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Geometric Complex Analysis

In Honor of Kang-Tae Kim's 60th Birthday, Gyeongju, Korea, 2017

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Professor Kang-Tae Kim

Preface

The 12th Korean Conference on Several Complex Variables (the KSCV12 Symposium) was held at Gyeongju in Korea during the week of July 3–7, 2017. This event was organized on the occasion of Kang-Tae Kim's 60th birthday. Kang-Tae has contributed to the geometric theory in several complex variables (SCV) and, more importantly, to the community of SCV.

Since the International Conference on Several Complex Variables held on June 23–27, 1997, at Pohang University of Science and Technology, the series of the KSCV Symposium has been held to provide a forum for the researchers to discuss and share newest advances in complex analysis and geometry. The sixth and tenth symposia (KSCV6, KSCV10) were held in 2002 and 2014 as satellite conferences to the Beijing ICM and the Seoul ICM, respectively. During the past 12 symposia, many world-renowned scholars in the field have participated and shown their great progresses, and simultaneously, many young researchers including this volume editors have been educated and encouraged. This success of the symposia was clearly due to Kang-Tae's generous efforts for more than 20 years. We are delighted to dedicate this volume to Kang-Tae Kim.

The volume collects research papers and survey articles of participants of the KSCV12 Symposium and also the KSCV11 Symposium (July 4–8, 2016, Gyeongju in Korea). Editors would like to express their sincere gratitude to all participants, especially contributing authors.

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April 2018


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THE 11TH KOREAN CONFERENCE ON SEVERAL COMPLEX VARIABLES
July 4-8, 2016
The Kolon Hotel in Gyeong-Ju, Korea
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The 12th Korean Conference on Several Complex Variables
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Contents

On a Hyperconvex Manifold Without Non-constant Bounded Holomorphic Functions

Masanori Adachi

Abstract An example is given of a hyperconvex manifold without non-constant bounded holomorphic functions, which is realized as a domain with real-analytic Levi-flat boundary in a projective surface.

Keywords Hyperconvexity · Liouville property · Levi-flat · Grauert tube
Ergodicity

1 Introduction

In geometric complex analysis, hyperbolicity and parabolicity of non-compact complex manifolds are key properties governing behavior of holomorphic functions. Stoll [24] introduced the notion of *parabolic manifold* to investigate value distribution of holomorphic functions in several variables. We recall this notion using the formulation of Aytuna and Sadullaev [6]:

Definition 1.1 A complex manifold X is said to be *parabolic* if X does not admit non-constant bounded plurisubharmonic function. We say that X is *S-parabolic* if it possesses a plurisubharmonic exhaustion φ that satisfies the homogeneous complex Monge–Ampère equation $(i\partial\overline{\partial}\varphi)^n = 0$ on $X\setminus K$ for some compact subset $K \subset X$.

S-parabolic manifolds are parabolic, and their model case is $\mathbb{C}^n$ equipped with the exhaustion $\log \|z\|$. We refer the reader to Aytuna and Sadullaev [6] for the detail.

On the other hand, it would also be of interest to investigate non-compact complex manifolds that are not parabolic in the sense above but enjoy some weaker parabolicity. Myrberg [21] gave such an example in one dimensional setting, namely, an open Riemann surface of infinite genus that has smooth boundary component, hence, not

Dedicated to Professor Kang-Tae Kim on the occasion of his 60th birthday.

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parabolic, but on which all the bounded holomorphic functions are constant. This celebrated example was the driving force toward the classification theory of Riemann surfaces (cf. Heins [14]).

In several complex variables, this sort of intermediate parabolicity actually appears too. See Aytuna and Sadullaev [6] for an example of unbounded pseudoconvex domain in $\mathbb{C}^n$ containing countably many copies of $\mathbb{C}$ and having plurisubharmonic defining function but no bounded holomorphic function except for constant functions. The purpose of this article is to remark another kind of example of non-parabolic Stein manifold without non-constant bounded holomorphic function, which the author hopes to be useful for further study.

Theorem *There exists a hyperconvex manifold that does not possess any non-constant bounded holomorphic function and is realized as a domain with real-analytic Levi-flat boundary.*

Here hyperconvexity is defined as

Definition 1.2 A complex manifold X is said to be *hyperconvex* if it admits strictly plurisubharmonic bounded exhaustion.

Recall that a function on a topological space X, $\varphi\colon X \to [-\infty, c)$, is said to be *bounded exhaustion* if all the sublevel sets $\{x \in X \mid \varphi(x) < b\}$, $b < c$, are relatively compact in X. For example, any C^2-smoothly bounded pseudoconvex domains in Stein manifolds is hyperconvex (Diederich and Fornæss [9]. Later the required smoothness was relaxed to C^1 by Kerzman and Rosay [17], then to Lipschitz boundary by Demailly [7]). Clearly, a hyperconvex manifold is not parabolic, but Theorem states that it can satisfy the Liouville property.

Now we explain the construction of the manifold claimed in Theorem. Let Σ be a compact Riemann surface of genus ≥ 2 and fix its uniformization $\Sigma = \mathbb{D}/\Gamma$ by a Fuchsian group Γ acting on the unit disk $\mathbb{D}$. We make Γ act on the bidisk $\mathbb{D} \times \mathbb{D}$ diagonally but with conjugated complex structure for second factor, namely, for each $\gamma \in \Gamma$ and $(z, w) \in \mathbb{D} \times \mathbb{D}$, we let

$$\gamma \cdot (z, w) := (\gamma z, \overline{\gamma \overline{w}}).$$

We shall show that the quotient space $X := \mathbb{D} \times \mathbb{D}/\Gamma$ enjoys the desired property.

This example has two origins. One is the work by Diederich and Ohsawa [10], where holomorphic $\mathbb{D}$-bundles over compact Kähler manifolds are shown to be weakly 1-complete. Such a holomorphic $\mathbb{D}$-bundle is canonically embedded in the associated holomorphic $\mathbb{CP}^1$-bundle as a pseudoconvex domain with real-analytic Levi-flat boundary. In our case, the first and the second projection endow X structures of $\mathbb{D}$-bundle over Σ and $\overline{\Sigma}$, the quotient of $\mathbb{D}$ by the conjugated action of Γ, respectively. Hence, X has two realization as domains in ruled surfaces $Y := \mathbb{D} \times \mathbb{CP}^1/\Gamma$ and $Y' := \mathbb{CP}^1 \times \mathbb{D}/\Gamma$, where the action of Γ is the same as above thanks to the fact $\mathrm{Aut}(\mathbb{D}) \subset \mathrm{Aut}(\mathbb{CP}^1)$. The Levi-flat boundaries of X in Y and Y' are denoted by $M = \mathbb{D} \times \partial\mathbb{D}/\Gamma$ and $M' = \partial\mathbb{D} \times \mathbb{D}/\Gamma$ respectively. In summary, we have two natural ways to realize X in larger complex manifolds Y and Y' and the real-analytic boundaries M and M' are inequivalent CR manifolds in general (Mitsumatsu [20]). For further background on $\mathbb{D}$-bundles, we refer the reader to a recent study by Deng and Fornæss [8].

Another origin is the Grauert tube of maximal radius in the sense of Guillemin and Stenzel [13] and Lempert and Szőke [19]. Since the conjugated diagonal set $\{(z, \overline{z}) \mid z \in \mathbb{D}\} \subset \mathbb{D} \times \mathbb{D}$ is preserved under the action of Γ, its quotient S is totally-real submanifold of real dimension two and isomorphic to Σ as real-analytic manifold. Namely, X is a complexification of Σ. Not only that, we can find a plurisubharmonic *bounded* exhaustion that satisfies the homogeneous complex Monge–Ampère equation on $X \backslash S$.

In Sect. 2, we first confirm that our X coincides with the Grauert tube of Σ, then show the hyperconvexity of X. In Sect. 3, after explaining that the Liouville property of X is actually a corollary of Hopf's ergodicity theorem, we shall give another proof for the Liouville property using the plurisubharmonic bounded exhaustion. In Sect. 4, some open questions are posed.

2 Grauert Tube and Its Hyperconvexity

First we recall the notion of Grauert tube in the sense of Guillemin–Stenzel and Lempert–Szőke.

Fact 2.1 (Guillemin and Stenzel [13], Lempert and Szőke [19]) *Let (M, g) be a compact real-analytic Riemannian manifold of dimension n. Denote by $\rho\colon TM \to \mathbb{R}_{\geq 0}$ the length function, and we identify M with the zero section of TM. Then, there exists $R \in (0, \infty]$ and unique complex structure on $X := \{v \in TM \mid \rho(v) < R\}$ such that*

(1) ρ enjoys the homogeneous complex Monge–Ampère equation $(i\partial\overline{\partial}\rho)^n = 0$ on $X \backslash M$;
(2) ρ^2 is strictly plurisubharmonic on X;
(3) $i\partial\overline{\partial}(\rho^2)$ agrees with g on TM.

This X above is called *the Grauert tube of M of radius R*. Since our Σ is endowed with the hyperbolic metric of constant Gaussian curvature -1, whose fundamental form is

$$g(z) = \frac{2i dz \wedge d\overline{z}}{(1 - |z|^2)^2},$$

Lempert and Szőke [19, Theorem 4.3] yields an upper bound of the radius R of the Grauert tube of Σ, $R \leq \pi/2$.

Proposition 2.2 *The complex manifold X defined in Sect. 1 is biholomorphic to the Grauert tube of Σ of radius $\pi/2$, which is maximum possible, whose length function agrees with*

$$\rho(z,w) := \arccos\sqrt{\delta} \quad where \quad \delta(z,w) := 1 - \left|\frac{w-\overline{z}}{1-zw}\right|^2.$$

Proof First note that $\delta\colon \mathbb{D}\times\mathbb{D}\to(0,1]$ is invariant under the action of Γ and induces a real-analytic function on X. Hence, $\rho\colon X\to[0,\pi/2)$ is well-defined bounded exhaustion and $\rho^{-1}(0)=S=\{(z,\overline{z})\mid z\in\mathbb{D}\}/\Gamma$, which we identified with Σ. Moreover, ρ^2 is C^∞-smooth function on X since

$$\rho(z,w) = \arcsin\left|\frac{w-\overline{z}}{1-zw}\right|.$$

In view of Lempert and Szőke [19, Theorem 3.1], it suffices to confirm that ρ satisfies the three conditions in Fact 2.1. From direct computation, we have

$$i\partial\overline{\partial}(-\log\delta) = \frac{idz\wedge d\overline{z}}{(1-|z|^2)^2} + \frac{idw\wedge d\overline{w}}{(1-|w|^2)^2},$$
$$\frac{i\partial(-\log\delta)\wedge\overline{\partial}(-\log\delta)}{1-\delta} = \frac{idz\wedge d\overline{z}}{(1-|z|^2)^2} + \frac{idw\wedge d\overline{w}}{(1-|w|^2)^2} + \frac{i\varepsilon dz\wedge d\overline{w} + i\overline{\varepsilon}dw\wedge d\overline{z}}{(1-|z|^2)(1-|w|^2)}$$

on $X\backslash S$, where $\varepsilon = -(w-\overline{z})(\overline{w}-z)^{-1}$. Hence, it follows that

$$\overline{\partial}\rho = \frac{1}{2}\sqrt{\frac{\delta}{1-\delta}}\,\overline{\partial}(-\log\delta),$$
$$i\partial\overline{\partial}\rho = \frac{1}{2}\sqrt{\frac{\delta}{1-\delta}}\left(i\partial\overline{\partial}(-\log\delta) - \frac{1}{2}\frac{i\partial(-\log\delta)\wedge\overline{\partial}(-\log\delta)}{1-\delta}\right)$$
$$= \frac{1}{4}\sqrt{\frac{\delta}{1-\delta}}\left(\frac{idz\wedge d\overline{z}}{(1-|z|^2)^2} + \frac{idw\wedge d\overline{w}}{(1-|w|^2)^2} - \frac{i\varepsilon dz\wedge d\overline{w} + i\overline{\varepsilon}dw\wedge d\overline{z}}{(1-|z|^2)(1-|w|^2)}\right),$$

and it is now clear that $(i\partial\overline{\partial}\rho)^2 = 0$ on $X\backslash S$. To check remaining two points, we compute on $X\backslash S$

$$i\partial\overline{\partial}(\rho^2) = 2(\rho i\partial\overline{\partial}\rho + i\partial\rho\wedge\overline{\partial}\rho)$$
$$= \frac{1}{2}\left(\rho\sqrt{\frac{\delta}{1-\delta}} + \delta\right)\left(\frac{idz\wedge d\overline{z}}{(1-|z|^2)^2} + \frac{idw\wedge d\overline{w}}{(1-|w|^2)^2}\right)$$
$$+ \frac{1}{2}\left(-\rho\sqrt{\frac{\delta}{1-\delta}} + \delta\right)\frac{i\varepsilon dz\wedge d\overline{w} + i\overline{\varepsilon}dw\wedge d\overline{z}}{(1-|z|^2)(1-|w|^2)}.$$

It follows that $i\partial\overline{\partial}(\rho^2) > 0$ on X, and g agrees with the restriction of

$$i\partial\overline{\partial}(\rho^2) = \frac{idz\wedge d\overline{z}}{(1-|z|^2)^2} + \frac{idw\wedge d\overline{w}}{(1-|w|^2)^2} = \frac{idz\wedge d\overline{z} + dw\wedge d\overline{w}}{(1-|z|^2)^2}$$

on S as Riemannian metric. The proof is completed. □

Remark 2.3 Kan [16] gave another realization of the Grauert tube of Σ extending the construction of Lempert [18].

Next we shall confirm that our X is hyperconvex.

Proposition 2.4 *The function $-\sqrt{\delta}$ is strictly plurisubharmonic bounded exhaustion on X. Hence, X is hyperconvex.*

Proof From the computation in the proof of Proposition 2.2, we have

$$\begin{aligned}\frac{i\partial\overline{\partial}(-\sqrt{\delta})}{\sqrt{\delta}/2} &= i\partial\overline{\partial}(-\log\delta) - \frac{1}{2}i\partial(-\log\delta)\wedge\overline{\partial}(-\log\delta)\\ &= \frac{1+\delta}{2}\left(\frac{idz\wedge d\overline{z}}{(1-|z|^2)^2} + \frac{idw\wedge d\overline{w}}{(1-|w|^2)^2}\right) + \frac{1-\delta}{2}\frac{i\varepsilon dz\wedge d\overline{w} + i\overline{\varepsilon}dw\wedge d\overline{z}}{(1-|z|^2)(1-|w|^2)}\end{aligned}$$

and this is positive definite everywhere on X. □

Remark 2.5 We may extend δ smoothly on a neighborhood of X in Y and also a neighborhood in Y' and regard $-\delta$ as a defining function of X in Y and X in Y'. Proposition 2.4 shows, by its definition, that $-\delta$ has the Diederich–Fornæss exponent $1/2$, which is the maximum possible value for relatively compact domains with Levi-flat boundary in complex surfaces (Fu and Shaw [11] and Adachi and Brinkschulte [2]. See also Demailly [7, Théorème 6.2]).

3 Proofs of the Liouville Property

Let us observe that the Liouville property of X is actually a corollary of Hopf's ergodicity theorem ([15]. See also Tsuji [26], Garnett [12] and Sullivan [25]).

Fact 3.1 (Hopf [15]) *Let $\Sigma = \mathbb{D}/\Gamma$ be a Riemann surface of finite hyperbolic area. Then, the diagonal action of Γ on $\partial\mathbb{D}\times\partial\mathbb{D}$ is ergodic with respect to its Lebesgue measure. Namely, for any Lebesgue measurable subset $E\subset\partial\mathbb{D}\times\partial\mathbb{D}$ invariant under the diagonal action of Γ has Lebesgue measure zero or full Lebesgue measure.*

We use the following Fatou type theorem.

Fact 3.2 (cf. Tsuji [26, Theorem IV.13]) *Let f be a bounded holomorphic function on $\mathbb{D}\times\mathbb{D}$. Then, there exists a measurable function $\tilde{f}\colon\partial\mathbb{D}\times\partial\mathbb{D}\to\mathbb{C}$ such that for almost all $(z_0, w_0)\in\partial\mathbb{D}\times\partial\mathbb{D}$,*

$$\lim_{(z,w)\to(z_0,w_0)} f(z,w) = \tilde{f}(z_0,w_0)$$

where z and w approach to z_0 and w_0 non-tangentially respectively. Moreover, f is a constant function if $\tilde{f}$ is constant on a subset of positive measure.

Theorem 3.3 *Any bounded holomorphic function on X is constant.*

Proof (*First proof of Theorem* 3.3) Let f be a bounded holomorphic function on $X = \mathbb{D} \times \mathbb{D}/\Gamma$. From Fact 3.2, f as a function on $\mathbb{D} \times \mathbb{D}$ has boundary value $\tilde{f}$ on $\partial\mathbb{D} \times \partial\mathbb{D}$ which is invariant under the action of Γ. Then, the function $(z, w) \mapsto \tilde{f}(z, \overline{w})$ on $\partial\mathbb{D} \times \partial\mathbb{D}$ is invariant under the diagonal action of Γ. Fact 3.1 implies that $\tilde{f}$ is constant almost everywhere, and we conclude by Fact 3.2. □

We shall give another proof, which does not rely on Fact 3.1 and explains how the bounded exhaustion ρ controls the growth of holomorphic functions on X.

Proof (*Second proof of Theorem* 3.3) Let f be a bounded holomorphic function on X. We shall show without using Fact 3.1 that the boundary value function $\tilde{f}$ on $\partial\mathbb{D} \times \partial\mathbb{D}$ is constant almost everywhere. Then the rest of the proof is the same as in the first proof.

We apply the integration formula used in Adachi and Brinkschulte [3] with the maximal plurisubharmonic function ρ on $X \setminus S$ used in Proposition 2.2. Namely, we integrate

$$i\partial\overline{\partial}|f|^2 \wedge d\rho \wedge d^c\rho + |f|^2(i\partial\overline{\partial}\rho)^2 = d(d^c|f|^2 \wedge i\partial\rho \wedge \overline{\partial}\rho + |f|^2 d^c\rho \wedge i\partial\overline{\partial}\rho)$$

on $\rho^{-1}(a, b)$, where our convention is $d^c := (\partial - \overline{\partial})/2i$. Since all the level sets $\rho^{-1}(c)$, $c \in (0, \pi/2]$, are smooth, for any $a, b \in (0, \pi/2)$, $a < b$, we have

$$\int_{\rho^{-1}(a,b)} i\partial\overline{\partial}|f|^2 \wedge d\rho \wedge d^c\rho = \int_{\rho^{-1}(b)} |f|^2 d^c\rho \wedge i\partial\overline{\partial}\rho - \int_{\rho^{-1}(a)} |f|^2 d^c\rho \wedge i\partial\overline{\partial}\rho.$$

Denoting by M_t the boundary of $\{x \in X \mid \rho(x) < t\} = \{x \in X \mid \delta(x) > \cos^2 t\}$ and rewriting in δ instead of ρ yield

$$\int_{\delta^{-1}(\beta,\alpha)} i\partial\overline{\partial}|f|^2 \wedge \frac{d\delta \wedge d^c\delta}{\delta(1-\delta)} = \frac{1}{\sin^2 b} \int_{M_b} |f|^2 d^c(-\delta) \wedge i\partial\overline{\partial}(-\log\delta) - \frac{1}{\sin^2 a} \int_{M_a} |f|^2 d^c(-\delta) \wedge i\partial\overline{\partial}(-\log\delta) \quad (1)$$

where $\alpha := \cos^2 a$ and $\beta := \cos^2 b$.

Now we look at behavior of terms in Eq. (1) when $b \nearrow \pi/2$, that is, $\beta \searrow 0$. For its RHS, we compute the first term using a smooth trivialization

$$\iota_t \colon R \times \partial\mathbb{D} \to M_t, \quad (z, e^{i\theta}) \mapsto \left(z, \frac{(\sin t)e^{i\theta} + \overline{z}}{1 + z(\sin t)e^{i\theta}}\right)$$

for $t \in (0, \pi/2]$ where R is a fundamental domain of the action of Γ on $\mathbb{D}$. It follows that

$$\begin{aligned}
&\frac{\beta}{\sin^2 b}\int_{M_b}|f|^2 d^c(-\log\delta)\wedge i\partial\overline{\partial}(-\log\delta)\\
&\quad=\frac{\beta}{\sin^2 b}\int_{M_b}|f|^2\left(\frac{idz\wedge d\overline{z}\wedge\frac{1}{2i}\left(\frac{\overline{w}-z}{1-zw}dw-\frac{w-\overline{z}}{1-\overline{z}\overline{w}}d\overline{w}\right)}{(1-|z|^2)^2(1-|w|^2)}\right.\\
&\qquad\left.+\frac{idw\wedge d\overline{w}\wedge\frac{1}{2i}\left(\frac{\overline{z}-w}{1-zw}dz-\frac{z-\overline{w}}{1-\overline{z}\overline{w}}d\overline{z}\right)}{(1-|w|^2)^2(1-|z|^2)}\right)\\
&\quad=\frac{1}{\sin^2 b}\int_{R\times\partial\mathbb{D}}|\iota_b^* f|^2\frac{idz\wedge d\overline{z}\wedge 2(\sin^2 b)d\theta}{(1-|z|^2)^2}\le 4\pi^2\sup_X|f|^2(2g-2)<\infty
\end{aligned}$$

where g is the genus of Σ. Therefore, the LHS should be finite; on the other hand,

$$\int_{\delta^{-1}(\beta,\alpha)} i\partial\overline{\partial}|f|^2\wedge\frac{d\delta\wedge d^c\delta}{\delta(1-\delta)}=\int_\beta^\alpha\frac{d\tau}{\tau(1-\tau)}\int_{M_{\arccos\sqrt{\tau}}} i\partial f\wedge\overline{\partial f}\wedge d^c(-\delta),$$

and the integrability requires

$$\lim_{t\nearrow\pi/2}\int_{M_s} i\partial f\wedge\overline{\partial f}\wedge d^c(-\delta)=0$$

as we will see below that this limit exists.

We can compute this limit in two ways. Note that

$$\left|\frac{\partial f}{\partial z}\right|\le\frac{\sup|f|}{1-|z|},\left|\frac{\partial f}{\partial w}\right|\le\frac{\sup|f|}{1-|w|}$$

On $\mathbb{D}\times\mathbb{D}$ from Cauchy's estimate, and, thanks to Fatou's theorem, we obtain the boundary value functions of f_z and f_w on $\mathbb{D}\times\partial\mathbb{D}$ and $\partial\mathbb{D}\times\mathbb{D}$ respectively, which are CR functions. By abuse of notation, we express the boundary value functions by the same symbols. Using the trivialization ι_t of M_t, the bounded convergence theorem yields

$$\begin{aligned}
0&=\lim_{t\nearrow\pi/2}\int_{M_t} i\partial f\wedge\overline{\partial f}\wedge d^c(-\delta) &&(2)\\
&=\lim_{t\nearrow\pi/2}\int_{R\times\partial\mathbb{D}}\iota_t^*\left(i\partial f\wedge\overline{\partial f}\wedge d^c(-\delta)\right)\\
&=\int_{R\times\partial\mathbb{D}}\iota_{\pi/2}^*\left(i\partial f\wedge\overline{\partial f}\wedge d^c(-\delta)\right)\\
&=\int_M\left|\frac{\partial f}{\partial z}\right|^2 idz\wedge d\overline{z}\wedge\frac{1-|z|^2}{|1-ze^{i\varphi}|^2}d\varphi
\end{aligned}$$

where we used the coordinate $(z,e^{i\varphi})\in\mathbb{D}\times\partial\mathbb{D}$ for $\iota_{\pi/2}(R\times\partial\mathbb{D})\subset M=\mathbb{D}\times\partial\mathbb{D}/\Gamma$. Using another trivialization κ_t of M_t,

$$\kappa_t\colon \partial\mathbb{D}\times R' \to M_t, \quad (e^{i\theta'}, w) \mapsto \left(\frac{(\sin t)e^{i\theta'}+\overline{w}}{1+w(\sin t)e^{i\theta'}}, w\right)$$

for $t \in (0, \pi/2]$ where R' is a fundamental domain of the conjugated action of Γ on $\mathbb{D}$, we similarly have

$$0 = \int_{M'} \left|\frac{\partial f}{\partial w}\right|^2 i dw \wedge d\overline{w} \wedge \frac{1-|w|^2}{|1-we^{i\varphi'}|^2} d\varphi' \tag{3}$$

where we used the coordinate $(e^{i\varphi'}, w) \in \partial\mathbb{D}\times\mathbb{D}$ for $\kappa'_{\pi/2}(\partial\mathbb{D}\times R') \subset M' = \partial\mathbb{D}\times\mathbb{D}/\Gamma$.

Equations (2) and (3) imply that the boundary value functions $f(z, e^{i\varphi})$ and $f(e^{i\varphi'}, w)$ are constant functions in z and w for almost all $e^{i\varphi}$ and $e^{i\varphi'} \in \partial\mathbb{D}$ since these functions are holomorphic in z and w respectively. Now it follows that $\tilde{f}(z, w) = f(e^{i\varphi'}, e^{i\varphi})\colon \partial\mathbb{D}\times\partial\mathbb{D} \to \mathbb{C}$ agrees with a constant function almost everywhere, and we finish this proof. □

Remark 3.4 The integration formula used in the proof is equivalent to Demailly's Lelong–Jensen formula [7]. Exploiting this formula, a notion of Hardy space for hyperconvex domains in $\mathbb{C}^n$, *Poletsky–Stessin Hardy spaces*, was introduced in Alan [4] and Poletsky and Stessin [23] independently (cf. Alan and Göğüş [5]). The proof above actually shows the triviality of L^2 Hardy space of $X \subset Y, Y'$.

Remark 3.5 Yet another proof for the Liouville property which does not employ Fact 3.2 can be obtained by a method similar to [1], which will be discussed in the author's forthcoming article. As in [1], we may show that all the weighted Bergman space of order > -1 of $X \subset Y, Y'$ is infinite dimensional in spite of the fact that its L^2 Hardy space is trivial.

4 Open Problems

We shall pose two open problems for further study.

Problem 1 Do other Grauert tubes of finite maximal radius give similar example of hyperconvex manifolds without non-constant bounded holomorphic function?

Problem 2 Is there any domain with Levi-flat boundary having positive Diederich–Fornæss index and non-constant bounded holomorphic function?

Problem 2 is a variant of an open problem raised by Sidney Frankel (cf. Ohsawa [22]), to classify Levi-flat hypersurfaces that bound domains with non-constant bounded holomorphic functions.

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CR-Geometry and Shearfree Lorentzian Geometry

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Abstract We study higher dimensional versions of shearfree null-congurences in conformal Lorentzian manifolds. We show that such structures induce a subconformal structure and a partially integrable almost CR-structure on the leaf space and we classify the Lorentzian metrics that induce the same subconformal structure. In the last section we survey some known applications of the correspondence between almost CR-structures and shearfree null-congurences in dimension 4.

Keywords Conformal Lorentzian manifold · CR-structure · Subconformal structure · Shearfree null-congurences

It is well known that CR-manifolds are intimately related with conformal Lorentzian manifolds by the Fefferman metric [1, 4, 7]. However there exist also other natural constructions of Lorentzian metrics on sphere or line bundles over CR-manifolds associated with the underlying CR-structure. Such correspondences have been used in both ways: to describe special algebraic solutions to Einstein's equation in 4-dimensional Lorentzian space using CR-structures and also to interpret CR-phenomena in terms of general relativity.

More precisely, let M be a 3-dimensional CR-manifold with contact distribution H and CR-structure $J: H \to H$. Following Cartan, the CR-stucture of M can be (locally) encoded as a choice of a real 1-form λ and a complex 1-form μ such that

(i) $\lambda \wedge \mu \wedge \bar{\mu} \neq 0$,
(ii) $H = \ker \lambda$,
(iii) $\mu|_H \circ J = \mathrm{i}\mu|_H$ for all $X \in H$.

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Then any other pair (λ', μ') of 1-forms defines the same CR-structure if it is related to (λ, μ) by

$$\lambda' = f\lambda, \qquad \mu' = \phi\mu + \psi\lambda$$

where f is a non-vanishing real function, ϕ is a non-vanishing complex function and ψ is an arbitrary complex function.

We assume that the CR-structure $(M, H, J) = (M, [\lambda, \mu])$ is Levi-nondegenerate, i.e. $d\lambda \wedge \lambda \neq 0$. In this case we can choose the pair of forms such that

$$\begin{aligned} d\lambda &= \mathrm{i}\mu \wedge \bar{\mu} + c\mu \wedge \lambda + \bar{c}\bar{\mu} \wedge \lambda \,, \\ d\mu &= a\mu \wedge \lambda + b\bar{\mu} \wedge \lambda. \end{aligned} \tag{1}$$

Recall that the Fefferman metric is a conformal class of Lorentzian metrics defined on the circle bundle $\mathfrak{M} = H^{1,0}/\mathbb{R}^+$ where $H^{1,0}$ is the i-eigenbundle of J on $H \otimes C$. Using a coframe (μ, λ) that satisfies (1) and the trivialisation $\mathfrak{M} \ni m|_p = \mathrm{e}^{-t-\mathrm{i}r}\,\partial|_p \mapsto (p, r)$ where $r \in [0, 2\pi)$, the Fefferman metric is defined by

$$g = \mathrm{e}^{2t}\left[\mu\bar{\mu} + \frac{1}{3}\lambda\left(2dr - \mathrm{i}c\mu + \mathrm{i}\bar{c}\bar{\mu} - \left(\frac{c_{\bar{\mu}} + \bar{c}_{\mu}}{4} - \frac{3\mathrm{i}(a - \bar{a})}{4}\right)\lambda\right)\right] \tag{2}$$

where, by abuse of notation, $\mu, \bar{\mu}, \lambda$ denote the pull-backs of the corresponding 1-forms on M, dr is the differential of the coordinate function r, a, c are the pull-backs of the structure functions from (1), e^{2t} is a conformal scaling factor and

$$dc = c_{\mu}\mu + c_{\bar{\mu}}\bar{\mu} + c_{\lambda}\lambda, \qquad d\bar{c} = \bar{c}_{\mu}\mu + \bar{c}_{\bar{\mu}}\bar{\mu} + \bar{c}_{\lambda}\lambda.$$

It can be shown, e.g. by using the canonical Cartan connection, that the conformal Fefferman metric (2) does not depend on the choice of the pair (λ, μ). It is obvious that the fundamental vector field ∂_r is a conformal Killing null vector field of the Fefferman metric.

The Fefferman construction is universal in the sense that it provides a unique conformal Lorentzian space for any Levi-nondegenerate CR-manifold, but, in general, the resulting conformal metrics cannot be rescaled to Einstein metrics.

An alternative, more flexible approach has been introduced by physicists. They consider a line bundle $\mathfrak{M}$ over a CR-manifold M together with a family of Lorentzian metrics, which takes in a trivialising chart $M \times \mathbb{R}$ the form

$$g = \mathrm{e}^{2t}\left[\mu\bar{\mu} + \lambda\left(dr + W\mu + \bar{W}\bar{\mu} + H\lambda\right)\right] \tag{3}$$

where r is the fibre variable and t, W, H are arbitrary functions.

The fundamental vector field $p = \partial_r$ is again null with respect to all metrics from the family g but it is not, in general, conformal Killing. Instead it satisfies the following condition of shearfreeness:

$$\mathcal{L}_p g = \rho g + g(p, \cdot) \vee \psi$$

where ρ is some function and ψ is some 1-form. This condition controls only the change of g on $\ker \lambda$ and hence is somewhat weaker than p being conformal Killing.

In this paper we describe a generalisation of the correspondence between higher dimensional CR-geometry, subconformal geometry and shearfree Lorentzian geometry.

1 Subconformal and CR-Manifolds

Throughout this paper we use the notion of a CR-manifold as a shorthand for a Levi-nondegenerate, partially integrable almost CR-manifold of hypersurface type as defined below:

Definition 1 A CR-manifold M is a contact manifold with contact distribution H and a smooth family of endomorphisms $J_x : H_x \to H_x$ with $J_x^2 = -\,\mathrm{id}$. We assume that (M, H, J) is partially integrable, i.e. the complex eigen-distribution $H^{1,0} \subset H \otimes \mathbb{C}$ of J with eigenvalue i satisfies

$$[H^{1,0}, H^{1,0}] \subseteq H \otimes \mathbb{C}.$$

Remark Partial integrability is equivalent to the following property: If λ is a contact form for the contact distribution H then

$$d\lambda(JX, JY) = d\lambda(X, Y)$$

for any sections X, Y of H. This property is also equivalent to $d\lambda(\cdot, J\cdot)$ being symmetric.

Definition 2 A subconformal manifold is a contact manifold M with contact distribution H that is endowed with a conformal class of subRiemannian metrics $[g_H]$.

In this article we will only consider orientable subconformal manifolds. In this case there exists a global contact 1-form λ such that $H = \ker \lambda$.

For $\dim M = 3$ subconformal manifolds are essentially the same as CR-manifolds. More precisely, the conformal metric on the contact distribution induces two mutually conjugate complex structures that rotate vectors by an angle $\frac{\pi}{2}$. Vice versa, the conformal structure can be recovered from either of these complex structures by making multiplication by complex numbers conformal mappings on the distribution.

In higher dimensions the relation between subconformal and CR-manifolds is less obvious.

Theorem 1 *Let $(M, H, [g_H])$ be an orientable subconformal manifold. Then M inherits two mutually conjugated partially integrable almost CR-structures J and $-J$.*

Proof Choose a contact form λ. Let $A = g^{-1} d\lambda|_H$, i.e. $d\lambda|_H = g(A\cdot, \cdot)|_H$. Then A is non-degenerate and skew-symmetric, hence A^2 is symmetric and negative definite. Define $J = \sqrt{-A^{-2}}A$. It follows that J depends smoothly on the coordinates of M. A different choice of the contact form λ affects only the sign of J. We show that J, and hence $-J$, define partially integrable almost CR-structures.

Since A and A^2 commute, the eigenspaces of A^2 at each point are invariant for A and, since $\sqrt{-A^{-2}}$ is diagonalisable with the same eigenspaces as A^2, A commutes with $\sqrt{-A^{-2}}$.

Therefore,

$$J^2 = \sqrt{-A^{-2}}A\sqrt{-A^{-2}}A = -A^{-2}A^2 = -\,\mathrm{id}\,.$$

To prove partial integrability let X, Y be two sections of H. Since $\sqrt{-A^{-2}}$ is symmetric, then

$$\begin{aligned} d\lambda(JX, JY) &= g(A\sqrt{-A^{-2}}AX, \sqrt{-A^{-2}}AY) = g(-A^{-2}A^2X, AY) \\ &= -g(X, AY) = -g(AY, X) = -d\lambda(Y, X) = d\lambda(X, Y). \end{aligned}$$

This proves partial integrability. □

The theorem above indicates that CR-structures in higher dimensions are weaker structures than subconformal ones. There are many different conformal structures that induce the same almost CR-structure. E.g. different subconformal structures can be obtained from a strictly pseudoconvex CR-structure (M, H, J) by additionally prescribing different $d\lambda$-orthogonal decompositions of the distribution H

$$H = \oplus H_j$$

and positive functions α_j. Then let $A|_{H_j} = \alpha_j J|_{H_j}$ and $g = d\lambda \circ A^{-1}$.

The extremal choices of the decomposition of H are on the one hand the trivial decomposition $H = H$ and on the other hand the decomposition into complex one-dimensional H_j. The former choice is equivalent to the CR-structure while the latter one induces a much more rigid geometric structure.

2 Shearfree Congruences

Definition 3 A shearfree congruence is a $(2n+2)$-dimensional Lorentzian manifold $(\mathfrak{M}, g)$ equipped with a foliation into integral curves of a nowhere vanishing vector field p such that

(i) p is null, i.e. $g(p, p) = 0$,
(ii) $\mathcal{L}_p g = \rho g + \theta \vee \psi$, where $\theta = g(p, \cdot)$, ρ is a function and ψ is a 1-form. This condition means that the local flow of p preserves the distribution

$$p^{\perp} = \{X \in T\mathfrak{M} \colon g(X, p) = 0\}$$

and the degenerate subconformal metric the metric $[g|_{p^{\perp}}]$ on $p^{\perp}$.

We call p a shearfree vector field (with respect to $(\mathfrak{M}, g)$) if it satisfies conditions (i) and (ii) above.

It can be shown that the conditions (i) and (ii) in the definition above imply that the vector field p is geodetic, i.e.

$$\nabla_p p = \beta p,$$

where ∇ is the Levi-Civita connection of g and β is some function. (See Proposition 1 below.) Hence a shearfree congruence is in fact a foliation of $\mathfrak{M}$ into null-geodesics, which can be interpreted as light rays.

Notice that shearfreeness of p depends only on the conformal class of g and is preserved under scaling of p.

We define also a global conformal version of shearfree congruences.

Definition 4 Let $(\mathfrak{M}, [g])$ be a $(2n + 2)$-dimensional conformal Lorentzian manifold with a shearfree vector field p and assume that the flow of p generates a free action of $G = \mathbb{R}$ or $G = S^1$ so that the orbit space by $M = \mathfrak{M}/G$ is a manifold and the canonical projection $\pi \colon \mathfrak{M} \to M$ is a principal G-bundle. We call the $(\mathfrak{M}, [g], p, M)$ a Robinson-Trautman space (RT-space) of type G.

Example 1. For a Lorentzian metric g any conformal Killing null vector field p is shearfree.
2. For the Lorentzian metrics (3) $p = \partial_r$ is a shearfree vector field on the trivial $\mathbb{R}$-bundle $M \times \mathbb{R}$.

Definition 5 A shearfree congruence is called diverging if the function ρ in (ii) does not vanish; it is called distinguished in the opposite case, i.e. if $\rho = 0$.
A shearfree vector field p is said to be autoparallel if

$$\nabla_p p = 0.$$

By rescaling the Lorentzian metric g, a shearfree congruence can be made distinguished, locally, and by rescaling the shearfree vector field p it can be made autoparallel at the same time. If $\mathfrak{M}$ is an RT space of type $\mathbb{R}$ then this can be achieved globally.

We summarise some properties of shearfree congruences.

Proposition 1 *(i) A shearfree vector field p is geodetic, i.e.* $\nabla_p p = \beta p$.
(ii) Locally, a Lorentzian metric g and a shearfree vector field p can be rescaled, so that g becomes distinguished and p becomes autoparallel at the same time. On an RT-space of type $\mathbb{R}$ *this can be achieved globally.*

(iii) Being autoparallel is equivalent to $p \lrcorner d\theta = 0$*, which in turn is equivalent to*

$$\mathcal{L}_p \theta = d(p \lrcorner \theta) + p \lrcorner d\theta = 0.$$

Proof (Sketch of the proof) For the proof of (i) and (iii) the notion of the Nomizu operator is convenient. For any vector field X the Nomizu operator L_X is defined as

$$L_X : Y \mapsto -\nabla_Y X,$$

where ∇ is the covariant derivative of the Levi-Civita connection for g. It is well known that

$$g^{-1}\mathcal{L}_X g = -L_X - L_X^*, \tag{4}$$

$$2g^{-1}d\theta = -L_X + L_X^*, \tag{5}$$

where $\theta = g(X, \cdot)$ and L_X^* is the g-adjoint of L_X.

The Nomizu operator satisfies

$$L_p^* p = 0 \tag{6}$$

for any null vector field p, because of

$$g(L_p^* p, X) = g(p, L_p X) = g(p, -\nabla_X p) = -\frac{1}{2} X g(p, p) = 0.$$

Now, statement (i) follows from

$$\begin{aligned} g(\nabla_p p, X) &= -g(L_p p, X) = -g(L_p p + L_p^* p, X) = \mathcal{L}_p g(p, X) \\ &= \rho g(p, X) + \psi(p)\theta(X) = g(\rho p, X) + \psi(p) g(p, X) = g((\rho + \psi(p))p, X), \end{aligned}$$

for all X, hence $\nabla_p p = \beta p$ with $\beta = \rho + \psi(p)$.

The condition that the shearfree congruence is distinguished can be achieved by scaling g by a factor t that is a solution of $\partial_p \log t = -\rho$. The condition $\nabla_p p = 0$ can be achieved by scaling p by a factor s that is a solution of $\partial_p \log s = -\beta$. These PDE can be solved locally, or globally in the case of an RT-space of type $\mathbb{R}$.

The equivalence of $\nabla_p p = 0$ and $p \lrcorner d\theta = 0$ follows from (6) because $\nabla_p p = -L_p p = 0$ can be written as $0 = -g(L_p p, X) = -g(L_p p - L_p^* p, X) = d\theta(p, X)$, that is $p \lrcorner d\theta = 0$, which is equivalent to

$$\mathcal{L}_p \theta = p \lrcorner d\theta + d(p \lrcorner \theta) = p \lrcorner d\theta = 0.$$

□

3 Shearfree Congruences and Their Orbit Spaces

In this section let $(\mathfrak{M}, [g], p)$ be an RT-space.

Definition 6 We say that $(\mathfrak{M}, [g], p)$ is *twisting* if $(d\theta)^n \wedge \theta \neq 0$, where $\theta = g(p, \cdot)$.

Notice that the notion of being twisting is invariant under scaling of p and g and therefore it is well-defined. Indeed, both scalings result in a scaling of θ, hence

$$d(\alpha\theta) \equiv \alpha\, d\theta (\mathrm{mod}\, \theta),$$

and

$$(d\alpha\theta)^n \wedge \alpha\theta = \alpha^{n+1}(d\theta)^n \wedge \theta.$$

Since the notion of shearfreeness of p is invariant with respect to rescalings of p we can replace p in the definition of a twisting RT-space by its equivalence class $[p]$.

We will show that the orbit space M of a twisting RT-space carries a canonical subconformal structure and hence a CR-stucture.

Definition 7 An RT-structure $(\mathfrak{M}, [g], [p])$ and a subconformal structure $(H, [g_H])$ with contact distribution H and subconformal metric $[g_H]$ on the orbit space M are called compatible if for any contact form λ on M with Reeb vector field Z

(i) $\ker \pi^*\lambda = p^\perp = \{X \in T\mathfrak{M} \colon g(X, p) = 0\}$ and
(ii) $\pi^* g_H^\lambda|_{p^\perp}$ is conformally equivalent to $g|_{p\perp}$. Here g_H^λ is the extension of g_H to the degenerate metric on M with $Z = \ker g_H^\lambda$. That is

$$g = P^2(\pi^* g_H^\lambda + g(p, \cdot) \vee \psi)$$

for some positive function P^2 and some 1-form ψ.

Theorem 2 *Let $(\mathfrak{M}, [g], [p])$ be a twisting RT-space. Then there exists a unique compatible subconformal structure on the orbit space M.*

Proof Let $U \in T_Q M$. Then we call $u \in T_q\mathfrak{M}$ a lift of U if $\pi(q) = Q$ and $\pi_* u = U$. A compatible contact distribution $H_Q \subset T_Q M$ must satisfy the condition $\theta(u) = g(p, u) = 0$ for any lift u of any $U \in H_Q$. This proves the uniqueness of the contact structure. We show that this condition does not depend on the choice of the lift. Let u_0 and u_1 be two lifts at q_0 and q_1, respectively, connected by a path $u(t)$, where t is the time parameter of the flow of the vector field p. Then, with respect to some local trivialisation,

$$u(t) = U + \alpha(t) p$$

and

$$\frac{d}{dt}\theta(u(t)) = \mathcal{L}_p g(u(t), p) = \rho g(u(t), p) + \theta(u(t))\psi(p) = (\rho + \psi(p))\theta(u(t)).$$

It follows that $\theta(u(t)) = Ce^{\int \rho+\psi(p)dt}$ and therefore either equals zero for all t or nowhere.

We show that H is a contact distribution. Let λ be a form that annihilates H. Then $\pi^*\lambda = \alpha\theta$, where α is a non-vanishing function. Since $\pi^*(d\lambda)^n \wedge \lambda = \alpha^{n+1} d\theta^n \wedge \theta \neq 0$ it follows $(d\lambda)^n \wedge \lambda \neq 0$. The conformal metric g_H on H_Q is uniquely determined by

$$g_H(U, V) = g(u, v)$$

for $U, V \in H_Q$ and any lifts $u, v \in p^{\perp}$ at the same base point q. We show that this definition does not depend on the choice of the lifts. Let

$$u(t) = U + \alpha(t)p, \qquad v(t) = V + \beta(t)p$$

be two paths connecting two pairs of lifts (u_0, v_0) and u_1, v_1 with respect to some trivialisation. Then,

$$\frac{d}{dt} g(u(t), v(t)) = \mathcal{L}_p g(u(t), v(t)) = \rho g(u(t), v(t)),$$

where ρ depends on t but not on $u(t)$ and $v(t)$ It follows that $g(u(t), v(t))$ scales along the path by a multiplier that does not depend on the path. Hence $g_H(U, V)$ is well-defined as a conformal metric. □

The theorem below describes the RT-structures that are compatible with a given subconformal structure on their orbit space.

Theorem 3 *Let $\pi\colon \mathfrak{M} \to M$ be a line bundle over a subconformal manifold $(M, H, [g_H])$ and p any non-vanishing vertical vector field. Then $(\mathfrak{M}, [g], [p])$ is a twisting RT-structure compatible with $(M, H, [g_H])$ if and only if*

$$g = P^2(\pi^* g_H^{\lambda} + \pi^*\lambda \vee \psi) \tag{7}$$

where λ is a contact form on M, P is a positive function on $\mathfrak{M}$ and ψ is a 1-form on $\mathfrak{M}$.

Proof Assume g has the form (7). Then

(i) g is Lorentzian, and $g|_{p^{\perp}}$ is conformally equivalent to $\pi^* g_H^{\lambda}|_{p^{\perp}}$,
(ii) p is null, and
(iii) $\mathcal{L}_p g = 2P \frac{\partial P}{\partial t}(g_H + \pi^*\lambda \vee \psi) + P^2(\pi^*\lambda \vee \mathcal{L}_p\psi) = 2P\frac{\partial P}{\partial v} g + \pi^*\lambda \vee \tilde{\psi}$, i.e. p is shearfree for g.

Therefore, $(\mathfrak{M}, [g], [p])$ is an RT-space compatible with $(M, H, [g_M])$.

It remains to show that any conformal Lorentzian metric that satisfies (i)–(iii) has the form (7). Condition (i) means that there exists a positive function P on M such that

$$g|_{p^{\perp}} = P^2 \pi^* g_H^{\lambda}|_{p^{\perp}}.$$

Consider the symmetric 2-form

$$T = g - P^2 \pi^* g_H^\lambda$$

for some choice of the contact form λ on M. Then $T(u, v) = 0$ for any $u, v \in T_q M$ such that $g(v, p) = 0$. Let z be a lift of the Reeb vector field Z. We can choose z such that $g(z, z) = 0$.

Consider the 1-forms

$$\theta = g(p, \cdot) = \gamma \pi^* \lambda, \qquad \psi' = g(z, \cdot).$$

We have $\theta(z) = g(z, p) = \gamma \lambda(Z) = \gamma$.

If $u = u' + \alpha z$ is the decomposition of a vector field u on M such that $u' \in p^\perp$ then

$$\theta(u) = \alpha g(p, z) = \alpha \gamma \pi^*(Z) = \alpha \gamma,$$

hence

$$\alpha = \frac{1}{\gamma} \theta(u) = \pi^* \lambda(u).$$

For two vector fields u, v on $\mathfrak{M}$ with $u = u' + \alpha z$, $v = v' + \beta z$ where $u', v' \in p^\perp$ we have

$$T(u, v) = \alpha g(z, v) + \beta g(u, z) = \frac{1}{\gamma}(\theta(u)\psi'(v) + \theta(v)\psi'(u)) = \pi^* \lambda \vee \psi'(u, v).$$

It follows

$$g = P^2 \pi^* g_H^\lambda + T = P^2(\pi^* g_H^\lambda + \pi^* \lambda \vee \psi)$$

where $\psi = \frac{1}{P^2} \psi'$. □

4 Applications of Shearfree Congruences in Dimension 4

In this section we survey some applications of shearfree congruences in dimension 4. The correspondence between 4-dimensional shearfree congruences and 3-dimensional CR-manifolds has been known by physicists and has been exploited in both directions (see, e.g., [5, 10] and references therein).

In [3] a 3-parametric family of Ricci flat Lorentzian 4-manifolds with shearfree congruence, which include the Kerr metric, have been constructed. It is given by

$$g = P^2 \mu \bar{\mu} + \lambda(dr + W\mu + \bar{W}\bar{\mu} + H\lambda),$$

where

$$\begin{aligned}
\mu &= dz, \\
\lambda &= du - 2\,\mathrm{Im}\,\frac{((a+b)|z|^2 + b)dz}{z(1+|z|^2)^2}, \\
P^2 &= \frac{r^2}{(1+|z|^2)^2} + \frac{(b-a)+(b+a)|z|^2}{(1+|z|^2)^4}, \\
W &= \frac{2\,\mathrm{i}\,az}{(1+|z|^2)^2}, \\
H &= \frac{2(mr+b^2)(1+|z|^2)^2 - 2ab(1-|z|^4)}{r^2(1+|z|^2)^2 + (b-a+(b+a)|z|^2)^2} - 1.
\end{aligned}$$

Here $z = x + \mathrm{i}y, u, r$ are coordinates in $\mathbb{R}^4$ and a, b, m are real parameters. The metric g is singular for $z = 0$ if $b \neq 0$ and for $r = 0$ and $|z|^2 = \frac{a-b}{a+b}$ if $|b| \leq |a|$. The corresponding RT-space $(\mathfrak{M}, [g], [\partial_r])$ is twisting, unless $a = b = 0$. For $b = 0$, the metric g is the Kerr rotating black hole with mass m and the angular momentum parameter a; if $a = b = 0$ the metric g describes the Schwarzschild black hole with mass m. For $m = a = 0$ this is the Taub-NUT vacuum metric. The orbit spaces M can be identified with $\mathbb{C} \times \mathbb{R}$ with coordinates (z, u). If $b \neq 0$ we have to delete the singular line $z = 0$. The induced subconformal structures are $(M, [\lambda], [\mu\bar{\mu}])$ and the CR-structures are defined by $(M, [\lambda, \mu])$. Notice that the parameter m only appears in the function H and does not affect the family of CR-manifolds. All resulting CR-manifolds can be embedded into $\mathbb{C}^2$ with coordinates (z, w) as

$$v = \mathrm{Im}\, w = \frac{-2a}{1+|z|^2} + 2b \log \frac{|z|^2}{1+|z|^2}.$$

This is the trivial Levi-flat CR-manifold $v = 0$ for the Schwarzschild solution, a spherical CR-manifold (with singularity at 0) in the Taub-NUT case and a non-spherical Sasakian manifold for the Kerr solution.

It is a natural question to ask, how analytic properties of a CR-manifold are reflected in a corresponding shearfree congruence. The papers [5, 8] by Lewandowski, Nurowski, Tafel and Hill, Lewandowski, Nurowski feature a fascinating approach to the local embeddability problem for 3-dimensional CR-manifolds: Let M be a 3-dimensional manifold with a CR-structure that is given by a pair of 1-forms (μ, λ) as above. Then the local embeddability problem reduces to finding two functionally independent CR-functions f, g, i.e. functions that satisfy

$$\bar{\partial} f = \bar{\partial} g = 0, \text{ and } df \wedge dg \neq 0,$$

where $(\partial, \bar{\partial}, \partial_0)$ is a dual frame to the coframe $(\mu, \bar{\mu}, \lambda)$. Using a Frobenius type result (see e.g. [3, 5]), one CR-function is constructed from a complex 1-form ϕ such that

$$d\phi \wedge \phi = 0 \text{ and } \phi \wedge \bar{\phi} \neq 0.$$

Such 1-form can be obtained as a structure form of the Levi-Civita connection as a consequence of the vanishing of certain components of the complexified Ricci curvature. The latter condition is, in a sense, vanishing of a $\bar{\partial}$ derivative.

According to a result by Jacobowitz [6], the existence of a second, functionally independent, CR-function can be related to a non-vanishing closed section of the canonical bundle of M. Here, the canonical bundle is simply the complex rank 1 line bundle spanned by the 2-form $\mu \wedge \lambda$. It is clear that this does not depend on the choice of the pair (μ, λ). The theorem below is a special case of Jacobowitz's result.

Theorem 4 (Jacobowitz, 1987) *If near some point x, (M^3, H, J) has a CR-function ζ with $d\zeta \neq 0$ and its canonical bundle has a closed non-zero section then (M^3, H, J) is embeddable on some neighbourhood of x.*

The condition of the existence of a non-zero closed section of the canonical bundle has a nice interpretation in general relativity. Recall that a 2-form F is a solution of the Maxwell equation in vacuum if

$$dF = d * F = 0.$$

The solution F is called a plane wave if it is null, i.e. $g(F, F) = g(F, *F) = 0$. Let $\mathcal{F} = F - \mathrm{i} * F$. According to a result by Robinson [9] the 2-forms $\mathcal{F}$ that correspond to plane waves have a representation

$$\mathcal{F} = \theta \wedge \epsilon$$

where $\theta = g(p, \cdot)$, $\epsilon = g(e, \cdot)$, p is a shearfree null vector field and $e = e_1 + \mathrm{i}e_2$ is a complex vector field with $g(p, e) = 0$, $g(e_i, e_j) = \delta_{ij}$. Vice versa, if p is a shearfree vector field and e a complex vector field as above, then

$$\theta \wedge \epsilon$$

is a plane wave.

A plane wave solution $\mathcal{F} = \theta \wedge \epsilon$ is said to be aligned with the shearfree congruence if $\mathcal{F} = \Phi\mu \wedge \lambda$, where μ and λ are the lifts of the corresponding forms on the underlying CR-manifold. Hence the existence of a non-zero closed section of the canonical bundle translates into the existence of an aligned plane wave solution of the Maxwell equation.

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Complex Surfaces with Many Holomorphic Automorphisms

Rafael B. Andrist

Abstract The density property for a complex manifold states in a mathematically precise way that it has a "large" group of holomorphic automorphisms. After a brief survey of the geometric consequences and the known classes of manifolds with the density property, we focus on affine algebraic surfaces with the density property, in particular on so-called Gizatullin surfaces.

Keywords Holomorphic automorphisms · Holomorphic flexibility
Elliptic manifolds · Density property · Andersen-Lempert theory · Gizatullin surfaces

1 Automorphisms of Complex-Euclidean Space

The systematic study of the holomorphic self-maps $\mathbb{C}^n \to \mathbb{C}^n$ started with the work of Rosay and Rudin [32] in 1988. They introduced holomorphic shears and overshears:

Definition 1
- A *holomorphic shear* in direction of the j-th coordinate is a holomorphic map $F\colon \mathbb{C}^n \to \mathbb{C}^n$ given by

$$z \mapsto (z_1, \dots, z_{j-1}, z_j + f(z_1, \dots, z_{j-1}, z_{j+1}, \dots z_n), z_{j+1}, \dots, z_n)$$

  where $f\colon \mathbb{C}^{n-1} \to \mathbb{C}$ is a holomorphic function.
- A *holomorphic overshear* in direction of the j-th coordinate is a holomorphic map $G\colon \mathbb{C}^n \to \mathbb{C}^n$ given by

$$z \mapsto (z_1, \dots, z_{j-1}, z_j \cdot g(z_1, \dots, z_{j-1}, z_{j+1}, \dots z_n), z_{j+1}, \dots, z_n)$$

  where $g\colon \mathbb{C}^{n-1} \to \mathbb{C}^*$ is a nowhere vanishing holomorphic function.

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Remark 1 It is straightforward to see that a holomorphic shear resp. over-shear has an inverse that is also a holomorphic shear resp. over-shear, hence all such maps are holomorphic automorphisms of $\mathbb{C}^n$.

Remark 2 The notion of *shear* resp. *overshear* can be generalized to arbitrary manifolds, see e.g. Varolin [33], as follows: Let V be a complete vector field and let $f \in \ker V$, then $f \cdot V$ is again a complete vector field and called a *shear* w.r.t. V. If $f \in \ker V^2$, then $f \cdot V$ is again a complete vector field and called an *overshear* w.r.t. V. Using this terminology, the shears and overshears of $\mathbb{C}^n$ are just the flow maps of shear and overshear vector fields w.r.t. the standard basis vector fields $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$. Explicit formulas for the flow maps are given in [6, Lemma 3.3].

Rosay and Rudin [32] showed that a lot of geometric constructions in $\mathbb{C}^n$ could be realized by compositions of holomorphic shears and overshears. This led to the natural question whether every holomorphic automorphism of $\mathbb{C}^n$ is a (possibly infinite) composition of holomorphic shears or overshears.

This question was answered in two remarkable papers by Andersén [1] in 1990 and Andersén and Lempert [3] in 1992:

Theorem 1 ([1, Theorem C]) *The subgroup generated by the holomorphic shears is dense in the group of volume-preserving holomorphic automorphisms of* $\mathbb{C}^n$, $n \geq 2$, *w.r.t. the compact-open topology. Here, we call a holomorphic map F volume-preserving if* $JF \equiv 1$.

Theorem 2 ([3, Theorem 1.3]) *The subgroup generated by the holomorphic overshears is dense in the group of holomorphic automorphisms of* $\mathbb{C}^n$, $n \geq 2$, *w.r.t. the compact-open topology.*

Andersén and Lempert [3] also showed that in both cases there exist examples which are a limit of compositions of shears resp. overshears but cannot be finite compositions of shears resp. overshears.

Their proof was based on the following theorem (which they had established for star-shaped domains).

Theorem 3 ([19, Theorem 1.1] and [20]) *Let* $\Omega \subset \mathbb{C}^n$ *be an open subset of* $\mathbb{C}^n$, $n \geq 2$. *Let* $\varphi\colon \Omega \times [0,1] \to \mathbb{C}^n$, $\varphi(z,t) = \varphi_t(z)$, *be a* $\mathcal{C}^1$*-smooth map such that*

1. $\varphi_0\colon \Omega \hookrightarrow \mathbb{C}^n$ *is the natural inclusion,*
2. $\varphi_t\colon \Omega \to \mathbb{C}^n$ *is a holomorphic injection for every* $t \in [0,1]$,
3. $\varphi_t(\Omega) \subseteq \mathbb{C}^n$ *is a Runge subset of* $\mathbb{C}^n$ *for every* $t \in [0,1]$.

Then for any compact $K \subset \Omega$ *and any* $\varepsilon > 0$ *there exists a continuous family* $\Phi\colon \mathbb{C}^n \times [0,1] \to \mathbb{C}^n$, $\Phi(z,t) = \Phi_t(z)$, *such that*

1. $\Phi_0 = \mathrm{id}_{\mathbb{C}^n}$,
2. Φ_t *is a holomorphic automorphism of* $\mathbb{C}^n$ *for every* $t \in [0,1]$,
3. $\|\Phi_1 - \varphi_1\|_K < \varepsilon$.

Moreover, Φ_1 *can be chosen to be a finite composition of shears and overshears.*

Proof (*Proof of Theorem* 2) Since every holomorphic automorphism of $\mathbb{C}^n$ can be connected through an isotopy to its linear part and since $\mathrm{GL}_n(\mathbb{C})$ is path-connected, we find for every automorphism φ_1 a family φ_t satisfying the assumptions of Theorem 3 with $\Omega = \mathbb{C}^n$. Note that $\varphi_t(\mathbb{C}^n) = \mathbb{C}^n$ trivially satisfies the Runge condition. □

2 Density Property

It was an observation by Varolin [34, 35] that the crucial ingredient in Theorem 3 is the so-called density property which he introduced:

Definition 2 Let X be a complex manifold. We say that X has the *density property* if the Lie algebra generated by the complete holomorphic vector fields on X is dense in the Lie algebra of all holomorphic vector fields on X.

Remark 3 There exists also another important notion, the so-called *volume density property*: Let X be a complex manifold. A *holomorphic volume form* ω on X is a nowhere vanishing form of maximal rank. Note that the existence of a holomorphic volume form ω on X is equivalent to saying that the canonical bundle of X is trivial.

We say that (X, ω) has the *volume density property* if the Lie algebra generated by the ω-preserving complete holomorphic vector fields on X is dense in the Lie algebra of all ω-preserving holomorphic vector fields on X.

If both X and ω are assumed to be algebraic, then ω is unique up to multiplication by a constant. The algebraic surfaces we are going to study later usually do not have a (algebraically) trivial canonical bundle, hence the notion of volume density property cannot be studied in theses cases. We therefore limit ourselves in this survey to the density property alone.

Theorem 3 can now be reformulated as follows.

Theorem 4 *Let $\Omega \subset X$ be an open subset of a Stein manifold X with the density property. Let d be a metric on X. Let $\varphi\colon \Omega \times [0, 1] \to X$, $\varphi(z, t) = \varphi_t(z)$, be a $\mathcal{C}^1$-smooth map such that*

1. *$\varphi_0\colon \Omega \hookrightarrow X$ is the natural inclusion,*
2. *$\varphi_t\colon \Omega \to X$ is a holomorphic injection for every $t \in [0, 1]$,*
3. *$\varphi_t(\Omega) \subseteq X$ is a Runge subset of X for every $t \in [0, 1]$.*

Then for any compact $K \subset \Omega$ and any $\varepsilon > 0$ there exists a continuous family $\Phi\colon X \times [0, 1] \to X$, $\Phi(z, t) = \Phi_t(z)$ such that

1. *$\Phi_0 = \mathrm{id}_X$,*
2. *Φ_t is a holomorphic automorphism of X for every $t \in [0, 1]$,*
3. *$\sup_{z \in K} d(\Phi_1(z), \varphi_1(z)) < \varepsilon$.*

Moreover, Φ_1 can be chosen to be a finite composition of time-1 maps corresponding to complete holomorphic vector fields that generate a dense Lie subalgebra of the Lie algebra of all holomorphic vector fields.

Among the rather easy consequences of this theorem are following [34, 35] important ones:

- On a Stein manifold X with the density property, there exist finitely many complete holomorphic vector fields that span the tangent space in every point. This implies that such a manifold is elliptic in the sense of Gromov [24] and hence an Oka manifold, see the textbook of Forstnerič [18] for more details.
- The holomorphic automorphisms of a Stein manifold with the density property act *multi-transitively* on it, i.e. for any $m \in \mathbb{N}$ the distinct images of m distinct points can be prescribed.
- If X is a Stein manifold with the density property of dimension n, then there exist Fatou–Bieberbach domains of the first and the second kind:

1. There exists a holomorphic injection $\mathbb{C}^n \hookrightarrow X$.
2. There exists a holomorphic injection $X \hookrightarrow X$ that is not onto.

More involved constructions using in particular the theorem above can be used to show the following:

Theorem 5 ([5, 8, 17]) *Let Y be a Stein manifold of complex dimension n and let X be a Stein manifold with the density property. If* $\dim X \geq 2n$*, then there exists a proper holomorphic immersion $Y \looparrowright X$. If* $\dim X \geq 2n+1$*, then there exists a proper holomorphic embedding $Y \hookrightarrow X$.*

For more results and applications concerning proper embeddings we refer to the recent survey [16].

Since Theorem 4 has many interesting consequences, it is of course important to know a large class of examples.

In retrospective, the result of Andersén and Lempert [3] can be restated as $\mathbb{C}^n$ having the density property. This is in fact a special case of a more general result:

Theorem 6 ([25, Theorem 3], [12, Theorem A], [27, Theorem 1.3]) *Let G be a connected linear algebraic group and $H \subset G$ a subgroup. If $X := G/H$ is an affine manifold, then X has the density property if it is not algebraically isomorphic to $\mathbb{C}$ or $(\mathbb{C}^*)^n$.*

For the case of $\mathbb{C}^n$, $n \geq 2$, we would like to give a very short proof for the density property that can be found in [25, Corollary 2.2]. The strategy of the proof, in particular Equation (1), motivates more elaborate concepts that are used in the proof of the general case.

Theorem 7 *For $n \geq 2$, $\mathbb{C}^n$ has the density property.*

Proof Given two holomorphic vector fields V and W on a complex manifold X, and holomorphic functions f, g, h on X, the following formula is straightforward to verify:

$$[fV, hgW] - [hfV, gW] = fg \cdot (V(h) \cdot W + W(h) \cdot V) \ . \tag{1}$$

This formula becomes very useful under the following extra assumptions: $f \in \ker V$, $g \in \ker W$ and $V^2(h) = 0$ and $W(h) = 0$. By Remark 2 we know that all four vector fields on the l.h.s. are shears resp. overshears of V resp. W. If we then further assume that V and W are complete, then the r.h.s. is in the Lie algebra generated by the complete holomorphic vector fields. To prove the density property of $\mathbb{C}^n$, we only need to choose $V = \frac{\partial}{\partial z_1}$, $W = \frac{\partial}{\partial z_2}$, $f = z_2^{j_2}, \ldots z_n^{j_n}$, $g = z_1^{j_1}$ and $h = z_1$ and we obtain all monomials in front of $\frac{\partial}{\partial z_2}$. This way, we obtain all polynomial vector fields, and by taking limits also all holomorphic vector fields. □

Only very few affine algebraic manifolds with the density property that are not homogeneous spaces of a complex Lie group, are known:

Theorem 8 ([26]) *Let X be a Stein manifold with the density property and let $f: X \to \mathbb{C}$ be a holomorphic function. We define X_f as follows:*

$$X_f := \{(x, y, z) \in \mathbb{C} \times \mathbb{C} \times X \ : \ xy - f(z) = 0\} .$$

If f has a smooth reduced zero-fibre, then X_f has the density property.

Example 1 The best-studied case is if $X = \mathbb{C}$ and f is a polynomial with simple roots. The surface X_f is then called a *Danielewski surface*. If $\deg f \geq 3$, the Danielewski surface X_f is not a homogeneous space of a complex Lie group, but has the density property according to the preceding theorem.

For a more comprehensive overview over the density property and related properties we refer the reader to the recent survey of Kaliman and Kutzschebauch [28].

3 Quasi-Homogeneous Surfaces

In the previous section we have seen that certain homogeneous spaces provide a large class of examples of Stein manifolds with the density property. However, only very few of these examples are complex surfaces. In this section we narrow our focus to complex algebraic surfaces with the density property. We know that the holomorphic automorphisms must act transitively, hence it is natural to study the surfaces admitting a transitive group action. However, this still seems a too large class of surfaces to deal with and we will concentrate on the algebraic surfaces whose algebraic automorphisms act (almost) transitively. These were studied by Gizatullin and Danilov [22, 23] in the 70s.

Definition 3 A normal affine-algebraic surface is called *quasi-homogeneous* if its group of algebraic automorphisms acts with an open orbit whose complement is at most finite.

Based on the works of Gizatullin [22], Bertin [10], Bandman and Makar-Limanov [9], and Dubouloz [13] the following characterization is possible:

Theorem 9 ([15, Theorem 4.3]) *For a normal complex-affine surface X that is not algebraically isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ and $\mathbb{C} \times \mathbb{C}^*$, the following are equivalent:*

1. *X is quasi-homogeneous.*
2. *X admits a completion by a* zigzag *i.e. by a linear chain of smooth rational curves.*
3. *The Makar-Limanov invariant (see below) of X is trivial.*

Definition 4 A complex surface satisfying the equivalent conditions of Theorem 9 is called a *Gizatullin surface*.

The *Makar-Limanov invariant* $\mathrm{ML}(X)$ of an algebraic variety X is defined as follows. Let $\mathrm{LND}(X)$ denote the set of all algebraic vector fields on X that have a complete, algebraic flow map.[1] We then set

$$\mathrm{ML}(X) := \bigcap_{V \in \mathrm{LND}(X)} \ker V \ .$$

The set $\mathrm{LND}(X)$ always contains the zero vector field whose kernel contains all functions. The constants are of course in the kernel of any vector field, hence we call the Makar-Limanov invariant of X *trivial*, if $\mathrm{ML}(X) = \mathbb{C}$. The triviality of the Makar-Limanov invariant means that there must be "many" complete vector fields with algebraic flows which is a strong hint towards the density property, albeit it does not imply it.

Let us elaborate a bit more the notion of a *zigzag*. A *zigzag* is a simple normal crossing divisor Γ consisting of smooth rational curves such that its dual graph is linear. According to the characterization of Gizatullin surfaces above, any Gizatullin surface X admits a completion $\bar{X}$ such that its boundary $\Gamma = \bar{X} \setminus X$ is a zigzag. Moreover, this zigzag can always be chosen to be a so-called *standard zigzag* [22, 23] whose linear dual graph can be described by $[[0, 0, -r_2, \ldots, -r_d]]$ where $d \geq 2$ and $r_j \geq 2$ for $j = 2, \ldots, d$. The numbers given in $[[0, 0, -r_2, \ldots, -r_d]]$ are the self-intersection numbers of the the involved projective lines.

Regarding the density property of these surfaces, following results are known so far:

- $d = 1$, resp. a zigzag $[[0, 0]]$ corresponds to $\mathbb{C}^2$ which has the density property as we have seen in the previous section.
- $d = 2$, resp. a zigzag $[[0, 0, -r_2]]$ corresponds to a Danielewski surface which has the density property, see [26].
- A zigzag $[[0, 0, -2, \ldots, -2]]$ corresponds to a Danilov–Gizatullin surface, uniquely determined by this zigzag, which has the density property [11].
- $d = 3$ resp. a zigzag $[[0, 0, -r_2, -r_3]]$ corresponds to a Gizatullin surface that has the density property [7].

[1] In the algebraic category, these are precisely the locally nilpotent derivations, abbreviated as LND, see e.g. the textbook of Freudenburg [21, Sect. 1].

A Gizatullin surface always admits two non-conjugate $\mathbb{C}^+$-actions, see e.g. Flenner, Kaliman and Zaidenberg [14]. For the concluding result we want to consider Gizatullin surfaces with a priori any possible standard zigzag as boundary divisor, but we make a restriction regarding the $\mathbb{C}^+$-actions: We require that there exists a $\mathbb{C}^+$-action on the surface such that its quotient is a $\mathbb{C}^1$-fibration with a single reduced degenerate fibre. One can always find a fibration with a single degenerate fibre, but in general this fibre will not be reduced.

Theorem 10 ([4, Corollary 1.6]) *Let X be a smooth Gizatullin surface with a $\mathbb{C}^+$-action such that its quotient is a $\mathbb{C}^1$-fibration with a single reduced degenerate fibre. Then X has the density property.*

This theorem contains all the previous examples listed above as special cases.

4 Open Questions

Questions 1 Does $\mathbb{C}^* \times \mathbb{C}^*$ have the density property?

The Lie group $\mathbb{C}^* \times \mathbb{C}^*$ comes with a natural, bi-invariant volume form $\omega = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$. All known holomorphic automorphisms of $\mathbb{C}^* \times \mathbb{C}^*$ preserve ω. On the other hand, there is—besides an incomplete proof [31], see [32, p. 79] and [2, p. 1080]—no evidence that there should not be any other holomorphic automorphisms of $\mathbb{C}^* \times \mathbb{C}^*$. It remains a long-standing open question whether $\mathbb{C}^* \times \mathbb{C}^*$ has the density property or not. A positive answer would also have interesting implications in complex dynamics since it would enable the construction of a Fatou–Bieberbach domain in $\mathbb{C}^2$ that avoids both coordinate axes.

Questions 2 Do all smooth Gizatullin surfaces have the density property?

All smooth Gizatullin surfaces investigated so far have the density property. However, we have positive results only for those that admit such a $\mathbb{C}^+$-action with a reduced exceptional fibre. Due to a work of Kovalenko [29] we know that there exist families of smooth Gizatullin surfaces that do not fall into this class.

Questions 3 Can we give a list of all algebraic surfaces with the density property? Can we classify the surfaces in this list up to biholomorphisms or up to algebraic isomorphisms?

Since even an affine-algebraic surface that is homogeneous w.r.t. its group of holomorphic automorphisms may have a discrete group of algebraic automorphisms, a smooth algebraic surface with the density property is not necessarily a Gizatullin surface.

Questions 4 What can be said about the density property for singular affine-algebraic surfaces?

Although required in the original definition, the definition of the density property can be stated also for singular varieties. For normal algebraic varieties with the (algebraic) density property there exists a meaningful theory that has been developed in [30]. However, only very few examples, namely certain toric varieties, are known. Singular Gizatullin surfaces are again natural candidates.

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Fatou Components for Conservative Holomorphic Surface Automorphisms

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Abstract We consider holomorphic maps of surfaces of complex dimension two. Such a map is said to be conservative if it preserves volume. We discuss the properties of these maps and present a number of open problems.

Keywords Fatou set · Holomorphic automorphism · Conservative map · Rotation domain

1 Introduction

The dynamics of automorphisms of complex surfaces is an active area where there has been a productive interaction between questions of dynamics and complex analysis. We will describe some questions and problems which arise out of complex dynamics but which have a strong component of complex analysis. We will discuss two general cases. The first is polynomial automorphisms of $\mathbb{C}^2$. By Friedland and Milnor [16], it suffices to restrict our attention to compositions of *generalized Hénon mappings*, which have the form

$$f(x, y) = (y, p(y) - \delta x) \tag{1.1}$$

where $p(y)$ is a monic polynomial of degree at least 2, and $\delta \in \mathbb{C}$ is a nonzero scalar. Some basic dynamical properties of a general Hénon map are given in [6, 15, 20, 21]. An intriguing aspect here is that the mappings themselves are simple to write down, but the dynamical questions have led to a rather elaborate theory. The results obtained have been most successful/complete in the case of dissipative maps. This paper is devoted to an outline of the conservative (volume-preserving) case, which corresponds to $|\delta| = 1$. Our theme will be that in the conservative case there seem to be more open questions than proved results.

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The second case we will discuss is rational surface automorphisms.[1] The dynamics of a positive entropy automorphism of a compact, complex surface may be studied along the same lines as the complex Hénon maps (see Cantat [10]). Here we consider the case where X is a rational surface (that is, X is birationally equivalent to $\mathbb{P}^2$). In the Appendix, we explain why we do not consider other complex surfaces. Consider the family of birational maps of $\mathbb{P}^2$ given by:

$$f_{a,b}(x, y) = \left(y, \frac{y+a}{x+b} \right) . \tag{1.2}$$

The map $f_{a,b}$ may not look like an automorphism because the line $\{y + a = 0\}$ is exceptional: it is mapped to the point $(-a, 0)$. However, let $\pi : X \to \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at the point $(-a, 0)$, and let L denote the lift of the line $\{y + a = 0\}$ to X. If we lift $f_{a,b}$ to a map of X, then the line L is no longer exceptional. Although this new map is not yet an automorphism, there is an infinite family of cases (see Theorem 9.1) where $f_{a,b}$ induces an automorphism after suitable further blowups.

All Hénon maps have constant Jacobian, which means that the holomorphic 2-form $\eta = dx \wedge dy$ satisfies $f^*(\eta) = \delta\eta$. If we extend f to a birational map of $\mathbb{P}^2$, then η becomes a meromorphic 2-form, with a pole of order 3 at infinity. When we consider an automorphism f of a compact, complex surface X, we require that there is a meromorphic 2-form η on X such that $f^*\eta = \delta\eta$ for some $|\delta| = 1$.

2 Fatou Set of a Conservative Hénon Map

A traditional starting place for complex dynamics is the dichotomy between the Fatou and Julia sets: the regions of regular and chaotic dynamics. If $f : X \to X$ is a holomorphic map, we define the *(forward) Fatou set* $\mathcal{F}^+$ to be the largest open subset of X such that the iterates form a *normal family*. This means that any sequence $\{f^{n_j}\}$ has a subsequence which converges uniformly on compact subsets of $\mathcal{F}$. This is equivalent to pre-compactness in the compact-open topology (which is the topology of uniform convergence on compact subsets). The Fatou set is also the set of points which are Lyapunov stable. If f is invertible, then the *backward Fatou set* $\mathcal{F}^-$ is defined similarly, with f replaced by f^{-1}.

In analogy with the case of a one-dimensional polynomial map, we define the *escape locus* U^+ and *boundedness locus* K^+ by

$$K^+ = \{q \in \mathbb{C}^2 : \{f^n(q) : n \geq 0\} \text{ is bounded}\}, \quad U^+ := \mathbb{C}^2 - K^+$$

[1] A Hénon map cannot be expressed as an automorphism of a compact surface. This is because by [16], the dynamical degree of a Hénon map is an integer greater than 1, whereas the dynamical degree of an automorphism of a compact surface cannot be rational, unless it is 1 (see [9]).

as well as the set $J^+ := \partial K^+$. It follows that the forward Fatou set is $\mathcal{F}^+ = \mathbb{C}^2 - J^+$. Similarly, we define K^-, U^- and J^- using f^{-1} in place of f, and we set

$$J := J^+ \cap J^-, \quad K := K^+ \cap K^- .$$

Let us recall that the *Shilov boundary* of a compact set $X \subset \mathbb{C}^2$, written $\partial_s X$, is the smallest closed set such that

$$\max_{x \in X} |P(x)| = \max_{x \in \partial_s X} |P(x)| .$$

By the Maximum Principle, $\partial_s X \subset \partial X$. We define $J^* := \partial_s K$.

In dimension 1, the set K is called the *filled Julia set*, and the *Julia set* itself is $J := \partial K$. For complex Hénon maps, the sets K^+, K^- and K are analogues of the filled Julia set, and the sets J^+, J^-, J and J^* are all analogues of the Julia set. While J^* may seem to be the most exotic of these sets, it has a number of natural characterizations (see [8] for (1) and [5] for (2)): (1) it is the support of the invariant (equilibrium) measure μ, and (2) it is the closure of the set of periodic points of saddle type. It is easily seen that $J^* \subset J$; and equality $J^* = J$ holds in a number of cases, but the problem of determining whether these two sets are always equal has remained elusive.

We recall the following basic result:

Theorem 2.1 ([16]) *If f is a volume-preserving (composition of) complex Hénon maps, then*

$$\mathrm{int}(K^+) = \mathrm{int}(K^-) = \mathrm{int}(K) .$$

This set is bounded, and if Ω is any connected component of $\mathrm{int}(K)$*, then there is an N such that $f^N(\Omega) = \Omega$.*[2]

A consequence of Theorem 2.1 is that $\mathcal{F}^\pm = U^\pm \cup \mathrm{int}(K)$. A *Fatou component* is a connected component of $\mathcal{F}^+$. We observe that since Ω is the set of normality of a sequence of polynomial mappings, it is a Runge domain, which means in particular that it is polynomially convex.

Although the unbounded set U^+ is a Fatou component, it is different from the others because it is the basin of attraction of a point at infinity, whereas, as we will see in the next section, all other Fatou components are rotation domains. Henceforth, we will consider only bounded Fatou components.

One of the most basic quadratic maps $p_c(z) = z^2 + c$ corresponds to c in the interior of the main cardioid of the Mandelbrot set. In this case, p_c has an attracting fixed point z_c, and the interior of $K(p_c)$ is the immediate basin of z_c. The interior of $K(p_c)$ is a Fatou component, and $K(p_c)$ itself is the closure of this component.

[2]A component Ω is said to be *wandering* if $f^n(\Omega) \cap \Omega = \emptyset$ for all nonzero $n \in \mathbb{Z}$. It is an open question, in the dissipative case $|\delta| < 1$, whether polynomial automorphisms can have wandering Fatou components.

In fact, $K(p_c)$ is topologically equivalent to a closed disk, and its boundary $J(p_c)$ is a Jordan curve.

Let p_c be as above, and let $f_\delta(x, y) = (y, p_c(y) - \delta x)$ be the Hénon map associated with p_c. It was shown in [15, 21] that if $|\delta| > 0$ is sufficiently small, then f_δ has an attracting fixed point, and the interior of K^+ is the basin of attraction. Thus K^+ is the closure of a Fatou component. In fact, J^+ is a topological 3-manifold, and (K^+, J^+) is a topological manifold-with-boundary. Our first question is whether anything similar can happen in the conservative case:

Question 1 *If Ω is a Fatou component of a conservative Hénon map, is it possible that $\overline{\Omega} = K$? Or is $\overline{\Omega}$ always a strict subset of K?*

3 Rotation Domains

We suppose that Ω is a conservative Fatou component with $f(\Omega) = \Omega$, and we define the set of all limits of convergent subsequences

$$\mathcal{G} := \{g = \lim_{n_j \to \infty} f^{n_j} : \Omega \to \overline{\Omega}\} .$$

If $g = \lim_{n_j \to \infty} f^{n_j}$ is such a limit, then g must preserve volume, and thus it is locally invertible. It follows that $g : \Omega \to \Omega$. Further, we may pass to a subsequence such that $m_j := n_{j+1} - 2n_j \to \infty$ and $h := \lim_{j \to \infty} f^{m_j}$ exists. It follows that

$$g \circ h = \lim_{j \to \infty} f^{n_j} \circ f^{m_j} = \lim_{j \to \infty} f^{n_{j+1} - n_j} = \mathrm{id}$$

and thus $h = g^{-1}$, so $\mathcal{G}$ is a group. Since Ω is a Fatou component, it follows that $\mathcal{G}$ is compact in the compact-open topology. By a Theorem of H. Cartan (see Narasimhan [28]), it follows that $\mathcal{G}$ is a Lie group. Thus the connected component $\mathcal{G}_0$ of the identity must be a (real) torus.

Theorem 3.1 ([7]) *$\mathcal{G}_0$ is isomorphic to $\mathbb{T}^\rho$ with $\rho = 1$ or 2.*

We conclude that Ω is invariant under a nontrivial torus of rotations, so we call it a *rotation domain*, and we refer to ρ as the *rank* of the rotation domain.

Question 2 *Does a rotation domain necessarily contain a fixed point?*

Rank 1. We first discuss the case of rank 1. In this case, $\mathcal{G}_0 \cong \mathbb{T}^1$, and $\mathcal{G}_0$ is generated by the real part of a holomorphic vector field $\mathcal{V}$ on Ω. It follows that the restriction of f to Ω is part of the flow generated by $\mathcal{V}$, so $f|_\Omega = \exp(t_1 \Re(\mathcal{V}))$ for some $t_1 > 0$. The zeros of $\mathcal{V}$ correspond to the fixed points of f in Ω; by [16], there are a total of d fixed points (counted with multiplicity) in $\mathbb{C}$. Suppose that $\omega \in \Omega$, and $\mathcal{V}(\omega) \neq 0$. Since $\mathcal{G}$ is compact, the orbit $\mathcal{G} \cdot \omega$ is a closed curve. It follows that the orbit $\mathcal{G} \cdot \omega$ is contained in a Riemann surface $\mathcal{R}_\omega$ and a proper map $\varphi_\omega : \mathcal{R}_\omega \to \Omega$. Now $\mathcal{R}_\omega$ is a Riemann surface which carries a $\mathbb{T}^1$ of automorphisms, so it follows that $\mathcal{R}_\omega$ must either be the disk or an annulus. We may write $\mathcal{R}_\omega$ as $\{|\zeta| < 1\}$ or $\{r_1 < |\zeta| < r_2\}$. Then the restriction $f|_{\mathcal{R}_\omega}$ is given by $\zeta \mapsto \alpha\zeta$, where $\alpha = \exp(2\pi i a)$, $a > 0$, and a is the rotation number of f on the curve $\mathcal{G} \cdot \omega$. The fixed points of f are isolated, so a must be irrational. Further, since a depends continuously on ω, we conclude that a must be constant. We call this the *rotation number of* Ω, written $\mathrm{rot}(\Omega)$.

The question arises whether the (abstract) torus action on Ω is equivalent to a more familiar circle action. Let $D \subset \mathbb{C}^2$ be a connected open set. We say that D is a (p, q) *domain* if $(e^{ip\theta}z, e^{iq\theta}w) \in D$ whenever $(z, w) \in D$ and $\theta \in \mathbb{R}$.

Question 3 *Suppose that Ω is a rank 1 rotation domain. Is there a (p, q)-domain $D \subset \mathbb{C}^2$ and a biholomorphic $\Phi : \Omega \to D$ satisfying $L \circ \Phi = \Phi \circ f$, with $L = \begin{pmatrix} \alpha^p & 0 \\ 0 & \alpha^q \end{pmatrix}$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$?*

If there is such a domain D, then we may take p and q to be relatively prime, and $\alpha = e^{2\pi i r}$, where $r = \mathrm{rot}(\Omega)$.

Question 4 *Can the case $pq < 0$ occur in Question 3?*

4 Reinhardt Domains

Let $D \subset \mathbb{C}^2$ be a connected open set. We say that D is a *Reinhardt domain* if $(e^{i\theta}z, e^{i\phi}w) \in D$ for all $(z, w) \in D$ and all $\theta, \phi \in \mathbb{R}$. A Reinhardt domain is determined by its logarithmic image

$$\log(D) := \{(\xi, \eta) = (\log|z|, \log|w|) \in \mathbb{R}^2 : (z, w) \in D\} .$$

A classical result asserts that D is holomorphically convex if and only if $\log(D)$ is convex. If Ω is a rank 2 rotation domain, then by [1] the $\mathcal{G}$-action on Ω may be conjugated to the standard linear action on $\mathbb{C}^2$:

Theorem 4.1 ([1]) *There are a Reinhardt domain $D \subset \mathbb{C}^2$, a linear map $L = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$, $|\alpha_1| = |\alpha_2| = 1$, and a biholomorphic map $\Phi : \Omega \to D$ such that $\Phi \circ f = L \circ \Phi$.*

Question 5 *What are the Reinhardt domains that can arise as rank 2 rotation domains?*

Note that Reinhardt D is the biholomorphic model of Ω, whereas Ω is a subset of dynamical space. Thus the boundary of D is logarithmically convex and rather "tame", whereas we expect that Ω may have a possibly "wild", fractal boundary.

The torus action on D has no fixed point except the origin $(0, 0)$. In our case, Ω is polynomially convex, and for polynomially convex sets we have $H_2(\Omega; \mathbb{Z}) = 0$. There are two possibilities: a polynomially convex Reinhardt domain is either:

(1) topologically equivalent to a ball (in which case it contains the fixed point $(0, 0)$), or
(2) topologically equivalent to disk $\times$ annulus (in which case it contains an invariant annulus inside one of the coordinate axes).

In either case, $\log(D)$ will be an unbounded, convex subset of $\mathbb{R}^2$. It seems hard to imagine that the Reinhardt model can be a domain that is "familiar," so we ask (expecting the answer "no"):

Question 6 *Can there be a "Siegel bidisk"? That is, can Ω be biholomorphic to the bidisk $\Delta^2 := \{(x, y) \in \mathbb{C}^2 : |x| < 1, |y| < 1\}$? More generally, can Ω be an analytic polyhedron?*

Question 7 *Can there be a "Siegel ball"? That is, can Ω be biholomorphic to the standard ball $\mathbb{B}^2 := \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 < 1\}$? More generally, can the boundary of Ω (or its Reinhardt model) be smooth?*

5 Existence of Rotation Domains

Let a linear map $L = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ be given, and consider a map

$$F(x, y) = L \begin{pmatrix} x \\ y \end{pmatrix} + \sum_{j+k \geq 2} F_{jk} x^j y^k, \quad F_{jk} \in \mathbb{C}^2 . \tag{5.1}$$

Consider a power series

$$\Phi = (x, y) + \sum_{j+k \geq 2} \Phi_{jk} x^j y^k, \quad \Phi_{jk} \in \mathbb{C}^2 \tag{5.2}$$

and the power series equation

$$\Phi \circ F = L \circ \Phi .$$

A *resonance* is a relation of the form $\lambda_j = \lambda_1^{m_1} \lambda_2^{m_2}$, where $j = 1$ or 2, and $m_1, m_2 \geq 0$, $m_1 + m_2 \geq 2$. Note that if $\lambda_1 = \alpha^p$, $\lambda_2 = \alpha^q$, then there are infinitely many resonances whenever $pq < 0$. If $pq > 0$, there is a resonance if and only if α is a root of unity. If there is no resonance, then it is possible to solve algebraically for the coefficients Φ_{jk}. In fact, the solution Φ_{jk} is a rational function in the coefficients F_{pq} with $p < j$ and $q < k$. The solution Φ is a formal power series, and convergence is a question. A classic theorem of C.L. Siegel (see Herman [19] and Pöschel [29] for subsequent developments) asserts:

Theorem 5.1 (Siegel, …) *If λ_1 and λ_2 are "sufficiently irrational", then the power series Φ converges in a neighborhood of the origin.*

The convergence of such a series has been much studied and is an example of a "small divisor" problem. Let us restrict ourselves to noting that the condition "sufficiently irrational" holds for almost every choice of (λ_1, λ_2) with $|\lambda_1| = |\lambda_2| = 1$. We say that λ_1 and λ_2 are *multiplicatively independent* if $\lambda_1^{m_1}\lambda_2^{m_2} = 1$ implies $m_1 = m_2 = 0$. Otherwise, there exists α such that $\lambda_1 = \alpha^p$, $\lambda_2 = \alpha^q$. (If $pq < 0$, there is a resonance between λ_1 and λ_2, and the "sufficiently irrational" hypothesis of Siegel's Theorem is not met.) When we apply Siegel's Theorem, the rotation domain Ω will have rank 2 if the λ_j's are multiplicatively independent, and rank 1 otherwise.

If a quadratic Hénon map has a fixed point whose differential is diagonalizable with eigenvalues λ_j, $j = 1, 2$, then it is conjugate to

$$H_{\lambda_1,\lambda_2} : (x, y) \mapsto (\lambda_1 x, \lambda_2 y) + (1, -1)(\lambda_1 x + \lambda_2 y)^2 . \tag{5.3}$$

If H_{λ_1,λ_2} is of the form (5.3), and if the eigenvalues $|\lambda_1| = |\lambda_2| = 1$ are suitable for Siegel's Theorem, then H_{λ_1,λ_2} is linearizable in a neighborhood U of the origin. It follows that U is contained in a Fatou component (rotation domain) $\Omega = \Omega_{\lambda_1,\lambda_2}$ with $H(\Omega) = \Omega$. Although the domain $\Omega_{\lambda_1,\lambda_2}$ is nonempty for almost every choice of $|\lambda_1| = |\lambda_2| = 1$, it is unstable, because the roots of unity are dense in $|\lambda_1| = |\lambda_2| = 1$, and whenever λ_1 and λ_2 are roots of unity, H_{λ_1,λ_2} cannot be linearized, and thus $\Omega_{\lambda_1,\lambda_2} = \emptyset$. In dimension 1, the dependence of the radius of a Siegel disk on the multiplier λ is related to the Brjuno function (see [23, 24]).

Let us revisit Question 4 above in the special case where Ω contains a fixed point. In this case, the eigenvalues have a resonance at the fixed point. Thus the generic polynomial map cannot be linearized. Question 4 asks whether *every* map with this resonance will fail to be linearizable.

We close this section with an easy complement to Siegel's Theorem. This is Proposition 5.1, which allows us to conclude that any Fatou component with a fixed point could have been constructed a *posteriori* by linearization. Let us suppose that Ω is a rotation domain, and suppose that $(0, 0) \in \Omega$ is a fixed point for F. Then $DF(0, 0)$ is linearly conjugate to a matrix $L = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $|\lambda_1| = |\lambda_2| = 1$. Let us consider the sequence of maps

$$\Phi_n = n^{-1} \left(Id + L^{-1}F + \cdots + L^{-n+1}F^{n-1} \right) .$$

Now F^n is bounded on any compact subset of Ω, so the Φ_n are bounded there, too. Thus $\{\Phi_n\}$ is a normal family of mappings, and we can let $\tilde{\Phi} := \lim_{j\to\infty} \Phi_{n_j}$ denote any sub-sequential limit. It is easy to see that $\tilde{\Phi}$ linearizes F in a neighborhood of $(0, 0)$:

Proposition 5.1 $\tilde{\Phi} \circ F = L \circ \tilde{\Phi}$.

Question 8 *Is there a construction that gives a Fatou component without starting at a fixed point?*

6 Nonexistence of Rotation Domains

The simplest resonance is where one of the eigenvalues is 1. Suppose f is a Hénon map with a fixed point z_0. If the eigenvalues of $Df(z_0)$ are 1 and λ, then f cannot be locally linearized at z_0 because the linear map with eigenvalues 1 and λ has a curve of fixed points, corresponding to the multiplier 1, but the fixed points of f are isolated. The next simplest case is where the eigenvalues are α and $\alpha^{-1} = \bar{\alpha}$, and thus $|\alpha| = 1$. This leads to an infinite number of resonances, each of which is a possible obstruction to linearization.

Much of the early interest in Hénon maps arose from the real, area-preserving case (see [18]). A complex Hénon map preserves $\mathbb{R}^2$ when it has real coefficients. In this case, the jacobian is $\delta = \pm 1$, which means that a periodic point has multipliers λ and $\pm\lambda^{-1}$, which forces a resonance. A fixed point for a generic area-preserving map with $|\lambda| = 1$ will be of "twist" type. The classical KAM theory asserts that near such a fixed point, there will be a positive measure set of rotation numbers ω, and for each ω there will be an invariant "KAM curve" γ_ω. For a complex Hénon map, γ_ω has a complexification to an annulus $\tilde{\gamma} \subset K \subset \mathbb{C}^2$. The "standard picture" of a twist map does not allow γ_ω to be inside the Fatou set.

We must ask whether, contrary to the generic picture, there is a map that can have a rotation domain. We expect that the answer to the following question is "no" in each case:

Question 9 *Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an area-preserving real Hénon map.*

(i) Is it possible for h to have a linearizable fixed (periodic) point?
(ii) Is it possible for the Fatou set $\mathcal{F}(h)$ to be nonempty?
(iii) Is it possible for $\mathcal{F}(h) \cap \mathbb{R}^2 \neq \emptyset$?

7 Computer Pictures: The Ushiki Approach

Computers have been very useful in illustrating theorems and motivating new results in the study of complex Hénon maps. The most useful computer picture has been the *unstable slice* picture, which was introduced and widely used by J. Hubbard. To make such a picture, you start with a saddle fixed (or periodic) point q. The "unstable slice" is $W^u(q) \cap K^+$. The unstable manifold $W^u(q)$ is conformally equivalent to $\mathbb{C}$, and it is not hard to compute the uniformization $\psi_q : \mathbb{C} \to W^u(q) \subset \mathbb{C}^2$. The rate of escape function (Green function of K^+) is given by

$$G^+(x, y) := \lim_{n\to\infty} \frac{1}{d^n} \log^+ \|f^n(x, y)\| .$$

In order view the unstable slice, we simply look at the level sets of G^+ and its harmonic conjugate. We refer the reader to [22] for an extended discussion of this and related pictures. Two important features of this picture are:

(6.1) The unstable slice $K^+ \cap W^u(q)$ is invariant under f, and the resulting picture in the ζ-plane $\mathbb{C}$ is invariant under a complex scaling $\zeta \mapsto \beta\zeta$.

(6.2) The restriction of G^+ to the unstable manifold is subharmonic, and it is is harmonic outside K^+. Thus the level sets will be compatible with the maximum principle and the mean value property.

While the unstable slice picture is quite useful in the dissipative case, the unstable slice cannot "see" the Fatou set of a conservative map since $W^u(q) \cap \Omega = \emptyset$ for any bounded Fatou component Ω. (For if a point belongs to Ω, then its backward/forward orbit stays away from the boundary; but if a point belongs to $W^u(q)$, its backward orbit converges to $q \notin \Omega$.) The inability of the stable/unstable manifolds to "penetrate" the Fatou set may be a reason why the conservative case seems more difficult than the dissipative case.

We say that a map f is *reversible by an involution* $\tau = \tau^{-1}$ if $\tau \circ f \circ \tau = f^{-1}$ (see [17]). For instance, if $\delta = 1$, then $f(x, y) = (y, p(y) - x)$, and f is reversible by the involution $(x, y) \mapsto (y, x)$. If h is a polynomial automorphism, then the (constant) Jacobian determinant of $h^{-1} \circ f \circ h$ is the same as that of f. Thus f cannot be reversible by a (holomorphic) polynomial automorphism unless $\delta = \pm 1$.

Theorem 7.1 (Ushiki) *A Hénon map is reversible by the (anti-holomorphic) involution $\tau(x, y) = (\bar{y}, \bar{x})$ if and only if it has the form*

$$f(x, y) = (y, \beta p(y) - \beta^2 x)$$

where $p(y)$ is a real polynomial, and $|\beta| = 1$.

The fixed point set of τ is the conjugate diagonal:

$$\mathrm{Fix}(\tau) = \Delta' := \{(\zeta, \bar{\zeta}) : \zeta \in \mathbb{C}\} .$$

The Ushiki approach is to look at maps that are reversible under τ and look at the slice of the interior of K by the conjugate diagonal Δ'. The new pictures will not have the same "feel" as the unstable slice pictures because properties (6.1) and (6.2) above do not hold. However, these slices are well suited to reversible maps, which seem to be a rich source of rotational behavior. Restricting to reversible maps also has the benefit of reducing the dimension of the (real) parameter space to two.

Ushiki has made a number of pictures of $K \cap \Delta'$ and has found a number of interesting phenomena. One of Ushiki's parameter values is used to make Figs. 1 and 2, which show two slices of K for the reversible map $f(x, y) = (y, e^{i\theta}(y^2 + \alpha) - e^{2i\theta}x)$. Figure 1 is the slice of K by the conjugate diagonal $(\zeta, \bar{\zeta})$

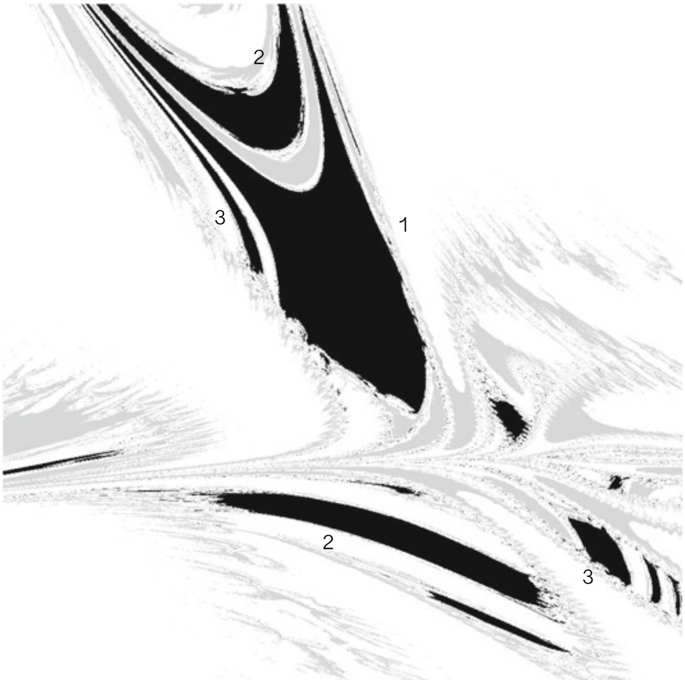

Fig. 1 Hénon map $f(x, y) = (y, e^{i\theta}(y^2 + \alpha) - e^{2i\theta}x)$: $\alpha = 0.269423$, $\theta = 1.02773$. Slice of K by the conjugate diagonal Δ'

for $0.2 \le \Re(\zeta) \le 0.6$, $-0.2 \le \Im(\zeta) \le 0.4$; Fig. 2 shows the slice of K by the "horizontal" line $(\zeta, 0.4 - 0.1\,i)$, $0.3 \le \Re(\zeta) \le 0.56$, $-0.05 \le \Im(\zeta) \le 0.3$. Points of $K = \{G^+ + G^- = 0\}$ are black, and other points are white/gray according to the value of $G^+ + G^- > 0$. Note, however, that the function $G^+ + G^-$ has no dynamic invariance.

The black components labeled '1' in Figs. 1 and 2 represent the same component of the interior of K: points from each of these regions have the same orbits. The point $(\zeta_1, \bar{\zeta}_1)$, $\zeta_1 = 0.396 + 0.19i$ is taken from region '1'. The orbit of this point is shown in Fig. 3 under two projections: $\pi_1(x, y) = (Re(x), Re(y))$ on the left, and $\pi_2(x, y) = (Re(x - y), Im(y))$ on the right. The closure of the orbit appears to be union of three 2-tori. Region '1' contains a point of period 3 and appears to be the slice of a 3-cycle of rotation domains of rank 2.

The black components labeled '2' in Figs. 1 and 2, represent the same connected component of the interior of K in $\mathbb{C}^2$. The point $(\zeta_2, \bar{\zeta}_2)$, $\zeta_2 = 0.36 + 0.298i$, is taken from region '2', and Fig. 3 shows the projection of points from the orbit of this point. Recall that by Theorem 3.1, the closure of an orbit is either a closed curve or a 2-torus. Thus it would appear that the orbits in Fig. 3 are increasing to something whose closure would be a (connected) 2-torus. If this is in fact the case, then region '2' is contained in an invariant Fatou component Ω_2 which has rank 2. On the other

Fig. 2 Same Hénon map as Fig. 1. Slice of K by complex line

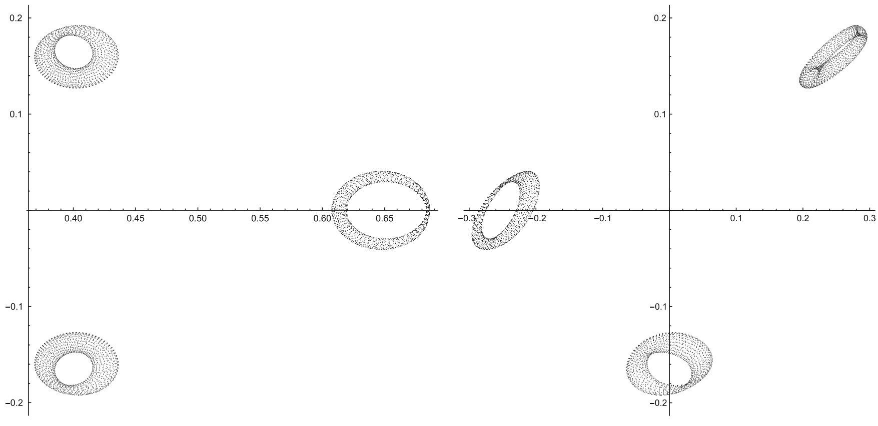

Fig. 3 First 5000 iterations of a point from region '1'; two projections

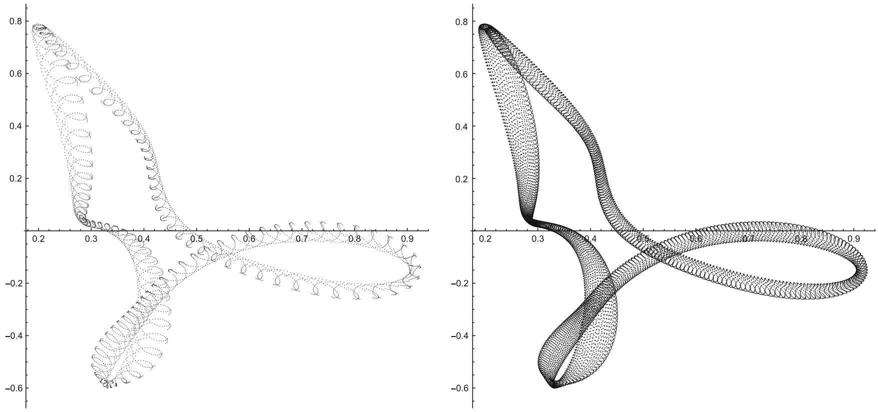

Fig. 4 5000 and 50000 iterations of a point from region '2'

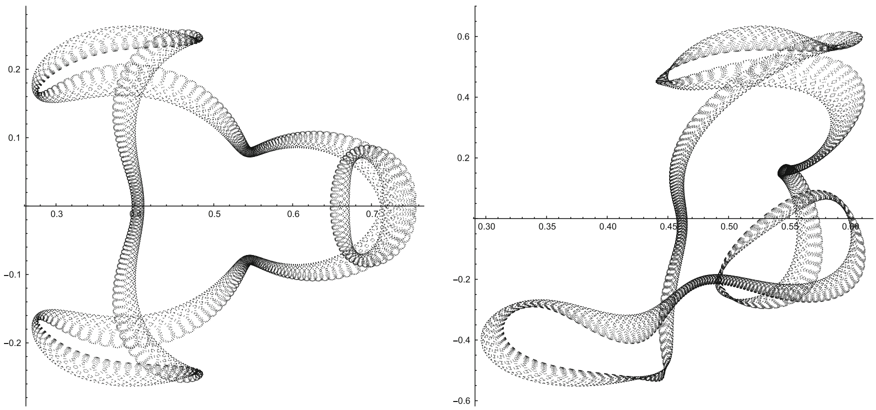

Fig. 5 Orbit (40000 points) from region '3': two projections

hand, Ω_2 cannot contain a fixed point because the 2 fixed points of f are both of saddle type. Thus Ω_2 would appear to be "exotic", which means that it is a rotation domain without a fixed point. There are also other components, such as '3', which appear to belong to "exotic" rotation domains, which was a motivation for Ushiki to find Hénon maps like this.

Problem 10 *Prove mathematically that there are Hénon maps with exotic rotation domains.*

It may appear surprising that the 2-tori in Figs. 3 and 4 are long and thin. If '1', '2', and '3' actually represent rotation domains of rank 2, then as was noted in Sect. 2, each of these domains is uniformized by a Reinhardt domain D, and each D has nonempty intersection with one or both coordinate axes. The axes are not generic for the torus action: the f-orbits of these points are dense in closed curves. So for a point of D which is close to one of the axes, we expect the orbit to be dense in a long, thin torus which looks almost like a closed curve such as in Figs. 4 and 5.

Question 11 *Can a Hénon map have infinitely many rotation domains? Can it have an infinite number of rotation domains with fixed (periodic) points?*

This question is motivated by the fact that there seem to be so many black components in Fig. 1 and any one Fatou component should not create many slice components because of the following rank 2 phenomenon:

Theorem 7.2 *Let f be reversible by τ, and let $\Omega = f(\Omega)$ be a rank 2 Fatou component with $\Omega \cap \Delta' \neq \emptyset$. If Ω contains a fixed point, then $\Omega \cap \Delta'$ is connected; otherwise it has exactly two connected components.*

Proof Since f is τ-reversible, $\tau(\Omega)$ is a Fatou component. Further, since $\Omega \cap \Delta' \neq \emptyset$, it follows that $\Omega \cup \tau(\Omega)$ is connected. Thus $\Omega = \tau(\Omega)$ is invariant under τ. Now we let $D \subset \mathbb{C}^2_{z_1,z_2}$ be the Reinhardt model for Ω, and the conjugacy takes f to a diagonal map $L = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ with $|\mu_1| = |\mu_2| = 1$. Let $\hat{\tau}$ denote the map of D induced by this conjugacy. Thus $\hat{\tau}$ is an anti-holomorphic and reverses L. Since L generates the standard torus action on $\mathbb{C}^2$, it follows that $\hat{\tau}$ reverses the torus action. Now we may write $\hat{\tau}(z) = (\hat{\tau}_1, \hat{\tau}_2)$, where each $\hat{\tau}_j = \sum a_\alpha \bar{z}^\alpha$ is a Laurent series in the anti-holomorphic variables $(\bar{z}_1, \bar{z}_2)$. The property of reversing the torus action means that for all $\theta_1, \theta_2 \in \mathbb{R}$, we have

$$\hat{\tau}(e^{i\theta_1} z_1, e^{i\theta_2} z_2) = (e^{-i\theta_1} \hat{\tau}_1, e^{-i\theta_2} \hat{\tau}_2) \ .$$

By checking the coefficients of the Laurent series, we conclude that $\hat{\tau}(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$.

The fixed point set of $\hat{\tau}$ is $\mathbb{R}^2$, so $\Omega \cap \Delta'$ corresponds to $D \cap \mathbb{R}^2$. If Ω has a fixed point, then D must contain the origin, and thus $D \cap \mathbb{R}^2$ is a connected set (which is logarithmically convex). On the other hand, if Ω does not have a fixed point, then D must be disjoint from one of the coordinate axes, say z_1. Thus $D \cap \mathbb{R}^2$ consists of two (logarithmically convex) open sets, which are symmetric under the map $(x_1, x_2) \mapsto (x_1, -x_2)$. This completes the proof. □

Question 12 *Can a Hénon map have exactly one rotation domain? Or can the existence of one rotation domain cause the existence of another?*

The essence of this question is whether there might be some phenomenon for rotation domains in the complex domain which is reminiscent of the island chains for twist maps: one twist causes the existence of others.

8 Herman Rings for Dissipative Maps?

For dissipative maps, $\mathrm{int}(K^+)$ can be unbounded, and it is not known whether every Fatou component $\Omega \subset \mathrm{int}(K^+)$ is necessarily periodic. Let us suppose that $f(\Omega) =$

Ω. We say that Ω is *recurrent* if there is a point $z_0 \in \Omega$ whose ω-limit set contains a point $z_1 \in \Omega$. In other words, there is a point $z_0 \in \Omega$ so that all the forward iterates do not converge to $\partial\Omega$. In the dissipative case, it is shown in [7] that every recurrent, periodic Fatou component Ω is a basin; there are 3 possibilities:

(1) Ω is the basin of an attracting fixed point;
(2) $\Omega = \mathcal{B}(\mathcal{S})$ is the basin of a Siegel disk; that is, $\varphi : \{|\zeta| < 1\} \to \mathcal{S} \subset \mathbb{C}^2$ is a holomorphic imbedding, and $f|_\mathcal{S}$ is conjugate to an irrational rotation;
(3) $\Omega = \mathcal{B}(\mathcal{A})$ is the basin of an annulus $\mathcal{A} \cong \{r_1 < |\zeta| < r_2\}$, and $f|_\mathcal{A}$ is conjugate to an irrational rotation.

We may linearize maps (5.3) at the origin with suitable eigenvalues λ_j, $j = 1, 2$, and show that cases (1) and (2) can occur. However the possibility of case (3) remains an unanswered question. In case (3), we have the intriguing situation that $\mathcal{B}(\mathcal{A})$ is biholomorphically equivalent to the product $\{r_1 < |\zeta| < r_2\} \times \mathbb{C}$, yet it must also be polynomially convex.

Let us suppose that Ω is a rank 2 rotation domain for the case $|\delta| = 1$, and let D be its Reinhardt model. If Ω has no fixed point, then $D \subset \mathbb{C}^2_{z,w}$ must intersect exactly one of the axes, say $\{w = 0\}$, and $A := D \cap \{w = 0\}$ must be an annulus. In other words, the condition that Ω does not have a fixed point is equivalent to the condition that Ω contains a (unique) proper, invariant annulus.

Question 13 *What happens to the invariant annulus $\mathcal{A}$ inside an exotic rotation domain if we perturb the map slightly to become dissipative? Can it "persist", or does it always "disappear"?*

The point here is that if $\mathcal{A}$ does not disappear, then we would have an example of case (3).

9 Rational Surface Automorphisms Preserving a 2-Form

The rational surface maps with invariant 2-form have been classified by Diller and Lin [12]. Rational surface automorphisms with invariant 2-forms have been given by [2, 4, 11, 26, 30]. The polynomial

$$\chi_n(t) = t^n(t^3 - t - 1) + t^3 + t^2 - 1 \tag{9.1}$$

is related to the family of rational surface automorphisms (1.2). It may be factored as $\chi_n = C_n S_n$, where C_n is a product of cyclotomic factors (all of whose zeroes are roots of unity), and S_n is a Salem polynomial, which means that it has two real roots λ and λ^{-1}, with $\lambda > 1$, and all other roots have modulus 1.

Theorem 9.1 ([2]) *There is a blowup $\pi : X \to \mathbb{P}^2$ at $n + 3$ points such that $f_{a,b}$ lifts to an automorphism of X if and only if $f^n_{a,b}(-a, 0) = (-b, -a)$. If in addition $n \geq 7$, then the entropy of the automorphism $f_{a,b}$ is* $\log \lambda$*, where $\lambda > 1$ is the largest real root of S_n.*

Some of the automorphisms given in this Theorem have invariant curves and some do not (see [3]). The curve $\{y = x^3\}$ is a cubic with a cusp at infinity. We will let $\mathcal{C}$ denote the image of this cubic under a linear automorphism of $\mathbb{P}^2$, and we let $\eta = dx \wedge dy/p(x, y)$ denote a 2-form with a simple pole along $\mathcal{C}$. McMullen [26] (see also [3] for a different approach) shows that among the maps $f_{a,b}$, there is a map with invariant curve for every root of S_n.

Theorem 9.2 ([26]) *If δ is a root of S_n, then there is an automorphism $f_{a,b}$ which leaves the cubic curve $\mathcal{C}$ invariant and which satisfies $f^*_{a,b}\eta = \delta\eta$.*

Let us change notation and write f_δ for the map in Theorem 9.2. We note that for each $n \geq 7$, all but two of the roots of S_n have modulus 1, which means that all but two of the f_δ are conservative automorphisms.

Theorem 9.3 (Ushiki) *The maps f_δ with $|\delta| = 1$ are conjugate to*

$$(x, y) \mapsto \left(y, \frac{y + \alpha}{x + i\beta} + i\beta\right) \tag{9.2}$$

with $\alpha, \beta \in \mathbb{R}$. This map is reversible by the involution $\tau(x, y) = (\bar{y}, \bar{x})$, and $\mathcal{C} \cap$ $\mathrm{Fix}(\tau)$ is a real curve.

Now let $\mathcal{F}^+$ denote the Fatou set, and let Ω be a Fatou component such that $f(\Omega) = \Omega$. We again consider the set of limits, $\mathcal{G}$, as in Sect. 1, and we find that the connected component of the identity $\mathcal{G}_0$ is a torus $\mathbb{T}^\rho$ with $\rho = 1$ or 2 (see [4] for details). Thus all periodic Fatou components are rotation domains of rank either 1 or 2. In particular, we see that $\mathcal{F}^- = \mathcal{F}^+$, so in the case of a conservative automorphism we will denote it simply as $\mathcal{F}$.

The multipliers of Df_δ at the cusp point of $\mathcal{C}$ are δ^{-2}, δ^{-3}, so when $|\delta| = 1$ each f_δ has a rotation domain of rank 1. It was shown in [26] and in [3] that in many cases the maps f_δ also have rank 2 rotation domains. We ask whether this always happens:

Question 14 *Does f_δ always have a rotation domain in addition to the one centered at the cusp point? In other words, is the Fatou set for f_δ always disconnected?*

Let $p_2 \in \mathcal{C}$ denote the other fixed point (not the cusp) of f. The eigenvalues of Df at p_2 are δ and δ^{-n}, so there is a resonance. By Pöschel [29] there are two invariant complex disks passing through p_2. Ushiki's computer work suggests more: an answer of "yes" the following:

Question 15 *Does the rotation domain containing the cusp contain the whole curve $\mathcal{C}$? In particular, is f_δ linearizable at p_2?*

An immediate consequence of "yes" would be that f_δ can have no wandering Fatou component, since the invariant volume form $\eta \wedge \bar{\eta}$ is bounded outside a neighborhood of $\mathcal{C}$. Some analogous resonant fixed points for (other) rational surface automorphisms were shown in [4] to be linearizable.

Further Questions All of the questions that we have asked about Fatou components of Hénon maps apply equally to surface automorphisms. In order to avoid duplication, we do not re-state them here. We note moreover that these same questions are interesting also in the case where f is merely birational (invertible but not required to be everywhere regular) and conservative.

How can we draw a computer picture of the Fatou set of a conservative surface automorphism? In the case of Hénon maps, it suffices to draw the set K^+ or K where orbits are bounded, but there seems to be nothing analogous for other automorphisms. The other useful object, the Green function G^+, exists only because there is a super-attracting invariant curve (the line at infinity). One alternate approach is to consider the maximal Lyapunov exponent of a point $p \in X$:

$$\Lambda(p) = \limsup_{n\to\pm\infty} \frac{1}{|n|} \log \|Df^n(p)\|.$$

Here $\|\cdot\|$ denotes the operator norm of Df with respect to any norm on the tangent space of X; the limit is independent of choice of norm.

If p belongs to the Fatou set, then Df^n is bounded in a neighborhood of p for all $n \geq 0$, and thus $\Lambda(p) = 0$. The converse, however, is not always true. For example, consider the automorphisms $f_{a,b}$ as in Theorem 9.1 for $n = 6$ (that is, X is obtained by blowing up $\mathbb{P}^2$ at 9 points). These automorphisms have the property that $\deg(f^n) \sim n^2$, and thus the derivative of f^n grows quadratically. Thus $\Lambda = 0$, and $f_{a,b}$ has zero entropy. On the other hand, these examples have an invariant fibration on which f acts as a "twist", so $\mathcal{F} = \emptyset$.

Let us restrict our attention now to the case of an automorphism with positive entropy $\log\lambda > 0$. In this case, there are positive closed currents $T^\pm$ which are invariant in the sense that $f^*T^\pm = \lambda^{\pm1}T^\pm$. Further, the wedge product of these currents defines a measure $\mu := T^+ \wedge T^-$, which is the unique measure of maximal entropy. The reader is referred to Cantat [10] for a presentation of this material. In particular, (see [10, Theorem 7.5]), we have:

Theorem 9.4 (Dinh-Sibony, Moncet, Ueda) *For an automorphism of positive entropy, $\mathcal{F} := \mathcal{F}^+ \cap \mathcal{F}^-$ is the complement of the support of $T^+ + T^-$, modulo periodic curves.*

We would now like to give another description of the Fatou set in terms of the support of $T^+ + T^-$. There are two Lyapunov exponents $\lambda_1 \geq \lambda_2$ with respect to μ. The larger one is given by

$$\lambda_1 = \lim_{n\to\infty} \frac{1}{n} \int \log \|Df^n(p)\| \, \mu(p) \, .$$

In the conservative case, we have $\lambda_1 + \lambda_2 = 0$, so the larger exponent determines the smaller one. In general (see [14]) we have

$$\lambda_1 \geq \frac{\log \lambda}{2} > 0 > -\frac{\log \lambda}{2} \geq \lambda_2 \,.$$

Thus μ is a hyperbolic measure, and by Pesin Theory, there are stable/unstable manifolds $W^{s/u}(x)$ through μ a.e. point.

Theorem 9.5 (Dujardin) *For μ a.e. x, the Pesin manifold $W^{s/u}(x)$ is dense in the support of $T^{+/-}$.*

Proof It suffices to consider T^+. This is generated by submanifolds in the "strongly laminar" sense of [13], which also gives a definition of "being in the same leaf", and shows that this is an equivalence relation on the submanifolds of T^+. By [13, Corollary 5.8] the space of leaves is ergodic with respect to the trace measure of T^+: for each measurable subset of leaves, either it or its complement has measure zero. Now let $\{U_i\}$ be a neighborhood basis for the support of T^+. It follows that the set of leaves that do not intersect U_i has trace measure zero. Taking a countable union of sets of zero measure, we see that the set of leaves which intersect every U_i has full measure. Now consider the leaves which are Pesin manifolds. This is a measurable, invariant set, which must have full trace measure, by the product structure of μ. Thus almost every Pesin manifold intersects every U_i.

Theorem 9.6 *Let U be an open set such that $\Lambda(p) < \frac{\log \lambda}{2}$ for all $p \in U$. Then U is disjoint from the support of $T^+ + T^-$.*

Proof It suffices to consider the case where U intersects the support of T^+. By Theorem 9.5, there is a Pesin manifold $W^s(x)$ and a point $p \in W^s(x) \cap U$. If $t \in T_p W^s(x)$ is a nonzero tangent vector, then $\log |Df^n(p)t|$ decreases like $n\lambda_2(x)$. Since f is conservative, it follows that $\log \|Df^n(p)\|$ grows like $n\lambda_1(x)$. We conclude that $\Lambda(p) \geq \frac{\log \lambda}{2}$.

Theorems 9.4 and 9.6 justify computing the set $\{\Lambda(p) > 0\}$ as the complement of $\mathcal{F}$.

Figures 6 and 7 For Figs. 6 and 7, we have used a map corresponding to $n = 7$ in Theorem 9.1. The Salem polynomial S_7 has degree 10. Eight of the roots of S_7 have modulus 1, and we may pair them up $\{\delta_j, \bar{\delta}_j\}$, $1 \leq j \leq 4$, where $f_{\bar{\delta}_j}$ is conjugate to the inverse of f_{δ_j}. This gives four essentially distinct conservative maps. When $n = 7$, the automorphism f_δ is obtained from $\mathbb{P}^2$ by blowing up 10 points, and the entropy is $\log 1.17628$, which was shown in [26] to be the minimum possible for an automorphism of a compact, complex surface. Figure 6 shows the map f_δ with $\delta \sim -0.2344 + 0.9721i$. The left hand image shows the slice by conjugate diagonal points $p_2 + (\zeta, \bar{\zeta})$, where p_2 is the resonant fixed point of Question 15, and $-2.4 \leq \Re(\zeta), \Im(\zeta) \leq 2.4$. The right hand frame of Fig. 6 is a detail, centered at p_2.

The coloring of Figs. 6 and 8 is opposite from Figs. 1 and 2, where the Fatou set was the interior of the black; here the white regions correspond to the Fatou set. Or perhaps more precisely, Figs. 1 and 2 show the basin of infinity in shades of white/gray, while Figs. 6 and 8 use shades of gray to show the Julia set. The region

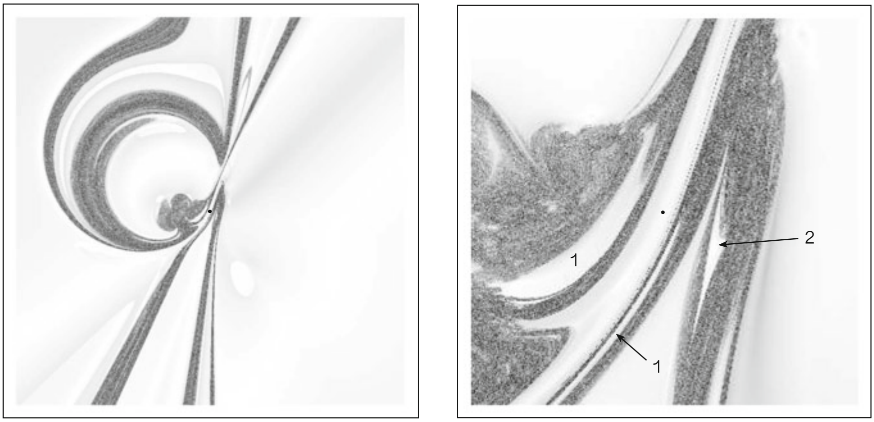

Fig. 6 Lyapunov exponent of map (8.2) with $\alpha \sim 0.5695$, $\beta \sim 0.3977$. Slice by conjugate diagonal Δ' (left); 'dot' is fixed point p_2. $8\times$ zoom about p_2 (right)

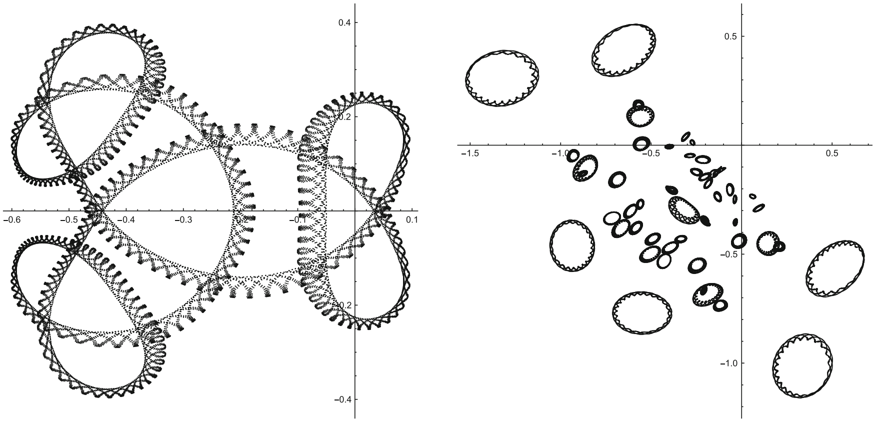

Fig. 7 Two orbits from Fig. 6. Region '1' (left); region '2' (right)

'1' on the right hand of Fig. 6 (whose slice by the conjugate diagonal appears to contain two components) was brought to our attention by Ushiki. An orbit from '1' is given on the left hand frame of Fig. 7. It suggests that '1' is part of a connected Fatou component Ω_1 of rank 2. If this is so, then Ω_1 must be "exotic" since it cannot contain a fixed point. The reason for this is the fixed points of f_δ are p_1 and p_2 which have been discussed earlier. Their eigenvalues are not multiplicatively independent, and thus they cannot be contained in a rotation domain of rank 2. The right hand frame of Fig. 7 indicates that '2' is contained in a rank 2 Fatou component of period 50. The number 50 is seen because an orbit of f^2 has visibly fewer components, and an orbit of f^{10} appears to consist of 5 components.

Another family of birational maps is given by

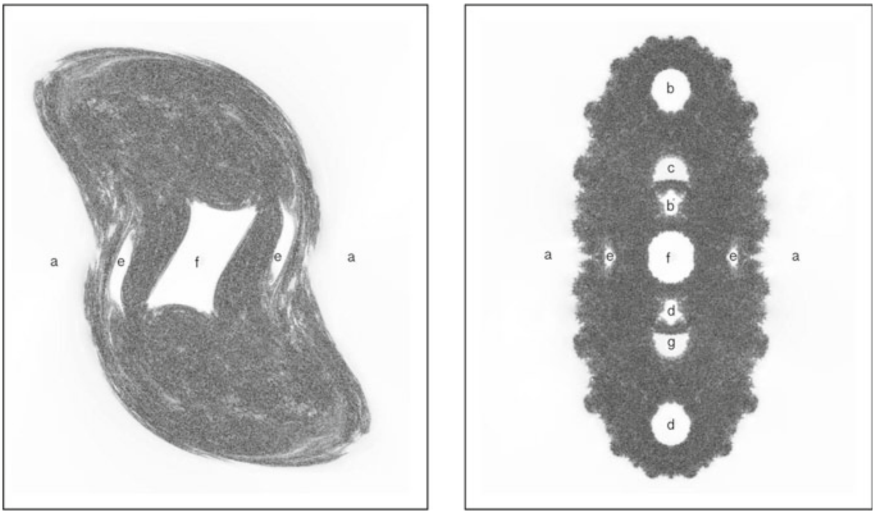

Fig. 8 Lyapunov exponent for map $(x, y) \mapsto (y, \beta(\sqrt{2}y + 1/y) - \beta^2 x)$. $\beta \sim 0.4174 + .9086i$ satisfies $\chi_{4,1}(\beta^2) = 0$. Slice by conjugate diagonal Δ' (left) and by complex diagonal (right)

$$g_{c,\beta}(x, y) = \big(y, \beta(c\, y + 1/y) - \beta^2 x\big)\,, \quad \beta^2 = \delta\ . \tag{9.3}$$

This map is reversible under $\tau(x, y) = (\bar{y}, \bar{x})$ if $c \in \mathbb{R}$ and $|\delta| = 1$ and preserves the 2-form $dx \wedge dy$. Associated with (9.3) is the family of polynomials

$$\chi_{n,m}(t) = 2\sum_{k=0}^{m} t^{kn} - \sum_{k=0}^{mn} t^k s. \tag{9.4}$$

As was the case with (9.1), we may factor $\chi_{n,m} = C_{n,m} S_{n,m}$, where $C_{n,m}$ is a product of cyclic polynomials, and $S_{n,m}$ is a Salem polynomial. The roots of this Salem polynomial give us automorphisms of the form (9.3).

Theorem 9.7 ([4]) *Suppose that $n \geq 4$, $m \geq 1$ or $n = 3$, $m \geq 2$. Let $\delta = \beta^2$ be a root of $S_{n,m}$, and let $c = 2\cos(j\pi/n)$, where $(j, n) = 1$. Then there is a blowup $\pi : X \to \mathbb{P}^2$ such that $g_{c,\beta}$ induces an automorphism g_X of X, and the entropy of g_X is $\log(\lambda_{n,m}) > 0$, where $\lambda_{n,m}$ is the largest root of $S_{n,m}$.*

The construction of g_X differs from the construction for the maps (1.2) and (9.2) because the blowups to make the space X are iterated to height 3. This is because $dx \wedge dy$ has a pole of order 3 along the line at infinity. However, by a general result of [12], there is a birational conjugacy under which the invariant 2-form of (9.3) will be transformed to another 2-form with only simple poles.

Figures 8 and 9 Fig. 8 gives a map from Theorem 9.7 with $n = 4$, $m = 1$, $j = 1$. If $L \subset X$ denotes the strict transform of the line at infinity, then L is invariant under g_X, and g_X acts as a rotation of period 4 on L. There is a rank 1 rotation domain

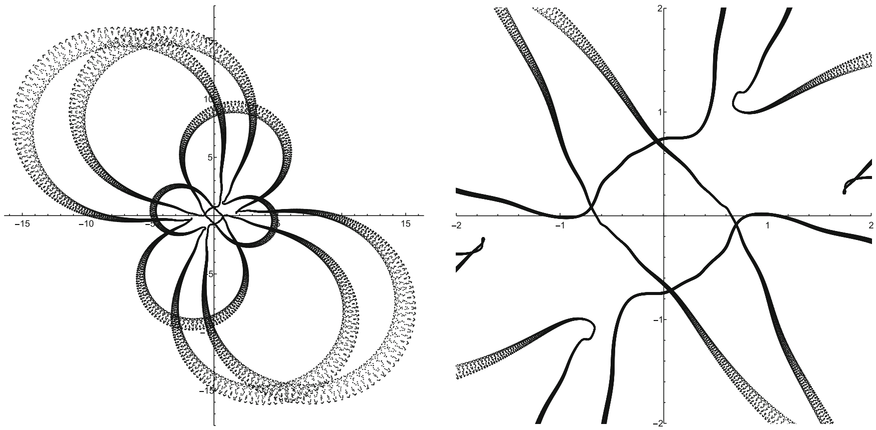

Fig. 9 Projection of an orbit of a point from region ‘e’ in Fig. 8 (left); detail (right)

$\Omega_L \supset L$, and the induced group is $\mathcal{G}(\Omega_L) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z}$. The global nature of Ω_L is discussed in [4]. Figure 8 shows two slices of the Fatou set (white). On the left are the points for a conjugate diagonal slice $\{(\zeta, \bar{\zeta}), -1.4 < \Re(\zeta), \Im(\zeta) < 1.4\}$, and on the right a complex diagonal slice $\{(\zeta, \zeta), -1.8 < \Re(\zeta), \Im(\zeta) < 1.8\}$. The regions marked ‘a’ correspond to Ω_L. The two fixed points in $\mathbb{C}^2$ were shown in [4] to be rotational of rank 2, and these are contained in the components ‘b’ and ‘d’. The computer evidence suggests that the two regions ‘b’ are connected inside $\mathbb{C}^2$ and are disjoint from the two components ‘d’. The components ‘c’ and ‘g’ appear to be rank 2 rotation domains of period 6, and component ‘f’ appears to be rank 2 with period 5. The projection of 220000 points of an orbit from ‘e’ is shown on the left half of Fig. 9. The detail on the right hand side lends evidence that the closure of the orbit is connected. The component ‘e’ was shown to us by Ushiki as a possible exotic rank 2 rotation domain. The reason why it might be exotic (why it cannot contain a fixed point) is that there are only four fixed points. Two of them are contained in Ω_L and one in each domain ‘b’ and ‘d’. However, Ω_L has rank 1 and cannot intersect ‘e’. Further ‘e’ is invariant under complex conjugation, so if ‘e’ intersects component ‘b’, then it must intersect ‘d’, which contradicts the apparent disjointness of ‘b’ and ‘d’.

Appendix. Compact Surface Automorphisms

We have considered only rational surface automorphisms in our discussions above. We have not considered other surfaces because of Theorem 10.1 of [10], which we summarize as follows:

Theorem 9.8 ([10, 27]) *Suppose that X is a compact complex surface and that $F \in Aut(X)$ has positive entropy. Then there are three possibilities for X:*

(i) $X = \mathbb{C}^2/\mathcal{L}$ *is a complex torus.*
(ii) X *is a* $K3$ *surface (or certain quotients).*
(iii) X *is a rational surface. In this case,* $\pi : X \to \mathbb{P}^2$ *is obtained from* $\mathbb{P}^2$ *by blowing up.*

In case of the torus, every automorphism F lifts to an affine map. It must preserve the lattice, so its determinant must be ± 1. If F has positive entropy, then the eigenvalues must be $|\lambda_1| < 1 < |\lambda_2|$. Thus it is hyperbolic, and it follows in this case that $\mathcal{F} = \emptyset$.

In the case of a $K3$ surface, McMullen [25] has shown that rotation domains can exist. He constructs lattices with a lattice automorphism which satisfies the conditions of the Torelli Theorem. The Torelli Theorem gives the existence of a $K3$ surface X with an automorphism F which will have the given behavior F^* on the cohomology lattice. Knowledge of F^* and the holomorphic Lefschetz Index Formula give the existence of a unique fixed point as well as the values of the eigenvalues λ_1, λ_2 of DF. He shows that the eigenvalues have modulus 1 and are multiplicatively independent and thus are "suitably irrational" for Siegel's Theorem. This automorphism F thus has a rank 2 rotation domain with a fixed point.

As was noted in [25], this $K3$ surface X is necessarily non-algebraic and cannot be exhibited explicitly. For if X is an algebraic $K3$ surface, then an automorphism $F \in Aut(X)$ cannot have a rank 2 rotation domain with a fixed point since the jacobian $\delta = \lambda_1\lambda_2$ would be a root of unity.

Question 16 *Can an automorphism of an algebraic* $K3$ *surface have a rotation domain? Equivalently, can it have a nonempty Fatou set?*

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A Twistor Transform for the Kobayashi Metric on a Convex Domain

John Bland, Dan Burns and Kin-Kwan Leung

Abstract Suppose that D is a bounded strictly linearly convex domain in $\mathbb{C}^2$ with smooth boundary. Lempert showed that for each $x \in D$ and $v \in T_x D$, there is a unique Kobayashi extremal disk such that its boundary lies on ∂D. Using Penrose's twistor theory, we show that the moduli space of Kobayashi extremal disks carries an anti-self-dual structure as in work of Hitchin. For D circular, the moduli space of extremal disks avoiding the origin has an Einstein-Weyl structure as in the work of LeBrun and Mason. We discuss these results here for the model example of the ball $\mathbb{B}^2 \subset \mathbb{C}^2$, and indicate those points where the general case still needs details filled in, the object of work in progress.

Keywords Kobayashi geodesics · Twistor transform · Anti-self-dual metrics · Einstein-Weyl structures

1 Introduction

The notion of a geometric transform goes back to Radon, who noticed that by viewing the projectivized tangent bundle $\mathbb{P}T\triangle$ over the unit disk $\triangle$ as an incidence variety, there is a double fibration—one onto $\triangle$, the set of points, and the second onto the space of lines.

Dedicated to Kang-Tae Kim on the occasion of his 60th birthday.

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Guillemin pursued these ideas in the study of Zoll metrics on S^2, that is, metrics for which the geodesics are all simple closed curves and of equal length. In this case, there are again two fibrations on the projectivized tangent bundle $\mathbb{P}TS^2$: one is the standard projection onto S^2, and the other is onto the moduli space of (unoriented) geodesics. The Zoll condition guarantees that the moduli space is a smooth manifold.

Penrose was the first person to introduce these ideas into the study of spacetime, when he observed that there is a natural family of conformal spheres in Minkowski space [9, 10]. He also observed that the total space of these spheres inherits a natural complex structure, which fibres over a dual complex manifold with fibres corresponding to the spheres. In the case in question, the total space is the grassmannian of the complex two planes in $\mathbb{C}^4$, and the dual complex manifold is $\mathbb{P}^3$, or the lines (passing through the origin) in $\mathbb{C}^4$. Under this transform, the dual manifold is a complex manifold with a family of rational curves, and the (compactified, complexified) Minkowski space is the space of rational curves, with its metric determined by the intersection properties of the curves; thus relatively rigid complex analytic data is transformed into metric data. Conversely, starting with the metric data on Minkowski space, Penrose infinitesimalizes his family of spheres into the bundle of almost complex structures compatible with the metric space structure, and his twistor space is the total space of this bundle with an adapted integrable complex structure (under certain curvature conditions). The twistor transform is described by this double fibration, and it transforms complex analytic data into geometric data.

Lebrun and Mason initiated an investigation into these ideas in the case when the family of rational curves in the complex manifold is replaced by a family of disks with boundary on a fixed maximally totally real submanifold. In [3], the complex manifold was $\mathbb{P}^2$ and the totally real manifold was $\mathbb{RP}^2$; the dual space of the disks is S^2, and the geometric structure induced by the intersection properties is a Zoll connection. The beautiful feature of this is that local deformations of the Zoll connection correspond to deformations of the totally real $\mathbb{RP}^2$ in $\mathbb{P}^2$, allowing them to give more precise information concerning the space of all Zoll metrics in a neighborhood of the standard round metric on S^2.

In [6], they studied the case where the complex manifold is the quadric in $\mathbb{P}^3$ and the totally real 2-manifold is S^2. This situation can be identified as a 2-to-1 branched cover of the case in the previous paragraph. In this case, the space of disks is a three manifold, and the geometric structure induced by the intersection properties is an Einstein Weyl structure. (See below for precise definitions.) Again, deformations of the Einstein-Weyl structure correspond to deformations of the totally real S^2 in Q^2.

The next case they considered was when the complex manifold was $\mathbb{P}^3$ and the totally real 3-manifold was $\mathbb{RP}^3$. In [4], the space of disks is a 4-manifold $S^2 \times S^2$, and the intersection properties on the space of disks induces a self-dual conformal structure. Deformations of the self-dual conformal structure correspond to deformations of the totally real $\mathbb{RP}^3 \subset \mathbb{P}^3$. The three cases above are described in Hitchin [2] for the moduli space of $\mathbb{P}^1$'s instead of holomorphic disks.

Our investigation begins with the work of Lempert [8]. He showed that the Kobayashi metric on a strongly convex domain $D \subset \mathbb{C}^n$ induces a foliation of the projectivized holomorphic tangent bundle by holomorphic disks which are totally geodesic for the Kobayashi metric, analogous to the real geodesic flow. Notice that the Kobayashi metric is only a Hermitian Finsler metric, but the foliation is smooth, and the space of geodesics is a smooth manifold. His method of proof involves the lift of the disks to a dual map to the projectivized cotangent bundle, and he showed that the Kobayashi disks were precisely those disks for which the lift mapped the boundary into a maximally totally real submanifold.

In this paper, we begin the investigation of applying the ideas of LeBrun and Mason to the study of the Kobayashi metric. We will state the results for the general case, and prove them carefully in the model situation of the unit ball in $\mathbb{C}^2$; our proof in the special case will introduce the main ideas of the proof for the general case, namely the Kobayashi metric on strongly linearly convex domains in $\mathbb{C}^2$. Similarly, we study the inverse to the Penrose transform for the Kobayashi metric here, proving the structure of the transform in the model case, but here there is less information a priori concerning the signature (2, 2) metric on the moduli space of suitable holomorphic disks, and so some properties of this transform are not established yet in full generality. Our work here differs from that of LeBrun-Mason [4] in two ways: our manifolds are non-compact, and they are of the opposite duality to theirs. This makes a significant difference when the underlying manifold is complex, and the metric is pseudo-Kähler of signature (1, 1). Surprisingly, such manifolds seem more rigid in some ways than their compact cousins in [4], since the standard model here is irreducible.

The layout of the paper is as follows. Section 2 recalls Lempert's theory of Kobayashi disks from [7] in the form we will use. Section 3 gives the two main statements, the first about the correspondence between Kobayashi metrics and anti-self-dual signature (2, 2) manifolds, and the second on the correspondence between S^1-invariant domains and a.s.d. (2, 2) manifolds and three dimensional Einstein-Weyl spaces. Section 4 treats the proof of the first result for the standard model case $D = \mathbb{B}^2$, $X = (\mathbb{P}^2)^* \backslash \mathbb{B}^2$ in a way extendible to the case of general D. We identify the twistor space in the model case in Sect. 5, and in Sect. 6 we prove the model case results for the relationship between the moduli space X and Einstein-Weyl structure. Section 7 closes with some last remarks and a few further questions.

2 Kobayashi Disks

Let D be a bounded domain in $\mathbb{C}^n$ and $\triangle$ be the unit disk in $\mathbb{C}$. For $z \in D$, $v \in T_z D$, $v \neq 0$, we let

$$\|v\|_K = \inf\{\lambda^{-1} : \exists f : \triangle \to D \text{ is holomorphic such that } f(0) = z,\ f'(0) = \lambda v,\ \lambda > 0\}.$$

We say that $\|v\|_K$ is the (infinitesimal) Kobayashi length of v. For a linearly convex domain D, $\|\cdot\|_K$ induces a norm on $T_z D$; that is, $\|\cdot\|_K$ is a Hermitian Finsler metric on D. We say that the image of minimizer f is an extremal disk with respect to z and v.

Lempert [7] showed the following

Theorem 2.1 *Let $D \subset \mathbb{C}^n$ be a smoothly bounded strictly linearly convex domain. Then*

- *For all $z \in D$ and $v \in T_z D$, $v \neq 0$, there exists a unique extremal map f with respect to z and v. The corresponding extremal disk $f(\triangle)$ is extremal for any point $w = f(\zeta) \in f(\triangle)$ and the direction $f'(\zeta)$.*
- *All the extremal maps are proper isometric imbeddings, and can be smoothly extended to the closed unit disk $\overline{\triangle}$.*
- *The extremal disks $f(\triangle)$ passing through a point $z \in D$ form a complex foliation of $D \backslash \{z\}$.*
- *The field of holomorphic tangent planes of ∂D in $f(\partial\triangle)$ can be holomorphically extended to the interior of the disk $f(\triangle)$ and define a holomorphic co-vector field on $f(\triangle)$.*

Let ρ be a strictly plurisubharmonic defining function for D and let f be extremal. Let $\zeta \in \partial\triangle$, and define $p(\zeta) = \zeta\rho_z(f(\zeta)) \cdot f'(\zeta)$. Then $p > 0$. The dual map $\tilde{f}$ of f is defined as

$$\tilde{f}(\zeta) = \frac{1}{p(\zeta)}\zeta\rho_z(f(\zeta)), \zeta \in \partial\triangle\ .$$

The lift $\tilde{f}$ is independent of the defining function ρ, and from the formula for $\tilde{f}$, we have $\tilde{f}(\zeta) \cdot f'(\zeta) = 1$ for all $\zeta \in \partial\triangle$. Note that $\tilde{f}$ is a complex number times ρ_z, which is the holomorphic tangent hyperplane to ∂D. Since f is an extremal disk, $\tilde{f}$ extends holomorphically to the interior of $\triangle$ by the above theorem.

The extremal disks are (complex) geodesics with respect to the (infinitesimal) Kobayashi metric, and they lift holomorphically (Gauss map) to the projectivized tangent bundle $\mathbb{P}TD$, where they define a foliation. The leaf space is a 4-dimensional real manifold, which corresponds to the moduli space of Kobayashi extremal disks.

Thus, for a smoothly bounded strictly linearly convex domain $D \subset \mathbb{C}^n$, the Kobayashi metric induces a foliation of $\mathbb{P}(TD)$ by complex geodesics, and we have a double fibration:

$$\begin{array}{ccc} & \mathbb{P}(TD) & \\ \pi \swarrow & & \searrow \tau \\ D & & X \end{array} \tag{2.1}$$

where

- π is the standard projection map;
- X is the space of complex geodesics for the Kobayashi metric in D;
- τ is the map to the leaf space for the geodesic foliation.

In this article, we discuss the geometric structure on X which is induced by this double fibration when $n = 2$.

3 Statement of Results

In this section, we state the main results; the remainder of the paper will provide a proof of these results in the case of the standard model, that is, the case when $D = \mathbb{B}^2 \subset \mathbb{C}^2$, and $X = (\mathbb{P}^2)^* \backslash \mathbb{B}^2$, the complement of the unit ball in the dual projective space.

Theorem 3.1 *X admits a neutral signature* $(++--)$ *anti-self-dual conformal structure.*

We recall that a conformal structure is an equivalence class of metrics $[g]$ up to conformal equivalence. The Weyl curvature is the conformally invariant portion of the usual curvature tensor, and the space X with a conformal class of metrics $[g]$ is said to be *anti-self-dual (a.s.d.)* if the Weyl curvature for $[g]$, considered as a bundle valued two form, is anti-self-dual, that is, it lies in the negative eigenspace of the Hodge star operator on two forms.

In the case that D is also a circular domain, then the free S^1 action on $D_0 := D\backslash\{0\}$ induces a free S^1 action on $X_0 = \tau(\mathbb{P}(TD_0))$. We first state the next theorem and then define the terms used in the theorem.

Theorem 3.2 *$M = X_0/S^1$ admits a complete Einstein-Weyl structure in the sense of LeBrun Mason; that is, M is a smooth, space-time oriented, conformally compact, globally hyperbolic, Lorentzian Einstein-Weyl 3-dimensional space-time* $(M, [g], \nabla)$.

$(M, [g], \nabla)$ is *Einstein-Weyl* if $[g]$ is a conformal class of metrics on M and ∇ is a compatible torsion-free connection such that the trace free part of the symmetric Ricci tensor vanishes; that is

(1) there exists a one-form α such that $\nabla g = \alpha \otimes g$ (compatibility), and
(2) ${R^k}_{ikj} + {R^k}_{jki} = f\, g_{ij}$, for some function f (Einstein-Weyl condition).

M is *conformally compact* if there is a metric in the conformal class which extends smoothly to infinity, and the conformal factor α satisfies $\alpha - 2du/u$ extends smoothly to the boundary, where u is a defining function for the boundary.

These results are inspired by the work of LeBrun and Mason. Indeed, Lempert's characterization of the extremal disks [7] was that the lift of f to $\mathbb{P}T^*D$ by $(f, [\tilde{f}])$ is proper and has boundary contained in a totally real submanifold; this is the context in which LeBrun and Mason work in [3, 4, 6].

4 The Standard Model

For a smoothly bounded strongly convex domain $D \subset \mathbb{C}^2$, the Kobayashi metric induces a foliation of $\mathbb{P}(TD)$ by complex geodesics, and we have a double fibration:

$$
\begin{array}{ccc} & \mathbb{P}(TD) & \\ {}^{\pi}\swarrow & & \searrow^{\tau} \\ D & & X \end{array} \tag{4.1}
$$

where

- π is the standard projection map;
- X is the space of complex geodesics for the Kobayashi metric in D;
- τ is the map to the leaf space for the geodesic foliation.

In this section, we prove the following theorems in the special case where $D = \mathbb{B}^2 \subset \mathbb{C}^2$:

Theorem 4.1 *X admits a neutral signature $(++--)$ anti-self-dual conformal structure.*

The proof of this in this special case will introduce the main ideas needed for the general case.

4.1 The Space X

Let (u, v) be the standard coordinates for $\mathbb{B}^2$, where we may also consider this to be the affine coordinates for $\mathbb{C}^2 \subset \mathbb{P}^2$ with projective coordinates $[u, v, w]$ and choosing the affine corresponding to $w = 1$.

A Kobayashi disk in $\mathbb{B}^2$ is simply the intersection of a complex line in $\mathbb{C}^2$ with the ball. Let $[x, y, z] \in (\mathbb{P}^2)^*$ be a point in the dual projective space corresponding to the complex line defined by $ux + vy = z$. Then the space X of Kobayashi disks corresponds to the lines with nonempty intersection with the unit ball; that is $X = \mathbb{P}^2_+ := \{[x, y, z] : |x|^2 + |y|^2 \geq |z|^2\}$.

4.2 The Tangent Space $T_{\mathbf{x}}X$

Since a point $\mathbf{x} \in X$ corresponds to a geodesic (given by the fibre $\tau^{-1}(\mathbf{x})$), infinitesimal variations of $\mathbf{x}$ correspond to Jacobi fields for the geodesic. The normal bundle to the fibre (a disk in D) is the line bundle $\mathcal{O}(1)$, restricted to the disk. The tangent space $T_{\mathbf{x}}X$ corresponds to holomorphic sections of $\mathcal{O}(1)$ restricted to the disk, which may be identified as the two complex dimensional space of linear functions restricted to the disk.

Since the fibre is a disk in $\mathbb{P}^1$, there is a natural involution on $\mathbb{P}^1$ given by reflection across the boundary. This induces an involution on sections of $\mathcal{O}(1)$, or a real structure. Explicitly, in the case that the line is given by $v = 0$, and the disk is $|u| < 1$, the reflection is $u \mapsto u/|u|^2 = 1/\bar{u}$ and the involution on sections is $au + b \mapsto \bar{a}\bar{u} + \bar{b} \mapsto \bar{a}/u + \bar{b} \cong \bar{a} + \bar{b}u$. In particular, notice that the section $au + b$ vanishes at $u = -b/a$, and the conjugate section $\bar{a} + \bar{b}u$ vanishes at $u = -\bar{a}/\bar{b} = 1/\left(\overline{-b/a}\right)$.

There is a natural conformal structure induced on $T_{\mathbf{x}}$ by declaring those sections which vanish simultaneously with their conjugate section to be null directions; explicitly, the zero must be on the boundary of the disk, and the coefficients satisfy the real quadratic relation $|a|^2 = |b|^2$. This quadratic condition can also be expressed as the vanishing of the determinant given by the coefficients of the linear function and its conjugate; it induces a hermitian conformal structure of signature (2, 2).

Remark 4.2 The proof in the general case follows Lempert more closely; rather than considering the tangent lift of the complex geodesic to the projectivized tangent bundle, we consider the lift by the dual map to the cotangent bundle: for f extremal, we lift f to $\mathbb{P}T^*D$ by $(f, [\tilde{f}])$. Geometrically, this is a map from the disk to $\mathbb{B}^2 \times (\mathbb{P}^2)^*$ specifying a point in $\mathbb{B}^2$ along with a hyperplane through that point; that is, we can view

$$(f, [\tilde{f}]) : \Delta \to \{((u, v), (x, y, z) \in \mathbb{B}^2 \times (\mathbb{P}^2)^* : xu + yv = z\}. \tag{4.2}$$

This extends to a holomorphic map from $\mathbb{P}^1$ to the incidence variety in $\mathbb{P}^2 \times (\mathbb{P}^2)^*$, and has normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. This fits into standard twistor theory, wherein vanishing of a section of the normal bundle is a quadratic condition on the four complex dimensional space of sections, and hence induces a conformal structure. The real sections are those which satisfy Lempert's reality condition: the boundary of the unit disk is mapped to boundary points in D with $[\tilde{f}]$ corresponding to the holomorphic cotangent space to ∂D at the corresponding point.

This same incidence variety comes up in a different context in the next section; more will be said about it at that time.

4.3 The Bundle of α Planes

We identify an S^1 bundle of null 2-planes over X. For $\mathbf{x} \in X$, the fibre meets the ball in a Kobayashi disk, and the boundary of the ball in a circle S^1. For each point on this circle, the Jacobi fields (linear functions) which vanish at this boundary point define a real two dimensional subspace which we refer to as an α plane. This is a two dimensional subspace of the tangent space $T_x X$, and the family of these α planes define an S^1 bundle F over X. The bundle F of α planes is foliated by a family of α surfaces.

We describe the situation explicitly. For $(u, v) \in \partial\mathbb{B}^2$, the space of lines through this point is

$$\Sigma_{(u,v)} := \{(x, y) \in \mathbb{P}^2_+ : ux + vy = 1\}. \tag{4.3}$$

This can be identified as the complex line in $\mathbb{P}^2_+$ which is tangent to the boundary 3-sphere $\partial\mathbb{P}^2_+$ at the corresponding point $(x, y) = (\bar{u}, \bar{v})$; this complex line is uniquely determined by the holomorphic tangent direction at $(\bar{u}, \bar{v})$. Since every point on $\Sigma_{(u,v)}$ corresponds to a Kobayashi disk through the fixed boundary point $(\bar{u}, \bar{v})$, any tangent vector to $\Sigma_{(u,v)}$ satisfies the nullity condition, and $\Sigma_{(u,v)}$ is a totally null surface. At any point along this surface, the tangent space is an α plane, and this in fact describes all α planes; indeed, through any point $\mathbf{x} \in X$, there is an S^1 of complex lines through $\mathbf{x}$ which are tangent to the boundary sphere $\partial\mathbb{P}^2_+$ and holomorphic. The S^1 variable defining the family of α planes is naturally identified with the boundary of the Kobayashi disk corresponding to the fibre over $\mathbf{x}$.

Remark 4.3 We note that at this point, we could restrict our attention and simply consider the boundary of the domain S^3 with a four parameter family of circles. Then X is the parameter space for this family of circles, and the conformal structure is identified in terms of the intersection properties of these circles. The fact that these circles are canonically identified with the boundaries of Kobayashi disks then becomes the 'hidden' property that guarantees that the incidence relation does indeed define a quadratic form on the tangent space of X.

4.4 The Conformal Structure on X

We determine the conformal structure on X obtained by identifying the α planes as null planes through simple Euclidean geometry and the identification of the α surfaces as the holomorphic lines which are tangent to the boundary.

Using the affine given by $z = 1$, we adopt the framing for the complex tangent space

$$\mathcal{X} = x\partial_x + y\partial_y \; ; \qquad Z = \bar{y}\partial_x - \bar{x}\partial_y. \tag{4.4}$$

Let $r^2 = |x|^2 + |y|^2$. Then the point (x, y) corresponds to the Kobayashi disk through $(u, v) = (\bar{x}, \bar{y})/r^2$ with tangent direction $(-y, x)$. The disk meets the boundary in points of norm 1, and these points may be succinctly expressed by the matrix equation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \bar{x} & -y \\ \bar{y} & x \end{pmatrix} \begin{pmatrix} 1 \\ e^{-i\theta}\sqrt{r^2-1} \end{pmatrix} / r^2 \, , \qquad e^{i\theta} \in S^1. \tag{4.5}$$

The alpha surfaces to which these points correspond are the holomorphic tangent planes to $\partial\mathbb{P}^2_+$ at the corresponding point, and they pass through the conjugate point in the dual space; hence, for each boundary point, the real direction vector from the point (x, y) to the boundary point is (mixing vector notation which identifies the affine $\mathbb{C}^2 \subset (\mathbb{P}^2)^*$ with a vector space $\mathbb{C}^2$)

$$\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \begin{pmatrix} 1 \\ e^{i\theta}\sqrt{r^2-1} \end{pmatrix} / r^2 = (\sqrt{r^2-1}\mathcal{X} + e^{i\theta} Z)(\sqrt{r^2-1}/r^2). \tag{4.6}$$

Proposition 4.4 *The conformal structure of the anti-self-dual metric is given by*

$$g = \frac{|\eta|^2}{r^2 - 1} - |\omega|^2 \tag{4.7}$$

where η, ω are the coframe dual to $\mathcal{X}$, Z.

Proof The distribution defining the null alpha planes is defined by the real span of the vectors $\sqrt{r^2-1}\mathcal{X} + e^{i\theta}Z$. Since these are null vectors, the proposition follows. □

Remark 4.5 We could have followed a different derivation using the $SU(2, 1)$ symmetry of the unit ball and some Kähler geometry.

Consider the potential $\log(|x|^2 + |y|^2 - |z|^2)$ for the $SU(2, 1)$ invariant metric $\partial\bar{\partial}\log(|x|^2 + |y|^2 - |z|^2)$. Since the potential restricts to any holomorphic line tangent to the boundary as a harmonic function, the metric vanishes on this line and must define a mixed signature metric for which these are null surfaces. A routine computation establishes the following:

Proposition 4.6 *The $SU(2, 1)$ invariant metric is given by*

$$G = \frac{r^2}{r^2-1} g = \frac{r^2}{(r^2-1)^2}|\eta|^2 - \frac{r^2}{r^2-1}|\omega|^2 \,. \tag{4.8}$$

Remark 4.7 We address the terminology of α planes. The conformal structure is defined by the indefinite Kähler metric (4.7). Using the Hodge star for indefinite metrics, we see that the Kähler form $\left(ir^2/(r^2-1)\right)\left[(\eta \wedge \bar{\eta})/(r^2-1) - \omega \wedge \bar{\omega}\right]$ is anti-self-dual.

On the other hand, the tangent space to the totally null surfaces is of the form $(e_1 - ie_2) + (e_3 - ie_4)$ where $(e_1 - ie_2)$, $(e_3 - ie_4)$ are of type $(1, 0)$ in the standard complex structure, and $\|e_1\|^2 = \|e_2\|^2 = 1$, $\|e_3\|^2 = \|e_4\|^2 = -1$. Then the oriented two planes tangent to the surface are of the form $(e_1 + e_3) \wedge (e_2 + e_4)$ which is self-dual. Hence, the surfaces are α surfaces.

4.5 *Integrability of the Distribution of α Planes*

We know from our earlier considerations that the bundle F of α planes is foliated by α surfaces. We now identify the corresponding rank 2 distribution on F, and directly verify integrability for this distribution.

Proposition 4.8 *The bundle F of null α-planes is a trivial S^1 bundle over X, and the distribution defined by $\sqrt{r^2-1}\mathcal{X} + e^{i\theta}Z - i\left(r^2/2\sqrt{r^2-1}\right)\partial_\theta$ is integrable.*

Proof Standard computations give the following commutators:

$$
\begin{aligned}
[\mathcal{X}, Z] &= -Z & [\mathcal{X}, \bar{Z}] &= \bar{Z} \\
[\bar{\mathcal{X}}, Z] &= Z & [\bar{\mathcal{X}}, \bar{Z}] &= -\bar{Z} \\
[\mathcal{X}, \bar{\mathcal{X}}] &= 0 & [Z, \bar{Z}] &= \mathcal{X} - \bar{\mathcal{X}}.
\end{aligned}
$$

Introducing the notation $\mathcal{V} = \sqrt{r^2 - 1}\mathcal{X}$, these become

$$
\begin{aligned}
[\mathcal{V}, Z] &= -\sqrt{r^2 - 1}Z & [\mathcal{V}, \bar{Z}] &= \sqrt{r^2 - 1}\bar{Z} \\
[\bar{\mathcal{V}}, Z] &= \sqrt{r^2 - 1}Z & [\bar{\mathcal{V}}, \bar{Z}] &= -\sqrt{r^2 - 1}\bar{Z} \\
[\mathcal{V}, \bar{\mathcal{V}}] &= \frac{-r^2}{2\sqrt{r^2 - 1}}(\mathcal{V} - \bar{\mathcal{V}}) & [Z, \bar{Z}] &= \frac{1}{\sqrt{r^2 - 1}}(\mathcal{V} - \bar{\mathcal{V}}).
\end{aligned}
$$

Then a direct computation shows that

$$
[\mathcal{V} + e^{i\theta} Z - \frac{ir^2}{2\sqrt{r^2 - 1}}\partial_\theta, \bar{\mathcal{V}} + e^{-i\theta}\bar{Z} + \frac{ir^2}{2\sqrt{r^2 - 1}}\partial_\theta] = 0 \; . \tag{4.9}
$$

□

4.6 The Twistor Space: Definiton

The bundle of null alpha planes naturally forms the boundary of a bundle of complex α planes, obtained by formally replacing $e^{\pm i\theta}$ with $\sigma^{\pm 1}$. We have the following

Theorem 4.9 *The twistor space is the trivial disk bundle $\mathcal{B}$ of complex α-planes. The distribution $H_{\mathcal{B}}$ defined by*

$$
\sqrt{r^2 - 1}\mathcal{X} + \sigma Z + \frac{1}{2}\frac{r^2}{\sqrt{r^2 - 1}}\sigma\partial_\sigma, \quad \sqrt{r^2 - 1}\bar{\mathcal{X}} + \frac{1}{\sigma}\bar{Z} - \frac{1}{2}\frac{r^2}{\sqrt{r^2 - 1}}\sigma\partial_\sigma, \quad \partial_{\bar{\sigma}} \; , \tag{4.10}
$$

is integrable, and defines a complex structure on $\mathcal{B}$.

The dual formulation for this complex structure is the kernel of the one forms

$$\sqrt{r^2-1}\omega - \sigma\eta; \qquad \sigma\sqrt{r^2-1}\bar{\omega} - \bar{\eta}; \qquad \frac{d\sigma}{\sigma} - \frac{r^2(\eta-\bar{\eta})}{2(r^2-1)}. \tag{4.11}$$

Proof The computation of integrability is formal, following the same computation as in the previous proposition. □

We refer to the distribution defined by the frame (4.10) as $H_{\mathcal{B}}$. The twistor space is $\mathcal{B}$ with the complex structure defined by this integrable distribution; we denote this by the pair $(\mathcal{B}, H_{\mathcal{B}})$.

Note that when $\sigma = 0$, the $(0, 1)$ vector space is spanned by $\mathcal{X}, \bar{Z}, \partial_{\bar{\sigma}}$. It follows that the 0-section is a complex submanifold of $\mathcal{B}$ diffeomorphic to $\mathbb{P}^2_+$; further, under the duality map $(x, y) \mapsto (x, y)/r^2$, this is biholomorphic to the unit ball with the origin blown up, where the $\mathbb{P}^1$ at infinity corresponds to lines through the origin, and transforms under the duality map to the blow up of the origin in the unit ball.

Since $SU(2, 1)$ acts transitively on points in $\mathbb{B}^2$, mapping another point to the origin under $SU(2, 1)$ corresponds to mapping the dual $\mathbb{P}^1 \subset \mathbb{P}^2_+$ to the rational curve at infinity under the dual transformation, and obtaining a new holomorphic section of the twistor space $(\mathcal{B}, H_{\mathcal{B}})$ for which this is the conjugate holomorphic $\mathbb{P}^1$ at infinity.

5 The Twistor Space

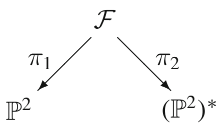
In the last section, we identified the twistor space as a smooth manifold $\mathcal{B}$ with an integrable distribution $H_{\mathcal{B}}$. In this section, we identify this complex manifold.

5.1 The Flag Variety

Begin with the incidence variety $\mathcal{F} \subset \mathbb{P}^2 \times (\mathbb{P}^2)^*$ defined by

$$\mathcal{F} = \{[\mathfrak{u}, \mathfrak{v}, \mathfrak{w}], [\mathfrak{x}, \mathfrak{y}, \mathfrak{z}] \subset \mathbb{P}^2 \times (\mathbb{P}^2)^* : \mathfrak{u}\mathfrak{x} + \mathfrak{v}\mathfrak{y} = \mathfrak{z}\mathfrak{w}\} \tag{5.1}$$

with the involution $([\mathfrak{u}, \mathfrak{v}, \mathfrak{w}], [\mathfrak{x}, \mathfrak{y}, \mathfrak{z}]) \mapsto (\overline{[\mathfrak{x}, \mathfrak{y}, \mathfrak{z}]}, \overline{[\mathfrak{u}, \mathfrak{v}, \mathfrak{w}]})$. The fixed point set of this involution is a totally real three sphere S. This set is a graph over $\{[\mathfrak{x}, \mathfrak{y}, \mathfrak{z}] \in (\mathbb{P}^2)^* : |\mathfrak{x}|^2 + |\mathfrak{y}|^2 = |\mathfrak{z}|^2\}$; equivalently, it is a graph over $\{[\mathfrak{u}, \mathfrak{v}, \mathfrak{w}] \in (\mathbb{P}^2) : |\mathfrak{u}|^2 + |\mathfrak{v}|^2 = |\mathfrak{w}|^2\}$. We have a double fibration:



(5.2)

The real involution takes a point to its dual point, and interchanges the fibres.

$\mathcal{F}$ is canonically identified with $\mathbb{P}(T\mathbb{P}^2)$ and with $\mathbb{P}(T\mathbb{P}^{2*})$; as such, the maps π_1, π_2 may be identified with the canonical projection maps. This identification is canonical, but it is not holomorphic. In fact, we will show that the complex structure on the twistor space agrees with that induced by this identification.

Remark 5.1 At the risk of some confusion, we point out that this double fibration may be viewed as an extension of the double fibration in (4.1) to a map on $\mathbb{P}(T\mathbb{P}^2)$, and at any point in $\mathbb{P}(T\mathbb{P}^2)\backslash S$, there are two rational curves corresponding to the the fibres of each projection. The real involution takes a point to its dual point, and interchanges the fibres; when restricted to $\mathbb{P}(TD)$, for D the ball $\mathbb{B}^2$, the real involution maps $\mathbb{P}(TD)$ to the twistor space $\mathcal{B}$.

5.2 Rational Curves in $\mathcal{F}$

In [1], Biquard defines a four complex parameter family of rational curves in $\mathcal{F}$ as follows. Choose a point $P \in \mathbb{P}^2$ and a point $L \in (\mathbb{P}^2)^*$ such that $P \notin L$. Then for every $[\mathfrak{u}, \mathfrak{v}, \mathfrak{w}] \in L$, there is a unique line $[\mathfrak{x}, \mathfrak{y}, \mathfrak{z}]$ satisfying $P \in [\mathfrak{x}, \mathfrak{y}, \mathfrak{z}]$, $[\mathfrak{u}, \mathfrak{v}, \mathfrak{w}] \in [\mathfrak{x}, \mathfrak{y}, \mathfrak{z}]$; we will denote the line so defined as $\Sigma_{P,L}$. We will say that the line is *real* if it is invariant under the involution; in particular, if $([\mathfrak{u}, \mathfrak{v}, \mathfrak{w}], [\mathfrak{x}, \mathfrak{y}, \mathfrak{z}]) \in \Sigma_{P,L}$ then $(\overline{[\mathfrak{x}, \mathfrak{y}, \mathfrak{z}]}, \overline{[\mathfrak{u}, \mathfrak{v}, \mathfrak{w}]}) \in \Sigma_{P,L}$. We note that the line is real iff $P = \bar{L}$.

Under the natural identification $\mathbb{P}(T\mathbb{P}^2) \cong \mathcal{F}$, then 'loosely' speaking, the complex lines $\Sigma_{P,L}$ in $\mathbb{P}(T\mathbb{P}^2)$ are lines L in $\mathbb{P}^2$ with a 'direction field' along L of directions focusing through the point P. The line is real if the point P is dual to the line L. The totally real set $S \subset \mathbb{P}(T\mathbb{P}^2)$ is the graph (or direction field) over the real $S^3 \subset \mathbb{P}^2$ corresponding to the holomorphic tangent direction at the corresponding point.

Proposition 5.2 *The complement of the real set S is foliated by real lines.*

Proof To show this, we work in the flag variety and observe that for any point $([\mathfrak{u}, \mathfrak{v}, \mathfrak{w}], [\mathfrak{x}, \mathfrak{y}, \mathfrak{z}]) \in (\mathcal{F}\backslash S)$, there is a unique pair P, L where $L = \bar{P}$ such that $[\mathfrak{u}, \mathfrak{v}, \mathfrak{w}] \in L$, and $P \in [\mathfrak{x}, \mathfrak{y}, \mathfrak{z}]$; this is a pair of equations for L: namely $[\mathfrak{u}, \mathfrak{v}, \mathfrak{w}] \cdot L = 0$, $[\mathfrak{x}, \mathfrak{y}, \mathfrak{z}] \cdot \bar{L} = 0$; these are independent equations precisely when $([\mathfrak{u}, \mathfrak{v}, \mathfrak{w}], [\mathfrak{x}, \mathfrak{y}, \mathfrak{z}])$ is not a real point, that is $([\mathfrak{u}, \mathfrak{v}, \mathfrak{w}], [\mathfrak{x}, \mathfrak{y}, \mathfrak{z}]) \notin S$. □

5.3 The Real Disks

The real lines provide a natural family of holomorphic disks in $\mathcal{F}$ with boundary on the totally real manifold S.

Proposition 5.3 *If a real line meets the set S, it meets it in either a point or a circle.*

Proof Notice that a real line $\Sigma_{P,L}$ meets S iff the projection $\pi_1(\Sigma_{P,L})$ meets the unit ball $\{[\mathfrak{u}, \mathfrak{v}, \mathfrak{w}] \in (\mathbb{P}^2) : |\mathfrak{u}|^2 + |\mathfrak{v}|^2 \le |\mathfrak{w}|^2\}$; in this case, it meets the ball in a disk, and the boundary in a circle (or a point in the limiting case). □

Proposition 5.4 *The real line $\Sigma_{P,L}$ meets S iff the line is of the form $L = [\bar{p}, \bar{q}, \lambda/\sqrt{1+|\lambda|^2}]$ for $(p, q) \in S^3$, $\lambda \ge 0$.*

Proof The real lines will meet the ball $\{[\mathfrak{u}, \mathfrak{v}, \mathfrak{w}] \in (\mathbb{P}^2) : |\mathfrak{u}|^2 + |\mathfrak{v}|^2 \le |\mathfrak{w}|^2\}$ iff the line is of the form $L = [\bar{p}, \bar{q}, \lambda/\sqrt{1+|\lambda|^2}]$ for $(p, q) \in S^3$, $\lambda \ge 0$; the line will meet S in a point in case $\lambda = \infty$. □

Proposition 5.5 *The real lines which meet S can be parametrized by $[\bar{p}, \bar{q}, \lambda/\sqrt{1+|\lambda|^2}] \in (\mathbb{P}^2)^*$, $\sigma \in \mathbb{P}^1$:*

$$\begin{pmatrix} \mathfrak{u} \\ \mathfrak{v} \end{pmatrix} = \begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix} \begin{pmatrix} \lambda \\ \sigma \end{pmatrix} / \sqrt{1+|\lambda|^2}\,, \qquad \mathfrak{w} = 1, \tag{5.3}$$

$$\begin{pmatrix} \mathfrak{x} \\ \mathfrak{y} \end{pmatrix} = \begin{pmatrix} \bar{p} & -q \\ \bar{q} & p \end{pmatrix} \begin{pmatrix} \lambda \\ \sigma^{-1} \end{pmatrix} / \sqrt{1+|\lambda|^2}\,, \qquad \mathfrak{z} = 1. \tag{5.4}$$

The points $|\sigma| = 1$ map to S.

5.4 The Twistor Space: Identification

Let $\mathcal{U} \subset \mathcal{F}$ be the subset defined by $\{([\mathfrak{u}, \mathfrak{v}, \mathfrak{w}], [\mathfrak{x}, \mathfrak{y}, \mathfrak{z}]) \in \mathcal{F} : |\mathfrak{u}|^2 + |\mathfrak{v}|^2 < |\mathfrak{w}|^2, |\mathfrak{x}|^2 + |\mathfrak{y}|^2 > |\mathfrak{z}|^2\}$. Our next proposition shows that $\mathcal{U}$ is a natural candidate for our twistor space.

Proposition 5.6 *The complex manifold $\mathcal{U} \subset \mathcal{F}$ is foliated by holomorphic disks with boundary in S.*

Proof It is easily checked that when we restrict the parameterization in Proposition (5.5) to the set $|\sigma| < 1$, then we are restricting to points in $\mathcal{F}$ which project under π_1 to points inside the unit ball $\{[\mathfrak{u}, \mathfrak{v}, \mathfrak{w}] \in (\mathbb{P}^2) : |\mathfrak{u}|^2 + |\mathfrak{v}|^2 \le |\mathfrak{w}|^2\}$. These disks foliate $\mathcal{U}$ and have boundaries in the fixed point set of the involution; their centres lie in the set $\mathfrak{z} = 0$; indeed, when $\sigma = 0$, $[\mathfrak{u}, \mathfrak{v}, \mathfrak{w}] = [\lambda p, \lambda q, \sqrt{1+|\lambda|^2}]$ and $[\mathfrak{x}, \mathfrak{y}, \mathfrak{z}] = [-q, p, 0]$. □

Remark 5.7 Once again, at the risk of confusion, we observe that under the identification of (5.2) with (4.1), the total space of the real lines which meet the ball is diffeomorphic to $\mathbb{P}(T\mathbb{P}^2_+)$ and the set $\mathcal{U}$ is diffeomorphic to the associated unit disk bundle $\mathcal{B}$.

Theorem 5.8 *The twistor space* $(\mathcal{B}, H_{\mathcal{B}})$ *is biholomorphic to* $\mathcal{U}$.

Proof Consider the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \left(\frac{1}{r^2} \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \begin{pmatrix} 1 \\ \sigma\sqrt{r^2-1} \end{pmatrix}, \ \frac{1}{r^2} \begin{pmatrix} \bar{x} & -y \\ \bar{y} & x \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{r^2-1}/\sigma \end{pmatrix} \right) = \left(\begin{pmatrix} \mathfrak{u} \\ \mathfrak{v} \end{pmatrix}, \begin{pmatrix} \mathfrak{x} \\ \mathfrak{y} \end{pmatrix} \right). \tag{5.5}$$

First note that since the incidence relation $\mathfrak{u}\mathfrak{x} + \mathfrak{v}\mathfrak{y} = 1$ holds on this image, this is a map $\mathcal{B} \to \mathcal{F}$. Also, since $(\mathfrak{x}, \mathfrak{y}) = \overline{(\mathfrak{u}, \mathfrak{v})}$ when $|\sigma| = 1$, the boundaries of the disks are mapped to S.

Next note that the map is a bijection preserving the foliation by holomorphic disks. We check that the fibre over $[x, y, 1] \in \mathbb{P}^2_+$ maps to the real line $\Sigma_{P,L}$ where $L = [x, y, 1]$. Setting $[p, q, \lambda/\sqrt{1+|\lambda|^2}] = [x, y, 1]$, we solve: $(x, y) = r(p, q), r = \sqrt{1+|\lambda|^2}/\lambda$. Then (5.5) maps the disk (x, y) to the corresponding real line (5.3), (5.4).

We check that this map is holomorphic relative to the complex structure defined by the distribution $H_{\mathcal{B}}$. This follows easily from observing that if we treat the image as the product of a matrix with a vector and differentiate, we obtain

$$\begin{aligned}
\mathcal{X}\left(\frac{1}{r^2} \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \begin{pmatrix} 1 \\ \sigma\sqrt{r^2-1} \end{pmatrix} \right) &= \frac{-1}{r^2} \left(\begin{pmatrix} 0 & -\bar{y} \\ 0 & \bar{x} \end{pmatrix} \begin{pmatrix} 1 \\ \sigma\sqrt{r^2-1} \end{pmatrix} \right. \\
&\qquad \left. - \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{\sigma r^2}{2\sqrt{r^2-1}} \end{pmatrix} \right) \\
&= \frac{-1}{r^2} \begin{pmatrix} -\bar{y} \\ \bar{x} \end{pmatrix} \sigma\sqrt{r^2-1} + \frac{1}{r^2} \begin{pmatrix} -\bar{y} \\ \bar{x} \end{pmatrix} \frac{\sigma r^2}{2\sqrt{r^2-1}} , \\
Z\left(\frac{1}{r^2} \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \begin{pmatrix} 1 \\ \sigma\sqrt{r^2-1} \end{pmatrix} \right) &= \frac{-1}{r^2} \begin{pmatrix} -\bar{y} \\ \bar{x} \end{pmatrix} , \\
\sigma\partial_\sigma \left(\frac{1}{r^2} \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \begin{pmatrix} 1 \\ \sigma\sqrt{r^2-1} \end{pmatrix} \right) &= \frac{1}{r^2} \begin{pmatrix} -\bar{y} \\ \bar{x} \end{pmatrix} \sigma\sqrt{r^2-1} .
\end{aligned}$$

Then since the right hand sides are all multiples of the column vector $(-\bar{y}, \bar{x})^T$, solving the equations amounts to solving linear equations for the coefficients of $\mathcal{X}, Z, \sigma\partial_\sigma$.

Similarly

$$\bar{\mathcal{X}}\left(\frac{1}{r^2}\begin{pmatrix} x & -\bar{y} \\ y & \bar{x}\end{pmatrix}\begin{pmatrix} 1 \\ \sigma\sqrt{r^2-1}\end{pmatrix}\right) = \frac{-1}{r^2}\left(\begin{pmatrix} x & 0 \\ y & 0\end{pmatrix}\begin{pmatrix} 1 \\ \sigma\sqrt{r^2-1}\end{pmatrix}\right.$$
$$\left. - \begin{pmatrix} x & -\bar{y} \\ y & \bar{x}\end{pmatrix}\begin{pmatrix} 0 \\ \frac{\sigma r^2}{2\sqrt{r^2-1}}\end{pmatrix}\right)$$
$$= \frac{-1}{r^2}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\bar{y} \\ \bar{x}\end{pmatrix}\frac{\sigma}{2\sqrt{r^2-1}}\,,$$

$$\bar{Z}\left(\frac{1}{r^2}\begin{pmatrix} x & -\bar{y} \\ y & \bar{x}\end{pmatrix}\begin{pmatrix} 1 \\ \sigma\sqrt{r^2-1}\end{pmatrix}\right) = \frac{1}{r^2}\begin{pmatrix} x \\ y \end{pmatrix}\sigma\sqrt{r^2-1}\,,$$

$$\sigma\partial_\sigma\left(\frac{1}{r^2}\begin{pmatrix} x & -\bar{y} \\ y & \bar{x}\end{pmatrix}\begin{pmatrix} 1 \\ \sigma\sqrt{r^2-1}\end{pmatrix}\right) = \frac{1}{r^2}\begin{pmatrix} -\bar{y} \\ \bar{x} \end{pmatrix}\sigma\sqrt{r^2-1}\,.$$

In this case, the right hand sides are all linear combinations of the column vectors $(-\bar{y}, \bar{x})^T$, $(x, y)^T$, and solving the equations amounts to solving linear equations for the coefficients of $\mathcal{X}$, Z, $\sigma\partial_\sigma$.

This shows that $(\mathfrak{u}, \mathfrak{v})$ are holomorphic relative to the complex structure defined by $H_{\mathcal{B}}$. The computations for $(\mathfrak{x}, \mathfrak{y})$ essentially follow by conjugation (and where we replace σ by $1/\sigma$ when $|\sigma| \neq 1$). □

Remark 5.9 Observe that when $|\sigma| = 1$, the image of the mapping is the totally real 3 sphere sitting inside $S^3 \times S^3$ as the graph of the conjugation map. Hence, it maps the 5 manifold F to a 3 manifold, and the fibres of the map are the real surfaces corresponding to the alpha surfaces. The first two vectors in the distribution $H_{\mathcal{B}}$ are complex conjugates of each other when restricted to F, and tangent to the alpha surfaces. Then complexifying the equations by allowing σ to be a complex parameter leads to the result.

6 Einstein-Weyl Spaces

In the case that D is also a circular domain, then the free S^1 action on $D_0 := D\backslash\{0\}$ induces a free S^1 action on $X_0 = \tau(\mathbb{P}(TD_0))$. We have the following:

Theorem 6.1 *$M = X_0/S^1$ admits a complete Einstein-Weyl structure in the sense of LeBrun-Mason; that is, M is a smooth, space-time oriented, conformally compact, globally hyperbolic, Lorentzian Einstein-Weyl 3-dimensional space-time* $(M, [g], \nabla)$.

Once again, we prove this in the model case when D is the unit ball $\mathbb{B}^2$. We begin by explicitly computing the maps involved and the S^1 action.

First, we parametrize the Kobayashi disks; for $(p,q) \in S^3$, $t \in (0,1)$, $|\sigma| < 1$, we have

$$((p,q),t,\sigma) \mapsto \cos t \begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix} \begin{pmatrix} 1 \\ \sigma \tan t \end{pmatrix} = \begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix} \begin{pmatrix} \cos t \\ \sigma \sin t \end{pmatrix}. \tag{6.1}$$

The S^1 action on the unit ball corresponds to the action $((p,q),t,\sigma) \mapsto (e^{i\theta}(p,q),t,e^{2i\theta}\sigma)$.

In these coordinates, the space X is obtained by $(x,y) = (p,q)/\cos t$, where $r^2 = 1/\cos^2 t$ and $r^2 - 1 = \tan^2 t$. $\mathcal{B}$ is parametrized by $((x,y),\sigma) = ((p,q)/\cos t, \sigma)$, and the mapping $\mathcal{B} \to \mathcal{U}$ is given by

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} \mapsto & \left(\cos t \begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix} \begin{pmatrix} 1 \\ \sigma \tan t \end{pmatrix}, \quad \cos t \begin{pmatrix} \bar{p} & -q \\ \bar{q} & p \end{pmatrix} \begin{pmatrix} 1 \\ \tan t/\sigma \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix} \begin{pmatrix} \cos t \\ \sigma \sin t \end{pmatrix}, \quad \begin{pmatrix} \bar{p} & -q \\ \bar{q} & p \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t/\sigma \end{pmatrix} \right) = \left(\begin{pmatrix} \mathfrak{u} \\ \mathfrak{v} \end{pmatrix}, \begin{pmatrix} \mathfrak{x} \\ \mathfrak{y} \end{pmatrix} \right). \end{aligned}$$

The S^1 action on D_0 induces the standard Hopf S^1 action on $X_0 = (\mathbb{C}^2 \backslash \mathbb{B}^2) \subset (\mathbb{P}^2)^*$. (It also induces an S^1 action on the affine in $\mathcal{F}$ in which $\mathfrak{w} = 1$, $\mathfrak{z} = 1$, although we are not really interested in this at present.)

We consider the anti-self-dual metric $|\eta|^2/(r^2-1) - |\omega|^2$. Here, $|\omega|^2$ is S^1 invariant, and descends to the Fubini Study metric on S^3/S^1. On the other hand, η descends to dr/r. Using $r = 1/\cos t$, this gives that $|\eta|^2$ descends to $(\tan^2 t)dt^2$. Finally, since $(r^2-1) = \tan^2 t$, we see that the conformal metric $|\eta|^2/(r^2-1) - |\omega|^2$ descends under the S^1 action to the $dt^2 - ds^2$ on $(0,1) \times S^2$ where ds^2 is the standard round metric. The conformally equivalent metric $\cosh^2 t \left(dt^2 - ds^2\right)$ is the de Sitter metric of signature $(1,2)$; we have established the corollary.

7 Final Remarks and Questions

One of the motivating forces behind the current work was the work by LeBrun-Mason [5] in which they identify Einstein-Weyl structures with certain foliations by holomorphic disks with boundaries in a totally real submanifold. Their investigation in this regard then led to work in a similar vein in which they identify self dual metrics with related foliations by holomorphic disks with boundaries in a totally real submanifold.

Lempert's groundbreaking work on the Kobayashi metric in [7] also relied on a family of holomorphic disks with boundary in a totally real manifold. The main aim of this article is to draw the connection between these two relatively separate areas of mathematics: the Kobayashi metric on convex domains, which is a smooth hermitian Finsler metric, and a.s.d. metrics of signature $(2,2)$. From a complex analytic viewpoint, it is remarkable that both sides in the double fibration (2.1) inherit such elegant geometric structures. This relationship appears to be particularly interesting in the case when the domain admits an S^1 symmetry, and the space $X_0/S^1 = M$ is an Einstein-Weyl space.

We make here some observations concerning this correspondence, and suggest areas to be investigated. First, we note that it should by now be clear that the argument above generalizes to the proof of the general theorem. In the general case, it is less clear what the a.s.d. metric looks like on the moduli space of Kobayashi extremal disks. This is work in progress. In general, we would like more clarity as to the details of the correspondence. Here are some slightly more specific and natural sample questions which arise.

1. The complex domain comes equipped with the Kobayashi distance, or equivalently, for any fixed point $a \in D$, the solution to the homogeneous Monge-Ampére equation. (This is sometimes referred to as the pluri-subharmonic Green's function.) Since this provides a canonical (family of) exhaustion(s) of the domain which generalizes the function $\log r$, does this lead to a natural duality map to parametrize the dual space X? More generally, how is this (family of) exhaustion(s) expressed in terms of the geometry of X?
2. When D is circular (hence, defined by a norm), we may restrict our consideration to the punctured domain D_0. In this case, the circular symmetry extends to a symmetry on X_0 and, as in [5], the a.s.d. metric reduces under this symmetry to an Einstein-Weyl structure. How does the geometry pass back and forth? In particular, what information does the norm defining D carry for the Einstein-Weyl structure? Does the exhaustion which it defines have particularly nice properties on M?
3. Does this picture allow us to say anything more about the dependence of the Kobayashi metric upon the base point?

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A Degenerate Donnelly–Fefferman Theorem and its Applications

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Abstract We prove a degenerate Donnelly–Fefferman theorem. Applications to local non-integrability of plurisubharmonic functions and L^2 boundary decay estimates of the Bergman kernel are given.

Keywords Donnelly–Fefferman estimate · Plurisubharmonic function · Bergman kernel

1 Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$ and φ a plurisubharmonic (psh) function on Ω. Let ψ be a C^2 strictly psh function on Ω which satisfies

$$ri\partial\bar{\partial}\psi \geq i\partial\psi \wedge \bar{\partial}\psi \tag{1.1}$$

for some $r > 0$. Donnelly and Fefferman [9] showed that for any $\bar{\partial}$-closed (0, 1)-form v on Ω there exists a solution of $\bar{\partial}u = v$ satisfying

$$\int_\Omega |u|^2 e^{-\varphi} \leq C_0 r \int_\Omega |v|^2_{i\partial\bar{\partial}\psi} e^{-\varphi} \tag{1.2}$$

where C_0 is an absolute constant, provided that the RHS of (1.2) is finite.

Berndtsson [1] observed that (1.2) is a formal consequence of the classical Hörmander L^2-estimates. Furthermore, he showed that if (1.1) holds for some $r < 1$ then the $L^2(\Omega, \varphi)$ minimal solution of $\bar{\partial}u = v$ satisfies a stronger estimate:

Dedicated to Professor Kang-Tae Kim on the occasion of his 60-th birthday.

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$$\int_\Omega |u|^2 e^{\psi-\varphi} \le \frac{6}{(1-r)^2}\int_\Omega |v|^2_{i\partial\bar\partial\psi} e^{\psi-\varphi}. \tag{1.3}$$

Here $L^2(\Omega,\varphi)$ stands for the set of measurable functions f on Ω satisfying $\int_\Omega |f|^2 e^{-\varphi} < \infty$. This estimate turns out to be quite useful in studying the Bergman kernel (see e.g. [3, 6, 7]). The optimal constant is known to be $4r/(1-r)^2$ (see [4]).

The degenerate case $r = 1$ was considered by the author [5], in order to derive the cerebrated Ohsawa-Takegoshi extension theorem from Hörmander's L^2-estimates. In this note, we will show

Theorem 1.1 *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $\varphi \in PSH(\Omega)$. Let ψ be a continuous psh function on Ω which satisfies*

$$i\partial\bar\partial\psi \ge i\partial\psi \wedge \bar\partial\psi. \tag{1.4}$$

Let $\eta : \mathbb{R} \to \mathbb{R}$ be a C^2 function satisfying $0 < \eta' \le 1/2$. Let u_φ be the $L^2(\Omega,\varphi)$-minimal solution of $\bar\partial u = v$. Then the following properties hold:

(1) If $\psi > 0$ and $\eta'' \le 0$, then

$$\int_\Omega \eta'(\psi)|u_\varphi|^2 e^{\psi-\eta(\psi)-\varphi} \le 10\int_\Omega \frac{1}{\eta'(\psi)}|v|^2_{i\partial\bar\partial\psi} e^{\psi-\eta(\psi)-\varphi}. \tag{1.5}$$

(2) If $\psi < 0$ and $\eta' \ge -4\eta''$, then

$$\int_\Omega \eta'(-\psi)|u_\varphi|^2 e^{\psi+\eta(-\psi)-\varphi} \le 192\int_\Omega \frac{1}{\eta'(-\psi)}|v|^2_{i\partial\bar\partial\psi} e^{\psi+\eta(-\psi)-\varphi}. \tag{1.6}$$

Warning: if ψ is not C^2 then $|v|^2_{i\partial\bar\partial\psi}$ should be understood as the infimum of all non-negative locally bounded functions H satisfying

$$i\bar v \wedge v \le Hi\partial\bar\partial\psi$$

in the sense of distributions, as noticed by Blocki (see e.g. [4]).

The primary motivation of formulating a theorem as above is to improve some results recently obtained by the author in [6, 7]. We will first show

Theorem 1.2 *Let $\varphi < 0$ be a psh function in a neighborhood $U \ni 0$ with $c_0(\varphi) > 0$, where $c_0(\varphi)$ is the complex singularity exponent of φ at 0, i.e., the supremum of $c \ge 0$ such that $e^{-c\varphi}$ is L^1 in a neighborhood of 0. Let $\kappa : \mathbb{R} \to (0,\infty)$ be a non-increasing C^1 function satisfying*

(1) $\int_0^\infty \kappa(t)dt = \infty$;
(2) $\kappa(t) \ge -5\kappa'(t)$ for $t \gg 1$.

Then for any neighborhood $V \ni 0$ one has

$$|\varphi|^{-1}\kappa(\log(-\varphi))e^{-c_0(\varphi)\varphi} \notin L^1(V).$$

The weaker statement that $e^{-c_0(\varphi)\varphi} \notin L^1(V)$ for any $V \ni 0$ is known as the openness conjecture of Demailly-Kollár [8], which was proved by Berndtsson [2] (see also [6, 10] for different proofs and improvements). The special case $\kappa(t) = t^\alpha$, $\alpha > 0$, was already verified in [6]. Notice that one may choose

$$1/\kappa(t) = t \prod_{j=1}^{k} \log_j t, \quad \log_j t := \overbrace{\log\log\cdots\log}^{j \text{ times}} t$$

for large t. It is not known whether the artificial condition (2) can be removed.

We also have the following L^2 boundary decay estimate of the Bergman kernel $K_\Omega(z, w)$:

Theorem 1.3 *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Let ρ be a negative continuous psh function on Ω. Set*

$$\Omega_t = \{z \in \Omega : -\rho(z) > t\}, \quad t > 0.$$

Let $0 < a < 1$ be given. For every $0 < \alpha < 1$, there exists a constant $C_\alpha > 0$ such that

$$\int_{-\rho \le \varepsilon} |K_\Omega(\cdot, w)|^2 \le C_\alpha\, K_{\Omega_a}(w)(\varepsilon/a)|\log \varepsilon|^{2+\alpha} \tag{1.7}$$

for all $w \in \Omega_a$ and $\varepsilon \ll a$.

In [7], the following weaker estimate was proved:

$$\int_{-\rho \le \varepsilon} |K_\Omega(\cdot, w)|^2 \le C_\alpha\, K_{\Omega_a}(w)(\varepsilon/a)^\alpha \tag{1.8}$$

for all $w \in \Omega_a$ and $\varepsilon \le \varepsilon_\alpha a$. We remark that L^2 boundary decay estimates of the Bergman kernel play a key role in [7]. For example, with (1.8) replaced by (1.7), one may verify similarly as Theorem 1.7 of [7] a slightly stronger result as follows.

Theorem 1.4 *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $\alpha(\Omega) > 0$. For every $0 < r < 1$, there exists a constant $C_r > 0$ such that*

$$\frac{|K_\Omega(z, w)|^2}{K_\Omega(z)K_\Omega(w)} \le C_r \min\left\{\frac{\nu(z)}{\mu(w)}, \frac{\nu(w)}{\mu(z)}\right\}, \quad z, w \in \Omega, \tag{1.9}$$

where $\mu := |\varrho|/(1 + |\log|\varrho||)$ and $\nu := |\varrho|(1 + |\log|\varrho||)^{n+2+r}$.

Here $\alpha(\Omega)$ is the hyperconvexity index of Ω and ϱ is the relative extremal function of a (fixed) closed ball $\overline{B} \subset \Omega$. We refer to [7] for definitions.

2 Proof of Theorem 1.1

We follow Berndtsson's method as [1] or [3]. Assume first that ψ is C^2 and strictly psh on $\overline{\Omega}$. For the first case, we define $\phi := \psi - \eta(\psi)$. Clearly, we have

$$i\partial\bar\partial\phi = (1-\eta'(\psi))i\partial\bar\partial\psi - \eta''(\psi)i\partial\psi\wedge\bar\partial\psi \\ \geq (1-\eta'(\psi))i\partial\bar\partial\psi. \tag{2.1}$$

Since $L^2(\Omega,\varphi+\phi)$ coincides with $L^2(\Omega,\varphi)$ as sets, we have $u_\varphi e^\phi \perp \mathrm{Ker}\,\bar\partial$ in $L^2(\Omega,\varphi+\phi)$, i.e., $u_\varphi e^\phi$ is the $L^2(\Omega,\varphi+\phi)$-minimal solution of the equation

$$\bar\partial u = \bar\partial(u_\varphi e^\phi).$$

It follows from Hörmander's L^2-estimates that

$$\int_\Omega |u_\varphi|^2 e^{\phi-\varphi} \leq \int_\Omega |\bar\partial(u_\varphi e^\phi)|^2_{i\partial\bar\partial\phi} e^{-\phi-\varphi} \\ \leq \int_\Omega \left(1+\frac{2}{\eta'(\psi)}\right)|v|^2_{i\partial\bar\partial\phi} e^{\phi-\varphi} \\ + \int_\Omega \left(1+\frac{\eta'(\psi)}{2}\right)|\bar\partial\phi|^2_{i\partial\bar\partial\phi}|u_\varphi|^2 e^{\phi-\varphi}. \tag{2.2}$$

By (2.1), we have

$$|\bar\partial\phi|^2_{i\partial\bar\partial\phi} \leq \frac{(1-\eta'(\psi))^2}{1-\eta'(\psi)}|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \leq 1-\eta'(\psi). \tag{2.3}$$

Substituting (2.3) into (2.2), we obtain

$$\frac{1}{2}\int_\Omega \left(\eta'(\psi)+\eta'(\psi)^2\right)|u_\varphi|^2 e^{\phi-\varphi} \leq \int_\Omega \left(1+\frac{2}{\eta'(\psi)}\right)|v|^2_{i\partial\bar\partial\phi} e^{\phi-\varphi},$$

from which (1.5) immediately follows since $\eta' \leq 1/2$ and $i\partial\bar\partial\phi \geq \frac{1}{2}i\partial\bar\partial\psi$.

For the second case, we define $\phi := \psi + \eta(-\psi)$. Since $\eta' \geq -4\eta''$, we have

$$i\partial\bar\partial\phi = (1-\eta'(-\psi))i\partial\bar\partial\psi + \eta''(-\psi)i\partial\psi\wedge\bar\partial\psi \\ \geq \left(1-\frac{5}{4}\eta'(-\psi)\right)i\partial\bar\partial\psi,$$

so that

$$|\bar\partial\phi|^2_{i\partial\bar\partial\phi} \leq \frac{(1-\eta'(-\psi))^2}{1-\frac{5}{4}\eta'(-\psi)}|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \leq \frac{1-\frac{3}{2}\eta'(-\psi)}{1-\frac{5}{4}\eta'(-\psi)} \tag{2.4}$$

for $\eta' \le 1/2$. Similar as above, we have

$$\begin{aligned}\int_\Omega |u_\varphi|^2 e^{\phi-\varphi} &\le \int_\Omega |\bar\partial(u_\varphi e^{\phi})|^2_{i\partial\bar\partial\phi} e^{-\phi-\varphi}\\ &\le \int_\Omega \left(1+\frac{8}{\eta'(-\psi)}\right)|v|^2_{i\partial\bar\partial\phi}e^{\phi-\varphi}\\ &\quad + \int_\Omega \left(1+\frac{\eta'(-\psi)}{8}\right)|\bar\partial\phi|^2_{i\partial\bar\partial\phi}|u_\varphi|^2 e^{\phi-\varphi}.\end{aligned}$$

This inequality combined with (2.4) gives

$$\frac{1}{8}\int_\Omega \frac{\eta'(-\psi)}{1-\frac{5}{4}\eta'(-\psi)}|u_\varphi|^2 e^{\phi-\varphi} \le \int_\Omega \frac{1+\frac{8}{\eta'(-\psi)}}{1-\frac{5}{4}\eta'(-\psi)}|v|^2_{i\partial\bar\partial\psi}e^{\phi-\varphi},$$

form which (1.6) follows since $0 < \eta' \le 1/2$.

For the general case, let Ω be exhausted by an increasing sequence of pseudoconvex domains $\Omega_j \subset\subset \Omega$, $j = 1, 2, \ldots$. Notice that the condition (1.4) is equivalent to say that $\Psi := -e^{-\psi}$ is psh. One may choose C^∞ strictly psh functions $\Psi_j < 0$ on $\overline{\Omega}_j$ such that $\Psi_j \downarrow \Psi$. It turns out that the smooth function $\psi_j := -\log(-\Psi_j)$ satisfies (1.4) and $\psi_j \downarrow \psi$. Since we have verified (1.5) and (1.6) for ψ_j on Ω_j, so it suffices to let $j \to \infty$. We leave the details of the argument to the reader.

3 Proof of Theorem 1.2

We apply the approach developed in [6]. Set

$$\eta(t) = \log\int_0^t \kappa(s)ds, \quad t > 0.$$

Clearly, we have $\eta(t) \to \infty$ as $t \to \infty$, and

$$\eta'(t)e^{\eta(t)} = \kappa(t). \tag{3.1}$$

As κ is non-increasing, we have

$$\int_0^t \kappa(s)ds \ge t\kappa(t), \tag{3.2}$$

so that

$$\eta'(t) = \frac{\kappa(t)}{\int_0^t \kappa(s)ds} \le 1/t. \tag{3.3}$$

Since $\kappa(t) \geq -5\kappa'(t)$ for $t \gg 1$, it follows that

$$\begin{aligned} -\eta''(t) &= -\frac{\kappa'(t)}{\int_0^t \kappa(s)ds} + \left(\frac{\kappa(t)}{\int_0^t \kappa(s)ds}\right)^2 \\ &\leq \frac{1}{\int_0^t \kappa(s)ds}\left(-\kappa'(t) + \frac{\kappa(t)}{t}\right) \quad \text{(by (3.2))} \\ &\leq \frac{\eta'(t)}{4} \quad \text{if } t \gg 1. \end{aligned}$$

Now fix a number $c_1 > c_0(\varphi)$. Set

$$E := \left\{z \in U : e^{-c_1\varphi} \text{ is not } L^1 \text{in any neighborhood of } z\right\}.$$

It follows from Bombieri's theorem that E is an analytic subset in U. Clearly, $0 \in E$. Shrinking U if necessary, we find $f_1, \ldots, f_m \in \mathcal{O}(U)$ such that $E \cap U = \cap_j f_j^{-1}(0)$ and

$$\tilde{\varphi} := \log \sum_j |f_j|^2 < 0$$

on U; moreover, there exists a decreasing sequence of smooth strictly psh functions $\varphi_k < 0$ such that $\varphi_k \downarrow \varphi$ on U. Set

$$\begin{aligned} \psi_k &= -\log\left(-\varphi_k - \tilde{\varphi}\right), \\ \psi &= -\log\left(-\varphi - \tilde{\varphi}\right). \end{aligned}$$

Clearly, both ψ_k and ψ satisfy (1.4). By (3.3), we have $\eta'(-\psi_k) \leq 1/2$ for all large k and some small U. Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth function satisfying $\chi|_{[0,\infty)} = 0$ and $\chi|_{(-\infty,-\log 2]} = 1$. Set

$$\lambda_{k,\varepsilon} = \chi(\log \eta(-\psi_k) + \log \varepsilon), \quad 0 < \varepsilon \ll 1.$$

Suppose that there exists a (pseudoconvex) domain V with $0 \in V \subset\subset U$ such that

$$\int_V |\varphi|^{-1}\kappa(-\psi)e^{-c_0(\varphi)\varphi} < \infty.$$

By Theorem 1.1/(2) and (3.1), we have a solution $u_{k,c,\varepsilon}$ of $\bar{\partial}u = \bar{\partial}\lambda_{k,\varepsilon}$ on V for every $0 < c < c_1$, which verifies

$$\begin{aligned}&\int_V \kappa(-\psi_k)|u_{k,c,\varepsilon}|^2 e^{\psi_k - c\varphi}\\ &\leq 192\int_V \frac{\kappa(-\psi_k)}{\eta'(-\psi_k)^2}|\bar\partial\lambda_{k,\varepsilon}|^2_{i\partial\bar\partial\psi_k} e^{\psi_k-c\varphi}\\ &\leq C_0\varepsilon^2\int_{V\cap\{1/(2\varepsilon)\leq\eta(-\psi_k)\leq 1/\varepsilon\}} |\varphi_k|^{-1}\kappa(-\psi_k)e^{-c\varphi}\end{aligned}$$

for some absolute constant $C_0 > 0$, since the function

$$|\bar\partial\lambda_{k,\varepsilon}|^2_{i\partial\bar\partial\psi_k} \leq \sup|\chi'|^2\frac{\eta'(-\psi_k)^2}{\eta(-\psi_k)^2} \leq C_0\varepsilon^2\eta'(-\psi_k)^2$$

on its support, and

$$e^{\psi_k} = \frac{1}{|\varphi_k + \tilde\varphi|} \leq \frac{1}{|\varphi_k|}.$$

Since $-\psi_k \geq \log(-\tilde\varphi)$ and η is an increasing function, it follows that

$$\begin{aligned}&V\cap\{\eta(-\psi_k)\leq 1/\varepsilon\}\\ &\subset V\cap\{\eta(\log(-\tilde\varphi))\leq 1/\varepsilon\} =: S_\varepsilon.\end{aligned}$$

Since $\eta(t)\to\infty$ as $t\to\infty$, we have $S_\varepsilon \subset\subset U\setminus E$. Since κ is non-increasing, $-\psi_k \geq \log(-\tilde\varphi)$ and $\varphi_k \geq \varphi$, we also have

$$\begin{aligned}&|\varphi_k|^{-1}\kappa(-\psi_k)e^{-c\varphi}\\ &\leq \left\{|\varphi_k|^{-1}\kappa(\log(-\varphi_k))e^{(c_1-c)\varphi_k}\right\}e^{-(c_1-c)\varphi_k}e^{-c\varphi}\\ &\leq \mathrm{const}_{c,c_1}\, e^{-c_1\varphi}\in L^1(S_\varepsilon).\end{aligned}$$

It follows from the dominated convergence theorem that

$$\begin{aligned}&\int_{V\cap\{\eta(-\psi_k)\leq 1/\varepsilon\}}|\varphi_k|^{-1}\kappa(-\psi_k)e^{-c\varphi}\\ &\to\int_{V\cap\{\eta(-\psi)\leq 1/\varepsilon\}}|\varphi|^{-1}\kappa(-\psi)e^{-c\varphi}\quad(k\to\infty)\\ &\to\int_{V\cap\{\eta(-\psi)\leq 1/\varepsilon\}}|\varphi|^{-1}\kappa(-\psi)e^{-c_0(\varphi)\varphi}\quad(c\to c_0(\varphi)).\end{aligned}$$

The function $f_{k,c,\varepsilon} := \lambda_{k,\varepsilon} - u_{k,c,\varepsilon}$ is holomorphic on V and satisfies

$$\int_V |f_{k,c,\varepsilon}|^2\kappa(-\psi_k)e^{\psi_k-c\varphi} \leq \mathrm{const.}\int_V|\varphi|^{-1}\kappa(-\psi)e^{-c_0(\varphi)\varphi}$$

and

$$
\begin{aligned}
&\int_V |f_{k,c,\varepsilon}-1|^2\kappa(-\psi_k)e^{\psi_k}\\
\leq\ & 2\int_{V\cap\{\eta(-\psi_k)\geq\frac{1}{2\varepsilon}\}}\kappa(-\psi_k)e^{\psi_k}\\
&+2\int_V\kappa(-\psi_k)|u_{k,c,\varepsilon}|^2e^{\psi_k-c\varphi}\\
\leq\ & \text{const.}\,|V\cap\{\eta(-\psi)\geq 1/(2\varepsilon)\}|\\
&+\text{const.}\,\varepsilon^2\int_V|\varphi|^{-1}\kappa(-\psi)e^{-c_0(\varphi)\varphi}\\
=:\ & \delta_\epsilon
\end{aligned}
$$

provided $k\geq k_\varepsilon\gg 1$ and $|c-c_0(\varphi)|\leq\tau_\varepsilon\ll 1$. Since $\psi_k\geq\psi$ and κ is non-increasing, it follows that

$$\int_V|f_{k,c,\varepsilon}|^2\kappa(-\psi)e^{\psi-c\varphi}\leq\text{const.}\int_V|\varphi|^{-1}\kappa(-\psi)e^{-c_0(\varphi)\varphi} \tag{3.4}$$

$$\int_V|f_{k,c,\varepsilon}-1|^2\kappa(-\psi)e^{\psi}\leq\delta_\epsilon. \tag{3.5}$$

Since $\kappa(t)\geq -5\kappa'(t)$ for $t\gg 1$, i.e. $(\log\kappa(t))'\geq -1/5$, so we have $\kappa(t)\geq$ const.$e^{-t/5}$ and

$$\kappa(-\psi)e^{\psi}\geq\text{const.}\,e^{2\psi}=\frac{\text{const.}}{|\varphi+\tilde{\varphi}|^2}. \tag{3.6}$$

By Schwarz's inequality, we obtain

$$
\begin{aligned}
\left(\int_V|f_{k,c,\varepsilon}-1|\right)^2 &\leq\int_V\frac{|f_{k,c,\varepsilon}-1|^2}{|\varphi+\tilde{\varphi}|^2}\int_V|\varphi+\tilde{\varphi}|^2\\
&\leq\text{const.}\,\delta_\varepsilon\int_V|\varphi+\tilde{\varphi}|^2.
\end{aligned}
$$

in view of (3.5), (3.6). As $\varphi+\tilde{\varphi}$ is psh on $\overline{V}$, it is well-known that $\varphi+\tilde{\varphi}\in L^p(V)$ for any $p\geq 1$ (cf. [11], Theorem 4.1.8). Since $\delta_\varepsilon\to 0$ as $\varepsilon\to 0$, it follows that for certain smaller neighborhood W of 0 (independent of k, c, ε) we have $|f_{k,c,\varepsilon}|\geq 1/2$ on W provided $\varepsilon\ll 1$. Thus by (3.4), we have $\kappa(-\psi)e^{\psi-c\varphi}\in L^1(W)$ for some $c>c_0(\varphi)$, so that for any $0<\alpha<1$,

$$\begin{aligned}\int_W e^{-\alpha c\varphi} &\le \left(\int_W \frac{e^{-c\varphi}}{|\varphi+\tilde{\varphi}|^2}\right)^{\alpha}\left(\int_W |\varphi+\tilde{\varphi}|^{\frac{2\alpha}{1-\alpha}}\right)^{1-\alpha}\\ &\le \text{const.}\left(\int_W \kappa(-\psi)e^{\psi-c\varphi}\right)^{\alpha}\left(\int_W |\varphi+\tilde{\varphi}|^{\frac{2\alpha}{1-\alpha}}\right)^{1-\alpha}\\ &< \infty\end{aligned}$$

in view of (3.6). Since α can be arbitrarily close to 1, we get a contradiction to the definition of $c_0(\varphi)$. Thus we have verified that $|\varphi|^{-1}\kappa(-\psi)e^{-c_0(\varphi)\varphi}$ is not L^1 in any neighborhood of 0. Since $-\psi \ge \log(-\varphi)$, we conclude the proof.

4 Proof of Theorem 1.3

Set $A^2(\Omega,\varphi) = L^2(\Omega,\varphi)\cap\mathcal{O}(\Omega)$. Let P_φ denote the orthogonal projection from $L^2(\Omega,\varphi)$ to $A^2(\Omega,\varphi)$. We first prove the following

Proposition 4.1 *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Let $\psi > 0$ be a continuous psh function on $\overline{\Omega}$ satisfying (1.4). Let η be as the case* (1) *of Theorem 1.1 and set $\phi = \psi - \eta(\psi)$. For every $f \in L^2(\Omega,\varphi-\phi)$, we have*

$$\int_\Omega |P_\varphi(f)|^2 e^{\phi-\varphi} \le 12\left(\sup_\Omega \frac{1}{\eta'(\psi)}\right)^2 \int_\Omega |f|^2 e^{\phi-\varphi}. \tag{4.1}$$

Proof We will follow the argument in [3]. Let us first verify that

$$P_\varphi(f) = P_\varphi(e^{-\phi}P_{\varphi+\phi}(e^\phi f)).$$

To see this, simply note that for any $g \in A^2(\Omega,\varphi)$,

$$\begin{aligned}(P_\varphi(f),g)_\varphi &= (f,g)_\varphi = (e^\phi f,g)_{\varphi+\phi}\\ &= (P_{\varphi+\phi}(e^\phi f),g)_{\varphi+\phi}\\ &= (e^{-\phi}P_{\varphi+\phi}(e^\phi f),g)_\varphi\\ &= (P_\varphi(e^{-\phi}P_{\varphi+\phi}(e^\phi f)),g)_\varphi.\end{aligned}$$

Set $h := e^{-\phi}P_{\varphi+\phi}(e^\phi f)$. Since $P_\varphi(h) = h - u_\varphi$ with u_φ being the $L^2(\Omega,\varphi)$-minimal solution of $\bar\partial u = \bar\partial h$, it follows from the proof of Theorem 1.1/(1) that

$$\begin{aligned}
&\int_\Omega \eta'(\psi)|P_\varphi(f)|^2 e^{\phi-\varphi}\\
&\le 2\int_\Omega \eta'(\psi)|h|^2 e^{\phi-\varphi} + 2\int_\Omega \eta'(\psi)|u_\varphi|^2 e^{\phi-\varphi}\\
&\le \int_\Omega |P_{\varphi+\phi}(e^\phi f)|^2 e^{-\phi-\varphi} + 2\int_\Omega \eta'(\psi)|u_\varphi|^2 e^{\phi-\varphi} \quad (\text{for } \eta' \le 1/2)\\
&\le \int_\Omega |f|^2 e^{\phi-\varphi} + 10\int_\Omega \frac{1}{\eta'(\psi)}|\bar\partial h|^2_{i\partial\bar\partial\phi} e^{\phi-\varphi}.
\end{aligned}$$

Since $\bar\partial h = -h\bar\partial\phi$, it follows from (2.3) that $|\bar\partial h|^2_{i\partial\bar\partial\phi} \le |h|^2$ and

$$\int_\Omega \frac{1}{\eta'(\psi)}|\bar\partial h|^2_{i\partial\bar\partial\phi} e^{\phi-\varphi} \le \sup_\Omega \frac{1}{\eta'(\psi)}\int_\Omega |h|^2 e^{\phi-\varphi} \le \sup_\Omega \frac{1}{\eta'(\psi)}\int_\Omega |f|^2 e^{\phi-\varphi}.$$

Since

$$\int_\Omega \eta'(\psi)|P_\varphi(f)|^2 e^{\phi-\varphi} \ge \left(\sup_\Omega \frac{1}{\eta'(\psi)}\right)^{-1}\int_\Omega |P_\varphi(f)|^2 e^{\phi-\varphi},$$

we conclude the proof. □

Proof (*Proof of Theorem* 1.3) Set $\psi := -\log(-\rho+\varepsilon)$ and $\eta(t) = \alpha\log t$ where $0 < \alpha < 1$. Clearly, one has $\eta'' < 0$. Since $\psi < -\log\varepsilon$, it follows that

$$\frac{1}{\eta'(\psi)} \le \frac{|\log\varepsilon|}{\alpha}.$$

Let $P_\Omega(f)(z) := \int_\Omega K_\Omega(z,\cdot)f(\cdot)$ denote the standard Bergman projection. If one applies $f = \chi_{\Omega_a}K_{\Omega_a}(\cdot,w)$ where χ_{Ω_a} denotes the characteristic function on Ω_a, then $K_\Omega(z,w) = P_\Omega(f)(z)$ and Proposition 4.1 implies that

$$\int_\Omega |K_\Omega(\cdot,w)|^2 e^{\psi-\eta(\psi)} \le 12\frac{|\log\varepsilon|^2}{\alpha^2}\int_{\Omega_a}|K_{\Omega_a}(\cdot,w)|^2 e^{\psi-\eta(\psi)}.$$

Since $\psi-\eta(\psi) \ge |\log 2\varepsilon| - \alpha\log|\log 2\varepsilon|$ on $\Omega\backslash\Omega_\varepsilon$ for $\varepsilon \ll 1$ and $\psi-\eta(\psi) \le |\log a|$ on Ω_a, it follows that

$$\int_{-\rho\le\varepsilon} |K_\Omega(\cdot,w)|^2 \le C_\alpha\, K_{\Omega_a}(w)(\varepsilon/a)|\log\varepsilon|^{2+\alpha}.$$

□

Remark The argument above is different from the original method in [7], which was suggested by one of the referees of that paper.

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Variation of Kähler-Einstein Metrics on Pseudoconvex Domains

Young-Jun Choi

Abstract The variation of Kähler-Einstein metrics on a family of canonically polarized compact Kähler manfolds is shown to be positive by Schumacher. In this survey, we introduce the variation of the Kähler-Einstein metrics on a family of bounded pseudoconvex domains and discuss the idea how to prove the positivity of the variation.

Keywords Kähler-Einstein metric · A family of pseudoconvex domains
Variation of Kähler-Einstein metrics

1 Introduction

On a family of canonically polarized compact Kähler manifolds, the variation of the Kähler-Einstein metrics is represented by a curvature form of the relative canonical line bundle. In [12], Schumacher has proved that the variation of the Kähler-Einstein metrics on a family of canonically polarized compact Kähler manifolds is semi-positive. He has also proved that it is strictly positive if the family is effectively parametrized. This celebrated theorem implies many important applications on the moduli space of canonically polarized compact Kähler manifolds, especially the extension of curvature forms and line bundles. Moreover, it also gives a nice curvature formula of the Weil-Petersson metric on the moduli space [12, 13].

In this survey, we will explain the brief idea of the Schumacher's theorem and how to apply his method to a *family of bounded pseudoconvex domains in* $\mathbb{C}^n$, which means a domain D in $\mathbb{C}^{n+1}$ such that every fiber

$$D_s := \{z \in \mathbb{C}^n : (z, s) \in D\}$$

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is a bounded pseudoconvex domain. If D is a smooth domain and every fiber is a bounded strongly pseudoconvex domain with smooth boundary, then the family is called a *family of bounded strongly pseudoconvex domains*. (In this paper, smoothness means C^∞.)

First we introduce the variation of the Kähler-Einstein metrics on a family of bounded strongly pseudoconvex domains. Let $(z, s) \in \mathbb{C}^n \times \mathbb{C}$ be the standard coordinates and D be a smooth domain in $\mathbb{C}^{n+1}$ such that every fiber D_s is a bounded strongly domainwith smooth boundary. Cheng and Yau's theorem [2] implies that there exists a unique complete Kähler metric $h_{\alpha\bar{\beta}}(z, s) := h^s_{\alpha\bar{\beta}}(z)$ on each fiber D_s which satisfies the following:

$$\begin{aligned}-(n+1)h_{\alpha\bar{\beta}}(z, s) &= \mathrm{Ric}_{\alpha\bar{\beta}}(z, s) \qquad \text{(the Ricci tensor)}\\ &= -\frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}} \log\det\left(h_{\gamma\bar{\delta}}(z, s)\right)_{1\le\gamma,\delta\le n}.\end{aligned}$$

This complete Kähler metric is called a *Kähler-Einstein metric*, whose Ricci curvature is a negative constant $-(n+1)$. This constant could be any negative number; $-(n+1)$ is chosen for the convenience. Then on each fiber D_s,

$$h(z, s) := \frac{1}{n+1} \log\det\left(h_{\gamma\bar{\delta}}(z, s)\right)_{1\le\gamma,\delta\le n}$$

is a potential function of the Kähler-Einstein metric $h_{\alpha\bar{\beta}}(\cdot, s)$. We can consider h as a smooth function on D (Sect. 3 in [3]). This fiberwise global potential is the *variation of the Kähler-Einstein metrics* on a family of bounded strongly pseudoconvex domains. It is an immediate consequence of the Kähler-Einstein conditions that the restriction of h to each fiber D_s is strictly plurisubharmonic. But it is not obvious that it is also plurisubharmonic or strictly plurisubharmonic in the base direction (the s-direction).

Theorem 1.1 *With the above notations, if D is a strongly pseudoconvex domain in $\mathbb{C}^{n+1}$, then h is strictly plurisubharmonic.*

In case of a general bounded pseudoconvex domain, Cheng and Yau also constructed a unique almost complete Kähler-Einstein metric which means a limit of Kähler-Einstein metrics on relatively compact subdomains [2]. In [7], Mok and Yau proved that this metric is, in fact, complete. Hence we can consider the variation $h(z, s)$ of Kähler-Einstein metrics on a family of bounded pseudoconvex domains. By an approximation process, we have the following corollary.

Corollary 1.2 *If D is pseudoconvex, then h is plurisubharmonic.*

Originally the variation of Kähler-Einstein metrics is strongly related with the Kodaira-Spencer class [12, 13]. In contrast to a family of compact Kähler manifolds, a family of pseudoconvex domains does not have Kodaira-Spencer theory. However, the geodesic curvature (which is defined in Sect. 2) is defined by the variation of Kähler-Einstein metrics. So it is natural to ask what happens if the geodesic curvature vanishes. The following theorem answers this question.

Theorem 1.3 *Suppose that the fiber dimension n is greater than or equal to* 3. *If the geodesic curvature vanishes, then the family is locally trivial.*

We will all the time consider only the case of one dimensional base, but the computations are easily generalized to the case of a higher dimensional base. Throughout this paper we use small Greek letters, $\alpha, \beta, \ldots = 1, \ldots, n$ for indices on $z \in \mathbb{C}^n$ unless otherwise specified. For a properly differentiable function f on $\mathbb{C}^n \times \mathbb{C}$, we denote by

$$f_\alpha = \frac{\partial f}{\partial z^\alpha} \quad \text{and} \quad f_{\bar\beta} = \frac{\partial f}{\partial z^{\bar\beta}}.$$

where $z^{\bar\beta}$ means $\overline{z^\beta}$. If there is no confusion, we always use the Einstein convention. For a complex manifold X, we denote by $T'X$ the complex tangent vector bundle of X of type $(1, 0)$.

2 Prelimiaries

2.1 The Kähler-Einstein Metric on a Strongly Pseudoconvex Domain

Let Ω be a bounded strongly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$. Then there exists a smooth defining function ρ of Ω satisfying that $\left(\rho_{\alpha\bar\beta}\right) > 0$ in $\bar\Omega$. Denote by $g = -\log(-\rho)$. Then g is a strictly plurisubharmonic function defined in Ω. Direct calculations [2] show that

$$g_{\alpha\bar\beta} = \frac{\rho_{\alpha\bar\beta}}{-\rho} + \frac{\rho_\alpha \rho_{\bar\beta}}{\rho^2}\ , \tag{2.1}$$

and the inverse is

$$g^{\bar\beta\alpha} = (-\rho)\left(\rho^{\bar\beta\alpha} + \frac{\rho^{\bar\beta}\rho^\alpha}{\rho - |\partial\rho|^2}\right), \tag{2.2}$$

where

$$(\rho^{\alpha\bar\beta}) = \left(\rho_{\alpha\bar\beta}\right)^{-1}, \ \ \rho^\alpha = \rho^{\alpha\bar\beta}\rho_{\bar\beta}, \text{ and } |\partial\rho|^2 = \rho^{\alpha\bar\beta}\rho_\alpha\rho_{\bar\beta}.$$

It is also easy to see that

$$g^{\bar\beta\alpha} g_\alpha g_{\bar\beta} = \frac{|\partial\rho|^2}{|\partial\rho|^2 - \rho} \le 1.$$

It follows that the metric $g_{\alpha\bar{\beta}}$ is a complete Kähler metric. The Ricci tensor of $g_{\alpha\bar{\beta}}$ is given by

$$R_{\alpha\bar{\beta}} = -(n+1)g_{\alpha\bar{\beta}} - \frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}} \log\det(\rho_{\gamma\bar{\delta}})(-\rho + |\partial\rho|^2). \tag{2.3}$$

Notice that $\det(\rho_{\alpha\bar{\beta}})(-\rho + |d\rho|^2)$ is a positive smooth function in $\bar{\Omega}$. It follows that

$$\frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}} \log\left(\det(\rho_{\gamma\bar{\delta}})(-\rho + |\partial\rho|^2)\right)$$

is a tensor whose length with respect to $\sum g_{\alpha\bar{\beta}} dz^\alpha dz^{\bar{\beta}}$ is close to zero near the boundary of Ω. Hence $(\Omega, g_{\alpha\bar{\beta}})$ is a complete Kähler manifold whose Ricci tensor is "asymptotically Einstein".

The following theorem due to Cheng and Yau gives a solution of the complex Monge-Ampère equation [2].

Theorem 2.1 (Simple Version) *For any $K > 0$ and $F \in C^\infty(\overline{\Omega})$, there exists a unique $u \in C^\infty(\Omega)$ satisfying the following relations:*

$$\begin{aligned} &\det(g_{\alpha\bar{\beta}} + u_{\alpha\bar{\beta}}) = e^{Ku} e^F \det(g_{\alpha\bar{\beta}}) \\ &\frac{1}{c}(g_{\alpha\bar{\beta}}) \le (g_{\alpha\bar{\beta}} + u_{\alpha\bar{\beta}}) \le c(g_{\alpha\bar{\beta}}). \end{aligned} \tag{2.4}$$

Moreover, if all the data are analytic, the solution is also analytic.

Theorem 2.1 implies that if we set

$$K = n+1 \quad \text{and} \quad F_\rho = -\log\det(\rho_{\alpha\bar{\beta}})(-\rho + |\partial\rho|^2),$$

the new metric tensor $h_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} + u_{\alpha\bar{\beta}}$ is the unique complete Kähler-Einstein metric on Ω with Ricci curvature is $-(n+1)$. Cheng and Yau also have computed the boundary behavior of the solution of complex Monge-Ampère equation [2].

Theorem 2.2 (Simple version) *Suppose that $F = \xi(-\rho)^{n+1}$, $\xi \in C^\infty(\bar{\Omega})$. If u is a solution of (2.4), then*

$$\left| \frac{\partial^{|A|+|B|}}{\partial z^A \partial \bar{z}^B} u \right| = O(|\rho|^{n+1/2-|A|+|B|-\varepsilon})$$

for $\varepsilon > 0$, where A and B are multi-indices. In particular, $u_{\alpha\bar{\beta}} = O(|\rho|^{n-3/2-\varepsilon})$.

2.2 Horizontal Lifts and Geodesic Curvatures

Definition 2.3 Let τ be a real $(1, 1)$-form on D which is positive definite on each fiber D_s. We denote by $v := \partial/\partial s$ the holomorphic coordinate vector field.

1. A vector field v_τ of type $(1, 0)$ is called a *horizontal lift* of v along D_s if v_τ satisfies the following:
   (i) $\langle v_\tau, w\rangle_\tau = 0$ for all $w \in T'D_s$,
   (ii) $d\pi(v_\tau) = v$.
2. The *geodesic curvature* $c(\tau)$ of τ is defined by the norm of v_τ with respect to the sesquilinear form $\langle\cdot,\cdot\rangle_\tau$ induced by τ, namely,

$$c(\tau) = \langle v_\tau, v_\tau\rangle_\tau .$$

Note that under the holomorphic coordinate (z, s), τ is written by

$$\tau = \sqrt{-1}\left(\tau_{s\bar{s}}ds \wedge d\bar{s} + \tau_{s\bar{\beta}}ds \wedge dz^{\bar{\beta}} + \tau_{\alpha\bar{s}}dz^\alpha \wedge d\bar{s} + \tau_{\alpha\bar{\beta}}dz^\alpha \wedge dz^{\bar{\beta}}\right).$$

Then the horizontal lift v_τ and the geodesic curvature $c(\tau)$ can be written by the following:

$$v_\tau = \frac{\partial}{\partial s} - \tau_{s\bar{\beta}}\tau^{\bar{\beta}\alpha}\frac{\partial}{\partial z^\alpha} \quad \text{and} \quad c(\tau) = \tau_{s\bar{s}} - \tau_{s\bar{\beta}}\tau^{\bar{\beta}\alpha}\tau_{\alpha\bar{s}},$$

where $(\tau^{\bar{\beta}\alpha})$ is the inverse of $(\tau_{\alpha\bar{\beta}})$. Then it is well known that

$$\frac{\tau^{n+1}}{(n+1)!} = c(\tau) \cdot \frac{\tau^n}{n!} \wedge \sqrt{-1}ds \wedge d\bar{s}. \tag{2.5}$$

Remark 2.4 Since τ is positive definite when restricted to D_s, (2.5) implies that if $c(\tau) > 0$ (≥ 0), then τ is a positive (semi-positive) real $(1, 1)$-form.

3 The Geodesic Curvature of the Real (1, 1)-Form Defined by a Defining Function

In this section, we consider a family of bounded strongly pseudoconvex domains. Since every fiber D_s is a bounded smooth strongly pseudoconvex domain, we can take a defining function φ of D which satisfies the following conditions:

(i) $\left(\varphi_{\alpha\bar{\beta}}(\cdot, s)\right) > 0$ on each fiber $\overline{D_s}$.
(ii) $\partial_z\varphi \neq 0$ on ∂D.

We denote by $g = -\log(-\varphi)$. By the same computation in Sect. 2.1, we have $g^{\alpha\bar{\beta}} g_\alpha g_{\bar{\beta}} \leq 1$ on each fiber D_s. It follows that $g_{\alpha\bar{\beta}}$ is a complete Kähler metric on D_s. Define the real (1, 1)-form G on D by $G = \sqrt{-1}\partial\bar{\partial}g$. A direct computation gives the following:

$$g_{s\bar{\beta}} g^{\bar{\beta}\alpha} = \varphi_{s\bar{\beta}} \left(\varphi^{\bar{\beta}\alpha} + \frac{\varphi^{\bar{\beta}}\varphi^{\alpha}}{\varphi - |\partial_z \varphi|^2} \right) + \frac{\varphi^{\alpha}\varphi_s}{|\partial_z\varphi|^2 - \varphi}.$$

This equation shows that the horizontal lift v_G with respect to G is smoothly extended up to the boundary and $v_G(\varphi)|_{\partial D} = 0$, i.e., $v_G|_{\partial D}$ is tangent to ∂D. On the other hand, the geodesic curvature of G is given by

$$c(G) = \langle v_G, v_G \rangle_G \,.$$

It follows from the definition of the Levi form that

$$c(G) = \mathcal{L}g\,(v_G, \bar{v}_G) = \mathcal{L}\left(-\log(-\varphi)\right)(v_G, \bar{v}_G) = \frac{1}{-\varphi}\mathcal{L}\varphi(v_G, \bar{v}_G) + \frac{1}{\varphi^2}\left|\partial\varphi(v_G)\right|^2,$$

where $\mathcal{L}$ is the Levi form. Hence we have the following:

(i) Since v_G is tangent to ∂D, $\partial\varphi(v_G)|_{\partial D} = 0$.
(ii) If D is pseudoconvex, then $c(G) \geq 0$.
(iii) If D is strongly pseudoconvex, then $c(G) \to \infty$ as a point goes to the boundary of order greater than or equal to 1.

4 Variation of Bounded Strongly Pseudoconvex Domains

In this section, we shall discuss the idea of the proof of Theorem 1.1. Note that we only consider the fiber dimension is greater than or equal to 2. One dimensional fiber case is already proved by Berndtsson, Maitani-Yamaguchi and Yamaguchi [1, 9, 10].

Let D be a family of bounded strongly pseudoconvex domains and $h_{\alpha\bar{\beta}}(z, s)$ be the unique complete Kähler-Einstein metric on each fiber D_s. Then the variation $h : D \to \mathbb{R}$ is defined by

$$h(z, s) = \frac{1}{n+1} \log\det\left(h_{\gamma\bar{\delta}}(z, s)\right)_{1 \leq \gamma, \delta \leq n}.$$

The real (1, 1)-form H defined by $H = \sqrt{-1}\partial\bar{\partial}h$ satisfies that the restriction on each fiber D_s is positive-definite by the Kähler-Einstein condition. Hence h is strictly plurisubharmonic if and only if the geodesic curvature $c(H) > 0$ by Remark 2.4. Therefore, it is enough to show that $c(H)$ is positive on a fixed fiber D_s.

Schumacher has proved that the geodesic curvature $c(H)$ satisfies a certain elliptic partial differential equation on each fiber (For the proof, see [12] or [3]).

Theorem 4.1 (Schumacher) *The following elliptic equation holds on each fiber D_s:*

$$-\Delta c(H) + (n+1)c(H) = \left|\bar{\partial} v_H\right|^2, \tag{4.1}$$

where $\Delta = \Delta_{h_{\alpha\bar{\beta}}}$ is the Laplace-Beltrami operator and $\left|\bar{\partial} v_H\right|^2$ is the pointwise norm with respect to the Kähler-Einstein metric $h_{\alpha\bar{\beta}}$ on D_s.

If fibers are compact, Theorem 4.1 immediately implies that the minimum of $c(H)$ is non-negative, i.e., H is positive semi-definite in D. This is how Schumacher proved the positivity of the variation of Kähler-Einstein metrics on a family of canonically polarized compact Kähler manifolds. In our case, fibers are not compact. So it is not guaranteed that there is a minimum point of $c(H)$.

But if we suppose that $c(H)$ is bounded from below on D_s, then the almost maximum principle due to Yau [14] says that there exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subset D_s$ such that

$$\lim_{k\to\infty} \nabla c(H)(x_k) = 0, \quad \liminf_{k\to\infty} \Delta c(H)(x_k) \geq 0, \quad \text{and}$$
$$\lim_{k\to\infty} c(H)(x_k) = \inf_{x\in D_s} c(H)(x).$$

It follows that

$$(n+1)c(H)(x_k, s) = \left|\bar{\partial} v_H\right|^2 + \Delta c(H)(x_k, s) \geq 0.$$

Taking $k \to \infty$, we have $c(H) \geq 0$. Moreover, if we suppose $c(H) \to \infty$ as $x \to \partial D_s$, then this prevents the function $c(H)$ from being zero. In fact, according to a theorem of Kazdan and De Turck [5], Kähler-Einstein metrics are real analytic on holomorphic coordinates, and by the Implicit Function Theorem, depend in a real-analytic way upon holomorphic parameters. This also applies to the function $c(H)$. Moreover, we have the following:

Proposition 4.2 ([11]) *Let ω be a Kähler form in $\mathbb{C}^n$. Let f and g be non-negative smooth functions on $U \subset \mathbb{C}^n$. Suppose*

$$-\Delta_\omega f + Cf = g$$

holds for some positive constant C. If $f(0) = 0$, then f and g vanish identically in a neighborhood of $0 \in \mathbb{C}^n$.

The real analyticity of $c(H)$ and Proposition 4.2 say that $c(H)$ is either identically zero, or never zero. Since $c(H)(x) \to \infty$ as $x \to \partial D_s$, $c(H)$ is strictly positive, i.e., h is strictly plurisubharmonic.

Therefore, the key point of Theorem 1.1 is investigating the boundary behavior of $c(H)$. Recall that the Kähler-Einstein metric on a strongly pseudoconvex domain is constructed by the perturbation of the reference metric which is defined by a defining function (see Sect. 2.1). Hence the variation function h can also be written by $h(z,s) = g(z,s) + u(z,s)$, where $g = -\log(-\varphi)$ and $u(z,s)$ is the solution of complex Monge-Ampère equation on each fiber D_s. Here $g_{\alpha\bar{\beta}}(z,s)$ is the reference metric on each fiber D_s. If we define the geodesic curvature $c(G)$ of $G = \sqrt{-1}\partial\bar{\partial}g$, then $c(G)$ is positive and blows up near the boundary by Sect. 3. Note that the computations in Sect. 3 are only available if the Levi form of the defining function is strictly positive up to the boundary. But it is not difficult to show that all statements in Sect. 3 hold with any choice of defining functions [3]. Therefore the following proposition completes the proof.

Proposition 4.3 ([3]) *Suppose that $\varphi(\cdot,s)$ is an approximate solution on D_s, then the geodesic curvatures $c(G)$ and $c(H)$ blow up near the boundary of the same order. More precisely, we have*

$$\frac{c(H)}{c(G)}(x) \to 1 \quad as \quad x \to \partial D_s.$$

Since $\varphi(\cdot,s)$ is an approximate solution, we could use the boundary behavior of u in Theorem 2.2. But this is not enough for showing Proposition 4.3 because we also need the boundary behavior of the s-derivatives of u. The main idea is differentiating the family of complex Monge-Ampère equations, i.e., Equation (2.4). Taking logarithm and differentiating (2.4) with respect to s, it follows that

$$\Delta_{h_{\alpha\bar{\beta}}} u_s - K u_s = F_s - \left(\Delta_{h_{\alpha\bar{\beta}}} - \Delta_{g_{\alpha\bar{\beta}}}\right) g_s.$$

Then the Schauder estimates gives the desired estimates. For the detail, see [3].

5 Variation of Bounded Pseudoconvex Domains

In this section, we discuss the variation of Kähler-Einstein metrics on general bounded pseudoconvex domains. First we recall the construction of the Kähler-Einstein metric on a bounded pseudoconvex domain. And we will prove Corollary 1.2 in the next subsection.

5.1 Kähler-Einstein Metric on a Bounded Pseudoconvex Domain

Let Ω be a bounded pseudoconvex domain. Then there exists a smooth strictly plurisubharmonic exhaustion function ψ such that for $N \in \mathbb{N}$,

$$\Omega^N := \{z \in \Omega : \psi(z) < N\}$$

is relatively compact in Ω. By the Sard theorem, we may assume that Ω^N is a bounded strongly pseudoconvex domain with smooth boundary. It is also obvious that $\{\Omega^N\}$ is a increasing union of Ω. Then Cheng and Yau's theorem implies that there exists a unique complete Kähler-Einstein metric $h^N_{\alpha\bar{\beta}}$ on Ω^N with Ricci curvature $-(n+1)$. By the Schwarz lemma for volume form due to Mok and Yau [7], we have that $\{\det(h^N_{\alpha\bar{\beta}})\}$ is a decreasing sequence, more precisely,

$$\det(h^N_{\alpha\bar{\beta}}) \geq \det(h^{N'}_{\alpha\bar{\beta}}) \quad \text{for} \quad N < N'.$$

From the Kähler-Einstein condition, $\log\det(h^N_{\alpha\bar{\beta}})$ is a strictly plurisubharmonic function on Ω^N. It follows that $\left\{\log\det(h^N_{\alpha\bar{\beta}})\right\}_{N\in\mathbb{N}}$ is a decreasing sequence of plurisubharmonic functions. This implies that the sequence converges to a plurisubharmonic function h. It is proved that $h_{\alpha\bar{\beta}}$ is the unique complete Kähler-Einstein metric with Ricci curvature $-(n+1)$ by Cheng-Yau and Mok-Yau [2, 7].

5.2 Plurisubharmonicity of the Variation

Let D be a bounded pseudoconvex domain in $\mathbb{C}^{n+1}$ such that every fiber D_s is a bounded pseudoconvex domain. By Mok and Yau's theorem, there exists a unique complete Kähler-Einstein metric $h_{\alpha\bar{\beta}}(z,s)$ with Ricci curvature $-(n+1)$. If we define the function $h : D \to \mathbb{R}$ by

$$h(z,s) = \frac{1}{n+1}\log\det\left(h_{\gamma\bar{\delta}}(z,s)\right)_{1\leq\gamma,\delta\leq n}$$

then h is strictly plurisubharmonic on each fiber D_s. Since D is a pseudoconvex domain, there exists a strictly plurisubharmonic exhaustion function ψ on D. Let $D^N = \{(z,s) \in \mathbb{C}^{n+1} : \psi(z,s) < N\}$ for $N \in \mathbb{N}$. Then we have the following:

- $D^N \subset\subset D$ and D is increasing union of $\{D^N\}$,
- each D^N is a bounded smooth strongly pseudoconvex subdomain in D.

Denote by $D^N_s = D^N \cap D_s$. Then there exists a unique complete Kähler-Einstein metric $h^N_{\alpha\bar{\beta}}(z,s)$ on each D^N_s. Define a function $h^N : D^N \to \mathbb{R}$ by

$$h^N(z,s) = \frac{1}{n+1}\log\det\left(h^N_{\gamma\bar{\delta}}(z,s)\right)_{1\le\gamma,\delta\le n}$$

for every $N \in \mathbb{N}$. Then we know that h^N is a smooth strictly plurisubharmonic function on D^N by Theorem 1.1. On each fiber D_s, $h^N(\cdot, s)$ forms a decreasing sequence which converges to $h(\cdot, s)$. It follows that the sequence $\{h^N\}$ is a decreasing sequence which converges to h on D. This implies that h is the limit of a decreasing sequence of plurisubharmonic functions, in particular h is plurisubharmonic. This proves Corollary 1.2.

6 Local Trivility

We conclude by mentioning the local triviality of a family of smooth bounded strongly pseudoconvex domains. Let D be a smooth domain in $\mathbb{C}^{n+1}$ such that every fiber is a bounded strongly pseudoconvex domain with smooth boundary. Since the computation is local, we may assume that $\pi(D) = U$ is the standard unit disc in $\mathbb{C}$. Suppose that the geodesic curvature $c(H)$ of H vanishes in D. Then (4.1) implies that $\left|\bar{\partial} v_H\right|$ vanishes, i.e., v_H is a holomorphic vector field on D. Thus we have a holomorphic vector field v_H on D such that $d\pi(v_H) = \partial/\partial s$. Moreover, if $|v_H(\varphi)| = O(|\varphi|)$, then the flow gives diffeomorphisms from one fiber to another fiber. (For the details, see [4].) This comes from the boundary behavior of the solution u if the fiber dimension is greater than or equal to 3.

Theorem 6.1 ([4]) *Suppose that the fiber dimension n is greater than or equal to* 3. *If the geodesic curvature $c(H)$ vanishes on D, then D is biholomorphic to $D_0 \times U$.*

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Extension of Holomorphic Functions and Cohomology Classes from Non Reduced Analytic Subvarieties

Jean-Pierre Demailly

Abstract The goal of this survey is to describe some recent results concerning the L^2 extension of holomorphic sections or cohomology classes with values in vector bundles satisfying weak semi-positivity properties. The results presented here are generalized versions of the Ohsawa–Takegoshi extension theorem, and borrow many techniques from the long series of papers by T. Ohsawa. The recent achievement that we want to point out is that the surjectivity property holds true for restriction morphisms to non necessarily reduced subvarieties, provided these are defined as zero varieties of multiplier ideal sheaves. The new idea involved to approach the existence problem is to make use of L^2 approximation in the Bochner-Kodaira technique. The extension results hold under curvature conditions that look pretty optimal. However, a major unsolved problem is to obtain natural (and hopefully best possible) L^2 estimates for the extension in the case of non reduced subvarieties—the case when Y has singularities or several irreducible components is also a substantial issue.

Keywords Compact Kähler manifold · Singular hermitian metric
Coherent sheaf cohomology · Dolbeault cohomology · Plurisubharmonic function
L^2 estimates · Ohsawa–Takegoshi extension theorem · Multiplier ideal sheaf

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1 Introduction and Main Results

The problem considered in these notes is whether a holomorphic object f defined on a subvariety Y of a complex manifold X can be extended as a holomorphic object F

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of the same nature on the whole of X. Here, Y is a subvariety defined as the the zero zet of a non necessarily reduced ideal $\mathcal{I}$ of $\mathcal{O}_X$, the object to extend can be either a section $f \in H^0(Y, E_{|Y})$ or a cohomology class $f \in H^q(Y, E_{|Y})$, and we look for an extension $F \in H^q(X, E)$, assuming suitable convexity properties of X and Y, suitable L^2 conditions for f on Y, and appropriate curvature positivity hypotheses for the bundle E. When Y is not connected, this can be also seen as an interpolation problem—the situation where Y is a discrete set is already very interesting.

The prototype of such results is the celebrated L^2 extension theorem of Ohsawa–Takegoshi [49], which deals with the important case when $X = \Omega \subset \mathbb{C}^n$ is a pseudoconvex open set, and $Y = \Omega \cap L$ is the intersection of Ω with a complex affine linear subspace $L \subset \mathbb{C}^n$. The accompanying L^2 estimates play a very important role in applications, possibly even more than the qualitative extension theorems by themselves (cf. Sect. 4 below). The related techniques have then been the subject of many works since 1987, proposing either greater generality [12, 34, 42–46, 48, 52], alternative proofs [3, 9], improved estimates [37, 55] or optimal ones [4, 5, 24].

In this survey, we mostly follow the lines of our previous papers [8, 14], whose goal is to pick the weakest possible curvature and convexity hypotheses, while allowing the subvariety Y to be non reduced. The ambient complex manifold X is assumed to be a Kähler and *holomorphically convex* (and thus not necessarily compact); by the Remmert reduction theorem, the holomorphic convexity is equivalent to the existence of a proper holomorphic map $\pi : X \to S$ onto a Stein complex space S, hence arbitrary relative situations over Stein bases are allowed. We consider a holomorphic line bundle $E \to X$ equipped with a singular hermitian metric h, namely a metric which can be expressed locally as $h = e^{-\varphi}$ where φ is a *quasi-psh* function, i.e. a function that is locally the sum $\varphi = \varphi_0 + u$ of a plurisubharmonic function φ_0 and of a smooth function u. Such a bundle admits a curvature current

$$\Theta_{E,h} := i\partial\overline{\partial}\varphi = i\partial\overline{\partial}\varphi_0 + i\partial\overline{\partial}u \tag{1.1}$$

which is locally the sum of a positive $(1,1)$-current $i\partial\overline{\partial}\varphi_0$ and a smooth $(1,1)$-form $i\partial\overline{\partial}u$. Our goal is to extend sections that are defined on a non necessarily reduced complex subspace $Y \subset X$, when the structure sheaf $\mathcal{O}_Y := \mathcal{O}_X/\mathcal{I}(e^{-\psi})$ is given by the multiplier ideal sheaf of a quasi-psh function ψ with *neat analytic singularities*, i.e. locally on a neighborhood V of an arbitrary point $x_0 \in X$ we have

$$\psi(z) = c \log \sum |g_j(z)|^2 + v(z), \qquad g_j \in \mathcal{O}_X(V),\ \ v \in C^\infty(V). \tag{1.2}$$

Let us recall that the multiplier ideal sheaf $\mathcal{I}(e^{-\varphi})$ of a quasi-psh function φ is defined by

$$\mathcal{I}(e^{-\varphi})_{x_0} = \Big\{ f \in \mathcal{O}_{X,x_0}\,;\ \exists U \ni x_0\,,\ \int_U |f|^2 e^{-\varphi} d\lambda < +\infty \Big\} \tag{1.3}$$

with respect to the Lebesgue measure λ in some local coordinates near x_0. As is well known, $\mathcal{I}(e^{-\varphi}) \subset \mathcal{O}_X$ is a coherent ideal sheaf (see e.g. [16]). We also denote by $K_X = \Lambda^n T_X^*$ the canonical bundle of an n-dimensional complex manifold X; in the case of (semi)positive curvature, the Bochner-Kodaira identity yields positive curvature terms only for (n, q)-forms, so the best way to state results is to consider the adjoint bundle $K_X \otimes E$ rather than the bundle E itself. The main qualitative statement is given by the following result of [8].

Theorem 1.1 *Let E be a holomorphic line bundle over a holomorphically convex Kähler manifold X. Let h be a possibly singular hermitian metric on E, ψ a quasi-psh function with neat analytic singularities on X. Assume that there exists a positive continuous function $\delta > 0$ on X such that*

$$\Theta_{E,h} + (1+\alpha\delta)i\partial\overline{\partial}\psi \geq 0 \qquad \textit{in the sense of currents, for } \alpha = 0, 1. \tag{1.4}$$

Then the morphism induced by the natural inclusion $\mathcal{I}(he^{-\psi}) \to \mathcal{I}(h)$

$$H^q(X, K_X \otimes E \otimes \mathcal{I}(he^{-\psi})) \to H^q(X, K_X \otimes E \otimes \mathcal{I}(h)) \tag{1.5}$$

is injective for every $q \geq 0$. In other words, the morphism induced by the natural sheaf surjection $\mathcal{I}(h) \to \mathcal{I}(h)/\mathcal{I}(he^{-\psi})$

$$H^q(X, K_X \otimes E \otimes \mathcal{I}(h)) \to H^q(X, K_X \otimes E \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi})) \tag{1.6}$$

is surjective for every $q \geq 0$.

Remark 1.2 (A) When h is smooth, we have $\mathcal{I}(h) = \mathcal{O}_X$ and

$$\mathcal{I}(h)/\mathcal{I}(he^{-\psi}) = \mathcal{O}_X/\mathcal{I}(e^{-\psi}) := \mathcal{O}_Y$$

where Y is the zero subvariety of the ideal sheaf $\mathcal{I}(e^{-\psi})$. Hence, the surjectivity statement can be interpreted an extension theorem with respect to the restriction morphism

$$H^q(X, K_X \otimes E) \to H^q(Y, (K_X \otimes E)_{|Y}). \tag{1.7}$$

In general, the quotient sheaf $\mathcal{I}(h)/\mathcal{I}(he^{-\psi})$ is supported in an analytic subvariety $Y \subset X$, which is the zero set of the conductor ideal

$$\mathcal{J}_Y := \mathcal{I}(he^{-\psi}) : \mathcal{I}(h) = \big\{f \in \mathcal{O}_X;\ \ f \cdot \mathcal{I}(h) \subset \mathcal{I}(he^{-\psi})\big\}, \tag{1.8}$$

and (1.6) can thus also be considered as a restriction morphism.

(B) A surjectivity statement similar to (1.7) holds true when (E, h) is a holomorphic vector bundle equipped with a smooth hermitian metric h. In that case, the required curvature condition (1.4) is a semipositivity assumption

$$\Theta_{E,h} + (1+\alpha\delta)i\partial\overline{\partial}\psi \otimes \mathrm{Id}_E \geq 0 \qquad \text{in the sense of Nakano, for } \alpha = 0, 1. \tag{1.9}$$

(This means that the corresponding hermitian form on $T_X \otimes E$ takes nonnegative values on all tensors of $T_X \otimes E$, even those that are non decomposable.)

(C) The strength of our statements lies in the fact that no strict positivity assumption is made. This is a typical situation in algebraic geometry, e.g. in the study of the minimal model program (MMP) for varieties which are not of general type. Our joint work [17] contains some algebraic applications which we intend to reinvestigate in future work, by means of the present stronger qualitative statements.

(D) Notice that if one replaces (1.4) by a strict positivity hypothesis

$$\Theta_{E,h} + i\partial\overline{\partial}\psi \geq \varepsilon\omega \qquad \text{in the sense of currents, for some } \varepsilon > 0, \tag{1.10}$$

then Nadel's vanishing theorem implies $H^q(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(he^{-\psi})) = 0$ for $q \geq 1$, and the injectivity and surjectivity statements are just trivial consequences.

(E) By applying convex combinations, one sees that condition (1.4) takes an equivalent form if we assume the inequality to hold for α varying in the whole interval [0,1]. □

We now turn ourselves to the problem of establishing L^2 estimates for the extension problem, along the lines of [49]. The reader will find all details in [14].

Definition 1.3 If ψ is a quasi-psh function on a complex manifold X, we say that the singularities of ψ are log canonical along the zero variety $Y = V(\mathcal{I}(e^{-\psi}))$ if $\mathcal{I}(e^{-(1-\varepsilon)\psi})_{|Y} = \mathcal{O}_{X\restriction Y}$ for every $\varepsilon > 0$.

In case ψ has log canonical singularities, it is easy to see by the Hölder inequality and the result of Guan-Zhou [25] on the "strong openness conjecture" that $\mathcal{I}(\psi)$ is a reduced ideal, i.e. that $Y = V(\mathcal{I}(\psi))$ is a reduced analytic subvariety of X. If ω is a Kähler metric on X, we let $dV_{X,\omega} = \frac{1}{n!}\omega^n$ be the corresponding Kähler volume element, $n = \dim X$. In case ψ has log canonical singularities along $Y = V(\mathcal{I}(\psi))$, one can also associate in a natural way a measure $dV_{Y^\circ,\omega}[\psi]$ on the set $Y^\circ = Y_{\text{reg}}$ of regular points of Y as follows. If $g \in \mathcal{C}_c(Y^\circ)$ is a compactly supported continuous function on Y° and $\widetilde{g}$ a compactly supported extension of g to X, we set

$$\int_{Y^\circ} g\, dV_{Y^\circ,\omega}[\psi] = \limsup_{t\to-\infty} \int_{\{x\in X\,,\ t<\psi(x)<t+1\}} \widetilde{g}e^{-\psi}\, dV_{X,\omega}. \tag{1.11}$$

By the Hironaka desingularization theorem, one can show that the limit does not depend on the continuous extension $\widetilde{g}$, and that one gets in this way a measure with smooth positive density with respect to the Lebesgue measure, at least on an (analytic) Zariski open set in Y°. In case Y is a codimension r subvariety of X defined by an equation $\sigma(x) = 0$ associated with a section $\sigma \in H^0(X, S)$ of some hermitian vector bundle (S, h_S) on X, and assuming that σ is generically transverse to zero along Y, it is natural to take

$$\psi(z) = r \log |\sigma(z)|^2_{h_S}. \tag{1.12}$$

One can then easily check that $dV_{Y^\circ,\omega}[\psi]$ is the measure supported on $Y^\circ = Y_{\mathrm{reg}}$ such that

$$dV_{Y^\circ,\omega}[\psi] = \frac{2^{r+1}\pi^r}{(r-1)!}\,\frac{1}{|\Lambda^r(d\sigma)|^2_{\omega,h_S}} dV_{Y,\omega} \quad \text{where} \quad dV_{Y,\omega} = \frac{1}{(n-r)!}\,\omega^{n-r}_{|Y^\circ}. \tag{1.13}$$

For a quasi-psh function with log canonical singularities, $dV_{Y^\circ,\omega}[\psi]$ should thus be seen as some sort of (inverse of) Jacobian determinant associated with the logarithmic singularities of ψ. In general, the measure $dV_{Y^\circ,\omega}[\psi]$ blows up (i.e. has infinite volume) in a neighborhood of singular points of Y. Finally, the following positive real function will make an appearance in several of our estimates:

$$\gamma(x) = \exp(-x/2) \ \text{ if } x \geq 0, \qquad \gamma(x) = \frac{1}{1+x^2} \ \text{ if } x \leq 0. \tag{1.14}$$

The first generalized L^2 estimate we are interested in is a variation of Theorem 4 in [45]. One difference is that we do not require any specific behavior of the quasi-psh function ψ defining the subvariety: any quasi-psh function with log canonical singularities will do; secondly, we do not want to make any assumption that there exist negligible sets in the ambient manifold whose complements are Stein, because such an hypothesis need not be true on a general compact Kähler manifold—one of the targets of our study.

Theorem 1.4 (L^2 estimate for the extension from reduced subvarieties) *Let X be a holomorphically convex Kähler manifold, and ω a Kähler metric on X. Let (E, h) be a holomorphic vector bundle equipped with a smooth hermitian metric h on X, and let $\psi : X \to [-\infty, +\infty[$ be a quasi-psh function on X with neat analytic singularities. Let Y be the analytic subvariety of X defined by $Y = V(\mathcal{I}(e^{-\psi}))$ and assume that ψ has log canonical singularities along Y, so that Y is reduced. Finally, assume that the Chern curvature tensor $\Theta_{E,h}$ is such that the sum*

$$\Theta_{E,h} + (1+\alpha\delta)\, i\partial\overline{\partial}\psi \otimes \mathrm{Id}_E$$

is Nakano semipositive for some $\delta > 0$ and $\alpha = 0, 1$. Then for every holomorphic section $f \in H^0(Y^\circ, (K_X \otimes E)_{|Y^\circ})$ on $Y^\circ = Y_{\mathrm{reg}}$ such that

$$\int_{Y^\circ} |f|^2_{\omega,h} dV_{Y^\circ,\omega}[\psi] < +\infty,$$

there exists an extension $F \in H^0(X, K_X \otimes E)$ whose restriction to Y° is equal to f, such that

$$\int_X \gamma(\delta\psi)\,|F|^2_{\omega,h} e^{-\psi} dV_{X,\omega} \le \frac{34}{\delta}\int_{Y^\circ} |f|^2_{\omega,h} dV_{Y^\circ,\omega}[\psi].$$

Remark 1.5 Although $|F|^2_{\omega,h}$ and $dV_{X,\omega}$ both depend on ω, it is easy to see that the product $|F|^2_{\omega,h} dV_{X,\omega}$ actually does not depend on ω when F is a $(n, 0)$-form. The same observation applies to the product $|f|^2_{\omega,h} dV_{Y^\circ,\omega}[\psi]$, hence the final L^2 estimate is in fact independent of ω. Nevertheless, the existence of a Kähler metric (and even of a complete Kähler metric) is crucial in the proof, thanks to the techniques developed in [2, 10]. The constant 34 is of course non optimal; the technique developed in [24] provides optimal choices of the function γ and of the constant in the right hand side. □

We now turn ourselves to the case where non reduced multiplier ideal sheaves and non reduced subvarieties are considered. This situation has already been considered by Popovici [52] in the case of powers of a reduced ideal, but we aim here at a much wider generality, which also yields more natural assumptions. For $m \in \mathbb{R}_+$, we consider the multiplier ideal sheaf $\mathcal{I}(e^{-m\psi})$ and the associated non necessarily reduced subvariety $Y^{(m)} = V(\mathcal{I}(e^{-m\psi}))$, together with the structure sheaf $\mathcal{O}_{Y^{(m)}} = \mathcal{O}_X/\mathcal{I}(e^{-m\psi})$, the real number m being viewed as some sort of multiplicity—the support $|Y^{(m)}|$ may increase with m, but certainly stabilizes to the set of poles $P = \psi^{-1}(-\infty)$ for m large enough. We assume the existence of a discrete sequence of positive numbers

$$0 = m_0 < m_1 < m_2 < \ldots < m_p < \ldots$$

such that $\mathcal{I}(e^{-m\psi}) = \mathcal{I}(e^{-m_p\psi})$ for $m \in [m_p, m_{p+1}[$ (with of course $\mathcal{I}(e^{-m_0\psi}) = \mathcal{O}_X$); they are called the *jumping numbers* of ψ. The existence of a discrete sequence of jumping numbers is automatic if X is compact. In general, this still holds on every relatively compact open subset

$$X_c := \{x \in X\,,\ \rho(x) < c\} \Subset X,$$

but requires some of uniform behaviour of singularities at infinity in the non compact case. We are interested in extending a holomorphic section

$$\begin{aligned} f \in H^0(Y^{(m_p)}, \mathcal{O}_{Y^{(m_p)}}(K_X \otimes E_{|Y^{(m_p)}}) \\ := H^0(Y^{(m_p)}, \mathcal{O}_X(K_X \otimes_{\mathbb{C}} E) \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}(e^{-m_p\psi})). \end{aligned}$$

[Later on, we usually omit to specify the rings over which tensor products are taken, as they are implicit from the nature of objects under consideration]. The results are easier to state in case one takes a nilpotent section of the form

$$f \in H^0(Y^{(m_p)}, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(e^{-m_{p-1}\psi})/\mathcal{I}(e^{-m_p\psi})).$$

Then $\mathcal{I}(e^{-m_{p-1}\psi})/\mathcal{I}(e^{-m_p\psi}))$ is actually a coherent sheaf, and one can see that its support is a reduced subvariety Z_p of $Y^{(m_p)}$. Therefore $\mathcal{I}(e^{-m_{p-1}\psi})/\mathcal{I}(e^{-m_p\psi}))$ can be seen as a vector bundle over a Zariski open set $Z_p^\circ \subset Z_p$. We can mimic formula (1.11) and define some sort of infinitesimal "m_p-jet" L^2 norm $|J^{m_p} f|^2_{\omega,h}\, dV_{Z_p^\circ,\omega}[\psi]$ (a purely formal notation), as the measure on Z_p° defined by

$$\int_{Z_p^\circ} g\, |J^{m_p} f|^2_{\omega,h}\, dV_{Z_p^\circ,\omega}[\psi] = \limsup_{t\to-\infty} \int_{\{x\in X\,,\ t<\psi(x)<t+1\}} \widetilde{g}\, |\widetilde{f}|^2_{\omega,h} e^{-m_p\psi}\, dV_{X,\omega} \tag{1.15}$$

for any $g \in \mathcal{C}_c(Z_p^\circ)$, where $\widetilde{g} \in \mathcal{C}_c(X)$ is a continuous extension of g and $\widetilde{f}$ a smooth extension of f on X such that $\widetilde{f} - f \in \mathcal{I}(m_p\psi) \otimes_{\mathcal{O}_X} \mathcal{C}^\infty$ (this measure again has a smooth positive density on a Zariski open set in Z_p°, and does not depend on the choices of $\widetilde{f}$ and $\widetilde{g}$). We extend the measure as being 0 on $Y_{\rm red}^{(m_p)} \smallsetminus Z_p$, since $\mathcal{I}(e^{-m_{p-1}\psi})/\mathcal{I}(e^{-m_p\psi}))$ has support in $Z_p^\circ \subset Z_p$. In this context, we introduce the following natural definition.

Definition 1.6 We define the restricted multiplied ideal sheaf

$$\mathcal{I}'(e^{-m_{p-1}\psi}) \subset \mathcal{I}(e^{-m_{p-1}\psi})$$

to be the set of germs $F \in \mathcal{I}(e^{-m_{p-1}\psi})_x \subset \mathcal{O}_{X,x}$ such that there exists a neighborhood U of x satisfying

$$\int_{Y^{(m_p)}\cap U} |J^{m_p} F|^2_{\omega,h}\, dV_{Y^{(m_p)},\omega}[\psi] < +\infty.$$

This is a coherent ideal sheaf that contains $\mathcal{I}(e^{-m_p\psi})$. Both of the inclusions

$$\mathcal{I}(e^{-m_p\psi}) \subset \mathcal{I}'(e^{-m_{p-1}\psi}) \subset \mathcal{I}(e^{-m_{p-1}\psi})$$

can be strict (even for $p = 1$).

One of the geometric consequences is the following "quantitative" surjectivity statement, which is the analogue of Theorem 1.4 for the case when the first non trivial jumping number $m_1 = 1$ is replaced by a higher jumping number m_p.

Theorem 1.7 *With the above notation and in the general setting of Theorem 1.4 (but without the hypothesis that the quasi-psh function ψ has log canonical singularities), let $0 = m_0 < m_1 < m_2 < \ldots < m_p < \ldots$ be the jumping numbers of ψ. Assume that*

$$\Theta_{E,h} + i(m_p + \alpha\delta)\partial\overline{\partial}\psi \otimes \mathrm{Id}_E \geq 0$$

is Nakano semipositive for $\alpha = 0, 1$ and some $\delta > 0$.

(a) *Let*

$$f \in H^0(Y^{(m_p)}, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}'(e^{-m_{p-1}\psi})/\mathcal{I}(e^{-m_p\psi}))$$

be a section such that

$$\int_{Y^{(m_p)}} |J^{m_p} f|^2_{\omega,h}\, dV_{Y^{(m_p)},\omega}[\psi] < +\infty.$$

Then there exists a global section

$$F \in H^0(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}'(e^{-m_{p-1}\psi}))$$

which maps to f under the morphism $\mathcal{I}'(e^{-m_{p-1}\psi}) \to \mathcal{I}(e^{-m_{p-1}\psi})/\mathcal{I}(e^{-m_p\psi})$, such that

$$\int_X \gamma(\delta\psi)\, |F|^2_{\omega,h}\, e^{-m_p\psi} dV_{X,\omega}[\psi] \le \frac{34}{\delta} \int_{Y^{(m_p)}} |J^{m_p} f|^2_{\omega,h}\, dV_{Y^{(m_p)},\omega}[\psi].$$

(b) *The restriction morphism*

$$H^0(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}'(e^{-m_{p-1}\psi})) \\ \to H^0(Y^{(m_p)}, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}'(e^{-m_{p-1}\psi})/\mathcal{I}(e^{-m_p\psi}))$$

is surjective.

If E is a line bundle and h a singular hermitian metric on E, a similar result can be obtained by approximating h. However, the L^2 estimates then require to incorporate h into the definition of the multiplier ideals, as in Theorem 1.1 (see [13]). Hosono [28] has shown that one can obtain again an optimal L^2 estimate in the situation of Theorem 1.7, when $\mathcal{I}(e^{-m_p\psi})$ is a power of the reduced ideal of Y.

Question 1.8 It would be interesting to know whether Theorem 1.1 can be strengthened by suitable L^2 estimates, without making undue additional hypotheses on the section f to extend. The main difficulty is already to define the norm of jets when there is more than one jump number involved. Some sort of "Cauchy inequality" for jets would be needed in order to derive the successive jet norms from a known global L^2 estimate for a holomorphic section defined on the whole of X. We do not know how to proceed further at this point.

2 Bochner-Kodaira Estimate with Approximation

The crucial idea of the proof is to prove the results (say, in the form of the surjectivity statement), only up to approximation. This is done by solving a $\overline{\partial}$-equation

$$\overline{\partial} u_\varepsilon + w_\varepsilon = v$$

where the right hand side v is given and w_ε is an error term such that $\|w_\varepsilon\| = O(\varepsilon^a)$ as $\varepsilon \to 0$, for some constant $a > 0$. A twisted Bochner-Kodaira-Nakano identity introduced by Donnelly and Fefferman [20], and Ohsawa and Takegoshi [49] is used for that purpose. The technology goes back to the fundamental work of Bochner [6], Kodaira [30–32], Akizuki-Nakano [1, 38], Kohn [21], Andreotti-Vesentini [2], Hörmander [26, 27]. The version we need uses in an essential way an additional correction term, so as to allow a weak positivity hypothesis. It can be stated as follows.

Proposition 2.1 (see [14, Proposition 3.12]) *Let X be a complete Kähler manifold equipped with a (non necessarily complete) Kähler metric ω, and let (E, h) be a Hermitian vector bundle over X. Assume that there are smooth and bounded functions $\eta,\ \lambda > 0$ on X such that the curvature operator*

$$B = B^{n,q}_{E,h,\omega,\eta,\lambda} = [\eta\,\Theta_{E,h} - i\,\partial\overline{\partial}\eta - i\lambda^{-1} d\partial\eta \wedge \overline{\partial}\eta,\ \Lambda_\omega] \in C^\infty(X, \mathrm{Herm}(\Lambda^{n,q}T^*_X \otimes E))$$

satisfies $B + \varepsilon I > 0$ for some $\varepsilon > 0$ (so that B can be just semi-positive or even slightly negative; here I is the identity endomorphism). Given a section $v \in L^2(X, \Lambda^{n,q}T^*_X \otimes E)$ such that $\overline{\partial}v = 0$ and

$$M(\varepsilon) := \int_X \langle (B + \varepsilon I)^{-1} v, v\rangle\, dV_{X,\omega} < +\infty,$$

there exists an approximate solution $f_\varepsilon \in L^2(X, \Lambda^{n,q-1}T^*_X \otimes E)$ and a correction term $w_\varepsilon \in L^2(X, \Lambda^{n,q}T^*_X \otimes E)$ such that $\overline{\partial}u_\varepsilon = v - w_\varepsilon$ and

$$\int_X (\eta + \lambda)^{-1} |u_\varepsilon|^2\, dV_{X,\omega} + \frac{1}{\varepsilon}\int_X |w_\varepsilon|^2\, dV_{X,\omega} \le M(\varepsilon).$$

Moreover, if v is smooth, then u_ε and w_ε can be taken smooth.

In our situation, the main part of the solution, namely u_ε, may very well explode as $\varepsilon \to 0$. In order to show that the equation $\overline{\partial}u = v$ can be solved, it is therefore needed to check that the space of coboundaries is closed in the space of cocycles in the Fréchet topology under consideration (here, the L^2_{loc} topology), in other words, that the related cohomology group $H^q(X, \mathcal{F})$ is Hausdorff. In this respect, the fact of considering $\overline{\partial}$-cohomology of smooth forms equipped with the C^∞ topology on the one hand, or cohomology of forms $u \in L^2_{\mathrm{loc}}$ with $\overline{\partial}u \in L^2_{\mathrm{loc}}$ on the other hand, yields the same topology on the resulting cohomology group $H^q(X, \mathcal{F})$. This comes from the fact that both complexes yield fine resolutions of the same coherent sheaf $\mathcal{F}$, and the topology of $H^q(X, \mathcal{F})$ can also be obtained by using Čech cochains with respect to a Stein covering $\mathcal{U}$ of X. The required Hausdorff property then comes from the following well known fact.

Lemma 2.2 *Let X be a holomorphically convex complex space and $\mathcal{F}$ a coherent analytic sheaf over X. Then all cohomology groups $H^q(X, \mathcal{F})$ are Hausdorff with respect to their natural topology (induced by the Fréchet topology of local uniform convergence of holomorphic cochains).*

In fact, the Remmert reduction theorem implies that X admits a proper holomorphic map $\pi : X \to S$ onto a Stein space S, and Grauert's direct image theorem shows that all direct images $R^q\pi_*\mathcal{F}$ are coherent sheaves on S. Now, as S is Stein, Leray's theorem combined with Cartan's theorem B tells us that we have an isomorphism $H^q(X, \mathcal{F}) \simeq H^0(S, R^q\pi_*\mathcal{F})$. More generally, if $U \subset S$ is a Stein open subset, we have

$$H^q(\pi^{-1}(U), \mathcal{F}) \simeq H^0(U, R^q\pi_*\mathcal{F}) \tag{2.1}$$

and when $U \Subset S$ is relatively compact, it is easily seen that this a topological isomorphism of Fréchet spaces since both sides are $\mathcal{O}_S(U)$ modules of finite type and can be seen as a Fréchet quotient of some direct sum $\mathcal{O}_S(U)^{\oplus N}$ by looking at local generators and local relations of $R^q\pi_*\mathcal{F}$. Therefore $H^q(X, \mathcal{F}) \simeq H^0(S, R^q\pi_*\mathcal{F})$ is a topological isomorphism and the space of sections in the right hand side is a Fréchet space. In particular, $H^q(X, \mathcal{F})$ is Hausdorff. □

3 Sketch of Proof of the Extension Theorem

The reader may consult [8, 14] for more details. After possibly shrinking X into a relatively compact holomorphically convex open subset $X' = \pi^{-1}(S') \Subset X$, we can suppose that $\delta > 0$ is a constant and that $\psi \leq 0$ (otherwise subtract a large constant to ψ). As $\pi : X \to S$ is proper, we can also assume that X admits a finite Stein covering $\mathcal{U} = (U_i)$. Any cohomology class in

$$H^q(Y, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$$

is represented by a holomorphic Čech q-cocycle with respect to the covering $\mathcal{U}$

$$(c_{i_0 \ldots i_q}), \qquad c_{i_0 \ldots i_q} \in H^0\big(U_{i_0} \cap \ldots \cap U_{i_q}, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi})\big).$$

By the standard sheaf theoretic isomorphisms with Dolbeault cohomology (cf. e.g. [15]), this class is represented by a smooth (n, q)-form

$$f = \sum_{i_0, \ldots, i_q} c_{i_0 \ldots i_q} \rho_{i_0} \overline{\partial} \rho_{i_1} \wedge \ldots \wedge \overline{\partial} \rho_{i_q}$$

by means of a partition of unity (ρ_i) subordinate to (U_i). This form is to be interpreted as a form on the (non reduced) analytic subvariety Y associated with the ideal sheaf $\mathcal{J} = \mathcal{I}(he^{-\psi}) : \mathcal{I}(h)$ and the structure sheaf $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{J}$. We get an extension as a smooth (no longer $\overline{\partial}$-closed) (n, q)-form on X by taking

$$\widetilde{f} = \sum_{i_0, \ldots, i_q} \widetilde{c}_{i_0 \ldots i_q} \rho_{i_0} \overline{\partial} \rho_{i_1} \wedge \ldots \wedge \overline{\partial} \rho_{i_q}$$

where $\widetilde{c}_{i_0 \ldots i_q}$ is an extension of $c_{i_0 \ldots i_q}$ from $U_{i_0} \cap \ldots \cap U_{i_q} \cap Y$ to $U_{i_0} \cap \ldots \cap U_{i_q}$. Without loss of generality, we can assume that ψ admits a discrete sequence of "jumping numbers"

$$0 = m_0 < m_1 < \cdots < m_p < \cdots \quad \text{such that}\, \mathcal{I}(m\psi) = \mathcal{I}(m_p\psi) \,\text{for}\, m \in [m_p, m_{p+1}[. \tag{3.1}$$

Since ψ is assumed to have analytic singularities, this follows from using a log resolution of singularities, thanks to the Hironaka desingularization theorem (by the much deeper result of [25] on the strong openness conjecture, one could even possibly eliminate the assumption that ψ has analytic singularities). We fix here p such that $m_p \leq 1 < m_{p+1}$, and in the notation of [14], we let $Y = Y^{(m_p)}$ be defined by the non necessarily reduced structure sheaf $\mathcal{O}_Y = \mathcal{O}_X / \mathcal{I}(e^{-\psi}) = \mathcal{O}_X / \mathcal{I}(e^{-m_p\psi})$.

We now explain the choice of metrics and auxiliary functions η, λ for the application of Proposition 2.1, following the arguments of [14, Proof of Theorem 2.14, p. 217]. Let $t \in \mathbb{R}^-$ and let χ_t be the negative convex increasing function defined in [14, (5.8 $*$), p. 211]. Put $\eta_t := 1 - \delta \cdot \chi_t(\psi)$ and $\lambda_t := 2\delta \frac{(\chi_t^2(\psi))^2}{\chi_t''(\psi)}$. We set

$$\begin{aligned} R_t &:= \eta_t(\Theta_{E,h} + i\partial\overline{\partial}\psi) - i\partial\overline{\partial}\eta_t - \lambda_t^{-1} i\partial\eta_t \wedge \overline{\partial}\eta_t \\ &= \eta_t(\Theta_{E,h} + (1 + \delta\eta_t^{-1}\chi_t'(\psi))i\partial\overline{\partial}\psi) + \frac{\delta \cdot \chi_t''(\psi)}{2} i\partial\psi \wedge \overline{\partial}\psi. \end{aligned}$$

Note that $\chi_t''(\psi) \geq \frac{1}{8}$ on $W_t = \{t < \psi < t+1\}$. The curvature assumption (1.4) implies

$$\Theta_{E,h} + (1 + \delta\eta_t^{-1}\chi_t'(\psi))\, i\partial\overline{\partial}\psi \geq 0 \qquad \text{on } X.$$

As in [14], we find

$$R_t \geq 0 \qquad \text{on } \; X \tag{3.2}$$

and

$$R_t \geq \frac{\delta}{16} i\partial\psi \wedge \overline{\partial}\psi \qquad \text{on } \; W_t = \{t < \psi < t+1\}. \tag{3.3}$$

Let $\theta : [-\infty, +\infty[\to [0, 1]$ be a smooth non increasing real function satisfying $\theta(x) = 1$ for $x \leq 0$, $\theta(x) = 0$ for $x \geq 1$ and $|\theta'| \leq 2$. By using a blowing up process, one can reduce the situation to the case where ψ has divisorial singularities. Then we still have

$$\Theta_{E,h} + (1 + \delta\eta_t^{-1}\chi_t'(\psi))(i\partial\overline{\partial}\psi)_{\mathrm{ac}} \geq 0 \qquad \text{on } X,$$

where $(i\partial\overline{\partial}\psi)_{\mathrm{ac}}$ is the absolutely continuous part of $i\partial\overline{\partial}\psi$. The regularization techniques of [19] and [13, Theorem 1.7, Remark 1.11] produce a family of singular metrics $\{h_{t,\varepsilon}\}_{k=1}^{+\infty}$ which are smooth in the complement $X \smallsetminus Z_{t,\varepsilon}$ of an analytic set, such that $\mathcal{I}(h_{t,\varepsilon}) = \mathcal{I}(h)$, $\mathcal{I}(h_{t,\varepsilon}e^{-\psi}) = \mathcal{I}(he^{-\psi})$ and

$$\Theta_{E,h_{t,\varepsilon}} + (1 + \delta\eta_t^{-1}\chi_t'(\psi))\, i\partial\overline{\partial}\psi \geq -\frac{1}{2}\varepsilon\omega \qquad \text{on } X.$$

The additional error term $-\frac{1}{2}\varepsilon\omega$ is irrelevant when we use Proposition 2.1, as it is absorbed by taking the hermitian operator $B + \varepsilon I$. Therefore for every $t \in \mathbb{R}^-$, with the adjustment $\varepsilon = e^{\alpha t}$, $\alpha \in]0, m_{p+1} - 1[$, we can find a singular metric $h_t = h_{t,\varepsilon}$ which is smooth in the complement $X \setminus Z_t$ of an analytic set, such that $\mathcal{I}(h_t) = \mathcal{I}(h)$, $\mathcal{I}(h_t e^{-\psi}) = \mathcal{I}(he^{-\psi})$ and $h_t \uparrow h$ as $t \to -\infty$. We now apply the L^2 estimate of Proposition 2.1 and observe that $X \smallsetminus Z_t$ is complete Kähler (at least after we shrink X a little bit as $X' = \pi^{-1}(S')$, cf. [10]). As a consequence, one can find sections u_t, w_t satisfying

$$\overline{\partial} u_t + w_t = v_t := \overline{\partial}\big(\theta(\psi - t)\cdot \widetilde{f}\,\big) \tag{3.4}$$

and

$$\begin{aligned}\int_X (\eta_t + \lambda_t)^{-1} |u_t|^2_{\omega,h_t} e^{-\psi} dV_{X,\omega} + \frac{1}{\varepsilon}\int_X |w_t|^2_{\omega,h_t} e^{-\psi} dV_{X,\omega} \\ \leq \int_X \langle (R_t + \varepsilon I)^{-1} v_t, v_t\rangle_{\omega,h_t}\, e^{-\psi} dV_{X,\omega}.\end{aligned} \tag{3.5}$$

One of the main consequence of (3.3) and (3.5) is that, for $\varepsilon = e^{\alpha t}$ and α well chosen, one can infer that the error term satisfies

$$\lim_{t\to-\infty} \int_X |w_t|^2_{\omega,h_t} e^{-\psi} dV_{X,\omega} = 0.$$

One difficulty, however, is that L^2 sections cannot be restricted in a continuous way to a subvariety. In order to overcome this problem, we play again the game of returning to Čech cohomology by solving inductively $\overline{\partial}$-equations for w_t on $U_{i_0} \cap \ldots \cap U_{i_k}$, until we reach an equality

$$\overline{\partial}\big(\theta(\psi - t)\cdot \widetilde{f} - \widetilde{u}_t\big) = \widetilde{w}_t := -\sum_{i_0,\ldots,i_{q-1}} s_{t,i_0\ldots i_q} \overline{\partial}\rho_{i_0} \wedge \overline{\partial}\rho_{i_1} \wedge \ldots \wedge \overline{\partial}\rho_{i_q} \tag{3.6}$$

with holomorphic sections $s_{t,I} = s_{t,i_0\ldots i_q}$ on $U_I = U_{i_0} \cap \ldots \cap U_{i_q}$, such that

$$\lim_{t\to-\infty} \int_{U_I} |s_{t,I}|^2_{\omega,h_t} e^{-\psi} dV_{X,\omega} = 0.$$

Then the right hand side of (3.6) is smooth, and more precisely has coefficients in the sheaf $\mathcal{C}^\infty \otimes_{\mathcal{O}} \mathcal{I}(he^{-\psi})$, and $\widetilde{w}_t \to 0$ in C^∞ topology. A priori, $\widetilde{u}_t$ is an L^2 (n,q)-form equal to u_t plus a combination $\sum \rho_i s_{t,i}$ of the local solutions of $\overline{\partial} s_{t,i} = w_t$, plus $\sum \rho_i s_{t,i,j} \wedge \overline{\partial}\rho_j$ where $\overline{\partial} s_{t,i,j} = s_{t,j} - s_{t,i}$, plus etc ..., and is such that

$$\int_X |\widetilde{u}_t|^2_{\omega,h_t} e^{-\psi} dV_{X,\omega} < +\infty.$$

Since $H^q(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(he^{-\psi}))$ can be computed with the L^2_{loc} resolution of the coherent sheaf, or alternatively with the $\overline{\partial}$-complex of $(n, \bullet)$-forms with coefficients in $\mathcal{C}^\infty \otimes_{\mathcal{O}} \mathcal{I}(he^{-\psi})$, we may assume that $\widetilde{u}_t \in \mathcal{C}^\infty \otimes_{\mathcal{O}} \mathcal{I}(he^{-\psi})$, after playing again with Čech cohomology. Lemma 2.2 yields a sequence of smooth (n, q)-forms σ_t with coefficients in $\mathcal{C}^\infty \otimes_{\mathcal{O}} \mathcal{I}(h)$, such that $\overline{\partial}\sigma_t = \widetilde{w}_t$ and $\sigma_t \to 0$ in C^∞-topology. Then $\widetilde{f}_t = \theta(\psi - t) \cdot \widetilde{f} - \widetilde{u}_t - \sigma_t$ is a $\overline{\partial}$-closed (n, q)-form on X with values in $\mathcal{C}^\infty \otimes_{\mathcal{O}} \mathcal{I}(h) \otimes \mathcal{O}_X(E)$, whose image in $H^q(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$ converges to $\{f\}$ in C^∞ Fréchet topology. We conclude by a density argument on the Stein space S, by looking at the coherent sheaf morphism

$$R^q\pi_*\big(\mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(h)\big) \to R^q\pi_*\big(\mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi})\big).$$

□

Proof of the quantitative estimates. We refer again to [14] for details. One of the main features of the above qualitative proof is that we have not tried to control the solution u_t of our $\overline{\partial}$-equation, in fact we only needed to prove that the error term w_t converges to zero. However, to get quantitative L^2 estimates, we have to pay attention to the L^2 norm of u_t. It is under control as $t \to -\infty$ only when f satisfies the more restrictive condition of being L^2 with respect to the residue measure $dV_{Y^\circ,\omega}[\psi]$. This is the reason why we lose track of the solution when the volume of the measure explodes on Y_{sing}, or when there are several jumps involved in the multiplier ideal sheaves.

4 Applications of the Ohsawa–Takegoshi Extension Theorem

The Ohsawa–Takegoshi extension theorem is a very powerful tool that has many important applications to complex analysis and geometry. We will content ourselves by mentioning only a few statements and references.

4.1 Approximation of Plurisubharmonic Functions and of Closed (1,1)-Currents

By considering the extension from points (i.e. a 0-dimensional connected subvariety $Y \subset X$), even just locally on coordinates balls, one gets a precise Bergman kernel estimate for Hilbert spaces attached to multiples of any plurisubharmonic function. This leads to regularization theorems [11] that have many applications, such as the Hard Lefschetz theorem with multiplier ideal sheaves [19], or extended vanishing theorems for pseudoeffective line bundles [7]. The result may consult [13] for a survey of these questions. Another consequence is a very simple and direct proof of Siu's result [53] on the analyticity of sublevel sets of Lelong numbers of closed positive currents.

4.2 Invariance of Plurigenera

Around 2000, Siu [54] proved that for every smooth projective deformation $\pi : \mathcal{X} \to S$ over an irreducible base S, the plurigenera $p_m(t) = h^0(X_t, K_{X_t}^{\otimes m})$ of the fibers $X_t = \pi^{-1}(t)$ are constant. The proof relies in an essential way on the Ohsawa–Takegoshi extension theorem, and was later simplified and generalized by Păun [50]. It is remarkable that no algebraic proof of this purely algebraic result is known!

4.3 Semicontinuity of Log Singularity Exponents

In [18], we proved that the log singularity exponent (or log canonical threshold) $c_x(\varphi)$, defined as the supremum of constants $c > 0$ such that $e^{-c\varphi}$ is integrable in a neighborhood of a point x, is a lower semicontinuous function with respect to the topology of weak convergence on plurisubharmonic functions. Guan and Zhou [25] recently proved our "strong openness conjecture", namely that the integrability of $e^{-\varphi}$ implies the integrability of $e^{-(1+\varepsilon)\varphi}$ for $\varepsilon > 0$ small; later alternative proofs have been exposed in [33, 51].

4.4 Proof of the Suita Conjecture

In [5] Błocki determined the value of the optimal constant in the Ohsawa–Takegoshi extension theorem, a result that was subsequently generalized by Guan and Zhou [24]. In complex dimension 1, this result implies in its turn a conjecture of N. Suita, stating that for any bounded domain D in $\mathbb{C}$, one has $c_D^2 \le \pi K_D$, where $c_D(z)$ is the logarithmic capacity of $\mathbb{C} \setminus D$ with respect to $z \in D$ and K_D is the Bergman kernel on the diagonal. Guan and Zhou [24] proved that the equality occurs if ond only if D is conformally equivalent to the disc minus a closed set of inner capacity zero.

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Group Actions in Several Complex Variables: A Survey

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Abstract We give a survey on some recent developments about group actions in several complex variables, including rigidity of the automorphism groups of the invariant domains in Stein homogenous spaces under complex reductive groups, and extension and rigidity of the proper holomorphic mappings of the domains in $\mathbb{C}^n$ with symmetries.

Keywords Group action · Rigidity · Orbit convexity

In this note, we will give a survey on two kinds of recent results about group actions in several complex variables that were mainly obtained in [22, 23, 45], one is about the rigidity of the automorphism groups of invariant domains in certain Stein homogenous spaces, another one is about the orbit convexity of the invariant Stein domains, and extension and rigidity properties of the proper holomorphic mappings of domains in $\mathbb{C}^n$ which is invariant to some group actions.

In the first two sections, we discuss the rigidity of the automorphism groups of the invariant domains in Stein homogenous spaces under complex reductive groups. In

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the last two sections, we consider the orbit convexity of the invariant Stein domains, and extension and rigidity of the proper holomorphic mappings of invariant domains in $\mathbb{C}^n$.

1 Automorphism Groups of Special Invariant Domains

Let $A = \{z \in \mathbb{C}; a < |z| < b\}$ be an annulus in the complex plane. It is obvious that the holomorphic automorphism group $Aut(A)$ of A contains the circle group S^1, given by rotations. It is known that the whole automorphism group of A is $Aut(A) = S^1 \rtimes \mathbb{Z}_2$, where the $\mathbb{Z}_2$ part is given by the reflection $z \mapsto ab/z$. So other than the obvious symmetry, A has no additional hidden positive dimensional symmetry. This phenomenon is referred to the rigidity property in the present note.

In the language of group actions, A is a S^1-invariant domain in $\mathbb{C}^*$, which is the complexification of S^1. Our aim here is to generalize the above result to a general framework of group actions.

Let G be a compact connected Lie group and K a compact subgroup of G. Let $G_{\mathbb{C}}$, $K_{\mathbb{C}}$ be the complexifications of G and K respectively. Then $G_{\mathbb{C}}/K_{\mathbb{C}}$ is a homogenous Stein manifold and $G_{\mathbb{C}}$ acts on $G_{\mathbb{C}}/K_{\mathbb{C}}$ naturally. Let D be a G-invariant domain (connected open set) in $G_{\mathbb{C}}/K_{\mathbb{C}}$. Then obviously G is an subgroup of the automorphism group $Aut(D)$ of D. Motivated by the case of an annulus, it is natural to ask whether D has no other positive dimensional symmetry, namely, $\dim Aut(D) = \dim G$? In Sect. 2, we will see that under reasonable conditions the answer to the above question is yes.

For two domains Ω_1 and Ω_2 in $\mathbb{C}^n$, $f : \Omega_1 \to \Omega_2$ is holomorphic. (1) If f is bijective, we call f is biholomorphic. (2) If for any compact subset $A \subset \Omega_2$, $f^{-1}(A)$ is compact, we call f is a proper holomorphic maping (from Ω_1 to Ω_2). Biholomorphic mappings must be proper holomorphic mappings. For two general domains, there are few results for such mappings.

In this paper, we also survey recent results on biholomorphic and proper holomorphic mappings between special domains. We consider the domains with compact Lie groups actions. These special domains are studied extensively, for example, Reinhardt domains, circular domains, quasicircular domains and so on. For the general results about Lie group action, we refer to [33, 55]. In the following, we focus on domains in $\mathbb{C}^n$ which are invariant under compact Lie groups actions.

2 Rigidity of Automorphism Groups of Certain Invariant Domains

Let G be a real Lie group. The (universal) complexification of G is defined to be a complex Lie group $G_{\mathbb{C}}$ with a real Lie group morphism $j : G \to G_{\mathbb{C}}$ such that for any real Lie group morphism f from G to a complex Lie group H there is a unique complex Lie group morphism $\tilde{f} : G_{\mathbb{C}} \to H$ such that $\tilde{f} \circ j = f$. It is known that the complexification of a real Lie group always exists and unique up to isomorphism. If G is compact, $G_{\mathbb{C}}$ is complex reductive and j is injective. For examples, $S^1_{\mathbb{C}} = \mathbb{C}^*$, $SL(n, \mathbb{R})_{\mathbb{C}} = SU(n)_{\mathbb{C}} = SL(n, \mathbb{C})$, and $U(n)_{\mathbb{C}} = GL(n, \mathbb{C})$.

We now assume that G is a connected compact Lie group and K a closed subgroup of G. Then K is also a Lie subgroup of G and $X = G/K$ is a compact homogenous space. Since $G_{\mathbb{C}}$, $K_{\mathbb{C}}$ are reductive, the quotient space $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ is a homogenous Stein manifold, which can be viewed as a complexification of X. There is a natural holomorphic action of $G_{\mathbb{C}}$ on $X_{\mathbb{C}}$ given by the left translations. Let $D \subset X_{\mathbb{C}}$ be a G-invariant domain. We will study the holomorphic automorphism group $Aut(D)$ of D.

When D is relatively compact in $G_{\mathbb{C}}/K_{\mathbb{C}}$, the third author proved in [57] the following rigidity-type result.

Theorem 2.1 ([57]) *Let G be a connected compact Lie group, K a closed subgroup of G, $D \subset\subset G_{\mathbb{C}}/K_{\mathbb{C}}$ a G-invariant domain, then $Aut(D)$ is compact.*

The above result was proved by G. Fels and L. Geatti for the special case when (G, K) is a Riemannian symmetric pair [29]. In this special case, $X_{\mathbb{C}}$ is the so-called complex symmetric space in the sense of Borel [14], namely for all $x \in X$ there exists an involution $s_x \in \mathrm{Aut}(X)$ which has x as an isolated fixed point.

In the present section, we shall discuss a more detailed description of $Aut(D)$ aiming at the following question. The automorphism group $Aut(D)$ of D obviously contains G, if the action is effective. A natural question is when there are not additional positive dimensional symmetries. If the answer to the question is positive, the automorphism groups could be thought of having a kind of rigidity. In general, the answer is not positive. For example, taking a G-domain in $G_{\mathbb{C}}$ which is also a $G \times G$-domain, the automorphism group of such a domain contains $G \times G$ which is bigger than G.

For the case of annuli in $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, we have already seen in the first section that the answer to the above question is yes. It is known that this result can be generalized to Reinhardt domains in $(\mathbb{C}^*)^n$, which can be seen as higher dimensional analogues of annuli.

Let T^n be the n-dimensional torus, it is a compact Lie group with (universal) complexification $(\mathbb{C}^*)^n$, where $\mathbb{C}^*$ is the punctured complex line with the usual multiplication as the group structure. There is a natural action of T^n on $(\mathbb{C}^*)^n$ defined by the group multiplication, and a Reinhardt domain D is by definition a domain invariant under this action.

The automorphism group $Aut(D)$ of D obviously contains T^n. On the other hand, the rigidity property for $Aut(D)$ with $D \subset\subset (\mathbb{C}^*)^n$, asserts that the identity component $Aut(D)^0$ of $Aut(D)$ is exactly T^n. This result was established by some different methods in several papers in the 1980s (see [2, 3, 39, 47]).

We note that the case of Reinhardt domains is a special case in the above setting when $G = T^n$ and K is trivial. We shall discuss the rigidity question in a general setting and extend the well-known results for Reinhardt domains to the general case of invariant domains in the homogeneous Stein spaces of several important classes, for example, in the complexifications of isotropy irreducible spaces, including those of irreducible Riemannian symmetric spaces of compact type which are complex symmetric spaces.

Returning to our setting of $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$, for a G-invariant domain D in $X_{\mathbb{C}}$, there is a morphism from G to $Aut(D)$. We denote the image of G in $Aut(D)$ by $\breve{G}$. In general, $\breve{G}$ can be identified with a quotient group of G. If the action is effective, $\breve{G}$ is isomorphic to G.

The following theorem is to relate the automorphism group of D to the isometry group of (X, g) for certain homogenous Riemannnian metric on $X = G/K$.

Theorem 2.2 ([23]) *Let $D \subset X_{\mathbb{C}}$ be a G-invariant domain and let K be connected. If W is a connected compact subgroup of $Aut(D)$ containing $\breve{G}$, then W can be naturally realized as a subgroup of the isometry group* $\mathrm{Iso}(X, g)$ *of (X, g), where g is some G-invariant Riemannian metric on X. Furthermore, if D is relatively compact in $X_{\mathbb{C}}$, then the identity component $Aut(D)^0$ of $Aut(D)$ can be realized as a closed subgroup of* $\mathrm{Iso}(X, g)$.

Although K is required here to be connected, it is shown in [23] that this constraint can essentially be removed. This is important in various applications of Theorem 2.2.

The well-known result on Reinhardt domains mentioned above is an immediate corollary of Theorem 2.2 since the identity component of the isometry group of the torus T^n with any T^n-invariant Riemannian metric is precisely T^n. In particular, the automorphism group of an annulus is just the isometry group of the circle T^1. This gives a new point of view in the approach to the well-known result for Reinhardt domains.

In the case when (G, K) is a Riemannian symmetric pair, we have

Theorem 2.3 *Let $G, K, D, \breve{G}$ as in Theorem 2.2 (K not necessarily connected). Assume that (G, K) is a Riemannian symmetric pair. Then $\breve{G}$ is a maximal connected compact subgroup of $Aut(D)$. In particular, if the identity component $Aut(D)^0$ of $Aut(D)$ is itself compact, then $Aut(D)^0 = \breve{G}$.*

Combining Theorem 2.3 with Theorem 2.1, we get

Corollary 2.4 *With the assumptions and notations as in Theorem 2.3. If D is relatively compact in $X_{\mathbb{C}}$. Then $Aut(D)^0 = \breve{G}$.*

Irreducible symmetric spaces are special cases of isotropy irreducible spaces. Theorem 2.3 and Corollary 2.4 actually extend to the more general case when G/K is an isotropy irreducible space.

We recall some basic notions about isotropy irreducible spaces. Let (M, g) be a Riemannian manifold and $I_p(M)$ be the group of isometries of M fixing $p \in M$. M is called isotropy irreducible at p if the isotropy representation of $I_p(M)$ on $T_p(M)$ is irreducible, and called strongly isotropy irreducible at p if the isotropy representation of the identity component of $I_p(M)$ on T_pM is irreducible.

We call M is (strongly) isotropy irreducible if it is (strongly) isotropy irreducible at each point. As an example, an irreducible Riemannian symmetric space is isotropy irreducible. By the definition, the universal covering of an (strongly) isotropy irreducible space is also (strongly) isotropy irreducible.

It turns out that an isotropy irreducible Riemannnian manifold must be homogenous, so it can be represented as the form G/K with a Lie group and G a compact subgroup K of G.

Conversely, given a (not necessarily connected) Lie group G and a compact subgroup K of G with G/K connected such that K acts irreducibly on $\mathfrak{g}/\mathfrak{k}$, then any two G-invariant Riemannnian metrics on G/K differ only by a scalar multiplication, and any G-invariant Riemannnian metric on G/K makes G/K an isotropy irreducible space, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K respectively. Just as in the case of symmetric spaces, the representation in Klein form G/K of an isotropy irreducible space may not be unique.

Following Wang and Ziller [50], we call G/K an isotropy irreducible homogenous space if K acts irreducibly on $\mathfrak{g}/\mathfrak{k}$, and a strongly isotropy irreducible homogenous space if K^0 acts irreducibly on $\mathfrak{g}/\mathfrak{k}$.

The complete classification of strongly isotropy irreducible spaces and simply connected isotropy irreducible spaces has been finished.

As important examples of isotropy irreducible homogenous spaces, irreducible symmetric spaces were classified by Cartan [15, 16]. Non-symmetric strongly isotropy irreducible homogeneous spaces were classified independently by Manturov [41–43] in 1961 and by Wolf [51] in 1968, and completed by Kräimer [37] in 1975. General simply connected isotropy irreducible homogenous spaces which are not strongly isotropy irreducible were completely classified by M. Wang and W. Ziller in 1991 [50].

Theorem 2.5 ([23]) *Let $X = G/K$ be a strongly isotropy irreducible space, where G is a connected compact Lie group and K is a closed subgroup of G. Let $D \subset\subset G_{\mathbb{C}}/K_{\mathbb{C}}$ be a G-invariant domain. Assume both G_2 and $Spin(7)$ are not the universal covering group $\tilde{G}$ of G. Then $Aut(D)^0 = \check{G}$.*

Theorem 2.6 ([23]) *Let $X = G/K$ be an isotropy irreducible homogeneous space which is not strongly isotropy irreducible, where G is a connected compact Lie group, $K \neq \{e\}$ is a closed subgroup of G. Let $D \subset\subset G_{\mathbb{C}}/K_{\mathbb{C}}$ be a G-invariant domain. Then $Aut(D)^0 = \check{G}$.*

The exceptional case when $K = \{e\}$ in Theorem 2.6 is also discussed in [23].

The above results about the case of isotropy irreducible homogeneous spaces could be extended to a more general case of the so-called "asystatic" transitive actions. A transitive action of the group G on G/K is called asystatic, if the group of all automorphisms of the homogeneous G-space G/K is discrete, i.e., if $N_G(K)/K$ is discrete. Another case of such actions are given by the space G/T, where T is the maximal compact torus in G. The isometric groups of the asystatic spaces also have rigidity property (see [31]).

It is also natural to consider invariant domains which are hyperbolic in the sense of Kobayashi but may be not relatively compact. The automorphism groups of such domains are always Lie groups which properly act on the domains [36]. In this section, hyperbolicity of a complex manifold always means the hyperbolicity in the sense of Kobayashi, that is, the Kobayashi pseudo-distance on the manifold is in fact a distance.

In contrast to the case of relatively compact invariant domains, the automorphism group of a hyperbolic invariant domain may be non-compact. Nevertheless one might also expect that the identity component of the automorphism group is compact and has a similar rigidity property.

For hyperbolic Reinhardt domains D in $(\mathbb{C}^*)^n$ it is known that the identity component of $Aut(D)$ is just the n-dimensional torus T^n [39], and there are examples showing that $Aut(D)$ may have infinitely many components. We have established similar rigidity results for the automorphism groups of hyperbolic invariant domains in the complexifications of isotropy-irreducible spaces.

Theorem 2.7 ([23]) *Let $X = G/K$ be a strongly isotropy irreducible space. Let $D \subset G_{\mathbb{C}}/K_{\mathbb{C}}$ be a hyperbolic G-invariant domain, and assume that universal covering group of G is neither G_2 nor $Spin(7)$. Then $Aut(D)^0 = \breve{G}$.*

Theorem 2.8 *Let $X = G/K$ be an isotropy irreducible homogeneous space which is not strongly isotropy irreducible. Suppose that $K \neq \{e\}$. Let $D \subset G_{\mathbb{C}}/K_{\mathbb{C}}$ be a hyperbolic G-invariant Stein domain. Then $Aut(D)^0 = \breve{G}$.*

The special case of $K = \{e\}$ for Theorem 2.8 is as follows: G is a connected compact simple Lie group, and G/K is isometric to G with some bi-invariant Riemannian metric and $G_{\mathbb{C}}/K_{\mathbb{C}} = G_{\mathbb{C}}$. In this case, the proof of Theorem 2.8 given in [23] shows that $Aut(D)^0 = \{(g, h) \in G \times G; gDh = D\}$. In particular, if D is a hyperbolic G bi-invariant domain, then $Aut(D)^0 = G \times G$.

In Theorem 2.8, the assumption of the Steinness of the considered domains is necessary. The reason is that the envelope of holomorphy of a hyperbolic invariant domain may not be hyperbolic in general (see [23] for an example).

The basic idea in our proofs of the theorems in this section is to relate the automorphism groups of the invariant domains with the isometric groups of minimal orbits in these domains and use the known results on the isometric groups of isotropy irreducible homogeneous spaces. The proofs are based on Theorem 2.1 and a result [56] on the univalence of envelope of holomorphy of certain invariant domains.

3 Orbit Convexity

In the present section, we will survey some recent results about orbit convex, we refer to the survey [46] for more information.

First, we recall some definitions about group actions for compact group actions.

Definition 3.1 Let G be a Lie group. A complex space X together with a group homomorphism ρ from G into the group $Aut(X)$ is called a complex G-space if the action $G \times X \to X$, $(g, x) \mapsto \rho(g)(x)$ is continuous.

We write $g \cdot x$ or gx for $\rho(g)(x)$.

If G is a complex Lie group, and the action $G \times X \to X$, $(g, x) \mapsto \rho(g)(x)$ is holomorphic, we call X a holomorphic G-space.

For a subset $A \subseteq X$, A is called G-invariant if $G \cdot A := \{g \cdot x : x \in A, g \in G\} = A$.

A holomorphic mapping $\psi : X \to Y$ between two complex G-spaces X and Y is called equivariant if $\psi(g \cdot x) = g \cdot \psi(x)$ for all $g \in G, x \in X$.

We denote by $Hol_G(X, Y)$ the set of all holomorphic equivariant mappings from X to Y.

Let G be a compact Lie group, G act linearly on $\mathbb{C}^n$, i.e., there is a continuous representation $\rho : G \to GL(\mathbb{C}^n)$.

Denote by $\mathcal{O}(\mathbb{C}^n)^G$ be the set of G-invariant entire functions, i.e.

$$\mathcal{O}(\mathbb{C}^n)^G := \{f \in \mathcal{O}(\mathbb{C}^n) : f \circ \rho(g) = f \ \ for\ all\ g \in G\}.$$

Let the compact Lie group G act linearly on $\mathbb{C}^n$ and G act trivially on $\mathbb{C}$, then $\mathcal{O}(\mathbb{C}^n)^G = Hol_G(\mathbb{C}^n, \mathbb{C})$.

Let G be a compact Lie group and $G^{\mathbb{C}}$ be the universal complexification of G. If $\mathfrak{g}$ is the Lie algebra of G, and $exp : \mathfrak{g} \to G$ is the exponential map, then the first Cartan decomposition theorem says that, $G^{\mathbb{C}} = G \cdot exp(i\mathfrak{g})$, and $G \cap exp(i\mathfrak{g}) = \{e\}$, where e is the identity of G.

Definition 3.2 (*see* [33, 56]) Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, and let $G^{\mathbb{C}}$ be the complexification of G. Let X be a $G^{\mathbb{C}}$-space. A G-invariant subset U of X is said to be orbit convex if for each $z \in U$ and $v \in i\mathfrak{g}$, such that $exp(v) \cdot z \in U$, then $exp(tv) \cdot z \in U$ for all $t \in [0, 1]$.

It is easy to use the following proposition to make invariant orbit convex domain.

Proposition 3.1 (see [33]) *Let X be a holomorphic $G^{\mathbb{C}}$-space, $D \subset X$ be a $G^{\mathbb{C}}$-domain, and φ be a G-invariant plurisubharmonic function on D. Then the G-set $D_0 = \{x \in D : \varphi(x) < 1\}$ is orbit convex.*

Zhou [55] proved the following two theorems.

Theorem 3.2 *Let $(S^1)^k$ act linearly on $\mathbb{C}^n$ and $\mathcal{O}(\mathbb{C}^n)^{(S^1)^k} = \mathbb{C}$. Let Ω be an $(S^1)^k$-invariant domain of holomorphy in $\mathbb{C}^n$ which contains 0. Then Ω is orbit convex.*

Theorem 3.3 *Let $(S^1)^k$ act linearly on $\mathbb{C}^n$, all linear actions on $\mathbb{C}^n$ of $(S^1)^k$ are of the form*

$$\rho : (S^1)^k \to GL(\mathbb{C}^n), \qquad \rho(s_1, \cdots, s_k) = diag(s_1^{m_{11}} \cdots s_k^{m_{1k}}, \cdots, s_1^{m_{n1}} \cdots s_k^{m_{nk}}),$$

where all m_{ij} are integer. Let Ω be an $(S^1)^k$-invariant domain of holomorphy in $\mathbb{C}^n$ which contains 0*. If all $m_{ij} \geq 0$, then Ω is orbit convex.*

The condition $0 \in \Omega$ can not be dropped, as the following example explains.

Recall that a domain $\Omega \in \mathbb{C}^n$ is called Hartogs domain if $(z', z_n) \in \Omega$ and $s \in S^1$, then $(z', sz_n) \in \Omega$.

Tarabusi and Trapani [49] gave the following example which shows that not every invariant pseudoconvex domain is orbit convex.

Example 3.1 Let

$$\begin{aligned} \Omega = \{(z_1, z_2, w) \in \mathbb{C}^3 : 0 < log|z_1| < 1, \\ log|w| < argz_1 < log|w| + 1, |z_2| < argz_1 < 4\pi\}, \end{aligned} \tag{1}$$

then Ω is a pseudoconvex Hartogs domain, but it is not orbit convex.

According to the above theorem and other results, A. G. Sergeev and X. Y. Zhou (see [46]) conjectured the following:

Conjecture 3.1 *Let K be a compact Lie group and act linearly on $\mathbb{C}^n$. Then any K-invariant domain of holomorphy Ω in $\mathbb{C}^n$ which contains* 0 *is orbit convex.*

Ning et al. [45] proved the above conjecture is true under the additional assumption $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$. As a result, any pseudoconvex circular domain which contains 0 is complete.

4 Biholomorphic and Proper Holomorphic Mappings Between Special Domains

For domains with compact Lie group action, the biholomorphisms and proper holomorphic mappings have special properties. In this section, we survey some results by Cartan [17], Kaup [35], Heinzner [32], Bell [6, 7], Ning et al. [45] et al. about biholomorphic mappings or proper holomorphic mappings between some special domains in $\mathbb{C}^n$. We give these results from the point view of the group actions.

A classical theorem of Cartan [17] asserts that:

Theorem 4.1 *If $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping between bounded circular domains in $\mathbb{C}^n$ which contain 0, and if $f(0) = 0$, then f is a linear mapping.*

Cartan's theorem was later generalized by Kaup [35] to biholomorphic mappings between some special invariant domains with particular S^1 actions.

A domain Ω in $\mathbb{C}^n$ is called a $(p_1, \cdots, p_n)$-domain if it is stable under the transformations

$$(z_1, \cdots, z_n) \mapsto (t^{p_1} z_1, \cdots, t^{p_n} z_n)$$

where $t \in S^1 := \{s \in \mathbb{C} : |s| = 1\}$ and $p_1, \cdots, p_n$ are integers. When all p_i are equal, the $(p_1, \cdots, p_n)$-domain is just the circular domain.

Kaup [35] proved the following:

Theorem 4.2 *If $0 \in \Omega$ is a bounded $(p_1, \cdots, p_n)$-domain and $p_i > 0$ for all i, then every automorphism of Ω extends holomorphically to an open neighborhood of the topological closure $\overline{\Omega}$ of Ω.*

A $(p_1, \cdots, p_n)$-domain with $p_i > 0$ for all i is called quasicircular domain.

Later on, P. Heinzner extended Kaup's result to the domains invariant with respect to some actions of any compact Lie groups in [32].

Heinzner [32] proved the following:

Theorem 4.3 *If $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$, then for any bounded G-invariant domain Ω in $\mathbb{C}^n$ which contains 0, any automorphism f of Ω can be extended holomorphically to an open neighborhood of $\overline{\Omega}$, and in addition if $f(0) = 0$, then f is a polynomial mapping.*

If S^1 acts on $\mathbb{C}^n$ by

$$(z_1, \cdots, z_n) \mapsto (t^{p_1} z_1, \cdots, t^{p_n} z_n)$$

where $t \in S^1$ and $p_1, \cdots, p_n$ are integers with $p_i > 0$ for $1 \leq i \leq n$. Then $\mathcal{O}(\mathbb{C}^n)^{S^1} = \mathbb{C}$. Therefore, Kaup's theorem is a special case of Heinzner's.

The above three authors all dealt with the automorphisms or biholomorphic mappings on some group invariant domains.

Bell [7] considered a kind of more general mappings than biholomorphic mappings, the so-called proper holomorphic mappings, but between more special domains (i.e., bounded circular domains containing 0) than what Kaup and Heinzner dealt with.

Bell [7] proved the following:

Theorem 4.4 *For any proper holomorphic mapping $f : \Omega_1 \to \Omega_2$, where Ω_1 and Ω_2 are bounded circular domains containing 0 in $\mathbb{C}^n$.*

(1) Let $K(z, w)$ be the Bergman kernel function of Ω_1. If for each compact subset D of Ω_1, there is an open set $G = G(D)$ containing $\overline{\Omega}_1$ such that $K(z, w)$ extends holomorphically with respect to $z \in G$ for all $w \in D$, then f can be extended holomorphically to an open neighborhood of $\overline{\Omega}_1$.
(2) If $f^{-1}(0) = \{0\}$, then f is a polynomial mapping.

Ning et al. [45] extended the above results due to Cartan, Kaup, Heinzner and Bell et al. to proper holomorphic mappings between some invariant domains w.r.t. arbitrary compact Lie groups. They found that the assumption (1) of S. Bell on the Bergman kernel functions in his above theorem is redundant.

Ning et al. [45] proved the following:

Theorem 4.5 *Let G_j be compact Lie groups, act linearly on $\mathbb{C}^n$ with $\mathcal{O}(\mathbb{C}^n)^{G_j} = \mathbb{C}$, and Ω_j be G_j-invariant bounded domains in $\mathbb{C}^n$ which contains 0 for $j = 1, 2$. Suppose that $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping. Then*

(1) f *can extend holomorphically to an open neighborhood of* $\overline{\Omega}_1$.
(2) If in addition $f^{-1}(0) = \{0\}$, *then* f *is a polynomial mapping.*

From the above theorem or Heinzner [32], if $f : \Omega_1 \to \Omega_2$ is biholomorphic, $f(0) = 0$, then f is a polynomial mapping. It is natural to ask what is the degree of such f? For results about this problem, see [20, 44, 53, 54].

S. Bell's transformation formula, which is proved by Bell [6] about proper holomorphic mappings, is useful in the study of proper holomorphic mappings.

Theorem 4.6 (see [6]) *Suppose that* $f : \Omega_1 \to \Omega_2$ *is a proper holomorphic mapping between bounded domains* Ω_1 *and* Ω_2 *contained in* $\mathbb{C}^n$. *Let* P_i *denote the Bergman projections associated to* Ω_i *for* $i = 1, 2$, *and let* $u = Det[f']$. *Then the Bergman projection transform formula holds:*

$$P_1(u \cdot \phi \circ f) = u \cdot (P_2\phi) \circ f$$

for all ϕ *in* $L^2(\Omega_2)$.

The proofs of both of the Theorems 4.4, 4.5 need Bell's transformation formula. The study of the Bergman kernel is important. The Bergman kernel of the circular domain is relatively easier than that of the domain invariant w.r.t. general Lie group action.

In the proof of Theorem 4.5, Ning, Zhang and Zhou extended the idea of Bell [7] to the setting of group actions, by using S. Bell's transformation formula for the Bergman projections and by combining with some results about compact Lie group actions.

The following two propositions show some properties of the Bergman kernel of the invariant domain, which are important in the proof of Theorem 4.5.

Proposition 4.7 *Let* G *be a compact Lie group which acts linearly on* $\mathbb{C}^n$ *with* $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$. *Suppose that* $0 \in \Omega$ *is a bounded* G*-invariant domain in* $\mathbb{C}^n$. *Let* $K(z, w)$ *be the Bergman kernel function of* Ω. *For each compact subset* D *of* Ω, *there is an open set* $G = G(D)$ *depending on* D *and containing* $\overline{\Omega}$ *such that* $K(z, w)$ *extends holomorphically with respect to* z *in* G *for all* $w \in D$.

Proposition 4.8 *Let* G *be a closed subgroup of* $U(n)$ *with* $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$. *Let* $0 \in \Omega$ *be a bounded* G*-invariant domain in* $\mathbb{C}^n$. *Suppose* $K(z, w)$ *is the Bergman kernel function associated to* Ω. *Then* $K^{\bar{\alpha}}(z, 0)$ *is a polynomial with* $deg\ K^{\bar{\alpha}}(z, 0) \le a|\alpha|$, *where* a *is some constant.*

For any multi-index β, *there are constants* $c_{\beta,\alpha}$ *such that*

$$z^\beta = \sum_{|\alpha| \le a|\beta|} c_{\beta,\alpha} K^{\bar{\alpha}}(z, 0).$$

Fefferman [28] proved that any biholomorphic mapping between bounded strictly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary can extend to the boundary smoothly.

After that, Bell and Catlin [10], Diederich and Fornaess [25] proved that: let Ω_1 and Ω_2 be two bounded pseudoconvex domains with smooth (or analytic) boundary. If Ω_1 satisfies condition R, then any proper holomorphic mapping from Ω_1 to Ω_2 can extend smoothly (or holomorphically) to $\overline{\Omega_1}$.

It should be noted that we need neither the boundary to be smooth nor the domain to be pseudoconvex in the above Theorem 4.5.

It has been proved by Heinzner [32] that: Let G be a compact Lie group, which acts linearly on $\mathbb{C}^n$ and $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$. If two bounded G-invariant domains Ω_1 and Ω_2 which both contain 0 are biholomorphically equivalent, then there is a polynomial biholomorphism $f : \Omega_1 \to \Omega_2$ which fixes 0.

Bell [5] asked: if two Reinhardt domains are related by a proper holomorphic mapping, then there is such a map which is a polynomial mapping? See [13, 40] for partial answer to this question. By Berteloot and Pinchuk [13], it is not hard to deduce that for two bounded Reinhardt domains in $\mathbb{C}^2$ contain 0, if there is a proper holomorphic mapping between them, then there is a polynomial proper mapping between them.

More generally, one may ask the following question: Let G be a compact Lie group, which acts linearly on $\mathbb{C}^n$ and $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$. For any two bounded invariant domains Ω_1 and Ω_2 which contain 0, if $f : \Omega_1 \to \Omega_2$ is proper holomorphic, is there such a map which is a polynomial mapping?

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Boundary Asymptotics of the Relative Bergman Kernel Metric for Elliptic Curves IV: Taylor Series

Robert Xin Dong

Abstract For a Legendre family of elliptic curves, the two-term asymptotic expansion of the relative Bergman kernel metric near the degenerate boundary is obtained by an approach based on the Taylor series of Abelian differentials and Riemann periods. Namely, the curvature form has hyperbolic growth in the transversal direction with an explicit second term at the node. For another nodal degenerate family of elliptic curves, the result turns out to be the same. But for two cusp cases, it is either trivial with a constant period or reducible to the Legendre family case. The proofs do not depend on special elliptic functions, and work also for higher genus cases. In the last part, we discuss invariant properties on curves.

Keywords Variation of Bergman kernel · Degeneration of elliptic curve
Node · Cusp

1 Introduction

On a connected complex manifold, the Bergman kernel is a reproducing kernel of the space of L^2 holomorphic top degree forms. It is determined by the complex structure and plays big roles in many deep results in Several Complex Variables and Complex Geometry. For a line bundle L equipped with a Hermitian metric h, the Bergman kernel is defined as $B := \sum_j |s_j|_h^2$, which is independent of choices of the complete orthonormal basis $\{s_j\}_j$ of $H^0(X, L)$. In this paper we consider the canonical bundle, and study asymptotic behaviours of the variations of Bergman kernels at the limiting case near degenerate boundaries by changing the complex structure. The variation of the Bergman kernels was initially studied by Maitani and Yamaguchi [19], who used the Green function as defining functions for potential-theoretically hyperbolic Riemann surfaces, combined the Bergman-Schiffer formula and derived the log-plurisubharmonicity of the Bergman kernels. Later their result was generalised to higher dimensional cases by Berndtsson [2] using L^2 estimates for the $\bar{\partial}$ operator.

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After that, arbitrary dimensional Stein manifolds and projective algebraic manifolds cases were decisively solved in [3, 6, 22, 23] and so on (see also [15, 16]). Recently in [5, 7, 8, 18], it was shown that this log-plurisubharmonic variation of Bergman kernels is intimately related to the extension of holomorphic functions with (optimal) L^2 estimates, which is originally due to Ohsawa and Takegoshi [20].

For a holomorphic family of Riemann surfaces X_λ parametrized by $\lambda \in \mathbb{C}$, each fiberwise Bergman kernel is locally written as $B_\lambda = k_\lambda(z)dz \wedge d\bar{z}$, in some coordinate z for some function k_λ. Then, the above variation results for Bergman kernels guarantee that $\log k_\lambda(z)$ is plurisubharmonic with respect to the parameter λ, if X_λ is smooth. Notice that $\log k_\lambda(z)$ induces the so-called relative Bergman kernel metric, whose curvature is denoted as $\sqrt{-1}\,\partial\bar{\partial}\log k_\lambda(z)$. In particular, the transversal curvature, denoted as $L_{\lambda,z} := \sqrt{-1}\,\partial_\lambda\bar{\partial}_\lambda \log k_\lambda(z)$, is semi-positivity, i.e.,

$$L_{\lambda,z} \geq 0. \tag{1.1}$$

Thus, if X_{λ_0} is `singular`, a natural question is to characterise asymptotic behaviours of $L_{\lambda,z}$ and $\log k_\lambda(z)$, as $\lambda \to \lambda_0$. In the affine coordinate $(x, y) \in \mathbb{C}^2$, the so-called Legendre family of elliptic curves $X^{(1)}_\lambda := \{y^2 = x(x-1)(x-\lambda)\} \cup \{\infty\}$ gives a general description of genus one compact Riemann surfaces, whose moduli space is $\mathbb{C} \setminus \{0, 1\}$. As λ tends to the moduli space boundary, i.e., $\{0, 1, \infty\}$, $X^{(1)}_\lambda$ degenerates to a singular curve with a node. By using the Weierstrass-$\wp$ function's coordinate parameterization and the elliptic modular lambda function's Taylor expansion, the author in [10, 11] observed that the curvature form of the relative Bergman kernel metric $L^{(1)}_{\lambda,z}$ has `hyperbolic growth` near 0 (asymptotic to the Poincaré metric of the punctured unit disk), and $L^{(1)}_{\lambda,z}$ coincides with the Poincaré metric of $\mathbb{C} \setminus \{0, 1\}$.

Theorem 1.1 *In the local coordinate z on $X^{(1)}_\lambda$, write its Bergman kernel as $B^{(1)}_\lambda = k^{(1)}_\lambda(z)dz \wedge d\bar{z}$, for $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Then, as $\lambda \to 0$,*

$$L^{(1)}_{\lambda,z} = \frac{\sqrt{-1}d\lambda \wedge d\bar{\lambda}}{|\lambda|^2(-\log|\lambda|^2)^2}\left(1 + \frac{2\log 16}{\log|\lambda|} + \mathrm{O}\left(\frac{1}{(\log|\lambda|)^2}\right)\right).$$

As $\lambda \to 0$, $X^{(1)}_\lambda$ degenerates to a singular curve $X^{(1)}_0 := \left\{y^2 = x^2(x-1)\right\} \cup \{\infty\}$. The second term in the asymptotic expansion contains more "logarithmic" information, which slows down the growth order at infinity. The case of other boundary points 1 and ∞ was studied in [12]. In fact, the above result describes the monodromy around the origin and are related to Schmid's Nilpotent Orbit Theorem [17, 21] from the variation of Hodge structures. The proof of Theorem 1.1 highly depend on special elliptic functions, and is difficult to be carried over to more general cases.

The motivation of this paper is to answer the following question. What happens to other families of curves (possibly elliptic, degenerating to a singular one with a node or cusp) where the special elliptic function method cannot apply? To answer this question, firstly we provide an alternative proof to Theorem 1.1, based on the Taylor series expansion of Abelian differentials and Riemann periods. We could see that this approach could determine accurately not only the first term but also the second term exactly as the elliptic function method does. By exploring this alternative approach, we obtain a new result on the asymptotic behaviours of the relative Bergman kernel metric for another nodal family of curves $X_\lambda^{(2)} := \left\{y^2 = (x-1)(x^2-\lambda)\right\}$, which degenerates as $\lambda \to 0$ to the same singular curve $X_0^{(1)}$ as above.

Theorem 1.2 *In the local coordinate z on $X_\lambda^{(2)}$, write its Bergman kernel as $B_\lambda^{(2)} = k_\lambda^{(2)}(z)dz \wedge d\bar{z}$, for $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Then, as $\lambda \to 0$,*

$$L_{\lambda,z}^{(2)} \sim \frac{\sqrt{-1}d\lambda \wedge d\bar{\lambda}}{|\lambda|^2(-\log|\lambda|^2)^2}.$$

Here "$\sim$" means that the ratio of both sides tends to 1, as $\lambda \to 0$. Secondly, for the cusp degeneration cases, two families of degenerate Riemann surfaces, namely $X_\lambda^{(3)} := \left\{y^2 = x(x^2-\lambda)\right\}$ and $X_\lambda^{(4)} := \left\{y^2 = x(x-\lambda)(x-\lambda^2)\right\}$, are considered. They have constant and non-constant periods, respectively. Information on both the singularity and the complex structure contributes to the determination of various boundary behaviors of Bergman kernels as we will see from these two examples.

Theorem 1.3 *In the local coordinate z on $X_\lambda^{(3)}$, write its Bergman kernel as $B_\lambda^{(3)} = k_\lambda^{(3)}(z)dz \wedge d\bar{z}$, for $\lambda \in \mathbb{C} \setminus \{0\}$. Then, as $\lambda \to 0$,*

$$L_{\lambda,z}^{(3)} \equiv 0.$$

The vanishing of the curvature form $L_{\lambda,z}^{(3)}$ implies that $\log k_\lambda^{(3)}(z)$ is harmonic in λ, and in this case all the fibers are in fact biholomorphically equivalent to each other. In the theorem below, we found that $X_\lambda^{(4)}$ is reducible to the $X_\lambda^{(1)}$ case (with the appearance of hyperbolic growth), since one can change coordinates holomorphically to make the reduction. This interesting connection between nodal and cuspidal cases in some way strengthens the importance of a Legendre family.

Theorem 1.4 *In the local coordinate z on $X_\lambda^{(4)}$, write its Bergman kernel as $B_\lambda^{(4)} = k_\lambda^{(4)}(z)dz \wedge d\bar{z}$, for $\lambda \in \mathbb{C} \setminus \{0\}$. Then, as $\lambda \to 0$,*

$$L_{\lambda,z}^{(4)} \sim \frac{\sqrt{-1}d\lambda \wedge d\bar{\lambda}}{|\lambda|^2(-\log|\lambda|^2)^2}.$$

Thirdly, based on Taylor series, this alternative approach is free from elliptic functions, and really works for higher genus nodal or cuspidal curves. Moreover, probably due to the uniformization theorem, the results for higher genus curves (see [13, 14]) turn out to be quite different from genus one curves. Finally, we find distinct λ_1 and λ_2 with the same absolute value such that $\log k_{\lambda_1}^{(1)}(z) \neq \log k_{\lambda_2}^{(1)}(z)$, although the asymptotic expansion result in Theorem 1.1 seems to suggest that $\log k_\lambda^{(1)}(z)$ or $L_{\lambda,z}^{(1)}$ is S^1-invariant, i.e., depending only on $|\lambda|$ (cf. [4]).

Theorem 1.5 *Under the same assumption as in Theorem 1.1., for $\lambda \in \mathbb{D} \setminus \{0\}$, it follows that $L^{(1)}_{\lambda,z}$ is not S^1-invariant in general.*

2 Preliminaries

On an elliptic curve $E := \{y^2 = p_\lambda(x)\}$, $p_\lambda(x)$ being a polynomial of x depending on λ of degree 3 or 4, there exists a globally defined basis $\omega := dx/y$ for the Hilbert space of L^2 holomorphic 1-forms. Locally, ω (which depends on λ, x and y) may be written under some u-coordinate as $\omega = f_\lambda(u)du$, where $f_\lambda(u)$ is a holomorphic in $u \in \mathbb{C}$. After normalising by the L^2 inner product $\omega_0 := C_\lambda^{-0.5}\omega$ will then become an orthonormal basis, where

$$C_\lambda := \frac{\sqrt{-1}}{2} \int_E \omega \wedge \overline{\omega} \; > 0 \tag{2.1}$$

is a positive real number depending only on λ. By definition, the Bergman kernel of the canonical bundle on E is just $K_\lambda := \omega_0 \wedge \overline{\omega_0} = C_\lambda^{-1} \cdot \omega \wedge \overline{\omega} \overset{locally}{=\!=\!=} C_\lambda^{-1} \cdot |f_\lambda(u)|^2 du \wedge d\bar{u}$, a $(1,1)$-form whose local coefficient depends on both λ and the coordinate u. By the transformation law of the Bergman kernel, the Bergman metric is invariant under holomorphic change of coordinates, i.e.

$$\partial_\lambda \bar{\partial}_\lambda \log C_\lambda^{-1} = \partial_\lambda \bar{\partial}_\lambda \log k_\lambda(\cdot), \tag{2.2}$$

for any local coefficient $k_\lambda(\cdot)$ of the Bergman kernel. Equation (2.2) means that only C_λ matters for the curvature form, and thus we will focus on the asymptotic behaviors of C_λ as $\lambda \to 0$.

2.1 A Legendre Family of Elliptic Curves

Particularly, if $p_\lambda(x) := x(x-1)(x-\lambda)$, $\lambda \in \mathbb{C} \setminus \{0, 1\}$, we get a Legendre family of elliptic curves X_λ. For small $\lambda \to 0$, a double covering of the Riemann sphere can be made by cutting itself from 0 to λ, and from 1 to ∞. Then, we get two cycles δ and γ forming a homologous basis of the elliptic curve, and containing $\{0, \lambda\}$ and $\{\lambda, 1\}$, respectively.

Proposition 2.1 *For a Legendre family of elliptic curves, it follows that*

$$C_\lambda = \mathrm{Im}\left(\int_\gamma \omega \cdot \int_\delta \overline{\omega}\right) = \mathrm{Im}\left(\frac{\int_\gamma \omega}{\int_\delta \omega}\right) \cdot \left|\int_\delta \omega\right|^2 > 0. \tag{2.3}$$

Proof Firstly, consider the path integral $p(\cdot) := \int_{z_0}^{\cdot} \omega$ defined on a parallelogram $X_{\lambda,\text{cut}}$ with two sides A and B, which are identified in order to obtain X_λ (regarded as a complex torus). Notice that p is well-defined (independent of paths) and holomorphic, since ω is a holomorphic 1-form and $X_{\lambda,\text{cut}}$ is simply connected. Secondly, (by Cauchy Integral Theorem) $p(\cdot + A) - p(\cdot) = \int_\delta \omega$ and $p(\cdot + B) - p(\cdot) = \int_\gamma \omega$, implying p is not doubly periodic on X_λ. Moreover, $\frac{\partial p}{\partial u} = f_\lambda(u)$, for some local u-coordinate and thus $\partial p = \omega$. Make the following computation $d(p \cdot \overline{\omega}) = \partial p \wedge \overline{\omega} + p \cdot \partial(\overline{f_\lambda(u)du}) = \omega \wedge \overline{\omega}$, and apply Stokes' theorem to it on $X_{\lambda,\text{cut}}$. Finally, it holds that $\int_{\partial X_{\lambda,\text{cut}}} p \cdot \overline{\omega} = \int_{X_{\lambda,\text{cut}}} d(p \cdot \overline{\omega}) = \int_{X_{\lambda,\text{cut}}} \omega \wedge \overline{\omega} = \int_{X_\lambda} \omega \wedge \overline{\omega}$, which implies that

$$
\begin{aligned}
C_\lambda &= \frac{\sqrt{-1}}{2} \int_{\partial X_{\lambda,\text{cut}}} p \cdot \overline{\omega} \\
&= \frac{\sqrt{-1}}{2} \left(\int_\delta p(u) \overline{f_\lambda(u)} d\overline{u} - \int_\delta p(u+B) \overline{f_\lambda(u+B)} d\overline{u} \right. \\
&\qquad \left. + \int_\gamma p(u+A) \overline{f_\lambda(u+A)} d\overline{u} - \int_\gamma p(u) \overline{f_\lambda(u)} d\overline{u} \right) \\
&= \frac{\sqrt{-1}}{2} \left(\int_\delta (p(u) - p(u+B)) \overline{f_\lambda(u)} d\overline{u} + \int_\gamma (p(u+A) - p(u)) \overline{f_\lambda(u)} d\overline{u} \right) \\
&= \frac{\sqrt{-1}}{2} \left(\int_\delta \left(- \int_\gamma \omega \right) \overline{f_\lambda(u)} d\overline{u} + \int_\gamma \left(\int_\delta \omega \right) \overline{f_\lambda(u)} d\overline{u} \right) \\
&= \frac{\sqrt{-1}}{2} \left(- \int_\gamma \omega \cdot \int_\delta \overline{\omega} + \int_\gamma \overline{\omega} \cdot \int_\delta \omega \right) > 0.
\end{aligned}
$$

□

2.2 Other Families of Elliptic Curves

Firstly, if $p(x) = (x-1)(x^2 - \lambda)$, on the elliptic curve $X_\lambda^{(2)}$ ($\lambda \in \mathbb{C} \setminus \{0, 1\}$), δ is a big circle centered at the origin containing $-\sqrt{\lambda}$ and $\sqrt{\lambda}$, and γ contains $\sqrt{\lambda}$ and 1. Secondly, if $p(x) = x(x^2 - \lambda)$, on the elliptic curve $X_\lambda^{(3)}$ ($\lambda \in \mathbb{C} \setminus \{0\}$), δ contains $-\sqrt{\lambda}$ and 0, and γ contains 0 and $\sqrt{\lambda}$. Thirdly, if $p(x) = x(x-\lambda)(x-\lambda^2)$, on the elliptic curve $X_\lambda^{(4)}$ ($\lambda \in \mathbb{C} \setminus \{0\}$), δ contains 0 and λ^2, and γ contains λ^2 and λ. Finally, let us recall that the Maclaurin series of the function $1/\sqrt{1-x}$ is $1 + x/2 + \mathrm{O}(x^2)$ for $|x| < 1$. When $|a| < |s|$, it holds that $1/\sqrt{s-a} = \left\{1 + a/2s + \mathrm{O}(a^2/s^2)\right\} / \sqrt{s}$.

3 New Proof of Theorem 1.1 (The First Term)

To get the first term in Theorem 1.1, we study the period of $X_\lambda^{(1)}$.

Lemma 3.1 *For $X_\lambda^{(1)}$, let C_λ be defined as in (2.1). Then, as $\lambda \to 0$,*

$$\log C_\lambda^{-1} \sim -\log(-\log|\lambda|).$$

Proof The numerator and the denominator in (2.3) will be estimated, respectively. By the two-sheeted construction of $X_\lambda^{(1)}$, we know that as $\lambda \to 0$,

$$\begin{aligned}
\int_\gamma \omega &= -2\int_\lambda^1 \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} \\
&= -2\int_\lambda^1 \frac{dx}{\sqrt{x(x-1)}}\cdot\frac{1}{\sqrt{x}}\left\{1+\mathrm{O}(\frac{\lambda}{x})\right\} \\
&= -2\int_\lambda^1 \frac{dx}{x\sqrt{x-1}} + \int_\lambda^1 \frac{dx}{x^2\sqrt{x-1}}\cdot \mathrm{O}(\lambda) \\
&= -2\int_\lambda^1 \frac{dx}{x\sqrt{x-1}} + \mathrm{O}(\lambda)\cdot\left(\left.\frac{\sqrt{x-1}}{x}\right|_\lambda^1 + \frac{1}{2}\int_\lambda^1 \frac{dx}{x\sqrt{x-1}}\right) \\
&= (-2+\mathrm{O}(\lambda))\cdot\int_\lambda^1 \frac{dx}{x\sqrt{x-1}} + \mathrm{O}(\lambda)\cdot\frac{\sqrt{\lambda-1}}{\lambda} \\
&= (-4+\mathrm{O}(\lambda))\cdot \arctan\left(\sqrt{x-1}\right)\Big|_\lambda^1 + \mathrm{O}(1) \\
&= (4+\mathrm{O}(\lambda))\cdot \arccos\frac{1}{\sqrt{\lambda}} + \mathrm{O}(1) \\
&= (4\sqrt{-1}+\mathrm{O}(\lambda))\cdot\log\left(\sqrt{\frac{1}{\lambda}}+\sqrt{\frac{1}{\lambda}-1}\right) + \mathrm{O}(1) \\
&\sim 4\sqrt{-1}\cdot\log\left(2\sqrt{\frac{1}{\lambda}}\right) + \mathrm{O}(1) = -2\sqrt{-1}\cdot\log\lambda + \mathrm{O}(1) \sim -2\sqrt{-1}\cdot\log\lambda.
\end{aligned}$$

Next, for $|\lambda| << |x| << 1$, it holds that $1/\sqrt{x-1} = \left(1+x/2+\mathrm{O}(x^2)\right)/\sqrt{-1}$. So,

$$\begin{aligned}
\int_\delta \omega &= \int_\delta \frac{dx}{\sqrt{x(x-1)}}\cdot\frac{1}{\sqrt{x}}\left\{1+\mathrm{O}(\frac{\lambda}{x})\right\} \\
&\sim \int_\delta \frac{dx}{x\sqrt{x-1}} = \int_\delta \frac{\left(1+x/2+\mathrm{O}(x^2)\right)dx}{x\sqrt{-1}} = \int_\delta \frac{dx}{x\sqrt{-1}},
\end{aligned}$$

since the higher order terms of the second to last integrand are holomorphic. Notice that δ contains $\{0, \lambda\}$, so the integration of x^{-1} along the circle δ is equal to $2\pi\sqrt{-1}$. Therefore, as $\lambda \to 0$,

$$\frac{\int_\gamma \omega}{\int_\delta \omega} \sim \frac{-2\sqrt{-1}\cdot \log\lambda}{2\pi} = \frac{\log\lambda}{\pi\sqrt{-1}},$$

implying that

$$\operatorname{Im}\left(\frac{\log\lambda}{\pi\sqrt{-1}}\right) = -\frac{\log|\lambda|}{\pi} > 0,$$

which by (2.3) finishes the proof. □

By Lemma 3.1 and (2.2), it follows as $\lambda \to 0$ that

$$\begin{aligned}\partial_\lambda\bar{\partial}_\lambda \log k_\lambda(\cdot) &\sim -\partial_\lambda\bar{\partial}_\lambda \log(-\log|\lambda|)\\ &= -\frac{\partial_\lambda\bar{\partial}_\lambda(\log|\lambda|) - \partial_\lambda(\log|\lambda|)\wedge\bar{\partial}_\lambda(\log|\lambda|)}{(\log|\lambda|)^2} = \frac{\partial_\lambda(\log|\lambda|^2)\wedge\bar{\partial}_\lambda(\log|\lambda|^2)}{4(\log|\lambda|)^2}\\ &= \frac{\partial_\lambda(|\lambda|^2)\wedge\bar{\partial}_\lambda(|\lambda|^2)}{4|\lambda|^4(\log|\lambda|)^2} = \frac{\bar{\lambda}d\lambda\wedge\lambda d\bar{\lambda}}{4|\lambda|^4(\log|\lambda|)^2} = \frac{d\lambda\wedge d\bar{\lambda}}{4|\lambda|^2(\log|\lambda|)^2},\end{aligned}$$

where $k(\cdot)$ is the coefficient of the Bergman kernel under some local coordinate. Thus, the the leading term (with hyperbolic growth) in Theorem 1.1 is shown. For the leading term asymptotics of the period, it is very nicely written in [9], where it is remarked that the sub-leading terms of the period are holomorphic with respect to λ.

4 New Proof of Theorem 1.1 (The Second Term)

The proof of Lemma 3.1 shows for the period $\tau(\lambda)$ of the Legendre family of elliptic curve that $\tau(\lambda) \sim \log\lambda/(\sqrt{-1}\pi)$, as $\lambda \to 0$. In this section, we will first determine its precise second term (which requires a little trick), and then obtain information on the Bergman kernel as stated in Theorem 1.1. Recall from [1, p. 280] the following two-term asymptotic formula of the elliptic modular lambda function. Rather than using the Weierstrass-$\wp$ function, we provide a proof by using the Taylor series expansion of Abelian differentials.

Proposition 4.1 (Two-term asymptotic formula of the period) *Let* $\tau(\lambda)$ *be the period of the elliptic curve* $X_\lambda^{(1)}$, *for* $\lambda \in \mathbb{C}\setminus\{0, 1\}$. *Then, as* $\lambda \to 0$,

$$\tau(\lambda) \sim \frac{\log\lambda - \log 16}{\sqrt{-1}\pi}.$$

Proof Firstly, for any real number $\epsilon > 1$ we separate the numerator in the proof of Lemma 3.1 as $-2\int_1^s \omega$ and $-2\int_s^t \omega$. As $t \to \infty$, it holds that

$$\int_1^\epsilon \omega = \int_1^\epsilon \frac{dx}{\sqrt{x(x-1)(x-t)}} \sim \int_1^\epsilon \frac{dx}{\sqrt{-t}\cdot\sqrt{x(x-1)}}$$
$$=\frac{1}{\sqrt{-t}} \log\left(2\sqrt{x(x-1)}+2x-1\right)\Big|_1^\epsilon = \frac{1}{\sqrt{-t}} \log\left(2\sqrt{\epsilon(\epsilon-1)}+2\epsilon-1\right)$$

and

$$\int_\epsilon^t \frac{dx}{x\sqrt{x-t}} = \frac{2}{\sqrt{t}} \arctan\sqrt{\frac{x-t}{t}}\Bigg|_\epsilon^t = \frac{-2}{\sqrt{t}}\sqrt{-1}\log\left(\sqrt{\frac{t}{\epsilon}}+\sqrt{\frac{t}{\epsilon}-1}\right)$$
$$= \frac{-2\sqrt{-1}}{\sqrt{t}} \log\left(\sqrt{\frac{t}{\epsilon}}\left(1+\sqrt{1-\frac{\epsilon}{t}}\right)\right)$$
$$= \frac{-2\sqrt{-1}}{\sqrt{t}} \left(\frac{1}{2}\log\frac{t}{\epsilon}+\log\left(1+\sqrt{1-\frac{\epsilon}{t}}\right)\right)$$
$$= \frac{-2}{\sqrt{t}}\sqrt{-1}\left(\frac{1}{2}\log\frac{t}{\epsilon}+\log\left(2-\frac{1}{2}\cdot\frac{2\epsilon}{t}-\frac{1}{8}\cdot\frac{4\epsilon^2}{t^2}+\mathrm{O}(\frac{\epsilon^3}{t^3})\right)\right)$$
$$\sim \frac{-2}{\sqrt{t}}\sqrt{-1}\left(\frac{1}{2}\log\frac{t}{\epsilon}+\log 2\right) = \frac{1}{\sqrt{-t}}\left(\log t + 2\log 2 - \log\epsilon\right).$$

Secondly, $x \geq \epsilon$ implies that

$$\left(\sqrt{\epsilon-1}\cdot\sqrt{x}\cdot(\sqrt{\epsilon}+\sqrt{\epsilon-1})\right)^{-1} \geq \left(\sqrt{x-1}\cdot\sqrt{x}\cdot\left(\sqrt{x}+\sqrt{x-1}\right)\right)^{-1} > 0,$$

which means

$$\left(\frac{\sqrt{\epsilon}}{\sqrt{\epsilon-1}}\right)\frac{1}{\sqrt{x}} = \left(\frac{1}{\sqrt{\epsilon-1}\cdot\sqrt{x}\cdot(\sqrt{\epsilon}+\sqrt{\epsilon-1})}\right)+\frac{1}{\sqrt{x}} \geq \frac{1}{\sqrt{x-1}} > \frac{1}{\sqrt{x}}.$$

Thus, $\int_\epsilon^t \omega = \int_\epsilon^t \frac{dx}{\sqrt{x(x-1)(x-t)}}$ can be squeezed by two terms A and B, namely

$$A := \int_\epsilon^t \frac{dx}{x\sqrt{(x-t)}} \sim \frac{1}{\sqrt{-t}}\left(\log t + 2\log 2 - \log\epsilon\right),$$

$$B := \left(\frac{\sqrt{\epsilon}}{\sqrt{\epsilon-1}}\right)\cdot A \sim \left(\frac{\sqrt{\epsilon}}{\sqrt{\epsilon-1}}\right)\cdot\frac{1}{\sqrt{-t}}\left(\log t + 2\log 2 - \log\epsilon\right).$$

So, the numerator can be squeezed by $-2(A+\int_1^\epsilon \omega)$ and $-2(B+\int_1^\epsilon \omega)$. On the one hand,

$$-2\left(A+\int_1^\epsilon \omega\right) \sim \frac{-2\left(\log t+2\log 2-\log \epsilon\right)}{\sqrt{-t}} - \frac{2\log\left(2\sqrt{\epsilon(\epsilon-1)}+2\epsilon-1\right)}{\sqrt{-t}}$$
$$\sim \frac{-2}{\sqrt{-t}}\left((\log t+2\log 2)+\log\left(2\sqrt{\frac{\epsilon-1}{\epsilon}}+2-\frac{1}{\epsilon}\right)\right).$$

On the other hand,

$$-2\left(B+\int_1^\epsilon \omega\right) \sim -2\left(\left(\frac{\sqrt{\epsilon}}{\sqrt{\epsilon-1}}\right)\cdot\frac{1}{\sqrt{-t}}\left(\log t+2\log 2-\log\epsilon\right)+\frac{\log\epsilon}{\sqrt{-t}}\right)$$
$$=\left(\frac{\sqrt{\epsilon}}{\sqrt{\epsilon-1}}\right)\cdot\frac{-2}{\sqrt{-t}}\left(\log t+2\log 2-\log\epsilon\right)+\frac{-2}{\sqrt{-t}}\log\left(2\sqrt{\epsilon(\epsilon-1)}+2\epsilon-1\right)$$
$$=\frac{-2}{\sqrt{-t}}\left(\left(\frac{\sqrt{\epsilon}}{\sqrt{\epsilon-1}}\right)\left(\log t+2\log 2-\log\epsilon\right)+\log\left(2\sqrt{\epsilon(\epsilon-1)}+2\epsilon-1\right)\right)$$
$$=\frac{-2}{\sqrt{-t}}\left(\left(\frac{\sqrt{\epsilon}}{\sqrt{\epsilon-1}}\right)\left(\log t+\log 4\right)+\log\left(2\sqrt{\epsilon(\epsilon-1)}+2\epsilon-1\right)-\frac{\sqrt{\epsilon}\cdot\log\epsilon}{\sqrt{\epsilon-1}}\right).$$

Thirdly, since ϵ is arbitrary, letting $\epsilon\to\infty$ we see that the numerator is asymptotic to

$$\frac{-2}{\sqrt{-t}}\left(\log t+\log 4+\log(2+2)\right)=\frac{-2}{\sqrt{-t}}\left(\log t+\log 16\right),$$

as $t\to\infty$. Taking the inverse $\lambda=1/t$, as $\lambda\to 0$, we know that the numerator is asymptotic to $-2\sqrt{-1}\left(\log\lambda-\log 16\right)$. Comparing it with the denominator, we have proved Proposition 4.1. □

Finally, by Proposition 4.1, (2.3) and (2.2), it follows as $\lambda\to 0$ that

$$\log C_\lambda^{-1}=-\log\left(-\log|\lambda|+\log 16\right)+O(1),$$

$$\partial_\lambda\bar\partial_\lambda\log k_\lambda(\cdot)\sim\frac{d\lambda\wedge d\bar\lambda}{4|\lambda|^2(\log|\lambda|-\log 16)^2},$$

To obtain the second term in Theorem 1.1, since

$$\partial(\log|\lambda|)=\frac{1}{2}\partial(\log(\lambda\bar\lambda))=\frac{\bar\lambda d\lambda}{2|\lambda|^2}=\frac{d\lambda}{2\lambda}\quad\text{and}\quad\bar\partial(\log|\lambda|)=\frac{d\bar\lambda}{2\bar\lambda},$$

it suffices to make the following computations.

$$\frac{1}{4|\lambda|^2(\log|\lambda|-\log 16)^2}-\frac{1}{4|\lambda|^2(\log|\lambda|)^2}$$
$$=\frac{2\log|\lambda|\cdot\log 16-(\log 16)^2}{4|\lambda|^2(\log|\lambda|-\log 16)^2(\log|\lambda|)^2}\sim\frac{(\log 16)}{2|\lambda|^2(\log|\lambda|)^3}.$$

5 Proof of Theorem 1.2

Proof (*Proof of Theorem* 1.2) By the construction of γ on $X_\lambda^{(2)}$, we know as $\lambda \to 0$ that

$$\begin{aligned}\int_\gamma \omega =& -2\int_{\sqrt{\lambda}}^1 \omega = -2\int_{\sqrt{\lambda}}^1 \frac{dx}{\sqrt{(x-1)(x^2-\lambda)}}\\ \sim& -2\int_{\sqrt{\lambda}}^1 \frac{dx}{x\sqrt{x-1}} = -2\cdot 2\sqrt{-1}\log\left(\sqrt{\frac{1}{x}}+\sqrt{\frac{1}{x}-1}\right)\Bigg|_{\sqrt{\lambda}}^1\\ =&4\sqrt{-1}\log\left(\sqrt{\frac{1}{\sqrt{\lambda}}}+\sqrt{\frac{1}{\sqrt{\lambda}}-1}\right) \sim 4\sqrt{-1}\log\left(\frac{1}{\sqrt{\sqrt{\lambda}}}\right) = -\sqrt{-1}\log\lambda.\end{aligned}$$

Since δ is a big circle containing $\sqrt{\lambda}$ and $-\sqrt{\lambda}$, on $X_\lambda^{(2)}$ it is equivalent to say that $-\delta$ contains only 1 and ∞. We then make changes of variables by setting $s = \frac{1}{x}$, $t = \frac{1}{\lambda}$ and denote the corresponding big circle by $-\tilde{\delta}$ which contains 1 and 0. Then, it follows that

$$\begin{aligned}\int_\delta \omega =& \int_\delta \frac{dx}{\sqrt{(x-1)(x^2-\lambda)}} = -\int_{-\tilde{\delta}} \frac{-s^{-2}ds}{\sqrt{(\frac{1}{s}-1)(\frac{1}{s^2}-\frac{1}{t})}}\\ =& \int_{-\tilde{\delta}} \frac{ds}{\sqrt{s(s-1)(\frac{s^2}{t}-1)}} = \int_{-\tilde{\delta}} \frac{\sqrt{t}\cdot ds}{\sqrt{s(s-1)(s^2-t)}} \sim \int_{-\tilde{\delta}} \frac{\sqrt{t}\cdot ds}{s\sqrt{-t}} = 2\pi,\end{aligned}$$

as $\lambda \to 0$ ($t \to \infty$). Therefore, we know that

$$\frac{\int_\gamma \omega}{\int_{\tilde{\delta}} \omega} \sim \frac{-\sqrt{-1}\log\lambda}{2\pi}.$$

By (2.2) and (2.3) we have finished the proof of Theorem 1.2. □

6 Proof of Theorem 1.3

We will show that $X_\lambda^{(3)}$ has a constant period, for all $\lambda \in \mathbb{C}\setminus\{0\}$.

Proof (*Proof of Theorem* 1.3) For $X_\lambda^{(3)}$, two cycles can be chosen such that δ is a big circle centered at the origin which contains $-\sqrt{\lambda}$ and 0, and γ satisfies that

$$\int_{\gamma} \omega = -2 \int_0^{\sqrt{\lambda}} \omega = -2 \int_0^{\sqrt{\lambda}} \frac{dx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})}}.$$

Since ω is holomorphic along δ, by Cauchy Integral Theorem, we can choose a path which is homologous to δ and consists of two circles c_1 and c_2 (of radius r, centered at $-\sqrt{\lambda}$ and 0, respectively) and two straight lines l_1 and l_2 connecting almost $-\sqrt{\lambda}+r$ and $-r$. Near $-\sqrt{\lambda}$, using polar coordinate we denote x by $-\sqrt{\lambda}+re^{\sqrt{-1}\theta}$, $\theta \in [\beta, 2\pi - \beta]$, for small $\beta > 0$. Then,

$$\begin{aligned}\int_{c_1} \omega &= \int_{\beta}^{2\pi-\beta} \frac{re^{\sqrt{-1}\theta}\sqrt{-1}d\theta}{\sqrt{(-\sqrt{\lambda}+\epsilon e^{\sqrt{-1}\theta}-1)(-\sqrt{\lambda}+\epsilon e^{\sqrt{-1}\theta}-\sqrt{\lambda})re^{\sqrt{-1}\theta}}} \\ &= \int_{\beta}^{2\pi-\beta} \frac{\sqrt{re^{\sqrt{-1}\theta}}\sqrt{-1}d\theta}{\sqrt{(-\sqrt{\lambda}+re^{\sqrt{-1}\theta}-1)(-\sqrt{\lambda}+re^{\sqrt{-1}\theta}-\sqrt{\lambda})}} \to 0,\end{aligned}$$

as $r \to 0$. Similarly, we can get $\int_{c_2} \omega \to 0$. Since r is arbitrary and ω changes the sign when switching between l_1 and l_2, we know that

$$\begin{aligned}\int_{\delta} \omega = \int_{c_1} \omega + \int_{c_2} \omega + \int_{l_1} \omega + \int_{l_2} \omega &= \int_{l_1} \omega + \int_{l_2} \omega \\ &= -2 \int_{-\sqrt{\lambda}}^{0} \omega = -2 \int_{-\sqrt{\lambda}}^{0} \frac{dx}{\sqrt{x(x^2-\lambda)}}.\end{aligned}$$

After making changes of variables by setting $s = -x$, we know that

$$\int_{\delta} \omega = -2 \int_{\sqrt{\lambda}}^{0} \frac{-ds}{\sqrt{-s(s^2-\lambda)}} = -2 \int_{\sqrt{\lambda}}^{0} \frac{\sqrt{-1}ds}{\sqrt{s(s^2-\lambda)}} = 2 \int_0^{\sqrt{\lambda}} \frac{\sqrt{-1}ds}{\sqrt{s(s^2-\lambda)}},$$

which implies

$$\frac{\int_{\gamma} \omega}{\int_{\delta} \omega} = \frac{-2\int_0^{\sqrt{\lambda}} \omega}{2\sqrt{-1}\int_0^{\sqrt{\lambda}} \omega} \equiv \sqrt{-1}.$$

Although their ratio is constant, we still can determine the asymptotics of the numerator and the denominator, first up to an multiplier and then with a precise constant. For example, we observe that $x+\sqrt{\lambda}$ is bounded on γ by $\sqrt{\lambda}$ and $2\sqrt{\lambda}$ which have the same order of growth. The antiderivative of the remaining term can be written down, namely

$$\int_0^{\sqrt{\lambda}} \frac{dx}{\sqrt{x(x-\sqrt{\lambda})}} = \log\left(2\sqrt{x(x-\sqrt{\lambda})} + 2x - \sqrt{\lambda}\right)\Big|_0^{\sqrt{\lambda}} = \pi\sqrt{-1}.$$

Therefore, we conclude that both the numerator and the denominator have the order of growth $\mathrm{O}\left(\lambda^{-1/4}\right)$. We determine them precisely as below and see their relations with a Legendre family.

$$\begin{aligned}\int_\delta \omega = -2\int_{-\sqrt{\lambda}}^{0} \frac{dx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})}} \\ \overset{q=x+\sqrt{\lambda}}{=\!=\!=} -2\int_0^{\sqrt{\lambda}} \frac{dq}{\sqrt{q(q-\sqrt{\lambda})(q-2\sqrt{\lambda})}} \\ \overset{q=v\cdot\sqrt{\lambda}}{=\!=\!=} \int_0^1 \frac{-2\sqrt{\lambda}\cdot dv}{\sqrt{v\cdot\sqrt{\lambda}\cdot(v\cdot\sqrt{\lambda}-\sqrt{\lambda})\cdot(v\cdot\sqrt{\lambda}-2\sqrt{\lambda})}} \\ = \frac{-2}{\sqrt{\sqrt{\lambda}}}\int_0^1 \frac{dv}{\sqrt{v(v-1)(v-2)}} := \frac{\alpha}{\sqrt{\sqrt{\lambda}}}.\end{aligned}$$

We know that this constant α is one period (the denominator part) of a Legendre family of elliptic curve $X_2^{(1)}$. Also, we know that $\left|\int_\delta \omega\right|^2 = |\alpha|^2 \cdot \sqrt{|\lambda|}$. Finally by (2.3) and (2.2), for all λ,

$$\frac{\partial^2(\log k_\lambda(\cdot))}{\partial\lambda\partial\bar{\lambda}} = \frac{\partial^2\left\{\mathrm{Im}\left(\sqrt{-1}\right)\cdot|\alpha|^2\cdot\sqrt{|\lambda|}\right\}^{-1}}{\partial\lambda\partial\bar{\lambda}} \equiv 0.$$

□

7 Proof of Theorem 1.4

For the elliptic curve $X_\lambda^{(4)}$ with a non-constant period, we estimate the numerator and the denominator, obtained from two cycle δ containing 0 and λ^2, and γ containing λ^2 and λ, respectively.

Proof (*Proof of Theorem* 1.4) Firstly, as $\lambda \to 0$, it follows that

$$\int_{\delta}\omega=\int_{\delta}\frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}}\overset{x=\lambda^2u}{=\!=\!=}\int_{\tilde\delta}\frac{\lambda^2du}{\sqrt{\lambda^2u(\lambda^2u-\lambda)(\lambda^2u-\lambda^2)}}$$
$$=\frac{1}{\sqrt\lambda}\int_{\tilde\delta}\frac{du}{\sqrt{u(u-1)(\lambda u-1)}}\overset{t=1/\lambda}{=\!=\!=}\frac{\sqrt t}{\sqrt\lambda}\int_{\tilde\delta}\frac{du}{\sqrt{u(u-1)(u-t)}}$$
$$\sim\frac{\sqrt t}{\sqrt\lambda}\int_{\tilde\delta}\frac{du}{u\sqrt{-t}}=\frac{\sqrt t}{\sqrt\lambda}\frac{2\pi\sqrt{-1}}{\sqrt{-t}}\sim\frac{2\pi}{\sqrt\lambda},$$

where $\tilde\delta$ is a cycle containing 0 and 1 (not containing $1/\lambda$ and ∞). Secondly, it holds that

$$\int_\gamma\omega=-2\int_{\lambda^2}^{\lambda}\frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}}\overset{x=\lambda u}{=\!=\!=}\frac{2}{-\sqrt\lambda}\int_\lambda^1\frac{du}{\sqrt{u(u-1)(u-\lambda)}}$$
$$\underset{t=\frac{1}{\lambda}}{\overset{u=\frac{1}{s}}{=\!=\!=}}\frac{2}{-\sqrt\lambda}\int_t^1\frac{-s^{-2}ds}{\sqrt{\frac1s(\frac1s-1)(\frac1s-\frac1t)}}=\frac{2}{\lambda}\int_1^t\frac{ds}{\sqrt{s(s-1)(s-t)}}$$
$$\sim\frac{2}{\lambda}\int_1^t\frac{ds}{s\sqrt{s-t}}=\frac{2}{\lambda}\cdot\frac{-2\sqrt{-1}}{\sqrt t}\log\left(\sqrt{\frac ts}+\sqrt{\frac ts-1}\right)\Bigg|_1^t$$
$$=\frac{-4\sqrt{-1}}{\sqrt\lambda}\left(-\log\left(\sqrt t+\sqrt{t-1}\right)\right)\sim\frac{-2\sqrt{-1}\log\lambda}{\sqrt\lambda},$$

as $\lambda\to0$. Thirdly, $\tau\sim\log\lambda/(\sqrt{-1}\pi)$ and $\operatorname{Im}\tau\sim-\log|\lambda|/\pi>0$. By (2.3) it follows that, $C_\lambda\sim(-4\pi\cdot\log|\lambda|)/|\lambda|$, as $\lambda\to0$. By (2.2), hyperbolic growth appears again for $L_\lambda^{(4)}$. □

8 Proof of Theorem 1.5

For each $\lambda\in\mathbb{D}\setminus\{0\}$, the asymptotic formula in Theorem 1.1 for $X_\lambda^{(1)}$ seems to suggest that $\log k_\lambda^{(1)}(z)$ or $L_{\lambda,z}^{(1)}$ is S^1-invariant, i.e., depending only on $|\lambda|$. If this is the case, then $\log k_\lambda^{(1)}(z)$, which is subharmonic with respect to λ, would be convex with respect to $\log|\lambda|$. However, we prove that there exist distinct $\lambda_1,\lambda_2\in\mathbb{D}\setminus\{0\}$ with $|\lambda_1|=|\lambda_2|$, such that $\log k_{\lambda_1}^{(1)}(z)\neq\log k_{\lambda_2}^{(1)}(z)$.

Proof (*Proof of Theorem* 1.5) We take $\tau_1:=b\sqrt{-1}$ and $\tau_2:=1+c\sqrt{-1}$, where b and c are positive real numbers. First, we observe that $0<\lambda(\tau_1)<1$ and $\lambda(\tau_2)<0$ are also real numbers. Moreover, as b and c tend to $+\infty$, $\lambda(\tau_1)$ and $\lambda(\tau_2)$ tend to 0 along the real axis. Then, take $c>>1$, so that both $\lambda(\tau_1)$ and $\lambda(\tau_2)$ are contained in the unit disk $\mathbb{D}$. Assume that $\lambda(\tau_1)=-\lambda(\tau_2)$, and we want to prove that $b\neq c$. Recall that

$$\lambda(\beta+1) = \frac{\lambda(\beta)}{\lambda(\beta)-1} = 1 + \frac{1}{\lambda(\beta)-1}. \quad (\Longrightarrow \lambda(\beta) - 1 = \frac{1}{\lambda(\beta+1)-1})$$

Then, $1 + \frac{1}{\lambda(c\sqrt{-1})-1} = \lambda(1 + c\sqrt{-1}) = \lambda(\tau_2) = -\lambda(\tau_1) = -\lambda(b\sqrt{-1})$. If $b = c$, then we get that

$$1 + \frac{1}{\lambda(c\sqrt{-1})-1} = -\lambda(c\sqrt{-1}),$$

implying that $0 = \lambda(b\sqrt{-1}) = \lambda(\tau_1)$, which is a contradiction. Similarly as in [10–12], the Bergman kernel B_λ in the local coordinate z (induced from the complex plane via elliptic function) can be written as $B_\lambda = \log k_\lambda(\cdot) dz \wedge \bar{z} = 1/\mathrm{Im}\, \tau(\lambda) dz \wedge \bar{z}$. Therefore, $\log k_{\lambda_1}(\cdot) \neq \log k_{\lambda_2}(\cdot)$ for the above distinct λ_1 and λ_2 with the same absolute value. This show that the relative Bergman kernel metric for a Legendre family of elliptic curves is not S^1-invariant with respect to the base in general. □

It is known that two elliptic curves are isomorphic over an algebraically closed field if and only if they have the same so-called J-invariant. For $X_\lambda^{(1)}$, it is possible to find in $\mathbb{D} \setminus \{0\}$ distinct points with the same J-invariant, which is unchanged when λ is replaced by for example $1 - \lambda$.

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Holomorphic Embeddings and Immersions of Stein Manifolds: A Survey

Franc Forstnerič

Abstract In this paper we survey results on the existence of holomorphic embeddings and immersions of Stein manifolds into complex manifolds. Most of them pertain to proper maps into Stein manifolds. We include a new result saying that every continuous map $X \to Y$ between Stein manifolds is homotopic to a proper holomorphic embedding provided that $\dim Y > 2 \dim X$ and we allow a homotopic deformation of the Stein structure on X.

Keywords Stein manifold · Embedding · Density property · Oka manifold

1 Introduction

In this paper we review what we know about the existence of holomorphic embeddings and immersions of Stein manifolds into other complex manifolds. The emphasis is on recent results, but we also include some classical ones for the sake of completeness and historical perspective. Recall that Stein manifolds are precisely the closed complex submanifolds of Euclidean spaces $\mathbb{C}^N$ (see Remmert [96], Bishop [21], and Narasimhan [92]; cf. Theorem 2.1). Stein manifolds of dimension 1 are open Riemann surfaces (see Behnke and Stein [19]). A domain in $\mathbb{C}^n$ is Stein if and only if it is a domain of holomorphy (see Cartan and Thullen [31]). For more information, see the monographs [62, 73, 78, 83].

Dedicated to Kang-Tae Kim for his sixtieth birthday.

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In Sect. 2 we survey results on the existence of proper holomorphic immersions and embeddings of Stein manifolds into Euclidean spaces. Of special interest are the minimal embedding and immersion dimensions. Theorem 2.4, due to Eliashberg and Gromov [43] (1992) and Schürmann [101] (1997), settles this question for Stein manifolds of dimension >1. It remains an open problem whether every open Riemann surface embeds holomorphically into $\mathbb{C}^2$; we describe its current status in Sect. 2.3. We also discuss the use of holomorphic automorphisms of Euclidean spaces in the construction of wild holomorphic embeddings (see Sects. 2.4 and 4.3).

It has recently been discovered by Andrist et al. [15, 16, 46] that there is a big class of Stein manifolds Y which contain every Stein manifold X with $2\dim X < \dim Y$ as a closed complex submanifold (see Theorem 3.4). In fact, this holds for every Stein manifold Y enjoying Varolin's *density property* [103, 104]: the Lie algebra of all holomorphic vector fields on Y is spanned by the $\mathbb{C}$-complete vector fields, i.e., those whose flow is an action of the additive group $(\mathbb{C}, +)$ by holomorphic automorphisms of Y (see Definition 3.3). Since the domain $(\mathbb{C}^*)^n$ enjoys the volume density property, we infer that every Stein manifold X of dimension n admits a proper holomorphic immersion to $(\mathbb{C}^*)^{2n}$ and a proper pluriharmonic map into $\mathbb{R}^{2n}$ (see Corollary 3.5). This provides a counterexample to the Schoen-Yau conjecture [100] for any Stein source manifold (see Sect. 3.3).

The class of Stein manifolds (in particular, of affine algebraic manifolds) with the density property is quite big and contains most complex Lie groups and homogeneous spaces, as well as many nonhomogeneous manifolds. This class has been the focus of intensive research during the last decade; we refer the reader to the recent surveys [86] and [62, Sect. 4.10]. An open problem posed by Varolin [103, 104] is whether every contractible Stein manifold with the density property is biholomorphic to a Euclidean space.

In Sect. 4 we recall a result of Drinovec Drnovšek and the author [39, 41] to the effect that every smoothly bounded, strongly pseudoconvex Stein domain X embeds properly holomorphically into an arbitrary Stein manifold Y with $\dim Y > 2\dim X$. More precisely, every continuous map $\overline{X} \to Y$ which is holomorphic on X is homotopic to a proper holomorphic embedding $X \hookrightarrow Y$ (see Theorem 4.1). The analogous result holds for immersions if $\dim Y \geq 2\dim X$, and also for every q-complete manifold Y with $q \in \{1, \dots, \dim Y - 2\dim X + 1\}$, where the Stein case corresponds to $q = 1$. This summarizes a long line of previous results. In Sect. 4.2 we mention a recent application of these techniques to the *Hodge conjecture* for the highest dimensional a priori nontrivial cohomology group of a q-complete manifold [59]. In Sect. 4.3 we survey recent results on the existence of *complete* proper holomorphic embeddings and immersions of strongly pseudoconvex domains into balls. Recall that a submanifold of $\mathbb{C}^N$ is said to be *complete* if every divergent curve in it has infinite Euclidean length.

In Sect. 5 we show how the combination of the techniques from [39, 41] with those of Slapar and the author [57, 58] can be used to prove that, if X and Y are Stein manifolds and $\dim Y > 2\dim X$, then every continuous map $X \to Y$ is homotopic to a proper holomorphic embedding up to a homotopic deformation of the Stein structure on X (see Theorem 5.1). The analogous result holds for immersions if $\dim Y \geq$

$2 \dim X$, and for q-complete manifolds Y with $q \le \dim Y - 2\dim X + 1$. A result in a similar vein, concerning proper holomorphic embeddings of open Riemann surfaces into $\mathbb{C}^2$ up to a deformation of their conformal structures, is due to Alarcón and López [12] (a special case was proved in [32]); see also Ritter [97] for embeddings into $(\mathbb{C}^*)^2$.

I have not included any topics from Cauchy-Riemann geometry since it would be impossible to properly discuss this major subject in the present survey of limited size and with a rather different focus. The reader may wish to consult the recent survey by Pinchuk et al. [93], the monograph by Baouendi et al. [18] from 1999, and my survey [48] from 1993. For a new direction in this field, see the papers by Bracci and Gaussier [26, 27].

We shall be using the following notation and terminology. Let $\mathbb{N} = \{1, 2, \ldots\}$. We denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the unit disc in $\mathbb{C}$, by $\mathbb{D}^n \subset \mathbb{C}^n$ the Cartesian product of n copies of $\mathbb{D}$ (the unit polydisc in $\mathbb{C}^n$), and by $\mathbb{B}^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z|^2 = |z_1|^2 + \cdots + |z_n|^2 < 1\}$ the unit ball in $\mathbb{C}^n$. By $\mathcal{O}(X)$ we denote the algebra of all holomorphic functions on a complex manifold X, and by $\mathcal{O}(X, Y)$ the space of all holomorphic maps $X \to Y$ between a pair of complex manifolds; thus $\mathcal{O}(X) = \mathcal{O}(X, \mathbb{C})$. These spaces carry the compact-open topology. This topology can be defined by a complete metric which renders them Baire spaces; in particular, $\mathcal{O}(X)$ is a Fréchet algebra. (See [62, p. 5] for more details.) A compact set K in a complex manifold X is said to be $\mathcal{O}(X)$-*convex* if $K = \widehat{K} := \{p \in X : |f(p)| \le \sup_K |f| \text{ for every } f \in \mathcal{O}(X)\}$.

2 Embeddings and Immersions of Stein Manifolds into Euclidean Spaces

In this section we survey results on proper holomorphic immersions and embeddings of Stein manifolds into Euclidean spaces.

2.1 Classical Results

We begin by recalling the results of Remmert [96], Bishop [21], and Narasimhan [92] from the period 1956–1961.

Theorem 2.1 *Assume that X is a Stein manifold of dimension n.*

(a) *If $N > 2n$ then the set of proper embeddings $X \hookrightarrow \mathbb{C}^N$ is dense in $\mathcal{O}(X, \mathbb{C}^N)$.*
(b) *If $N \ge 2n$ then the set of proper immersions $X \hookrightarrow \mathbb{C}^N$ is dense in $\mathcal{O}(X, \mathbb{C}^N)$.*
(c) *If $N > n$ then the set of proper maps $X \to \mathbb{C}^N$ is dense in $\mathcal{O}(X, \mathbb{C}^N)$.*
(d) *If $N \ge n$ then the set of almost proper maps $X \to \mathbb{C}^N$ is residual in $\mathcal{O}(X, \mathbb{C}^N)$.*

A proof of these results can also be found in the monograph by Gunning and Rossi [78].

Recall that a set in a Baire space (such as $\mathcal{O}(X, \mathbb{C}^N)$) is said to be *residual*, or a set *of second category*, if it is the intersection of at most countably many open everywhere dense sets. Every residual set is dense. A property of elements in a Baire space is said to be *generic* if it holds for all elements in a residual set.

The density statement for embeddings and immersions is an easy consequence of the following result which follows from the jet transversality theorem for holomorphic maps. (See Forster [45] for maps to Euclidean spaces and Kaliman and Zaidenberg [87] for the general case. A more complete discussion of this topic can be found in [62, Sect. 8.8].) Note also that maps which are immersions or embeddings on a given compact set constitute an open set in the corresponding mapping space.

Proposition 2.2 *Assume that X is a Stein manifold, K is a compact set in X, and $U \Subset X$ is an open relatively compact set containing K. If Y is a complex manifold such that* $\dim Y > 2\dim X$*, then every holomorphic map $f\colon X \to Y$ can be approximated uniformly on K by holomorphic embeddings $U \hookrightarrow Y$. If* $2\dim X \le \dim Y$ *then f can be approximated by holomorphic immersions $U \to Y$.*

Proposition 2.2 fails in general without shrinking the domain of the map, for otherwise it would yield nonconstant holomorphic maps of $\mathbb{C}$ to any complex manifold of dimension > 1 which is clearly false. On the other hand, it holds without shrinking the domain of the map if the target manifold Y satisfies a suitable holomorphic flexibility property, in particular, if it is an *Oka manifold*. See [62, Chap. 5] for the definition of this class of complex manifolds and [62, Corollary 8.8.7] for the mentioned result.

In the proof of Theorem 2.1, parts (a)–(c), we exhaust X by a sequence $K_1 \subset K_2 \subset \cdots$ of compact $\mathcal{O}(X)$-convex sets and approximate the holomorphic map $f_j\colon X \to \mathbb{C}^N$ in the inductive step, uniformly on K_j, by a holomorphic map $f_{j+1}\colon X \to \mathbb{C}^N$ whose norm $|f_{j+1}|$ is not too small on $K_{j+1}\backslash K_j$ and such that $|f_{j+1}(x)| > 1 + \sup_{K_j} |f_j|$ holds for all $x \in bK_{j+1}$. If the approximation is close enough at every step then the sequence f_j converges to a proper holomorphic map $f = \lim_{j\to\infty} f_j\colon X \to \mathbb{C}^N$. If $N > 2n$ then every map f_j in the sequence can be made an embedding on K_j (immersion in $N \ge 2n$) by Proposition 2.2, and hence the limit map f is also such.

A more efficient way of constructing proper maps, immersions and embeddings of Stein manifolds into Euclidean space was introduced by Bishop [21]. He showed that any holomorphic map $X \to \mathbb{C}^n$ from an n-dimensional Stein manifold X can be approximated uniformly on compacts by *almost proper* holomorphic maps $h\colon X \to \mathbb{C}^n$; see Theorem 2.1(d). More precisely, there is an increasing sequence $P_1 \subset P_2 \subset \cdots \subset X$ of relatively compact open sets exhausting X such that every P_j is a union of finitely many special analytic polyhedra and h maps P_j properly onto a polydisc $a_j\mathbb{D}^n \subset \mathbb{C}^n$, where $0 < a_1 < a_2 < \ldots$ and $\lim_{j\to\infty} a_j = +\infty$. We then obtain a proper map $(h, g)\colon X \to \mathbb{C}^{n+1}$ by choosing $g \in \mathcal{O}(X)$ such that for every $j \in \mathbb{N}$ we have $g > j$ on the compact set $L_j = \{x \in \overline{P}_{j+1} \setminus P_j : |h(x)| \le a_{j-1}\}$;

since $\overline{P}_{j-1} \cup L_j$ is $\mathcal{O}(X)$-convex, this is possible by inductively using the Oka-Weil theorem. One can then find proper immersions and embeddings by adding a suitable number of additional components to (h, g) (any such map is clearly proper) and using Proposition 2.2 and the Oka-Weil theorem inductively.

The first of the above mentioned procedures easily adapts to give a proof of the following interpolation theorem due to Acquistapace et al. [2, Theorem 1]. Their result also pertains to Stein spaces of bounded embedding dimension.

Theorem 2.3 ([2, Theorem 1]) *Assume that X is an n-dimensional Stein manifold, X' is a closed complex subvariety of X, and $\phi\colon X' \hookrightarrow \mathbb{C}^N$ is a proper holomorphic embedding for some $N > 2n$. Then the set of all proper holomorphic embeddings $X \hookrightarrow \mathbb{C}^N$ that extend ϕ is dense in the space of all holomorphic maps $X \to \mathbb{C}^N$ extending ϕ. The analogous result holds for proper holomorphic immersions $X \to \mathbb{C}^N$ when $N \geq 2n$.*

This interpolation theorem fails when $N < 2n$. Indeed, for every $n > 1$ there exists a proper holomorphic embedding $\phi\colon \mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^{2n-1}$ such that $\mathbb{C}^{2n-1} \setminus \phi(\mathbb{C}^{n-1})$ is Eisenman n-hyperbolic, so ϕ does not extend to an injective holomorphic map $f\colon \mathbb{C}^n \to \mathbb{C}^{2n-1}$ (see [62, Proposition 9.5.6]; this topic is discussed in Sect. 2.4). The answer to the interpolation problem for embeddings seems unknown in the borderline case $N = 2n$.

2.2 Embeddings and Immersions into Spaces of Minimal Dimension

After Theorem 2.1 was proved in the early 1960s, one of the main questions driving this theory during the next decades was to find the smallest number $N = N(n)$ such that every Stein manifold X of dimension n embeds or immerses properly holomorphically into $\mathbb{C}^N$. The belief that a Stein manifold of complex dimension n admits proper holomorphic embeddings to Euclidean spaces of dimension smaller than $2n + 1$ was based on the observation that such a manifold is homotopically equivalent to a CW complex of dimension at most n; this follows from Morse theory (see Milnor [91]) and the existence of strongly plurisubharmonic Morse exhaustion functions on X (see Hamm [79] and [62, Sect. 3.12]). This problem, which was investigated by Forster [45], Eliashberg and Gromov [77] and others, gave rise to major new methods in Stein geometry. Except in the case $n = 1$ when X is an open Riemann surface, the following optimal answer was given by Eliashberg and Gromov [43] in 1992, with an improvement by one for odd values on n due to Schürmann [101].

Theorem 2.4 ([43, 101]) *Every Stein manifold X of dimension n immerses properly holomorphically into $\mathbb{C}^M$ with $M = \left[\frac{3n+1}{2}\right]$, and if $n > 1$ then X embeds properly holomorphically into $\mathbb{C}^N$ with $N = \left[\frac{3n}{2}\right] + 1$.*

Schürmann [101] also found optimal embedding dimensions for Stein spaces with singularities and with bounded embedding dimension.

The key ingredient in the proof of Theorem 2.4 is a certain major extension of the Oka-Grauert theory, due to Gromov whose 1989 paper [76] marks the beginning of *modern Oka theory*. (See [54] for an introduction to Oka theory and [62] for a complete account.)

Forster showed in [45, Proposition 3] that the embedding dimension $N = \left[\frac{3n}{2}\right] + 1$ is the minimal possible for every $n > 1$, and the immersion dimension $M = \left[\frac{3n+1}{2}\right]$ is minimal for every even n, while for odd n there could be two possible values. (See also [62, Proposition 9.3.3].) In 2012, Ho et al. [82] found new examples showing that these dimensions are optimal already for Grauert tubes around compact totally real submanifolds, except perhaps for immersions with odd n. A more complete discussion of this topic and a self-contained proof of Theorem 2.4 can be found in [62, Sects. 9.2–9.5]. Here we only give a brief outline of the main ideas used in the proof.

One begins by choosing a sufficiently generic almost proper map $h\colon X \to \mathbb{C}^n$ (see Theorem 2.1(d)) and then tries to find the smallest possible number of functions $g_1, \ldots, g_q \in \mathcal{O}(X)$ such that the map

$$f = (h, g_1, \ldots, g_q)\colon X \to \mathbb{C}^{n+q} \tag{2.1}$$

is a proper embedding or immersion. Starting with a big number of functions $\tilde{g}_1, \ldots, \tilde{g}_{\tilde{q}} \in \mathcal{O}(X)$ which do the job, we try to reduce their number by applying a suitable fibrewise linear projection onto a smaller dimensional subspace, where the projection depends holomorphically on the base point. Explicitly, we look for functions

$$g_j = \sum_{k=1}^{\tilde{q}} a_{j,k}\tilde{g}_k, \qquad a_{j,k} \in \mathcal{O}(X), \quad j = 1, \ldots, q \tag{2.2}$$

such that the map (2.1) is a proper embedding or immersion. In order to separate those pairs of points in X which are not separated by the base map $h\colon X \to \mathbb{C}^n$, we consider coefficient functions of the form $a_{j,k} = b_{j,k} \circ h$ where $b_{j,k} \in \mathcal{O}(\mathbb{C}^n)$ and $h\colon X \to \mathbb{C}^n$ is the chosen base almost proper map.

This outline cannot be applied directly since the base map $h\colon X \to \mathbb{C}^n$ may have too complicated behavior. Instead, one proceeds by induction on strata in a suitably chosen complex analytic stratification of X which is equisingular with respect to h. The induction steps are of two kinds. In a step of the first kind we find a map $g = (g_1, \ldots, g_q)$ (2.2) which separates points on the (finite) fibres of h over the next bigger stratum and matches the map from the previous step on the union of the previous strata (the latter is a closed complex subvariety of X). A step of the second kind amounts to removing the kernel of the differential dh_x for all points in the next stratum, thereby ensuring that $df_x = dh_x \oplus dg_x$ is injective there. Analysis of the immersion condition shows that the graph of the map $\alpha = (a_{j,k})\colon X \to \mathbb{C}^{q\tilde{q}}$ in (2.2) over the given stratum must avoid a certain complex subvariety Σ of $E = X \times \mathbb{C}^{q\tilde{q}}$

with algebraic fibres. Similarly, analysis of the point separation condition leads to the problem of finding a map $\beta = (b_{j,k})\colon \mathbb{C}^n \to \mathbb{C}^{q\tilde{q}}$ avoiding a certain complex subvariety of $E = \mathbb{C}^n \times \mathbb{C}^{q\tilde{q}}$ with algebraic fibers. In both cases the projection $\pi\colon E\backslash\Sigma \to X$ is a stratified holomorphic fibre bundle all of whose fibres are Oka manifolds. More precisely, if $q \geq \left[\frac{n}{2}\right] + 1$ then each fibre $\Sigma_x = \Sigma \cap E_x$ is either empty or a union of finitely many affine linear subspaces of E_x of complex codimension > 1. The same lower bound on q guarantees the existence of a continuous section $\alpha\colon X \to E\backslash\Sigma$ avoiding Σ. Gromov's Oka principle [76] then furnishes a holomorphic section $X \to E\backslash\Sigma$. A general Oka principle for sections of stratified holomorphic fiber bundles with Oka fibres is given by [62, Theorem 5.4.4]. We refer the reader to the original papers or to [62, Sect. 9.3–9.4] for further details.

The classical constructions of proper holomorphic embeddings of Stein manifolds into Euclidean spaces are coordinate dependent and hence do not generalize to more general target manifolds. A conceptually new method has been found recently by Ritter and the author [55]. It is based on a method of separating certain pairs of compact polynomially convex sets in $\mathbb{C}^N$ by Fatou-Bieberbach domains which contain one of the sets and avoid the other one. Another recently developed method, which also depends on holomorphic automorphisms and applies to a much bigger class of target manifolds, is discussed in Sect. 3.

2.3 Embedding Open Riemann Surfaces into $\mathbb{C}^2$

The constructions described so far fail to embed open Riemann surfaces into $\mathbb{C}^2$. The problem is that the subvarieties Σ in the proof of Theorem 2.4 may contain hypersurfaces, and hence the Oka principle for sections of $E\backslash\Sigma \to X$ fails in general due to hyperbolicity of its complement. It is still an open problem whether every open Riemann surface embeds as a smooth closed complex curve in $\mathbb{C}^2$. (By Theorem 2.1 it embeds properly holomorphically into $\mathbb{C}^3$ and immerses with normal crossings into $\mathbb{C}^2$. Every compact Riemann surface embeds holomorphically into $\mathbb{CP}^3$ and immerses into $\mathbb{CP}^2$, but very few of them embed into $\mathbb{CP}^2$; see [75].) There are no topological obstructions to this problem—it was shown by Alarcón and López [12] that every open orientable surface S carries a complex structure J such that the Riemann surface $X = (S, J)$ admits a proper holomorphic embedding into $\mathbb{C}^2$.

There is a variety of results in the literature concerning the existence of proper holomorphic embeddings of certain special open Riemann surfaces into $\mathbb{C}^2$; the reader may wish to consult the survey in [62, Sect. 9.10–9.11]. Here we mention only a few of the currently most general known results on the subject. The first one from 2009, due to Wold and the author, concerns bordered Riemann surfaces.

Theorem 2.5 ([60, Corollary 1.2]) *Assume that X is a compact bordered Riemann surface with boundary of class $\mathscr{C}^r$ for some $r > 1$. If $f\colon X \hookrightarrow \mathbb{C}^2$ is a $\mathscr{C}^1$ embedding that is holomorphic in the interior $\mathring{X} = X\backslash bX$, then f can be approximated uniformly on compacts in $\mathring{X}$ by proper holomorphic embeddings $\mathring{X} \hookrightarrow \mathbb{C}^2$.*

The proof relies on techniques introduced mainly by Wold [105–107]. One of them concerns exposing boundary points of an embedded bordered Riemann surface in $\mathbb{C}^2$. This technique was improved in [60]; see also the exposition in [62, Sect. 9.9]. The second one depends on methods of Andersén-Lempert theory concerning holomorphic automorphisms of Euclidean spaces; see Sect. 2.4. A proper holomorphic embedding $\mathring{X} \hookrightarrow \mathbb{C}^2$ is obtained by first exposing a boundary point in each of the boundary curves of $f(X) \subset \mathbb{C}^2$, sending these points to infinity by a rational shear on $\mathbb{C}^2$ without other poles on $f(X)$, and then using a carefully constructed sequence of holomorphic automorphisms of $\mathbb{C}^2$ whose domain of convergence is a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^2$ which contains the embedded complex curve $f(X) \subset \mathbb{C}^2$, but does not contain any of its boundary points. If $\phi\colon \Omega \to \mathbb{C}^2$ is a Fatou-Bieberbach map then $\phi \circ f : X \hookrightarrow \mathbb{C}^2$ is a proper holomorphic embedding. A complete exposition of this proof can also be found in [62, Sect. 9.10].

The second result due to Wold and the author [61] (2013) concerns domains with infinitely many boundary components. A domain X in the Riemann sphere $\mathbb{P}^1$ is a *generalized circled domain* if every connected component of $\mathbb{P}^1\backslash X$ is a round disc or a point. Note that $\mathbb{P}^1\backslash X$ contains at most countably many discs. By the uniformization theorem of He and Schramm [80, 81], every domain in $\mathbb{P}^1$ with at most countably many complementary components is conformally equivalent to a generalized circled domain.

Theorem 2.6 ([61, Theorem 5.1]) *Let X be a generalized circled domain in $\mathbb{P}^1$. If all but finitely many punctures in $\mathbb{P}^1\backslash X$ are limit points of discs in $\mathbb{P}^1\backslash X$, then X embeds properly holomorphically into $\mathbb{C}^2$.*

The paper [61] contains several other more precise results on this subject.

The special case of Theorem 2.6 for plane domains $X \subset \mathbb{C}$ bounded by finitely many Jordan curves (and without punctures) is due to Globevnik and Stensønes [68]. Results on embedding certain Riemann surfaces with countably many boundary components into $\mathbb{C}^2$ were also proved by Majcen [90]; an exposition can be found in [62, Sect. 9.11]. The proof of Theorem 2.6 relies on similar techniques as that of Theorem 2.5, but it uses a considerably more involved induction scheme for dealing with infinitely many boundary components, clustering them together into suitable subsets to which the available analytic methods can be applied. The same technique gives the analogous result for domains in tori.

There are a few other recent results concerning embeddings of open Riemann surfaces into $\mathbb{C} \times \mathbb{C}^*$ and $(\mathbb{C}^*)^2$, where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Ritter showed in [98] that, for every circular domain $X \subset \mathbb{D}$ with finitely many boundary components, each homotopy class of continuous maps $X \to \mathbb{C} \times \mathbb{C}^*$ contains a proper holomorphic map. If $\mathbb{D}\backslash X$ contains finitely many punctures, then every continuous map $X \to \mathbb{C} \times \mathbb{C}^*$ is homotopic to a proper holomorphic immersion that identifies at most finitely many pairs of points in X (Lárusson and Ritter [88]). Ritter [97] also gave an analogue of Theorem 2.5 for proper holomorphic embeddings of certain open Riemann surfaces into $(\mathbb{C}^*)^2$.

2.4 Automorphisms of Euclidean Spaces and Wild Embeddings

There is another line of investigations that we wish to touch upon. It concerns the question how many proper holomorphic embeddings $X \hookrightarrow \mathbb{C}^N$ of a given Stein manifold X are there up to automorphisms of $\mathbb{C}^N$, and possibly also of X. This question was motivated by certain famous results from algebraic geometry, such as the one of Abhyankar and Moh [1] and Suzuki [102] to the effect that every polynomial embedding $\mathbb{C} \hookrightarrow \mathbb{C}^2$ is equivalent to the linear embedding $z \mapsto (z, 0)$ by a polynomial automorphism of $\mathbb{C}^2$.

It is a basic fact that for any $N > 1$ the holomorphic automorphism group $\mathrm{Aut}(\mathbb{C}^N)$ is very big and complicated. This is in stark contrast to the situation for bounded or, more generally, hyperbolic domains in $\mathbb{C}^N$ which have few automorphism; see Greene et al. [74] for a survey of the latter topic. Major early work on understanding the group $\mathrm{Aut}(\mathbb{C}^N)$ was made by Rosay and Rudin [99]. This theory became very useful with the papers of Andersén and Lempert [14] and Rosay and the author [56] in 1992–93. The central result is that every map in a smooth isotopy of biholomorphic mappings $\Phi_t : \Omega = \Omega_0 \to \Omega_t$ $(t \in [0, 1])$ between Runge domains in $\mathbb{C}^N$, with Φ_0 the identity on Ω, can be approximated uniformly on compacts in Ω by holomorphic automorphisms of $\mathbb{C}^N$ (see [56, Theorem 1.1] or [62, Theorem 4.9.2]). The analogous result holds on any Stein manifold with the density property; see Sect. 3. A comprehensive survey of this subject can be found in [62, Chap. 4].

By twisting a given submanifold of $\mathbb{C}^N$ with a sequence of holomorphic automorphisms, one can show that for any pair of integers $1 \le n < N$ the set of all equivalence classes of proper holomorphic embeddings $\mathbb{C}^n \hookrightarrow \mathbb{C}^N$, modulo automorphisms of both spaces, is uncountable (see [35]). In particular, the Abhyankar-Moh theorem fails in the holomorphic category since there exist proper holomorphic embeddings $\phi\colon \mathbb{C} \hookrightarrow \mathbb{C}^2$ that are nonstraightenable by automorphisms of $\mathbb{C}^2$ [53], as well as embeddings whose complement $\mathbb{C}^2 \setminus \phi(\mathbb{C})$ is Kobayashi hyperbolic [29]. More generally, for any pair of integers $1 \le n < N$ there exists a proper holomorphic embedding $\phi\colon \mathbb{C}^n \hookrightarrow \mathbb{C}^N$ such that every nondegenerate holomorphic map $\mathbb{C}^{N-n} \to \mathbb{C}^N$ intersects $\phi(\mathbb{C}^n)$ at infinitely many points [49]. It is also possible to arrange that $\mathbb{C}^N \setminus \phi(\mathbb{C}^n)$ is Eisenman $(N-n)$-hyperbolic [23]. A more comprehensive discussion of this subject can be found in [62, Sect. 4.18].

By using nonlinearizable proper holomorphic embeddings $\mathbb{C} \hookrightarrow \mathbb{C}^2$, Derksen and Kutzschauch gave the first known examples of nonlinearizable periodic automorphisms of $\mathbb{C}^n$ [34]. For instance, there is a nonlinearizable holomorphic involution on $\mathbb{C}^4$.

In another direction, Baader et al. [17] constructed an example of a properly embedded disc in $\mathbb{C}^2$ whose image is topologically knotted, thereby answering a questions of Kirby. It is unknown whether there exists a knotted proper holomorphic embedding $\mathbb{C} \hookrightarrow \mathbb{C}^2$, or an unknotted proper holomorphic embedding $\mathbb{D} \hookrightarrow \mathbb{C}^2$ of the disc.

Automorphisms of $\mathbb{C}^2$ and $\mathbb{C}^* \times \mathbb{C}$ were used in a very clever way by Wold in his landmark construction of non-Runge Fatou-Bieberbach domains in $\mathbb{C}^2$ [108] and of non-Stein long $\mathbb{C}^2$'s [109]. Each of these results solved a long-standing open problem. More recently, Wold's construction was developed further by Boc Thaler and the author [22] who showed that there is a continuum of pairwise nonequivalent long $\mathbb{C}^n$'s for any $n > 1$ which do not admit any nonconstant holomorphic or plurisubharmonic functions. (See also [62, 4.21].)

3 Embeddings into Stein Manifolds with the Density Property

3.1 Universal Stein Manifolds

It is natural to ask which Stein manifolds, besides the Euclidean spaces, contain all Stein manifolds of suitably low dimension as closed complex submanifolds. To facilitate the discussion, we introduce the following notions.

Definition 3.1 Let Y be a Stein manifold.

(1) Y is *universal for proper holomorphic embeddings* if every Stein manifold X with $2 \dim X < \dim Y$ admits a proper holomorphic embedding $X \hookrightarrow Y$.
(2) Y is *strongly universal for proper holomorphic embeddings* if, under the assumptions in (1), every continuous map $f_0 \colon X \to Y$ which is holomorphic in a neighborhood of a compact $\mathcal{O}(X)$-convex set $K \subset X$ is homotopic to a proper holomorphic embedding $f_0 \colon X \hookrightarrow Y$ by a homotopy $f_t \colon X \to Y$ $(t \in [0, 1])$ such that f_t is holomorphic and arbitrarily close to f_0 on K for every $t \in [0, 1]$.
(3) Y is (strongly) *universal for proper holomorphic immersions* if condition (1) (resp. (2)) holds for proper holomorphic immersions $X \to Y$ from any Stein manifold X satisfying $2 \dim X \leq \dim Y$.

In the terminology of Oka theory (cf. [62, Chap. 5]), a complex manifold Y is (strongly) universal for proper holomorphic embeddings if it satisfies the basic Oka property (with approximation) for proper holomorphic embeddings $X \to Y$ from Stein manifolds of dimension $2 \dim X < \dim Y$. The dimension hypotheses in the above definition are justified by Proposition 2.2. The main goal is to find good sufficient conditions for a Stein manifold to be universal. If a manifold Y is Brody hyperbolic [28] (i.e., it does not admit any nonconstant holomorphic images of $\mathbb{C}$) then clearly no complex manifold containing a nontrivial holomorphic image of $\mathbb{C}$ can be embedded into Y. In order to get positive results, one must therefore assume that Y enjoys a suitable holomorphic flexibility (anti-hyperbolicity) property.

Problem 3.2 Is every Stein Oka manifold (strongly) universal for proper holomorphic embeddings and immersions?

Recall (see e.g. [62, Theorem 5.5.1]) that every Oka manifold is strongly universal for not necessarily proper holomorphic maps, embeddings and immersions. Indeed, the cited theorem asserts that a generic holomorphic map $X \to Y$ from a Stein manifold X into an Oka manifold Y is an immersion if $\dim Y \geq 2 \dim X$, and is an injective immersion if $\dim Y > 2 \dim X$. However, the Oka condition does not imply universality for *proper* holomorphic maps since there are examples of (compact or noncompact) Oka manifolds without any closed complex subvarieties of positive dimension (see [62, Example 9.8.3]).

3.2 Manifolds with the (Volume) Density Property

The following condition was introduced in 2000 by Varolin [103, 104].

Definition 3.3 A complex manifold Y enjoys the (holomorphic) *density property* if the Lie algebra generated by the $\mathbb{C}$-complete holomorphic vector fields on Y is dense in the Lie algebra of all holomorphic vector fields in the compact-open topology.

A complex manifold Y endowed with a holomorphic volume form ω enjoys the *volume density property* if the analogous density condition holds in the Lie algebra of all holomorphic vector fields on Y with vanishing ω-divergence.

The algebraic density and volume density properties were introduced by Kaliman and Kutzschebauch [85]. The class of Stein manifolds with the (volume) density property includes most complex Lie groups and homogeneous spaces, as well as many nonhomogeneous manifolds. We refer to [62, Sect. 4.10] for a more complete discussion and an up-to-date collection of references on this subject. Another recent survey is the paper by Kaliman and Kutzschebauch [86]. Every complex manifold with the density property is an Oka manifold, and a Stein manifold with the density property is elliptic in the sense of Gromov (see [62, Proposition 5.6.23]). It is an open problem whether a contractible Stein manifold with the density property is biholomorphic to a complex Euclidean space.

The following result is due to Andrist and Wold [16] in the special case when X is an open Riemann surface, to Andrist et al. [15, Theorems 1.1, 1.2] for embeddings, and to the author [46, Theorem 1.1] for immersions in the double dimension.

Theorem 3.4 ([15, 16, 46]) *Every Stein manifold with the density or the volume density property is strongly universal for proper holomorphic embeddings and immersions.*

To prove Theorem 3.4, one follows the scheme of proof of the Oka principle for maps from Stein manifolds to Oka manifolds (see [62, Chap. 5]), but with a crucial addition which we now briefly describe.

Assume that $D \Subset X$ is a relatively compact strongly pseudoconvex domain with smooth boundary and $f : \overline{D} \hookrightarrow Y$ is a holomorphic embedding such that $f(bD) \subset Y \backslash L$, where L is a given compact $\mathcal{O}(Y)$-convex set in Y. We wish to approximate f uniformly on $\overline{D}$ by a holomorphic embedding $f' : \overline{D'} \hookrightarrow Y$ of a certain bigger strongly pseudoconvex domain $\overline{D'} \Subset X$ to Y, where D' is either a union of D with a small convex bump B chosen such that $f(\overline{D \cap B}) \subset Y \backslash L$, or a thin handlebody whose core is the union of D and a suitable smoothly embedded totally real disc in $X \backslash D$. (The second case amounts to a change of topology of the domain, and it typically occurs when passing a critical point of a strongly plurisubharmonic exhaustion function on X.) In view of Proposition 2.2, we only need to approximate f by a holomorphic map $f' : \overline{D'} \to Y$ since a small generic perturbation of f' then yields an embedding. It turns out that the second case involving a handlebody easily reduces to the first one by applying a Mergelyan type approximation theorem; see [62, Sect. 5.11] for this reduction. The attachment of a bump is handled by using the density property of Y. This property allows us to find a holomorphic map $g : \overline{B} \to Y \backslash L$ approximating f as closely as desired on a neighborhood of the attaching set $\overline{B \cap D}$ and satisfying $g(\overline{B}) \subset Y \backslash L$. (More precisely, we use that isotopies of biholomorphic maps between pseudoconvex Runge domains in Y can be approximated by holomorphic automorphisms of Y; see [56, Theorem 1.1] and also [62, Theorem 4.10.5] for the version pertaining to Stein manifolds with the density property.) Assuming that g is sufficiently close to f on $\overline{B \cap D}$, we can glue them into a holomorphic map $f' : \overline{D'} \to Y$ which approximates f on $\overline{D}$ and satisfies $f'(\overline{B}) \subset Y \backslash L$. The proof is completed by an induction procedure in which every induction step is of the type described above. The inclusion $f'(\overline{B}) \subset Y \backslash L$ satisfied by the next map in the induction step guarantees properness of the limit embedding $X \hookrightarrow Y$. Of course the sets $L \subset Y$ also increase and form an exhaustion of Y.

The case of immersions in double dimension requires a more precise analysis. In the induction step described above, we must ensure that the immersion $f : \overline{D} \to Y$ is injective (an embedding) on the attaching set $\overline{B \cap D}$ of the bump B. This can be arranged by general position provided that $\overline{B \cap D}$ is very thin. It is shown in [46] that it suffices to work with convex bumps such that, in suitably chosen holomorphic coordinates on a neighborhood of $\overline{B}$, the set B is a convex polyhedron and $\overline{B \cap D}$ is a very thin neighborhood of one of its faces. This means that $\overline{B \cap D}$ is small thickening of a $(2n-1)$-dimensional object in X, and hence we can easily arrange that f is injective on it. The remainder of the proof proceeds exactly as before, completing our sketch of proof of Theorem 3.4.

3.3 On the Schoen-Yau Conjecture

The following corollary to Theorem 3.4 is related to a conjecture of Schoen and Yau [100] that the disc $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ does not admit any proper harmonic maps to $\mathbb{R}^2$.

Corollary 3.5 *Every Stein manifold X of complex dimension n admits a proper holomorphic immersion to $(\mathbb{C}^*)^{2n}$, and a proper pluriharmonic map into $\mathbb{R}^{2n}$.*

Proof The space $(\mathbb{C}^*)^n$ with coordinates $z = (z_1, \ldots, z_n)$ (where $z_j \in \mathbb{C}^*$ for $j = 1, \ldots, n$) enjoys the volume density property with respect to the volume form

$$\omega = \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n}.$$

(See Varolin [104] or [62, Theorem 4.10.9(c)].) Hence, [46, Theorem 1.2] (the part of Theorem 3.4 above concerning immersions into the double dimension) furnishes a proper holomorphic immersion $f = (f_1, \ldots, f_{2n}) \colon X \to (\mathbb{C}^*)^{2n}$. It follows that the map

$$u = (u_1, \ldots, u_{2n}) \colon X \to \mathbb{R}^{2n} \quad \text{with} \quad u_j = \log |f_j| \quad \text{for} \quad j = 1, \ldots, 2n \qquad (3.1)$$

is a proper map of X to $\mathbb{R}^{2n}$ whose components are pluriharmonic functions. □

Corollary 3.5 gives a counterexample to the Schoen-Yau conjecture in every dimension and for any Stein source manifold. The first and very explicit counterexample was given by Božin [25] in 1999. In 2001, Globevnik and the author [52] constructed a proper holomorphic map $f = (f_1, f_2) \colon \mathbb{D} \to \mathbb{C}^2$ whose image is contained in $(\mathbb{C}^*)^2$, i.e., it avoids both coordinate axes. The associated harmonic map $u = (u_1, u_2) \colon \mathbb{D} \to \mathbb{R}^2$ (3.1) then satisfies $\lim_{|\zeta| \to 1} \max\{u_1(\zeta), u_2(\zeta)\} = +\infty$ which implies properness. Next, Alarcón and López [11] showed in 2012 that every open Riemann surface X admits a conformal minimal immersion $u = (u_1, u_2, u_3) \colon X \to \mathbb{R}^3$ with a proper (harmonic) projection $(u_1, u_2) \colon X \to \mathbb{R}^2$. In 2014, Andrist and Wold [16, Theorem 5.6] proved Corollary 3.5 in the case $n = 1$.

Comparing Corollary 3.5 with the above mentioned result of Globevnik and the author [52], one is led to the following question.

Problem 3.6 Let X be a Stein manifold of dimension $n > 1$. Does there exist a proper holomorphic immersion $f \colon X \to \mathbb{C}^{2n}$ such that $f(X) \subset (\mathbb{C}^*)^{2n}$?

More generally, one can ask which type of sets in Stein manifolds can be avoided by proper holomorphic maps from Stein manifolds of sufficiently low dimension. In this direction, Drinovec Drnovšek showed in [37] that any closed complete pluripolar set can be avoided by proper holomorphic discs; see also Borell et al. [24] for embedded discs in $\mathbb{C}^n$. Note that every closed complex subvariety is a complete pluripolar set.

4 Embeddings of Strongly Pseudoconvex Stein Domains

4.1 *The Oka Principle for Embeddings of Strongly Pseudoconvex Domains*

What can be said about proper holomorphic embeddings and immersions of Stein manifolds X into arbitrary (Stein) manifolds Y? If Y is Brody hyperbolic [28], then no complex manifold containing a nontrivial holomorphic image of $\mathbb{C}$ embeds into Y. However, if $\dim Y > 1$ and Y is Stein then Y still admits proper holomorphic images of any bordered Riemann surface [39, 64]. For domains in Euclidean spaces, this line of investigation was started in 1976 by Fornæss [44] and continued in 1985 by Løw [89] and the author [47] who proved that every bounded strongly pseudoconvex domain $X \subset \mathbb{C}^n$ admits a proper holomorphic embedding into a high dimensional polydisc and ball. The long line of subsequent developments culminated in the following result of Drinovec Drnovšek and the author [39, 41].

Theorem 4.1 ([41, Corollary 1.2]) *Let X be a relatively compact, smoothly bounded, strongly pseudoconvex domain in a Stein manifold $\widetilde{X}$ of dimension n, and let Y be a Stein manifold of dimension N. If $N > 2n$ then every continuous map $f : \overline{X} \to Y$ which is holomorphic on X can be approximated uniformly on compacts in X by proper holomorphic embeddings $X \hookrightarrow Y$. If $N \geq 2n$ then the analogous result holds for immersions. The same conclusions hold if the manifold Y is strongly q-complete for some $q \in \{1, 2, \ldots, N - 2n + 1\}$, where the case $q = 1$ corresponds to Stein manifolds.*

In the special case when Y is a domain in a Euclidean space, this is due to Dor [36]. The papers [39, 41] include several more precise results in this direction and references to numerous previous works. Note that a continuous map $f : \overline{X} \to Y$ from a compact strongly pseudoconvex domain which is holomorphic on the open domain X, with values in an arbitrary complex manifold Y, can be approximated uniformly on $\overline{X}$ by holomorphic maps from small open neighborhoods of $\overline{X}$ in the ambient manifold $\widetilde{X}$, where the neighborhood depends on the map (see [40, Theorem 1.2] or [62, Theorem 8.11.4]). However, unless Y is an Oka manifold, it is impossible to approximate f uniformly on $\overline{X}$ by holomorphic maps from a fixed bigger domain $X_1 \subset \widetilde{X}$ independent of the map. For this reason, it is imperative that the initial map f in Theorem 4.1 be defined on all of $\overline{X}$.

One of the main techniques used in the proof of Theorem 4.1 are special holomorphic peaking functions on X. The second tool is the method of holomorphic sprays developed in the context of Oka theory; this is essentially a nonlinear version of the $\overline{\partial}$-method.

Here is the main idea of the proof of Theorem 4.1. Choose a strongly q-convex Morse exhaustion function $\rho \colon Y \to \mathbb{R}_+$. (When $q = 1$, ρ is strongly plurisubharmonic.) By using the mentioned tools, one can approximate any given holomorphic map $f : \overline{X} \to Y$ uniformly on compacts in X by another holomorphic map

$\tilde{f}\colon \overline{X} \to Y$ such that $\rho \circ \tilde{f} > \rho \circ f + c$ holds on bX for some constant $c > 0$ depending only on the geometry of ρ on a given compact set $L \subset Y$ containing $f(\overline{X})$. Geometrically speaking, this means that we lift the image of the boundary of X in Y to a higher level of the function ρ by a prescribed amount. At the same time, we can ensure that $\rho \circ \tilde{f} > \rho \circ f - \delta$ on X for any given $\delta > 0$, and that $\tilde{f}$ approximates f as closely as desired on a given compact $\mathcal{O}(X)$-convex subset $K \subset X$. By Proposition 2.2 we can ensure that our maps are embeddings. An inductive application of this technique yields a sequence of holomorphic embeddings $f_k\colon \overline{X} \hookrightarrow Y$ converging to a proper holomorphic embedding $X \hookrightarrow Y$. The same construction gives proper holomorphic immersions when $N \geq 2n$.

4.2 On the Hodge Conjecture for *q*-Complete Manifolds

A more precise analysis of the proof of Theorem 4.1 was used by Smrekar, Sukhov and the author [59] to show the following result along the lines of the Hodge conjecture.

Theorem 4.2 *If Y is a q-complete complex manifold of dimension N and of finite topology such that $q < N$ and the number $N + q - 1 = 2p$ is even, then every cohomology class in $H^{N+q-1}(Y;\mathbb{Z})$ is Poincaré dual to an analytic cycle in Y consisting of proper holomorphic images of the ball $\mathbb{B}^p \subset \mathbb{C}^p$. If the manifold Y has infinite topology, the same result holds for elements of the group $\mathscr{H}^{N+q-1}(Y;\mathbb{Z}) = \lim_j H^{N+q-1}(M_j;Z)$ where $\{M_j\}_{j\in\mathbb{N}}$ is an exhaustion of Y by compact smoothly bounded domains.*

Note that $H^{N+q-1}(Y;\mathbb{Z})$ is the highest dimensional a priori nontrivial cohomology group of a q-complete manifold Y of dimension N. We do not know whether a similar result holds for lower dimensional cohomology groups of a q-complete manifold. In the special case when Y is a Stein manifold, the situation is better understood thanks to the Oka-Grauert principle, and the reader can find appropriate references in the paper [59].

4.3 Complete Bounded Complex Submanifolds

There are interesting recent constructions of properly embedded complex submanifolds $X \subset \mathbb{B}^N$ of the unit ball in $\mathbb{C}^N$ (or of pseudoconvex domains in $\mathbb{C}^N$) which are *complete* in the sense that every curve in X terminating on the sphere $b\mathbb{B}^N$ has infinite length. Equivalently, the metric on X, induced from the Euclidean metric on $\mathbb{C}^N$ by the embedding $X \hookrightarrow \mathbb{C}^N$, is a complete metric.

The question whether there exist complete bounded complex submanifolds in Euclidean spaces was asked by Paul Yang in 1977. The first such examples were provided by Jones [84] in 1979. Recent results on this subject are due to Alarcón and the author [6], Alarcón and López [13], Drinovec Drnovšek [38], Globevnik [65–67], and Alarcón et al. [9, 10]. In [6] it was shown that any bordered Riemann surface admits a proper complete holomorphic immersion into $\mathbb{B}^2$ and embedding into $\mathbb{B}^3$ (no change of the complex structure on the surface is necessary). In [9] the authors showed that properly embedded complete complex curves in the ball $\mathbb{B}^2$ can have any topology, but their method (using holomorphic automorphisms) does not allow one to control the complex structure of the examples. Drinovec Drnovšek [38] proved that every strongly pseudoconvex domain embeds as a complete complex submanifold of a high dimensional ball. Globevnik proved [65, 67] that any pseudoconvex domain in $\mathbb{C}^N$ for $N > 1$ can be foliated by complete complex hypersurfaces given as level sets of a holomorphic function, and Alarcón showed [3] that there are nonsingular foliations of this type given as level sets of a holomorphic function without critical points. Furthermore, there is a complete proper holomorphic embedding $\mathbb{D} \hookrightarrow \mathbb{B}^2$ whose image contains any given discrete subset of $\mathbb{B}^2$ [66], and there exist complex curves of arbitrary topology in $\mathbb{B}^2$ satisfying this property [9]. The constructions in these papers, except those in [3, 65, 67], rely on one of the following two methods:

(a) Riemann-Hilbert boundary values problem (or holomorphic peaking functions in the case of higher dimensional domains considered in [38]);
(b) holomorphic automorphisms of the ambient space $\mathbb{C}^N$.

Each of these methods can be used to increase the intrinsic boundary distance in an embedded or immersed submanifold. The first method has the advantage of preserving the complex structure, and the disadvantage of introducing self-intersections in the double dimension or below. The second method is precisely the opposite—it keeps embeddedness, but does not provide any control of the complex structure since one must cut away pieces of the image manifold to keep it suitably bounded. The first of these methods has recently been applied in the theory of minimal surfaces in $\mathbb{R}^n$; we refer to the papers [4, 5, 7] and the references therein. On the other hand, ambient automorphisms cannot be applied in minimal surface theory since the only class of self-maps of $\mathbb{R}^n$ $(n > 2)$ mapping minimal surfaces to minimal surfaces are the rigid affine linear maps.

Globevnik's method in [65, 67] is different from both of the above. He showed that for every integer $N > 1$ there is a holomorphic function f on the ball $\mathbb{B}^N$ whose real part $\Re f$ is unbounded on every path of finite length that ends on $b\mathbb{B}^N$. It follows that every level set $M_c = \{f = c\}$ is a closed complete complex hypersurface in $\mathbb{B}^N$, and M_c is smooth for most values of c in view of Sard's lemma. The function f is constructed such that its real part grows sufficiently fast on a certain labyrinth $\Lambda \subset \mathbb{B}^N$, consisting of pairwise disjoint closed polygonal domains in real affine hyperplanes, such that every curve in $\mathbb{B}^N \setminus \Lambda$ which terminates on $b\mathbb{B}^N$ has infinite length. The advantage of his method is that it gives an affirmative answer to Yang's question in all dimensions and codimensions. The disadvantage is that one cannot control the topology or the complex structure of the level sets. By using instead holomorphic automorphisms in order to push a submanifold off the labyrinth Λ, Alarcón et al. [10] succeeded to obtain partial control of the topology of the embedded submanifold, and complete control in the case of complex curves [9]. Finally, by using the method of constructing noncritical holomorphic functions due to Forstnerič [50], Alarcón [3] improved Globevnik's main result from [65] by showing that every closed complete complex hypersurface in the ball $\mathbb{B}^n$ $(n > 1)$ is a leaf in a nonsingular holomorphic foliation of $\mathbb{B}^n$ by closed complete complex hypersurfaces.

By using the labyrinths constructed in [10, 65] and methods of Andersén-Lempert theory, Alarcón and the author showed in [8] that there exists a complete injective holomorphic immersion $\mathbb{C} \to \mathbb{C}^2$ whose image is everywhere dense in $\mathbb{C}^2$ [8, Corollary 1.2]. The analogous result holds for any closed complex submanifold $X \subsetneqq \mathbb{C}^n$ for $n > 1$ (see [8, Theorem 1.1]). Furthermore, if X intersects the ball $\mathbb{B}^n$ and K is a connected compact subset of $X \cap \mathbb{B}^n$, then there is a Runge domain $\Omega \subset X$ containing K which admits a complete injective holomorphic immersion $\Omega \to \mathbb{B}^n$ whose image is dense in $\mathbb{B}^n$.

4.4 Submanifolds with Exotic Boundary Behaviour

The boundary behavior of proper holomorphic maps between bounded domains with smooth boundaries in complex Euclidean spaces has been studied extensively; see the recent survey by Pinchuk et al. [93]. It is generally believed, and has been proved under a variety of additional conditions, that proper holomorphic maps between relatively compact smoothly bounded domains of the same dimension always extend smoothly up to the boundary. In dimension 1 this is the classical theorem of Carathéodory (see [30] or [94, Theorem 2.7]). On the other hand, proper holomorphic maps into higher dimensional domains may have rather wild boundary behavior. For example, Globevnik [63] proved in 1987 that, given $n \in \mathbb{N}$, if $N \in \mathbb{N}$ is sufficiently large then there exists a continuous map $f : \overline{\mathbb{B}}^n \to \overline{\mathbb{B}}^N$ which is holomorphic in $\mathbb{B}^n$ and satisfies $f(b\mathbb{B}^n) = b\mathbb{B}^N$. Recently, the author [51] constructed a properly embedded holomorphic disc $\mathbb{D} \hookrightarrow \mathbb{B}^2$ in the 2-ball with arbitrarily small area (hence it is the zero set of a bounded holomorphic function on $\mathbb{B}^2$ according to Berndtsson [20]) which extends holomorphically across the boundary of the disc, with the exception of one boundary point, such that its boundary curve is injectively immersed and everywhere dense in the sphere $b\mathbb{B}^2$. Examples of proper (not necessarily embedded) discs with similar behavior were found earlier by Globevnik and Stout [69].

5 The Soft Oka Principle for Proper Holomorphic Embeddings

By combining the technique in the proof of Theorem 4.1 with methods from the papers by Slapar and the author [57, 58] one can prove the following seemingly new result.

Theorem 5.1 *Let* (X, J) *and* Y *be Stein manifolds, where* $J: TX \to TX$ *denotes the complex structure operator on* X*. If* $\dim Y > 2\dim X$ *then for every continuous map* $f: X \to Y$ *there exists a Stein structure* J' *on* X*, homotopic to* J*, and a proper holomorphic embedding* $f': (X, J') \hookrightarrow Y$ *homotopic to* f*. If* $\dim Y \geq 2\dim X$ *then* f' *can be chosen a proper holomorphic immersion having only simple double points. The same holds if the manifold* Y *is* q*-complete for some* $q \in \{1, 2, \ldots, \dim Y - 2\dim X + 1\}$*, where* $q = 1$ *corresponds to Stein manifolds.*

Intuitively speaking, every Stein manifold X embeds properly holomorphically into any other Stein manifold Y of dimension $\dim Y > 2\dim X$ up to a change of the Stein structure on X. The main result of [58] amounts to the same statement for holomorphic maps (instead of proper embeddings), but without any hypothesis on the target complex manifold Y. In order to obtain *proper* holomorphic maps $X \to Y$, we need a suitable geometric hypothesis on Y in view of the examples of noncompact (even Oka) manifolds without any closed complex subvarieties (see [62, Example 9.8.3]).

The results from [57, 58] were extended by Prezelj and Slapar [95] to 1-convex source manifolds. For Stein manifolds X of complex dimension 2, these results also stipulate a change of the underlying $\mathscr{C}^\infty$ structure on X. It was later shown by Cieliebak and Eliashberg that such change is not necessary if one begins with an integrable Stein structure; see [33, Theorem 8.43 and Remark 8.44]. For the constructions of exotic Stein structures on smooth orientable 4-manifolds, in particular on $\mathbb{R}^4$, see Gompf [70–72].

Proof (Sketch of proof of Theorem 5.1) In order to fully understand the proof, the reader should be familiar with [58, proof of Theorem 1.1]. (Theorem 1.2 in the same paper gives an equivalent formulation where one does not change the Stein structure on X, but instead finds a desired holomorphic map on a Stein Runge domain $\Omega \subset X$ which is diffeotopic to X. An exposition is also available in [33, Theorem 8.43 and Remark 8.44] and [62, Sect. 10.9].)

We explain the main step in the case $\dim Y > 2\dim X$; the theorem follows by using it inductively as in [58]. An interested reader is invited to provide the details.

Assume that $X_0 \subset X_1$ is a pair of relatively compact, smoothly bounded, strongly pseudoconvex domains in X such that there exists a strongly plurisubharmonic Morse function ρ on an open set $U \supset \overline{X_1 \backslash X_0}$ in X satisfying

$$X_0 \cap U = \{x \in U : \rho(x) < a\}, \quad X_1 \cap U = \{x \in U : \rho(x) < b\},$$

for a pair of constants $a < b$ and $d\rho \neq 0$ on $bX_0 \cup bX_1$. Let $L_0 \subset L_1$ be a pair of compact sets in Y. (In the induction, L_0 and L_1 are sublevel sets of a strongly q-convex exhaustion function on Y.) Assume that $f_0 \colon X \to Y$ is a continuous map whose restriction to a neighborhood of $\overline{X}_0$ is a J-holomorphic embedding satisfying $f_0(bX_0) \subset Y \backslash L_0$. The goal is to find a new Stein structure J_1 on X, homotopic to J by a smooth homotopy that is fixed in a neighborhood of $\overline{X}_0$, such that f_0 can be deformed to a map $f_1 \colon X \to Y$ whose restriction to a neighborhood of $\overline{X}_1$ is a J_1-holomorphic embedding which approximates f_0 uniformly on $\overline{X}_0$ as closely as desired and satisfies

$$f_1(\overline{X_1 \backslash X_0}) \subset Y \backslash L_0, \qquad f_1(bX_1) \subset Y \backslash L_1. \tag{5.1}$$

An inductive application of this result proves Theorem 5.1 as in [58]. (For the case $\dim X = 2$, see [33, Theorem 8.43 and Remark 8.44].)

By subdividing the problem into finitely many steps of the same kind, it suffices to consider the following two basic cases:

(a) *The noncritical case:* $d\rho \neq 0$ on $\overline{X_1 \backslash X_0}$. In this case we say that X_1 is a *noncritical strongly pseudoconvex extension* of X_0.
(b) *The critical case:* ρ has exactly one critical point p in $\overline{X_1 \backslash X_0}$.

Let $U_0 \subset U_0' \subset X$ be a pair of small open neighborhoods of $\overline{X}_0$ such that f_0 is an embedding on U_0'. Also, let $U_1 \subset U_1' \subset X$ be small open neighborhoods of $\overline{X}_1$.

In case (a), there exists a smooth diffeomorphism $\phi \colon X \to X$ which is diffeotopic to the identity map on X by a diffeotopy which is fixed on $U_0 \cup (X \backslash U_1')$ such that $\phi(U_1) \subset U_0'$. The map $\tilde{f}_0 = f_0 \circ \phi \colon X \to Y$ is then a holomorphic embedding on the set U_1 with respect to the Stein structure $J_1 = \phi^*(J)$ on X (the pullback of J by ϕ). Applying the lifting procedure in the proof of Theorem 4.1 and up to shrinking U_1 around $\overline{X}_1$, we can homotopically deform $\tilde{f}_0$ to a continuous map $f_1 \colon X \to Y$ whose restriction to U_1 is a J_1-holomorphic embedding $U_1 \hookrightarrow Y$ satisfying conditions (5.1).

In case (b), the change of topology of the sublevel sets of ρ at the critical point p is described by attaching to the strongly pseudoconvex domain $\overline{X}_0$ a smoothly embedded totally real disc $M \subset X_1 \backslash X_0$, with $p \in M$ and $bM \subset bX_0$, whose dimension equals the Morse index of ρ at p. As shown in [33, 42, 58], M can be chosen such that $\overline{X}_0 \cup M$ has a basis of smooth strongly pseudoconvex neighborhoods (handlebodies) H which deformation retract onto $\overline{X}_0 \cup M$ such that X_1 is a noncritical strongly pseudoconvex extension of H. Furthermore, as explained in [58], we can homotopically deform the map $f_0 \colon X \to Y$, keeping it fixed in some neighborhood of $\overline{X}_0$, to a map that is holomorphic on H and maps $H \backslash \overline{X}_0$ to $L_1 \backslash L_0$. By Proposition 2.2 we can assume that the new map is a holomorphic embedding on H. This reduces case (b) to case (a).

In the inductive construction, we alternate the application of cases (a) and (b). If $\dim Y \geq 2 \dim X$ then the same procedure applies to immersions. □

A version of this construction, for embedding open Riemann surfaces into $\mathbb{C}^2$ or $(\mathbb{C}^*)^2$ up to a deformation of their complex structure, can be found in the papers by Alarcón and López [12] and Ritter [97]. However, they use holomorphic automorphisms in order to push the boundary curves to infinity without introducing self-intersections of the image complex curve. The technique in the proof of Theorem 4.1 will in general introduce self-intersections in double dimension.

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Variation of Schiffer and Hyperbolic Spans Under Pseudoconvexity

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Abstract This is a survey article on our works related to the variational formulas for Schiffer spans and hyperbolic spans. A planar open Riemann surface admits the Schiffer span. Shiba showed that an open Riemann surface R of genus one admits the hyperbolic span $\sigma(R)$, which gives a generalization of the Schiffer span and the size of ideal boundary of R. We establish the variational formula of $\sigma(R(t))$ for the deformed open Riemann surface $R(t)$ of genus one with complex parameter t, and give some applications to the rigidity theorems under pseudoconvexity.

Keywords Pseudoconvexity · Moduli disk · Principal function · Span

1 Introduction

Schiffer [18] introduced the notion of span to study the theory of univalent functions of multiply connected plane domains, and it has been playing an important role in the theory of conformal mapping of planar Riemann surfaces. To deal with non-planar Riemann surfaces equally, we shall have to take account of holomorphic mappings into other Riemann surfaces as well as holomorphic functions, and also have to generalize the notion of span. For our purposes it is enough to consider conformal embeddings, namely, injective holomorphic mappings. A closing of an open Riemann surface of finite genus is, roughly speaking, a conformal embedding of the open Riemann surface into closed one of the same genus which induces the prescribed correspondence between their canonical homology bases. Shiba [19] studied the case of genus one and showed that the set of moduli of closings of an open torus is a closed disk in the upper half plane, and that the diameter of this moduli disk gives a close analogue of the Schiffer span.

Dedicated to Professor Kang-Tae Kim on his sixtieth birthday.

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We are mainly concerned with a two-dimensional Stein manifold $\mathcal{R}$ and a holomorphic mapping $\pi : \mathcal{R} \to \Delta$ into a disk Δ in $\mathbb{C}_t$, and look at its fibers $\pi^{-1}(t) =: R(t)$ with $t \in \Delta$ whose irreducible components are Riemann surfaces. Therefore the function theory of one complex variable is very involved. Under the above situation, Nishino [16] showed that if each $R(t)$, $t \in \Delta$, is planar and conformally equivalent to $\mathbb{C}$, then $\mathcal{R}$ is biholomorphic to $\Delta \times \mathbb{C}$. Yamaguchi [22] extended the above result as follows: if $R(t)$, $t \in \Delta$, is planar and of class O_G (namely, parabolic), then $\mathcal{R}$ is biholomorphic to a univalent domain in $\Delta \times \mathbb{P}$. The solution of this problem is reduced by showing the superharmonicity of variations of Robin constants $\lambda(t)$ for $R(t)$. Under certain regularity assumptions, the equality variation formula of the second order for $\lambda(t)$ was established by Maitani and Yamaguchi [15] as follows:

$$\frac{\partial^2 \lambda(t)}{\partial t \partial \bar{t}} = -\frac{1}{\pi} \int_{\partial R(t)} k_2(t, z) \left| \frac{\partial g(t, z)}{\partial z} \right|^2 ds_z - \frac{4}{\pi} \iint_{R(t)} \left| \frac{\partial^2 g(t, z)}{\partial \bar{t} \partial z} \right|^2 dxdy.$$

Here, $k_2(t, z)$ is the *Levi curvature* for $\partial \mathcal{R}$ (more precisely, see (3.1)) and $g(t, z)$ is the Green function. This theory of the variation of Robin constants is generalized to domains over $\mathbb{C}^n$, and moreover to complex manifolds (see Yamaguchi [23]; Levenberg and Yamaguchi [14]; Kim et al. [12]). Combining the above variation formula with Suita's formula [21], Maitani-Yamaguchi showed the variational property of the Bergman kernel $K(t, \zeta)$ under a pseudoconvex variation $\mathcal{R}$, namely, $\log K(t, \zeta)$ is plurisubharmonic on $\mathcal{R}$. It was extended by Berndtsson (see [2, 3]) in general, which recently led to beautiful and strong results (see Guan and Zhou [5] and Berndtsson and Lempert [4]). In [9] we studied the metric deformations induced by Schiffer and harmonic spans on planar open Riemann surfaces, and showed the same phenomenon as a variational property of the Bergman metric $K(t, \zeta)|d\zeta|^2$ under pseudoconvexity.

2 Preparations from One Complex Variable

We shall recall elementary properties of principal functions and the related spans on the potential theory of one complex variable (cf. Ahlfors and Sario [1]; Sario and Nakai [17] for general references).

2.1 Schiffer Spans

Let R be a bordered Riemann surface with a finite number of C^ω-smooth contours C_j $(j = 1, \ldots, \nu)$ in a larger Riemann surface $\widetilde{R}$. Let $a \in R$ and let $U_a = \{|z - \zeta| < r_a\}$ be the local coordinate of a neighborhood of a in R, where z is a variable and ζ is a fixed point in whole paper. We consider all harmonic functions $p(z)$ on $R \backslash \{a\}$ with singularity $\mathrm{Re}\, \frac{1}{z-\zeta}$ at a normalized so that $\lim_{z \to \zeta}(p(z) - \mathrm{Re}\, \frac{1}{z-\zeta}) = 0$. Then there are two uniquely determined functions $p_i(z)$ $(i = 1, 0)$ with the following boundary conditions (L_i): for $j = 1, \ldots, \nu$,

$$(L_1) \quad p_1(z) = c_j \text{ (constant) on } C_j \quad \text{and} \quad \int_{C_j} \frac{\partial p_1(z)}{\partial n_z} ds_z = 0;$$

$$(L_0) \quad \frac{\partial p_0(z)}{\partial n_z} = 0 \quad \text{on } C_j.$$

We have $p_i(z) = \text{Re}\left\{\frac{1}{z-\zeta} + \sum_{n=1}^{\infty} A_n^i (z-\zeta)^n\right\}$ at ζ $(i = 1, 0)$. $p_i(z)$ and $\alpha_i := \text{Re}\{A_1^i\}$ are called the L_i-principal function and the L_i-constant for (R, a) with respect to the local coordinate U_a, respectively (simply, for (R, ζ)), and $s := \alpha_0 - \alpha_1$ is called the *Schiffer span* for (R, ζ). In the special case when R is a *planar* Riemann surface, the Schiffer span s for (R, ζ) has the geometric meaning as follows. Let $\mathcal{P}(R)$ be the set of all univalent functions $P(z)$ on R with the given singularity $\frac{1}{z-\zeta}$ at ζ which are normalized so that $\lim_{z\to\zeta}(P(z) - \frac{1}{z-\zeta}) = 0$. Namely, $P(z)$ has the following expression

$$P(z) = \frac{1}{z-\zeta} + 0 + \sum_{n=1}^{\infty} A_n (z-\zeta)^n \qquad \text{at } \zeta.$$

For $w = P(z) \in \mathcal{P}(R)$, let $\mathcal{E}_P$ denote the Euclidean area of $\mathbb{P}_w \backslash P(R)$, and set $\mathcal{E}(R) = \sup\{\mathcal{E}_P \mid P(z) \in \mathcal{P}(R)\}$. Koebe constructed two special functions $P_0(z)$ and $P_1(z)$ in $\mathcal{P}(R)$ which are the horizontal slit mapping and the vertical slit mapping for (R, a) with respect to U_a (simply, for (R, ζ)), respectively. Grunsky introduced the functions $P_0(z) \pm P_1(z)$, and studied their extremal properties. Schiffer discovered that the quantity $s := A_1^0 - A_1^1$ is always real and nonnegative, where $P_i(z) = \frac{1}{z-\zeta} + \sum_{n=1}^{\infty} A_n^i (z-\zeta)^n$ at ζ $(i = 1, 0)$. Then, Schiffer [18] proved that $M(z) := \frac{1}{2}(P_0(z) + P_1(z))$ belongs to $\mathcal{P}(R)$, the set $\mathbb{C}_w \backslash M(R)$ consists of convex domains, and the complementary area $\mathcal{E}(R)$ satisfies $\mathcal{E}(R) = \mathcal{E}_M = \frac{\pi}{2}s$. Further, such function is uniquely determined. By the standard approximation argument we define the *Schiffer span* s for (R, ζ) and the maximizing function $M(z)$ of s for any planar Riemann surface.

2.2 Hyperbolic Spans

Let R be a bordered Riemann surface of genus one and $\chi = \{A, B\}$ be a fixed canonical homology basis of R modulo dividing cycles C_j $(j = 1, \ldots, \nu)$. Consider a triplet $(\hat{R}, \hat{\chi}, \hat{\imath})$ consisting of a (closed) torus $\hat{R}$, a canonical homology basis $\hat{\chi} = \{\hat{A}, \hat{B}\}$ of $\hat{R}$, and a conformal embedding $\hat{\imath}$ of R into $\hat{R}$ such that $\hat{\imath}(A)$ (resp., $\hat{\imath}(B)$) is homologous to $\hat{A}$ (resp., $\hat{B}$) in $\hat{R}$. We say that two such triplets $(\hat{R}, \hat{\chi}, \hat{\imath})$ and $(\hat{R}', \hat{\chi}', \hat{\imath}')$ are equivalent if there is a conformal mapping f of $\hat{R}$ onto $\hat{R}'$ with $f \circ \hat{\imath} = \hat{\imath}'$ on

R. Each equivalence class is called a *closing* of the marked open torus (R, χ) and denoted by $[\hat{R}, \hat{\chi}, \hat{i}]$. The closing $[\hat{R}, \hat{\chi}, \hat{i}]$ carries a unique holomorphic differential $\phi^{\hat{R}}$ with $\int_{\hat{A}} \phi^{\hat{R}} = 1$.

$$\tau[\hat{R}, \hat{\chi}, \hat{i}] := \int_{\hat{B}} \phi^{\hat{R}}$$

is called the *modulus* of $[\hat{R}, \hat{\chi}, \hat{i}]$. We denote by $\mathcal{C}(R, \chi)$ the set of closings of (R, χ) and put

$$\mathfrak{M}(R, \chi) = \{\tau = \tau[\hat{R}, \hat{\chi}, \hat{i}] \in \mathbb{C} \mid [\hat{R}, \hat{\chi}, \hat{i}] \in \mathcal{C}(R, \chi)\}.$$

Then $\mathfrak{M}(R, \chi)$ lies in the upper half plane $\mathbb{H}$. Heins [11] proved that $\mathfrak{M}(R, \chi)$ is a compact set. Shiba showed the following:

Theorem 2.1 (Shiba [19]) *$\mathfrak{M}(R, \chi)$ is a closed disk in $\mathbb{H}$, namely, $\mathfrak{M}(R, \chi) = \{\tau \in \mathbb{H} \mid |\tau - \tau^*| \le \rho\}$.*

We call $\mathfrak{M}(R, \chi)$ the *moduli disk* for (R, χ). The Euclidean radius ρ of $\mathfrak{M}(R, \chi)$ depends on R and χ. ρ represents the size of the ideal boundary of R and gives a generalization of the Schiffer span for planar Riemann surfaces. The upper half plane carries the Poincaré metric, and the set $\mathfrak{M}(R, \chi)$ is again a closed disk with respect to this metric. Hence it makes sense to refer to the hyperbolic diameter σ of $\mathfrak{M}(R, \chi)$. It is invariant under any change of canonical homology basis χ of R. Since σ is determined solely by R itself, σ is referred to as the *hyperbolic span* of R (cf. Shiba [20]).

To show Theorem 2.1, the following L_s-canonical differentials have played a key role in characterizing the closings of (R, χ). Shiba showed that, for $-1 < s \le 1$, there uniquely exists a holomorphic differential ϕ_s on R such that $\int_A \phi_s = 1$ and ϕ_s satisfies the following properties on C_j:

$$\int_{C_j} \phi_s = 0 \quad \text{and} \quad \text{Im}\, [e^{-\frac{\pi i}{2} s} \phi_s] = 0 \text{ on } C_j \quad (j = 1, \ldots, \nu).$$

In other words, for a thin tubular neighborhood V_j of C_j, the branch on V_j of the abelian integral $\varPhi_s(z) = \int_\zeta^z \phi_s$ is a single-valued holomorphic function on V_j such that $\varPhi_s(C_j)$ is parallel slit region. We call ϕ_s the L_s-canonical differential for (R, χ) (see also [1, 13]). In the special case when $s = 1, 0$, the L_1-canonical differential ϕ_1 for (R, χ) satisfies $\text{Re}\, \phi_1 = 0$ on C_j, the L_0-canonical differential ϕ_0 satisfies $\text{Im}\, \phi_0 = 0$ on C_j, and $\int_A \phi_1 = \int_A \phi_0 = 1$. We set

$$\tau_1 = \int_B \phi_1(z), \qquad \tau_0 = \int_B \phi_0(z).$$

Then τ_1 (resp., τ_0) is the top (resp., lowest) point of $\mathfrak{M}(R, \chi)$, so that $\rho = \text{Im}\, \tau_1 - \text{Im}\, \tau_0$ and $\sigma = \log \frac{\text{Im}\, \tau_1}{\text{Im}\, \tau_0}$.

3 Variational Formulas for Schiffer and Hyperbolic Spans

3.1 *Smooth Variation of Domains*

Let Δ be a disk in $\mathbb{C}_t$. Let $\pi : \widetilde{\mathcal{R}} \to \Delta$ be a holomorphic family of open Riemann surfaces $\widetilde{R}(t) = \pi^{-1}(t)$, $t \in \Delta$, such that $\widetilde{R}(t)$ is irreducible and non-singular in $\widetilde{\mathcal{R}}$. Let $\pi|_{\mathcal{R}} : \mathcal{R} \to \Delta$ be a sub-holomorphic family of $\widetilde{\mathcal{R}}$ such that each fiber $R(t) = (\pi|_{\mathcal{R}})^{-1}(t) \Subset \widetilde{R}(t)$ and $R(t)$ is a bordered Riemann surface of genus g $(0 \leq g < \infty)$. If $\partial R(t)$ in $\widetilde{R}(t)$ consists of a finite number of C^ω smooth contours $C_j(t)$ $(j = 1, \ldots, \nu)$, then we say that $\mathcal{R}$ is a *smooth variation* in $\widetilde{\mathcal{R}}$ by regarding $\mathcal{R}$ as the variation $\mathcal{R} : t \in \Delta \to R(t)$. Each $C_j(t)$ is oriented by $\partial R(t) = C_1(t) + \cdots + C_\nu(t)$. We can take a canonical homology basis $\{A_l(t), B_l(t)\}_{l=1}^g$ of $R(t)$ such that $A_l(t)$ and $B_l(t)$ vary continuously with $t \in \Delta$ in $\mathcal{R}$. We note that g and ν are independent of $t \in \Delta$. For a C^2 defining function $\varphi(t, z)$ of the boundary $\partial\mathcal{R}$ of $\mathcal{R}$ in $\widetilde{\mathcal{R}}$, we set

$$k_2(t,z) = \left(\frac{\partial^2\varphi}{\partial t \partial \bar{t}} \left| \frac{\partial \varphi}{\partial z} \right|^2 - 2\mathrm{Re} \left\{ \frac{\partial^2 \varphi}{\partial \bar{t} \partial z} \frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial \bar{z}} \right\} + \left| \frac{\partial \varphi}{\partial t} \right|^2 \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} \right) \left| \frac{\partial \varphi}{\partial z} \right|^{-3} \tag{3.1}$$

on $\partial\mathcal{R}$. The function $k_2(t, z)$ on $\partial\mathcal{R}$ is due to Maitani and Yamaguchi in [15, (7)] which is based on [14, (1.3)]. Note that $k_2(t, z)$ does not depend on the choice of defining functions $\varphi(t, z)$ of $\partial\mathcal{R}$. We see that $\mathcal{R}$ is a smooth variation if and only if there exists a C^ω defining function $\varphi(t, z)$ of $\partial\mathcal{R}$ such that $\frac{\partial\varphi}{\partial z} \neq 0$.

3.2 *Variation of Schiffer Spans*

Let $\mathcal{R}$ be a smooth variation in $\widetilde{\mathcal{R}}$. Assume that there exists a section $\mathbf{a} := \{a(t) \in R(t) \mid t \in \Delta\}$ of $\mathcal{R}$ over Δ, and there exists a π-local coordinate $\mathcal{U} := \Delta \times \{|z - \zeta| < r\}$ of a neighborhood $\mathcal{V}$ of $\mathbf{a}$ in $\mathcal{R}$ such that $\mathbf{a}$ corresponds to $\Delta \times \{\zeta\}$. Let $t \in \Delta$ be fixed. Then each $R(t)$, $t \in \Delta$, carries the L_i-principal function $p_i(t, z)$ $(i = 1, 0)$ and L_i-constant $\alpha_i(t)$ $(i = 1, 0)$ for $(R(t), \zeta)$. Under the smooth variation $\mathcal{R}$, we established the following variational formulas for $\alpha_i(t)$ for $(R(t), \zeta)$.

Lemma 3.1

$$\frac{\partial^2 \alpha_1(t)}{\partial t \partial \bar{t}} = -\left(\frac{1}{\pi} \int_{\partial R(t)} k_2(t,z) \left| \frac{\partial p_1(t,z)}{\partial z} \right|^2 ds_z + \frac{4}{\pi} \iint_{R(t)} \left| \frac{\partial^2 p_1(t,z)}{\partial \bar{t} \partial z} \right|^2 dxdy \right).$$

$$\frac{\partial^2 \alpha_0(t)}{\partial t \partial \bar{t}} = \frac{1}{\pi}\int_{\partial R(t)} k_2(t,z)\left|\frac{\partial p_0(t,z)}{\partial z}\right|^2 ds_z + \frac{4}{\pi}\iint_{R(t)}\left|\frac{\partial^2 p_0(t,z)}{\partial \bar{t}\partial z}\right|^2 dxdy$$
$$-\frac{2}{\pi}\mathrm{Im}\left\{\sum_{l=1}^{g}\frac{\partial}{\partial t}\left(\int_{A_l(t)} * dp_0(t,z)\right)\frac{\partial}{\partial \bar{t}}\left(\int_{B_l(t)} * dp_0(t,z)\right)\right\}.$$

Here, $k_2(t,z)$ is the same as (3.1).

In the case of the deformed *planar* bordered Riemann surface $R(t)$, Lemma 3.1 immediately implies the following variational formula for the Schiffer span $s(t) := \alpha_0(t) - \alpha_1(t)$ for $(R(t), \zeta)$.

Lemma 3.2 *If $R(t)$ is a planar bordered Riemann surface, then we have:*

$$\frac{\partial^2 s(t)}{\partial t \partial \bar{t}} = \frac{1}{\pi}\int_{\partial R(t)} k_2(t,z)\left(\left|\frac{\partial p_1(t,z)}{\partial z}\right|^2 + \left|\frac{\partial p_0(t,z)}{\partial z}\right|^2\right) ds_z$$
$$+\frac{4}{\pi}\iint_{R(t)}\left(\left|\frac{\partial^2 p_1(t,z)}{\partial \bar{t}\partial z}\right|^2 + \left|\frac{\partial^2 p_0(t,z)}{\partial \bar{t}\partial z}\right|^2\right) dxdy.$$

Here, $k_2(t,z)$ is the same as (3.1).

Theorem 3.3 ([8])

(i) If $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$, then the L_1-constant $\alpha_1(t)$ for $(R(t), \zeta)$ is superharmonic on Δ;
(ii) If $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$ and each $R(t)$, $t \in \Delta$, is planar, then the L_0-constant $\alpha_0(t)$ for $(R(t), \zeta)$ is subharmonic on Δ;
(iii) If $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$ and each $R(t)$, $t \in \Delta$, is planar, the Schiffer span $s(t)$ for $(R(t), \zeta)$ is **logarithmically** *subharmonic on Δ, namely,* $\log s(t)$ *is subharmonic on Δ.*

To apply them to Stein manifolds, we need to study the general (non-smooth) variation $\mathcal{R}$ under pseudoconvexity, namely, the variation $\mathcal{R} : t \in \Delta \to R(t)$ is pseudoconvex in $\widetilde{\mathcal{R}}$ such that neither $\partial R(t)$, $t \in \Delta$, is always C^ω smooth nor the variation $\partial\mathcal{R} : t \in \Delta \to \partial R(t)$ is C^ω smooth with $t \in \Delta$. Through the investigation [7], in [8] we applied the *logarithmically* subharmonicity of $s(t)$ to show the simultaneous uniformization of moving *planar* Riemann surfaces $R(t)$ such that there exists no non-constant holomorphic function with finite Dirichlet integral on $R(t)$. It led to an extension of the main theorem in [6].

3.3 Variation of Hyperbolic Spans

Let $\mathcal{R}$ be a smooth variation in $\widetilde{\mathcal{R}}$. We assume that $R(t)$ is a bordered Riemann surface of genus one with C^ω smooth boundary $\partial R(t)$ in $\widetilde{R}(t)$. For $t \in \Delta$, we fix a canonical homology basis $\chi(t) = \{A(t), B(t)\}$ of $R(t)$ modulo dividing cycles such that $A(t)$ and $B(t)$ move continuously in $\mathcal{R}$ with $t \in \Delta$. Then each $R(t)$, $t \in \Delta$, admits the L_s-canonical differential $\phi_s(t,z)$ $(s = 1, 0)$ for $(R(t), \chi(t))$ such that $\int_{A(t)} \phi_s(t,z) = 1$. We set $\tau_s(t) = \int_{B(t)} \phi_s(t,z)$, and write $\phi_s(t,z) = f_s(t,z)dz$ by use of the local parameter of $\overline{R(t)}$. Under the smooth variation $\mathcal{R}$, we established the following variational formulas for Im $\tau_s(t)$:

Lemma 3.4

$$\frac{\partial^2 \mathrm{Im}\, \tau_1(t)}{\partial t \partial \bar{t}} = \frac{1}{2}\int_{\partial R(t)} k_2(t,z)|f_1(t,z)|^2|dz| + \left\| \frac{\partial \phi_1(t,z)}{\partial \bar{t}} \right\|^2_{R(t)},$$
$$\frac{\partial^2 \mathrm{Im}\, \tau_0(t)}{\partial t \partial \bar{t}} = -\left(\frac{1}{2}\int_{\partial R(t)} k_2(t,z)|f_0(t,z)|^2|dz| + \left\| \frac{\partial \phi_0(t,z)}{\partial \bar{t}} \right\|^2_{R(t)}\right).$$

Here, $k_2(t,z)$ is the same as (3.1).

Theorem 3.5 ([10]) *If $\mathcal{R}$ is a pseudoconvex domain in $\widetilde{\mathcal{R}}$, then*

(i) the hyperbolic span $\sigma(t)$ for $R(t)$ is subharmonic on Δ,
(ii) $\sigma(t)$ is harmonic on Δ if and only if a holomorphic family $\pi : \mathcal{R} \to \Delta$ is trivial; $\mathcal{R} \approx \Delta \times R(0)$.

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A Survey on the Weighted Log Canonical Threshold and the Weighted Multiplier Ideal Sheaf

Pham Hoang Hiep

Abstract We present some recent results and propose a list of questions on the weighted log canonical threshold and the weighted multiplier ideal sheaf. This survey is dedicated to Prof. Kang-Tae Kim on the Occasion of His 60th Birthday.

Keywords Weighted log canonical threshold · Weighted multiplier ideal sheaves

1 Introduction

Let Ω be a domain in $\mathbb{C}^n$ and φ in the set PSH(Ω) of plurisubharmonic functions on Ω. Following Demailly and Kollár [6], we introduce the log canonical threshold of φ at a point $z_0 \in \Omega$:

$$c_\varphi(z_0) = \sup\left\{c > 0 : e^{-2c\varphi} \text{is } L^1(dV_{2n}) \text{ on a neighborhood of } z_0\right\} \in (0, +\infty],$$

where dV_{2n} denotes the Lebesgue measure in $\mathbb{C}^n$. It is an invariant of the singularity of φ at 0. We refer to [3–6, 13, 17, 19] for further information about this number.

For every non-negative Radon measure μ on a neighbourhood of $0 \in \mathbb{C}^n$, we introduce the weighted log canonical threshold of φ with weight μ at z_0:

$$c_{\varphi,\mu}(z_0) = \sup\left\{c \geq 0 : e^{-2c\varphi} \text{is } L^1(\mu) \text{ on a neighborhood of } z_0\right\} \in [0, +\infty].$$

Following Nadel [15], we denote by the multiplier ideal sheaf $\mathcal{I}(\varphi)$ the sheaf of germs of holomorphic functions $f \in \mathcal{O}_{\mathbb{C}^n,z}$ such that

$$\int_U |f|^2 e^{-2\varphi} dV_{2n} < +\infty,$$

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on some neighborhood U of z. It is known by [15] that this is a coherent ideal sheaf over Ω.

For every non-negative Radon measure μ on Ω, we denote the weighted multiplier ideal sheaf of φ with weight μ by $\mathcal{I}(\varphi, \mu)$ the sheaf of germs of holomorphic functions $f \in \mathcal{O}_{\mathbb{C}^n, z}$ such that $\int_U |f|^2 e^{-2\varphi} d\mu < +\infty$ on some neighborhood U of z.

In this survey, we present some recent results on the weighted log canonical threshold and the weighted multiplier ideal sheaf. We also propose a list of questions in this topic.

2 Strong Openness Conjecture for Weighted Log Canonical Thresholds

In [6], Demailly and Kollár stated the following openness conjecture:

Openness Conjecture. *The set* $\{c > 0 : e^{-2c\varphi}$ *is* L^1 *on a neighborhood of* $z_0\}$ *equals the open interval* $(0, c_\varphi(z_0))$.

In 2005, this conjecture was proved in dimension 2 by Favre and Jonsson (see [7]). In 2013, Berndtsson (see [1]) completely proved it in arbitrary dimension. For every holomorphic function f on Ω, we introduce the weighted log canonical threshold of φ with weight f at z_0:

$$c_{\varphi, f}(z_0) = \sup\left\{c > 0 : \ |f|^2 e^{-2c\varphi} \text{is } L^1 \text{ on a neighborhood of } z_0\right\} \in (0, +\infty].$$

Guan and Zhou used the L^2-extension theorem of Ohsawa and Takegoshi in combination with the curve selection lemma, to prove the **strong openness conjecture** posed by Demailly, i.e. the analogue openness statement for weighted thresholds $c_{\varphi, f}(z_0)$ (see [9]). Later, Guan and Zhou (see [8]) used and developed the sophisticated version of L^2-extension theorem of Ohsawa and Takegoshi to give proofs of Demailly-Kollár's conjecture and Jonsson-Mustata's conjecture. In [12], we applied the original version of the L^2-extension theorem (see [16]) to the members of a standard basis for a multiplier ideal sheaf of holomorphic functions associated with a plurisubharmonic function φ. In this way, by means of a simple induction on dimension, we showed an effective version of the semicontinuity theorem for weighted log canonical thresholds (see [12]):

Theorem 2.1 *Let f be a holomorphic function on an open set Ω in $\mathbb{C}^n$ and let* $\varphi \in \mathrm{PSH}(\Omega)$.

(i) Assume that $\int_{\Omega'} e^{-2c\varphi} dV_{2n} < +\infty$ on some open subset $\Omega' \subset \Omega$ and let $z_0 \in \Omega'$. Then there exists $\delta = \delta(c, \varphi, \Omega', z_0) > 0$ for $\psi \in \mathrm{PSH}(\Omega')$ such that $\|\psi - \varphi\|_{L^1(\Omega')} \leq \delta$ implies $c_\psi(z_0) > c$. Moreover, as ψ converges to φ in $L^1(\Omega')$, the function $e^{-2c\psi}$ converges to $e^{-2c\varphi}$ in $L^1_{loc}(\Omega')$.

(ii) Assume that $\int_{\Omega'} |f|^2 e^{-2c\varphi} dV_{2n} < +\infty$ *on some open subset* $\Omega' \subset \Omega$. *When* $\psi \in \mathrm{PSH}(\Omega')$ *converges to* φ *in* $L^1(\Omega')$ *with* $\psi \leq \varphi$, *the function* $|f|^2 e^{-2c\psi}$ *converges to* $|f|^2 e^{-2c\varphi}$ *in* $L^1_{loc}(\Omega')$.

This theorem is used to obtain some consequences as the strong openness conjecture and a theorem of convergence from below:

Corollary 2.2 (**Strong openness**) *For any plurisubharmonic function* φ *on a neighborhood of a point* $z_0 \in \mathbb{C}^n$, *the set*

$$\{c > 0 : \ |f|^2 e^{-2c\varphi} \textit{is } L^1 \textit{ on a neighborhood of } z_0\}$$

is an open interval $(0, c_{\varphi,f}(z_0))$.

Corollary 2.3 (**Convergence from below**) *If* $\psi \leq \varphi$ *converges to* φ *in a neighborhood of* $z_0 \in \mathbb{C}^n$, *then* $c_{\psi,f}(z_0) \leq c_{\varphi,f}(z_0)$ *converges to* $c_{\varphi,f}(z_0)$.

Now, we propose two following questions:

Question 2.4 (Semicontinuity for weighted log canonical thresholds). What condition on μ is necessary and sufficient, such that if

$$\int_{\Omega'} e^{-2c\varphi} d\mu < +\infty,$$

on some open subset $\Omega' \subset \Omega$ and let $z_0 \in \Omega'$ then there exists

$$\delta = \delta(c, \varphi, \mu, \Omega', z_0) > 0,$$

for $\psi \in \mathrm{PSH}(\Omega')$ such that $\|\psi - \varphi\|_{L^1(\Omega')} \leq \delta$ implies $c_{\psi,\mu}(z_0) > c$. Moreover, as ψ converges to φ in $L^1(\Omega')$, the function $e^{-2c'\psi}$ converges to $e^{-2c'\varphi}$ in $L^1_{loc}(\Omega', d\mu)$ for all $c' < c$?

Question 2.5 (Strong openness for weighted log canonical thresholds). What condition on μ is necessary and sufficient, such that for any plurisubharmonic function φ on a neighborhood of a point $z_0 \in \mathbb{C}^n$, the set

$$\{c > 0 : \ e^{-2c\varphi} \text{is } L^1(d\mu) \text{ on a neighborhood of } z_0\}$$

is an open interval $(0, c_{\varphi,\mu}(z_0))$?

3 ACC Conjecture for Weighted Log Canonical Thresholds

In [6], Demailly and Kollár stated the following ACC conjecture discussed in [14] and [18]:

ACC Conjecture *Let* $\{f_j\}_{j\geq 1}$ *be a sequence of holomorphic functions on neighborhoods of* $0 \in \mathbb{C}^n$*. Assume that*

$$c_{\log|f_1|}(0) \leq \ldots \leq c_{\log|f_j|}(0) \leq \ldots$$

Then there exists $j_0 \geq 1$ *such that*

$$c_{\log|f_j|}(0) = c_{\log|f_{j+1}|}(0), \forall j \geq j_0.$$

In 2009, this conjecture was proved by De Fernex, Ein and Mustata (see [2] and also see [10]). Now, we propose the following question:

Question 3.1 (ACC conjecture for weighted log canonical thresholds.) What condition on μ is necessary and sufficient, such that if $\{f_j\}_{j\geq 1}$ is a sequence of holomorphic functions on neighborhoods of $0 \in \mathbb{C}^n$ with

$$c_{\log|f_1|,\mu}(0) \leq \ldots \leq c_{\log|f_j|,\mu}(0) \leq \ldots,$$

then there exists $j_0 \geq 1$ such that

$$c_{\log|f_j|,\mu}(0) = c_{\log|f_{j+1}|,\mu}(0), \forall j \geq j_0?$$

4 Some Questions on the Weighted Multiplier Ideal Sheaf

In [11], we proved the following result concerning stability of the weighted log canonical threshold:

Theorem 4.1 *Let* $\{\varphi_j\}_{j\geq 1} \subset \mathrm{PSH}(\Omega)$ *and* $\varphi \in \mathrm{PSH}(\Omega)$ *be such that* $\varphi_j \to \varphi$ *in* $L^1_{loc}(\Omega)$*. Then*

$$\underline{\lim}_{j\to\infty} c_{||z||^{2t}dV_{2n}}(\varphi_j) \geq c_{||z||^{2t}dV_{2n}}(\varphi), \;\; \forall t \in (-n, 1].$$

Theorem 4.1 is used to obtain a result concerning stability of the multiplier ideal sheaf:

Corollary 4.2 *Let* $\{\varphi_j\}_{j\geq 1} \subset \mathrm{PSH}(\Omega)$ *and* $\varphi \in \mathrm{PSH}(\Omega)$ *be such that* $\varphi_j \to \varphi$ *in* $L^1_{loc}(\Omega)$*. Then the following two statements hold:*

(i) If $\varphi_j \leq \varphi$ *for all* $j \geq 1$*, then for* $\Omega' \Subset \Omega$ *there exists* $j_0 \geq 1$ *such that* $\mathcal{I}(\varphi_j) = \mathcal{I}(\varphi)$ *on* Ω' *for all* $j \geq j_0$*.*
(ii) If $\{z_1, \ldots, z_n\} \in \mathcal{I}(\varphi)_0$ *then there exists* $j_0 \geq 1$ *such that* $\{z_1, \ldots, z_n\} \in \mathcal{I}(\varphi_j)_0$ *for all* $j \geq j_0$*.*

Now, we propose two questions related to the coherent and the stability of the weighted multiplier ideal sheaf:

Question 4.3 (Coherent of the weighted multiplier ideal sheaf). What condition on μ is necessary and sufficient, such that the weighted multiplier ideal sheaf $\mathcal{I}(\varphi, \mu)$ is coherent?

Question 4.4 (Stability of the weighted multiplier ideal sheaf). What condition on μ is necessary and sufficient, such that if $\{\varphi_j\}_{j\geq 1} \subset \mathrm{PSH}(\Omega)$ and $\varphi \in \mathrm{PSH}(\Omega)$ with $\varphi_j \to \varphi$ in $L^1_{loc}(\Omega)$ then the following two statements hold:

(i) If $\varphi_j \leq \varphi$ for all $j \geq 1$, then for $\Omega' \Subset \Omega$ there exists $j_0 \geq 1$ such that $\mathcal{I}(\varphi_j, \mu) = \mathcal{I}(\varphi, \mu)$ on Ω' for all $j \geq j_0$.
(ii) If $\{z_1, ..., z_n\} \in \mathcal{I}(\varphi, \mu)_0$ then there exists $j_0 \geq 1$ such that $\{z_1, ..., z_n\} \in \mathcal{I}(\varphi_j, \mu)_0$ for all $j \geq j_0$?

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A CR Embedding Problem for an Algebraic Levi Non-degenerate Hypersurface into a Hyperquadric

Xiaojun Huang and Ming Xiao

Abstract We give a survey on the current developments of the embeddability problem of a Levi non-degenerate hypersurface into its model, i.e., hyerquadrics. We also formulate and study a local sums-of-squares problem, and make connections with the embeddability problem.

Keywords Hyperquadrics · Spheres · CR embedding

1 Introduction

Let M be a smooth real hypersurface. We say that M is a Levi non-degenerate hypersurface in $\mathbb{C}^{n+1}$ at $p \in M$ with signature $\ell \leq n/2$ if there is a local holomorphic change of coordinates, that maps p to the origin, such that in the new coordinates, M is defined near 0 by an equation of the form:

$$r = v - |z|_\ell^2 + o(|z|^2 + |zu|) = 0 \tag{1}$$

Dedicated to Kang-Tae Kim on the occasion of his 60th birthday

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Here, we write $u = \mathrm{Re}(w), v = \mathrm{Im}(w)$ and $|z|_\ell^2 =< z, \bar{z} >_\ell = -\sum_{j\le\ell}|z_j|^2 + \sum_{j=\ell+1}^n |z_j|^2$. When $\ell = 0$, we regard $\sum_{j\le\ell}|z_j|^2 = 0$ and p is a strongly pseudoconvex point of M.

The compact model of Levi non-degenerate hypersurfaces with signature ℓ is the boundary of the generalized ℓ-ball:

$$\mathbb{B}_\ell^{n+1} := \left\{ [z_0, \ldots, z_{n+1}] \in \mathbb{P}^{n+1} : -\sum_{j=0}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n+1} |z_j|^2 < 0 \right\}.$$

The boundary $\partial\mathbb{B}_\ell^{n+1}$ is locally holomorphically equivalent to the hyperquadric $\mathbb{H}_\ell^{n+1} \subset \mathbb{C}^{n+1}$ of signature ℓ defined by $\Im z_{n+1} = -\sum_{j=1}^{\ell}|z_j|^2 + \sum_{j=\ell+1}^{n}|z_j|^2$, where $(z_1, \ldots, z_{n+1})$ is the coordinates of $\mathbb{C}^{n+1}$.

When $\ell = 0$, it reduces to the standard sphere (compact model) or the Heisenberg hypersurface (non-compact model).

A long-standing question in Complex Analysis is to understand when a (compact) Levi non-degenerate hypersurface of signature $\ell \ge 0$ can be holomorphically embedded into its model, namely, the generalized ball with the same signature. Along these lines, there have been much work done in the past 40 years and we refer to a recent paper of Huang-Xiao [11] for a detailed discussion on history and references. In this article, we summarize some of the work done in Huang-Zaitsev [13], Huang-Li-Xiao [10] and Huang-Xiao [11] and also connect this study with others such as writing a positive real algebraic functions as the sum of norm square of holomorphic functions.

This article is based on the talk given at the 12th Korean Conference on Several Complex Variables—in honor of Kang-Tae Kim's 60th birthday.

We start by mentioning a famous theorem of Fornæss obtained in 1976, which states as follows:

Theorem (Fornaess [6]) *Any compact strongly pseudoconvex hypersurface M can be smoothly embedded into a certain compact strongly convex hypersurface of a larger dimension.*

It is not clear from any known proof of the Fornæss embedding Theorem whether, when M is real analytic (or real algebraic), we can embed M into a compact real analytic (or real algebraic) strongly convex hypersurface of larger dimension or not. It indicates, however, that some type of embedding theorem into manifolds of a more special class is possible.

On the other hand, Forstneric [7] proved in 1986 (See also [5]) the following theorem:

Theorem (Forstneric [7]) *A generic real analytic strongly pseudoconvex hypersurface can not be locally (transversally) holomorphicaly embedded into a hyperquadric (or a generalized sphere) of any dimension.*

A more recent result by Forstneric [8] states even more: Most real-analytic hypersurfaces do not admit a transversal holomorphic embedding even into a merely algebraic hypersurface of higher dimension. Forstneric's argument is based on a Baire category theory, which roughly says that the set of real analytic strongly pseudoconvex hypersurfaces is much larger than the set of holomorphic maps sending them into spheres. It cannot, however, be used to produce any specific examples. In 2006, Zaistev [21] constructed explicitly for the first time a germ of a real analytic strongly pseudoconvex hypersurface which can not be holomorphically embedded into a sphere of any dimension.

The situation in the algebraic category is very different. Webster proved in 1970's the following well-known theorem, which is in sharp contrast with the above mentioned result of Forstneric.

Theorem (Webster [19]) *Every real-algebraic Levi-nondegenerate hypersurface $M \subset \mathbb{C}^n$ admits a transversal holomorphic embedding into a non-degenerate hyperquadric (likely with a much bigger signature) in complex space $\mathbb{C}^N$ of sufficietly large dimension N.*

In Webster's theorem, the embedding dimension N depends on both the source dimension n and the degree of the defining function of M. Indeed, in a recent paper of Kossovskiy-Xiao [15], the following non-embeddability result has been established: For any integers $N > n \geq 1$, there exists $\mu = \mu(n, N)$ such that a Zariski generic real-algebraic hypersurface $M \subset \mathbb{C}^{n+1}$ of any degree $k \geq \mu$ is not transversally holomorphically embeddable into a hyperquadric in $\mathbb{C}^N$.

Moreover, in Webster's theorem, it is crucial that the signature of the target hyperquadric is allowed to be bigger than that of the source manifold. Webster observed in the degree two case, one can always embed a real ellipsoid into the sphere in a complex space of one dimension higher.

Example of Webster [18–20]: Let $M \subset \mathbb{C}^{n+1}$ be a non-spherical ellipsoid, i.e., defined in real coordinates $z_j = x_j + iy_j$, $j = 1, ..., n+1$, by $\sum_{j=1}^{n+1}(\frac{x_j^2}{a_j^2} + \frac{y_j^2}{b_j^2}) = 1$. After a simple scaling (if necessary), the ellipsoid M can be described by the equation of the form:

$$\sum_{j=1}^{n+1}(A_j z_j^2 + A_j \overline{z_j^2}) + \sum_{j=1}^{n+1}|z_j|^2 = 1, \ 0 \leq A_1 \leq \cdots \leq A_{n+1} < 1.$$

Let $G : \mathbb{C}^{n+1} \to \mathbb{C}^{n+2}$ be given by $G = \left(z_1, \ldots, z_{n+1}, i(1 - 2\sum_{j=1}^{n+1} A_j z_j^2)\right)$. Then it holomorphically embeds M into the Heisenberg hypersurface $\mathbb{H}_0^{n+2} \subset \mathbb{C}^{n+2}$.

With all these said, the following had been remained to be a long-standing folklore open question in the field:

Question *Is every compact real-algebraic strongly pseudoconvex real hypersuraface in $\mathbb{C}^n$ holomorphically embeddable into a sphere of sufficiently large dimension?*

The question is closely related to write a positive polynomial in a certain sense to a sum of squares of the absolute values of holomorphic functions. (See the end of Sect. 2.)

2 Algebraic Perturbation of Kohn-Nirenberg Hypersurfaces

The above open problem has been answered in the negative through the work in Huang-Zaitsev [13], Huang-Li-Xiao [10] (See also [4, 14], etc) and finally in Huang-Xiao [11]. Indeed, we constructed a very concrete example based on small perturbations of the famous Kohn-Nirenberg hypersurfaces (see [16]), which we describe below.

Consider the following family of compact real-algebraic strongly pseudoconvex real hypersurfaces:

$$M_\epsilon = \{(z,w) \in \mathbb{C}^2 : \varepsilon_0(|z|^8 + c\mathrm{Re}|z|^2 z^6) + |w|^2 + |z|^{10} + \epsilon|z|^2 - 1 = 0\}. \quad (2)$$

Here, $0 < \epsilon < 1,\ 2 < c < \frac{16}{7}$, and $\varepsilon_0 > 0$ is a sufficiently small number such that M_ε is smooth for all $0 \le \epsilon < 1$.

One can compute to verify that, for any $0 < \epsilon < 1$, M_ϵ is strongly pseudoconvex. M_ϵ is a small algebraic deformation of the famous Kohn-Nirenberg domain:

The Kohn-Nirenberg Domain:

$$M_{KN} = \{(z,w) \in \mathbb{C}^2 : \varepsilon_0(|z|^8 + c\mathrm{Re}|z|^2 z^6) + |w|^2 + |z|^{10} - 1 = 0\},$$

with $2 < c < \frac{16}{7}$.

It is a compact real algebraic pseudo-convex hypersurface with exactly one weakly pseudo-convex point at $p^* = (z,w) = (0,1)$. The significant nature of M_{KN} is that any local complex analytic curve in a neighborhood of p^* must stay in both sides of M_{KN} near p^*.

The theorem we proved in [11], based on the work in [10, 13], is as follows:

Main Theorem (Huang-Xiao [11]) *Let M_ε be defined as above with $\epsilon, \epsilon_0 > 0$ sufficiently small. For any $N \in \mathbb{N}$, a smooth CR map from an open piece of M_ε into the standard sphere $S^{2N-1} = \partial\mathbb{B}^N \subset \mathbb{C}^N$ must be a constant. Hence, there is no smooth CR embedding from any open piece of M_ε into S^{2N-1}.*

The theorem answers, in the negative, the above mentioned question.

As observed in [10], M_ϵ can be holomorphically transversally embedded into the generalized sphere in $\mathbb{C}^6$ with one negative Levi eigenvalue as follows:

Observe that $\mathrm{Re}(|z|^2 z^6) = \frac{1}{4}(|z^7 + z|^2 - |z^7 - z|^2)$. Thus the map

$$F(z,w) = (\sqrt{\varepsilon_0}z^4, \frac{1}{2}\sqrt{\varepsilon_0 c}(z^7+z), w, z^5, \sqrt{\epsilon}z, \frac{1}{2}\sqrt{\varepsilon_0 c}(z^7-z))$$

holomorphically embeds M_ϵ into the generalized sphere in $\mathbb{C}^6$ defined by $\mathbb{S}^{11} = \{(Z_1,\ldots,Z_6) \in \mathbb{C}^6 : \sum_{j=1}^5 |Z_j|^2 - |Z_6|^2 = 1\}$. Indeed, M_ϵ can be holomorphically embedded into $\mathbb{S}^9 = \{(Z_1,\ldots,Z_5) \in \mathbb{C}^5 : \sum_{j=1}^4 |Z_j|^2 - |Z_5|^2 = 1\}$. See Remark 2.4.

To end this section, we make connections of the embeddability problem with a Hermitian version of Hilbert's 17th problem. The problem we introduce here, which is more of local nature, differs from the one studied in [2, 3], etc.

Let $p \in \mathbb{C}^n$ and U a simply connected open set containing p. Let $r(z,\overline{z})$ be a real analytic (real-valued) function in $z \in U$. Then there exist holomorphic functions $f_1,\ldots,f_r, g_1,\ldots,g_s$ and h defined in U with each $f_i(0) = 0$ and $g_j(0) = 0$ such that the decomposition holds:

$$r(z,\overline{z}) = \mathrm{Im}(h) + \sum_{i=1}^{s} |f_i|^2 - \sum_{j=1}^{t} |g_j|^2. \tag{3}$$

Moreover, $f_1,\ldots,f_s, g_1,\ldots,g_t$ are $\mathbb{C}-$linearly independent. See [17] for more details and also [2] which emphasizes more on the real polynomial setting.

In general, s, t may be infinite. When s, t are finite, then we say $r(z,\overline{z})$ has finite rank and signature (s,t) at p. Moreover, in the decomposition (3), h is uniquely determined up to a constant term. And $(f_1,\ldots,f_s,g_1,\ldots,g_t)$ is also uniquely determined up to a complex linear transformation in $U(s,t)$. Thus, the pair (s,t) is uniquely determined at p and does not depend on the choice of the decomposition in (3). See [17] for more details and the proof.

Lemma 2.1 *Let U be simply connected and $r(z,\overline{z})$ be finite rank as above. Let $p, q \in U$. Then the signatures of $r(z,\overline{z})$ at p and at q are identical. In particular, when $r(z,\overline{z})$ is a real polynomial, then it has the same signature at every point in $\mathbb{C}^n$.*

Proof This fact is contained in [17]. For self-containedness, we will sketch a proof based on the setup above. Assume the signature of $r(z,\overline{z})$ is (s,t) at p. Then there exist holomorphic functions $f_1,\ldots,f_s,g_1,\ldots,g_t$ and h such that (3) holds, where all $f_i(0) = 0$ and $g_j(0) = 0$. Moreover, $f_1,\ldots,f_s,g_1,\ldots,g_t$ are $\mathbb{C}-$linearly independent. Note for a holomorphic function f in U, we have

$$|f(z)|^2 = |f(z) - f(q) + f(q)|^2 = |f(z)-f(q)|^2 + \mathrm{Im}\left(2\sqrt{-1}f(z)\overline{f(q)} - \sqrt{-1}|f(q)|^2\right).$$

We apply this equation to each $|f_i(z)|^2$ and $|g_j(z)|^2$ and then (3) can be rewritten as,

$$r(z,\overline{z}) = \mathrm{Im}(\phi(z)) + \sum_{i=1}^{s} |f_i(z) - f_i(q)|^2 - \sum_{j=1}^{t} |g_j(z) - g_j(q)|^2$$

for some holomorphic function $\phi(z)$ in U. Note $f_i(z) - f_i(q)$ and $g_j(z) - g_j(q)$ vanish at q. Moreover, the linear independence of $\{f_1, \ldots, f_s, g_1, \ldots, g_t\}$ is equivalent to that of $\{f_1(z) - f_1(q), \ldots, f_s(z) - f_s(q), g_1(z) - g_1(q), \ldots, g_t(z) - g_t(q)\}$. By the definition of signature, $r(z, \overline{z})$ is also of signature (s, t) at q and we thus have proved the first assertion in the lemma. The second part of the lemma also follows easily. □

Let $\Lambda(U)$ be the set of real analytic functions of finite rank in U. Note $\Lambda(U)$ forms a ring. We pass this notion to the germ level at p. Let $R(z, \overline{z})$ be a germ of real analytic function at p. We say $R(z, \overline{z})$ is of finite rank at p if there is a neighborhood V of p such that $R(z, \overline{z}) \in \Lambda(V)$. Note the signature (s, t) of $R(z, \overline{z})$ does not depend on the choice of V. We will simply say $R(z, \overline{z})$ is finite rank and of signature (s, t) at p. Write $\Lambda(p)$ for the set of germs of real-analytic functions with finite rank at p. Note $\Lambda(p)$ is also a ring.

Recall in a simply connected domain, the signature of $R(z, \overline{z})$ does not depend on the choice of the base point. Assume $R(z, \overline{z})$ is real analytic over a connected domain U, and $p, q \in U$, $p \neq q$. We can choose a path $\gamma \in U$ connecting p, q and a small tube neighborhood V of γ that is simply connected. Then the signature of $R(z, \overline{z})$ is identical at p, q. This yields the following remark (one can also see [17]).

Remark 2.2 Let U be a connected open set in $\mathbb{C}^n$ and $R(z, \overline{z})$ real analytic in U with finite rank at some $p \in U$. Then $R(z, \overline{z})$ has finite rank and the same signature at every point in U.

Remark 2.3 Assume $r(z, \overline{z}) = 0$ defines a real hypersurface M in U. If $r(z, \overline{z})$ is of finite rank (s, t) with the decomposition in (3), then M admits a transversal holomorphic mapping into $\mathbb{H}_t^{s+t+1}$ given by $\Phi := (g_1, \ldots, g_s, f_1, \ldots, f_r, -h)$. Recall in Webster's ellipsoid example, the defining function of an ellipsoid is given by

$$\mathrm{Im}(2\sqrt{-1}A_j z_j^2 - \sqrt{-1}) + \sum_{j=1}^{n+1} |z_j|^2 = 0.$$

Clearly it is of signature $(n + 1, 0)$.

Remark 2.4 Let $0 < \epsilon_0 \ll \epsilon \ll 1$. Let $\rho = \epsilon_0(|z|^8 + c\mathrm{Re}|z|^2 z^6) + |w|^2 + |z|^{10} + \epsilon|z|^2 - 1$. Recall it is the defining function of the non-embeddable hypersurfaces in our main theorem. Note

$$\epsilon_0 c\mathrm{Re}(|z|^2 z^6) + \epsilon|z|^2 = -\frac{\epsilon_0^2 c^2}{4(\epsilon + \epsilon_0 c)}\left|z^7 - z\right|^2 + \frac{1}{\epsilon + \epsilon_0 c}\left|\frac{\epsilon_0 c}{2}(z^7 + z) + \epsilon z\right|^2.$$

We have ρ is of rank (4, 1). Note also ρ is strongly plurisubharmonic.

When $R(z, \overline{z})$ is of signature $(s, 0)$ at p with $s > 0$ and finite, we say $R(z, \overline{z})$ is a (finite) sum of squares (modulo pure terms) at p. We will write $\mathcal{P}(p) \subset \Lambda(p)$ for the set of (germs of) nonzero functions that are (finite) sums of sqaures (modulo pure terms) at p.

The following fact is a consequence of Remark 2.2.

Remark 2.5 Let U be a connected open set in $\mathbb{C}^n$ and $R(z,\overline{z})$ real analytic in U. Then $R(z,\overline{z}) \in \mathcal{P}(p)$ for some $p \in U$ if and only if $R(z,\overline{z}) \in \mathcal{P}(q)$ for every $q \in U$.

Let $r(z,\overline{z})$ be a germ of real analytic function at p with finite rank. Let $\mathcal{I}(r,p)$ be the ideal in $\Lambda(p)$ generated by $r(z,\overline{z})$. Assume $r(z,\overline{z})$ is strongly plurisubharmonic and $r(p,\overline{p}) = 0$. Then the local non-embeddality problem into spheres can be reduced to a real-algebraic geometry problem as follows.

Question *Characterize the class of (strongly plurisubharmonic) real analytic functions $r(z,\overline{z})$ at p such that $\mathcal{I}(r,p) \cap \mathcal{P}(p) = \emptyset$.*

Our main theorem yields the following result.

Corollary 2.6 *Let ρ be in Remark 2.4 and $\rho(p,\overline{p}) = 0$. Then $\mathcal{I}(\rho,p) \cap \mathcal{P}(p) = \emptyset$.*

Proof We will prove by contradiction. Suppose $\mathcal{I}(\rho,p) \cap \mathcal{P}(p) \neq \emptyset$. Then there exist some neighborhood V of p and a real analytic function $H(z,\overline{z})$ defined in V such that $H(z,\overline{z})\rho(z,\overline{z})$ is a sum of sqaures (modulo pure terms). That is, there exists $\mathbb{C}-$linearly independent holomorphic functions $f_1, \ldots, f_s$ and h such that

$$H(z,\overline{z})\rho(z,\overline{z}) = \mathrm{Im}(h) + \sum_{i=1}^{s} |f_i|^2.$$

Consequently, $\Phi := (f_1, \ldots, f_s, -h)$ gives a map in V that sends the hypersurface $M := \{z \in V : \rho(z,\overline{z}) = 0\}$ into the Heisenberg surface $\mathbb{H}_0^{s+1}$, which is locally biholomorphic to the unit sphere in $\mathbb{C}^{s+1}$. By our main theorem, Φ is a constant map. Thus $H(z,\overline{z})\rho(z,\overline{z})$ is constant in V. Since $\rho(p,\overline{p}) = 0$, we have $H(z,\overline{z})\rho(z,\overline{z}) \equiv 0$. This is a contradiction. Thus $\mathcal{I}(\rho,p) \cap \mathcal{P}(p) = \emptyset$. This establishes the corollary. □

The following signature stability result follows easily from Corollary 2.6.

Corollary 2.7 *(1) Let U be a connected open set in $\mathbb{C}^2$ with nonempty intersection with M_ϵ. Let $H(z,\overline{z})$ be a non-zero real analytic function over U. Assume the signature of $H(z,\overline{z})\rho(z,\overline{z})$ is (s,t) with s,t integers, then $s > 0, t > 0$ (at every point in U).*

(2) Let $H(z,\overline{z})$ be a non-zero real polynomial over $\mathbb{C}^2$. Then the signature (s,t) of $H(z,\overline{z})\rho(z,\overline{z})$ satisfies $s > 0, t > 0$ (at every point in $\mathbb{C}^2$).

Proof Suppose $t = 0$. Then $H(z,\overline{z})\rho(z,\overline{z})$ is a sum of sqaures (modulo pure terms) at every point in U (See Remark 2.5). In particular, $H(z,\overline{z})r(z,\overline{z}) \in \mathcal{P}(p)$ for $p \in U \cap M_\epsilon$. It contradicts with the non-embeddability of any open piece of M_ϵ into the sphere (See the proof of Corollary 2.6). Thus we must have $t > 0$. We can similarly consider the signature of $-H(z,\overline{z})\rho(z,\overline{z})$ to conclude $s > 0$. This establishes part (1). Part (2) follows easily from part (1). □

Remark 2.8 The assumption that $U \cap M_\epsilon \neq \emptyset$ in Corollary 2.7 cannot be dropped. Indeed, when $U \cap M_\epsilon = \emptyset$, we can take $H(z,\overline{z}) = \frac{||z||^2}{\rho(z,\overline{z})}$, then the conclusion in Corollary 2.7 fails.

The above discussion motivates the following question: Let $r(z,\overline{z})$ be a strongly plurisubharmonic real polynomial in $\mathbb{C}^n (n \geq 2)$, and assume $\{r(z,\overline{z}) = 0\}$ defines a real hypersurface passing through $p \in \mathbb{C}^n$. Assume $r(z,\overline{z})$ has signature (s,t) with $s > 0, t > 0$. Is it true that $\mathcal{I}(r,p) \cap \mathcal{P}(p) = \emptyset$?

Our result also has applications to the study of the holomorphic isometric embedding problems. Let U be a connected open subset in $\mathbb{C}^2$ and ω a (1,1)-form defined in $V \subset U$ as follows:

$$\omega = \sqrt{-1}\partial\overline{\partial}\log(1 + R(z,\overline{z})), \quad z \in V, \tag{4}$$

where $R(z,\overline{z})$ is a (non-constant) real analytic function in U.

Corollary 2.9 *Let ω be as above which takes the form in (4). Let $(\mathbb{CP}^N, \omega_{FS})$ be the complex projective space equipped with the Fubini-Study metric. Assume there is a holomorphic map $F : (V,\omega) \to (\mathbb{CP}^N, \omega_{FS})$ satisfying*

$$F^*(\omega_{FS}) = \omega.$$

Then $R(z,\overline{z}) \in \mathcal{P}(q)$ for every $q \in U$.

Proof We will first prove $1 + R(z,\overline{z}) \in \mathcal{P}(p)$ for some $p \in V$. Write in the homogeneous coordinates $F = [F_0, \ldots, F_N]$. Fix $p \in V$. Composing F with a self-isometry of $\mathbb{CP}^N$ if necessary, we can assume $F(p) = [1, 0, \ldots, 0]$. Then by shrinking V if necessary, in the affine coordinates of $\mathbb{CP}^N$, we can write F as $\hat{F} = (\hat{F}_1, \ldots, \hat{F}_N)$, where $\hat{F}_i = \frac{F_i}{F_0}$. In particular, $\hat{F}_i(p) = 0$. By the metric-preserving assumption, we have

$$\partial\overline{\partial}\log(1 + R(z,\overline{z})) = \partial\overline{\partial}\log(1 + ||\hat{F}||^2). \tag{5}$$

Assume in V that

$$1 + R(z,\overline{z}) = c(1 + \mathrm{Re}(2h) + ||f||^2 - ||g||^2). \tag{6}$$

Here c is a positive constant, $f = (f_1, \ldots, f_s)$, $g = (g_1, \ldots, g_t)$, and h are holomorphic maps satisfying $f(0) = 0, g(0) = 0, h(0) = 0$. Moreover, $f_1, \ldots, f_s$, $g_1, \ldots, g_t$ are $\mathbb{C}-$linearly independent. Then (5) yields

$$\partial\overline{\partial}\log(1 + \mathrm{Re}(2h) + ||f||^2 - ||g||^2) = \partial\overline{\partial}\log(1 + ||\hat{F}||^2) \tag{7}$$

By shrinking V if necessary, we assume $|h(z)| < 1$ for every $z \in V$. We will prove by contradiction that in the signature (s,t) we must have $t = 0$. Suppose $t > 0$.

We next write

$$1 + \mathrm{Re}(2h) + ||f||^2 - ||g||^2 = (1 + h(z))(1 + \overline{h(z)})(1 + T(z, \overline{z}))$$

for some real analytic function $T(z, \overline{z})$ in V. Then it is easy to see that $T(z, \overline{z})$ has only mixed terms (in $z - p, \overline{z - p}$) in its Taylor expansion at $(p, \overline{p})$.

Note $\partial\overline{\partial}\log(1 + h(z))(1 + \overline{h(z)}) = 0$. It follows from (7) that

$$\partial\overline{\partial}\log(1 + T(z, \overline{z})) = \partial\overline{\partial}\log(1 + ||\hat{F}||^2).$$

Since both $T(z, \overline{z})$ and $||\hat{F}||^2$ have only mixed terms in their Taylor expansions. It follows that

$$1 + T(z, \overline{z}) = 1 + ||\hat{F}||^2.$$

That is,

$$\frac{1 + \mathrm{Re}(2h) + ||f||^2 - ||g||^2}{|1 + h|^2} = 1 + ||\hat{F}||^2.$$

This yields

$$||\frac{f}{1 + h}||^2 - ||\frac{g}{1 + h}||^2 = |\frac{h}{1 + h}|^2 + ||\hat{F}||^2.$$

Note the functions $\{\frac{f_i}{1+h}, \frac{g_j}{1+h}\}_{1\le i\le s, 1\le j\le t}$ are still $\mathbb{C}-$linearly independent. The left hand side of the above equation thus has signature (s, t) with $t > 0$, while the right hand side is a sum of squares. This is a contradiction. Hence $t = 0$. So $1 + R(z, \overline{z}) \in \mathcal{P}(p)$ and thus $R(z, \overline{z}) \in \mathcal{P}(p)$. By Remark 2.5, we have $R(z, \overline{z}) \in \mathcal{P}(q)$ for every $q \in U$. □

Let ρ be as in Remark 2.4. Assume ω be a Kähler metric defined in $V \subset \mathbb{C}^2$ of the form:

$$\omega = \sqrt{-1}\partial\overline{\partial}\log(1 + H(z, \overline{z})\rho(z, \overline{z})), \tag{8}$$

where $H(z, \overline{z})$ is a (non-zero) global real analytic function (of finite rank) in $\mathbb{C}^2$. See the following example.

Example 2.10 Let $H(z, \overline{z}) \equiv \frac{1}{2}$, then

$$\omega = \sqrt{-1}\partial\overline{\partial}\log(2 + \rho(z, \overline{z}))$$

defines an algebraic (1, 1)-form in $\mathbb{C}^2$ and in particular defines a (positive definite) metric in a sufficiently small neighborhood V of 0. Similarly, let $H(z, \overline{z}) \equiv \frac{1}{2} + P(z, \overline{z})$, where $P(z, \overline{z})$ is a homogeneous real polynomial of degree at least 3, then ω in (8) defines a (positive definite) metric in a sufficiently small neighborhood V of 0.

We have the following consequences of Corollary 2.7 and 2.9.

Corollary 2.11 *Let (V, ω) be as in (8) with $H(z, \overline{z})$ a (non-zero) global real analytic function (with finite rank) in $\mathbb{C}^2$. Then (V, ω) cannot be locally holomorphically isometrically embedded into $(\mathbb{P}^N, \omega_{FS})$ for any N.*

Example 2.12 Define $\omega = i\partial\overline{\partial}\log(2+\rho)$ over $\mathbb{C}^2$. Then $(\mathbb{C}^2, \omega)$ can not be locally holomorphically isometrically embedded into $(\mathbb{P}^N, \omega_{FS})$ for any N.

3 Ideas of the Proof of the Main Theorem

Let F be a non-constant smooth CR map from an open piece of M_ϵ into $\mathbb{S}^{2N-1}$. By an algbebraicity theorem due to the first author in [9], it follows that F is Nash-algebraic. A main step is to prove that F is rational— making use of a monodromy argument based on the invariant property of Segre foliations.

Let $M \subset U (\subset \mathbb{C}^n)$ be a closed real-analytic subset defined by a family of real-valued real analytic functions $\{\rho_\alpha(Z, \overline{Z})\}$, where Z is the coordinates of $\mathbb{C}^n$. Assume that the complexification $\rho_\alpha(Z, W)$ of $\rho_\alpha(Z, \overline{Z})$ is holomorphic over $U \times conj(U)$ with

$$conj(U) := \{W : \overline{W} \in U\}$$

for each α. Then the complexification $\mathcal{M}$ of M is the complex-analytic subset in $U \times conj(U)$ defined by $\rho_\alpha(Z, W) = 0$ for each α. Then for $W \in \mathbb{C}^n$, the Segre variety of M associated with the point W is defined by $Q_W := \{Z : (Z, \overline{W}) \in \mathcal{M}\}$.

Write $\mathcal{M}_\epsilon$ for the complexification of M_ϵ and write $\mathcal{M}'$ for the complexification of $\partial\mathbb{B}^N$.

Write Q^ϵ_p for the Segre variety of M_ϵ associated with the point p, and write Q'_q for the Segre variety of $\partial\mathbb{B}^N$ associated with the point q. For any $p \in \mathbb{C}^2$, write $p = (z_p, w_p)$ or $p = (\xi_p, \eta_p)$.

The following lemma from [13] can be proved by using the Thom transversality:

> Let $U \subset \mathbb{C}^n$ be a simply connected open subset and $\mathcal{S} \subset U$ be a closed complex analytic subset of codimension one. Then for $p \in U \setminus \mathcal{S}$, the fundamental group $\pi_1(U \setminus \mathcal{S}, p)$ is generated by loops obtained by concatenating (Jordan) paths $\gamma_1, \gamma_2, \gamma_3$, where γ_1 connects p with a point arbitrary close to a smooth point $q_0 \in \mathcal{S}$, γ_2 is a loop around $\mathcal{S}$ near q_0 and γ_3 is γ_1 reversed.

Making use of the above lemma, we have the following:

Lemma (Huang-Zaitsev [13]) *Let M_ϵ be defined as before and let $p_0 \in M_\epsilon$. Let $\mathcal{S}$ be a complex analytic hyper-variety in $\mathbb{C}^2$ not containing p_0. Let $\gamma \in \pi_1(\mathbb{C}^2 \setminus \mathcal{S}, p_0)$ be obtained by concatenation of $\gamma_1, \gamma_2, \gamma_3$ as described in the Lemma, where γ_2 is a small loop around $\mathcal{S}$ near a smooth point $q_0 \in \mathcal{S}$ with $w_{q_0} \neq 0$. Then γ can be slightly and homopotically perturbed to a loop $\widetilde{\gamma} \in \pi_1(\mathbb{C}^2 \setminus \mathcal{S}, p_0)$ such that there exists a null-homotopic loop $\lambda \in \pi_1(\mathbb{C}^2 \setminus \mathcal{S}, p_0)$ with $(\lambda, \overline{\widetilde{\gamma}})$ contained in the complexification $\mathcal{M}_\epsilon$ of M_ϵ. Also, for an element $\hat{\gamma} \in \pi_1(\mathbb{C}^2 \setminus \mathcal{S}, p_0)$ with a similar* *property described above, after a small perturbation to $\hat{\gamma}$ if needed, we can find a null-homotopic loop in $\hat{\lambda} \in \pi_1(\mathbb{C}^2 \setminus \mathcal{S}, p_0)$ such that $(\hat{\gamma}, \overline{\hat{\lambda}}) \subset \mathcal{M}_\epsilon$.*

Notice that F is complex algebraic (possibly multi-valued). In particular, any branch of F can be holomorphically continued along a path not cutting a certain proper complex algebraic subset $\mathcal{S} \subset \mathbb{C}^2$. We need only to prove the single-valued property for F assuming that $\mathcal{S}$ is a hyper-complex analytic variety. Seeking a contradiction, suppose not. Then we can find a point $p_0 \in U \subset M_\epsilon$, $p_0 = (z_0, w_0)$ with $w_0 \neq 0$, a loop $\gamma \in \pi_1(\mathbb{C}^2 \setminus \mathcal{S}, p_0)$ obtained by concatenation of $\gamma_1, \gamma_2, \gamma_3$ as in Lemma, where γ_2 is a small loop around $\mathcal{S}$ near a smooth point $q_0 \in \mathcal{S}$, such that when we holomorphically continue F from a neighborhood of p_0 along γ one round, we will obtain another branch $F_2(\neq F)$ of F near p_0. Obviously, we can assume q_0 is a smooth point of some branching hypervariety $\mathcal{S}' \subset \mathcal{S}$ of F. We next proceed in two steps:

Case I: (See [10] for more details here): If we can find a loop γ as above such that the corresponding $\mathcal{S}' \neq \{w = 0\}$, by perturbing γ if necessary, we can make $w_{q_0} \neq 0$. By the Lemma, after slightly perturbing γ if necessary, there exists a null-homotopic loop λ in $\pi_1(\mathbb{C}^2 \setminus \mathcal{S}, p_0)$ with $(\gamma, \overline{\lambda})$ contained in the complexification $\mathcal{M}_\epsilon$ of M_ϵ We know that $(F, \overline{F}) := (F(\cdot), \overline{F(\overline{\cdot})})$ sends a neighborhood of $(p_0, \overline{p_0})$ in $\mathcal{M}_\epsilon$ into $\mathcal{M}'$. Applying the analytic continuation along the loop $(\gamma, \overline{\lambda})$ in $\mathcal{M}_\epsilon$ for $\sum_{j=1}^{N}(F_{(j)}\overline{F_{(j)}(\overline{\cdot})}) - 1$, one concludes by the uniqueness of analytic functions that $(F_2, \overline{F})$ also sends a neighborhood of $(p_0, \overline{p_0})$ in $\mathcal{M}_\epsilon$ into $\mathcal{M}'$. Consequently, we get $F_2(Q_p) \subset Q'_{F(p)}$ for $p \in M_\epsilon$ near p_0. In particular, we have the following:

$$F_2(p) \in Q'_{F(p)}, \quad \forall p \in M_\epsilon, \ p \approx p_0.$$

Now applying the holomorphic continuation along the loop $(\lambda, \overline{\gamma})$ in $\mathcal{M}_\epsilon$ for (F_2, F), we get by uniqueness of analytic functions that $(F_2, \overline{F_2})$ sends a neighborhood of (p_0, p_0) in $\mathcal{M}_\epsilon$ into $\mathcal{M}'$. Hence, we also have

$$F_2(p) \in Q'_{F_2(p)}, \quad \forall p \in M_\epsilon, \ p \approx p_0.$$

In particular, $F_2(p) \in \partial\mathbb{B}^N$. Combining this, and noting that for any $q \in \partial\mathbb{B}^N$, $\partial\mathbb{B}^N \cap Q'_q = q$, we get $F_2(p) = F(p)$ for any $p \in M_\epsilon$ near p_0. Thus $F_2 \equiv F$ in a neighborhood of p_0 in $\mathbb{C}^2$, which is a contradiction.

Case II: Now, suppose $W := \{w = 0\}$ is the only branching locus of the algebraic extension of F. Since W is smooth and $\pi_1(\mathbb{C}^2 \setminus W) = \mathbb{Z}$, we get the cyclic branching property for F. Now, we notice that W cuts M_ϵ transversally at a certain point $p^* =: (z_0, 0)$. When we will continue along loops inside $T^{(1,0)}_{p^*} M_\epsilon$ near p^*, we recover all branches of $F(z, w)$. Since any loop inside $T^{(1,0)}_{p^*} M_\epsilon$ near p^* can be easily homotopically deformed into loops in M_ϵ near p^*, we conclude that we recover all branches of F near p^* by continuing any branch of F near p^* along loops inside $M_\epsilon \setminus W$ near p^*. Hence, we are now reduced to the local situation first encountered in

Huang-Zaitsev [13]. Hence, with an argument using the invariant property of Segre varieties, for $Z(\neq) \approx p^*$ and two branches F_1 and F_2 of F near $Z \in M \setminus W$, we have $F_1(Z), F_2(Z) \in Q'_{F_1(Z)} \cap Q'_{F_2(Z)}$. As above, we see that $F_1(Z) = F_2(Z)$. We thus conclude that F is single-valued.

Since F is algebraic, it is rational. Once we know that F is a rational map from M_ϵ into the sphere, by a theorem of Chiappari [1], we know that F extends to a holomorphic map from a neighborhood of D_ϵ and properly maps D_ϵ into the ball.

The next step is the application of the Kohn-Nirenberg condition:

Lemma *Let $p_0 = (0, 1) \in M_\epsilon$. There exists $\widetilde{\epsilon} > 0$ such that for each $0 < \epsilon < \widetilde{\epsilon}$, $Q_{p_0} \cap M_\epsilon$ is a real subvariety of dimension one.*

It suffices to show that there exists $q \in Q_{p_0}$ such that $q \in D_\epsilon$. Note that $Q_{p_0} = \{(z, w) : w = 1\}$. Set

$$\psi(z, \epsilon) = \varepsilon_0(|z|^8 + c\mathrm{Re}|z|^2 z^6) + |z|^{10} + \epsilon|z|^2,\ 0 \leq \epsilon < 1.$$

Note $q = (\mu_0, 1) \in D_\epsilon$ if and only if $\psi(\mu_0, \epsilon) < 0$. Now Set $\phi(\lambda, \epsilon) = \varepsilon_0\lambda^8(1 - c) + \lambda^{10} + \epsilon\lambda^2, 0 \leq \epsilon < 1$. First we note there exists small $\lambda' > 0$, such that $\phi(\lambda', 0) < 0$. Consequently, we can find $\widetilde{\epsilon} > 0$ such that for each $0 < \epsilon \leq \widetilde{\epsilon}$, $\phi(\lambda', \epsilon) < 0$. Write $\mu_0 = \lambda' e^{i\frac{\pi}{6}}$. It is easy to see that $\psi(\mu_0, \epsilon) < 0$ if $0 < \epsilon \leq \widetilde{\epsilon}$.

The next simple lemma we need is the following.

Lemma *Let $M := \{Z \in \mathbb{C}^n : \rho(Z, \overline{Z}) = 0\}, n \geq 2,$ be a compact, connected, strongly pseudo-convex real-algebraic hypersurface. Assume that there exists $p \in M$ such that the associated Segre variety Q_p is irreducible and Q_p intersects M at infinitely many points. Let F be a holomorphic rational map sending M into the unit sphere $\mathbb{S}^{2N-1}$ in some $\mathbb{C}^N$. Then F is a constant map.*

Write $\mathcal{S}$ as the singular set of F, then it does not pass through M. Write Q'_q for the Segre variety of $\mathbb{S}^{2N-1}$ at q. For any $p \approx M$, $F(Q_p \setminus \mathcal{S}) \subset Q'_{F(p)}$. Note that $\mathcal{S} \cap Q_p$ is either empty or a Zariski closed subset of Q_p. Notice that Q_p is connected as it is irreducible. We conclude by unique continuation that if $\widetilde{p} \in Q_p$ and F is holomorphic at $\widetilde{p}$, then $F(\widetilde{p}) \in Q'_{F(p)}$. In particular, if $\widetilde{p} \in Q_p \cap M$, then $F(\widetilde{p}) \in Q'_{F(p)} \cap \mathbb{S}^{2N-1} = \{F(p)\}$. That is, $F(\widetilde{p}) = F(p)$.

Notice that $Q_p \cap M$ is a compact set and contains infinitely many points. Let $\hat{p}$ be an accumulation point of $Q_p \cap M$. Clearly, by what we argued above, F is not one-to-one in any neighborhood of $\hat{p}$. This shows that F is constant. Indeed, suppose F is not a constant map. We then conclude that F is a holomorphic embedding near $\hat{p}$ by a standard Hopf lemma type argument for both M_ϵ and $\mathbb{S}^{2N-1}$ are strongly pseudo-convex.

Now, let ϵ, ε_0 be sufficient small such that all these lemmas hold. Let F be a holomorphic map defined in a small neighborhood U of $p \in M_\epsilon$ that sends an open piece of M_ϵ into $\mathbb{S}^{2N-1}$. Then F is a rational map. Pick $p_0 = (0, 1) \in M_\epsilon$. Notice that the associated Segre variety $Q_{p_0} = \{(z, 1) : z \in \mathbb{C}\}$ is an irreducible complex variety in $\mathbb{C}^2$. Then it follows that F is a constant. We have thus sketched the proof of the main Theorem.

4 Non-embeddable Examples in the Positive Signature Case

Let n, ℓ be two integers with $1 < \ell \leq n/2$. For any ϵ, define

$$M_\epsilon := \{[z_0, \ldots, z_{n+1}] \in \mathbb{P}^{n+1} :$$

$$||z||^2 \left(-\sum_{j=0}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n+1} |z_j|^2 \right) + \epsilon \left(|z_1|^4 - |z_{n+1}|^4 \right) = 0 \Big\} .$$

Here $||z||^2 = \sum_{j=0}^{n+1} |z_j|^2$ as usual. For $\epsilon = 0$, M_ϵ reduces to the generalized sphere with signature ℓ, which is the boundary of the generalized ball.

For $0 < \epsilon << 1$, M_ϵ is a compact smooth real-algebraic hypersurface with Levi form non-degenerate of the same signature ℓ.

Theorem (Huang-Zaitsev [13]) *There is an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, the following holds: M_ϵ is a smooth real-algebraic hypersurface in $\mathbb{P}^{n+1}$ with non-degenerate Levi form of signature ℓ at every point. There does not exist any holomorphic embedding from any open piece of M_ℓ into $\mathbb{H}_\ell^{N+1}$.*

When $0 < \epsilon << 1$, since M_ϵ is a small algebraic deformation, we see that M_ϵ must also be a compact real-algebraic Levi non-degenerate hypersurface in $\mathbb{P}^{n+1}$ with signature ℓ diffeomorphic to the generalized ball $\mathbb{B}_\ell^{n+1} \subset \mathbb{P}^{n+1}$.

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Recognizing G/P by Varieties of Minimal Rational Tangents

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Abstract We give an overview of the problem of biholomorphic equivalence of germs of free unbendable rational curves on complex manifolds, describing varieties of minimal rational tangents as a key invariant of the equivalence problem. Major examples are provided by lines on a rational homogeneous space G/P with a simple Lie group G and a maximal parabolic subgroup P. We review the previous works when P is associated to a long root and discuss recent progress when G/P is a symplectic Grassmannian, which is the most prominent example of the case when P is associated to a short root.

Keywords Minimal rational curves · Cartan connection · Rational homogeneous space

1 Equivalence Problem for Germs of Unbendable Free Rational Curves

A smooth (or immersed) rational curve $(\mathbb{P}^1 \cong) C \subset M$ on a complex manifold M is *free* if its normal bundle is semi-positive. We will say that a free rational curve is *unbendable*, if its normal bundle N_C is of the form

$$N_C \cong \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}, \ n = \dim M$$

for some nonnegative p. The name comes from the fact that among free rational curves, unbendable ones are exactly those which do not have infinitesimal deformations fixing two distinct points:

Dedicated to Kang-Tae Kim on his sixtieth birthday.

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$$H^0(C, N_C \otimes \mathcal{O}(-2)) = 0.$$

Unbendable free rational curves arise naturally in geometric problems. Recall that on a projective algebraic manifold or a compact Kähler manifold, a rational curve of minimal degree, with respect to either an ample line bundle or a Kähler metric, through a general point of the manifold, is called a minimal rational curve (see [7] or [4]). The following result (e.g. Theorem 1.2 in [4]) going back to Mori provides many examples of unbendable free rational curves.

Theorem 1.1 *A general minimal rational curve through a general point of a projective algebraic (or compact Kähler) manifold is an unbendable free rational curve.*

We are interested in the following equivalence problem for germs of unbendable free rational curves.

Problem 1.2 Given two unbendable free rational curves $C_1 \subset M_1$ and $C_2 \subset M_2$ with $\dim M_1 = \dim M_2$, when is the germ of C_1 in M_1 biholomorphic to the germ of C_2 in M_2? In other words, when can we find open neighborhoods

$$C_1 \subset U_1 \subset M_1 \quad \text{and} \quad C_2 \subset U_2 \subset M_2$$

with a biholomorphism $f : U_1 \to U_2$ satisfying $f(C_1) = C_2$?

Why do we care about Problem 1.2? Of course, the biholomorphic equivalence of germs of complex submanifolds is a basic question in complex geometry. But the case of unbendable free rational curves is of particular interest because of the following result from [8].

Theorem 1.3 *Let X_1 and X_2 be Fano manifolds with $b_2 = 1$. Assume that the space of minimal rational curves through a fixed general point of X_1 (resp. X_2) is irreducible and general minimal rational curves $C_1 \subset X_1$ and $C_2 \subset X_2$ have biholomorphic germs. Then the two manifolds X_1 and X_2 are biholomorphic.*

In other words, under some topological conditions, the germ of a minimal rational curve in a Fano manifold determines the Fano manifold. Thus results on Problem 1.2 can be useful in showing two Fano manifolds are biholomorphic.

2 Varieties of Minimal Rational Tangents

An obvious necessary condition for Problem 1.2 is that the normal bundles of C_1 and C_2 are isomorphic:

$$N_{C_1} \cong \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p} \cong N_{C_2}.$$

When $p = 0$, i.e., when the normal bundles are trivial, we have the following result, which is a direct consequence of the basic deformation theory of rational curves.

Theorem 2.1 *A smooth rational curve $C \subset M$ with trivial normal bundle has a holomorphic tubular neighborhood, i.e., a neighborhood $U \subset M$ with a biholomorphism*

$$U \cong (C \times \Delta^{n-1})$$

sending C to $C \times 0$.

By Theorem 2.1, the interesting case of Problem 1.2 is when $p > 0$. From now on, we make the assumption that $p > 0$.

In the study of minimal rational curves, we have the notion of varieties of minimal rational tangents (to be abbreviated to VMRT) as discussed in [4, 7]. For a fixed irreducible family of minimal rational curves on a projective manifold X, its VMRT at a point $x \in X$ is the subvariety $\mathcal{C}_x \subset \mathbb{P}T_x X$ of the projectivized tangent space at x consisting of the tangent directions to minimal rational curves through x. We can introduce a similar notion for any unbendable free rational curve in the following way.

Given an unbendable free rational curve $C \subset M$ on a complex manifold M with $p > 0$ and a point $x \in C$, the deformations of C fixing x form a p-dimensional family of curves:

$$\{x \in C_t \subset M,\ t \in \Delta^p,\ C_0 = C\}.$$

The tangent spaces of these curves form a p-dimensional germ of complex submanifold in $\mathbb{P}T_x M$

$$\mathcal{C}_{x,C} := \{\mathbb{P}T_x C_t, t \in \Delta^p\} \subset \mathbb{P}T_x M \cong \mathbb{P}^{n-1}.$$

This local projective submanifold $\mathcal{C}_{x,C} \subset \mathbb{P}T_x M$ through the point $[\mathbb{P}T_x C] \in \mathbb{P}T_x M$ will be called the VMRT of C at x. If $(C_1 \subset M_1) \cong (C_2 \subset M_2)$ are biholomorphic germs, deformations of C_1 fixing a point is biholomorphic to deformations of C_2 fixing some point. Thus we immediately have the following necessary condition for the equivalence.

Lemma 2.2 *If two unbendable free rational curves $C_1 \subset M_1$ and $C_2 \subset M_2$ have biholomorphically equivalent germs, then for each $x_1 \in C_1$, there must be $x_2 \in C_2$ with a projective isomorphism*

$$(\mathcal{C}_{x_1,C_1} \subset \mathbb{P}T_{x_1} M_1) \cong (\mathcal{C}_{x_2,C_2} \subset \mathbb{P}T_{x_2} M_2).$$

In other words, the projective isomorphism type of the VMRT $\mathcal{C}_{x,C} \subset \mathbb{P}^{n-1}$ is an invariant of the germ $C \in M$ at x. Is it a good invariant? Is it easy to compute?

For a projective algebraic manifold X, the VMRT of a minimal rational curve $C \subset X$ at a general point $x \in X$ is just the germ of the VMRT $\mathcal{C}_x$ at the point $\mathbb{P}T_x C$. As the VMRT $\mathcal{C}_x$ can be described explicitly in many examples, we have many situations where $\mathcal{C}_{x,C}$ can be computed.

Example 2.3 Let $S = \mathbf{Gr}(a, \mathbb{C}^{a+b})$ be the Grassmannian of a-dimensional subspaces in $\mathbb{C}^{a+b}$. Then S is covered by lines under the Plücker embedding

$$S = \mathbf{Gr}(a, \mathbb{C}^{a+b}) \subset \mathbb{P} \wedge^a \mathbb{C}^{a+b},$$

and any two lines are isomorphic to each other by the natural action of $\mathbf{GL}(\mathbb{C}^{a+b})$ on S. Lines are exactly minimal rational curves of S. There is a natural tensor decomposition $TS \cong \mathcal{U} \otimes \mathcal{Q}$ where $\mathcal{U}$ and $\mathcal{Q}$ are vector bundles on S of rank a and b, respectively. The VMRT at a point $s \in S$ turns out to be the set of decomposable tensors:

$$\mathbb{P}\mathcal{U}_s \times \mathbb{P}\mathcal{Q}_s \cong \mathcal{C}_s \subset \mathbb{P}T_s S.$$

Thus for any line $s \in \ell \subset S$, the VMRT $\mathcal{C}_{s,\ell}$ is the germ of the Segre variety $\mathcal{C}_s$ at the point $\mathbb{P}T_s\ell$.

3 Recognizing G/P Associated to a Long Root by VMRT

In the previous section, we have seen that VMRT is an invariant of the germ of an unbendable free rational curve, which is often easy to compute. We will turn to the question whether it is an effective invariant in distinguishing germs. The first result along this line is the following result of N. Mok [10]. Although Mok had not stated the result in this form, this is the essential point of what was proved in [10] for the Grassmannians.

Theorem 3.1 *Let $S = \mathbf{Gr}(a, \mathbb{C}^{a+b})$ be the Grassmannian with $a, b \geq 2$. Let $C \subset M$ be a general unbendable free rational curve through a general point $x \in M$ on a complex manifold M. Assume that the VMRT $\mathcal{C}_{x,C}$ is projectively isomorphic to the VMRT $\mathcal{C}_{s,\ell}$ of a line $\ell \subset S$. Then the germ $C \subset M$ is biholomorphic to the germ $\ell \subset S$.*

In other words, the VMRT at a general point determines the germ completely! Combining Theorem 3.1 with Theorem 1.3, we have the following corollary.

Corollary 3.2 *Let X be a Fano manifold with $b_2(X) = 1$. Assume that the VMRT at a general point of X for a family of minimal rational curves on X is projectively isomorphic to the VMRT at a point for the family of lines on $\mathbf{Gr}(a, \mathbb{C}^{a+b})$. Then X is biholomorphic to $\mathbf{Gr}(a, \mathbb{C}^{a+b})$.*

In [3, 10], these results have been generalized to a rational homogeneous space $S = G/P$ such that $G = \mathrm{Aut}_0(S)$ is a complex simple Lie group and P is a maximal parabolic subgroup associated to a long simple root of G. Classical examples of such homogeneous spaces are Grassmannians and orthogonal Grassmannians.

The VMRT of such a homogeneous space S can be described as follows. There exists a minimal projective embedding $S = G/P \subset \mathbb{P}^N$, analogous to the Plücker embedding of Grassmannians, under which lines cover S and all lines are isomorphic to each other by the action of G. For the base point $s \in S$ corresponding to $P \subset G$, the isotropy representation of P on $T_s S$ has a unique irreducible subspace. The VMRT

$\mathcal{C}_s \subset \mathbb{P}T_s S$ is exactly the highest weight orbit of the irreducible subspace. For any point s on a line $\ell \subset S$, the VMRT $\mathcal{C}_{s,\ell}$ is the germ of the homogeneous variety $\mathcal{C}_s$ at the point $\mathbb{P}T_s\ell$. Theorem 3.1 has been generalized in [3, 10] as follows.

Theorem 3.3 *Let $S = G/P$ be the homogenous space (different from projective space) with a complex simple Lie group G and a maximal parabolic subgroup P associated to a long simple root. Let $C \subset M$ be a general unbendable free rational curve through a general point $x \in M$ on a complex manifold M. Assume that the VMRT $\mathcal{C}_{x,C}$ is projectively isomorphic to the VMRT $\mathcal{C}_{s,\ell}$ of a line $\ell \subset S$ and the Pfaffian system spanned by VMRT's in a neighborhood of x is bracket-generating. Then the germ $C \subset M$ is biholomorphic to the germ $\ell \subset S$.*

The bracket-generating condition in the above theorem is automatically satisfied in the following generalization of Corollary 3.2.

Corollary 3.4 *Let X be a Fano manifold with $b_2(X) = 1$. Assume that the VMRT through a general point of X for a family of minimal rational curves on X is projectively isomorphic to the VMRT at a point for the family of lines on G/P associated to a long simple root. Then X is biholomorphic to G/P.*

4 VMRT of Lines on Symplectic Grassmannians

A strict analogue of Theorem 3.3 does not hold for G/P associated to a short root. The typical example of such G/P is a symplectic Grassmannian. Let us briefly examine this example. We recommend Sect. 5 of [11] for an extensive discussion.

Let ω be a symplectic form, i.e., a nondegenerate antisymmetric 2-form, on $\mathbb{C}^{2n}$. Fix a natural number k, $1 < k < n$, and let $\mathcal{S}^\omega$ be the space of ω-isotropic k-dimensional subspaces of $\mathbb{C}^{2n}$. This is called a symplectic Grassmannian and is a homogeneous space G/P where G is the symplectic group $\mathrm{Sp}(\mathbb{C}^{2n}, \omega)$ and P is associated to a short root of G. Under the Plücker embedding $\mathcal{S}^\omega \subset \mathbb{P} \wedge^k \mathbb{C}^{2n}$, lines cover $\mathcal{S}^\omega$.

Unlike the case of G/P associated to a long root, there are two distinct types of lines on $\mathcal{S}^\omega$ under $\mathrm{Sp}(\mathbb{C}^{2n}, \omega)$. The VMRT $\mathcal{C}_s \subset \mathbb{P}T_s\mathcal{S}^\omega$ of lines through a point $s \in \mathcal{S}^\omega$ turns out to be isomorphic to the blow-up of $\mathbb{P}^{2n-k-1}$ along a linear subspace $\mathbb{P}^{2n-2k-1} \subset \mathbb{P}^{2n-k-1}$, with a suitable projective embedding. The exceptional divisor $\mathcal{E}_s \subset \mathcal{C}_s \subset \mathbb{P}T_s\mathcal{S}^\omega$ of the blow-up spans a linear subspace $\mathcal{D}^\omega_s \subset T_s\mathcal{S}^\omega$. This determines a Pfaffian system $\mathcal{D}^\omega \subset T\mathcal{S}^\omega$. The Frobenius bracket (Levy form)

$$\wedge^2 \mathcal{D}^\omega \to T\mathcal{S}^\omega / \mathcal{D}^\omega$$

is surjective and contains the information of ω.

Most of these geometric properties of a symplectic Grassmannian hold for a presymplectic Grassmannian, which is defined as follows. Let σ be a presymplectic form, i.e., a degenerate antisymmetric 2-form on $\mathbb{C}^{2n}$ with the null space $\mathrm{Null}(\sigma) \neq 0$.

The space $\mathcal{S}^\sigma$ of σ-isotropic k-dimensional subspaces of $\mathbb{C}^{2n}$ is called a presymplectic Grassmannian. It is a singular projective variety. Under the Plücker embedding $\mathcal{S}^\sigma \subset \mathbb{P} \wedge^k \mathbb{C}^{2n}$, lines cover $\mathcal{S}^\sigma$. If $\dim \mathrm{Null}(\sigma) \leq 2n - 2k$, all lines through a general point $s \in \mathcal{S}^\sigma$ are contained in the smooth locus of $\mathcal{S}^\sigma$ and the VMRT at a general point of a general line in $\mathcal{S}^\sigma$ is projectively isomorphic to the VMRT at a general point of a general line in $\mathcal{S}^\omega$. The exceptional divisors $\mathcal{E}_s \subset \mathcal{C}_s$ at $s \in \mathcal{S}^\sigma$ spans a Pfaffian system $\mathcal{D}^\sigma \subset T\mathcal{S}^\sigma$. The Frobenius bracket

$$\wedge^2 \mathcal{D}^\sigma \to TS^\sigma/\mathcal{D}^\sigma$$

is surjective and retains the information of $\mathrm{Null}(\sigma)$. This implies that the Pfaffian systems $\mathcal{D}^\sigma$ and $\mathcal{D}^\omega$ are not locally equivalent. The difference between these two Pfaffian systems is like the difference between Levi-degenerate CR-structures and Levi-nondegenerate CR-structures.

In conclusion, the VMRT of a general line on the smooth locus of $\mathcal{S}^\sigma$ is projectively isomorphic to the VMRT of a general line on $\mathcal{S}^\omega$, but the germ of a general line in $\mathcal{S}^\sigma$ is not biholomorphic to the germ of a general line in $\mathcal{S}^\omega$, because the germs determine $\mathcal{D}^\sigma$ and $\mathcal{D}^\omega$. This shows that the analogue of Theorem 3.3 fails when P is associated to a short root.

5 Recognizing Symplectic Grassmannians by VMRT

From the discussion in the previous section, a natural question arises: to what extent VMRT isomorphic to that of a symplectic Grassmannian controls the biholomorphic type of the germ of an unbendable free rational curve? Furthermore, as the variety $\mathcal{S}^\sigma$ is singular, it does not give a counterexample to the analogue of Corollary 3.4 for symplectic Grassmannians. So we may ask: does the analogue of Corollary 3.4 hold for symplectic Grassmannians? These questions have been puzzling experts ever since the works [3, 10] appeared. It is stated as one of the most tantalizing questions regarding VMRT in [5].

Recently, in a joint work [6] with Qifeng Li, we have answered these questions as follows.

Theorem 5.1 *Let $C \subset M$ be a general unbendable rational curve through a general point $x \in M$ on a complex manifold M. Assume that the VMRT $\mathcal{C}_{x,C}$ is projectively isomorphic to the VMRT $\mathcal{C}_{s,\ell}$ of a general line $\ell \subset \mathcal{S}^\omega$. Then the germ $C \subset M$ is biholomorphic either to the germ of $\ell \subset \mathcal{S}^\omega$, or to the germ of a general line $\ell' \subset \mathcal{S}^\sigma$ for some presymplectic form σ with* $\dim \mathrm{Null}(\sigma) \leq 2n - 2k$.

In other words, the VMRT at a general point determines the germ of the curve, up to finitely many possibilities, and they are exactly those described in the previous section. Combining Theorem 5.1 with a suitable singular version of Theorem 1.3, we obtain the following.

Corollary 5.2 *Let X be a Fano manifold with $b_2(X) = 1$. Assume that the VMRT at a general point of X for a family of minimal rational curves on X is projectively isomorphic to the VMRT at a point for the family of lines on $\mathcal{S}^\omega$. Then X is biholomorphic to $\mathcal{S}^\omega$.*

As a matter of fact, analogues of Theorem 5.1 and Corollary 5.2 hold for odd-symplectic Grassmannians, i.e., the space of isotropic subspaces in $\mathbb{C}^{2n+1}$ with respect to a maximally nondegenerate antisymmetric form (see [9] for a precise definition).

Other than symplectic Grassmannians, there are exactly two rational homogeneous spaces with $b_2 = 1$ associated to a short root. They are of the form G/P with G of type F_4 and P associated to one of the two short roots of G. We expect that our methods can be applied to give analogues of Theorem 5.1 and Corollary 5.2 for these two remaining cases.

6 Strategy of the Proof of Theorem 5.1

To prove Theorem 5.1, as well as Theorems 3.1 and 3.3, we need to construct an open immersion $f : U \to G/G^0$ from a neighborhood U of a rational curve to a homogeneous space G/G^0, for a suitable complex Lie group G and a closed subgroup G^0. For example, we have $G = \mathrm{SL}(\mathbb{C}^{a+b})$ for the Grassmannian, $G = \mathrm{Sp}(\mathbb{C}^{2n}, \omega)$ for the symplectic Grassmannian and $G = \mathrm{Sp}(\mathbb{C}^{2n}, \sigma)$ for the presymplectic Grassmannian. Note that the last group is not reductive. To construct such f, we first consider its infinitesimal version. This is the concept of a Cartan connection.

For a homogeneous space G/G^0, let $\mathfrak{g}^0 \subset \mathfrak{g}$ be the Lie algebras of $G^0 \subset G$. Recall (see [12]) that a Cartan connection of type G/G^0 on a manifold U is a G^0-principal bundle $\mathcal{P} \to U$ equipped with a $\mathfrak{g}$-valued 1-form θ such that

(1) $\theta(\zeta_A) = A$ for the fundamental vector field ζ_A on $\mathcal{P}$ associated to an element $A \in \mathfrak{g}^0$;
(2) $R_a^*\theta = \mathrm{Ad}(a^{-1})\theta$ for the right action R_a of $a \in G^0$ on $\mathcal{P}$;
(3) $\theta_y : T_y\mathcal{P} \to \mathfrak{g}$ is an isomorphism at each point $y \in \mathcal{P}$.

We say that θ satisfies the Maurer-Cartan equation if $\mathrm{d}\theta + \frac{1}{2}[\theta, \theta] = 0$.

The prototypical example of a Cartan connection is the homogeneous space G/G^0 itself. In fact, view the quotient map $G \to G/G^0$ as a G^0-principal bundle on the homogeneous space G/G^0. Then the Maurer-Cartan form on G gives a Cartan connection of type G/G^0 which satisfies the Maurer-Cartan equation. The following classical result of E. Cartan gives a partial converse.

Theorem 6.1 *Constructing an open immersion $f : U \to G/G^0$ is equivalent to constructing a Cartan connection of type G/G^0 on the manifold U that satisfies the Maurer-Cartan equation.*

Thus our problem becomes constructing a Cartan connection in a neighborhood of an unbendable free rational curve, that satisfies the Maurer-Cartan equation. It turns out that the Maurer-Cartan equation is automatic in our situation by the following fact, which uses an observation from [1].

Lemma 6.2 *Let $C \subset M$ be an unbendable free rational curve. If VMRT at some $x \in C$ spans $T_x M$, then any Cartan connection on a neighborhood of C satisfies the Maurer-Cartan equation.*

To construct a Cartan connection, we need to build certain natural principal bundles in a neighborhood of our unbendable free rational curve. The starting point is the following result of N. Mok from [10].

Lemma 6.3 *For any two points x, y on an unbendable free rational curve $C \subset M$, the second fundamental form of $\mathcal{C}_{x,C}$ at $\mathbb{P}T_x C$ is isomorphic to that of $\mathcal{C}_{y,C}$ at $\mathbb{P}T_y C$.*

For a certain type of VMRT, Lemma 6.3 implies that $\mathcal{C}_{x,C}$ and $\mathcal{C}_{y,C}$ are isomorphic as projective subvarieties. This is the case for VMRT's of G/P for P associated to a simple root, either long or short. Thus for an unbendable free rational curve C in a complex manifold M, if the VMRT $\mathcal{C}_{x,C}$ of a general point $x \in M$ is isomorphic to that of G/P, so is $\mathcal{C}_{y,C}$ for every point $y \in C$. Thus we obtain a fiber subbundle of $\mathbb{P}TU$ on a neighborhood $U \subset M$ of C whose fibers are all isomorphic to $\mathcal{C}_{s,\ell}$ for a general line $\ell \subset G/P$. From this fiber subbundle of $\mathbb{P}TU$, we can construct a G_0-principal bundle on a neighborhood $U \subset M$ of C, where G_0 is the image of G^0 in the graded automorphism group of a natural Pfaffian system on the homogeneous space G/G^0.

The main technical difficulty is that the group G_0 is much smaller than G^0. So the key issue is how to prolong the given G_0-principal bundle to the desired G^0-principal bundle. When the G_0-principal bundle is modeled on G/P, this technical problem was settled by Tanaka in [13] (also see [2]), constructing Tanaka connections for parabolic geometries. The Chern-Moser-Tanaka connection for strictly pseudoconvex (Levi-nondegenerate) CR-structures is an example. In the work of [3, 10], in the case of G/P associated with long root, Tanaka connection is used to construct the map to the homogeneous space G/P. However, Tanaka's prolongation procedure used Kostant's harmonic theory and cannot be adapted when G is not reductive and (G, G^0) has degeneracy, as in the case of presymplectic Grassmannians. The difficulty is similar to that of Levi-degenerate CR-structures.

In [6], we overcome this technical difficulty by the following general result.

Theorem 6.4 *Given a G_0-structure $\mathcal{P}_0$ on a manifold U, we can prolong it to a G^0-principal bundle with a Cartan connection, if for all $\ell \geq 1$,*

$$H^0(U, C^{\ell,2}(\mathcal{P}_0)/\partial C^{\ell,1}(\mathcal{P}_0)) = 0.$$

Here $C^{\ell,2}(\mathcal{P}_0)/\partial C^{\ell,1}(\mathcal{P}_0)$ denotes the vector bundle associated to $\mathcal{P}_0$ via the Spencer complex $\partial : C^{\ell,1}(\mathfrak{g}_0) \to C^{\ell,2}(\mathfrak{g}_0)$. This construction works for a very general G/G^0, but the vanishing condition seldom holds locally. It is a kind of pseudoconcavity of the manifold U. In the case of Theorem 5.1, the vanishing is checked by exploiting the geometry of unbendable rational curves and the projective geometry of the VMRT of presymplectic Grassmannians. See [6] for details.

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Semicontinuity Theorems for Holomorphic and CR Automorphism Groups

Jae-Cheon Joo

Abstract We review the semicontinuity property for automorphism groups of CR manifolds proved in Joo (Pac J Math 285:225–241, 2016 [19]) which belongs to the line of research on the semicontinuity for Riemmanian isometry groups and holomorphic automorphism groups of domains Ebin (Proceedings of symposia in pure mathematics, pp 11–40, 1968 [6]), Greene and Kim (Math Z 277:909–916, 2014 [10]), Greene et al. (Pac J Math 262:365–395, 2013 [12]), Greene and Krantz (Math Ann 261:425–446, 1982 [13]), Kim (Arch Math (Basel) 49:450–455, 1987 [21]), Krantz (Real Anal Exch 36, 421–433, 2010/11 [22]). A brief explanation of the proof and some related open problems as well as the motivation of the study are provided.

Keywords Semicontinuity property · Automorphism groups · CR structures
CR Yamabe equation

1 Introduction

One of the general ideas in geometry is that a deformation of a geometric structure easily destroys the symmetry. This means a symmetry group of a geometric structure cannot be very smaller than those of nearby structures. There is a variety of theorems about this property. One of the strongest version was obtained by D. Ebin for abstract compact Lie group actions on compact manifolds.

Theorem 1.1 ([6]; cf. [11, 12, 14]) *Let M be a compact C^∞-smooth manifold and let G_k $(k = 1, 2, \ldots)$ and G_0 be compact subgroups of* Diff(M). *Suppose $G_k \to G_0$ in C^∞-topology as $k \to \infty$. Then G_k is isomorphic to a subgroup of G_0 for every sufficiently large k. Moreover, the isomorphism can be obtained by the conjugation by a diffeomorphism φ_k of M which converges to the identity map in C^∞-sense, that is, $\varphi_k \circ G_k \circ \varphi_k^{-1}$ is a subgroup of G_0.*

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Here $\mathrm{Diff}(M)$ represents the group of C^∞-smooth diffeomorphisms on M. As an immediate consequence, Ebin proved the following semicontinuity property for Riemannian structures.

Theorem 1.2 ([6]; cf. [21]) *Let M be a C^∞-smooth compact manifold and let $\{g_j : j = 1, 2, ...\}$ be a sequence of C^∞-smooth Riemannian structures which converges to a Riemannian metric g_0 in C^∞-sense. Then for each sufficiently large j, there exists a diffeomorphism $\varphi_j : M \to M$ suh that $\varphi_j \circ I_j \circ \varphi_j^{-1}$ is a Lie subgroup of I_0, where I_j and I_0 represent the groups of isometries for g_j and g_0, respectively.*

Thanks to Theorem 1.1, one need to show that $I_j \to I_0$ as $j \to \infty$ in order to prove Theorem 1.2. That is, it is needed to show every sequence $\{F_j \in I_j\}$ has a convergent subsequence. The subsequential convergence follows easily from the Arzela-Ascoli theorem.

The geometric structure of a complex analytic space is the complex structure, and the symmetry group in this case is the group of holomorphic automorphisms—biholomorphic self-mappings—on a given analytic space. Therefore it is natural to ask what kind of semicontinuity theorem holds true for automorphism groups under deformations of complex structures. It seems nontrivial to answer this general question. But if we restrict our attention to the case of bounded domains, then we may expect a similar result with Theorem 1.2, since every holomorphic automorphism preserves the invariant Bergman metric. R. E. Greene and S. G. Krantz provided an affirmative solution for strongly pseudoconvex cases.

Theorem 1.3 ([13]) *Let Ω_k $(k = 1, 2, ...)$ and Ω_0 be bounded strongly pseudoconvex domains in $\mathbb{C}^n$ with C^∞-smooth boundary. Suppose that $\mathrm{Aut}(\Omega_0)$ is compact and that Ω_k converges to Ω_0 in C^∞-sense, that is, there exists a diffeomorphism ψ_k defined on a neighborhood of $\overline{\Omega}_0$ into $\mathbb{C}^n$ such that $\psi_k(\Omega_0) = \Omega_k$ and $\psi_k \to Id$ in C^∞-sense on $\overline{\Omega}_0$. Then for every sufficiently large k, there exists a diffeomorphism $\phi_k : \Omega_k \to \Omega_0$ such that $\phi_k \circ \mathrm{Aut}(\Omega_k) \circ \phi_k^{-1}$ is a Lie subgroup of $\mathrm{Aut}(\Omega_0)$.*

The idea of the proof of Theorem 1.3 is the construction of compact manifolds M containing Ω_k as a relatively compact subset, and a sequence of Riemnannian metrics $\{g_k\}$ such that the isometry group of g_k contains $\mathrm{Aut}(\Omega_k)$ as a subgroup for each k. Then Theorem 1.2 yields the desired conclusion. Construction of such an auxiliary Riemannian structure is rather technical and the original way of construction strongly depends on the geometric structure of the Bergman kernel and metric. Two essential ingredients for the construction of such structures are the smooth extension of automorphisms up to the boundary and the existence of a point which is contained in a relatively compact subset of Ω_k for every sufficiently large k. Such a point is called a *stably interior point*.

Recently, it turned out that the use of the Bergman geometry is not very essential for the proof of the semicontinuity. For instance, in [12], the authors make use of Lempert's theory on the stationary discs ([24, 25]) instead of Fefferman's extension theorem ([7]) in order to get extensions of automorphisms to the boundary. They proved the following theorem.

Theorem 1.4 ([12]) *Let Ω_0 be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with $C^{k,\alpha}$-smooth boundary ($k \geq 4, \alpha > 0$). Suppose* $\mathrm{Aut}(\Omega_0)$ *is compact. Then there exists a $C^{k,\alpha}$-neghborhood $\mathcal{N}$ of Ω_0 such that for each $\Omega \in \mathcal{N}$, there is a C^{k-1}-diffeomorphism $f : \Omega \to \Omega_0$ with the property that* $f \circ \mathrm{Aut}(\Omega) \circ f^{-1} \subset \mathrm{Aut}(\Omega_0)$.

In [10], R. E. Greene and K. -T. Kim also proved that the existence of a stably interior point can be shown by the scaling argument of domains. With this observation, they proved a semicontinuity property for a class of domains where the scaling argument works well.

In the next section, we review a CR version of Theorem 1.3 which is another generalization of Theorem 1.3, especially in the intrinsic geometric point of view.

2 Case of Deformations of CR Structures

If we denote by $J_k = \psi_k^* J_0$ in Theorem 1.3, where J_0 is the standard complex structure of $\mathbb{C}^n$, then J_k is a sequence of complex structure on a neighborhood of $\overline{\Omega}_0$ tending to J_0. Therefore, Theorem 1.3 can be regarded as an intrinsic semicontinuity result on a fixed domain under deformations of complex structures. Indeed, it turned out by R. Hamilton that every deformation of complex structures on a bounded strongly pseudoconvex domain is obtained by deformations of domains. See [15, 16]. On the other hand, the original proof of Theorem 1.3 is quite extrinsic as we have seen in Sect. 1. Therefore, it would be natural to ask if there exists a generalization in an intrinsic point of view. Thinking domains does not seem a good idea for intrinsic generalization, since domains are already extrinsic objects. Instead, we consider deformations of strongly pseudoconvex CR structures on odd dimensional smooth manifolds. Here and in the sequel, a CR manifold is a smooth manifold M of real dimension $2n + 1$ for some positive integer n with a rank $2n$ subbundle of TM carrying a complex structure. More precisely, An *almost CR structure* on M means a smooth section J of the bundle of endomorphisms of a rank $2n$ subbundle H satisfying $J_p \circ J_p = Id_{H_p}$ for each $p \in M$. The subbundle H is called the *CR distribution*. An almost CR structure is said to be *integrable* if it satisfies the formal integrability condition

$$[\Gamma(H_{1,0}), \Gamma(H_{1,0})] \subset \Gamma(H_{1,0}),$$

where $H_{1,0}$ stands for the i-eigensubspace of J in the complexification of H. An integrable almost CR structure is simply called a *CR structure*. A CR strucutre is said to be *strongly pseudoconvex* if its CR distribution H is a contact distribution and for a contact form θ, the *Levi form* $\mathcal{L}_\theta$ defined by

$$\mathcal{L}_\theta(Z, \overline{W}) := -i\, d\theta(Z, \overline{W})$$

Theorem 2.1 ([19]) *Let $\{J_k : k = 1, 2, \ldots\}$ be a sequence of C^∞-smooth strongly pseudoconvex CR structures on a compact differentiable manifold M of dimension $2n + 1$, which converges to a C^∞-smooth strongly pseudoconvex CR structure J_0 on M in C^∞-sense. Suppose that the CR automorphism group $\mathrm{Aut}_{CR}(M, J_0)$ is compact. Then there exists $N > 0$ and a diffeomorphism $\varphi_k : M \to M$ for each $k > N$ such that $\varphi_k \circ \mathrm{Aut}_{CR}(M, J_k) \circ \varphi_k^{-1}$ is a Lie subgroup of $\mathrm{Aut}_{CR}(M, J_0)$.*

One should notice that Theorem 2.1 implies Theorem 1.3, since every holomorphic automorphism on a bounded strongly pseudoconvex domain gives rise to a CR automorphism of the boundary, and the compact-open topology of the group of the holomorphic automorphisms on the domain coincides with the C^∞-topology on the group of CR automorphisms on the boundary if the group is compact (cf. [1]).

Thanks to Theorem 1.1, it suffices to prove the following proposition for the proof.

Proposition 2.2 *Let $\{J_k : k = 1.2. \ldots\}$ be a sequence of strongly pseudoconvex CR structures on a compact manifold M which tends to a strongly pseudoconvex CR structure J_0 as in Theorem 2.1. Suppose that $\mathrm{Aut}_{CR}(M, J_0)$ is compact. Then every sequence $\{F_k \in \mathrm{Aut}_{CR}(M, J_k) : k = 1, 2, \ldots\}$ admits a subsequence converging to an element $F \in \mathrm{Aut}_{CR}(M, J_0)$ in C^∞-sense.*

To show a subsequential convergence of a sequence of mappings, we need to get estimates of derivatives. If we are dealing with holomorphic mappings, the Cauchy estimates play the role. In PDE theoretic point of view, the derivative estimates of the holomorphic mappings follow from the ellipticity of $\bar{\partial}$-operator. On the other hand, it is hopeless to expect such a general estimates of derivatives for arbitrary CR mappings, since $\bar{\partial}_b$ operator does not enjoy elliptic property. However, we are dealing with only CR automorphisms which resembles conformal mappings in Riemannian geometry. The CR transformation formula of the Webster scalar curvature which is called the CR Yamabe equation can be used to get derivative estimates of CR automorphisms. Before discussing the idea of proof of Proposition 2.2, we recall the pseudohermitian geometry and the CR Yamabe equation.

We call a fixed contact form for the CR distribution of a strongly pseudoconvex CR structure a *pseudohermitian structure*. Let $\{W_\alpha : \alpha = 1, ..., n\}$ be a local frame, that is, W_α's are sections of $H^{1,0}$ which form a pointwise basis for $H_{1,0}$. We call a collection of 1-forms $\{\theta^\alpha\}$ the *admissible coframe* of $\{W_\alpha\}$, if they are sections of $(H^{1,0})^*$ and satisfy

$$\theta^\alpha(W_\beta) = \delta^\alpha_\beta, \quad \theta^\alpha(T) = 0,$$

where T is the vector field uniquely determined by

$$\theta(T) = 1, \quad T \lrcorner d\theta = 0,$$

which is called the *characteristic vector field* for θ. let $g_{\alpha\bar{\beta}} = \mathcal{L}_\theta(W_\alpha, W_{\bar{\beta}})$. Then

$$d\theta = 2i g_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}},$$

where $\{\theta^\alpha\}$ is the admissible coframe for $\{W_\alpha\}$.

Theorem 2.3 ([29]) *There exist a local* 1*-form* $\omega = (\omega_\beta{}^\alpha)$ *and local functions* $A^\alpha{}_\beta$ *uniquely determined by*

$$d\theta^\alpha = \theta^\beta \wedge \omega_\beta{}^\alpha + A^\alpha{}_{\bar\beta}\,\theta \wedge \theta^{\bar\beta},$$

$$dg_{\alpha\bar\beta} = \omega_{\alpha\bar\beta} + \omega_{\bar\beta\alpha}, \quad A_{\alpha\beta} = A_{\beta\alpha}.$$

Here and the sequel, we lower or raise index by $(g_{\alpha\bar\beta})$ and $(g^{\alpha\bar\beta}) = (g_{\alpha\bar\beta})^{-1}$. A connection ∇ defined by

$$\nabla W_\alpha = \omega_\alpha{}^\beta \otimes W_\beta, \quad \nabla T = 0$$

is called the *pseudohermitian connection* or the *Webster connection* for θ. The functions $A^\alpha{}_\beta$ are called the coefficients of the *torsion tensor* **T**. Let

$$d\omega_\alpha{}^\beta - \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta \equiv R_\alpha{}^\beta{}_{\gamma\bar\sigma}\,\theta^\gamma \wedge \theta^{\bar\sigma} \mod \theta, \theta^\gamma \wedge \theta^\sigma, \theta^{\bar\gamma} \wedge \theta^{\bar\sigma}.$$

We call $R_\alpha{}^\beta{}_{\gamma\bar\sigma}$ the coefficients of the *Webster curvature tensor* **R**. Contracting indices, we obtain the coefficients $R_{\alpha\bar\beta}$ of the Webster Ricci curvature Ric and the Webster scalar curvature S as follows:

$$R_{\alpha\bar\beta} = R_\gamma{}^\gamma{}_{\alpha\bar\beta}, \quad S = R_{\alpha\bar\beta}\, g^{\alpha\bar\beta}.$$

Let u be a positive smooth function on M and let $\tilde\theta = u^{2/n}\theta$ be a conformal change of the contact form. Then it turns out that

$$L_\theta u := (b_n \Delta_\theta + S)\, u = \widetilde{S}\, u^{p-1}, \tag{2.1}$$

where $b_n = p = 2 + 2/n$, Δ_θ is the sublaplacian of θ and $\widetilde{S}$ is the Webster scalar curvature of $\tilde\theta$ (see [17, 18, 23]). Equation (2.1) is called the *CR Yamabe equation* and the subelliptic linear operator L_θ is called the *CR Laplacan* for θ. The *CR Yamabe problem* is to find a positive smooth function u which makes $\widetilde{S}$ constant. Although there are still some interesting problems on the CR Yamabe equation, the CR Yamabe problem was completely solved. See [8, 9, 17, 18].

The most important invariant in dealing with the CR Yamabe equation is the *CR Yamabe invariant*. It is defined by the infimum of the total scalar curvature under the constraint of unit volume, that is, the CR Yamambe invariant $Y(M)$ is

$$Y(M) = \inf \left\{ \int_M S\,\theta \wedge d\theta^n : \int_M \theta \wedge d\theta^n = 1 \right\}.$$

The proof of Proposition 2.2 also depends on the sign of the CR Yamabe invariants. Choosing a subsequence, we can assume either $Y_k(M) \leq 0$ for all k or they are all positive, where $Y_k(M)$ denotes the CR Yamabe invariants for J_k. In the former case, it turns out that

- There is a unique θ_k with unit volume such that $S_k = Y_k(M)$ for each k, where S_k is the Webster scalar curvature for θ_k.
- Every CR automorphism for J_k preserves θ_k as well.

See [17, 18]. Moreover, one can show that

- $\theta_k \to \theta_0$ as $k \to \infty$ in C^∞-smooth sense

by a Sobolev-type embedding theorem. Therefore, if we let g_k be a Riemannian metrix on M defined by

$$g_k = \theta_k \otimes \theta_k + d\theta_k(\cdot, J_k\cdot),$$

then $g_k \to g_0$ as $k \to \infty$, and a CR automorphism for J_k is an isometry for g_k. Then Theorem 1.2 yields the conclusion in this case.

Let us assume now $Y_k(M) > 0$ for all $k = 1, 2, \ldots$. Assume also a sequence of mappings $\{F_k\}$ consisting of CR automorphisms for J_k is divergent. Let $\{\theta_k\}$ be pseudohermitian structures for J_k tending to θ_0. Choosing a subsequence, we may assume that $|F_k'(x_k)| \to \infty$ and $x_k \to x_0$ in M as $k \to \infty$. The idea of the proof in this case is detecting a bubbling phenomenon. Let U be a small neighborhood of x_0. We can also assume that all x_k's are in this neighborhood. Let $y_k \in M \setminus \overline{F_k(U)}$. Denote the Green function with pole at y_k for L_{θ_k} by G_k. The existence the Green function is guaranteed by the positivity of the CR Yamabe invariant (cf. [5]). Let $\tilde{\theta}_k = G_k^{2/n}\theta_k$. Then $\tilde{\theta}_k$ is defined well on $M \setminus \{y_k\}$ and has trivial scalar curvature. Therefore, if $F_k^*\tilde{\theta}_k = u_k^{2/n}\theta_k$, then the function u_k satisfies

$$L_{\theta_k} u_k = 0 \tag{2.2}$$

for every k. The assumption that $|F_k'(x_k)| \to \infty$ implies that $u_k(x_k) \to \infty$ as well. Then the Harnack inequality for linear subellitic Eq. (2.2) implies $u_k(x) \to \infty$ for every $x \in U$, by shrinking U if necessary. This means that the image $F_k(U)$ is strictly growing as k grows. One can show that the limit of $F_k(U)$ is indeed $M \setminus \{y_0\}$, where $y_0 = \lim_{k\to\infty} y_k$. One can also show that the pseudohermitian curvature tensor and torsion tensor are tending to 0 along this sequence. As a conclusion, $M \setminus \{y_0\}$ must be CR equivalent to the Heisenberg group. Applying the theorem on removal of singularity (cf. [17]), we conclude that (M, J_0) must be CR equivalent to the standard unit sphere, which is a contradiction to the hypothesis on the compactness of the CR automorphism group.

3 Remarks and Open Problems

3.1 Deformations of the Sphere

Indeed, the compactness of $\mathrm{Aut}(\Omega_0)$ is not assumed in [13]. The celebrated theorem by B. Wong and J. -P. Rosay [27, 30] implies there is only one case of noncompact automorphism group—the unit ball. In case Ω is a small deformation of the unit ball and is not biholomorphic to the ball, the authors of [13] used the exponential map of the Bergman metric to realize $\mathrm{Aut}(\Omega)$ as a subgroup of the unitary group. In CR case, it also turns out that the unit sphere is a unique compact strongly pseudoconvex CR manifold with noncompact automorphism group (see [28]). If J is a small deformation of the standard CR structure of the sphere and if the dimension is greater than 3, then the CR manifold can be realized as a boundary of strongly pseudoconvex domain in the complex Euclidean space (see [4]), while such an embedding property fails in 3-dimensional case (cf. [26]). An intrinsic theory of the semicontinuity for the case of deformations of the sphere, including 3-dimensional case, has not been established yet. One reasonable attempt may be to study actions of CR automorphisms on *normal forms* of deformed structures. A full description of the normal form of a small deformation of the standard sphere was done by J. Bland and T. Duchamp in [2, 3] in terms of the Fourier coefficients of the circle action.

3.2 Almost CR Cases

Even if we do not assume the integrability, Webster's pseudohermitian connection and curvature tensors can be defined in canonical way (see [20]). We say an almost complex structure is *partially integrable* if the non-integrability tensor stays in the complex direction, that is,

$$[\Gamma(H_{1,0}), \Gamma(H_{1,0})] \subset \Gamma(H_{1,0} \oplus H_{0,1}).$$

It turns out that if the almost complex structure is partially integrable, then the conformal transformation formula of the Webster scalar curvature still follows the CR Yamabe equation. Following the same argument with the proof of Theorem 2.1, one can see the theorem is also valid even in the class of partially integrable almost CR structures. On the other hand, the CR Yamabe equation is not valid any longer, if we do not assume partial integrability. The transformation formula of the Webster scalar curvature is too complicated to be simplified like the CR Yamabe equation. In [20], we constructed an auxiliary subconformal structure in order to generalize Schoen's work [28] on the classification of strongly pseudoconvex CR manifolds with nonproperly acting automorphism groups. The same idea can be used to generalize Theorem 2.1 to almost CR cases. But unfortunately, we succeeded in the construction of the auxiliary subconformal structure only in low dimensional cases. Construction in higher dimensional cases and the full generalization of Theorem 2.1 to the class of almost CR structures are still open.

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On $\mathcal{N}_p$-Spaces in the Ball

Le Hai Khoi

Abstract We present a short survey on the so-called $\mathcal{N}_p$-spaces in the ball. Basic properties as well as the structure of $\mathcal{N}_p$-spaces are given. Weighted composition operators and composition operators on these spaces are also studied. Some open problems are provided.

Keywords $\mathcal{N}_p$-space

1 Introduction

For a positive integer n, if $z = (z_1, \ldots, z_n)$, $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$, then we define the inner product $\langle z, \zeta \rangle = z_1\bar{\zeta}_1 + \cdots + z_n\bar{\zeta}_n$ and norm $|z| = (z_1\bar{z}_1 + \cdots + z_n\bar{z}_n)^{1/2}$. Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$ under this norm, with $\mathbb{S}$ as its boundary.

The space $\mathcal{O}(\mathbb{B})$ consists of all holomorphic functions in $\mathbb{B}$ with the compact-open topology, and $H^\infty(\mathbb{B})$ is the Banach space of all bounded functions in $\mathcal{O}(\mathbb{B})$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{B}} |f(z)|$.

Let $p > 0$, the Bergman space $A^2(\mathbb{B})$ is defined as

$$A^2(\mathbb{B}) = \left\{ f \in \mathcal{O}(\mathbb{B}) : \|f\|_{A^2} = \left(\int_{\mathbb{B}} |f(z)|^2 dV(z) \right)^{1/2} < \infty \right\},$$

where dV is the normalized Lebesgue volume measure on $\mathbb{B}$ so that $V(\mathbb{B}) = 1$, while the Bergman-type space (sometimes also called the Beurling-type space) $A^{-p}(\mathbb{B})$ is defined as

Dedicated to Prof. Kang-Tae Kim on the occasion of his 60th birthday.

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$$A^{-p}(\mathbb{B}) = \left\{ f \in \mathcal{O}(\mathbb{B}) : |f|_p = \sup_{z\in\mathbb{B}} |f(z)|(1-|z|^2)^p < \infty \right\}.$$

The Bergman spaces and Bergman-type spaces attract a great attention of mathematicians. These spaces, on one hand play an important role in both function theoretic and operator theoretic development of function spaces, and on the other hand have a closed connection to Bloch spaces which appear as the images of the bounded functions under the Bergman projections. Bloch spaces also play the role of the dual spaces of the Bergman spaces.

Also among the Bergman and Bergman-type spaces, the so-called Q_p-spaces are of great interest. These spaces on the open unit disk $\mathbb{D}$ of the complex plane were introduced by Aulaskari et al. [1]. For $p > 0$, the Q_p-space consists of functions in $\mathcal{O}(\mathbb{D})$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1-|\sigma_a(z)|^2)^p \, dA(z) < \infty.$$

Here $\sigma_a(z) = (a-z)/(1-\overline{a}z)$ is the automorphism of $\mathbb{D}$ that changes 0 and a, and dA is the Lebesgue area measure on the plane, normalized so that $A(\mathbb{D}) = 1$. It is known that Q_p-spaces coincide with the classical Bloch space $\mathcal{B}$ for $p \in (1, \infty)$; Q_1 is equal to BMOA, the space of holomorphic functions on $\mathbb{D}$ with bounded mean oscillation; and for $p \in (0, 1)$, the Q_p-spaces are all different.

If in the definition of the Q_p-space, $f'(z)$ is replaced by $f(z)$, then we have the so-called $\mathcal{N}_p$-space. This space consists of functions in $\mathcal{O}(\mathbb{D})$ for which

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 (1-|\sigma_a(z)|^2)^p \, dA(z) < \infty.$$

Some properties of $\mathcal{N}_p$-spaces are: for $p > 1$, the $\mathcal{N}_p$-space coincides with A^{-1} and for $p \in (0, 1]$, the $\mathcal{N}_p$-spaces are all different.

We note that the Q_p-spaces have been studied intensively for over a decade. The background on these spaces can be found in the book by Xiao [18]. So due to both the similarities and the differences between $\mathcal{B}$ and A^{-1}, it is natural to study the $\mathcal{N}_p$-spaces. For the unit disk $\mathbb{D}$ these spaces were first introduced and studied by Palmberg in [12] and then by Ueki in [17]. In these works, different properties of weighted composition operators and composition operators on $\mathcal{N}_p$-spaces in the unit disk are also studied.

The question to ask is: how about $\mathcal{N}_p$-spaces in higher dimensions? The aim of this paper is to present a short survey on these spaces.

One of the important topics presented in this survey concerns weighted composition operators. Let φ denote a non-constant holomorphic self-mapping of $\mathbb{B}$, while ψ will be a holomorphic function on $\mathbb{B}$, which is not identically equal to zero. These functions induce a linear operator $W_{\psi,\varphi} : \mathcal{O}(\mathbb{B}) \to \mathcal{O}(\mathbb{B})$, which is called a *weighted composition operator* with symbols ψ and φ defined by

$$W_{\psi,\varphi}(f)(z) = \psi(z) \cdot (f \circ \varphi(z)), f \in \mathcal{O}(\mathbb{B}), z \in \mathbb{B}.$$

Observe that if ψ is identically 1, then $W_{\psi,\varphi} = C_\varphi$ is the *composition operator*, and if φ is the identity, then $W_{\psi,\varphi} = M_\psi$ is the *multiplication operator*.

Composition operators and weighted composition operators acting on spaces of holomorphic functions in the unit disk $\mathbb{D}$ of the complex plane have been studied quite well. We refer the readers to the monographs [2, 15] for detailed information. Composition operators on $A^{-p}(\mathbb{D})$ have also been intensively studied (see, e.g., [4, 20] and references therein).

This survey is organized as follows. Section 2 deals with basic properties of $\mathcal{N}_p$-spaces in connection with the Bergman-type spaces $A^{-q}(\mathbb{B})$. In particular, an embedding theorem for $\mathcal{N}_p(\mathbb{B})$-spaces and $A^{-q}(\mathbb{B})$ is provided, together with other useful properties. Several results on the structure of $\mathcal{N}_p$-spaces such as multipliers, $\mathcal{M}$-invariance, the closure of all polynomials, $\mathcal{N}_p$-norm via Carleson measures, … are also given. The result about differences of $\mathcal{N}_p$ for small values of p is obtained in this section. In Sect. 3 weighted composition operators $W_{\psi,\varphi}$ between $\mathcal{N}_p$ and Bergman-type spaces A^{-q} are studied. Criteria for boundedness and compactness of these operators, the compact differences and the estimate for essential norm of weighted composition operators $W_{u,\varphi}$ are given. Section 4 studies composition operators C_φ acting between $\mathcal{N}_p$-spaces, which is one of the most interesting (and quite difficult) topics. The characterizations of the boundedness and compactness of these operators are obtained. In Sect. 5 some open problems are provided. We also note that the main results in this survey can be found in [5–9].

The important remark is that although the N_p-spaces are closely related to the Q_p-spaces, the approach and techniques used for investigating $\mathcal{N}_p$-spaces are quite different from those of Q_p-space. Moreover, they have their own interest.

Throughout the paper, $d\sigma$ denotes the normalized surface measure on the boundary $\mathbb{S}$ of $\mathbb{B}$. For $a, b \in \mathbb{R}$, $a \lesssim b$ ($a \gtrsim b$, respectively) means there exists a positive number C, which is independent of a and b, such that $a \leq Cb$ ($a \geq Cb$, respectively). Moreover, if both $a \lesssim b$ and $a \gtrsim b$ hold, then we write $a \simeq b$.

2 Basic Properties of $\mathcal{N}_p$-Spaces in the Ball

The $\mathcal{N}_p$-space in $\mathbb{B}$ is introduced in [5]. It is defined as follows

$$\mathcal{N}_p(\mathbb{B}) = \left\{ f \in \mathcal{O}(\mathbb{B}) : \|f\|_p = \sup_{a\in\mathbb{B}} \left(\int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) \right)^{1/2} < \infty \right\},$$

where $dV(z)$ is the normalized volume measure over $\mathbb{B}$ and $\Phi_a \in \mathrm{Aut}(\mathbb{B})$ is the involutive automorphism of the unit ball that interchanges 0 and $a \in \mathbb{B}$ (see, e.g. [13, Chap. 2]).

2.1 $\mathcal{N}_p$-Spaces and Bergman-Type Spaces

Several basic properties of $\mathcal{N}_p(\mathbb{B})$-spaces can be obtained, in connection with the Bergman-type spaces $A^{-q}(\mathbb{B})$. In particular, a continuous embedding for $\mathcal{N}_p(\mathbb{B})$-spaces and $A^{-q}(\mathbb{B})$, together with other useful properties, are given in the following theorem.

Theorem 2.1 ([5]) *The following statements hold.*

(a) For $p > q > 0$, we have $H^\infty(\mathbb{B}) \hookrightarrow \mathcal{N}_q(\mathbb{B}) \hookrightarrow \mathcal{N}_p(\mathbb{B}) \hookrightarrow A^{-\frac{n+1}{2}}(\mathbb{B})$.
(b) For $p > 0$, if $p > 2k - 1$, $k \in (0, \frac{n+1}{2}]$, then $A^{-k}(\mathbb{B}) \hookrightarrow \mathcal{N}_p(\mathbb{B})$. In particular, when $p > n$, $\mathcal{N}_p(\mathbb{B}) = A^{-\frac{n+1}{2}}(\mathbb{B})$.
(c) $\mathcal{N}_p(\mathbb{B})$ is a functional Banach space with the norm $\|\cdot\|_p$, and moreover, its norm topology is stronger than the compact-open topology.
(d) For $0 < p < \infty$, $\mathcal{B}(\mathbb{B}) \hookrightarrow \mathcal{N}_p(\mathbb{B})$, where $\mathcal{B}(\mathbb{B})$ is the Bloch space in $\mathbb{B}$.

Here if X and Y are two topological vector spaces, then the notation $X \hookrightarrow Y$ means the continuous embedding of X into Y.

The following "probe" functions in $\mathcal{N}_p$-spaces are used very often in the sequel.

Lemma 2.2 *For each $w \in \mathbb{B}$, set*

$$k_w(z) = \left(\frac{1 - |w|^2}{(1 - \langle z, w\rangle)^2}\right)^{\frac{n+1}{2}}, \ z \in \mathbb{B}.$$

Then $k_w \in \mathcal{N}_p(\mathbb{B})$ and $\sup\limits_{w\in\mathbb{B}} \|k_w\|_p \leq 1$.

Such k_w is a normalized reproducing kernel function in the Bergman space $A^2(\mathbb{B})$. We have $k_w \in \mathcal{N}_p$ and $\sup\limits_{w\in\mathbb{B}} \|k_w\|_p \leq 1$. Note that for all $w \in \mathbb{B}$, we have $k_w(w) = (1 - |w|^2)^{-(n+1)/2}$.

2.2 Multipliers and $\mathcal{M}$-Invariance of $\mathcal{N}_p$-Spaces

We first describe the space $\mathrm{Mult}(\mathcal{N}_p)$ of multipliers of $\mathcal{N}_p$. Recall that a function $u : \mathbb{B} \to \mathbb{C}$ is a *multiplier* of $\mathcal{N}_p$ if uf belongs to $\mathcal{N}_p$ for all f in $\mathcal{N}_p$. An application of the the closed graph theorem shows that for any $u \in \mathrm{Mult}(\mathcal{N}_p)$, the multiplication operator M_u is bounded on $\mathcal{N}_p$.

Proposition 2.3 ([9]) *For any $p > 0$, we have* $\mathrm{Mult}(\mathcal{N}_p) = H^\infty(\mathbb{B})$. *Furthermore, for any $u \in H^\infty(\mathbb{B})$, $\|M_u\| = \|u\|_\infty$.*

By using weighted composition operators with particular symbols, we can obtain an alternate description of the norm in $\mathcal{N}_p$. Namely, we note that for any $\Phi \in \mathrm{Aut}(\mathbb{B})$, by [13, Theorem 2.2.5], there exists a unitary operator U such that $\Phi = U\Phi_a$, where $a = \Phi^{-1}(0)$. This shows that $|\Phi(z)| = |\Phi_a(z)|$ for all $z \in \mathbb{B}$. Let W_Φ denote the weighted composition operator $W_{k_a,\Phi}$. Then we have

$$\|f\|_p = \sup\left\{\|W_\Phi f\|_{A_p^2} : \Phi \in \mathrm{Aut}(\mathbb{B})\right\}. \tag{2.1}$$

Here A_p^2 is the weighted Bergman space over $\mathbb{B}$ defined by

$$A_p^2(\mathbb{B}) = \left\{f \in \mathcal{O}(\mathbb{B}) : \|f\|_{A_p^2} = \left(\int_{\mathbb{B}} |f(z)|^2(1-|z|^2)^p\, dV(z)\right)^{1/2} < \infty\right\}.$$

We have the following result.

Theorem 2.4 ([9]) *For any automorphism Ψ of the unit ball $\mathbb{B}$, the weighted composition operator W_Ψ is a surjective isometry on $\mathcal{N}_p$.*

Next, recall that a space $\mathcal{X}$ of functions defined on $\mathbb{B}$ is said to be *Moebius-invariant*, or simply $\mathcal{M}$*-invariant*, if $f \circ \Phi \in \mathcal{X}$ for every $f \in \mathcal{X}$ and every $\Phi \in \mathrm{Aut}(\mathbb{B})$ (see, e.g., [13]). As a corollary to Theorem 2.4, we obtain

Corollary 2.5 *The space $\mathcal{N}_p$ is $\mathcal{M}$-invariant. Moreover, for any $\Phi \in \mathrm{Aut}(\mathbb{B})$, we have*

$$\|C_\Phi\| = \|M_{1/k_a}\| = \left(\frac{1+|a|}{1-|a|}\right)^{\frac{n+1}{2}}, \tag{2.2}$$

where $a = \Phi^{-1}(0)$.

2.3 The Closure of All Polynomials in $\mathcal{N}_p$

It is natural to consider what the closure of all the polynomials is in $\mathcal{N}_p$-spaces. We introduce the little space $\mathcal{N}_p^0$ of $\mathcal{N}_p$, which is defined as follows

$$\mathcal{N}_p^0(\mathbb{B}) = \left\{f \in \mathcal{N}_p : \lim_{|a|\to 1^-} \int_{\mathbb{B}} |f(z)|^2(1-|\Phi_a(z)|^2)^p dV(z) = 0\right\}.$$

The main result to present is that the closure of all the polynomials on $\mathbb{B}$ coincides with the little space $\mathcal{N}_p^0$.

Proposition 2.6 ([9]) *The following assertions hold.*

(a) *$\mathcal{N}_p^0(\mathbb{B})$ is a closed subspace of $\mathcal{N}_p(\mathbb{B})$, and hence it is a Banach space.*
(b) *For any $p > 0$, we have $A_p^2(\mathbb{B}) \subset \mathcal{N}_p^0(\mathbb{B})$.*

Theorem 2.7 ([9]) *Suppose $f \in \mathcal{N}_p$. Then $f \in \mathcal{N}_p^0$ if and only if*

$$\|f_r - f\|_p \to 0 \ \ as \ \ r \to 1^-,$$

where $f_r(z) = f(rz)$ for all $z \in \mathbb{B}$.

As a corollary to Theorem 2.7, we obtain

Corollary 2.8 *The set of polynomials is dense in $\mathcal{N}_p^0$.*

2.4 $\mathcal{N}_p$-Norm via Carleson Measures

Recall (see, e.g., [19]) that for $\xi \in \mathbb{S}$ and $r > 0$, a Carleson tube at ξ is defined as

$$Q_r(\xi) = \{z \in \mathbb{B} : |1 - \langle z, w\rangle| < r\}.$$

A positive Borel measure μ in $\mathbb{B}$ is called a *p-Carleson measure* if there exists a constant $C > 0$ such that

$$\mu(Q_r(\xi)) \leq Cr^p$$

for all $\xi \in \mathbb{S}$ and $r > 0$. Moreover, if

$$\lim_{r\to 0} \frac{\mu(Q_r(\xi))}{r^p} = 0$$

uniformly for $\xi \in \mathbb{S}$, then μ is called a *vanishing p-Carleson measure*.

The following result describes a relationship between functions in $\mathcal{N}_p$ as well as $\mathcal{N}_p^0$ and Carleson measures.

Proposition 2.9 ([9]) *Let $p > 0$ and $f \in \mathcal{O}(\mathbb{B})$. Define $d\mu_{f,p}(z) = |f(z)|^2(1-|z|^2)^p$ $dV(z)$. The following assertions hold.*

1. *$f \in \mathcal{N}_p$ if and only if $\mu_{f,p}$ is a p-Carleson measure.*
2. *$f \in \mathcal{N}_p^0$ if and only if $\mu_{f,p}$ is a vanishing p-Carleson measure.*

Moreover, it holds

$$\|f\|_p^2 \simeq \sup_{r\in(0,1),\xi\in\mathbb{S}} \frac{\mu_{f,p}(Q_r(\xi))}{r^p} = \sup_{r\in(0,1),\xi\in\mathbb{S}} \frac{1}{r^p}\int_{Q_r(\xi)} |f(z)|^2(1-|z|^2)^p \, dV(z). \tag{2.3}$$

Lemma 2.10 *Let $p = n + 1 + \alpha > 0$ and μ be a finite positive Borel measure on $\mathbb{B}$. Then the following conditions are equivalent.*

(a) μ is a vanishing p-Carleson measure.

(b) *For each* $s > 0$,

$$\lim_{|z|\to 1^-} \int_{\mathbb{B}} \frac{(1-|z|^2)^s d\mu(w)}{|1-\langle z, w\rangle|^{p+s}} = 0. \tag{2.4}$$

(c) *For some* $s > 0$, (2.4) *holds.*

2.5 Differences of $\mathcal{N}_p$ for Small Values of p

In this section, we show that for p small, that is $0 < p \leq n$, all $\mathcal{N}_p$-spaces are different. This together with Theorem 2.1(b) gives a complete relationship between $\mathcal{N}_p$-spaces for all $p > 0$.

We prove this fact by a construction. In [14], the authors constructed a sequence of homogeneous polynomials $(P_k)_{k\in\mathbb{N}}$ satisfying $\deg(P_k) = k$,

$$\|P_k\|_\infty = \sup_{\xi\in\mathbb{S}} |P_k(\xi)| = 1, \text{ and} \left(\int_{\mathbb{S}} |P_k(\xi)|^2 d\sigma(\xi)\right)^{1/2} \geq \frac{\sqrt{\pi}}{2^n}. \tag{2.5}$$

Note that the homogeneity of P_k implies that $|P_k(z)| \leq |z|^k$ for all $z \in \mathbb{B}$.

Let $\{m_k\}_{k=0}^\infty$ be a sequence of positive integers such that $m_{k+1}/m_k \geq c$ for all $k \geq 0$, where $c > 1$ is a constant. Let

$$f(z) = \sum_{k=0}^\infty b_k P_{m_k}(z) \text{ for } z \in \mathbb{B}. \tag{2.6}$$

Such a function is said to belong to the *Hadamard gap class*. A characterization for a Hadamard gap class function to be in a weighted Bergman space was given in [16]. In the following result, we obtain an estimate for the $\mathcal{N}_p$-norm and A^{-q}-norm of f. These results are higher dimensional versions of [12, Theorem 3.3].

Theorem 2.11 ([9]) *Let f be defined as in* (2.6). *Let p be a positive real number. Then the following statements hold:*

(a) *For* $0 < p \leq n$, *we have* $\|f\|_p^2 \simeq \sum_{k=0}^\infty \frac{|b_k|^2}{m_k^{p+1}}$.

(b) *For any* $q > 0$, *we have* $|f|_q \simeq \sup_k \frac{|b_k|}{m_k^q}$.

(Here, $\|f\|_p$ *and* $|f|_q$ *denote the norm of f in the spaces* $\mathcal{N}_p$ *and* A^{-q}, *respectively).*

Note that Theorem 2.11, for $n = 1$, contains the corresponding results in [12] as particular cases.

Corollary 2.12 *If* $0 < p_1 < p_2 \leq n$, *then we have*

$$\mathcal{N}_{p_1}(\mathbb{B}) \subsetneq \mathcal{N}_{p_2}(\mathbb{B}) \subsetneq A^{-\frac{n+1}{2}}(\mathbb{B}).$$

3 Weighted Composition Operators Between $\mathcal{N}_p(\mathbb{B})$ and $A^{-q}(\mathbb{B})$

In this section, we consider the weighted composition operators acting from $\mathcal{N}_p(\mathbb{B})$ to $A^{-q}(\mathbb{B})$.

3.1 Boundedness

Note that if the weighted composition operator $W_{u,\varphi}$ maps $\mathcal{N}_p(\mathbb{B})$ into $A^{-q}(\mathbb{B})$, then an application of the closed graph theorem shows that $W_{u,\varphi}$ is automatically bounded from $\mathcal{N}_p(\mathbb{B})$ into $A^{-q}(\mathbb{B})$.

Theorem 3.1 ([5]) *Let $\varphi\colon \mathbb{B} \to \mathbb{B}$ be a holomorphic mapping, $u\colon \mathbb{B} \to \mathbb{C}$ a holomorphic mapping and $p, q > 0$. The weighted composition operator $W_{u,\varphi}$: $\mathcal{N}_p(\mathbb{B}) \to A^{-q}(\mathbb{B})$ is bounded if and only if*

$$\sup_{z\in\mathbb{B}} |u(z)| \frac{(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{\frac{n+1}{2}}} < \infty. \tag{3.1}$$

Note that when $n = 1$, we obtain Theorem 3 of [17] as a particular case.

3.2 Compactness

We have the following criteria for compactness of weighted composition operators $W_{u,\varphi}$.

Theorem 3.2 ([5]) *Let $\varphi\colon \mathbb{B} \to \mathbb{B}$ be a holomorphic mapping, $u\colon \mathbb{B} \to \mathbb{C}$ a holomorphic mapping and $p, q > 0$. The weighted composition operator $W_{u,\varphi}$: $\mathcal{N}_p(\mathbb{B}) \to A^{-q}(\mathbb{B})$ is compact if and only if*

$$\lim_{r\to 1^-} \sup_{|\varphi(z)|>r} |u(z)| \frac{(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{\frac{n+1}{2}}} = 0. \tag{3.2}$$

When $n = 1$, Theorem 3.2 contains Corollary 2 in [17] as a particular case.

As it is seen from the theorems, the results in higher dimensions depend essentially on the dimension n of the space $\mathbb{C}^n$.

As a corollary of the obtained results above, we have the following criteria for composition operators.

Corollary 3.3 *Let $\varphi\colon \mathbb{B} \to \mathbb{B}$ be a holomorphic mapping and $p, q > 0$. The composition operator C_φ acting from $\mathcal{N}_p(\mathbb{B}) \to A^{-q}(\mathbb{B})$*

1. is bounded if and only if

$$\sup_{z\in\mathbb{B}} \frac{(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{\frac{n+1}{2}}} < \infty.$$

2. is compact if and only if

$$\lim_{r\to 1^-} \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{\frac{n+1}{2}}} = 0.$$

When $n = 1$, Corollary 3.3 contains Theorems 4.1 and 4.3 in [12] as particular cases.

3.3 Compact Difference

We study the compactness of the difference of two bounded weighted composition operators acting from $\mathcal{N}_p(\mathbb{B})$ into $A^{-q}(\mathbb{B})$.

Let φ_1, φ_2 be two holomorphic self-mappings on $\mathbb{B}$, $u_1, u_2 : \mathbb{B} \to \mathbb{C}$ two holomorphic mappings, and $p, q > 0$. Let further, W_{u_1,φ_1} and W_{u_2,φ_2} be two weighted composition operators acting from $\mathcal{N}_p(\mathbb{B})$ into $A^{-q}(\mathbb{B})$.

Recall that the *pseudo-hyperbolic metric* in the ball is defined as

$$\rho(z, w) = |\Phi_w(z)|, \quad z, w \in \mathbb{B}.$$

It is a true metric (see, e.g., [3]). Also it is easy to verify, in particular, that $\rho(0, w) = |w|$, $\rho(\Phi_w(z), w) = |z|$.

The following lemmas play an important role in the study of compact differences. They also have their own interest.

Lemma 3.4 *For $z, w \in \mathbb{B}$, if $\rho(z, w) \le \frac{1}{2}$, then*

$$\frac{1}{6} \le \frac{1-|z|^2}{1-|w|^2} \le 6.$$

Lemma 3.5 *For $f \in \mathcal{N}_p(\mathbb{B})$ and $z, w \in \mathbb{B}$, we have*

$$|f(z) - f(w)| \le M\, \|f\|_p \max\left\{\frac{1}{(1-|z|^2)^{\frac{n+1}{2}}}, \frac{1}{(1-|w|^2)^{\frac{n+1}{2}}}\right\} \rho(z, w).$$

Here $M = \frac{6^{\frac{n+1}{2}} \cdot 2^{p+n+1}(3+2\sqrt{3})\sqrt{n}}{3^{p/2}}$.

Inspiring by the pseudo-hyperbolic metric in the unit ball, for two holomorphic mappings $\varphi, \psi \colon \mathbb{B} \to \mathbb{B}$, we define

$$\rho_{\varphi,\psi}(z) = \left|\Phi_{\varphi(z)}(\psi(z))\right|, \quad z \in \mathbb{B}.$$

Evidently, $\rho_{\varphi,\psi} = \rho_{\psi,\varphi}$.

Now we are ready to formulate the following result.

Theorem 3.6 ([6]) *Let* $\varphi_1, \varphi_2 : \mathbb{B} \to \mathbb{B}$ *be two holomorphic mappings,* $u_1, u_2 : \mathbb{B} \to \mathbb{C}$ *two holomorphic mappings and* $p, q > 0$*. Let further,* W_{u_1,φ_1} *and* W_{u_2,φ_2} *be two weighted composition operators acting from* $\mathcal{N}_p(\mathbb{B})$ *into* $A^{-q}(\mathbb{B})$*. Then* $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ *is compact if and only if the following conditions are satisfied:*

(i)

$$\lim_{r\to 1^-} \sup_{|\varphi_k(z)|>r} \left\{ \frac{|u_k(z)|(1-|z|^2)^q}{(1-|\varphi_k(z)|^2)^{\frac{n+1}{2}}} \rho_{\varphi_1,\varphi_2}(z) \right\} = 0 \quad (k = 1, 2);$$

(ii)

$$\lim_{r\to 1^-} \sup_{\min\{|\varphi_1(z)|,|\varphi_2(z)|\}>r} \Bigg[|u_1(z) - u_2(z)| \times \min\left\{ \frac{(1-|z|^2)^q}{(1-|\varphi_1(z)|^2)^{\frac{n+1}{2}}}, \frac{(1-|z|^2)^q}{(1-|\varphi_2(z)|^2)^{\frac{n+1}{2}}} \right\} \Bigg] = 0.$$

3.4 Essential Norm

Let us denote by $\mathcal{K} := \mathcal{K}(\mathcal{N}_p, A^{-q})$ the set of all compact operators acting from $\mathcal{N}_p$ into A^{-q}. Then the essential norm of $W_{u,\varphi}$ is defined as follows:

$$\|W_{u,\varphi}\|_e = \inf_{K\in\mathcal{K}} \left\{ \|W_{u,\varphi} - K\| \right\}.$$

Obviously, the essential norm of a compact operator is zero.

Upper bound of the essential norm We need some auxiliary results.

Lemma 3.7 *Suppose* φ *is a self-mapping of* $\mathbb{B}$ *such that* $\|\varphi\|_\infty < 1$ *and* $u \in \mathcal{N}_p$*. Then the weighted composition operator* $W_{u,\varphi} : \mathcal{N}_p \longrightarrow \mathcal{N}_p$ *is compact.*

In the case z and w are multiple of each other, Lemma 3.5 can be simplified as in the following result.

Lemma 3.8 *Let* f *be in* $\mathcal{N}_p$*. For any* $a \in \mathbb{B}$ *and any* $\kappa \in (0, 1)$*, we have*

$$|f(a) - f(\kappa a)| \le \frac{M(1-\kappa)|a|}{1-\kappa|a|^2} \cdot \frac{\|f\|_p}{(1-|a|^2)^{\frac{n+1}{2}}} \le \frac{M\|f\|_p}{(1-|a|^2)^{\frac{n+1}{2}}}. \tag{3.3}$$

Consequently, for any $0 < r < 1$*, we have*

$$\sup_{|a|\le r} |f(a) - f(\kappa a)| \le \frac{Mr(1-\kappa)\|f\|_p}{(1-r^2)^{\frac{n+3}{2}}}. \tag{3.4}$$

Here M is the constant from Lemma 3.5.

Now we can have an estimate for the upper bound of the essential norm of $W_{u,\varphi}$.

Theorem 3.9 ([7]) *Let p and q be two positive numbers. Let $\varphi\colon \mathbb{B} \to \mathbb{B}$ be a holomorphic self-mapping and $u : \mathbb{B} \to \mathbb{C}$ be a holomorphic function. Suppose that $W_{u,\varphi}$ is a bounded operator acting from $\mathcal{N}_p$ to A^{-q}. Then*

$$\|W_{u,\varphi}\|_e \le M \lim_{r\to 1^-} \sup_{|\varphi(z)|>r} \frac{|u(z)|(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{\frac{n+1}{2}}},$$

where M is the constant from Lemma 3.5.

Lower bound of the essential norm We now discuss the estimation for the lower bound of $\|W_{u,\varphi}\|_e$. We will make use of weakly convergent sequences in the Bergman space A^2. The following lemma plays an important role.

Lemma 3.10 *Suppose $\{f_m\}_{m\ge 1} \subset A^2$ is a sequence that converges weakly to zero in A^2. Then $\{f_m\}_{m\ge 1}$ converges weakly to zero in $\mathcal{N}_p$ as well.*

Corollary 3.11 *Let $\{w_m\}_{m\in\mathbb{N}} \subset \mathbb{B}$ and $|w_m| \to 1$ as $m \to \infty$, then $\{k_{w_m}\}$ converges weakly to zero in $\mathcal{N}_p$.*

Theorem 3.12 ([7]) *Let p and q be two positive numbers. Let $\varphi\colon \mathbb{B} \to \mathbb{B}$ be a holomorphic self-mapping and $u : \mathbb{B} \to \mathbb{C}$ be a holomorphic function. Suppose that $W_{u,\varphi}$ is a bounded operator acting from $\mathcal{N}_p$ to A^{-q}. Then*

$$\|W_{u,\varphi}\|_e \ge \lim_{r\to 1^-} \sup_{|\varphi(z)|>r} \frac{|u(z)|(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{\frac{n+1}{2}}}.$$

In conclusion, combining Theorems 3.9 and 3.12, we obtain a full description of the essential norm of $W_{u,\varphi}$.

Theorem 3.13 *Let p and q be two positive numbers. Let $\varphi\colon \mathbb{B} \to \mathbb{B}$ be a holomorphic self-mapping and $u : \mathbb{B} \to \mathbb{C}$ be a holomorphic function. Suppose that $W_{u,\varphi}$ is a bounded operator acting from $\mathcal{N}_p$ to A^{-q}. Then*

$$\|W_{u,\varphi}\|_e \simeq \lim_{r\to 1^-} \sup_{|\varphi(z)|>r} \frac{|u(z)|(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{\frac{n+1}{2}}}.$$

Theorem 3.13 provides us a characterization of compact weighted composition operators from $\mathcal{N}_p$ to A^{-q} as in Theorem 3.2.

Corollary 3.14 *Suppose that $W_{u,\varphi}$ is a bounded operator acting from $\mathcal{N}_p$ to A^{-q} as in Theorem 3.13. Then $W_{u,\varphi}$ is compact if and only if*

$$\lim_{r\to 1^-} \sup_{|\varphi(z)|>r} \frac{|u(z)|(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{\frac{n+1}{2}}} = 0.$$

4 Composition Operators Between $\mathcal{N}_p$ and $\mathcal{N}_q$

One of the most interesting (and quite difficult) topics is to study properties of (weighted) composition operators between $\mathcal{N}_p$-spaces. In this section, we present the results for composition operators C_φ acting from $\mathcal{N}_p$ into $\mathcal{N}_q$. Note that the one dimensional case was considered by Palmberg in [12, Theorem 4.2]. A sufficient condition and a necessary condition for C_φ to be bounded on $\mathcal{N}_p$ of the unit disk were given there. These conditions involve the generalized Nevanlinna counting function introduced by Shapiro.

4.1 Boundedness of Composition Operators from $\mathcal{N}_p$ to $\mathcal{N}_q$

We begin with the case φ is a univalent holomorphic self-mapping of the unit ball $\mathbb{B}$.

Theorem 4.1 ([8]) *Let φ be a univalent holomorphic self-mapping of $\mathbb{B}$ and p be any positive real number. Suppose that*

$$\delta = \inf\{|J\varphi(z)| : z \in \mathbb{B}\} > 0,$$

where $J\varphi$ the complex Jacobian of φ. Then $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_p$ is a bounded operator with $\|C_\varphi\| \leq \delta^{-1}$ for all $p > 0$.

Corollary 4.2 *Suppose $A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is an invertible linear operator and b is a vector in $\mathbb{C}^n$ such that $\varphi(z) = Az + b$ is a self-mapping of $\mathbb{B}$. Then $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_p$ is bounded for all $p > 0$.*

For general self-mappings φ of the unit ball, we provide a necessary condition and a sufficient condition for C_φ to be bounded from $\mathcal{N}_p$ into $\mathcal{N}_q$. We make use of a sequence of homogeneous polynomials $\{P_k\}$ in (2.5). We also need the inequality

$$1 + \sum_{k=0}^{\infty} 2^{k\gamma} x^{2^k} \gtrsim (1-x)^{-\gamma} \text{ for } 0 \leq x < 1, \tag{4.1}$$

where γ is a positive number.

We have the following necessary condition and sufficient condition for the boundedness of $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$.

Theorem 4.3 ([8]) *Let p and q be two positive numbers and φ a holomorphic self-mapping of $\mathbb{B}$. If*

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \frac{(1-|\Phi_a(z)|^2)^q}{(1-|\varphi(z)|^2)^{n+1}}\, dV(z) < \infty, \tag{4.2}$$

then $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ is bounded.

Conversely, if $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ is bounded, then for any $0 < \varepsilon \le p+1$,

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \frac{(1-|\Phi_a(z)|^2)^q}{(1-|\varphi(z)|^2)^{p+1-\varepsilon}}\, dV(z) < \infty. \tag{4.3}$$

An application of Theorem 4.3 immediately gives the following result.

Corollary 4.4 *Let φ be a holomorphic self-mapping of $\mathbb{B}$ such that $\|\varphi\|_\infty < 1$. Then $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ is bounded for all $p, q > 0$.*

We note that in fact the operator C_φ in the corollary is compact.

4.2 Compactness of Composition Operators from $\mathcal{N}_p$ to $\mathcal{N}_q$

First we have the following equivalent conditions for the compactness of C_φ.

Proposition 4.5 *Let p, q be positive numbers and φ be a holomorphic self-mapping of $\mathbb{B}$ such that the composition operator $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ is bounded. Then the following statements are equivalent.*

(a) C_φ *is a compact operator from* $\mathcal{N}_p$ *into* $\mathcal{N}_q$.

(b) $\lim_{t\to 1^-} \left(\sup_{a\in\mathbb{B}, f\in\mathbb{B}_{\mathcal{N}_p}} \left[\int_{|z|>t} |f(\varphi(z))|^2 (1-|\Phi_a(z)|^2)^q\, dV(z) \right] \right) = 0.$

(c) $\lim_{t\to 1^-} \left(\sup_{a\in\mathbb{B}, f\in\mathbb{B}_{\mathcal{N}_p}} \left[\int_{|\varphi(z)|>t} |f(\varphi(z))|^2 (1-|\Phi_a(z)|^2)^q\, dV(z) \right] \right) = 0.$

For a particular case of φ, we can also have a compactness of C_φ.

Proposition 4.6 *Let φ be a holomorphic self-mapping of $\mathbb{B}$ such that $\|\varphi\|_\infty < 1$. Then $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ is compact for all $p, q > 0$.*

Now we can formulate a necessary condition and a sufficient condition for the compactness of C_φ. These conditions are quite useful in applications.

Theorem 4.7 ([8]) *Let $p, q \in (0, n]$ be two positive numbers and φ a holomorphic self-mapping of $\mathbb{B}$ such that $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ is bounded. If*

$$\lim_{t\to 1^-} \sup_{a\in\mathbb{B}} \int_{|\varphi(z)|>t} \frac{(1-|\Phi_a(z)|^2)^q}{(1-|\varphi(z)|^2)^{n+1}}\, dV(z) = 0, \tag{4.4}$$

then C_φ is compact.

Conversely, if $C_\varphi : \mathcal{N}_p \to \mathcal{N}_q$ is compact, then for $0 < \varepsilon \le p + 1$,

$$\lim_{t\to 1^-} \sup_{a\in\mathbb{B}} \int_{|\varphi(z)|>t} \frac{(1-|\Phi_a(z)|^2)^q}{(1-|\varphi(z)|^2)^{p+1-\varepsilon}}\, dV(z) = 0. \tag{4.5}$$

As applications of Theorems 4.3 and 4.7, we have the following result.

Corollary 4.8 *Suppose* $k > 0, p, q, r \in (0, n), r \ge q, \varepsilon \in (0, q+1)$ *and* φ *is a holomorphic self-mapping of* $\mathbb{B}$. *The following statements hold*

1. *If* $C_\varphi : A^{-k(q+1-\varepsilon)} \longrightarrow A^{-k(n+1)}$ *is a bounded operator, then* $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_r$ *is a bounded operator;*
2. *If* $C_\varphi : A^{-k(q+1-\varepsilon)} \longrightarrow A^{-k(n+1)}$ *is a compact operator, then* $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_r$ *is a compact operator.*

5 Some Open Problems

1. In Sect. 2, the differences of $\mathcal{N}_p$ spaces for small values of p are obtained by using the Hadamard gap class functions. In connection with this, it is interesting to develop this topic to a more general class of functions and a more general class of spaces as well.

Problem 5.1 Study Hadamard gap series for classes which are more general than $\mathcal{N}_p$-spaces.

We note that some development on $\mathcal{N}_p$-type functions with Hadamard gaps and also Hadamard gap series in weighted-type spaces on the unit ball is presented in [10, 11].

2. In Sect. 3, several properties of weighted composition operators $W_{u,\varphi}$ acting from $\mathcal{N}_p(\mathbb{B})$ to $A^{-q}(\mathbb{B})$ have been investigated. It is natural to ask for about weighted composition operators acting $\mathcal{N}_p(\mathbb{B})$ to other function spaces, which are subspaces of the space $\mathcal{O}(\mathbb{B})$ of holomorphic functions in the unit ball $\mathbb{B}$.

Problem 5.2 Let $\mathcal{H}$ be a subspace of the space $\mathcal{O}(\mathbb{B})$. Study properties of weighted composition operators $W_{u,\varphi}$ acting between the $\mathcal{N}_p$-space and $\mathcal{H}$.

3. In Sect. 4, composition operators C_φ acting between $\mathcal{N}_p$-spaces have been studied. However, there still is a "gap" between necessary and sufficient conditions. Also weighted composition operators have not been investigated. So we have the following problems.

Problem 5.3 What are "sharp" criteria for $C_\varphi : \mathcal{N}_p \to \mathcal{N}_q$ to be bounded, compact, etc.?

Problem 5.4 Investigate weighted composition operators $W_{u,\varphi}$ acting between $\mathcal{N}_p$-spaces.

We note that Problems 5.3–5.4 for even Q_p-spaces are still unsolved.

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Einstein Metrics on Strictly Pseudoconvex Domains from the Viewpoint of Bulk-Boundary Correspondence

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Abstract We present an overview of the correspondence between asymptotically complex hyperbolic Einstein metrics and CR structures on the boundary at infinity, which is the complex version of that between Poincaré-Einstein metrics and conformal structures, with the main focus on existence results. We also propose several open problems.

Keywords Einstein equation · Strictly pseudoconvex domains · CR structures

1 Introduction

This is a survey discussing some aspects of the correspondence proposed by Biquard [5, 6] (see also [7]) between Einstein metrics on the interior of a manifold-with-boundary and strictly pseudoconvex CR structures on the boundary. Here, by "CR structures," we shall mean not only integrable almost CR structures but also certain nonintegrable ones, which will be described later in this section.

We may regard this correspondence as a differential-geometric interpretation and generalization of the classical complex-analytic correspondence between strictly pseudoconvex domains and CR manifolds arising as their boundaries. Let us start with describing this viewpoint.

Fefferman's mapping theorem [19] states that any biholomorphism $\Omega_1 \to \Omega_2$ between smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^n$ ($n \geq 2$) extends to a diffeomorphism between the closures of the domains. This extension automatically restricts to a CR-diffeomorphism from $\partial\Omega_1$ to $\partial\Omega_2$. Conversely, any CR-diffeomorphism $\partial\Omega_1 \to \partial\Omega_2$ necessarily extends to a biholomorphism $\Omega_1 \to \Omega_2$

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by the Bochner–Hartogs theorem [11]. Such phenomena for domains in $\mathbb{C}^n$ generalize to those in Stein manifolds by the works of Bedford et al. [3] and Kohn and Rossi [35], and as a consequence, if $\mathcal{D}$ denotes the set of all smoothly bounded strictly pseudoconvex domains in Stein manifolds, then biholomorphism classes of domains in $\mathcal{D}$ and CR-diffeomorphism classes of the boundaries of domains in $\mathcal{D}$ are in a one-to-one correspondence:

$$\mathcal{D}/\sim_{\text{bihol}} \cong \{\partial\Omega \mid \Omega \in \mathcal{D}\} / \sim_{\text{CR-diffeo}} . \tag{1.1}$$

The classical approach toward the correspondence (1.1) from differential geometry uses the Bergman metric. However, here we would rather make use of the Einstein metric of Cheng and Yau [17] in the following theorem, because the Einstein equation has an advantage that it makes sense without complex structures (recall that we are going to include some nonintegrable CR structures on the boundary into our consideration).

Theorem 1.1 (Cheng and Yau [17]) *Let Ω be a smoothly bounded strictly pseudoconvex domain in a Stein manifold of dimension $n \geq 2$. Then there exists a complete Kähler-Einstein metric with negative Einstein constant on Ω, which is unique up to homothety.*

The metric is determined by the complex structure of Ω. On the other hand, the asymptotic behavior of the metric at the boundary is mostly determined by the local CR geometry of $\partial\Omega$, as discussed by Fefferman [24] and Graham [26]. In this sense, the Cheng–Yau metric is a link that realizes (to some extent) the correspondence (1.1). We will introduce the notion of "asymptotically complex hyperbolic Einstein metrics" in the next section, for which the Cheng–Yau metrics serve as model examples.

Now we illustrate the class of almost CR structures that we consider. Let M be a (connected) differentiable manifold of dimension $2n-1$, where $n \geq 2$, and H a contact distribution over M. We say, in this article, that an almost CR structure J on H (i.e., $J \in \Gamma(\mathrm{End}(H))$ satisfying $J^2 = -\mathrm{id}$) is *compatible* when the *Levi form*

$$h_\theta(X, Y) := d\theta(X, JY), \qquad X,\ Y \in H$$

is symmetric in X and Y for some (hence any) contact 1-form θ annihilating the distribution H. (If $H^\perp \subset T^*M$ is oriented, then H has a natural $CSp(n-1)$-structure, and fixing a compatible almost CR structure amounts to a reduction of the structure group of H from $CSp(n-1)$ to $CU(p, n-1-p)$ for some p. The term "compatible" refers to the compatibility of J to the $CSp(n-1)$-structure of H in this setting.) Integrable almost CR structures are always compatible as is well known, but there are more compatible structures (except for the three-dimensional case, in which any almost CR structure is automatically integrable). It can be easily checked that J is compatible if and only if

$$[\Gamma(H_J^{1,0}), \Gamma(H_J^{1,0})] \subset \Gamma(H_\mathbb{C}), \tag{1.2}$$

where $H_{\mathbb{C}}$ is the complexification of H and $H_{\mathbb{C}} = H_J^{1,0} \oplus H_J^{0,1}$ is the eigenbundle decomposition with respect to J (note that (1.2) is not a trivial condition since $H_{\mathbb{C}} \subsetneq T_{\mathbb{C}}\partial\Omega$). Because of (1.2), compatible almost CR structures are also called *partially integrable* in the literature (e.g., [13, 14, 43–45]).

The usual notion of strict pseudoconvexity naturally extends to compatible almost CR structures. Namely, a compatible almost CR structure J is said to be *strictly pseudoconvex* if the Levi form h_θ has definite signature. In the following we shall always assume the strict pseudoconvexity, and a contact form θ is always taken so that h_θ is positive definite.

Each asymptotically complex hyperbolic (ACH for short) Einstein metric is "associated" to, or "fills inside" of, a manifold equipped with a strictly pseudoconvex compatible almost CR structure, as the Cheng–Yau metric does. Relationships between Einstein metrics and geometric structures on the boundary have been more actively studied in the setting of Poincaré-Einstein (or AH-Einstein) metrics and conformal structures, partly because of physical interests. Furthermore, the cases of Poincaré-Einstein and ACH-Einstein metrics are generalized to a broader perspective involving "asymptotically symmetric Einstein metrics" and "parabolic geometries," which is illustrated in [5, 6, 9]. The term "bulk-boundary correspondence" in the title of this article is intended to indicate this very general correspondence, most part of which is yet to be unveiled.

2 Asymptotically Complex Hyperbolic Einstein Metrics

In order to motivate our definition of ACH metrics, let us first observe the fact that the leading part of the asymptotic behavior of the Cheng–Yau metric g at the boundary can be described in terms of the CR structure of $\partial\Omega$.

From the proof of its existence, it is known that g is expressed (after a normalization) as

$$g_{i\overline{j}} = \partial_i\partial_{\overline{j}} \log\frac{1}{\varphi} = \frac{(\partial_i\varphi)(\partial_{\overline{j}}\varphi)}{\varphi^2} - \frac{\partial_i\partial_{\overline{j}}\varphi}{\varphi}, \tag{2.1}$$

where $\varphi \in C^\infty(\Omega) \cap C^{n+1,\alpha}(\overline{\Omega})$ is some defining function of Ω, i.e., $\Omega = \{\varphi > 0\}$ and $d\varphi$ is nowhere vanishing on $\partial\Omega$, where $\alpha \in (0, 1)$ is arbitrary (to be precise, [38] is responsible for this optimal boundary regularity). Because of (2.1), one can take a diffeomorphism of the form

$$\Phi = (\pi, \rho) \colon \mathcal{U} \to \partial\Omega \times [0, \varepsilon), \tag{2.2}$$

where $\mathcal{U}$ is an open neighborhood of $\partial\Omega$ in $\overline{\Omega}$, such that $\pi : \mathcal{U} \to \partial\Omega$ restricts to the identity map on $\partial\Omega$ and the Cheng–Yau metric g satisfies

$$g \sim \Phi^* g_\theta, \qquad g_\theta = \frac{1}{2}\left(\frac{d\rho^2}{\rho^2} + \frac{\theta^2}{\rho^2} + \frac{h_\theta}{\rho}\right) \tag{2.3}$$

as $\rho \to 0$, where θ is some contact form on $\partial\Omega$ annihilating the natural contact distribution and h_θ is the associated Levi form. The meaning of (2.3) can be understood as, for example, that $|g - \Phi^* g_\theta|_{\Phi^* g_\theta}$ uniformly tends to 0 as $\rho \to 0$. More is true actually: it follows from the asymptotic expansion established in [38] that the C^k norm of $\rho^{-1}(g - \Phi^* g_\theta)$, defined geometrically by $\Phi^* g_\theta$, is finite for any $k \geq 0$. We note that there exists, for any choice of θ, a diffeomorphism Φ with respect to which (2.3) holds—there is no preferred choice of θ. We also remark that in the literature the model metric g_θ is sometimes expressed as

$$g_\theta = \frac{1}{2}\left(4\frac{dx^2}{x^2} + \frac{\theta^2}{x^4} + \frac{h_\theta}{x^2}\right)$$

by introducing a new coordinate $x = \sqrt{\rho}$.

Observing the asymptotic behavior (2.3) of the Cheng–Yau metric, we define as follows. Metrics of this type are firstly considered by Epstein et al. [18], in which the meromorphic continuation of the resolvent of the Laplacian (on functions) is studied.

Definition 2.1 Let $\overline{X}$ be a compact smooth manifold-with-boundary with $\dim_{\mathbb{R}} \overline{X} = 2n$, where $n \geq 2$, and X its interior. A Riemannian metric g defined on X is called an *asymptotically complex hyperbolic metric* (or *ACH metric*) when there exists a diffeomorphism Φ like (2.2) such that g satisfies (2.3) with respect to some contact distribution H over ∂X, a strictly pseudoconvex compatible almost CR structure J on H, and a contact form θ in the sense that

$$g - \Phi^* g_\theta \in C^{2,\alpha}_\delta(X, S^2T^*X)$$

for some $\delta > 0$ and arbitrary $\alpha \in (0, 1)$. Here $C^{k,\alpha}_\delta(X, S^2T^*X)$ denotes the space of C^k symmetric 2-tensors σ on X such that $\rho^{-\delta/2}\sigma$ has finite $C^{k,\alpha}$ norm with respect to $\Phi^* g_\theta$ (this space depends on Φ and H, but not on J). The almost CR structure J, or the triple $(\partial X, H, J)$, is called the *conformal infinity* of g.

Our fundamental questions on ACH metrics are the following. For a given $\overline{X}$, does there exist an Einstein ACH metric on X with prescribed conformal infinity? If there does, how many are there essentially (i.e., up to the action of diffeomorphisms)?

Let us focus on the existence problem for the moment. The Cheng–Yau theorem (Theorem 1.1) provides many examples of Einstein ACH metrics, but for general infinity, only perturbative results are known. Such results are given by Roth [46], Biquard [6], and the present author [44], which we shall now discuss.

In [6, 46], general perturbation theory is established. Roth considered deformations of the Cheng–Yau metrics, while Biquard worked on those of arbitrary Einstein ACH metrics. It was shown that, in the both works, that if the given Einstein metric g has negative sectional curvature everywhere, then compatible almost CR structures nearby the conformal infinity of g are also “fillable” with Einstein metrics. More precisely, the following theorem holds.

Theorem 2.2 (Biquard [6]) *Let g be an Einstein ACH metric on X, whose conformal infinity is denoted by $(\partial X, H, J_0)$. Suppose that g has negative sectional curvature. Then, if $\mathcal{J}$ is a sufficiently small $C^{2,\alpha}$ neighborhood of J_0 in the space of compatible almost CR structures on H, any $J \in \mathcal{J}$ is the conformal infinity of some Einstein ACH metric on X.*

In particular, Theorem 2.2 is applicable to the complex hyperbolic metric on the unit ball B^n in $\mathbb{C}^n$ (note also that it is the Cheng–Yau metric of B^n). Leaving the contact distribution H unchanged is not an additional restriction, because contact structures of closed manifolds are rigid.

Here is a very brief sketch of the construction (which is discussed more in the next section). We first assign to each $J \in \mathcal{J}$ an approximate ACH solution g_J of the Einstein equation which satisfies

$$\mathrm{Ric}(g_J) + (n+1)g_J \in C^{0,\alpha}_\delta(X, S^2T^*X)$$

for some $\delta > 0$ (δ must be independent of J). We can do it in such a way that g_J is smooth in J and g_{J_0} equals the original metric g. Then we use functional analysis to show that, making $\mathcal{J}$ smaller if necessary, for each $J \in \mathcal{J}$ one can find $\sigma \in C^{2,\alpha}_\delta(X, S^2T^*X)$ for which $g'_J = g_J + \sigma$ satisfies $\mathrm{Ric}(g'_J) = -(n+1)g'_J$. Since the modification term σ belongs to $C^{2,\alpha}_\delta$, g'_J is still an ACH metric whose conformal infinity is J.

The negative curvature assumption is an easy sufficient condition that makes this plan work. However, in practice, it is a nontrivial matter to check whether this condition is satisfied for a given g. The following theorem shows that it is unnecessary for the Cheng–Yau metrics, (at least) except for the two-dimensional case.

Theorem 2.3 (Matsumoto [44]) *Let Ω be a smoothly bounded strictly pseudoconvex domain in a Stein manifold of dimension $n \geq 3$, and $\mathcal{J}$ a sufficiently small $C^{2,\alpha}$ neighborhood of the induced CR structure J_0 in the space of compatible almost CR structures on the natural contact distribution over $\partial\Omega$. Then for each $J \in \mathcal{J}$, there is an Einstein ACH metric on Ω with conformal infinity J.*

There are such perturbation theorems also for Poincaré-Einstein metrics. The possibility of deforming the real hyperbolic metric is shown by Graham and Lee [28], and in [6] it was pointed out that the negative curvature assumption is sufficient. Lee [37] showed that a weaker curvature assumption suffices when the boundary conformal structure has nonnegative Yamabe constant.

In [6], the local uniqueness of Einstein ACH metrics is also discussed. By shrinking $\mathcal{J}$ if necessary, the Einstein metric g'_J constructed for each $J \in \mathcal{J}$ is the unique Einstein metric modulo diffeomorphism action in a neighborhood of g'_J in $g'_J + C^{2,\alpha}_\delta(X, S^2T^*X)$ (for any $\delta > 0$). Probably the following refined question can be asked: is there a neighborhood of g in the unweighted Hölder space $C^{2,\alpha}(X, S^2T^*X)$ in which there is only one Einstein metric for each conformal infinity? To the author's knowledge, this is not yet settled so far.

3 Ideas of the Proofs of Theorems 2.2 and 2.3

The two theorems in the previous section are reduced to the vanishing of the L^2 kernel of the "linearized gauged Einstein operator" acting on symmetric 2-tensors, which is

$$P = \frac{1}{2}(\nabla^*\nabla - 2\mathring{R}), \tag{3.1}$$

where g is the given Einstein ACH metric and $\mathring{R}$ denotes the pointwise linear action of the curvature tensor of g. Let us see how this reduction is carried out.

It is natural to study the linearization of the Einstein equation in order to deform Einstein metrics. However, if we consider the Einstein equation itself, we encounter a difficulty that originates from the diffeomorphism invariance of the equation. One usually introduces an additional term to break this "gauge invariance." Here we set, following [6],

$$\mathcal{E}_g(g') := \mathrm{Ric}(g') + (n+1)g' + \delta^*_{g'}\mathcal{B}_g(g'), \qquad \mathcal{B}_g(g') := \delta_g g' + \frac{1}{2} d\,\mathrm{tr}_g\, g'.$$

As long as we consider g' in a small neighborhood of g in $g + C^{2,\alpha}_\delta(X, S^2T^*X)$, any solution of $\mathcal{E}_g(g') = 0$ automatically satisfies $\mathcal{B}_g(g') = 0$, and hence it becomes an Einstein metric.

We apply the implicit function theorem to the mapping

$$\mathcal{J} \times C^{2,\alpha}_\delta(X, S^2T^*X) \to C^{0,\alpha}_\delta(X, S^2T^*X), \qquad (J, \sigma) \mapsto \mathcal{E}_{g_J}(g_J + \sigma)$$

at $(J_0, 0)$, where g_J is a family of approximate solutions as described in the previous section. If the linearization of $\sigma \mapsto \mathcal{E}_g(g + \sigma)$ at $\sigma = 0$, which is the operator (3.1), is invertible, then for each $J \in \mathcal{J}$ sufficiently close to J_0 there exists $\sigma \in C^{2,\alpha}_\delta(X, S^2T^*X)$ satisfying $\mathcal{E}_{g_J}(g_J + \sigma) = 0$. Thus it suffices to prove that (3.1) is an isomorphism as the mapping

$$P \colon C^{2,\alpha}_\delta(X, S^2T^*X) \to C^{0,\alpha}_\delta(X, S^2T^*X) \tag{3.2}$$

for sufficiently small $\delta > 0$.

An essential part is to show that (3.2) is an isomorphism for small $\delta > 0$ if and only if

$$P \colon H^2(X, S^2T^*X) \to L^2(X, S^2T^*X) \tag{3.3}$$

is isomorphic, where $H^2(X, S^2T^*X)$ denotes the L^2 Sobolev space of order 2, which is actually the domain of P seen as an unbounded operator on $L^2(X, S^2T^*X)$. It is easy to show that P is a self-adjoint unbounded operator, and hence (3.3) is isomorphic if the L^2 kernel vanishes. The equivalence of (3.2) and (3.3) being isomorphic follows by a certain parametrix construction, which makes good use of the geometry of ACH metrics, explained in [6]. The exposition on the Poincaré-Einstein case in [37] is also useful.

Consequently, it suffices to show that the L^2 kernel of P is trivial. When g has negative sectional curvature, the vanishing can be proved by the following Bochner technique. Note that any element of the kernel must be trace-free, because if $\sigma = ug$ for some $u \in C^\infty(X)$ then $P\sigma = (\nabla^*\nabla u + 2(n+1)u)g$, and the operator $\nabla^*\nabla + 2(n+1)$ acting on functions has trivial L^2 kernel. Now if a general symmetric 2-tensor σ is regarded as a 1-form with values in T^*X, then P satisfies the Weitzenböck formula below given in terms of the exterior covariant differentiation D (see [4, 12.69]):

$$2P\sigma = (DD^* + D^*D)\sigma - \mathring{R}\sigma + (n+1)\sigma.$$

Moreover, there is also a pointwise estimate valid for trace-free σ [4, 12.71] that

$$\langle \mathring{R}\sigma, \sigma\rangle \leq (n + 1 + 2(n-1)K_{\max})|\sigma|^2,$$

where $K_{\max}$ is the maximum of the sectional curvatures at a point. Since the assumption implies that the sectional curvature is bounded from above by a negative constant (by virtue of the asymptotic complex hyperbolicity), one can deduce that the L^2 kernel of P is trivial in this case.

When g is the Cheng–Yau metric of a smoothly bounded strictly pseudoconvex domain Ω in a Stein manifold, we argue as follows based on Koiso's observations [36] (see also Besse [4, Sect. 12.J]). The Kählerness of g implies that P respects the type decomposition of σ into hermitian and anti-hermitian parts, and due to the Einstein condition, P on each type becomes a familiar operator. If σ is hermitian, then one may regard it as a $(1,1)$-form and we have

$$2P\sigma = (dd^* + d^*d)\sigma + 2(n+1)\sigma.$$

This shows the vanishing of the hermitian part of the L^2 kernel. On anti-hermitian symmetric 2-tensors, by regarding them as $(0,1)$-forms with values in the holomorphic tangent bundle $T^{1,0}\Omega$, we obtain

$$2P\sigma = (\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial})\sigma.$$

Therefore it suffices to show that there are no nontrivial L^2 harmonic $(0,1)$-forms with values in $T^{1,0}\Omega$. This allows one to restate the problem in terms of cohomology: the isomorphicity of (3.2) for small $\delta > 0$ follows if the L^2 Dolbeault cohomology $H^{0,1}_{(2)}(\Omega, T^{1,0}\Omega)$ vanishes.

It is a consequence of classical theory on Stein manifolds that the compactly supported cohomology $H^{0,1}_c(\Omega, T^{1,0}\Omega)$ vanishes. Moreover, it can be observed that the following sequence involving the inductive limit $\varinjlim_K H^{0,1}_{(2)}(\Omega \setminus K; T^{1,0}(\Omega \setminus K))$, where K runs through all compact subsets of Ω, is exact:

$$\cdots \to H_c^{0,1}(\Omega; T^{1,0}\Omega) \to H_{(2)}^{0,1}(\Omega; T^{1,0}\Omega) \\ \to \varinjlim_K H_{(2)}^{0,1}(\Omega \setminus K; T^{1,0}(\Omega \setminus K)) \to H_c^{0,2}(\Omega; T^{1,0}\Omega) \to \cdots .$$

Therefore, $H_{(2)}^{0,1}(\Omega, T^{1,0}\Omega) = 0$ follows if

$$\varinjlim_K H_{(2)}^{0,1}(\Omega \setminus K, T^{1,0}(\Omega \setminus K)) = 0 \tag{3.4}$$

holds. We show (3.4) by proving

$$H_{(2)}^{0,1}(\mathcal{U}, T^{1,0}\mathcal{U}) = 0, \tag{3.5}$$

where $\mathcal{U}$ is a sufficiently narrow collar neighborhood of $\partial\Omega$ intersected with Ω. The vanishing (3.5) is attacked by the usual technique of L^2 estimate, but one needs to be careful because boundary integrals along the inner boundary of $\mathcal{U}$, which is strictly pseudoconcave, comes into play. The L^2 estimate so obtained is actually sufficient to prove (3.5) only when $n \geq 4$. When $n = 3$, one needs to work with a weighted L^2 cohomology instead.

4 Problems

An obvious problem related to Theorem 2.3 is to clarify what happens in the two-dimensional case. The author expects (perhaps optimistically) that finally one can simply remove the assumption $n \geq 3$ from the theorem, for it is at least true for the unit ball by Theorem 2.2, and there seems to be no reason to expect that it fails for general strictly pseudoconvex domains.

A more challenging issue about existence is how we can construct Einstein ACH metrics for compatible almost CR structures which are far from those that are known to be "fillable." The corresponding problem in the Poincaré-Einstein setting is also a long-standing one. One should be aware that there is a recent *nonexistence* result by Gursky and Han [29] in the latter setting.

Turning to the uniqueness of Einstein fillings for a given conformal infinity, in the Poincaré-Einstein case, an example of Hawking and Page [30] exhibits that it fails in general (see Anderson [1] for further explanation). A similar nonuniqueness example for ACH-Einstein metrics will be of great interest, as well as uniqueness results under some assumption. There is also a room for further investigations about local uniqueness as mentioned at the end of Sect. 2.

A typical application of Poincaré-Einstein metrics is the construction of conformally invariant objects on the boundary, and there is a similar story for ACH-Einstein metrics. For this purpose the determination of the asymptotic behavior of the metric in terms of the boundary geometry is important, and its formal aspects are studied by Fefferman and Graham [21, 22] for the Poincaré-Einstein metrics, by Fefferman [24] and Graham [26] for the Cheng–Yau metrics (as mentioned in Sect. 1), and by Biquard and Herzlich [8] and the author [43] (see also [42]) for general ACH-Einstein metrics. There is a tremendous amount of literature regarding constructions of conformal invariants based on [21, 22], while in the CR case such constructions are discussed in, e.g., [2, 8, 12, 15, 16, 20, 23, 25–27, 31–34, 39–41, 45, 47–49]. Further developments along this line are anticipated. It would be also very interesting if there is some invariant construction that needs global considerations on Einstein metrics in an essential way.

Now let us take notice of the fact that the Cheng–Yau metrics come with complex structures with respect to which they are Kähler. As a problem without any counterpart in the Poincaré-Einstein setting, it may be interesting to look for a canonical way to determine a good almost complex structure on a manifold equipped with an ACH-Einstein metric. That is to say, Riemannian metrics may not be the "best" filling geometric structure inside CR manifolds. It seems to the author that this idea is backed up by the fact that the Einstein deformation problem is recast in the proof of Theorem 2.3 in terms of harmonic (0, 1)-forms with values in the holomorphic tangent bundle.

Finally, the author would like to remark once again that it should also be fruitful to examine geometries modelled on other symmetric spaces. Interested readers are referred to Biquard [5, 6], Biquard and Mazzeo [9, 10], and references therein.

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Variation of Numerical Dimension of Singular Hermitian Line Bundles

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Abstract The purpose of this paper is to give two supplements for vanishing theorems: One is a relative version of the Kawamata-Viehweg-Nadel type vanishing theorem, which is obtained from an observation for the variation of the numerical dimension of singular hermitian line bundles. The other is an analytic injectivity theorem for log canonical pairs on surfaces, which can be seen as a partial answer for Fujino's conjecture.

Keywords Injectivity theorem · Vanishing theorem · Singular hermitian metrics Multiplier ideal sheaves · Numerical dimension · Log canonical singularities

1 Introduction

The injectivity theorem is one of the most important generalizations of the Kodaira vanishing theorem, and it plays an important role in complex geometry in the last decades, in which analytic methods and algebraic geometric methods have been nourishing each other. After Tankeev's pioneer work in [31], Kollár in [21] and [22] established the celebrated injectivity theorem for semi-ample line bundles on projective manifolds by Hodge theory. Enoki in [8] generalized Kollár's injectivity for semi-positive line bundles on compact Kähler manifolds by the theory of harmonic integrals, and Takegoshi in [29] gave a relative version of Enoki's injectivity for Kähler morphisms. We recently obtained a further generalization of them for pseudo-effective line bundles with singular hermitian metrics by a combination of the theory of harmonic integrals and L^2-methods for $\overline{\partial}$-equations (for example see [15, 25, 26, 28]).

Dedicated to Professor Kang-Tae Kim on the occasion of his 60th birthday.

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In this paper, as an application of [26], we prove a relative version of the Kawamata-Viehweg-Nadel type vanishing theorem (Theorem 1.2) for Kähler morphisms, by the vanishing theorem on compact Kähler manifolds in [4], the solution of the strong openness conjecture in [16] (see also [17, 18, 23]), and an observation for the variation of the numerical dimension of singular hermitian line bundles (Theorem 1.1). Moreover, as an application of [27], we give an affirmative answer for Fujino's conjecture (Conjecture 1.3) in the two dimensional cases (Theorem 1.4).

Theorem 1.1 (Variation of numerical dimensions, cf. [26, Proposition 1.6]) *Let $\pi\colon X \to B$ be a smooth proper Kähler morphism from a complex manifold X to a complex manifold B, and let T be a positive d-closed $(1, 1)$-current on X. Then there is a subset $C \subset B$ of Lebesgue measure zero with the following property: For an arbitrary point $b \in B \setminus C$, the restriction $T|_{X_b}$ of T to the fiber $X_b := \pi^{-1}(b)$ is well-defined, and the numerical dimension $\mathrm{nd}(X_b, T|_{X_b})$ of $T|_{X_b}$ on X_b does not depend on $b \in B \setminus C$.*

Theorem 1.2 (Relative vanishing theorem of Kawamata-Viehweg-Nadel type) *Let $\pi\colon X \to \Delta$ be a surjective proper Kähler morphism from a complex manifold X to an analytic space Δ, and (F, h) be a singular hermitian line bundle on X with semi-positive curvature.*

Then we have

$$R^q\pi_*(K_X \otimes F \otimes \mathcal{I}(h)) = 0 \quad \textit{for every } q > n - \mathrm{nd}_{\mathrm{rel}}(F, h),$$

where n is the relative dimension of $\pi\colon X \to \Delta$ and $\mathrm{nd}_{\mathrm{rel}}(F, h) = \mathrm{nd}_{\mathrm{rel}}(\sqrt{-1}\Theta_h(F))$ is the relative numerical dimension defined by Definition 2.1. Here K_X is the canonical bundle on X, $\mathcal{I}(h)$ is the multiplier ideal sheaf of h, $R^q\pi_*(\bullet)$ is the q-th direct image sheaf.

Ambro and Fujino proved an injectivity theorem for log canonical (lc for short) pairs by Hodge theory (see [1, 2, 9], [10, Sect. 6], [12, 13]). It is a natural and quite interesting problem to ask whether the injectivity theorem for lc pairs can be generalized from semi-ample line bundles to semi-positive line bundles, which was first posed by Fujino.

Conjecture 1.3 ([14, Conjecture 2.21], cf. [11, Problem 1.8]) *Let D be a simple normal crossing divisor on a compact Kähler manifold X and F be a semi-positive line bundle on X (that is, it admits a smooth hermitian metric with semi-positive curvature). Assume that there is a section $s \in H^0(X, F^m)$ such that the zero locus $s^{-1}(0)$ contains no lc centers of the lc pair (X, D). Then, the multiplication map induced by the tensor product with s*

$$H^q(X, K_X \otimes D \otimes F) \xrightarrow{\otimes s} H^q(X, K_X \otimes D \otimes F^{m+1})$$

is injective for every q.

In [27], we proved the above conjecture in the case of purely log terminal pairs by developing techniques in [28, 30] (see [5, 19, 24] for another interesting approach). In this paper, as an application of [27], we affirmatively solve Fujino's conjecture on surfaces without any assumptions (Theorem 1.4).

Theorem 1.4 *Let D be a simple normal crossing divisor on a compact Kähler surface X and F be a semi-positive line bundle on X. Assume that there is a section $s \in H^0(X, F^m)$ such that the zero locus $s^{-1}(0)$ contains no irreducible components of D. Then the same conclusion as in Conjecture 1.3 holds.*

Remark 1.5 The assumption of Theorem 1.4 for the zero locus $s^{-1}(0)$ is weaker than that of Conjecture 1.3, since the zero locus $s^{-1}(0)$ may contain $D_i \cap D_j$, where D_i is an irreducible component of $D = \sum_{i \in I} D_i$.

2 Proof of the Results

2.1 Proof of Theorem 1.1

We first recall the definition of the numerical dimension of (possibly) singular hermitian line bundles with semi-positive curvature (more generally positive d-closed $(1,1)$-currents) in [4]. For a positive d-closed $(1,1)$-current T on a compact Kähler manifold X, we can take a family $\{T_k\}_{k=1}^{\infty}$ of d-closed $(1,1)$-currents (called an equi-singular approximation with analytic singularities) with the following properties:

(0) T_k is a d-closed $(1,1)$-current representing the same cohomology class $\{T\}$.
(1) T_k has analytic singularities.
(2) $T_k \geq -\varepsilon_k \omega$, where $\varepsilon_k \searrow 0$ and ω is a fixed Kähler form on X.
(3) T_{k+1} is more singular than T_k.
(4) For rational numbers $\delta > 0$ and $m > 0$, there exists an integer k_0 such that

$$\mathcal{I}((m(1+\delta)T_k)) \subset \mathcal{I}(mT) \text{ for } k \geq k_0.$$

Here, for a d-closed $(1,1)$-current S, the multiplier ideal $\mathcal{I}(S)$ can be defined by the set of holomorphic functions f such that $|f|^2 e^{-\varphi}$ is locally integrable, where φ is a local potential function of S. Then, by the same way as in [4], the numerical dimension $\mathrm{nd}(T)$ of T is defined by

$$\mathrm{nd}(T) := \mathrm{nd}(X, T) := \max\{\nu \in \mathbb{Z}_{\geq 0} \mid \liminf_{k \to \infty} \int_X (T_{k,\mathrm{ac}})^{\nu} \wedge \omega^{n-\nu} > 0\},$$

where n is the dimension of X and T_{ac} is the absolutely continuous part of T (see [3] for the definition). Note that the above numerical dimension can be expressed by the growth of the dimension of the space of global sections, in the case where

X is a projective manifold and T is a semi-positive curvature current of a singular hermitian line bundle (see [4, Proposition 4.3]).

Proof (Proof of Theorem 1.1) By replacing B with an open subset in B (if necessarily), we may assume that X is a Kähler manifold. Further we may assume that there is an equisingular approximation $\{T_k\}_{k=1}^{\infty}$ of T satisfying properties (0)–(4) on a (non-compact) manifold X, since Demailly's approximation theorem (see [6], [7]), which plays a crucial role to obtain such an equisingular approximation, still works on a relatively compact set in X. By an observation for multiplier ideal sheaves and Fubini's theorem, we have the following claim:

Claim 2.1 *There is a subset $C \subset B$ of Lebesgue measure zero with the following property: For an arbitrary point $b \in B \setminus C$, the restriction $T|_{X_b}$ (resp. $T_k|_{X_b}$) of T (resp. T_k) to the fiber $X_b := \pi^{-1}(b)$ is well-defined (that is, the restriction of its potential function is not identically $-\infty$), and $\{T_k|_{X_b}\}_{k=1}^{\infty}$ is also an equisingular approximation of $T|_{X_b}$ on X_b.*

Proof (Proof of Claim 2.1) We can easily check that properties (0)–(3) for $T_k|_{X_b}$ and $T|_{X_b}$ still hold on a fiber X_b, if the restriction of them to X_b is well-defined. In general, for a quasi-psh function φ on X, we have the restriction formula $\mathcal{I}(\varphi|_{X_b}) \subset \mathcal{I}(\varphi)|_{X_b}$ by the Ohsawa-Takegoshi L^2-extension theorem. Further, by Fubini's theorem, we can show that the converse inclusion holds for almost all $b \in B$, that is, the subset

$$\{b \in B \mid \varphi|_{X_b} \text{ is not well-defined or } \mathcal{I}(\varphi|_{X_b}) \neq \mathcal{I}(\varphi)|_{X_b}\}$$

has Lebesgue measure zero. Indeed, for a holomorphic function f on a (sufficiently small) open set U in X, Fubini's theorem yields

$$\int_U |f|^2(z,b)e^{-\varphi(z,b)} = \int_{b\in B}\Big(\int_{z\in X_b\cap U} |f|^2(z,b)e^{-\varphi(z,b)}\Big),$$

where (z,b) is a coordinate on U such that $b = \pi(z,b)$ gives a local coordinate on B. If the left hand side converges, the integrand $\int_{z\in X_b\cap U} |f|^2(z,b)e^{-\varphi(z,b)}$ also converges for almost all $b \in B$. This implies that the above subset has Lebesgue measure zero since multiplier ideal sheaves are coherent sheaves.

Now we define the subset C by the union of

$$\{b \in B \mid T|_{X_b} \text{ is not well-defined or } \mathcal{I}(mT|_{X_b}) \neq \mathcal{I}(mT)|_{X_b}\} \text{ and}$$
$$\{b \in B \mid T_k|_{X_b} \text{ is not well-defined or } \mathcal{I}(m(1+\delta)T_k|_{X_b}) \neq \mathcal{I}(m(1+\delta)T_k)|_{X_b}\},$$

where δ and m run through positive rational numbers. Then the union C also has Lebesgue measure zero since C is a countable union of subsets of Lebesgue measure zero. Therefore it follows that the restriction $T_k|_{X_b}$ gives an equisingular approximation of $T|_{X_b}$ on the fiber X_b for an arbitrary point $b \in B \setminus C$. □

Since T_k has analytic singularities, we can take a modification $f_k : X_k \to X$ such that $f_k^*(T_k) = P_k + [E_k]$, where P_k is a smooth semi-positive $(1,1)$-form on X_k and

$[E_k]$ is the integration current of an effective $\mathbb{R}$-divisor E_k. We consider the restriction of f_k to $X_{k,b} := f_k^{-1}(X_b)$

$$f_{k,b} := f_k|_{f_k^{-1}(X_b)} : X_{k,b} := f_k^{-1}(X_b) \to X_b.$$

Let Z_k be a subvariety of X such that $f_k : X_k \setminus f_k^{-1}(Z_k) \to X \setminus Z_k$ is an isomorphism. Since the subset $C_k := \{b \in B \mid X_b \subset Z_k \text{ or } X_{k,b} \subset E_k\}$ is a proper subvariety of B, by replacing C with $\cup_{k=1}^{\infty} C_k \cup C$ (if necessarily), we may assume that the restriction $f_{k,b} : X_{k,b} \to X_b$ is a modification and the fiber $X_{k,b}$ is not contained in E_k for every $b \in B \setminus C$. Then, for every $b \in B \setminus C$, we have

$$f_{k,b}^*(T_k|_{X_{k,b}}) = f_k^*(T_k)|_{X_{k,b}} = P_k|_{X_{k,b}} + [E_k|_{X_{k,b}}].$$

Then, for a Kähler form ω on X and a non-negative integer d, we can see that

$$\begin{aligned}\int_{X_b} (T_k|_{X_b})_{\mathrm{ac}}^d \wedge \omega|_{X_b}^{n-d} &= \int_{X_b \setminus \mathrm{Sing}\, T_k|_{X_b}} (T_k|_{X_b})^d \wedge \omega|_{X_b}^{n-d} \\ &= \int_{X_{k,b} \setminus \mathrm{Supp} E_k} (P_k|_{X_{k,b}})^d \wedge f_{k,b}^* \omega|_{X_b}^{n-d} \\ &= \int_{X_{k,b}} (P_k^d \wedge f_k^* \omega^{n-d})|_{X_{k,b}},\end{aligned}$$

where n is the dimension of X_b.

Now we consider the push-forward $(\pi \circ f_k)_*(P_k^d \wedge f_k^* \omega^{n-d})$ of the smooth (n, n)-form. This push-forward is a d-closed $(0, 0)$-current, and thus it must be a constant function on B. Let C_k' be a subvariety of B such that $\pi \circ f_k$ is a smooth morphism over $B \setminus C_k'$. Then the push-forward $(\pi \circ f_k)_*(P_k^d \wedge f_k^* \omega^{N-d})$ is a smooth function whose value at $b \in B \setminus C_k'$ is given by the fiber integral. Therefore, replacing C with $\cup_{k=1}^{\infty} C_k' \cup C$ again, we can check that

$$(\pi \circ f_k)_*(P_k^d \wedge f_k^* \omega^{n-d})(b) = \int_{X_{k,b}} (P_k^d \wedge f_k^* \omega^{n-d})|_{X_{k,b}} = \int_{X_b} (T_k|_{X_b})_{\mathrm{ac}}^d \wedge \omega|_{X_b}^{n-d}$$

for $b \in B \setminus C$ by the above argument. The left hand side does not depend on $b \in B \setminus C$ since $(\pi \circ f_k)_*(P_k^d \wedge f_k^* \omega^{n-d})$ is a constant function. Hence we obtain the desired conclusion. □

2.2 *Proof of Theorem 1.2*

As an application of [4, Theorem 1.3], [16, Theorem 1.1], and [26, Theorem 1.1], we prove Theorem 1.2 by the same argument as in [26, Theorem 1.7]. We first define the relative numerical dimension by using Theorem 1.1.

Definition 2.1 *(Relative numerical dimension)* Let $\pi\colon X \to \Delta$ be a surjective proper Kähler morphism from a complex manifold X to an analytic space Δ, and let T be a positive d-closed $(1,1)$-current on X. For a Zariski open set B over which π is smooth, by taking $C \subset B$ of Lebesgue measure zero satisfying the property of Theorem 1.1, we define the relative numerical dimension

$$\mathrm{nd}_{\mathrm{rel}}(T) := \mathrm{nd}(X_b, T|_{X_b})$$

for $b \in B \setminus C$. (See Sect. 2.1 for the definition of the usual numerical dimension.)

Proof (Proof of Theorem 1.2) For a Zariski open set B in C over which π is smooth, we take $C \subset B$ of Lebesgue measure zero with the property of Theorem 1.1. We replace C with $C \cup (\Delta \setminus B)$. Then, by the argument of Claim 2.1, we obtain the additional property:

$$\mathrm{nd}(F|_{X_b}, h|_{X_b}) = \mathrm{nd}_{\mathrm{rel}}(F, h) \text{ and } \mathcal{I}(h|_{X_b}) = \mathcal{I}(h)|_{X_b} \text{ holds for every } b \in \Delta \setminus C.$$

For $q > n - \mathrm{nd}_{\mathrm{rel}}(F, h)$ and for $b \in \Delta \setminus C$, we have the vanishing theorem

$$H^q(X_b, \mathcal{O}_{X_b}(K_X \otimes F) \otimes \mathcal{I}(h)) = H^q(X_b, \mathcal{O}_{X_b}(K_X \otimes F) \otimes \mathcal{I}(h|_{X_b})) = 0$$

on a fiber X_b by [4, Theorem 1.3] and [16, Theorem 1.1]. In particular, for the Zariski open set Δ' in Δ defined by

$$\Delta' := \{b \in \Delta \mid \pi \text{ is smooth at } b \text{ and } R^q\pi_*(K_X \otimes F \otimes \mathcal{I}(h)) \text{ is locally free at } b\},$$

we can see that $R^q\pi_*(K_X \otimes F \otimes \mathcal{I}(h))_b = 0$ for every $b \in \Delta' \setminus C$ by the flat base change theorem. Hence we have $R^q\pi_*(K_X \otimes F \otimes \mathcal{I}(h)) = 0$ on Δ'. We obtain the desired conclusion since $R^q\pi_*(K_X \otimes F \otimes \mathcal{I}(h))$ is torsion free by [26, Theorem 1.1]. □

2.3 Proof of Theorem 1.4

We finally prove Theorem 1.4 as an application of [27].

Proof (Proof of Theorem 1.4) The conclusion is obvious in the case $q = 0$. Further we can easily check the conclusion in the case $q = 2$ by the Serre duality. Indeed, by the Serre duality, we have

$$H^2(X, K_X \otimes D \otimes F) = H^0(X, \mathcal{O}_X(D \otimes F)^*) = 0$$

unless s is a non-vanishing section. Hence it is enough to consider the case $q = 1$.

Let α be a cohomology class $\alpha \in H^q(X, K_X \otimes D \otimes F)$ satisfying $s\alpha = 0 \in H^q(X, K_X \otimes D \otimes F^{m+1})$. By [27, Theorem 1.6], it is sufficient to show that α

belongs to the image of the morphism

$$\theta_D : H^q(X, K_X \otimes F) \to H^q(X, K_X \otimes D \otimes F)$$

induced by the effective divisor D.

For the irreducible decomposition $D = \sum_{i \in I} D_i$ of D, we define the divisors D_J and D_K by

$$D_J := \sum_{j \in J} D_j \text{ and } D_K := D - D_J, \text{ where } J := \{i \in I \mid D_i \cap s^{-1}(0) \neq \emptyset\}.$$

We consider the commutative long exact sequence induced by the standard short exact sequence:

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
H^q(X, K_X \otimes D_J \otimes F) & \xrightarrow{\otimes s} & H^q(X, K_X \otimes D_J \otimes F^{m+1}) \\
\downarrow{\scriptstyle \theta_K} & & \downarrow \\
H^q(X, K_X \otimes D \otimes F) & \xrightarrow{\otimes s} & H^q(X, K_X \otimes D \otimes F^{m+1}) \\
\downarrow{\scriptstyle r_K} & & \downarrow \\
H^q(D_K, K_{D_K} \otimes D_J \otimes F) & \xrightarrow{f := \otimes s|_{D_K}} & H^q(D_K, K_{D_K} \otimes D_J \otimes F^{m+1}) \\
\downarrow & & \downarrow
\end{array} \tag{2.1}$$

Here θ_K (resp. r_K) is the morphism induced by the effective divisor D_K (resp. the restriction to D_K). It follows that $f(r_K(\alpha)) = 0$ from the assumption $s\alpha = 0$. On the other hand, the morphism f admits the inverse map since the section s is non-vanishing on D_K by the definition of D_K. Therefore we can find $\beta \in H^q(X, K_X \otimes D_J \otimes F)$ such that $\alpha = \theta_K(\beta)$.

For a given index $i \in J$, we consider $r_i(\beta) \in H^q(D_i, K_{D_i} \otimes \hat{D}_i \otimes F)$, where $\hat{D}_i := D_J - D_i$ and r_i is the morphism induced by the restriction to D_i. It follows that $\deg F|_{D_i} > 0$ since $s^{-1}(0)$ intersects with D_i by the definition of J, and thus we have $H^q(D_i, K_{D_i} \otimes \hat{D}_i \otimes F) = 0$ by the vanishing theorem on the curve D_i. In particular, we can take $\beta' \in H^q(X, K_X \otimes \hat{D}_i \otimes F)$ such that $\beta = \theta_i(\beta')$. For $j \in J$ with $j \neq i$, it can be seen that

$$r_j(\beta') \in H^q(D_j, K_{D_j} \otimes \hat{D}_{ij} \otimes F) = 0$$

by using the vanishing theorem again, where $\hat{D}_{ij} := D_J - (D_i + D_j)$. Hence we can take $\beta'' \in H^q(X, K_X \otimes \hat{D}_{ij} \otimes F)$ such that $\beta = \theta_i(\theta_j(\beta'')) = \theta_{ij}(\beta'')$, where θ_{ij} is the morphism induced by the effective divisor $D_i + D_j$. By repeating this process, we can conclude that $\alpha = \theta_K(\beta) = \theta_K(\theta_J(\gamma)) = \theta_D(\gamma)$ for some $\gamma \in H^q(X, K_X \otimes F)$. This completes the proof by [27, Theorem 1.6]. □

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Schottky Group Actions in Complex Geometry

Christian Miebach and Karl Oeljeklaus

Abstract This paper discusses some recent progress on Schottky group actions on compact and non-compact complex manifolds.

Keywords Flag manifold · Schottky group action · non-Kähler manifold · Stein quotient

1 Introduction

In this article we survey a number of results concerning Schottky group actions on complex manifolds. As an abstract group a Schottky group is isomorphic to the free group F_r of rank $r \geq 1$, and a Schottky group action on the complex manifold X yields an F_r-invariant connected open subset $\mathcal{U}$ of X on which action of F_r is free and properly discontinuous. Our motivation for the study of Schottky group actions stems from the fact that the quotient manifolds $\mathcal{U}/F_r$ have very interesting properties.

Having discussed the historical background of Schottky groups in complex geometry in Sect. 1, we review the main results of [18] in Sect. 2. Section 3 contains a new result concerning Schottky group actions on the unit ball of $\mathbb{C}^n$, which has been obtained independently by Stefan Nemirovski. To the best of our knowledge this is the first time that complex analytic properties of Schottky quotients are discussed in a non-compact setting. In the last section we list a number of open problems in this circle of ideas.

Dedicated to Kang-Tae Kim at the occasion of his 60th birthday.

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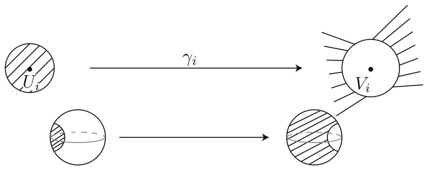
2 Historical Remarks on Schottky Actions

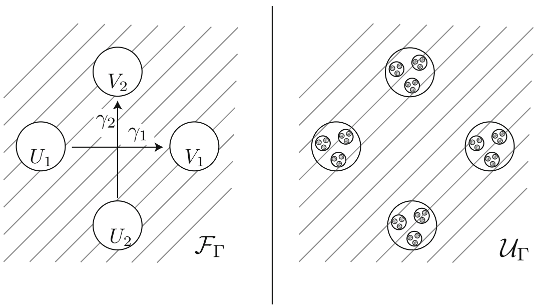
In 1877, Schottky constructed holomorphic actions of free groups on the Riemann sphere in the following way: Take $(U_1, V_1), \dots, (U_r, V_r)$ to be r pairs of open subsets of $\mathbb{P}_1$ bounded by Jordan curves, such that $\overline{U_1}, \overline{V_1}, \dots, \overline{U_r}, \overline{V_r}$ are pairwise disjoint and such that there are loxodromic elements $\gamma_1, \dots, \gamma_r \in \mathrm{Aut}(\mathbb{P}_1)$ with

$$\gamma_i(U_i) = \mathbb{P}_1 \setminus \overline{V_i}, \quad i = 1, \dots, r.$$



The *Schottky group* $\Gamma := \langle \gamma_1, ..., \gamma_r \rangle \subset \mathrm{PSL}(2, \mathbb{C})$ is free and non-abelian. Define

$$\mathcal{F}_\Gamma := \mathbb{P}_1 \setminus \bigcup_{i=1}^{r} (U_i \cup V_i) \text{ and } \mathcal{U}_\Gamma := \bigcup_{\gamma \in \Gamma} \gamma\left(\mathcal{F}_\Gamma\right).$$



Then Γ is a free group of r generators and it acts freely and properly discontinuously on $\mathcal{U}_\Gamma$ with $\mathcal{F}_\Gamma$ as a fundamental domain. Furthermore the quotient $Q_\Gamma := \mathcal{U}_\Gamma / \Gamma$ is a compact Riemann surface R_r of genus r and every such surface R_r can be obtained this way, [14]. The complement $\mathbb{P}_1 \setminus \mathcal{U}_\Gamma$ is a Cantor set and $\mathcal{U}_\Gamma$ is not simply-connected. In 1967 Maskit proved that a subgroup $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$ is

Schottky if and only if it is finitely generated, free, has nonempty domain of discontinuity and all non-trivial elements are conjugate to some $\gamma = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $|\lambda| \neq 1$, see [17]. A natural idea is to generalize this construction to higher dimensions which has been done since the beginning of the 1980s.

In [19] Nori gave an explicit construction of Schottky groups acting on $\mathbb{P}_N$ for every odd N. Here a subgroup Γ of $\mathrm{Aut}(\mathbb{P}_N)$ is called a *Schottky group* if there exist $2r$ open subsets $U_1, \ldots, U_r, V_1, \ldots, V_r$ of $\mathbb{P}_N$, being the interiors of their pairwise disjoint topological closures, such that Γ is generated by r automorphisms $\gamma_1, \ldots, \gamma_r \in \mathrm{Aut}(\mathbb{P}_N)$ which satisfy

$$\gamma_j(U_j) = \mathbb{P}_N \setminus \overline{V_j} \quad \text{for all } 1 \leq j \leq r.$$

In this situation standard arguments show that Γ is freely generated by $\gamma_1, \ldots, \gamma_r$ and hence isomorphic to the free group of rank r. Defining

$$\mathcal{F}_\Gamma := \mathbb{P}_N \setminus \bigcup_{j=1}^{r} (U_j \cup V_j) \quad \text{and} \quad \mathcal{U}_\Gamma := \Gamma \cdot \mathcal{F}_\Gamma,$$

it is not hard to show that Γ acts properly on the connected open subset $\mathcal{U}_\Gamma$ of $\mathbb{P}_N$ with fundamental domain $\mathcal{F}_\Gamma$ and that we obtain a compact complex manifold $Q_\Gamma := \mathcal{U}_\Gamma / \Gamma$ as quotient. Moreover, if $N > 1$, then $\mathcal{U}_\Gamma$ is simply connected and thus the fundamental group of Q_Γ is isomorphic to Γ.

In [20] Seade and Verjovsky use inversions at certain hypersurfaces of $\mathbb{P}_N$ with N odd in order to construct discrete subgroups of $\mathrm{Aut}(\mathbb{P}_N)$ which contain Nori's Schottky groups as subgroups of index 2. On the other hand, in [5] Cano shows that there are no Schottky groups acting on $\mathbb{P}_N$ for even N.

In dimension 3 a more general approach to Schottky quotients has been developed by Ma. Kato. A compact complex 3-fold is of class L if it contains an open set biholomorphic to a neighborhood of a complex line in the complex projective space $\mathbb{P}_3$. This notion was introduced by Ma. Kato (see [10, 11]) who studied manifolds of class L in great detail in several articles. An important subclass of class L is given by compact complex manifolds whose universal covering is biholomorphic to a domain in $\mathbb{P}_3$ containing a complex line. In his paper [12], Kato constructed many examples of such manifolds which he calls "manifolds of Schottky type" using the method of "Klein combination" which produces given two such manifolds a third one still having a domain of $\mathbb{P}_3$ as universal covering. These spaces are quotients by discrete subgroups of the automorphism group of $\mathbb{P}_3$, but these groups are only implicitly constructed. In [15] Lárusson has also contributed to the theory of 3-dimensional manifolds of Schottky type.

3 Schottky Group Actions on Flag Manifolds

In this section we review the main results from our paper [18].

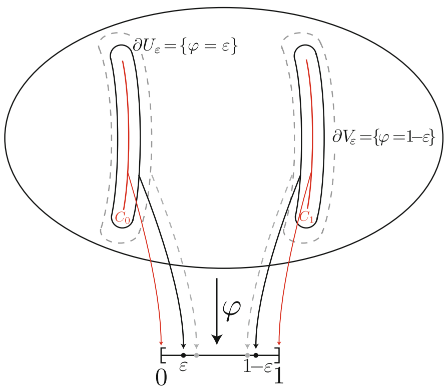
3.1 Construction of Schottky Groups Acting on Flag Manifolds

Let us recall Nori's construction of Schottky group actions on an odd-dimensional complex projective space $\mathbb{P}_{2n+1}$ with $n \geq 0$. Writing homogeneous coordinates of $\mathbb{P}_{2n+1}$ as $[z : w]$ with $z = (z_0, \ldots, z_n)$ and $w = (w_0, \ldots, w_n)$ we define

$$C_0 := \big\{[z : w] \in \mathbb{P}_{2n+1};\ w = 0\big\} \quad \text{and} \quad C_1 := \big\{[z : w] \in \mathbb{P}_{2n+1};\ z = 0\big\}$$

as well as a smooth function $\varphi\colon \mathbb{P}_{2n+1} \to [0, 1]$ by

$$\varphi[z : w] := \frac{\|w\|^2}{\|z\|^2 + \|w\|^2}.$$



Note that $C_j = \varphi^{-1}(j)$ for $j = 0, 1$. For $0 < \varepsilon < \frac{1}{2}$ we define open neighborhoods of C_0, C_1 by $U_\varepsilon := \{\varphi < \varepsilon\}$ and $V_\varepsilon := \{\varphi > 1 - \varepsilon\}$, respectively. A direct calculation shows that the automophism $\gamma \in \mathrm{Aut}(\mathbb{P}_{2n+1})$ given by

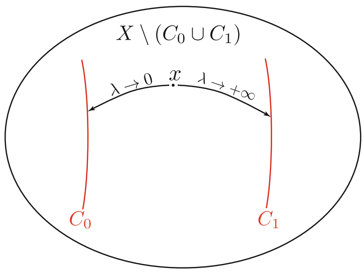
$$\gamma\big([z : w]\big) := [z : \lambda w]$$

satisfies $\gamma(U_\varepsilon) = \mathbb{P}_{2n+1} \setminus \overline{V}_\varepsilon$ if $|\lambda| = \frac{1-\varepsilon}{\varepsilon}$. Choosing ε sufficiently small and conjugating γ by automorphisms $f_1, \ldots, f_r \in \mathrm{Aut}(\mathbb{P}_{2n+1})$ we obtain Nori's examples of Schottky groups acting on $\mathbb{P}_{2n+1}$.

Since Nori and Cano settled the question of existence of Schottky groups acting on complex projective space, the problem that motivated the research leading to [18] was to determine all complex flag manifolds that admit Schottky group actions. Here, by flag manifold we mean a complex homogeneous manifold $X = G/P$ where G is a connected semisimple complex Lie group and P is a parabolic subgroup of G.

From now on we replace complex projective space by an arbitrary complex flag manifold X. Motivated by Nori's construction we define a *Schottky pair in* X to be a pair of disjoint connected compact complex submanifolds C_0, C_1 of X for which there exists a holomorphic $\mathbb{C}^*$-action on X given by the one parameter family $\{\varphi_\lambda\}_{\lambda\in\mathbb{C}^*}$ of automorphisms of X such that

(1) the fixed point set $X^{\mathbb{C}^*}$ coincides with $C_0 \cup C_1$ and
(2) $\mathbb{C}^*$ acts freely and properly on $X \setminus (C_0 \cup C_1)$.



Such a pair is the basic ingredient in order to define a generator of a Schottky group: We may choose suitable open neighborhoods U of C_0 and V of C_1 and an automorphism φ_{λ_0} such that $\varphi_{\lambda_0}(U) = X \setminus \overline{V}$. Hence, we may construct a Schottky group of arbitrarily high rank as soon as we find "many" disjoint Schottky pairs. More precisely, we say that the Schottky pair (C_0, C_1) is *movable* if for every $r \geq 1$ there are automorphisms $f_1, \ldots, f_r \in \mathrm{Aut}(X)$ such that $f_k(C_j)$ are pairwise disjoint for all $1 \leq k \leq r$ and $j = 0, 1$.

In order to construct Seade's and Verjovsky's extended Schottky groups we need an involutive automorphism $s \in \mathrm{Aut}(X)$ such that $s(C_0) = C_1$ and $s \circ \varphi_\lambda = \varphi_{\lambda^{-1}} \circ s$ for all $\lambda \in \mathbb{C}^*$. Note that in Nori's examples such an involution is defined by $s[z : w] = [w : z]$. In particular, its fixed point set X^s is the hypersurface

$$H := \left\{[z : w] \in \mathbb{P}_{2n+1};\ \|z\|^2 - \|w\|^2 = 0\right\}.$$

The crucial observation which eventually leads to the construction of Schottky pairs in certain flag manifolds X is that H is an orbit of the non-compact real form $G_0 = \mathrm{PSU}(n+1, n+1)$ of $G = \mathrm{Aut}(\mathbb{P}_{2n+1}) \cong \mathrm{PSL}(2n+2, \mathbb{C})$. Indeed, combining a theorem of Akhiezer (cf. [1]) with Matsuki duality (cf. [4]) we can prove the following, see [18, Proposition 3.2].

Proposition 3.1 *Let $X = G/P$ be a complex flag manifold. If there exists a non-compact real form G_0 of G having a compact orbit of real codimension 1 in X, then X admits a Schottky pair.*

Proposition 3.1 suggests the following strategy for the construction of Schottky groups acting on a flag manifold X. First classify couples $(G/P, G_0)$ where $X = G/P$ is a flag manifold and G_0 is a non-compact real form of G having a compact hypersurface orbit in X, then determine which of the corresponding Schottky pairs are actually movable in order to produce Schottky groups of arbitrarily high rank acting on X.

This program leads to the following result, see [18, Theorems 4.2 and A.1].

Theorem 3.2 *The pairs $(G/P, G_0)$ of flag manifolds $X = G/P$ and real forms G_0 of G having a compact hypersurface orbit in X are the following:*

(1) $G_0 = \mathrm{SU}(p, q)$ *acting on* $X = \mathbb{P}_{p+q-1}$*;*
(2) $G_0 = \mathrm{Sp}(p, q)$ *acting on* $X = \mathbb{P}_{2(p+q)-1}$*;*
(3) $G_0 = \mathrm{SU}(1, n)$ *acting on the Graßmannian* $X = \mathrm{Gr}_k(\mathbb{C}^{n+1})$ *of k-dimensional linear subspaces in* $\mathbb{C}^{n+1}$*;*
(4) $G_0 = \mathrm{SO}^*(2n)$ *acting on the quadric* $X = Q_{2n-2} = \{[z : w] \in \mathbb{P}_{2n-1};\ z_0 w_0 + \cdots + z_{n-1} w_{n-1} = 0\}$*;*
(5) $G_0 = \mathrm{SO}(1, 2n)$ *acting on the manifold* $X = \mathrm{IGr}_n(\mathbb{C}^{2n+1})$ *of n-dimensional linear subspaces of* $\mathbb{C}^{2n+1}$ *which are isotropic with respect to a non-degenerate symmetric bilinear form on* $\mathbb{C}^{2n+1}$*;*
(6) $G_0 = \mathrm{SO}(2, 2n)$ *acting on* $X = \mathrm{IGr}_{n+1}(\mathbb{C}^{2n+2})^0$.

The Schottky pairs giving rise to Schottky group actions on X of arbitrary rank r are precisely the ones on the odd-dimensional projective space $\mathbb{P}_{2n+1}$ where $G_0 = \mathrm{SU}(n+1, n+1)$, on $\mathbb{P}_{4n+3}$ where $G_0 = \mathrm{Sp}(n+1, n+1)$, on the quadric Q_{4n-2} where $G_0 = \mathrm{SO}^*(4n)$ and on the isotropic Graßmannian $\mathrm{IGr}_n(\mathbb{C}^{2n+1})$ where $G_0 = \mathrm{SO}(1, 2n)$.

Remark The movable Schottky pairs obtained by Theorem 3.2 yield Schottky group actions on irreducible flag manifolds.[1] If $X = G/P$ is one of these flag manifolds, and if P' is a parabolic subgroup of P, we obtain a Schottky group action on $X' = G/P'$ by lifting the one on X.

In the rest of this subsection we will describe the movable Schottky pairs in greater detail. In particular we will show that in every case there is an involutive automorphism $s \in \mathrm{Aut}(X)$ such that X^s is a compact hypersurface orbit of G_0. Consequently, it is possible to construct discrete subgroups of $\mathrm{Aut}(X)$ which contain Schottky groups as subgroups of index 2 as it is done in [20]. In addition we may extend Kato's notion of complex-analytic connected sum to these cases.

We have already discussed Nori's Schottky groups acting on $\mathbb{P}_{2n+1}$. In these cases the Schottky pairs are linked by a holomorphic involution.

[1]A flag manifold $X = G/P$ is called *irreducible* if P is a maximal parabolic subgroup of G, i.e., if P is not properly contained in any proper parabolic subgroup of G.

Example Let us consider the case of the quadric $X = Q_{4n-2}$, compare [18, Sect. 4.2]. In this case a movable Schottky pair is given by

$$C_0 = \{[z : w] \in Q_{4n-2};\ w = 0\} \quad \text{and} \quad C_1 = \{[z : w] \in Q_{4n-2};\ z = 0\}.$$

The map defined by $s[z : w] = [w : z]$ defines a holomorphic involution of $X = Q_{4n-2}$ whose fixed point set in X coincides with the compact $\mathrm{SO}^*(4n)$-orbit in X.

Example In the case of the isotropic Graßmannian $X = \mathrm{IGr}_n(\mathbb{C}^{2n+1})$ a movable Schottky pair is given as follows, see [18, Sect. 4.4]. The set of n-dimensional isotropic linear subspaces of $\mathbb{C}^{2n+1}$ which lie in $\mathbb{C}^{2n} \times \{0\}$ is isomorphic to $\mathrm{IGr}_n(\mathbb{C}^{2n})$ and has two connected components C_0 and C_1 which form a movable Schottky pair in X. There is a holomorphic action of the group $\mathrm{O}(2n, \mathbb{C})$ on X such that $C_0 \cup C_1$ is an orbit. Moreover, we may find an element of order 2 in $\mathrm{O}(2n, \mathbb{C}) \setminus \mathrm{SO}(2n, \mathbb{C})$ which exchanges C_0 and C_1 and therefore yields a holomorphic involution of X with the desired properties.

3.2 Analytic and Geometric Properties of Q_Γ

Let $\Gamma \subset G$ be a Schottky group acting on the flag manifold $X = G/P$. We suppose throughout the following that $\dim X = d \geq 2$ and that $\dim C_j \leq d - 2$. This implies that the maximal domain $\mathcal{U}_\Gamma$ on which Γ acts properly is simply connected. Let $p\colon \mathcal{U}_\Gamma \to Q_\Gamma$ be the universal covering of the compact quotient manifold Q_Γ.

The general strategy to investigate holomorphic objects on Q_Γ, such as meromorphic functions or holomorphic differential forms, is to pull them back to $\mathcal{U}_\Gamma$ and then to use classical extension theorems in order to extend them to the whole of X, see [18, Sect. 5].

Since the fundamental group of Q_Γ is isomorphic to Γ, a free group of rank $r \geq 1$, it is well known that Q_Γ cannot be Kähler. Due to the fact that $\mathcal{U}_\Gamma$ contains many of the rational curves of X, the quotient manifold Q_Γ is rationally connected. Since holomorphic d-forms extend from $\mathcal{U}_\Gamma$ to X, the Kodaira dimension of Q_Γ is $-\infty$.

Moreover, meromorphic functions also extend from $\mathcal{U}_\Gamma$ to X. Hence, $\mathcal{M}(Q_\Gamma)$ can be identified with the set of Γ-invariant rational functions on X. Since Γ-invariant rational functions on X are automatically invariant under the Zariski closure of Γ in G, an application of Rosenlicht's theorem yields the following.

Theorem 3.3 *The algebraic dimension $a(Q_\Gamma)$ coincides with the codimension of a generic H-orbit in X where H denotes the Zariski closure of Γ in G. In particular, $a(Q_\Gamma) = 0$ if and only of H has an open orbit in X.*

Although a generic Schottky group acting on X is Zariski dense in G so that Q_Γ has algebraic dimension 0, we have given examples where $a(Q_\Gamma)$ is positive for every X admitting movable Schottky pairs, see [18, Sect. 6.1].

The following assumptions on the dimensions of X are necessary for us to extend higher cohomology classes from $\mathcal{U}_\Gamma$.

Dimension Assumption 3.4 *From now on let X be either $\mathbb{P}_{2n+1}$ with $n \geq 3$ or Q_{4n+2} with $n \geq 2$ or Q_{2n+1} with $n \geq 3$ or X_n with $n \geq 4$.*

Theorem 3.5 *Let Γ be a Schottky group of rank r acting on X with associated quotient $\pi\colon \mathcal{U}_\Gamma \to Q_\Gamma$. Then the Picard group $H^1(Q_\Gamma, \mathcal{O}^*)$ of Q_Γ is isomorphic to $(\mathbb{C}^*)^r \times \mathbb{Z}$.*

The factor $(\mathbb{C}^*)^r$ contains the topologically trivial holomorphic line bundles on Q_Γ, given by a representation of Γ in $\mathbb{C}^*$, while the factor $\mathbb{Z}$ comes from a hyperplane bundle in X.

In closing we note the following result about the deformation theory of Q_Γ. We observe that Γ acts on the Lie algebra $\mathfrak{g}$ of G via the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ and we denote by $\mathfrak{g}^\Gamma$ the subspace of Γ-fixed points.

Theorem 3.6 *The Kuranishi space of versal deformations of Q_Γ is smooth at Q_Γ and of complex dimension $(r-1)\dim\mathfrak{g} + \dim\mathfrak{g}^\Gamma$. Moreover, the automorphism group $\mathrm{Aut}(Q_\Gamma)$ admits as Lie algebra $\mathfrak{g}^\Gamma$.*

Note that the automorphism group of Q_Γ can be rather large. For example, it is even possible that Q_Γ is almost homogeneous, see [18, Examples 6.4 and 6.5].

4 Schottky Groups Acting on the Unit Ball

In this section we consider Schottky subgroups Γ of the automorphism group of the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ and show that the quotient manifolds $\mathbb{B}_n$ are Stein.

4.1 Construction of Schottky Actions

Let $\mathbb{B}_n$ denote the unit ball of $\mathbb{C}^n$. Recall that every holomorphic automorphism of $\mathbb{B}_n$ extends to the boundary $\partial\mathbb{B}_n = S^{2n-1}$ and is called *elliptic* or *parabolic* or *hyperbolic* if it has a fixed point in $\mathbb{B}_n$ or exactly one or exactly two fixed points in $\partial\mathbb{B}_n$, respectively, see [7, Sect. 6.2.1], p. 203.

Let $\{\varphi_t\}_{t\in\mathbb{R}}$ be a one parameter group of hyperbolic automorphisms of $\mathbb{B}_n$. Then, for every $z \in \mathbb{B}_n$, we have

$$\lim_{t\to\pm\infty} \varphi_t(z) = x_\pm \in \partial\mathbb{B}_n$$

where $x_\pm$ are the two joint fixed points of $\{\varphi_t\}$. Consequently, we may choose open neighborhoods $U_\pm$ of $x_\pm$ in $\partial\mathbb{B}_n$ and $t_0 \in \mathbb{R}$ such that

$$\varphi_{t_0}(U_-) = \partial\mathbb{B}_n \setminus \overline{U_+}.$$

This shows that we can construct a Schottky group action on the unit sphere S^{2n-1} (compare [18, Remark 4.6]) that extends to a holomorphic group action on the unit ball.

Remark Such a construction can also be found in [9]. Note that there are also Schottky groups containing parabolic elements, see [16].

As a consequence we obtain a discrete subgroup Γ of $\mathrm{Aut}(\mathbb{B}_n)$ which is freely generated by r hyperbolic automorphisms for every $r \geq 1$. We define its limit set $\Lambda_\Gamma \subset \partial\mathbb{B}_n$ as the set of accumulation points of $\Gamma \cdot z_0$ in $\partial\mathbb{B}_n$ for $z_0 \in \mathbb{B}_n$. Note that this set is independent of the choice of $z_0 \in \mathbb{B}_n$. Moreover, Γ acts freely and properly on $\mathbb{B}_n \cup \Omega_\Gamma$ where $\Omega_\Gamma := \partial\mathbb{B}_n \setminus \Lambda_\Gamma$ is the discontinuity domain of Γ, see [3, Proposition 3.2.6].

4.2 Schottky Quotients are Stein

Let $X := \mathbb{B}_n/\Gamma$ and $\partial X := \Omega_\Gamma/\Gamma$ be the corresponding quotient manifolds. Since ∂X is a compact spherical CR manifold, the union $\overline{X} := X \cup \partial X$ is a compact complex manifold with strictly pseudoconvex boundary ∂X. We shall see that X is a Stein manifold.

Remark Note that Γ is geometrically finite in the sense of [3], that generically Γ is Zariski-dense in the real algebraic group $\mathrm{Aut}(\mathbb{B}_n)$ and that Λ_Γ is a Cantor set in $\partial\mathbb{B}_n$.

Lemma 4.1 *The quotient manifold X is strongly pseudoconvex. In particular, X is a proper modification of a Stein space and therefore holomorphically convex.*

Proof Every point $x \in \partial X$ has an open neighborhood $V \subset \overline{X}$ that is biholomorphically equivalent to an open neighborhood U of some boundary point of $\mathbb{B}_n$. Since $\partial\mathbb{B}_n$ is smooth and strictly pseudoconvex, there exists a strictly plurisubharmonic function $\rho_V \in \mathcal{C}^\infty(V)$ such that $\lim_{x\to\partial X\cap V} \rho_V(x) = \infty$. Using a smooth partition of unity, we may find a function $\rho \in \mathcal{C}^\infty(X)$ which is strictly plurisubharmonic near ∂X such that $\lim_{x\to\partial X} \rho(x) = \infty$ holds. Since $\overline{X}$ is compact, ρ is an exhaustion function on X which is strictly plurisubharmonic outside some compact subset of X, i.e., X is strongly pseudoconvex. □

In order to show that X is Stein we must exclude that X contains compact analytic subsets of positive dimension.

Lemma 4.2 *The quotient manifold X does not contain compact analytic subsets of positive dimension.*

Proof Suppose that $A \subset X$ is an irreducible compact analytic subset of dimension $\dim A = k > 0$.

Applying the Cartan-Leray spectral sequence to the universal covering $\mathbb{B}_n \to X$ it follows that $H_{2k}(X, \mathbb{Z}) = \{0\}$ for all $k \geq 1$. Indeed, since we have $E_2^{p,q} \cong H_p\big(\Gamma, H_q(\mathbb{B}_n)\big)$ and Γ is free, the only non-zero terms are $E_2^{0,0}$ and $E_2^{1,0}$, which proves the claim.

Consequently, $[A] \in H_{2k}(X, \mathbb{Z})$ is trivial, hence $A = \partial M$ for some real submanifold M of X.

On the other hand, the Bergman metric of the unit ball is invariant under $\mathrm{Aut}(\mathbb{B}_n)$ and thus descends to X. Let ω denote the corresponding Kähler form on X. Then we obtain

$$0 < \int_A \omega^k = \int_M d\omega^k = 0,$$

a contradiction. □

Remark The argument in the proof of Lemma 4.2 shows that, whenever a free group acts properly on a contractible Stein manifold which carries an invariant Kähler form, the quotient manifold does not contain compact analytic sets of positive dimension.

Combining Lemmas 4.1 and 4.2 we obtain the following

Theorem 4.3 *Let* $\Gamma \subset \mathrm{Aut}(\mathbb{B}_n)$ *be a Schottky group acting on* $\mathbb{B}_n$*. Then the quotient manifold* $X = \mathbb{B}_n/\Gamma$ *is Stein.*

Remark Extending the action of Γ to $\mathbb{P}_n \supset \mathbb{B}_n$, we obtain a holomorphic action of the free group Γ of arbitrary rank $r > 0$ on $\mathbb{P}_n$ which has domain of discontinuity $\mathbb{B}_n$.

Remark Stefan Nemirovski announced Theorem 4.3 during the workshop "Lie Groups, Invariant Theory and Complex Geometry" that took place on June 1st and 2nd 2017 at the University of Duisburg-Essen, Germany. His proof of Lemma 4.2 however uses a fundamental domain for the action of Γ constructed with the help of bisectors and relies on their complex-analytic properties.

5 Open Problems

5.1 Cohomology Groups in Small Dimensions

Recall that we had to make the Dimension Assumption 3.4 in order to calculate the dimensions of several cohomology groups.

Problem 1 Calculate the cohomology groups of the quotient manifolds Q_Γ for which the initial flag manifold X is isomorphic to the projective spaces $\mathbb{P}_3$, $\mathbb{P}_5$ and the quadrics Q_3, Q_5 and Q_6.

5.2 Generalizations of Blanchard Manifolds

Following Ma. Kato, a *Blanchard manifold* is a compact complex manifold Y whose universal covering is isomorphic to the complement of a single projective line in $\mathbb{P}_3$. The fundamental group $\Gamma := \pi_1(Y)$ is isomorphic to $(\mathbb{Z}^4, +)$. It is natural to look for generalisations of this situation in higher dimensions. One has first the following negative result.

Proposition 5.1 *Let C be a linearly embedded subspace $\mathbb{P}_k$ of $\mathbb{P}_n$. If $k < (n-1)/2$, then there is no infinite discrete group of automorphisms of $\Omega := \mathbb{P}_n \setminus C$ acting properly on Ω.*

Proof We may suppose that $C = \{[x] \in \mathbb{P}_n;\ x_{k+1} = \cdots = x_n = 0\}$. A neighborhood basis of C is given by the open sets of the form

$$U_\varepsilon := \left\{[x:y] \in \mathbb{P}_n;\ \varepsilon\|x\|^2 - (1-\varepsilon)\|y\|^2 > 0\right\}, \quad \varepsilon > 0,$$

where $x = (x_0, \ldots, x_k)$ and $y = (x_{k+1}, \ldots, x_n)$. One verifies directly that the sets U_ε are strictly k-convex. In particular, U_ε does not contain any compact analytic subset of dimension bigger than k.

Suppose now for a moment that there exists an infinite discrete group acting holomorphically and properly on Ω. Then for arbitrary $0 < \varepsilon < 1$ and for all but finitely many $\gamma \in \Gamma$ we have $\gamma(C') \subset U_\varepsilon$ where $C' := \{[x:y] \in \mathbb{P}_n;\ x = 0\} \cong \mathbb{P}_{n-k-1}$. It follows from the above discussion that $n - k - 1 \leq k$, which is equivalent to $k \geq (n-1)/2$. □

In order to generalize the notion of Blanchard manifold one may consider the following problem.

Problem 2 (a) Is it possible that $\Omega = \mathbb{P}_n \setminus \mathbb{P}_k$ covers a compact complex manifold if $n - 2 \geq k \geq (n-1)/2$, and $n > 3$?

(b) Let X be one of the flag manifolds admitting an action of a Schottky group of arbitrary rank r and let $C := C_0$ be a connected component of a movable Schottky pair. Determine all the cases when $X' := X \setminus C$ is the universal covering of a compact complex manifold, i.e. when X' admits a free and properly discontinuous co-compact action of a discrete group of automorphisms.

5.3 Generalized Kato Theory

In a series of papers Ma. Kato developed the theory of compact complex 3-folds of class L using in particular the methods of connected sums and Klein combinations, which allow to produce given two manifolds of class L a third one of class L.

Let X be one of the flag manifolds admitting an action of a Schottky group of arbitrary rang r and let $C := C_0 \subset X$ be a connected component of a movable

Schottky pair. By imitating Kato's definition, one may define a manifold of class C as a compact complex manifold containing an open domain which is biholomorphic to an open neighborhood of C in X. As in Kato's case it is still possible to produce connected sums and Klein combinations of manifolds of class C.

Problem 3 Develop a theory of manifolds of class C.

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Some Recent Results on Holomorphic Isometries of the Complex Unit Ball into Bounded Symmetric Domains and Related Problems

Ngaiming Mok

Abstract In his seminal work Calabi established the foundation on the study of holomorphic isometries from a Kähler manifold with real analytic local potential functions into complex space forms, e.g., Fubini-Study spaces. This leads to interior extension results on germs of holomorphic isometries between bounded domains. General results on boundary extension were obtained by Mok under assumptions such as the rationality of Bergman kernels, which applies especially to holomorphic isometries between bounded symmetric domains in their Harish-Chandra realizations. Because of rigidity results in the cases where the holomorphic isometry is defined on an irreducible bounded symmetric domain of rank ≥ 2, we focus on holomorphic isometries defined on the complex unit ball $\mathbb{B}^n$, $n \geq 1$. We discuss results on the construction, characterization and classification of holomorphic isometries of the complex unit ball into bounded symmetric domains and more generally into bounded homogeneous domains. Furthermore, in relation to the study of the Hyperbolic Ax-Lindemann Conjecture for not necessarily arithmetic quotients of bounded symmetric domains, such holomorphic isometric embeddings play an important role. We also present some differential-geometric techniques arising from the study of the latter conjecture.

Keywords Holomorphic isometry · Bergman kernel · Bounded symmetric domain · Functional transcendence theory

The subject of holomorphic isometries between Kähler manifolds is a classical topic in complex differential geometry going back to Bochner and Calabi. Especially, starting from the seminal work of Calabi [5], in which questions of existence, uniqueness and analytic continuation of holomorphic isometries of Kähler manifolds into space forms such as the Euclidean and the Fubini-Study spaces were systematically studied, tools have been developed, notably using normalized potential functions called *diastases* defined in [5], for the study of germs of holomorphic isometries between

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Kähler manifolds. The author was led to consider such questions especially for bounded symmetric domains equipped with the Bergman metric, in part to answer questions concerning commutants of modular correspondences on such domains raised by Clozel and Ullmo [14], who reduced rigidity problems in this context to questions in complex differential geometry including one on holomorphic isometries. These questions led the author to systematically study germs of holomorphic isometries between bounded domains in Euclidean spaces. Embedding a bounded domain by means of an orthonormal basis of the Hilbert space $H^2(U)$ of square integrable holomorphic functions into the Fubini-Study space $\mathbb{P}^\infty$ of countably infinite dimension, analytic continuation of germs of holomorphic isometries with respect to scalar multiples of the Bergman metric already follows from Calabi [5], and the author has been focusing on boundary behavior of such analytic continuation as the domain of definition traverses the boundary of bounded domains under the assumption that Bergman kernels are rational.

In [24] the author wrote a survey article "*Geometry of holomorphic isometries and related maps between bounded domains*", giving a historical account starting with Calabi [5], explaining the motivation of the author's works in the subject area, and posing a number of open questions. There the related maps include holomorphic measure-preserving maps from a bounded symmetric domain into a Cartesian power of the domain in the terminology of Clozel and Ullmo [14]. In recent years the author has continued to work on various problems on the existence, uniqueness, characterization and classifications on holomorphic isometries especially between bounded symmetric domains (and sometimes more generally bounded homogeneous domains) with respect to scalar multiples of the Bergman metric. While the reader would benefit from reading the current article in conjunction with [24], we note that the current article is *not a survey* on the subject, but rather an annotated account of the overall structure of works of the author and collaborators in the years since [24] in the research area, focusing on the topic of holomorphic isometries between bounded symmetric domains and leaving aside other types of related holomorphic mappings. Statements of our principal results will be given together with brief discussions on the context, motivation and methodology, while we will mention recent developments on the topic from other researchers, and refer the reader to consult their original articles. As such the current article serves more as a *Leitfaden* on the author and his collaborators' recent works on the topic, together with an excursion on possible future links of the study of holomorphic isometries with other domains of research, notably with the theory of geometric structures and substructures, functional transcendence theory and the geometry of flag domains. For a discussion in those direction we refer the reader to Sect. 6 on "*Perspectives and concluding remarks*" and to the last two paragraphs of Sect. 5.

1 Introduction, First Examples and Background Results

Given complex manifolds (X, ω_X) and (Y, ω_Y) equipped with respective Kähler forms, a holomorphic mapping $f : X \to Y$ is a holomorphic isometry if and only if $f^*\omega_Y = \omega_X$. If there exist global potential functions so that $\omega_X = \sqrt{-1}\partial\overline{\partial}\varphi_X$ and $\omega_Y = \sqrt{-1}\partial\overline{\partial}\varphi_Y$, then the holomorphic map $f : X \to Y$ is an isometry if and only if $\sqrt{-1}\partial\overline{\partial}(\varphi_X - f^*\varphi_Y) = 0$, i.e., $h(x) := \varphi(x) - \varphi_Y(f(x))$ is a pluriharmonic function, equivalently locally the real part of a holomorphic function. This simplification applies when we consider Bergman metrics on bounded domains U, since by definition there are global potential functions given by the $\log(k_U)$, where $k_U(x) = K_U(x, x)$ for the Bergman kernel $K_U(z, w)$ on U. To verify that f is a holomorphic isometry it is sufficient to check that $\varphi_X - f^*\varphi_Y$ is a constant, and this applies to give first examples of nonstandard holomorphic isometries from disks to polydisks. Equip the upper half-plane $\mathscr{H}$ with the Poincaré metric $ds^2_{\mathscr{H}} = \mathrm{Re}\frac{d\tau\otimes d\overline{\tau}}{(\mathrm{Im}\tau)^2}$ of constant Gaussian curvature -1 and $\mathscr{H}^2$ with the product metric. Then, the proper holomorphic map $f : \mathscr{H} \to \mathscr{H}^2$ given by $f(\tau) = \left(\sqrt{\tau}, i\sqrt{\tau}\right)$ is a holomorphic isometric embedding. More generally, we have the p-th root map given by

Proposition 1.1 (Mok [25]) *Let $p \geq 2$ be a positive integer and $\gamma = e^{\frac{\pi i}{p}}$. Then, the proper holomorphic mapping $f : \left(\mathscr{H}, ds^2_{\mathscr{H}}\right) \to \left(\mathscr{H}, ds^2_{\mathscr{H}}\right)^p$ defined by*

$$f(\tau) = \left(\tau^{\frac{1}{p}}, \gamma\tau^{\frac{1}{p}}, \ldots, \gamma^{p-1}\tau^{\frac{1}{p}}\right)$$

is a holomorphic isometric embedding.

Proposition 1.1 results simply from the fact that we have for $p \geq 2$ the trigonometric identity $\sin\theta \sin\left(\frac{\pi}{p} + \theta\right) \cdots \sin\left(\frac{(p-1)\pi}{p} + \theta\right) = c_p \sin(p\theta)$ for some positive constant c_p.

For a positive integer g denote by $M_s(g, \mathbb{C})$ the complex vector space of symmetric g-by-g matrices, and write $\mathscr{H}_g \subset M_s(g, \mathbb{C})$ for the Siegel upper half-plane of genus g defined by $\mathscr{H}_g = \left\{\tau \in M_s(g, \mathbb{C}) : \mathrm{Im}(\tau) > 0\right\}$. Another early example of a nonstandard holomorphic isometric embedding is given by the following map of the upper half-plane $\mathscr{H}$ into $\mathscr{H}_3$, together with a verification that it does not arise from p-th root maps. From now on for a domain $U \subset \mathbb{C}^n$ biholomorphic to a bounded domain, we will denote by ds^2_U the Bergman metric on U. Note that when U is a homogeneous domain (U, ds^2_U) is Kähler-Einstein and its Ricci curvatures are equal to -1.

Proposition 1.2 (Mok [25]) *For $\zeta = \rho e^{i\varphi}$, $\rho > 0$, $0 < \varphi < \pi$, n a positive integer, we write $\zeta^{\frac{1}{n}} := \rho^{\frac{1}{n}} e^{\frac{i\varphi}{n}}$. Then, the holomorphic mapping $G : \mathscr{H} \to M_s(3, \mathbb{C})$ defined by*

$$G(\tau) = \begin{bmatrix} e^{\frac{\pi i}{6}}\tau^{\frac{2}{3}} & \sqrt{2}e^{-\frac{\pi i}{6}}\tau^{\frac{1}{3}} & 0 \\ \sqrt{2}e^{-\frac{\pi i}{6}}\tau^{\frac{1}{3}} & i & 0 \\ 0 & 0 & e^{\frac{\pi i}{3}}\tau^{\frac{1}{3}} \end{bmatrix}$$

maps $\mathscr{H}$ into $\mathscr{H}_3$, and $G : \left(\mathscr{H}, 2ds^2_{\mathscr{H}}\right) \to \left(\mathscr{H}_3, ds^2_{\mathscr{H}_3}\right)$ is a holomorphic isometry.

In what follows for an integer $p \geq 2$ we write $\rho_p : \mathscr{H} \to \mathscr{H}^p$ for the p-th root map as given by $f(\tau)$ in Proposition 1.1. We denote by $\iota : \mathscr{H}^p \to \mathscr{H}_p$ the standard inclusion of $\mathscr{H}^p$ into the Siegel upper half-plane $\mathscr{H}_p$ of genus p as a set of diagonal matrices given by $\iota(\tau_1, \cdots, \tau_p) = \operatorname{diag}(\tau_1, \cdots, \tau_p)$.

Proposition 1.3 (Mok [25]) *The two holomorphic isometric embeddings $F, G : (\mathscr{H}, 2ds^2_{\mathscr{H}}) \hookrightarrow (\mathscr{H}_3, ds^2_{\mathscr{H}_3})$, $F := \iota \circ \rho_3$, are not congruent to each other. In fact, for any holomorphic isometric embedding $h : \mathscr{H} \hookrightarrow \mathscr{H} \times \mathscr{H} \times \mathscr{H}$, and for $H := \iota \circ h$, the two holomorphic embeddings $G, H : (\mathscr{H}, 2ds^2_{\mathscr{H}}) \hookrightarrow (\mathscr{H}_3, ds^2_{\mathscr{H}_3})$ are incongruent to each other.*

Here two holomorphic isometries $f, g : (D, \lambda ds^2_D) \to (\Omega, ds^2_\Omega)$ are said to be congruent to each other if and only if there exist $\varphi \in \operatorname{Aut}(D)$ and $\psi \in \operatorname{Aut}(\Omega)$ such that $g = \psi \circ f \circ \varphi$.

The main result of Mok [25] is the following theorem on the analytic continuation of germs of holomorphic isometries with respect to multiples of the Bergman metric under the assumption that Bergman kernels of both the domain and the target are rational.

Theorem 1.1 (Mok [25]) *Let $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ be bounded domains. Let $\lambda > 0$ and $f : (D, \lambda ds^2_D; x_0) \to (\Omega, ds^2_\Omega; y_0)$ be a germ of holomorphic isometry with respect to Bergman metrics up to a normalizing constant. Assume that the Bergman metrics on D and Ω are complete, that the Bergman kernel $K_D(z, w)$ on D extends to a rational function in $(z, \overline{w})$, and that analogously the Bergman kernel $K_\Omega(\xi, \eta)$ extends to a rational function in $(\xi, \overline{\eta})$. Then, f extends to a proper holomorphic isometric embedding $F : (D, \lambda ds^2_D) \hookrightarrow (\Omega, ds^2_\Omega)$. Moreover,* Graph$(f) \subset D \times \Omega$ *extends to an affine-algebraic subvariety $V \subset \mathbb{C}^n \times \mathbb{C}^N$.*

Note that if in Theorem 1.1 we weaken the hypothesis to assuming that $K_D(z, w)$ extends to a meromorphic function in $(z, \overline{w})$ to a neighborhood of $\overline{D} \times D'$ and likewise $K_\Omega(\xi, \eta)$ extends to a meromorphic function in $(\xi, \overline{\eta})$ to a neighborhood of $\overline{\Omega} \times \Omega'$, where for a Euclidean domain $G \subset \mathbb{C}^n$ we write $G' = \{z \in \mathbb{C}^n : \overline{z} \in G\}$, Theorem 1.1 holds for the germ of holomorphic isometry f when the last sentence is replaced by the statement that Graph$(f) \subset D \times \Omega$ extends to a complex-analytic subvariety on some neighborhood of $\overline{D} \times \overline{\Omega}$.

Consider the special case where D and Ω are complete circular domains, e.g., bounded symmetric domains in their Harish-Chandra realizations, and let $f : (D, \lambda ds^2_D; 0) \to (\Omega, ds^2_\Omega; 0)$ be a holomorphic isometry. From $f(0) = 0$ and the invariance of the Bergman kernels K_D and K_Ω under the circle group action, expanding in Taylor series at $0 \in D$ we have actually the equality $\log K_\Omega(f(z), f(z)) = \lambda \log K_\Omega(z, z) + a$ for some constant a, and hence by polarization the holomorphic functional identities $(\mathbf{I_{w_0}})$ $\log K_\Omega(f(z), f(w_0)) = \lambda \log K_\Omega(z, w_0) + a$ in z for any w_0 belonging to a sufficiently small neighborhood of $0 \in D$. If K_D and K_Ω are rational as assumed then differentiating the identities $(\mathbf{I_{w_0}})$ one can remove the logarithm and equivalently consider an infinite system $(\mathbf{J_{w_0}})$ of *algebraic* holomorphic identities. If now we replace $f(z)$ by the complex variables $\zeta = (\zeta_1, \cdots, \zeta_N)$, we obtain for each $w_0 \in U$ a subset $V_{w_0} \subset \mathbb{C}^n \times \mathbb{C}^N$ consisting of all (z, ζ) satisfying $(\mathbf{J_{w_0}})$, then the common zero set $V := \bigcap \{V_{w_0} : w_0 \in U\} \subset \mathbb{C}^n \times \mathbb{C}^N$ is an affine-algebraic subvariety containing Graph(f), and the difficulty was to prove that Graph$(f) \subset V$ is an open subset, so that $V \supset$ Graph(f) yields the desired extension, and that was precisely what was established in [25] under some simplifying assumption. In general, we prove that there exists an algebraic subvariety $V' \subset V$ defined using extremal functions for the Bergman kernel such that Graph$(f) = V' \cap \Omega$. The same argumentation applies after modification to the general situation in Theorem 1.1 where D and Ω need not be circular domains and where $x_0 \in D$ and $y_0 \in \Omega$ are arbitrary base points to yield a proof of the theorem.

We note that in the special case concerning commutants of modular correspondences raised in Clozel and Ullmo [14], the unsolved case was when $D = \mathbb{B}^n$, $n \geq 2$, and $\Omega = \mathbb{B}^n \times \cdots \times \mathbb{B}^n$ (with p factors), and the germ of holomorphic isometry up to scaling factors is given $f = (f^1, \cdots f^p)$, $f^k : (\mathbb{B}^n; 0) \to (\mathbb{B}^n; 0)$, such that $\det(df^k) \not\equiv 0$. In that case the corresponding special case in Theorem 1.1 is much easier, and was already obtained by the author in Mok [22] with the stronger conclusion that each f^k, $1 \leq k \leq p$ extends to an automorphism of $\mathbb{B}^n$, an assertion that follows after analytic continuation of each f^k across $\partial\mathbb{B}^n$ has been established by means of Alexander's theorem [1].

2 Existence and Classification Results

With reference to Theorem 1.1, the extension theorem for germs of holomorphic isometries, we will now specialize to the case where $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ are bounded symmetric domains in their Harish-Chandra realizations. As mentioned, in these cases the Bergman kernel $K_D(z, w)$ resp. $K_\Omega(\xi, \eta)$ is a rational function in $(z, \overline{w})$ resp. $(\xi, \overline{\eta})$. A bounded symmetric domain is always of nonpositive holomorphic bisectional curvature. If we restrict to the case where D is irreducible, as observed in Clozel and Ullmo [14], it follows from the proofs of Mok [20, 21] on Hermitian metric rigidity that any germ of holomorphic isometry $f : (D, ds_D^2; x_0) \to (\Omega, ds_\Omega^2; y_0)$ must necessarily be totally geodesic whenever D is of rank ≥ 2. Thus, the interesting case is where $D \cong \mathbb{B}^n$ is the n-dimensional complex unit ball, $n \geq 2$, in which case $\left(\mathbb{B}^n, ds_{\mathbb{B}^n}^2\right)$ is of strictly negative holomorphic bisectional curvature. The first examples were holomorphic isometric embeddings of the Poincaré disk into bounded symmetric domains, and for some time it was unknown whether nonstandard holomorphic isometries could exist when $n \geq 2$. This was raised as Problem 5.1.3 in [24]. When Ω is itself a complex unit ball $\mathbb{B}^N$, it follows from Umehara [47] that any germ of holomorphic isometry $f : \left(\mathbb{B}^n, \lambda ds_{\mathbb{B}^n}^2; x_0\right) \to \left(\mathbb{B}^N, ds_{\mathbb{B}^N}^2; y_0\right)$ must necessarily be totally geodesic. In fact it was proven in [47] that any germ of Kähler-Einstein complex submanifold on $\left(\mathbb{B}^N, ds_{\mathbb{B}^N}^2\right)$ must necessarily be totally geodesic. For the case where Ω is an irreducible bounded symmetric domain of rank ≥ 2, a priori we have the following restriction on the maximal dimension of a holomorphically and isometrically embedded complex unit ball. Here for the formulation we normalize the scalar multiple of the Bergman metric, which is Kähler-Einstein, such that minimal disks are of constant Gaussian curvature -2. The canonical Kähler-Einstein metric on Ω chosen this way will be denoted by h, and those on a complex unit ball $\mathbb{B}^m$ will be denoted by g, or by g_n when the dimension n is important for the discussion.

Theorem 2.1 (Mok [29]) *Let $\Omega \subset \Sigma$ be the Borel embedding of an irreducible bounded symmetric domain Ω into its dual Hermitian symmetric manifold Σ of the compact type, where* $\mathrm{Pic}(\Sigma) \cong \mathbb{Z}$*, generated by the positive line bundle $\mathcal{O}(1)$. Let g resp. h be the canonical Kähler-Einstein metric on $\mathbb{B}^n$ resp. Ω normalized so that minimal disks on $\mathbb{B}^n$ resp. Ω are of constant Gaussian curvature -2. Let $p = p(\Omega)$ be the nonnegative integer such that $K_\Sigma^{-1} \cong \mathcal{O}(p+2)$. Suppose $F : (B^n, g) \to (\Omega, h)$ is a holomorphic isometry (which is necessarily a proper holomorphic isometric embedding). Then $n \le p + 1$.*

For a uniruled projective manifold X equipped with a minimal rational component $\mathcal{K}$ and for a standard minimal rational curve ℓ (assumed smooth for convenience) belonging to $\mathcal{K}$ we have the Grothendieck decomposition $T(X)|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$ for some $p \ge 0$ independent of the choice of ℓ. From the deformation theory of rational curves, for a general point $x \in X$ and denoting by $\mathscr{C}_x(X)$ the variety of minimal rational tangents (VMRT) of $(X, \mathcal{K})$ at x, i.e., the variety of projectivizations of tangents to minimal rational curves passing through x (cf. Mok [30]), p is exactly the dimension of $\mathscr{C}_x(X)$. When X is of Picard number 1 and $\mathcal{K}$ pertains to a uniruling by rational curves of degree 1, as is the case of irreducible Hermitian symmetric spaces Σ of the compact type, we have $K_X^{-1} \cong \mathcal{O}(p+2)$. Since Σ is in particular homogeneous, the positive integer p in Theorem 2.1 is exactly the dimension of the VMRT at any point of Σ. We have also

Theorem 2.2 (Mok [29]) *Let $n \ge 1$, $\lambda > 0$, and $F : (\mathbb{B}^n, \lambda g) \to (\Omega, h)$ be a holomorphic isometry such that $\overline{F(\mathbb{B}^n)} \cap Reg(\partial\Omega) \ne \emptyset$. Then, $\lambda = 1$ and $n \le p + 1$.*

The basis of both Theorems 2.1 and 2.2 lies in the structure of $\partial\Omega$ (cf. Wolf [49]). Writing $G_0 = \mathrm{Aut}_0(\Omega)$ and $r := \mathrm{rank}(\Omega)$, $\partial\Omega$ decomposes into r disjoint union of G_0-orbits E_i, $1 \le i \le r$, such that $E_{k+1} \subset \overline{E_k}$. E_1 is the same as the smooth part $\mathrm{Reg}(\partial\Omega)$, which is foliated by maximal complex submanifolds of dimension $N - p - 1$. Consider the strictly plurisubharmonic function $\varphi_\Omega(z) := \frac{1}{p+2} \log K_\Omega(z, z)$, where $K_\Omega(z, w)$ stands for the Bergman kernel on Ω. Then, $\varphi_\Omega = -\log(-\rho_\Omega)$ where ρ_Ω is a real-analytic defining function of $\partial\Omega$ at any point on $\mathrm{Reg}(\partial\Omega)$.

By Theorem 1.1, for any holomorphic isometry F from the complex unit ball $\mathbb{B}^n$ to Ω with respect to scalar multiples of the Bergman metric, $\mathrm{Graph}(F)$ must necessarily extend to an affine-algebraic variety $V \subset \mathbb{C}^n \times \mathbb{C}^N$. From the assumptions in either Theorem 2.1 or Theorem 2.2 one can deduce that for a general point $a \in \partial\mathbb{B}^n$ there exists an open neighborhood G of a in $\mathbb{C}^n$ such that $F|_{G\cap\mathbb{B}^n}$ extends holomorphically to a holomorphic embedding $F^\sharp$ of G onto an n-dimensional complex submanifold $Z \subset U$ of some open neighborhood U of $b = f(a) \in \mathrm{Reg}(\partial\Omega)$. (For b sufficiently general the embedding can be chosen such that $(F^\sharp)^*\rho_\Omega$ is a defining function of $\mathbb{B}^n$ at a (i.e., it has nonzero gradient at a) although this fact need not be used in what follows.) From the strict pseudoconvexity of $\mathbb{B}^n$ it follows that the nonnegative

Levi form $\sqrt{-1}\partial\overline{\partial}\rho_\Omega|_{T_b^{1,0}(Z\cap\partial\Omega)}$ has $n-1$ positive eigenvalues, and the dimension estimate $n \le p+1$ follows from the foliated structure of $\mathrm{Reg}(\partial\Omega)$ described in the last paragraph, proving both Theorems 2.1 and 2.2.

Note that for a local strictly pseudoconvex domain U with smooth boundary defined by $\rho < 0$, where $d\rho$ is nowhere zero and ρ is strictly plurisubharmonic, letting θ be the Kähler metric with Kähler form $\sqrt{-1}\partial\overline{\partial}(-\log(-\rho))$, by a computation of Klembeck [17] (U, θ) is asymptotically of constant holomorphic sectional curvature -2 along the real hypersurface $\rho = 0$.

Going in the opposite direction we proved in [29] the existence of nonstandard holomorphic isometric embeddings of the $(p+1)$-dimensional complex unit ball into Ω.

Theorem 2.3 (Mok [29]) *Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank ≥ 2 and denote by Σ the irreducible Hermitian symmetric manifold of the compact type dual to Ω. Denoting by $\delta \in H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}$ the positive generator of the second integral cohomology group of Σ, we write $c_1(\Sigma) = (p+2)\delta$. Then, there exists a nonstandard proper holomorphic isometric embedding $F : \left(\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}}\right) \hookrightarrow \left(\Omega, ds^2_\Omega\right)$. More precisely, letting $\Omega \subset \Sigma$ be the Borel embedding of Ω into its dual Hermitian symmetric space of the compact type Σ, and denoting by $\mathscr{V}_x$ the union of all minimal rational curves on Σ passing through a point $x \in \Sigma$, for any smooth boundary point $q \in Reg(\partial\Omega)$, the intersection $V_q := \mathscr{V}_q \cap \Omega$ is the image of a holomorphic isometric embedding $F_q : \left(\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}}\right) \hookrightarrow \left(\Omega, ds^2_\Omega\right)$.*

The proof of Theorem 2.3 is geometric, and it follows from the aforementioned computation of Klembeck [17] on asymptotic curvature behavior along strictly pseudoconvex boundary points. The bounded symmetric domain $\Omega \Subset \mathbb{C}^N$ in its Harish-Chandra realization is a complete circular domain such that, given any complex line ℓ passing through $0 \in \Omega$, $\ell \cap \Omega$ is a disk centered at $0 \in \ell$ of radius between 1 and $\sqrt{r}$, $r := \mathrm{rank}(\Omega)$, and it is of radius 1 if and only if $[T_0\ell] \in \mathscr{C}_0(\Sigma)$. Consider $V_0 = \mathscr{V}_0 \cap \Omega$. Then $V_0 = \mathscr{V}_0 \cap \mathbb{B}^N$, and thus $\partial V_0 \subset \mathscr{V}_0$ is a strictly pseudoconvex real hypersurface. When $r \ge 2$, $\mathscr{V}_0$ is smooth except for the isolated singularity at 0. By Klembeck [17], the normalized Bergman metric $h := \frac{1}{p+2} ds^2_\Omega$ is asymptotically of constant holomorphic sectional curvature -2 along ∂V_0. Fix any line ℓ_0 passing through 0 such that $[T_0(\ell_0)] \in \mathscr{C}_0(\Sigma)$, pick any boundary point q of the minimal disk $\Delta_0 := \ell_0 \cap \Omega$, and consider a real one-parameter subgroup $\left\{\Phi_t : t \in \mathbb{R}\right\}$ of transvections in $\mathrm{Aut}_0(\Omega)$ fixing Δ_0 as a set such that $\Phi_t(0)$, $t \ge 0$, traverses a geodesic ray and converges to q in the Euclidean topology as $t \to \infty$. Then, $\Phi_t(V_0) = V_{\Phi_t(0)}$ and $V_{\Phi_t(0)}$ converges to V_q as subvarieties as $t \to \infty$. As a Kähler submanifold of (Ω, h), the local differential geometry of $(V_q, h|_{V_q})$ is identical to the asymptotic geometry of $\left(V_0, h|_{\mathrm{Reg(V_0)}}\right)$, hence $(V_q, h|_{V_q})$ is of constant holomorphic sectional curvature -2, and it must be the image of a holomorphic isometry of $(\mathbb{B}^{p+1}, g)$ into (Ω, h) by Theorem 1.1 (In this case it already follows from Calabi [5]). Since $ds_{\mathbb{B}^{p+1}} = (p+2)g$ and $ds_\Omega = (p+2)h$, we have equivalently that $F : \left(\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}}\right) \hookrightarrow \left(\Omega, ds^2_\Omega\right)$ is a holomorphic isometric embedding.

After constructing examples of holomorphic isometric embeddings from complex unit balls into irreducible bounded symmetric domains Ω as in Theorem 2.3, and given the dimension estimates on the complex unit balls on which such isometries may be defined, it is natural to study the set of all holomorphic isometries from $\left(\mathbb{B}^{p+1}, g\right)$ into (Ω, h). For this reason the author proposes an approach which reduces the problems to questions in the theory of geometric substructures (VMRT-substructures) on projective manifolds uniruled by projective lines. We note that this approach, which will be discussed in Sect. 3, applies in principle to all irreducible bounded symmetric domains excepting those of type IV, i.e., the n-dimensional Lie spheres D_n^{IV}, $n \geq 3$, and for that reason for some time it was not clear whether one should expect nonstandard holomorphic isometries of $F : \left(\mathbb{B}^{n-1}, ds^2_{\mathbb{B}^{n-1}}\right) \hookrightarrow (D_n^{IV}, ds^2_{D_n^{IV}})$ other than those defined by cones of minimal rational curves as given in Theorem 2.3. It turns out that such examples do exist. In fact, by a manipulation of polarized forms of functional equations arising from equating potential functions for Kähler metrics as in [25], Chan and Mok [9] was able to completely classify and describe all holomorphic isometries of complex unit balls into D_n^{IV} as given in the following theorem and its corollary.

The irreducible bounded symmetric domain in $\mathbb{C}^n$, $n \geq 3$, of type IV is given by

$$D_n^{IV} = \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^2 < 2,\ \sum_{j=1}^n |z_j|^2 < 1 + \left|\frac{1}{2}\sum_{j=1}^n z_j^2\right|^2\right\}$$

and the Kähler form corresponding to the Bergman metric $ds^2_{D_n^{IV}}$ on D_n^{IV} is given by $\omega_{ds^2_{D_n^{IV}}} = -n\sqrt{-1}\partial\overline{\partial}\log\left(1 - \sum_{j=1}^n |z_j|^2 + \left|\frac{1}{2}\sum_{j=1}^n z_j^2\right|^2\right)$. We have $h = \frac{1}{n}ds^2_{D_n^{IV}}$. For $\mathbf{v} \in M(1, n; \mathbb{C})$ we write $V_{\mathbf{v}} \subseteq \mathbb{C}^n$ for the affine-algebraic subvariety defined by $\sum_{j=1}^n v_j z_j - \frac{1}{2}\sum_{j=1}^n z_j^2 = 0$, and we write $\Sigma_{\mathbf{v}} := V_{\mathbf{v}} \cap D_n^{IV}$. Manipulating the functional equations as in Mok [25] relating Bergman kernels $K_{\mathbb{B}^m}(z, w)$ and $K_{D_n^{IV}}(\xi, \eta)$ via holomorphic isometries $F : (\mathbb{B}^m, \lambda ds^2_{\mathbb{B}^m}) \hookrightarrow (D_n^{IV}, ds^2_{D_n^{IV}})$, $\xi := F(z)$, $\eta = F(w)$, we were able to completely classify holomorphic isometries from complex unit balls into type-IV domains with respect to scalar multiples of the Bergman metric, as follows.

Theorem 2.4 (Chan and Mok [9]) *Let $F : (\mathbb{B}^m, \lambda ds^2_{\mathbb{B}^m}) \hookrightarrow (D_n^{IV}, ds^2_{D_n^{IV}})$ be a holomorphic isometric embedding, where $n \geq 3$ and $m \geq 1$ are integers. Then, either $\lambda = \frac{n}{m+1}$ or $\lambda = \frac{2n}{m+1}$ and we have the following.*

(1) If $\lambda = \frac{n}{m+1}$, then $1 \leq m \leq n-1$ and $F = \widetilde{f} \circ \rho$ for some holomorphic isometric embedding $\widetilde{f} : (\mathbb{B}^{n-1}, g_{n-1}) \hookrightarrow (D_n^{IV}, h)$ and some totally geodesic holomorphic isometric embedding $\rho : (\mathbb{B}^m, g_m) \hookrightarrow (\mathbb{B}^{n-1}, g_{n-1})$.

(2) If $\lambda = \frac{n}{m+1}$ and $m = n-1$, then F is congruent to a nonstandard holomorphic isometric embedding $\widehat{F}_{\mathbf{c}} : (\mathbb{B}^{n-1}, g_{n-1}) \hookrightarrow (D_n^{IV}, h)$ such that $\widehat{F}_c(\mathbb{B}^{n-1})$ is the irreducible component of $\Sigma_{\mathbf{c}}$ containing $\mathbf{0}$ for some $\mathbf{c} \in M(1, n; \mathbb{C})$ satisfying $\mathbf{c}\overline{\mathbf{c}}^t = 1$. In addition, F is congruent to $F_q : \left(\mathbb{B}^{n-1}, ds^2_{\mathbb{B}^{n-1}}\right) \hookrightarrow \left(D_n^{IV}, h\right)$ for

$q \in \mathrm{Reg}(\partial\Omega)$ (as given in Theorem 2.3) if and only if F is congruent to $\widehat{F}_{\mathbf{c}}$ for some $\mathbf{c}$ satisfying $\mathbf{c}\mathbf{c}^t = 0$.

(3) If $\lambda = \frac{2n}{m+1}$, then $m = 1$ and $F : (\Delta, n ds^2_{\Delta}) \hookrightarrow \left(D_n^{IV}, ds^2_{D_n^{IV}}\right)$ is totally geodesic.

Corollary 2.1 (Chan and Mok [9]) *Let $F : (\mathbb{B}^m, g_m) \hookrightarrow (D_n^{IV}, h)$ be a holomorphic isometric embedding, where $1 \le m \le n-2$ and $n \ge 3$. Then F is induced by some holomorphic isometric embedding $\widetilde{f} : (\mathbb{B}^{n-1}, g_{n-1}) \hookrightarrow (D_n^{IV}, h)$ via slicing of $\mathbb{B}^{n-1}$. More precisely $F = \widetilde{f} \circ \rho$ for some totally geodesic holomorphic isometric embedding $\rho : (\mathbb{B}^m, g_m) \hookrightarrow (\mathbb{B}^{n-1}, g_{n-1})$.*

Upmeier et al. [48], Xiao and Yuan [50] have independently obtained the classification result in Theorem 2.4 (on type-IV domains) for the case of $m = n-1$ and they give explicit parametrizations of the maps. Functional equations were made use of in [50] while in [48] the authors studied operators on Hilbert spaces induced by holomorphic isometries and made use of Jordan algebras. Moreover, for an arbitrary bounded symmetric domain Ω of rank ≥ 2 they gave an interesting characterization in terms of Jordan algebras of the holomorphic isometric embedding $F_q : \left(\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}}\right) \hookrightarrow \left(\Omega, ds^2_{\Omega}\right)$, as defined in Theorem 2.3, among all holomorphic isometric embeddings of $\mathbb{B}^{p+1}$ into Ω, where $p = p(\Omega)$.

Finally, we state a result with a sketch of the proof concerning bounded homogeneous domains which can be established along the lines of argument of Theorem 2.3. The proof of the latter theorem involves an argument ascertaining the convergence of a sequence of subvarieties which are the images of $V_0 = \mathscr{V}_0 \cap \Omega$ by a sequence of automorphisms, thereby obtaining in the limit a complex submanifold V_q which reflects the asymptotic geometry of V_0 near strictly pseudoconvex boundary points. For this type of argument to work on a bounded symmetric domain Ω, it is sufficient to have from the very beginning a local complex submanifold $Z \subset U$ on a neighborhood of a smooth boundary point $b \in \partial U$ such that Z intersects $\partial\Omega$ transversally along $Z \cap \partial\Omega$, $Z \cap \Omega \subset Z$ is strictly pseudoconvex along $Z \cap \partial\Omega$, a sequence of points $x_n \in Z$, $n \geq 1$, converging to b, and a sequence of automorphisms $\Phi_n \in \mathrm{Aut}(\Omega)$ such that $\Phi_n(x_n) = x_0$, where $x_0 \in \Omega$ is some fixed base point. It then follows from the fact that $(Z \cap \Omega, g|_{Z\cap\Omega})$ is asymptotically of holomorphic sectional curvature -2 that $Z_n := \Phi_n(Z \cap \Omega) \subset \Phi_n(U \cap \Omega)$ converges as a subvariety to some Kähler submanifold $Z_\infty \subset \Omega$ of constant holomorphic sectional curvature -2. Now the same set-up can be applied to the class of bounded homogeneous domains. These are bounded domains biholomorphic to homogeneous Siegel domains of the first or second kind constructed by Pyatetskii-Shapiro [42], which are biholomorphic to bounded domains via canonical isomorphisms (cf. Xu [51]) and the description in Mok [28, Sect. 5]). A bounded homogeneous domain $\mathscr{D}$ is weakly pseudoconvex and there is in a canonical realization $\mathscr{D} \subset \mathbb{C}^N$ a dense subset of smooth boundary points in the semi-algebraic boundary $\partial\mathscr{D}$. Moreover, the Bergman kernel $K_{\mathscr{D}}(\xi, \eta)$ is a rational function in $(\xi, \overline{\eta})$, from which one deduces from the aforementioned "*rescaling*" argument the existence of proper holomorphic isometric embeddings of the complex unit ball whose dimension is equal to the number of positive eigenvalues of the Levi form of a smooth defining function on the complex tangent spaces at such points. We have

Theorem 2.5 *Let $\mathscr{D} \Subset \mathbb{C}^N$ be a canonical realization of a bounded homogeneous domain. Let $b_0 \in Reg(\partial\mathscr{D})$ and let ρ be a smooth local defining function of $\mathscr{D}$ on a neighborhood of $b \in \mathbb{C}^N$. Suppose for b lying on some neighborhood of b_0 on $\partial\mathscr{D}$ the Levi form $\sqrt{-1}\partial\overline{\partial}\rho$ restricted to the complex tangent space $T_b^{1,0}(\partial\mathscr{D})$ has exactly s positive eigenvalues. Then, there exists a proper holomorphic isometric embedding $F : (\mathbb{B}^{s+1}, g) \hookrightarrow (\mathscr{D}, h)$ with respect to the normalized canonical Kähler-Einstein metric g resp. h on $\mathbb{B}^{s+1}$ resp. $\mathscr{D}$ such that* $\mathrm{Graph}(F) \subset \mathbb{C}^{s+1} \times \mathbb{C}^N$ *extends to an affine-algebraic subvariety.*

Here as before $(\mathbb{B}^{s+1}, g)$ is normalized so that minimal disks are of constant Gaussian curvature -2. On the other hand, $(\mathscr{D}, h)$ is normalized so that the Kähler form ω_h is given by $\omega_h = \sqrt{-1}\partial\overline{\partial}(-\log(-\rho'))$ where ρ' is a smooth defining function of D at a general point of $\partial\mathscr{D}$.

Regarding the very first examples of holomorphic isometric embeddings from the unit disk into polydisks, viz., the p-th root maps and maps obtained from them by means of composition, those maps remain the only known holomorphic isometric embeddings. It is a tempting yet challenging problem to ask whether the p-th root maps are in a certain sense the generators of the set $\mathbf{HI}(\Delta, \Delta^p)$ of all holomorphic isometric embeddings of the unit disk into polydisks. This was Problem 5.1.2 of Mok [24] which remains unsolved. For the more accessible problem of characterization the p-th root map, we have recently the work of Chan [6] which completed a partial result of Ng [36] solving the problem for $p = 2$ and for p odd, settling in the affirmative a characterization problem for the p-th root map in terms of sheeting numbers.

Theorem 2.6 (Chan [6], Ng [36] for $p = 2$ and for p odd) *Let $p \geq 2$ be an integer. If $f : (\Delta, ds^2_\Delta) \hookrightarrow (\Delta^p, ds^2_{\Delta^p})$ is a holomorphic isometric embedding with sheeting number $n = p$, then f is the p-th root embedding up to reparametrization.*

Here by Theorem 1.1 $\mathrm{Graph}(f)$ extends to an irreducible subvariety $V \subset \mathbb{P}^1 \times (\mathbb{P}^1)^p$, and the sheeting number n is by definition the sheeting number of the canonical projection $\pi : V \to \mathbb{P}^1$ onto the first factor. By a reparametrization of f we mean the composition $\psi \circ f \circ \varphi$, where $\varphi \in \mathrm{Aut}(\Delta)$ and $\psi \in \mathrm{Aut}(\Delta^p)$ (which includes permutations of the components). On top of Theorem 2.6, $\mathbf{HI}(\Delta, \Delta^p)$ is now completely determined for $p \leq 4$ (Ng [36] for $p = 2, 3$ and Chan [7] for $p = 4$).

Concerning holomorphic isometries of the complex unit ball into irreducible bounded symmetric domains Ω, other than the rank-1 case $\Omega \cong \mathbb{B}^n$, in which case all such maps are totally geodesic, and the case of Lie spheres $\Omega \cong D_n^{IV}$, $n \geq 3$, where there is a complete classification given by Theorem 2.4, other than some examples (as given in Sect. 1) not much is known about the set $\mathbf{HI}(\mathbb{B}^m, \Omega)$ of holomorphic isometries up to normalizing constants from the complex unit ball to Ω. On the existential side by considering holomorphic isometries of $\Delta \times \mathbb{B}^m$ into Ω for Ω of rank ≥ 2 not biholomorphic to a Lie sphere Chan and Yuan [12] have now obtained new examples of maps from complex unit balls into Ω incongruent to those obtained by restriction from $F_q : \mathbb{B}^{p+1} \hookrightarrow \Omega$ as given in Theorem 2.3 constructed from cones of minimal rational curves.

3 Structural and Uniqueness Results

Following up on the discussion on holomorphic isometries of the complex unit ball into irreducible bounded symmetric domain Ω in Sect. 2, Chan and Mok [9] have obtained the following general results on *bona fide* holomorphic isometric embeddings of the complex unit ball into Ω, i.e., for holomorphic isometries with respect to the Bergman metric or with respect to the normalized canonical Kähler-Einstein metrics (subject to the requirement that minimal disks are of constant Gaussian curvature -2) without normalizing constants, in which we consider $\Omega \subset \Sigma$ canonically as an open subset of its dual Hermitian symmetric space Σ of the compact type by the Borel embedding.

Theorem 3.1 (Chan and Mok [9]) *Let $f : (\mathbb{B}^n, g_n) \hookrightarrow (\Omega, h)$ be a holomorphic isometric embedding, where $n \geq 1$ and $\Omega \Subset \mathbb{C}^N$ is an irreducible bounded symmetric domain of rank ≥ 2 in its Harish-Chandra realization. Let $\Omega \subset \Sigma$ be the Borel embedding of Ω into its dual Hermitian symmetric space of the compact type Σ. Denote by $\iota : \Sigma \hookrightarrow \mathbb{P}\big(\Gamma(\Sigma, \mathcal{O}(1))^*\big)$ the minimal canonical projective embedding of Σ. Then, $f(\mathbb{B}^n)$ is an irreducible component of a complex-analytic subvariety $V \subseteq \Omega$ satisfying $\iota(V) = P \cap \iota(\Omega)$ for some projective linear subspace $P \subset \mathbb{P}\big(\Gamma(\Sigma, \mathcal{O}(1))^*\big)$.*

For $\mathbf{HI}_1(\mathbb{B}^m, \Omega)$, i.e., *bona fide* holomorphic isometric embeddings from $\mathbb{B}^{p+1}$ into Ω, $p = p(\Omega)$, our belief is that $F_q : \mathbb{B}^{p+1} \hookrightarrow \Omega$ of Theorem 2.3 are the only holomorphic isometries whenever Ω is not biholomorphic to a Lie sphere, which we confirm in the rank-2 cases, as follows.

Theorem 3.2 (Mok and Yang [33]) *Let $\Omega \subset \Sigma$ be the Borel embedding of an irreducible bounded symmetric domain Ω of rank 2 not biholomorphic to any type-IV domain $D_n^{IV}, n \geq 3$. Let $F : \big(\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}}\big) \hookrightarrow \big(\Omega, ds^2_{\Omega}\big)$ be a holomorphic isometric embedding, $p := p(\Omega)$, and write $Z := f(\mathbb{B}^{p+1})$. Then, there exists $q \in Reg(\partial\Omega)$ such that $Z = \mathscr{V}_q \cap \Omega$, where $\mathscr{V}_q \subset \Sigma$ is the union of minimal rational curves on Σ passing through q, and $F : \big(\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}}\big) \hookrightarrow \big(\Omega, ds^2_{\Omega}\big)$ is congruent to the holomorphic isometric embedding $F_q : \big(\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}}\big) \hookrightarrow \big(\Omega, ds^2_{\Omega}\big)$ given in Theorem 2.3.*

Recall here that p is given by $c_1(\Sigma) = (p+2)\delta$ and equivalently by $p = \dim(\mathscr{C}_x(\Sigma))$ for the VMRT $\mathscr{C}_x(\Sigma)$ at any point $x \in \Sigma$.

Theorem 3.2 covers the case where Ω is a type-I domain $D^I(2, q), q \geq 3$ (which is dual to the Grassmannian $G(2, q)$), the type-II domain $D^{II}(5, 5)$ (which is dual to the 10-dimensional orthogonal Grassmannian $G^{II}(5, 5)$) and the 16-dimensional exceptional domain D^V (which is of type E_6). The analogous uniqueness results when

Ω is a type-I domain $D^I(3, q)$, $q \geq 3$, or when Ω is the 27-dimensional exceptional domain D^{VI} (which is of type E_7) have also been established, cf. Yang [52]. The first open case is that of the type-III domain $D^{III}(3, 3)$ (which is biholomorphic to the Siegel upper half-plane $\mathscr{H}_3$ and dual to the 6-dimensional Lagrangian Grassmannian $G^{III}(3, 3)$), the only remaining irreducible bounded symmetric domain of rank 3.

Here is a sketch of an approach that the author has proposed for proving uniqueness up to reparametrization of nonstandard holomorphic isometries $F : \big(\mathbb{B}^{p+1}; g\big) \hookrightarrow (\Omega, h)$, $p = p(\Omega)$, for irreducible bounded symmetric domains Ω of rank $\geq$2 not biholomorphic to a type-IV domain D_n^{IV}, $n \geq 3$. Recall that $\Omega \subset \Sigma$ is the Borel embedding, and that for $x \in \Sigma$, $\mathscr{V}_x$ is a union of minimal rational curves (i.e., projective lines on Σ with respect to the first canonical embedding $\nu : \Sigma \hookrightarrow \mathbb{P}(\Gamma(\Sigma, \mathcal{O}(1))^*)$). $\mathscr{V}_x$ has an isolated singularity x, and $\mathscr{V}_x$ is homogeneous under the stabilizer $H \subset \mathrm{Aut}(\Sigma)$ of $\mathscr{V}_x$. Thus, for $y_1, y_2 \in \mathscr{V}_x - \big\{x\big\}$, there exists $\varphi \in \mathrm{Aut}(\Sigma)$ such that $\varphi(y_1) = \varphi(y_2)$ and such that $d\varphi(T_{y_1}(\mathscr{V}_x)) = T_{y_2}(\mathscr{V}_x)$. Given $Z = F(\mathbb{B}^{p+1})$, the author proposed to show that at a general point $z \in Z$, the inclusion $(T_z(Z) \subset T_z(\Sigma))$ is transformed to $(T_y(\mathscr{V}_x) \subset T_y(\Sigma))$ for a smooth point y on $\mathscr{V}_x$ by $d\psi$ for some $\psi \in \mathrm{Aut}(\Sigma)$ such that $\psi(z) = y$. Our strategy consists more precisely of (a) identifying the isomorphism class of the inclusion $(T_z(Z) \subset T_z(\Sigma))$ under the action of $\mathrm{Aut}(\Sigma)$ as described; (b) reconstructing Z as an open subset of some $\mathscr{V}_x$, and (c) proving that $x = q \in \partial\Omega$.

The approach turns out to work for the rank-2 cases in Theorem 3.2 and the rank-3 cases in Yang [52]. Step (a) was established using methods of local differential geometry based on the Gauss equations for the holomorphic isometry $F : \big(\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}};\big) \hookrightarrow \big(\Omega, ds^2_\Omega\big)$. Step (b) is implemented by means of techniques of reconstructing germs of complex submanifolds $(S; x_0)$ equipped with sub-VMRT structures (cf. Mok and Zhang [34]) modeled on certain uniruled projective subvarieties of classical Fano manifolds of Picard number 1. In general such reconstruction consists of proving linear saturation (i.e., the property that the germ of a projective line $(\ell; x)$ tangent to S at $x \in S$ must necessarily lie on $(S; x_0)$) by verifying certain nondegeneracy conditions expressed in terms of second fundamental forms, followed by a process of adjunction of minimal rational curves, as introduced in Mok and Zhang [34] and discussed in the expository article Mok [30]. In the case at hand the models are the Schubert subvarieties $\mathscr{V}_x \subset \Sigma$ which are uniruled by projective lines outside the isolated singularity $x \in \mathscr{V}_x$ (with the exception of Lagrangian Grassmannians for which the method does not apply). Once we have identified Z as an open subset of some $\mathscr{V}_x$, it follows from Theorem 1.1 that $x \notin \Omega$ since $Z \subset \Omega$ must be nonsingular. From the identity theorem for real-analytic functions it follows easily that $x \in \mathrm{Reg}(\partial\Omega)$.

4 Boundary Behavior of Holomorphic Isometries

Regarding the boundary behavior of holomorphic isometric embeddings of the Poincaré disk into a bounded symmetric domain, we have proved the following general result on the boundary behavior of locally closed holomorphic curves on a bounded symmetric domain Ω when the holomorphic curves exit $\partial\Omega$.

Theorem 4.1 (Chan and Mok [10]) *Let $b_0 \in \partial\Delta$, U be an open neighborhood of a point b_0 in $\mathbb{C}$, $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain in its Harish-Chandra realization, and let $\mu : U \hookrightarrow \mathbb{C}^N$ be a holomorphic embedding such that $\mu(U \cap \Delta) \subset \Omega$ and $\mu(U \cap \partial\Delta) \subset \partial\Omega$. Then, μ is asymptotically totally geodesic at a general point $b \in U \cap \partial\Delta$. More precisely, denoting by $\sigma(z)$ the second fundamental form of $\mu(U \cap \Delta)$ in (Ω, ds^2_Ω) at $z = \mu(w)$, for a general point $b \in U \cap \partial\Delta$ we have* $\lim_{w \in U\cap\Delta,\ w\to b} \|\sigma(\mu(w))\| = 0$.

Since by Theorem 1.1 the graph of any holomorphic isometry (with respect to scalar multiples of the Bergman metric) between bounded symmetric domains in their Harish-Chandra realizations extends to an affine-algebraic variety, it follows from Theorem 4.1 that we have the following result on holomorphic isometries from the unit disk to bounded symmetric domains.

Theorem 4.2 *Let $f : (\Delta, \lambda ds^2_\Delta) \hookrightarrow (\Omega, ds^2_\Omega)$ be a holomorphic isometric embedding, where λ is a positive constant and $\Omega \Subset \mathbb{C}^N$ is a bounded symmetric domain in its Harish-Chandra realization. Then, f is asymptotically totally geodesic at a general point $b \in \partial\Delta$.*

As a consequence of Theorem 4.2, we have

Theorem 4.3 (Chan and Mok [10], Clozel [13] for the classical cases) *Let D and Ω be bounded symmetric domains, $\Phi : Aut_0(D) \to Aut_0(\Omega)$ be a group homomorphism, and $F : D \to \Omega$ be a Φ-equivariant holomorphic map. Then, F is totally geodesic.*

Theorem 4.1 was first proved by Mok [27] under the stronger assumption that $\mu(U \cap \partial\Delta) \subset \mathrm{Reg}(\partial\Omega)$. In that case we obtained at the same time the estimate that for a general point $b \in U \cap \partial\Delta$, there exists a relatively compact open neighborhood U_0 of b in U and a constant $C \geq 0$ such that the estimate $\|\sigma(\mu(w))\| \leq C(1 - \|w\|)$ holds for $w \in U_0 \cap \Delta$. The proof in Mok [27] is direct and elementary. Although it is tempting to generalize the arguments of [27] to the general situation where $\mu(U \cap \Delta)$ exits an arbitrary stratum of the boundary (in its decomposition into $\mathrm{Aut}_0(\Omega)$-orbits), the problem is more delicate than it appears, and in Chan and Mok [10]) we presented instead an indirect proof involving rescaling and the use of the Poincaré-Lelong equation adopting a methodology which is of independent interest in its own right.

The proof in [10], which is sketched below, is by argument by contradiction, and that is the reason why an asymptotic estimate of $\|\sigma\|$ is lacking in general. Write $Z = \mu(U \cap \Delta)$ and $Z^\sharp = \mu(U)$. For a general point $b \in U$, $Z^\sharp$ is smooth at $\mu(b)$ and the restriction of the Bergman metric of Ω on Z is of asymptotically constant Gaussian curvature at $\mu(b)$. Suppose for the sake of argument by contradiction that μ is not asymptotically totally geodesic at b. By rescaling one extracts a holomorphic mapping F of the Poincaré disk which reflects the asymptotic behavior of μ at b. In particular F is a holomorphic isometric embedding since μ is asymptotically of constant Gaussian curvature. We may further rescale F if necessary and assume that the holomorphic isometry F is as "uniform" as one desires (e.g., we may require that the norm of the second fundamental form to be constant), and we obtain a contradiction to the existence of a certain "rescaled" hypothetical and nonstandard holomorphic isometric embedding of the Poincaré disk by reducing it to the case where $Z' := F(\Delta)$ lies on a tube domain of rank $s \le r := \mathrm{rank}(\Omega)$, and where nonzero vectors tangents to Z' are of rank s, and by applying the Poincaré-Lelong equation to the logarithm of the (constant) norm of some "tautological" section of an $\mathrm{Aut}(\Omega)$-homogeneous holomorphic line bundle over Z' (cf. [10]). In the case of type-III domains (which are biholomorphic to Siegel upper half-planes) the tautological section is a "twisted determinant" on tangents to the curve when tangent vectors are identified with symmetric matrices.

Theorem 4.2 should be contrasted with the existence result Theorem 2.3. There, for Ω an irreducible bounded symmetric domain of rank ≥ 2 and for $q \in \mathrm{Reg}(\partial\Omega)$ the holomorphic isometric embedding $F_q : \left(\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}}\right) \hookrightarrow \left(\Omega, ds^2_\Omega\right)$ such that $F_q(\mathbb{B}^{p+1}) = V_q = \mathscr{V}_q \cap \Omega$ is never asymptotically totally geodesic.

5 Zariski Closures of Images of Algebraic Sets Under the Uniformization Map

For a bounded symmetric domain Ω denote by $\Omega \Subset \mathbb{C}^N \subset \Sigma$ the standard inclusions incorporating both the Harish-Chandra embedding $\Omega \Subset \mathbb{C}^m$ into a Euclidean space and the Borel embedding $\Omega \subset \Sigma$ into its dual Hermitian symmetric space of the compact type. A subvariety $S \subset \Omega$ is said to be an irreducible algebraic (sub)set if and only if it is an irreducible component of the intersection $\mathscr{V} \cap \Omega$ for some projective subvariety $\mathscr{V} \subset \Sigma$. An algebraic subset $S \subset \Omega$ is by definition the union of a finite number of irreducible algebraic subsets of Ω. In the case where $\Omega = \mathbb{B}^n$, $n \ge 2$, note that a totally geodesic complex submanifold of $\mathbb{B}^n$ is precisely a non-empty intersection of the form $\Pi \cap \mathbb{B}^n$, where $\Pi \subset \mathbb{P}^n$ is a projective linear subspace of $\mathbb{P}^n$.

Any totally geodesic complex submanifold $\Xi \subset \Omega$ is an open subset of its dual Hermitian symmetric space of the compact type Θ, $\Theta \subset \Sigma$, so that $\Xi \subset \Omega$ is an example of an algebraic subset. In connection with problems on a "dual" projective geometry on quotients $X_\Gamma := \mathbb{B}^n/\Gamma$ of the complex unit ball by a torsion-free lattice $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$, the author was led first of all to study Zariski closures of images of totally geodesic complex submanifolds $S \subset \Omega$ under the uniformization map $\pi : \mathbb{B}^n \to X_\Gamma := \mathbb{B}^n/\Gamma$. From a geometric perspective the same problem can be raised when $\mathbb{B}^n$ is replaced by a bounded symmetric domain Ω. It transpires that similar questions were raised in number theory and functional transcendence theory. In fact, it was conjectured that the Zariski closure of the image of an algebraic subset $S \subset \Omega$ under the uniformization map $\pi : \Omega \to X_\Gamma := \Omega/\Gamma$ must necessarily be a totally geodesic subset when the lattice Γ is *arithmetic*. The latter is known as the Hyperbolic Ax-Lindemann Conjecture, and it is one of the two components for giving an unconditional proof of the André-Oort Conjecture following the scheme of proof of Pila and Zannier [41]. (See last two paragraphs of Sect. 5 for more details.) Here X_Γ is equipped with a canonical quasi-projective structure as given by Baily and Borel [3]. From a purely geometric perspective there is no reason why one needs to restrict to arithmetic lattices, although assuming Ω to be irreducible by Margulis [19] nonarithmetic lattices $\Gamma \subset \mathrm{Aut}(\Omega)$ only occur in the rank-1 cases, i.e., in the cases of $\Omega = \mathbb{B}^n, n \geq 1$. Focusing on the rank-1 cases and using methods of complex differential geometry we have proven the following theorems.

Theorem 5.1 (Mok [31]) *Let $n \geq 2$ be an integer and let $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ be a torsion-free lattice. Denote by $X_\Gamma := \mathbb{B}^n/\Gamma$ the quotient manifold, of finite volume with respect to the canonical Kähler-Einstein metric $ds^2_{X_\Gamma}$ induced from the Bergman metric $ds^2_{\mathbb{B}^n}$. Let $\pi : \mathbb{B}^n \to X_\Gamma$ be the universal covering map and denote by $S \subset \mathbb{B}^n$ an irreducible algebraic subset. Then, the Zariski closure $Z \subset X_\Gamma$ of $\pi(S)$ in X_Γ is a totally geodesic subset.*

Theorem 5.2 (Mok [31]) *Let A be any set of indices and $\Sigma_\alpha \subset X_\Gamma$, $\alpha \in A$, be a family of closed totally geodesic subsets of X_Γ of positive dimension. Write $E := \bigcup \{\Sigma_\alpha : \alpha \in A\}$. Then, the Zariski closure of E in X_Γ is a union of finitely many totally geodesic subsets.*

For a possibly *nonarithmetic* lattice $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$, X_Γ in the above is endowed a canonical quasi-projective structure defined by a compactification result of Mok [26]. The latter result is deduced from L^2-estimates of $\overline{\partial}$ and from the compactification theorem of Siu and Yau [43] yielding a Moishezon compactification of complete Kähler manifolds of finite volume and of pinched strictly negative sectional curvature.

Proposition 5.1 (Mok [26]) *Writing X for X_Γ in the notation of the preceding theorems, there exists a projective variety $\overline{X}_{\min}$ such that $X = \overline{X}_{\min} - \{p_1, \cdots, p_m\}$, where each p_i, $1 \leq i \leq m$, is a normal isolated singularity of $\overline{X}_{\min}$.*

In [31] we introduce a new framework into problems for functional transcendence on not necessarily arithmetic finite-volume quotients of bounded symmetric domains, although the methods were only applied there to the rank-1 case. In what follows let $\Omega \Subset \mathbb{C}^N$ be any possibly reducible bounded symmetric domain, G_0 be the identity component of $\mathrm{Aut}(\Omega)$, $\Gamma \subset G_0$ be a torsion-free lattice, and $\Omega \subset \Sigma$ be the Borel embedding. Write G for the identity component of $\mathrm{Aut}(\Sigma)$. Let now $\mathscr{V} \subset \Sigma$ be an irreducible subvariety and $S \subset \Omega$ be an irreducible algebraic subset, $\dim(S) =: s$, which is an irreducible component of $\mathscr{V} \cap \Omega$, and for the ensuing discussion assume for convenience that the reduced subvariety $\mathscr{V} \subset \Sigma$ corresponds to a smooth point of some irreducible component $\mathcal{K}$ of the Chow space $\mathrm{Chow}(\Sigma)$ of Σ. Let $\rho : \mathscr{U} \to \mathcal{K}$, $\mu : \mathscr{U} \to \Sigma$ be the universal family of $\mathcal{K}$, and write $\mu_0 : \mathscr{U}_0 := \mathscr{U}|_{\mu^{-1}(\Omega)} \to \Omega$ be the restriction of $\mathscr{U}$ over Ω. Then G acts on $\mathscr{U}_0$ and by restriction G_0 acts on $\mathscr{U}_0$.

Write $X_\Gamma := \Omega/\Gamma$, which is equipped with a canonical quasi-projective structure, $\pi_\Gamma : \Omega \to X_\Gamma$ for the uniformization map and define $Z \subset \overline{\pi(S)}^{\mathscr{Z}ar}$ for the

Zariski closure of $\pi(S)$ in X_Γ, which admits a canonical quasi-projective structure. $\mu_0 : \mathscr{U}_0 \to \Omega$ descends to X_Γ to give a locally homogeneous holomorphic fiber bundle $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$. In case $\Gamma \subset G_0$ is cocompact, then $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ is projective. When $\Gamma \subset G_0$ is a nonuniform lattice, by the differential-geometric method of compactification of Mok and Zhong [35] we have a quasi-projective compactification $\overline{\mu_\Gamma} : \overline{\mathscr{U}_\Gamma} \to \overline{X_\Gamma}$. In this context the meromorphic foliation $\mathscr{F}$ on $\mathscr{U}$ defined by tautological liftings of $\mathscr{W} \subset \Sigma$ (of members $\mathscr{W}$ belonging to $\mathcal{K}$) extends meromorphically to $\overline{\mathscr{U}_\Gamma}$. (The proof of the extension was only written for the rank-1 case, but can be strengthened using [35] to the general case.) The simplifying assumption that $\mathscr{V}$ corresponds to a smooth point of $\mathcal{K}$ implies that $\mathscr{F}$ is holomorphic at a general point of $\mathscr{V}$. We take the Zariski closure of the tautological lifting $\mathscr{S} \subset \mathscr{U}_\Gamma$ to obtain $\mathscr{Z} = \overline{\mathscr{S}}^{\mathscr{Z}ar} \subset \mathscr{U}_\Gamma$ and we have $Z = \mu_\Gamma(\mathscr{Z})$. We proved in [31] that $\mathscr{Z}$ is saturated with respect to the foliation $\mathscr{F}$. Let now $\widetilde{Z} \subset \Omega$ be an irreducible component of $\pi_\Gamma^{-1}(Z)$. From the saturation of $\mathscr{Z}$ under $\mathscr{F}$ it follows that $\widetilde{Z}$ is in a neighborhood of a general point $x \in S$ the union of an analytic family Φ of (connected open subsets of) members of $\mathcal{K}$, and the union of the members of Φ then contains the germ of an s-dimensional complex submanifold Ξ of Σ at $b_0 \in \partial\widetilde{Z}$.

Now we restrict to the rank-1 situation covered by Theorems 5.1 and 5.2. At a general point $b \in \xi \cap \partial\mathbb{B}^n$, $\Xi \cap \mathbb{B}^n \subset \Xi$ is *strictly pseudoconvex* at b, and from a computation of Klembeck [17] $\widetilde{Z}$ is asymptotically of constant negative holomorphic sectional curvature. By a comparison with curvatures of $(\mathbb{B}^n, g)$ we conclude that $\widetilde{Z} \subset \mathbb{B}^n$ is asymptotically totally geodesic. Now $Z = \widetilde{Z}/\Gamma'$ for some infinite subgroup $\Gamma' \subset \Gamma$. In case $\Gamma \subset G_0$ is cocompact let $U \Subset \widetilde{Z}$ be an open relatively compact subset such that $\pi(U) = Z$. From strict pseudoconvexity of $\widetilde{Z}$ at b there exists a sequence of elements $\gamma_n \in \Gamma'$ such that $\gamma_n(x)$ converges to b for any $x \in U$, and we conclude that $\widetilde{Z} \subset \mathbb{B}^n$ is totally geodesic, which gives Theorems 5.1 and 5.2 in the cocompact case. For the modification in this last step to the case of a nonuniform lattice $\Gamma \subset G_0$ we refer the reader to [31].

Here we get the total geodesy of $\widetilde{Z} \Subset \mathbb{B}^n$ directly. A slight reformulation of this last step makes it applicable to the higher rank situation, in the event that $\widetilde{Z}$ happens to be strictly pseudoconvex at b, as follows. From the asymptotic curvature property of $\widetilde{Z}$ at b we conclude that $\widetilde{Z}$ is the image of a holomorphic isometry $F : \mathbb{B}^s \to \Omega$ with respect to multiples of the Bergman metric. By Mok [26], $\widetilde{Z} \subset \Omega$ is algebraic. In other words, both the domain and the target of the covering map $\pi_\Gamma|_{\widetilde{Z}} : \widetilde{Z} \to Z$ are algebraic. If the lattice $\Gamma \subset G_0$ is arithmetic, then we can conclude that $\widetilde{Z} \subset \Omega$ and $Z \subset X_\Gamma$ are totally geodesic, by Ullmo and Yafaev [45]. Without the arithmeticity assumption but using the special property that $\widetilde{Z}$ is strictly pseudoconvex at a general point of $\partial\widetilde{Z}$, one can easily show that $\widetilde{Z}$ is homogeneous under an algebraic subgroup of G_0. Since $\widetilde{Z}$ admits a quotient of finite volume, this implies that $\widetilde{Z}$ is a holomorphic isometric copy of a bounded symmetric domain, and hence totally geodesic by Chan and Mok [10, Theorem 5.19].

For the reader who may like to see how complex differential geometry could interact with questions in diophantine geometry and functional transcendence, here is a digression around the Hyperbolic Ax-Lindemann Conjecture. We start with a special case of Ax's Theorem (Ax [2]) on $\mathbb{C}^n$, with Euclidean coordinates $(z_1, \cdots z_n)$. Let $V \subset \mathbb{C}^n$ be an m-dimensional irreducible affine algebraic subvariety. Assume that 0 is a smooth point on V and that $(z_1, \cdots, z_m)$ serve as holomorphic local coordinates on V at 0. For $1 \leq k \leq n$ define $f_k = 2\pi i z_k|_V$. Define $\pi : \mathbb{C}^n \to (\mathbb{C}^*)^n$ by $\pi(z_1, \cdots, z_n) = (e^{2\pi i z_1}, \cdots, e^{2\pi i z_n})$. Then, by [2] the transcendence degree of the field of functions on V generated by $\left\{f_1, \cdots, f_n; e^{f_1}, \cdots, e^{f_n}\right\}$ is equal to $n + m$ unless V lies on a $\mathbb{Q}$-hyperplane of $\mathbb{C}^n$, i.e., unless $\pi(V)$ is contained in an algebraic torus, equivalently a totally geodesic subvariety of $(\mathbb{C}^*)^n$ with respect to the Kähler metric on $(\mathbb{C}^*)^n$ induced from the (translation-invariant) Euclidean metric ds^2_{euc} on $\mathbb{C}^n$ by the uniformization map π. In particular, when the n component functions of $\pi|_V$ are algebraically dependent, it follows readily that the Zariski closure of $\pi(V) \subset (\mathbb{C}^*)^n$ is an algebraic torus $T \subsetneq (\mathbb{C}^*)^n$. This gives the Ax-Lindemann Theorem for the exponential map.

If for Ax-Lindemann we replace $\mathbb{C}^n$ by a bounded symmetric domain and denote now by $\pi : \Omega \to \Omega/\Gamma =: X_\Gamma$ the uniformization map for a torsion-free *arithmetic* lattice $\Gamma \subset \mathrm{Aut}(\Omega)$, and replace $V \subset \mathbb{C}^n$ by an algebraic subset $S \subset \Omega$, the analogous conjectural statement was that the Zariski closure of $\pi(S)$ in X_Γ is necessarily a totally geodesic subset, commonly called the Hyperbolic Ax-Lindemann Conjecture. Pila and Zannier [41] adopted a strategy for conjectures regarding special points. For the André-Oort Conjecture regarding Zariski closures of sets of special points on Shimura varieties it led to a reduction of the conjecture into two components, a number-theoretic component concerning lower bounds for the sizes of Galois orbits and a geometric component which is precisely the Hyperbolic Ax-Lindemann Conjecture, a conjecture by now confirmed by Ullmo and Yafaev [46] in the cocompact case, by Pila and Tsimerman [39] for $\mathscr{A}_g$, and by Klingler et al. [18] in the general case. A crucial ingredient is o-minimal geometry, especially counting arguments on rational points of Pila and Wilkie [40] in a model-theoretic context (cf. also Bombieri and Pila [4]), and methods of Peterzil and Starchenko [37] on tame complex analysis. Complex differential geometry entered into play in [18], where volume estimates of Hwang and To [16] on subvarieties on bounded symmetric domains were used in an essential way. Applications of hyperbolic Ax-Lindemann to number theory are given in Ullmo [44], and uses of o-minimality in functional transcendence are expounded in Pila [38]. For the broader context of problems in arithmetic and geometry related to unlikely intersection, we refer the reader to Zannier [53].

6 Perspectives and Concluding Remarks

In this article the author has been discussing various aspects of his recent works in part with collaborators on the topic of holomorphic isometries in Kähler geometry, together with some references to related recent results by other researchers. Our discussion was on research problems intrinsic to complex differential geometry and also on the use of holomorphic isometries in other contexts. To gauge how research on the topic could develop in the future the author would venture to examine it (a) from the point of view of a complex differential geometer, (b) in connection with applications to other subject areas in mathematics and (c) with an eye on identifying new directions of research. As is the flavor of this article, these are only reflections from the author prompted by his own research involvement and does not represent a comprehensive overview on the subject.

The study of holomorphic isometries on Kähler geometry is by its very definition intrinsic to complex differential geometry, and the subject took shape from works of Bochner and Calabi, notably the seminal work of Calabi [5]. In the tradition of classical differential geometry and focusing on bounded symmetric domains, recent progress on the subject include existence and uniqueness results, structural and characterization theorems and results on the asymptotic geometry of holomorphic isometric embeddings into bounded symmetric domains. The structure of the full set of holomorphic isometries between two given bounded symmetric domains is a natural object of study, but so far only in very special cases are we in a position to classify such maps. For instance, in the case of maps from the Poincaré disk into polydisks, while classical techniques involving functional identities arising from diastases and the study of branched coverings of compact Riemann surfaces have led to some neat classification results in low dimensions, the complexity of such objects grows very fast with dimensions, and it appears that in this and other characterization problems new tools are necessary for general results.

As an example, in the study of holomorphic isometries from complex unit balls of maximal dimension into an irreducible bounded symmetric domain, the geometric theory of varieties of minimal rational tangents has been a source of methods both for existence results and for structural results notably in the reconstruction of images of holomorphic isometric embeddings via the method of geometric substructures. Another new source of methods is the use of Jordan algebras and operator theory on Hilbert spaces in Upmeier et al. [48]. It is for instance tempting to study holomorphic isometric embeddings from the Poincaré disk into polydisks via linear isometries between Hilbert spaces of functions, and the methods developed could shed light on factorization problems especially the question whether the p-th root maps are the “generators” of holomorphic isometries between polydisks. The space of holomorphic isometries from the complex unit ball into a bounded symmetric domain forms a real algebraic variety, and the result that images of *bona fide* holomorphic isometries arise from linear sections with respect to the minimal canonical embedding gives an effective bound on the number of parameters for its description. From Chan [8] one would expect that the same is valid for holomorphic isometries with other normalizing constants, and a confirmation of that belief in the general case would be a unifying result for bounded symmetric domains.

The author was led to consider holomorphic isometries of Kähler manifolds and related topics (such as holomorphic measure-preserving maps) in order to answer questions concerning modular correspondences on finite-volume quotients of bounded symmetric domains, notably questions in arithmetic dynamics raised by Clozel and Ullmo [14] concerning commutants of such correspondences. Thus, it was around bounded symmetric domains that the author started his investigation, and it is gratifying to see that the study of holomorphic isometries from complex unit balls into bounded symmetric domains finds its way into problems in functional transcendence theory. A primary issue in functional transcendence theory is the question of generating algebraic subsets from processes which are a priori complex-analytic, and the use of holomorphic isometries provides such a means, viz., from the algebraicity of such maps due to the rationality of Bergman kernels. The link between holomorphic isometries with the uniformization map arises when some *strictly pseudoconvex* complex tangential directions are picked up as complex-analytic families of algebraic subsets traverse the boundary of bounded symmetric domains, and the asymptotic curvature behavior is then recaptured through rescaling arguments. Further input from complex differential geometry into problems in functional transcendence theory could involve the study of asymptotic geometric behavior as complex-analytic families of algebraic subsets exit the boundary of bounded symmetric domains (or more generally flag domains, cf. second last paragraph). In this regard, the work of Chan and Mok [10] on the asymptotic curvature behavior when holomorphic curves exit boundaries of bounded symmetric domains is a step in this direction, and for the general question the fine structure of boundaries of bounded symmetric domains in their Harish-Chandra realizations (cf. Wolf [49]) will enter into play.

In other directions holomorphic isometries in Kähler geometry are a source of examples for further study in other areas of mathematics. In several complex variables they may give new examples of proper holomorphic maps exhibiting new qualitative behavior (e.g., Chan et al. [11]). They may give algebraic subsets of bounded symmetric domains admitting interesting geometric substructures (e.g., holomorphic isometries of complex unit balls into Lie spheres give rise to possibly degenerate holomorphic conformal structures), and they are also related to the study of some special Schubert varieties (cf. [29, 32]). The latter examples could motivate differential-geometric characterizations of (open subsets of) wider classes of Schubert varieties.

In the seminal paper of Calabi [5] complex space forms endowed with pseudo-Kähler metrics were already studied. Beyond bounded symmetric domains it would be natural, both from the point of view of developing the theory of holomorphic isometries in a pseudo-Kählerian context, and in connection with the study of functional transcendence, to generalize to the study of quotients of large classes of flag domains (cf. Fels et al. [15]), especially those admitting invariant pseudo-Kähler metrics and compatible filtrations of the holomorphic tangent bundles. These include the period domains for complex variations of Hodge structures, but ought to be broader in scope, and it will be interesting to construct horizontal algebraic subvarieties of such flag domains and to study *horizontal* holomorphic isometries into such flag domains and their connection to various questions in functional transcendence.

Finally, research on holomorphic isometries in Kähler geometry may have acted as a *catalyst* to highlight the relevance of complex differential geometry in the study of Shimura varieties which are by their very definition arithmetic quotients of bounded symmetric domains. Years ago the author had harbored the hope that Bergman metrics could be made use of in treating number-theoretic problems on Shimura varieties. With the advance of methods from diophantine geometry and model theory and the schematic reduction of outstanding problems on special points to issues involving unlikely intersection, the hope is already reality. From the point of view of complex differential geometry the study of holomorphic isometries serves perhaps as a path through the fascinating territory of Shimura varieties and their generalizations, and its proper role in tackling problems in functional transcendence will depend on future interaction between analytic, algebraic and model-theoretic perspectives in the study of such varieties.

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Ergodic Theory for Riemann Surface Laminations: A Survey

Viêt-Anh Nguyên

Abstract We survey some recent developments in the ergodic theory for hyperbolic Riemann surface laminations. The emphasis is on singular holomorphic foliations. These results not only illustrate the strong similarities between the ergodic theory of maps and that of Riemann surface laminations, but also indicate the fundamental differences between these two theories.

Keywords Riemann surface lamination · Leafwise Poincaré metric
Positive harmonic currents · Ergodic theorems · Lyapunov exponents
Hyperbolic entropy

1 Introduction

These notes are based on a series of lectures given by the author at KIAS and at the KSCV Symposium 11 in Gyeongju in 2016. The purpose is to review some developments in the ergodic theory of laminations by hyperbolic Riemann surfaces. In particular, we focus on the ergodic theory of singular holomorphic foliations. The emphasis is on recent results, but we also include some classical ones for the sake of completeness and historical perspective.

There is a well-known connection between Riemann surface laminations and the dynamics of iterations of continuous maps. In the meromorphic context, this becomes a link between singular holomorphic foliations by Riemann surfaces in dimension $k \geq 2$ and the dynamics of iterations of meromorphic maps in dimension $k - 1$. On the one side, the abstract ergodic theory of maps has reached maturity

Dedicated to Professor Kang-Tae Kim for his sixtieth birthday

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with remarkable achievements like the Oseledec-Pesin theory. The ergodic theory of meromorphic maps is, however, much less developed. Indeed, it has only been studied intensively during the last three decades, see the survey of Dinh-Sibony [20]. On the other side, the ergodic theory for hyperbolic Riemann surface laminations, and in particular, for the subclass consisting of singular holomorphic foliations by Riemann surfaces, is only in the early stages of development and faces a range of challenges in finding its own way. In this paper we describe some recent approaches to this new theory. We hope that the ideas reviewed in this notes will be developed and expanded in the future. In writing these notes, we are inspired by the surveys and lecture notes of Deroin [14], Fornæss-Sibony [26], Ghys [30], Hurder [34], Zakeri [54] etc. In particular, we are largely influenced by the survey of Fornæss-Sibony [26] which gives an introduction to harmonic currents on singular foliations as developed by themselves. Harmonic currents are the analog of invariant measures in discrete dynamics. Their approach opens new avenues in studying the interplay between geometry, topology and dynamics in the theory of hyperbolic Riemann surface laminations.

In Sect. 2, we will recall basic facts on Riemann surface laminations (without and with singularities), singular holomorphic foliations. The hyperbolicity and the leafwise Poincaré metric will be introduced. As consequences, we will develop the heat diffusions and define the notion of harmonic measures for hyperbolic Riemann surface laminations. We also recall from [26] the notion of positive harmonic currents directed by a Riemann surface lamination (possibly with singularities), and compare it with the notion of positive harmonic currents on complex manifolds. We give a short digression to the isolated singularities for singular holomorphic foliations. Singular holomorphic foliations by Riemann surfaces in $\mathbb{P}^k$ $(k > 1)$ provide a large family of examples where all the above notions apply. In the light of recent results of Jouanolou [37], Lins Neto-Soares [45], Glutsyuk [32], Lins Neto [44], Brunella [4], and Loray-Rebelo [42], we will describe the properties of a generic holomorphic foliation in $\mathbb{P}^k$ with a given degree $d > 1$.

In Sect. 3 we will introduce a function η which measures the ratio between the ambient metric and the leafwise Poincaré metric of a lamination. This function plays an important role in the study of laminations by hyperbolic Riemann surfaces. We also introduce the class of Brody hyperbolic laminations. This class contains not only all compact laminations by hyperbolic Riemann surfaces, but it also includes many singular holomorphic foliations. We then state some recent results on the regularity of Brody hyperbolic laminations which arise from our joint-works with Dinh and Sibony in [17, 18].

In Sect. 4 we study the mas-distribution for directed positive harmonic currents in the local and global settings. Applications to the recurrence phenomenon of a generic leaf will be considered. The material for this section is mainly taken from [16, 48].

A fundamental contribution to the ergodic theory of laminations/foliations has been made by Garnett in [29] where she introduces the notion of directed harmonic measures and considers the diffusions of the heat equation in the Riemannian context. This idea is further developed by Candel [7]. In Sect. 5 we introduce the diffusions of the heat equation for laminations (possibly with singularities) with respect to a positive harmonic current directed by a lamination. This approach allows us in [16] to extend the classical theory of Garnett [29] and Candel [7] to Riemann surface laminations with singularities or to foliations with not necessarily bounded geometry. We present two kinds of ergodic theorems for such currents: one associated to the heat diffusions and one of geometric nature close to Birkhoff's averaging on orbits of a dynamical system.

In Sect. 6 we present a notion of hyperbolic entropy, using hyperbolic time, for laminations by hyperbolic Riemann surfaces. When the lamination is compact and transversally smooth, we state some theorems on the finiteness of the hyperbolic entropy. A notion of metric entropy is also introduced for directed positive harmonic measures. This section is based on our joint-works with Dinh and Sibony in [17, 18].

Section 7 is devoted to the Lyapunov theory for hyperbolic Riemann surface laminations. The central objects of this theory are the cocycles which are modelled on the holonomy cocycle of a foliation. We state the Oseledec multiplicative ergodic theorem for laminations. Next, we apply it to smooth compact laminations by hyperbolic Riemann surfaces and to compact singular holomorphic foliations by Riemann surfaces. After all, we characterize geometrically the Lyapunov exponents of a smooth cocycle with respect to a harmonic measure. This section is a synthesis of our several works in [46, 47, 49].

Some open problems develop in the course of the exposition. Finally, since the choice of the material reflects the limited knowledge of the author on the ergodic theory of hyperbolic Riemann surface laminations, we note that many topics are not included here. The author apologizes in advance for omissions or undue biases, and will welcome comments of suggested inclusions.

Main notaion Throughout the paper, $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$, $r\mathbb{D}$ denotes the disc of center 0 and of radius r, and $\mathbb{D}_R \subset \mathbb{D}$ is the disc of center 0 and of radius R with respect to the Poincaré metric on $\mathbb{D}$, i.e. $\mathbb{D}_R = r\mathbb{D}$ with $R := \log[(1+r)/(1-r)]$. Poincaré metric on a hyperbolic Riemann surface, in particular on $\mathbb{D}$ and on the leaves of a hyperbolic Riemann surface lamination, is given by a positive $(1,1)$-form that we denote by g_P. The associated distance is denoted by dist_P. Given a Riemann surface lamination $(X, \mathscr{L})$, a leaf through a point $x \in X$ is often denoted by L_x. Recall that $d^c := \frac{i}{2\pi}(\overline{\partial} - \partial)$ and $dd^c = \frac{i}{\pi}\partial\overline{\partial}$.

2 Basic Results

2.1 Riemann Surface Laminations

Let X be a locally compact space. A *Riemann surface lamination* $(X, \mathscr{L})$ is the data of a *(lamination) atlas* $\mathscr{L}$ of X with (laminated) charts

$$\Phi_p : \mathbb{U}_p \to \mathbb{B}_p \times \mathbb{T}_p.$$

Here, $\mathbb{T}_p$ is a locally compact metric space, $\mathbb{B}_p$ is a domain in $\mathbb{C}$, $\mathbb{U}_p$ is an open set in X, and Φ_p is a homeomorphism, and all the changes of coordinates $\Phi_p \circ \Phi_q^{-1}$ are of the form

$$x = (y, t) \mapsto x' = (y', t'), \quad y' = \Psi(y, t), \quad t' = \Lambda(t),$$

where Ψ, Λ are continuous functions and Ψ is holomorphic in y.

The open set $\mathbb{U}_p$ is called a *flow box* and the Riemann surface $\Phi_p^{-1}\{t = c\}$ in $\mathbb{U}_p$ with $c \in \mathbb{T}_p$ is a *plaque*. The property of the above coordinate changes insures that the plaques in different flow boxes are compatible in the intersection of the boxes. Two plaques are *adjacent* if they have non-empty intersection.

A *leaf* L is a minimal connected subset of X such that if L intersects a plaque, it contains that plaque. So a leaf L is a Riemann surface immersed in X which is a union of plaques. For every point $x \in X$, denote by L_x the leaf passing through x. A subset $M \subset X$ is called *leafwise saturated* if $x \in M$ implies $L_x \subset M$.

We say that a Riemann surface lamination $(X, \mathscr{L})$ is *smooth* if each map Ψ above is smooth with respect to y, and its partial derivatives of any order with respect to y and $\bar{y}$ are jointly continuous with respect to (y, t).

We are mostly interested in the case where the $\mathbb{T}_i$ are closed subsets of smooth real manifolds and the functions Ψ, Λ are smooth in all variables. In this case, we say that the lamination $(X, \mathscr{L})$ is *transversally smooth*. If, moreover, X is compact, we can embed it in an $\mathbb{R}^N$ in order to use the distance induced by a Riemannian metric on $\mathbb{R}^N$.

We say that a transversally smooth Riemann surface lamination $(X, \mathscr{L})$ is a *smooth foliation* if X is a manifold and all leaves of $\mathscr{L}$ are Riemann surfaces immersed in X.

We say that a Riemann surface lamination $(X, \mathscr{L})$ is a *holomorphic foliation* if X is a complex manifold (of dimension k) and there is an atlas $\mathscr{L}$ of X with charts

$$\Phi_i : \mathbb{U}_i \to \mathbb{B}_i \times \mathbb{T}_i,$$

where the $\mathbb{T}_i$'s are open sets of $\mathbb{C}^{k-1}$ and all above maps Ψ, Λ are holomorphic.

Many examples of abstract compact Riemann surface laminations are constructed in [8, 9, 30]. Suspensions of group actions give already a large class of laminations without singularities.

2.2 Hyperbolicity and Leafwise Poincaré Metric

Consider now a Riemann surface lamination $(X, \mathscr{L})$.

Definition 2.1 A leaf L of $(X, \mathscr{L})$ is said to be *hyperbolic* if it is a hyperbolic Riemann surface, i.e., it is uniformized by $\mathbb{D}$. $(X, \mathscr{L})$ is said to be *hyperbolic* if its leaves are all hyperbolic.

For every $x \in X$ such that L_x is hyperbolic, consider a universal covering map

$$\phi_x : \mathbb{D} \to L_x \qquad \text{such that } \phi_x(0) = x. \tag{2.1}$$

This map is uniquely defined by x up to a rotation on $\mathbb{D}$. Then, by pushing forward the Poincaré metric g_P on $\mathbb{D}$ via ϕ_x, we obtain the so-called *Poincaré metric* on L_x which depends only on the leaf. The latter metric is given by a positive (1, 1)-form on L_x that we also denote by g_P for the sake of simplicity.

2.3 Heat Diffusions and Harmonic Measures

Let $(X, \mathscr{L})$ be a hyperbolic Riemann surface lamination. The leafwise Poincaré metric g_P induces the corresponding Laplacian Δ on leaves (see formula (5.1) for $\beta := g_P$ below). For every point $x \in X$, consider the *heat equation* on L_x

$$\frac{\partial p(x,y,t)}{\partial t} = \Delta_y p(x,y,t), \qquad \lim_{t\to 0+} p(x,y,t) = \delta_x(y), \qquad y \in L_x,\ t \in \mathbb{R}^+.$$

Here δ_x denotes the Dirac mass at x, Δ_y denotes the Laplacian Δ with respect to the variable y, and the limit is taken in the sense of distribution, that is,

$$\lim_{t\to 0+} \int_{L_x} p(x,y,t) f(y) g_P(y) = f(x)$$

for every smooth function f compactly supported in L_x.

The smallest positive solution of the above equation, denoted by $p(x, y, t)$, is called *the heat kernel*. Such a solution exists because (L_x, g_P) is complete and of bounded geometry (see, for example, [9, 11]). The heat kernel $p(x, y, t)$ gives rise to a one parameter family $\{D_t :\ t \geq 0\}$ of diffusion operators defined on bounded measurable functions on X by

$$D_t f(x) := \int_{L_x} p(x,y,t) f(y) g_P(y), \qquad x \in X. \tag{2.2}$$

This family is a semi-group, that is,

$$D_0 = \text{id} \quad \text{and} \quad D_t\mathbf{1} = \mathbf{1} \quad \text{and} \quad D_{t+s} = D_t \circ D_s \quad \text{for } t, s \geq 0, \tag{2.3}$$

where $\mathbf{1}$ denotes the function which is identically equal to 1.

We denote by $\mathscr{C}(X, \mathscr{L})$ the space of all functions f defined and compactly supported on X which are leafwise $\mathscr{C}^2$-smooth and transversally continuous, that is, for each laminated chart $\Phi_p : \mathbb{U}_p \to \mathbb{B}_p \times \mathbb{T}_p$ and all $r, s \in \mathbb{N}$ with $r + s \leq 2$, the derivatives $\frac{\partial^{r+s}(f\circ\Phi_p^{-1})}{\partial y^r \partial \bar{y}^s}$ exist and are jointly continuous in (y, t).

Definition 2.2 Let Δ be the Laplacian on Δ, that is, the aggregate of the leafwise Laplacians $\{\Delta_x\}_{x\in X}$.

A positive Borel measure μ on X is said to be *quasi-harmonic* if

$$\int_X \Delta u \, d\mu = 0$$

for all functions $u \in \mathscr{C}(X, \mathscr{L})$.

A quasi-harmonic measure μ is said to be *harmonic* if μ is finite and μ is D_t-*invariant* for all $t \in \mathbb{R}^+$, i.e.,

$$\int_X D_t f d\mu = \int_X f d\mu, \qquad f \in \mathscr{C}(X, \mathscr{L}), \; t \in \mathbb{R}^+.$$

2.4 Positive Harmonic Currents on Complex Manifolds

Let M be a complex manifold of dimension k. A (p, p)-form on M is *positive* if it can be written at every point as a combination with positive coefficients of forms of type

$$i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_p \wedge \overline{\alpha}_p$$

where the α_j are $(1, 0)$-forms. A (p, p)-current or a (p, p)-form T on M is *weakly positive* if $T \wedge \varphi$ is a positive measure for any smooth positive $(k-p, k-p)$-form φ. A (p, p)-current T is *positive* if $T \wedge \varphi$ is a positive measure for any smooth weakly positive $(k-p, k-p)$-form φ. If M is given with a Hermitian metric β and T is a positive (p, p)-current on M, $T \wedge \beta^{k-p}$ is a positive measure on M. The mass of $T \wedge \beta^{k-p}$ on a measurable set E is denoted by $\|T\|_E$ and is called *the mass of* T *on* E. *The mass* $\|T\|$ of T is the total mass of $T \wedge \beta^{k-p}$ on M.

A (p, p)-current on M is *harmonic* if $dd^cT = 0$ in the weak sense (namely, $T(dd^c f) = 0$ for all compactly smooth $(k-p-1, k-p-1)$-forms f on M).

In this article, for every $r > 0$ let B_r denote the ball of center 0 and of radius r in $\mathbb{C}^k$. The following local property of positive harmonic currents is discovered by Skoda [51].

Proposition 2.3 (Skoda [51]) *Let* T *be a positive harmonic* (p, p)*-current in a ball* B_{r_0}. *Define* $\beta := dd^c\|z\|^2$ *the standard Kähler form where* z *is the canonical coordinates on* $\mathbb{C}^n$. *Then the function* $r \mapsto \pi^{-(k-p)} r^{-2(k-p)} \|T \wedge \beta^{k-p}\|_{B_r}$ *is increasing on* $0 < r \le r_0$. *In particular, it is bounded on* $]0, r_1]$ *for any* $0 < r_1 < r_0$.

The limit of the above function when $r \to 0$ is called *the Lelong number* of T at 0. The above proposition shows that Lelong number always exists and is finite positive.

The next simple result allows for extending positive harmonic currents through isolated points.

Proposition 2.4 (Dinh-Nguyen-Sibony [16, Lemma 2.5]) *Let T be a positive current of bidimension $(1, 1)$ with compact support on a complex manifold M of dimension > 1. Assume that dd^cT is a negative measure on $M \setminus E$ where E is a finite set. Then T is a positive harmonic current on M.*

2.5 Directed Positive Harmonic Currents

Let $(X, \mathscr{L})$ be a Riemann surface lamination. Let $\mathscr{C}^{1,1}(X, \mathscr{L})$ denote the space of all forms f of bidegree $(1, 1)$ defined on leaves of the lamination and compactly supported on X such that f is transversally continuous. The last continuity condition means that for each laminated chart $\Phi_p : \mathbb{U}_p \to \mathbb{B}_p \times \mathbb{T}_p$, the form $f \circ \Phi_p^{-1}$ is jointly continuous in (y, t). For each chart $\Phi_p : \mathbb{U}_p \to \mathbb{B}_p \times \mathbb{T}_p$, the complex structure on $\mathbb{B}_p$ induces a complex structure on the leaves of X. Therefore, the operator d and d^c can be defined so that they act leafwise on forms as in the case of complex manifolds. So we get easily that $dd^c : \mathscr{C}(X, \mathscr{L}) \to \mathscr{C}^{1,1}(X, \mathscr{L})$. A form $f \in \mathscr{C}^{1,1}(X, \mathscr{L})$ is said to be *positive* if its restriction to every plaque is a positive $(1, 1)$-form in the usual sense.

Definition 2.5 (Garnett [29], *see also* Sullivan [52]) A *directed current* on $(X, \mathscr{L})$ (or equivalently, a *current directed by the lamination* $(X, \mathscr{L})$) is a linear continuous form on $\mathscr{C}^{1,1}(X, \mathscr{L})$. Let T be a directed current.

- T is said to be *positive* if $T(f) \geq 0$ for all positive forms $f \in \mathscr{C}^{1,1}(X, \mathscr{L})$.
- T is said to be *harmonic* if $dd^cT = 0$ in the weak sense (namely, $T(dd^cg) = 0$ for all functions $g \in \mathscr{C}(X, \mathscr{L})$).

We have the following decomposition.

Proposition 2.6 *Let T be a directed positive harmonic current on $(X, \mathscr{L})$. Let $\mathbb{U} \simeq \mathbb{B} \times \mathbb{T}$ be a flow box which is relatively compact in X. Then, there is a positive Radon measure ν on $\mathbb{T}$ and for ν-almost every $t \in \mathbb{T}$, there is a positive harmonic function h_t on $\mathbb{B}$ such that if K is compact in $\mathbb{B}$, the integral $\int_K \|h_t\|_{L^1(K)} d\nu(t)$ is finite and*

$$T(f) = \int_{\mathbb{T}} \Big(\int_{\mathbb{B}} h_t(y) f(y, t) \Big) d\nu(t)$$

for every form $f \in \mathscr{C}^{1,1}(X, \mathscr{L})$ compactly supported on $\mathbb{U}$.

2.6 Directed Positive Harmonic Currents Versus Harmonic Measures

Recall that a positive finite measure μ on the σ-algebra of Borel sets in X is said to be *ergodic* if for every leafwise saturated Borel measurable set $Z \subset X$, $\mu(Z)$ is equal to either $\mu(X)$ or 0. A directed positive harmonic current T is said to be *extremal* if $T = T_1 + T_2$ for directed positive harmonic current T_1, T_2 implies that $T_1 = \lambda T$ for some $\lambda \in [0, 1]$. The following result relates the notions of harmonic measures and directed positive harmonic currents (see [16, 47]).

Theorem 2.7 *Let $(X, \mathscr{L})$ be a hyperbolic Riemann surface lamination.*

(i) If X is compact, then each quasi-harmonic measure is harmonic.
(ii) The map $T \mapsto \mu = T \wedge g_P$ which is defined on the convex cone of all directed positive harmonic currents is one-to-one and its image is contained in the convex cone of all quasi-harmonic measures μ. If, moreover, X is compact, then this map is a bijection from the convex cone of all directed positive harmonic currents T onto the convex cone of all harmonic measures μ.
(iii) If T is an extremal directed positive harmonic current and $\mu := T \wedge g_P$ is finite, then μ is ergodic.

2.7 Riemann Surface Laminations with Singularities, Singular Holomorphic Foliations and Examples

We call *Riemann surface lamination with singularities* the data $(X, \mathscr{L}, E)$ where X is a locally compact space, E a closed subset of X and $(X \setminus E, \mathscr{L})$ is a Riemann surface lamination. The set E is *the singularity set* of the lamination. In order to simplify the presentation, we will mostly consider the case where X is a closed subset of a complex manifold M of dimension $k \geq 1$ and E is a locally finite subset of X. We assume that M is endowed with a Hermitian metric g_M. We also assume that the complex structures on the leaves of the lamination coincide with the ones induced by M, that is, the leaves of $(X \setminus E, \mathscr{L})$ are Riemann surfaces holomorphically immersed in M.

We say that $\mathscr{F} := (X, \mathscr{L}, E)$ is a *singular foliation* (resp. *singular holomorphic foliation*) if X is a complex manifold and $E \subset X$ is a closed subset such that $\overline{X \setminus E} = X$ and $(X \setminus E, \mathscr{L})$ is a foliation (resp. a holomorphic foliation). E is said to be the *set of singularities* of the foliation $\mathscr{F}$. We say that $\mathscr{F}$ is compact if X is compact.

Definition 2.8 Let $Z = \sum_{j=1}^k F_j(z) \frac{\partial}{\partial z_j}$ be a holomorphic vector field defined in a neighborhood U of $0 \in \mathbb{C}^k$. Consider the holomorphic map $F := (F_1, \ldots, F_k) : U \to \mathbb{C}^k$. We say that Z is

(1) *singular at* 0 if $F(0) = 0$.
(2) *generic linear* if it can be written as

$$Z(z) = \sum_{j=1}^k \lambda_j z_j \frac{\partial}{\partial z_j}$$

where λ_j are non-zero complex numbers.

(3) *with non-degenerate singularity at* 0 if Z is singular at 0 and the eigenvalues $\lambda_1, \ldots, \lambda_k$ of the Jacobian matrix $DF(0)$ are all nonzero.
(4) *with hyperbolic singularity at* 0 if Z is singular at 0 and the eigenvalues $\lambda_1, \ldots, \lambda_k$ of the Jacobian matrix $DF(0)$ satisfy $\lambda_j \neq 0$ and $\lambda_i/\lambda_j \notin \mathbb{R}$ for all $1 \leq i \neq j \leq k$.

The integral curves of Z define a singular holomorphic foliation on U. The condition $\lambda_j \neq 0$ implies that the foliation has an isolated singularity at 0.

Let $\mathscr{F} = (X, \mathscr{L}, E)$ be a singular holomorphic foliation such that E is an analytic subset of X with $\operatorname{codim}(E) \geq 2$. Then $\mathscr{F}$ is given locally by holomorphic vector fields and its leaves are locally, integral curves of these vector fields, and the singularities of $\mathscr{F}$ coincide with the singular set of these vector fields. We say that a singular point $a \in E$ is *linearizable* (resp. *hyperbolic*) if there is a local holomorphic coordinate system of X near a on which the leaves of $\mathscr{F}$ are integral curves of a generic linear vector field (resp. of a holomorphic vector field admitting 0 as a hyperbolic singularity). In dimension 2 (i.e. $\dim X = 2$), if a is a hyperbolic singularity, then there is a local holomorphic coordinates system of X near a on which the leaves of $\mathscr{F}$ are integral curves of a vector field $Z(z_1, z_2) = \lambda_1 z_1 \frac{\partial}{\partial z_1} + \lambda_2 z_2 \frac{\partial}{\partial z_2}$, where λ_1, λ_2 are some nonzero complex numbers with $\lambda_1/\lambda_2 \notin \mathbb{R}$. In particular, a is a linearizable singularity. The analytic curves $\{z_1 = 0\}$ and $\{z_2 = 0\}$ are called *separatrice* at a.

Now we discuss singular holomorphic foliations on $\mathbb{P}^k$ with $k \geq 2$. Let $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ denote the canonical projection. Let $\mathscr{F}$ be a singular holomorphic foliations on $\mathbb{P}^k$, It can be shown that $\pi^*\mathscr{F}$ is a singular foliation on $\mathbb{C}^{k+1}$ associated to a vector field Z of the form

$$Z := \sum_{j=0}^{k} F_j(z) \frac{\partial}{\partial z_j},$$

where the F_j are homogeneous polynomials of degree $d \geq 1$. We call d the *degree* of the foliation. A point $x \in \mathbb{P}^k$ is a singularity of $\mathscr{F}$ if $F(x)$ is colinear with x, i.e., if x is either an indeterminacy point or a fixed point of $f = [F_0 : \ldots : F_k]$ as a meromorphic map in $\mathbb{P}^k$. For $d \geq 2$, let $\mathscr{F}_d(\mathbb{P}^k)$ be the space of singular holomorphic foliations of degree d in $\mathbb{P}^k$. Using the above form of Z, we can show that $\mathscr{F}_d(\mathbb{P}^k)$ can be canonically identified with a Zariski open subset of $\mathbb{P}^N$, where $N := (d + k + 1)\frac{(d+k-1)!}{(k-1)!d!} - 1$ (see [4]). The next result describes the typical properties of a generic foliation $\mathscr{F} \in \mathscr{F}_d(\mathbb{P}^k)$.

Theorem 2.9 *Let* $d, k > 1$.

(1) (Jouanolou [37], Lins Neto-Soares [45]). *There is a real Zariski dense open set* $\mathcal{H}(d) \subset \mathscr{F}_d(\mathbb{P}^k)$ *such that for every* $\mathscr{F} \in \mathcal{H}(d)$, *all the singularities of* $\mathscr{F}$ *are hyperbolic and* $\mathscr{F}$ *do not possess any invariant algebraic curve.*
(2) (Glutsyuk [32], Lins Neto [44]). *If all the singularities of a foliation* $\mathscr{F} \in \mathscr{F}_d(\mathbb{P}^k)$ *are non-degenerate, then* $\mathscr{F}$ *is hyperbolic.*

(3) (Brunella [4]). *If all the singularities of a foliation $\mathscr{F} \in \mathscr{F}_d(\mathbb{P}^k)$ are hyperbolic and $\mathscr{F}$ do not possess any invariant algebraic curve, then $\mathscr{F}$ admits no nontrivial directed positive closed current.*

Moreover, Loray-Rebelo [42] constructed a nonempty open set $\mathcal{U}(d)$ of $\mathscr{F}_d(\mathbb{P}^k)$ such that every leaf of $\mathscr{F} \in \mathcal{U}(d)$ is dense.

Now we come to the notion of directed positive harmonic currents on singular Riemann surface laminations.

Definition 2.10 (Berndtsson-Sibony [1], Fornæss-Sibony [25, 26]) Let $(X, \mathscr{L}, E)$ be a Riemann surface lamination with singularities, where X is a closed subset of a complex manifold M and the leaves of $(X\setminus E, \mathscr{L})$ are Riemann surfaces holomorphically immersed in M. A *directed harmonic current* on $(X, \mathscr{L}, E)$ is a positive harmonic current T of bidimension $(1, 1)$ on M such that the support of T is contained in X and that the restriction of T on $X\setminus E$ is a directed harmonic current on the Riemann surface lamination $(X, \mathscr{L})$ in the sense of Definition 2.5.

The existence of directed positive harmonic currents for compact (nonsingular) laminations was proved by Garnett [29]. The case of compact singular Riemann surface laminations was proved by Berndtsson-Sibony under reasonable assumptions.

Theorem 2.11 (Berndtsson-Sibony [1], *see also* [26, Theorem 23]) *Let $(X, \mathscr{L}, E)$ be a singular Riemann surface lamination as in the assumption of Definition 2.10. Assume moreover that X is compact and E is locally pluripolar in M. Then there is a nonzero directed positive harmonic current T. In particular, if the set E does not support any nonzero positive harmonic current (e.g. if $\Lambda_2(E) = 0$, where Λ_2 denotes the two dimensional Hausdorff measure), then the restriction of such a current T on $X\setminus E$ induces a nonzero directed positive harmonic current on $(X\setminus E, \mathscr{L})$.*

When a leaf L_x is hyperbolic, an average on L_x was introduced by Fornæss-Sibony [25] (see also [26, Corolary 3]). It allows another construction of directed positive harmonic currents. By Theorem 2.9, Theorem 2.11 applies to every generic foliation in $\mathbb{P}^k$ with a given degree $d > 1$.

2.8 General Laminations/Foliations

We formulate some general notions of laminations and foliations. Although they are not the main topic of this article, some results presented here could be extended to these general objects. Let $l \geq 1$ be an integer.

An *l-dimensional lamination* $(X, \mathscr{L})$ is the data of a locally compact space X and a *(lamination) atlas* $\mathscr{L}$ of with (laminated) charts

$$\Phi_p : \mathbb{U}_p \to \mathbb{B}_p \times \mathbb{T}_p.$$

Here, $\mathbb{T}_p$ is a locally compact metric space, $\mathbb{B}_p$ is a domain in $\mathbb{R}^l$, $\mathbb{U}_p$ is an open set in X, and Φ_p is a homeomorphism, and all the changes of coordinates $\Phi_p \circ \Phi_q^{-1}$ are of the form

$$x = (y, t) \mapsto x' = (y', t'), \quad y' = \Psi(y, t), \quad t' = \Lambda(t),$$

where Ψ, Λ are continuous functions. We call *l-dimensional lamination with singularities* the data $(X, \mathscr{L}, E)$ where X is a locally compact space, E is a closed subset of X and $(X \setminus E, \mathscr{L})$ is an l-dimensional lamination. The set E is *the singularity set* of the lamination.

An *l-dimensional Riemannian foliation* $(X, \mathscr{L})$ is the data of a *(lamination) atlas* $\mathscr{L}$ of a Riemannian manifold X of dimension $k \geq l$ with (laminated) charts

$$\Phi_p : \mathbb{U}_p \to \mathbb{B}_p \times \mathbb{T}_p.$$

Here, $\mathbb{T}_p$ is a domain in $\mathbb{R}^{k-l}$, $\mathbb{B}_p$ is a domain in $\mathbb{R}^l$, $\mathbb{U}_p$ is an open set in X, and Φ_p is a homeomorphism, and all the changes of coordinates $\Phi_p \circ \Phi_q^{-1}$ are of the form

$$x = (y, t) \mapsto x' = (y', t'), \quad y' = \Psi(y, t), \quad t' = \Lambda(t),$$

where Ψ, Λ are smooth functions. We call *l-dimensional Riemannian foliation with singularities* the data $(X, \mathscr{L}, E)$ where X is a Riemannian manifold X of dimension $k \geq l$, E is a closed subset of X and $(X \setminus E, \mathscr{L})$ is an l-dimensional Riemannian foliation. The set E is *the singularity set* of the foliation.

By considering classical constructions such as projective limits and suspensions, Fornæss-Sibony-Wold obtain in [28] examples of laminations possessing many, or few, directed positive closed or directed positive harmonic currents. They also give examples of compact laminations by complex manifolds of dimension ≥ 2 with no nonzero directed positive harmonic current. This is in contrast with the Riemann surface case, where such a current always exists.

3 Regularity of the Leafwise Poincaré Metric

Let $(X, \mathscr{L}, E)$ be a hyperbolic Riemann surface lamination with singularities. Let g_P be the leafwise Poincaré metric for the lamination $(X \setminus E, \mathscr{L})$ given in Sect. 2.2. Let g_X be a Hermitian metric on the leaves which is transversally smooth. We can construct such a metric on flow boxes and glue them using a partition of unity. We have

$$g_X = \eta^2 g_P \quad \text{where} \quad \eta(x) := \|D\phi_x(0)\|. \tag{3.1}$$

Here, ϕ_x is defined in (2.1), and for the norm of the differential $D\phi_x$ we use the Poincaré metric on $\mathbb{D}$ and the Hermitian metric g_X on L_x.

The extremal property of the Poincaré metric implies that

$$\eta(x) = \sup \big\{ \|(D\phi)(0)\|, \quad \phi : \mathbb{D} \to L \text{ holomorphic such that } \phi(0) = x \big\}.$$

Using a map sending $\mathbb{D}$ to a plaque, we see that the function η is locally bounded from below on $X \setminus E$ by a strictly positive constant. When X is compact and $E = \varnothing$, the classical Brody lemma (see [40, p. 100]) implies that η is also bounded from above.

The continuity of the function η was studied by Candel, Ghys, Verjovsky, see [6, 30, 53]. The survey [26] establishes the result as a consequence of Royden's lemma. Indeed with his lemma, Royden proved the upper-semicontinuity of the infinitesimal Kobayashi metric in a Kobayashi hyperbolic manifold (see [40, p. 91 and p. 153]). The following theorem gives refinements of the previous results.

Theorem 3.1 (Dinh-Nguyen-Sibony [17]) *Let $(X, \mathscr{L})$ be a transversally smooth compact lamination by hyperbolic Riemann surfaces. Then the Poincaré metric on the leaves is Hölder continuous, that is, the function η defined in* (3.1) *is Hölder continuous on X. Moreover, the exponent of Hölder continuity can be estimated in geometric terms.*

The main tool of the proof of Theorem 3.1 is to use Beltrami's equation in order to compare universal covering maps of any leaf L_y near a given leaf L_x. More precisely, for $R > 0$ let $\mathbb{D}_R$ be the disc of center 0 with radius R with respect to the Poincaré metric on $\mathbb{D}$ (see Main Notations in Sect. 1). We first construct a non-holomorphic parametrization ψ from $\mathbb{D}_R$ to L_y which is close to a universal covering map $\phi_x : \mathbb{D} \to L_x$ for all R large enough. Next, precise geometric estimates on ψ allow us to modify it, using Beltrami's equation. We then obtain a holomorphic map that we can explicitly compare with a universal covering map $\phi_y : \mathbb{D} \to L_y$.

Next, we investigate the regularity of the leafwise Poincaré metric g_P of a compact singular holomorphic foliation. Here an important difficulty emerges: a leaf of the foliation may visit singular flow boxes without any obvious rule. We are interested in the following class of laminations.

Definition 3.2 (Dinh-Nguyen-Sibony [18]) A hyperbolic Riemann surface lamination with singularities $(X, \mathscr{L}, E)$ with X compact is said to be *Brody hyperbolic* if there is a constant $c_0 > 0$ such that

$$\|D\phi(0)\| \leq c_0$$

for all holomorphic maps ϕ from $\mathbb{D}$ into a leaf.

Remark 3.3 It is clear that if the lamination is Brody hyperbolic then its leaves are hyperbolic in the sense of Kobayashi. Conversely, the Brody hyperbolicity is a consequence of the non-existence of holomorphic non-constant maps $\mathbb{C} \to X$ such that out of E the image of $\mathbb{C}$ is locally contained in leaves, see [26, Theorem 15].

On the other hand, Lins Neto proved in [44] that for every holomorphic foliation of degree larger than 1 in $\mathbb{P}^k$, with non-degenerate singularities, there is a smooth metric with negative curvature on its tangent bundle, see also Glutsyuk [32]. Hence, these foliations are Brody hyperbolic. Consequently, holomorphic foliations in $\mathbb{P}^k$ are generically Brody hyperbolic, see Theorem 2.9 (1).

Denote by $\log^\star(\cdot) := 1 + |\log(\cdot)|$ a log-type function, and by dist the distance on X induced by the Hermitian metric g_X. The following result is a counterpart of Theorem 3.1 in the context of singular holomorphic foliations.

Theorem 3.4 (Dinh-Nguyen-Sibony [18]) *Let $(X, \mathscr{L}, E)$ be a Brody hyperbolic singular holomorphic foliation on a Hermitian compact complex manifold X. Assume that the singular set E is finite and that all points of E are linearizable. Then, there are constants $c > 0$ and $0 < \alpha < 1$ such that*

$$|\eta(x) - \eta(y)| \leq c\Big(\frac{\max\{\log^\star \operatorname{dist}(x, E), \log^\star \operatorname{dist}(y, E)\}}{\log^\star \operatorname{dist}(x, y)}\Big)^\alpha$$

for all x, y in $X \setminus E$.

To prove this theorem, we analyze the behavior of a leaf near singularities and get an explicit estimate on the modulus of continuity of the Poincaé metric on leaves. The following estimates are crucial in our method. They are also useful in other problems.

Proposition 3.5 (Dinh-Nguyen-Sibony [18]) *Under the hypotheses of Theorem 3.4, there exists a constant $c_1 > 1$ such that*

$$c_1^{-1} s \log^\star s \leq \eta(x) \leq c_1 s \log^\star s$$

for $x \in X \setminus E$ and $s := \operatorname{dist}(x, E)$.

The Poincaré metric on the leaves of a hyperbolic foliation is a fundamental object which is extremely delicate to understand. As we see in Theorem 3.4, the regularity in the direction transverse to the foliation is quite weak. This is partly due to the presence of the singularities. We end the section with the following open question.

Problem 3.6 Let $(X, \mathscr{L}, E)$ be a compact singular holomorphic foliation by hyperbolic Riemann surfaces. Assume that every point $a \in E$ is a non-degenerate singularity. Study the regularity of the function η. In case $\dim(X) = 2$, we may investigate the problem when the singularities are not necessarily non-degenerate.

4 Mass-Distribution of Directed Positive Harmonic Currents

Let $(X, \mathscr{L}, E)$ be a singular holomorphic foliation and let T be a positive harmonic current on $X \setminus E$. By Proposition 2.4 its mass with respect to any Hermitian metric on

X is finite. We call *Poincaré mass* of T the mass of T with respect to Poincaré metric g_P on $X\setminus E$, i.e. the mass of the positive measure $m_P := T \wedge g_P$. A priori, Poincaré mass may be infinite near the singular points. The following proposition gives us a criterion for the finiteness of this mass. It can be applied to generic foliations in $\mathbb{P}^k$ (see Theorem 2.9).

Proposition 4.1 (Dinh-Nguyen-Sibony [16]) *Let $(X, \mathscr{L}, E)$ be a singular holomorphic foliation. If $a \in E$ is a linearizable singularity, then any positive harmonic current on X has locally finite Poincaré mass near a.*

The proof of this result is based on the finiteness of the Lelong number of T at a (see Proposition 2.3). In dimension 2 we have a more precise result when T is directed and the singular point is hyperbolic.

Theorem 4.2 (Nguyen [48]) *Let $(X, \mathscr{L}, E)$ be a singular holomorphic foliation with* $\dim X = 2$. *If $a \in E$ is a hyperbolic singularity, then for any directed positive harmonic current T on X which does not give mass to any of the two separatrices at a, the Lelong number of T at a vanishes.*

An immediate consequence of Theorem 4.2 is the following result on the Lelong numbers of a directed harmonic current.

Corollary 4.3 *Let $\mathscr{F} = (X, \mathscr{F}, E)$ be a singular holomorphic foliation with X a compact complex surface. Assume that all the singularities are hyperbolic and that the foliation has no invariant analytic curve. Then for every harmonic current T directed by $\mathscr{F}$, the Lelong number of T vanishes everywhere in X.*

The above corollary can be applied to every generic foliation in $\mathbb{P}^2$ with a given degree $d > 1$ (see Theorem 2.9).

We can apply Corollary 4.3 to study the recurrence of a generic leaf. More specifically, let T be a positive harmonic current directed by a singular holomorphic foliation $(X, \mathscr{L}, E)$ with X a compact complex surface. Assume that all the singularities are hyperbolic and that the foliation has no invariant analytic curve. Consider the positive measure $m_P := T \wedge g_P$. We know by Proposition 4.1 that m_P is a finite measure. Given a point $x \in X$ and a m_P-generic point $a \in X\setminus E$, we want to know how often the leaf L_a visits the ball $B(x, r)$ as $r \searrow 0$. Here $B(x, r)$ denotes the open ball with center x and radius r with respect to a fixed metric on X.

Let us introduce some more notation and terminology. Denote by $r\mathbb{D}$ the disc of center 0 and of radius r with $0 < r < 1$. In the Poincaré disc $(\mathbb{D}, \omega_P)$, $r\mathbb{D}$ is also the disc of center 0 and of radius

$$R := \log \frac{1+r}{1-r}. \tag{4.1}$$

So, we will also denote by $\mathbb{D}_R$ this disc, and by $\partial\mathbb{D}_R$ its boundary which is also the Poincaré circle of center 0 and radius R.

Together with Dinh and Sibony, we introduce in [48] the following indicator.

Definition 4.4 For each $r > 0$, the *visibility of a point* $a \in X\setminus E$ *within distance* r *from a point* $x \in M$ is the number

$$N(a,x,r) = \limsup_{R\to\infty} \frac{1}{R}\int_0^R \Big(\int_{\theta=0}^1 \mathbf{1}_{B(x,r)}\big(\phi_a(s_t e^{2\pi i\theta})\big)d\theta\Big)dt \in [0,1],$$

where $\mathbf{1}_{B(x,r)}$ is the characteristic function associated to the set $B(x,r)$, and s_t is defined by the relation $t = \log\frac{1+s_t}{1-s_t}$, that is, $s_t\mathbb{D} = \mathbb{D}_t$.

Geometrically, $N(a,x,r)$ is the average, as $R\to\infty$, over the hyperbolic time $t\in[0,R]$ of the Lebesgue measure of the set $\{\theta\in[0,1]:\ \phi_a(s_t e^{2\pi i\theta})\in B(x,r)\}$. The last quantity may be interpreted as the portion which hits $B(x,r)$ of the Poincaré circle of radius t with center a spanned on the leaf L_a.

We combine Corollary 4.3 and the so-called geometric ergodic theorem (Theorem 5.4) which will be presented in the next section. Consequently, we obtain the following upper bound on the visibility of a generic point.

Theorem 4.5 *We keep the above hypothesis and notation. Then for m_P-almost every point $a \in X\setminus E$ and for every point $x\in X$, we have that*

$$N(a,x,r) = \begin{cases} o(r^2), & x\in X\setminus E;\\ o(|\log r|^{-1}), & x\in E.\end{cases}$$

Here are some open questions.

Problem 4.6 Let $\mathscr{F} = (X,\mathscr{L},E)$ be a compact singular holomorphic foliation by hyperbolic Riemann surfaces. Let $a\in E$ be a non-degenerate singularity. Find sufficient conditions on $\mathscr{F}$ and a so that any positive harmonic current on X has locally finite Poincaré mass near a.

Problem 4.7 Can one generalize Theorem 4.2 to higher dimensions?

Problem 4.8 Find an effective lower bound on the visibility of a generic point.

5 Heat Equation and Ergodic Theorems

In this section we will report some recent techniques used to obtain ergodic theorems for Riemann surface laminations.

5.1 Ergodic Theorems Associated with the Heat Diffusions

In collaboration with Dinh and Sibony [16], we introduce the heat equation relative to a positive harmonic current and apply it to the directed positive harmonic currents of a Riemann surface laminations with singularities. This permits to construct the heat diffusion with respect to various Laplacians that could be defined almost everywhere with respect to the positive harmonic current.

More concretely, let $(X, \mathscr{L}, E)$ be a compact Riemann surface lamination with singularities, where X is a (compact) subset of a complex manifold M and the leaves of $(X\setminus E, \mathscr{L})$ are Riemann surfaces holomorphically immersed in M. For simplicity, fix a Hermitian form g_M on M. Let T be a directed positive harmonic current. So, $T \wedge g_M$ is a positive measure. Consider a positive $(1, 1)$-form β which is defined almost everywhere on M with respect to $T \wedge g_P$. We say that T is *β-regular* if $T \wedge \beta$ is of finite mass and $i\tau \wedge \overline{\tau} \wedge T \leq T \wedge \beta$, where τ is a $(1, 0)$-form defined almost everywhere with respect to $T \wedge g_P$ such that $\partial T = \tau \wedge T$. Under the notation of Proposition 2.6, we see that $\tau = h_t^{-1}\partial h_t$ on the plaque passing through $t \in \mathbb{T}$ for ν-almost every $t \in \mathbb{T}$. The following result gives a typical example of β-regularity with $\beta := g_P$.

Proposition 5.1 *Let $(X, \mathscr{L}, E)$ be a lamination as above. Let $\mathscr{F}$ be a singular holomorphic foliation on M such that the restriction of $\mathscr{F}$ on $X\setminus E$ induces $\mathscr{L}$ and that all points of E are linearizable singularities of $\mathscr{F}$. Then every directed harmonic current T on $(X, \mathscr{L}, E)$ is g_P-regular.*

Using Proposition 4.1, Proposition 5.1 follows essentially from Proposition 3 and Theorem 9 in [27].

We define the Laplacian Δ_β by

$$(\Delta_\beta u)T \wedge \beta := dd^c u \wedge T \text{ for } u \in \mathscr{C}_0^\infty(M). \tag{5.1}$$

We will extend the definition of Δ_β to larger spaces, suitable for developing L^2-techniques. To this end, let m_β denote the measure $T \wedge \beta$ and consider the Hilbert space $L := L^2(m_\beta)$. We also introduce the Hilbert space $H = H^1_\beta(T) \subset L^2(m_\beta)$ associated with T and β as the completion of $\mathscr{C}_0^\infty(M)$ with respect to the Dirichlet norm

$$\|u\|^2_{H^1_\beta} := \int |u|^2 T \wedge \beta + i \int \partial u \wedge \overline{\partial} u \wedge T.$$

Using the assumption that T is β-regular, we can show that there exists a semi-group of contractions $S(t) : L \to L$, $t \in \mathbb{R}^+$ such that for every function $u_0 \in H$, $u(t, \cdot) := S(t)u_0$ satisfies

$$\frac{\partial u(t, \cdot)}{\partial t} = \Delta_\beta u(t, \cdot) \text{ and } u(0, \cdot) = u_0.$$

Recall that a family $S(t) : L \to L$, $t \in \mathbb{R}_+$, is a *semi-group of contractions* if $S(t + t') = S(t) \circ S(t')$ and if $\|S(t)\| \leq 1$ for all $t, t' \geq 0$.

To prove this result we use functional analysis (Hille-Yosida theorem, Lax-Milgram theorem etc.). We also use Stokes' theorem on M. It is worthy noting that Garnett [29] and Candel [7] also solve the heat equation. But they consider the case without singularities. Moreover, they solve the equation pointwise, that is, in the space of smooth functions. So their methods are quite diffrent from ours. Indeed, we solve the equation with respect to a positive harmonic current, in a suitable L^2-space.

The following result is an ergodic theorem associated to the heat diffusions.

Theorem 5.2 (Dinh-Nguyen-Sibony [16]) *We keep the above hypothesis and notation. Then*

(1) the measure m_β is $S(t)$-invariant (that is, $\langle S(t)u, m_\beta\rangle = \langle u, m_\beta\rangle$ for every $u \in L$), and $S(t)$ is a positive contraction in $L^p(m_\beta)$ for all $1 \le p \le \infty$ (that is, $\|S(t)u\|_{L^p(m_\beta)} \le \|u\|_{L^p(m_\beta)}$ for every $u \in L$);

(2) for all $u_0 \in L^p(m_\beta)$, $1 \le p < \infty$, the average

$$\frac{1}{R}\int_0^R S(t)u_0 dt$$

converges pointwise m_β-almost everywhere and also in $L^p(m)$ to an $S(t)$-invariant function u_0^* when R goes to infinity. Moreover, u_0^* is constant on the leaf L_a for m_β-almost every a. If m_β is an extremal harmonic measure, then u is constant m_β-almost everywhere.

Combining Proposition 5.1 and Theorem 5.2, we obtain the following relation between harmonic measures and directed positive harmonic currents which is a complement to Theorem 2.7(ii).

Proposition 5.3 *Let $(X, \mathscr{L}, E)$ be a lamination as above. Let $\mathscr{F}$ be a singular holomorphic foliation on M such that the restriction of $\mathscr{F}$ on $X \setminus E$ induces $\mathscr{L}$ and that all points of E are linearizable singularities of $\mathscr{F}$. Then on $(X, \mathscr{L}, E)$ the map $T \mapsto \mu = T \wedge g_P$ is a bijection from the convex cone of all directed positive harmonic currents T onto the convex cone of all harmonic measures μ.*

5.2 *Geometric Ergodic Theorems*

In this subsection, we will give an analogue of Birkhoff's ergodic theorem in the context of a compact Riemann surface lamination $(X, \mathscr{L}, E)$ with singularities. Our ergodic theorem is of geometric nature and it is close to Birkhoff's averaging on orbits of a dynamical system. Here the averaging is on hyperbolic leaves and the time is the hyperbolic time.

Let $(X, \mathscr{L}, E)$ be a Riemann surface lamination with singularities which is embedded in a complex manifold M as in Sect. 5.1. Let T be a directed positive harmonic current on $(X, \mathscr{L}, E)$ such that T is g_P-regular. A leaf L_x is called *parabolic* if it is not hyperbolic. We assume that T has no mass on the union of parabolic leaves and that $m_P := T \wedge g_P$ is a probability measure. So by Theorem 2.7, m_P is a quasi-harmonic measure on X with respect to g_P.

For any point $x \in X\setminus E$ such that the corresponding leaf L_x is hyperbolic, let $\phi_x : \mathbb{D} \to L_x$ be given by (2.1). Denote by $r\mathbb{D}$ the disc of center 0 and of radius r with $0 < r < 1$. Recall from (4.1) that in the Poincaré metric, this is also the disc of center 0 and of radius $R := \log \frac{1+r}{1-r}$, and we will also denote by $\mathbb{D}_R$ this disc. For all $0 < R < \infty$, consider

$$\begin{aligned} m_{x,R} &:= \frac{1}{M_R}(\phi_x)_*\Big(\log^+ \frac{r}{|\zeta|} g_P\Big), \\ \tau_{x,R} &:= \frac{1}{M_R}(\phi_x)_*\Big(\log^+ \frac{r}{|\zeta|}\Big). \end{aligned} \tag{5.2}$$

where $\log^+ := \max\{\log, 0\}$, g_P denotes also the Poincaré metric on $\mathbb{D}$ and

$$M_R := \int \log^+ \frac{r}{|\zeta|} g_P = \int \log^+ \frac{r}{|\zeta|} \frac{2}{(1-|\zeta|^2)^2} i d\zeta \wedge d\bar{\zeta}.$$

So, $m_{x,R}$ (resp. $\tau_{x,R}$) is a probability measure (resp. a directed positive current of bidimension $(1, 1)$) which depends on x, R; but does not depend on the choice of ϕ_x.

Theorem 5.4 (Dinh-Nguyen-Sibony [16]) *We keep the above hypothesis and notation. Assume in addition that the current T is extremal. Then for almost every point $x \in X$ with respect to the measure $m_P := T \wedge g_P$, the measure $m_{x,R}$ defined above converges to m_P when $R \to \infty$.*

To prove the theorem, our main ingredient is a delicate estimate on the heat kernel of the Poincaré disc (see [16, p. 370, line 8-] for its statement). This estimate allows us to deduce the the desired result from the ergodic theorem associated to the heat diffusions (Theorem 5.2).

Remark 5.5 Let $(X, \mathscr{L})$ be a compact lamination by Riemann surfaces without singularities. Let T be a positive harmonic current directed by the lamination which is extremal, with full mass on hyperbolic leaves and with Poincaré mass 1. Then, the conclusions of Theorems 5.2 and 5.4 are still valid. The proofs are essentially the same but we need to use a finite partition of unity for X instead of applying Stokes' theorem for M. Moreover, Theorem 5.2 still holds for compact smooth p-dimensional laminations in the sense of Sect. 2.8.

5.3 *Unique Ergodicity Theorems*

In [25] Fornæss and Sibony develop the theory of harmonic currents of finite energy. They introduce a notion of energy for positive harmonic currents of bidegree $(1, 1)$ on a compact Kähler manifold (X, ω) of dimension $k \geq 2$. This allows to define $\int_X T \wedge T \wedge \omega^{k-2}$ for every positive harmonic current T of bidegree $(1, 1)$ on X.

This theory applies to directed positive harmonic currents on singular holomorphic foliations on compact Kähler surfaces.

In [25, 27] Fornæss and Sibony also develop a geometric intersection theory for directed positive harmonic currents on singular holomorphic foliations on $\mathbb{P}^2$.

Combining these two theories, they obtain the following remarkable unique ergodicity result for singular holomorphic foliations without invariant algebraic curves.

Theorem 5.6 (Fornæss-Sibony [27]) *Let $\mathscr{F}$ be a singular holomorphic foliation in $\mathbb{P}^2$ whose singularities are all hyperbolic. Assume that $\mathscr{F}$ has no invariant algebraic curve. Then $\mathscr{F}$ has a unique directed positive harmonic current of mass* 1. *Moreover, this unique current T is not closed. In particular, for every point x outside the singularity set of $\mathscr{F}$, the current $\tau_{x,R}$ defined in* (5.2) *converges to T when $R \to \infty$.*

The case where $\mathscr{F}$ possesses invariant algebraic curves has recently been answered.

Theorem 5.7 (Dinh-Sibony [23]) *Let $\mathscr{F}$ be a singular holomorphic foliation in $\mathbb{P}^2$ whose singularities are all hyperbolic. Assume that $\mathscr{F}$ admits a finite number of invariant algebraic curves. Then any directed positive harmonic current is a linear combination of the currents of integration on these curves. In particular, all directed positive harmonic currents are closed.*

Theorem 5.7 is surprising even in the special case where $\mathscr{F}$ admits the line at infinity L_∞ as an invariant curve. Let $\mathscr{F}$ be a generic foliation of a given degree $d > 1$ with this property. By Khudai-Veronov [36], all leaves (except L_∞) of $\mathscr{F}$ are dense. So by intuition from Theorem 5.6 one could expect that there should be a directed harmonic current with support $\mathbb{P}^2$. However, Theorem 5.7 says that this intuition is false.

To prove Theorem 5.7 we need to to show that if T is a positive harmonic current directed by $\mathscr{F}$ having no mass on any leaf, then T is zero. For this purpose, Dinh and Sibony [23] develop a theory of densities of positive dd^c-closed $(1, 1)$-currents in a compact Kähler surface. A related theory was previously developed by these authors in [21] for positive closed currents defined on compact Kähler manifolds. Applications of these theories in complex dynamics of higher dimension could be found in [19, 22, 24] etc.

Problem 5.8 Are there any versions of Theorems 5.6 and 5.7 for compact Kähler surfaces?

6 Topological and Metric Entropies for Hyperbolic Riemann Surface Laminations

Ghys-Langevin-Walczak introduced in [31] a notion of geometric entropy for compact transversally smooth Riemannian foliations $(X, \mathscr{L})$ (see also Candel-Conlon [8] and Walczak [55] for recent expositions). They prove that this entropy is always finite. In fact, their notion is related to the entropy of the holonomy pseudogroup, which depends on the chosen generators. It also depends on the choice of the metric on the ambient manifold X. The basic idea is to quantify how much leaves get far apart transversally. The transverse regularity of the metric on leaves and the lack of singularities play a role in the finiteness of the entropy.

In [17] Dinh-Nguyen-Sibony introduce a general notion of entropy, which permits to describe some natural situations in dynamics and in laminations/foliation theory. This new notion of entropy contains a large number of classical situations. In particular, it also includes Riemannian foliations with singularities. Another interesting fact is that this entropy is related to an increasing family of distances as in Bowen's point of view [2]. This allows, for example, for introducing other dynamical notions like metric entropy, local entropies etc.

Let X be a metric space endowed with a distance dist_X. Consider a family $\mathscr{D} = \{\text{dist}_t\}$ of distances on X indexed by $t \in \mathbb{R}^+$. We can also replace $\mathbb{R}^+$ by $\mathbb{N}$ and in practice we often have that $\text{dist}_0 = \text{dist}_X$ and that dist_t is increasing with respect to $t \geq 0$.

Let Y be a non-empty subset of X. Denote by $N(Y, t, \epsilon)$ the minimal number of balls of radius ϵ with respect to the distance dist_t needed to cover Y. Define the *entropy* of Y with respect to $\mathscr{D}$ by

$$h_{\mathscr{D}}(Y) := \sup_{\epsilon>0} \limsup_{t\to\infty} \frac{1}{t} \log N(Y, t, \epsilon). \tag{6.1}$$

When $Y = X$ we will denote by $h_{\mathscr{D}}$ this entropy.

Observe that when dist_t is increasing, $N(Y, t, \epsilon)$ is increasing with respect to $t \geq 0$. Moreover,

$$\limsup_{t\to\infty} \frac{1}{t} \log N(Y, t, \epsilon)$$

is increasing when ϵ decreases. So, in the above definition, we can replace $\sup_{\epsilon>0}$ by $\lim_{\epsilon\to 0^+}$. If $\mathscr{D} = \{\text{dist}_t\}$ and $\mathscr{D}' = \{\text{dist}'_t\}$ are two families of distances on X such that $\text{dist}'_t \geq A\,\text{dist}_t$ for all t with a fixed constant $A > 0$, then $h_{\mathscr{D}'} \geq h_{\mathscr{D}}$.

A subset $F \subset X$ is said to be (t, ϵ)-*separated* if for all distinct points x, y in F we have $\text{dist}_t(x, y) > \epsilon$. Let $M(Y, t, \epsilon)$ denote the maximal number of points in a (t, ϵ)-separated family $F \subset Y$. We record here a simple relation between $N(Y, t, \epsilon)$ and $M(Y, t, \epsilon)$.

Lemma 6.1 *We have*

$$N(Y, t, \epsilon) \leq M(Y, t, \epsilon) \leq N(Y, t, \epsilon/2).$$

An important consequence of Lemma 6.1 is that we can formulate the entropy of a subset $Y \subset X$ using $M(Y, t, \epsilon)$ instead of $N(Y, t, \epsilon)$:

$$h_{\mathscr{D}}(Y) = \sup_{\epsilon>0} \limsup_{t\to\infty} \frac{1}{t} \log M(Y, t, \epsilon).$$

Let $(X, \mathscr{L})$ be a hyperbolic Riemann surface lamination, where X is a metric space endowed with a distance dist_X. Recall that for every $x \in X$, a universal covering map ϕ_x of the leaf L_x with $\phi_x(0) = x$ is given in (2.1). For every universal covering map ψ of the leaf L_x with $\psi(x) = x$, there is $\theta \in \mathbb{R}$ such that ψ is equal to the map $\mathbb{D} \ni \xi \mapsto \phi_x(e^{i\theta}\xi)$, in other words, those maps ψ are unique up to a rotation on $\mathbb{D}$. Define the family of distances $\mathscr{D} := \{\text{dist}_t\}$:

$$\text{dist}_t(x, y) := \inf_{\theta \in \mathbb{R}} \sup_{\xi \in \mathbb{D}_t} \text{dist}_X(\phi_x(e^{i\theta}\xi), \phi_y(\xi)). \tag{6.2}$$

The metric dist_t measures how far two leaves get apart before the hyperbolic time t. It takes into account the time parametrization like in the classical case where one measures the distance of two orbits before time n, by measuring the distance at each time $i < n$. So, we are not just concerned with geometric proximity.

Definition 6.2 (Dinh-Nguyen-Sibony [17]) The *hyperbolic entropy* of a hyperbolic Riemann surface $(X, \mathscr{L})$, denoted by $h(\mathscr{L})$, is the entropy, computed by (6.1), of X with respect to the family $\mathscr{D} := \{\text{dist}_t\}$ which is given by (6.2).

So, the value of the entropy is unchanged under homeomorphisms between laminations which are holomorphic along leaves. The advantage here is that the hyperbolic time we choose is canonical. The notion of hyperbolic entropy can be extended to (l-dimensional) Riemannian foliations with singularities, or more generally (l-dimensional) laminations with singularities, and a priori it is bigger than or equal to the geometric entropy of Ghys, Langevin and Walczak (see [17, p. 584] for the definition of the latter entropy and its relation with the hyperbolic entropy).

Theorem 6.3 (Dinh-Nguyen-Sibony [17]) *Let $(X, \mathscr{L})$ be a transversally smooth compact lamination by hyperbolic Riemann surfaces. Embed the lamination in an $\mathbb{R}^N$ in order to use the distance* dist_X *induced by a Riemannian metric on $\mathbb{R}^N$. Then,* $2 \le h(\mathscr{L}) < \infty$.

The following proposition gives a simple criterion for the finiteness of the hyperbolic entropy. We will need it for the proof of Theorem 6.3.

Proposition 6.4 *Assume that there are positive constants A and m such that for every $\epsilon > 0$ small enough X admits a covering by less than $A\epsilon^{-m}$ balls of radius ϵ for the distance* dist_X. *Assume also that*

$$\text{dist}_t \le e^{ct+d}\, \text{dist}_X + \varphi(t)$$

for some constants $c, d \ge 0$ and a function φ with $\varphi(t) \to 0$ as $t \to \infty$. Then, the entropy $h_{\mathscr{D}}$ is at most equal to mc.

We will apply Proposition 6.4 in order to prove Theorem 6.3. So it is necessary to estimate the distance dist_t between leaves. For this purpose, we use the Beltrami equation as in the proof of the transverse regularity of the Poincaré metric in Theorem 3.1.

To study the finiteness of the entropy for singular holomorphic foliations is a very hard matter. A satisfactory answer is only obtained for complex surfaces.

Theorem 6.5 (Dinh-Nguyen-Sibony [18]) *Let $(X, \mathscr{L}, E)$ be a singular foliation by Riemann surfaces on a compact Hermitien complex surface X. Assume that the singularities are linearizable and that the foliation is Brody hyperbolic. Then, its hyperbolic entropy $h(\mathscr{L})$ is finite.*

The proof of this theorem is quite delicate and requires a careful analysis of the dynamics around the singularities. We deduce from the above theorem and Theorem 2.9 the following corollary. It can be applied to foliations of degree at least 2 with hyperbolic singularities.

Corollary 6.6 (Dinh-Nguyen-Sibony [18]) *Let $(\mathbb{P}^2, \mathscr{L}, E)$ be a singular foliation by Riemann surfaces on the complex projective plane $\mathbb{P}^2$ endowed with the Fubini-Study metric. Assume that the singularities are linearizable. Then, the hyperbolic entropy $h(\mathscr{L})$ of $(\mathbb{P}^2, \mathscr{L}, E)$ is finite.*

Consider an abstract setting of a metric space (X, dist_X) endowed with a family $\mathscr{D} := \{\mathrm{dist}_t\}_{t\geq 0}$ of distances. Let m be a probability measure on X. For positive constants ϵ, δ and t, let $N_m(t, \epsilon, \delta)$ be the minimal number of balls of radius ϵ relative to the metric dist_t whose union has at least m-measure $1-\delta$. *The (metric) entropy* of m is defined by the following formula

$$h_{\mathscr{D}}(m) := \lim_{\delta\to 0}\lim_{\epsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log N_m(t, \epsilon, \delta).$$

We have the following general property.

Lemma 6.7 *For any probability measure m on X, we have*

$$h_{\mathscr{D}}(m) \leq h_{\mathscr{D}}(\mathrm{supp}(m)),$$

where $h_{\mathscr{D}}(\mathrm{supp}(m))$ is computed using formula (6.1). In particular, the metric entropy of a probability measure is dominated by the entropy of the whole space.

As in Brin-Katok's theorem [3], we can introduce the *local entropies* of m at $x \in X$ by

$$h^+_{\mathscr{D}}(m, x, \epsilon) := \limsup_{t\to\infty} -\frac{1}{t}\log m(B_t(x, \epsilon)), \qquad h^+_{\mathscr{D}}(m, x) := \sup_{\epsilon>0} h^+_{\mathscr{D}}(m, x, \epsilon),$$

and

$$h^-_{\mathscr{D}}(m, x, \epsilon) := \liminf_{t\to\infty} -\frac{1}{t}\log m(B_t(x, \epsilon)), \qquad h^-_{\mathscr{D}}(m, x) := \sup_{\epsilon>0} h^-_{\mathscr{D}}(m, x, \epsilon),$$

where $B_t(x, \epsilon)$ denotes the ball centered at x of radius ϵ with respect to the distance dist_t.

Note that in the case of an ergodic invariant measure associated with a continuous map on a metric compact space, the above notions of entropies coincide with the classical entropy of m, see Brin-Katok [3].

Recall that in (6.2) we have associated to $(X, \mathscr{L})$ a special family of distances $\mathscr{D} := \{\text{dist}_t\}_{t\geq 0}$. Therefore, we can associate to m a metric entropy and local entropies defined as above in the abstract setting. Recall also that a harmonic probability measure m is *extremal* if all harmonic probability measures m_1, m_2 satisfying $m_1 + m_2 = 2m$ are equal to m. We have the following result.

Theorem 6.8 (Dinh-Nguyen-Sibony [17]) *Let $(X, \mathscr{L})$ be a compact transversally smooth lamination by hyperbolic Riemann surfaces. Let m be a harmonic probability measure. Then, the local entropies $h^{\pm}$ of m are constant on leaves. In particular, if m is extremal, then $h^{\pm}$ are constant m-almost everywhere.*

We reproduce from [17] some fundamental problems concerning metric entropies for Riemann surface laminations. Assume, in the first two problems that $(X, \mathscr{L})$ is a compact transversally smooth lamination by hyperbolic Riemann surfaces. However, the problems can be stated in a more general setting.

Problem 6.9 Consider extremal harmonic probability measures m. Is the following *variational principle* always true

$$h(\mathscr{L}) = \sup_m h(m)\ ?$$

Even when this principle does not hold, it would be of interest to consider the quantity

$$h(\mathscr{L}) - \sup_m h(m)$$

and to explain the role of the hyperbolic time in this number.

Problem 6.10 If m is as above, is the identity $h^+(m) = h^-(m)$ always true?

We think that the answer is affirmative and gives an analog of the Brin-Katok theorem.

Notice that there is a notion of entropy for harmonic measures introduced by Kaimanovich [38]. Consider a metric g of bounded geometry on the leaves of the lamination. Then, we can consider the heat kernel $p(t, \cdot, \cdot)$ associated to the Laplacian determined by this metric. If m is a harmonic probability measure on X, Kaimanovich defines the entropy of m as

$$h_K(m) := \int dm(x)\Big(\lim_{t\to\infty} -\frac{1}{t}\int p(t,x,y)\log p(t,x,y)g(y)\Big).$$

He shows that the limit exists and is constant m-almost everywhere when m is extremal.

This notion of entropy has been extensively studied for the universal covering of a compact Riemannian manifold, see e.g. Ledrappier [41].

Problem 6.11 It is interesting to find relations between Kaimanovich entropy and our notions of entropy. Moreover, studying these relations in the context of singular holomorphic foliations is also an important question.

In Kaimanovich's entropy, the transverse spreading is present through the variation of the heat kernel from leaf to leaf. Therefore, a question naturally arises whether one can make this dependence more explicit.

Here is an open problem from [18].

Problem 6.12 If $\mathscr{F}$ is a generic element in $\mathscr{F}_d(\mathbb{P}^k)$ (with $d > 1$ and $k > 2$), is the hyperbolic entropy of $\mathscr{F}$ finite? The same question is asked for a singular holomorphic foliation on a compact Hermitian complex manifold. This is a generalization of Theorem 6.5 to higher dimensions.

Finally, the following problem seems of interest.

Problem 6.13 Is the lower bound in Theorem 6.3 optimal? If yes, study the class of all transversally smooth compact lamination $(X, \mathscr{L})$ by hyperbolic Riemann surfaces such that $h(\mathscr{L}) = 2$.

7 Lyapunov Theory for Hyperbolic Riemann Surface Laminations

The purpose of this section is to present some recent results obtained in our works [46, 47].

7.1 Brownian Motion and Wiener Measures

We start with Garnett's theory of leafwise Brownian motion in [29] (see also [7, 9]). Our presentation follows [47]. We first recall the construction of the Wiener measure W_0 on the Poincaré disc $(\mathbb{D}, g_P)$. Let Ω_0 be the space consisting of all continuous paths $\omega : [0,\infty) \to \mathbb{D}$ with $\omega(0) = 0$. A *cylinder set (in Ω_0)* is a set of the form

$$C = C(\{t_i, B_i\} : 1 \le i \le m) := \{\omega \in \Omega_0 : \omega(t_i) \in B_i, \quad 1 \le i \le m\},$$

where m is a positive integer and the B_i's are Borel subsets of $\mathbb{D}$, and $0 < t_1 < t_2 < \cdots < t_m$ is a set of increasing times. In other words, C consists of all paths $\omega \in \Omega_0$ which can be found within B_i at time t_i. Let $\mathscr{A}_0$ be the σ-algebra on Ω_0 generated by all cylinder sets. For each cylinder set $C := C(\{t_i, B_i\} : 1 \leq i \leq m)$ as above, define

$$W_x(C) := \Big(D_{t_1}(\mathbf{1}_{B_1} D_{t_2-t_1}(\mathbf{1}_{B_2} \cdots \mathbf{1}_{B_{m-1}} D_{t_m-t_{m-1}}(\mathbf{1}_{B_m}) \cdots))\Big)(x), \tag{7.1}$$

where, $\mathbf{1}_{B_i}$ is the characteristic function of B_i and D_t is the diffusion operator given by (2.2) where $p(x, y, t)$ therein is the heat kernel of the Poincaré disc. It is well-known that W_0 can be extended to a unique probability measure on $(\Omega_0, \mathscr{A}_0)$. This is the *canonical Wiener measure* at 0 on the Poincaré disc.

Let $(X, \mathscr{L})$ be a hyperbolic Riemann surface lamination endowed with the leafwise Poincaré metric g_P. Let $\Omega := \Omega(X, \mathscr{L})$ be the space consisting of all continuous paths $\omega : [0, \infty) \to X$ with image fully contained in a single leaf. This space is called *the sample-path space* associated to $(X, \mathscr{L})$. Observe that Ω can be thought of as the set of all possible paths that a Brownian particle, located at $\omega(0)$ at time $t = 0$, might follow as time progresses. For each $x \in X$, let $\Omega_x = \Omega_x(X, \mathscr{L})$ be the space of all continuous leafwise paths starting at x in $(X, \mathscr{L})$, that is,

$$\Omega_x := \{\omega \in \Omega : \omega(0) = x\}.$$

For each $x \in X$, the following mapping

$$\Omega_0 \ni \omega \mapsto \phi_x \circ \omega \quad \text{maps} \quad \Omega_0 \quad \text{bijectively onto} \quad \Omega_x, \tag{7.2}$$

where $\phi_x : \mathbb{D} \to L_x$ is given in (2.1). Using this bijection we obtain a natural σ-algebra $\mathscr{A}_x$ on the space Ω_x, and a natural probability (Wiener) measure W_x on $\mathscr{A}_x$ as follows:

$$\mathscr{A}_x := \{\phi_x \circ A : A \in \mathscr{A}_0\} \quad \text{and} \quad W_x(\phi_x \circ A) := W_0(A), \qquad A \in \mathscr{A}_0, \tag{7.3}$$

where $\phi_x \circ A := \{\phi_x \circ \omega : \omega \in A\} \subset \Omega_x$.

7.2 Cocycles

The notion of (multiplicative) cocycles have been introduced in [46] for (l-dimensional) laminations. For the sake of simplicity we only formulate this notion for Riemann surface laminations in this article. In the rest of the section we make the following convention: $\mathbb{K}$ denotes either the field $\mathbb{R}$ or $\mathbb{C}$. Moreover, given any integer $d \geq 1$, $\mathrm{GL}(d, \mathbb{K})$ denotes the general linear group of degree d over $\mathbb{K}$ and $\mathbb{P}^d(\mathbb{K})$ denotes the $\mathbb{K}$-projective space of dimension d.

Definition 7.1 (Nguyen [46, Definition 3.2]) A $\mathbb{K}$-*valued cocycle* (of rank d) is a map $\mathcal{A}: \ \Omega \times \mathbb{R}^+ \to \mathrm{GL}(d, \mathbb{K})$ such that

(1) (*identity law*) $\mathcal{A}(\omega, 0) = \mathrm{id}$ for all $\omega \in \Omega$;
(2) (*homotopy law*) if $\omega_1, \omega_2 \in \Omega_x$ and $t_1, t_2 \in \mathbb{R}^+$ such that $\omega_1(t_1) = \omega_2(t_2)$ and $\omega_1|_{[0,t_1]}$ is homotopic to $\omega_2|_{[0,t_2]}$ (that is, the path $\omega_1|_{[0,t_1]}$ can be deformed continuously on L_x to the path $\omega_2|_{[0,t_2]}$, the two endpoints of $\omega_1|_{[0,t_1]}$ being kept fixed during the deformation), then

$$\mathcal{A}(\omega_1, t_1) = \mathcal{A}(\omega_2, t_2);$$

(3) (*multiplicative law*) $\mathcal{A}(\omega, s + t) = \mathcal{A}(\sigma_t(\omega), s)\mathcal{A}(\omega, t)$ for all $s, t \in \mathbb{R}^+$ and $\omega \in \Omega$;
(4) (*measurable law*) the *local expression* of $\mathcal{A}$ on each laminated chart is Borel measurable. Here, the *local expression* of $\mathcal{A}$ on the laminated chart $\Phi : \mathbb{U} \to \mathbb{D} \times \mathbb{T}$, is the map $A : \ \mathbb{D} \times \mathbb{D} \times \mathbb{T} \to \mathrm{GL}(d, \mathbb{K})$ defined by

$$A(y, z, t) := \mathcal{A}(\omega, 1),$$

where ω is any leafwise path such that $\omega(0) = \Phi^{-1}(y, t)$, $\omega(1) = \Phi^{-1}(z, t)$ and $\omega[0, 1]$ is contained in the simply connected plaque $\Phi^{-1}(\cdot, t)$.

A cocycle $\mathcal{A}$ on a smooth Riemann surface lamination $(X, \mathscr{L})$ is called a *smooth* if, for each laminated chart Φ, the local expression A is smooth with respect to (y, z) and its partial derivatives of any total order with respect to (y, z) are jointly continuous in (y, z, t).

The cocycles of rank 1 have been investigated by several authors (see, for example, Candel [7], Deroin [13], etc.). The holonomy cocycle (or equivalently the normal derivative cocycle) of the regular part of a singular holomorphic foliation by hyperbolic Riemann surfaces $(X, \mathscr{L}, E)$ with $\dim_{\mathbb{C}} X = n$ is a typical example of $\mathbb{C}$-valued cocycles of rank $n - 1$. These cocycles capture the topological aspect of the considered foliations. Moreover, we can produce new cocycles from the old ones by performing some basic operations such as the wedge product and the tensor product (see [46, Sect. 3.1]).

7.3 Oseledec Multiplicative Ergodic Theorem

Now we are in the position to state the Oseledec multiplicative ergodic theorem for hyperbolic Riemann surface laminations.

Theorem 7.2 (Nguyen [46, Theorem 3.11]) *Let $(X, \mathscr{L})$ be a hyperbolic Riemann surface lamination. Let μ be a harmonic measure which is also ergodic.*

Consider a cocycle $\mathcal{A}: \ \Omega \times \mathbb{R}^+ \to \mathrm{GL}(d, \mathbb{K})$. *Assume that the following integrability condition is satisfied for some real number* $t_0 > 0$:

$$\int_{x\in X} \Big(\int_{\Omega_x} \sup_{t\in[0,t_0]} \log^+ \|\mathcal{A}(\omega, t)\| dW_x(\omega)\Big) d\mu(x) < \infty, \qquad (7.4)$$

where $\log^+ := \max(0, \log)$. *Then there exist a leafwise saturated Borel set* $Y \subset X$ *of total* μ*-measure and a number* $m \in \mathbb{N}$ *together with* m *integers* $d_1, \ldots, d_m \in \mathbb{N}$ *such that the following properties hold:*

(i) For each $x \in Y$ *there exists a decomposition of* $\mathbb{K}^d$ *as a direct sum of* $\mathbb{K}$*-linear subspaces*

$$\mathbb{K}^d = \oplus_{i=1}^m H_i(x),$$

such that $\dim H_i(x) = d_i$ *and* $\mathcal{A}(\omega, t)H_i(x) = H_i(\omega(t))$ *for all* $\omega \in \Omega_x$ *and* $t \in \mathbb{R}^+$. *Moreover,* $x \mapsto H_i(x)$ *is a measurable map from* Y *into the Grassmannian of* $\mathbb{K}^d$. *For each* $1 \le i \le m$ *and each* $x \in Y$, *let* $V_i(x) := \oplus_{j=i}^m H_j(x)$. *Set* $V_{m+1}(x) \equiv \{0\}$.

(ii) There are real numbers

$$\chi_m < \chi_{m-1} < \cdots < \chi_2 < \chi_1,$$

and for each $x \in Y$, *there is a set* $F_x \subset \Omega_x$ *of total* W_x*-measure such that for every* $1 \le i \le m$ *and every* $v \in V_i(x)\setminus V_{i+1}(x)$ *and every* $\omega \in F_x$,

$$\lim_{t\to\infty, t\in\mathbb{R}^+} \frac{1}{t} \log \frac{\|\mathcal{A}(\omega, t)v\|}{\|v\|} = \chi_i. \qquad (7.5)$$

Moreover,

$$\lim_{t\to\infty, t\in\mathbb{R}^+} \frac{1}{t} \log \|\mathcal{A}(\omega, t)\| = \chi_1 \qquad (7.6)$$

for each $x \in Y$ *and for every* $\omega \in F_x$.

Here $\|\cdot\|$ *denotes the standard Euclidean norm of* $\mathbb{K}^d$.

The above result is the counterpart, in the context of hyperbolic Riemann surface laminations, of the classical Oseledec multiplicative ergodic theorem for maps (see [39, 50]). In fact, Theorem 3.11 in [46] is much more general than Theorem 7.2. Indeed, the former is formulated for l-dimensional laminations and for leafwise Riemannian metrics which satisfy some reasonable geometric conditions.

Assertion (i) above tells us that the Oseledec decomposition exists for all points x in a leafwise saturated Borel set of total μ-measure and that this decomposition is holonomy invariant. Observe that the Oseledec decomposition in (i) depends only on $x \in Y$, in particular, it does not depend on paths $\omega \in \Omega_x$.

The decreasing sequence of subspaces of $\mathbb{K}^d$ given by assertion (i):

$$\{0\} \equiv V_{m+1}(x) \subset V_m(x) \subset \cdots \subset V_1(x) = \mathbb{K}^d$$

is called the *Lyapunov filtration* associated to $\mathcal{A}$ at a given point $x \in Y$.

The numbers $\chi_m < \chi_{m-1} < \cdots < \chi_2 < \chi_1$ given by assertion (ii) above are called the *Lyapunov exponents* of the cocycle $\mathcal{A}$ with respect to the harmonic measure μ. It follows from formulas (7.5) and (7.6) above that these characteristic numbers measure heuristically the expansion rate of $\mathcal{A}$ along different vector-directions v and along leafwise Brownian trajectories. In other words, the stochastic formulas (7.5)–(7.6) highlight the dynamical character of the Lyapunov exponents.

7.4 Applications to Compact Smooth Laminations and Compact Singular Foliations

Let $\mathcal{A} : \Omega \times \mathbb{R}^+ \to \mathrm{GL}(d, \mathbb{K})$ be a smooth cocycle defined on a smooth hyperbolic Riemann surface lamination $(X, \mathscr{L})$. Observe that the map $\mathcal{A}^{*-1} : \Omega \times \mathbb{R}^+ \to \mathrm{GL}(d, \mathbb{K})$, defined by $\mathcal{A}^{*-1}(\omega, t) := \big(\mathcal{A}(\omega, t)\big)^{*-1}$, is also a cocycle, where A^* (resp. A^{-1}) denotes as usual the transpose (resp. the inverse) of a square matrix A.

We define two functions $\bar{\delta}(\mathcal{A})$, $\underline{\delta}(\mathcal{A}) : X \to \mathbb{R}$ as well as four quantities $\bar{\chi}_{\max}(\mathcal{A})$, $\underline{\chi}_{\max}(\mathcal{A})$, $\bar{\chi}_{\min}(\mathcal{A})$, $\underline{\chi}_{\min}(\mathcal{A})$ as follows. Fix a point $x \in X$, an element $u \in \mathbb{K}^d \setminus \{0\}$ and a simply connected plaque K of $(X, \mathscr{L})$ passing through x. Consider the function $f_{u,x} : K \to \mathbb{R}$ defined by

$$f_{u,x}(y) := \log \frac{\|\mathcal{A}(\omega, 1)u\|}{\|u\|}, \qquad y \in K,\ u \in \mathbb{K}^d \setminus \{0\}, \tag{7.7}$$

where $\omega \in \Omega$ is any path such that $\omega(0) = x$, $\omega(1) = y$ and that $\omega[0, 1]$ is contained in K. Then define

$$\bar{\delta}(\mathcal{A})(x) := \sup_{u \in \mathbb{K}^d :\ \|u\|=1} (\Delta f_{u,x})(x) \ \text{ and } \ \underline{\delta}(\mathcal{A})(x) := \inf_{u \in \mathbb{K}^d :\ \|u\|=1} (\Delta f_{u,x})(x), \tag{7.8}$$

where Δ is, as usual, the Laplacian on the leaf L_x induced by the leafwise Poincaré metric g_P on $(X, \mathscr{L})$ (see formula (5.1) for $\beta := g_P$). We also define

$$\begin{aligned}
\bar{\chi}_{\max} &= \bar{\chi}_{\max}(\mathcal{A}) := \int_X \bar{\delta}(\mathcal{A})(x)\,d\mu(x),\\
\underline{\chi}_{\max} &= \underline{\chi}_{\max}(\mathcal{A}) := \int_X \underline{\delta}(\mathcal{A})(x)\,d\mu(x);\\
\underline{\chi}_{\min} &= \underline{\chi}_{\min}(\mathcal{A}) := -\bar{\chi}_{\max}(\mathcal{A}^{*-1}),\\
\bar{\chi}_{\min} &= \bar{\chi}_{\min}(\mathcal{A}) := -\underline{\chi}_{\max}(\mathcal{A}^{*-1}).
\end{aligned} \tag{7.9}$$

Note that our functions $\bar{\delta}$, $\underline{\delta}$ are the multi-dimensional generalizations of the operator δ introduced by Candel [7].

We are in the position to state effective integral estimates on the Lyapunov exponents.

Theorem 7.3 (Nguyen [46, Theorem 3.12]) *Let $(X, \mathscr{L})$ be a compact smooth lamination by hyperbolic Riemann surfaces. Let μ be a harmonic probability measure which is ergodic.*

Let $\mathcal{A}: \ \Omega \times \mathbb{R}^+ \to \mathrm{GL}(d, \mathbb{K})$ be a smooth cocycle. Let

$$\chi_m < \chi_{m-1} < \cdots < \chi_2 < \chi_1$$

be the Lyapunov exponents of the cocycle $\mathcal{A}$ with respect to μ, given by Theorem 7.2. Then the following inequalities hold

$$\underline{\chi}_{\max} \le \chi_1 \le \bar{\chi}_{\max} \quad \textit{and} \quad \underline{\chi}_{\min} \le \chi_m \le \bar{\chi}_{\min}.$$

This theorem generalizes some results of Candel [6] and Deroin [13] who treat the case $d = 1$. Under the assumption of Theorem 7.3, the integrability condition (7.4) follows from some well-known estimates of the heat kernels of the Poincaré disc and the fact that the lamination is compact and is without singularities. In fact, we improve the method of Candel in [7].

The holonomy cocycle (or equivalently, the normal derivative cocycle) of a foliation is a very important object which encodes dynamical as well as geometric and analytic informations of the foliation. Exploring this object allows us to understand more about the foliation itself. On the other hand, recall that the main examples of holomorphic foliations by curves are those in the complex projective space $\mathbb{P}^k$ of arbitrary dimension (in which case there are always singularities) or in algebraic manifolds, and that the typical properties of a generic foliation of a given degree $d > 1$ in $\mathbb{P}^k$ are described by Theorem 2.9. Therefore, the following fundamental question arises naturally:

Question *Can one define the Lyapunov exponents of an ergodic harmonic measure μ on a compact singular holomorphic hyperbolic foliation $\mathscr{F} = (X, \mathscr{L}, E)$?*

By Proposition 5.3, this question can be rephrased for directed harmonic currents on the foliation. We have recently obtained the following affirmative answer to this question for generic foliations in dimension two.

Theorem 7.4 (Nguyen [49, Theorem 1.1]) *Let $\mathscr{F} = (X, \mathscr{L}, E)$ be a holomorphic Brody hyperbolic foliation with hyperbolic singularities E in a Hermitian compact complex projective surface X. Let $\mathcal{A}$ be the holonomy cocycle of the foliation. Let T be a positive harmonic current directed by $\mathscr{F}$ which does not give mass to any invariant analytic curve. Consider the corresponding harmonic measure $\mu := T \wedge g_P$, where g_P is as usual the leafwise Poincaré metric. Then the integrability condition (7.4) is satisfied for all $t_0 > 0$.*

Here is an immediate consequence of this theorem.

Corollary 7.5 *Under the hypotheses and notation of Theorem 7.4, assume in addition that the measure μ is ergodic. Then T admits the (unique) Lyapunov exponent $\lambda(T)$ given by the formula*

$$\lambda(T) := \int_X \Big(\int_{\Omega_x} \log \|\mathcal{A}(\omega, 1)\| dW_x(\omega)\Big) d\mu(x).$$

Moreover, for μ-almost every $x \in X$, we have

$$\lim_{t\to\infty} \frac{1}{t} \log \|\mathcal{A}(\omega, t)\| = \lambda(T)$$

for almost every path $\omega \in \Omega$ with respect to the Wiener measure at x which lives on the leaf passing through x.

Consider a singular foliation by curves $\mathscr{F} = (\mathbb{P}^2, \mathscr{L}, E)$ on the complex projective plane $\mathbb{P}^2$ such that all the singularities of $\mathscr{F}$ are hyperbolic and that $\mathscr{F}$ has no invariant algebraic curve. By Remark 3.3 we know that $\mathscr{F}$ is Brody hyperbolic. Moreover, Theorem 5.6 tells us that the harmonic current T is unique up to a multiplicative constant. So the convex cone of all harmonic currents of $\mathscr{F}$ is just a real half-line, and hence all these currents are extremal. Therefore, by Proposition 5.3 the measure $T \wedge g_P$ is ergodic. Consequently, Corollary 7.5 applies and gives us the following result. It can be applied to every generic foliation in $\mathbb{P}^2$ with a given degree $d > 1$.

Corollary 7.6 *Let $\mathscr{F} = (\mathbb{P}^2, \mathscr{L}, E)$ be a singular foliation by curves on the complex projective plane $\mathbb{P}^2$. Assume that all the singularities are hyperbolic and that $\mathscr{F}$ has no invariant algebraic curve. Let T be the unique harmonic current tangent to $\mathscr{F}$ such that $\mu := T \wedge g_P$ is a probability measure. Let $\mathcal{A}$, be as in the statement of Theorem 7.4. Then the conclusion of this theorem as well as that of Corollary 7.5 hold. In particular, $\mathscr{F}$ admits a unique Lyapunov exponent.*

The novelty of the last corollary is that the (unique) Lyapunov exponent of such a foliation $\mathscr{F}$ is intrinsic and canonical.

The proof of Theorem 7.4 consists of two steps. Let g_X be a Hermitian metric on X and let dist denote the distance on X induced by g_X. In the first step we show that Theorem 7.4 follows from the new integrability condition (7.10).

$$\text{(new integrability condition):} \quad \int_X |\log \text{dist}(x, E)| \cdot (T \wedge g_P)(x) < \infty. \qquad (7.10)$$

This new condition has the advantage over the old one (7.4), since the former does not involve the somewhat complicating Wiener measures, and hence it is easier to handle than the latter.

For this purpose we study the behavior of the holonomy cocycle near the singularities with respect to the leafwise Poincaré metric. Roughly speaking, this step quantifies the expansion speed of the hololomy cocycle in terms of the ambient metric g_X when one travels along unit-speed geodesic rays. One of the main ingredients is a detailed analysis of the behaviour of the leafwise Poincaré metric near hyperbolic singularities which has been carried out in [16–18].

The second main step is then devoted to the proof of inequality (7.10). The main difficulty is that known estimates (see, for example, [16]) on the behavior of T near linearizable singularities, only give a weaker inequality

$$\int_X |\log \operatorname{dist}(x, E)|^{1-\delta} \cdot (T \wedge g_P)(x) < \infty, \qquad \forall \delta > 0. \tag{7.11}$$

So (7.10) is the limiting case of (7.11). The proof of (7.11) relies on the finiteness of the Lelong number of T at every point which has been established in Proposition 2.3. Recall that Theorem 4.2 sharpens the last estimate by showing that the Lelong number of T vanishes at every hyperbolic singular point $x \in E$. Nevertheless, even this better estimate does not suffice to prove (7.10).

The new idea in [49] is that we use a cohomological argument which exploits fully the assumption that X is projective. This assumption imposes a stronger mass-clustering condition on harmonic currents than (7.11).

The condition of Brody hyperbolicity seems to be indispensable for the integrability of the holonomy cocycle. Indeed, Hussenot [35, Theorem A] finds out the following remarkable property for a class of Ricatti foliations $\mathscr{F}$ on $\mathbb{P}^2$. For every $x \in \mathbb{P}^2$ outside invariant curves of every foliation in this class, it holds that

$$\limsup_{t \to \infty} \frac{1}{t} \log \|\mathcal{A}(\omega, t)\| = \infty$$

for almost every path $\omega \in \Omega_x$ with respect to the Wiener measure at x which lives on the leaf passing through x. By Theorem 2.9, these foliations are hyperbolic since all their singular points have non-degenerate linear part. Nevertheless, neither of them is Brody hyperbolic because they all contain integral curves which are some images of $\mathbb{P}^1$ (see Remark 3.3).

Problem 7.7 (*see also* [15, 35]) Is the Lyapunov exponent of a generic foliation with a given degree $d > 1$ in $\mathbb{P}^2$ positive/negative/zero?

Problem 7.8 Does Theorem 7.4 still hold if the ambient compact projective manifold X is of dimension > 2?

7.5 Geometric Characterization of Lyapunov Exponents

To find a geometric interpretation of these characteristic quantities, our idea consists in replacing the Brownian trajectories by the more appealing objects, namely, the *unit-speed geodesic rays*. These paths are parameterized by their length (with respect to the leafwise Poincaré metric). Therefore, we characterize the Lyapunov exponents in terms of the expansion rates of $\mathcal{A}$ along the geodesic rays.

Let $(X, \mathscr{L})$ be a hyperbolic Riemann surface lamination. Recall from (2.1) that $(\phi_x)_{x\in X}$ is a given family of universal covering maps $\phi_x : \mathbb{D} \to L_x$ with $\phi_x(0) = x$. For every $x \in X$, the set of all unit-speed geodesic rays $\omega : [0, \infty) \to L_x$ starting at x (that is, $\omega(0) = x$), can be described by the family $(\gamma_{x,\theta})_{\theta\in[0,1)}$, where

$$\gamma_{x,\theta}(R) := \phi_x(e^{2\pi i\theta} r_R), \qquad R \in \mathbb{R}^+, \tag{7.12}$$

and r_R is uniquely determined by the equation $r_R\mathbb{D} = \mathbb{D}_R$ (see (4.1)). The path $\gamma_{x,\theta}$ is called the *unit-speed geodesic ray* at x with the leaf-direction θ. Unless otherwise specified, the *space of leaf-directions* $[0, 1)$ is endowed with the Lebesgue measure. This space is visibly identified, via the map $[0, 1) \ni \theta \mapsto e^{2\pi i\theta}$, with the unit circle $\partial\mathbb{D}$ endowed with the normalized rotation measure.

We introduce the following notions of expansion rates for cocycles.

Definition 7.9 Let $\mathcal{A}$ be a $\mathbb{K}$-valued cocycle and $R > 0$ a time.

The *expansion rate* of $\mathcal{A}$ at a point $x \in X$ in the leaf-direction θ at time R along the vector $v \in \mathbb{K}^d\setminus\{0\}$ is the number

$$\mathscr{E}(x, \theta, v, R) := \frac{1}{R}\log\frac{\|\mathcal{A}(\gamma_{x,\theta}, R)v\|}{\|v\|}.$$

The *expansion rate* of $\mathcal{A}$ at a point $x \in X$ in the leaf-direction θ at time R is

$$\begin{aligned}\mathscr{E}(x, \theta, R) := \sup_{v\in\mathbb{K}^d\setminus\{0\}} \mathscr{E}(x, \theta, v, R) &= \sup_{v\in\mathbb{K}^d\setminus\{0\}} \frac{1}{R}\log\frac{\|\mathcal{A}(\gamma_{x,\theta}, R)v\|}{\|v\|}\\ &= \frac{1}{R}\log\|\mathcal{A}(\gamma_{x,\theta}, R)\|.\end{aligned}$$

Given a $\mathbb{K}$-vector subspace $\{0\} \neq H \subset \mathbb{K}^d$, the *expansion rate* of $\mathcal{A}$ at a point $x \in X$ at time R along the vector space H is the interval $\mathscr{E}(x, H, R) := [a, b]$, where

$$\begin{aligned}a &:= \inf_{v\in H\setminus\{0\}} \int_0^1 \left(\frac{1}{R}\log\frac{\|\mathcal{A}(\gamma_{x,\theta}, R)v\|}{\|v\|}\right) d\theta,\\ b &:= \sup_{v\in H\setminus\{0\}} \int_0^1 \left(\frac{1}{R}\log\frac{\|\mathcal{A}(\gamma_{x,\theta}, R)v\|}{\|v\|}\right) d\theta.\end{aligned}$$

Notice that $\mathscr{E}(x, \theta, v, R)$ (resp. $\mathscr{E}(x, \theta, R)$) expresses geometrically the expansion rate (resp. the maximal expansion rate) of the cocycle when one travels along the unit-speed geodesic ray $\gamma_{x,\theta}$ up to time R. On the other hand, $\mathscr{E}(x, H, R)$ represents the smallest closed interval which contains all numbers

$$\int_0^1 \Big(\frac{1}{R}\log\frac{\|\mathcal{A}(\gamma_{x,\theta}, R)v\|}{\|v\|}\Big)d\theta,$$

where v ranges over $H \setminus \{0\}$. Note that the above integral is the average of the expansion rate of the cocycle when one travels along the unit-speed geodesic rays along the vector-direction $v \in H$ from x to the Poincaré circle with radius R and center x spanned on L_x.

We say that a sequence of intervals $[a(R), b(R)] \subset \mathbb{R}$ indexed by $R \in \mathbb{R}^+$ converges to a number $\chi \in \mathbb{R}$ and write $\lim_{R\to\infty}[a(R), b(R)] = \chi$, if $\lim_{R\to\infty} a(R) = \lim_{R\to\infty} b(R) = \chi$.

Now we are able to state the main result of this subsection.

Theorem 7.10 (Nguyen [47]) *Let $(X, \mathscr{L})$ be a compact smooth hyperbolic Riemann surface lamination and T a directed positive harmonic current which is also extremal. Let $\mu := T \wedge g_P$ be the (positive finite Borel) measure associated to T. Consider a smooth cocycle $\mathcal{A}: \ \Omega \times \mathbb{R}^+ \to \mathrm{GL}(d, \mathbb{K})$. Then there is a leafwise saturated Borel set Y of total μ-measure which satisfies the conclusion of Theorem 7.2 and the following additional geometric properties:*

(i) For each $1 \le i \le m$ and for each $x \in Y$, there is a set $G_x \subset [0, 1)$ of total Lebesgue measure such that for each $v \in V_i(x) \setminus V_{i+1}(x)$,

$$\lim_{R\to\infty} \mathscr{E}(x, \theta, v, R) = \chi_i, \qquad \theta \in G_x.$$

Moreover, the maximal Lyapunov exponent χ_1 satisfies

$$\lim_{R\to\infty} \mathscr{E}(x, \theta, R) = \chi_1, \qquad \theta \in G_x.$$

(ii) For each $1 \le i \le m$ and each $x \in Y$,

$$\lim_{R\to\infty} \mathscr{E}(x, H_i(x), R) = \chi_i.$$

Here $\mathbb{K}^d = \oplus_{i=1}^m H_i(x)$, $x \in Y$, is the Oseledec decomposition given by Theorem 7.2 and $\chi_m < \chi_{m-1} < \cdots < \chi_2 < \chi_1$ are the corresponding Lyapunov exponents.

Theorem 7.10 gives a geometric meaning to the stochastic formulas (7.5)–(7.6).

Let $\mathscr{F} = (M, \mathscr{L}, E)$ be a transversally smooth (resp. holomorphic) singular foliation by Riemann surfaces in a Riemannian manifold (resp. Hermitian complex manifold) M. Consider a leafwise saturated, compact set $X \subset M \setminus E$ whose leaves are all hyperbolic. So the restriction of the foliation $(M \setminus E, \mathscr{L})$ to X gives an inherited compact smooth hyperbolic Riemann lamination $(X, \mathscr{L})$. Moreover, the holonomy cocycle of $(M \setminus E, \mathscr{L})$ induces, by restriction, an inherited smooth cocycle on $(X, \mathscr{L}|x)$. Hence, Theorem 7.10 applies to the latter cocycle. Recall that a *minimal set* of $\mathscr{F}$ is a leafwise saturated subset of $M \setminus E$ which is also a closed subset of M and which contains no proper subset with these properties. In particular, when $(M, \mathscr{L}, E)$ is a singular holomorphic foliation on a compact Hermitian complex manifold M of dimension n, the last theorem applies to the induced holonomy cocycle of rank $n-1$ associated with every minimal set X whose leaves are all hyperbolic.

The proof of Theorem 7.10 (i) relies on the theory of Brownian trajectories on hyperbolic spaces. More concretely, some quantitative results on the boundary behavior of Brownian trajectories by Lyons [43] and Cranston [12] and on the shadow of Brownian trajectories by geodesic rays are our main ingredients. This allows us to replace a Brownian trajectory by a unit-speed geodesic ray with uniformly distributed leaf-direction. Hence, Part (i) of Theorem 7.10 will follow from Theorem 7.3.

To establish Part (ii) of Theorem 7.10 we need two steps. In the first step we adapt to our context the so-called *Ledrappier type characterization of Lyapunov spectrum* which was introduced in [46]. This allows us to show that a similar version of Part (ii) of Theorem 7.10 holds when the expansion rates in terms of geodesic rays are replaced by some heat diffusions associated with the cocycle. The second step shows that the above heat diffusions can be approximated by the expansion rates. To this end we establish a new geometric estimate on the heat diffusions which relies on the proof of the geometric Birkhoff ergodic theorem (Theorem 5.4).

Problem 7.11 Is Theorem 7.10 still true if $(X, \mathscr{L})$ is the whole regular part of a singular holomorphic foliation $\mathscr{F}$ by hyperbolic Riemann surfaces on a compact complex manifold M and $\mathcal{A}$ is the holonomy cocycle? We can begin with the case where M is a surface.

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A Report on Pseudoconvexity Properties of Analytic Families of $\mathbb{C}$-Bundles and $\mathbb{D}$-Bundles

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Abstract Naturally arising analytic families of pseudoconvex manifolds are locally psuedoconvex in some cases and not in the other. Such a phenomenon is studied in connection to the L^2 extension theorem.

Keywords Pseudoconvex · Analytic family · L^2 extension theorem

1 Introduction

In the theory of several complex variables, important existence theorems are based on the pseudoconvexity properties of complex manifolds. A celebrated example is a characterization of closed complex submanifolds of $\mathbb{C}^n$, or equivalently Stein manifolds, as manifolds admitting strictly plurisubharmonic exhaustion functions (cf. [4, 10, 23, 35]). To generalize the principal argument of this result, which solves Cousin's additive problem on such manifolds, Andreotti and Grauert introduced q-complete manifolds and q-convex manifolds in [1], Stein manifolds being equivalent to 1-complete manifolds. On the other hand, a classical existence theorem of meromorphic functions on compact Riemann surfaces was generalized by Kodaira [16] as an embedding theorem into $\mathbb{CP}^n$ for compact complex manifolds with positive line bundles. The main ingredient of Kodaira's theory is a cohomology vanishing theorem, which is also a solution to Cousin's problem. Grauert observed in [12] that Kodaira's embedding theorem can be recovered from a finite-dimensionality theorem for the analytic sheaf cohomology on 1-convex manifolds. Kodaira's cohomology vaishing theorem was also generalized to 1-convex manifold admitting C^∞ plurisubharmonic exhaustion functions, which he called *weakly 1-complete manifolds*. There was actually a need to investigate this class in characterizing the inverse of monoidal

Dedicated to Kang-Tae Kim on his sixtieth birthday

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transform in terms of the topological property of the normal bundles (cf. [9, 20]). As for the extension of the Andreotti-Garuert theory to weakly 1-complete manifolds, see [26, 27], for instance.

It turned out much later that the cohomology theory on weakly 1-complete manifolds are useful in the the study of the domains with Levi flat boundary such as disc bundles over compact Kähler manifolds (cf. [29]). Accordingly, it is desirable to extend it so that it is available also for analytic families of weakly 1-complete manifolds. However, in contrast to the compact case as in [11, 17], not so much is known about analytic families of 1-complete manifolds except for the case of Riemann surfaces (cf. [15, 19]) and recent results on the parameter dependence of the Bergman kernels (cf. [2, 3, 14, 18]). In both cases, the families are required to be locally pseudoconvex (see Sect. 2 below). So we would like to know, among the naturally arising families of weakly 1-complete manifolds, which one is weakly 1-complete and which one is not. In the present article, we shall report on a recent activity in [31, 32] concerning this question.

2 Bundles Over Compact Kähler Manifolds

Let X and T be two complex manifolds and let $\pi : X \to T$ be a holomorphic map. We shall say that π is *locally pseudoconvex* if every point $t \in T$ has a neighborhood U such that $\pi^{-1}(U)$ is weakly 1-complete. As is well known, the answer is affirmative if π is a local homeomorphism and T is Stein (cf. [34]), and not so in general even if T is Stein (cf. [5, 8, 36]). In this context, it might be worthwhile to recall that a Stein morphism is analytically a bundle map if fibers are $\mathbb{C}$ (cf. [25]). (See also [24].) We shall restrict ourselves to the cases where the fibers of π are analytic fiber bundles over compact complex manifolds whose fibers are either $\mathbb{C}$ or the unit disc $\mathbb{D}$. First we recall when such fiber bundles over a compact complex manifold say M are weakly 1-complete. Since $\mathrm{Aut}(\mathbb{C})$ consists of polynomials of degree one, analytic $\mathbb{C}$-bundles are those fiber bundles whose transition functions are of the form $\zeta_j = a_{jk}(z)\zeta_k + b_{jk}(z)$ with respect to an open covering $M = \cup_j U_j$. Here ζ_j denotes the fiber coordinate over U_j and $z \in U_j \cap U_k$. We shall denote by L_0 the line bundle associated to the 1-cocyle a_{jk}.

Proposition 2.1 *Let M be a compact complex manifold and let $L \to M$ be an analytic fiber bundle whose fibers are $\mathbb{C}$. Then L_0 is weakly 1-complete if L_0 is seminegative.*

Here a line bundle is said to be seminegative if it admits a fiber metric h_j whose curvature form $-\partial\bar{\partial}\log h_j$ is seminegative. The proof is straighforward from the definition. If $\dim M = 1$, much more strict implications are known to hold. The following is essentially due to Ueda [37].

Theorem 2.1 *If* $\dim M = 1$, L *is weakly 1-complete if and only if* L_0 *is seminegative.*

The ingredient of the proof of Theorem 2.1 is contained in the following.

Theorem 2.2 *Topologically trivial analytic affine line bundles over compact Kähler manifolds are weakly 1-complete.*

Proof Let M be a compact Kähler manifold, let $L \to M$ be a topologically trivial analytic affine line bundle. Then, since M is Kähler, one can find an open covering $\{U_j\}$ of M and local trivializations of L such that the transition relations are of the form $\zeta_j = e^{\sqrt{-1}\theta_{jk}}\zeta_k + a_{jk}(z)$ for some $\theta_{jk} \in \mathbb{R}$ and $a_{jk} \in \mathcal{O}(U_j \cap U_k)$, where $\mathcal{O}(U)$ denotes the set of holomorphic functions on U (by a standard application of the $\partial\bar{\partial}$-lemma). Applying the Kähler condition again, by replacing $\{U_j\}$ by its refinement if necessary, one can find $a_j, b_j \in \mathcal{O}(U_j)$ such that

$$a_{jk} = a_j + \overline{b_j} - e^{\sqrt{-1}\theta_{jk}}(a_k + \overline{b_k})$$

holds on $U_j \cap U_k$. This is also a consequence of the $\partial\bar{\partial}$-lemma (cf. [6], Lemma 2). For simplicity we put $h_j = a_j + \overline{b_j}$. The system h_j is naturally identified with a pluriharmonic section of the bundle $L \to M$. Then it is straightforward that the function

$$\Phi = |\zeta_j - h_j|^2$$

is a well defined plurisubharmonic exhaustion function on L. The plurisubharmonicity of can be seen from

$$\begin{aligned}\partial\bar{\partial}\Phi &= d\zeta_j d\overline{\zeta}_j - d\zeta_j\overline{\partial h_j} - d\overline{\zeta}_j\bar{\partial}h_j + \partial h_j\overline{\partial h_j} + \partial\overline{h}_j\bar{\partial}h_j \\ &\geq \partial\overline{h}_j\bar{\partial}h_j.\end{aligned}$$

□

The following was obtained in [7] as a continuation of Theorem 2.2.

Theorem 2.3 *Analytic* $\mathbb{D}$*-bundles over compact Kähler manifolds are weakly 1-complete.*

By these results, a similarity is apparent between the divisors with topologically trivial normal bundle and Levi flat hypersurfaces. In [29, 30], Theorem 2.2 and Theorem 2.3 are complemented respectively by the following results where the similarity still persists.

Theorem 2.4 *Let* X *be a compact Kähler manifold and let* Y *be an effective divisor of* X *whose normal bundle is topologically trivial. Then the complement of the support of* Y *does not admit a* C^∞ *plurisubharmmonic exhaustion function whose Levi form has at least 3 positive eigenvalues everywhere outside a compact set.*

Theorem 2.5 *Let X be a compact Kähler manifold and let $\Sigma \subset X$ be a real analytic Leviat hypersurface. Then $X \setminus \Sigma$ does not admit a C^∞ plurisubharmonic exhaustion function whose Levi form has at least 3 positive eigenvalues everywhere outside some compact subset of $X \setminus \Sigma$.*

For the families of affine line bundles, the following was proved in [32].

Theorem 2.6 *Let T be a complex manifold, let $p : S \to T$ be a proper holomorphic map with smooth one-dimensional fibers, and let $q : L \to S$ be an analytic affine line bundle. Then $p \circ q : L \to T$ is locally pseudoconvex if one of the following conditions is satisfied.*

(i) *Fibers L_t $(t \in T)$ of $p \circ q$ are of negative degrees over the fibers S_t of p.*
(ii) *L_t are topologically trivial over S_t and not analytically equivalent to line bundles.*
(iii) *$L \to S$ is a $U(1)$-flat line bundle.*

For the families of $\mathbb{D}$-bundles, we have shown in advance to Theorem 2.6 its counterpart in the following form.

Theorem 2.7 (cf. [31]) *Let $p : S \to T$ be as in Theorem 2.6 and let $\mathcal{D}$ be an analytic $\mathbb{D}$-bundle over S. Then $\mathcal{D}$ is weakly 1-complete if T is Stein.*

The parallerism between $\mathbb{C}$-bundles and $\mathbb{D}$-bundles stops here because it turned out that one cannot drop the assumptions in Theorem 2.6. At this point, which will be discussed below, the so-called Ohsawa-Takegoshi theorem entered the argument unexpectedly.

3 A Counterexample

Let A be a complex torus of dimension one (i.e. an elliptic curve), say $A = (\mathbb{C} \setminus \{0\}/\mathbb{Z})$, where the action of $\mathbb{Z}$ on $\mathbb{C} \setminus \{0\}$ is given by $z \mapsto e^m z$ for $m \in \mathbb{Z}$. Over the product $A \times \mathbb{C}$ as an analytic family of compact Riemann surfaces over $\mathbb{C}$, we define an affine line bundle $F : A \times \mathbb{C}$ the quotient of the trivial bundle $((\mathbb{C} \setminus \{0\} \times \mathbb{C}) \times \mathbb{C}) \times \mathbb{C} \to (\mathbb{C} \setminus \{0\} \times \mathbb{C})$ by the action of $\mathbb{Z}$ defined by $(z, t, \zeta) \mapsto (e^m z, t, \zeta + mt)$. Suppose that F is locally pseudoconvex with respect to the map $\pi : F \to \mathbb{C}$ induced by the projection to the second factor of $A \times \mathbb{C}$. Then there will exist a neighborhood $V \ni 0$ such that $\pi^{-1}(V)$ is weakly 1-complete. Then, since the canonical bundle of F is obviously trivial, holomorphic functions on $\pi^{-1}(t)$ must be holomorphically extendable by the L^2 extension theorem in [28]. (See also [33].) But this will mean that $\pi^{-1}(0)$ can be blown down to $\mathbb{C}$ in F because the otherbers of π are equivalent to $(\mathbb{C} \setminus \{0\})^2$. This contradicts that the normal bundle of the divisor $\pi^{-1}(0)$ is trivial.

4 Proof of Theorem 2.6

Since the conclusion is obvious when (i) or (iii) is the case, let us assume (ii). Then, as in the proof of Theorem 2.2, one can find a system of fiber coordinates ζ_j of the bundle over S and a system of C^∞ functions h_j which are harmonic on the fibers of $S \to T$ such that $|\zeta_j - h_j|$ is globally defined on L. Note that h_j are nonconstant on the fibers of p by assumption. Then it is easy to see that, for any Stein open set $V \subset T$, there exist a C^∞ positive plurisubharmonic exhaustion function ψ on V and a positive C^∞ function ϕ on $q^{-1}(p^{-1}(V))$ satisfying $\phi \le \psi \circ p \circ q$ such that $\phi + |\zeta_j - h_j|^2$ is strictly plurisubharmonic on each fiber of $p \circ q$. Then it is easy to verify by a direct computation that one can find a convex increasing function λ on $\mathbb{R}$ such that $(1 + \phi + |\zeta_j - h_j|^2) \cdot \lambda \circ \psi \circ p \circ q$ is strictly plurisubharmonic everywhere. □

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Regularity of Kobayashi Metric

Giorgio Patrizio and Andrea Spiro

Abstract We review some recent results on existence and regularity of Monge-Ampère exhaustions on the smoothly bounded strongly pseudoconvex domains, which admit at least one such exhaustion of sufficiently high regularity. A main consequence of our results is the fact that the Kobayashi pseudo-metric κ on an appropriate open subset of each of the above domains is actually a smooth Finsler metric. The class of domains to which our result apply is very large. It includes for instance all smoothly bounded strongly pseudoconvex complete circular domains and all their sufficiently small deformations.

Keywords Monge-Ampère equations · Pluricomplex green functions
Manifolds of circular type · Kobayashi metric · Deformations of complex structures

1 Introduction

In this note, providing the necessary background, we survey some recent results about the existence of regular Monge-Ampère exhaustions which, in turn, imply regularity properties for the Kobayashi metric. More precisely, for domains D admitting a smooth Monge-Ampère exhaustion centered at a point $z_o \in D$ (that is, a strictly plurisubharmonic $\mathcal{C}^0$ exhaustions $\tau : \overline{D} \to [0, 1]$, which is $\mathcal{C}^\infty$ at all points, with only

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Dedicated to Kang-Tae Kim for his sixtieth birthday

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possible exception at the minimum set $\{\tau = 0\} = \{z_o\}$, and such that $u = \log \tau$ has a logarithmic singularity at z_o and satisfies the complex Monge-Ampère equation $(\partial\bar{\partial}u)^n = 0$ at all other points), our results show that there exists an open neighborhood D′CD of t_o such that for each $z \in D'$ there exists an analogous smooth Monge-Ampère exhaustion, centered at such point [23]. One of the main consequence of this result is that the Kobayashi pseudo-metric on each such domain is actually a smooth complex Finsler metric on an open subset D′CD. Since any smoothly bounded strongly pseudoconvex complete circular domain and any of its sufficiently small smooth deformations have at least one Monge-Ampère exhaustion and our proof shows how to determine when D′CD, our result reveals that there exists a new large class of domains, on which the Kobayashi metric has extremely high regularity properties. In fact, the results in [23] are proven for closed strongly pseudoconvex domains with Monge-Ampère exhaustions of class $\mathcal{C}^{r,\alpha}$, $r \geq 4, \alpha > 0$, and imply regularity for the Kobayashi pseudo-metric also under such weaker regularity assumptions.

The structure of the paper is the following. In Sect. 1 we recall a few basic properties of Monge-Ampère exhaustions and Kobayashi metrics. In Sect. 2 we present our results with a short description of their proofs. Finally, in Sect. 3 we present some open questions that might be addressed using the results in Sect. 2.

2 Monge-Ampère Exhaustions and Kobayashi Metrics

2.1 *Monge-Ampère Exhaustions and Monge-Ampère Foliations*

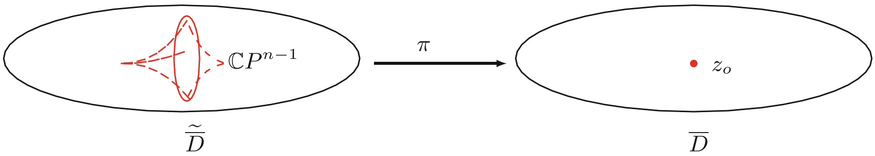
Let $D \subset \mathbb{C}^n$ be a bounded domain with boundary of class $\mathcal{C}^{k,\alpha}$, $k \geq 2$, $\alpha > 0$. A *Monge-Ampère exhaustion* for $\overline{D}$ is a continuous exhaustion $\tau : \overline{D} \to [0, 1]$ satisfying the following conditions:

(i) The boundary ∂D coincides with the level set $\{\tau = 1\}$ while the level set $\{\tau = 0\}$ consists of exactly one interior point z_o, called *center* of the exhaustion.

(ii) The map τ is of class $\mathcal{C}^{k,\alpha}$ on $\overline{D} \setminus \{z_o\}$ and, in general, is only continuous at z_o. However, if $\pi : \widetilde{\overline{D}} \to \overline{D}$ denotes the blow up of $\overline{D}$ at the center z_o, we assume that the $\mathcal{C}^{k,\alpha}$ lifted map

$$\widetilde{\tau} = \tau|_{\overline{D}\setminus\{0\}} \circ \pi : \widetilde{\overline{D}} \setminus \pi^{-1}(\{z_o\}) \to (0, 1]$$

admits a $\mathcal{C}^{k,\alpha}$-extension to the whole $\widetilde{\overline{D}}$.



(iii) On $\overline{D} \setminus \{z_o\}$ the following differential conditions hold:

(a) $2i\partial\bar{\partial}\tau > 0$;
(b) $2i\partial\bar{\partial}\log\tau \geq 0$;
(c) $(\partial\bar{\partial}\log\tau)^n = 0$ (*homogeneous complex Monge-Ampère equation*).

Note that (a)–(c) imply that, at each point z, the kernel of the 2-form $\partial\bar{\partial}\log\tau_z$ is 1-dimensional and that each level set $\{\tau = c\}$ is a strongly pseudoconvex real hypersurface.

(iv) In proximity of the center, $\log\tau$ goes as $\log\tau(z) \simeq \log(\|z - z_o\|) + O(1)$.

The closure of a domain $D \subset \mathbb{C}^n$, for which there is at least one Monge-Ampère exhaustion, is called *(closed) domain of circular type*.

Remark 1 The above definition, given here for domains in $\mathbb{C}^n$, can be easily extended and stated in full generality for complex manifolds with boundary. We refer to [23] for details.

Up to a few minor changes, the above notion of domain of circular type coincides with the one introduced by the first author in [17] to capture the most crucial properties of the following two important classes of domains.

Class A. Let $\overline{D} \subset \mathbb{C}^n$ be the closure of a complete circular domain, i.e. of a domain $D = \{z : \mu(z) < 1\}$ determined by a defining function $\mu : \mathbb{C}^n \longrightarrow [0, +\infty)$ satisfying the condition

$$\mu(\lambda z) = |\lambda|\mu(z) \qquad \text{for all } \lambda \in \mathbb{C}\ .$$

Assume that the function μ, called the *Minkowski function* of D, has the following two properties:

(a) one (and, consequently, all) of the level sets $\{\mu = c\}$ for $0 < c \leq 1$ is a strongly pseudoconvex hypersurface,
(b) μ is of class $\mathcal{C}^{k,\alpha}$, with $k \geq 2$, $\alpha > 0$, on $\mathbb{C}^n \setminus \{0\}$.

One can then directly see that the square $\tau := (\mu|_{\overline{D}})^2$ is a Monge-Ampère exhaustion for $\overline{D}$, centered at $z_o = 0$, so that $\overline{D}$ is of circular type.

Class B. Let $\overline{D} \subset \mathbb{C}^n$ be the closure of a strictly linearly convex domain D with boundary of class $\mathcal{C}^{k,\alpha}$ for some $k \geq 4$, $\alpha > 0$. Let also $\delta : D \times D \to \mathbb{R}$ be the Kobayashi pseudodistance of D and, for each given $z_o \in D$, set

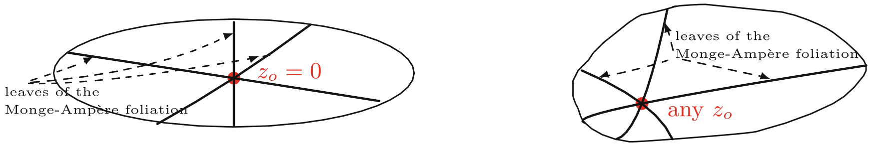
$$\tau^{(z_o)} : \overline{D} \longrightarrow [0, 1]\ , \qquad \tau^{(z_o)}(w) := \tanh^2(\delta(z_o, w)). \tag{2.1}$$

Lempert's theory of Kobayashi distance on convex domains [12] shows that *for each $z_o \in D$ the corresponding real function* (2.1) *is a Monge-Ampère exhaustion for $\overline{D}$*, *centered at* z_o. Thus, any such domain is of circular type. Note that, in contrast with the previous construction, for such domains *any* z_o occurs as the center of a Monge-Ampère exhaustion.

An arbitrary domain $\overline{D}$ of circular type has always the following crucial features. For any $z \in \overline{D} \setminus \{z_o\}$, let $\mathcal{Z}_z$ be the 1-dimensional kernel

$$\mathcal{Z}_z := \ker(2i\partial\bar{\partial}\log\tau)|_z \subset T_z D\ .$$

Being formed by the kernels of a closed 2-form, the distribution $\mathcal{Z}$ is integrable and the (closures of) its integral leaves form a singular foliation of D, called *Monge-Ampère foliation*. The leaves are complex curves, each of them biholomorphic to the unit disk Δ in $\mathbb{C}$, passing through the center z_o of the Monge-Ampère exhaustion [17, 18, 20].



strongly pseudoconvex circular domain   strictly linearly convex domain

The existence of such foliation has important consequences on the Kobayashi pseudo-metrics of domains of circular type, which we now review.

2.2 *Kobayashi Pseudo-Metrics*

Let D be a bounded domain of $\mathbb{C}^n$ and z_o a point of D. We recall that the *Kobayashi (infinitesimal) pseudo-metric* of D at z_o is the real valued function on $T_{z_o}D \simeq \mathbb{C}^n$, defined by

$$\kappa_{z_o} : T_{z_o}D \setminus \{\underline{0}\} \longrightarrow [0, +\infty), \qquad \kappa_{z_o}(\underline{v}) := \inf_{f \in \mathcal{A}_{(z_o, \underline{v})}} \|f_*^{-1}(\underline{v})\|\ ,$$

where $\mathcal{A}_{(z_o,\underline{v})}$ denotes the set of all holomorphic maps from Δ to D, passing through z_o and tangent to $\underline{v}$, i.e.

$$\mathcal{A}_{(z_o,\underline{v})} := \left\{ f : \Delta \to D \ \text{ holom. with } f(0) = z_o\ ,\ f_*\left(\lambda \frac{\partial}{\partial x}\bigg|_0\right) = \underline{v} \text{ for } \lambda \in \mathbb{R} \right\}.$$

The literature on the Kobayashi pseudometric is vast (see e.g. [9–11] and references therein). Here we just recall two simple—but crucial—facts.

(1) It satisfies the so-called *distance decreasing property*, i.e. for any holomorphic map $F : D \longrightarrow D'$ between two domains D, D' and for each $z_o \in D$, the Kobayashi pseudo-metrics κ_{z_o} and $\kappa_{F(z_o)}$ of D and D', respectively, satisfy the inequality

$$\kappa_{F(z_o)}(F_*(\underline{v})) \leq \kappa_{z_o}(\underline{v}) \qquad \text{for each } \ \underline{v} \in T_{z_o}D\ .$$

It follows immediately that if F is a biholomorphism, then the equality holds, meaning that the Kobayashi pseudo-metric is a (very important) biholomorphic invariant.

(2) For each $z_o \in D$ and $\underline{v} \in T_{z_o}D$,

$$\kappa_{z_o}(\lambda \underline{v}) = |\lambda|\, \kappa_{z_o}(\underline{v}) \qquad \text{for each } \lambda \in \mathbb{C}\,.$$

This yields that the *indicatrix* of the Kobayashi pseudo-metric at z_o, that is the set

$$I_{z_o} := \{\underline{v} \in T_{z_o}D \setminus \{0\} \ : \kappa_{z_o}(\underline{v}) < 1\} \cup \{\underline{0}\}\,, \tag{2.2}$$

is always a balanced domain of $T_{z_o}D \simeq \mathbb{C}^n$.

The situations where I_{z_o} is a *strongly pseudoconvex domain* for each point z_o are of particular interest, because in those cases the Kobayashi pseudo-metric $\kappa : TD \to [0, +\infty)$ is a (possibly non-smooth) complex Finsler metric.

2.3 Finsler Metrics

Let us shortly recall the definitions of real and complex Finsler metrics. Given an n-dimensional real manifold M, with tangent bundle TM, a (smooth) *real Finsler metric* is a continuous map

$$F : TM \longrightarrow [0, +\infty)\,,$$

which is $\mathcal{C}^\infty$ on $TM^o := TM \setminus \{\text{zero section}\}$ and such that

(a) for every non-negative *real* number λ and every vector $(x, \underline{v}) \in T_xM$ at some point $x \in M$ one has that

$$F(x; \lambda\underline{v}) = |\lambda| F(x; \underline{v})\,,$$

(b) for each $x_o \in M$, the indicatrix $I_{x_o} := \{(x_o, \underline{v}) : F(x_o; \underline{v}) < 1\}$ is a strictly linearly convex domain of $T_{x_o}M \simeq \mathbb{R}^n$.

In other words, a smooth real Finsler metric is a norm function on all tangent spaces of M, which is smoothly depending on the base points and on the (non-zero) tangent vectors (when such dependence is not $\mathcal{C}^\infty$, it is usually said that F is a "non-smooth" Finsler metric). In fact, a very simple example of Finsler metric on a manifold M is given by the norm function

$$F(x; \underline{v}) := \sqrt{g_x(\underline{v}, \underline{v})}\,,$$

determined by some fixed Riemannian metric g on M. But many other examples, for which there is no associated Riemannian metric, can be easily constructed. Indeed, in order to define a Finsler metric, it suffices to fix a linearly convex indicatrix I_x in each tangent space, smoothly depending on the base point x, and use it to define a norm function. If the assigned indicatrices are not linearly equivalent to quadrics, the corresponding Finsler metric cannot be associated with any Riemannian metric.

The notion of complex Finsler metric is very similar. If M is a complex manifold of complex dimension n, a (smooth) *complex Finsler metric* on M is a continuous real valued map

$$F : TM \longrightarrow [0, +\infty) ,$$

which is $\mathcal{C}^\infty$ on $TM^o := TM \setminus \{\text{zero section}\}$ and satisfies the following analogues of (a) and (b):

(a′) for every *complex* number λ and every vector $(x, \underline{v}) \in T_xM$, $x \in M$,

$$F(x; \lambda\underline{v}) = |\lambda| F(x; \underline{v});$$

(b′) for each $x_o \in M$, the indicatrix $I_{x_o} := \{(x_o, \underline{v}) : F(x_o; \underline{v}) < 1\}$ is a strongly pseudoconvex domain of $T_{x_o}M \simeq \mathbb{C}^n$.

As before, a very simple example of complex Finsler metric on a complex manifold M is given by the norm function

$$F(x; \underline{v}) := \sqrt{h_x(\underline{v}, \underline{v})} ,$$

determined by an Hermitian metric h on M. But many other examples can be easily determined, for which there is no associated Hermitian metric. In perfect analogy with the real case, complex Finsler metrics are uniquely determined by the associated family of indicatrices.

As for the Kobayashi metric, also the literature on Finsler metrics is enormous. For an introduction to this important and interesting area, the reader might take a look at standard texts as [1, 3, 5] and references therein.

2.4 Kobayashi Metrics and Monge-Ampère Foliations

Coming back to the Kobayashi pseudo-metric κ of a domain $D \subset \mathbb{C}^n$, if the indicatrices (2.2) are smooth and strongly pseudoconvex and if they smoothly depend on their base points, then κ is a complex Finsler metric. But here come two of the most unfriendly features of Kobayashi pseudo-metrics.

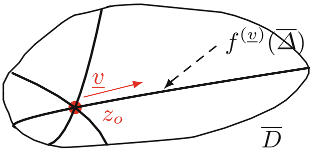
- For a generic domain, the indicatrix (2.2) is usually *not strongly pseudoconvex* and the function $\kappa : TD^o = TD \setminus \{\text{zero section}\} \to [0, +\infty)$ is often no better than $\mathcal{C}^0$.
- Domains, for which κ can be explicitly computed—hence analyzed in greater detail—are not easy to find.

Nonetheless, for domains of circular type things are somehow nicer.

Let $\overline{D} \subset \mathbb{C}^n$ be a strongly pseudoconvex domain of circular type, equipped with a Monge-Ampère exhaustion τ with center z_o. As it is shown in [18], for each non-zero tangent vector $\underline{v} \in T_{z_o}D$ at the center z_o, there exists a *unique* proper holomorphic disk $f^{(\underline{v})}: \overline{\Delta} \to \overline{D}$, whose image $f^{(\underline{v})}(\overline{\Delta})$ coincides with the closure of a leaf of the Monge-Ampère foliation of τ and tangent to $\underline{v}$ at the origin, i.e. satisfying the conditions

$$f(0) = z_o\ , \qquad f_*\left(\lambda \frac{\partial}{\partial x}\bigg|_0\right) = \underline{v} \qquad \text{for some } \lambda \in \mathbb{R}\ . \tag{2.3}$$

Here λ is uniquely determined by $\underline{v}$ and is non-zero. Let us denote it by $\lambda^{(\underline{v})}$.



A crucial relation between the Monge-Ampère foliation of $\overline{D}$ and its Kobayashi pseudo-metric is represented by the following two facts [19]:

(a) for each $\underline{v} \in T_{z_o}D \setminus \{0\}$ one has that $\kappa_{z_o}(\underline{v}) = \lambda^{(\underline{v})}$;
(b) the indicatrix I_{z_o} at the center of κ is strongly pseudoconvex.

This yields to the following couple of nice properties.

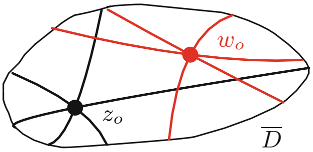
- If $D \subset \mathbb{C}^n$ is a smoothly bounded, strongly pseudoconvex circular domain, then at $z_o = 0$ the indicatrix $I_{z_o=0}$ is smooth and strongly pseudoconvex. In fact, it is linearly equivalent to the domain D.
- If $D \subset \mathbb{C}^n$ is a smoothly bounded strictly linearly convex domain, then at each point $z_o \in D$, the indicatrix I_{z_o} is smooth and strongly pseudoconvex and it smoothly depends on the base point. In other words, κ is a smooth complex Finsler metric.

3 The Phenomenon of Propagation of Regularity

3.1 Domains with a lot of Monge-Ampère Exhaustions

In this section, we study the properties of the domains of circular type admitting a one-parameter family of Monge-Ampère exhaustions, centered at the points of a given curve. We will shortly see that the existence of such Monge-Ampère exhaustions is equivalent to the existence of a special one-parameter family of homeomorphisms of the domain into itself, with nice regularity properties outside the curve and satisfying a special set of differential constraints. On the basis of these facts, all main results presented in this note will be built.

Domains with Two Monge-Ampère Exhaustions Let $\overline{D} \subset \mathbb{C}^n$ be a domain of circular type and assume that it admits at least *two* distinct Monge-Ampère exhaustions, the first centered at z_o and the second at w_o, and consequently with *two* Monge-Ampère foliations, one for z_o, the other for w_o.

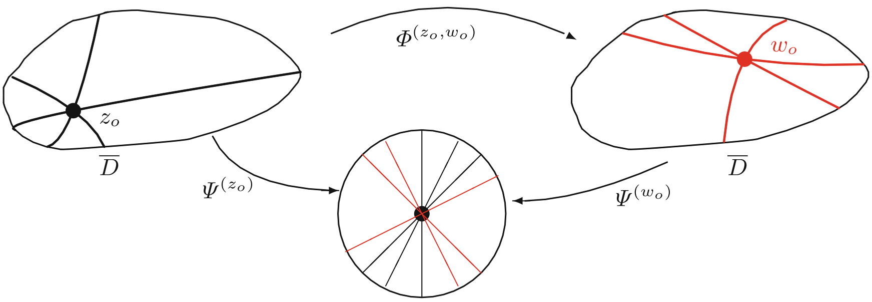


Applying various results by the first author on the so-called *circular representation* [18], it is possible to show that the existence of a Monge-Ampère exhaustion $\tau^{(z_o)}$ of class $\mathcal{C}^{k,\alpha}$, $k \geq 4$, $\alpha > 0$, off the center z_o, implies also the existence of a "straightening" homeomorphism $\Psi^{(z_o)} : \overline{D} \longrightarrow \overline{\mathbb{B}}^n$ onto the closed unit ball of $\mathbb{C}^n$ centered at 0, with the following nice properties [20, 23]:

(a) $\Psi^{(z_o)}(z_o) = 0$ and each level set $\{\tau^{(z_o)}(z) = c\}, 0 < c \leq 1$, is mapped diffeomorphically onto the sphere $\{|z| = c\}$ of radius c. The restriction of $\Psi^{(z)}$ on each such level set is *in general not a CR map*, but nonetheless maps the real contact distributions underlying the two CR structures one into the other.

(b) The disks of the Monge-Ampère foliation through z_o are mapped bijectively onto the straight disks of $\mathbb{B}^n$ through 0. Each restriction of $\Psi^{(z_o)}$ along one such a disk is a biholomorphism.

(c) $\Psi^{(z_o)}$ is of class $\mathcal{C}^{k-2,\alpha}$ on $\overline{D} \setminus \{z_o\}$. Further, if we denote by $\widetilde{\overline{D}}^{(z_o)}$ and $\widetilde{\overline{\mathbb{B}}}^n$ the blow-ups at z_o and 0 of the domains, then the restriction $\Psi^{(z_o)}|_{\overline{D}\setminus\{z_o\}}$ admits a unique $\mathcal{C}^{k-2,\alpha}$-extension to a map between $\widetilde{\overline{D}}^{(z_o)}$ and $\widetilde{\overline{\mathbb{B}}}^n$.



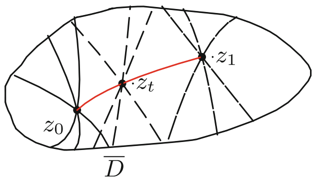
Due to this, if we have *two* Monge-Ampère exhaustions, we may compose the *two* associated "straightening" homeomorphisms and get a homeomorphism from $\overline{D}$ into itself $\Phi^{(z_o,w_o)} := (\Psi^{(w_o)})^{-1} \circ \Psi^{(z_o)} : \overline{D} \longrightarrow \overline{D}$ such that

(α) $\Phi^{(z_o,w_o)}$ transforms the center z_o into the center w_o and maps each level set of $\tau^{(z_o)}$ into the corresponding one of $\tau^{(w_o)}$. The restriction of $\Phi^{(z_o,w_o)}$ on each level set maps one to the other the contact structures underlying the CR structures of those hypersurfaces.

(β) The disks of the Monge-Ampère foliation through z_o are mapped bijectively onto the disks of the Monge-Ampère foliation through w_o and on each of them the restriction of $\Phi^{(z_o,w_o)}$ is a biholomorphism.

(γ) the map $\Phi^{(z_o,w_o)}$ is of class $\mathcal{C}^{k-2,\alpha}$ on $\overline{D}\setminus\{z_o\}$ and, considering the blow-ups $\overset{\simeq}{\overline{D}}^{(z_o)}$ and $\overset{\simeq}{\overline{D}}^{(w_o)}$ at z_o and w_o, the restriction $\Phi^{(z_o,w_o)}|_{\overline{D}\setminus\{z_o\}}$ admits a unique $\mathcal{C}^{k-2,\alpha}$-extension that goes from $\overset{\simeq}{\overline{D}}^{(z_o)}$ to $\overset{\simeq}{\overline{D}}^{(w_o)}$.

Domains with a One-Parameter Family of Monge-Ampère Exhaustions Let us now address the case of a closed domain of circular type $\overline{D}$ admitting a *one-parameter family of Monge-Ampère exhaustions*, centered at the points z_t, $t \in [0, 1]$, of a smooth curve of D.



The remarks of previous section imply that in this case $\overline{D}$ is equipped with a 1-parameter family of homeomorphisms $\Phi^{(t)} : \overline{D} \longrightarrow \overline{D}$, $t \in [0, 1]$, such that:

(1) $\Phi^{(t)}$ is $\mathcal{C}^{k-2,\alpha}$ on $\overline{D}\setminus\{z_0\}$, with $\Phi^{(t)}|_{\overline{D}\setminus\{z_0\}}$ with a $\mathcal{C}^{k-2,\alpha}$-extension to a map from the blow up $\overset{\simeq}{\overline{D}}^{(z_0)}$ at z_0 onto the blow up $\overset{\simeq}{\overline{D}}^{(z_t)}$ at z_t;

(2) $\Phi^{(t)}$ maps z_0 into z_t, and sends the level sets of $\tau^{(z_o)}$ onto the corresponding level sets of $\tau^{(w_o)}$ by contact transformations. In particular, it induces a $\mathcal{C}^{k-2,\alpha}$ contact map from ∂D into itself and $\tau^{(t)} = \tau^{(0)} \circ (\Phi^{(t)})^{-1}$.

(3) The disks of the Monge-Ampère foliation through z_0 are biholomorphically mapped by $\Phi^{(t)}$ onto the disks of the Monge-Ampère foliation through z_t.

Since each of the associated lifts between blow-ups is $\mathcal{C}^{k-2,\alpha}$ with $k-2 \geq 2$, we may consider the pull-backed tensor fields on $\overset{\simeq}{\overline{D}}^{(z_0)}$ defined by

$$J_t := (\Phi^{(t)-1})_*(J_o)\ , \qquad t \in [0, 1]\ , \tag{3.1}$$

where J_o stands for the standard complex structure of the blow up $\overset{\simeq}{\overline{D}}^{(z_t)}$ at z_t. The J_t are tensor fields of type $(1, 1)$, they verify the condition $J_t^2|_z = -I$ and their Nijenhueis tensors are identically vanishing, being each J_t a pull-back of the *integrable* complex structure J_o. Further, each of them is of class $\mathcal{C}^{k-1,\alpha}$ with $k-1 \geq 1$ and $\alpha > 0$. Hence, by Newlander-Nirenberg Theorem [8, 13–15, 25], each J_t is a *non-standard* integrable complex structure. We call them *non-standard* simply because in general the $\Phi^{(t)}$ *are not biholomorphisms* and, consequently, the pull-backs of the standard complex J_o by such maps are different from J_o.

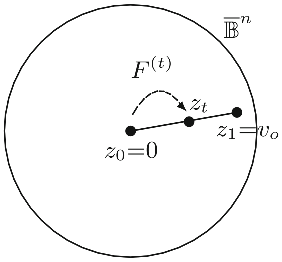
If the curve of centers z_t is at least $\mathcal{C}^1$ and if we select the diffeomorphisms $\Phi^{(t)}$ in such a way that the family $\Phi^{(t)}$ is differentiable with respect to the parameter t, we may also consider the one-parameter family of vector field on $\overline{D}\setminus\{z_0\}$, defined by

$$X_t|_z := (\Phi^{(t)-1})_* \left(\left. \frac{d\Phi^{(t)}}{dt} \right|_{(t,z)} \right) \qquad \text{for each } z \in \overline{D} \setminus \{z_0\} . \tag{3.2}$$

For getting a physical intuition of such vector fields, consider the one-parameter family of maps $\Phi^{(t)}$ as a fluid motion and the coordinates of the points of $\overline{D}$ as Lagrange coordinates for the fluid. In this way X_t can be interpreted as the *velocity field of the flow at time t in Lagrangian coordinates*.

We will shortly see that all crucial information about the $\Phi^{(t)}$ is encoded in the one-parameter family of pairs (X_t, J_t), $t \in [0, 1]$, which we call the *fundamental pair* for the one-parameter family $\Phi^{(t)}$.

A Simple Model Example Assume that $\overline{D} = \overline{\mathbb{B}}^n$, let $v_o \neq 0$ in $\mathbb{B}^n$ and denote by $z_t = t\cdot v_o$, $t \in [0, 1]$, the points of the segment between 0 and v_o. Since $\mathbb{B}^n$ is homogeneous, we may consider a smooth family of automorphisms $F^{(t)} \in \mathrm{Aut}\,(\overline{\mathbb{B}^n}, J_o)$, mapping the origin into the points z_t.



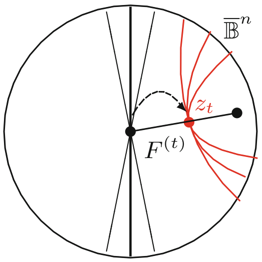
$\overline{\mathbb{B}}^n$ is clearly a circular domain with Minkowski function $\| \cdot \|$ and

$$\tau^{(0)} : \overline{\mathbb{B}}^n \longrightarrow [0, 1] \,, \qquad \tau^{(0)}(w) := \|w\|^2$$

is a Monge-Ampère exhaustion for $\mathbb{B}^n$ centered at 0. The corresponding Monge-Ampère foliation is made by the straight radial disks $f^{(\underline{v})}(\overline{\Delta})$, images of the maps $f^{(\underline{v})}(\zeta){:=}\zeta\cdot\underline{v}$ with $\|\underline{v}\| = 1$. Using the fact that each $F^{(t)}$ is a biholomorphism from $\mathbb{B}^n$ into itself, one can directly check that the exhaustions

$$\tau^{(t)} = \tau^{(0)} \circ F^{(t)-1}$$

are all Monge-Ampère exhaustions, each of them centered at a different z_t. The corresponding Monge-Ampère foliations are made of the images under the maps $F^{(t)}$ of the straight radial disks through the origin.



We are in the situation considered in the previous section: There is a one-parameter family of Monge-Ampère exhaustions $\tau^{(t)}$, centered at the points of a curve z_t. In this special case, we may take as maps $\Phi^{(t)} : \overline{\mathbb{B}}^n \to \overline{\mathbb{B}}^n$ the biholomorphisms $\Phi^{(t)} = F^{(t)}$ and obtain as fundamental pair

$$J_t = (F^{(t)-1})_*(J_o) = J_o \ , \qquad X_t = (F^{(t)-1})_* \left(\frac{dF^{(t)}}{dt} \right) .$$

This situation is however somehow peculiar, because, in contrast with the generic case, here each map $\Phi^{(t)} = F^{(t)}$ is a biholomorphism.

3.2 The Propagation of Regularity Theorem

We are now ready to state the main result and provide a short description of the tools which are used to obtain it. Basically, all is built upon the next two lemmas.

The Main Lemmas If $\Phi^{(t)}:\overline{D}\to\overline{D}$ is one of the families of homeomorphisms considered in Sect. 3.1, the associated fundamental pair (X_t, J_t), $t \in [0, 1]$, satisfies an important set of differential constraints, which we are now going to list. In order to prevent diversions of reader's attention from the most crucial aspects, we give here just intuitive descriptions of such constraints. The interested reader is referred to [23] for the detailed expressions.

The constraints that any fundamental pair satisfies are:

(A) The restrictions of each J_t to the tangent spaces of the leaves of the Monge-Ampère foliations of $\tau^{(0)}$ coincide with the restrictions to the same spaces of the standard complex structure J_o. This is due to the fact that the restriction of $\Phi^{(t)}$ on each such a leaf is a biholomorphism.
(B) The one-parameter families X_t and J_t satisfy $\frac{dJ_t}{dt} = \mathcal{L}_{X_t} J_t$ with $J_{t=0} = J_o$.
(C) Each J_t has identically vanishing Nijenhueis tensor, as pointed in Sect. 3.1.
(D) Each vector field X_t satisfies special boundary conditions at ∂D and on its limit behavior near z_0. They are determined by the fact that each $\Phi^{(t)}$ maps diffeomorphically ∂D into itself and, at the same time, it admits a $\mathcal{C}^{k-2,\alpha}$ extension between the blow ups at z_0 and z_t.

The explicit expression for the limit behavior of X_t at z_0 mentioned in (D) depends in a non trivial way *on the tangent vector of the curve* z_t *at the time* t. Such dependence is quite technical and we refer to [23] for explicit details.

All this motivates the next.

Definition 1 Let z_t, $t \in [0, 1]$, be a $\mathcal{C}^1$ curve z_t in D starting from the center z_0. We call *abstract fundamental pair guided by* z_t any one-parameter family of pairs (X_t, J_t), formed by vector fields on $\overline{D} \setminus \{z_0\}$ and almost complex structures J_t on $\widetilde{\overline{D}}^{(z_0)}$ satisfying the constraints (A)–(D).

If there exists a one-parameter family $\tau^{(t)}$ of Monge-Ampère exhaustions, centered at the points z_t and with associated maps $\Phi^{(t)}$ having (X_t, J_t) as fundamental pair, we call the pair a *concrete fundamental pair*.

We may now state our two main lemmas [23].

Lemma 1 *Let* $\tau = \tau^{(0)}$ *be a Monge-Ampère exhaustion on* $\overline{D}$*, centered at* z_0 *and of class* $\mathcal{C}^{k,\alpha}$*,* $k \geq 4$*,* $\alpha > 0$*, on* $\overline{D} \setminus \{z_0\}$*. Any* abstract *fundamental pair* (X_t, J_t) *of class* $\mathcal{C}^{k-2,\alpha}$ *is* concrete *and is associated with the maps* $\Phi^{(t)}$ *of a one-parameter family of Monge-Ampère exhaustions* $\tau^{(t)}$ *of class* $\mathcal{C}^{k-2,\alpha}$ *off the centers.*

Lemma 2 *Let* $\tau = \tau^{(0)}$ *be a Monge-Ampère exhaustion on* $\overline{D}$ *as in the previous lemma and denote by* z_t*,* $t \in [0, 1]$*, the points of a radius of a holomorphic disk* $f(\Delta)$ *of the Monge-Ampère foliation through* z_0 *(i.e.* z_t *has the form* $z_t = f(tv)$ *for some fixed* $\|v\| = 1$*). Then there exists a value* $\lambda_0 \in (0, 1]$ *such that for all* $\lambda \in (0, \lambda_0)$ *there is an* abstract *(hence* concrete*) fundamental pair* (X_t, J_t) *of class* $\mathcal{C}^{k-2,\alpha}$ *guided by the curve* $z_{\lambda t}$*.*

The proof of Lemma 1 essentially consists of two parts. One first shows that, for a given abstract pair (X_t, J_t), one can solve the differential problem in $\Phi^{(t)}$, given by the Eq. (3.2) and the initial conditions $\Phi^{(t=0)} = \mathrm{Id}$. For proving this, the key idea is to observe that the *non-linear* Eq. (3.2) is equivalent to a *quasi-linear* equation on the inverse maps $\Psi^{(t)} = (\Phi^{(t)})^{-1}$, for which the existence of solutions can be proved with little effort. Secondly, one tries to show that the compositions $\tau^{(t)} := \tau^{(0)} \circ (\Phi^{(t)})^{-1}$ satisfy all conditions for being Monge-Ampère exhaustions. The only non immediate points of this check are reduced to prove that the maps $\Phi^{(t)}$ have uniformly bounded Jacobians at the points where they are differentiable. This is first proved for the points in ∂D and then shown for all other points, using an argument based on the Maximum Principle for harmonic functions.

The starting point for Lemma 2 is given by a preliminary result, which shows that the constraints (A) and (D) are satisfied if and only if the vector fields X_t of an abstract fundamental pair must have a very special form, with very few degrees of freedom. With this the proof boils down to showing the existence of abstract pairs (X_t, J_t), in which the X_t have the above mentioned special form and J_t has to satisfy the remaining constraints (B) and (C). Since (C) is actually a consequence of (B), everything reduces to proving the existence of a solution to the differential problem (B) with X_t in special form. Note that the explicit expression for a vector field X_t

in special form involves the complex structure J_t and the curve z_t. This makes the differential equation in (B) *non-linear* in the tensor field J_t. The proof is obtained by first proving the existence of solutions in the real-analytic category and then getting the result by an approximation argument.

An immediate consequence of the above two lemmas is the fact that, for each sufficiently small straight segment z_t in a disk of a Monge-Ampère foliation of sufficiently high regularity, there is a one-parameter family of Monge-Ampère exhaustions $\tau^{(t)} : \overline{D} \to [0, 1]$, centered at the points z_t. Thus, the next theorem follows.

Theorem 1 (Propagation of regularity) *Let $\overline{D} \subset \mathbb{C}^n$ be a closed strongly pseudoconvex domain of circular type, with a Monge-Ampère exhaustion $\tau : \overline{D} \longrightarrow [0, 1]$, with center z_o and of class $\mathcal{C}^{k,\alpha}$ on $\overline{D} \setminus \{z_o\}$ for some $k \geq 4$ and $\alpha > 0$. Then there is an open neighborhood $D' CD$ of z_o such that any other point z of D' is center of a Monge-Ampère exhaustion $\tau^{(z)}:\overline{D} \to [0, 1]$ of class $\mathcal{C}^{k-2,\alpha}$ on $\overline{D} \setminus \{z\}$. The dependence of $\tau^{(z)}$ on z is $\mathcal{C}^{k-2,\alpha}$.*

Combining this with the previously described properties of Kobayashi metric of domains of domains of circular type, we immediately get the following;

Corollary 1 *If $\overline{D}$ admits a Monge-Ampère exhaustion, which is $\mathcal{C}^\infty$ off the center, its Kobayashi pseudo-metric $\kappa : TD \to [0, +\infty)$ is a smooth Finsler metric on an appropriate open subset $D' CD$.*

Note that the proof provides also an explicit description of the points of a D′, thus a method to determine when D′ = D.

From this we see that the class of domains in $\mathbb{C}^n$, for which the Kobayashi pseudometric is actually a Finsler metric not only contains all smoothly bounded, strictly linearly convex domains, as Lempert proved, but it is much larger than that. It includes for instance all smoothly bounded, strongly pseudoconvex circular domains satisfying appropriate conditions and, further, all sufficiently small deformations of such domains [2, 21]. This is indeed a consequence of the property that any smooth deformation $\overline{D}''$ of a given circular domain $\overline{D}$ is biholomorphic to an (abstract) manifold with boundary, given by equipping $\overline{D}$ with an appropriate deformed complex structure $J \neq J_o$. If the deformation is sufficiently small, stability properties of the equations for stationary disks imply that $(\overline{D}, J)$ (hence, also $\overline{D}''$) admits a special foliation, made of J-stationary disks passing through $x_o = 0$ (see e.g. [21], Prop. 3.4). Then, the regularity of the data and the stability property of the conditions which give D′ = D imply that these disks form the Monge-Ampère foliation of an exhaustion $\tau : \overline{D}'' \to [0, 1]$, which makes $\overline{D}''$ a domain of circular type to which Corollary 1 applies.

We conclude stressing that all proofs in [23] are actually given in the category of complex manifolds with boundary and are valid also for abstract strongly pseudoconvex manifolds, regardless of their embeddability in $\mathbb{C}^n$.

4 Concluding Remarks

The above described results can be taken as starting points for various lines of further investigation. Some of them can be shortly described as follows.

(1) By Theorem 1, the existence of a single Monge-Ampère exhaustion τ_o of class $\mathcal{C}^{k,\alpha}$, $k \geq 4, \alpha > 2$, off the center implies the existence of an infinity of other Monge-Ampère exhaustions of lower regularity, smaller by two orders. Such loss of regularity is due to a technical tool used in the proof, namely the use of the so-called *normalization maps*. Other than this, we do not see any intuitive reason for such loss of regularity and we expect that the main results can be refined on this aspect. Any such improvement would be quite valuable.

Note also that if τ_o is a Monge-Ampere exhaustion, then the logarithm $u_o = \log \tau_o$ is a pluricomplex Green function, with pole at the center of τ_o. Hence, improvements of our results in the described direction might give useful information on pluricomplex Green functions and enrich the theory of such functions that has been so far developed (for the known regularity properties of pluricomplex Green functions, see e.g. [4, 6, 7]).

(2) Corollary 1 implies that satisfying appropriate conditions if $\overline{D}$ is a closed domain of circular type with a smooth Monge-Ampère exhaustion, then $\overline{D}$ is completely determined by the Finsler invariants of its Kobayashi metric, namely by the Finsler curvature and all Finsler covariant derivatives up to an appropriate order [24]. On the other hand, the same domain is equipped also with another important sequence of invariants, the *Bland and Duchamp invariants*, which are tensor fields on the blow up of $\overline{D}$ at the center of a Monge-Ampère exhaustion, which describe how the CR structures of the level sets $\{\tau = c\}$ evolve when c varies between 0 and 1 [2, 20]. Also these invariants completely determine $\overline{D}$ up to biholomorphisms.

We feel that it is possible to determine explicit relations between the Finsler and the Bland and Duchamp invariants. Finding such relations would very likely lead to a deep insight on the intrinsic properties of strictly linearly convex domains and, more generally, of all domains of circular type.

(3) From Lempert theory, we know that on any closed, smoothly bounded, strictly linearly convex domain $\overline{D}$, the disks of the Monge-Ampère foliations are *complex geodesics* for the Kobayashi metric of the domain. For other types of closed domains of circular type with smooth Monge-Ampère exhaustions, the disks of the Monge-Ampère foliations are surely *extremal disks* for the Kobayashi metric at the centers but there is no manifest reason for them to be complex geodesics. Since the property of being a complex geodesic can be nicely described in terms of Finsler covariant derivatives (see e.g. [1]), writing down the explicit relations between the Bland and Duchamp invariants and the Finsler invariants might be helpful to characterize the convexifiable domains (i.e. those that are biholomorphic to some strictly linearly convex domain) in terms of their Bland and Duchamp invariants. Combining this with the so far known techniques for constructing domains with prescribed Bland and Duchamp invariants [2, 22], all this would pave the way towards a useful characterization of convexifiable domains of $\mathbb{C}^n$.

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A Note on Poincaré's Polyhedron Theorem in Complex Hyperbolic Space

Li-Jie Sun

Abstract Poincaré's polyhedron theorem gives geometrical conditions on a domain constructed with spherical sides so that the group generated by some elements which permute those sides is discrete. The polyhedron we construct in complex hyperbolic space is bounded by bisectors. We will see a particular form originally proposed by Mostow, and will prove it in the same fashion with real hyperbolic case. Then we will apply it to investigation of the discreteness of complex hyperbolic ultra-ideal triangle groups.

Keywords Poincaré's polyhedron theorem

1 Introduction

It has been well known about real hyperbolic geometry. Many results from real hyperbolic geometry carry over to the complex hyperbolic case, such as Jørgensen's inequality and Shimizu's lemma (see [2, 8, 9, 12]). The unit ball in $\mathbb{C}^2$ has a natural metric of constant negative holomorphic sectional curvature (which we normalized to be -1), called the Bergman metric. As such it forms a model of $\mathbf{H}^2_{\mathbb{C}}$ analogous to the unit disk model of real hyperbolic plane $\mathbf{H}^2_{\mathbb{R}}$. However the real sectional curvature is no longer constant, but is pinched between -1 and $-1/4$. Besides that, in $\mathbf{H}^2_{\mathbb{C}}$ there does not exist totally geodesic real hypersurface, unlike in real hyperbolic space. This increases the difficulty of constructing fundamental polyhedra for discrete subgroups of complex hyperbolic isometries.

Poincaré first proved it for Fuchsian groups, that is for SL(2, $\mathbb{R}$), and later he proved it for Kleinian groups, that is for SL(2, $\mathbb{C}$). Many refinements and discussions of that theorem are now available, see [4, 10]. All versions assume that the group acts on a space of constant sectional curvature. Falbel and Zocca state a Poincaré's polyhedron theorem for complex hyperbolic geometry (see [5]). The fundamental

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domain has faces that are $\mathbb{C}$-spheres foliated by $\mathbb{C}$-circle along an $\mathbb{R}$-circle. Schwartz constructed the fundamental domain, whose sides are $\mathbb{R}$-spheres foliated by arcs of $\mathbb{R}$-circle around a $\mathbb{C}$-circle, see [14]. Both $\mathbb{R}$-spheres and $\mathbb{C}$-spheres are generalizations of bisectors. Deraux, Parker and Paupert in [3] provide efficient geometric and computational tools for constructing fundamental domains for discrete groups acting on the complex hyperbolic plane. In [13], the authors showed that the sufficient condition of the complex hyperbolic triangle groups of type $(3, 3, n)$ to be discrete. The method they used to obtain the conclusion is fundamental polyhedra for coset decompositions(see [3]). In this paper, we mainly prove a special complex hyperbolic version of Poincaré's polyhedron theorem and then will apply it to consider the discreteness of complex hyperbolic ultra-ideal triangle groups.

The article is organized as follows. In Sect. 1, we recall some fundamentals on complex hyperbolic space. In Sect. 2, we mainly prove the Poincaré's polyhedron theorem due to G. D. Mostow. In Sect. 3, we will introduce complex hyperbolic ultra-ideal triangle groups and talk about the discreteness by using Poincaré's polyhedron theorem. Finally, in Sect. 4, we simply give the statement of fundamental polyhedron for coset decompositions.

2 Complex Hyperbolic Space $\mathbf{H}^2_{\mathbb{C}}$

We use [6, 11] as the general references about the complex hyperbolic space. Let $\mathbb{C}^{2,1}$ denote the vector space $\mathbb{C}^3$ equipped with the Hermitian form

$$\langle z, w\rangle = z_1\overline{w}_1 + z_2\overline{w}_2 - z_3\overline{w}_3$$

of signature (2, 1). We denote by $\mathbb{CP}^2$ the complex projectivisation of $\mathbb{C}^{2,1}$ and by $\mathbb{P} : \mathbb{C}^{2,1} \setminus \{0\} \to \mathbb{CP}^2$ the natural projectivisation map. We call a vector $z \in \mathbb{C}^{2,1}$ *negative, null,* or *positive*, if $\langle z, z\rangle$ is negative, zero, or positive respectively. On the chart of $\mathbb{C}^{2,1}$ with $z_3 \neq 0$ the projectivisation map $\mathbb{P}$ is given by

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{pmatrix} z_1/z_3 \\ z_2/z_3 \end{pmatrix} \in \mathbb{C}^2.$$

The *complex hyperbolic 2-space* $\mathbf{H}^2_{\mathbb{C}}$ is defined as the complex projectivisation of the set of negative vectors in $\mathbb{C}^{2,1}$. It is called the standard projective model of the complex hyperbolic space. Its boundary $\partial\mathbf{H}^2_{\mathbb{C}}$ is defined as the complex projectivisation of the set of null vectors in $\mathbb{C}^{2,1}$. By taking $z_3 = 1$, one can also consider the *unit ball model* $\{(z_1, z_2) \in \mathbf{H}^2_{\mathbb{C}} : |z_1|^2 + |z_2|^2 < 1\}$, whose boundary is the unit sphere $\mathbb{S}^3$. We define the *standard lift* of $z = (z_1, z_2) \in \mathbb{C}^2$ to be point $\tilde{z} = [z_1, z_2, 1] \in \mathbb{C}^{2,1}$. This definition could be extended to include the point ∞. We define the standard lift of ∞ to be

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{C}^{2,1}.$$

Let $x, y \in \mathbf{H}^2_{\mathbb{C}}$ be points corresponding to vectors $\tilde{x}, \tilde{y} \in \mathbb{C}^{2,1} \setminus \{0\}$. Then the *Bergman metric* ρ on $\mathbf{H}^2_{\mathbb{C}}$ is given by

$$\cosh^2\left(\frac{\rho(x,y)}{2}\right) = \frac{\langle \tilde{x}, \tilde{y}\rangle \langle \tilde{y}, \tilde{x}\rangle}{\langle \tilde{x}, \tilde{x}\rangle \langle \tilde{y}, \tilde{y}\rangle}.$$

Since the Bergman metric is given in terms of the Hermitian form $\langle \cdot, \cdot \rangle$, it is easy to obtain that if A is unitary with respect to $\langle \cdot, \cdot \rangle$, then A acts isometrically on complex hyperbolic space. The automorphism group $\mathrm{PU}(2, 1)$ of $\mathbf{H}^2_{\mathbb{C}}$ is the projectivisation of the group $\mathbf{U}(2, 1)$ of complex linear transformations on $\mathbb{C}^{2,1}$, which preserve the Hermitian form. Actually every isometry of $\mathbf{H}^2_{\mathbb{C}}$ is either holomorphic or anti-holomorphic. Moreover, each holomorphic isometry of $\mathbf{H}^2_{\mathbb{C}}$ is given by a matrix in $\mathrm{PU}(2, 1)$ and each anti-holomorphic isometry is given by complex conjugation followed by a matrix in $\mathrm{PU}(2, 1)$.

A *complex line* in $\mathbb{CP}^2$ is a complex projectivisation of a two dimensional complex subspace of $\mathbb{C}^{2,1}$. Given two points x and y in $\mathbf{H}^2_{\mathbb{C}} \cup \partial \mathbf{H}^2_{\mathbb{C}}$, with lifts $\tilde{x}$ and $\tilde{y}$ to $\mathbb{C}^{2,1}$ respectively, the complex span of $\tilde{x}$ and $\tilde{y}$ projects to the complex line passing through x and y. We call the intersection of a complex line with $\mathbf{H}^2_{\mathbb{C}}$ to be a *complex geodesic* C, which can be uniquely determined by a positive vector $p \in \mathbb{C}^{2,1}$, i.e. $C = \mathbb{P}(\{z \in \mathbb{C}^{2,1} | \langle z, p\rangle = 0\})$. We call p a *polar vector* to C.

The involution (complex reflection of order 2) in C is represented by an element $I_C(z)$ that is given by

$$I_C(z) = -z + 2\frac{\langle z, p\rangle}{\langle p, p\rangle} p, \qquad z \in \mathbb{C}^{2,1},$$

where p is a polar vector of C. There is a one-to-one correspondence between complex geodesics and chains. Therefore we also say that $I_C(z)$ is an involution on ∂C. We normalize the polar vector $p = [w_1, w_2, w_3]^t$ of a complex geodesic C such that $|\langle p, p\rangle| = 2$, then the involution I_C has a matrix representation in $\mathrm{PU}(2, 1)$ as follows

$$\begin{bmatrix} -1 + |w_1|^2 & w_1\overline{w_2} & -w_1\overline{w_3} \\ \overline{w_1}w_2 & -1 + |w_2|^2 & -w_2\overline{w_3} \\ \overline{w_1}w_3 & \overline{w_2}w_3 & -1 - |w_3|^2 \end{bmatrix}.$$

The boundary $\partial \mathbf{H}^2_{\mathbb{C}}$ is homeomorphic to S^3, here the representations we would like to choose is $\mathbb{C} \times \mathbb{R} \cup \{\infty\}$, with points either ∞ or (z, r) ($z \in \mathbb{C}$ and $r \in \mathbb{R}$). Denote $\mathbb{C} \times \mathbb{R} \cup \{\infty\}$ by $\mathfrak{H}$. The homeomorphism which takes S^3 to $\mathfrak{H}$ is given by the following projection

$$(z_1, z_2) \mapsto \left(\frac{z_1}{1+z_2}, -\mathrm{Im}\left(\frac{1-z_2}{1+z_2}\right)\right),$$
$$(0, -1) \mapsto \infty .$$

There are many stereographic projections base from the points in the boundary. It suffices to choose one example based at $(0, -1)$.

Recall that a chain is the intersection of a complex geodesic with $\partial\mathbf{H}^2_{\mathbb{C}}$. For $z \in \mathbb{C}$, the *z-chain* is the chain having polar vector $[1, -\bar{z}, \bar{z}]^t$. The z-chain is the vertical chain in $\mathfrak{H}$ through the point $(z, 0)$. For $s \in \mathbb{R}$, $r \in \mathbb{R} - \{0\}$ the (s, r)*-chain* is the chain having polar vector $[0, 1 + r^2 + is, 1 - r^2 - is]^t$. The (s, r)-chain is the circle with radius r centered at the origin in $\mathbb{C} \times \{s\} \subset \mathfrak{H}$. It is easy to show that the only chains through ∞ are vertical. Other chains are various ellipses (possibly circles) which project to circles via the projection $\mathbb{C} \times \mathbb{R} \to \mathbb{C}$. Specifically, the unit circle in $\mathbb{C} \times \{0\}$ and vertical lines (with the point at infinity) are all chains. For more details about chains, the reader may refer to [6, §4.3].

Let $z_1, z_2 \in \mathbf{H}^2_{\mathbb{C}}$ be two distinct points. We define the *equidistant half-space* $\mathcal{H}(z_1, z_2)$ to be

$$\begin{aligned}\mathcal{H}(z_1, z_2) &= \{z \in \mathbf{H}^2_{\mathbb{C}} : \rho(z, z_1) < \rho(z, z_2))\}\\ &= \{z \in \mathbf{H}^2_{\mathbb{C}} : |\langle \tilde{z}, \tilde{z_1}\rangle| < |\langle \tilde{z}, \tilde{z_2}\rangle|\}.\end{aligned}$$

The boundary of $\mathcal{H}(z_1, z_2)$ is the *equidistant hypersurface or bisector* $V(z_1, z_2)$:

$$V(z_1, z_2) = \{z \in \mathbf{H}^2_{\mathbb{C}} : \rho(z, z_1) = \rho(z, z_2)\}.$$

3 Complex Hyperbolic Versions of Poincaré's Polyhedron Theorem

Consider Poincaré's polyhedron theorem in $\mathbf{H}^2_{\mathbb{C}}$ and the holomorphic isometry group $\mathrm{PU}(2, 1)$ of $\mathbf{H}^2_{\mathbb{C}}$. A subset $D \subset \mathbf{H}^2_{\mathbb{C}}$ is defined to be a polyhedron if D is the intersection of finitely many equidistant half-spaces $\mathcal{H}$ in $\mathbf{H}^2_{\mathbb{C}}$. The boundary of equidistant half-space in $\mathbf{H}^2_{\mathbb{C}}$ is bisector denoted by B. The set $a = B \cap \partial D$ is called side of D. It is codimsional one face.

In what follows Gusevskii and Parker give conditions on D such that the group G, generated by the identifications of the sides of D is discrete and D is a fundamental polyhedron for G. We prove it in the same fashion with Theorem 9.8.4 in [1] and IV. F of [10].

Theorem 3.1 ([7, Theorem 6.2]) *Assume that the finite sided polyhedron* $D \subset \boldsymbol{H}^2_{\mathbb{C}}$ *satisfies the following conditions:*

(1) For each side a, there exist another side a' *and* $g_a \in \mathrm{PU}(2, 1)$ *such that* $g_a(a) = a'$, $g_{a'} = g_a^{-1}$ *(the isometries* g_a *are called the side pairing transformations),*

(2) $g_a(D) \cap D = \emptyset$ *for any side pairing transformation,*
(3) *For any two sides a and b of D, either they are tangent at some point lying on the boundary of* $\boldsymbol{H}^2_{\mathbb{C}}$*, or their closures are disjoint,*
(4) *The induced metric on* $\overline{D}$ *modulo the finite set consisting of the side pairing transformations is complete.*

Then the group G generated by the side pairing transformations is discrete, D is a fundamental polyhedron for G.

Proof We endow G, $\overline{D}$ with the discrete and natural topology respectively, and define the relation $\sim$ on $G \times \overline{D}$ by

$$(g, x) \sim (h, y)$$

if and only if either:

(i) $g = h, x = y$; or
(ii) $x \in a, y = g_a(x), g = hg_a$.

We endow $G \times \overline{D}$ with the product topology. And the relation $\sim$ extends to an equivalence relation $*$ on $G \times \overline{D}$ by defining

$$(g, x) * (h, y)$$

if and only if there exist some (g_j, x_j) such that

$$(g, x) = (g_1, x_1) \sim (g_2, x_2) \sim \cdots \sim (g_n, x_n) = (h, y).$$

The equivalence class containing (g, x) is denoted by $\langle g, x \rangle$ and let X^* be $G \times \overline{D}$, factored by the equivalence relation $*$. We endow X^* with the usual identification topology such that the natural projection from $G \times \overline{D}$ to X^* is continuous.

Each f in G induces a map $f^* : X^* \to X^*$ by the rule

$$f^* : \langle g, x \rangle \mapsto \langle fg, x \rangle$$

and this is obviously well defined. It is easy to check that the group G^* of all such f^* is a group of bijection of X^* onto itself and $f \mapsto f^*$ is a homomorphism of G onto G^*.

Introduce the natural maps $\alpha : X^* \to \mathbf{H}^2_{\mathbb{C}}$, $\beta : G \times \overline{D} \to X^*$, and $\gamma : G \times \overline{D} \to \mathbf{H}^2_{\mathbb{C}}$, given by

$$\alpha\langle g, x \rangle = g(x),$$

$$\beta(g, x) = \langle g, x \rangle,$$

$$\gamma(g, x) = g(x).$$

Note that $\gamma = \alpha\beta$. If U is an open set in $\mathbf{H}^2_{\mathbb{C}}$, then

$$\gamma^{-1}(U) = \bigcup_g \{g\} \times (g^{-1}(U) \cap \overline{D})$$

which is open in $G \times \overline{D}$. Therefore γ is continuous, so is α. By the condition (2) with (3), we know that

$$\langle Id, x\rangle = \begin{cases} \{(Id, x)\}, & x \in D, \\ \{(Id, x), (g_a^{-1}, g_a(x))\}, & x \in a \subset \partial D. \end{cases} \tag{1}$$

i.e. each point $z \in \overline{D}$ has a finite equivalent class

$$\langle Id, z\rangle = \{(g_1, z_1), (g_2, z_2)\}. \tag{2}$$

One also needs to note that every equivalent class $\langle f, x\rangle$ is finite since $\langle f, x\rangle = f^*\langle Id, x\rangle$. In addition, if we set $N_i = \{y \in \overline{D} : \rho(y, z_i) < \varepsilon\}$ $(i = 1, 2)$ for sufficiently small $\varepsilon > 0$, then $B(z, \varepsilon) = \bigcup_i g_i(N_i)$. Then it is easy to see that $G(\overline{D})$ is open. Let us write

$$W = \bigcup_i (g_i, N_i), \qquad V = \beta(W). \tag{3}$$

Obviously V is open in X^* because W is a union of equivalent classes. Each $f \in G$ induces a map $\tilde{f} : G \times \overline{D} \to G \times \overline{D}$ by the rule

$$\tilde{f} : (g, x) \mapsto (fg, x).$$

Note that the $\tilde{f}$ are homeomorphism of $G \times \overline{D}$ onto itself, the group of such $\tilde{f}$ is isomorphic to G and

$$\beta\tilde{f} = f^*\beta, \qquad \gamma\tilde{f} = f\gamma.$$

Therefore $\alpha f^*(V) = \alpha f^*\beta(W) = f\gamma(W) = B(f(x), \varepsilon)$, i.e. α maps each $f^*(V)$ to an open set.

We claim that $f^*(V)$ are a base for the topology of X^*. Taking any open set A of X^*, we suppose that $\langle f, z\rangle \in A$. Then we know that

$$\langle f, z\rangle = \{(fg_1, z_1), (fg_2, z_2)\}.$$

Because β is continuous,

$$\beta^{-1}(A) = \bigcup_{h \in G} (h, A_h),$$

where each A_h is open in $\overline{D}$. For enough small ε, Eq. (3) yields to

$$\tilde{f}(W) = \bigcup_j (f g_j, N_j) \subseteq \beta^{-1}(A)$$

and so

$$f^*(V) = f^*\beta(W) = \beta\tilde{f}(W) \subseteq A.$$

As $(Id, z) \in W$, so $\langle f, z\rangle \in f^*\beta(W) = f^*(V)$. We see that $\alpha : X^* \to \mathbf{H}^2_{\mathbb{C}}$ is open, because previously we have shown that α maps each $f^*(V)$ to an open set.

Without loss of generality, we assume that $u, v \in f^*(V)$ and $\alpha(u) = \alpha(v)$, then choose $u', v' \in \tilde{f}(W)$ with $\beta(u') = u$, $\beta(v') = v$. Hence

$$\gamma(u') = \alpha\beta(u') = \alpha(u) = \alpha(v) = \gamma(v')$$

and then u', v' are in the same equivalent class by (1). Therefore α is injective from $f^*(V)$ to $f(\gamma(W))$.

So far α is a local homeomorphism. In the following we will prove that α is a homeomorphism.

We declare that X^* is a Hausdorff space. Take two distinct points $\langle f, x\rangle$, $\langle g, y\rangle$ from X^*. We assume that

$$\langle f, x\rangle = \{(f_1, x_1), (f_2, x_2)\}$$
$$\langle g, y\rangle = \{(g_1, y_1), (g_2, y_2)\}$$

by (2). Then like (3) we take the neighborhoods like U_i of x_i, and neighborhoods of V_i of y_i for $i = 1, 2$. Then we consider whether that $W_x = (f_1, U_1) \cup (f_2, U_2)$ and $W_y = (g_1, V_1) \cup (g_2, V_2)$ intersect. If there exist f_i, f_j such that $f_i = g_j$, then $U_i \neq V_j$ since $\langle f, x\rangle \neq \langle g, y\rangle$. i.e. $W_x \cap W_y = \varnothing$. It is obvious that $W_x \cap W_y = \varnothing$ if any f_i is not same as f_j. It follows that $\beta(W_x) \cap \beta(W_y) = \varnothing$. Hence X^* is a Hausdorff space.

We denote the quotient space $\overline{D}$ modulo the finite set consisting of the side pairing transformations by D^*, and give the canonical projection $\pi : \overline{D} \to D^*$ which is surjection. Given two points $x^*, y^* \in D^*$, usually we define a pseudo metric ρ_0^* on D^* by $\rho_0^* = \inf \rho(x, y)$, which is taken over all x, y satisfying $x \in \pi^{-1}(x^*)$, $y \in \pi^{-1}(y^*)$. However ρ_0^* does not satisfy triangle inequality. Therefore we define metric ρ^* on D^* in the following form

$$\rho^*(x^*, y^*) = \inf \sum \rho(x_i, x_{i+1}), \tag{4}$$

where the infimum is taken over all finite sequences $\{x_1, \cdots, x_k\}$ with $\pi(x_1) = x^*$, $\pi(x_k) = y^*$.

Now we define a map $\sigma : X^* \to D^*$ by the composition of projection on the second factor of $G \times \overline{D}$ with the projection $\pi : \overline{D} \to D^*$. It is easy to check that σ is well defined. We choose an arbitrary point y from $\alpha(X^*)$, let $\omega(t)$ be the (real) geodesic satrting from point y with arc length parameter t. Since α is local homeomorphism, there exists $0 < t_0$ such that for $0 \leqslant t < t_0$, $w(t)$ could be lifted to $\tilde{\omega}(t)$ of X^*. However ω may not be continuously lifted to X^* for $0 \leqslant t \leqslant t_0$. Take a sequence $\{t_m\}$ of numbers, which satisfy $t_m \to t_0$ as $m \to \infty$ and $\tilde{\omega}(t)$ lies in an open set of the form $f^*(V)$ for $t_m \leqslant t \leqslant t_{m+1}$.

By considering (1) and (3), one could easily show that for any two points $z_1^*, z_2^* \in f^*(V)$, $\rho^*(\sigma(z_1^*), \sigma(z_2^*)) \leqslant \rho(\alpha(z_1^*), \alpha(z_2^*))$. It yields to

$$\rho^*(\sigma(\tilde{\omega}(t_m)), \sigma(\tilde{\omega}(t_{m+1}))) \leqslant \rho(\omega(t_m), \omega(t_{m+1})).$$

Since D^* is complete, we know that $\sigma(\tilde{\omega}(t_m)) \to z^* \in D^*$ as $m \to \infty$, actually $\sigma(\tilde{\omega}(t)) \to z^*$ as $t \to t_0$. Assume that $z^* = \pi(z)$, obviously $z \in \overline{D}$.

If $z \in D$, then there exists a neighborhood U_z of z in D. Setting $U^* = \pi(U_z)$, we see $\sigma^{-1}(U^*) = \bigcup \langle f, U_z \rangle$, by taking over all $f \in G$. Obviously $\langle f, U \rangle \cap \langle h, U \rangle = \emptyset$, if $f \neq h$.

If z lies in the interior of one side a of D, then we suppose that $z' = g_a(z)$. There exist neighborhoods U_1, U_2 of z, $g_a(z)$ respectively, such that $U_1, U_2 \subseteq \overline{D}$ and $U_1 \cap U_2 = \emptyset$. Assume that $U^* = \pi(U_1) \cup \pi(U_2)$. Each connected component Ω_J of $\sigma^{-1}(U^*)$ consists of the union of two half balls. Actually each Ω_J is a neighborhood of a point of the form $\langle f, z \rangle$ in X^*, i.e. Ω_J is of the form $f^*(V)$, where V is as (3). Also we could easily show that $f^*(V) \cap h^*(V) = \emptyset$ for $f \neq h$.

Therefore there is a neighborhood U^* of z^* which satisfies $\sigma(\tilde{\omega}(t)) \subseteq U^*$ for t sufficient close to t_0, such that $\tilde{\omega}(t)$ will lie in a relative compact set of the form $f^*(V)$. Then we could continuously define the lift $\tilde{\omega}(t_0)$. By repeating this process, one can lift the whole geodesic $\omega(t)$, which is also unique. Then α is homeomorphism. It is not difficult to show that $g(D) \cap h(D) = \emptyset$ for distinct elements $g, h \in D$, and the tessellation of $\overline{D}$ under the action G is the whole space $\mathbf{H}^2_{\mathbb{C}}$.

The assertion now can be obtained. □

4 Application to Verify the Discreteness of Triangle Groups

Definition 4.1 Given three complex geodesics $C_1,\ C_2,\ C_3$, consider the representation $\Gamma : \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathrm{PU}(2, 1)$. We use I_i to denote the inversion in the complex geodesic C_i $(i = 1, 2, 3)$. Then Γ is using to denote the triangle groups generated by $I_1,\ I_2,\ I_3$. Define a complex triangle group to be an ultra-ideal triangle group, if each pair of the three complex geodesics do not intersect in $\mathbf{H}^2_{\mathbb{C}}$. Ultra-ideal triangle group is of type $[l_1,\ l_2,\ l_3]$ $(l_1,\ l_2,\ l_3 > 0)$, if the distances of each pair complex geodesics are $l_1,\ l_2,\ l_3$ respectively.

In the following we will consider one kind of isosceles ultra-ideal triangle. Assume that the three chains corresponding to $C_1,\ C_2,\ C_3$ are $re^{i\theta}$- chain, (x, y)-chain and $(-x, y)$-chain $(r > 0,\ x,\ y \in \mathbb{R},\ \theta \in [0, 2\pi])$, i.e. they have polar vectors

$$p_1 = \begin{pmatrix} 1 \\ -re^{-i\theta} \\ re^{-i\theta} \end{pmatrix}, \quad p_2 = \begin{pmatrix} 0 \\ 1+y^2+ix \\ 1-y^2-ix \end{pmatrix}, \quad p_3 = \begin{pmatrix} 0 \\ 1+y^2-ix \\ 1-y^2+ix \end{pmatrix}.$$

respectively. The involutions in the complex chains $\partial C_1, \partial C_2, \partial C_3$ are respectively as follows

$$I_1 = \begin{pmatrix} 1 & -2re^{i\theta} & -2re^{i\theta} \\ -2re^{-i\theta} & 2r^2-1 & 2r^2 \\ 2re^{-i\theta} & -2r^2 & -2r^2-1 \end{pmatrix},$$

$$I_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1+x^2+y^4}{2y^2} & \frac{-1+x^2+y^4-2ix}{2y^2} \\ 0 & -\frac{-1+x^2+y^4+2ix}{2y^2} & -\frac{1+x^2+y^4}{2y^2} \end{pmatrix},$$

$$I_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1+x^2+y^4}{2y^2} & \frac{-1+x^2+y^4+2ix}{2y^2} \\ 0 & -\frac{-1+x^2+y^4-2ix}{2y^2} & -\frac{1+x^2+y^4}{2y^2} \end{pmatrix}.$$

And we will see that

$$\cosh^2\left(\frac{\rho(C_1, C_2)}{2}\right) = \frac{r^2}{y^2}, \qquad \cosh^2\left(\frac{\rho(C_1, C_3)}{2}\right) = \frac{r^2}{y^2},$$
$$\cosh^2\left(\frac{\rho(C_2, C_3)}{2}\right) = 1 + \frac{x^2}{y^4},$$

therefore it is necessary to assume that $r > |y|$.

Consider the polyhedron D to be intersection of equidistant half-spaces $\mathcal{H}_i$ ($i = 1, 2, 3$) based at the origin $z_0 = [o, o]^t \in \mathbb{B}^2$, where

$$\mathcal{H}_i = \{w \in \mathbf{H}^2_{\mathbb{C}} : \rho(w, z_0 < \rho(w, I_i(z_0))\}.$$

We denote the bisector by B_i which is the boundary of $\mathcal{H}_i$. The codimensional-1 face $a_i = B_i \cap \partial D$ is called side of D.

Take $r = 1,\ \theta = \frac{\pi}{3},\ x = 1,\ y = \frac{1}{2}$ for example. By using Mathematica 10.0, we could check that $B_i \cap B_j = \emptyset$, whenever $i \neq j$. For each side S_i, the inversion I_i maps it onto itself, but not as the identification. It is also easy to see that $I_i(D) \cap D = \emptyset$. Also, for each point v in the interior of a_i, there exists neighborhood U_v such that $U_v \subseteq \overline{D} \cup I_i(\overline{D})$. By seeing the proof of Theorem 3.1, one could obtain that $\Gamma = \langle I_1, I_2, I_3 \rangle$ is discrete.

5 Fundamental Polyhedron for Coset Decompositions

In order to apply Poincaré's polyhedron theorem to other type of triangle groups, we consider fundamental polyhedron for coset decompositions, where the polyhedron is invariant under a non-trivial subgroup. In the following we only think about the finite-sided polyhedron D which constructed by intersection of equidistant half-spaces. As stated at [3], we will call closed cells of a polyhedron as facets, refer to facets of codimension one, two, three and four as sides, ridges, edges and vertices.

We suppose that side parings are just as statement of Sect. 2 and the polyhedron D is preserved by a finite group $H \subseteq \mathrm{PU}(2,1)$. Let Γ be the group generated by H and the side pairing transformations. We assume that the side pairing transformation g satisfy, for each side a and $h \in H$ we have $g_{h(a)} = hg_a h^{-1}$.

Considering one ridge r_1, we know that $r_1 = a_1 \cap a'$ where a_1, a' are two sides. There exists a side pairing transformation g_{a_1} which maps a_1 to a'_1. Then there is a side a_2 such that ridge $r_2 = g_{a_1}(r_1) = a_2 \cap a'_1$. We repeat the previous process and will obtain a sequence of ridges r_i, sides a_i and side pairing transformation g_{a_i} so that $r_i = a_i \cap a'_{i-1}$. Note that there are positive integer m and an element $h \in H$ such that $hr_{m+1} = r_1$. We call that $(r_1, \cdots, r_{m+1})$ is the ridge cycle of r_1 and define the cycle transformation to be $T = h \circ g_{a_m} \cdots g_{a_1}$. Then it yields to

$$T = T(r_1) = h \circ g_{a_m} \cdots g_{a_1} \circ (r_1) = r_1 \ .$$

In the same fashion with the proof of Theorem 2.1, one can proof the following theorem simply.

Theorem 5.1 *Assume that the finite sided polyhedron $D \subset \boldsymbol{H}^2_{\mathbb{C}}$ satisfies the following conditions:*

(1) For each side a, there exist another side a' and $g_a \in \mathrm{PU}(2,1)$ such that $g_a = a'$, $g_{a'} = g_a^{-1}$ (the isometries g_a are called the side pairing transformations);
(2) $g_a(D) \cap D = \emptyset$;
(3) Any point $z \in D^*$, $\pi^{-1}(z)$ is a finite set, where D^* is obtained by identifying the sides of D; i.e. there is a surjection $\pi : \overline{D} \to D^*$ where $\pi(x) = \pi(y)$ if there is a side pairing transformation g with $g(x) = y$;
(4) cycle conditions;
(5) existence of consistent horoball to guarantee completeness of D^*.

Then the group G generated by H the side pairing transformations is discrete, a fundamental polyhedron for G is obtained by D with a fundamental domain for H.

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