{\varepsilon}dw\wedge d\overline{z}}{(1-|z|^2)(1-|w|^2)}\right), \end{aligned}$$

and it is now clear that $(i\partial\overline{\partial}\rho)^2 = 0$ on $X\backslash S$. To check remaining two points, we compute on $X\backslash S$

$$\begin{aligned} i\partial\overline{\partial}(\rho^2) &= 2(\rho i\partial\overline{\partial}\rho + i\partial\rho\wedge\overline{\partial}\rho)\\ &= \frac{1}{2}\left(\rho\sqrt{\frac{\delta}{1-\delta}} + \delta\right)\left(\frac{idz\wedge d\overline{z}}{(1-|z|^2)^2} + \frac{idw\wedge d\overline{w}}{(1-|w|^2)^2}\right)\\ &\quad + \frac{1}{2}\left(-\rho\sqrt{\frac{\delta}{1-\delta}} + \delta\right)\frac{i\varepsilon dz\wedge d\overline{w} + i\overline{\varepsilon}dw\wedge d\overline{z}}{(1-|z|^2)(1-|w|^2)}. \end{aligned}$$

It follows that $i\partial\overline{\partial}(\rho^2) > 0$ on X, and g agrees with the restriction of

$$i\partial\overline{\partial}(\rho^2) = \frac{idz\wedge d\overline{z}}{(1-|z|^2)^2} + \frac{idw\wedge d\overline{w}}{(1-|w|^2)^2} = \frac{idz\wedge d\overline{z} + dw\wedge d\overline{w}}{(1-|z|^2)^2}$$

on S as Riemannian metric. The proof is completed. □

Remark 2.3 Kan [16] gave another realization of the Grauert tube of Σ extending the construction of Lempert [18].

Next we shall confirm that our X is hyperconvex.

Proposition 2.4 *The function $-\sqrt{\delta}$ is strictly plurisubharmonic bounded exhaustion on X. Hence, X is hyperconvex.*

Proof From the computation in the proof of Proposition 2.2, we have

$$\begin{aligned}\frac{i\partial\overline{\partial}(-\sqrt{\delta})}{\sqrt{\delta}/2} &= i\partial\overline{\partial}(-\log\delta) - \frac{1}{2}i\partial(-\log\delta)\wedge\overline{\partial}(-\log\delta)\\ &= \frac{1+\delta}{2}\left(\frac{i dz\wedge d\overline{z}}{(1-|z|^2)^2}+\frac{i dw\wedge d\overline{w}}{(1-|w|^2)^2}\right)+\frac{1-\delta}{2}\frac{i\varepsilon dz\wedge d\overline{w}+i\overline{\varepsilon}dw\wedge d\overline{z}}{(1-|z|^2)(1-|w|^2)}\end{aligned}$$

and this is positive definite everywhere on X. □

Remark 2.5 We may extend δ smoothly on a neighborhood of X in Y and also a neighborhood in Y' and regard $-\delta$ as a defining function of X in Y and X in Y'. Proposition 2.4 shows, by its definition, that $-\delta$ has the Diederich–Fornæss exponent $1/2$, which is the maximum possible value for relatively compact domains with Levi-flat boundary in complex surfaces (Fu and Shaw [11] and Adachi and Brinkschulte [2]. See also Demailly [7, Théorème 6.2]).

3 Proofs of the Liouville Property

Let us observe that the Liouville property of X is actually a corollary of Hopf's ergodicity theorem ([15]. See also Tsuji [26], Garnett [12] and Sullivan [25]).

Fact 3.1 (Hopf [15]) *Let $\Sigma = \mathbb{D}/\Gamma$ be a Riemann surface of finite hyperbolic area. Then, the diagonal action of Γ on $\partial\mathbb{D}\times\partial\mathbb{D}$ is ergodic with respect to its Lebesgue measure. Namely, for any Lebesgue measurable subset $E\subset\partial\mathbb{D}\times\partial\mathbb{D}$ invariant under the diagonal action of Γ has Lebesgue measure zero or full Lebesgue measure.*

We use the following Fatou type theorem.

Fact 3.2 (cf. Tsuji [26, Theorem IV.13]) *Let f be a bounded holomorphic function on $\mathbb{D}\times\mathbb{D}$. Then, there exists a measurable function $\tilde{f}\colon\partial\mathbb{D}\times\partial\mathbb{D}\to\mathbb{C}$ such that for almost all $(z_0, w_0)\in\partial\mathbb{D}\times\partial\mathbb{D}$,*

$$\lim_{(z,w)\to(z_0,w_0)} f(z,w) = \tilde{f}(z_0,w_0)$$

where z and w approach to z_0 and w_0 non-tangentially respectively. Moreover, f is a constant function if $\tilde{f}$ is constant on a subset of positive measure.

Theorem 3.3 *Any bounded holomorphic function on X is constant.*

Proof (*First proof of Theorem* 3.3) Let f be a bounded holomorphic function on $X = \mathbb{D} \times \mathbb{D}/\Gamma$. From Fact 3.2, f as a function on $\mathbb{D} \times \mathbb{D}$ has boundary value $\tilde{f}$ on $\partial\mathbb{D} \times \partial\mathbb{D}$ which is invariant under the action of Γ. Then, the function $(z, w) \mapsto \tilde{f}(z, \overline{w})$ on $\partial\mathbb{D} \times \partial\mathbb{D}$ is invariant under the diagonal action of Γ. Fact 3.1 implies that $\tilde{f}$ is constant almost everywhere, and we conclude by Fact 3.2. □

We shall give another proof, which does not rely on Fact 3.1 and explains how the bounded exhaustion ρ controls the growth of holomorphic functions on X.

Proof (*Second proof of Theorem* 3.3) Let f be a bounded holomorphic function on X. We shall show without using Fact 3.1 that the boundary value function $\tilde{f}$ on $\partial\mathbb{D} \times \partial\mathbb{D}$ is constant almost everywhere. Then the rest of the proof is the same as in the first proof.

We apply the integration formula used in Adachi and Brinkschulte [3] with the maximal plurisubharmonic function ρ on $X \backslash S$ used in Proposition 2.2. Namely, we integrate

$$i\partial\overline{\partial}|f|^2 \wedge d\rho \wedge d^c\rho + |f|^2(i\partial\overline{\partial}\rho)^2 = d(d^c|f|^2 \wedge i\partial\rho \wedge \overline{\partial}\rho + |f|^2 d^c\rho \wedge i\partial\overline{\partial}\rho)$$

on $\rho^{-1}(a, b)$, where our convention is $d^c := (\partial - \overline{\partial})/2i$. Since all the level sets $\rho^{-1}(c)$, $c \in (0, \pi/2]$, are smooth, for any $a, b \in (0, \pi/2)$, $a < b$, we have

$$\int_{\rho^{-1}(a,b)} i\partial\overline{\partial}|f|^2 \wedge d\rho \wedge d^c\rho = \int_{\rho^{-1}(b)} |f|^2 d^c\rho \wedge i\partial\overline{\partial}\rho - \int_{\rho^{-1}(a)} |f|^2 d^c\rho \wedge i\partial\overline{\partial}\rho.$$

Denoting by M_t the boundary of $\{x \in X \mid \rho(x) < t\} = \{x \in X \mid \delta(x) > \cos^2 t\}$ and rewriting in δ instead of ρ yield

$$\begin{aligned}\int_{\delta^{-1}(\beta,\alpha)} i\partial\overline{\partial}|f|^2 \wedge \frac{d\delta \wedge d^c\delta}{\delta(1-\delta)} &= \frac{1}{\sin^2 b} \int_{M_b} |f|^2 d^c(-\delta) \wedge i\partial\overline{\partial}(-\log\delta) \\ &\quad - \frac{1}{\sin^2 a} \int_{M_a} |f|^2 d^c(-\delta) \wedge i\partial\overline{\partial}(-\log\delta)\end{aligned} \tag{1}$$

where $\alpha := \cos^2 a$ and $\beta := \cos^2 b$.

Now we look at behavior of terms in Eq. (1) when $b \nearrow \pi/2$, that is, $\beta \searrow 0$. For its RHS, we compute the first term using a smooth trivialization

$$\iota_t \colon R \times \partial\mathbb{D} \to M_t, \quad (z, e^{i\theta}) \mapsto \left(z, \frac{(\sin t)e^{i\theta} + \overline{z}}{1 + z(\sin t)e^{i\theta}}\right)$$

for $t \in (0, \pi/2]$ where R is a fundamental domain of the action of Γ on $\mathbb{D}$. It follows that

$$
\begin{aligned}
&\frac{\beta}{\sin^2 b}\int_{M_b}|f|^2 d^c(-\log\delta)\wedge i\partial\overline{\partial}(-\log\delta)\\
&\quad=\frac{\beta}{\sin^2 b}\int_{M_b}|f|^2\left(\frac{idz\wedge d\overline{z}\wedge\frac{1}{2i}\left(\frac{\overline{w}-z}{1-zw}dw-\frac{w-\overline{z}}{1-\overline{z}\overline{w}}d\overline{w}\right)}{(1-|z|^2)^2(1-|w|^2)}\right.\\
&\qquad\left.+\frac{idw\wedge d\overline{w}\wedge\frac{1}{2i}\left(\frac{\overline{z}-w}{1-zw}dz-\frac{z-\overline{w}}{1-\overline{z}\overline{w}}d\overline{z}\right)}{(1-|w|^2)^2(1-|z|^2)}\right)\\
&\quad=\frac{1}{\sin^2 b}\int_{R\times\partial\mathbb{D}}|\iota_b^* f|^2\frac{idz\wedge d\overline{z}\wedge 2(\sin^2 b)d\theta}{(1-|z|^2)^2}\leq 4\pi^2\sup_X|f|^2(2g-2)<\infty
\end{aligned}
$$

where g is the genus of Σ. Therefore, the LHS should be finite; on the other hand,

$$
\int_{\delta^{-1}(\beta,\alpha)} i\partial\overline{\partial}|f|^2\wedge\frac{d\delta\wedge d^c\delta}{\delta(1-\delta)}=\int_\beta^\alpha\frac{d\tau}{\tau(1-\tau)}\int_{M_{\arccos\sqrt{\tau}}} i\partial f\wedge\overline{\partial f}\wedge d^c(-\delta),
$$

and the integrability requires

$$
\lim_{t\nearrow\pi/2}\int_{M_s} i\partial f\wedge\overline{\partial f}\wedge d^c(-\delta)=0
$$

as we will see below that this limit exists.

We can compute this limit in two ways. Note that

$$
\left|\frac{\partial f}{\partial z}\right|\leq\frac{\sup|f|}{1-|z|},\ \left|\frac{\partial f}{\partial w}\right|\leq\frac{\sup|f|}{1-|w|}
$$

On $\mathbb{D}\times\mathbb{D}$ from Cauchy's estimate, and, thanks to Fatou's theorem, we obtain the boundary value functions of f_z and f_w on $\mathbb{D}\times\partial\mathbb{D}$ and $\partial\mathbb{D}\times\mathbb{D}$ respectively, which are CR functions. By abuse of notation, we express the boundary value functions by the same symbols. Using the trivialization ι_t of M_t, the bounded convergence theorem yields

$$
\begin{aligned}
0&=\lim_{t\nearrow\pi/2}\int_{M_t} i\partial f\wedge\overline{\partial f}\wedge d^c(-\delta)\\
&=\lim_{t\nearrow\pi/2}\int_{R\times\partial\mathbb{D}}\iota_t^*\left(i\partial f\wedge\overline{\partial f}\wedge d^c(-\delta)\right)\\
&=\int_{R\times\partial\mathbb{D}}\iota_{\pi/2}^*\left(i\partial f\wedge\overline{\partial f}\wedge d^c(-\delta)\right)\\
&=\int_M\left|\frac{\partial f}{\partial z}\right|^2 idz\wedge d\overline{z}\wedge\frac{1-|z|^2}{|1-ze^{i\varphi}|^2}d\varphi
\end{aligned}
\tag{2}
$$

where we used the coordinate $(z,e^{i\varphi})\in\mathbb{D}\times\partial\mathbb{D}$ for $\iota_{\pi/2}(R\times\partial\mathbb{D})\subset M=\mathbb{D}\times\partial\mathbb{D}/\Gamma$. Using another trivialization κ_t of M_t,

$$\kappa_t \colon \partial\mathbb{D} \times R' \to M_t, \quad (e^{i\theta'}, w) \mapsto \left(\frac{(\sin t)e^{i\theta'} + \overline{w}}{1 + w(\sin t)e^{i\theta'}}, w \right)$$

for $t \in (0, \pi/2]$ where R' is a fundamental domain of the conjugated action of Γ on $\mathbb{D}$, we similarly have

$$0 = \int_{M'} \left| \frac{\partial f}{\partial w} \right|^2 i dw \wedge d\overline{w} \wedge \frac{1 - |w|^2}{|1 - we^{i\varphi'}|^2} d\varphi' \tag{3}$$

where we used the coordinate $(e^{i\varphi'}, w) \in \partial\mathbb{D} \times \mathbb{D}$ for $\kappa'_{\pi/2}(\partial\mathbb{D} \times R') \subset M' = \partial\mathbb{D} \times \mathbb{D}/\Gamma$.

Equations (2) and (3) imply that the boundary value functions $f(z, e^{i\varphi})$ and $f(e^{i\varphi'}, w)$ are constant functions in z and w for almost all $e^{i\varphi}$ and $e^{i\varphi'} \in \partial\mathbb{D}$ since these functions are holomorphic in z and w respectively. Now it follows that $\tilde{f}(z, w) = f(e^{i\varphi'}, e^{i\varphi}) \colon \partial\mathbb{D} \times \partial\mathbb{D} \to \mathbb{C}$ agrees with a constant function almost everywhere, and we finish this proof. □

Remark 3.4 The integration formula used in the proof is equivalent to Demailly's Lelong–Jensen formula [7]. Exploiting this formula, a notion of Hardy space for hyperconvex domains in $\mathbb{C}^n$, *Poletsky–Stessin Hardy spaces*, was introduced in Alan [4] and Poletsky and Stessin [23] independently (cf. Alan and Göğüş [5]). The proof above actually shows the triviality of L^2 Hardy space of $X \subset Y, Y'$.

Remark 3.5 Yet another proof for the Liouville property which does not employ Fact 3.2 can be obtained by a method similar to [1], which will be discussed in the author's forthcoming article. As in [1], we may show that all the weighted Bergman space of order > -1 of $X \subset Y, Y'$ is infinite dimensional in spite of the fact that its L^2 Hardy space is trivial.

4 Open Problems

We shall pose two open problems for further study.

Problem 1 Do other Grauert tubes of finite maximal radius give similar example of hyperconvex manifolds without non-constant bounded holomorphic function?

Problem 2 Is there any domain with Levi-flat boundary having positive Diederich–Fornæss index and non-constant bounded holomorphic function?

Problem 2 is a variant of an open problem raised by Sidney Frankel (cf. Ohsawa [22]), to classify Levi-flat hypersurfaces that bound domains with non-constant bounded holomorphic functions.

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