K. Ramasubramanian *Editor*

Gaṇitānanda

Selected Works of Radha Charan Gupta on History of Mathematics

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R. C. Gupta receives the Kenneth O. May Prize (2009)

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समर्पणम्

दीप्तं गुप्तवरेण गुप्तगणितं व्याख्या-सुकान्त्या भृशं
लेखालोक इहानयेद्विकसनं विद्वन्मनःपद्मके।
लेखांस्तान् विविधानिहैककरणं कार्तज्ञ्यभावैर्मुदा
कृत्वा प्रस्तुतिरर्प्यते हि विदुषां सेवेयमादीयताम्॥

Many hitherto hidden facts about [Indian] mathematics were brought to light by R. C. Gupta through the light of his brilliant expositions. May the sun [light] of these expositions blossom the lotus minds of scholars. Having compiled these expositions, we present them to the scholarly world with immense pleasure and a deep sense of gratitude. May this offering be gracefully accepted.

Foreword

It is a great pleasure as well as a privilege to be invited to write these few words about this collection of the selected works of Prof. Radha Charan Gupta, one of the great Indian scholars on the history of Indic mathematics. Dr. Gupta began his career as a mathematics teacher in a college and was inspired by Datta & Singh's book on the *History of Hindu Mathematics*, which he came across in 1963, to pursue the history of Indic mathematics. He went on to complete his Ph.D. thesis under the guidance of Prof. T. A. Sarasvati Amma, whose book on *Geometry in Ancient and Medieval India* represents a landmark in Indic historical studies of mathematics.

Dr. Gupta dedicated the rest of his life to carrying out a whole series of meticulous studies on Indic works in mathematics—studies which have stood the test of time and won him the 7th Kenneth O. May Prize in the year 2009, awarded by the International Commission on the History of Mathematics—the first Indian to get the Prize. Among his most notable contributions are the analyses of Parameśvara's third-order series approximations for the sine, and the methods of Govindasvāmin for interpolating in sine tables. He has contributed articles to *Studies in History of Exact Sciences*, volume in honour of David Pingree, edited by Burnett et al., and *Writing the History of Mathematics: Its Historical Development*, edited by Dauben and Scriba. He has over 500 publications and has served the world of scholarship in many different ways.

His personal qualities of simplicity, modesty, frugality and generosity match the passion and extraordinary care with which he has pursued the history of mathematics. He is at once both an outstanding scholar and a sage, in the classical Indic tradition. I congratulate the editor of this volume Prof. K. Ramasubramanian for his effort in producing a collection of selected works of Dr. Gupta. The volume is appropriately called *Gaṇitānanda* (roughly the Joy of Mathematics)—and I cannot help recalling how the great Bhāskara promised that his algorithms will bring *ānanda* to all *gaṇakas*. I am sure this book will do the same.

We need more of the kind of authentic papers that Dr. Gupta has written with such dedication. Such papers are rare, and are generally scattered over the scholarly literature, so it is convenient to collect them into a volume and make them easily available. Prof. Ramasubramanian deserves our gratitude for his effort.

October 2015 Roddam Narasimha

Preface to the Second Edition

In the year 2015, the Annual Conference of the Indian Society for History of Mathematics (ISHM) was held at IIT Bombay. As the 80th birthday of Prof. R. C. Gupta also happened to be in the same year, there was a suggestion from the then Administrative Secretary S. L. Singh[1] and the President of the Society Prof. S. G. Dani that we could take the opportunity to felicitate Prof. R. C. Gupta on this occasion for the immense contributions he has made to the studies in the history of mathematics as well as to the society. From this germinated the idea of bringing out a volume containing selected works of Prof. Gupta. Since the time interval between the emergence of this idea and the conference was hardly two months, it was not possible to approach any professional publisher to bring out this volume. Hence, the volume was brought out as a publication of ISHM, and only a few copies were printed, primarily for the distribution among the participants of the conference and a few others.

Since this was a limited edition, subsequently it was felt that it would be highly desirable to bring out the volume through a reputed publisher and make it commercially available for various institutions and individuals to purchase and get benefited from its contents. Professor Dani took the lead in this direction and got in touch with Springer. They readily agreed to publish the volume and requested that I, as an editor of the volume, should consider adding a few more articles to the volume. However, it occurred to me, I thought that it would be more appropriate to seek inputs from Prof. Gupta himself in this regard. When I approached him, he gave a list of articles that could be considered to be included in this revised and enlarged edition. From this list, ten articles have been typeset and included in this volume. The inclusions have been done in such a way that we may find one article added per part of the volume either at the beginning or at the end, except for Part I and Part VIII of the previous edition.

[1] Professor Shyam Lal Singh, born on January 20, 1942, took his last breath at the age of 75 on October 2, 2017. His dedication to the promotion of studies in the history of mathematics in India and his selfless service to the society can never be overstated. For more details, see *Gaṇita Bhāratī* 39.1 (2017).

While Part I of the previous edition remains unchanged, Part VIII is included with two very interesting articles, one on Prof. P. C. Sengupta—a doyen among the historians of Indian mathematics and astronomy—and the other on Prof. Augustus De Morgan, who was a brilliant teacher, a prolific writer and a scholar par excellence on mathematics, logic, and philosophy. Incidentally, during one of the conversations with Prof. Gupta, I came to know that he travelled all the way to Calcutta to meet the direct heirs of Prof. Sengupta in order to get an authentic biographical sketch about their father, which he has nicely written up in the form of a short article.

Another valuable addition that has been made to this volume is the inclusion of an up-to-date bibliography of Prof. Gupta. In 1996, Prof. Takao Hayashi prepared a bibliography of the writings of Prof. Gupta which was published in *Historia Scientiarum*. Then again in 2011, he compiled an updated version of it which was published in *Gaṇita Bhāratī*. Those two years incidentally happened to be the 60th and 75th birth anniversaries of Prof. Gupta. It was so kind of Prof. Gupta that he gave us the necessary materials to further update and present an up-to-date bibliography of his writings while working on this revised edition. This bibliography is provided at the end of this volume.

While preparing this edition, as done in the earlier edition, much attention was paid to eliminate the typos that had inadvertently crept into the earlier publications, particularly in the Sanskrit passages quoted by the author as well as in the transliteration of Sanskrit terms. It is worth mentioning the suggestion from Prof. Gupta that this volume be supplemented with an exhaustive index. Efforts were taken in this direction, and significant progress was also made. But considering the current trend of both the individuals and the institutions preferring to procure digital versions of the texts, it was suggested by the publisher that efforts towards preparing an index may not be worthwhile. Hence, the idea of adding an index to the volume was dropped.

Finally, I would like to acknowledge all those who contributed in various ways to the production of this revised and enlarged edition of *Gaṇitānanda*. First, I would like to thank Prof. Dani of ISHM and Mr. Shamim Ahmad of Springer for having taken interest in bringing out this revised edition. Working with Shamim has been a great pleasure as he has been very gentle, highly encouraging and extremely cooperative. Secondly, I would like to thank all my project staff, namely Ms. Sushama Sonak, Ms. Sreelekshmy Ranjit, Ms. Lalita Hotkar, Dr. Dinesh Mohan Joshi and Mr. G. Periasamy, for their editorial assistance. In particular, Sushama needs a special mention for being kind enough to work quite late in the evenings on a few days in order to meet the deadline for the submission of the manuscript.

All these project staff are supported under the auspices of Science and Heritage Initiative (SandHI) at IIT Bombay. But for the financial support made available, I cannot imagine venturing into the task of producing this volume, as well as others that were produced in the past, or are in the pipeline. In this connection, I would like to express my sincere gratitude to Ms. Amita Sharma (former Additional Secretary, MHRD) and Prof. Devang Khakhar (Director, IIT Bombay) for getting this project sanctioned and letting me work with it within the norms of the system in an unfettered system.

Thanks are also due to Mr. Aditya Kolachana and Dr. K. Mahesh for their technical assistance in handling the bibliography using LaTeX. Mr. Devaraja Adiga and Mr. Prasad Jawalgekar also helped by proofreading a few articles. Last but not least, I am grateful to Prof. R. C. Gupta, who notwithstanding his old age enthusiastically responded to a few queries over the phone—in spite of the oddities of the connections!

हेविलम्बि-चैत्र-कृष्ण-चतुर्थी
कल्यब्दः ५११९
April 4, 2018

K. Ramasubramanian
IIT Bombay

Preface to the First Edition

The study of the Indian tradition of *Gaṇita* and *Jyotiṣa* (Mathematics and Astronomy) was taken up by many great Indian savants in the modern period, starting with Pandit Bāpūdeva Śāstrī (1821–1900), and Mahāmahopādhyāya Sudhākara Dvivedī (1855–1910). Both these highly talented scholars, after mastering the traditional Indian texts, effortlessly got acquainted with the modern mathematical tradition. In fact, besides teaching traditional texts, they were also teaching Euclidean geometry (*rekhāgaṇita*) in the Government Sanskrit College.[2] It was Bāpūdeva Śāstrī who first drew the attention of modern scholars about the occurrence of a differential formula ($\delta \sin\theta = \cos\theta \cdot \delta\theta$) in Bhāskara's *Siddhāntaśiromaṇī*, while he was translating the work into English along with Lancelot Wilkinson.

The contributions of *Mahāmahopādhyāya* Pandit Sudhākara Dvivedī, who was the successor of Pandit Bāpūdeva Śāstrī to teach astronomy and mathematics in Benaras since 1890, are indeed phenomenal. He brought out authentic editions of several important works, such as *Siddhānta-tattvaviveka, Śiṣyadhīvṛddhidatantra, Pañcasiddhāntikā, Bṛhatsaṃhitā, Triśatikā, Brāhmasphuṭasiddhānta, Grahalāghava, Sūryasiddhānta* and *Gaṇitakaumudī* (published later by his son Padmākara Dvivedī), when he was working in the *Sarasvatī Bhavana Granthālaya* (Library), now merged with Sampurnanand Sanskrit University. Dvivedī also added his own commentary in lucid Sanskrit to most of these texts. He also did pioneering work in producing text books in Hindi and Sanskrit on modern mathematics. The stupendous work done by him in such a short span of life that he lived (55 years) is both amazing and inspiring!

Early decades of the twentieth century saw yet another remarkable scholar Bibhutibhusan Datta (1888–1958). True to his name[3] Datta was blessed with remarkable ability to deeply penetrate into any subject and bring out its essence. *Ancient Hindu Geometry: The Science of the Sulba* based on the lectures that he gave in 1932, and the two volume *History of Hindu Mathematics* that he wrote along with A. N. Singh, are indeed all time classics in the history of Indian mathematics.

[2]It is in this college that the famous western indologists James R. Ballantyne, Ralph T. H. Griffith, George Thibaut and Arthur Venis, served as Principals in succession from 1846–1918.

[3]The term Bibhutibhusan when taken as a *bahuvrīhi* compound means 'for whom talents are an ornament' or equivalently 'who is adorned with talents'.

P. C. Sengupta (1876–1962) and A. A. Krishnaswami Ayyangar (1892–1953) are two of the outstanding scholars who enriched our understanding of the history of mathematics and astronomy through their writings in the early part of twentieth century.[4] Soon there emerged a host of other scholars such as T. S. Kuppanna Sastry (1900–1982), K. S. Shukla (1918–2007) and K. V. Sarma (1919–2005) who meticulously brought out excellent editions of important primary sources, generally accompanied with translations and detailed mathematical notes as well. They also wrote scholarly articles on various topics that are highly insightful.

The tradition that was set by such stalwarts was carried forward by many other erudite scholars. Notable among them are: C. N. Srinivasiengar (1901–1972), C. T. Rajagopal (1903–1978), S. N. Sen (1918–1992), T. A. Sarasvati Amma (1918–2000), B. V. Subbarayappa (b. 1925), S. R. Sarma (b. 1937) and A. K. Bag (b. 1937). All these scholars, through their painstaking efforts, have made substantial contributions which have greatly enhanced our understanding of the development of Indian mathematics and astronomy.[5]

Professor Radha Charan Gupta (RCG), whose selected works are presently being brought out as a volume, is a shining star in the galaxy of such illustrious scholars. He was born in Jhansi,[6] to Chote Lal and Bino Bai in the Śālivāhana Śaka year 1857 on the *Śrāvaṇa-pūrṇimā* day, which corresponds to August 14, 1935.[7] He had his education up to Intermediate (currently higher secondary) in Jhansi itself. Even during early 1950s there was not much facility to pursue higher studies in Science in Jhansi, and one had to go to Lucknow (approx. 300 km from Jhansi) for that purpose. While RCG wanted to pursue his studies, his father seemed to have reservations due to financial constraints. However, being keen on what he wanted to achieve, RCG managed to get a merit scholarship (Rs. 60 per month) to continue his studies in Lucknow University.

While he was doing Bachelor's in Lucknow University, RCG got married to Savitri Devi in 1953.[8] Continuing his studies further, RCG successfully completed his Masters degree with flying colors in 1957.[9] As he was already married, and had

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[4]Among other works of these two scholars, the translation and detailed mathematical notes of *Khaṇḍakhādyaka* brought out by Sengupta in 1934, and the (24 page) article on *Cakravāla* method, written by Ayyangar in 1930 are truly exceptional!

[5]Needless to say that this scholarly enterprise also owes a great deal to the monumental efforts and contributions of several European scholars such as H. T. Colebrooke, Lancelot Wilkinson, Rev. Ebenzer Burgess, George Thibaut in the nineteenth century and David Pingree and his disciples in the second half of the twentieth century.

[6]A historic city of northern India, in the region of Bundelkhand, in the southern part of Uttar Pradesh, India.

[7]It was clarified by Prof. Gupta himself that the date of October 26, 2015 given as his date of birth in the citation of Kenneth O. May prize is erroneous.

[8]During personal conversations RCG mentioned that in those days the practice of dowry was rampant. Since his father was in need of money to pay the dowry to get his sister (Shanti) married, RCG was obliged to get married so early, so that the dowry that was received in his marriage could be used for the dowry to get Shanti married.

[9]RCG was awarded Devi Sahay Mishra Gold Medal, for securing the first position in the Master's examination.

to support his family, he immediately took up the job of a lecturer, deferring his desire to pursue further studies leading to a doctoral degree. This in fact turned out to be a blessing in disguise for him. In 1963–1964, RCG got acquainted with the great scholar Dr. Sarasvati Amma. Under her guidance he completed his Ph.D. dissertation on “Trigonometry in Ancient and Medieval India” in 1970–1971.

The pursuit of history of mathematics that RCG started in the early 1960s, has been continuing till date unabated. Inexorably drawn as it were into the subject, he has published several hundreds of articles on a variety of topics related to history of science over a period of nearly five decades, starting from the late 1960s. From the day he stepped on to the hill of history of mathematics, by a divine providence, he has been trekking up the hill all the way!

This volume contains 46 selected articles of RCG written on a variety of important topics in Indian astronomy and mathematics. These articles have been organized in 9 parts, whose short summary can be gleaned from the back cover of the volume. The articles appearing in each of these parts have been arranged in chronological order of their appearance in journals/books.

In this volume, except for correcting a few obvious typographical errors in the text, in the equations, and in the Sanskrit verses, the articles are essentially reproduced as in the original (including the style of footnotes, references, etc.). Other editorial corrections such as replacing or inserting a Sanskrit word in the quoted passages that were carried out are indicated in footnotes with ‘*–ed.*’ at the end of them. Though all the papers have been entirely retyped, most of the figures (many of which are hand-drawn by RCG himself) have been simply scanned and reproduced.

Venturing into the task of highlighting the key points made by RCG in each of his articles, would make the preface too lengthy. To avoid this, I shall confine myself to merely highlighting some of the special and distinctive features that characterize all his writings.

Brevity and clarity: This seems to be one of the hallmarks of RCG’s articles. It is hard to find any of his articles being discursive or verbose. By adopting a trenchant, yet simple style, he seems to have set a trend that is worth emulating.

Building on factual findings: His articles are filled with factual findings, without making exaggerated claims. Given an opportunity, RCG does not spare those who try to make superfluous statements. For instance, in his article “In the name of Vedic mathematics” he observes:

> It is a common experience that as soon as a scientific discovery is made known, someone would come out with the claim that it was already known to the Vedic sages. To give a sample we take an example…

Precision in translation: RCG did not have the privilege of receiving any formal training in Sanskrit, while he was a student in school or college. Neither does he hail from a family of pandits. However, the great erudition that he has acquired in making precise translations of Sanskrit passages is highly commendable. During one of my visits to his place, when I asked him about how he gained such a mastery in Sanskrit, he took me to his library and showed a set of Sanskrit dictionaries. I was amazed to notice marginal notes that he had made by painstakingly going through *every* relevant entry of these dictionaries.

Sobriety of judgement: RCG has been extremely careful while analysing the contribution made by ancients. Wherever he finds any inaccuracy in the ancient work, he doesn't shy or hesitate to point out the lacunae in their treatment. At the same time he also makes it a point to add that "it will not be fair to judge those books by modern standards" (see article in Part VIII of the volume, on Pandit Sudhākara Dvivedī).

Critical but not harsh: Though he has been quite critical in his analysis, he carefully avoids making harsh statements. In fact, his general approach to history of science, can be gleaned from the following statement he makes in the preamble to his article "In the name of Vedic mathematics", appearing in Part II of the volume:

> Excitement is predominating over real understanding of the matter. Emotional feelings are overtaking rationality needed to know the correct situation involved in the issue. Logic and cool thinking is necessary for assessment of historical significance, scientific value, educational utility, and…

RCG also seems to be quite adept in subtly pointing out the errors noticed in other works, instead of being blunt or unceremonious in making remarks.

Meticulous in giving references: One of the hallmarks of RCG is that he is extremely meticulous in providing references to the source texts and secondary works that he has consulted.

Always up-to-date: With the advancement in technology, today it may be easy to have access to various journals merely by the click of a button. But in those days (40–50 years ago) when RCG was a young researcher, it would have been extremely difficult to keep oneself informed on the research activities going on at different places, particularly about the current work being done abroad. But, somehow, RCG seems to have managed to stay in touch with many scholars abroad and thereby keep himself up-to-date, as is evident from the references provided in his articles.

Aware of the historical context: Yet another interesting feature of RCG's articles is that, wherever possible he tries to make a link to the relevant historical information. This besides making the reading of the article more lively, also sometimes helps us develop an understanding of the context in which certain things happen, certain ideas get developed, etc. For instance, posing himself the question as to why Pandit Sudhākara Dvivedī has not written a commentary on *Śiṣyadhīvṛddhidatantra*, as he has done it for most of the other texts, RCG observes:

> There is no commentary, possibly because the text itself was simple and expository, or more probably because the editor was lacking peace of mind due to the sad loss of his father which he mentioned in the words:
>
> यातेदिवं पितरि तद्विरहज्वरेण सन्तापतप्तहृदयेन सुधाकरेण। संशोधितम् ...

Balanced approach: While scholars like David Pingree have written extensively on transmission of scientific ideas into India, not many have attempted to study the transmission of ideas outwards from India. Some of the articles contained in Part IX throw light on this aspect. It is also interesting to note that, while narrating about the transmission of mathematical ideas of Indian origin into other cultures, RCG also points out how in the process of transmission, the ideas themselves get transformed and assume new dimensions.

Having said this much about RCG's unique and scholarly style of writing, now I would like to focus on the unique contributions made by him to promote studies in history of science in India. From his fairly early teaching days, RCG has been deeply concerned about the need to foster and promote scholarship on history of science in India. In his article on "The Study of History of Mathematical Sciences in India" (Part II of the volume), he observes:

> There are no adequate arrangements in India for imparting sufficient formal education and training to produce competent historians of science. A few special papers on History of Science are taught in some universities, but there are hardly any full graduate or post-graduate degree courses exclusively in the field. …
>
> Indian scholars must cultivate greater historical sense. Also we must have real love for records and try to preserve them. In foreign (western) countries, the papers, notes, correspondence of scientists is preserved and catalogued (in various libraries). In India, the material is often disposed of as waste paper (also cf. the practice of worshipping very costly idols and then do *visarjana*).

Elsewhere in the article he remarks:

> It seems History of Mathematics is not a dead but dynamic subject. Before 1930, it was mostly just the glory of the Greeks. After that the picture changed by findings in Babylonian mathematics. Further dimensions have been added to ancient period by the researches in prehistoric and megalithic times and by theories of ritual origin of sciences. Now medieval period is being enriched by studies and publications of Arabic mathematics. However, it is better to wait rather than give final and immature judgements in a hurry. Let us pool material; building may be erected later on. It is wise to look before leap.

The above passages more or less summarize RCG's understanding of the nature of history of science, and the role of serious scholarly studies in taking the subject forward. RCG not merely wrote on what needs to be done to promote history of science in India, he indeed devoted his entire life and all his resources for this cause. He was instrumental in forming the Indian Society for History of Mathematics, and also served as the founder-editor of *Gaṇita Bhāratī*—the official journal of the Society for over 25 years from 1979. It was indeed most appropriate that he was honoured with the coveted Kenneth O. May Prize in the year 2009, the highest international honour that a historian of mathematics can seek.

Before I conclude, I would like to quote a verse from Bhartṛhari:

रत्नैर्महाब्धेस्तुतुषुर्न देवाः न भेजिरे भीमविषेण भीतिम् ।
सुधां विना न प्रययुर्विरामं न निश्चितार्थाद्विरमन्ति धीराः ॥

> The divine beings were not contented with the gems (that emerged out) of the ocean; nor were they frightened by the terrible poison (which arose from it); they did not rest till they obtained the nectar (for which they commenced their activity). Indeed, resolute men do not relax till they reach their determined goal.

The nectar for RCG is achieving a truly global view of history of mathematics. The enthusiasm that he has, even in an advanced age of eighty plus, for history of mathematics is indeed amazing. I only wish and pray that he be blessed with many more years of healthy life, so that the world of history of mathematics is guided and benefited by all that emerges from the pen of this erudite scholar!

मन्मथ-आश्वयुज-कृष्ण-चतुर्थी
कल्यब्दः ५११६
October 30, 2015

K. Ramasubramanian
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Acknowledgements

The idea of bringing out this volume got germinated only during the first week of September 2015. It took a while for the idea to get concretized, and the actual work of typing the articles commenced only by mid September. Within a period of one-and-a-half months, the volume containing around 500 pages had to be prepared in camera-ready form and handed over to the publisher. This would have been impossible without a dedicated team working towards achieving this. It would be amiss to present the volume to the readers without acknowledging the numerous people who were involved in the production of the volume, either directly or indirectly, at various stages.

Germination of the idea: The idea to bring out the volume is closely linked with the idea of dedicating the 2015 Indian Society for History of Mathematics (ISHM) Conference to Prof. R. C. Gupta (RCG) on the occasion of his 80th birthday. This idea to dedicate to RCG—though formally conveyed to me by Prof. S. G. Dani (the President of ISHM)—seems to have originated from Prof. S. L. Singh (the Administrative Secretary of ISHM), when the latter came to know that RCG is completing 80 years of age this year. I would like to express my sincere gratitude to both Profs. Dani and Singh, for encouraging me in taking the idea forward, and to come up with this volume—a challenging, yet academically enriching exercise. Especially, I would like to acknowledge Prof. Dani's timely help in copy-editing some sections of the volume in the final stages of its preparation.

Organizing the content: A preliminary list of articles that could be considered for inclusion in the volume, was given by RCG himself on the day (September 5) I met him at his residence in Jhansi, to propose the idea of the volume, and also invite him for the conference. When I shared the list with Prof. M. D. Srinivas (MDS) of the Centre for Policy Studies, he suggested a few additions to it, besides also outlining how the articles could be organized under different sections. After a few iterations between the two of us, and in consultation with RCG, the contents of the volume were finalized. I profusely thank MDS for his valuable inputs in this regard. In this context, I would also like to thank Profs. M. S. Sriram, and Dani for their advice at different stages.

Typing the material: It would have been impossible to bring out the volume in such a short period, but for the immense help from G. Periasamy. He typed most of the articles in the volume by working tirelessly almost 15 hours a day, and completed the task assigned to him in about a month's time. We abundantly thank him for the same, and also deeply appreciate his dedicated service. Thanks are also due to Dinesh Mohan Joshi and Sreelekshmy Ranjit for sharing the work with Periasamy and typing a few articles.

Proof-reading and page-making: Among the several people who contributed towards this, firstly I would like to acknowledge the enthusiastic help from Keshav Melnad. His sincere efforts towards compiling the entire document, page-making as well as checking the correctness of the Sanskrit verses is indeed commendable. Secondly, for her meticulous proof-reading of the entire manuscript, and assisting Keshav for implementing the corrections, I would like to thank Anupriya Aggarwal. Sincere thanks are also due to Prasad Jawalgekar, Dinesh Mohan Joshi and Sushama Sonak for the second round of proof-reading different sections of the manuscript, which mirrors the quality of this publication. Thanks are also due to Devaraja Adiga and K. Mahesh for going through the Sanskrit passages in the final stages of the proof-reading and also helping out with other sundry tasks associated with the publication.

Technical assistance: The articles contained in the volume, besides having Roman letters, also contain several passages in Sanskrit (Devanāgarī script), transliterations of innumerable Sanskrit terms, diagrams, mathematical equations, tables, and so on. But for the cheerful technical assistance provided by Aditya Kolachana, by way of installing suitable software as well as suggesting different editors that can be employed to handle all the above requirements with great facility, it would have been impossible to get the entire thing typeset in a short span of time, without hassle. The entire team would like to thank Aditya for his timely and enthusiastic help.

Granting permissions: As the articles that constitute the volume have already been published in various journals, I, as the editor of the volume, had to seek the necessary copyright permissions from the publishers for reproducing the articles both in printed and electronic format. It is my pleasant duty to place on record thanks to all the publishers (listed below) of journals/books for readily granting permissions to reproduce the articles free of cost.

1. Brill (Book, *Studies in History of Exact Sciences*)
2. Elsevier (Journal, *Historia Mathematica*)
3. Indian National Science Academy (Journal, *IndianJournal of History of Science*)
4. Indian Society for History of Mathematics (Journal, *Gaṇita Bhāratī*)
5. Wiley (Journal, *Centaurus*)

In particular I would like record my special thanks to Tom Archibald of Simon Fraser University, one of the co-Editors-in-Chief of *Historia Mathematica*, for sending his recommendations to the manager of Elsevier, to get the necessary permissions granted.

Composing verses: The beautiful verses that are found in the *Samarpaṇam* (dedication) and the *Praśastiḥ* (at the end of the volume), are essentially compositions of K. Mahesh. Thanks to him for coming up with these nice verses in the metre *Śārdūlavikrīḍitam*.

Facilitating correspondence with RCG: Since RCG does not use computers, whenever I had to consult with him from an editorial standpoint, I had to seek the help of Dr. Brajesh Dixit of Jhansi (who was introduced to me by Prof. Krishna Gandhi) to act as liaison. Brajesh cheerfully assisted me in several respects, and has even agreed to accompany RCG during his forthcoming visit to IIT Bombay for the conference. My sincere thanks to Brajesh for all the help he has rendered, both during my visits to Jhansi as well as in facilitating correspondence with RCG.

Cover design: I don't have words to thank Sharanya Ganesh for readily conceding to my request to design the cover page in a very short span of time. Thanks are also due to Prof. Rajendra Bhatia, ISI New Delhi, for his valuable suggestions both in the preparation of the cover and the content of the volume.

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I would also like to place on record, my thanks to Venugopal D. Heroor (a chartered engineer by profession, but deeply interested in history of mathematics) for voluntarily and willingly sharing a file containing the bio-bibliographical sketch of RCG. This was found useful particularly in preparing the *Praśasti* of RCG. My sincere appreciations are also due to Ms. Sivakami of IRCC and her team, for promptly acting on our files, and in particular Amod Pansare, for ably and efficiently handling the project.

Before I conclude, I would like to express my sincere gratitude to Prof. Roddam Narasimha, for agreeing to inaugurate the conference, felicitate RCG, and also write a foreword to the second edition. I would be failing in my duty if I do not acknowledge RCG, for being very kind to offer his valuable advice at various stages, and also for being extremely cooperative in quickly referring back to the manuscripts for the correct reading, whenever I needed it (even late in the evenings). The field of history of mathematics surely owes all of the above a debt for furthering the study of ancient Indian mathematics!

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About the Editor

K. Ramasubramanian is Professor at the Cell for the Indian Science and Technology in Sanskrit, Department of Humanities and Social Sciences, Indian Institute of Technology Bombay, India. He holds a doctorate degree in theoretical physics, a master's degree in Sanskrit and a bachelor's degree in engineering—a weird but formidable combination of subjects to do multidisciplinary research. He was honoured with the coveted title 'Vidvat Pravara' by the Shankaracharya of Sringeri Sharada Peetham, Karnataka, India, for completing a rigorous course in *Advaita Vedānta* (a 14-semester programme), in 2003. He is one of the authors who prepared detailed explanatory notes of the celebrated works *Gaṇita-Yuktibhāṣā* (rationales in mathematical astronomy) *Tantrasaṅgraha* and *Karaṇapaddhati*, which brought out the seminal contributions of the Kerala School of Astronomy and Mathematics. The prestigious *Maharshi Badrayan Vyas Samman* was conferred upon him by the president of India, in 2008. He is a recipient of several other awards and coveted titles as well. Since 2013, he serves as an elected council member of the International Union of History and Philosophy of Science and Technology. He is also a fellow of The Indian National Science Academy.

Part I

The Oeuvre of Radha Charan Gupta

A Portrait of the Life of R. C. Gupta

K. Ramasubramanian

1 Introduction

Though I have known Prof. R. C. Gupta (RCG) for more than two decades through his writings, till recently I did not have an occasion to interact with him closely.[1] However, by divine providence, I have had an opportunity to make a couple of visits to his residence in Jhansi during the previous two months.

My first visit to meet RCG was on September 5, 2015. This was primarily to inform him about the proposed Indian Society of History of Mathematics (ISHM) Annual Conference to be held at IIT Bombay between November 14 and 16, 2015, and invite him for the same. I was particularly interested in his presence at the conference, as we had plans to dedicate this conference to him on the occasion of his 80th birthday, as well as bring out a volume containing his selected articles. RCG gladly accepted the invite and also quickly prepared a list of articles that could be considered to be included in the volume.

Prof. R.C. Gupta with Prof. K. Ramasubramanian

[1]Meeting him at conferences was not possible as RCG did not travel much, particularly after his 'formal' retirement in 1995, around which period I had just entered into the field. Correspondence through e-mails was not possible either, as he does not use computers at all!.

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Subsequently I met him two more times, once on September 25, and again on October 18 to discuss and finalize the contents of the proposed volume. During these visits, besides having fruitful academic discussions, I also had the privilege of talking to him at length on several issues of common interest. During the conversations, RCG recounted some interesting anecdotes from his life, including those that played a crucial role in turning him into an outstanding researcher in history of mathematics, rather than merely retiring as an ordinary mathematics teacher.

These conversations also provided me an opportunity to discover in him many divine virtues (*daivī sampat*) such as honesty, simplicity, nobility, generosity and more importantly fixity of purpose. In short, the noble native that I could discover in him enhanced my respect for him manifold. For the benefit of those who have not had an opportunity to interact with RCG, I would like to share some of my observations.

2 The Role of Providence in Shaping his Career

Though RCG started his career as an ordinary mathematics teacher in a college, it was providence that seems to have driven him to take up serious studies in history of mathematics and thereby turn him into an extraordinary historian of mathematics. Narrating how he was drawn, inexorably as it were, into the area of history of mathematics, RCG said, "three events played a crucial role in shaping me, into what I am today":

1. The first and the foremost was reading a review of the book on 'History of Hindu Mathematics'[2] by Datta and Singh in 1963. The charges that were levelled against the authors as being ignorant of historical matters and their theories being filled with grossest errors of fact made me curious as well as restless. So, I decided to take up research in history of mathematics myself.
2. Getting acquainted with the remarkable scholar T. A. Sarasvati Amma[3] in 1963–64, who was working at the same place (Ranchi) where I was employed, and subsequently completing Ph.D. under her guidance.[4]
3. Becoming a member of the International Commission on History of Mathematics, and the International Mathematical Union, and serving in these bodies for a fairly long period (1972–97), which helped me develop a better world view in this area.

The arduous task of engaging himself to do meticulous research in Indian mathematics, that RCG took upon himself in the late 1960s, is still shouldered by him, unabated in spirit (even at the advanced age of 80+ years). He has published more than 500 papers till now, excelling all his predecessors in the field, and is still pro-

[2] *Mathematical Review* 26, (1963) p. 1142.

[3] Whose outstanding doctoral dissertation got published as *Geometry in Ancient and Medieval India*, Motilal Banarsidass, Delhi, 1979.

[4] By submitting my dissertation on 'Trigonometry in Ancient and Medieval India'.

lific. In this connection, I am reminded of a beautiful verse in the *Nītiśatakam* of Bhartṛhari[5]:

> प्रारभ्यते न खलु विघ्नभयेन नीचैः प्रारभ्य विघ्नविहता विरमन्ति मध्याः।
> विघ्नैः पुनः पुनरपि प्रतिहन्यमानाः प्रारब्धमुत्तमगुणा न परित्यजन्ति॥
>
> No task is undertaken, out of the fear of obstacles, by the lowly people. Those of the middle class, do undertake a task, but give it up when faced with impediments. [However] men of the noblest calibre (*uttama-guṇa*) never abandon a task once they have undertaken it, in spite of being assailed repeatedly by difficulties.

3 Passion to Discover the Truth

During my second visit to meet RCG, in a particular context, there was a discussion on *Bhūtasaṅkhyā* system of representing numbers. Then I shared with him that for a long time I had not been in a position to figure out the rationale behind the use of the word नृप (lit. king) to refer to the number 16. Though RCG did not have an answer to that immediately, he said: "it should not be difficult to find that out".[6] Then as we both left for lunch, the discussion on the matter got suspended.

A couple of days later, after returning to IIT Bombay, when I called RCG to inform him about the travel arrangements that had been made for his forthcoming visit in November, the first news that he shared with me was that he had found the names of the 16 kings. Thanking him profusely for that, I told him that I would note down the names when I would meet him the next.

On my next visit, when I expressed my desire to note down the names, he swiftly walked to his library, pulled out a volume[7] and showed me where the 16 names appear. Needless to say I was overjoyed as I got the answer for the question that was nagging my mind for a long time.

Also as I flipped through the volume, (essentially a Compendium of technical terms in Astronomy and Astrology), I was amazed to notice marginal notes made almost in every page of the volume, which is truly encyclopedic in nature.

In addition to the rare collection of articles, books, dictionaries, encyclopedias, etc., his library contains huge collection of data cards on a variety of topics related to history of astronomy and mathematics. Figure 1 depicts a sample of the card that RCG had prepared (on September 29, 2015) soon after discovering the basis for the mnemonic नृप referring to the number sixteen.

The word 'Eureka' in the top left corner of the card clearly mirrors the joy that would have been experienced by RCG in discovering the truth.

---

[5]One of the greatest poets of all times, who through his immortal contributions enriched several branches of Sanskrit literature.

[6]I was a bit surprised at his cool and spontaneous positive response, because I was expecting him to second my thought by saying: "yes, these things are difficult to find out …".

[7]ज्योतिश्शब्दकोश by Mukunda Sarma, published by Mukundasram, Amola, Garhwal, Uttaranchal (formerly UP), 1967.

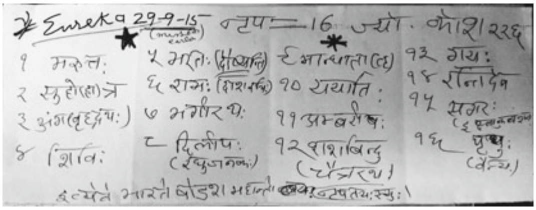

Fig. 1 Sample of data card

4 A Unique Collection of Data Cards

The amount of information that has been meticulously collected by RCG over the past five decades, and stored in the form of data cards, is simply astounding. In my estimate, the number of cards in his possession, all handwritten, would easily be of the order of a few tens of thousands, or it may even cross a lakh. RCG's style of making these cards, the mode of storing them, updating them, and more importantly easily tracing a card when required, was indeed a sight to watch. Some characteristic features about these cards that I would like to share are:

- All these cards are essentially of the same size (8 $\times$ 4 inches) mostly generated by cutting one-sided A4 sheets it into 3 parts.
- They are stored in the form of bundles wrapped in old postal envelopes (Fig. 2) and tied with a thin piece of old cloth, fashioned in the form of a ribbon.
- Each bundle is titled by topic (Āryabhaṭa, Bhāskara I, Bibliography, Transits, Misc., etc.), and the size of the bundle varies depending upon data collected on that topic. For instance, the cards related to the famous mathematician Bhāskarācārya were stored in 2–3 bundles each containing a few hundred cards.[8]
- As RCG keeps adding cards to these bundles (see Fig. 1, for the one added on 29–9–15), as and when he gathers more information, the bundles keep growing fatter.[9]
- The information in these data cards is not merely confined to providing details regarding source works, secondary works, journals, encyclopedia, etc., but also extend to a variety of other things related to other studies currently going at different places. For instance, when I met him the first time, and told him about the students who had completed Ph.D. with me, he immediately pulled out a piece of paper, noted down the names of the students, along with the topics of their dissertation, turned that piece of paper into a card and inserted it into a bundle marked '*Transit*'.

---

[8]The data collected by RCG seems to be quite comprehensive. Name any topic in history of mathematics, RCG may be able to pull out a bundle (piled up in cartons) in no time and present a card giving the necessary information!.

[9]Naturally, a few cards in these bundles would be decades-old and brittle, while others relatively new.

Fig. 2 Bundles of data cards

Out of curiosity, when I opened a bundle titled ‘Miscellaneous’, to my pleasant surprise, I found several oft-quoted verses jotted down in some of those cards, *with proper references*. Meticulously noting down the references is indeed a hallmark of RCG! So much so, this data card collection, in my view, forms the most valuable and unique feature of RCG’s library.

5 Frugal and Simple, Yet Quite Generous

Anyone who pays a visit to RCG’s residence in Jhansi, would quickly realize that he greatly values a frugal and simple life style. The choice of RCG to lead such a frugal life, is born out of certain mature understanding and appreciation of higher principles of life, and definitely not out of necessity.

There is a fine line between frugality and miserliness. Having said this, I must also add that though RCG leads a quite frugal and simple life, he has been extremely generous when it comes to serving a cause. With his meagre earnings as a mathematics teacher,[10] and having no additional source of income, RCG has donated substantial amounts of money to various institutions,[11] to establish endowments, to organize lectures in History of Mathematics and to create awareness among the masses. A remarkable gesture indeed!

It may not be out of context to remind ourselves of an interesting verse from *Vidura-nīti* (verse 53) in Mahābhārata:

द्वाविमौ पुरुषौ राजन् स्वर्गस्योपरि तिष्ठतः। प्रभुश्च क्षमया युक्तः दरिद्रश्च प्रदानवान् ॥

These two, O king, dwell (as it were) in regions higher than the heaven. viz., a powerful man endowed with forgiveness, and the poor man who is incredibly generous.

[10]RCG told me (the author of the article) that he started his teaching career as a lecturer with a salary of around Rs. 250 per month and retired as professor with a salary of around Rs. 9000 per month.

[11]To name a few: National Academy of Sciences (NASI), Association of Mathematics Teachers of India (AMTI), Astronomical Society of India, Calcutta Mathematical Society, Kerala Mathematical Association, Sree Sarada Educational Society.

6 Concluding Remarks

Most of us, not knowing how to effectively manage our time between work and family, hardly find any time to pursue other interests seriously and thereby end up retiring as exhausted individuals, having nothing much to look back and cherish in the life. But here is an individual who could find sufficient time to earnestly pursue his research in history of mathematics, and also bag the Kenneth O. May prize, the highest honour bestowed upon a historian, notwithstanding the fact that by profession he was a teacher of mainstream mathematics, which hardly had anything to do with history.

The secret behind his success got incidentally revealed during one of the private conversations with him at his place. In a particular context his wife, Savitri Devi (a very pious lady) remarked (in Hindi), "most of the time he is so engrossed in research that he hardly finds time to participate in social and religious functions". RCG quickly pitched in and said, not that I do not want to participate, but:

> मेरा तन मन धन सब `हिस्टोरि आफ़ मेथमेटिक्स' मे लगा हुआ है।
>
> My body, mind, as well as all the resources at my disposal are inexorably drawn towards history of mathematics.

From what has been previously narrated in the article, it should be quite evident that there is no figurative element or boast in the statement above. Prof. R. C. Gupta not merely worked in the area of history of mathematics, but seems to have revelled in it. Based on the close interactions that I had with him, I can confidently say that his passion to have 'truly' global view of history of mathematics is unparalleled, and his enthusiasm to further the cause even at his advanced age of 80 plus is indeed incredible!

A Birthday Tribute to R. C. Gupta

Christoph J. Scriba

The internationally renowned historian of mathematics, Radha Charan Gupta, celebrated his 60th birthday on October 26, 1995.[†] According to custom in his native India, this implies official retirement from his post as Professor of Mathematics at the Birla Institute of Technology (BIT) in Mesra, Ranchi. His many friends, as well as the many colleagues who have been in contact with him, realize, however, that "retirement" and "official retirement" are two different things; they look forward to benefiting from the continuing labours of this active and productive historian of mathematics towards the promotion of the discipline both in his native country and on an international level.

Born in Jhansi, Uttar Pradesh, Gupta graduated from Lucknow University in 1955. Two years later, he passed the M.Sc. examination (with a major in mathematics) in the first rank from the same university, and in 1965, he earned a diploma in mechanical engineering from the School of Careers in London. Ranchi University awarded him a Ph.D. for his research in the history of mathematics in 1971. His achievements were further acknowledged in 1986 with an honorary doctorate in the history of science from the World University (U.S.A.).

After teaching at Lucknow Christian College for a year, he joined the staff of the Birla Institute of Technology in 1958. He served there in the ranks of first assistant professor and then associate professor prior to his promotion to full professor of mathematics in 1982. Beginning in 1979, he was Professor-in-charge of the Research Center for the History of Science at BIT.

R. C. Gupta has travelled widely in India and abroad and has given presentations of his research before many audiences. In 1977, he addressed the British Society for the History of Mathematics at Cambridge and attended the XVth International Congress on the History of Science in Edinburgh. Three years later, he lectured in

Historia Mathematica, 23 (1996), pp. 117–120.

[†]The editor of the volume was informed by Prof. R. C. Gupta himself that somehow this wrong date has gone into records, and that he was actually born on Śrāvaṇa-pūrṇimā (श्रावण-पूर्णिमा) of the year 1935, which corresponds to August 14 (October 26 is as per H.S. Certificate).

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Germany, the USA, and Canada. He also participated in the XVIIth International Congress of Mathematicians in Berkeley, California, in 1985.

In his several hundred papers—among them a series of 16 popular articles, entitled “Glimpses of Ancient Indian Mathematics”—Gupta has always striven to deepen and broaden our knowledge and understanding of the development of mathematics on the Indian subcontinent. He is thus a successor to his compatriots, B. B. Datta (1888–1958) and A. N. Singh (1901–1954). Their important book, *History of Hindu Mathematics* (Lahore, 1935, 1938), although now some 60 years old, remains a standard reference work for those of us who are unable to read the native Indian languages. It is to be hoped that Gupta’s forthcoming book on Indian mathematics (in Hindi) will soon be translated for a wider readership. In the meantime, we can gain from his insights by reading the pages of *Gaṇita Bhāratī*, the official journal of the Indian Society for the History of Mathematics, which he founded in 1979 and which he continues to edit. As the cumulative index (in volume 13 (1991)) for volumes 1 through 12 of that journal reveals, Gupta has published many articles, notes, and reviews there under both his own name and the pseudonym “Ganitanand.” One recent article, “The Chronic Problem of Ancient Indian Chronology” (*Gaṇita Bhāratī* 12 (1990), 17–26), is characteristic of his scholarship. For reasons of space, the selected bibliography below is limited to some of the more extensive papers that Gupta has published in English.

R. C. Gupta’s scientific achievements have received acknowledgement in many ways during his fruitful and ongoing career. Most recently, he was elected Fellow of the National Academy of Sciences in India in 1991, President of the Association of Mathematics Teachers of India in 1994, and Corresponding Member of the International Academy of the History of Science in 1995. He has also represented India on the Executive Committee of the International Commission on the History of Mathematics for many years. An active sportsman, he has won numerous medals and prizes for his athletic prowess. May his health continue and enable him to pursue his researches in the history of mathematics for many years to come.

Selected Bibliography of Radha Charan Gupta by Takao Hayashi

Abbreviations used: *GB, Gaṇita Bhāratī* (Bulletin of the Indian Society for History of Mathematics); *HM*, *Historia Mathematica*; *HS*, *Historia Scientiarum* [Japan]; *IJHS*, *Indian Journal of History of Science; IS, Indological Studies (Journal of the Department of Sanskrit, University of Delhi); JAS, Journal of the Asiatic Society.*

1. Bhāskara I’s Approximation to Sine, *IJHS* 2 (1967), 121–136.
2. Second-Order Interpolation in Indian Mathematics up to the Fifteenth Century, *IJHS* 4 (1969), 86–98.
3. Fractional Parts of Āryabhaṭa’s Sines and Certain Rules Found in Govindasvāmin’s *Bhāṣya* on the *Mahābhāskarīya*, *IJHS* 6 (1971), 51–59.

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Professor R. C. Gupta Receives the Kenneth O. May Prize

Kim Plofker

Professor Radha Charan Gupta, who founded and nurtured *Gaṇita Bhāratī* as editor for over a quarter century, was awarded the Kenneth 0. May prize of 2009, jointly with Prof. Ivor Grattan—Guinness of UK—by the International Commission for the History of Mathematics (ICHM). The prize, which includes a bronze medal designed by the Canadian sculptor Salius Jaskus, was instituted in 1989 and is given every four years; it was named after the mathematician and historian Kenneth 0. May who founded the ICHM and its journal *Historia Mathematica* and is awarded in recognition of scholarly work in the history of mathematics. Dirk Struik (USA), Adolf P. Yushkevieh (Soviet Union), Christoph J. Scriba (Germany), Hans Wussing (Germany), René Taton (France), Ubiratan d'Ambrosio (Brazil), Lam Lay Yong (Singapore) and Henk Bos (The Netherlands) have been the earlier recipients of the award. As Prof. Gupta could not be present at the 23rd International Congress of History of Science and Technology held in Budapest, Hungary, in 2009, at which the original awards ceremony was held, the award was presented to him at the International Congress of Mathematicians, at Hyderabad on August 2010, at its closing ceremony on 27 August 2010.

Professor Gupta has been a scholar and researcher par excellence in the area of history of mathematics; the corpus of his works exceeds 500 items. Among his groundbreaking works are his analysis of Parameśvara's third-order series approximation for the sine function, in the fifteenth century ("An Indian form of third-order Taylor series approximation of the sine", *Historia Mathematica* 1 (1974), 287–289) and his examination of the eighth-century methods of Govindasvāmin for interpolating in sine tables ("Fractional parts of Āryabhaṭa's sines and certain rules found in Govindasvāmin's *Bhāṣya* on the *Mahābhāskarīya*", *Indian Journal of History of Science* 6 (1971), 51–59). Prof. Gupta's notable recent publications include "Historiography of Mathematics in India" (in *Writing the History of Mathematics: Its*

Gaṇita Bhāratī, Vol. 31, Nos. 1–2 (2009), pp. 115–118.

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Historical Development (Basel, 2002), pp. 307–315 (Chap. 18) edited by J. Dauben and C. J. Scriba), "Area of a bow-figure in India" (in *Studies in the History of the Exact Sciences in Honour of David Pingree*, Leiden, 2004, pp. 517–532 edited by Burnett et al.) and "A little-known nineteenth century study of *Gaṇita-sāra-saṇgraha*", *Arhat Vacana* 14, 2–3 (2002), 101–102.

Radha Charan Gupta was born on 26 October 1935 in Jhansi, Uttar Pradesh, in north India. He received his B.Sc. and M.Sc. degrees from Lucknow University in 1955 and 1957, respectively, earning a first-place medal in the M.Sc. mathematics examination. He received his Ph.D. in the history of mathematics from Ranchi University in 1971, for his dissertation work carried out with T. A. Sarasvati Amma, the renowned historian of Indian mathematics and author of the book *Geometry in Ancient and Medieval India* (Delhi, 1979), in honour of whom Prof. Gupta later endowed the annual Memorial Lecture of the Kerala Mathematical Association. Prof. Gupta's career as a teacher began in 1957 at Lucknow Christian College where he served as Lecturer during 1957–58, following his M.Sc. In 1958, he entered the faculty in mathematics of the Birla Institute of Technology (BIT), Ranchi, where he spent the rest of his regular career. In 1979, he was appointed as Head of the Research Center for the History of Science at BIT. He was made full professor in 1982 and emeritus professor in 1995, following the mandatory retirement at the age of 60 years. He currently continues to be active in research and other academic activities under the aegis of the Gaṇita Bhāratī Institute, from his home at his native city of Jhansi.

Prof. Gupta's research work, which started in the late 1960s, has focused on ancient Indian mathematics, particularly the development of trigonometry, including interpolation rules and infinite series for trigonometric functions. Besides skilfully analysing many hitherto unknown mathematical formulas expressed in elliptical Sanskrit verses, Prof. Gupta has published several key papers on the remarkable mathematical discoveries of the Jaina tradition; this has been a yeoman service especially in the case of the many works that have been almost inaccessible to anyone not closely linked with the Jaina canon. Prof. Gupta has adopted this "bridge-building" approach in many other respects as well: explaining Sanskrit algorithms for a modern mathematical audience, surveying twentieth-century Indian doctoral research on history of mathematics, tracing the influence of Indian mathematical discoveries in foreign traditions, and expounding Jaina, Buddhist or Hindu cosmological theories in the context of early Indian work with transfinite quantities. He has combined scrupulous textual scholarship and expert mathematical commentary with clear and comprehensible exposition, serving the needs of general audiences and specialist researchers alike. It may be worthwhile to quote here the following lines from the citation read out on the occasion of the awards ceremony of the Kenneth 0. May prize, at the Budapest function in 2009, about Professor Gupta: "No scholar in the twentieth century has done more to advance widespread understanding of the development of Indian mathematics—and that, in a century that spanned (most of) the working lifetimes of researchers such as S. Dvivedi, B. Datta and A. N. Singh, K. S. Shukla, A. K. Bag, Sarasvati Amma, David Pingree and K. V. Sarma, is saying something."

Apart from carrying out excellent research and dedicated teaching service, Prof. Gupta has contributed in numerous other ways towards enhancing the awareness of the history of mathematics in general and of ancient Indian mathematics in particular. In 1979, he started *Gaṇita Bhāratī* and fostered it as editor for over a quarter century, until 2005. A large number of high-quality articles appeared in the journal over the years of his editorship. He himself contributed regularly articles and reviews to the journal, some under his own name and others with the pen name "Ganitanand" ("joy of mathematics"). His pedagogical publications and lectures, in English and Hindi, as well as his sponsorship of numerous endowed lectures have greatly increased the prominence of history of mathematics in Indian mathematics education and scholarship.

His endeavours received recognition in various ways. In 1991, he was elected Fellow of the National Academy of Sciences, India, and in 1994, he became President of the Association of Mathematics Teachers of India, a position which he continues to hold. In February 1995, he was elected as Corresponding Member of the International Academy of History of Science. In May 2002, he was elected Effective Member of the same Academy (No. E305).

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Part II

On Studies in History of Mathematics

On the Date of Śrīdhara

H. T. Colebrooke seems to be the first modern scholar who studied the work of Śrīdhara. He had an incomplete copy of *Pāṭīgaṇitasāra* from which he often quoted parallel passages in his translation (1817) of the *Līlāvatī*. But he did not take any risk of dating Śrīdhara in a narrower range than the very safe span from Āryabhaṭa I (b. 476 AD) to Bhāskara II (c. 1150). Mabel Duff (1890) gave astronomer Śrīdhara's date as 691 AD. In spite of G. R. Kaye's refutation, this date may not be far from Śrīdhara's birth-year. Sudhakara Dvivedi (1892) dated his work at about 991 AD on the false identification of the mathematician Śrīdhara with the author of *Nyāyakandalī* (Śaka 913). Kaye (1912–13) committed a double mistake of accepting this wrong identification and further regarding the date as birth-year, thereby pulling the date of *Triśatikā* to about 1020!

S. B. Dikshit (1896) had found a reference to Śrīdhara by name in an old manuscript of Mahāvīra's *Gaṇita-sāra-saṅgraha* (c. 850) and so put the former before the latter. The modern editions of the *GSS* do contain the said quoted rule but not as quotation from Śrīdhara or others. On the other hand the said rule is not found in or reported from any available portions of Śrīdhara's extant works. However, Dvivedi seems to have partially accepted Dikshit's observation when he stated (1899) that the rule might be from the lost algebra of Śrīdhara. Royal Asiatic Society, Bombay Ms. No. 230 of *GSS* also ends with the words *(ABORI,* Vol. 31, p. 268)

क्रमादित्युक्तं श्रीधराचार्यैरिति भद्रं भूयात् ।......

The similarity of several rules and of many other features between the works of Śrīdhara and Mahāvīra is accepted by scholars. Both may have drawn from a third and common source which is not known nor likely to be known. But most of the scholars considered Mahāvīra as borrower (he himself named his work as a

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Gaṇita Bhāratī, Vol. 9, Nos. 1–4 (1987), pp. 54–56. Also, see *Gaṇita Bhāratī*, Vol. 25 (2003), pp. 146–149.

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“Collection”). Thus placing somewhat midway between Brahmagupta (628 AD) and Mahāvīra, Śrīdhara’s date was accepted as circa 750 AD by B. Datta and A. N. Singh in the famous *History of Hindu Mathematics* (1935–38). This date or at least placement before Mahāvīra was also accepted by B. Mishra (1946), N. C. Jain (1947), S. Singh (1950), etc. However, the possibility of dating Śrīdhara before Brahmagupta as suggested sometimes, e.g. by S. Singh, is to be ruled out because Brahmagupta’s line

स्थूलफलं त्रिचतुर्भुजबाहुप्रतिबाहुयोगदलघातः। (*BSS*, XII, 21)

is quoted in Śrīdhara’s *Pāṭīgaṇita*, rule 112 (but without naming).

The practice of placing Śrīdhara in the eighth century was upset by K. S. Shukla’s emphasis on regarding Śrīdhara to be posterior to Mahāvīra. His arguments are mostly based on a comparative study of works of these two ancient mathematicians (see *Pāṭīgaṇita*, Introduction, pp. xxxix–xliii) and have some flaws. For instance, absence of certain rules of Śrīdhara in Mahāvīra (p. xl), may be deliberate, just as the former himself omitted the earlier rule of eleven (p. xv), rather than due to anteriority of the latter. Similarly, concise statements of certain rules in Śrīdhara may be true, namely Mahāvīra elaborated Śrīdhara’s brief rules. In fact, the several and varietyful arguments given by Datta (1932), S. Singh and others for Mahāvīra being a borrower from Śrīdhara are more forceful and convincing than those of Shukla and U. Asthana. Still, due to established authority of Shukla in the field, scholars quoted and used his assigned date circa 850 to 950 for Śrīdhara.

Luckily the priority of Śrīdhara over Mahāvīra is now shown in a different manner (independent of the mutual relation between them) by D. Pingree’s discovery of a genuine quotation by Govindasvāmin (c. 800–850) of a *sūtra* from Śrīdhara’s *Triśatikā*. Earlier, Pingree had dated Śrīdhara between 850 and 1050 (*CESS*, series A, Vol. I, 1970, p. 53) or in nineth century (*DSB*, Vol. 12, 1975, p. 597).

It may be pointed out that Shukla, following earlier scholars, dated Āryabhaṭa II at circa 950 AD, while Pingree put him between 950 and 1100 in *CESS* (A), I, p. 53. However, the date of Āryabhaṭa II’s *Mahāsiddhānta* has been now pushed forward to about 1500 due to researches of Roger Billard and others. After all, history of mathematics is not a dead subject. Another interesting possibility may be pointed out. Shukla, in the meantime, also sees “reasons to suspect that Govindasvāmin was either anterior to or senior contemporary of Haridatta” (683 AD) (see Shukla’s edition of *Āryabhaṭīya* with commentary of Bhāskara I etc., Delhi, 1976, (p. lxxxviii). If this comes to be true, it will bring Śrīdhara very close to Brahmagupta! To me this possibility seems to be remote and we may stay safely by placing Śrīdhara in the eighth century, be it beginning, middle or end.

The date circa 799 AD was assigned to Śrīdhara by N. C. Jain by equating him to the Jaina author of *Jyotirjñānavidhi* (799). And to reconcile salutations ‘Sivam’ and ‘Jinam’—of the different manuscripts—it has been suggested that the same Śrīdhara, after writing mathematical works, may have turned a Jaina towards the end of his life. Anyway, there seems to be a lot of scope and need to carry out vigorous research on the various aspects of Śrīdhara. Ph.D. theses of S. Singh and U. Asthana need to be examined, revised and augmented (a third thesis on Śrīdhara was abandoned partly because the candidate’s findings disproved those which the supervisor made).

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In the Name of Vedic Mathematics

For quite some time and especially in recent days there is much talk about "Vedic Mathematics" among the mathematicians as well as among others. People hear about lot of activities of various types which are going on in India and abroad in the field of Vedic Mathematics. From school children to university professors and from scholars to government officials all are involved in programmes ranging from elementary education to higher research in Vedic Mathematics. Its effectiveness in propagating the grandeur of ancient India's glorious past and its utility in playing significant role in modern computer-oriented fast mathematical calculations are being much talked in various circles and mass media. Its name is working miracles. At the same time there is lot of confusion among historians, indologists, and other scholars about it. Excitement is predominating over real understanding of the matter. Emotional feelings are overtaking rationality needed to know the correct situation involved in the issue. Logic and cool thinking is necessary for assessment of historical significance, scientific value, educational utility, and genuineness of other claims.

In this note we shall describe some clearly distinct categories of activities which have been going on in the name of "Vedic Mathematics".

1 The Real Ancient Vedic Mathematics

In this category comes the genuine Vedic Mathematics as found in the *Vedas* or Vedic literature in general. The four *Vedas* consisting of various Vedic *Saṃhitās*, the several *Brāhmaṇas, Āraṇyakas* and *Upaniṣads* are the basic literary sources for this type of Vedic Mathematics. The original works on the six *Vedāṅgas* form the rich primary sources for research in Vedic science including mathematics. Some other works such as the *Prātiśākhyas*, and available ancient original Upvedic texts are also very much part of the ancient Vedic *vāṅmaya* in general sense. The mathematics of the Vedic Age as reconstructed by using logical-mathematical implications, historical deductions,

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justified empirical generalizations, and other historiographical techniques will also come under this category. The mathematical knowledge of the Vedic Indians or of Vedic Āryans of antiquity (whether settled or on the move) as gleaned and gathered by corroborative research using literary, historical, epigraphical, archaeological, and other sources (whether Indian or foreign) is also to be included.

One of the best studies of this true category of Vedic Mathematics is *The Science of Śulba* by Bibhutibhusan Datta (1888–1958) published by the Calcutta University in 1932. The author has correctly stated in the preface that the book is more specifically concerned with "Vedic Science of Geometry" which mostly grew and developed for and from the construction of Vedic altars needed to perform Vedic sacrifices and other rituals. Datta's article on "Vedic Mathematics" appeared 5 years later in *The Cultural Heritage of India,* Vol. III (Calcutta, 1937).

For the last 30 years A. Seidenberg has been making serious study of Vedic Mathematics as part of his general theory of ritual origin of mathematics. His paper on "The origin of mathematics" (*Archive Hist. Exact Sciences*, 18, 1978, 301–342) includes a special Appendix on "Vedic Mathematics". Another of his paper on "The Geometry of the Vedic Rituals" appeared recently in the book *Agni* edited by F. Staal (Delhi 1984).

The contents of researches and findings in this category of "Vedic Mathematics" have great historical significance. They provide guiding material for studying and knowing science and culture of India some 2000 to 5000 years ago.

2 The Fantastic Vedic Mathematics

In this category we place those mathematical achievements which are claimed to be found in the *Vedas,* but actually the claims are either baseless or grossly exaggerated. The achievements, especially of modern mathematics, are part of more general claims of all types of modern and advanced science which is "already found in our *Vedas*" as the fanatics put it. In fact protagonists of this type easily assert that

भूतं भव्यं भविष्यं[†]च सर्वं वेदात् प्रसिद्ध्यति ।

All[†] that was, is, and will be (in future) can be derived from the *Veda*.

Accordingly, they believe that all modern discoveries and inventions of science (including mathematics) are hidden secretly in the *Vedas*. It is a common experience that as soon as a scientific discovery is made known, some one would come out with the claim that it was already known to the Vedic sages.

To give a sample we take an example. It is well known that the sequence of odd numbers, 1, 3, 5, ... up to 33, is mentioned in the *Yajurveda* (XVIII, 24). A scholar considered these as the first differences of the square numbers 0, 1, 4, 9, ... although

[†]The reading in the original published article was: "भवद् भविष्यच्च". It has been changed as above in consultation with the original source work (मनुस्मृति, XII. 97) –*ed*. Also see p. 36.

no such thing is mentioned in the text. But then he quotes the following modern formulas

$$y = \log x; \qquad \frac{dy}{dx} = x^{-1}; \qquad \frac{d^2y}{dx^2} = -x^{-2}, \text{ etc.}$$

and (by a sort of false analogy) jumps to conclude that "From this it is clear that Indians had already known differential calculus through the *Veda*" (See *Siddhānta-śiromaṇi, Grahagaṇita*, ed. by K. D. Joshi, Part III, Varanasi, 1964; *Prākkathana*, p. 8).

In another work (*Some Positive Sciences in the Vedas*, 1961) it is claimed that area of a triangle is dealt in the *Atharvaveda* (VIII, 9, 2) and surface of a cylinder in the *Ṛgveda* (I, 105, 17). Also without giving any reference a large number of *bhūtasaṃkhyās* are stated to be used by Vedic people.

Examples of fantastic claims can be easily multiplied. In fact western historians generally forewarn the reader that "there are a number of books in which the contributions from India are grossly overrated" (C. B. Boyer, *Hist. of Math.,* 1968; p. 229).

It is said that during the period of struggle for independence, the motivation of inflating the scientific achievements of the *Vedas* to somehow boost up the morale of the people might be understood. But the tendency continues and certain type of scholars want to make the public believe that the *Vedas* contain the whole of science, even, "all the knowledge needed by mankind" (p. xiii of Tirthaji's book mentioned below). Let the readers go through the issues of the journal *Vaidika-Gaṇita* published from Sonepat, Haryana (1st issue dated 1986) to know more about similar claims.

It must be noted, however, that by above criticism, we are not denying that *Vedas*, especially seeing their antiquity, are rich source for knowledge and wisdom.

3 Tirthaji's Modern System of Vedic Mathematics

Jagadguru Śaṅkarācārya Swami Bharati Krishna Tirthaji's book entitled *"Vedic Mathematics"* or '*Sixteen Simple Mathematical Formulae from the Vedas*' *(For One-line Answers to All Mathematical Problems*) was first published in 1965. It has created a new category of "Vedic Mathematics."

The 16 *Sūtras* (aphorisms) along with 13 *sub-sūtras* given in the book provide many algorithmic devices which enable us in a simple way to make fast calculations related to certain problems of elementary mathematics mostly arithmetic. The techniques help in rapid computations and in this respect the book has great educational and research value. The author has contributed a very significant and original mathematical system in a general way. This merit, the pontifical authority of the author, and the sensational title have made the circulation of the book quite wide. The system has become popular.

The objectionable things about the book or system are the name "Vedic Mathematics" given to it and the claim that the 16 formulas are from the *Vedas*. Both are deceptive and false and are responsible for creating lot of confusion and misunderstanding. The book or the *sūtras* have nothing to do with the *Vedas*.

The *sūtras* (or formulas) were composed by the author himself who lived from 1884 to 1960. Hence, as mildly stated by Manjula Trivedi, a disciple of the author, "these formulae are not to be found in the present recensions of *Atharvaveda;* they were actually reconstructed [by the author] on the basis of intuitive revelation from materials scattered here and there in the *Atharvaveda*" (p. x).

Only thing is that the author composed the 16 aphorisms in Sanskrit *sūtra* style, and put a stamp "From the Vedas" on them. But the language is clearly modern and not Vedic Sanskrit. Hence the author's claim that they are "contained in the *Pariśiṣṭa* (the Appendix portion) of the *Atharvaveda*" (p. xv) can be tolerated only by regarding them,† following a suggestion of V. S. Agrawal (see p. 6 of his Foreword) as a new *Pariśiṣṭa* added according to the tradition of formulating subsidiary apocryphal texts. Thus we find that the title "Vedic Mathematics" of the book or formulas have no more worth than that of a fiction. An expert Sanskrit scholar even says that "to glorify the *Vedas* and Hindu culture by these false claims will only create revulsion of feeling when the truth is known" (*Vishveshvaranand Indological Jour.,* Vol. IV, 1966, p. 109).

However, by above criticism we are not undermining the mathematical excellence of the book or its intrinsic importance. As A. P. Nicholas puts it, Tirthaji's system is "one of the most delightful chapters of the twentieth century mathematical history" (*Gaṇita Bhāratī*, Vol. 6, p. 37). Its novelty has inspired several scholars to delve into the new area of research created by it and found more significant results through its approach. Its popularity is increasing both in India and outside. Lot of activities are taking place. Literature in the field has been growing. There is even an international journal called *Vaidic Gaṇit* (Vol. I dated Nagpur, 1985).

†For obvious reasons, the original reading: "regarded them", has been changed as above. *–ed.*

Foreign Reviews and Evaluation of Indian Works on History of Science

Although studies and research in the field of history of exact sciences in India have been going on for the last two centuries, no comprehensive and authentic chronological history of Indian mathematics has been written so far.[1] Whatever books and monographs are available at present, they are not only inadequate but far from being satisfactory. Their coverage is not up-to-date and the treatment is poor. There is an urgent need to write a history of mathematics in India in the true sense of the word, and this is a national task.[2]

Although assessment of any work might be considered a matter of personal opinion of the reviewer, and thus there may be some bias, but there are some international norms and standard practices by which works are judged by professional experts, and we cannot altogether shut our eyes to the consistent evaluation and opinions of foreign scholars and reviewers.

Here we bring to notice of all concerned some serious shortcomings in Indian publications so that due attention be paid to rectify them or justify them by scholarly counter-refutations. It is hoped that things will be taken sportively as a *śāstrārtha*.

Forewarning the reader, Boyer in his *History of Mathematics* states[3] that "there are a number of books in which the contributions from India are grossly overrated". He mentions B. K. Sarkar's *Hindu Achievements in Exact Sciences* (1918) as one such book.

The History of Hindu Mathematics by B. Datta and A. N. Singh (Parts I and II Lahore, 1935 and 1938) is considered to be the most standard book on the subject. But even this is said to have unreliable features.[4] While reviewing the single volume edition of the work (Bombay, 1962), G. J. Toomer[5] of Oxford charged the Indian authors to be ignorant of historical matters, prejudiced against admitting that there was any influence on Indian civilization from outside, and to rest their theories often on the "grossest errors of facts".

Gaṇita Bhāratī, Vol. 11, Nos. 1–4 (1989), pp. 50–54.

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L. V. Gurjar's *Ancient Indian Mathematics and Vedha* (Poona, 1947) is pronounced as a "miserable book" by Archibald.[6] E. B. Allen in his review[7] of the book noted many deviations of the texts cited and lack of references for many quotations. Another phase of its "worthlessness" is brought out elsewhere.[8] Gurjar has completely misinterpreted Brahmagupta's rule for the volume of frustum-like solids.[9]

J. Filliozat wrote in the preface to his monograph on Indian medicine (Paris, 1949) that[10] "Indian scholars, moved by national pride, are prone to maintain that their sciences in high antiquity surpassed even those of today". Joseph Needham,[11] while reviewing the papers presented at the Symposium on History of Science and Technology in India and South Asia (held under INSA, etc., in 1950), said that "the papers too marked a chauvinistic tendency, an effort to minimize foreign influences on Indian science and emphasize all outward transmissions."

Jaggi's work on history of science and technology in India[12] has been reviewed by David Pingree in *Isis* (Vol. 61, Part 3, 1970, 407–408), the standard international journal of professionals in the field. But the review is very unfavourable. The Indian author has been criticized to have an "uncritical attitude towards his sources", and to depend on frequently the "worst translations" for primary sources. Parts of the work are said to be written at "primitive level" and bibliographies are described as "grossly insufficient and out-of-date" for serious students. The reviewer goes to the extent of saying that the author does "not have the competence" to write good work in the field. The volumes are considered unfit to be recommended.

The reviewer took the opportunity to blame other Indian scholars also by stating that the author "shares with many of his countrymen an unfortunate tendency to read ancient Sanskrit texts as if their authors were striving, somewhat obscurely, to express precisely what one now finds in a text on physics, astronomy, or chemistry."

Before pronouncing the foreign reviewer as biased, the scholars will do well if they read the review of Jaggi's *Scientists of Ancient India* (Delhi, 1966) because this review is by two Indian reviewers who have exposed lot of weak points of the author and his work.[13]

The voluminous Concise *History of Science in India* published by the Indian National Science Academy (New Delhi, 1971) is often described by Indians as a prestigious publication. It was reviewed by W. A. Blanpied in the *J. for Hist. of Astronomy,* Vol. 6 (1975), 135–137. According to the reviewer, the volume has a number of deficiencies. Several of the authors are considered "to have scant expertise in historical methodology or familiarity with history of science outside India". Blanpied states that "the book's endeavour to demonstrate a one-to-one correspondence between classical Indian and modern-Western disciplinary divisions has led to a number of serious distortions." He adds that several of the contributing authors "have been led into attempts of dubious validity to demonstrate the independent, parallel development of western scientific concepts while ignoring several indigenous concepts and techniques that had no western analogue".

However, the contributions of one author (S. N. Sen) are considered by the reviewer to be free from the "myopia and frequent defensiveness" of the other chapters. But even Sen (like others) is said to focus on North Indian Science (e.g. the

Tamil tradition is not even mentioned). Of course, the treatment given to the history of mathematics has similar limitations and many other weaknesses.

The Science and Technology in Medieval India—A Bibliography of Source Materials in Sanskrit, Arabic and Persian compiled by A. Rahman and others is another massive volume published by INSA (New Delhi, 1982). However, it is said to suffer from serious defects both in conception and in execution in a recent review.[14] The reviewer, who is expert in Sanskrit, Arabic and Persian, and a specialist in bibliographic research, judges that the compilers were neither familiar with relevant literature nor particularly careful in their scholarship. According to him the compilers failed "to consult standard bibliographical and biographical reference works." The reviewer has pointed out several other defects such as "listing of non-existing authors and works", imposition of the inherently absurd western classification of sciences on the Indian *śāstras*, etc.

The History of Astronomy in India recently published by INSA (New Delhi, 1985) does not fare better. It has almost all same and similar defects as pointed out for above two works. Most of the authors are unaware of the modern work done on the history of both Indian and non-Indian science. In a personal correspondence with the editor of a journal, a foreign scholar wrote that he found the book to be filled with errors and confusions; indeed, some of the contributions are thoroughly disgraceful. Apart from Sen and Ansari, no contributor took any notice of Billard's new methodology and numerous other achievements which are fundamental and were published more than a decade earlier.

In fact the book is merely a collection of isolated articles on selected topics and not a history of astronomy in India in any sense. The bibliography does not mention the most comprehensive recent work, the *History of Ancient Mathematical Astronomy* by Otto Neugebauer (1975). No doubt the 12 page errata lists hundreds of printing errors but this represents only a fraction of the actual errors and omissions.

More instances of the poor quality work on history of exact sciences published in India can be cited. Lack of international perspective, and of awareness about current research work in the field is a general defect in the work of most Indians which is published here. For fantastic claims about Indian achievements one may refer to some recent works.[15]

Indians must give a serious thought to the point as to why, in spite of so much expenditure in the research and publication in the field, the situation is deplorable. But whatever be that, no one will disagree with a recent historian,[16] that the history of Indian mathematics, "still awaits a more reliable and scholarly treatment."

References and Notes

1. Among the early studies may be cited the works of (i) R. Burrow, "A Proof that the Hindoos had the Binomial Theorem", *Asiatick Researches*, II (1790), 487–497; (ii) S. Davies, "On the Astronomical Computations of the Hindus", *Ibid*., 225–287; and (iii) J. Playfair, "Observations on the Trigonometrical Tables of the Brahmins", *Transactions of the Royal Society of*

Edinburgh, IV (1978), 83–106. Among Indian scholars of nineteenth century we may mention (iv) Bapudeva Sastri, "Bhāskara's Knowledge of the Differential Calculus", *J. Asiatic Soc. of Bengal*, 27, (1858), 213–216; (v) Bhau Daji, "On Ancient Sanskrit Numerals", same journal, 32 (1863), 161–167; and (vi) Bhagvanlal Indraji, "On Ancient Nagari Numeration from an Inscription at Naneghat", *J. Bombay Branch of Royal Asiatic Soc.*, 12 (1876), 404–406.
2. For a general perspective on this need, see R. C. Gupta, "Swami Vidyaranya and the History and Historiography of Mathematics in India", *The Mathematics Student*, 55 (1987), 117–122.
3. Carl B. Boyer, *History of Mathematics*, Wiley, New York, 1968, p. 229.
4. See R. C. Archibald in *American Math. Monthly*, 56 (1949), p. 70. The unreliable features are set forth by Otto Neugebauer while reviewing the first part of the work in *Quellen and Studun*, Vol. 3B, 1936, pp. 263–271.
5. See *The Mathematical Reviews*, Vol. 26, 1963, p. 1142.
6. *Amer. Math. Monthly*, 56 (1949), p. 70.
7. *Math. Reviews*, Vol. 9, No. 2 (Feb. 1948), p. 73.
8. See Archibald, *Amer. Math. Monthly*, Vol. 56, p. 70, and *Nature*, Vol. 161 (April 17, 1948), p. 580.
9. See Gurjar's book, pp. 88–89. For correct interpretation, see R. C. Gupta, Brahmagupta's Rule for the Volume of Frustum-like Solids, *The Math. Education*, VI (1972), Sec. B, 117–120.
10. As quoted by Joseph Needham in *Nature*, Vol. 168 (July 14, 1951), pp. 64 ff.
11. Needham, *Ibid*.
12. O. P. Jaggi, *History of Science and Technology in India*, Vols. I, II; Atma Ram, Delhi, 1969.
13. The review, by H. N. Gupta and A. K. Bag, is published in the *Indian J. Hist. Science*, 2 (1967), 61–62.
14. *Gaṇita Bhāratī*, Vol. 7 (1985), 44–46.
15. (a) D. Chattopadhyaya, *History of Science and Technology in India: The Beginnings*. KLM, Calcutta, 1986; pp. 398–403. (b) *Ganitanand*, "In the Name of Vedic Mathematics", *Gaṇita Bhāratī*, 10 (1988), 75–78.
16. Howard Eves, *Introduction to History of Mathematics*, New York, 1969; p. 182. Quotation is valid for 1989 also.

The Study of History of Mathematical Sciences in India

1 Introduction

Part I (1935) and Part II (1938) of the famous *History of Hindu Mathematics* by B. B. Datta and A. N. Singh were published from Lahore (now in Pakistan), while the promised Part III was never published in its authors' lifetime. In 1962 a single volume edition (actually a mere reprint of Parts I and II) was brought out by the Asia Publishing House, Mumbai. This was reviewed by G. J. Toomer of Oxford (*Math. Reviews*, Vol. 26, 1963, p. 1142). He charged the authors of being ignorant of "historical matters" and of having "prejudice against admitting that there was any influence on Indian civilization from outside". Their theories have been said to be often based on the "grossest errors of fact".

Such remarks against a work which was (and still) taken to be a standard book on the subject arose curiosity in my mind and I decided to study the history of mathematics seriously. In 1964 Dr. T. A. Sarasvati Amma completed her doctoral work under the great Sanskrit scholar Dr. V. Raghavan. Her thesis was on *Geometry in Ancient and Medieval India* (published, Delhi, 1979). She supervised my doctoral thesis on Trigonometry in Ancient and Medieval India (Ranchi Univ. 1970/71, unpublished). In 1972, I was appointed India's representative on the International Commission on History of Mathematics which started the *Historia Mathematica* journal in 1974.

As a significant event, India's first artificial satellite was launched in 1975 from a cosmodrome in Russia. It was named ARYABHATA after the great Indian revolutionary scientist Āryabhaṭa I (born AD 476) whose 1500th birth anniversary was also celebrated throughout the country during 1976. Due to these events, a great interest was rekindled amongst the Indian scientists to study history of science.

On the initiative of late Prof. U. N. Singh, the Indian Society for History of Mathematics was formed in 1976 (*HM*, Vol. 5, 465–466). Soon the Society accepted

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Gaṇita Bhāratī, Vol. 23, Nos. 1–4 (2001), pp. 1–11; Expanded version of a talk given during the NISTADS-INSA Workshop on History of Science Research in India. New Delhi, 2000.

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the idea of starting its Bulletin and appointed me as the editor. It was named *Gaṇita Bhāratī* and the first volume (2 issues) was printed and distributed from Ranchi in 1979. Since then the *Gaṇita Bhāratī* has been appearing continuously and, in fact, serves as an international journal in the field of history of mathematics with particular reference to India. It has a special column (a teamwork) on "Notices of Select Current Publications" as a service to scholars. It is a refereed journal. I try to associate and involve a large number of scholars with *GB* and follow an open policy.

My contact and correspondence with scholars in India and abroad (both as the editor of *GB* and a member of the ICHM) gave me ample opportunities to gain experience and enrich my knowledge in the field of History of Mathematics. This in turn, also made the *GB* richer in information and varietyful articles which successfully played the double role of making foreigners more familiar with mathematical sciences of ancient and medieval India and in making Indian scholars familiar with the work, publications, and other activities going on outside India to some extent.

In fact the output of studies and researches during the last 40 years have grown so much so to provide enough material for writing a national *History of Mathematics in India* easily in 10 or 12 volumes (similar to the famous *History of Hindu Mathematics* or, better, in chronological order). Of course the Indus Valley achievements are still quite uncertain because Indus scripts have not yet been fully deciphered.

Knowledge in any field may be compared with an ocean. Even small articles on some topics have been expanded to give more historical information and deeper analysis. Moreover, new findings and features have been brought to light. Examples are the use of $\pi = \frac{25}{8}$ in ancient Indian (*Mānava Śulba-sūtra*), 4th order pandiagonal magic square in Varāhamihira's *Bṛhat-saṃhitā* (sixth century AD), secant method in medieval India, and a number of new researches in the Jaina School of Indian Mathematics (both in *Laukika* and *Pāralaukika-gaṇita*). The history of Indian mathematics is greatly enriched by recent doctoral theses of foreign scholars such as Takao Hayashi, Yukio Ohashi, Kim Plofker, and Agatha Keller. Lot of bio-bibliographical material has been also brought to surface in *GB* issues.

It seems History of Mathematics is not a dead but dynamic subject. Before 1930, it was mostly just the glory of the Greeks. After that the picture changed by findings in Babylonian mathematics. Further dimensions have been added to ancient period by the researches in prehistoric and megalithic times and by theories of ritual origin of sciences. Now medieval period is being enriched by studies and publications of Arabic mathematics. However, it is better to wait rather than give final and immature judgements in a hurry. Let us pool material; building may be erected later on. It is wise to look before leap.

2 Ancient Indian Chronology

Chronology is the backbone of history. For ancient India, the problem of a sound chronology continues to be a serious matter. The most important point to keep in mind in this regard is that the date of a historical event and that of a work which records that event may be widely apart. For example the dates of the astronomical events found recorded or described in the *Vedāṅga-jyotiṣa* (= *VJ*) may be quite earlier than the date of composition of the work. There are also necessary questions of internal consistency and collaboration from other sources. Without these considerations, there will be no coherency as is clear from the following so many divergent dates suggested for the *VJ* (for reference see present author's paper in *GB*, Vol. 12, 1990, pp. 19–20):

(i) V. R. Lele, 26000 BC (as quoted by S. B. Dikshit).
(ii) B. N. Narahari Achar pleads for 1800 BC (*see Indian J. Hist. Sci.* 35, 2000, 173–195.)
(iii) H. T. Colebrooke, P. C. Sengupta, and K. S. Shukla, about 1400 BC
(iv) T. S. Kuppanna Sastry and Gorakh Prasad, 1200 BC (This date is quite popular).
(v) Ramatosh Sarkar, 600 BC and Chronology Committee of 1950, (see below), 500 BC
(vi) S. N. Sen, 400 BC (Also Pingree for *Ṛg* version).
(vii) Maxmuller 300 BC; A. K. Bag, 200 BC
(viii) A. Weber, c. 400 AD (Also Pingree for the *Yajur* recension).

When *VJ* is dated so differently, the divergence for other Vedic Corpus (*Saṃhitās, Brāhmaṇas,* etc.) is bound to be wider. Even the so-called "astronomical methods" also yield different results because of the different identification of stars, different parameters used, and also due to cyclic nature of some events.

Another difficulty is created by assigning divine origin to almost all ancient Indian sciences. For instance, it is stated in the *Nārada-purāṇa* (Chap. 54) that *Jyotiṣa* was enunciated by Brahmā in antiquity, and *Garuḍa-purāṇa* (59.1) says that *Jyotiṣa* science of 4 lakh stanzas was communicated to god Rudra by god Keśava (see *GB*, 12, p. 18). *Yantras* (including magic squares) are said to be first taught by lord Śiva, and the game of the chess by lord Kṛṣṇa (to Rādhā). It is said that the attribution of the divine origin to works or subjects was due to philosophical attitude (of being indifferent to worldly matters) of the Indian mind. But often the practice is for securing importance and antiquity for one's own works and findings. Anyway, in the absence of the knowledge of the real author or actual date of a work, we cannot know the true history of science.

An important point to remember in this connection is that many of the ancient works (especially epics and *purāṇas*) are of composite nature; i.e., different portions were composed by different writers of different times.

The difficulty of early chronology in India was realized as early as in 1950 when the first noted symposium on History of Sciences in South Asia was organized by INSA at Delhi. At that time a Chronological Committee of senior historians of science was formed. It recommended the following working chronology:

Age of *Ṛgveda*	2000 BC–1500 BC
Age of *Saṃhitās* and *Brāhmaṇas*	1500 BC–800 BC
Dharmasūtras	600 BC–200 BC
Vedāṅga-jyotiṣa (Present text)	500 BC
Śulbasūtras	500 BC and later.
Mahābhārata and *Rāmāyaṇa*	200 BC–200 AD

For some sort of uniformity, scholars should adhere to above dates. Of course, the whole matter should also be reviewed by a fresh committee to be formed jointly by INSA, ICHR, etc., to reduce controversies.

3 Importance of History of Science

"History makes a man wise" is a common saying. The study of history of science should certainly make scientists wiser. It cannot only save us from repeating mistakes of the past but we can learn a lot from them. Our curiosity or urge for knowing everything, including past, gave rise to historical disciplines. Then how we can neglect the subject of History of Science in this era of science, technology, computers and information.

We are all enjoying the fruits and comforts provided by science and technology. But these facilities are the results of hard labour, often painful, of hundreds of scientists working over centuries. It is our moral duty to remember these pioneers. In fact the story of development of science (including mathematics) and technology is quite fascinating in itself. Often it will be found to be thrilling.

Mathematical sciences have always played a significant role in the development of science and technology and provide a rational organization of natural phenomena. Due to nature of mathematics, the History of Mathematics needs special treatment by mathematicians themselves but with adequate knowledge of linguistics and historical methods.

The greatest use of History of Mathematics is in mathematics education. Its study makes the subject more enlightened and increases the understanding of mathematics itself. One can know clearly as to how and why mathematics is created, grows, develops, changes, abstracted and generalized. According to George Sarton, "the main duty of the historian of mathematics, as well as his fondest privilege, is to explain the humanity of mathematics, to illustrate its greatness, beauty and dignity." Through the study of history of mathematics, one can also correct a lot of miscredits prevailing in education.

While addressing the British Association for the Advancement of Science in 1890, J. W. L. Glaisher rightly remarked that "no subject loses more than mathematics by any attempt to dissociate it from its history." Thus History of Mathematics should be a serious concern of each student, teacher, and researcher of mathematics. Specialists in History of Mathematics should also study History of Science for a broader base and developmental links.

By now enough studies and researches in the field of History of Science have been (and still being) carried out to treat it as a separate discipline (with specialized societies, journals etc.). In fact, separate departments of History of Science flourish in many universities in the world. Even quite a few full-fledged institutions also exist. Recent examples are those Dibner Institute in U. S. A. and Max Planck Institute in Germany. More specialized institutions are also there e.g., Needham Research Institute of Cambridge, U. K., which is devoted to Chinese Sciences.

In this race of institutionalization, India is lagging far behind compared to its size and needs (because it has a continuous scientific tradition of 5000 years). There is an utter lack of job opportunities for historians of science in India—a fact which in turn detracts young brilliant scholars. However, some funds are available for doing research in History of Science in India (with various types of fellowships). Moreover, teachers and professors must take interest in the history of their own subject or discipline. They can make use of history of the relevant topics in their lectures, textbooks, and offer to teach courses in the History of Science (general or particular).

4 Problems of History of Science Scholars in India

There are no adequate arrangements in India for imparting sufficient formal education and training to produce competent historians of science. A few special papers on History of Science are taught in some universities, but there are hardly any full graduate or postgraduate degree courses exclusively in the field. Most of the Indian scholars in the field have managed to acquire knowledge and efficiency in that interdisciplinary field by self-study. Quite a good number of them have successfully completed their Ph.D. especially in the specialized area of History of Mathematics. Many modern scientists have developed interest in History of Science (specially in their respective science subject) but knowledge of languages and of historical methods is also needed.

A major practical difficulty faced by Indian working scholars is the cost of publications both books and journals. When *Historia Mathematica* was started in 1974, its annual subscription was \$6 (and dollar itself was cheap). Now the subscription is more than a hundred dollars of much higher rates. How can the Indian scholar subscribe the standard journals—*HM, AHES, JHA, SCIAMVS,* etc. Only very few libraries in the vast India could afford them.

Some foreign books are equally beyond individual's affordable capacity. The important *Bakhshālī Manuscript* (by T. Hayashi) costs Rs. 5000/- and the very useful *Encyclopedia of History of Science in Non-Western Cultures* (1977) is priced USD 420 (or Rs. 20000/-). Indian publications have also become costly for poor Indian pockets. *Indian J. Hist. Sci.* costed Rs. 10/- yearly when started in 1966; now the annual subscription is Rs. 700/- (and it is not a private business). Some charitable institutions are still mindful of Indian scholars (recent voluminous edition of *Gaṇita-sāra-saṅgraha* from Hombuja Jain Math, costs only Rs. 750/-).

The case about acquiring photocopies and transcripts of manuscripts is more horrible. Very few Indian scholars succeed in getting these. Even it is difficult to get replies of requests (perhaps manuscripts libraries are too busy). Foreign scholars face less difficulty in this matter (even when they visit or write to Indian libraries). National Commission on History of Science (under INSA) should have a knowledgeable person who should be able to provide prompt information to scholars regarding publications, journals, institutions, grants, and other facilities in the field. INSA should also maintain an up-to-date core library with relevant back volumes and modern facilities (at present even *GB* set is not there in INSA!).

Lastly, something about historical attitude and temperament. It is said that ancient India could not produce historians like Herodotus (note that epic and *purāṇic* tradition is mixed with myths). The reason is philosophical, i.e. more attention to spiritual matters. Indian scholars must cultivate greater historical sense. Also we must have real love for records and try to preserve them. In foreign (western) countries, the papers, notes, correspondence of scientists is preserved and catalogued (in various libraries). In India, the material is often disposed off as waste paper (also *cf.* the practice of worshipping very costly idols and then do *visarjana*—sinking in water).

5 Nature and Type of Research

Many scholars believe that to establish national and racial superiority in the past is the sole purpose of research in the field of history of science even if such a thing was not there in reality. So attempts to glorify one's national and racial achievements by exaggerated and inflated claims are frequently made in historical studies. Often such tendencies of making hyperbolic claims are sought to be justified even at the cost of scholarly norms and standards. Actually, the purpose of history of science studies should be truly educational and it should not be used as an instrument for making priority claims. We should try to know the intellectual framework for scientific inventions and to learn why discoveries took place.

In the case of history of science in India, the *Vedas* are frequently proclaimed to contain the whole stock of modern and advanced science (including mathematics). In fact, champions of this Vedic orthodoxy assert that, in the words of Manu,

> भूतं भव्यं भविष्यं च सर्वं वेदात्प्रसिद्ध्यति। (मनुस्मृति, XII. 97)
>
> All that was, is, and will be (in future) can be derived from the Veda.

Also

> शब्दः स्पर्शश्च रूपं च रसो गन्धश्च पञ्चमः।
> वेदादेव प्रसूयन्ते ------------॥ (मनुस्मृति, XII. 98)

It is common experience that as soon as (not before) a scientific discovery or invention is made known, someone would come out with the claim that it was already known to the Vedic sages. Examples of claim include knowledge of differential calculus, Newton's law of gravitation, Einstein's equation, $E = mc^2$, Dirac delta function, proof of Fermat's Last theorem, etc. (see *GB*, Vol. 16, 9–10; and 20, p. 114).

The usual methodology for the Vedic claims for modern science is to give arbitrary meaning to certain words and to have forced interpretation without caring for coherency, consistency, and collaboration. For example, *Indra* is taken to mean 'Sun' in one place and 'electricity' at another by Hansraj (in *Science* in *Vedas*), 'autumn equinox' by R. Krishnamurty (*Math. Student*, **23**, 77–81), 'celestial point at 180° from Sun' (Holay in *Ganit Bikash*, **28**, 36), etc. The popular *Vedic Mathematics* book by Swami Bharati Krishna Tirthaji claims to give "One-line Answers to All Mathematical Problems"! Actually the book has nothing to do with *Vedas*, Vedic literature, or Vedic times, but has some educational value.

The so-called research projects have been going on (especially at INSA) since long. Now an evaluation is needed so that we can assess whether they were or are worth the money and time spent. May be that some modifications will yield better results. In particular, we should also conduct a survey to find as to what happened to the research scholars and the projects themselves. A consolidated list of all the History of Science research projects sponsored by INSA during 1960–2000 is now available in the *IJHS*, Vol. 35, No. 4 (Dec., 2000) which also has the Cumulative Index of the journal, Vols. 1–35. But the preservation of the project reports is not satisfactory as scholars would like to consult the submitted reports.

I believe that research in History of Science is a serious pursuit and not a holiday pastime meant for retired persons or waiting room-like job for scholars. It is also not just a side-hobby but full time devoted work. Without keeping in mind the sober nature of research in History of Science, it is not easy to get the projects completed in entirety (like that on *Śulbasūtras*) or at all (like that on *Brāhmasphuṭa-siddhānta*). Why other parts of S. N. Sen's *Bibliography of Sanskrit Works* are not published?

Past experience shows that some invited projects succeed when right choice is made (*cf.* INSA publications on the *Āryabhaṭīya*). Now there is a national need for following type of projects (either team work or single competent scholar):

(i) Multi-volume chronological History of Science in India.
(ii) Encyclopedia of History of Science in India (similar to that of Arabic Sciences or that of History of Science, edited by H. Selin).
(iii) History of each discipline (e.g. mathematics) in India.
(iv) Source books of each discipline.
(v) Special or specific monographs related to Bibliography, Glossary, etc., or Jaina School.

6 World Perspectives and Scenario

George Alfred Léon Sarton (1884–1956), the Belgium born late Professor of History of Science at Harvard (U. S. A.) is often called the father of History of Science. After getting doctorate in mathematics and celestial mechanics, he decided to devote his life "to explain the development of science across the ages and around the earth, the growth of man's knowledge of nature and himself". In 1912, he founded and supported the *Isis*, an International Journal of History of Science (*Isis* is the name of an Egyptian goddess). In 1915, he migrated to U. S. A. where the History of Science Society was founded in 1924. His monumental *Introduction to the History of Science* (3 Vols. in 5 parts, Baltimore 1927–48) is still useful. For modern scholars, the multi-volume *Dictionary of Scientific Biography* (a famous work) will be found to be very informative.

Noteworthy accounts of Indian mathematical sciences in European languages are already found in the eighteenth century. In 1772, Guillaume Le Gentil (1725–1792) published an account of Indian astronomy in his "Mémorie sur I'Inde" which was included in the *Histoire de l'Academie Royale des Sciences* (Paris, 1772), J. S. Bailly in his *Traité* (1787) emphasized the antiquity, originality, and methodology of Indian exact sciences, and this attracted attention of many Europeans.

Meanwhile the Asiatic Society was founded in 1784 in Calcutta by William Jones to study history, arts, crafts, sciences, and literature of not only the peoples of Asia in general, but of India in particular. Soon the English translations of *Gītā* and *Śakuntalā* stirred Europe. Jones's announcement (1786) of Sanskrit's affinity to European languages brought a revolution in the whole approach to history of entire human race! A new chapter in the historiography of Indian mathematical sciences began when H. T. Colebrooke's *Algebra with Arithmetic and Mensuration from the Sanscrit of Bramegupta and Bháscara̲* was published (London, 1817).

C. M. Whish's paper (1835) on South Indian mathematics, E. Burgess' translation (1860) of *Sūrya-siddhānta*, A. Weber's translation (1862) of *Vedāṅga-jyotiṣa*, H. Kern's edition (1874) of *Āryabhaṭīya*, G. Thibaut's translations (1875, 1888) of *Śulba-sūtra*, and *Pañcasiddhāntikā*, etc. all unfolded Indian exact sciences. Thus we find that Western scholars were keenly bringing to light the treasures of Indian traditional mathematics and astronomy hidden in ancient Sanskrit works.

On the other hand Indian scholars were busy in writing native works on modern (western) sciences to educate their fellow beings "towards the cultivation of Western science." The works were based on European scientific works or were their translations (into native languages) and this renaissance continued throughout the nineteenth century. Scores of such works can be cited but a few illustrative examples should suffice.

M. Ḥusain Isfahanī wrote his *Risālah-i-Hai'yat-i-Angrezī* in 1797. It is on European astronomical system. About 1824, Maulvī Abdur Rahīm translated European books in mathematics into Persian for Calcutta Madrasa. Yogadhyāna Miśra's *Kṣetratattva-pradīpikā* (1826) was based on Hutton's work. Ghulam Ḥusain Jaunpurī's *Jāmi'i Bahādur Khānī* (1833/1834) is a Persian encyclopedic work which contains European methods along with logarithms and trigonometrical tables. Nānā Apte, Kṛṣṇa Godbole, and others translated Euclid's Elements into Marathi (nineteenth century.).

Syed Ahmed Khan founded the Aligarh Science Society in 1864 while the Indian Association for the Cultivation of Science was formed by M. L. Sircar (1876). The *Golaprakāśa* (in Sanskrit) of Nīlāmbara Jha (1823–1883) is on trigonometry and spherical geometry. It is said to be an enlargement of an English textbook. Sudhākara Dvivedī was a pioneer in this regard. His *Dīrghavṛtta-lakṣaṇam* (1881) on ellipse, *Calanakalana* (1886) and *Calarāśikalana* on differential and integral calculus, and *Samīkaraṇa-mīmāṃsā* (1897) on the theory of equations are significant works.

However, it must be noted that there were several original and noteworthy contributions by Indian scholars to mathematical sciences. Ram Chandra of Delhi published a *Treatise of Problems of Maxima and Minima Solved by Algebra* (Calcutta, 1850). His talent was soon discovered by Augustus De Morgan of U. K. Ashutosh Mukherji's first research paper was on elliptic functions (1886). Ganesh Prasad was a great mathematician of north India during this period. Concurrently, a sort of synthesis of the Indian tradition of mathematical sciences with the universal or the world heritage of mathematics was slowly but steadily going on. The present vast stock of mathematics is like an ocean into which different national streams poured and continue to pour their mathematics.

7 Concluding Remarks and Epilogue

In India, the main official body to promote activities and research in the field of History of Science is the National Commission for History of Science (NCHS) working at I.N.S.A., New Delhi, and they get funds for the purpose. To popularize study and research in the field, the NCHS may think of bringing out a *Directory of Indian Historians of Science* as well as a *Handbook* which should be able to provide necessary introductory information and guidance to scholars and aspirants in the field. If possible a separate full-fledged Section on History of Science may be started at INSA to give formal recognition to the discipline similar to Sections on Mathematics, Physics, etc.

There was an Indian Society for History of Science which was formed in 1957 (*see BNISI* No. 21, article by S. N. Sen). In 1974 the Indian Association for the History and Philosophy of Science was formed, but due to restricted attitude of its executive official, the activities could not yield desired result. Now nothing is being heard of the IAHPS (perhaps it has ceased working). So there is an urgent need to form an ISHS again. Of course separate history societies/associations for different disciplines have their own importance (for mathematics, the ISHM is there; for astronomy, an ISHA has been recently formed at Hyderabad).

There exist a number of world prizes in the field of History of Science (for some details, *see GB*, Vol. 2. pp. 62–63). One of them is rightly named after George Sarton. For History of Mathematics, the ICHM awards Kenneth O. May Medal during each International Congress History of Science. Some such prizes are needed to be established in India. However, a number of endowment lectures in history of mathematics have already been started in India. Recently, the National Academy of Sciences, India has instituted History of Science Lecture Award. It seems enough encouragement and recognition is available for work in the field of History of Science in the world.

In spite of all this, very few Indians study the World History of Science. In fact most of research, studies, and publications in India in the field of history of science are confined to India. One reason is lack of facilities. But the main reason may be that of languages. True research is possibly only if one can consult the original sources. So interested Indians will have to pick up working knowledge of Chinese, Greek, Latin, etc. as the case may be. National history of any science subject may be easy, but the world history of the same subject is a tough thing. But I hope that the interested serious scientist will accept the challenge.

Historical Notes: Kali Chronograms of Nārāyaṇa Bhaṭṭatiri

Nārāyaṇa Bhaṭṭatiri (sixteenth–seventeenth century AD), son of Māṭrdatta, is one of the greatest scholar-poets of Kerala. He composed many works on diverse subjects both literary as well as technical in Sanskrit. He was a Nambutiri Brāhmin hailing from the family of Melpattur situated not far from the bank of the river Bharatappuzha. According to his grammatical work (see below), he learned *Mīmāṃsā* from his father, Vedas from Mādhava, logic from Dāmodara and grammar from Acyuta who was a great authority in the subject of *Vyākaraṇa-śāstra*.

In addition to grammar, Acyuta (a member of the Piṣāraṭi community), was a scholar of astronomy, astrology, poetics and medicine. He was a pupil of Jyeṣṭhadeva, the author of the famous Malayalam work *Yuktibhāṣā* on astronomy and mathematics, and was patronized by the king Ramavarman of Prakasavisaya who ruled from 1595 to 1607 of the Common Era (=AD).

Acyuta Piṣāraṭi wrote *Praveśaka* on grammar, *Horāsāroccaya* on astrology, a Malayalam commentary on *Veṇvāroha* of Mādhava of Saṅgamagrāma (not the same as the Mādhava mentioned above), and half a dozen works on astronomy. Pingree's *Census*[1] descriptively mentions these as

1. *Karaṇottama* (with auto-commentary)
2. *Uparāgakriyākrama*
3. *Sphuṭanirṇaya*
4. *Chayāṣṭaka*
5. *Uparāgaviṃiśati* and
6. *Rāśigolasphuṭānīti*

[1]Full references are given at the end.

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Of these, the concluding verse of the *Uparāgakriyākrama* contains the Kali chronogram (Sarma, p. 16) प्रोक्तः प्रवयसो ध्यानात् (*proktaḥ pravayaso dhyānāt*) which gives (in the usual *kaṭapayādi* system) in number 01714262.

This has been taken (as explained in a commentary) to yield the date of composition of the work on the Kali day 1714262 (of the present *Kaliyuga*). Ever since Iyer (p. 44) took this to correspond to 1593 AD, many scholars (such as Pingree and Sarma) accepted it, but K. K. Raja (*Adyar Library Bulletin* No. 27, p. 157) mentioned it as 1592 which is correct. The exact date worked out by the author (RCG) of the present article comes out to be Monday, July 10, 1592 (Julian). This working is based on assuming the usually accepted date Friday, February 18, 3102 BC as the first day of present *Kaliyuga*.

According to a popular Kerala tradition, when Acyuta died, his pupil Nārāyana composed a *caramaśloka* (obituary verse) in his memory. There is slight difference in the text found in various sources (e.g. Pingree, p. 37 and Raja, p. 125), but the quoted fourth line

विद्यात्मा स्वरसर्पदद्य भवतामाधारभूरच्युतः ॥
vidyātmā svarasarpadadya bhavatāmādhārabhūracyutaḥ

The phrase constituted by the first seven syllables, namely *vidyātmā svarasarpat*, which literally means "the learned soul passed to heaven", also has the common Kali chronogram. The chronogram represents the Kali day number 1724514 thereby mentioning the definite date of Acyuta's death. Unfortunately, here also the corresponding year is wrongly given as 1621 AD by various scholars such as Iyer, Pingree, Raja, Sarma etc. The correct date works out to be Friday, August 4, 1620 (Julian) or August 14, 1620 (Gregorian) as difference being 10 days here.

Nārāyaṇa's *Nārāyaṇīyam* is his most popular work and is one of the finest religious lyrics in Sanskrit literature. It primarily deals with the themes of the famous *Bhāgavata-purāṇa* (including its *sāṃkhya* doctrine) as well as presents a condensed version of *Rāmāyaṇa*. Its date of composition is expressed by the following interesting Kali chronogram given at the end.

आयुरारोग्यसौख्यम्
āyurārogyasaukhyam

This on the one hand is a wish or prayer for longevity (*āyuḥ*), health (*ārogya*) and happiness (*saukhya*), and on the other hand it represents the Kali day number 1712210 expressed in the usual *Kaṭapayādi* system.[2] The corresponding Julian date is November 27, 1586 as correctly given by Raja (pp. 126 and 130), the week day being Sunday.

Nārāyaṇa was not only fond of forming such chronograms but was an expert in creating them with literary gymnastics. A popular tradition in Kerala ascribes him the following verse (Raja p. 130 with slight correction):

[2] *In the Concept of Śūnya* (Delhi, 2003, p. 41), the number appears wrongly as 171211 (paper by K. V. Sarma), and the same also in *IJHS*, 34(4), 1999, p. 274.

नदीपुष्टिरसह्या नु न ह्यसारं पयोऽजनि ।
निजात् कुटीरात् सायाह्ने नष्टार्थाः प्रययुर्जनाः ॥
nadīpuṣṭirasahyā nu na hyasāram payo'jani |
nijāt kuṭīrāt sāyāhne naṣṭārthāḥ prayayurjanāḥ ||

The verse describes[3] the catastrophe of the devastating flood in the Bharatappuzha river in the following words:

> The flood in the river was unbearable and there came down an abundance of water. By the evening, the people (living nearby) fled from their huts, having lost all their belongings.

But the more interesting part in the verse is that its each four lines (*pādas*) represent the same number 01721180 in the *Kaṭapayādi* system. However, here we have to note that while the usual right to left convention is followed in the first and third *pādas*, the opposite left to right convention is to be observed in the other *pādas*. This is a good example showing that the same person may follow different conventions at the same time! The Julian date of the tragic event corresponding to the above 1721180th Kali day was Wednesday, June 19, 1611.

The *Prakriyā-sarvasva* stands at the top of the scientific works (i.e., those devoted to *Śāstras* or technical Sanskrit) of Nārāyaṇa. It is said to be an original recast of Paṇinian *sūtras* on the Sanskrit grammar (*Aṣṭādhāyī*). According to Raja (p. 129), two Kali chronograms found in one of its introductory verses are:

यत्नः फलप्रसूः स्यात् (*yatnaḥ phalaprasūḥ syāt*) and
कृतरागरसोद्य (*kṛtarāgarasodya*)

The first of these represents the Kali day 1723201 and the second the Kali day 1723261, their difference being only of 60 days. The corresponding Julian dates are Monday, 30 December, 1616 and Friday 28 February, 1617. In Gregorian these days will fall in January and March in 1617 and not in 1616 as Raja states. Regarding ancient dates, there has to be always a clear mention or understanding as to whether they are in Julian or Gregorian to avoid confusion. It may be mentioned that although in Italy the Gregorian reform was adopted in 1582, it was adopted much later (in 1752) in England.

Nārāyaṇa is also said to have coined the chronogram *Bālakalatram saukhyam* (बालकलत्रं सौख्यम्) as printed in Raja's book (p. 121). This gives Kali Day 1723133 and corresponds to the Julian date 23 October, 1616 (Wednesday). However, there seems to be some confusion apparently because Raja mentions the Kali day numbers as 1729133 which will correspond to the date 28 March, 1633 (Thursday). Of course, the latter number can be easily obtained by taking the second *la* (ल) in the above chronogram as *ḷa* (ळ) of the Malayalam as this denotes 9 (instead of 3) in the extended *Kaṭapayādi* system. The story goes that when Acyuta asked *Nārāyaṇa* to give an alternative chronogram, the pupil formed the new one as

लिङ्गव्याधिरसह्यः (*liṅga vyādhirasahyaḥ*)

[3]Several typographical errors that had crept into this verse (as well as many other places), in the earlier printed version of the article, were corrected. –*ed*.

which represents the same Kali Day 1729133. The date of 1633 might had been the then proposed date for completion of *Prakriyā-sarvasva*.

Nārāyaṇa composed *Caturaṅga-ślokas* on the game of chess (Raja, p. 148) whose oriental name *śataraṃja* is clearly derived from the Sanskrit name. His *Śūkta-ślokas* are said to give various statistics about the *Ṛgveda*. The technique used is described in the opening verse and is based on the *Kaṭapayādi* system with some changes. Here the letter *na* means 10 (and not the usual 0) and the conjoint letter *kṣa* (क्ष) means 12 (not 6). This shows that variation in the system already started. The famous *Vedic Mathematics* by Swami Bharati Krishna Tirthaji uses *kṣa* = 0.

Above, the dates in AD of many Kali chronograms have been given. The converse problem of finding the Kali day or chronogram for a given date is also there. On a current Nth Kali day, the *gata ahargaṇa* (elapsed number of Kali days) is $(N - 1)$. For example, the epoch of *Karaṇa-kuthūhala* is Thursday, the 24 February, 1183 (Julian) which corresponds to the Kali day number $N = 1564738$ and on this Kali day the (*gata*) *Ahargaṇa* is 1564737 (Rao and Uma, p. S171). Of course the *Ahargaṇa* number also represents the $(N - 1)$th Kali day and so on. In essence, the day by day counting of civil days from the first day of *Kaliyuga* is involved.

Important Indian astronomical works contain methods of finding *Ahargaṇa* on any lunar *tithi*. Minor deviations or errors can be corrected if week day is known. But often mistaken results are found. For instance D. A. Somayaji (*IJHS*, Vol. 20, 164–165) finds the *ahargaṇa* upto *Aṣādha-bahula-amāvasyā, Śaka* 1906 as the number 1857473. But according to Rao and Uma (pp. S171–S174) the *ahargaṇa* for the said Gregorian date 28 July, 1984, comes out to be 1857444 days! How Nārāyaṇa got the Kali day numbers for forming his chronograms is also worth investigating.

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Part III

Mathematics in Ancient Period

On Some Mathematical Rules from the *Āryabhaṭīya*

The paper deals with the controversies which arise due to different interpretations of certain mathematical rules as found in the *Āryabhaṭīya* of Āryabhaṭa I (born 476 AD). The first half of the sixth stanza (Kern's edition) from the *Gaṇitapāda* part of the work gives the area of a triangle as half the product of its base and altitude. But even in this respect a controversy exists as to whether the text gives the rule for the general triangle or for the isosceles triangle only. The second half of the same stanza is generally interpreted to contain the wrong expression for the volume of a tetrahedron as half the product of it areal base and altitude. However, some scholars have attempted to interpret is differently so as to yield the correct formula.

$$\text{Volume of tetrahedron} = \text{area of base} \times \frac{\text{altitude}}{3}$$

The first half of the next stanza (no. 7) gives the correct rule

$$\text{Area of a circle} = \frac{\text{circumference} \times \text{diameter}}{4}$$

The second half is generally taken to contain the wrong formula

$$\text{Volume of sphere} = A\sqrt{A}$$

where A is the area of its (greatest) circular section.

However, by giving very unusual interpretations, some scholars maintain that the rule in the text is not about the volume of a sphere but rather about the surface of a hemisphere for which it is made to give a correct expression!

Another controversy is about the interpretation of the 28th stanza from the *Golapāda* part of the work. P. C. Sengupta has translated it in such a way as to discredit Āryabhaṭa for not knowing the correct rule for finding the altitude of the Sun at any time of the day. However, it is pointed out here that the explanation given by Parameśvara and the observations made by Pṛthūdaka show that Āryabhaṭa knew the correct rule.

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1 Introduction

The Sanskrit work *Āryabhaṭīya* (hereafter abbreviated as *AB*) is from the pen of the well-known Indian astronomer and mathematician, Āryabhaṭa I (born 476 AD) who is also called the elder Āryabhaṭa in order to distinguish him from his name sake, Āryabhaṭa II or the younger Āryabhaṭa (tenth century). According to an interpretation of the author's own statement in *AB*, III (*kālakriyā-pāda*), 10 (p. 58),[1] the work was composed at a young age of 23 years. This interpretation is clearly given by Sūryadeva Yajvan (born 1191) in the introductory passage to his *AB* commentary called *Bhaṭa-prakāśikā* (=*BP*),[2] and by Parameśvara (c. 1380–1460) in his *AB* commentary called *Bhaṭa-dīpikā* (=*BD*) (p. 58). However, the matter is not free from objections. For instance, Sengupta accepted this interpretation when he translated[3] that *AB*, but later on he gave another interpretation[4] and said that "we are not justified in concluding that the *AB* was composed when Āryabhaṭa was only 23 years old."

The *AB* has four parts or sections of which the second is called *Gaṇita-pada* and deals with mathematics. However, the author has presented only selected topics from the mathematical knowledge of his time in India in the form of condensed rules. According to the commentator Bhāskara I (629 AD),[5] the *AB* contains only " a bit of mathematics".

Due to brevity, elliptic language, and poetic form of the rules, the difficulty of understanding the text increased with the passage of time. Already in the nineteenth century, the work was considered to be[6]

वज्रमयपर्वतादपि कठिनतरः।
More hard than even the rocky mountain.

It is therefore not surprising to find that some of the AB rules have been differently interpreted by various commentators, translators, and other scholars, thereby leading to several controversies. The present paper deals with four such rules.

2 Rule for the Area of a Triangle

The first half of *AB*, II. 6 (p.23) says:

त्रिभुजस्य फलशरीरं समदलकोटीभुजार्धसंवर्गः।
The area of a triangle is the product of the *samadalakoṭī* and half of the base.

Interpretation of the rule depends on the meaning of the word *sama-dala-koṭi*. The *BP* (p. 39) and *BD* (p. 23) both take it to mean altitude, the perpendicular side common to the two triangles formed by it. Bhāskara I also illustrates the rule by taking examples of all types of triangles.[7]

However, Nīlakaṇṭha Somasutvan's (c. 1500) *Bhāṣya (=NAB)* on the *AB* says[8] that the rule is meant for equilateral triangle and explains that the common altitude *(koṭi)* is designated by the work *sama-dala* here because it divides the base into two equal halves.

Giving weight to earlier commentators, modern scholars like Sengupta (AB transl. p. 15) and Shukla[9] take the rule to mean the general formula

$$\text{Area} = \frac{1}{2}\,(\text{base} \times \text{altitude}). \tag{1}$$

G. R. Kaye,[10] however, takes the word to mean 'perpendicular that bisects the base' (cf. *NAB*). B. Mohan,[11] listing several possible meanings of the word *sama*, finally takes it to mean, like *BD*, 'common'.

3 Rule for the Volume of a Pyramid

The second half of AB, II.6 (p. 23) says:

ऊर्ध्वभुजा तत्संवर्गार्धं स घनः षडश्रिरिति ॥

Half the product of that (area of the triangular base) and the vertical height is the volume of a six-edged solid (or tetrahedron).

So that this rule implies the volume of a pyramid with triangular base as

$$\text{Volume} = \text{Area of base} \times \frac{\text{height}}{2}, \tag{2}$$

which is wrong, the correct formula being

$$\text{Volume} = \text{Area of base} \times \frac{\text{height}}{3}. \tag{3}$$

Bhāskara I (629 AD) is stated[12] to have made little or no improvement on the result (2). However, Brahmagupta (628 AD), a great Indian mathematician of the same time, has given the correct rule in his *Brāhma-sphuṭa-siddhānta*, XII. 44 which says[13]

क्षेत्रफलं वेधगुणं समखातफलं, हृतं त्रिभिः सूच्याः।

The volume of a pit of uniform depth is the (sectional) area multiplied by the depth; (this) divided by three is (the volume) of *sūcī* (a tapering figure, i.e., the pyramid or cone).

Hence the formula (3) is implied here. The *BP* (pp. 41–42) says that the height of the tetrahedron involved in the above *AB* rule lies along the height at the centre or middle (*madhya*) of the base and gives the following formula for its computation

द्विघ्ना कर्णकृतिर्भक्ता त्रिभिरूर्ध्वभुजाकृतिः।

Two times the square of the edge divided by three is the square of the height [of a regular tetrahedron].

That is

$$h^2 = \left(\frac{2}{3}\right) s^2, \tag{4}$$

which is correct for a regular tetrahedron of each edge s. In fact, the *BP* takes the *AB* rule as meant for such pyramids only. The *BD* (p. 23) also gives the usual rule for finding the *lamba* (altitude) of any triangle of given sides. But, for finding the height of a tetrahedron, it gives the following line

लम्ब-तदर्धयोर्वर्गान्तरपदम्-अत्रोर्ध्वबाहुर्भवति।

The square root of the difference of the squares of the *lamba* (of the triangular base) and of its half is the vertical height (of the triangular pyramid) here.

That is

$$h = \sqrt{p^2 - \left(\frac{p}{2}\right)^2}, \tag{5}$$

where p is a *lamba* (altitude) in the triangular base.

The result (5) is not true for regular tetrahedron whose every face is an equilateral triangle. But it is certainly true for the tetrahedron whose base is any triangle (with an altitude p) and whose one slant face is congruent to the base. Further, this face is so inclined that the slant height equals p and that the perpendicular from the vertex falls on the middle point of the altitude p of the base. Does the *BD* really intend to deal only with pyramids which have the above-mentioned property that is not found even in the simple regular tetrahedron, or, the formula (5) is the result of confusing the centre of the triangular base with the middle point of the altitude in the base?

The *NAB* (part I, p. 35) takes the six-edged solid to mean, like *BP*, a regular tetrahedron whose faces are equilateral triangles. Not only that the Sanskrit line for (4) is quoted from Sūryadeva Yajvan (author of *BP*), but a long and systematic derivation of it is presented.

However, like other earlier Āryabhaṭan scholars he accepted the wrong formula (2) without demur and even justified it. Even the very late commentator Kodaṇḍarāma (c. 1850) made no fuss about it (see No. 6 at the end in References).

Some modern scholars have tried to interpret the Sanskrit text in such a way as to yield the correct formula (3), thereby defending Āryabhaṭa I. For instance, Conrad Mueller[14] translates *phala-śarīram* as a "a solid obtained from the area (of a triangle)", that is, a special pyramid which, when unfolded gives a triangle and *ṣaḍaśri* as "a prism equivalent to six (of mentioned) pyramids" (i.e. a cube). The stanza *AB*, II. 6 is then taken to give the lateral area and volume of a regular triangular pyramid. The details are as follows:

Lateral surface (S) of the pyramid

$$\begin{aligned} S &= \text{Area of the unfolded triangle} \\ &= 4A, \end{aligned}$$

where A is the area of each face. Then,

$$\text{Volume (of } \textit{ṣaḍaśri}) = S \times \frac{h}{2}$$
$$\text{or} \qquad 6V = S \times \frac{h}{2},$$

where V is the volume of the pyramid. From these we readily get

$$V = A \times \frac{h}{3},$$

which is the correct rule (3). The details have again appeared in some recent publications of Kurt Elfering.[15] These peculiar interpretations are not supported by any known ancient commentator, and the argument that the correct formula was surely known to Āryabhaṭa because the Indians were familiar with Greek science, needs justifications.[16]

No doubt the Greeks not only knew the correct rule (3) but have proved it several centuries before the date of *AB*,[17] but the errors continued not only in India but abroad also. Maimonides (1135–1204) in his *Moreh N Vokheem* speaks of those who think the cone to be half of the cylinder with the same base and height.[18] Analogy, which was one of the tempting methods for arriving at empirical formulas (see the next section), with the formula (1) seems to be responsible for the error.

4 Rule for the Volume of a Sphere

The second half of the *AB*, II. 7 (p. 24) says:

> तन्निजमूलेन हतं घनगोलफलं निरवशेषम् ॥
>
> That (i.e. the area of a circle mentioned in the first half) multiplied by its own (square) root is the volume of a sphere (whose greatest section is above circle), without any remainder (i.e. exactly).

So that, the volume of a sphere is

$$V = A\sqrt{A}, \tag{6}$$

where A is the area of the greatest circular section and which, by the first half of the verse, is given by

$$A = \left(\frac{\text{circumference}}{2}\right) \times \left(\frac{\text{diameter}}{2}\right) \tag{7}$$
$$= C \times \frac{D}{4} = \pi R^2. \tag{8}$$

The formula (6) is wrong although *AB* calls it exact. More surprising is the fact that the commentaries, of Bhāskara I,[19] *BP* (p. 43), *BD* (p. 24), *NAB* (part I, p. 39), all fail to point out the inaccuracy. Elsewhere the *BD* (p. 25) says:

घनगोलेऽपि वृत्तफलस्य मूलम् उच्छ्रायः।

In (the case of a) sphere, the (square) root of the area of the (great) circle is the (effective?) height.

The implied understanding may be put as

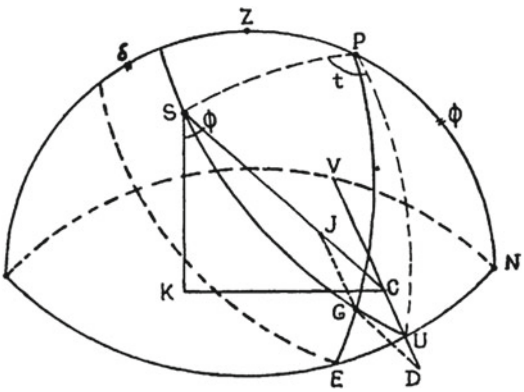
$$\text{Volume} = \text{effective height} \times \text{sectional area}, \tag{9}$$

which can yield correct results if, corresponding to a sectional area taken, the effective height is properly chosen. For example, in the case of a cube of uniform sectional area A, the effective height is the square root of A and the formula (6) is valid for a cube. An analogy with a cube seems to be the way of arriving at (6) for a sphere. This view is supported by *NAB* (part I, p. 39).

According to some new translation,[20] the text is taken to give the curved surface of a hemisphere (and not the volume of a sphere), the result being obtained by taking the radius and circumference for the two factors (*tat* and *nijamūla*) mentioned in the text and forming the specified product.

5 Altitude of the Sun at Any Time

In the accompanying diagram, UGS is a part of the diurnal circle of the sun whose position, at any time t, is at S. The line UV is the rising-setting line, and GH (not shown) is the line of intersection of the diurnal circle and the six o'clock circle. It is clear from the figure that the altitude is given by



$$SK = SC \cos\ \phi = SC \times \frac{(R \cos\ \phi)}{R}, \tag{10}$$

where SC is the perpendicular from S on UV. The main problem, therefore, is the computation of SC. Now *AB*, IV. 28 (p. 89) states:

> स्वाहोरात्रेष्टज्यां क्षितिजादवलम्बकाहतां कृत्वा ।
> विष्कम्भार्धे विभक्ते दिनस्य गतशेषयोः शङ्कुः ॥ 28 ॥
>
> After multiplying the *iṣṭajyā* (measured) from the horizon in (the sun's) own diurnal circle, by the *avalambaka* (sine of the colatitude), divide by the semi-diameter. The result is the (great) gnomon (sine of the sun's altitude) corresponding to the (part of the) day elapsed or to be elapsed.

That is,

$$SK = iṣṭajyā \times \frac{(R\cos\ \phi)}{R}. \tag{11}$$

Now, Sengupta (*AB* translation, pp. 47–48) takes

$$iṣṭajyā = R\sin t \times \frac{(R\cos\delta)}{R}, \tag{12}$$

thereby discrediting Āryabhaṭa for not knowing the correct rule. However, the *BD* (p. 89) clearly explains that (for northern hemisphere)[21]

$$\begin{aligned} iṣṭajyā &= (\text{earth-Sine}) + [R\sin(t - cara)]\frac{(R\cos\delta)}{R} \\ &= GD + SJ \\ &= SC, \qquad \text{as required.} \end{aligned}$$

The remark of Pṛthūdaka (860 AD) in his commentary on the *BSS*, III. 25–26 also indicates that Āryabhaṭa I knew the correct rule.[22] The *NAB* (part III, pp. 56–57) explains the subject in detail.

References and Notes

1. The page reference to *AB* text and to the *Bhaṭa-dīpikā* (=*BD*) commentary of Parameśvara on it are as per the edition by H. Kern Brill, Leiden, 1874.
2. See the extract from a manuscript of *BP* quoted by Kern in the preface (p. x) of his *AB* edition. Other references to *BP* are as per the edition by K. Sarma who has very kindly made a few pages available for my use (New Delhi 1976).
3. Sengupta, P. C.'s English translation of the *AB* was published in the *J. Deptt. letters* (Calcutta Univ.) Vol. 16 (1927) article 6 pp. 1–56.
4. See the preface (pp. XVIII–XX) to his Engish transaltion of Brahmagupta's *Khaṇda-khādyaka*, Calcutta Univ. Calcutta 1934.
5. Shukla K. S. "Hindu Mathematics in the Seventh Century as found in Bhāskara I's Commentary on the *AB*", (first in a series of four articles) *Gaṇita*, 22, No. 1 (June 1971) p. 129.
6. See the opening verse (No. 8) of the Telugu commentary *Sudhātaraṇga* by Kodaṇḍarāma (1807–1883) edited by V. Lakshminarayana Sastri, Government Oriental Manuscript Library, Madras, 1956, p. 1.

7. Shukla. K. S. "Hindu Math. in the Seventh Century (Part two)", *Gaṇita*, 22, No. 2 (Dec. 1971, pp. 62–63.)
8. NAB, Part I (*Gaṇita*), p. 28, The part I as also part II (*Kāla*) were edited by K. Sambasiva Sastri, Trivandrum Sanskrit Series Nos. 101 and 110, Trivandrum, 1930 and 1931. The part III (*Gola*)was edited by S. K. Pillai, Trivandrum, 1957 (*TSS* 185).
9. Shukla, K. S. *op. cit.* (under Ref. 7 above), p. 63.
10. Kaye, G. R. " Notes on Indian Mathematics. No. 2: Āryabhaṭa" *J. Asiatic. Soc. Bengal* (N. S.), Vol. 4, No. 3 (March, 1903), p. 120.
11. Mohan, B. *History of Mathematics* (in Hindi), Hindi Samiti, Lucknow, 1965, pp. 271–273.
12. Shukla, *op. cit.* (under Ref. 7 above), p. 63
13. See the *BSS* edited by R. S. Sharma and his team, Vol. III, p. 869; Indian Inst. of Astronomical and Sanskrit Research, New Delhi, 1966.
14. Muller Conrad "Volumen und Oberflache der Kugel bei Āryabhaṭa I" *Deutsche Math.*, 5 (1940), pp. 244–255, as quoted in the *Math. Reviews*, 2 (1941), p. 114
15. See his article in *Rechenpfenninge* (Festschrift fur Kurt Vogel) pp. 57–67, Munich, 1968; and his German book *Die Mathematik des Āryabhaṭa I*, pp. 68–71, Munich 1975.
16. See *Math. Reviews*, 39 (1970), p. 720.
17. Smith, D. E. *History of Mathematics* (Dover, New York, 1958), Vol. I, p. 91, states that Archimedes (c. 225 BC) credits Eudoxus (c. 370 BC) for proving the rule.
18. H. Midonick (editor), *The Treasury of Mathematics*, Vol. I, p. 199 (Penguin Books, 1968).
19. Shukla. K.S. *op. cit.* (under Ref. 7 above), p. 64.
20. See Elfering's publications mentioned above (under Ref. 15). The Sanskrit word *tat* (that) is taken to mean 'radius' in the former (1968) and 'circumference') in the latter (1975) translation.
21. The earth-Sine (*kujyā*) is the distance between the rising-setting line UV and the line joining the points of intersection of the diurnal circle and the six o'clock circle (i.e. line G H). A rule for finding it is given in *AB*, IV, 26 (p. 87) which further adds that it is responsible for finding the variation in the day (which is caused by *cara* or the ascensional difference).
22. See the *BSS* edited by Sharma (Ref. 13 above), Vol. II, p 297. The remark is (after making slight correction):
    सैव स्वाहोरात्रेष्टज्योच्यते स्वक्षितिजात् आर्यभटादिष्वस्मत्सिद्धान्ते च छेद इत्यभिधीयते ।

Decimal Denominational Terms in Ancient and Medieval India

1 Introduction

Although the choice of ten as a base for numeration is not the best, God favoured it by giving us ten fingers. In India, ten has been the basis for counting since very early days. Later on it served the base for the place-value system of numerals which was invented in India about two thousand years ago. Specific names are found for numbers which are equal in value to 10^n, where $n = 0, 1, 2, \ldots$. Later on these decuple terms were used as names for various notational places when positive integral numbers were written in decimal place-value system.

It is remarkable that Indians developed terminology for denominations up to very high orders in comparison to other ancient nations. In this connection we quote al-Bīrūnī (Vol. I, p. 174) (c. 1030 AD):

> I have studied the names of the orders of the numbers in various languages with all kinds of people with whom I have been in contact, and have found that no nation goes beyond the thousand. The Arabs too stop with the thousand, which is certainly the most correct and the most natural thing to do. I have written a separate treatise on this subject. Those, however, who go beyond the thousand in their numeral system are the Hindus.....

This leads us to ask the following questions:

1. Why was stopping at the thousand considered most correct and natural by al-Bīrūnī?
2. Was he ignorant of the Greek term *myriad* (from myrioi, "countless") which stood for ten-thousand? Which is his separate treatise on the subject?
3. What about the terms for the large numbers used by Archimedes (died 212 BC) in his book entitled *Principles* which dealt with the naming of numbers (Heath, p. xxxvi)? I leave the matter and come to the main topic.

Gaṇita Bhāratī, Vol. 5, Nos. 1–4 (1983), pp. 8–15.

K. Ramasubramanian (ed.), *Gaṇitānanda*,
https://doi.org/10.1007/978-981-13-1229-8_10

2 Decuple Terms in Vedic Literature

Decuple terms up to *sahasra* ("thousand") frequently occur in *Ṛg-Veda,* the oldest of the four Vedas. The next term *ayuta* in the sense of "ten thousand" is also found in it side by side with *sahasra* at several places especially in the Eighth Maṇḍala e.g. 1, 5 (p. 1107); 2, 41 (p. 1117); 21, 18 (p. 1185); 34, 15 (p. 1224); and 46, 22 (p. 1257). The next three (decuple) terms namely (see below) *niyuta, prayuta,* and *arbuda* are also found in it, but none of these is said to denote a number (Satya Prakash, p. 367).

A definite list of 13 decuple terms occurs in the *Yajur-Veda* (Vājasaneyī recension), XVIII, 2 (p. 270) as follows:

> *eka; daśa; śata; sahasra; ayuta; niyuta; prayuta* ($= 10^6$); *arbuda; nyarbuda; samudra; madhya; anta; and parārdha* ($= 10^{12}$).

We shall call this set as Medhātithi's List after the name of the associated Vedic seer. The same set is found in the *Taittirīya-saṃhitā*, IV. 4, 11, 4 of the Black Yajur-Veda (Macdonell and Keith, p. 342; and Mishra, p. 17) and also in the *Śatapatha Brāhmaṇa,* IX. 1, 2, 16 (*Ibid.,* p. 342; and Suryakant, p. 189).

The popularity of Medhātithi's List is also shown by the fact that with slight variations it occurs in several other places in the Vedic literature. Some references are:

(a) *Kāṭhaka-saṃhitā,* XVII, 10, where *niyuta* and *prayuta* interchange places (Macdonell and Keith, p. 342; Datta and Singh, I, 10).
(b) *Maitrāyaṇī-saṃhitā*, II, 8.14, where we have *ayuta* for 10^4 as well as for 10^6 (*Ibid.,* p. 342; and Kapadia, p. XLVIII).
(c) *Kāṭhaka-saṃhitā*, XXXIX, 6, where there is an interchange like (a) above, and then, after *nyarbuda* ($= 10^8$), a new term *badva* ($= 10^9$) intervenes so that the subsequent four terms take the list to 10^{13} here (*Ibid.,* 342; and Suryakant, 189)
(d) *Pañcaviṃśa*-(or *tāṇḍa-brāhmaṇa*, XVII, 14.2 where the last four terms of Medhātithi's List are replaced by *nikharvaka, badva, akṣita,* and go ($= 10^{12}$) respectively) *Ibid.,* 342; and Kapadia, XLIX).
(e) *Jaiminīya-upaniṣad-brāhmaṇa,* I. 10, 28, 29, seems to replace the last four terms, like (d) above, by *nikharva, padma* ($= 10^{10}$), *akṣiti* ($= 10^{11}$), and ends with the phrase *vyomāntaḥ* which might imply a term or two (*vyoma* and *anta*) further (*Ibid.,* 342; Suryakant, 189).
(f) *Śāṅkhāyana-śrauta-sūtra,* XV, 10.7, where, similarly, the last four terms are replaced by five namely, *nikharva* or *nikharvāda* ($= 10^9$; *samudra*; *salila*; *antya*; and *ananta* ($= 10^{13}$); *Ibid.,* 342; *Ibid.,* 189; Datta and Singh, I, 10).

Now we give a Vedic extension (of Medhātithi's List) which is not mentioned in many histories of Indian mathematics. According to Datta and Singh (I, 9), the *Taittirīya-saṃhitā,* VII, 2.20, contains just the above list of 13 terms. But according to Gurjar (p. 16), Ram Behari (p. 3 and cover page 3), and Mishra (p. 18), etc., the set of decuple terms given here extends Medhātithi's List to 7 more terms as follows:

> *Uṣas* ($= 10^{13}$); *vyuṣṭi; udeṣyat (deśyat* according to Gurjar); *udyat; udita; suvarga;* and *loka* ($= 10^{19}$).

These terms are not interpreted in the above manner apparently by Macdonell and Keith (p. 342) and by Suryakant (p. 189), but Mishra (p. 18) quotes the words of the commentator, Bhattabhāskara, in support of the above numerical extension.

3 Decimal Notational Names in Scientific Works

We have already shown that Medhātithi's List was not followed uniformly beyond *ayuta*. Due to vast distances and intervals of time, and also due to different Vedic schools, the names of the decuple terms varied. Sometimes the same term denoted different numbers in different texts. No care seems to have been taken to maintain the order of terms especially in the Epic literature. For instance, the *Mahābhārata, Sabhāparva*, 65, 3–4 (Gita Press ed., Vol. I, p. 889) lines

अयुतं प्रयुतं चैव शङ्कुं पद्मं तथार्बुदम् ।
खर्वं शङ्खं निखर्वं च महापद्मं च कोटयः ॥ ३ ॥
मध्यं चैव परार्धं च सपरं चात्र पण्यताम् ।

in which Yudhiṣṭhira describes his wealth, present a complete mixing of various decuple terms.

Obviously there was a need for fixing the order of the terms and possibly standardizing the terminology. This seems to have been attempted by the authors of the astronomical and mathematical works. Āryabhaṭa I (b. 476 AD) in his *Āryabhaṭīya,* II. 2 (p. 33) confined to just 10 terms (why?) which are same as in Medhātithi's list up to 10^6 beyond which the former's terms are *koṭi, arbuda* (= 10^8 here), and *vṛnda* ("flock"), the last, for 10^9. According to al-Bīrūnī (I, 176). "The book of Āryabhaṭa of Kusumapura" gives, after *koṭi, padma* (= 10^8) and *parapadma* (= 10^9).

But stopping at the tenth "order" could not be accepted especially when higher orders were already reached earlier. Also there was a need to standardize the number of terms to be given in common texts for general use. Various considerations led to favour the traditional number eighteen as the choice for the purpose. Accordingly a list of 18 decuple terms came to be accepted as the standard practice. Al-Bīrūnī (I, 174) says that the extension to the 18th order was done for "religious reasons". According to the *Viṣṇu-purāṇa,* VI, 3, 4–5 (pp. 513–514), the 18th place of counting (in decuple scale) is called *parārdha* (= 10^{17}) which represents half of that period at the end of which there is the *prākṛta-pralaya* ("annihilation of nature") when the visible world shrinks into the invisible one.

Whatever be that, the names up to 18th denominational place are found in standard works on Indian mathematics by Śrīdhara, Bhāskara II, etc. But who was the first to do so? Lists of "eighteen places of numeration" do appear in the Puranic literature, e.g. *Vāyu-purāṇa,* Chap. 63 (pp. 408–410) gives two sets of names. But due to uncertainty about the dates of the *Purāṇas* and due to their composite character, we cannot be sure of the originality of their lists.

According to al-Bīrūnī (I, 177), the *Puliśa-siddhānta* has the following names;

> *eka, daśa, śata, sahasra, ayuta, niyuta, prayuta, koṭi, arbuda, kharva, kharva* (repeated), *nikharva, mahāpadma, śaṅku, samudra, madhya, antya, parārdha.*

Half this list is same as that of Āryabhaṭa I. But then the use of *kharva* for 10th as well as for the 11th order is a confusion which is to be removed. D. Pingree (p. 186) who collected and translated the fragments of the said work is silent over the matter. He calls the work as "The Later *Puliśa-siddhānta*" and places it in eighth century AD (*Ibid.,* p. 172). Since the original forms of the above and of the *Older Puliśa-siddhānta* which is summarized by Varāhamihira (sixth century AD) in his *Pañca-siddhāntikā* are not extant, we cannot be sure of their contents. According to Pingree (p. 172), the *Puliśa-siddhānta* mentioned by al-Bīrūnī above, was already known to *Pṛthūdaka* (864 AD).

A definite list of 18 denominational places is given by Śrīdhara who was placed circa 750 AD by Datta and Singh but now placed after Mahāvīra (c. 850) by Shukla (p. XLII). Whatever be that, Śrīdhara's set of 18 names became almost the standard Indian list of 18 decuple terms which also served as a sort of ideal for subsequent writers. Of course this idealism was not confined to names of numbers only; Śrīdhara was the model for ancient Indian mathematics in general—"*Gaṇite Śrīdharācāryaḥ*"—as an old popular saying puts it. His list is as follows:

> *eka, daśa, śata, sahasra, ayuta, lakṣa (or laksya), prayuta, koṭi, arbuda, abja (or abda), kharva, nikharva, mahāsaroja, śaṅku (or śaṅkha), saritāṃ-pati, antya, madhya,* and *parārdha* ($= 10^{17}$).

The Sanskrit text is found in Śrīdhara's *Pāṭī-gaṇita*, verses 7–8 (Shukla, text p. 5) and *verbatim* in his *Triśatikā,* verses 2–3. The variants *lakṣya* and *śaṅkha* are quoted by Shukla while *abda* and *śaṅkha* are found in the *Kriyākramakarī* commentary (p. 8) on the *Līlāvatī* edited by K. V. Sarma (V.V.R.I. Hoshiarpur, 1975). Similar lists are found in the following works:

- Al-Bīrūnī's *India* (I, 175) except for the use of *nyarbuda* for *arbuda,* and of synonyms *padma, mahāpadma, samudra* for the 10th, 13th, 15th terms respectively; also *antya* and *madhya* interchange places.
- Śrīpati's *Gaṇita-Tilaka* (c. 1040), I, 2–3 (Kapadia, p. I) except *padma* for *abja*, and *samudra* for 10^{14}.
- *Līlāvatī,* verses 10–11 (pp. 11–12) of Bhāskara II (1150 AD) except for the use of synonyms *mahāpadma* and *jaladhi* for 13th and 15th orders respectively. The *Kriyākramakarī* (mentioned above) version gives *abda* for *abja* besides the above two changes.
- the ancient anonymous commentary (Shukla, text p. 5) on *Pāṭīgaṇita* gives a list similar to the *Līlāvatī.* Does this indicate that the commentary is to be dated after 1150 AD?
- Hemachandra's *Abhidhāna-cintāmaṇi* (twelfth century), III, 537–538 (Kapadia, p. XLVIII) except *mahāmbuja* for *mahāsaroja.*

- *Gaṇita-Kaumudī,* 1, 2–3 (p. 1) of Nārāyaṇa Paṇḍita (1356 AD) except for the use of the synonyms *saroja, mahābja,* and *pārāvāra* for the 10th, 13th, and 15th orders respectively, Cf. (i) above.

4 Lore of Large Numbers

While the practice of giving a list of 18 decuple terms was generally accepted by many leading Indian mathematical authors, some other scholars, especially those living in the southern part, included terms extending to very high orders. One such extended list was given by Mahāvīra (c. 850), a Jaina writer who lived during the reign of Amoghavarṣa I, the Raṣṭrakūṭa monarch of Karnataka and Maharashtra (815–877). In his *Gaṇita-sāra-saṅgraha,* I, 63–68 (p. 8) is found a list of 24 decuple terms, the last being *mahākṣobha* (= 10^{23}). The list is well-known (*see* Datta and Singh, I, 13; Kapadia, XVI–XVII; Shukla, transl., 2–3). Also see below.

Actually Mahāvīra could carry out the task by employing a fewer words than one would expect by resorting to the prefixes *daśa* and *mahā* ("great"). Thus he avoided *ayuta, niyuta* and *prayuta* by using *daśa-sahasra, lakṣa,* and *daśa-lakṣa* respectively. In fact, according to al-Bīrūnī (I, 176), *daśa-sahasra* and *daśa-lakṣa* were the "popular" names and their equivalents, *ayuta* and *prayuta* were "rarely used". The question arises whether Mahāvīra followed the practice because it was already popular in his times, or the *daśa*-system became popular because of Mahāvīra. Anyway the system is even more popular now.

Shukla (transl., 3) has brought out a big list of 36 decuple terms from the *Gaṇita-śāstra* of Pāvaḷūri Mallikārjuna. Up to 24th place, the names are same as those given by Mahāvīra except for the mutual interchange of places of *kṣoṇī* and *mahā-kṣoṇī* with the next pair of *śaṇkha* and *mahāśaṅkha*, respectively. Beyond 24th order, Mallikārjuna's terms are:

> *nidhi, mahā-nidhi, parārdha, parata, ananta, sāgara, avyaya, aprameya, atula, ameya, bhūri,* and *mahā-bhūri* (= 10^{35}).

Although Shukla has not given the date and manuscript reference, we guess that the above Mallikārjuna is same as Pāvulūri Mallana (c. 1100?) who wrote a Telugu version or adaptation of the *Gaṇita-sāra-saṅgraha* (see *Census of the Exact Sciences in Sanskrit* under Mahāvīra). The *Gaṇita-śāstra* mentioned by Shukla may be same as that whose author is given as Mallaya (=Mallana) and which is manuscript No. 551 (in Telugu script) in E. Hultzsch's *Reports of Sanskrit Manuscripts in Southern India,* Vol. I, Madras, 1895 (see S. N. Sen's *Bibliography,* p. 140; and *Census,* A, 4, 365). According to K. R. Rajagopalan (*Bhavan's Journal* dated Nov. 15, 1959), Mallana has mentioned king Pratāparudra (1158–94) and Rājāditya's work (?) (see below). Also note another scholar, Mallikārjuna Sūri (fl. 1178), who wrote a Telugu commentary on *Sūrya-siddhānta* (and another in Sanskrit).

Now we mention another Jaina author, Rājāditya (c.1190 AD), whose *Vyavahāra-gaṇitam* (in Kannada) extends the list of decimal place-names to 40th order. This list is not well-known to the historians of mathematics and is as follows (p. 3 of the work):

> (1) *ekam,* (2) *dāham,* (3) *śatam,* (4) *sābira,* (5) *dāsābira,* (6) *lakṣa,* (7) *dālakṣa,* (8) *koṭi,* (9) *dākoṭi,* (10) *śatakoṭi,* (11) *arbuda,* (12) *nyarbuda,* (13) *kharva,* (14) *mahākharva,* (15) *padma,* (16) *mahāpadma,* (17) *kṣoṇī,* (18) *mahākṣoṇī,* (19) *śaṅkha,* (20) *mahāśaṅkha,* (21) *kṣiti,* (22) *mahākṣiti,* (23) *kṣobha,* (24) *mahākṣobha,* (25) *nadī,* (26) *mahānadī,* (27) *naga,* (28) *mahānaga,* (29) *ratha,* (30) *mahāratha,* (31) *hari,* (32) *mahāhari,* (33) *phaṇi,* (34) *mahāphaṇi,* (35) *kratu,* (36) *mahākratu,* (37) *sāgara,* (38) *mahāsāgara,* (39) *parimita,* and (40) *mahāparimita* ($= 10^{39}$).

It will be seen that the first 24 names in the above list are exactly same as those given by Mahāvīra except for Kannada influence, i.e. using *dāham* (or *dā*) and *sābir* for *daśa* and *sahasra*, respectively.

Lastly, we may point out that a list of 29 place-names* is said to be given by Yallaya (c. 1480) in his commentary on the *Āryabhaṭīya*, II, 2. Shukla (transl., p. 3) has already published the list. The first 24 names are the same as those given by Mahāvīra except that Yallaya goes back to the *ayuta* and *prayuta* for *daśasahasra* and *daśalakṣa*, respectively; and, like Pāvaḷūri Mallikārjuna, the pair *kṣoṇī* and *mahākṣoṇi* interchanges places with the pair *śaṅkha* and *mahāśaṅkha* respectively. Beyond 24th, the terms are: *parārdha, sāgara, ananta, cintya* and *bhūri* ($= 10^{28}$). Yallaya belonged to Skandasomeśva (in Āndhra region) and was influenced in giving his list by the Telugu writer Pāvaḷūri Mallikārjuna rather than the Kannada writer Rājāditya. Yallaya points out that some people use the term *saṅkṛti* for *parārdha* (10^{24}) and makes the surprising statement that the people of Āndhra and Karṇataka "call the number (10^{10}) *(arbuda)* by the denomination *śatakoṭī*" which is otherwise equal to 10^9 only (see *Āryabhaṭīya*) (pp. XLIII–XLIV).

In the end we mention that Kapadia (pp. XX and XLIX) has quoted a peculiar list of 24 decuple terms from a Buddhist work called *Abhidhānappadīpikā,* and another (incomplete) list of 60 decuple terms from Rahul Sankrityayan's commentary on Vasubandhu's *Abhidharmakośa,* III, 94. We leave these for a future discussion. See now *Vijñāna Parisad Anusandhāna Patrikā* 47.1(2004), 1–6.

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*The no. 29 is the average of 18 (items in Śrīdhara's list) and 40 (items in Rājāditya's list)!-(RCG).

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New Indian Values of π from the *Mānava-śulba-sūtra*

1 Introduction

India's oldest written works are the *Vedas*. To assist their proper study, there are six ancillary texts, called *Vedāṅgas* ("limbs of the veda"), namely *Śikṣā* (Phonetics), *Kalpa* (Ritualistics), *Vyākaraṇa* (Grammar), *Nirukta* (Etymology), *Chandas* (Prosody) and *Jyotiṣa* (Astronomy and Astrology). The *Kalpa* deals with rules and methods for performing vedic rituals, sacrifices and ceremonies, and is divided into three categories, namely *Śrauta, Gṛhya* and *Dharma*. The *Śrauta-sūtras*, especially those which belong to the various *Saṃhitās* of the *Yajurveda*, often include treatises that give rules concerning the mensuration and construction of fireplaces and altars, and also deal with allied ecclesiastical matters.

These treatises are often found as separate works and are called *Śulba-sūtras*. They represent, in the coded form, the much older and traditional Indian mathematics developed for construction and transformation of Vedic altars of various forms. The *Śulba-sūtras* are thus the oldest geometrical treatises which are also simply called *Śulbas*. The word *śulba* literally means a cord, rope or string and is derived from the basic root *śulb* (or *śulv*), "to mete out" or "to measure".

The names of the 10 *Śulba-sūtras* are known, namely *Baudhāyana, Āpastamba, Satyāṣāḍha* (whose text is said to be identical with that of *Āpastamba*), *Kātyāyana, Mānava, Maitrāyaṇī* (which is said to be another recension of the *Mānava*), *Vārāha, Vādhūla, Maśaka* and *Hiraṇyakeśī*. They are variously dated, and their exact times of composition or compilation are controversial. The *Baudhāyana Śulba-sūtra* (= *BSS*) is the oldest of them and is generally placed between 800 BC and 500 BC. The *Āpastamba Śulba-sūtra* (= *ASS*), *Kātyāyana Śulba-sūtra* (= *KSS*), and the *Mānava Śulba-sūtra* (= *MSS*) are the other important old works.[1]

Following A. J. E. M. Smeur[2] I shall distinguish between π and π' defined respectively by

Centaurus, 1988 Vol. 31: pp. 114–125.

K. Ramasubramanian (ed.), *Gaṇitānanda*,
https://doi.org/10.1007/978-981-13-1229-8_11

$$\frac{\text{area of circle}}{\text{square on the radius}} = \pi \quad \text{and} \quad \frac{\text{circumference of a circle}}{\text{diameter}} = \pi'.$$

2 Some Rules Related to Circle and Square

For transforming a square into a circle of equal area (approximately), the *BSS* 2.9 (p. 19) prescribes the following rule:

> If it is desired to turn a square into a circle, half the diagonal-cord is stretched from the centre towards the east (which is along the right bisector of a side). By one-third of that which lies beyond (the side) combined with the remainder the desired circle is drawn.

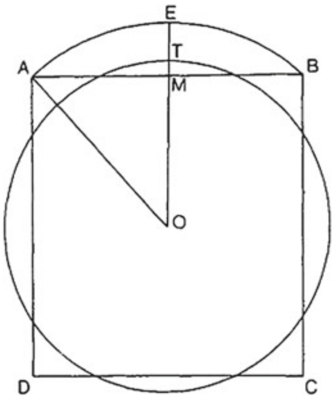

Fig. 1 Baudhāyana rule

That is, if (see Fig. 1)

$$MT = \frac{ME}{3} \tag{1}$$

where ME is that part of OE $(= OA$, the half diagonal) which lies beyond the side AB $(= a)$ then OT $(= r)$ is the radius of the required circle. It can be easily seen that

$$r = \frac{a(2 + \sqrt{2})}{6}. \tag{2}$$

The same rule (but in different wordings) is found in the *ASS* 3.2 (p. 40), *KSS* 3.11 (p. 56) and *MSS* 1.8 (p. 58). Since the true area of the obtained circle will be πr^2, the above rule implies the following approximation of π

$$\pi = \frac{a^2}{r^2} = 18(3 - 2\sqrt{2}). \tag{3}$$

It may be pointed out that the same rule is repeated in *MSS* 11.10 (p. 66) although the language used here is not so clear as in *MSS* 1.8.

The *MSS* 11.13 (p. 66) gives a rough rule for getting the circumference of a circle quickly. It says:

विष्कम्भः पञ्चभागश्च विष्कम्भस्त्रिगुणश्च यः। स मण्डलपरिक्षेपो न वालमतिरिच्यते ॥

The fifth part of the diameter and three times the diameter is the perimeter of a circle. Not even a hair's length is extra.

That is, the perimeter p is related to the diameter d by

$$p = \frac{d}{5} + 3d = \left(\frac{16}{5}\right) d \tag{4}$$

which is an upper bound of the circle's circumference. The relation (4) implies the approximation

$$\pi' = \frac{p}{d} = \frac{16}{5}. \tag{5}$$

Although simple, the above rule is quite differently (and wrongly) interpreted by van Gelder[3] and Kulkarni.[4] According to these scholars, the rule states that the side length of a square of an area equal to that of a given circle is 13 parts out of the 15 parts into which the diameter of the circle is divided. That is,

$$\left(\frac{13d}{15}\right)^2 = \frac{\pi d^2}{4} \tag{6}$$

which implies

$$\pi = \frac{676}{225}. \tag{7}$$

The *BSS* 2.11 (p. 19), *ASS* 3.3 (p. 41) and *KSS* 3.12 (p. 56) all contain rules which will give (6), and the two above scholars perhaps thought (incorrectly) that the same should be the case with *MSS* 11.13 which otherwise gives (4), and not (6).

About the *Gārhapatya* fire altar, the MSS 13.6 (p. 68) says:

A square or a circle is the twofold form of the *Gārhapatya*. Construct the square of (side) one *vyāyāma*, and the circle of (radius) half *puruṣa*.

Now according to *MSS* 4.4 and 4.5 (p. 60) itself, a *vyāyāma* is equal to 96 *aṅgulas* (finger-breadths) and a *puruṣa* is equal to 120 *aṅgulas*. So that the areas of the two forms of the *Gārhapatya* fire altar will be as follows.

$$\text{Square form} = 96^2 = 9216 \text{ sq.}\textit{aṅg}.$$

$$\text{Circle form} = \pi \cdot 60^2 = 3600\,\pi \text{ sq.}\textit{aṅg}.$$

Even with the simple and low value $\pi = 3$, the above two areas are not equal (even approximately)[5]. It is also possible to interpret the second part of the above rule (*MSS* 13.6) as follows:

> Construct the square of measure one (square) *vyāyāma*, and the circle of (area) half (square) *puruṣa*.

This will yield the areas:

$$\text{Square form} = 96 \times 96 = 9216 \text{ sq.}\textit{aṅg}.$$

$$\text{Circle form} = (120 \times 120)/2 = 7200 \text{ sq.}\textit{aṅg}.$$

These cannot either be regarded approximately equal[6]. These *BSS* 7.4, and 7.5 (p. 25), and *ASS* 7.3 (p. 44) state that the *Gārhapatya* fire altar is known to be of one *vyāyāma* measure, and that it is a square according to some and a circle according to others. This ancient Indian tradition of a twofold form is also very well expressed in the *Āpasatamba Śrauta-sūtra*, XVI, 14.1, as[7]

गार्हपत्यचितेरायतनं व्यायाममात्रं चतुरस्रं परिमण्डलं वा।

> The extent of the *Gārhapatya* altar is a square or circle and measures one square *vyāyāma*.

These statements from the two ancient works show that the areas of the two forms of the *Gārhapatya* should be equal. Modern scholars such as Thibaut[8], Datta[9] and others also believe in the equality of the areas. In spite of all this, the *MSS* 13.6 rule does not, as shown above, lead to equal areas. In fact, the equalization of the area will mean

$$3600\pi = 9212$$

which will imply the very unlikely value

$$\pi = \frac{64}{25} = 2.56. \tag{8}$$

However, if we equate the perimeters (instead of areas) of the two forms, namely a square of side one *vyāyāma* and a circle of radius half *puruṣa*, we get

$$4 \times 96 = 2\pi' \cdot 60$$

giving exactly the same value of π' as in (5). Thus, we find that the rules in *MSS* 11.13 and *MSS* 13.6 use the same approximation ($\pi' = 3.2$). It may also be pointed out that besides the equality of areas and perimeters, there seems to be a third type, namely equality of breadths. For example, the ancient scholiast Dvārkānātha Yajva, while commenting on *BSS* 7.5, says:[10]

परिमण्डलपक्षे व्यायामार्धेन परिमण्डलकरणम्। तत्र चतुररत्न्यायामः।

> In the case of circular form [of the *Gārhapatya*], the circle should be drawn with half *vyāyāma* (as radius). There the breadth is four *aratnis* [= 4 × 24 *aṅgulas*].

Thus, the diameter of the circle will be equal to the side of the square but neither their areas nor perimeters will be equal.

Without quoting the Sanskrit text, Datta writes[11] that the "*MSS* (I. 27) states that a square of two by two cubits is equivalent to a circle of one cubit and three *aṅgulas*". Since a cubit (*aratni*) is taken to have 24 *aṅgulas*, the rule gives

$$4 = \pi \left(1 + \frac{3}{24}\right)^2 \tag{9}$$

which yields the approximation

$$\pi = \frac{256}{81}. \tag{10}$$

This value is also implied in some old Indian rules which are quoted by Khadilkar[12] and which are equivalent to

$$a = d - \left(\frac{2d}{18}\right) = d - \frac{d}{9} \tag{11}$$

where a is the side of the square equal in area to a circle of diameter d. Khadilkar adds that the same value ($\pi = 3.1605$ nearly) is also given by the *MSS* but does not quote any reference. It may be pointed out that the Egyptian *Rhind Mathematical Papyrus* (circa 1650 BC). Problem 50 uses the second form of the rule (11) for finding the area of a round field of diameter 9 *khet*.[13]

According to Mazumdar's account of *MSS*:[14]

(i) *Gārhapatya* altar is a circle of radius 14 *aṅgulas* less 1 *yava*.
(ii) *Āhavanīya* altar is a square of side 24 *aṅgulas*.
(iii) *Dakṣinī* altar is a semi-circle of radius $19\frac{1}{2}$ *aṅgulas*.

Apparently these statements are from the commentary of Śivadāsa (see below) in an illustration to *MSS* 1.8 and not from the text or *MSS*. The areas of the above three altars are to be the same. But Mazumdar made a mistake in stating[15] that an *aṅgula* has 8 *yavas*. This mistake led Kulkarni to take the radius of a circle as 13 plus $\frac{7}{8}$ *aṅgulas* and get, by[16]

$$\pi \left(\frac{111}{8}\right)^2 = 24^2 \tag{12}$$

the wrong value

$$\pi = \frac{36864}{12321} = 2.99 \text{ nearly}, \tag{13}$$

The correct relation is, according to *MSS* 4.4 (p. 60), that one *aṅgula* equals 6 *yavas*. So that the radius of the circle will be 13 plus $\frac{5}{6}$ *aṅgulas* as given by Śivadāsa.[17] Hence, the correct equation will be

$$\pi \left(\frac{83}{6}\right)^2 = 24^2 \tag{14}$$

which gives

$$\pi = \frac{20736}{6889} = 3.01 \text{ nearly} \tag{15}$$

Consideration of the semicircular altar will give

$$\left(\frac{\pi}{2}\right) \cdot \left(\frac{39}{2}\right)^2 = 24^2 \tag{16}$$

which yields

$$\pi = \frac{512}{169} = 3.03 \text{ nearly} \tag{17}$$

3 New Values of π

Sen and Bag's remark[18] that in *MSS* 11.14 and 11.15 "ordinary squares are drawn without any mathematical significance" shows that they have not understood the rules fully. We shall present a simple and meaningful interpretation of these two verses. *MSS* 11.14 (p. 66) states:

दशधा छिद्य विष्कम्भं त्रिभागानुद्धरेत्ततः। तेन यच्चतुरस्रं स्यान्मण्डले तदपप्रथिः॥

After dividing the diameter (of a circle) into ten (equal) parts, leave out three parts there from. The Square which is drawn with the remainder (as a side) has its extension upto the circle.

That is, if d is the diameter of a circle, then $7\frac{d}{10}$ will be (approximately) the side of the inscribed square or

$$AB = \frac{7d}{10}. \tag{18}$$

Since the actual value of AB is $\frac{d}{\sqrt{2}}$, the above implies the simple approximation

$$\sqrt{2} = \frac{10}{7}. \tag{19}$$

It may be remarked that, since $\frac{1}{\sqrt{2}}$ is the same as $\frac{\sqrt{2}}{2}$, the above approximation may be considered equivalent with $\sqrt{2} = \frac{7}{5}$ also (Fig. 2).

The next verse *MSS* 11.15 (p. 66) states:

चतुरस्रं नवधा कुर्याद् धनुःकोट्यस्त्रिधा त्रिधा। उत्सेधात्पञ्चमं लुम्पेत्पुरीषेणेह तत्समम्॥

Make a ninefold division of the square and [the elongated trisecting lines will] divide the arcual segments [of the circumscribing circle] into three parts each. Leave out the fifth part from the altitude [OE]. The circular disc formed by the remainder [OT] as radius is equal to that [square].

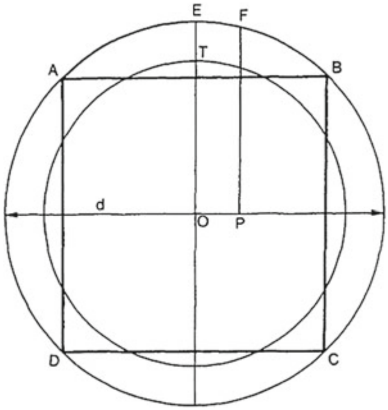

Fig. 2 *Mānava Śulba-sūtra* rule

That is, if (see Fig. 2)

$$r = OT = OE - \left(\frac{OE}{5}\right) = \left(\frac{4}{5}\right) OE \tag{20}$$

then (approximately)

$$\pi r^2 = \text{area of the square} = a^2 \tag{21}$$

where a is the actual side of the square. Now the actual value of OE is $\frac{a}{\sqrt{2}}$, so that the above rule (20) gives

$$r = \frac{4a}{5\sqrt{2}}. \tag{22}$$

Hence, by (21) we get the approximation

$$\pi = \frac{a^2}{r^2} = \left(\frac{5\sqrt{2}}{4}\right)^2 = \frac{25}{8}. \tag{23}$$

However, if we use the approximation (18) of the previous verse with $OE = \frac{d}{2}$, then from (20) $r = \left(\frac{4}{5}\right) \cdot \left(\frac{d}{2}\right)$, and from (18) $a = \frac{7d}{10}$. Hence,

$$\pi = \frac{a^2}{r^2} = \frac{49}{16}. \tag{24}$$

Another value is obtained if we use the approximation (19) in (22). Thus (22) will be

$$r = \frac{14a}{25} \tag{25}$$

yielding

$$\pi = \frac{a^2}{r^2} = \frac{625}{196} \tag{26}$$

which gives a value in excess, but better than (5).

Now we shall give a possible and simple derivation of the basic relation (20) for the suggested method of circling the square. The traditional older method found in all the four important *Śulba-sūtras* (Sect. 2) is represented by the relation (2)

$$r = \frac{a\left(2 + \sqrt{2}\right)}{6}.$$

Using the approximation (19) for $\sqrt{2}$, we get from (2)

$$r = \frac{4a}{7} = OT. \tag{27}$$

But (Fig. 1) $OE = OA = \frac{AC}{2} = \frac{(\sqrt{2}a)}{2}$ which, by using the same approximation (19), becomes

$$OE = \frac{5a}{7}. \tag{28}$$

Hence, by (27) and (28),

$$OT = \left(\frac{4}{5}\right) OE = OE - \left(\frac{OE}{5}\right)$$

which is height diminished by its own fifth part, and is the required relation (20). Moreover, the new rule is far better than the older traditional rule because the approximation (23) is more accurate than (3).

The translation of the *MSS* 11.15 as given by Sen and Bag[19] is incomplete and does not serve any purpose. Kulkarni has quoted the better translation of Van Gelder but he takes *PF* (instead of *OE*) for the height *(utsedha)*.[20] After making some tedious calculations (and an assumption for $\sqrt{17}$), he gets the approximation[21]

$$r = \frac{11a}{20} \tag{29}$$

which yields

$$\pi = \frac{400}{121} = 3.3 \text{ nearly} \tag{30}$$

instead of our (23). Since this value is quite high, he thinks that the original reading in the *MSS* text may be equivalent to

$$r = (utsedha) - \frac{(utsedha)}{6}$$

by which he finds r to be $\frac{55a}{96}$, and π as nearly 3.047.[22]

4 Table of Values of π

Sl.No.	Value of π	Reference/Remark
1	$\frac{64}{25} = 2.56$	*MSS* 13.6 (cf. Serial No. 13)
2	$\left(\frac{192}{111}\right)^2 = 2.99$ nearly	Wrong calculation by Mazumdar and Kulkarni (see Serial No. 4)
3	$\frac{676}{225} = 3.004$ nearly	*MSS* 11.13 misinterpreted by van Gelder (see Serial No. 13)
4	$\left(\frac{144}{83}\right)^2 = 3.01$ nearly	Śivadāsa's example (for Serial No. 2)
5	$\frac{512}{169} = 3.03$ nearly	Rule quoted by Mazumdar
6	$\left(\frac{96}{55}\right)^2 = 3.047$ nearly	Kulkarni's guess for *MSS* 11.15
7	$\frac{1296}{425} = 3.049$ nearly	Exact calculation for Kulkarni's guess (Serial No. 6)
8	$\frac{49}{16} = 3.063$ nearly	*MSS* = 11.15 with *MSS* 11.14
9	$18(3-2\sqrt{2}) = 3.094$ nearly	*MSS* 1.8 and *MSS* 11.10
10	$\frac{25}{8} = 3.125$	Our interpretation of MSS 11.15
11	$\frac{256}{81} = 3.1605$ nearly	Rule quoted by Datta
12	$\frac{625}{196} = 3.18$ nearly	*MSS* 11.15 with *MSS* 11.14
13	$\frac{16}{5} = 3.2 = (\pi')$	*MSS* 11.13; also our new interpretation of *MSS* 13.6
14	$\frac{400}{121} = 3.3$ nearly	Kulkarni's interpretation and calculation for *MSS* 11.15
15	$\frac{225}{68} = 3.31$ nearly	Accurate calculation for Kulkarni's interpretation of *MSS* 11.15 (Serial No. 14)

Details and exact location of references are given in the main body of the present paper. It will be seen from the Table that the best value of π therein is $\frac{25}{8}$ (Serial No. 10). In France, this value was given by La Comme in 1836, and in England by

James Smith in 1860 (see *Mathematics Magazine*, Vol. 23. pp. 226–227). It is an old Babylonian value.

References and Notes

1. There exist various editions and translations of these four *Śulba-sūtras*. Unless otherwise mentioned, text and page references in the present paper are according to S. N. Sen and A. K. Bag, The *Śulbasūtras*, INSA, New Delhi, 1983
2. A. J. E. M. Smeur, "On the value equivalent to π in Ancient Mathematical Texts: A New Interpretation", *Archive for History of Exact Sciences*, Vol. 6 (1970), 249–270, p. 252.
3. J. M. Van Gelder (translator), The *Mānava Śrautasūtra*, New Delhi, 1960; p. 300.
4. R. P. Kulkarni, "The Value of π known to Śulbasūtrakāras", *Indian Jour. Hist. Science*. Vol. 13 (1978), 32–41, p. 34.
5. Van Gelder (ref. 3), p. 303, has given the two areas as 9216 and 11307 $a\dot{n}g^2$ respectively.
6. Of course the mean of 3600 π and 7200 will give a value nearly equal to 9216.
7. Quoted in S. D. Khadilkar (editor), *Kātyāyana Śulba Śūtra*, Poona , 1974, p. 60.
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10. V. Bhattacharya (editor), *Baudhāyanaśulbasūtram* (with two commentaries), Varanasi, 1979, p. 70. Also see Datta (ref. 9) pp. 180–181.
11. Datta (ref. 9), p. 149. The same rule is quoted by Kulkarni (ref. 4), p. 37, and by K. S. Shukla in the *Indian Jour. Hist. Science*, Vol. 15 (1980), p. 151. It may be that the rule belonged to the commentary part of the manuscript (C.U.23 = No. I. E. 17 of Asiatic Society of Bengal) consulted by Datta (ref. 9, p. 230).
12. Khadilkar (ref. 7), p. 66. The rules are quoted from the commentary of Mahīdhara on the *KSS*.
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14. N. K. Mazumdar, "Mānava Śulba-sūtram", *Jour. Dept. of Letters* (Calcutta University), Vol. 8 (1922), 327–342, p. 335.
15. *Ibid.,* p. 333.
16. Kulkarni (ref. 4), p. 37. There is a printing error also (R.H.S. should be 576).
17. See Van Gelder (ref. 5), p. 287.
18. Sen and Bag (ref. 1), p. 278.
19. *Ibid.,* p. 136.
20. Kulkarni (ref. 4), pp. 38–39. FP is parallel to EO and trisects AB.
21. *Ibid.,* p. 39. The actual or exact value (i.e. without assuming any approximation) of r will be $\frac{2\sqrt{17}a}{15}$ and that of π will be $\frac{225}{68}$ (= 3.31 nearly).
22. *Ibid.,* p. 40. He has given r in the unsimplified form as $\frac{165a}{288}$. The exact values of r and π will be $\frac{5\sqrt{17}a}{36}$ and $\frac{1296}{425}$ (= 3.049 nearly).

The *Lakṣa* Scale of the *Vālmīki Rāmāyaṇa* and Rāmā's Army

C. N. Srinivasiengar was perhaps the first historian of mathematics to give a modern exposition of the Lakṣa Scale as found in the *Yuddhakāṇḍa* (the Sixth Book) of the *Vālmīki Rāmāyaṇa*, the national epic of India. This is a numeration system in which counting (here beyond *Koṭi* or crore) proceeds by the scale factor of one lakh or 10^5. He quoted 11 lines of the relevant Sanskrit text, but nowhere mentioned the edition or even the recension of the *Rāmāyaṇa* which he used as his source. This created a difficulty for serious scholars especially in view of the fact that different recensions and versions of the epic are available with quite different chapter and verse numbering as well as with variant readings.

I have consulted four different editions published from Bombay, Lahore, Baroda and Gorakhpur (see details in the Bibliography). It was found that the text quoted by Srinivasiengar closely resembles that of the Bombay edition. Most probably it was perhaps this very edition or version which was used by him. His indicated reference vi, 28 also tallies.

However, on making a close comparison, it is found that the Bombay edition contains 12 relevant lines instead of 11 quoted by him. The 11th line of the original text is missing in his quoted set of verses, and this has made his exposition not only imperfect but wrong towards the end. The Gorakhpur edition also confirms the mistake of his omission. In this small article, attempt will be made to present a correct form of the *Lakṣa* Scale.

There is one more important point which was noted. The Bombay text uses the term *Śaṅkha* to denote the number 10^{12}, while the other editions have the word *Śaṅku* for the same purpose. This also indicates that Srinivasiengar perhaps used

Gaṇita Bhāratī, Vol. 12, Nos. 1–2 (1990), pp. 10–16. Also see *Indian Journal History of Science*, 43 (2008), pp. 79–82.

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the Bombay version. Of course, both the above terms (*Śaṅkha* and *Śaṅku*) were in common use to denote 10^{12} during ancient and medieval India. For example, they can be found in Sanskrit works on mathematics of various authors from Śrīdhara to Bhāskara II and their commentaries [Gupta, 1983, 11–12].

Since *Śaṅku* is found used more frequently than *Śaṅkha*, we shall give the full relevant text for the *Rāmāyaṇa Lakṣa* Scale from the Gorakhpur edition (Vol. II, p. 1124). The 12 lines from the 28th *Sarga* of the *Yuddhakāṇḍa* are as follows:

> शतं शतसहस्राणां कोटिमाहुर्मनीषिणः। शतं कोटिसहस्राणां शङ्कुरित्यभिधीयते ॥ ३३॥
> शतं शङ्कुसहस्राणां महाशङ्कुरिति स्मृतः। महाशङ्कुसहस्राणां शतं वृन्दमिहोच्यते ॥ ३४॥
> शतं वृन्दसहस्राणां महावृन्दमिति स्मृतम्। महावृन्दसहस्राणां शतं पद्ममिहोच्यते ॥ ३५॥
> शतं पद्मसहस्राणां महापद्ममिति स्मृतम्। महापद्मसहस्राणां शतं खर्वमिहोच्यते ॥ ३६॥
> शतं खर्वसहस्राणां महाखर्वमिति स्मृतम्। महाखर्वसहस्राणां समुद्रमभिधीयते ॥
> शतं समुद्रसाहस्रमोघ इत्यभिधीयते ॥ ३७॥ शतमोघसहस्राणां महौघ इति विश्रुतः॥

The language is simple and straightforward and may be translated verse by verse thus:

A hundred of hundred-thousand is said to be *Koṭi* by the learned.
A hundred of thousand-*koṭi* is termed *Śaṅku* (33).
A hundred of thousand-*śaṅku* is known as *Mahā-śaṅku*.
A hundred of thousand-*mahāśaṅku* is called *Vṛnda* (34).
A hundred of thousand-*vṛnda* is known as *Mahā-vṛnda*.
A hundred of thousand-*mahāvṛnda* is called *Padma* (35).
A hundred of thousand-*padma* is known as *Mahā-padma*.
A hundred of thousand-*mahāpadma* is called *Kharva* (36)
A hundred of thousand-*kharva* is known as *Mahā-kharva*.
Thousand-*mahākharva* is termed *Samudra*.
A hundred of thousand-*samudra* is termed *Ogha* (37).
A hundred of thousand-*ogha* is heard to be *Mahaugha*.

The meaning of these lines is given in the form of a table for better comprehension.

It may be noted that the continuity of the scale factor of one *lakṣa* or lakh is broken at one place in the above table. Perhaps this was done to attain the convenient sexagesimal power at the end. Another point is that the word *lakṣa* itself is not used in describing the above *Lakṣa* Scale Counting System from 10^7 to 10^{60}.

From the way in which the above counting system is described, it is clear that it is given as a traditional method of numeration of very large sets. It is used in the *Vālmīki Rāmāyaṇa* for narrating the strength of Rāma's army that reached *Laṅkā* after crossing the newly constructed bridge across the ocean (the bridge is said to be 100 *yojanas* long and 10 *yojanas* wide). The exact strength of the army was told to *Rāvaṅa* by one of his spies in the following words (*Sarga* 28, p. 1124)[†]:

[†] The Bombay edition is explicit समुद्रेण शतेनैव.

Table 1 India *Lakṣa* scale

100 lakh	= 1 *koṭi*	= 10^7
1 lakh *koṭi*	= 1 *śaṅku*	= 10^{12}
1 lakh *śaṅku*	= 1 *mahāśaṅku*	= 10^{17}
1 lakh *mahāśaṅku*	= 1 *vṛnda*	= 10^{22}
1 lakh *vṛnda*	= 1 *mahavṛnda*	= 10^{27}
1 lakh *mahāvṛnda*	= 1 *padma*	= 10^{32}
1 lakh *padma*	= 1 *mahāpadma*	= 10^{37}
1 lakh *mahāpadma*	= 1 *kharva*	= 10^{42}
1 lakh *kharva*	= 1 *mahākharva*	= 10^{47}
1000 *mahākharva*	= 1 *samudra*	=10^{50}
1 lakh *samudra*	= 1 *ogha*	= 10^{55}
1 lakh *ogha*	= 1 *mahaugha*	= 10^{60}

एवं कोटिसहस्रेण शङ्कूनां च शतेन च। महाशङ्कुसहस्रेण तथा वृन्दशतेन च॥३८॥
महावृन्दसहस्रेण तथा पद्मशतेन च। महापद्मसहस्रेण तथा खर्वशतेन च॥३९॥
समुद्रेण च तेनैव महौघेन तथैव च।† एष कोटिमहौघेन समुद्रसदृशेन च॥४०॥

In this way (the strength of Rāma's army is) a thousand *Koṭi* and a hundred *Śaṅku*; and a thousand *Mahā-śaṅku* plus a hundred *Vṛnda* (38); and a thousand *Mahā-Vṛnda* plus a hundred *Padma*; and a thousand *Mahā-padma* plus a hundred *Kharva* (39); and same (hundred) *Samudra* plus the same (number) of *Mahaugha*; and a *Koṭi Mahaugha*. It is like a sea.

That is, the strength of the army (see Table 1)

$$\begin{aligned} &= 1000 \cdot 10^7 + 100 \cdot 10^{12} + 1000 \cdot 10^{17} + 100 \cdot 10^{22} + 1000 \cdot 10^{27} \\ &\quad + 100 \cdot 10^{32} + 1000 \cdot 10^{37} + 100 \cdot 10^{42} + 100 \cdot 10^{50} \\ &\quad + 100 \cdot 10^{60} + 10^7 \cdot 10^{60} \end{aligned}$$

So that the strength is given by

$$\begin{aligned} N = &\ 10^{10} + 10^{14} + 10^{20} + 10^{24} + 10^{30} + 10^{34} + 10^{40} + 10^{44} \\ &+ 10^{52} + 10^{62} + 10^{67}, \end{aligned}$$

excluding the commander-in-chief (Sugrīva) and his (four) ministers.

It may be pointed out that the last line of the Sanskrit text admits other interpretations also. The translation given here is somewhat supported by the commentaries included in the Bombay edition (p. 230, under verse 41 there) and by the exposition of Srinivasiengar. According to the translation in the Gorakhpur edition, we should have 10^{69} instead of 10^{67} in the above representation of N. The last figure in N can also be replaced by $(10^{67} + 10^{57})$ according to another interpretation, and etc.

Table 2 Older *Lakṣa* scale

Denomination (number)	Lahore ed. term	Baroda edition (extra verses)
10^7	*Koṭi*	*Koṭi*
10^{12}	*Śaṅku*	*Śaṅku*
10^{17}	*Vṛnda*	*Mahāśaṅku*
10^{22}	*Mahāvṛnda*	*Vṛnda*
10^{27}	*Padma*	*Mahāvṛnda*
10^{32}	*Mahāpadma*	*Padma*
10^{37}	- - -	*Mahāpadma*
10^{42}	- - -	*Kharva*
10^{47}	- - -	*Samudra*
10^{52}	- - -	*Mahaugha*

Anyway, the number N is really very very high. Its hugeness may be visualized in an interesting manner as follows:

We know that the equatorial diameter of the Earth (on which we live) is about 7927 miles. Hence by using the usual formula ($S = 4\pi R^2$), the area of Earth's surface (both land and sea) can be easily found to be about 20 crore square miles. In square feet this will be

$$= 20 \times 10^7 \times (1760 \times 3)^2$$

which can be seen to be less than 10^{16} square feet.

Thus, if we calculate the area of ground needed for the army even at the bare rate of one square foot per warrior, an area equal to that of our Earth can hardly accommodate 10^{16} warriors (without bothering whether they get land or water). In this way, by looking at the various terms in N, we find that 10^{20} warriors will need 10000 Earths, 10^{24} warriors will need 10 crore Earths, and so on.

From this we get some idea of the hugeness of the monstrous number N. For human beings it is not possible to imagine the strength of Rāma's army. It all looks to be a world of super-human beings or gods. What to say of the present Sri Lanka, even our present globe of Earth was not sufficient to accommodate the army.

We get some less imaginative figures when we look into the critical editions of *Vālmīki Rāmāyaṇa* which are based on older manuscripts. The Lahore edition deals with the relevant matter in *Sarga* 4 of the *Yuddhakāṇḍa*. Verses 51–53 (p. 23) give a shorter list of names of terms which denote various denominations in the *Lakṣa* Scale. These are shown in Table 2—column II.

The short list in column II ends with *Mahāpadma* (= 10^{32}). It should be noted that, since *Mahāśaṅku* is missing here, the denominational values of terms beyond *Śaṅku* will be different here (for column II) than those in Table 1.

In terms of values in Table 2 (column II), the strength of Rāma's army is given in the Lahore edition (verses 54–55a) to be as

$$\begin{aligned} & 1000\ \textit{koṭi} + 100\ \textit{śaṅku} + 1000\ \textit{vṛnda} + 100\ \textit{mahāvṛnda} \\ & +1000\ \textit{padma} + 100\ \textit{mahāpadma} \\ = {} & 10^{10} + 10^{14} + 10^{20} + 10^{24} + 10^{30} + 10^{34} \end{aligned}$$

which is exactly same as represented by first six terms in N (which, of course, is a much bigger number).

Thus the basic definition of the *Lakṣa* Scale and the similarity of the method of describing Rāma's army are both found here also. The only difference is that the span of the table is short here, but the mathematical principle is essentially present. It appears that the old list of denominational names of the *Lakṣa* Scale was smoothed out and extended to *Mahaugha* subsequently, as happened in the case of the Medhātithi's List of Vedic decuple terms in Decimal Scale System [Gupta, 1983, pp. 9–10].

As far as the very critical and detailed Baroda edition of the *Yuddhakāṇḍa* is concerned, it is stated to be based on 34 manuscripts (instead of 10 in the Lahore edition). The relevant subject is dealt here in *Sarga* 19 (pp. 118–124). But most of the relevant verses are given in the footnotes, and not in the main body of the (accepted) text.

The strength of Rāma's army, as mentioned in the main text, verse 33 (p.123), is given to be

$$1000\ \textit{koṭi} + 100\ \textit{śaṅku} = 10^{10} + 10^{14}$$

The additional verses, which will make the army's strength same as given in the Lahore and Gorakhpur editions, are mentioned in foot-notes (p. 124). Moreover even the number $(10^{10} + 10^{14})$ is not small and, significantly, represents the first two terms in N.

Then there are only ten lines (instead of 12) defining the *Lakṣa* Scale which are mentioned in the foot-notes. Thus *Mahākharva* and *Ogha* are missing (but these are found separately given under material from other manuscripts). But the definition of the scale factor (in the footnote verses) is uniformly followed from *Koṭi* ($= 10^7$) to *Mahaugha*. The *Lakṣa* Scale from these verses is shown in Table 2 (column III).

It is interesting to note that this list of *Lakṣa* Scale permits a different interpretation of the 6 lines which we have given above from the Gorakhpur edition for the strength of Rāma's army. Of course, the same lines can also be found in the Baroda edition in which one line is given in the main text (verse 33) and the other as extra lines in foot-notes (p. 124). The crucial line is:

समुद्रेण च तेनैव महौघेन तथैव च ।

The new interpretation is based on taking the phrases '*tenaiva*' and '*tathaiva*' both to mean 'following the same pattern' (as in the previous four lines). This pattern is (as can be seen from those 4 lines)

$$1000x + 100y\,,$$

where x and y are various denominations of the *concerned Lakṣa* Scale, taken in pairs (that is, two at a time). By this rule, the strength of the army will be, using column III of Table 2, given by

$$N' = 10^{10} + 10^{14} + 10^{20} + 10^{24} + 10^{30} + 10^{34} + 10^{40} + 10^{44} + 10^{50} + 10^{54} + 10^{59}$$

Here the last term is from *Koṭi-Mahaugha*.

Whatever be different forms of Lakṣa Scale, the most important point is that the technical terms *Koṭi* and *Śaṅku* (of Śaṅkha) are found even in the oldest manuscripts (see verse no. 4 in the relevant *Sarga* of all the four editions.) And since the scale difference between these two is equal to

$$\frac{10^{12}}{10^{7}} = 10^{5} = \textit{lakṣa}$$

it is clear that the basic idea of the *Lakṣa* Scale is very old. Hence, as usually happens in the historical growth of science, the idea was then developed in a fuller *Lakṣa* Scale for numeration of very very large numbers from 10^7 to 10^{60}. The narration of Rāma's army also followed the same pattern as given in the older or original versions, whatever be the other historical, mythological and related matters.

Lastly, the date of composition of the *Vālmīki Rāmāyaṇa* is a difficult and controversial question (as usually happens with such works). Moreover, the date of Rāma (or Rāma's story), the date of Vālmīki (or of his original composition), and the date of the present form of the text of the epic are, historically speaking, all different things. Various scholars have placed the work from 600 BC to 400 AD which has been reasonably narrowed down to the period 200 BC–200 AD for the epic [Roy, 1963, p. 58]. Sengupta [1947, p. ix] considers the present text to be not earlier than circa 450 AD.

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The Chronic Problem of Ancient Indian Chronology

1 Introduction

Chronology is the backbone of history, and its knowledge is essential for a historian dealing with any period, culture area or subject. There cannot be a coherent history without a chronological order. Proper historical writing is not possible unless there is a sound chronology.

Unfortunately in case of India, the problem of chronology continues to be very serious especially with regard to the prehistoric and ancient periods. The dates of most of the important events and literary sources are full of serious controversies and divergent opinions. What to say about the absolute chronology, even a relative chronology is not free from challenges.

The situation has been creating hampering factors in dealing with Indian history and historiography whether of arts or of sciences. In this paper, we shall briefly highlight the facts and the situations, present problems and offer some preliminary suggestions.

2 Arbitrary and Controversial Dates

Let us see the matter in some details. Firstly, fantastic chronological claims are not lacking. The knowledge contained in the *Sūrya-siddhānta* is stated to be communicated some time before the end of the elapsed *Kṛtayuga* which occurred in 2163102 BC according to the work itself.[1] V. R. Lele observes that the Indian people "possessed upto-date knowledge of astronomy since 26000 years before Śaka or even before that date" (about 25922 BC).[2]

Dinanatha Sastri Chulet has attempted to show in his *Vedakāla Nirṇaya*[3] that the *Vedas* are more than 18000 years old. A recent writer brings down the claim of

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antiquity of the Vedic literature to a moderate figure 8000 BC[4] More moderate claims for the antiquity of the *Vedas* upto 4000 BC or earlier are met quite frequently. It seems that there is a sort of race for assigning earlier dates on any whimsical ground.

In this connection, it must be noted that the date of a book may be much later than the date of an event which it records. Clarifying this point, P. C. Sengupta writes[5]:

> .... the antiquity of the Vedic culture is one thing while the date of the present *Ṛgveda* (the oldest of the four Vedas) is another, the date of the (*Mahā*) *Bhārata* Battle is one thing while the date of the modern *Mahābhārata* (about 400 BC to 300 BC, according to him) is another, the date of Rāma or the Rāma story is one thing while the date of the modern (Vālmīki) *Rāmāyaṇa* (about AD 450 or later, according to him) is another.

Unless the above distinction is kept in mind, correct historical chronology cannot be visualized.

Another chronological difficulty is created by the fact that Hindus generally attribute a divine origin to all sciences. This practice automatically attaches a hoary past to the related works. Thus, the exposition of Chap. 54 (on astronomy and mathematics) of the *Nārada-purāṇa* commences with the line[6]:

> ज्यौतिषाङ्गं प्रवक्ष्यामि यदुक्तं ब्रह्मणा पुरा।
>
> (Sanandana says) I shall now set out the *Jyotiṣa* part which enunciated in antiquity by (god) Brahmā.

The *Garuḍa-purāṇa* (59.1) similarly states that the *Jyotiṣa* science of 4 lakhs stanzas was communicated to god Rudra by god Keśava.[7] The extant *Sūrya-siddhānta* is said to be based in the astronomy taught by the Sun (god) himself to Maya (demon) more than 20 lakhs years ago.[8] In fact, the astronomical works of all the eighteen classical expounders of *Jyotiṣa-śāstra* are considered to be *apauruṣeya* (non-human) writings, being attributed to ancient sages. Even the historical work *Āryabhaṭīya* of Āryabhaṭa I (born AD 476) is stated to be "the same as the ancient *Svāyambhuva* (that is, which was revealed to Brahmā)" according to the last verse of the work itself.[9]

These divine or superhuman attributions seem to be deliberately made in order to claim great antiquity and unquestionable authority for the works. Whatever be that, this practice does a great harm and injustice not only to history and historiography, but also to the authors themselves (who are deprived of the credit for their contributions). By it, we are unable to know the real science-authors who are supposed to be human beings. Anyway, if such is the state of affairs, how can we have a historical chronology?

3 Special Examples for Scientific Works

The *Śulba-sūtras* are a class of ancient Sanskrit works whose importance for history of mathematics is enormous. They are considered to be part of the Vedic literature (via the six *Vedāṅgas*). Ever since Datta[10] carried out his detailed study of these works, Indian historians of mathematics have been following him in assigning the time 800 BC to 500 BC to them (especially the four major *Śulba-sūtras*, namely *Baudhāyana*, *Āpastamba*, *Kātyāyana* and *Mānava*). However, some Western scholars, like A. B. Keith,[11] have placed them in the period 500 to 200 BC; S. Prakash[12] suggested, perhaps by combining the above two, the span 800 to 200 BC, while P. V. Kane[13] dated them between 800 and 400 BC After some consideration, Ramgopal[14] came to the conclusion that the first dates (800 to 500 BC) were alright.

On the other hand, one Indian scholar[15] pushed back the date of the *Śulba* period from 1200 to 800 BC And although scholars seem to be unanimous in considering *Baudhāyana Śulba-sūtra* to be the oldest such work, the problem is whether it should be assigned 500 BC, 800 BC, 1000 BC or even 1200 BC A variation of more than 500 years in the date of the same important work is bound to create serious difficulty in the history of not only Indian but of the world mathematics, keeping in view, e.g. the date of Thales (about 600 BC), the father of Greek mathematics.

A more interesting case is that of the *Vedāṅga-jyotiṣa* (=VJ). Three versions of *VJ* are well-known, namely *Ṛg*-, *Yajur*- and *Atharva*- (the last is, admittedly, of a later date). *Ṛg* and *Yajur* versions are similar and both are ancient. But even for these two, about a score of dates have been suggested which are so divergent as spread over a range of some 30000 years. Some selected assigned dates are as follows:

1. V. R. Lele says that the *VJ* was probably composed 28000 years ago.[16]
2. According to Sengupta,[17] the year 1429 BC is the true date of a *VJ* tradition.
3. H. T. Colebrooke got the date 1410 BC[18]
4. T. S. Kuppanna Sastry gives two dates namely 1370 BC or 1150 BC[19]
5. Gorakh Prasad favours 1200 BC[20]
6. A. N. Singh gives 1000 BC[21]
7. Ramatosh Sarkar's date for *VJ* is about 600 BC[22]
8. D. Pingree puts the *Ṛg* Recension in 400 BC but the *Yajur* Recension in 400 AD[23]
9. Bag says that "the modern scholars are more or less unanimous" in fixing the date of *VJ* in 200 BC[24]
10. A. Weber even suspects *VJ* to have been written in the fifth century AD[25]

The awkward part about these and many other divergent dates is that they often appear side by side in the same collected work. The situation leaves the readers in a confusing condition, and scholars have been expressing disgust about the matter. Some instances of this may be cited. As early as in 1951, Jospeh Needham, while reviewing the papers of the Symposium on history of Science and Technology in India and S. E. Asia, pointed out the divergence if the "quite unacceptably early" dating of the *VJ* at 1400 BC (by K. S. Shukla) from the dates 600 BC to 200 BC given by A. S. Altekar and R. C. Majumdar.[26] Some four decades later, N. Sivin reviewed[27] the papers of another conference (IAU Colloquium 91 on History of Oriental Astronomy). He also emphasized the serious disagreement of dates about landmarks by citing that S. K. Chatterjee dates *VJ* around 1300 BC but just 20 pages later S. N. Sen says that it was prepared around 400 BC

Thus, we do not seem to bother about any chronological agreement, coherency or even about the working dates recommended by a committee (see Sect. 5), and assign any date which we feel convenient or suitable. Obviously no historical purpose is served by such attitudes and practices. Sooner or later the situation has to be changed or improved, otherwise we will be simply wasting time, energy and resources without any progress or significant achievement in this connection.

4 Some Related Matters

As is known to scholars, some works are of composite nature. Such a work is not composition of a single author. Its different portions are pieces from the writings of different times, and of different authors whose identities are not disclosed. Great epics like *Vālmīki Rāmāyaṇa* and *Mahābhārata* and most of the (*Mahā)-purāṇas* (in their present forms) easily come under the category of composite works. For example, this is very true for the *Nārada-purāṇa* which was made a compendium of knowledge of different branches of Sanskrit literature by extracting portions and pieces from the classics on the subject. A large number of verses in this *Purāṇa* have been drawn from the *Sūrya-siddhānta* and the famous *Līlāvatī* (twelfth century AD).[28]

The extant *Sūrya-siddhānta*, the most popular work of Hindu astronomy, is an example of a scientific work of this nature. The original work, as Shukla clearly points out,[29] has been subjected to correction, emendation and modification, etc., from time to time, and the present *Sūrya-siddhānta* is the latest redaction or version of that work. In such a case, a more relevant chronological question is to ask as to which specific portions are assigned which dates, to find the date of the latest version, and broadly indicate the two limits of the time between which the whole can be placed.

What to say of sources of the different portions of a composite work, it is not often easy to sort out the portions and date them. Critical editions of such works are frequently useful in sorting out later additions from the earlier portions and assign some chronology to them, but there are limitations. Whatever be that date of a piece in a composite work cannot be taken to be the date of the whole work as such. Of course, the dating of works of known joint authorship is a different matter and may not present serious chronological problem. The *Kriyākramakarī* commentary[30] on the *Līlāvatī* was composed partly by Śaṅkara Vāriyar, and, after his death, the rest by Mahiṣamaṅgala Nārāyaṇa (about AD 1560).

It frequently happened, especially with religious works, that the subsequently written items were attributed to older names to get a stamp of authority. For example, many of the *Purāṇas* are attributed to Vyāsa. The interpretation or explanation that Vyāsa is not the name of a person but only a general title or designation does not solve the problem of authorship or of chronology. Then, there have been practices of adding appendices to existing texts and the tradition of formulating subsidiary apocryphal texts. The *Pariśiṣṭas* (Appendices) of the *Atharva-veda* are well-known.

Often the copyist of the manuscripts adds material at different places of the work, and we would be lucky if he himself mentions this. A good example is that of the copyist Govinda (son of Bhaṭṭa Vāhnika) who added four chapters by way of supplementing the material of the *Tripraśnādhikāra* part of the *Vaṭeśvara-siddhānta*.[31]

For ancient chronological matters, there is also the so-called Astronomical Methods of dating, that is, finding the date on the basis of the astronomical data or evidence found in literary and other works. But here also, due to different possible interpretations, we often do not get unique dates. In some cases, it has been shown that the data is not enough to determine a definite date.[32]

Moreover, the Astronomical Method is based on the fallacious assumption that the data recorded was found by the actual observations (which could not be very accurate due to limitations of the instruments, needs of those time, and etc.), and that the observed data was recorded as such without rounding-off.

Let us take an example. The value of the obliquity of the ecliptic, ϵ, as mentioned in almost all Hindu astronomical works (including *Sūrya-siddhānta*) is $24°$. We know that the obliquity has been decreasing at the rate of about 47 seconds per century. A simple formula for finding its value in the year t AD is[33]

$$\epsilon = 23°27'8''.26 - 0''.4684\ (t - 1900)$$

By using this, it can be found that ϵ was $24°$ when $t = -2309$, that is, in 2310 BC Thus, it may be claimed that the *Sūrya-siddhānta* was composed near this date. In fact, this is how Samuel Davis argued as early as in 1789 although his date was slightly different.[34] Such conclusions were alright if $24°$ was the exactly recorded value of the actually (and accurately) made observations for which there is no definite evidence. Moreover, $24°$ is the value of obliquity used by the Hindus for about 2000 years (if it had been found by observations from time to time, the situation would have been different).

5 Chronology Committee

Serious-minded historians have always been worried about the chronic problem of Indian chronology, more so about the very divergent views held by various scholars. The difficulty was also realized as early as 1950 when the first significant symposium on History of Sciences in South Asia was organized at Delhi by INSA (which was then called NISI). At the very outset of the Symposium, a Chronology Committee was appointed under the Chairmanship of the noted historian R. C. Majumdar.[35] It met on November 5, 1950 and, after lengthy discussions, recommended the following chronological tables as a working hypothesis.[36]

Age of the *Ṛgveda*	2000 BC–1500 BC
Age of *Saṃhitās* and *Brāhmaṇas*	1500 BC–800 BC
Age of Old *Upaniṣads*	900 BC–500 BC
Vedāṅga-jyotiṣa (Present text)	500 BC
Śulba-sūtras	500 BC and later
Dharmasūtras	600 BC–200 BC
Mahābhārata; also *Manusmṛti* and *Rāmāyaṇa*	200 BC–200 AD

It should be noted that this table is said to have been adopted by the General Meeting of the Symposium on November 7, 1950. Although about 30 leading historians and scientists of India participated in the Symposium, the suggested dates seem to have no significant effect on subsequent writings on history of science in India. In fact, most of the present Indian historians of science seem to be unaware of the Committee and its recommendations.

Seeing the continuation of confusions and controversies in ancient Indian chronology, the need is even more now for following some norms. One way to meet the demand is that bodies like the Indian Council for Historical Research and INSA's National Commission on History of Science should jointly appoint a permanent Chronology Committee which should play the leading role in the matter. It should examine the problem in detail and suggest a revised and longer unambiguous working table which the scholars should follow. Wide and continuous publicity should be given. Appropriate revisions and changes may be made from time to time. Those who differ from the suggested dates should clarify the reasons (which should not be whimsical) thereof as an accepted norm.

One of the serious charges against many of the Indian writers is that of claiming fantastic early chronology especially when it is on arbitrary grounds. No such complaint can be there if some suggested working chronology is followed. There is also no need or use of making false claim of antiquity by putting an older stamp (such as prefixing the word 'Vedic' to a later composition). The currently popular book *Vedic Mathematics* by Tirthaji (1884–1960) is a recent example of such work. A forced chronology or exaggerated claim will not stand the test of time.

Of course, some controversies about dates of a few specific authors or works may remain as isolated problems even in later periods. Instances of such cases include the date of the *Bakhshālī Manuscript* (fourth to seventh century AD), date of Śrīdhara (seventh to nineth century), and data of Āryabhata II's *Mahāsiddhānta*.[37] It is hoped that these minor chronological problems will be settled soon, just as the accepted date (beyond any doubt) of Bhāskara I's commentary on the *Āryabhaṭīya* is now settled as AD 629 (instead of AD 522 as believed half a century ago).

Finally before closing this discussion, a few general points may be given for consideration:

1. An earlier chronology does not necessarily or automatically imply a borrowing by later writers or civilizations.
2. While using doubtful chronological limits, it is better to be critical and take the safer dates especially when serious or significant conclusions are drawn.

3. If possible, support, corroboration or even confirmation may be sought from archaeological, epigraphical and similar other historical sources. Note that consistency is a sound criterion and a great merit.
4. It will be safe (and therefore better) if emphasis is not given on *apauruṣeya* works especially for making historical chronology and conclusions.
5. Current research and publications (giving latest findings) should be consulted as 'ignorance is no bliss'. If possible, examine the things critically instead of accepting them blindly.

If attention is paid to what has been discussed in this paper, then scholars will not be groping in darkness about chronology. Otherwise, more and more controversies will go on creating more and more problems and confusions (perhaps with little hope of synthesis); and we would be doing research which is one sided or is a world of our own (perhaps like that of frogs in a well). A good example about such further confusions has just come in the knowledge of the present author at the time of closing this article. It is this:

> While C. S. Upasak saw the evolution of the Brahmi script to go back to 1000 BC only, S. C. Kak is of the view that it evolved probably out of the Harappan script "perhaps in the first half of the second millennium BC" (say about 1500 BC).[38] On the other hand, but just at the same time, L. C. Jain,[39] while mentioning the 'fierce controversy' about origin of Brāhmi script, still banks and builds his research on the following two 'inescapable conclusions'–
>
> (i) Brahmi script was invented in the third century BC.
>
> (ii) Indians (except the people of N. W.) did not have any written letters, whatever, before, that time,[40] that is before the time of Aśoka.

References and Notes

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2. S. B. Dikshit, *Bharatiya Jyotish Sastra* (History of Indian Astronomy) translated into English by R. V. Vaidya, Part I, Manager of Publications (Govt. of India), Delhi, 1969, p. 122.
3. As quoted in Gorakh Prasad, *History of Indian Astronomy* (in Hindi), Publication Bureau (Govt. of U. P.), Lucknow, 1956, p. 249.
4. See the paper K. Venkatachaliyengar in *Scientific Heritage of India* edited by B. V. Subbarayappa and S. R. N. Murthy, Bangalore, 1988, p. 37.
5. P. C. Sengupta, *Ancient Indian Chronology*, Univ. of Calcutta, Calcutta, 1947, p. ix.
6. Srirama Sharma (editor) *Nārada-purāṇa*, Part II, Bareilly 1971, p. 318. In the second verse of Chap. 54, the science of Jyotiṣa (astronomy, astrology, and math.) is said to have been expounded in 4 lakh verses.
7. See the edition by Jivanada Bhattacharya, Calcutta, 1890, p. 145.
8. See Shukla (ref. 1), p. 14.
9. K. S. Shukla (editor and translator): *The Āryabhaṭīya* of Āryabhaṭa, INSA, New Delhi, 1976, p. 164.
10. B. Datta, *The Science of the Śulba*, Calcutta. 1932.
11. S. Prakash and U. Jyotishmati (editors), *The Śulba-sūtras*, Allahabad, 1979, p. 67.
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15. B. L. Upadhyaya, *Prācīna Bhārtīya Gaṇita* (in Hindi), New Delhi, 1971, p. 210.
16. As quoted in Dikshit (ref. 2), p. 92.
17. Sengupta, (ref. 5), p. 191.
18. See Dikshit, (ref. 2), p. 87.
19. K. V. Sarma (editor), *VJ* of Lagadha (with trans. and notes of T. S. Kuppanna Sastry), INSA, New Delhi, 1985, p. 13.
20. Prasad, (ref. 3), p. 45.
21. See *Proceedings of the NISI*, Vol. 18 (1952), p. 339.
22. See *J. Asiatic Soc.,* 30 (1988), p. 13.
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27. See *J. for History of Astronomy,* Vol. 20 (1989), pp. 136–138.
28. For details, see K. D. Nambiar, *Nārada-purāṇa: A Critical Study*, Varanasi, 1979, pp. 101–112. Also see p. ii for its date (eleventh to twelfth century).
29. Shukla, (ref. 1), p. 15.
30. Edited by K. V. Sarma, *VVRI*, Hoshiarpur, 1975.
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32. For example see, T. S. Kuppanna Sastry, "Determination of the Date of the *Mahābhārata*: The Possibility thereof", *Vishveshvaranand Indolog. J., 14* (1976), 1–9. For Śrī Rāma's time, see Sengupta, (ref. 5). p. xiii.
33. See e.g. Shukla, (ref. 9), p. 18.
34. See Prasad, (ref. 3), p. 258.
35. See *Proceedings of NISI,* 18 (1952), p. 331. Other members were: A. S. Altekar, P. C. Bagchi, S. L. Hora, D. S. Kothari and A. N. Singh.
36. *Ibid.,* p. 331. We have omitted few items.
37. The latest study of Takao Hayashi, *The Bakhshāli Manuscript*, Ph.D. Thesis, Brown Univ., 1985, suggests seventh century as the date of the work. On the date of Śrīdhara, see *Gaṇita Bhāratī*, Vol. 9 (1987), 54–56; and the *Mahāsidhānta*, see *GB*, Vol. 10 (1988), p. 29.
38. S. C. Kak, "Indus Writing", *The Mankind*, 30 (1989), p. 117.
39. L. C. Jain, "On the Origin of the Brahmi Script", *Mahāvīra Jayantī Smārikā*, No. 27 (1990), Sect. 5, pp. 8–9.
40. Thus the helpless readers are put to utter confusion as to whether the Brahmi script was invented in 300 BC or 1500 BC And these so divergent dates are being simultaneous given in the name of latest research of the century.

A Problem on Interest in the *Nārada-purāṇa*

The Sanskrit text of the *Nārada-purāṇa* (abbreviated *NP* hereafter) was, perhaps first, published by the Venkateshwara Press, Bombay in Śaka 1845 (or AD 1923). Chapter 54 of the *Pūrvabhāga* of *NP* is devoted to mathematics and astronomy (*gaṇita-jyotiṣa*). The next two chapters are on astrology (*phalita-jyotiṣa*). A somewhat emended text of this chapter with Hindi translation was published in 1954 by the Gita Press, Gorakhpur as part of their *Saṃkṣipta Nārada-Viṣṇu-purāṇāṅka* which was the special issue of *Kalyāṇa* for its 28th year. In fixing the reading of the Chap. 54 in this edition, the editor and translator Pt. Rāmanārāyaṇadatta was assisted by the famous Sītārāma Jhā of Kashi and others. Almost the same text of the chapter was included by Pt. Śrīrāma Śarmā in his edition and Hindi translation of the *NP* (Saṃskṛti Saṃsthāna, Bareilly, 1971).

Based on the Bombay text-edition, Chap. 54 of the *NP Pūrvabhāga* was translated into English by K. V. Sarma (wrongly spelt as Sharma) and M. R. Bhatt. The translation appeared in 1981 in the *NP* (translated by G. V. Tagare), Part II which was published as Ancient Indian Tradition and Mythology, Volume 16 (Delhi, 1981). On the other hand, the *NP* Sanskrit text of the Bombay edition has been brought out by Nag Publishers, Delhi, 1984.

Very recently, Takao Hayashi has published a paper entitled "The Mathematical Section of the *Nārada-purāṇa*" in the *Indo-Iranian Journal*, Vol. 36 (1993), pp. 1–28. This is a detailed study of the text and contents of Chap. 54 (mentioned above), verses 1 to 60, and contains an emended text of these verses. The first eleven and a half verses give general introduction and the remaining deal with mathematics (except the 60th which contains concluding remarks).

Most of these 'mathematical' verses and the rules contained therein resemble those found in the famous *Līlāvatī* (the most popular work on Hindu mathematics)

Gaṇita Bhāratī, Vol. 15, Nos. 1–4 (1993), pp. 67–69.

K. Ramasubramanian (ed.), *Gaṇitānanda*,
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of the renowned Bhāskarācārya or Bhāskara II (born AD 1114) whose 800th Death Anniversary is being observed this year (1993). Thus it is clear that the mathematical portion of the *NP* was incorporated in it after AD 1150 (which is the date of composition of the *Līlāvatī*), as other possibilities being very meagre. In fact the said portion may be much later than twelfth century. To H. H. Wilson, the extant *NP* appeared to be a compilation of the sixteenth or seventeenth century (see Gorakhpur edition mentioned above, p. 12). It is also to be noted that the astronomical portion of the *NP* (Chap. 54) resembles the modern *Sūrya-siddhānta* especially in Raṅganātha's version of the early seventeenth century.

In this note we are more concerned about a couplet or rule which has not been fully understood by scholars. The Sanskrit text reads (Hayashi, p. 16):

बहुराशिफलात् स्वल्पराशिमासफलं बहु ॥ ४० ॥
चेद् राशिविवरं मासफलान्तरहृतं चयः ॥ ४० $\frac{१}{२}$ ॥

(*NP*, *Pūrvabhāga*, Chap. 54, verses 40–41)

Hayashi admits that "the mathematical purport of this rule is not clear" to him (p. 22). As such his tentative translation, although literal, does not explain the matter clearly. As he indicates, the translation given by Sarma and Bhatt (Delhi translation of *NP*, p. 696) is not satisfactory as it misses the words *vivara* and *antara*. On the other hand the Gorakhpur and Bareilly editions do not contain these words at all!

Actually, I think that the rule contained in the above lines is related to a problem on simple interest. Two unequal principal amounts (*rāśis*) P_1 and P_2 ($P_1 > P_2$) are loaned or invested simultaneously at the interest-rates r_1 and r_2 per cent per month, respectively, so that the monthly interest (*māsaphala*) on P_1 will be

$$\frac{P_1 r_1}{100} = i_1 \text{ say,}$$

and on P_2 will be

$$\frac{P_2 r_2}{100} = i_2 \text{ say.}$$

If $i_2 > i_1$, then there will be a period of time ($= m$ months say) when the total amounts (capital+interest) will become equal, that is

$$P_1 + mi_1 = P_2 + mi_2. \tag{1}$$

I believe that the above Sanskrit lines simply given a rule to find m in this problem. My translation will be:

> If (*ced*) the monthly interest (i_2) on the smaller amount (P_2) is more than the (monthly) interest (i_1) on the greater amount (P_1) then the difference of the amounts divided by the difference of monthly interests gives the *caya* (number of months) (when the total amounts become equal).

That is

$$m = \frac{(P_1 - P_2)}{(i_2 - i_1)} \tag{2}$$

which is the correct solution of (1).

Example: An amount of Rs. 700/- was lent at the rate of 3% per month and, at the same time, another of Rs. 500/- was lent at the rate of 5% per month. After how many months will the amounts become equal sums (that is, the principal amount plus interest in each case)?

Here

$$P_1 = 700,\ i_1 = 700 \times \frac{3}{100} = 21,$$
$$P_2 = 500,\ i_2 = 500 \times \frac{5}{100} = 25.$$

Hence

$$\begin{aligned} m &= \frac{(700 - 500)}{(25 - 21)}, \qquad \text{by (2)} \\ &= \frac{200}{4} = 50 \text{ months.} \end{aligned}$$

Thus the *māsacaya*, the 'collection' or 'heap' or the number of required months is 50 (after which *each* amount will be 1750/-).

In the Sanskrit rule, *māsacaya* is indicated by *'caya'* only just as i_1 is indicated by *'phala'* instead of *māsaphala* (as is the case with i_2). Anyway, the keyword to understand the rule is the word *'māsaphala'* ('monthly fruit') used, quite appropriately, for 'monthly interest'.

Thus, whatever be the source for the Sanskrit lines, or whosoever be the author or compiler of the rule, it represents a simple interest-problem belonging to the category of equalization problems which were very popular and common in ancient Indian mathematics, e.g. see R. C. Gupta, "Some Equalization Problems from the Bakhśālī Manuscript", *IJHS*, 21 (1986), 51–61.

Who Invented the Zero?

1 Introduction

Obviously the answer depends on the meaning of ‘zero’. That is whether we mean the word zero or some concept of zero, the number zero or some symbol for zero, the mathematical zero or some philosophical zeroism. As a word for literal description, zero means a person or thing with no importance or independent existence. Ideas of neutrality, non-entity, and total absence may be indicated by zero. Nadir may be called as zero. अविद्यं जीवनं शून्यम् *(avidyaṃ jīvanaṃ śūnyam)* ‘Without learning the life is void (or zero)’ is an ancient Indian saying.

Technical meanings of zero are there. Zero hour is the scheduled time for a specific action or operation. Zero day, zero date, and zero year (i.e. the starting point of an era). Zero error is frequently applied to scientific instruments.

Discussion in this paper will be often viewed according to the following four broad (but not exclusive) categories.

(I) Since ancient times the ideas of emptiness or nothingness are represented by zero. This notion may not imply any numerical concept. It may indicate absence of a thing or vacantness of some kind. The emptiness of unfilled square cells (to be filled by letters or numbers) in a puzzle comes under this category as also the vacant entries in qualitative or quantitative tables. Such blank space may be indicated by a sign such as a cross, star, or a small circle (not necessarily zero number)[1] which will show that there is ‘nothing’ or ‘zero’ there. Empty space between other symbols may be denoted by a separation mark. When total receipts and total payments are found to be equal, the balance will be ‘nothing’ (or zero) which may be denoted by a sign. But actual subtraction of numbers or the concept of zero as a number may not be involved.

(II) In this category, we put such ideas and concepts which are found, e.g. in the practice of using zero as an indicator of the point of reckoning, measuring, or

Gaṇita Bhāratī, Vol. 17, Nos. 1–4 (1995), pp. 45–61.

K. Ramasubramanian (ed.), *Gaṇitānanda*,
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graduation, etc. It may be a mark for any datum level. For measuring distances along any line or curve (e.g. a road) we may use and mark a reference point and call it our zero. For the calibration of a scale we use a similar mark, e.g. zero temperature. In spiritual matter there is an ascent of the soul (zero level) to Brahma (infinite).

When such type of reckoning or measuring is combined with that in the possible opposite sense or that which is extended in the opposite direction, the zero is the meeting point (or bridge) between positive and negative quantities or levels. For instance, the mean sea level is the zero between the heights (above) and depths (below). Profits and losses may be plotted on the opposite sides of a zero. For chronological reckoning of historical events there can be a zero between BC and AD dates. The origin or the point (0,0,0) is the meeting point of positive and negative parts of the axes.

(III) Grown from the idea of emptiness, this category deals with the concept of zero as a symbol which denotes the absence of a denominational term in any positional numerical notation. The sign or indicator for zero may be merely a 'vacant space' or it may be some concrete symbol to avoid any visual confusion. Our present decimal place-value system is a very clear and ideal example of a positional numerical notation in which the oval symbol '0' is used to denote the zero. In this system any positive integer N (and zero itself) can be expressed as

$$\begin{aligned} N &= a_0 + 10 \cdot a_1 + 100 \cdot a_2 + \ldots\ldots 10^n \cdot a_n \\ &= \sum_{r=0}^{n} 10^r \cdot a_r \text{ where } a_r = 0 \text{ or } 1, 2, \ldots\ldots 9. \end{aligned}$$

In a fully developed place-value system, the zero must be able to play all the following three roles successfully:

(i) medial or internal, which is the classical role of a blank space, e.g. as in 205 or 2005, etc.
(ii) final or terminal, which is a more stringent role, e.g. in 250 or 2500.
(iii) initial, which is rather a superfluous role ordinarily, e.g.

$$025 = 0025 = 25 \text{ in value.}$$

For a perfect and ideal place-value system to base (radix, scale, period) b, there must be only one set of $(b - 1)$ nonzero independent numerical signs which must be capable of being used in all positions or places (representing various ranks or denominations or orders, or gradations) without any further modification or additional aiding device. Same zero sign should play all the above three roles. Earlier examples of place-value systems often lack these ideal conditions.

(IV) This category has concepts which treat zero as a number (going beyond the role of blank space filler or a reference point). Here the zero can be subjected (like other numbers) to mathematical operations (with some exceptions), and it obeys certain laws. Ancient mystical doctrine looked at zero as a sort of infinitely small quantity which does not affect another quantity (additively or subtractively).[2] So that this 'little zero' (*kṣudra*) may be defined by

$$N \pm 0 = N \tag{1}$$

When we subtract any (finite) numbers from itself, the result is zero number, i.e.

$$n - n = 0 \tag{2}$$

In this paper, we do not need arithmetic of zero in detail.

2 The Case of Ancient Egypt

The pictographic or hieroglyphic numerals of the Egyptians (about 3000 BC) formed a perfect juxtapositional numerical notation in which the principles of repetition and addition were used. The base was strictly ten; i.e., ten additive signs of any order would be represented by a single sign of the next higher order. A set of n independent signs would be able to represent all numbers less than 10^n. The sign for unit was a vertical stroke and that for ten looked like an inverted U. Thus 23 would appear as

$$|\,|\,|\cap\cap \quad \text{(read from right to left)}$$

The concept or sign for zero was totally absent as there was no need. The circular symbol "0" stood[3] for the Egyptian Horus eye fraction $\frac{1}{4}$.

Although simple, the above system would need a large number of signs to represent big numbers. To overcome this burden or defect, the Egyptians[*] introduced the principle of independent representation called encipherment or cipherization. This they developed when their hieratic ("sacred") script, a cursive form of hieroglyphs, was evolved. In hieratic numerals, a set of nine symbols was used to denote the numbers 1 to 9, but another (different) set for the sequence 10, 20, . . .90, yet another for the numbers 100, 200, . . .900, and so on. Hieratic numbers are found used in the *Reisner Papyri* (about 1880 BC), *Mascow Papyrus* (c. 1850 BC), *Rhind Papyrus* (c. 1650 BC) and other documents. Still the hieratic system of numerals was zero-less.

According to R. J. Gillings,[4] while giving certain results in the *Reisner Papyri,* "a blank space indicates zero." For instance a result (to be expressed in terms of the units: cubit, palm, and finger) may be $4c\ 2f$, leaving blank space for the absent or

[*]For contribution of Egypt, see B. Lumpkin, "Africa, Cradle of Mathematics", *Gaṇita Bhāratī* 19 (1997), pp. 1–10, especially 6–7.

zero palm (i.e. *o p*). The concept of zero is also found in some other contexts among the Egyptians. For instance, Dieter Arnold has shown that a zero symbol was used by the Egyptians to label or mark the ground level reference line at the medium pyramid belonging to the Old Kingdom.[5]

The symbol employed for zero (level) was the hieroglyph *nfr*. It is also claimed[6] that this symbol was also "used to express zero remainders in a monthly account sheet from the Middle Kingdom, dynasty XIII (c. 1770 BC)". It is explained that the account (in a double entry account sheet) was balanced. Income was added, and then, disbursement was totalled. Finally, total disbursement was subtracted from total income (for each column). Four columns had zero remainder which was shown by the symbol *nfr*.

3 Zero in Ancient Babylonia

By convention, the word 'Babylonia' applies to the whole region of Mesopotamia of ancient times with various civilizations. The antique Sumerian word *geš* for one was also used for sixty.[7] Similarly, the Sumerian sign for sixty was also same as that for one but bigger in size, so that 60 was treated a 'big 1'. This idea was perhaps the first step towards a positional sexagesimal notation. Recent findings of proto-cuneiform script from some tablets (c. 3000 BC) show the earliest development of the sexagesimal place-value notation and the concept of zero.[8] Clay tablets from the Old Babylonian period (2000 to 1600 BC) show a wedge shaped sign ($\triangledown$) for one and a hook or angle-bracket sign (𒌋) for ten. These two signs were used to form numerals up to 59 by the usual additive principle with decimal base.

For higher numbers, the Babylonians made use of the positional notation to base sixty. Thus in the clay tablet VAT 7858, the sequence 10, 20, 30, 40, 50 is continued by their symbols which can be read as (in our symbols with commas).[9]

1	(which stands for 60)
1, 10	(which stands for $1 \times 60 + 10$)
1, 20	(which stands for $1 \times 60 + 20$)
....	....
....	....
1, 50	(which stands for $1 \times 60 + 50$)
2	(which stands for 2×60)
2, 10	(which stands for $2 \times 60 + 10$)

But the Old Babylonians did not use a concrete sign for zero. However, they often left blank space when any medial denominational term was absent. Still this was a serious defect because we cannot be sure whether one or more terms are missing in the blank space left. Moreover, without a terminal zero sign the Babylonian system behaved as a "floating-point place-value system". For instance, ancient contexts show that while in one case, the triplet (commas put by us) 17, 46, 40 represents.[10]

$$17 \times 60^2 + 46 \times 60 + 40 = 64000,$$

and in another case, a triplet, namely 42, 25, 35, must be taken to represent[11]

$$42 + \frac{25}{60} + \frac{35}{60^2}.$$

In the *VAT* 7537 tablet, there is an expression of zero for the result of the subtraction process when the subtrahend (quantity to be subtracted) was equal to the minuend (or diminuend). However, it is said to indicate only the emptiness of the result (due to balancing) or "nothingness" (category I).[12]

About 500 BC, we do come across a zero sign (which looks like the sign for ten). This sign (say, Z) has been used in the text CBS 1535 in the following cases[13]

$$\begin{array}{ll} & (24, 30)^2 \text{ is given as } 10, Z, 15 \\ \text{and} & (42, 30)^2 \text{ as } 30, Z, 6, 15. \end{array}$$

A few centuries later, the Babylonians of the Seleucid period (312 to 64 BC) introduced some signs for zero. One of them was a punctuation mark used as a separation sign in literary or bilingual texts.[14] One commonly used zero symbol in this Neo-Babylonian period was the slanted double-wedge sign. Use of the angle-bracket sign (with the lower arm somewhat extended) was another zero symbol. This latter sign is comparable to that used in *CBS* 1535 (noted above).

Examples:

$$\begin{array}{ll} & Z, Z, 30 \text{ for } 0 + \frac{0}{60} + \frac{30}{60^2} \\ \text{and} & 2, 11, 46, Z \text{ for } 2 \times 60^3 + 11 \times 60^2 + 46 \times 60 + 0 \end{array}$$

Thus, we find that Babylonians of about 300 BC were freely using a sexagesimal place-value system with zero symbol in all the three roles. But the symbol was not standardized. Moreover, the numbers 1 to 59 were not represented by independent ciphers, but were additively composed from two signs on base ten (like the Egyptians). It was thus a mixed system.

4 Zero in Greek Mathematics and Astronomy

Although the principle of cipherization, that is, of independent representation, had been already applied by the Egyptians (2000 BC or earlier) in their hieratic numerals, it was limited to numbers upto thousands. About 700 BC, the Greek used the technique of encipherment systematically to cover numbers up to millions (and even beyond) by utilizing their alphabet. To the 24 letters of their classical alphabet, the Greeks added three archaic forms and cipherized the full set on decimal base. The letters from *alpha to theta* denoted numbers 1 to 9 (with *stigma = 6); iota* to *qoppa* denoted $10, 20, \ldots 90$; and *rho* to *sampi* stood for $100, 200, \ldots 900$, respectively. Higher sequences of similar type were formed with modification marks. Often bars were used to indicate numerical role of letters. For example, 202 would appear as $\sigma\beta$ (i.e. $200 + 2$), and 2534 as, $\beta\phi\lambda\delta$ (i.e. $2000 + 500 + 30 + 4$), where we have put a comma (instead of an accent) in the lower left corner of β to make it denote 2×1000 as per the scheme.

There was no need of zero in this system. Aristotle (c. 340 BC) rejects zero from being included among numbers because it could not be used in forming ratios.[15]

Five centuries later the famous astronomer Claudius Ptolemy (c. 150 AD) used the alphabetical notation (of decimal base) to express integers but followed the old (Babylonian) scheme for writing the fractions. His *Almagest* contains a table of chords in a circle of radius 60 units (or parts, p). His value of the (length of) chord subtending angle 72° at the centre is[16]

$$\begin{aligned} & \bar{o}, \overline{\lambda\beta}, \bar{\gamma} \\ \text{that is,} \quad & 70^p \;\; 32', 3'' \\ \text{or,} \quad & 70 + \frac{32}{60} + \frac{3}{60^2} \end{aligned}$$

Note that the Greek letter omicron "o" stands for 70. Another example is:

$$Crd\ 90° = \overline{\pi\delta},\ \overline{\nu\alpha},\ \bar{\iota} = 84^p, 51', 10''.$$

The first and the last entries in the table are[17]

$$Crd\ L' = \overline{\text{O}}, \lambda\alpha, \kappa\varepsilon \tag{3}$$

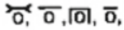
and

$$Crd\ \rho\pi = \rho\kappa, \overline{\text{O}}, \overline{\text{O}} \tag{4}$$

which mean

$$Crd\left(\frac{1}{2}\right)^{\circ} = O^p, 31', 25'' \quad \text{and} \quad Crd\ 180° = 120^p, O', O'',$$

respectively. Note that commas in all above values have been inserted by us for convenience, and there are no usual bars over the alphabetic numerals in (3) and (4) as they are already from a numerical table. In (3) and (4), the most important thing is the use of a special zero symbol ($\overline{\text{O}}$) to indicate the empty space caused by the absence of either the whole part or a fractional part. Equally significant to note is that the symbol is the decorated form of a sign (O) which is quite like ours. According to Neugebauer,[18] the same symbol is found in a papyrus belonging to the second century AD. Other manuscripts also have the same or similar signs:[†]



It is clear that the new zero symbols are various embellishments of the basic sign which is a small circle "O". And, as pointed out by several scholars this O is almost surely from the first letter (omicron) of the Greek word "Ouden" ($o\upsilon'\delta\varepsilon'\nu$) or "Oudemia" ($o\upsilon'\delta\varepsilon\mu\acute{\iota}\alpha$) which means "nothing" quite appropriately. But as an alphabetic numeral O (omicron) denoted 70, it was embellished or decorated to give a different look (and thereby avoid confusion) to serve a zero symbol. The crowning of the small o ("micron o") by a dumb-bell crown made it the crowned king (among the numerical signs) which was to govern the world of numerals for ever. Later on the embellishments were dropped and the bare o-like zero sign is found in manuscripts belonging to the Byzantine period (AD 300–600).[19]

5 The Chinese Case

Since the Chinese language is written in ideograms (or ideographs), its first nine numbers–words can also be treated as the basic nine number symbols or numerals 1 to 9 needed to form the base for the Chinese system of written numbers. The characters for the numbers ten, hundred, thousand, and ten thousand were *shi, bai, qian,* and *wan*, respectively. These characters (say S, B, Q, W in short) were also used to indicate the corresponding ranks or denominations in the numerical description or representation of numbers. For example, the numbers 14957 and 4085 would appear as

$$1\ W\ 4\ Q\ 9\ B\ 5\ S\ 7 \quad \text{and} \quad 4\ Q\ 8\ S\ 5 \ \text{ respectively.}$$

If some denominational term was absent (as in the 2nd example above), it was omitted altogether, without leaving even a blank space to indicate the situation. Anyway, the ancient Chinese written numerals (more than 2000 years old) formed a perfect example of a ranked or named system to base ten without any zero (which was not needed).

For carrying out computations the ancient Chinese employed the so-called rod numerals in which the principle of decimal positional notation was used. Two sets of basic numerals 1 to 9 were defined so that the digits in the adjacent positions could be differentiated. When a number had no digit of a particular rank or denomination, the position corresponding to that rank was left vacant (this continued to AD 700).[20]

[†]For an example from about AD 200, see Neugebauer's article in *A Scientific Humanist,* Philadelphia, 1988; pp. 301–304.

For instance the number 33406 would appear as

$$| \; | \; | \quad \equiv | \; | \; | \; | \quad T \text{ (here } T \text{ represents 6).}$$

This practice of leaving the vacant space doubtlessly reflects the idea of zero as indicator of emptiness or nothing (category I). But, as pointed out by Martzloff,[21] the existence of blank spaces on the counting board is not itself a special property of Chinese counting-rod system but rather of counting boards and abacuses (abaci) in general. According to Lam and Ang,[22] the vacant place of a rod numeral was described by the Chinese character *kong* ("empty"), but no historical example has been given to clarify or illustrate what exactly was the role played by it. That is, it is not clear whether *kong* was really a technical term (for zero) which could be repeated two (*as kong kong*) or more times for consecutive vacant places.

Moreover, it is mentioned by Martzloff[23] that according to the old mathematical manuscript Stein 930 from the Dunhuang caves (c. AD 900), "the presence of blank spaces was not automatically preserved in the written versions of the rod numeral system".

After the spread of Buddhism in China in the early centuries of the present era, there were lot of cultural contacts and exchanges between India and China. A large number of Indian scholars visited China with lot of Indian works of which many were translated into Chinese. These included *Lalitavistara* and *Abhidharmakośa* which have, respectively, centesimal and decimal scales of counting to very high orders. Indian systems of recording small and large numbers were adopted in China and did affect Chinese mathematics.[24] An Indian system of counting appeared in China in the *Ta Pao Chi Ching ("Mahāratnakūṭa-sūtra")* translated by Upaśūnya ("small zero") in AD 541, and Chinese children learnt mathematics from Buddhist textbooks.[25] The list given in the *Sui Shu* or *Official History of the Sui Dynasty* (seventh century) mentions about half a dozen Chinese translations of Indian works on mathematics and astronomy.[26]

More vigorous contacts and activities took place during the Thang dynasty (618–907). Gotama Siddha (Levensita) prepared the famous translation *Chiu Chih li* or *Jiu zhi li ("Navagraha Karaṇa")* from Sanskrit sources. Later on this was included in the *Khai-Yuan Chan Ching* (c. 725). Through it the Indian methods of calculation based on the decimal place-value system (with the zero symbol denoted by bindu or thick dot) were introduced in China.[27] For small number names *xū* ("void") and *kong* ("empty"), see Li and Du (ref. 24), p. 108.

Earlier, this zero symbol (*bindu*) has appeared in India as is clear from Subandhu's *Vāsavadattā* (sixth century), *Bakhshālī Manuscript* (seventh century), and from South-East Asian inscriptions under Indian influence, as well as from some other sources.[28] However, it is said that the mathematicians of China did not adopt the above Indian zero symbol. The circular zero symbol (which was also used in India and in S. E. Asian inscriptions) appeared in China very late (1247) but its source and other details are not known.

6 The Mayan Zero

Probably the ancient choice (although not the best) of base ten for counting was based on fingers. Similarly the choice of counting by 20 was perhaps based on fingers and toes. The Maya of central America had practised a vigesimal system of counting, more than 2000 years ago although they were not the first to do so. Earliest Maya village has been dated at 1000 BC by archaeologists. They had uncompounded words *hun, kal, bak, pic, calab, kinchil*, and *alau* for numbers 20^n, $n = 0, 1, 2, 3, 4, 5, 6$, respectively.

The Maya had a system of absolute chronology with zero date in 3114 BC[29] Their various units or periods of time were *kin (= 1 day), uinal (= 20 kins)* (month), *tun (= 18 uinals), katun (= 20 tuns), baktun (= 20 katuns),* then *pictun, calabtun, kinchiltun* and *alautun* each 20 times the preceding.[30]

As an example, the date on Stela E at Quiringua represents[31] 9 *baktuns*, 17 *katuns*, *tuns*, 0 *uinals*, 0 *kins* which gives 1418000 days (Reckoned from the Mayan epoch this is said to correspond to January 24, AD 771).

The usual Maya numerical notation for the numbers 1 to 19 uses two basic signs, namely a thick bar (which has value five) and a thick dot (which has value one). Then the usual ancient principles of repetition and addition are employed. For example 9 and 12 would appear as

respectively.

For numbers greater than 19 the principle of positional notation was used. It is said that the invention of this notation in Mesoamerica[32] may have been "the work of earlier peoples other than the Maya, such as those in Oaxaca, Tabasco, and Vera Cruz". In fact the bar-and-dot numerals are already found in the Oaxaca Valley inscriptions (c. 500 BC).[33] It is also shown that the Aztecs (of Mexico) had a positional numerical notation in which a corn glyph was used for zero.[34] The positional notation is also found in several Mesoamerican monuments (of non-Maya origin) dated from 36 BC to AD 162; and significantly, the notation was pure positional (i.e. without using signs for the ranks), but zero sign is not found.[35]

On dynastic monuments, the Maya used named positional system in which the time counts were expressed by prefixing numerals to glyphs representing *kin, uinal, tun, katun and baktun.* The oldest dated Maya monument of this type although without any zero glyph is Stela 29 at Tikal (AD 292), and second oldest is the Leiden Plaque (AD 320).[36] For use of zero glyphs, we should refer to Stela 18 and Stela 19 from Uaxactun bearing date 8, 16, 0, 0, 0 (in AD. 357).[37] Pure positional dates are also mentioned.[38]

In its abstract from the Mayan place-value system to base 20 (except in one place) with zero is quite simple, elegant, clear and systematic (and perhaps also original). The most common zero symbol employed is in the form of a shell design (ornate shell) resembling half open eye. We shall extract examples from a very good Mayan text called *Dresden Codex* (c. AD 1200), which is a copy or new recension of an earlier work (eighth century).[39] It contains a table (see its p. 24) which gives multiples of the synodic revolution of Venus (which was reckoned as 584 days).

Example (i): In this 5 × 584 or 2920 days appear as

(8)
(2)
(0)

That is, $8 \times (18 \times 20) + 2 \times 20 + 0 (= 2920)$

Example (ii): Here 45 × 584 or 26280 (days) shown as

(3)
(13)
(0)
(0)

That is, $3 \times (20 \times 18 \times 20) + 13 \times 360 + 0 \times 20 + 0$

Example (iii): Interestingly 25 × 584 or 14600 appears as

(2)
(0)
(10)
(0)

That is, $2 \times 7200 + 0 \times 360 + 10 \times 20 + 0$.

An important thing to note is that the medial (or internal) zero is depicted by a special symbol (different from terminal zeros). This practice is found in other cases also.[40] In fact, a number of forms (glyphs) or variations for zero are used with decorations, and Guitel suggested that Maya use of zero was dictated by practical, religious, and aesthetic reasons.[41]

Another numerical notation used by Maya was the system of head-variant numerals (with zero) in which portrait heads represented numbers.[42]

7 Claims for Indian Zero

The Indian decimal place-value notation (with zero) is now used all over the world but Boyer regards statements like "In the whole history of mathematics there has been no more revolutionary step than the one which the Hindus made when they invented the sign 0" as incorrect in historical details.[43] Here we present material with fresh findings for critical examination of opinions and difficulties thereto.

One difficulty is that the script used during the remarkable Indus Valley civilization (3rd millennium BC) has not been deciphered successfully. The system of Indus weights (as examined by A. S. Hemmy) was binary in the case of smaller ones and then decimal for the higher.[44] The Indus "decimal ruler" has remarkably marked accurate divisions of which only nine remain. The zero of the scale is indicated by a small circle, and a thick dot on the 6th line marks the next (quinary) division.[45] Some 3000 years later these two signs (small ∘ and •) were used as zero symbols by the Indians!

The Indus people (like the Egyptians) used vertical strokes (each of value one) to represent small numbers. But (unlike the Egyptians) groups of 10, 12, and 13 strokes are reportedly used.[46] This violates numeration to base ten. Moreover quite divergent views exist regarding other supposed numerical signs. For instance, the sign "U" is interpreted as

(i) letter *u* (उ) by F. Singh,
(ii) sacrificial pot by A. Parpola,
(iii) number 100 by J. E. Mitchiner,
(iv) number 20 by B. V. Subbarayappa, and
(v) number 10 by S. C. Kak.[47]

By correlating the round zero symbol to the loop of the fish sign ($\propto$) of Brāhmī and Indus script, Kak pushes the ancestry of the (present) zero sign to the third millennium BC However, no zero is claimed for Indus numerals themselves.[48]

A more serious difficulty is about the ancient Indian chronology. The Chronology Committee of 1950 (under Chairmanship of R. C. Majumdar) suggested following dates:

Age of the *Ṛgveda*	−−	2000 − −1500 BC
Other *Saṃhitās* and *Brāhmaṇas*	−−	1500 − −800 BC
Old *Upaniṣads*	−−	900 − −500 BC, etc.

But claims of *Vedas* and vedic times being many millennia earlier are not lacking.[49] Specific names of decuple terms found in the *Vedas* were used later on as designations of various notational places when the decimal positional system was evolved. The most popular was Medhātithi's list of 13 terms, the last three being *madhya* ($= 10^{10}$), *anta*, and *parārdha* ($= 10^{12}$). Later on this was extended to include 7 more terms namely,[50] *uṣas* ($= 10^{13}$) *vyuṣṭi, udeśyat, udyat, udita, suvarga,* and *loka* ($= 10^{19}$). But Mukherjee interprets these terms philosophically to contain idea of zero.[51] He further writes, like some other scholars, the Vedic number words in the form of modern numerals (with zeros) to advance claim of Vedic knowledge of place-value system with zero.

By giving a very peculiar interpretation of *Ṛgveda* Hymn (IV 58, 2–3) E. C. McClain in his *The Myth of Invariance* (Boulder, 1978) has claimed that "Indians must have known a positional number system" at that time.[52] T. M. P. Mahadevan (1969) and K. S. Shukla (1989) see the word *kṣudra* in *Atharvaveda,* XIX, 22–23, to refer to zero (along with *anṛca* as negative number).[53] *Kṣudra* means very small or tiny and led Tirthaji to take the letter *kṣa* (क्ष) as denoting zero in his famous "Vedic Mathematics" system. Of course we know that many words, such as *kham, vyoma, pūrṇa,* found in the *Vedas*, were used later on as word–numerals for zero.

The use of the thick dot sign for zero is claimed to be found in the text of the so-called Kashmirian *Atharva-veda* and in the marginal notes therein.[54] Surprisingly, the number one is represented by a square (in diamond form), and small circles in pair have been used to denote blank space. Although critical investigation of these unusual features of a peculiar work (or manuscript) is needed, the reviewer[55] has already refuted Mukherjee's claim of Indian invention of full place-value system in 3000 BC

Scholars (including Needham) have often talked about the contribution of philosophical and mystic ideas in the evolution of the concept and symbol for zero. Vedic philosophy of void and *māyā*, Buddhist theory of *śūnyavāda* (zeroism) as propounded by Nāgasena (first century BC) and Nāgārjuna (c. 200 AD) and the idea of *abhāva* ("absence") of Indian Nyāya system, etc., are all quoted for the genesis of zero.[56] But no definite and convincing linkage or links are available.

On the other hand the Sanskrit grammatic system of Pāṇini (c. 500 BC) has been claimed recently to contribute to the concept of zero in mathematical sense (i.e. involving positional analysis, operation of subtraction, process for going from maximum to minimum)[57] It is even said[58] that "he was the first man to use mathematical concept of 'zero' before mathematicians accepted it". His conception is presented in three forms, namely the linguistic zero, the *it* (इत्) zero, and the *anuvṛtti-zero*, but his idea of 'absence' (*lopa*, etc.) cannot be truly compared with a zero in a place-value system.[59]

According to Sadguru-śiṣya, the prosodist Piṅgala was a younger brother of Pāṇini, but usually Piṅgala is taken to flourish about 200 BC For computing 2^n, he gave a set of four *sūtras* one of which reads (VIII, 29 in his *Chandaḥ-śāstra*):

रूपे शून्यम् *(rūpe śūnyam)*

or "(Place) a zero ("śūnya") when unity is subtracted (from index or power)".

So it is believed that India possessed a zero symbol at that time[60] (but *śūnya* may mean blank space.).

The word *'thibuga'* used by Bhadrabāhu (c. 300 BC) has been found in a quoted *gāthā* and interpreted by Hemacandra to mean *bindu*. Some scholars try to see 'zero' of place-value notation in this.[61] The Jaina canonical work *Anuyogadvāra-sūtra* (c. 100 BC) is said to provide the "earliest literary evidence" of the use of the word "notational place" (see sūtra 142).[62] Now credit for inventing the place-value system (with zero) is also being given of Kundakunda (between 100 BC and 100 AD) who may be the possible author of relevant works (*Parikarma* and *Saṃta-kamma-paṃjiya* which are relevant.[63]

That the decimal place-value system was in use then in India is clear from reference to it by Vasumitra (first century AD) to illustrate that 'things are spoken in accordance with their states'. He says[64] "When the clay counting-piece is placed in the place of Units, it is denominated 'one', when placed in the place of Hundreds, it is denominated 'hundred', and in place of Thousands it is denominated a 'thousand'. Vasumitra was a Buddhist. Similar counting process is mentioned in ancient Jaina works.[65] In such positional process, the circular symbol (representing empty pit) would automatically denote zero. The use of zero symbol to fill the blank space[66] is also found in *Mahābandha* (c. AD 100).

There are direct and indirect evidences to show that the decimal place-value system was prevalent in India during the early centuries of present era. It is implied in the system of word–numerals[67] whose origin cannot be exactly dated. An early example is provided by the chronogram *viṣṇugraha* which David Pingree[68] has interpreted as *śaka* 71 or AD 149 (it can also mean *śaka* 91 or AD 169). Equally popular was the system of letter–numerals called *Kaṭapayādi Nyāya* which fully recognizes and employs the decimal place-value system with zero. Its promulgation is often attributed to Vararuci (c. AD 300).

The *Bakhshālī Manuscript* (a mathematical work) surely and explicitly uses the decimal place-value system with a zero symbol (dot or circle),[69] but its date is uncertain (being given from AD 200 to 1200). Same difficulty of definite dates is there in the case of a large number of *apauruṣeya* works (i.e. those which are attributed to gods and sages) including the often cited *Puliśa-siddhānta* which is available not in the original but in later versions.

A very good confirmation of the popularity of the decimal place-value system comes from the following line in the *Vyāsabhāṣya* (before AD 400)[70]: *yathā ekā rekhā śatasthāne śataṃ, daśasthāne daśa, ekañcaikasthāne,* i.e. "as the same one stroke (or numeral 1) denotes 100 in the hundreds place, 10 in tens place and one in units place". It seems that the use of the system had penetrated different literary circles. But due to peculiar Indian social set-up, the artisans (including engravers) got its knowledge only slowly.[71] If D. C. Sircar's guess is correct, the earliest epigraphic use of the zero symbol (small o) is in the Mankuwar stone inscription of Gupta year 109 or AD 428 (but still category I and not III).[72] Rules in the *Āryabhaṭīya* (AD 499) are based on place-value system with zero. Subsequent evidences are too many to be documented here (see references under note 28 below).

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44. E. Mackay, *Early Indus Civilization,* revised edition, New Delhi (Indolog Book Corper.), 1976, p. 103.
45. *Ibid.* 103 and plate XXIV, no. 9: G. Sarton, "A Hindu decimal ruler of 3rd millennium", *Isis*, 26 (1937), p. 305.
46. (a) B. Datta and A.N. Singh, *History of Hindu Mathematics,* Bombay, 1962; Vol. 1, pp. 19–20, (b) S.C. Kak, "Indus Writing", *The Mankind Quarterly* 30(1–2) (1989), p. 116; (c) B. B. Lal in *Vivekanand Commemoration Vol.* Madras, 1970, p. 199.
47. (a) Subbarayappa, *Indus Script*, Bangalore, ca. 1989, pp. 18–25; (b) Kak. "A Frequency Analysis of the Indus Script", *Cryptologia*, 12(3) (1988), p. 142.
48. Kak. "The Sign for Zero", *The Mankind Quarterly*, 30(3) (1990), p. 203.
49. R. C. Gupta, "The Chronic Problem of Ancient Indian Chronology", *Gaṇita Bhāratī*, 12 (1990) pp. 17 and 22; Also Mukherjee, *op. cit.,* (ref. 28), p. 35.

50. R. C. Gupta, "Decimal Denominational Terms in Ancient and Medieval India," *Gaṇita Bhāratī*, 5 (1983), pp. 9–10.
51. Mukherjee, *op cit.,* pp. 30–31.
52. As quoted by S. C. Kak in *Indian J. Hist. Sci.* 22 (1987), p. 220.
53. See *Indian J. Hist. Sci.* 4 (1969), p. 28, and *Mathematical Education*, 5(3) (1989), p. 132, *cf*. Mukherjee, p. 31.
54. Mukherjee, *op. cit.,* p. 55.
55. See *Gaṇita Bhāratī*, 15 (1993), p. 97.
56. Needham (ref. 20), p. 12; and M. D. Pandit, *Zero in Pāṇini*, Poona Univ, Pune, 1990, pp. 108–115.
57. Pandit, *op. cit.,* above, pp. 17 and 105–107.
58. *Ibid.*, p. 23.
59. *Ibid.*, pp. 15–16, 128–131; and the review of the book in *Gaṇita Bhāratī*, 14 (1992), 89–91.
60. Gupta, "An Ancient Method of Piṅgala for Finding a^n", *Bona Mathematica,* 2 (1991), 77–80, and Mukherjee (ref. 28), p. 43.
61. H.R. Kapadia (editor), *Gaṇita-tilaka*, Baroda, 1937, Introduction, pp. xxi–xxii.
62. Datta and Singh, *op. cit.,* (see ref. 46), p. 83.
63. L. C. Jain and A. Jain,"Was Ācārya Kundakunda the inventor of the decimal place-value system" (in Hindi), *Arhat Vacana*, 1(3) (1989), 7–15.
64. See *The Tattvasangraha of Śāntarakṣita with the Commentary of Kamalaśīla* (eighth century), translated into English by G. N. Jha, Vol. II, Baroda 1939, p. 862, where Vasumitra's words are quoted. Also Shukla's 'Bhūmikā' in Hindi transl. of ref. 46 (a), part I, Lucknow, 1956/1974.
65. See *Gaṇita Bhāratī*, 14 (1992), pp. 16–17.
66. Jain and Jain (ref. 63), p. 10.
67. Datta and Singh, ref. 46, pp. 58–59, and 86.
68. See *Jour. of Amer. Oriental Society,* 79 (1959), p. 282. But the interpretation and date of Yavaneśvara are controversial.
69. Datta and Singh, *op, cit.,* pp. 77–78; Mukherjee, see ref. 28, pp. 45–46 and 54. T. Hayashi in his Ph.D. Thesis on the work (Brown Univ. 1985), places it in AD seventh century.
70. B. N. Seal. *The Positive Sciences of the Ancient Hindus,* Delhi, 1958, pp. 50–51.
71. Thus S.L. Gokhale's *Indian Numerals*, (Poona, 1967) which documents forms from epigraphic and archaeological sources (upto AD 700) hardly refers to zero signs.
72. D. C. Sircar (*J. Ancient Indian Hist.,* III, 1969–70, pp. 133–135) has argued that the Mankuwar Buddha Image stone inscription (fifth century AD) has a small globular symbol for zero denoting absence of tens sign (and not a symbol for 20).

World's Longest Lists of Decuple Terms

1 Lists from Vedic Literature

Perhaps the practice of using fingers for counting was responsible for the choice of ten as a base for numeration. In India, ten has been the basis for counting since the very early days. Specific names are found in the Vedic literature for numbers which are equal in value to 10^n, where $n = 0, 1, 2, 3, \ldots$ A typical and definite list of 13 decuple terms occurs in the *Vājasaneyī-saṃhitā*, XVII, 2 of the *Śukla* (White) *Yajurveda* and is as follows.[1]

> *eka, daśa, śata, sahasra, ayuta, niyuta, prayuta* $(= 10^6)$, *arbuda, nyarbuda, samudra, madhya, anta, and parārdha* $(= 10^{12})$.

This set is called Medhātithi's List after the associated Vedic seer. Its popularity is shown by its occurrence (with slight variations) in several works of Vedic corpus.[2] Some of the instances are as follows

(i) *Kaṭha* or *Kāṭhaka-saṃhitā*, 17.10, where *niyuta* and *prayuta* interchange their places.
(ii) Same interchange is found in the *Kapiṣṭhala Kaṭha-saṃhitā*, 26.9.[3]
(iii) But *Kāṭhaka-saṃhitā*, 39.6, in addition to the above interchange, further inserts a new term *badva* for 10^9 thereby making *samudra, madhya anta and parārdha* to stand for 10^{10}, 10^{11}, 10^{12}, and 10^{13}, respectively.
(iv) The *Maitrāyaṇī-saṃhitā*, II, 8.14 has *ayuta* for 10^4 as well as for 10^6. The second *ayuta* may be emended to *niyuta* in conformity with (i), (ii) and (iii).
(v) The *Pañcaviṃśa* (or *Tāṇḍya*) *Brāhmaṇa*, XVII, 14.2, replaces the last four terms of the Medhātithi's List by *nikharvaka, badva, akṣita,* and *go* $(= 10^{12})$, respectively.[4]
(vi) The *Jaiminīya-upaniṣad-brāhmaṇa* I, 10.28–29 replaces the last four terms differently by *nikharva, padma* $(= 10^{10})$, *akṣiti* $(= 10^{11})$, and with the phrase, *vyomānta* $(= 10^{12})$.[5]

Gaṇita Bhāratī , Vol. 23, Nos. 1–4 (2001), pp. 83–90.

K. Ramasubramanian (ed.), *Gaṇitānanda*,
https://doi.org/10.1007/978-981-13-1229-8_16

(vii) *Śāṅkhāyana-śrauta-sūtra*, XV, 11.7, where we have, (after *nyarbuda*), *nikharvāda* (= *nikharva*, 10^9), *samudra, salila, antya* (= 10^{12}), and *ananta* (= 10^{13}).

(viii) The above referred *White Yajurveda mantra,* XVII, 2 (which contains the Medhātithi List) has been briefly mentioned in the *Śatapatha Brāhmaṇa*, IX, 1,2.16 from *ekā ca daśa* etc. to *antaśca parārdhaśca*.[6]

(ix) The *Atharvaveda* (*cf.* 8.8.7 and 20.8.24) seems to have a shortened list, something like *eka, daśa, śata, sahasra, ayuta, nyarbuda,* and *badva*.[7]

(x) *Taittirīya-saṃhitā* of the *Kṛṣṇa* (Black) *Yajurveda*, IV, 4, 11.4 contains the Medhātithi List of 13 decuple terms.[8]

Moreover, the *Taittirīya-saṃhitā*, VII, 2.20, contains what can be called as the Extended Medhātithi List. The extension part consists of the following 7 decuple terms, beyond the *parārdha* (= 10^{12}), of the above ordinary Medhātithi List:[9]

$$uṣas (= 10^{13}),\ vyuṣṭi,\ udeṣyat,\ udyat,\ udita,\ suvarga,\ and\ loka\ (= 10^{19}).$$

The above additional 7 decuple terms are not usually interpreted in this numerical manner by earlier scholars but Markandeya Mishra has quoted the words of the commentator Bhaṭṭabhāskara in support of the above extension.[10] Later on decuple terms were used as denominational names in the decimal place-value system.

2 Buddhist Lists

Buddhist religious and philosophical exposition needed very large numbers. In reply to a test-question put by the mathematician Arjuna, Prince Gautama narrated nicely a centesimal numeration system which was then prevalent in India. As described in the *Lalitavistara* (first century BC)[11], the first series of this counting system consisted of the 23 names from *ayuta* (= 100 *koṭi*) to *tallakṣana* (= 100^{23} *koṭi*) which stands for 10^{53} (see accompanying Table 1). Then follows 8 more such series. Thus the whole set will go upto the monstrous number[12]

$$10^{53+8\times 46} = 10^{421}.$$

Another interesting numeration system is found in Kāccāyana's Pali Grammar[13]. It is in the *koṭi* scale and, starting from *koṭi* (= 10^7), it goes upto *asaṃkheya* which stands for

$$10^{20\times 7} = 10^{140}.$$

It is this very numeration scale which is found in the Buddhist work styled as *Abhidhānappadīpikā*.[14] In forming names of decuple terms also, the Buddhists went far ahead of the Vedic lists.

Table 1 Buddhist Names of Numbers

Number	*Lalita-vistara* Centesimal scale	Vasubandhu's decuple term	Number	*Lalita-vistara* Centesimal Scale	Vasubandhu's decuple term
1	...	*eka*	10^{30}	—	*mahā-vyava pra-jña*
10	...	*daśa*	10^{31}	*hetuhila*	*hetu*
10^2	...	*śata*	10^{32}	—	*mahā-hetu*
10^3	...	*sahasra*	10^{33}	*karaku/karahu*	*karabh*
10^4	...	*prabheda*	10^{34}	—	*mahā°*
10^5	...	*lakṣa*	10^{35}	*hetvindriya*	*indra*
10^6	...	*atilakṣa*	10^{36}	—	*mahā°*
10^7	*koṭi*	*kauṭi (koṭi)*	10^{37}	*samāpta-lambha*	*samāpta*
10^8	...	*madya*	10^{38}	—	*mahā°*
10^9	*ayuta*	*ayuta*	10^{39}	*guṇanā-gati*	*gati*
10^{10}	...	*mahā-ayuta*	10^{40}	—	*mahāgati*
10^{11}	*niyuta*	*niyuta*	10^{41}	*niravadya*	*nimbaraja*
10^{12}	...	*mahā-niyuta*	10^{42}	—	*mahā°*
10^{13}	*kaṅkara*	*kaṅkara*	10^{43}	*mudrābala*	*mudrā*
10^{14}	...	*mahā°*	10^{44}	—	*mahā°*
10^{15}	*vivara*	*vivara*	10^{45}	*sarva-bala*	*bala*
10^{16}	...	*mahā°*	10^{46}	—	*mahā°*
10^{17}	*akṣobhya*	*akṣobhya*	10^{47}	*visaṃjña-gatī*	*saṃjñā*
10^{18}	...	*mahā°*	10^{48}	—	*mahā°*
10^{19}	*vivāha*	*vivāha°*	10^{49}	*sarva-saṃjñā*	*sarva-saṃjñā/bindu*
10^{20}	...	*mahā°*	10^{50}	—	*mahā°*
10^{21}	*utsaṅga*	*utsaṅga*	10^{51}	*vibhūtaṅgamā*	*vibhūti*
10^{22}	...	*mahā°*	10^{52}	—	*mahā°*
10^{23}	*bahula*	*bahula*	10^{53}	*tallakṣaṇa*	*tallakṣaṇa*
10^{24}	...	*mahā°*	10^{54}	—	*mahā°*
10^{25}	*nāgabala*	*vāhana*	10^{55}	—	*ogha*
10^{26}	...	*mahā°*	10^{56}	—	*mahā°/abbuda*
10^{27}	*tiṭilambha*	*tiṭibha*	10^{57}	—	*balākṣa*
10^{28}	...	*mahā°*	10^{58}	—	*mahā°*
10^{29}	*vyavasthāna-prajñapti*	*vyavasthāna-prajñapti*	10^{59}	—	*asaṃkhya*

The famous Buddhist scholar Vasubandhu (AD fifth century) in his commentary on his own *Abhidharmakośa* (III, 93–94) talks of 60 decimal denominational places.[15] He describes 52 decuple terms from *eka* to *asaṃkhyam* (= 10^{59}) stating that

अष्टकं मध्याद् विस्मृतम् ।

Eight (place-names) from *madhya* (middle) have been forgotten.

The super-commentator Yaśomitra states that to restore the full set of 60 terms, one should form suitable eight names oneself (to compensate for the loss caused by carelessness of the scribe). So I reconstructed[*] the whole Buddhist set of 60 decuple terms (see Table 1) taking the help of the terminology of names found in the *Lalitavistara* mostly. To have consistency, coherency, and uniformity of similar names (with similar values), I had to drop *prayuta* from Vasubandhu's list and make some other slight changes.[16]

3 Jaina and Other Lists

We have seen that there was a lack of uniformity in assigning numerical values to *ayuta, niyuta, prayuta*, etc., in the earlier lists. To avoid ambiguity and confusion, authors of astronomical and mathematical works usually gave definite lists of decimal denominational names. Āryabhaṭa I (born AD 476) in his *Āryabhaṭīya* (which is regarded to be the first extant Indian work of the *pauruṣeya* or historical type) gives the following list of 10 decuple terms.[17]

eka, daśa, śata, sahasra, ayuta, niyuta, prayuta, koṭi, arbuda and *vṛnda* (= 10^9).

Obviously this list is too short; so bigger lists were evolved. A definite list of 18 decuple terms is found in the *Pāṭīgaṇita* (verses 7–8) in *Triśatikā* (verses 2–3) of *Śrīdhara* (c. AD 750). The names are[18]

eka, daśa, śata, sahasra, ayuta, lakṣa (= 10^5), *prayuta, koṭi, arbuda, abja, kharva, nikharva, mahāsaroja, śaṅku (or śaṅkha), saritāṃ-pati, antya, madhya* and *parārdha* (= 10^{17}).

The choice of eighteen for the size of the set is noteworthy as 18 is a sacred Hindu number.[19] Anyway, like Śrīdhara and his works, his above list also became very popular in India. It is found (often with minor changes only) in the works of al-Bīrūnī, Bhoja, Kṣīrasvāmin, Śrīpati, Someśvara, Bhāskara II, Hemacandra, Nārāyaṇa Paṇḍita (AD 1356), etc.[20]

If the number 18 is sacred to Hindus, then 24 is more sacred to Jainas. So when Mahāvīra (c. 850) the famous Jaina mathematician, wrote his *Gaṇita-sāra-saṅgraha*, he gave in it a list of 24 decuple terms. The names are (*GSS*, I, 63–68).[21]

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[*]The Buddhist table is now restored by R. C. Gupta. See the Hindi Journal विज्ञान परिषद् अनुसन्धान पत्रिका (Allahabad), 47.1 (2004), pp. 3–6.

eka, daśa, śata, sahasra, daśa-sahasra, lakṣa, daśalakṣa, koṭi, daśakoṭi, śatakoṭi, arbuda, nyarbuda, kharva, mahā-kharva, padma, mahā°, kṣoṇī, mahā°, śaṅkha, mahā°, kṣiti, mahā°, kṣobha, and mahā-kṣobha (= 10^{23}).

Mallana (c. 1100) son of Śivvana translated the *Gaṇita-sāra-saṅgraha* (= *GSS*) from Sanskrit into Telugu. Being a Hindu, he naturally changed the deity's name 'Jina' (found in *GSS*) to 'Śiva'.[22] He also made another change in this spirit. To Mahāvīra's list of 24 decuple terms (as found in the *GSS*), he added further 12 terms more to make a total of 36 which is double of the Hindu sacred number 18. The names added by Mallana are[23]:

nidhi (= 10^{24}), *mahā-nidhi* (= 10^{25}), *parata, ananta, bhūri, mahā-bhūri, meru* (= 10^{30}), *mahā-meru, bahuśa, mahā-bahuśa, samudra,* and *mahā-samudra* or *sāgara* (= 10^{35}).

Shukla gives a different list of these additional 12 names from *Gaṇita-śāstra* of Pāvaḷūri Mallikārjuna (which is Sanskrit form of Mallana).[24] Whole matter needs further studies and a critical edition of Mallana's work(s).

Thus we find that a sort of religious rivalry gave us bigger lists of decuple terms. But it was not the end. Mallana was soon excelled by another Jaina author. Rājāditya wrote his *Vyavahāra-Gaṇitam* in Kannada in the twelfth century. In this work, he extended Mahāvīra's list to 40 terms (instead of Mallana's 36)! His addition consisted of the following 16 new terms.[25]

nadī (= 10^{24}), *mahā-nadī* (= 10^{25}), *naga, mahā°, ratha, mahā°, hari* (= 10^{30}), *mahā°, phaṇi, mahā°, kratu, mahā°, sāgara* (= 10^{36}), *mahā°, parimiti,* and *mahā-parimiti* (= 10^{39}).

We find that Rājāditya uses the suffix "mahā" (which was used earlier by Vasubandhu etc.) uniformly and consistently for economy. Popularity of Mahāvīra's list is also reflected.[26]

Finally, I give now the longest list of decuple terms which I have come to know recently. The list is a big extension of the traditional Hindi list of 19 terms which are:

(1) *eka*, (2) *daśa* (or *dasa*), (3) *sau*, (4) *hajāra*, (5) *dasa hajāra*, (6) *lākha*, (7) *dasa lākha*, (8) *karoḍa*, (9) *dasa karoḍa*, (10) *araba*, (11) *dasa araba*, (12) *kharaba*, (13) *dasa kharaba*, (14) *padma*, (15) *dasa padma*, (16) *nīla* (17) *dasa nīla*, (18) *śaṃkha*, and (19) *dasa śaṃkha* (= 10^{18}).

Beyond this set, the work entitled *Amalasiddhi* gives the following[27]:

(20) *kṣiti* (= 10^{19}), (21) *dasa kṣiti*, (22) *kṣobha*, (23) *dasa kṣobha*, (24) *riddhi*, (25) *dasa* —, (26) *siddhi*, (27) *dasa*—, (28) *nidhi*, (29) *dasa*— , (30) *kṣoṇi*, (31) *dasa*—, (32) *kalpa*, (33) *dasa*—, (34) *prāhi*, (35) *dasa*—, (36) *brahmāṇda*, (37) *dasa*—, (38) *rudra*, (39) *dasa*— , (40) *tāla*, (41) *dasa*—, (42) *bhāra* (= 10^{41}), (43) *dasa*—, (44) *burja*, (45) *dasa*—, (46) *ghaṇṭā*, (47) *dasa*—, (48) *nūla*, (49) *dasa*—, (50) *pacūra*, (51) *dasa*—, (52) *laya* (= 10^{51}), (53) *dasa*—, (54) *kāra*, (55) *dasa*—, (56) *apāra*, (57) *dasa*—, (58) *naṭa*, (59) *dasa*—, (60) *giri*, (61) *dasa giri* (= 10^{60}), (62) *mana* (= 10^{61}), (63) *dasa*—, (64) *bana*, (65) *dasa*—, (66) *śaṅkū*, (67) *dasa*—, (68) *bāpa*, (69) *dasa*—, (70) *bala*. (71) *dasa bala* (= 10^{70}) (72) *jhāḍa*, (73)*dasa*—, (74) *bhīra*, (75) *dasa*—, (76) *vajra*, (77) *dasa*—, (78) *loṭa*, (79) *dasa*—, (80) *naje*, (81) *dasa*—, (82) *paṭa*, (83) *dasa*—, (84) *tama*, (85) *dasa*—, (86) *drambha*, (87) *dasa*—, (88) *kaika*, (89) *dasa*—, (90) *amita*, (91) *dasa*—, (92) *gola* (= 10^{91}), (93) *dasa*—, (94) *parāmita*, (95) *dasa*—, (96) *ananta* (= 10^{95}), and (97) *dasa-ananta* (= 10^{96}).

Indeed this is world's longest list (so far known) of decuple terms or of the denominational names in decimal place-value system of numeration. *Amalasiddhi's* author and manuscripts etc. should be found out. Also note the influence of sacred numbers 18, 96 (which is four times 24), and 33 (see *note* 27 at the end) in the powers of 10 with which the lists end.

References and Notes

1. *Yajurveda* ed. by S. Sharma, Bareilly, 1962, p. 270.
2. For details, see (a) R. C. Gupta, "Decimal Denominational Terms in Ancient and Medieval India", *Gaṇita Bhāratī (=GB)*, 5 (1983), 8–15; (b) Takao Hayashi, The *Bakhshālī Manuscript,* Egbert Forsten, Groningen, 1995; p. 66. Also see *GB*, 22, 1–3.
3. See the *Brāhmaṇoddhāra Koṣa* ed. by Vishva Bandhu, Vishveshvaranand Indol. Research Series, Vol. 38 (Hosiarpur, 1966), p. 800).
4. Hayashi (ref. 2), p. 66. But the text quoted by H. R. Kapadia in the Introduction, p. XLIX, to his own ed. of *Gaṇita Tilaka* (Baroda, 1937), has interchange similar to (i) & (ii) also. Of course *nikharvaka* is same as *nikharva* (*GB*, 22, p. 2).
5. Or, we might say that here we have *vyoma* $= 10^{12}$, and *anta* $= 10^{13}$.
6. See *Śatapatha Brāhmaṇa* with commentaries of Sāyaṇa and Hari Svāmin, Part III, Nag Publishers, Delhi, 1990, pp. 2098–2099; and *J. Eggeling's* translation, Delhi, 1963, part IV. p. 172.
7. K. N. Sinha, "Mathematics from Vedic Saṃhitās", *Gaṇita-Bhāratī*, 22 (2000), 2–3.
8. *Ibid.,* p. 2 and Hayashi (ref. 2), p. 66, who follows A. A. Macdonell and A. B. Keith, *Vedic Index of Names and Subject* (1912), p. 342. But Keith in his translation (1914) of the Black *Yajurveda* wrongly extended the list to "a hundred hundred thousand million" i.e., 10^{13} because *parārdha* is 10^{12}. For Sanskrit text see Kapadia (ref. 4), p. XLVIII. Keith's 1914 transl. reprinted, Delhi, 1967; pp. 350–351.
9. See Gupta (ref. 2), pp. 9–10 for details and references. L.V. Gurjar, *Ancient Indian Mathematics and Vedha,* Poona, 1947, p. 16 gives *deśyat* in place of *udeṣyat*. Markandeya Mishra, "Mention of Numbers in the Vedas" (in Hindi), *Jyotish-kalp* (Lucknow), Vol. I, No. 2 (Nov. 1969), 17–19, also in II (2), 17–19.
10. Misra, *op. cit.* (ref. 9). The terms *vyuṣṭi* and *svarga* are also found in the *White Yajurveda*, XXII. 34 (ref. I above, p. 388) along with words for numbers 1,2,100, and 101 in the same tone (i.e. preceded by *'svāhā'*). Also the word *loka* occurs at the end of the *mantra* (*Ibid.,* XVII, 2, see ref. 1 above), which we have mentioned as a source for Medhātithi List, but not in numerical sense here.
11. P. L. Vaidya (ed.), *Lalitavistara*, The Mithila Institute, Darbhanga, 1958, p. 103.
12. According to K. Menninger, *Number Words and Number Symbols,* MIT Press, Cambridge, Mass., 1970, p. 137. The interpretation of Hayashi, ref. 2, is different.
13. B. Datta and A. N. Singh, *History of Hindu Mathematics,* Bombay, 1962; Vol. I, pp. 11–12.
14. As quoted by Kapadia (ref. 4), p. XLIX.
15. See *Abhidharmakośa and Bhāṣya* of Vasubandhu (with commentary of Yaśomitra), Part I, ed. by Sw. Dwarikadas Sastri, Varanasi, 1981, pp. 544–545.
16. Inserted names are *bohula, vyavasthāna-prajñapti, sarvasaṃjña* and *tallakṣaṇa* all from *Lalitavistara*; also alternatives *abbuda* and *bindu* from *Kāccāyana* (see Datta and Singh, ref. 13 above, p. 12), and *Ogha* from *Rāmāyaṇa* (see Hayashi, ref. 2, p. 69, or *GB*, 11, 1990, 10–16).
17. *Āryabhaṭīya*, II, 2 Sec Gupta, ref. 2 (a), p. 10 for details.
18. Gupta, *op. cit.,* p. 11, gives more informations.
19. Extension of list to 18th order for "religious reasons" is also pointed out by al-Bīrūnī who also mentions significance of the word *parārdha*. See *Alberuni's India* (Delhi, 1964), I, 174–175; and Gupta, p. 10.

20. See Gupta, ref. 2 (a), 11–12; and Hayashi, ref. (b), p. 70.
21. See the *Gaṇita-sāra-saṅgraha* ed., with English and Kannada translations, by Padmavathamma, Hombuja Jain Math, Hombuja, 2000, pp. 18–19. *Kṣityā* is changed to *kṣiti* by us for better reading.
22. On Mallana, see S.R. Sarma, "The *Pāvulūrigaṇitam* (popular name of Mallana's transl. of *GSS*), the first Telugu work on Mathematics", *Studien zur Indologie and Iranistik* Heft 13/14 (1987), 163–176, p. 169.
23. We have made slight changes in the names as found in the printed version of Mallana's above work (*Sārasaṃgraha-gaṇitamu*=transl. of *GSS*), Part I, Tirupati, 1952. Mallana's Surname Pāvuḷūrī is derived from his place's name Pāvulūru.
24. K.S. Shukla (ed. and tranls.), *Pāṭīgaṇita* of Śrīdharācārya, Lucknow, 1959, Transl. p. 3. Shukla does not give manuscript-reference, but see Gupta, ref. 2 (a), pp. 12–13 for some details and for Shukla's list also.
25. *Vyavahāra-gaṇita* (in Kannada) ed. by M.M. Bhat, G.O.M.L., Madras, 1955, p. 3. Also see Gupta, ref. 2(a), p. 13 for the full list of 40 terms with Kannada influence.
26. Yallaya (c. 1480) also extended Mahāvīra's list to 29 terms in his commentary on the *Āryabhaṭīya*. See Shukla, *op. cit.* (under ref. 24 above), transl. p. 3, and Gupta, ref. 2(a), p. 13 for some details. Extensions of Mahāvīra's list continued in South India even upto the end of seventeenth century.
27. The names of the work (*Amalasiddhi*) and of the decuple terms have been taken from Agaracandra Nāhāṭā's Hindi article on "Significant Time-reckoning in Jaina Canon," published on Śrimad Rajendrasūri Memorial Volume (Āhora & Bāgarā, 1957), pp. 564–579. Nahata gives another additional list of 15 terms from *kṣiti* (= 10^{19}) to *Oṃkāra śakti* (= 10^{33}) from *"Līlāvatī"*. But he does not give details of the two works.

Part IV

Jaina Mathematics and Astronomy

Circumference of the Jambūdvīpa in Jaina Cosmography

In Jain cosmography, the periphery of the Jambu Island is taken to be a circle of diameter 100,000 *yojanas*. The circumference of a circle of this size, as stated in Jain canonical and geographical works like the *Anuyogadvāra-sūtra* and *Triloka-sāra* etc. is equal to

316227 *yojanas*, 3 *krośas*, 128 *daṇḍas* and $13\frac{1}{2}$ *aṅgulas* nearly.

However, the *Tiloya-paṇṇatti* (between the fifth and the ninth century AD) gives a value (apparently quoted from the canonical work *Diṭṭhivāda*) of the circumference of the Jambūdvīpa as calculated upto a very fine unit of length called *avasannāsanna-skandha* where 8^{12} of these units make one *aṅgula* (finger-breadth). It is shown that the value was computed by making use of the following two approximate rules

- circumference $= \sqrt{10\ (\text{diameter})^2}$
- $\sqrt{a^2 + x} \ = a + (\frac{x}{2a})$.

The correctly carried out long numerical calculations leave a fractional remainder whose true interpretation has been obtained here.

1 Introduction

According to Jaina cosmography, the Jambūdvīpa ('Jambu Island') is circular in shape and has diameter of 100,000 *yojanas*. Umāsvāti's *Tattvārthādhigamasūtra* (= *TDS*), III, 9, for example, states.[1]

......योजनशतसहस्रविष्कम्भो जम्बूद्वीपः ॥ ९ ॥

The *Jambūdvīpa* is of diameter one hundred thousand *yojanas*'. That is,

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K. Ramasubramanian (ed.), *Gaṇitānanda*,
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$$D = 100,000 \text{ yojanas}. \tag{1}$$

Some other explicit references are:

- *Tiloya-paṇṇatti* (= *TP*), IV, 11 (Vol. I, p. 143) of Yativṛṣabha[2]
- *Tiloya-sāra* (= *TS*), *gāthā* 308 (p. 123) of Nemicandra (tenth century AD)[3]
- *Jambū-paṇṇatti-saṃgaho* (= *JPS*), I, 20 (p. 3) of Padmanandin[4]

The *Viṣṇu-purāṇa*, a non-Jaina work, also takes the Jambūdvīpa to be of the same shape and size[5].

The constancy of the ratio of the circumference of any circle to its diameter was recognized in all parts of the ancient world. This ratio is denoted by the Greek letter π(*pi*), so that the circumference C is given by

$$C = \pi D. \tag{2}$$

However, *pi* is not a 'simple' number. It is not only irrational but transcendental. Hence its true value cannot be expressed by an integer, fraction, surd, or by a terminating decimal. Thus, for any practical purpose, we can use only an approximate value of *pi*.

The simplest approximation to the exact formula (2) will be

$$C = 3D. \tag{3}$$

A rule equivalent to (3) is contained, for example, in *TS*, 17 (p. 9) which states

वासो तिगुणो परिही .... || १७ ||

Diameter multiplied by three is the circumference.*

Utilizing the crude formula (3), the circumference of the Jambūdvīpa will be given by

$$C = 300,000 \text{ yojanas}. \tag{4}$$

However, the Jainas knew the inaccuracy of the rough value given by (4). That is why they attempted to find an accurate value which is far better than (4).

The purpose of the present paper is to describe those values of C which were intended to be more accurate and explain as to how they were obtained.

For the purpose of comparison, we first find the correct modern value of C. Taking the true modern value of *pi*, correct upto 27 decimal places, and using (2), we get[6]

$$C = 314159.265,358,979,323,846,264,338,3 \text{ yojanas} \tag{5}$$

correct to 22 decimal places.

However, the form in which ancient values were expressed should not be expected to be of the type (5) which utilizes decimal fractions. For expressing fractional parts, the Jainas employed a series of sub-multiple units to a very very fine degree. Starting

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*Rule also in TP, V, 241 (p. 560).

with the *paramāṇu* ('extremely small particle') of an indeterminately small size and ending with the *yojana*, the *TP*, I, 102–106 (pp. 12–13) and I, 114–116 (p. 14), contains a system of linear units which we present in Table 1 below.[7]

From Table 1, it can be easily seen that

$$1 \textit{ yojana} = 5.3 \times 10^{16} \textit{ avasa} \text{ units roughly},$$

so that an *avasa* unit is of the order of about 10^{-17} of a *yojana* or of the order of about 10^{-22} with respect to the given diameter (1). That is why we must employ a decimal value correct to about 25 places in order to check or compare with another value which is specified upto the *avasa* unit together with the fractional remainder thereafter.

The value (5), which is in conformity with above consideration, can now easily be transformed and expressed in terms of the units of Table 1. We have done this by successively changing the value of the fractional part left into sub-units at each stage. This transformed form of the correct modern value of the circumference of the Jambūdvīpa is shown in Table 2.

Table 1 Units of length from the *Tiloya-paṇṇatti*

Infinitely many *paramāṇus*	=	*avasannāsanna skandha*
8 *avasa* units	=	1 *sannāsanna skandha*
8 *sannāsannas*	=	1 *truṭareṇu*
8 *truṭareṇus*	=	1 *trasareṇu*
8 *trasareṇus*	=	1 *rathareṇu*
8 *rathareṇus*	=	1 *uttama bhogabhūmi bālāgra*
8 *ut. bho. bālāgras*	=	1 *madhyama bhogabhūmi bālāgra*
8 *ma. bho. bālāgras*	=	1 *jaghanya bhogabhumi bālāgra*
8 *ja. bho. bālāgras*	=	1 *karma-bhūmi bālāgra*
8 *ka. bālāgras*	=	1 *likṣa*
8 *likṣas*	=	1 *yūka*
8 *yūkas*	=	1 *yava* (barley corn)
8 *yavas*	=	1 *aṅgula* (finger-breadth)
6 *aṅgulas*	=	1 *pāda*
2 *pādas*	=	1 *vitasti* (span)
2 *vitastis*	=	1 *hasta (fore arm or cubit)*
2 *hastas*	=	1 *rikkū* (or *kiṣku*)
2 *kiṣkus*	=	1 *daṇḍa* (staff) or *dhanuṣ* (bow)
2000 *daṇḍas*	=	1 *krośa*
4 *krośas*	=	1 *yojana*

Table 2 Circumference of the Jambūdvīpa of Diameter 100,000 yojanas

Sl. No.	Denomination or unit	By $C = \pi D$, with actual value of *pi*	By $C = \sqrt{10}D$, with actual value of $\sqrt{10}$	As Found in the *Tiloya paṇṇatti (TP)*	Area $= C.\frac{D}{4}$, with C from *TP*. (in square units)
1	*yojana*	314159	316227	316227	79056, 94150
2	*krośa*	1	3	3	1
3	*daṇḍa*	122	128	128	1553
4	*kiṣku*	1	0	0	0
5	*hasta*	1	0	0	0
6	*vitasti*	0	1	1	1
7	*pāda*	1	0	0	0
8	*aṅgula*	5	0	1	1
9	*yava*	5	7	5	6
10	*yūka*	4	3	1	3
11	*likṣa*	4	4	1	3
12	*ka. bālāgra*	3	7	6	2
13	*ja. bho. bālāgra*	2	4	0	7
14	*ma. bho. bālāgra*	3	3	7	3
15	*ut. bho. bālāgra*	6	5	5	7*
16	*rathareṇu*	7	5	1	4*
17	*trasareṇu*	4	2	3	2*
18	*truṭareṇu*	5	1	0	3*
19	*sannāsanna*	0	5	2	7
20	*avasa.* units	6	7	3	1
21	*kha-kha* fraction (or remainder)	$\frac{43}{100}$ nearly	$\frac{71}{100}$ nearly	$\frac{23213}{105409}$	$\frac{48455}{105409}$

2 The Jaina Value of the Circumference

Naturally, we need not expect the exact modern value of C (as calculated by us above) to be stated in any ancient Jaina work, because, like all other ancient peoples, the Jainas also used only approximate values of *pi* needed in the relation (2).

The Jainas commonly employed the following formula, which is better than (3),

$$C = \sqrt{10D^2} \tag{6}$$

$$\text{or} \quad C = \sqrt{10}D \tag{7}$$

There is no shortage of references to (6) or (7) in Jaina works. It occurs in the *Bhāṣya* (p. 170)[8] which accompanies the *TDS* under III, II. Some other references are:

1. *TP*, I, 117, first half (Vol. I, p. 14); *TP*, IV, 9 (vol. I, p. 143); etc.
2. *TS*, 96, first half (p. 41) and *TS*, 311, first half (p. 125).
3. *JPS*, I, 23 (p.3).
4. *Jyotiṣ-karaṇḍaka* (*gāthā* 185).[9]

By taking the value of $\sqrt{10}$ correct to 27 decimal places, we get, from (7) which is theoretically equivalent to (6),

$$C = 316227.766, 016, 837, 933, 199, 889, 354, 4 \textit{ yojanas}. \tag{8}$$

As before, we have converted this value in terms of the units of Table 1. The result obtained is shown in Table 2.

The value of the circumference of the Jambūdvīpa as found stated in the *TP*, IV, 50–57 (vol. I, p. 148)[10] is also given in Table 2. The *TP* value is slightly more than

$$C = 316227 \textit{ yojanas}, 3 \textit{ krośas}, \ 128 \textit{ daṇḍas}, \text{ and } 13\frac{1}{2} \textit{ aṅgulas}. \tag{9}$$

This simplified value which is rounded off to the nearest half of an *aṅgula* is found in many works including:

1. *Anuyogadvāra-sūtra*, 146, where it is given as the circumference in a *palya* of diameter one lac *yojana*.[11]
2. *Jīvājīvābhigama-sūtra*, 82 (without reference to Jambūdvipa).[12]
3. *TS*, 312 (p. 126) as an accurate value.
4. *JPS*, I, 21–22 (p. 3).

A glance at the Table 2 will show that the *TP* value does not fully agree with that which is accurately found by the Jaina formula (6) or (7). The latter value is slightly *less* than

$$C = 316227 \textit{ yojanas}, 3 \textit{ krośas}, \ 128 \textit{ daṇḍas}, \text{ and } 13 \textit{ aṅgulas}. \tag{10}$$

Thus, there is a divergence even between the frequently met and rounded off Jaina value, given by (9), and the one given by (10) which is based on the correct value of the square-root of ten to a desired degree.[13]

Naturally, we are keen to know the cause of disagreement between the two sets of values, particularly because the values are intended to give accuracy to a very fine degree of smallness. Is there some arithmetical error of calculation in extracting the square root, successively, to the desired degree? Or, the Jainas followed some different procedure? This we answer in the following pages.

3 How the Circumference was Obtained

For finding the square-root of a non-square positive integral number N, the following binomial approximation was frequently used during the ancient and medieval times.

$$\sqrt{N} \equiv \sqrt{(a^2 + x)} = a + \left(\frac{x}{2a}\right) \tag{11}$$

where a and x are positive integers, and the 'remainder' x is less than the 'divisor' $2a$; otherwise or alternately, we may use

$$\sqrt{N} \equiv \sqrt{(b^2 - y)} = b - \left(\frac{y}{2b}\right). \tag{12}$$

The approximation (11) was known to the Greek Heron of Alexandria (between c. 50–c. 250 AD)[14] and even to the ancient Babylonians.[15] The Chinese Sun Tzu (between 280 and 473 AD),[16] while extracting the square-root of 234567 by an elaborate method, finally said:[17]

"Thus we get 484 for the square-root in the above and 968 for the *hsia-fa*, the remainder being 311."

He gave the answer

$$484 + \left(\frac{311}{968}\right). \tag{13}$$

Thus, whatever be the method of Sun Tzu, the result (13) is equivalent to what we get by using (11).

The *Jaina Gem Dictionary* (pp. 154–155) gives the same rule, as represented by (11), for finding the square-root.[18] The *TP*, I, 117 (vol. I. p. 14) implies that the circumference of a circle of diameter one *yojana* was calculated to be $\frac{19}{6}$ *yojanas*. This is in agreement with the use of the rule (11), since

$$\sqrt{10} = \sqrt{(3^2 + 1)} = 3 + \left(\frac{1}{6}\right). \tag{14}$$

Now from (1) and (6) we get

$$\begin{aligned} C &= \sqrt{(100,000,000,000)} = \sqrt{(316227)^2 + 484471} \\ &= 316227 + \frac{484471}{2 \times 316227}\ \textit{yojanas} \end{aligned} \tag{15}$$

by applying the approximation (11). In the present case, therefore, we have

$$\begin{aligned} &\text{'divisor'} = 632454 \\ \text{and}\quad &\text{'remainder'} = 484471. \end{aligned}$$

The fractional *yojana* remainder, namely

$$\frac{484471}{632454}$$

when converted into *krośas*, will give

$$484471 \times \frac{4}{632454} krośas = 3 + \left(\frac{40522}{632454}\right) krośas. \tag{16}$$

The fractional *krośa* remainder, namely

$$\frac{40522}{632454}$$

can, similarly, be converted into the next lower sub-units (*daṇḍas*). The process can be continued likewise.

We shall easily get 128 *daṇḍas*, 1 *vitasti* (= 12 *aṅgulas*) and 1 *aṅgula* with the fractional *aṅgula* remainder to be equal to

$$\frac{407346}{632454} \tag{17}$$

which is equal to

$$\frac{67891}{105409}. \tag{18}$$

Thus, we see that the fractional *aṅgula*-remainder (18) is slightly more than half. In this way, we get the circumference of the Jambūdvīpa as given by (9).

However, if we want to carry out the evaluation to lower and lower units (as should be done in order to get a value comparable to that found in the *TP*), we easily have (putting 105409 equal to H);

(a) *aṅgula*-fraction, $\frac{67891}{105409} = 5 + \left(\frac{16083}{H}\right)$ *yavas*
(b) *yava*-fraction, $\frac{16083}{H} = 1 + \left(\frac{23255}{H}\right)$ *yūkas*
(c) *yūka*-fraction, $\frac{23255}{H} = 1 + \left(\frac{80631}{H}\right)$ *likṣas*
(d) *likṣa*-fraction, $\frac{80631}{H} = 6 + \left(\frac{12594}{H}\right)$ *ka. bālāgras*
(e) *ka. bāl.*-fraction, $\frac{12594}{H} = 0 + \left(\frac{100752}{H}\right)$ *ja. bho. bālāgras*
(f) *ja. bho. bāl.*-fraction, $\frac{100752}{H} = 7 + \left(\frac{68153}{H}\right)$ *ma. bho. bālāgras*
(g) *ma. bho. bāl.*-fraction, $\frac{68153}{H} = 5 + \left(\frac{18179}{H}\right)$ *ut. bho. bālāgras*
(h) *ut. bho. bāl.*-fraction, $\frac{18179}{H} = 1 + \left(\frac{40023}{H}\right)$ *ratharenus*
(i) *ratharenu*-fraction, $\frac{40023}{H} = 3 + \left(\frac{3957}{H}\right)$ *trasareṇus*
(j) *trasareṇu*-fraction, $\frac{3957}{H} = 0 + \left(\frac{31656}{H}\right)$ *truṭareṇus*

(k) *truṭareṇu*-fraction, $\frac{31656}{H} = 2 + \left(\frac{42430}{H}\right)$ *sannāsanna*

(l) *sannāsanna*-fraction, $\frac{42430}{H} = 3 + \left(\frac{23213}{H}\right)$ *avasa* units

Thus we have, finally, the *avasannāsanna* fractional remainder

$$= \frac{23213}{105409}. \tag{19}$$

In this way, we see that the above long calculation yields a value which is in complete agreement with the *TP* value right from the whole number of a *yojana* down to the lowest submultiple units defined in the text. Moreover, we have found out a meaning of the fraction (19), designated as *kha-kha* (or *ananta-ananta*, 'endlessly endless') term, which can yield measure in still smaller and smaller units of length (to be defined with the help of the infinitely small particles or *paramāṇus*) if desired.

That the above method is the actual one which was used by the Jainas is quite evident from the full agreement obtained above and is also confirmed by what is given by Madhava-candra in the commentary of his teacher's *TS* under the *gāthā* 311 (pp. 125–126) where the calculation has been carried out upto the fractional *aṅgula* remainder (17).

Once we know the circumference, the area of the Jambūdvīpa can be computed by using the well-known rule, for example see *TP*, IV, 9 (Vol. I, p. 143),

$$Area = C \cdot \frac{D}{4}. \tag{20}$$

The result of our computation of the area by using (20) and *TP* value of C is shown is Table 2. The contribution of the fraction (19)

$$\begin{aligned} &= 23213 \times \frac{25000}{105409} \text{ square } \textit{avasa} \text{ units} \\ &= 5505 + \left(\frac{48455}{105409}\right). \end{aligned} \tag{21}$$

The measures of various denominations (specifying the area) as found in the *TP*, IV, 58–64 (Vol. I, p. 149) agree with the corresponding value which we have computed, including the *kha-kha* fraction[19] given by the bracketed quantity in (21). This again confirms our calculations and interpretations.

Incidentally we have discovered that at least one line (or verse), which ought to be there to specify the numerical values (marked by asterisks in Table 2) of the four denominations from *ut. bho. bālāgras* to *truṭareṇus*, is missing in the printed text in the *TP* (between verses 61 and 62 in the fourth *mahādhikāra*) which we have consulted if not in the original manuscripts.

The contents of the manuscript entitled *jambūdvīpa-paridhi*[20] (Jambūdvīpa-Circumference'), which seems to be relevant to the subject of our present paper, are not known to me.

References and Notes

1. The *Sabhāṣya TDS* edited with the Hindi translation of Khubacandra, p. 163, Bombay, 1932 (Paramasruta Prabhavaka Jaina Mandala). The date of Umāsvāti (or Umāsvāmin) is about 40–90 AD according to J. P. Jain, *The Jaina Sources of the History of Ancient India*, p. 267, Delhi, 1964 (Munshi Ram Manohar Lal); and about fourth or fifth century according to Nathuram Premi, *Jaina Literature and History* (in Hindi), p. 547, Bombay, 1956 (Hindi Grantha Ratnakara).
2. The *TP* (Sanskrit, *Triloka-prajñapti*) in two vols. Part I (2nd ed., 1936) ed. by A. N. Upadhye and Hiralal Jain; Part II (1st ed., 1951) ed. by Jaina and Upadhye. Both published by the Jaina Sanskrit Samrakshaka Samgha, Sholapur (Jivaraj Jain Granthmala No. 1). According to Dr. Upadhye (*TP*, Vol. II, Intr., p. 7), the *TP* is to be assigned to some period between 473 AD and 609 AD However, the work may have acquired its present form as late as about the beginning of the ninth century (*TP*, Vol. II, Hindi Intr., p. 20).
3. The *TS* (Sanskrit, *Triloka-sāra*) ed., with the commentary of Mādhava-candra, by Manohar Lal Shastri, Bombay, 1918 (Manikachandra Digambara Jain Granthamala No. 12).
4. The *JPS* ed. by A. N. Upadhye and Hiralal Jain, Sholapur, 1958 (Jivaraj Jain Granthamala No. 7). According to the editors (*JPS*, Intr., p. 14), Padmanandin might have composed the *JPS* about 1000 AD
5. See the *Viṣṇu-purāṇa, aṃśa* 2, Chap. 2 (pp. 138–40), ed., with Hindi transl., by Munilal Gupta, Geeta Press, Gorakhpur, 4th ed., 1957. Also cf. *TP*, Vol. II, Hindi Intr., p. 83.
6. See Howard Eves, *An Introduction to the History of Mathematics,* p. 94, New York, 1959 (Holt, Rinehart and Winston).
7. Cf. L. C. Jain, *"Mathematics of the TP"* (in Hindi), prefixed with the Sholapur ed. of the *JPS*., p. 19.
8. Premi, *op. cit.,* pp. 524–529, believes that the *Bhāṣya* is by the author of *TDS* itself, while J. P. Jain, *op.cit.,* p. 135, says that 'no evidence of the existence of such a *Bhāṣya* prior to eighth century AD has yet been discovered'.
9. As quoted by R. D. Misra, "Mathematics of a circle etc." (in Hindi), *Jaina Siddhānta Bhāskara*, Vol. 15, no.2 (January 1949), p. 105: According to the commentator Malayagiri (c. 1200 AD), the *Jyotiṣ-karaṇḍaka* (of Pūrvācārya) was edited on the basis of the first Valabhi *vācanā* which took place c. 303 AD; *see* J. C. Jain *History of Prakrit Literature* (in Hindi), pp. 38 and 131, Chowkhamba Vidya Bhavan Varanasi, 1961.
10. In this connection, the *TP* mentions the work *Diṭṭhivāda* (Sanskrit, *Dṛṣtivāda*) from which the value is apparently quoted: (*see Babu Chotelal Jain Smriti Grantha,* Calcutta, 1967, English section, p. 292; and the *Anusandhāna Patrikā,* no. 2. April-June, 1973, p. 30 (Jaina Vishva Bharati, Ladnun).
11. *See* the *Mūlasuttāṇi* edited by Kanhaiya Lalji, pp. 561–562 (Gurukul Printing Press, Byavara, 1953.)
12. Quoted by H. R. Kapadia in the "Introduction", p. XLV, to his edition of the *Gaṇita-tilaka*, Oriental Institute, Baroda, 1937.
13. The comparison made by Dr. C. N. Srinivasiengar, *The History of Ancient Indian Mathematics,* p. 22 (World Press, Calcutta, 1967) is wrong because he takes one *dhanuṣ* (or *daṇḍa*) to be equal to 100 *aṅgulas* (instead of 96).
14. D. E. Smith, *History of Mathematics,* Vol. II, p. 254 (Dover reprint, New York, 1958).
15. C. B. Boyer, *A History of Mathematics,* p. 31 (Wiley, New York, 1968).

16. *See ISIS*, Vol. 61, part I, (1970), p. 92.
17. Y. Mikami, *The Development of Mathematics in China, and Japan,* p. 31, (Chelsea reprint, New York, 1961.)
18. Quoted by Kapadia (ed.), *op. cit.,* p. XLVI.
19. *See* L. C. Jain, *op. cit.,* pp. 49–50, for his comments on these *kha-kha* fractions.
20. *See the Catalogue of Manuscripts at Limbadi* (in Devānagarī), edited by Catura-vijaya, p. 61, serial no. 1014, Bombay, 1928 (Agamodaya Samiti).

Mādhavacandra's and Other Octagonal Derivations of the Jaina Value $\pi = \sqrt{10}$

$\sqrt{10}$ was one of the approximate values of π used in ancient and medieval times especially in Jaina works. K. Hunrath derived it from a dodecagon a century ago, and G. Chakravarti from an octagon about fifty years ago. An ancient derivation given by Mādhavacandra (c. 1000 AD) in his Sanskrit commentary on *Tiloya-sāra* of Nemicandra. (c. 975 AD) has been examined in detail especially in the light of expositions given by Chakravarti and Āryikā Viśuddhamatī recently. Some new and more plausible interpretations are advanced here regarding derivation of $\pi = \sqrt{10}$ based on octagons. Use of the process of averaging is also illustrated.

1 Introduction

It is well known that an ancient Indian rule for finding the perimeter (or circumference) p of a circle of diameter d can be expressed by the formula

$$p = \sqrt{10\, d^2}. \tag{1}$$

This rule is found used or adopted especially in the early Jaina canonical and other works.[1] Rules equivalent to (1) are also found in non-Jaina Indian as well as foreign works.[2] Several derivations of (1) have been suggested. According to Colebrooke (1755–1837),[3] Brahmagupta (c. 628 AD) is said to have obtained the value $\pi = \sqrt{10}$ by "inscribing in a circle of unit diameter regular polygons of 12, 24, 48, and 96 sides and calculating successively their perimeters which he found to be $\sqrt{9.65}$, $\sqrt{9.81}$, $\sqrt{9.86}$, $\sqrt{9.87}$, respectively, and to have assumed that as number of sides is increased indefinitely, the perimeter would approximate to $\sqrt{10}$". Hankel (1873) suggested the

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same method by taking $d = 10$ (instead of unity) but this explanation is considered doubtful by Hobson.[4]

Hunrath (c. 1883)[5] first calculated the arrow (= height of the segment) corresponding to the arc of a sixth part of the circumference of the circle as

$$h_6 = \frac{d(2 - \sqrt{3})}{4}. \tag{2}$$

Then he took the approximate value $\frac{5}{3}$ for $\sqrt{3}$ thereby getting

$$h_6 = \frac{d}{12} \tag{3}$$

and so the side of the inscribed dodecagon would be given by

$$\begin{aligned} s_{12}^2 &= h_6^2 + \left(\frac{1}{4}\right) \cdot s_6^2 \\ &= \left(\frac{d}{12}\right)^2 + \left(\frac{1}{4}\right) \cdot \left(\frac{d}{2}\right)^2 \\ &= \frac{10d^2}{144}. \end{aligned}$$

(Subscripts here denote the number of sides of the related inscribed regular polygon). Hence, finally (but approximately)

$$p^2 = (12s_{12})^2 = 10d^2$$

which is equivalent to (1). Hunrath's method is considered to be "most plausible" by Sarasvati Amma[6] (but see Sect. 3 below for a comment on this method).

Another method based on dodecagon has been recently suggested by Afzal Ahmad[7] but it is unsatisfactory since it chooses an arbitrary denominator in approximating $\sqrt{3}$.

However, Mādhavacandra Traividya (c. 1000 AD), an ancient Jaina writer, gave a different derivation which is based on a polygon of only 8 sides (instead of 12 and more employed in above methods). We discuss below in detail the various methods based on considerations of octagons.

2 Mādhavacandra's Derivation

Nemicandra, a famous *Digambara* Jaina author (c. 975 AD) composed his *Tiloya-sāra* (Sanskrit, *Triloka-sāra*) in Prakrit. Its *gāthā* 96 contains the formula (1) as[8]

विक्खं भवग्गदहगुणकरणी वट्टस्स परिरयो होदि।
(विष्कम्भवर्गदशगुणकरणिः वृत्तस्य परिधिः भवति।)

The square root of ten-times the square of the diameter becomes the circumference of the circle.

Mādhavacandra, pupil of Nemicandra, in his Sanskrit commentary on the *Triloka-sāra* gives the derivation *(vāsanā)* of (1) under the above *gāthā* in the following words:[9]

....एकयोजन-वृत्त-क्षेत्रं तत्प्रमाणेन चतुरस्रं कृत्वा भुज-कोट्योः कृत्योः परस्परं गुणयित्वा `विवि १ विवि १ समासे वि वि २, कर्णकृतिः तस्यामर्धितायां द्वितीयांशः तस्मिन्नर्धिते (पुनरप्यर्धितायां) चतुर्थांशः, तस्मिन्नर्धिते अष्टमांशं खण्डं, तत्रैकखण्डं गृहीत्वा भुजकोट्योः द्वाभ्यां समानछेदेन मेलनं कृत्वा एकखण्डस्य एतावति फले अष्टखण्डस्य किम्। वर्गराशेर्गुणकार-भागहरौ वर्गात्मकौ भवत इति न्यायेन इच्छाङ्कः वर्गरूपेण गुणकारो भवति। तयोर्गुणकारभागहारयोर्वज्रापवर्तने दशगुणिते विष्कम्भवासना भवति।

(Take) a circle of diameter one *yojana* and draw a square of the same dimension. By mutual (or self) multiplication obtain the squares of the base (horizontal side, say) and upright (perpendicular side) *1dd* and *1dd*; adding which we get $2d^2$, the square of the diagonal. Halving (each d) in it we get (the square of) half the diagonal, and by again halving, we get (the square of) the fourth part (of the diagonal); and by (once more) halving, we get (the square of) the eighth part of the diagonal.

Now take one of the (eight arcual) segment, and by reducing (the squares of) the *bhuja*, $\left(= \frac{2d^2}{16}\right)$ and *koṭi* $\left(= \frac{2d^2}{64}\right)$ to a common denominator, add them $\left(\text{to get } \frac{10d^2}{64}\right)$. If this is the result for one segment *(khaṇḍa)*, what will it be for the eight segments? By the rule *(nyāya)* that when a (to be operated) quantity is in square form, its multiplier and divisor should also be in the square form, the desired (proportionality) multiplier here should be in square form (*i.e.* $8^2 = 64$). So that by mutual (or cross) cancellation $\left(\text{in } 64 \times \frac{10d^2}{64}\right)$, the result will be $10d^2$. This gives the derivation of the above rule (starting with the word) *viṣkambha*.

From this almost literal translation of Mādhavacandra's passage, it is seen that many points in his derivation need explanation especially because he himself did not give any accompanying diagram which would have clarified the doubts. This situation has given rise to various interpretations by scholars. In the following sections we critically examine some of these interpretations.

3 G. Chakravarti's Computations

More than 50 years ago Chakravarti[10] found that Mādhavacandra's method consisted in equating the perimeter of the (inscribed regular) octagon to the circumference of the circle. In other words the arc WmR (see Fig. 1) was taken equal to the chord WR which is a side of the octagon. To find this Chakravarti gave the following calculations:

$$WY = \frac{OW}{\sqrt{2}} = \frac{d}{2\sqrt{2}}. \tag{4}$$

This is what Mādhavacandra called *bhuja*, for one *khaṇḍa* (segment or portion) and gave

$$(bhuja)^2 = \frac{2d^2}{16} \tag{5}$$

which is clearly equal to $(WY)^2$ exactly. However, Chakravarti took the value

$$\sqrt{2} = 1 + \left(\frac{1}{3}\right) = \frac{4}{3} \tag{6}$$

which he got from the *Śulbasūtra* approximation

$$\sqrt{2} = 1 + \left(\frac{1}{3}\right) + \left(\frac{1}{3.4}\right) - \left(\frac{1}{3.4.34}\right) \tag{7}$$

whose first two terms are believed to be based on the formula (which itself is based on a sort of linear interpolation)[11]

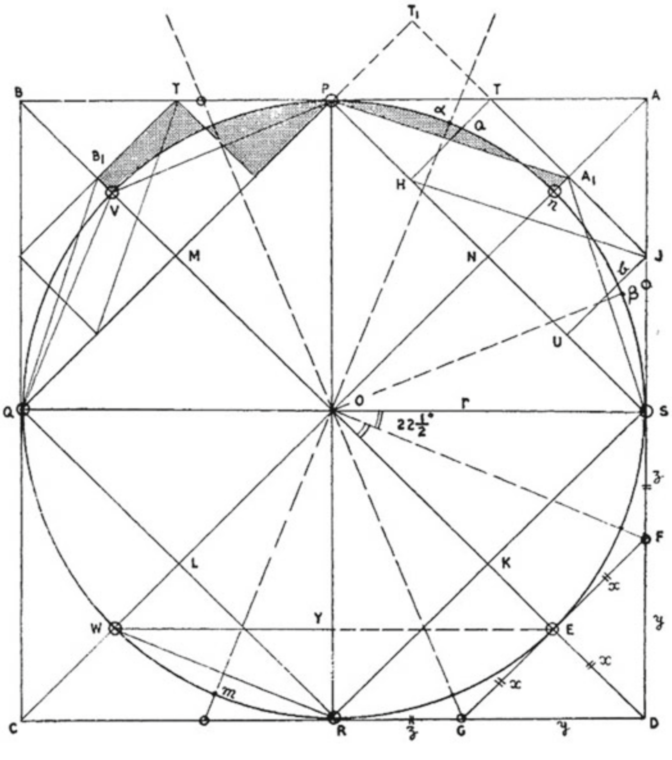

Fig. 1 Mādhavacandra's method

$$\sqrt{a^2 + x} = a + \frac{x}{(2a+1)}. \tag{8}$$

By using (6), Chakravarti then got, from (4),

$$WY = \frac{3d}{8} \quad \text{or} \quad (WY)^2 = \frac{9d^2}{64} \tag{9}$$

against the mathematically exact value (5) given by Mādhavacandra. Chakravarti then calculated

$$RY = OR - OY = \left(\frac{d}{2}\right) - WY = \frac{d}{8}$$

so that

$$(RY)^2 = \frac{d^2}{64}, \tag{10}$$

while Mādhavacandra took

$$(koṭi)^2 = \frac{2d^2}{64}. \tag{11}$$

However, the final results will be the same since

$$(WR)^2 = (WY)^2 + (RY)^2 = \left(\frac{9d^2}{64}\right) + \left(\frac{d^2}{64}\right) = \frac{10d^2}{64}$$

and also, from (5) and (11),

$$(bhuja)^2 + (koṭi)^2 = \left(\frac{8d^2}{64}\right) + \left(\frac{2d^2}{64}\right) = \frac{10d^2}{64}.$$

It must be noted that although Mādhavacandra's *bhuja* gives the exact value of WY, his value of *koṭi* does not represent the exact or true value of RY which is, otherwise, given by

$$RY = \left(\frac{d}{2}\right) - \left(\frac{d}{2\sqrt{2}}\right) = \frac{d(2-\sqrt{2})}{4}. \tag{12}$$

Of course Chakravarti's values of WY and RY, given by (9) and (10), are both approximate. Anyway, if WY is regarded *bhuja* in the right angled $\triangle WYR$, then RY must be called *koṭi* therein. Now Mādhavacandra's value of his *koṭi* also represents the length of half of OY or OL or CL, and WL is equal to YR. We may therefore say that he made the practically sound assumption that W is the middle point of CL.

However, there seems to be another reason as to why Mādhavacandra took his *koṭi* equal to half of his *bhuja*. We have, by using the true values[†] (4) and (12),

$$\frac{YR}{OY} = \frac{YR}{WY} = \sqrt{2} - 1, \text{ exactly.} \tag{13}$$

But according to the usual Jaina formula (of the binomial theorem type and based on completing the square, say)[12]

$$\sqrt{a^2 + x} = a + \left(\frac{x}{2a}\right) \tag{14}$$

so that,

$$\sqrt{2} = \sqrt{1^2 + 1} \simeq 1 + \frac{1}{2} = \frac{3}{2} \tag{15}$$

and thus, by (13)

$$\frac{YR}{OY} = \frac{1}{2} = \frac{WL}{OL} = \frac{WL}{CL}. \tag{16}$$

That is, YR is half of WY or WL is half of LR. In other words *koṭi* is half of *bhuja*. And this theoretical basis and practical interpretation seem to be quite plausible (see Sect. 4 below, for other interpretations). We have thus distinguished Mādhavacandra's method based on (14) from that of Chakravarti based on (8). It may be pointed out that Hunrath's value $\frac{5}{3}$ for $\sqrt{3}$ (see Sect. 1 above) is also based on the non-Jaina formula (8). To emphasize the contrast between the calculations of Mādhavacandra and Chakravarti we see that the value of $\left(\frac{WY}{RY}\right)^2$ is 4 according to the former but 9 according to the latter, while the correct value is, from (4) and (12), given by

$$\frac{(WY)^2}{(RY)^2} = 3 + 2\sqrt{2} \approx 5.8.$$

So Mādhavacandra's value is better. It should also be noted that although point Y trisects OR according to (16) in conformity with Mādhavacandra's values, he did not take the simplified values

$$YR = \frac{OR}{3} = \frac{d}{6}$$

$$\text{and} \quad OY = \frac{2 \cdot OR}{3} = \frac{2d}{6} = WY$$

[†]This also follows directly from the fact that

$$OY + YR = OR = OW = \sqrt{2} \cdot OY$$

$$\text{or} \quad YR = \sqrt{2} \cdot OY - OY = (\sqrt{2} - 1)\ OY.$$

.

as these would have led to

$$(WR)^2 = \left(\frac{2d}{6}\right)^2 + \left(\frac{d}{6}\right)^2 = \frac{10d^2}{72}$$

instead of the desired value $\frac{10d^2}{64}$.

Finally, one more thing may be pointed out. The method of *inscribed* polygon, when correctly followed, should lead to a value of π which is *less* than the actual value; but here we are getting $\sqrt{10}$ which is greater than the true value of π. The reason is that in finding the length of the side of the octagon both Mādhavacandra and Chakravarti overestimated it.

4 Āryikā Viśuddhamatī's Exposition

In her recent translation and exposition of Mādhavacandra's derivation, Viśuddhamatī[13] has correctly given the values of the squares of the *bhuja* and *koṭi* as mentioned in the commentary by the former and represented by (5) and (11). But from the diagrams accompanying her exposition it seems that she has taken the rectangle $THUJ$ as representing an *aṣṭamāṃśa* ("eighth part") which has also been drawn as shown separately. No doubt, the dimensions (i.e. the two mutually perpendicular sides HU and UJ) of this rectangle are the same as those given by Mādhavacandra. There are two difficulties in accepting this interpretation. Firstly, the arc *anb* contained in it is not the eighth part of the circumference of the circle (the eighth part is correctly given by the arc $\alpha n\beta$, instead of *anb*). Secondly, the practical equality of the arcual eighth part (*anb* or even $\alpha n\beta$) to the diagonal HJ of the rectangle is not clear from the diagram. So, this interpretation cannot be considered satisfactory, although it does not theoretically affect the derivation.

To remove the above difficulties, I suggest that Mādhavacandra's *bhuja*, be taken to represent the side $PN(= HU)$ and his *koṭi* be taken to represent the upright or perpendicular side $NA_1(= UJ)$ in the right-angled $\triangle PNA_1$ (or the rectangle PNA_1T_1). Here, the enclosed arc Pan is exactly equal to the eighth part of the circumference of the circle and the upright NA_1 is also the eighth part of the diagonal AC. Moreover, the equality of the resulting hypotenuse (or diagonal) PA_1 to the overlapping (or crossing) arc Pan. seems to be a practical approximation to the eyes. Of course, the final result in this new interpretation will be the same as found by Mādhavacandra (and Viśuddhamatī, and even Chakravarti) since

$$\begin{aligned} PA_1^2 &= PN^2 + NA_1^2 = (RN)^2 + \left(\frac{NA}{2}\right)^2 \\ &= (bhuja)^2 + (koṭi)^2 = \left(\frac{8d^2}{64}\right) + \left(\frac{2d^2}{64}\right) = \frac{10d^2}{64}, \end{aligned}$$

And so

$$p^2 \approx (8.PA_1)^2 = 10d^2, \text{ as desired.}$$

5 Applying the Process of Averaging

The process of averaging is known to be a popular and useful ancient technique especially when the exact result or derivation was unknown or difficult.[14] Even in matters of circling the square or squaring the circle, averaging has been suggested as an explanation of some of the Indian rules.[15] Here we shall confine to derivations based on considerations of octagons only.

From the figure we have

$$x = ED = (\sqrt{2} - 1)r,$$
$$y = FD = \sqrt{2}x = (2 - \sqrt{2})r$$
$$z = FS = r - y = (\sqrt{2} - 1)r = x.$$

These results also follow by using trigonometry, since $2x$ or $2z$ is the side of the circumscribed regular octagon and, from $\triangle s$ OEF and OSF

$$x = z = r \tan\left(\frac{45}{2}\right)^\circ = (\sqrt{2} - 1)r,$$

but we confine to more elementary and primitive methods and approach. Now the perimeter of the circumscribed octagon

$$= 8x + 8z = 16x,$$

so that

$$\pi = \frac{p}{2r} < \frac{16x}{2r} = 8(\sqrt{2} - 1). \tag{17}$$

Therefore,

$$\phi^2 < 64(3 - 2\sqrt{2}) < 11.$$

On the other hand by considering the inscribed regular octagon and using (4) and (12), we have

$$s_8^2 = WR^2 = (2 - \sqrt{2})r^2,$$

Now,

$$2\pi r > 8 \cdot s_8$$

or,

$$\begin{aligned}\pi^2 &> 16(2-\sqrt{2})\\ &= 9 + (23 - 16\sqrt{2}),\end{aligned}$$

from which it follows that

$$\pi^2 > 9.$$

Hence we have

$$9 < \pi^2 < 11,$$

and so by averaging

$$\pi^2 = 10$$

as desired and implied in (1).

Instead of perimeters, we may consider the areas of the two octagons. We see that the area of the circumscribed octagon

$$\begin{aligned}&= \text{square } ABCD - 4 \cdot \triangle GDF\\ &= (2r)^2 - 4 \cdot \left(\frac{y^2}{2}\right) = 8(\sqrt{2}-1)r^2.\end{aligned}$$

So that

$$\pi r^2 < 8(\sqrt{2}-1)r^2$$

giving the same inequality as (17), and hence here also

$$\pi^2 < 11.$$

On the other hand, let us approximate the area of the circle by the square $PQRS$ plus the four rectangles of the type $THUJ$ on the four sides. From the shaded areas in the octant POB_1 we see that the area left-out is more than the extra area included. Thus,

$$\begin{aligned}\pi r^2 &> \text{square } PQRS + 4 \times (\text{rectangle } THUJ)\\ &= 6 \times (\text{square } OMPN)\\ &= 6 \times \frac{r^2}{2} = 3r^2.\end{aligned}$$

So that

$$\pi^2 > 9.$$

Hence we get, again

$$9 < \pi^2 < 11,$$

and the desired result follows by averaging as before.

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Chords and Areas of Jambūdvīpa Regions in Jaina Cosmography

In Jaina works, the Jambūdvīpa ("Jambu Island") is circular and of diameter $D = 100000$ *yojanas* (*Tiloyapaṇṇattī* $=$ *TP*, IV. 11; Vol. II, p. 4; Kota, 1986). It is divided into 13 main regions by boundary lines which are all parallel to the east–west direction. Starting from southern end, their widths are $2^n\sigma$, where $n = 0, 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1, 0$, respectively (the *śalākā* $\sigma = \frac{D}{190}$). The central zone, called Videha, of width 64σ is thus bisected by the east–west diameter. The names of the regions of the southern half of the Jambūdvīpa are shown in Table 1 along with the lengths of their northern boundary chords (which cut out circular segments of various heights h).

The lengths of the chords were found out by the well-known ancient rule (equivalently given in *TP*, IV. 183, p. 51)

$$c^2 = 4h(D - h). \tag{1}$$

The calculations have been done in a simplified manner. In evaluating the square root of a rational number, a suitable integral denominator or divisor is separated and the square root of the large numerator is extracted to a whole number (leaving out the remaining portion). For example, the chord of the Bharata region is given by, using (1),

$$\begin{aligned} c &= \frac{\sqrt{756 \times 10^8}}{19} \\ &= \frac{\sqrt{(274954)^2 + 297884}}{19} \\ &= \frac{274954.54}{19} \qquad \text{nearly.} \end{aligned}$$

Gaṇita Bhāratī, Vol. 9, Nos. 1–4 (1987), pp. 51–53.

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https://doi.org/10.1007/978-981-13-1229-8_19

Table 1 Northern chords

Sl. no.	Region	Width	h	Integral value of chord	Actual part nearly	Fraction after leaving decimal portion	Textual value of fraction	*TP* reference
1	Bharata	σ	σ	14471	$\frac{5.54}{19}$	$\frac{5}{19}$	$\frac{5}{19}$	IV, 194; p. 56
2	Himavān	2σ	3σ	24932	$\frac{0.77}{19}$	0	$\frac{1}{19}$	IV, 1647; p. 471
3	Haimavata	4σ	7σ	37674	$\frac{15.26}{19}$	$\frac{15}{19}$	$\frac{16}{19}$	IV, 1722; p. 487
4	Mahāhimavān	8σ	15σ	53931	$\frac{6.07}{19}$	$\frac{6}{19}$	$\frac{6}{19}$	IV, 1742; p. 491
5	Hari	16σ	31σ	73901	$\frac{17.7}{19}$	$\frac{17}{19}$	$\frac{17}{19}$	IV, 1763; p. 495
6	Niṣadha	32σ	63σ	94156	$\frac{2.1}{19}$	$\frac{2}{19}$	$\frac{2}{19}$	IV, 1775; p. 498
7	South Videha	32σ	$\frac{D}{2}$	100000	0	0	0	IV, 1798; p. 503

But the value given in the text (*TP*, IV. 194, p. 56) is

$$c = \frac{274954}{19} = 14471 + \frac{5}{19},$$

omitting the (decimal) portion of the square root although greater than half. However, the text values do not uniformly confine to this (or any other) convention of rounding off. Table 1, we have listed the integral values, the (nearly) actual parts, the fractions obtained by leaving out the (decimal) portions of the square roots (even if greater than half), and the fractions as found in the *TP*.

However, we have noted a very significant uniformity in one matter. If we use the fractions obtained by leaving out the decimal portions (in the square roots) in calculating the areas of the regions (see below), there is perfect agreement with the text values of the areas. We show this now.

For finding the area of a segment of a circle, the *TP*, IV. 2401 (p. 636) contains the verbal rule equivalent to the empirical formula

$$S = \sqrt{10\left(\frac{ch}{4}\right)^2}. \tag{2}$$

Using this, the area of the Bharata region will be

$$
\begin{aligned}
A_1 = S_1 &= \sqrt{\left(\frac{10}{16}\right) \cdot \left(\frac{274954}{19}\right)^2 \cdot \left(\frac{10^4}{19}\right)^2} \\
&= \frac{\sqrt{4724, 9813, 8225 \times 10^7}}{361} \\
&= \frac{2173, 7022, 29}{361}.
\end{aligned}
$$

where we have, following the same above practice, neglected the decimal portion of the square root in the numerator. Thus, we have

$$
A_1 = 6021, 335 + \frac{294}{361},
$$

which is exactly same as given in *TP*, IV. 2402 (p. 636).

For the next segment formed jointly by the Bharata and Himavān regions, we will have

$$
\begin{aligned}
A_1 + A_2 = S_2 &= \sqrt{\frac{10}{16} \cdot \left(\frac{473709}{19}\right)^2 \cdot \left(3 \times \frac{10^4}{19}\right)^2} \\
&= \frac{\sqrt{1262, 2458, 8961 \times 10^9}}{361} \\
&= \frac{1123, 4971, 693}{361} \\
&= 3112, 1805 + \frac{88}{361},
\end{aligned}
$$

following the same practice or convention of extracting square root to completed (not nearest) whole number. In this way we get

$$
A_2 = S_2 - S_1 = 2510, 0469 + \frac{155}{361},
$$

which is exactly what is given in *TP*, IV. 2403 (p. 637).

Similarly we can find S_3, S_4, S_5 and S_6. The combined segment of the listed seven regions is simply equal to half the Jambūdvīpa whose area is already given earlier in *TP*, IV. 59 (p. 17) as 7905,6941,50 (square) *yojanas*. So we have S_7 equal to 3952,8470,75 units without taking trouble to find it by the above procedure (which otherwise yields the additional fraction $\frac{75}{361}$).

All these calculated values of the segmental areas are shown in Table 2 along with the resulting corresponding regional areas which are found to tally completely with those given in text (*TP*, IV. 2402–2407, pp. 636–638).

Table 2 Areas

Sl. no.	Region	Area of corresponding segment calculated by whole-number square root method S	Area of the region from previous column $A = \nabla S$
1	Bharata	$602, 1335 + \frac{294}{361}$	$602, 1335 + \frac{294}{361}$
2	Himavān	$3112, 1805 + \frac{88}{361}$	$2510, 0469 + \frac{155}{361}$
3	Haimavata	$1, 0973, 2502 + \frac{25}{361}$	$7861, 0696 + \frac{298}{361}$
4	Mahāhimavān	$3, 3660, 3542 + \frac{349}{361}$	$2, 2687, 1040 + \frac{324}{361}$
5	Hari	$9, 5324, 3109 + \frac{260}{361}$	$6, 1663, 9566 + \frac{272}{361}$
6	Niṣadha	$24, 6817, 2123 + \frac{211}{361}$	$15, 1492, 9013 + \frac{312}{361}$
7	South Videha	$39, 5284, 7075$	$14, 8467, 4951 + \frac{150}{361}$
		Total	$39, 5284, 7075$

The area of Mahāhimavān is not available as the relevant *gāthā* in the manuscript is eaten by the moths! For finding areas, the present writer has found another method which does not need extraction of square roots.

Reference

Tiloyapaṇṇattī edited with the Hindi translation of *Aryikā Viśuddhamatī* by C. P. Patni, Vol. II, Kota, 1986 (All India Digambara Jaina Mahāsabhā).

The First Unenumerable Number in Jaina Mathematics

1 Introduction

The definition of *asaṃkhyāta* ("unenumerable") numbers in the ancient Indian Jaina Schools is linked to their cosmography according to which the Jambūdvīpa ("Jambū Island") is circular in shape and has a diameter D_0 equal to one lakh *yojanas*. It is surrounded by a series of concentric rings (or annuli) of sea and land alternately (see Fig. 1). The width of the *N*th ring (whether land or sea), taking the Jambūdvīpa itself as the first ring, is given by

$$W_{N-1} = 2^{N-1}.W_0 \tag{1}$$

where the width of the first ring (= the Jambū Island, J.I.) is denoted and given by

$$W_0 = D_0 = 100,000 \textit{ yojanas}. \tag{2}$$

The diameter of outer boundary of the *N*th ring will be given by (see Fig. 2)

$$\begin{aligned} D_{N-1} &= W_0 + 2(W_1 + W_2 + \cdots + W_{N-1}) \\ &= W_0 + 2(2 + 2^2 + 2^3 + \cdots + 2^{N-1})W_0 \\ &= W_0 + 2(2^N - 2)W_0 \\ &= (2^{N+1} - 3)D_0. \end{aligned} \tag{3}$$

Now counting is done by means of tiny *sarṣapa* seeds with which a variable (*anavasthita*) pit is repeatedly filled and emptied by dropping the seeds one by one on the various rings of land and sea starting with the Jambū Island itself. Let

$$n = f(R) \tag{4}$$

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K. Ramasubramanian (ed.), *Gaṇitānanda*,
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where n is the total number of seeds required to fill the pit fully in the prescribed manner when its radius is R (the pit is cylindrical). Initially the pit has a radius R_0 equal to that of the Jambū Island and is over-filled with n_0 seeds (see below for calculation of n_0). By dropping these seeds one by one on the various successive rings we will be able to cover the first n_0 rings.

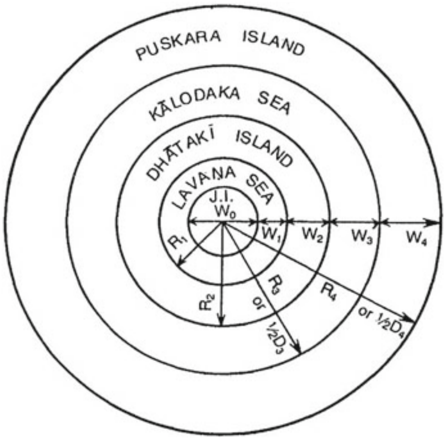

Fig. 1 Jaina cosmography

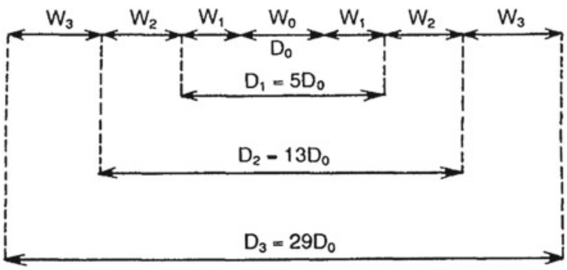

Fig. 2 Widths and diameters of rings

Let the radius of the last ring (n_0^{th}) reached at the end of the first operation of filling and emptying be R_1 which is now taken to be the radius of the freshly made variable pit. Suppose n_1 is the number of seeds required to over-fill it, and therefore

$$n_1 = f(R_1). \tag{5}$$

When dropped one by one these n_1 seeds will cover the next n_1 rings. At the end of this second operation of filling and emptying, the last ring reached will be the $(n_0 + n_1)$th whose radius is R_2 say. The number of seeds needed to over-fill the fresh pit of radius R_2 will be

$$n_2 = f(R_2). \tag{6}$$

Again the one-by-one dropping of seeds is continued and will cover the next n_2 rings. At the end of the third operation of filling and emptying, we will make the variable pit of radius R_3 (equal to that of the last ring reached) and over-fill it with n_3 seeds where

$$n_3 = f(R_3). \tag{7}$$

The above operation of filling and emptying the variable pit is to be performed n_0^3 times at the end of which we will reach the ring whose radius will be $R_{n_o{}^3}$. The pit with this radius will then have seeds whose number is given by

$$n_{n_o{}^3} = f(R_{n_o{}^3}). \tag{8}$$

This number is found to be very very large and is called *jaghanya-parīta-asaṃkhyāta* ("unenumerable of low enhanced order"). The actual calculation of the numerical value of this first giant number is very complicated and is somewhat different in the two main Jaina Schools (Digambara and Śvetāmbara). We present below the exposition based on the Digambara texts in simplified modern language and notation.

2 Value of n_0

The filling of the cylindrical variable pit (of radius r, and height or rather depth h) with seeds is done in such a manner that the over-filled tiny seeds from a right circular cone of height H. Thus the volume of the cylindrical portion will be (Fig. 3).

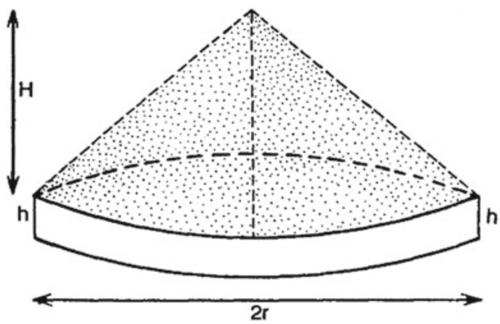

Fig. 3 Overfilled cylindrical pit

$$V_1 = \pi r^2 h \tag{9}$$

and that of the surmounting cone

$$V_2 = \left(\frac{1}{3}\right) \pi r^2 H. \tag{10}$$

The volume of each tiny seed assumed to be spherical and of radius *e*, (say), will be

$$v = \left(\frac{4}{3}\right) \pi e^3. \tag{11}$$

The number of seeds required to fill the cylindrical part (neglecting the interparticle empty space) will be

$$N_1 = \frac{V_1}{v} = \frac{3r^2 h}{4e^3} \tag{12}$$

and that for the conical part will be

$$N_2 = \frac{V_2}{v} = \frac{r^2 H}{4e^3}. \tag{13}$$

Now according to *Triloka-sāra* (*gāthā* 23) of Nemicandra (tenth century AD)[1]

$$H = \frac{\text{perimeter}}{11} \tag{14}$$

$$= \frac{2\pi r}{11}. \tag{15}$$

Thus

$$N_2 = \frac{\pi r^3}{22 e^3}. \tag{16}$$

The diameter of each tiny seed is taken to be equal to one *sarṣapa* (a very small unit of length). Hence

$$e = \frac{1}{2} \quad \textit{sarṣapa}. \tag{17}$$

On the other hand *r, h* and *H* are measured in *yojanas* where

$$\begin{aligned} 1 \ \textit{yojana} &= 4 \times 2000 \ \textit{daṇḍas} \\ &= 8000 \times 96 \ \textit{pramāṇa aṅgulas} \\ &= 8000 \times 96 \times 500 \quad \textit{utsedha aṅgulas} \\ &= 96 \times 400,0000 \times 64 \quad \textit{sarṣapas} \\ &= 6 \times 8^4 \times 10^6 \quad \textit{sarṣapas}. \end{aligned} \tag{18}$$

Now the initial value of r, that is r_0, is 50000 *yojanas* and h is constantly taken to be 1000 *yojanas* always (even when the radius of the pit varies). By (12) we get

$$N_1 = 6r^2h \tag{19}$$

when r and h are in *sarṣapas*. Thus

$$\begin{aligned} N_1 &= 6 \times (50000)^2 \times 1000 \times (10^6 \times 8^4 \times 6)^3 \\ &= 81 \times 2^{38} \times 10^{31}. \end{aligned} \tag{20}$$

Also from (12), (13) and (16) we get

$$\frac{N_2}{N_1} = \frac{V_2}{V_1} = \frac{H}{3h} \tag{21}$$

$$= \frac{2\pi r}{33h} \tag{22}$$

$$= \frac{100\pi}{33}. \tag{23}$$

Therefore

$$N_2 = \left(\frac{100\pi}{33}\right) N_1. \tag{24}$$

Thus we get the initial value of the total number of seeds to be

$$\begin{aligned} n_o = N_1 + N_2 &= (100\pi + 33)\frac{N_1}{33} \\ &= (100\pi + 33) \cdot \left(\frac{27}{11}\right) \cdot 2^{38} \times 10^{31}. \end{aligned} \tag{25}$$

More explicitly we have

$$N_1 = 222, 651, 104, 64 \times 10^{31} \tag{26}$$

$$= 2.22651 \times 10^{44} \quad \text{nearly} \tag{27}$$

and

$$N_2 = 9.51997751 N_1 \ \text{ nearly} \tag{28}$$

or

$$N_2 = 2.11963 \times 10^{45} \ \text{ nearly}. \tag{29}$$

Hence

$$n_0 = 2.34228 \times 10^{45} \text{ nearly.} \tag{30}$$

In ancient texts the above (modern) calculations were done in slightly different way. The value of V_1 was found by taking $\pi = 3$ (*Triloka-sāra*, *gāthā* 17). So the ancient value was

$$V_1' = 3r^2h. \tag{31}$$

Similarly for the conical volume, we have (*Triloka-sāra*, *gāthā* 22)

$$V_2' = \left(\frac{\text{perimeter}}{6}\right)^2 \cdot \left(\frac{\text{perimeter}}{11}\right) \tag{32}$$

$$= \frac{2\pi^3 r^3}{99} \tag{33}$$

$$= \frac{6r^3}{11} \quad \text{by taking } \pi = 3 \text{ (as implied).} \tag{34}$$

Therefore,

$$\frac{N_2'}{N_1'} = \frac{V_2'}{V_1'} = \frac{2r}{11h} = \frac{100}{11}. \tag{35}$$

Finally the value of v was not found by (11), but by using the following ancient formula (*Triloka-sāra*, *gāthā* 19)

$$v' = \left(\frac{9}{2}\right) e^3. \tag{36}$$

Thus

$$N_1' = \frac{V_1'}{v'} = \frac{2r^2h}{3e^3} = \left(\frac{8}{9}\right) N_1 \tag{37}$$

by (12). Hence we have, from (20),

$$N_1' = 9 \times 2^{41} \times 10^{31} \tag{38}$$

$$= 197,912,092,999,68 \times 10^{31} \tag{39}$$

which tallies exactly with the value as given in the *Triloka-sāra*, *gāthā* 21. Similarly the value of N_2', which can be easily found by using (35), tallies with the ancient value found in *gāthā* 25, and hence also that of $(N_1' + N_2')$ (*gāthā* 28). Approximately this ancient value is

$$n_0' = 1.997113 \times 10^{45} \text{ nearly.} \tag{40}$$

3 How the Operations Were Counted

We have stated above that the operation of filling-and-emptying the variable (*anavasthita*) pit was carried out n_0^3 times before the final over-filling was done to define the *jaghanya-parīta-asaṃkhyāta*. Since n_0 itself is a very large number, the question may be asked as to how it was possible to maintain a check or record of the number of operations completed at any stage. The answer is that the very ancient principle of counting by using positional places and counting pebbles or sticks (*śalākā*) was employed.

Let us take a simple case. Suppose we have to count 10^3 or 1000 operations. For this we take 30 pebbles and divide them into three groups *A, B, C*, each containing 10 pebbles, and place them near three separately marked places or pits *P, Q, R*, which may be called units, tens, and hundreds places, respectively.

When the said operations are over once, we place one pebble from group *A* into *P*. When the operation is completed once more, we place one more pebble from group *A* into *P*. Thus when the operation is performed 10 times, all the pebbles of group *A* will be inside *P*. At this stage, we take out all the 10 pebbles out of *P*, *but place one pebble from group B into the pit Q* to have an equivalent count of 10 operations (which is represented by a single pebble in the tens place).

When the next (that is, the 11th) operation is completed we again put one pebble from group *A* into *P*, and so on. Thus for every 10 operations, marked by putting 10 pebbles in *P*, we will place one pebble from *B* into *Q* (and at the same time take out the accumulated 10 from *P*). This can be done upto 10^2 or 100 operations when all the 10 pebbles of group *B* will be inside *Q*.

At this stage, we take out the 10 pebbles from *Q*, but place one pebble from group *C* into *R* at the same time. Being in hundreds place, this one pebble in *R* will mark those 100 operations. If we repeat the above process of 100 operations again, we will have two pebbles in *R* at the end of the 200 operations performed so far. It will be clear now that when there will accumulate all the 10 pebbles of group *C* into *R* (and there is none in *P* and *Q*), then it will mark the completion of $10^3 = 1000$ operations. Of course, if we still continue, we can go upto $(10 + 10^2 + 10^3)$ or 1110 operations whose number can be recorded by the 30 pebbles of the three groups when they are all inside their pits (Fig. 4 shows the stage when 123 operations have been completed).

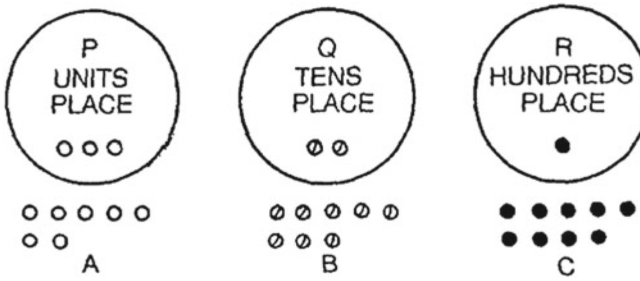

Fig. 4 Method of Counting

Exactly in a similar way, the ancient Jaina texts[2] ask us to make three pits, called *śalākā, paratīśalākā*, and *mahāśalākā Kuṇḍas*, each of which is equal (in shape and size) to the initial form of the variable (*anavasthita*) pit. Since the capacity of each of these auxiliary pits is n_0 seeds, we shall be able to count upto n_0^3 by the positional method described above. Actually, we have simplified the matter. The fact is that from the manner in which the texts describe the process of filling-and-emptying the three auxiliary pits, it can be concluded that the operation was performed n_0^3 times.

4 The Function *f(r)*

We will now discuss the capacity (in terms of number of seeds, *n*, which it can contain) when the radius of the variable pit is *r*. We see that *v* or *v'* is fixed because *e* is taken to be constant. Also *h* is constant (being taken to be 1000 *yojanas* always), and *H* varies linearly with *r*. Since

$$n = \frac{\text{vol. of cylinder and cone}}{v \text{ or } v'},$$

we will have from (9), (10) and (11), or (31), (32) and (36),

$$n = a_1 r^2 + a_2 r^3 \tag{41}$$

where a_1 and a_2 are certain constant. Initially the radius of the pit is 50000 *yojanas*, but its subsequent value will depend on the serial number *N* of the ring (of land or sea, counting from the Jambūdvīpa itself as the first ring) where the said operation of filling and emptying ends. It follows from (3) that the radius of the *N*th ring will be given by

$$R_{N-1} = (2^{N+1} - 3)R_0, \tag{42}$$

where R_0 is the radius of the Jambū Island. Prof. L. C. Jain[3] and Muni Mahendra Kumar Dvitīya[4] have mistakenly taken R_{N-1} to be equal to half of the *Valaya-vyāsa* (width of the ring) W_{N-1} given by (1). The more correct or proper things is to take it equal to R_{N-1} which is half the *sūcī-vyāsa* (diameter of the outer boundary of the ring) D_{N-1} given by (3).

Another mistake made by both the above scholars is to take the first term in (41) for making further calculations and neglect the second term which is in fact always far greater than the first term. If N_1 and N_2 denote the two terms in (41), we have

$$\frac{N_1}{N_2} = \frac{a_1}{a_2 \cdot r} = \frac{a_3}{r}, \quad \text{say.} \tag{43}$$

which becomes smaller and smaller since r goes on increasing very rapidly. Even the initial value of the ratio $\frac{N_1}{N_2}$ is, by (23),

$$= \frac{33}{100\pi} < \frac{11}{100} = 0.11. \tag{44}$$

Also see (35) in this connection.

As will be seen below, the number n_0^3 is so large that it cannot be expressed in terms of the usual number-words or digits or even by simple numerical power-notation. Therefore the central question is only to find a good lower bound for its value. Hence we can use some simplifications to have easy calculations.

From (42), it is clear that the radii of the successive rings do not form a geometrical progression. But we can write that equation as

$$\begin{aligned} R_{N-1} &= 2^N \left(2 - \frac{3}{2^N}\right) R_0 \\ &> 2^N R_0, \text{ if } N \geq 2. \end{aligned} \tag{45}$$

Now the initial case of Jambū Island ($N = 1$) can be treated separately, and for all other rings we can safely apply (45). Thus we take

$$r = k2^N \tag{46}$$

where k is constant, and N is equal to 2, 3, 4, 5, etc. According to (45) the constant k will be equal to r_0. But if we take $k = \frac{r_0}{2}$, we will get what is assumed by the above two mentioned scholars. However, we are able to take stronger case ($k = r_0$). It must be remembered that (46) is not to be used for $N = 1$, which is the case of Jambū Island for which we know separately that $r = r_0 = 50000$ *yojanas*.

In the like manner we take another general rule, namely

$$n = ar^p. \tag{47}$$

This will cover the first term of (41) when $a = a_1$ and $p = 2$ and the second term when $a = a_2$ and $p = 3$. Of course we know that the latter case ($p = 3$) will yield a far greater and therefore a better lower bound than the former. Thus, although the true form of the function *f(r)*, in the equation (4), is given by (41), we will first go ahead (see the next section) by assuming the rule (47) for finding all values of n, utilizing (46) also when the next successive value is found each time.

5 Expressions for *Jaghanya-parīta-asaṃkhyāta*

Before calculating various values of n, we recollect the very useful and important Jaina concept of *Vargita-saṃvargita* which will be needed. The (first) *vargita-saṃvargita* of x is defined by x^x, that is, by raising any number x to power (or index) x itself. It is denoted by

$$x\rceil^1, \quad \text{or simply} \quad x\rceil.$$

Thus

$$x\rceil = x^x = y, \quad \text{say} \tag{48}$$

The second *vargita-saṃvargita* of x is written and defined by

$$x\rceil^2 = y^y = (x^x)^{x^x} = z, \quad \text{say} \tag{49}$$

that is, the *vargita-saṃvargita* of the first *vargita-saṃvargita* of x is called the second *vargita-saṃvargita* of x, or

$$x\rceil^2 = (x\rceil)\rceil.$$

Similarly if we take the *vargita-saṃvargita* of $x\rceil^2$, that is of z, we will get the third *vargita-saṃvargita* of x. So that

$$x\rceil^3 = (x\rceil^2)^{(x\rceil^2)} = z^z = (y^y)^{y^y}. \tag{50}$$

Similarly for higher *vargita-saṃvargita*. In general the qth *vargita-saṃvargita* of x is written and defined by the relation

$$x\rceil^q = (x\rceil^{q-1})^{(x\rceil^{q-1})} \tag{51}$$

where $q = 2, 3, 4$, etc. (for including the case $q = 1$, we may define $x\rceil^0$ to be equal to x itself).

There are several interesting properties associated with the above concept and notation. One of them is the relation (which can be easily checked to any base)

$$\log(x\rceil^q) = (\log x) \cdot x \cdot (x\rceil^1) \cdot (x\rceil^2) \dots (x\rceil^{q-1}). \tag{52}$$

It is clear that we can obtain very rapidly increasing divergent sequence by applying the above definition. For example the various *vargita-saṃvargita* of the small number 2 will be

$$2\rceil = 2^2 = 4, \quad 2\rceil^2 = 4^4 = 256, \quad 2\rceil^3 = (256)^{256}$$

which is a number that contains 617 digits. Let the readers try to find the next figure of the sequence or ask some computer to do it.

With the above powerful notational tool in our hand, the calculation of the various successive values of n can be done quite easily. The numerical value of the number n_0 has already been found above in Sect. 2 (it is a 46-digit number). Here by (47) we will have the relation

$$n_0 = a \cdot (r_0)^p \tag{53}$$

where r_0 is equal to 50000 *yojanas*.

The radius r_1 of the n_0th ring (which will be reached at the end of the first operation of filling-and-emptying) will be, from (46), given by

$$r_1 = k \cdot 2^{n_0}. \tag{54}$$

The corresponding number n_1 (which represents the number of seeds in the variable pit) will then be, by (47),

$$n_1 = ar_1^p = a \cdot k^p \cdot 2^{p \cdot n_0} \tag{55}$$

by further using (54). These seeds will cover the next n_1 rings, thereby reaching the $(n_0 + n_1)$th ring. The radius of this ring will be, by (46),

$$r_2 = k \cdot 2^{n_0 + n_1}. \tag{56}$$

And the corresponding number of seeds in the variable pit (with radius r_2) will be, by (47),

$$n_2 = a \cdot r_2^p \tag{57}$$

$$= a \cdot k^p \cdot 2^{p \cdot n_0} \cdot 2^{p \cdot n_1} \tag{58}$$

$$= n_1 \cdot 2^{p \cdot n_1} \tag{59}$$

by using (55). Similarly the next pair of values (or r and n) will be

$$r_3 = k \cdot 2^{(n_0+n_1+n_2)} \tag{60}$$

and

$$\begin{aligned} n_3 &= a \cdot r_3^p && (61)\\ &= a \cdot k^p \cdot 2^{p \cdot n_0} \cdot 2^{p \cdot n_1} \cdot 2^{p \cdot n_2} && (62)\\ &= n_2 \cdot 2^{p \cdot n_2} && (63) \end{aligned}$$

and so on. In general, we have the recurrence relation

$$n_{i+1} = n_i \cdot 2^{p \cdot n_i} \tag{64}$$

where $i = 1, 2, 3, \ldots$. Therefore we will have

$$\begin{aligned} 2^{p \cdot n_{i+1}} &= 2^{(p \cdot n_i) \times 2^{p \cdot n_i}} && (65)\\ &= (2^{p \cdot n_i})^{(2^{p \cdot n_i})} \\ &= (2^{p \cdot n_i})\rceil && (66) \end{aligned}$$

where the right-hand side denotes the (first) *vargita-saṃvargita* of $2^{p \cdot n_i}$. Thus

$$2^{p \cdot n_2} = (2^{p \cdot n_1})^{2^{(p \cdot n_1)}} = c^c = c\rceil \tag{67}$$

where

$$c = 2^{p \cdot n_1} \tag{68}$$

Similarly,

$$2^{p \cdot n_3} = (2^{p \cdot n_2})^{(2^{p \cdot n_2})} = (c^c)^{c^c} = c\rceil^2. \tag{69}$$

In general, we will have

$$2^{p \cdot n_{i+1}} = c\rceil^i. \tag{70}$$

From this we can find the values of all the successive numbers n_2, n_3, n_4, etc., after finding c by using (55) and (68). In particular, we thus get theoretically the number $n_{n_0^3}$ also by taking

$$i = n_0^3 - 1. \tag{71}$$

Let us try a slightly different approach. By using (68), we can write (59) and (63) as

$$n_2 = n_1.c \tag{72}$$

and

$$n_3 = n_1 \cdot c \cdot 2^{p \cdot n_1 \cdot c} = n_1 \cdot c \cdot c^c, \quad by \ (68)$$
$$= n_1 \cdot c \cdot c\rceil. \tag{73}$$

Similarly, by (64) etc.,

$$n_4 = n_3 \cdot 2^{p \cdot n_3} = n_1 \cdot c \cdot c^c \cdot 2^{p \cdot n_1 \cdot c \cdot c\rceil}, \text{ by } (73)$$
$$= n_1 \cdot c \cdot c\rceil \cdot c\rceil^2. \tag{74}$$

In general we will have

$$n_{i+1} = n_1 \cdot c \cdot c\rceil \cdot c\rceil^2 \ldots c\rceil^{i-1}. \tag{75}$$

Now we will show that (70) and (75) are equivalent as ought to be. From (70) we get

$$p \cdot n_{i+1} \log 2 = \log(c\rceil^i)$$
$$= (\log c) \cdot c \cdot c\rceil \cdot c\rceil^2 \ldots c\rceil^{i-1}, \text{ by } (52).$$

This is easily seen to be same as (75) because from (68) we have

$$\log c = p \cdot n_1 \cdot \log 2. \tag{76}$$

Thus we see that the giant numbers n_{i+1} can be expressed by simple relations like (70) or (75). Even simpler lower bounds can be found from them.

For instance, we have, from (75),

$$n_{i+1} > n_1 . c\rceil^{i-1}$$

$$\text{or} \qquad \log\left(\frac{n_{i+1}}{n_1}\right) > \log c\rceil^{i-1} = (\log c) \cdot c \cdot c\rceil \cdot c\rceil^2 \ldots c\rceil^{i-2}$$

by using (52). Thus,

$$n_{i+1} > n_1 \cdot c^{c \cdot c\rceil \cdot c\rceil^2 \ldots c\rceil^{i-2}} > n_0 \cdot c^{c \cdot c\rceil \cdot c\rceil^2 \ldots c\rceil^{i-2}}. \tag{77}$$

A result somewhat equivalent to this inequality has been obtained by Muni Mahendra Kumar Dvitīya in a complicated manner.[5] Moreover, there are some mistakes (besides those of printing) in his expressions. For example, from the first two factors in the power in (77), namely

$$c \cdot c\rceil \quad \text{or} \quad c \cdot c^c.$$

he mistakenly guessed the next factor to be c^{c^c} instead of the actual one namely

$$c\rceil^2 \quad \text{or} \quad (c^c)^{(c^c)}$$

Thus he thought wrongly that the various factors (shown separated by dots) in the power or index in (77) are formed after c has been self-power-raised (*svaghātita*) 3, 4, 5, etc., times (after c^c) instead of being formed by the respective *vargita-saṃvargita* forms of c successively which is the actual case here. Due to this lapse, his conclusion about the number of c's which form the last factor in the power of (77) is also not correct. Instead of $(n_0^3 - 2)$ (which he found), the correct number of c's forming the last factor c^{i-2} will be

$$2^{i-2}, \quad \text{or} \quad 2^{n_0^3 - 3}, \quad \text{by} \quad (71).$$

6 Numerical Lower Bounds

We will try to compute the numerical value of the constant c on which depends the measure of *jaghanya-parīta-asaṃkhyāta*. From (55) and (68) we have

$$c = 2^{pak^p \cdot 2^{pn_0}}. \tag{78}$$

For finding better lower bound we take $p = 3$ (instead of 2), and $k = r_0$ (instead of $\frac{r_0}{2}$). Thus

$$c = 2^{3 \cdot a \cdot r_0^3 \cdot 2^{3 \cdot n_0}} = 2^{3n_0 \cdot 2^{3 \cdot n_0}} \tag{79}$$

by (53). We can write this as

$$c = 8^{n_0 \cdot 8^{n_0}}. \tag{80}$$

We can further change the base from 8 to 10 by writing the above as

$$c = 10^{(n_0 \log 8) \cdot 10^{(n_0 \log 8)}}. \tag{81}$$

Now the bracketed quantity is

$$n_0 \log 8 = 2 \cdot 1153 \times 10^{45} \quad \text{nearly} \tag{82}$$

by using the modern value of n_0 given by (30), and

$$n_0 \log 8 = 1 \cdot 8036 \times 10^{45} \quad \text{nearly} \tag{83}$$

if we use the ancient value given by (40). Since we are finding a lower bound, we should accept the value (83) which is also justified due to historical suitability. Also we have

$$1.8036 = 10^{0.25614} \quad \text{nearly} \tag{84}$$

$$2.1153 = 10^{0.3254} \quad \text{nearly.} \tag{85}$$

Thus by using (83) along with (84) in (81), we can very safely say that

$$c > 10^{10^{(n_0 \log 8)}} \tag{86}$$

$$> 10^{10^{10^{45.256}}} \tag{87}$$

or more simply we have

$$c > 10^{10^{10^{45}}}. \tag{88}$$

This value of the lower bound is better than that obtained in the *Viśva-prahelikā* (pp. 269–271) and which is

$$10^{10^{10^{43}}}. \tag{89}$$

Again we have

$$n_0^3 = 12.85 \times 10^{135} \quad \text{nearly (modern)}$$

$$\text{or} \qquad n_0^3 = 7.9656 \times 10^{135} \quad \text{nearly (ancient).}$$

With these values in mind, we find that the ancient Jaina number *jaghanya-parīta-asaṃkhyāta* was, from (70), still far greater than

$$\frac{\log \left[c \rceil^{(n_0^3 - 1)} \right]}{(\log 8)} \tag{90}$$

which is quite a compact expression for the lower bound.

References and Notes

1. See the *Triloka-sāra* with the commentary of Mādhavacandra and Hindi translation of Viśuddhamatī, edited by R. C. Jain and C. P. Patni, Shri Mahaviraji (Rajasthan), 1975, p. 30. The rules in *gāthās* 22 and 23 may be compared to those given by Brahmagupta in his *Brāhmasphuṭa-Siddhānta* (AD 628), XII, 50 (see the edition by R. S. Sharma and his team, Vol. III, p. 887, New Delhi, 1966).

2. See *Tiloyapaṇṇatti* (=*Trilokaprajñapti*) of Yativṛṣabha edited with the Hindi commentary of Viśuddhamatī by C. P. Patni, Vol. II, Kota, 1986, pp. 90–93, and *Triloka-sāra* (ref. 1), *gāthā* 14 and 15.
3. L. C. Jain, "Tiloyapaṇṇatti kā Gaṇita" (in Hindi), Essay in the *Jambūdvīpa-paṇṇati-saṃgaho*, edited, with the Hindi translation of Balacandra Siddhantasastri, by A. N. Upadhye and Hirala Jain, Sholapur, 1958, introductory pp. 102–103.
4. Muni Mahendra Kumar, H., *Viśva-prahelikā* (in Hindi), Bombay, 1969, pp. 263–266.
5. *Ibid.*, 265–268. It should be noted that with $p = 2$ and $k = \frac{r_0}{2}$ (as taken by Muniji), his quantity (or *ga*) is same as our c.

गोलपृष्ठ के लिये महावीर-फेरू सूत्र और विदेशों में उनकी झलक

सारांश

इस लेख में जिस व्यावहारिक सूत्र की चर्चा की गई है वह है

$$S = \frac{C^2}{4}$$

जहाँ S एक गोले का पृष्ठफल है तथा C उस गोले के केन्द्रीय खण्ड (*Section*) की परिधि है। भारत में इस सूत्र की जड़ गणितज्ञ महावीराचार्य (लगभग 850 ई.) के एक सरल नियम में निहित है और ठक्कुर फेरू (लगभग 1300 ई.) ने इसका कुछ सुधरा रूप दिया है। योरोप तथा जापान के (बाद के कुछ) ग्रन्थों में भी यह सूत्र पाया जाता है।

1 गोलीय खण्ड के पृष्ठफल के लिये महावीराचार्य का सूत्र-

प्राचीन काल से ही एक वृत्ताकार (circular) क्षेत्र का क्षेत्रफल A निकालने का विशुद्ध (exact) सूत्र

$$A = \frac{p \cdot \omega}{4} \tag{1}$$

विभिन्न सभ्यताओं में पाया जाता है। यहाँ p क्षेत्र का घेरा (परिधि) (perimeter) तथा ω क्षेत्र की चौड़ाई (width) या व्यास (diamter) है। दिलचस्प बात यह है कि इस सूत्र में समचतुर्भुज (square) का क्षेत्रफल भी सही सही आ जाता है जिसके लिये $p = 4a$, तथा $\omega = a$ (= भुजामान) है। प्राचीन काल में यह सूत्र अण्डाकार (oval) तथा पल्लवाकार या नेत्राकार (जैसे दोहरा वृत्तीयखण्ड) क्षेत्रों का व्यावहारिक फल निकालने में उपयोगी था (Gupta 2011, pp. 640–641)।

महान् दिगम्बर जैन गणितज्ञ महावीराचार्य ने अपने गणित सार संग्रह (= GSS) में आयतवृत्त (elongated circle or ellipse) के क्षेत्रफल निकालने के जो दो सूत्र दिये हैं (GSS, VII, 21 एवं 63) उन दोनों में ही सूत्र (1) का उपायोग किया गया है (Gupta 1974)। लेकिन गणितीय दृष्टि से उल्लेखनीय बात यह है कि उन्होंने जो व्यावहारिक सूत्र गोल नतोदर (concave) तथा उन्नतोदर

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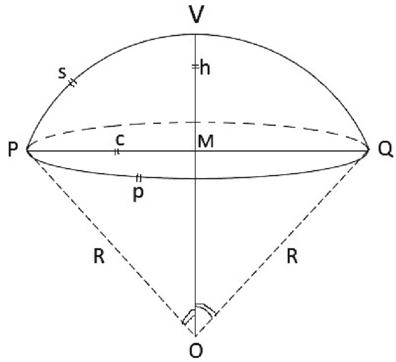
(convex) तलों के लिये दिया है, वह भी (1) का विस्तृत (extended) रूप मालूम पड़ता है। यह है (GSS, VII. 25): -

परिधेश्च चतुर्भागो विष्कम्भगुणः स विद्धि गणितफलम्।
चत्वाले कूर्मनिभे क्षेत्रे निम्नोन्नते तस्मात्॥ २५॥



चित्र.1 **एक गोले का खंड**

इस मूल सूत्र का अनुवाद इस प्रकार है (चित्र 1 देखें)। परिधि के चौथाई भाग तथा (वक्रीय) व्यास (PVQ = s) के गुणनफल को चत्वाल जैसे निम्न (नतोदर) और कूर्मपीठ जैसे उन्नत (उन्नतोदर) सतहों का क्षेत्रफल जानो।

अर्थात् p परिधि वाले आधार वृत्त (base circle) (जिसका व्यास PQ है) के ऊपर खड़े गोलीयखण्ड (spherical segment) के पृष्ठ का क्षेत्रफल

$$A = \frac{p \cdot s}{4} \qquad (2)$$

जहाँ s खण्ड का वक्रीय व्यास (curvilinear width) PVQ है।

सूत्र देने के बाद ही ग्रंथकर्ता ने प्रश्नरूप में दो उदाहरण दिये हैं जो इस प्रकार हैं-

(i) चत्वाल क्षेत्र का व्यास $s = 27$, और परिधि $p = 56$ है। उसका क्षेत्रफल क्या है? (उत्तर 378)

(ii) कूर्मनिभ क्षेत्र का विष्कम्भ $s = 15$, तथा परिधि $p = 36$ है। उसका व्यावहारिक क्षेत्रफल क्या है? (उत्तर 135)

दोनों उदाहरणों में परिधि p का मान अपने-अपने व्यास (विष्कम्भ) s के तिगुने से बहुत कम है। अतः स्पष्ट है कि सूत्र (2) में s का अर्थ आधार वृत्त का व्यास PQ नहीं है। फिर भी प्राध्यापक रंगाचार्य ने जब GSS का अंग्रेजी अनुवाद (मद्रास 1912) किया तो उसमें $s = PQ$ लिया जो उचित नहीं लगता। प्रा. लक्ष्मीचंद जैन ने अपने हिन्दी अनुवाद (शोलापुर 1963) में उनका ही अनुसरण किया।

सूत्र GSS, VII. 25 में आये `विष्कम्भ' (=व्यास) शब्द का अर्थ `वक्रीय व्यास' (curvilinear diameter or width) PVQ लेने की बात इस लेखक ने 1974 में प्रस्तुत (1975 में प्रकाशित) एक छोटे से शोधपत्र में की थी। प्रसन्नता की बात है कि अधिकतर विशेषज्ञों ने इस नये शाब्दिक अर्थ (interpretation) को मान लिया है (देखें Hayashi तथा जैन और अग्रवाल p. 49) आशा है कि विद्वान इस ओर ध्यान देंगे।

2 सूत्र (2) की शुद्धता का विवेचन-

महावीराचार्य ने स्वयं अपने सूत्र (2) को व्यावहारिक श्रेणी में रखा है न कि सूक्ष्म श्रेणी में। अर्थात् सूत्र से विशुद्ध या पूर्णशुद्ध (exact) परिणाम अपेक्षित नहीं है। यहाँ हम कुछ नये ढंग से उसकी शुद्धता की जाँच करेंगे। पहले हम प्राचीन जैन गणित में प्रयुक्त नियमों से गोलखण्ड का क्षेत्रफल निकालेंगे। सर्वविदित सामान्य सूत्र से-

$$p = \pi c \tag{3}$$

जहाँ $c = PQ$ (चित्र 1 देखें)। एक अन्य सटीक सूत्र से

$$4h\,(d-h) = c^2 \tag{4}$$

जिसमें h गोलीय खण्ड की ऊँचाई VM है तथा $d\,(=2R)$ संपूर्ण गोले का व्यास (=अर्धव्यास VO का दुगुना है। यही d उस ऊर्ध्ववृत्त (vertical circle) का भी व्यास है जिसका वृत्तीय खण्ड $PVQP$ है। इस वृत्तीयखण्ड (segment of circle) के लिये प्राचीन जैन गणित के एक विशेष नियम के अनुसार (Gupta 1979 तथा 1989)

$$s^2 = c^2 + (\pi^2 - 4)h^2 \tag{5}$$

जिसका उपयोग GSS में हुआ है (VII, 43 b तथा 73½)

यहाँ हमें p तथा s के मान मालूम हैं। इनकी मदद से पहले (3) से c का, फिर (5) से h का, और अन्त में (4) से d का मान प्राप्त हो जाएगा। आधुनिक गणित के अनुसार गोलखण्ड का सही (exact) क्षेत्रफल

$$A_0 = 2\,\pi\,R\,h. \tag{6}$$

इसमें (4) से $dh\,(=2Rh)$ का मान रखने से

$$A_0 = \pi\left(h^2 + \frac{c^2}{4}\right) \tag{7}$$

जो पूर्णतया सही (exact) है। अब व्यावहारिक सूत्र (5) का उपयोग करके

$$A_0 = \left(\frac{\pi c^2}{4}\right) + \frac{\pi(s^2 - c^2)}{(\pi^2 - 4)}. \tag{8}$$

अंत में (3) से c का मान रखकर सरलीकरण करने पर हमें प्राप्त होगा

$$A_0 = \frac{4\pi^2 s^2 + (\pi^2 - 8)p^2}{4\pi(\pi^2 - 4)}. \tag{9}$$

नियम (5) के इस्तेमाल के कारण अब सूत्र (9) से विशुद्धता की अपेक्षा नहीं करना चाहिए। फिर भी व्यावहारिक सूत्र (9) को लगाकर निकाले गए क्षेत्रफल की तुलना महावीराचार्य के अत्यन्त

सरल सूत्र (2) से प्राप्त मान से की जा सकती है। प्राचीन जैन गणित में सामान्यतया $\pi = 3$ (स्थूल) तथा $\pi = \sqrt{10}$ (सूक्ष्म) लेकर गणना की जाती है। संलग्न सारिणी (Table 1) में विभिन्न क्षेत्रफल दिये गए हैं।

सारिणि की तैयारी में पॉकेटगणक (calculator) का उपयोग किया गया है। यह ध्यान रहे कि महावीराचार्य के सूत्र (2) में π की सीधे कोई भागीदारी नहीं है। लेकिन (9) में π सक्रिय है। सूत्र (2), (3) तथा (6) से

$$\frac{A}{A_0} = \frac{(\pi c)s}{8\pi Rh} = \frac{cs}{8Rh}$$

Table 1. क्षेत्रफल सारिणी (Table of Areas)

उदाहरण	GSS से		सूत्र (2) से	सूत्र (9) से क्षेत्रफल			आधुनिका विधि
	s	p	क्षेत्रफल	$\pi = 3$	$\pi = \sqrt{10}$	सही π	से क्षेत्रफल
(i)	27	56	378	489.67	466.86	469.67	471.5
(ii)	15	36	135	156.60	152.74	153.28	155.3

इसमें सरल त्रिकोणमितीय संबंध जैसे (चित्र 1 देखें)
$s = 2R\theta;\ c = 2R\sin\theta;\ h = R(l - \cos\theta)$, लगाकर सरलीकरण करने पर हम पायेंगे कि

$$\frac{A}{A_0} = \frac{\left(\frac{\theta}{2}\right)}{\left[\tan\frac{\theta}{2}\right]}$$

जिसका मान यहाँ हमेशा धन (positive) और एक से कम है। अतः सूत्र (2) से प्राप्त क्षेत्रफल सदा सही मान से कम होगा।

3 महावीराचार्य के प्रश्न का आधुनिक हल-

किसी गोलखण्ड में p तथा s के मान ज्ञात होने पर उसके क्षेत्रफल निकालने की एक आधुनिक विधि इस प्रकार है। चित्र 1 में त्रिभुज MOP से

$$\text{तथा}\ \sin\theta = \frac{PM}{OP} = \frac{c}{2R} = \frac{p}{2\pi R} \tag{10}$$

$$\theta = \frac{(\text{arc } PV)}{R} = \frac{s}{2R}. \tag{11}$$

इन दो संबंधों से हमें कोण θ निकालने के लिये समीकरण (equation) मिलती हैं-

$$\sin\theta = \frac{p\,\theta}{\pi s}. \tag{12}$$

इस समीकरण (12) में कोण θ का मान रेडियन (radians) में है। यदि मान को अंशों (degrees) में लें तो समीकरण का रूप होगा

$$\sin\theta = \frac{p\,\theta}{180s}. \tag{13}$$

महावीराचार्य के पहले ($s = 27,\ p = 56$) तथा दूसरे ($s = 15,\ p = 36$) उदाहरणों के लिये कोण θ निकालने के लिये समीकरणें क्रमशः होगी

$$\sin\theta = \frac{14\theta}{1215}. \tag{14}$$

तथा

$$\sin\theta = \frac{\theta}{75}. \tag{15}$$

इन समीकरणों को प्राचीन या आधुनिक किसी विधि से हल किया जा सकता है। असकृतकर्म (interation) की विधि लगाकर प्राप्त हल (दो दशमलस्थान तक) हैं।

$$\theta = 86.64 \text{ अंश (प्रथम उदाहरण)} \tag{16}$$

तथा

$$\theta = 70.85 \text{ अंश (दूसरा उदाहरण)} \tag{17}$$

अब विशुद्ध (exact) सूत्र (6) से गोलखण्ड का सही क्षेत्रफल

$$A_0 = 2\pi\ (1 - \cos\theta)\ R^2 \tag{18}$$

जहाँ $h = (R - R\cos\theta)$ ले लिया गया है। इसमें

$$p = c\pi = 2\pi R\ \sin\theta$$

का उपयोग करके सरल करने पर

$$A_0 = \frac{p^2}{2\pi(1 + \cos\theta)}. \tag{19}$$

इस सरल सूत्र में θ के ऊपर निकाले गये आधुनिक मानों के लिये महावीराचार्य के दोनों उदाहरणों में क्षेत्रफल निकाला जा सकता है।एक दशमलव तक यह क्षेत्रफल 471.5 तथा 155.3 आते हैं जिन्हें सारणी में लिख दिया गया है।

कुछ व्यावहारिक सूत्र लगाकर हायाशी (p. 201) ने θ के जो स्थलमान (86.25 तथा 67.70) निकाले उनकी तुलना ऊपर निकाले गये आधुनिक मानों से की जा सकती है। इसके अतिरिक्त सूत्र (9) कि तरह, जैन परम्परा के सूत्रों को लगाकर $\sin\theta$ के लिये एक व्यापक (general) सूत्र भी निकाला जा सकता है। यह इस प्रकार है। चित्र 1 से

$$\sin\theta = \frac{c}{2R} = \frac{4ch}{4dh} = \frac{4ch}{(c^2 + 4h^2)}, \text{ सूत्र (4) से।}$$

अब इसमें (5) से h का मान रखने पर

$$\sin\theta = \frac{4c\sqrt{\frac{(s^2-c^2)}{(\pi^2-4)}}}{c^2+\frac{4(s^2-c^2)}{(\pi^2-4)}}.$$

इसमें (3) का उपयोग करके, सरल करने से

$$\sin\theta = \frac{4p\sqrt{(\pi^2-4)(\pi^2s^2-p^2)}}{4\pi^2s^2+(\pi^2-8)p^2}. \tag{20}$$

व्यावहारिक सूत्र (20) की विशेष बात यह है कि इसे लगाकर किसी भी उदाहरण में मनचाही π के मान के लिये सीधे कोण θ का मान निकाला जा सकता है। अर्थात् एक-एक करके पहले c, h तथा d निकालने की जरूरत नहीं है। एक नमूना पर्याप्त होगा। महावीराचार्य के द्वितीय उदाहरण ($s = 15, p = 36$) तथा सरलतम व्यावहारिक $\pi = 3$ लेने पर सूत्र (20) से

$$\sin\theta = \frac{12\sqrt{5}}{29}.$$

जो प्राचीन जैन गणित लगाकर प्राप्त सही (exact) मान देगा। दो दशमलव तक राउण्ड करने पर $\theta = 67.71$ अंश आता है।

4 संपूर्ण गोले (sphere) के लिये देश-विदेश में महावीर-फेरू सूत्र-

उन्नतोदर तलों में गोलार्ध की बाह्य सतह का मामला रोचक है। यदि इसके आधार वृत्त की परिधि C हो तो सूत्र (2) में

$$p = C,\ \text{तथा}\ s = \frac{C}{2}$$

होगा। अतः गोलार्ध का वक्रीय पृष्ठफल $\frac{C^2}{8}$ होगा।इस प्रकार हम कह सकते हैं कि एक संपूर्ण गोले (sphere) का गोलीय पृष्ठफल (spherical surface area)

$$S = \frac{C^2}{4} \tag{21}$$

जहाँ गोले के किसी भी महावृत्त (great circle) की परिधि C है।

सूत्र (21) को महावीराचार्य ने स्पष्ट (explicit) रूप में नहीं दिया है, लेकिन वह उनके सूत्र (2) का सीधा परिणाम है। बाद में ठक्कुर फेरू (लगभग 1300 ई.) ने अपनॆ ग्रंथ गणितसार कौमुदी में दो स्थलों (III. 65 तथा V. 25) पर (21) का थोड़ा सा सुधरा रूप इस प्रकार दिया है-

$$S_1 = \left(\frac{10}{9}\right)\left(\frac{C^2}{4}\right). \tag{22}$$

ध्यान देने योग्य बात यह है कि सुधारक गुणक $\left(\frac{10}{9}\right)$ भी फेरू ने महावीराचार्य से ग्रहण किया हुआ मालूम पड़ता है। महावीराचार्य ने इस गुणक का उपयोग करके अपने गोले के आयतन वाले सूत्र को परिवर्तित रूप में देने के लिये किया था (Gupta 2011, pp. 650–653)। अतः दोनों को श्रेय देते हुए हम (21) को महावीर-फेरू सूत्र कहेंगे, तथा (22) को उसका सुधारा (modified) रूप।

प्राचीन जैन गणित में वृत्त परिधि और व्यास संबंधित नियम ($C = \pi d$) का स्थूल (व्यावहारिक) रूप

$$C = 3d. \tag{23}$$

तथा सूक्ष्म रूप

$$C_1 = \sqrt{10d^2} \tag{24}$$

सर्वविदित है। अतः

$$C_1 = \left(\frac{\sqrt{10}}{3}\right)C. \tag{25}$$

जिससे स्पष्ट है कि यदि हम (21) को गोलपृष्ठ का स्थूल नियम कहें तो (22) को उसका सूक्ष्म कहेंगे। संभव है कि ठक्कुर फेरू ने (25) का उपयोग करके (22) प्राप्त किया हो। ज्ञातव्य है कि गोलपृष्ठ का शुद्ध (exact) सूत्र हैः-

$$S_0 = \frac{C^2}{\pi}. \tag{26}$$

सूत्र (21) के बारे में एक बहुत ही रोचक ऐतिहासिक तथ्य यह है कि वह बाद में अनेक विदेशी ग्रंथों में पाया जाता है। इटली (Italy) की ऐसी हस्तलिखित पोथियों (manuscripts and codices) की सूचना हम Simi और Rigatelli के लेख के आधार पर दे रहे हैं जो इस प्रकार है (Simi and Rigatelli, pp. 466–469):-

बोलोग्ना (Bologna) विश्वविद्यालय के पुस्तकालय के एक हस्तलिखित ग्रंथ Ms. 1612 (वर्ष 1464 ई.) का रचनाकार Piero Jachon di Maestro Antonio de Chapelam था जो वहीं का निवासी था। उसने सूत्र (21) को

$$S = \left(\frac{C}{2}\right)^2 \tag{27}$$

के रूप में दिया। यही सूत्र फ्लोरेन्स (Florence) के राष्ट्रीय पुस्तकालय के हस्तलिखित ग्रंथ Palat 575 (वर्ष 1460 ई. के लगभग) तथा वहीं के एक अन्य पुस्तकालय के Pluteo 30, 26 (वर्ष 1370) के लेखकों ने दिया है।

जापान में तो महावीर-फेरू सूत्र (21) ईसवीं की 17 वीं शताब्दी के अनेक ग्रंथों में पाया जाता है। जैसे इमामूरा चिशो (Imamura Chishō) के 1639 ई. में छपे Jugai-roku नाम के ग्रंथ में (Mikami, p. 296)। यह ग्रंथ मूल रूप में क्लासिकल चीनी भाषा में लिखा गया था जिसका बाद में जापानी भाषा में अनुवाद किया गया। इमामूरा के शिष्य Andō Yūyeki ने अपने गुरु के उक्त ग्रंथ को अपनी टीका के साथ 1660 ई. में छपाया।

सन् 1660 ईसवीं में ही सूत्र (21) जापान के इसोमारू कित्तोकू (Isomaru Kittoku) रचित जापानी ग्रंथ Ketsugi-shō में भी देखने को मिला। इसका दूसरा संस्करण 1684 ई. में छपा। इस संस्करण में लेखक ने स्पष्ट लिखा है कि उक्त सूत्र (21) की जानकारी निम्नलिखित जापानी ग्रंथकारों को थी (Smith and Mikami, pp. 74–75):-

(i) Mōri (1600 ई. के लगभग) (= मोरी या मओरी)
(ii) योशिदा (Yoshida) (1627 ई.) (मोरी के शिष्य)
(iii) इमामूरा (1639 ई.)
(iv) तकाहारा किस्सू (Takahara Kissu) (इसोमारु के गुरु जिन्हें योशिताने Yoshitane भी कहते थे)
(v) हिरागे (Hirage)
(vi) शिमोदा (Shimoda)
(vii) इत्यादि (जैसे सुमिदा Sumida) (Mikami p. 296)

दिलचस्प बात यह भी है कि इसोमारु ने सूत्र (21) को एक प्राचीन विधि बताया और माना था कि उनके पहले सही (exact) सूत्र जापान में किसी को भी मालूम नहीं था। बाद में इसोमारू ने बहुत ही सरल किन्तु बुद्धिपूर्ण विधि से सही सूत्र का पता लगा लिया था जो था

$$S_0 = 6 \times \frac{\text{(गोले का आयतन)}}{\text{(व्यास)}}.$$

यह सूत्र सही सूत्र (26) का ही एक अन्य रूप है। वास्तव में (26) तथा इसोमारू के रूपों को सरल करने पर उनका आधुनिक रूप

$$S_0 = 4\pi R^2$$

मिलता है जिसे सूत्र (6) में $h = 2R$ रखने से भी प्राप्त किया जा सकता है।

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Part V

Geometry

Brahmagupta's Formulas for the Area and Diagonals of a Cyclic Quadrilateral

1 Introduction

Let $ABCD$ be a plane (convex) quadrilateral with sides AB, BC, CD and DA equal to a, b, c and d, respectively. Let the figure be drawn in such a manner that we may consider, according to the traditional terminology, the side BC to be the base (*bhū*), the side AD to be the face (*mukha*), and the sides AB and DC to be the flank, sides (*bhujas* or arms) of the quadrilateral.

Since a quadrilateral is not uniquely defined by its four sides, its shape and size are not fixed. So that, by merely specifying the four sides, the question of finding its area does not arise. That is why Āryabhaṭa II (950 AD) in his *Mahāsiddhānta* XV. 70 says:[1]

कर्णज्ञानेन विना चतुरस्रे लम्बकं फलं यद्वा।
वक्तुं वाञ्छति गणको योऽसौ मूर्खः पिशाचो वा॥

> The mathematician who desires to tell the area or the altitude of a quadrilateral without knowing a diagonal, is either a fool or a devil.

Brahmagupta (628 AD)[2] in his *Brāhmasphuṭasiddhānta* (*BSS*) has given two rules (see below) for finding the area of a quadrilateral in terms of its four given sides. One of the rules is for getting a rough value of the area and the other for an accurate (*sūkṣma*) value. Now, Brahmagupta's formula for the area of a quadrilateral gives the exact value only when the quadrilateral is cyclic, although he has not specified this condition. But the condition may be taken to be understood, especially when we know (see below) that his expressions for the diagonals of the quadrilateral are also true only when the figure is cyclic, otherwise the diagonals have remained undefined. In fact, Brahmagupta does speak of the circum-circle (*koṇaspṛg-vṛtta*) and the circum-radius (*hṛdaya-rajju*) of triangle and quadrilateral in connection with some

The Mathematics Education, Vol. VIII, No. 2, 1974, pp. 33–36.

K. Ramasubramanian (ed.), *Gaṇitānanda*,
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other rules which are given between his rule for the area and that for the diagonals of the (cyclic) quadrilateral.[3]

2 Rules for the Area

The BSS XII. 21 (Vol. III, p. 816) states:

स्थूलफलं त्रिचतुर्भुजबाहुप्रतिबाहुयोगदलघातः।
भुजयोगार्धचतुष्टयभुजोनघातात् पदं सूक्ष्मम् ॥

The product of half the sums of (the two pairs of) the opposite sides of a triangle or a quadrilateral, gives the gross area. Set down half the sum of the sides in four places (and) diminish them by the (four) sides (respectively). The square–root of the product (of the four numbers) is the accurate area.

The formulas coded in the above verse may be expressed as:

$$\text{gross area} = \frac{1}{2}(a+c) \times \frac{1}{2}(b+d) \tag{1}$$

$$\text{accurate area} = \sqrt{(s-a)(s-b)(s-c)(s-d)} \tag{2}$$

$$\text{where} \qquad s = \frac{(a+b+c+d)}{2}. \tag{3}$$

The above formulas are stated to be applicable to the quadrilateral as well as to the triangle. In the latter case we have to take the face d to be zero. Thus, in the case of a triangle of sides $a, b, c,$ we have

$$\text{gross area} = \frac{b}{2} \times \frac{(a+c)}{2} \tag{4}$$

$$\text{accurate area} = \sqrt{s(s-a)(s-b)(s-c)} \tag{5}$$

$$\text{where} \qquad s = \frac{(a+b+c)}{2}. \tag{6}$$

We see that it does not matter much whether the formula (1) is used for cyclic or other quadrilaterals, since, after all, it is stated to be a rough one only. Formula (2) is know to give exact area only in the case of cyclic quadrilateral. However, the formula (5) is applicable to every triangle. But the formula (4) has now an additional defect of not yielding a unique (though rough) value of the area of a triangle, because we may get different results by regarding each of the sides a, b, c to be 'base' in turn.

Anyway, equivalent rules, which yield formulas (1) and (2), have been given by several subsequent Indian writers with or without some additional comments. Some of these will be noted now.

Śrīdhara in his *Pātīgaṇita* (*PG*)[4] has reproduced, word by word, *BSS* rule which gives the formula (1). However, he adds the following remarks immediately after quoting the rule.[5]

> But this result (1) is true only for those figures in which the difference between the altitude and the flank sides is small. In the case of other figures the above result is far removed from the truth; as for example, in the case of the triangle having 13 for the two (flank) sides and 24 for the base, the gross area is 156, whereas the correct area is 60 (*PG*, rule 112–114).

An ancient commentator of the *PG* even goes further and points out an interesting theoretical defect of the rule (1) or (4) other than its grossness. He says (p. 160) that the rule may yield a rough answer for the area even in the case of impossible figures, and gives the example of a triangle of base 20 and flank sides 13 and 7. Since the sum of the two sides is equal to the third (base), no triangle is possible, but the formula (4) will give 100 for its gross area.

The *Mahāsiddhānta* XV. 69 (p. 165) gives the *BSS* rule for the accurate area, but it is laid down there for a triangle only and not for a quadrilateral. Bhāskara II (AD 1150) in his *Līlāvatī*, rule 169, has also given the same rule but with the remark that it gives exact area for a triangle and inexact (*asphuṭa*) for a quadrilateral.[6]

3 Some Historical and Other Remarks

The approximate formula (1) was used outside India much before the date of Brahmagupta. The Babylonians of the ancient Mesopotamian valley are stated to have used it in finding the area of a quadrilateral.[7] The same formula can be gathered from the inscriptions (about 100 BC) found on the Temple of Horus at Edfu.[8] In this type of Egyptian mensurational mathematics, the triangles were regarded[9] as cases of quadrilaterals in which one side (the face) is made zero, just as what is met with in Brahmagupta.

The Chinese mathematical work *Wu-t'sao Suan-ching* (about fifth or sixth century) applies the formula (1) for computing the area of a quadrangular field whose eastern, western, southern and northern sides are given to be 35, 45, 25 and 15 paces respectively.[10] The formula (5) for the area of a triangle is generally called Heron's Formula, but, according to some medieval Arabic scholars, it was known even to Archimedes (third century BC).[11]

How Brahmagupta arrived at his formula (2), is difficult to say with certainty. For an exposition of the attempted proof, of this formula, as given by Gaṇeśa Daivajña in his commentary (1545 AD) on the *Līlāvatī*, a paper by M. G. Inamdar may be consulted.[12] The *Yukti-bhāṣā* (=*YB*, sixteenth century) also contains a proof of the same formula.[13]

4 Brahmagupta's Expressions for the Diagonals

The *BSS* XII. 28 (Vol. III, p. 836) states:

कर्णाश्रितभुजघातैक्यमुभयथान्योन्यभाजितं गुणयेत् ।
योगेन भुजप्रतिभुजवधयोः कर्णौ पदे विषमे ॥ २८ ॥

The sums of the products of the sides about the diagonals be both divided by each other; multiply (the quotients obtained) by the sum of the products of the opposite sides; the square-root (of the results) are the two diagonals (*viṣame?*).

That is

$$AC = \sqrt{(ac+bd)\frac{(ad+bc)}{(ab+cd)}} \tag{7}$$

$$\text{and} \qquad BD = \sqrt{(ac+bd)\frac{(ab+cd)}{(ad+bc)}}. \tag{8}$$

Brahmagupta's Sanskrit stanza, giving these diagonals, has been quoted verbatim by Bhāskara II in his *Līlāvatī*[14] with the remark that 'although indeterminate, the diagonals are sought to as determinate by Brahmagupta and others'. It may be noted that from (7) and (8), we immediately get

$$AC \times BD = a.c + b.d, \tag{9}$$

which is called the Ptolemy's theorem for cyclic quadrilateral after the famous Greek astronomer of the second century AD. The *YB* (pp. 232–33), however, follows the opposite procedure of deriving (7) and (8) from (9) and some other relations.

Brahmagupta's expressions for the diagonals are considered to be the 'most remarkable in Hindu geometry and solitary in its excellence' by a recent historian of mathematics.[15] The formula (8) is stated to be rediscovered[16] in Europe by W. Snell (about 1619 AD).

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On the Volume of a Sphere in Ancient India

1 Introduction

Seidenberg's paper "On the Volume of a Sphere" appeared in the *Archive for the History of Exact Sciences,* Vol. 39 (1988), pp. 97–119. He had conveyed it in January 1988 but could not see its proofs due to his death a few months later on May 3. The present article may be considered as a sequel to his paper although a version of it containing the basic findings was presented during the Seminar on Astronomy and Mathematics in Ancient and Medieval India, Calcutta, May 19–21, 1987.

The famous Greek mathematician Archimedes (third century BC) states in the preface of his treatise, *On the Sphere and Cylinder,* that the formula

$$\text{volume} = \frac{(\text{area of base}) \cdot (\text{altitude})}{3} \tag{1}$$

for any pyramid (including cone) was already known to Eudoxus (circa 370 BC), but that the formula

$$V = \left(\frac{2}{3}\right) \cdot \text{ (vol. of circumscribed cylinder)} = \left(\frac{4}{3}\right) \cdot \pi r^3 \tag{2}$$

for the sphere was unknown to geometers before his own time.[1] The Chinese work, *Chiu Chang Suan Shu* ("Nine Chapters on Mathematical Art"), of about 100 BC contains a rule which implies

$$V = \left(\frac{9}{16}\right) d^3 = \left(\frac{9}{2}\right) r^3 \tag{3}$$

for the sphere.[2] The commentator Liu Hui (third century) interprets this in a manner which implies[3]

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$$V = \left(\frac{\pi}{4}\right)^2 d^3 = \left(\frac{\pi^2}{2}\right) r^3. \tag{4}$$

And, although Tsu Keng-chih (fifth century) gave a derivation of the correct formula $\left(V = \frac{\pi d^3}{6}\right)$, the use of the crude relation (3) continued in China, through her Golden Era, by mathematicians such as Yang Hui (thirteenth century) and Chu Shi-chih (circa 1300).[4]

Archimedes gave two derivations of (2), one based on the method of balancing and the other on the method of exhaustion.[5] The Chinese proof utilized a sort of Cavalieri's Theorem.[6] Surprisingly, the process of averaging, which was quite popular in ancient and medieval times,[7] also leads to formula (2) if, in contents, the sphere is taken to be the mean between the inscribed double cone and the circumscribed cylinder. For, if a cone, a hemisphere, and a cylinder all have a common base and a common altitude (= r), then the mean of the volumes of the cylinder and the cone is

$$= \frac{1}{2}\left(\pi r^3 + \frac{\pi r^3}{3}\right) = \left(\frac{2}{3}\right)\pi r^3, \tag{5}$$

which is exactly the volume of the hemisphere. This also shows that the mere occurrence of the correct expression for the volume of a sphere in some work does not necessarily imply that a correct proof was also known.

For practical purpose, a fairly accurate formula (of the type $V = kr^3$) could be obtained by weighing a solid sphere of known density, or by comparing its weight with that of a cube of the same material.

2 Āryabhaṭa I's Rule

According to Sarasvati Amma,[8] we do not come across any authentic mention of the sphere in India before the time of Āryabhaṭa I (born AD 476). She is not sure as to whether the term *ghana-parimaṇḍala* means a cylinder or a sphere. But other scholars[9] have taken it to mean an elliptic cylinder in contradistinction to the term *pratara-maṇḍala* ("plane-ellipse") as found in the Jaina canonical work, *Bhagavatī-sūtra* (Sūtra 726).

Āryabhaṭa's peculiar rule is contained in his *Āryabhaṭīya*, II *(gaṇitapāda)*, 7, which is:[10]

समपरिणाहस्यार्धं विष्कम्भार्धहतम् एव वृत्तफलम् ।
तन्निजमूलेन हतं घनगोलफलं निरवशेषम् ॥

Half the circumference multiplied by half the diameter is the area of a circle, that multiplied by its own square-root is the volume of a sphere without remainder.

That is,

$$\left(\frac{c}{2}\right)\left(\frac{d}{2}\right) = \text{area of a circle, } A \text{ say.} \tag{6}$$

Then

$$A\sqrt{A} = \text{volume of the sphere, } V. \tag{7}$$

By taking πr^2 for A, scholars transform the working relation (7) to a mathematical formula, namely,

$$V = \pi\sqrt{\pi} \cdot r^3 \tag{8}$$

from which unnecessary conclusions are often drawn. Thus Smith,[11] by a comparison to correct expression (2), infers that the rule (8) "would make π equal to $\frac{16}{9}$, possibly an error for the $\left(\frac{16}{9}\right)^2$ of Ahmes", the scribe of the famous Egyptian *Rhind Mathematical Papyrus* (about 1600 BC). This is unjustified. First, because Āryabhaṭa's rule is based on a different consideration (see below) and so the question of comparing (8) with (2) does not arise. And second because Āryabhaṭa knew the far better value 3.1416 of π (*Āryabhaṭīya*, II, 10) and hence there is no need or ground of assuming his use of the bogus value $\pi = \frac{16}{9}$, or of imagining any possible confusion with the Egyptian value $\frac{256}{81}$.

Parameśvara (early fifteenth century) in his commentary on *Āryabhaṭīya*, II, 9 says:[12]

घनगोलेऽपि वृत्तफलस्य मूलम् उच्छ्रायः ॥

In a sphere also, the square-root of the area of a (great) circle is the (effective) altitude.

That is,

$$\sqrt{A} = \text{effective height or altitude, } h. \tag{9}$$

So that

$$V = A \cdot h. \tag{10}$$

Of course, exact volume can always be found by using correct effective height. However, the correct effective height here is $\frac{4r}{3}$, and not that which is given by (9).

Nīlakaṇṭha (about 1501) has explained the calculation of the volume of a sphere in terms of an effective side of an equivalent cube. In his commentary on *Āryabhaṭīya*, II, 7, he says:[13]

अस्मिन् फले मूलिते पुनस्तन्निर्मितचतुरश्रबाहुः स्यात् । एवं वृत्तक्षेत्रेण समचतुरश्रं सम्पादनीयम् । एवं घनगोलस्य समद्वादशाश्रत्वम् आपन्नस्यापि तच्चतुरश्रबाहुतुल्य एव द्वादश बाहवः । तस्मात् तद्बाहुघन एव गोलघनफलम्... ॥

Here the square-root of an area is the side of the square constructed (from that area). Thus from a circular figure an equivalent square can be made. Similarly, the cubic contents of a sphere too can be represented by an equivalent cube whose side is equal to the side of the square (base or face). Hence the cube of that side is the volume of the sphere.

However, Nīlakaṇṭha is not right in regarding $\sqrt{A}$ to be the side of the equivalent cube. The correct effective side is the cube root of the volume of the sphere. Anyway, it appears that Āryabhaṭa propounded, according to the discussion of the above commentators, his rule (7) by analogy to a cube for which the square-root of its square sectional area is actually both the effective height h and the effective side a. That is,

$$\text{cube's } V = A \cdot h = A\sqrt{A} = (\sqrt{A})^3 = a^3. \tag{11}$$

The commentaries of Parameśvara and Nīlakaṇṭha do not regard Āryabhaṭa's rule to be approximate. In fact, Parameśvara takes the descriptive expression *'niravaśeṣam'* ("without remainder") to mean *'sphuṭam'* ("exact"). Sūryadeva Yajvan (born 1191) in his commentary says:[14]

शास्त्रान्तरेषूपायान्तरदर्शनाद् एवम् अभिधानम् ।

Due to occurrence of a different (i.e. non-exact) method given in some other work, the rule is called here like that (i.e. exact).

Bhāskara 1 (629 AD) in his commentary explains the expression as follows:[15]

निरवशेषम् । न किञ्चिद् अनेन कर्मणा शिष्यते । येनान्येन कर्मणा घनगोलफलम्
आनयन्ति न तेन घनगोलफलं निरवशेषं भवति, व्यावहारिकत्वात् तस्य कर्मणः... ।

Niravaśeṣam (means that) nothing is left out by applying this method. By which it is made clear that the volume found by other method, the result is not *niravaśeṣam* because of the practical nature of that method.

Bhāskara I then quotes the other rule (see the next section). From the numerical examples solved by Bhāskara I and Sūryadeva Yajvan, no indication is given that Āryabhaṭa's rule was considered to be crude or rough. In fact, answers to the examples have not been worked out fully and are left in the form $\sqrt{N}$ possibly for fear of leaving out remainder after extracting the square-root up to certain stage thereby violating the *niravaśeṣam* claim.

The examples given by Bhāskara I correspond to d equal to 2, 5, and 10, while that of Sūryadeva to $d = 8$. But it is interesting to note that no units are mentioned at all for any of these linear measures.

We have seen that, although Āryabhaṭa's rule is very crude, he or his four commentators do not explicitly mention this crudeness or even its non-exactness. Shukla remarks[15] that "mathematicians and astronomers in northern India too regarded Āryabhaṭa I's formula as accurate and went on using it even in the second half of the ninth century AD", and he cites the case of Pṛthūdaka Caturveda (860 AD) who prescribed it in his commentary on the work of Brahmagupta (628 AD) who himself is silent on the matter. Bhāskara II (twelfth century) who knew the correct rule (see below) tried to defend Pṛthūdaka by saying that after all[16]

चतुर्वेदाचार्यः परमतम् उपन्यस्तवान् ॥

Professor Caturveda has given other's opinion.
(So he should not be blamed for the wrong rule.)

Lastly it may be pointed out that in some German translations, the Sanskrit text is interpreted in such manners as to yield not the wrong formula (7) for volume, but the correct formula for rather the surface of a hemisphere.[17]

3 The Jaina Tradition

While commenting on Āryabhaṭa's rule, Bhāskara I quotes the following empirical *(vyāvahārika)* rule:[18]

व्यासार्धघनं भित्वा नवगुणितम् अयोगुडस्य घनगणितम् ॥

The cube of the semi-diameter (when) halved and multiplied by nine gives the volume of a sphere.

That is

$$V = \left(\frac{9}{2}\right) \cdot \left(\frac{d}{2}\right)^3 = \left(\frac{9}{2}\right) \cdot r^3. \tag{12}$$

It is likely that Āryabhaṭa knew this rule and was led to designate his own rule as exact *niravaśeṣam* to proudly distinguish it from the above approximate (*sāvaśeṣam* or *sthūlam*) formula. It is also probably that (12) was already found in some early Jaina work as it is found in several subsequent works of the Jaina School. After all, Bhāskara I also quoted the well-known Jaina rule for perimeter of a circle[19]

$$C = \sqrt{10d^2},$$

while commenting similarly on *Āryabhaṭīya,* II, 10, which gives

$$C = \left(\frac{62832}{20000}\right) d.$$

Mahāvīra (about 850 AD), a famous Jaina mathematician, in his *Gaṇita-sāra-saṅgraha (=GSS),* VIII, 28, has given (12) in similar wording:[20]

व्यासार्धघनार्धगुणा नव गोलव्यावहारिकं गणितम् ॥

Half the cube of the semi-diameter multiplied by nine is the practical volume of a sphere.

That is,

$$V = \left(\frac{9}{2}\right) \cdot \left(\frac{d}{2}\right)^3, \tag{13}$$

which is further taken to be as

$$V = \left(\frac{3}{2}\right) \cdot \pi r^3 \tag{14}$$

by Sarasvati Amma since $\pi = 3$ for practical purposes according to Mahāvīra.

Subsequently the same formula (13) is also found in the following Jaina works:

(i) *Tiloya-sāra (= Triloka-sāra), gāthā* 19, of Nemicandra (about 975 AD).[21]

(ii) *Gaṇita-sāra* (in Prakrit), V, 25, of Ṭhakkura Pherū (about 1300 AD), where the rule is given in the form[22]

$$V = \left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}\right) \cdot d^3. \tag{15}$$

Indian mathematicians were aware of the roughness of the rule (although not stated so in the above two works) and believed it to be based on the crude value $\pi = 3$. However, the corresponding formula with general π was not (14) as thought by Sarasvati Amma; it was rather (4) although both are actually wrong or false. Our conjecture that (13) was believed by Mahāvīra to be based on (4) with $\pi = 3$ is shown by the fact that he gave the corresponding *sūkṣma* ("accurate") formula as

$$V' = \left(\frac{10}{9}\right) V, \tag{16}$$

which is obtained by adjusting the rough value V by means of his *sūkṣma* value $\pi' = \sqrt{10}$ in (4) since

$$\frac{V'}{V} = \left(\frac{\pi'}{\pi}\right)^2 = \frac{10}{9}.$$

That Mahāvīra used $\pi = 3$ as a rough value and $\pi' = \sqrt{10}$ as the accurate value is already known from his set of rough and accurate rules for the perimeter and area of a circle (*GSS*, VII, 19, 60), arc of a circular segment (*GSS*, VIII, 43, 45, 73),[23] etc. The original Sanskrit text as found in the manuscripts of the *GSS*, VIII, 28.5 runs as follows:[24]

तन्नवमांशं दशगुणम् अशेषसूक्ष्मं फलं भवति ॥

The ninth part of that (rough value V just found in the previous line) multiplied by ten becomes the accurate volume without remainder.

That is,

$$V' = \left(\frac{V}{9}\right) \cdot 10 = 5r^3. \tag{17}$$

However, since V already gives a value in excess of the correct volume, formula (17) has become bad to worse which is contrary to the intended improvement or accuracy. Hence the modern editors and translators (M. Rangacharya and L. C. Jain)[25] have adopted the following emended text for the above original of Mahāvīra:

तद्दशमांशं नवगुणम् अशेषसूक्ष्मं फलं भवति ॥

The tenth part of that (rough value V) multiplied by nine is the accurate volume with no remainder.

That is,

$$V' = \left(\frac{9V}{10}\right) = \left(\frac{81}{20}\right) r^3 = (4.05) r^3. \tag{18}$$

This yields a far better value than V and is quite comparable to the modern value

$$\left(\frac{4\pi}{3}\right) r^3 = (4.188) r^3 \text{ nearly.} \tag{19}$$

If Sarasvati Amma's interpretation of (13) as (14) were correct, the accurate formula expected from Mahāvīra would be

$$V_2 = \left(\frac{3}{2}\right) \cdot \sqrt{10 r^3} = 4.74 r^3 \text{ nearly,} \tag{20}$$

instead of (17) or (18).

There is another way of arriving at (12) and showing that it was based on taking $\pi^2 = 9$. The method is based on finding the surface area of the sphere by a rule given by Mahāvīra and as newly interpreted by the present writer in an earlier paper.[26] *GSS*, VII, 25 states:[27]

> परिधेश्च चतुर्भागो विष्कम्भगुणः स विद्धि गणितफलम् ।
> चत्वाले कूर्मनिभे क्षेत्रे निम्नोन्नते तस्मात् ॥
>
> Know that one fourth of the perimeter multiplied by the curvilinear breadth is the area of the concave or convex spherical surface resembling the sacrificial fire-pit or the back of a tortoise.

That is,

$$S = \left(\frac{p}{4}\right) \cdot b. \tag{21}$$

By applying this for a hemispherical surface, the area of the sphere's surface will be

$$S = 2\left(\frac{2\pi r}{4}\right) \cdot \pi r = \pi^2 r^2, \tag{22}$$

against the mathematically correct value $4\pi r^2$. If we now apply the empirical formula

$$V = \left(\frac{1}{2}\right) A h, \tag{23}$$

for the volume of a pyramid which, although wrong, was used often in India and abroad during olden days,[28] we get the volume of the sphere to be

$$V = \left(\frac{1}{2}\right) \cdot S \cdot r = \left(\frac{1}{2}\right) \cdot \pi^2 r^3, \tag{24}$$

by the usual method of regarding the sphere to consist of a large number of pyramids with their common vertex at the centre. And by taking π^2 equal to nine we get the required result (12).

There is yet another supporting evidence in this matter. The derivation *(vāsanā)* of (12) as given by Mādhavacandra (about 1000 AD) has been recently explained to be based on determining the effective depth of a hemispherical pit.[29] The area of the base will be $3r^2$ (with π equal to 3). The effective depth has been found to be $\frac{3r}{4}$. Hence

$$\text{volume of the hemisphere} = (3r^2)\cdot\left(\frac{3r}{4}\right) = \left(\frac{9}{4}\right)r^3, \tag{25}$$

which will give (12) by doubling.

I think that the effective depth has been found by analogy to the effective altitude of a semi-circle for which

$$\left(\frac{\pi}{2}\right)r^2 = 2r\cdot h,$$

which gives h equal to $\frac{\pi r}{4}$ (which will become $\frac{3r}{4}$ with $\pi = 3$). Hence (12) should be interpreted as

$$V = 2(3r^2)\cdot\left(\frac{3r}{4}\right) = 2\cdot(\pi r^2)\cdot\left(\frac{\pi r}{4}\right) = \left(\frac{\pi^2}{2}\right)r^3,$$

indicating the implied assumption π^2 equal to the rough value 9. Of course, if the effective depth $\frac{3r}{4}$ were found by, say, actual digging and converting the hemispherical pit into an equivalent cylindrical one, then (12) should be interpreted as

$$V_1 = 2(\pi r^2)h, \tag{26}$$

where h is $\frac{3r}{4}$, the mathematically correct value being $\frac{2r}{3}$ for equivalence *(samīkaraṇārtha)* as intended by Mādhavacandra.[30]

It is surprisingly interesting to note that if we apply formula (23) to averaging process similar to what we used in deriving the correct rule in (5), we get yet another empirical way of getting (12). For, the mean of the volumes of the circumscribed cylinder and the inscribed double cone will give

$$\left(\frac{1}{2}\right)\left(\pi r^2\cdot 2r + \frac{\pi r^2 .2r}{2}\right) = \left(\frac{3\pi}{2}\right)r^3, \tag{27}$$

which gives the required result (12) with $\pi = 3$. Although (26) and (27) are what Sarasvati Amma will read in (12), the last itself should be interpreted as (4) according to Liu Hui and Mahāvīra.

According to Wagner,[31] the derivation of (4) was believed by Liu Hui to be based on two reasoning:

(A) The volume of a cylinder inscribed in a cube is $\left(\frac{\pi}{4}\right)$ times the volume of the cube.

(B) The volume of the inscribed sphere is $\left(\frac{\pi}{4}\right)$ times the volume of the above inscribed cylinder.

Thus the volume of the sphere will be

$$V = \left(\frac{\pi}{4}\right)^2 d^3 = \left(\frac{\pi^2}{2}\right) r^3, \tag{28}$$

which yields (3) or (12) when π is taken as 3, the simple and universally used ancient value. Now it should be noted that, although assumption (A) is correct, assumption (B) is incorrect. As mentioned earlier, the correct ratio in (B) was already found by Archimedes to be $\frac{2}{3}$, instead of $\frac{\pi}{4}$. Thus the volume of a sphere will be given by

$$V = \left(\frac{2}{3}\right) \cdot \left(\frac{\pi}{4}\right) (\text{cube on diameter } d) = \left(\frac{\pi}{6}\right) d^3. \tag{28a}$$

Anyway, we note that the mistake of Liu Hui in taking $\frac{\pi}{4}$ for the correct value $\frac{2}{3}$ is comparable to Mādhavacandra's taking $\frac{3}{4}$ (instead of $\frac{2}{3}$) in finding the effective depth (see above). In this connection, Liu Hui has also considered the volume of an object called *ho-kai* ("box-lid") which is the intersection of two cylinders inscribed in the cube with perpendicular axes. His compatriot, Tsu Keng-chi, had finally shown that[32]

$$V = \left(\frac{\pi}{4}\right) \cdot (\text{vol. of box-lid}) = \left(\frac{\pi}{4}\right) \cdot \left(\frac{2}{3}\right) d^3, \tag{29}$$

giving the correct result for the sphere.

4 Śrīdhara's Rule

Śrīdhara (about 800 AD) in his *Triśatikā*, rule 56, states:[33]

गोलव्यासघनार्धं स्वाष्टादशभागसंयुक्तं गणितम् ॥

Half the cube of the diameter of a sphere combined with its own eighteenth part is the volume (of the sphere).

That is,

$$V = \left(\frac{d^3}{2}\right) + \left(\frac{1}{18}\right)\left(\frac{d^3}{2}\right) \tag{30}$$

$$= \left(\frac{19}{36}\right) d^3 = \left(\frac{38}{9}\right) r^3 = (4.22\ldots) r^3, \tag{31}$$

which is quite comparable to the true value (19). The same rule is found in *Siddhānta-śekhara*, XIII, 46 or Śrīpati (eleventh century),[34] and in *Mahā-siddhānta*, XVI, 108 of Āryabhaṭa II (fifteenth century ?).[35]

The usual derivation or rationale of (30) is given to be as follows (using $\sqrt{10}$ for π from *Triśatikā*, 45):[36]

$$V = \left(\frac{\pi}{6}\right) d^3 = \left(\frac{\sqrt{10}}{6}\right) d^3. \tag{32}$$

But by approximations we have

$$\sqrt{10} = \sqrt{3^2 + 1} = 3 + \frac{1}{6} = \frac{19}{6}. \tag{33}$$

Or, by using Śrīdhara's own rule as given in his *Pāṭīgaṇita* (rule 118),[37]

$$\sqrt{10} = \frac{\sqrt{360}}{6} = \frac{\left(\sqrt{19^2 - 1}\right)}{6} = \frac{19}{6} \tag{34}$$

Hence by (32) we get

$$V = \left(\frac{19}{36}\right) d^3 = \left(1 + \frac{1}{18}\right)\left(\frac{d^3}{2}\right), \tag{35}$$

the required equivalent of (30).

Recently it has been argued[38] that Śrīdhara also knew the better approximation $\frac{22}{7}$ for π. In that case we shall have

$$V = \left(\frac{\pi}{6}\right) d^3 = \left(\frac{22}{42}\right) d^3 = \left(1 + \frac{1}{21}\right)\left(\frac{d^3}{2}\right) \tag{36}$$

$$= \left(\frac{88}{21}\right) r^3 = (4.19) r^3. \tag{37}$$

Formula (36) as such is directly found in the *Līlāvatī, sūtra* 203, of Bhāskara II (twelfth century) as a rough *(sthūla)* value because the author knew not only the exact formula but also a still better value of π.[39]

The important question is whether Śrīdhara knew the exact form of the expression $\left(\frac{\pi}{6}\right) d^3$ for the volume, and then derived (30) from it as suggested above by modern scholars, or he got (30) directly by some empirical or approximate method. Also, if he knew the exact form, how did he obtain it? That is, whether it was some true mathematical demonstration or simply some other technique like the averaging process explained above in deriving (5) which was also possible because the correct formula for the volume of a pyramid (including a cone) was already known by his time in India. For instance, Brahmagupta in his *Brāmasphuṭa-siddhānta* (628 AD), XII, 44, gave the correct rule for the volume of *sūci* (a tapering figure, i.e. pyramid or cone), although his contemporary Bhāskara I did not give it.[40] Brahmagupta is silent on the volume of a sphere, and Śrīdhara's works are not fully known.

In case of a circle of diameter d, the areas of its circumscribed and inscribed squares are d^2 and $\frac{d^2}{2}$, respectively. Hence the area of the circle itself will be $k\left(\frac{d^2}{2}\right)$, where k lies between 1 and 2. In analogy (which is not mathematically correct) to this, the volume of a sphere might have been considered to lie between d^3 (which is the volume of the circumscribed cube) and $\frac{d^3}{2}$ (which is, however, not truly the volume of the inscribed cube). Thus

$$V = c\left(\frac{d^3}{2}\right), \qquad 1 < c < 2. \tag{38}$$

In fact most of the writers do first take the quantity $\frac{d^3}{2}$ and then prescribe formulas which are of the type (38). Now for the correct volume,

$$c = \frac{\pi}{3} = 1 + e, \text{ nearly (say)}, \tag{39}$$

where e is a small fraction, depending on the approximate value of π employed. Śrīdhara's rule has e equal to $\frac{1}{18}$ $\left(\text{which corresponds to } \pi = \frac{19}{6}\right)$ and Bhāskara II's rule has e equal to $\frac{1}{21}$ (corresponding to $\pi = \frac{22}{7}$). Both values of π implied here can be derived from $\sqrt{10}$ itself according to whether we approximate $\sqrt{a^2 + x}$ by

$$\left(a + \frac{x}{2a}\right), \text{ or } \left(a + \frac{x}{2a+1}\right),$$

both of which were used in ancient times.[41]

However, as far as Bhāskara II is concerned, we need not make any conjecture in this regard. He not only gave the correct formulas (*Līlāvatī, sūtra* 201),[42]

$$V = \frac{S \cdot d}{6} = 4\left(\frac{c \cdot d}{4}\right) \cdot \left(\frac{d}{6}\right), \tag{40}$$

but also mentioned the derivation of the volume by the usual simple method of dividing it into pyramids in a subsequent work *(Siddhānia-śiromaṇi, Golādhāya, Vāsanābhāṣya* under *Bhuvanakoṣa*, verses 58–61).[43]

References and Notes

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Kamalākara's Mathematics and Construction of *Kuṇḍas*

1 Introduction

Kamalākara was a great astronomer and mathematician of India and was a senior contemporary of the famous Newton in Europe. He belonged to a family of jyotiṣīs and was the second son of Nṛsimha who wrote a commentary called *Saurabhāṣya* on the *Sūrya-siddhānta* in AD 1611. Nṛsimha also wrote the *vasanāvārttika* commentary (AD 1621) on the *Siddhānta-śironmaṇi* of Bhāskara II (AD twelfth century). According to Sudhakara Dvivedi,[1] this commentary mentions a number of *yantras* (astronomical instruments) such as *mayūra-yantra, brahmacāri-yantra, haṃsa-yantra, vānara-yantra, śaravedha-yantra*.

Kamalākara studied astronomy and mathematics under his elder brother Divākara (born 1606) who wrote a commentary on the *Makaranda* (1478) and another called *Gaṇitatattva-cintāmaṇi* on his own *Jātakamārga* (1625). Kamalākara wrote a commentary called *Sauravāsānā* on the *Pūrvakhaṇḍa* of the *Sūrya-siddhānta, and Sauravāsanā* has been recently published[2] and refers to the *Siddhānta-tattva-viveka* (= *STV*, AD 1658) which is the main work of Kamalākara.

The *STV* contains 15 (including *Mānādhyāya* and *Upasaṃhāra*) chapters and is accompanied by an auto-commentary (= *STVC*) which consists of various derivations and explanations (*upapatti* and vāsanā). There is also the remaining (*"śeṣa"*) or supplementary part of the *STVC* called *śeṣavāsanā*. Apparently another commentary called *Tattvavivekodāharaṇa* by the author himself is reported by Pingree.[3]

Sudhakara Dvivedi[4] considers the *STV* to be the best among all the *siddhāntas* (Indian astronomical works in Sanskrit). It is respected, studied, and taught in the traditional system of education in India through the Sanskrit medium. Dvivedi was the first to edit and publish it.[5] Gangadhara Misra did prepare an edition in 1923, but it was published slowly. He included his own *Vāsanābhāṣya*, a detailed commentary in Sanskrit, in it along with the *Śeṣavāsanā*.[6]

Gaṇita Bhāratī, Vol. 20, Nos. 1–4 (1998), pp. 8–24.

K. Ramasubramanian (ed.), *Gaṇitānanda*,
https://doi.org/10.1007/978-981-13-1229-8_24

Due to being a staunch follower of the *Sūrya-siddhānta*, Kamalākara gave the following crude rules connecting the diameter D and circumference C in any circle[7]

$$C = \sqrt{10D^2} \tag{1}$$

$$D = \sqrt{\frac{C^2}{10}}. \tag{2}$$

Of course, such rules are quite old and are found in many ancient Indian works which are older than the extant *Sūrya-siddhānta*, the implied approximation $\pi = \sqrt{10}$ being usually called a Jaina value.[8] Kamalākara knew better values of π, but his adherence to $\pi = \sqrt{10}$ was a hurdle in attaining the intended accuracy in many cases.

Nevertheless, the *STV* contains a large number of novelties and these have been listed in Sanskrit, Hindi, and English.[9] Due to many new things, new topics, and alternatives in methodology, historians of science and scholars interested in traditional *jyotiṣa-śāstra* still read the *STV*. One such topic is related to the construction of the *agni-kuṇḍas* ("fire pits") which is the subject of the present chapter.

2 Havana-yajña and Agni-kuṇḍas

For attaining full benefits, certain religious acts such as building of temple, construction of a tank, a *mahādāna* ("great donation" e.g. *tulādāna*), some associated ceremonies are needed to be followed. One such ceremony is the performance of a *havana-yajña* in which oblations are offered to consecrated fire. An *agni-kuṇḍa* ("fire pit") is a pit of prescribed shape and size dug in the ground to hold the ritual fire. A *kuṇḍa* of specific shape is dug for specifically desired object or objective e.g. lotus-shaped *kuṇḍa* for rain and triangular *kuṇḍa* for the destruction of enemy!

Somewhat eight or ten different types of *kuṇḍas* are mentioned in traditionally typical works like *Śāradātilaka* (eleventh century?), *Maṇḍapa-kuṇḍa-siddhiḥ* (= *MKS*, 1619)[10] of Viṭṭhala Dīkṣita, and *Maṇḍapa-druma* (1654) of Mahādevasūri.[11] Kamalākara has dealt with 12 forms of *kuṇḍas*, namely

(i) *caturbhujam* ("four-sided") or square
(ii) *vṛtta* or circular
(iii) *ardha-candram* ("half-moon") or semicircle
(iv) *tribhujam* or (equilateral) triangle
(v) *yonikuṇḍa* I whose shape resembles the leaf of *pippal* tree of fig family (Ficus Religiosa)
(vi) *yonikuṇḍa* II
(vii) *ṣaḍasram* ("six-edged") or regular hexagon
(viii) *aṣṭāsram* or regular octagon
(ix) *padma-kuṇḍa* I or lotus-shaped No. 1
(x) -do- II

(xi) *pañcāsram* or regular pentagon
(xii) *saptāsram* or regular heptagon

The important point to note is that, whatever be the shape (out of the 12 above sections), the area enclosed by the curve in any ceremony is determined by the size of the *havana-yajña* i.e. by the number of *āhutis* (oblations) to be offered. Table 1 is prepared according to the *MKS*, II, 5 (p. 30) as further explained by its expositor B. Pathak (pp. 30–31).[12]

The units used are defined in *MKS*, 1, 3–4 (p. 3) and further explained by Pathak. These are presented in Table 2 after consolidation.

It must be noted that the *MKS* here defines the *hasta* as one-fifth of the full height of the *yajamāna* (or *yajña*-performer) when he stands (even on his toes) with his arms fully stretched upwards. Therefore, the cubit and other measures of Table 2 are not absolutely fixed. The *aṅgula* measure obtained in this way is called *dehāṅgula* (i.e. *aṅgula* as related to the body of the *yajña-kartā*). Moreover, there are many different types of even variable *aṅgulas* and several conventions to adopt them on different occasions.

Table 1 Fixing the area of a *kuṇḍa*

	Size of *Yajña*	No. of *Āhutis*	*Kuṇḍa*-Area
1.	*Śatārdha-nyūna*	Less than 50	No *kuṇḍa*-needed
2.	*Śatārdha-havana*	50 to 99	1 sq. *ratni*
3.	*Śata-havana*	100 – 999	1 sq. *aratni*
4.	*Śahasra-havana*	10^3 – 9999	1 sq. *hasta*
5.	*Ayuta-havana*	10^4 – 99999	2 sq. *hastas*
6.	*Lakṣa-havana*	10^5 – 999999	4 sq. *hastas*
7.	*Prayuta-havana*	10^6 – 9999999	6 sq. *hastas*
8.	*Koṫi-havana*	10^7 and above	8 or 16 sq. *hastas*

Table 2 (Linear measures)

1	*hasta* (cubit) = 24 *aṅgulas*
1	*aṅgula* = 8 *yavas* (barley corns)
1	*yava* = 8 *yūkās*
1	*yūkā* = 8 *līkṣās*
1	*līkṣā* = 8 *bālāgras* (tips of hair)
1	*bālāgra* = 8 *rathareṇus*
1	*rathareṇu* = 8 *trasareṇus*
1	*trasareṇu* = 8 *paramāṇus* (“atomic particles”)
1	*ratni* = 21 *aṅgulas*
1	*aratni* = $22\frac{1}{2}$ *aṅgulas*[13]

The dimensions of a *kuṇḍa* must be drawn accurately so that the prescribed area-measure or value is achieved. Otherwise what to say of attaining a desired object even adverse results might be caused. Several warnings are found e.g.[14]

(i) मानाधिके भवेन्मृत्युर्मानहीने दरिद्रता।

When the area is more (than the prescribed amount), there will be death; when the area is in deficit, there will be poverty.

(ii) खातेऽधिके भवेद्रोगो हीने धेनुधनक्षयः।

When the volume is in excess, there will be diseases; when in deficit, there will be loss of cattle and wealth.

The usual or traditional method of drawing the prescribed sectional curve was first to draw a square of the desired area and then convert or transform it into the prescribed shape of equal area with sufficient accuracy (as could be expected with possessed knowledge of that time, exactness being theoretically impossible in some cases). However, Kamalākara's method was different. By using relevant mathematical rules, he found two coefficients (*guṇakas*) for each type of 12 *kuṇḍas* he dealt with. If S is the area to be achieved for a *kuṇḍa*, the two coefficients β (called *bhuja-guṇaka*) and δ (called *vyāsa-guṇaka*) are defined for that *kuṇḍa* by the relations

$$b^2 = \beta S \tag{3}$$

$$d^2 = \delta S \tag{4}$$

so that

$$\frac{b^2}{\beta} = \frac{d^2}{\delta} = S \tag{5}$$

where b is the side (*bhuja*) and d the diameter (*vyāsa*) related to the figure to be drawn (see the next section for details). Kamalākara has expressed the values of the two coefficients in the usual ancient sexagesimal system (of fractions) and presented them in a tabular form (*STVC*, p. 167) which is produced here as Table 3.

Thus for any particular *yajña*, we know S from Table 1 and then find b and d by using (3) and (2) in conjunction with Table 3. The details for each *kuṇḍa* (with known b and d) are given below. The discussion does not cover the placement and orientation of the sacred pits.

3 Kamalākara's Calculations and Constructions

Kamalākara has dealt with the subject of *kuṇḍas* in three parts:

(i) *Gaṇita-prakāra* (*STV*, III, 105–141) contains some relevant rules and numerical results.

(ii) *Sādhana-prakāra* (*STV*, III, 142–146) contains the methods of construction.

(iii) *Vāsanā* (*STVC*, pp. 160–167) and *Śeṣa-vāsanā*, p. 12, contains derivations and calculations.

Table 3 (*Kuṇḍa* coefficients)

Sl. No.	*Kuṇḍa*	β	δ
1.	Square	1;0,0	2;0,0
2.	Circle	–	1;15,53
3.	Semicircle	–	2;31, 47
4.	Triangle	2;18,33	3;4,45
5.	Yoni I	0;40,43	0;54,17
6.	Yoni II	0;33,30	1; 7,1
7.	Hexagon	0;23,10	1;32,40
8.	Octagon	0;12,24	1;24,54
9.	Padma I	0,7,7	0;48,46
10.	Padma II	0;6,7	0;41,51
11.	Pentagon	0;34,51	1; 40, 56
12.	Heptagon	0;16,30	1;27,41

He begins by saying (*STV*, III, 105) that after knowing the rough methods which lead to inauspicious constructions as given by others, he is presenting the method for drawing the *kuṇḍa*-figures by using trigonometrical demonstration. We take his mathematical exposition of the *kuṇḍas* one by one.

3.1 Square *kuṇḍa*

$$S = b^2 \tag{6}$$

$$\text{and} \quad d^2 = 2b^2 = 2S \tag{7}$$

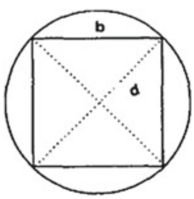

Fig. 1 Square *kuṇḍa*

Hence, $\beta = 1$, and $\delta = 2$ as in Table 3; also for $S = 1$ square *hasta* (or 576 sq. *aṅgulas*) b will be 24 *aṅgulas* and d will be $24\sqrt{2}$ units (Fig. 1). This is given in *STV*, III, 114 (p. 152) as

$$d = 33;56 \text{ } i.e. \text{ } 33 + \frac{56}{60} \text{ } a\dot{n}gulas$$

by taking square root closely (*āsanna-mūla*). In this *kuṇḍa*, the diameter d of the circumscribed circle is same as the diagonal of the square. The figure can be constructed directly on side b or after drawing the circle first. It seems that for irrational values, sufficiently accurate dimensions are taken for practical construction (as will be clear below also).

3.2 *Circular kuṇḍa (of diameter d)*

Here, there is no b or β, and

$$S = \frac{\pi d^2}{4}$$

$$\therefore d^2 = \left(\frac{4}{\pi}\right) S = \left(\sqrt{\frac{8}{5}}\right) S \tag{8}$$

which is given in *STV*, III, 115 (pp. 152–153) in equivalent forms because $\pi = \sqrt{10}$ for Kamalākara. Thus

$$\delta = \sqrt{\left(\frac{8}{5}\right)} = 1;15,53 \text{ nearly}$$

as given in Table 3. The actual coefficient is $\delta = \frac{4}{\pi} = 1;14,24$ nearly.

Also for unit-hasta square, (8) gives

$$d = \left[\left(\sqrt{\frac{8}{5}}\right) \times 576\right]^{\frac{1}{2}} = 27 \text{ } a\dot{n}gulas\text{, almost}$$

as given in *STV*, III, 116 (p. 153). Interestingly, this areal equivalence of a square of side 24 with a circle of diameter 27 implies the ancient Egyptian value π, namely $4 \times \left(\frac{8}{9}\right)^2$.

3.3 *Semicircular kuṇḍa*

Here also, there is no b or β and

$$S = \pi \frac{d^2}{8}, \text{ or } d^2 = \left(\frac{8}{\pi}\right) S \tag{9}$$

Thus with $\pi = \sqrt{10}$, we have $\delta = \frac{8}{\sqrt{10}} = 2; 31, 47$ nearly as in Table 3. Also we have, from (9), similarly

$$d^4 = \left(\frac{32}{5}\right) S^2 \tag{10}$$

which is found in STV, III, 117–118 (p. 153) with the numerical value $d = 38; 10$ for $S = 576$.

3.4 *Triangular kuṇḍa*

Here, b is the side of the equilateral triangle EFG and d is the diameter of the circumscribed circle. We have, area of the triangle,

$$S = \frac{\sqrt{3}b^2}{4} \tag{11}$$

or

$$b^2 = \left(\frac{4}{\sqrt{3}}\right) S \tag{12}$$

giving $\beta = \frac{4}{\sqrt{3}} = 2; 18, 33$ nearly as in Table 3, after neglecting fractions STV, III, 119–120 (pp. 155–156) states equivalent of (12) and also the correct formula

$$d = \sqrt{\left(\frac{4}{3}\right) b^2} \tag{13}$$

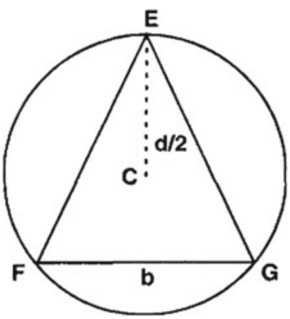

Fig. 2 Triangular *kuṇḍa*

From (12) and (13), we get

$$d^4 = \sqrt{\frac{(16S)^2}{27}} \tag{14}$$

which is also stated along with the above rule and from which follows

$$\delta = \frac{16\sqrt{3}}{9} = 3;4,45 \text{ nearly}$$

as in Table 3. For the unit-*hasta* square (i.e. $S = 576$*sq. aṅgulas*), *STV*, III, 121 (p. 156) states $b = 36;28$ and $d = 42;7$ which are correct approximations (Fig. 2).

3.5 *Yoni kuṇḍa No.1*

Construction of the first form of the *yoni-kuṇḍa* is given in *STV*, III, 147–149 (p. 158). In this, the starting figure is the equilateral $\triangle EFG$. Outer semicircles are described on all sides as diameters. Middle point here K of one of the semicircle is joined to F and G. Leaving out the arcs of the Chords FK and KG, we get the figure $FMELGKF$ of the *first yoni-kuṇḍa* in the form of a *pippal* leaf. *STVC*, pp. 161–162, gives area of $\triangle EFG$ (i.e. $\frac{\sqrt{3}b^2}{4} = (1;43,55)\cdot\frac{b^2}{4}$
area of two semicircles $\left(\text{i.e. } \frac{\sqrt{10}b^2}{4}\right) = (3;9,44)\cdot\frac{b^2}{4}$
area of $\triangle FKG$ $\left(\text{i.e. } \frac{b^2}{4}\right) = (1;0,0)\cdot\frac{b^2}{4}$.

Adding these, we get

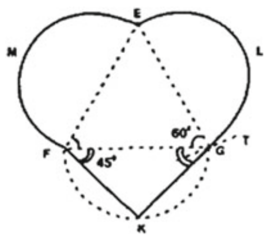

Fig. 3 *Yoni kuṇḍa* No.1

$$S = (5;53,39).\frac{b^2}{4} = \left(\frac{7073}{4800}\right)b^2 \tag{15}$$

$$\text{so that} \quad \beta = \frac{4800}{7073} = 0;40,43 \tag{16}$$

as given in *STV*, III, 122 (p. 154), *STVC*, p. 162, and Table 3. Again, if d is the diameter of the circumcircle of $\triangle EFG$, them by Eq. (13)

$$d^2 = \left(\frac{4}{3}\right) b^2 = \left(\frac{4}{3}\right) \cdot \left(\frac{4800}{7073}\right) S, \text{ by (15).}$$

Thus, we get

$$\delta = \frac{6400}{7073} = 0, 54, 17, \text{ as in Table 3.}$$

Using (15) for unit-*hasta* square, we get

$$b = \sqrt{\left(\frac{4800}{7073}\right) \cdot S} = 19; 46, 16$$

as given in the *STV*, III, 123 (p. 155) along with $d = 22; 50$ (which follows by using (13) or the value of δ).

It must be pointed out that the tangent GT to the semicircle GLE at G does not lie along the side GK (see Fig. 3). Thus, the curve is not smooth (although it is continuous) at F and G. This defect is also found in the usual traditional method.[15]

3.6 Yoni kuṇḍa No.2

The construction of this is described in *STV*, III, 150–151 (p. 158). In this, the basic figure is the square $EFGH$ of side b inscribed in a circle of diameter d (Fig. 4). Semicircles are described on sides EF and EH. Figure $EMFGHLE$ is the *pippal*-leaf-shaped *Yoni Kuṇḍa* No.2. As explained in *STVC*, p. 162, here we have

$$\begin{aligned} S &= \text{ square} + 2 \text{ semicircles; with } b = \frac{d}{\sqrt{2}} \\ &= \frac{d^2}{2} + \frac{\pi d^2}{8}; \text{ with } \pi \text{ as in No.1} \\ &= (1; 42, 26)\frac{d^2}{2} \end{aligned} \tag{17}$$

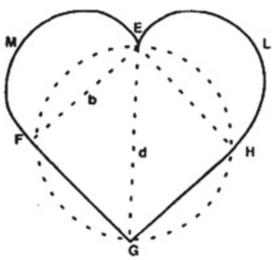

Fig. 4 *Yoni kuṇḍa* No.2

giving $\delta = \frac{2}{(1;47,26)} = \frac{3600}{3223} = 1;7,1$ as in *STV*, III, 124 (p. 155), *STVC*, p. 162, and Table 3 (Fig. 4).

Also $b^2 = \frac{d^2}{2} = \frac{\delta S}{2}$ which gives $\beta = \frac{\delta}{2} = 0;33,30$, nearly, as in Table 3. For the unit-*hasta* square ($S = 576$), the *STV*, III, 125, gives

$$d = 25;22,20 \text{ (The correct value being } 25;21,53\text{), and}$$
$$b = 17;56,8 \text{ which is correct.}$$

It should be noted that here the curve is smooth at F and H. But in the traditional construction, end G lies below outside the circle, and so it will make a defective leaf.[16]

3.7 Hexagonal *kuṇḍa*

In Fig. 5, let EF represent a side b of a regular polygon of n sides inscribed in a circle of diameter d with its centre at C. Modern forms of two simple relations are

$$b = d \sin\left(\frac{\pi}{n}\right) \tag{18}$$

$$S = \left(\frac{n\,d^2}{8}\right) \sin\left(\frac{2\pi}{n}\right). \tag{19}$$

In the case of a hexagon ($n = 6$), these become

$$b = \frac{d}{2} \tag{20}$$

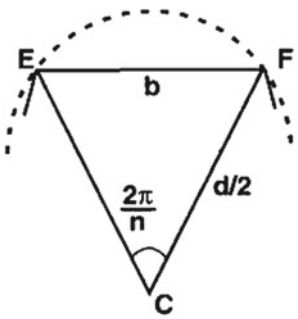

Fig. 5 Part of octagonal *kuṇḍa*

and

$$S^2 = \frac{27d^4}{64} \tag{21}$$

which are stated in the *STVC*, p. 162. Thus

$$d^2 = \left(\frac{8}{3\sqrt{3}}\right) S$$

leading to $\delta = 8 \times \frac{\sqrt{3}}{9} = 1; 32, 23$ which is slightly less than that in Table 3. *STV*, III 126–127 (p. 155) states some other rules which follow from (20) and (21). Equations (5) and (20) show that $\beta = \frac{\delta}{4}$ as indicated by data in Table 3. Lastly from (20) and (21), we get

$$b^2 = \left(\sqrt{\frac{4}{27}}\right) S = \left(\sqrt{\frac{4}{27}}\right) \times 576,$$

for $S = 576$; whence we get $b = 14; 53$ nearly, as in *STV.III*, 128.

3.8 Octagonal kuṇḍa

In this case, $n = 8$, and (18) gives

$$b = d \sin\left(\frac{\pi}{8}\right) = (0; 22, 57)\ d\ \text{nearly} \tag{22}$$

as stated in *STV*, III, 130 (p. 156) and as derived in *STVC*, p. 163 by using the ancient trigonometrical formula

$$R \sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{R}{2} \cdot (R - R\cos\theta)}, \qquad \text{with } \theta = \frac{\pi}{4}.$$

Then, the *STVC* also finds the area of the isosceles $\triangle EFC$ (of sides $\frac{d}{2}$, $\frac{d}{2}$, and b found above) to be (0;5,18) d^2. Thus

$$S = 8 \times \triangle EFC = (0; 42, 24)d^2. \tag{23}$$

Hence, we get

$$d^2 = \frac{S}{(0; 42, 24)} = \left(\frac{75}{53}\right) S = (1; 24, 54)S \tag{24}$$

as given in *STV*, III, 129 (p. 156), derived in *STVC*, p. 163, and as implied in Table 3 in the form of δ. From (22) and (24), we now get

$$b^2 = (0; 22, 57)^2 \times \left(\frac{75}{53}\right) S = (1; 12, 25)S \text{ nearly}$$

which compares well with that implied in Table 3. Using the above two results for unit-*hasta* square ($S = 576$), we easily get $d = 28; 33$, and $b = 10; 55$ as in *STV*, III, 131.

3.9 Lotus kuṇḍa No.1

A *padma* ("lotus") *kuṇḍa* has a flowery look with 8 equal petals (Fig. 6). The sides of a regular octagon are the bases (such as *EF*) of the petals. Different modes of formation of the petals give rise to different *lotus kuṇḍas*.

In Kamalākara's lotus kuṇḍa No. 1, a petal is drawn as follows (see STV, III, 152–156, p. 159):

The outer semicircle on a side $EF (= b)$ is divided into four equal parts (Fig. 7), two of which are arcs *EP* and *QF*. Tip *V* of the petal is obtained by taking

$$PV = QV = \frac{b}{2}. \tag{25}$$

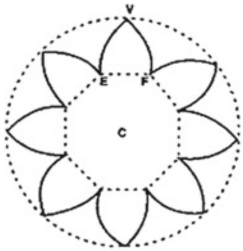

Fig. 6 Lotus *kuṇḍa* No.1

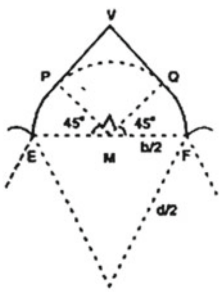

Fig. 7 Petal of lotus *kuṇḍa* No.1

The petal boundary on side EF consists of the circular arcs EP and FQ, and the (tangential) lines PV and QV. Calculations are explained in the $STVC$, pp. 163–164. M is the middle point of EF and is the centre of the semicircle.

Here,

$$\angle EMQ = \angle QMF = \frac{\pi}{4}$$
$$\therefore \angle PMQ = \frac{\pi}{2} \text{ or } 90^\circ$$
$$\text{Also } MP = MQ = PV = QV = \frac{b}{2} \text{ each}$$

$\therefore$ $PMQV$ is a square and (PV and QV are tangents).

Thus, we see that the area of the petal consists of a square of area $\frac{b^2}{4}$ and two sectoral triangles of the semicircle. Hence by adding the areas of the eight petals to the area of the inner octagon, we have

$$S = 2\left(\frac{\pi b^2}{4}\right) + 8\left(\frac{b^2}{4}\right) + \text{Octagonal area.}$$

Now

$$\begin{aligned}\frac{2\pi b^2}{4} &= \sqrt{4\left(\frac{pb^2}{4}\right)^2} = \sqrt{4\left(\frac{5}{8}\right)b^4}, \text{ as } p = \sqrt{10} \text{ here,}\\ &= \sqrt{4\left(\frac{5}{8}\right)(0;22,57)^4 d^4}, \text{ by eq. (22)}\\ &= (0;13,52,42)\ d^2 \qquad (26)\end{aligned}$$

as has been worked out step by step in the $STVC$, pp. 163–164. Again

$$\begin{aligned}8\left(\frac{b^2}{4}\right) &= 2b^2 = 2(0;22,57)^2 d^2, \text{ by (22),}\\ &= (0;17,33,24)\ d^2. \qquad (27)\end{aligned}$$

Also by Eq. (23),

$$\text{Octagonal area} = (0;42,24)\ d^2. \qquad (28)$$

Hence by adding (26), (27), and (28), we get

$$S = (1;3,50)\ d^2 \text{ nearly}$$

or

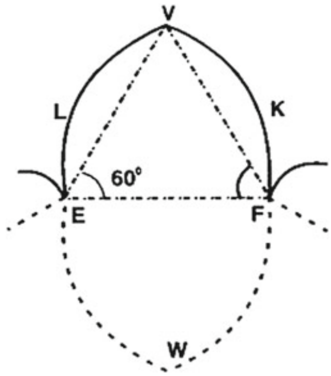

Fig. 8 Petal of lotus *kuṇḍa* No.2

$$d^2 = \frac{S}{(1;\,13,\,50)}$$
$$= \left(\frac{360}{443}\right) S = (0;\,48,\,46)\ S \tag{29}$$

as given in *STV*, III 132, *STVC*, p. 164, and Table 3. Again using (22) and (29), we have

$$b^2 = (0;\,22,\,57)^2(0;\,48,\,46)S$$
$$= (0;\,7,\,7)\ S \text{ nearly, giving } \beta \text{ of Table 3.}$$

when $S = 576$ square angulas, (29) will give

$d = \sqrt{\left(\frac{360}{443}\right) \times 576} = 21;\,38,\,7$ *aṅgulas*, as given on p. 164 of the *STVC*.[17] Again for $S = 576$, we also have from above

$$b = \sqrt{(0;\,22,\,57)^2(0;\,48,\,46) \times 576}$$
$$= 8;\,16,\,14 \text{ nearly.}$$

3.10 Lotus kuṇḍa No.2

Construction[18] for this *kuṇḍa* is given in *STV*, III, 157–159 (p. 159). For forming the petal on side $EF(= b)$, circular arcs $WELV$ and $WFKV$ are drawn by taking radius b and centres at F and E, respectively (Fig. 8). The petal $ELVKFE$ is called outer part of the *matsya* ("Fish Figure") $EVFWE$. Calculations are found in *STVC*, pp. 164–165.

There the square of the area of the equilateral triangle VEF is given (p. 165) i.e. $\frac{3b^4}{16} = (0; 11, 15)b^4$ which is exact. Extracting the square root, we then get $\triangle VEF \left(\text{i.e. } \frac{\sqrt{3}b^2}{4}\right) = (0; 25, 59)b^2$, very nearly.

Similarly, the area of the circle sector $EVKFE$ i.e. $\frac{\pi b^2}{6} = (0; 31, 37)b^2$, with $\pi = \sqrt{10}$.

Now, the area of the petal

$$\begin{aligned}
&= \text{sector } EVKFE + \text{ segment } ELVE \\
&= \text{sector } + (\text{sector } - \triangle VEF) \\
&= (0; 32, 37)b^2 + (0; 31, 37 - 0; 25, 59)b^2, \text{ by above} \\
&= (0; 37, 15)b^2. \qquad (30)
\end{aligned}$$

Also from Eq. (24),

$$\begin{aligned}
\text{area of octagon} &= \left(\frac{53}{75}\right) d^2 \\
&= \left(\frac{53}{75}\right) \frac{b^2}{(0; 22, 57)^2}, \text{ by using (22)} \\
&= (4; 50, 11)b^2. \qquad (31)
\end{aligned}$$

Adding to this area of 8 petals from (30), we get

$$\begin{aligned}
S &= (4; 50, 11)b^2 + 8(0; 37, 15)b^2 \\
&= (9; 48, 11)\ b^2 \ \text{ as in } STVC\text{, p. 165.}
\end{aligned}$$

From this, we also get

$$b^2 = \frac{S}{(9; 48, 11)} = (0; 6, 7, 14)S, \qquad (32)$$

giving $\beta = 0; 6, 7, 14$ as in STV, III, 134 (p. 156), $STVC$, p. 165, and Table 3. Again by using (22) and (32), we have

$$\begin{aligned}
d^2 &= (0; 6, 7, 14)\frac{S}{(0; 22, 57)^2} \\
&= (0; 41, 50)S \text{ nearly.}
\end{aligned}$$

This gives a value of δ quite near to that in Table 3.

3.11 Pentagonal kuṇḍa

In this case (see Fig. 5), $n = 5$ and we have $b = d \sin 36°$, cf. Eq. (18).

But from Kamalākara's sine table (see *STVC*, pp. 168–169), we see that

$$60 \sin 36° = 35;\ 16,\ 1,\ 36,\ 52$$
$$b = (0;\ 35,\ 16)\ d \text{ nearly} \tag{33}$$

as in *STV*, III, 138 and in *STVC*, p. 165. Then, the *lamba* (perpendicular) from C on *EF* as

$$lamba = \sqrt{\left(\frac{d}{2}\right)^2 - \left(\frac{b}{2}\right)^2} = (0;\ 24,\ 17)\ d, \text{ using (33).}$$

Now

$$S = 5 \times \triangle CEF = \frac{5(b \times lamba)}{2}$$
$$= (0;\ 35, 40)d^2 \tag{34}$$

or

$$d^2 = \frac{S}{(0;\ 35, 40)} = \left(\frac{180}{107}\right) S \tag{35}$$

as given in *STV*, III, 137 (p. 157) and *STVC*, p. 166. Also $\delta = \frac{180}{107} = 1;\ 40, 56,$ as in Table 3. Again from (33) and (35)

$$b^2 = (0;\ 35,\ 16)^2 \left(\frac{180}{107}\right) S = (0;\ 34,\ 52)S$$

the tabular value of the coefficient being 0; 34, 51. But for $S = 576$ square *aṅgulas*, we will have $b = \sqrt{(0;\ 34,\ 52) \times 576} = 18;\ 18,$ *aṅgulas* as in *STV*, III, 139 (p. 157) which also contains $d = 31;\ 9$ for same value of S. However, (35) gives $d = 31;\ 8$.

3.12 Heptagonal kuṇḍa

In this case ($n = 7$), we have (see Fig. 5)

$$b = d \sin\left(\frac{\pi}{7}\right) = \sin\left(\frac{180°}{7}\right)$$
$$= (0;\ 26,\ 1,\ 59)\ d \tag{36}$$

by modern calculator. But by using the sine table in the *STVC* (p. 168) and the usual linear interpolation, we will get

$$
\begin{aligned}
60 \sin\left(\frac{180^\circ}{7}\right) &= 60 \sin\left(25 + \frac{5}{7}\right)^\circ \\
&= 60 \sin 25^\circ + \left(\frac{5}{7}\right).(60 \sin 26^\circ - 60 \sin 25^\circ) \\
&= 25;\,21,25,33 + \left(\frac{5}{7}\right)(0;\,56,42,37) \\
&= 26;\,1,55,58 \\
\therefore \quad b &= (0;\,26,1,56)\; d, \ \text{nearly} \qquad (37)
\end{aligned}
$$

as given in *STV*, III, 141, (p. 157) and *STVC*, p. 166. To get better value, the *Śeṣavāsanā* (p. 12) asks us to get sine of $\frac{180}{7}$ degrees by interpolation between 25.5° (instead of 25°) and 26°, but further details are not mentioned.

Anyway, as in the case of pentagonal *kuṇḍa*, the *lamba* from C on EF (Fig. 5) can be found here also, and then the area of the isosceles $\triangle CEF$ of sides b, $\frac{d}{2}$, $\frac{d}{2}$. *STVC*, p. 166, correctly finds

$$S = 7 \times \triangle CEF = (0;\,41,3)d^2. \qquad (38)$$

Thus

$$d^2 = \frac{S}{(0;\,41,3)} = \left(\frac{1200}{821}\right) S \qquad (39)$$

as given in *STV*, III, 140 (p. 157) and *STVC*, p. 166. From (39), we find

$$\delta = \frac{1200}{821} = 1;\,27,41,52\ldots$$

which appears in Table 3 as 1;27,41. Also from (37) and (39)

$$
\begin{aligned}
b^2 &= (0;\,26,1,56)^2 \times \left(\frac{1200}{821}\right) S \qquad (40) \\
&= (0;\,16,30) S \ \text{nearly} \qquad (41)
\end{aligned}
$$

giving the coefficient β of Table 3. Finally for $S = 576$, we can find d and b using (39) and (40). The values given in *STVC*, p. 157 are $D = 29, 0, 56$ which is correct, and $b = 12;\,35,21$ which is nearly correct.

For application of the pentagonal and septagonal *kuṇḍas* to religious rituals, one may refer to the *Kuṇḍa-kādambarī*.[19]

4 Concluding Remarks

It may be mentioned that Kamalākara has not dealt with those two forms of *kuṇḍas* which have been called *viṣama-saḍasra* and *viṣama-aṣṭāsra* (cf. *MKS*, II, 13 and 16, pp. 44–55). Hayashi[20] calls these figures as irregular hexagon and irregular octagon, respectively. Actually, they are what we usually call hexagram or hexacle (Fig. 9) and octagram or octacle (Fig. 10). *MKS* rightly considers the hexagram to be *netra-ramyam* ("charming to eye") or beautiful.

Kamalākara has used the words *sama* ("equal") and *viṣama* ("unequal") with respect to the various sides of a polygon is *STV*, III, 111 (p. 152). He says that polygonal *kuṇḍas* with equal sides (and accurate area) lead to longevity, health, and prosperity, while *kuṇḍas* of unequal sides may cause the opposite.

As explained above, Kamalākara's Table 3 of coefficients will be able to find b and d corresponding to any given S by using the relation (5). In particular, we can apply it to the fundamental case of *ekahasta-kuṇḍa* i.e. when S is unit square *hasta* or 576 square *aṅgulas*. Due to the importance of this case, Kamalākara has invariably stated the values of b and d specifically and separately for each type of *kuṇḍa*. We collect and present these in Table 4.

Now it is clear from (5) that if the area S is increased to mS (i.e. m times its own value), then the corresponding values of b and d will be increased $\sqrt{m}$ times. Thus, we see that for S equal to m square *hastas* (or $m \times 576$ sq. *aṅgulas*), the values of b and d for any *kuṇḍa* will be $\sqrt{m}$ times those given in Table 4 which, therefore, provides an alternative method for finding b and d. Even a table of $\sqrt{m}$ (which we need in this method) for $m = 1$ to 10 (integral values only) is also available e.g. in

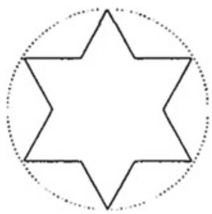

Fig. 9 Hexacle

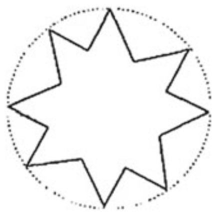

Fig. 10 Octacle

Table 4 *Bhuja* and *vyāsa* for 1 square *hasta* area

Sl. No.	*kuṇḍa shape*	*b in aṅgulas*	*d*	*STV Reference*
1.	Square	24;0,0	33;56	III, 114 (p. 152)
2.	Circle	–	27	III, 116 (p. 153)[21]
3.	Semicircle	–	38;10	III, 118
4.	Triangle	36;28	42;7	III, 121 (p. 154)
5.	Yoni No.1	19;46.16	22;50	III, 123 (p. 155)
6.	Yoni no.2	17;56,8	25;22,20	III, 125 (p. 155)[22]
7.	Hexagon	14;53	29;46	III, 128
8.	Octagon	10;55	28;33	III, 131 (p. 156)
9.	Lotus No. 1	8;16,31	21;38.7	*STV* (p. 164)[23]
10.	Lotus No. 2	7;39,55	20;3,12	III, 136 (p. 157)[24]
11.	Pentagon	18;18	31;9	III, 139 (p. 157)[25]
12.	Septagon	12;35,21	29;0,56	*STVC* (p. 157)[26]

MKS, II, 7 (pp. 32–34)[27] which gives the values of $24\sqrt{m}$ *aṅgulas* (and from which $\sqrt{m}$ follows easily). One must remember that *MKS* values are expressed in octonary fractions, while Kamalākara uses sexagesimal fractions.

Of course for the discussion of accuracy of any ancient tabular values and other numerical results, attention has to be paid to the method of multiplication, division, square rooting, etc., followed in ancient works (along with conventions of rounding off etc.) Kamalākara's claim of accuracy of many results are not theoretically sound because of his use of $\pi = \sqrt{10}$ and some other approximations.

Although there exist scores of works on *kuṇḍas*, the historical and mathematical aspects of the related geometrical figures have not been given the attention they deserve. The present paper forms an introductory study some what similar to what Hayashi (*ref.* 12 at the end) has done from Gaṇeśa's commentary on the *MKS* (but we have omitted Sanskrit texts). Relevant connected topics, like *maṇḍapas, vedis, mekhalas,* also need attention. Anyway, I have now added one more item on the interesting mathematics from Kamalākara's *STV*.[28]

Before closing this chapter, I am tempted to point out a relevant fact which seems to have been hidden by the editor and publisher of the *MKS* which has been consulted here (*ref.* 10 at the end). A small work called *"Vāstava-kuṇḍasiddhiḥ"* of 63 verses is attached at the end (see supplementary pages 1 to 7). The colophon describes this small work to be *"Baladevapāthaka-praṇītā"* i.e. as "composed by Baldev Pathak". But actually the 63 verses are exactly the reproduction of the *STV*, III, verses 105–168 (leaving out no. 166) without any acknowledgement![29]

References and Notes

1. S. Dvivedi, *Gaṇaka-Taranginī* (in Sanskrit), Benares, 1933, pp. 83–84. This was a posthumous edition (of the work first published in 1892) by Padmakara Dvivedi who also wrote the article "Kamalākara-bhaṭṭa" (in Hindi with English summary) in the *Proceedings of the Benares Math. Society,* (O.S.) 2 (1920), 67–80.
2. *Sūrya-siddhānta* with *Sauravāsanā* edited by late Sricandra Pandeya, Sampurnanand Sanskrit Univ., Varanasi, 1991. It has eleven chapters, although the colophon mentions 'ten' as is also mentioned in *CESS* (*A*) 2, p. 23 (Philadelphia, 1971), *STV* is referred in *Sauravāsanā* on pp. 14 and 104.
3. David Pingree, *Census of the Exact Sciences in Sanskrit,* series A. Vol. 2 or *CESS* (*A*) 2, p. 23, Raghunātha and Veṅkateśa are two other commentators of *STV*, *Ibid*., p. 21.
4. S. Dvivedi, Gaṇaka Taraṅginī (ref. 1 above), p. 99.
5. S. Dvivedi (editor), *STV*, 5 Parts, Benares, 1880–1885. It includes *STVC*. A revised edition by Muralidhara Jha (died 1929) was published, 1924–1935. The last or fifth part of this edition also contains the *Śeṣavāsanā* with notes by Muralidhara Thakur (or Thakkura), on supplementary pages 1 to 61. The revised edition has been recently reprinted in a single volume, Chaukhamba, Varanasi, 1991.
6. Misra's edition was published in three parts. The one mentioned in *CESS* (*A*) 2, pp. 22 and 85 is only the first part. Full details are as follows: Part I (chapters I–IV), Naval Kishore Press, Lucknow, 1929; Part II (chapter V–X), Mithila Press, Bhagalpur, 1935; Part III (chapters XI–XV), Khelarilal, Benares, 1941. The last part also has Misra's 23 essays attached at the end (supplementary pp. 1–7); Part I is not seen by me.
7. *STV*, I, 147 (p. 50) and *Sūrya-siddhānta*, I, 59 (ref. 2 above, p. 13).
8. See R. C. Gupta

   (i) "Circumference of the Jambūdvipa in Jaina Cosmography," *Indian J. Hist. Sci.* 10 (1975) 38–46 and
   (ii) "The Jaina Value of Pi and Its Transmission Abroad", *Arhad Vacana.* I (i) (1988), 15–18.

9. See S. Dvivedi's "*Bhūmikā*" in *STV* (1991) reprint, pp. 1–3) and P. Dvivedi's article (ref. 1) pp. 76–80.
10. *MKS* with the *Baladā* Sanskrit commentary and Hindi translation both by Baladev Pathak, published by his son Ganeshadatt, Benares, 1926.
11. The *Maṇḍapadruma* is edited by Shreekrishna Sarma, *Adyar Library Bulletin,* 22 (1–2) (1958), 119–157.
12. The table given by Takao Hayashi, "Ritual Application of Mensuration Rules in India etc", *Bulletin of the National Museum of Ethnology* (Osaka), 12 (1) (1987), 199–224, on p. 211 is somewhat different as the text and interpretation he mentions are at variation. Moreover *MKS*, III, 6 itself gives some other opinions.
13. Pathak, *MKS*, p. 4 (see ref. 10 above) says *ratni* is cubit measure with closed fist, and *aratni* is cubit measure upto the end of little finger and quotes. $aratni = (hand) - \left(\frac{hand}{16}\right) = 24 - 1.5$ *aṅgulas*.
14. See Gokulanātha's *Kuṇḍa-kādambarī* (1741) edited by Dharmanātha and Kumudanātha, Darabhaṅgā, 1982, p. 155–156, also Hayashi, ref. 12, p. 200 for another warning. However, Mahādeva's *Maṇḍapadruma* (see ref. 11), I, 2–5 (p. 128) regards the insistence on accuracy to be of no purpose.
15. See Hayashi (ref. 12 above), p. 212, where the drawn figure has same defect at the points *N* & *S*.
16. *Ibid.* Does such a defect imply any inauspiciousness?
17. But the value of *vyāsa* or *d* is given in *STV*, III, 133 (p. 156) as 21;38,25 *aṅgulas* which is wrong.
18. Kamalākara's value is 8;16,38 (*STV*, III, 133 p. 156) or 8;16,31 (*STVC*, p. 164).
19. Gokulanātha's *Kuṇḍa-kādambarī* (see ref. 14), p. 117.
20. Hayashi, ref. 12, pp. 216–218.

21. Correct value of d is 26.99 *aṅgulas* (using pocket calculator).
22. Correct value $d = 25;21,53$.
23. Correct $b = 8;16,14$. The values of b and d in *STV*, III, 133 (p. 156) are both wrong or misprinted.
24. Correct value of d is 20;2,24 (or 38).
25. Correct value of d is 31;8.
26. More correct value of b is 12;35,20.
27. See Hayashi, ref. 12, p. 211, Of course *MKS*'*s* purpose is different.
28. R. C. Gupta's earlier studies arc (i) "Sines and Cosines of multiple Arcs as Given by Kamalākara" *Indian J. Hist. Sci.,* 9 (1974), 143–150. (ii) "Sines of Sub-multiple Arcs as Found in the *STV*", *Ranchi Univ. Math. J.* 5 (1974), 21–27.
29. B. Pathak in his Hindi "Bhūmika" writes that the small work (mūlagrantha) is *samkalita* ("collected" or "added") with the *MKS*. So it is possible that someone else has already taken out the verses from *STV* and named them to form a separate work *Vāstavakuṇḍa siddhiḥ*. But the real author is Kamalākara who should have been given the credit.

Area of a Bow-Figure in India

1 Introduction

In Fig. 1, $PNQP$ is segment of a circle (i.e. circular disc) whose centre is at O and whose radius is $OP = OQ = r$. Due to the figure's resemblance to an archer's bow, the arc PNQ (= s in length) was called *cāpa* ('bow'), the chord $PQ (= c)$ was called *jyā* or *jīvā* ('bow-string'), and the segment's height $MN (= h)$ was called *bāṅa* or *śara* ('arrow') in ancient India. The *cāpakṣetra* ('bow-figure') or segment of a circle had great importance in Indian cosmography and geography, especially in the Jaina school. The Bharata-kṣetra (=Bhārata-varṣa or 'land of India') of those times was in the shape of a bow-figure which formed the southernmost part of the central continent or Jambūdvīpa ('Jambū Island') which is stated to be circular and of diameter one lac (100,000) *yojanas*. This cartographic description may be taken to represent the oldest map of India as part of Asia. The maximum north–south breadth of the country was $526\frac{6}{19}$ *yojanas*.

The exact relation between c and h for any segment of a circle of diameter d ($= 2r$) is

$$c = \sqrt{4h(d-h)} \tag{1}$$

which easily follows by applying the so-called Pythagorean theorem to the right-angled triangle OPM. An explicit verbal statement of (1) is found in the *Bhāṣya* on the *Tattvārthādigama-sūtra* (III, 11) of Umāsvāti.[1]

The usual method for finding the exact area A of the circular segment takes

[1] See the *Bhāṣya* under *sūtra* 11 of chapter III in [Umāsvāti 1932, 170]. He is placed in the first century AD by [Pingree 1970, 59]. But the date (and even authorship) is controversial.

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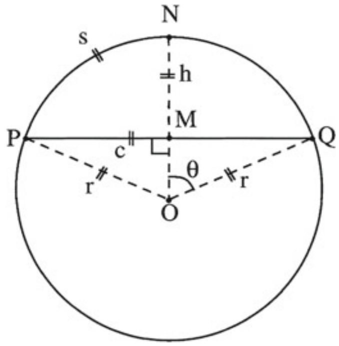

Fig. 1 Segment of a circle

$$A = \text{sector } OPNQ - \text{triangle } OPQ \tag{2}$$
$$= \frac{s \cdot r}{2} - \frac{c(r-h)}{2} \tag{3}$$
$$= \frac{r(s-c)}{2} + \frac{c \cdot h}{2}. \tag{4}$$

In terms of the semi-central angle θ subtended by arc at the centre, we have the formulas

$$s = 2r\theta, \tag{5}$$
$$c = 2r \sin\theta, \tag{6}$$
$$h = r(1 - \cos\theta), \tag{7}$$

where

$$\tan\left(\frac{\theta}{2}\right) = \frac{2h}{c}. \tag{8}$$

We see that the use of trigonometric functions and tables makes the computation of the arc-length s and of the area A quite straightforward when any two of the three parameters c, h and $d(= 2r)$ are known. But when trigonometry was not sufficiently developed, or when its proper use was unknown or avoided, the problem of finding s and A was difficult. In such a situation mathematicians had recourse to devising suitable empirical rules. Practical formulas were found for needful calculations.

In an earlier paper [Gupta 1979], the present author discussed the Indian rules for finding the arc of a circular segment. Most of the formulas used in ancient and medieval India were of the type

$$s = \sqrt{c^2 + kh^2} \tag{9}$$

where k was chosen such that the formula yielded the expected result for the semicircle (which is also a segment with $c = 2r$ and $h = r$). That is,

$$k = \pi^2 - 4. \tag{10}$$

The simplest approximation $\pi = 3$ gives $k = 5$. But the most commonly used value[2] of k was 6, which corresponds to the well-known Jaina approximation $\pi = \sqrt{10}$. For small arcs, Nīlakaṇṭha Somasutvan (c.1500 AD)[3] found the best formula of the type (9) to correspond to $k = \frac{16}{3}$. An altogether different formula.

$$s = \sqrt{10\left(\frac{c}{4} + \frac{h}{2}\right)^2} \tag{11}$$

is said [4] to be quoted by Bhāskara I in his commentary (AD 629) on the *Āryabhaṭīya* of Āryabhaṭa I (born AD 476).

2 Area of a Circular Segment in Other Ancient Civilizations

Some rules for computing the area of a segment of a circle are found in Babylonian tablets, but they have not been stated clearly therein nor have they been understood satisfactorily[5]. However, the present author was able [Gupta 2001] to assign some sensible meaning to certain procedures in a Babylonian text to arrive at a few empirical mensurational rules for the arc-length and area of a circular segment. On that basis the old Babylonian text BM 85194 (c. 1600 BC)[6] can be cited to infer that the Babylonians had used the formula

$$A = ch - kh^2 \tag{12}$$

for the area of the segment where k is $\frac{1}{2}$ or 1. The value $k = \frac{1}{2}$ is to be preferred if (12) is expected to give the exact result for a semicircle with the Babylonian value $\pi = 3$. Otherwise $k = 1$ yields better values of A for segments significantly smaller than a semicircle.

[2] See [Umāsvāti, 170], and [Gupta 1979, 91–2].

[3] [Gupta 1972], [Gupta 1972–73]; also [Gupta 1979, 93].

[4] [Shukla 1976, LVI, 74] (under II, 10).

[5] For an example see [Katz 1993, 20].

[6] [Van der Waerden 1983, 177–9].

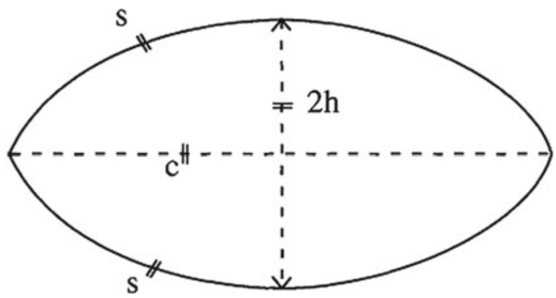

Fig. 2 The double-segment

Nevertheless there is no doubt that the most popular ancient rule for the area of a circular segment was

$$A = (c + h) \cdot \frac{h}{2}. \tag{13}$$

So far no direct evidence for the use of (13) is found in Babylonian texts. But it easily follows by using the newly discovered formula (for a segmental arc)[7]

$$s = c + h \tag{14}$$

suitably in the case of the figure formed (in the shape of a banana leaf) by the double-segment (see Fig. 2). For this purpose we use the well-known and universal ancient rule[8] for round figures

$$\text{area} = \frac{\text{perimeter} \times \text{width}}{4}. \tag{15}$$

When this is applied to the double-segment of area $2A$, we get, by (14),

$$2A = 2(c + h) \cdot \frac{2h}{4} \tag{16}$$

thereby getting the expected rule (13).

The formula (13) was used in a Demotic mathematical papyrus of Hellenistic Egypt. It is found in the Papyrus Cairo (JE 89127–30 and 89137–43) written during the third century BC[9] In its following equivalent form

$$A = \frac{ch + h^2}{2} \tag{17}$$

[7] [Gupta 2001] contains details.

[8] For circular areas the rule is found in Umāsvāti's *Bhāṣya*, 170 and in the *Jiu Zhang Suanshu* (I.32) (c. AD 100). See [Lam 1994, 13]; also see [Hϕyrup 1996, 21–3], and [Hayashi 1990, 5].

[9] [van der Waerden 1983, 39–40, 172–7].

the same rule (13) is found in the famous Chinese classic *Jiu Zhang Suanshu* ('Nine Chapters on mathematical Art') written in the Han Period (206 BC to 221 AD).[10]

The inaccuracy of the formula (13) was known to the Greek mathematician Heron (AD first century). He attributed the rule[11] to 'the ancients' and conjectured that it arose by taking $\pi = 3$ in which case it gives the correct area for the semicircle. He further says that those who wanted a better result applied the formula

$$A = (c + h) \cdot \frac{h}{2} + \frac{1}{14} \cdot \left(\frac{c}{2}\right)^2. \tag{18}$$

Since for a semicircle ($c = 2r$, $h = r$), this will give

$$A = \frac{11}{7} r^2, \tag{19}$$

it is clear that the correction term in (18) was added by those who accepted the Archimedean value $\pi = \frac{22}{7}$. However, Heron adds that the use of (18) should be restricted to the range where

$$2 \le \frac{c}{h} \le 3. \tag{20}$$

Where $\frac{c}{h} > 3$, Heron recommends the formula

$$A = \frac{2ch}{3} \tag{21}$$

which is based on assuming the circular arc (of the segment) to be approximated by a parabolic arc in the Archimedean style.[12] Further the *Mensurae (= De mensuris)* attributed to Heron contains two more formulas [Heath 1981, 330]

$$A = \frac{h}{2} \cdot (c + h) \left(1 + \frac{1}{16}\right) \tag{22}$$

$$A = \frac{h}{2} \cdot (c + h) \left(1 + \frac{1}{21}\right) \tag{23}$$

to be used for segments which are smaller and bigger than a semicircle, respectively (for semicircle itself the second rule gives exact result with $\pi = \frac{22}{7}$).

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[10] [Lam 1994, 13] and [van der Waerden 1983, 36–40].

[11] [Heath 1981, 330].

[12] When h is small we can neglect h^2 in (1) to get $\left(\frac{c}{2}\right)^2 = dh$ which becomes the parabola $y^2 = dx$ with a proper choice of coordinate axes.

The formula (18) is mentioned by the Roman agrimensor Columella (AD first century)[13] and is also found in the Hebrew work *Mishnat ha-Middot* which is attributed to Rabbi Nehemiah (c. AD 150).[14]

3 The Classical Rule (13) in India

It has been pointed out above that in Heron's view, the classical rule (13) based on $\pi = 3$, and it was modified to the form (18) by those who preferred value $\pi = \frac{22}{7}$. A similar thing happened in China. The formula

$$A = \frac{ch + h^2}{2} + (\pi - 3) \cdot \frac{c^2}{8} \tag{24}$$

is found in the *Siyuan yujian* (AD1303) of Zhi Shijie.[15] It is his improvement of the Chinese form (17) with $\pi = \frac{22}{7}$ and $\frac{157}{50}$.

In India, modification was done in a different manner which may be compared with those that are represented by (22) and (23). We shall describe them here. But first of all it may be mentioned that the classical rule (13) itself as such as is found in the *Gaṇita-sāra-saṅgraha* (VII, 43) of Mahāvīra (c. AD 850) as well as in the *Triloka-sāra* (*gāthā* 762) of Nemicandra (tenth century).[16] Both these authors use $\pi = 3$ as a rough approximation and hence specify the said rule also so which is quite natural. The rule (13) along with (14) is also found in the *Gaṇitakaumudī* (IV, 12) of Nārāyaṇa Paṇḍita (1356) for segments which are smaller than a semicircle.[17]

A rule given by Śrīdhara (c. AD 750) in his *Triśatikā* (*sūtra* 47) is perhaps the earliest modified form of (13) found in India. He says[18]

जीवाशरैक्यदलहतशरस्य वर्गं दशाहतं नवभिः।
विभजेदवाप्तमूलं प्रजायते कार्मुकस्य फलम् ॥

Take ten times the square of the product of the arrow and half the sum of the chord and arrow, and divide by nine. The square-root of the quotient (so obtained) gives the area of the bow-figure.

That is,

$$A = \sqrt{\left(h \cdot \frac{c+h}{2}\right)^2 \left(\frac{10}{9}\right)}. \tag{25}$$

[13] [Heath 1981, 303], and [Hϕyrup 1996, 13, 16].

[14] [Midonick 1968, 197]. For controversy about date and authorship of the Hebrew work, see [Katz 1993, 152] and [Hϕyrup 1996, 25].

[15] [Martzloff 1997, 327].

[16] [L. C. Jain 1963, 190], and [*Viśuddhamati* 1975, 597].

[17] [Hayashi 1990].

[18] [Dvivedi 1899, 35].

Clearly this is a modification of (13) based on an adjustment of π from the rough value $\pi = 3$ to the better value $\pi = \sqrt{10}$ which is used in the *Triśatikā* itself (*sūtra* 45) and the accompanying example. The formula (25) is also found in the Prakrit work *Gaṇita-sāra* (III, 46) of Ṭhakkura Pherū who, being a Jaina, describes $\pi = \sqrt{10}$ as exact (III, 43).[19]

In the *Mahā-siddhānta* (XV, 89) of Āryabhaṭa II the equivalent of formula (25) is given as a rough rule[20] and the corresponding value $\pi = \sqrt{10}$ is used in the preceding verse (XV, 88). For accurate area, he gives (XV, 93) a verbal rule equivalent to the formula (23) with the corresponding value $\pi = \frac{22}{7}$ in the preceding verse (XV, 92)[21]. Actually Āryabhaṭa II's form for (23) is

$$A = 22 \cdot \frac{c+h}{2} \cdot \frac{h}{21}. \tag{26}$$

It is clear that (25) and (26) imply a modification of (13) to some desired value of π, thereby yielding the general prototype form

$$A = (c+h) \cdot h \cdot \frac{\pi}{6}. \tag{27}$$

Now the Jainas frequently used the approximation formula [Gupta 1975, 43]

$$\sqrt{a^2 + x} = a + \frac{x}{2a}. \tag{28}$$

This will give the approximation $\pi = \frac{19}{6}$ for $\sqrt{10}$ (with $a = 3, x = 1$). The value $\pi = \frac{19}{6}$ itself is found[22] in the *Gaṇita-sāra* (III, 45) of Ṭhakkura Pherū. For such π, the typical rule (27) implies the formula

$$A = (c+h)\left(\frac{h}{2}\right)\left(1 + \frac{1}{18}\right) \tag{29}$$

which is fact reported to be found in the anonymous Indian work *Pañcaviṃśatikā* (c. 1400 or earlier) [Hayashi 1991, 399, 411, 436–7]. In addition to (28), the following formula was also used in India [Gupta 1985, 14, 17]:

$$\sqrt{a^2 + x} = a + \frac{x}{2a+1}. \tag{30}$$

[19] [Nahata & Nahata 1961, part II, 56].

[20] [Dvivedi 1910, 171] where the reading in the footnote is correct and accepted here.

[21] [Dvivedi 1910, 172] [Billard 1971, 157–62] shifts the date of the *Mahā-siddhānta* to early sixteenth century. Also see [Mercier 1993].

[22] The use of $\pi = \frac{19}{6}$ is found earlier in the *Tiloya-paṇṇatī*, I, 118 (see *Viśuddhamatī* 1984, 26]) and elsewhere (see [Hayashi 1991, 333–5]).

This easily enables us to get $\pi = \frac{22}{7}$ from $\pi = \sqrt{10}$ which was well known in India[23]. In turn we get (26) as a modification of (13).

It may be that practical geometers or surveyors found that (13) always yielded results in defect of the actual area for segments smaller than a semicircle or even for semicircle (because they knew that the actual value of π was greater than 3). So a modifying factor f might have been thought to be a remedy. That is, for better results, a suggested modification could be in the form

$$A = (c + h)\left(\frac{h}{2}\right) \cdot f \tag{31}$$

where f slightly greater than one, or more specifically of the form

$$f = 1 + \frac{1}{N} \tag{32}$$

where N is a suitable positive integer. The choice of $N = 21$ in (23) and of $N = 18$ in (29) was made from a consideration of approximation to π. If the same criterion is applied to $N = 16$ in (22), it would imply the value $\pi = \frac{51}{16}$ which is nowhere mentioned. So there may have been some other consideration for choosing $N = 16$. Of course for a chosen integer N, the corresponding implied value of π will be given by

$$\pi = 3 + \frac{3}{N}. \tag{33}$$

It is possible that a simpler integer was selected for making the calculations more convenient but without affecting the result much. In India, the case $N = 18$ and 21 are found in (29) and (26). It may be that for convenience of computation, the choice of $N = 20$ was found to be good because it is a decimally simple integer between 18 and 21 (in fact near the better $N = 21$). The new choice yields

$$A = (c + h)\left(\frac{h}{2}\right)\left(1 + \frac{1}{20}\right). \tag{34}$$

And it is interesting to note that this formula was in fact popular in India in the fifteenth century. It is said to have been used by Viṣṇu Paṇḍita (c. 1410)[24]. Gaṇeśa in his commentary (1545) on *Līlāvatī* (rule 213) states that it was known to his father Keśava (c.1500)[25]. The Sanskrit text for (34) is

[23] [Gupta 1991]. Alberuni credits Brahmagupta (fl.628) with a knowledge of $\pi = \frac{22}{7}$ (see [Sachau 1964, Vol. I, 168]). It was also known to Śrīdhara [Hayashi 1985, 755].

[24] [Datta & Singh 1980, 167].

[25] [Apte 1937, part II 218].

चापे फलं शरो ज्येषु योगार्धघ्नो नखांशयुक्

In a bow-figure, the area is the product of the arrow and half the sum of the chord and the arrow, increased by its 20th part.

That is,

$$A = h \cdot \frac{c+h}{2} + \frac{1}{20} \cdot \left(h \cdot \frac{c+h}{2} \right). \tag{35}$$

Interestingly, the same Sanskrit hemistich is found in the *Gaṇitapañcaviṃśī* (*sūtra* 25) which is attributed to Srīdhara but whose authorship of the extant work is said to be doubtful.[26] If Śrīdhara was the original author of (35), it is likely he got it from his own formula (25) by simplification and using typical ancient Indian techniques of surd-computation to produce[27]

$$\frac{\sqrt{10}}{3} = \frac{\sqrt{4000}}{60} = \frac{63}{60} = \frac{21}{20} = 1 + \frac{1}{20}, \tag{36}$$

the required form.

4 Special Jaina Rule for the Area of a Segment

In the Jaina school of Indian mathematics, the computation of the area of the bow-figure was carried out also in another way especially for cosmographical purposes. The method is based on a formula which is explicitly mentioned in the *Tiloya-paṇṇattī* (IV, 2401) of Yativṛṣabha in the following verbal statement:[28]

इसुपाद-गुणिद-जीवा गुणिदव्वा दसपदेन जं वग्गं ॥
मूलं चावायारे खेत्तेत्थं होदि सुहुम-फलं ॥

The square of the product of a quarter *iśu* (= h) and chord (= c) is multiplied by ten. The square-root of the result is the accurate (*suhuma*) area of the bow-figure.

That is,

$$A = \sqrt{10 \left(\frac{c \cdot h}{4} \right)^2}. \tag{37}$$

We can say that this formula is based on $\pi = \sqrt{10}$ because in the case of a semicircle ($c = 2r, h = r$) it gives the exact area $\frac{\pi r^2}{2}$ for that value of π . The formula (37)

[26] [Pingree 1979, 903]; and [Hayashi 1985, 751–6].

[27] For Śrīdhara's rule $\sqrt{N} = \frac{\sqrt{Na^2}}{a}$, see [Shukla 1959, 175] and *Triśatikā* rule 46.

[28] [*Viśuddhamatī* 1986, 636]. *Yativṛṣabha* is placed between AD 473 and 609.

is also found in the *Bṛhatkṣetrasamāsa* (I, 122) of Jinabhadra Gaṇi. (fl. 609 AD).[29] The first half of a *gāthā* quoted by Bhāskara I in his commentary (629 AD) on the *Āryabhaṭīya* (under II, 10) reads[30]

इसुपायगुणा जीवा दसिकरणि भवेद् विगणिय पदं

The product of the chord and a quarter of the arrow, when further multiplied by the square-root of ten, becomes the area of the bow-figure. That is,

$$A = \sqrt{10}\, c \cdot \frac{h}{4} \tag{38}$$

which is just a simplified form of (37). This form of the formula is also found in the *Gaṇita-sāra-saṅgraha* (VII, 70) of Mahāvīra[31] as an accurate rule, whereas (13) is given as an approximate rule (VII, 43). This is also the case with another Jaina work, the *Triloka-sāra* of Nemicandra (*gāthā* 762).[32]

By making use of (37) the areas of various geographical regions (into which the Jambū Island is divided) were obtained. These areas are found in the *Tiloyapaṇṇatti* (IV, 2402–9) itself and have been shown to be in complete agreement with those which the present writer computed by applying (37) and then simplifying the calculations in the Jaina style.[33] It seems that the formula (37) is older than the *Tiloyapaṇṇatti*.

It may be pointed out that the corresponding formula of the type (37), which is consistent with the rough value $\pi = 3$, would be

$$A = 3c \cdot \frac{h}{4}. \tag{39}$$

But it is significant to note that the Jaina authors of *Gaṇita-sāra-saṅgraha* and *Triloka-sāra* (both of which give $\pi = 3$ as a rough value) did not give the formula (39). Instead, both of them preferred to prescribe the popular classical rule (13) for rough calculation. Why was it so? This point is relevant because the formula (39) gives better results than (37) or (38) and even than[34]

$$A = \pi c \cdot \frac{h}{4} \tag{40}$$

except for segments which are nearly semicircular.

A more important question is the source or derivation of the peculiar rule (37). It is still likely that (37) was a modification of (39) from $\pi = 3$ to $\pi = \sqrt{10}$. So we consider the rationales of the simple rule (39).

[29] [Anupam Jain 1990, 163].

[30] [Shukla 1976, 73].

[31] [L. C. Jain 1963, 198].

[32] [Viśuddhamatī 1975, 597]; see Sect. 3.

[33] [Gupta 1987, 52–3].

[34] [Gupta 1989, 21–2].

Averaging[35] was a usual and simple method to get empirical results. In Fig. 3, situated on the same base PQ, the area of the inscribed triangle PNQ is $\frac{ch}{2}$, and that of the outer rectangle $PSRQ$ is ch. Taking the average of these two areas,[36] we get (39) by regarding the segmental area lying midway between those of the triangle and rectangle.

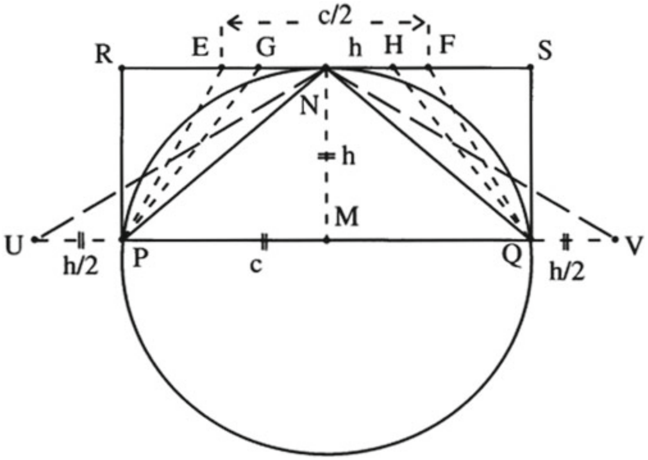

Fig. 3 Mean of two areas

Geometrically the above averaging process amounts to equating the area of the segment with that of the trapezoid $PEFQ$ where $EF = \frac{c}{2}$. Incidentally it may be pointed out that such a technique of replacing a given figure by a simpler figure that is assumed equivalent has often been taken as an explanation or rationale of ancient formulas. Thus the popular classical rule (13) follows by equating the area of the segment to that of the trapezoid $PGHQ$, where $GH = h$. On the other hand, [Eves 1983 11] obtained (13) by equating the segmental area to that of triangle UNV, where $UP = QV = \frac{h}{2}$.

5 Karavinda's Segment-Area Rules

In closing, some rules found in Karavinda's commentary on the *Āpastamba Śulbasūtra* may be mentioned. One verbal rule reads[37]

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[35] [Gupta 1981] contains a survey on averaging.

[36] Ancient mathematicians may have easily noted that the area of a semicircle ($\frac{3r^2}{2}$ with $\pi = 3$) is in fact the mean of the areas of the triangle ($= r^2$) and outer rectangle ($= 2r^2$) on the same base.

[37] [Srinivasachar & Narasimhachar 1931, 124].

शराहतस्तु कोदण्डो दलितो धनुषः फलम् ।

Half of (the product of) arc as multiplied by arrow is the area of the bow-figure.

That is,

$$A = \frac{s \cdot h}{2} \tag{41}$$

According to Datta,[38] the above Sanskrit line gives the accurate area of the sector of a circle. His interpretation is wrong because *śara* ('arrow') is usually taken as the height of the segment (and not the radius of the circle). However, he is right in pointing out that another rule

$$A = \frac{s}{2} \cdot \frac{h}{2} \tag{42}$$

which Karavinda quotes is incorrect.[39] If we apply the ancient rule (15) to the double-segment in Fig. 2, we get

$$2A = \frac{2s \cdot 2h}{4} \tag{43}$$

which at once gives (41). The analogy of segment with semicircle may be used by writing the latter's area as

$$A = \frac{\pi r^2}{2} \approx \frac{3r^2}{2} = \frac{3r \cdot r}{2} = \frac{3\,(2r \cdot r)}{4}.$$

In the last two expressions, replacing $3r$, $2r$ and r by s, c and h (as is true for the semicircle), we get (41) and (39) for the segment analogously.

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[38] [Datta 1991, 17]. Datta's interpretation is good when the bow-string PMQ (Fig. 1) is assumed to take the position POQ in the pulled state.

[39] Ibid., and [Srinivasachar & Narasimhachar 1931, 56]. [Chakravarti 1934, 28] says that the formula $A = \frac{ch}{2}$ is found in the *Śulbasūtras* but gives no details.

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Yantras or Mystic Diagrams: A Wide Area for Study in Ancient and Medieval Indian Mathematics

As an appliance, *yantra* may be an astronomical or surgical instrument, or a machine or mechanical device. In religion and mysticism, *yantra* is a diagram containing geometrical drawing and mystical symbols including *mantras*, letters, numbers and other figures. These mystic diagrams are used in worship, meditation, and ritual practices. They have been also used for protection against ill effects of evil spirits, diseases and planets, and even for *abhicāra* (malefic practices). Mathematical magic squares *(aṅka-yantras)* and other magical figures are also included in them.

The present paper deals with various aspects of *yantras* including traditional views, classification and technical terminology along with appropriate historical remarks. The famous and profound *śrīyantra* has been given special attention. Other *yantras* discussed include Gaṇeśa, Durgā, Rudra, Bhauma (related to planet Mars) and the beautiful *Sarvatobhadra-yantra*.

Detailed discussions of *yantras'* construction and of the mathematics involved are there in the paper. Full references to original Sanskrit texts and profuse illustrations are included here. There is a list of one hundred important *yantras* (with references) and a glossary of technical terms. It is hoped that this general study of *yantras* will motivate further studies and research and will serve to draw attention of scholars to the somewhat hitherto neglected area of the history of ancient and medieval science in India.

1 Introduction: *Yantras* in General

Regarding the Sanskrit word *yantra*, quite a few etymological connections and explanations are found in different works. Apte[1] gives the root *yantr* which means to check, restrain or fasten from which the verbal forms *yantrati, yantrayati* follow. According to Monier-Williams[2], the root *yantr* (*Dhātupāṭha* XXXII. 3) itself is rather a nominal verb from the word *yantra*. In general *yantra* is said to mean that which checks or restrains.

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According to *Vācaspatyam*[3] and Sanskrit *Dhātu Sāgara Taraṇiḥ*,[4] the word *yantra* is connected to the root *yantri* (to curb or check). It has been also connected with the root *yam* which is used in somewhat similar sense.[5] Rao[6] gives derivation of *yantra* from *yam* as well as from the above two verbal forms. Thus the Sanskrit word *yantra* usually means any appliance or apparatus, contrivance, or device, engine or machine, implement or instrument in general. Depending on the context, it may specifically denote an object of any of the above type in different areas of Indian sciences in a broad sense.

In *Gaṇita-jyotiṣa* (mathematical astronomy), the astronomical instruments have been called *yantras* in general. Yukio Ohashi's doctoral work[7] *A History of Astronomical Instruments in India* is very comprehensive on such *yantras*. The earliest of these are the *nara-yantra* or *śaṅku* (gnomon) and the *ghaṭikā* or *ghaṭī-yantra* a which is also called *jala-yantra* (clepsydra).

The traditional Siddhāntas (Sanskrit works on astronomy) deal with a number of astronomical *yantras* including the *gola-yantra* (celestial globe or armillary sphere). The staff-type instruments were *yaṣṭi-yantra, nalaka* or *nālakā, śalākā, śakaṭa*, etc. Under the round-type are put *cakra, dhanur* or *cāpa, turya* (qudrant), *bhugaṇa* or *nāḍīvalaya, kartarī, kapāla, pīṭha* and Āryabhaṭa's *chatra-yantra*.[8] Bhāskara II's *phalaka-yantra* (board instrument) is his own invention and his *dhī-yantra* is called *buddhi-yantra* by Munīśvara.[9] The *yantrarāja* (astrolabe) was indeed 'king' among *yantras*.

List of other Indian astronomical instruments include the *dhruvabhrama-yantra, diksādhan-yantra* (Padmanābha), *kaśā-yantra* (Hema), *pratoda* or *cābuka-yantra* and *sudhīrañjana* (Gaṇeśa). The Sanskrit manuscript *Yantra-prakāra* (in City Palace, Jaipur) is said to list more than a dozen astronomical instruments including *jayaprakāśa, krānti-vṛtta, palabhā-yantra, digaṃsa-yantra, śara-yantra, agrā-yantra, yāmyottara-bhitti, rāśī-valaya* and *Sudas Phakarī* (= *suds fakhrī*) also called *ṣaṣṭhāṃśa* (sextant).[10]

The Jaipur Observatory is the biggest and best preserved among the five observatories of Jai Singh and has two dozen instruments. According to *Zīj-i Muhammad Shāhī* (1733/1738 AD), Jai Singh himself invented the *Jayaprakāśa-yantra* (named after himself), *Rāmaprakāśa-yantra* (named after his grandfather Rāmasiṃha) and *Samrāṭ-yantra* (named after his guru Jagannātha Paṇḍita).[11]

In ancient and medieval times, mathematics was intimately connected with astronomy and the twin mathematical sciences contributed significantly to their mutual development. The theory and construction of astronomical *yantras* involved a lot of mathematics. So a study of the works on such *yantras* (instruments) and analysis of the principles on which the *yantras* were based cannot be neglected while dealing broadly with the history of mathematics of the time in India.

Since our concern here is more about the mathematics involved in some specific type of *yantras*, so a few other type of *yantras* are only briefly mentioned now. In the traditional *Rasāyana-śāstra*, different types of apparatus used in processing of medicines *(auṣadha)*, other preparations (*rasas*, etc.) were called *yantras*. Dozens of such *yantras* are known such as *dola-yantra, ḍeki-yantra* (for distillation), *bhūdhara-yantra, vidyādhara, ḍamarū, nālikā, ghaṭa-yantra*, etc.[12] In ancient Indian system of surgery *(śalya)*, the term *yantra* was applied to the surgical instruments. The *Suśruta-saṃhitā* mentions many such *yantras* such as *śālākya-yantra, tāla-yantra, saṃdaṃśa-yantra, nādi* (tabular) and there were also *upa-yantras* (accessory appliances).[13]

Among the various mechanical devices which were called *yantras*, mention may be made of the *kūpa-yantra* (for drawing water), *taila-yantra* (for extracting oil) and *dāru-yantra* (wooden puppets). The *Yantra-Sarvasva* of Bhāradvāja (manuscript at Baroda) is said to describe a few *yantras*.[14]

2 Mystic Diagrams

For certain meditation and ritual practices (especially in Buddhism and Tantric Hinduism), frequent use is made of a variety of diagrams with mystic and magical designs. These mystic diagrams (or figures) comprise some sort of graphical representations involving geometrical drawings and designs and are called *yantras*. Usually they contain a few particular numbers, letters or words which may form some *mantras* (mystic formulas) or their symbolic representations. Often figures and symbols representing objects and ideas which have religious, mystical and philosophical significance are also included in such *yantras*. Examples of such objects are the so frequently used lotus (*padma*) which is a symbol of purity, trident *(triśūla)* which represents the vector of energy, and *vajra* which is Indra's divine weapon and which is also a symbol of highest intellectual power in the Vajrayāna Buddhist School.[15]

These mystic diagrams *yantras* may be broadly classified into several categories such as *pūjana yantras, mantra-yantras, rakṣā yantras* and a type which are called malefic *yantras*. Of course the employment of *yantras* from a variety of objectives and other various purposes is so wide and divergent that it will be difficult to have an exhaustive and non-overlapping classification.

The *pūjana yantras* are used in worshipping or actualizing divinities. They are deity-specific, i.e. each divine form is associated with a *yantra* of its own. Thus *Durgā yantra* and *kālī yantra* are different. Even minor deities have their separate *yantras*. Often more than one version of *yantra* is associated with a deity, more so when the purpose of the *yantra* is different. A *dhyāna-yantra* may serve as a visual aid for the concentration of mind in meditation.

The *rakṣā yantras* are meant to provide protection for a variety of ills and dangers. Their wearing is said to pacify the troubles arising out of diseases and destroy the evil effect produced due to unfavourable position of astrological planets *(grahas)*. When such *yantras* are worn by a person on his body (as amulet or talisman), they are called *dhāraṇa yantras*. For a deity, the *pūjana* and *rakṣā yantras* may be different but it is often feasible to combine them.

The malefic *yantras* are used for *abhicāra* ("destructive magic") such as sorcery, witchcraft and black magic. Usually they are used for seven specific objectives:

stambhanam (arresting the movement or speech of opponent), *mohanam* (attracting affection by coercion), *uccāṭanam* (upsetting enemy by occult influence), *vaśī karaṇa* (controlling by magic and hypnotism), *Jṛmbhaṇam* (terrorizing opponent), *vidveṣaṇam* (causing enmity among friends) and *māraṇam* (causing death).

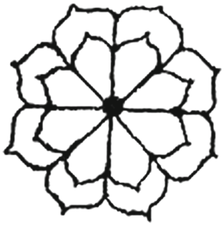


Fig. 1 Double lotus

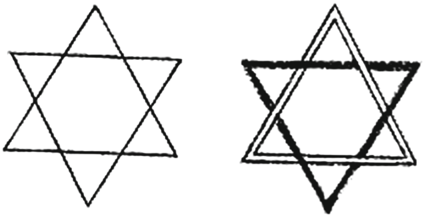


Fig. 2 Hexagram and Solomon's seal

To take a simple example, the *Puraścaryārṇava*[16] contains the statement *aṣṭadala-kamala-dvayātmakaṃ candrayantram*, 'The Moon mystic diagram consists, of the figure of double eight-petalled lotus'. This is shown in Fig. 1 which should be, as usual, enclosed by a decorated square called *bhūpura*, and which is comparable to the *yantra* of the sun.[17]

The hexagram (Fig. 2) is called *Ṣaṭkoṇa* ('six-angled') in Tantric literature and is the basic figure in many mystic diagrams especially for the malefic *yantras*.[18] It is interesting to note that Solomon's seal of the hexagram form has been used in the western culture also as an amulet especially against fever.[19]

Maṇḍala is another important term in connection with *yantras*. A simple figure consisting of square inscribed by a circle (which itself has an inscribed equilateral triangle) is called *kalaśasthāpanā maṇḍala*.[20] The term *maṇḍala* is also applied to special type of mystic diagrams which consists of concentric circles interwoven with lotus petals. *Maṇḍala* (*dkil-dkhoṛ* in Tibetan) as a mystic diagram is one of the most important objects of Lamaist meditation and worship.[21] The Tibetan *Śrīcakra-sambhāra maṇḍala* is dedicated to god Heruka who is personified as *Nirvāṇa*.[22]

Many of the *yantras* were in the form of what are now called Latin and Magic squares. They were called *aṅka-yantras* (numerical diagrams). The *Ṇamokāra yantra* (Fig. 3) is basically a Latin square.[23]

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

Fig. 3 *Ṇamokāra yantra*

The use of magic squares of order three for pacifying the astrological nine planets *(navagrahas)* has been prescribed by the legendary writer Garga. The nine magic squares for the purpose are shown in Fig. 4 concisely from which *yantras* associated with sun, moon, mercury, jupiter, venus, saturn, rāhu and ketu can be obtained by taking $x = 0$ to 8, respectively. In the *Bṛhaddaivajña-rañjana*[24], the verses containing these magic squares and credit to Garga, etc. are quoted from *Yantra-cintāmaṇi*.[25]

$6 + x$	$1 + x$	$8 + x$
$7 + x$	$5 + x$	$3 + x$
$2 + x$	$9 + x$	$4 + x$

Fig. 4 General *navagraha yantra*

Among the 4th-order *aṅka-yantras*, the available evidences show that the Indian had an early interest in pandiagonal magic squares.[26]

The *aṅka-yantras* of Fig. 5 was possible used by Varāhamihira (fifth century AD.)[27] and that of Fig. 6 was carved on the lintel of an eleventh century temple at Dudhai (then in Jhansi district) and is still found in an inscription at the famous Khajuraho (100 miles east of Jhansi).[28] It seems that early peoples were astonished to find the peculiarly wonderful arrangements of numerical figures in the form of magic squares. They were influenced, and attributed some magical powers to the arrangements. Hence they were frequently employed as *yantras* (mystical diagrams).

But soon the properties of magic squares attracted mathematicians both as a source of recreational mathematics and as a branch of pure mathematics (combinatorics). By now the subject of these *aṅka-yantras* is vast and their genesis and growth form a significant part of history of the development of mathematics.

10	3	13	8
5	16	2	11
4	9	7	14
15	6	12	1

Fig. 5 *An aṅka-yantra*

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Fig. 6 *A temple yantra*

3 Some Traditional Views and *yantras*

In ancient India, arts and sciences were handmaiden of religion. Almost all the sciences have been attributed to a divine origin. This attitude (and practice) automatically attaches a hoary past to the genesis and beginning of those sciences. It also puts a stamp of unquestionable authority on the so-called *apauruṣeya* works, i.e. those works which are attributed to ancient sages although they are composed by ordinary human beings.

Thus, the exposition of Chap. 54 (on astronomy and mathematics) in the *Nārada-purāṇa* commences with the line.[29]

ज्यौतिषाङ्गं प्रवक्ष्यामि यदुक्तं ब्रह्मणा पुरा ।

(Sanandana Says) I shall now set out the *Jyotiṣa* portion which was enunciated in antiquity by (god) Brahmā.

The *ghaṭīyantra* is attributed to the same god:[30] *mukhyaṃ tvamasi yantrāṇāṃ brahmaṇā nirmitaṃpurā*.

Nārāyaṇa Paṇḍita begins chapter on magic squares in his *Gaṇita-kaumudī* (1356 AD) by stating that the subject was taught to Manibhadra by Lord Śiva.[31] In fact all *yantras* or mystic diagrams, as explained by Mahindhara in his auto-commentary on *Mantra-mahodadhi* (XX.1, p. 180),[32] were told by Lord Śiva to his consort Gauri.

Another characteristic of religious domination of Indian history and culture is to trace the beginning of everything to *Vedas* which are taken to be the fountainhead of all knowledge whether past, present or future. And since Vedas themselves are regarded to be God's words, the origin of all *vidyās* (arts and sciences) whether sacred or secular are attributed a divine origin. Nīlakaṇṭha Caturdhara (seventeenth century AD) has claimed that the practice of generating *aṅka-yantras* (magic squares) is hinted in certain *Ṛg-vedic* verses.[33]

The Vedic tradition of construction and mensuration of plane geometrical figures existed in India since quite ancient times in connection with the erection of śrauta (i.e. Vedic) *agnis* and *citis* (fire-altars) which are dealt and discussed in the *Śulba-sūtras* in great details. Later on the mathematics of the plane geometrical diagrams is also met in the construction and calculation related to *Kuṇḍas* (fire-pits) and *maṇḍapas* of the *smārta* tradition which became somewhat more popular and practical in medieval India. Thus the mathematics needed for the construction of the *tāntric cakras, maṇḍalas* and *yantras* may be considered as a continuation and extension of the earlier traditions. It involved the application of the Indian geometrical knowledge related to plane figures including circles, triangles, polygons, lotuses and other flowery designs obtained by combining these figures in various ways. Such diagrams have specifically direct relevance to the history of geometrical knowledge in broad terms and reflects an aspect of application of ancient and medieval Indian mathematics in a field different from astronomy.

Most of the mystic diagrams to be considered here in detail are the *pūjana-yantras* used in worshipping various divinities. Their importance is clearly stated in the fact that[34]

विना यन्त्रेण चेत् पूजा देवता न प्रसीदति ।

A worship without *yantra* does not please the deity.

Correctness in forms as laid down and of dimensions as prescribed is significant while drawing the geometrical diagrams whether they are related to the *śrauta* or *smārta* or tantric rituals. Otherwise desired objective may not be achieved and even adverse effects might be caused. For instance, regarding the area of a *kuṇḍa*, a warning reads[35]

मानाधिक्ये भवेद् रोगो मानहीने दरिद्रता ।

When the area is more (than the prescribed amount), there will be disease; when it deficit, there will be poverty.

Similarly for drawing (or engraving) a mystical diagram, the straight lines must be made perfectly; otherwise, poverty may be caused instead of *lakṣmī* (wealth) as is reflected in the statement[36]

ऋजुलेखे भवेल्लक्ष्मीः, वक्ररेखे दरिद्रता ।

Interestingly, it is also said that a *yantra* (magic diagram) is to be drawn by freehand and not by the use of instruments.[37]

A noted attitude of Indian mind which affected the speedy growth and propagation (transfer and transmission) of all branches of ancient knowledge was the practice of *gopanīyatā* or protected and hidden secrecy. An astronomical correction, called *Bīja-saṃskāra*, was found in some manuscripts (which were used by its commentators Raṅganātha and Viśvanātha) of the *Śūrya-siddhānta*, the most famous Indian work on astronomy. The correction is stated to be *gopanīyam* as it is to be taught only to a well-tested pupil and not to others.[38] The sacred and mystic sciences of *tantra, mantra* and *yantra* are given such treatment of well-guarded protection more strictly. The *Ṣaḍakṣarī Vidyā* is not to be given to others even if one has to sacrifice his "state *(rājyam)*, son, wife, life, etc." so says the *Nārada-pañcarātra*.[39]

The recitation or muttering *(japa)* of a *mantra* (formula of prayer) is a significant Hindu method of worshipping any deity. Some of these *mantras* are to be written down (or engraved) on suitable plates of suitable materials. The resulting documents are called *mantra-yantras* (mystic diagrams of *mantra*) which are also used for the worship of the *mantras* themselves. The importance of *mantras* is clear from the ancient saying that *"Siddhavaidyastu-māntrikaḥ"* which implies that *mantras* were believed to have some role in medical treatment. In fact the triple path or means of *tantra, mantra* and *yantra* was used for sacred as well as secular objectives.

Although *mantras* are not to be translated, their original forms must be written and pronounced correctly. Incorrectly written *mantras* or their *bījākṣaras* (mystic or seed letters which serve as algebraic symbols) on the *mantras* may lead to adverse results. We need not only to have a correct understanding of construction of *yantras* but to know the correct meaning of the technical terms used. The language and symbology used in *tāntric* tradition of writing, worshipping and performing rituals is quite complicated. A handy glossary is required for reference.

4 Technical Terms and Symbols

Every art and science has its own terminology and symbology. Without a knowledge of relevant technical terms and symbols used in any specific area of study, a clear understanding of its various topics and matters is not possible. Some simple examples will be mentioned here for illustration taking the specific case of the technical term *'manu'* for expository clarification.

Scholars of History of Science are familiar with the usual various systems of expressing numbers using Sanskrit words and letters of alphabet. These include the popular *bhūta-saṃkhyās* (word-numerals), Āryabhaṭa I's special alphabetic system and the famous *Kaṭapayādi-nyāya* so frequently used in ancient and medieval Indian mathematical sciences.[40]

In Indian mythological history, mention is made of 14 successive progenitors and sovereigns of Earth who are called Manus. So, as a *bhūta-saṃkhyā*, the Sanskrit word *manu* is used for 14, just as *agni* (fire) stands for 3, *veda* for 4, etc.

Thus in a description of famous *Śriyantra* (see next section), we come across the line

मन्वस्रनागदलसंयुतषोडशारम्

(The *yantra*) has 14 corners with (lotuses) of 8 *(nāga)* and 16 petals.

One of the usual meanings of the word *manu* is *mantra* (formula). *Mantra Mahodadhi* X. 71 of Mahīdhara (1588 AD)[41] has the line

वेदरुद्राक्षरो मनुः

A *mantra (manu)* of 114 letter.

(Here 114 comes from *veda* = 4 and *rudra* = 11 written from right to left according to convention).

As a technical term *manu* is also used as a big period of time, there being 14 such *manus* in the bigger astronomical period called *Kalpa*. Āryabhaṭa I (born 476 AD) puts the equation as

> काहो मनवो ढ
>
> A day of Brahmā (or a *Kalpa*) has *ḍha* or 14 (*ḍha* = 14 according to Āryabhaṭa's system) *manus*.

It may be mentioned that at present we are living in the period of the 7th Manu (called *Vaivasvata*). In the above equation, Brahmā is denoted by the single letter *ka*. But in the vital word $Om(= a + u + m)$ which is symbol of Hindu Trinity, he is denoted by *ma*.[42] Also it may be noted that, as a combination of letters *ma* and *nu*, the phrase *manu* will denote the number 200025 according to Āryabhaṭa I's alphabetic system, but will stand for the number 05 according to the well-known *Kaṭapayādi* system.

For Tantric literature and for matters related to *tantra-mantra-yantra* in general, a special type of glossary is also needed. Various *mantras* (mystic formulas) are almost invariably inscribed on different *yantras* (mystic diagrams). Due to want of space, these *mantras* are frequently given in abbreviated forms which are usually called *bīja-mantra* and *bījākṣara* (seed or algebraic letters). These letters are evolved by certain syncopation and other processes. In fact we have works like *Mātṛkā-nighaṇṭu*, which are a sort of *ekākṣara koṣa* (dictionary of one-letter words). For example, words *bhṛguḥ* and *haṃsaḥ* both denote letter *sa*, and the letter *ha* is denoted by *nabhaḥ* (sky) and its synonyms.[43]

The set of five monograms or mystic letters representing germ or seed of *mantra* for the five metaphysical elements (pañca mahā-bhūtas) are *laṃ* for *kṣiti* (earth), *vaṃ* for *jala* (water), *raṃ* for agni (fire), *yaṃ* for *vāyu* (wind) and *haṃ* for *gagana* (sky, ether or space) as mentioned above. Complicated *tāntric* expressions are often used (especially for poetic use) to denote *bījas* and *bījamantras*. For example, take the phrase.[44]

> अग्नीन्दुशान्तियुग्वियत्
>
> which literally means "sky with fire, moon, and peace", but whose actual contextual meaning is entirely different. It is as follows: Here *agni* (fire) stands for the seed letter *r (reph), indu* (moon) for *anusvāra śānti* (for vowel), and *viyat* (sky) for *h*. So the above phrase means "The letter *h* with *r*, mātrā and anusvāra"

That is, the *śakti-bīja "hrīṃ"* (ह्रीम्).

This highly technical (sacred and secret) symbology must be noted in tantric context. Otherwise as the usual word-numerals, *agni* denotes 3, *indu* 1 and *viyat* 0.

As another example take the phrase[45]

> भृगुवह्नीन्दुयुङ्मनुः
>
> The actual meaning of this is: Letter *s* with *r*, vowel *au* and *anusvāra*, i.e. *sraum̥* (स्रौम्) *bīja*.

Here the Sanskrit word *manu* stands for the vowel *au* (औ). A possible explanation is that *au* is the 14th *mātrā* in the set of 16 *svaras* (vowels) of Devanāgarī alphabet.

The historians of science should also note some technical terminology and symbology regarding geometrical figures related to mystic diagrams. The usual isosceles (including equilateral) triangle with vertex upwards (Fig. 7a) is called *agni* (fire) or *Śiva* triangle. It is stated that[46]

अग्निरूर्ध्वमुखं त्रिकोणम्

Fire is represented by an upward triangle.

It may be pointed out that the Greek Pythagoreans represented the metaphysical element fire by a pyramid whose symbolic form can be taken to resemble Fig. 7a. The reverse triangle, i.e. the one with vertex downwards (Fig. 7b) is called *Śakti* or *yoni* triangle. The symmetrical combination of two equilateral triangles one of which is Śiva and the other is Śakti gives us the *ṣaṭkoṇa* (hexagram of Fig. 2) which is taken to represent the universe (produced from the primordial energy).

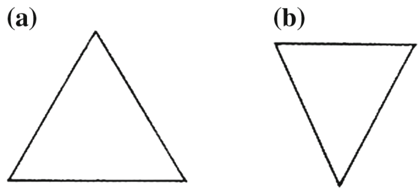


Fig. 7 The *Śiva* and *Śakti* triangles

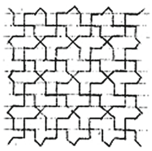


Fig. 8 A *svastika* floor design

The figure of *svastika* is considered auspicious in India. Its use has been noticed even in the Indus Valley motifs in antiquity. The figure of *svastika* has been used in constructing some *yantras* (mystic diagrams). The *Vṛhat Sarvatobhadra yantra* made from *svastikas* (Fig. 8)[47] yields a beautiful floor design which can be used for mathematically symmetrical tiling. Nārāyaṇa Paṇḍita in his *Gaṇita-kaumudī* (1356 AD) has given the name *'sarvatobhadra'* to magic figure *(aṅka-yantra)* which is obtained by filling (in Fig. 9) the 64 triangles by numbers 1 to 64 to obtain magically constant sum.[48]

Yantras or mystic diagrams are frequently enclosed or surrounded by what is called *bhūpura* (Earth-city or world-place) which is a square with openings on all the four sides or cardinal (Fig. 10).

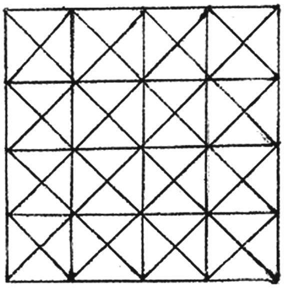


Fig. 9 Square divided into 64 cells

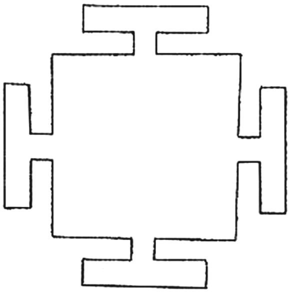


Fig. 10 *Bhūpura*

A very common figure on *yantras* is that of lotus *(padma)* with a number of petals, the most frequent number being 8 (representing the 8 cardinal directions and corner directions). Petals are usually of three types, namely,

(i) Round (Fig. 11),
(ii) Simply pointed (Fig. 12) and
(iii) Ogee form or inflectional (Fig. 1),

in which each side of a petal has a point of inflection where the curvature changes (sign). Some other symbols depicted on mystic diagrams include those for *triśula*

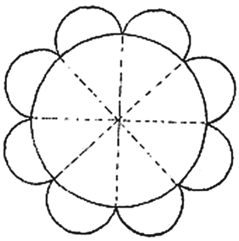


Fig. 11 Round-*petal* lotus

(trident), *vajra*, etc. It is often maintained that mystic language and symbols are needed to express the higher and deeper inner experience of the *yogic, tāntric* and spiritual mind.

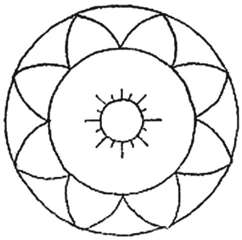


Fig. 12 Pointed-petal lotus

5 *Śrīyantra:* The Famous Mystic Diagram

According to the anonymous Sanskrit work *Yantroddhāra Sarvasva*,[49] there are as many as 10000 *yantras* or mystic diagrams. Among these the *Śrīyantra* is found to be most important and popular. It is the one which has drawn the widest attention of scholars. Indeed, it is the profoundest *yantra* and is significant from various points of view.

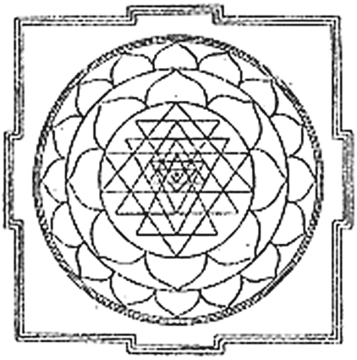


Fig. 13 *Śrīyantra*

As a basic geometrical diagram, the usual and most commonly depicted *Śrīyantra* is the plane (two-dimensional) type shown in Fig. 13 which shows its line diagram. The diagram consists of a central *bindu* (dot) surrounded by a bilaterally symmetrical figure composed of a set of nine interwoven primary isosceles triangles four of which are Śiva (vertex upwards) and five Śakti (apex downwards). The vertices of all the 9 triangles lie on the East (taken upwards) to West line of symmetry, and their bases and tops run from North to South.

The central design of the triangular complex is usually enclosed in a circle and surrounded by a lotus figure of 8 petals and then by another lotus of 16 petals situated similarly and symmetrically all around. Then a triplet of concentric circles is often made to surround the lotuses. Finally the whole pattern is enclosed in a three-lined square boundary (called *bhūpura*) with a gate on each of the cardinal sides.

It is clear from the diagram that mathematically the most complicated part of *Śrīyantra* is the inner triangular complex. Among the traditional constructions *(uddhāra-prakāra)*, a well-known classical method is that of Kaivalyāśrama which is given in his commentary on famous *Saundarya Laharī* attributed to Śaṅkarācārya. This may be briefly described as follows.[50]

Draw a circle (see Fig. 14) of desired size and divide the vertical diameter EW into 48 equal parts or units. Starting from E, draw 9 parallel chords of the circle (all perpendicular to EW) at respective distance of 6, 12, 17, 20, 23, 27, 30, 36 and 42 units. These are marked as A_1B_1, A_2B_2, etc. to A_9B_9 serially. Leaving out the third and seventh chords as they are, delete (or rule off) 3, 5, 16, 18, 16, 4 and 3 units of length at both ends of the 1st, 2nd, 4þ, 5þ, 6th 8þ and 9th chords, respectively. By this, these seven shortened chords become the line segments ($C_1D_1, C_2D_2, C_4D_4, C_5D_5, C_6D_6, C_8D_8$ and C_9D_9 symmetrically placed on the EW line (for completeness we may say C_3D_3 is A_3B_3 and C_7D_7 is A_7B_7).

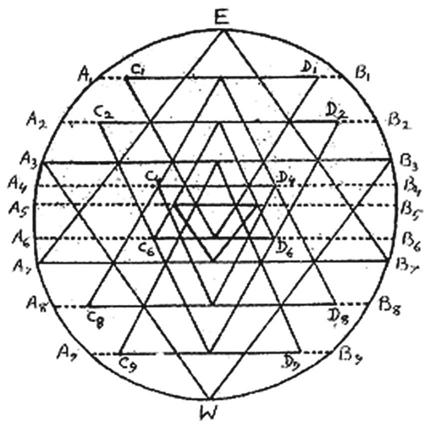


Fig. 14 Construction of *Śrīyantra*

Now the nine basic triangles of the triangular complex are formed as follows:

(i) Ends of the 1st segment C_1D_1 are joined to the midpoint of C_6D_6.
(ii) Ends of the 2nd segment C_2D_2 are joined to the midpoint of C_9D_9.
(iii) Ends of the chord A_3B_3 are joined to west point W.

(iv) Ends of segment C_4D_4 are joined to the midpoint of C_8D_8.
(v) Ends of C_5D_5 are joined to the mid-point of C_7D_7.
(vi) Ends of C_6D_6 to the midpoint of C_2D_2.
(vii) Ends of A_7B_7 to east point of E.
(viii) Ends of C_8D_8 to midpoint of C_1D_1.
(ix) Ends of C_9D_9 to the midpoint of C_3D_3.

It may be noted that the midpoints of the segments C_4D_4 and C_5D_5 are not used in forming any of the nine above primary triangles.

A slightly different version of the construction of *Śrīyantra* is found in *Tantra-samuccya* (śilpabhāgam)[51] in which the *kaṭapaya* system is used to specify the distances of the parallel chords from E, and for giving the amounts of deletion at their ends. In this version the amount of deletion is 4 units (instead of 5) for the 2nd chord and 19 units (instead of 18) for the 5th chord.

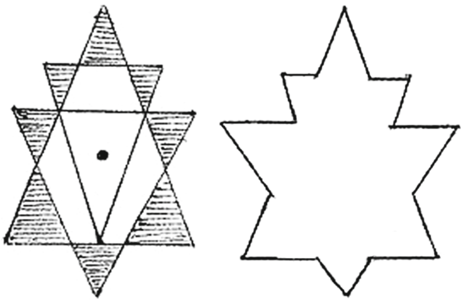
Recently the author (*RCG*) of the present article has found a new version of the construction which the chosen diameter is divided into 42 parts (instead of 48) and distances of the chords from E are taken to different. In this version the deletion of 8th and 9th chords are given to be 8 and 6 units. If these are taken as total deletions, then at one end of the said two chords, the deletion will be 4 and 3 units which are same as in Kaivalyāśrama's version.[52]

Lakṣmīdhara, another commentator, of *Saundarya Laharī*, calls the above construction of Kaivalyāśrama to be one of *saṃharakrama* (order of destruction).[53] He has another construction which is called to be of *sṛṣṭi-krama* (order of creation). In this, we start from the centre, i.e. with the construction of the small innermost *śakti* triangle around the *bindu* (dot) and then move outwards to construct other triangles to complete the desired set of 9 primary triangles.[54]

The method in the order of destruction is also said to be found in the *Tantrarāja* and that of creation in *Jnānārṇava*. Further,[55] there is mention of a third method called that of *sthiti-krama* (order of sustenance or protection) which is to be found in the work *Śubhagodaya*. These three orders match or correspond to the three stages or creation, protection and destruction *(pralaya)* in Hindu cosmological science *(sṛṣṭi-vijñāna)*.

Fig. 15 The 43 triangles of *Śrīyantra*

The mutual intersections of the nine basic primary triangles in the *saṃhāra*-order construction given above (Fig. 14) results in the formation of 43 smaller secondary triangles (Fig. 15). The inner most central *śakti* triangle, containing dot (*bindu*, as a symbol of single unseparated form of śiva and śakti), is surrounded by an enclosure *(āvaraṇa)* formed by 8 triangles arranged in a symmetric polygonal figure called *aṣṭakoṇa* (eight-angled) or *aṣṭāra*. The outer boundary of this enclosure (Fig. 16a) of 8 small triangles forms the figure of a re-entrant polygon with 8 angles (Fig. 16b).



(a) **(b)**

Fig. 16 The *aṣṭakona* (with *yoni* traingle) and re-entrant octagon

Figure 16a itself is important and is called *navakonamaka* mystic diagram. It is often used to form other *yantras*. After the enclosure of 8 small triangles, three more successive such enclosures or garlands are formed. They contain 10, 10 and 14 secondary triangles, respectively (see Fig. 15).

Thus the total number of secondary triangles:

$= 1 + 8 + 10 + 10 + 14 = 43$

as already mentioned above.

For the geometrical constituents *(avayavas)* of the *Śrīyantra* (Fig. 13), the following classical verse from *Rudrayāmala Tantra* is frequently quoted[56]

बिन्दुत्रिकोणवसुकोणदशारयुग्मं, मन्वस्त्रनागदलसंयुतषोडशारम् ।
वृत्तत्रयं च धरणीसदनत्रयं च, श्रीचक्रराजमुदितं परदेवतायाः ॥

The great *Śrīyantra* of the supreme deity consists of a *bindu* (dot), a central triangle, then enclosures formed of 8, 10, 10 and 14 triangles, and then surrounded three circles and three *bhūpuras*.

The above Sanskrit verse is also said to be found in the *Tripuropaniṣad*.[57] Due to its importance, the *Śrīyantra* is discussed in many ancient works especially tantric texts and related works. But it appears that a large variety of forms and constructions of this great *yantra* are available. This is not surprising for a vast country like India which has a continuous history and culture of thousands of years. Some differences arise from different interpretations of the Sanskrit technical terms.

Out of the ten components or constituents of *Śrīyantra* mentioned above, the enclosure formed by the triplet of circles *(vṛtta-trayaṃ)* is not accepted by some schools (e.g. Hayagrīva school). The remaining 9 constituents are usually called the nine *cakras* of the mystic diagram. But the term or word *cakra* is used in other senses also, e.g. even the 9 triangles of the triangular complex of the *Śrīyantra* have been called 9 *cakras*.[58]

The *Mantra-mahodadhi*[59] gives the construction of the *Śrīyantra* as follows:

बिंदुगर्भं त्रिकोणं तु कृत्वा चाष्टारमुद्धरेत् ॥ दशारद्वयमन्वस्राष्टारषोडशकोणकम् ॥

The accompanying figure in the edition used here by us shows that the word *āra* has been interpreted in the sense 'petalled lotus', and therefore the central innermost triangle (with *bindu* inside) is surrounded by three lotuses of 8, 10 and 10 petals (instead of angled polygons of Fig. 15). This interpretation is similar to that of the famous tantric or yogic *sahasrāra*, 'the 1000-petalled lotus'. Also, the *ṣoḍaśa-koṇakam* is drawn as a lotus with 16 angular petals. So we have a different *Śrīyantra* here which have teethed wheels.

In the Kaivalyāśrama's construction of *Śrīyantra* (given earlier in this very section) some small imperfections are found at a few intersections of lines which form the 43 secondary triangles. Of course, by drawing the mystic diagram on a smaller scale and with a little sleight of hand in drawing it, the imperfections become practically undetectable. The Mathematical aspect in attaining precision in the construction of theoretically ideal *Śrīyantra* has been discussed by Kulaichev.[60]

A technique for drawing a nearly perfect *Śrīyantra* within a square has been given by Bolton and Macleod who also mention that Alan West of the University of Leeds has produced a scheme to construct the *yantra* without any error.[61] A Nepalese version (dated 1700 AD) of *Śrīyantra* is reported to illustrate the occurrence of the mysterious pyramid angle of 51°51′ in the largest triangles of the *yantra*, thereby showing geometrical relationships involving the famous constant π (ratio of circumference to diameter in any circle).[62] The Tibetan '*Śrīcakra sambhāra maṇḍala*' diagram consists of a series of circles and lotuses.[63]

From the point of view of architectural construction, the *Gaurī yāmala Tantra* mentions four types of *Śrīyantra* as follows.[64]

चातुर्विध्यं हि चक्रस्य प्रस्ताराश्च भवन्ति हि । भूकूर्मपद्मप्रस्तारा मेरुश्चापि तथा विधः ॥

There are four *prastāras* (architectural forms) of *Śrīyantra*, namely *bhū, kūrma, padma,* and *meru*.

The *bhū* version is the plane version in which the full diagram lies in a horizontal plane. In the *kūrma* form (resembling the back of a tortoise), the triangular complex is drawn on the spherical surface with the help of spherical triangles. In the *meru* version, different constituents or enclosures (counted from outermost) lie in different horizontal planes at different heights like the mythical Mount Meru. The *padma* (lotus) form does not seem to be popular.

The history of *Śrīyantra* is claimed to go to Vedic times, and it is found mentioned in Buddhist inscriptions of Sumatra (seventh century AD).[65] Verse no. 11

of *Saundarya-laharī* (attributed to Ādi-Śaṅkarācārya)[66] is taken to refer to the *Śrīyantra*.

Elementary mathematics is involved in the solution of primary triangles formed by *Kaivalyāśrama's* method (Fig. 14). Let x be the distance, from E, of the chord along which the base or top of such a triangle lies. If k is the deletion on either end of the chord, then the length of the base or top of the triangle will be given by

$$b = 2(\sqrt{x(2R - x)} - k$$

where R is the radius of the circle (Fig. 14). If y is the distance of the chord on which the apex of the triangle lies, then the apex angle will be given by

$$\theta = 2\tan^{-1}\left(\frac{b}{2}|x-y|\right), \text{ here } |x-y| \text{ is modulus of } (x-y).$$

For example, for the *śakti* triangle of top C_1D_1 (lying along A_1B_1) and apex on C_6D_6, we have $R = 24, x = 6,\ y = 27$, and $k = 3$.

Using above formulas, we get $b = 25.75$ and $\theta = 63^\circ$ nearly. Thus the innermost triangle (Fig. 15), which contains the *bindu*, is nearly equilateral.

The high mathematical theory of the spherical type of *Śrīyantra* is reportedly found in the doctoral thesis on Plane and Spherical Triangular network by Dr. C. S. Rao (I.I.T., Bombay, 1993).[67] Also the *Śrīyantra* as an "Ancient Instrument to Control, the Psychophysiological State of Man" has been discussed in a joint paper by Kulaichev and Ramendic.[68] In fact, the *yantra* is regarded to be a complicated object whose study requires efforts by specialists from different fields of knowledge. Indeed *Śrīyantra* is rightly called *yantrarāja*, the king of mystical diagrams.

6 Other Selected *yantras*

As already mentioned, the total number of *yantras* or mystical diagrams is practically very large, and theoretically without any limit if we include the *aṅka yantras* (magic squares and other magic figures) also. The writer of the present article believes that for the authenticity and genuineness of a *yantra* found anywhere, the name of the ancient work and mention of the relevant Sanskrit text should be ensured. The *Jainendra Siddhānta Kośa* (ref. 23 at the end) contains a large number of *yantras* but neither the source nor the Sanskrit/Prakrit text is found there. Similar remark applies to the *Saundarya-Laharī* which we consulted (ref. 66) and which is supplemented with a large number of *yantras*. Huge collections of *yantras* are also found in several modern works[69] but original Sanskrit lines for their ancient *uddhāra* (construction and description) are generally missing.

The author (***RCG***) of the present article has collected a number of mystical and magical diagrams with relevant Sanskrit verses from various sources. A sample list of these is given in Appendix for illustration, a few of typical *yantras* are described in this section. For a broader panorama, the selection below is made full of variety. Abbreviations used are as follows:

MM = *Mantra-mahodadhi* (1588 AD) of Mahīdhara with Auto-commentary, Bombay, 1988.

PC = *Puraścaryārṇava* (1775), edited by M. Jha, Delhi, 1985 (see ref. 16).

YCD = *Yantra-cintāmaṇi* of Dāmodara (seventeenth century), ed. by H. G. Turstig, Stuttgart, 1988.

6.1 Gaṇeśa yantras

According to the Hindu tradition of *'Ādi pūjyo gaṇeśvaraḥ'*, the God Gaṇeśa is to be worshipped first in all religious work to avoid any hurdle *(vighna)* during the period. For this his Vighnarāja ('controller-king of hurdles') form may be selected. The corresponding *yantra* is described in the *Merutantra* as follows (PC, p. 1140) (Fig. 17):

चतुर्द्वारयुतं कुर्याच्चतुरस्रत्रयं शुभम् । तन्मध्येऽष्टदलं कार्यं पूजापीठं गणेशितुः ॥

For worshipping Lord Gaṇeśa, make an auspicious triple square (i.e. bhūpura) with four gates and in its middle make a lotus of eight petals.

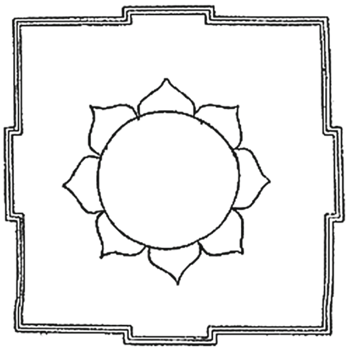


Fig. 17 *Gaṇeśa yantra*

The mystical diagram for the *śakti* and *virañci* forms of Gaṇeśa is said to be same. In the case of Mahāgaṇapati, the *karṇika* (pericarp) of the lotus contains a hexagram which itself has a triangle (*PC*, p. 1140), and with slight modifications, we get a few other forms.[70]

6.2 *Janana yantras*

भूर्जपत्रे लिखेत्सम्यक्त्रिकोणं रोचनादिभिः ॥ ९८ ॥
वारुणं कोणमारभ्य सप्तधा विभजेत्समम् ।
एवमीशाग्निकोणाभ्यां जायन्ते तत्र योनयः ॥ ९९ ॥
नववेदमितास्तत्र विलिख्योन्मातृकां क्रमात् ।
अकारादिहकारान्तमीशादिवरुणावधि ॥ १०० ॥

Make an equilateral triangle on birch paper with yellow ink etc. Starting with the west corner (taken downward) divide it sevenfold by equi-distant lines. Carry out similar division from NE and SE corners thereby generating 49 triangular cells *(yonis)* in which should be written the alphabet from *a* to *ha* serially from NE corner to the west corner.

The total number of *mantras* is said to be seven crore (MM, p. 224). But they all have some *doṣa* (lacuna). The number of various types of *doṣas* is fifty. For pacification of ill effects caused by the *doṣas* and for curing them, ten *saṃskāras* are prescribed. The first of which is called *Janana*. The mystical diagram used for the purpose is called *Janana yantra* (Fig. 18). *MM, XXIV*, 98–100 (p. 224) describes its method of construction as follows:

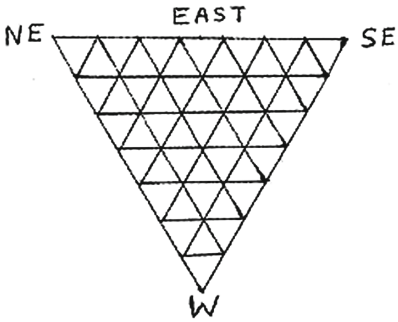


Fig. 18 *Janana yantra*

Thus the original equilateral triangle is divided into 49 small triangles by 18 equidistant lines (6 each parallel to the three sides). The cells are filled with 49 letters (16 vowels and 33 consonants) of the Sanskrit alphabet (not shown in Fig. 18). If we count the cells, we have (starting from W)

$1 + 3 + 5 + 7 + 9 + 11 + 13 = 49$

which leads easily and geometrically to

$1 + 3 + 5 + \ldots + (2n - 1) = n^2$

It may be mentioned that Sanskrit alphabets were scientifically devised separating vowels and consonants which were further classified scientifically according to place of pronunciation. In fact, India's linguistic sciences were quite advanced relatively.

6.3 A *Māraṇa yantra*

Astrology is a pseudoscience, but astrology of ancient times is significant for a study of history of astronomy. Similarly, association of magical properties with *yantras* may be superstitious and claims of their efficacy may be ridiculous. Yet here we are concerned with them only as ancient geometrical diagrams. A *māraṇa yantra* is mentioned in the YCD (p. 45) as follows:

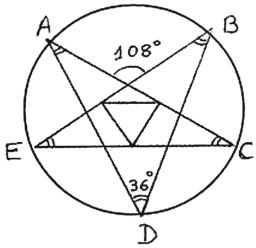


Fig. 19 *Māraṇa yantra*

साध्यनाम लिखेन् मध्ये स्तम्भस्तम्भेति सम्पुटम् ॥ ३३ ॥
ततस्त्रिकोणं सम्वेष्ट्य पञ्चकोणं तथोपरि ।

Write the intended name between the coupled word *stambasambha,* enclose it in a triangle and surround the whole by a pentagram.

That is, we get a diagram of Fig. 19 in which the writing of the phrase is omitted. The usual figure of a pentagram is shown with an apex at the top (i.e. at highest point). It was the emblem of the Greek Pythagorean school. The figure may be drawn with the help of a regular pentagon $ABCDE$ or by making angles on a line EC, etc. Did the Indians know to divide a circle into five equal parts? What is the nature of the inscribed triangle?

6.4 *Bālā-pūjana yantra*

This is described in MM, VIII.17 (p. 58) as

नवयोन्यात्मकं यन्त्रं बहिरष्टदलावृतम् । भूगृहेण पुनर्वीतं पूजनाय लिखेत्सुधीः ॥ १७ ॥

For worshipping the deity, the *Navayonyātmaka yantra* should be written and it should be surrounded by an eight-petalled lotus which should be enclosed further by *bhūpura*.

Thus the mystical diagram consists of a *navakoṇātmaka yantra* (Fig. 16a) surrounded by the usual lotus and *bhūpura*.

The geometrical diagram of the *Bālādhāraṇa yantra* (MM, VIII, 74–76; p. 62) is same except that the outermost single *bhūpura* is to be replaced by two *bhūpuras* with different orientations.

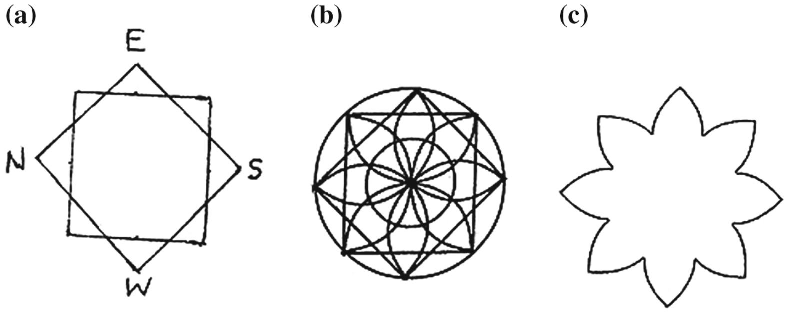


Fig. 20 Squares, lotus construction, and lotus

As explained in the *MM* commentary (p. 62) the *bhūpura* pair here consists of two squares one of whose vertices (or corners) lie along cardinal directions and those of the other along the intermediary directions (Fig. 20a). Incidentally, if eight semicircles are described on the eight equal sides of the squares inwardly, we get a flowery design (Fig. 20b) and finally an eight-petalled *padma* with simply pointed petals by mathematical method (Fig. 20c) (after deleting superfluous portions).[71]

6.5 Other yantras Based on Navakoṇaka yantra

The *Navakoṇātmaka (=navakoṇaka) yantra* was introduced above in section 5 as part of *Śrīyantra*. It consists of (see Fig. 16a) one central *tri-koṇa* (triangle) and eight surrounding outer triangles or outward angles *(koṇas)*. It is also called *navay-*

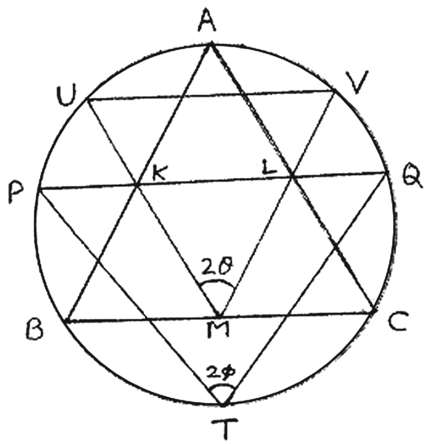


Fig. 21 A mini-*Śrīyantra*

onyātmaka (9-triangled) *yantra* and may even be called a mini *Śrīyantra*. Since this type of mystical diagram forms the main part in several other *yantras*, a simple construction was evolved for it. In a circle of desired size (Fig. 21), the equilateral triangle ABC with vertex upwards is inscribed. An isosceles triangle UMV is then constructed with apex.

M is at the midpoint of BC. These two triangles intersect at K and L also. The third inscribed triangle PQT is formed by producing KL both ways and joining the ends to the lowest point T of the circle.

Let r be the radius of the circle and 2a the side of the triangle ABC. If 2θ and 2ϕ are the angles at the apexes M and T of the other triangles, the following mathematical relations can be easily found.

Height of P above $T = PT\cos\phi$

$$= (AT\cos\phi).\cos\phi = 2r\cos^2\phi.$$

$\therefore$ Altitude of the triangle KBM

$$\begin{aligned} h &= 2r\cos^2\phi - MT = 2r\cos^2\phi - (2r - \sqrt{3}a) \\ &= \sqrt{3}a - 2r\sin^2\phi \end{aligned}$$

But from $\triangle KBM$, we also have

$$h\cot 60^\circ + h\cot(90^\circ - \theta) = BM = a.$$

Putting above value of h in this and using $r = \frac{2a}{\sqrt{3}}$, we finally get, on simplification,

$$(3 - 4\sin^2\phi)(1 + \sqrt{3}\tan\theta) = 3.$$

For the usual value $2\theta = 60^\circ$, we get $2\phi = 76^\circ$.

In addition to the *Bālā yantras* already mentioned, the mystical diagram of Fig. 21 is the central figure in the *Tripura Bhairavī* and *Dhanadā Devi yantras*.[72] The *PC* (pp. 1154–1155) quotes the Sanskrit verse for the *Tripura Bhairavī yantra* but interprets *navayonis* as 9 concentric triangles (see PC plate 12) instead of *navayonis* of Fig. 21. One form of *Durgā Pūjana yantra*[73] also is based on Fig. 21 (see below) For the Sanskrit text of *Dhanadā Devi yantras*, see *PC*, p. 1215.

6.6 *Durgā yantras*

Goddess Durgā is a popular deity. The construction of her *yantra* is described in the *Merutantra* as follows (*PC*, p. 1159):

अष्टपत्राम्बुजद्वन्द्वं चतुरस्रत्रयावृतम् । चतुर्द्वारसमायुक्तं कुङ्कुमादिभिरुद्धरेत् ॥

Construct, with *kuṇkuma* (saffron) etc. a pair of 8-petalled lotuses surrounded by three squares each with four gates.

That is, the *Durgā yantra* accordingly to the *Meru-tantra* consists of a usual double lotus (Fig. 1) enclosed in a triple *bhūpura*.

The *Durgā yantra* which is used in the *śatacaṇḍī* ceremony is usually called *Durgā-sapataśatī-mahāyantra*. It consists of a *śakti* equilateral triangle (apex downwards) circumscribed by a hexagram (Fig. 2), and then enclosing the latter in 8-petalled lotus surrounded by a *bhūpura*.[74] But the Sanskrit text (*MM*, p. 167), *tattvapatrāvṛta-tryasra-ṣaṭakoṇāṣṭadalānvite*, asks us to draw a 24-petalled lotus also (before *bhūpura*).[75]

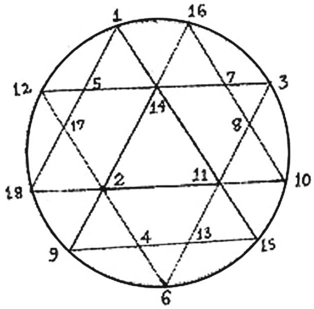


Fig. 22 *Durgā yantra*

Another form of *Durgā yantra* consists of the *Navakoṇaka* diagram (Fig. 21) surrounded by a triplet of circles and then by the usual lotus and *bhūpura*.[76] A beautiful rendering or modification of Fig. 21 is found in *Durgā yantra* designed by Penny Lea Morris Serferovich (Fig. 22).[77] The complex has 9 lines and 18 points of intersection (including vertices). The importance of the basic diagram was increased by Michael Keith by making it an *aṅka yantra* also. He filled the 18 points of intersection by consecutive numbers 1 to 18 in such a way that the sum along each of the 9 lines comes magically the same, namely, 41 (the magical constant).

6.7 *Rudra yantra*

This is described in *MM*, XVI. 78–79 (p. 143) as follows:

अष्टपत्रं षोडशारं चतुर्विंशतिपत्रकम् ॥
दन्तपत्रं ततः कुर्याच्चत्वारिंशद्दलं ततः । तद्बहिर्भूपुरं कुर्यात्तत्र रुद्रं प्रपूजयेत् ॥

Make lotuses (successively) of 8, 16 and 24 petals, then of 32, and then of 40 petals. Outside them make the *bhūpura*. In that *yantra*, the God Rudra should be worshipped.

Thus we have the Rudra mystical diagram as shown in Fig. 23. The same is said to be found in the *Skanda-purāṇa*.[78] It may be noted that the number of petals in the *yantra* forms the arithmetical progression 8, 16, 24, 32, 40.

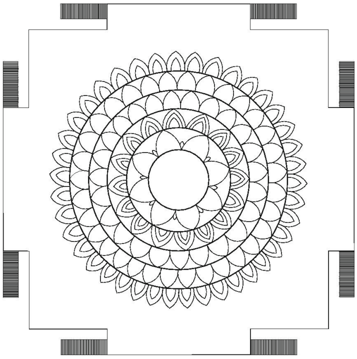


Fig. 23 *Rudra yantra*

There are *yantras* in which the number of petals in successive lotuses forms a geometrical progression. One such diagram is the *Vidyarājñi yantra* (*MM*, V. 32–33; p. 39) in which we come across the set 8, 16, 32 and 64.

6.8 Svayaṃvarakalā yantra

This is a sort of *ākarṣana yantra* (claimed to help in attaining the goal of marriage!). It is taken here to illustrate that often somewhat complicated mathematical figures are prescribed. *MM*, VI. 60–61 (p. 47) describes the *yantra* as follows:

त्रिकोणचतुरस्राङ्ककोणाष्टदलदिग्दलम् ।
दिक्कलादन्तपत्राणि चतुष्षष्टिदलं पुनः ॥
वृत्तत्रयं चतुर्द्वारयुक्तं धरणिकेतनम् ।

The above Sanskrit lines simply give a list of the mathematical objects which one has to construct to get the *yantra* for doing the *pūjā* for coercion. They are successively triangle, square and hexagram *(aṅgakoṇa = ṣaṭakoṇa)*; then lotuses of 8, 10 *(dik)*, 10, 16 *(kalā)*, 32 *(danta)* and 64 petals; then three circles, and finally the *bhūpura (dharaṇi-ketana)* with four gates.

Knowledge of elementary mensurational geometry is needed to draw the diagram. For example, for making square inscribed in the hexagram (Fig. 2), one has to draw a square in the hexagon space inside it (Fig. 24). If 2a and 2b are the sides of the hexagon and square in Fig. 24, it be shown that $b = (3 - \sqrt{3})a$. By considering angles, a square can be circumscribed by hexagon.

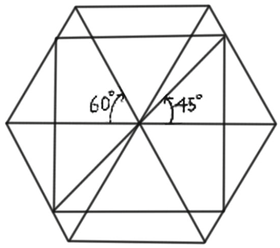


Fig. 24 Square inscribed in a hexagon

6.9 Bhauma yantra

The *aṅka-yantras* (magic squares) associated with the nine ancient astrological planets have been already mentioned in Sect. 2 (see Fig. 4). Similarly, there is a mystical diagram for each of the *navagrahas*. Some details on the subject have been already published by the present writer (see ref. no. 17 and the end). The mystical diagram of planet Mars is peculiar and is called *Maṅgala* or *Bhauma yantra*. It is briefly described here for illustration.

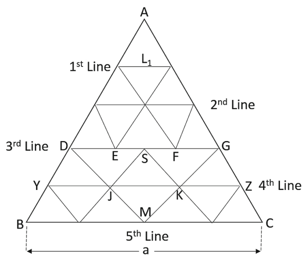


Fig. 25 *Bhauma* (*Mars*) *yantra*

The planet Mars has been associated with triangle and this played role in the evolution of its *yantra* (Fig. 25).[79] The MM, XV. 51 (p. 133) knows that it consists of 21 triangular cells.

The full details of the construction of the Mars *yantra* are described in the *merutantra* whose verses are quoted in PC, p. 1158. The Sanskrit text and its translation can be found in present author's paper mentioned above. Here we give a new translation as follows.[80]

"First construct an equilateral triangle (ABC) and then divide it into five parts (by equidistant lines parallel to the base). Mark the third line (DG) by points (E and F) of three equal division. Join (crossly) the ends of the first line to these points (E and F) of the third line. Join directly the ends of the second line to the same points. The already connected third line be bisected (at S), and the fourth and fifth lines be divided by two (J and K) and three (U, M, V) points. Join the ends (D and G) of the third line to the midpoint (M) of the fifth line, and the ends (Y and Z) of the fourth to its other points (U and V). The wise man should supply the pair of lines (SU and SV) for forming figure of two fishes (joined back to back at SM). Thus we get twenty one cells".

In this way the Mars *yantra* (Fig. 25) is obtained. Some involved crucial mathematics related to the construction is already published.[81] If DM and SU intersect on YZ at J, the BU will be $\frac{a}{5}$ and YJ will be $\frac{a}{4}$, where $BC = a$.

6.10 *Sarvatobhadra yantra*

The *Sarvatobhadra* mystical diagrams (*cakras, yantras, maṇḍalas*) are symmetrical from all four sides. They are indeed architecturally beautiful and considered auspicious. For constructing them, a big square is subdivided into a large number of small square cells like the chess board or ordinary graph paper often with cross lines (Fig. 9).

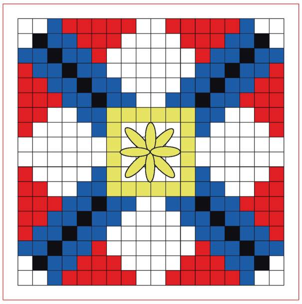


Fig. 26 A *Sarvatobhadra yantra*

A few Sanskrit texts for making the *Sarvatobhadra yantras* are quoted in the *Vācaspatyam*.[82] The text for the elaborate diagram of Fig. 26 is given from *Hemadri (Skande)* as follows:

प्रागुदीच्याङ्गता रेखाः कुर्यादेकोनविंशतिम् । खण्डेन्दुस्त्रिपादकोणे शृङ्खला पञ्चभिः पदैः ॥ १ ॥
एकादशपदा वल्ली भद्रन्तु नवभिः पदैः ।चतुर्विंशत्पदावापि परिधिर्विंशत्या पदैः ॥ २ ॥
मध्ये षोडशभिः कोष्ठैः पद्ममष्टदलम् । श्वेतेन्दुः शृङ्खला-कृष्णां वल्लीं नीलेन पूरयेत् ॥ ३ ॥
भद्रारुणा सितावापी परिधिः पीतवर्णकः । बाह्यन्तरदलैः श्वेता कर्णिका पीतवर्णिका ॥ ४ ॥

The last three lines mention the colours of the various regions of the *yantra*. Based on the above text, its construction can be concisely explained as follows:

Draw 19 equidistant lines from east to west and from north to south (These will form a square network of 18×18 or 324 small square cells). In the space of central *(madhya)* 16 cells, a padma (pink lotus) of 8 petals be made with yellow *karṇikā* (pericarp). Around it a square yellow belt (called *paridhi* or periphery) of 20 cells is made. Just outside this belt and on each of its 4 cardinal sides, a *vāpī* (like a square *kuṇḍa* with steps) of 24 white cells be constructed.

Starting from each corner of the *paridhi*, a chain *(śṛṅkhalā)* of 5 black cells is laid down along outward diagonal direction. At the end of each chain, angled tromino *(khaṇḍendu)* of 3 white cells is placed. Closely juxtaposed on each side of every *śṛṅkhalā* (chain) is a *vallī* (stepped creeper) constructed from 11 blue cells (So far 252 cells out of 324 have been filled). The remaining eight spaces (two on each side) are called *bhadras* (pyramid type nonaminos). On each side, the two *bhadras* are between *vāpī* and its adjacent *vallīs*. *Bhadras* are red *(aruṇa)*, and each has 9 cells. The space between lotus and *paridhi* is white. Finally the whole figure of 324 cells is to be surrounded by three square belts of white, red and black colours (these three squares may be said to form *bhūpura*).

Appendix I: *Yantra-śataka*

(list of 100 *yantras* or mystic geometrical diagrams)

Abbreviations used are: *MM* = *Mantra-mahodadhi* (ref. 32); *PC* = *Puraścar-yarṇava* (ref. 16); *YCD* = *Yantra-cintāmaṇi* of Dāmodara (ref. 18); Mishra (ref. 69); Varni (ref. 23); etc. (see references at the end).

1. *Agni-pūjaṇa-yantra* (***MM***, I. 113, p 7).
2. *Agni-stambhana-yantra* (***YCD***, No. 35, p. 37).
3. *Annapūrṇā-yantra* (***PC***, p. 1157) (from *Merutantra*). Cf. *Annapurṇeśvarī-yantra* (***MM***, IX. 9, p. 68).
4. *Bagalāmukhī-pūjana-yantra* (***MM***, X. 7, p. 78; ***PC***, p. 1156).
5. *Bagalāmukhī-stambhana-yantra* (***MM***, X. 25–26, p. 79).
6. *Bālā-pūjāna-yantra* (***MM***, VIII. 7, p. 58). Also see section 6 above.
7. *Bālā-dhāraṇa-yantra* (***MM***, VIII. 74–76, p. 62); Section 6 of this paper.
8. *Bandhamokṣa-karam-yantra* (***MM***, XX. 118–119, p. 188).
9. *Bhauma-yantra* (from *Merutantra*) (***PC***, p. 1158); Sec. 6 of this paper.
10. *Bhavānī-yantra* (***PC***, p. 1146).
11. *Bhutalipi-yantra* (***PC***, p. 1148) (from *Śāradātilaka-ṭīka*).
12. *Bhuvaneśvarī-yantra* (***PC***, p. 1154) (from *Śārada-tilaka*).
13. *Brahma-yantra* (***PC***, p. 1158).
14. *Brāhmī-yantra* (*Ibid*).
15. (Lord) *Buddha-yantra* (***PC***, p. 1145).
16. *(Planet) Budha-yantra* (***PC***, p. 1158).
17. *Caitanya-bhairavī-yantra* (from *Jñānārṇava*) (***PC***, p. 1155).
18. *Cāmuṇḍā-Mahālakṣmī-yantra* (***PC***, p. 1158).
19. *Cāmuṇḍā (Navadurgātmaka)-yantra* (***PC***, p. 1158).
20. *Candra*-(Moon)-*yantra* (***PC***, p. 1158).
21. *Chinnamastā pūjana-yantra* (***MM***, VI. 12, p. 45).
22. *Chinnamastā-yantra from Rudra-yāmala* (***PC***, p. 1155).

23. *Dakṣiṇā-murti-yantra* (from *Merutantra*) (*PC*, p. 1145).
24. *Dattātreya-yantra* (see *Kalyāṇa* Vol. 42, 1968 i.e. ref. 39; plate facing p. 544).
25. *Devamātṛka-yantra* (*YCD*, No. 30, p. 34).
26. *Dhanadā-devi-yantra* (*PC*, p. 1215; Mishra, p. 193).
27. *Dhūmāvatī-yantra* (*PC*, p. 1156).
28. *Dūramāraṇam-yantra* (*YCD*, No. 47, p. 44).
29. *Durga-yantra* (I) (*PC*, p. 1159). See Sect. 6 of the paper.
30. *Durga-yantra* (II). This is called *Caṇḍi-yantra* (*MM*, p. 167 and its figure no. 49). Also cf. Sec. 6 and Mishra, p. 79.
31. *Gaṇeśa-yantras* (*PC*, p. 1140 and Sect. 6 of present paper).
32. *Garuḍa-yantra* (*PC*, p. 1146 and Mishra, pp. 61–63).
33. *Gāyatrī-yantra* (see A. Avalan, *Iśopaniṣad*, Madras, 1952).
34. *Guhyakālī-yantra* (I) (*PC*, p. 1149–1150) (from *Mahakalasamhita*).
35. *Guhyakālī-yantra* (II) (Mishra, pp. 125–126).
36. *Hanumat-pūjana-yantra* (*PC*, p. 1147).
37. *Hanumat-dhāraṇa-yantra* (*MM*, XIII. 46–53, p. 116).
38. *Hayagrīva-yantra* (*PC*, p. 1145).
39. *Indra-yantra* (*PC*, p. 1158).
40. *Janana-yantra* (*MM*, XX IV. 98–101, p. 224). see Sec 6.
41. *Jayadam-yantra* (*MM*, XX. 53–57, p. 184).
42. *Jvaraharaṇa-yantra* (*YCD*, No. 60, p. 50; *MM*, p. 188).
43. *Kālarātri Dīpasthāpana-yantra* (*MM*, XVIII. 39, p. 158).
44. *Kālarātri Pūjana-yantra* (*MM*, XVIII. 13–14, p. 157).
45. *Kālī-yantras* (from *Kālītantra* etc) (*PC*, p. 1148–49 mentioning other works also; *MM*, III. 11, p. 23).
46. *Kalki-yantra* (*PC*, p. 1145).
47. *Kāmakalā-yantra* (from *Mahākāla-saṃhitā*) (*PC*, p. 1150).
48. *Kāmya-yantras* (from *Merutantra*) (*PC*, p. 1146–47).
49. *Kārtavīrya-dīpasthāpana-yantra* (*MM*, XVII. 64–81, p. 153–154).
50. *Kārtavīrya-pūjāyantra* (*MM*, XVII, 21–22, p. 150) *(= Arjuna-yantra).*
51. *Kaumārī-yantram* (*PC*, p. 1158).
52. *Krodha-śamana-yantra* (*YCD*, No. 18, p. 27).
53. *Kṛṣṇa-yantra* (I) (*PC*, p. 1145, and Mishra, p. 66).
54. *Kṛṣṇa-yantra* (II) (from *Gautamiyatantra*) (Mishra, p. 65–66).
55. *Kubera-yantra* (*PC*, p. 1158).
56. *Kubjikā-yantra* (*PC*, p. 1157).
57. *Kūrma-yantra* (*PC*, p. 1141, and p. 476 for *cakra*).
58. *Laghuśyamā-yantra* (*MM*, VIII. 121, p. 66).
59. *Lakṣmī-yantra* (*PC*, p. 1157, and Mishra, p. 189).
60. *Lalitā-yantra* (*MM*, XX. 74–79, p. 185).
61. *Mahāgaṇapati-yantra* (from *Merutantra*) (*PC*, p. 1140).
62. *Mahāmohana-yantra* (*YCD*, No. 1, p. 20).
63. *Mālā-yantra*(?) (*PC*, p. 1158).
64. *Māraṇa-yantras* (*YCD*, No. 49, p. 45; *MM*, XX. 97–98, p. 187). Also see Sec. 6 of present paper.

65. *Mātaṅgī-yantra* (*MM*, VII. 72, p. 55; *PC*, p. 1156–57).
66. *Mātṛkā-yantra* (from *Saradatilaka*) (*PC*, p. 1148).
67. *Matsya-yantra* (*PC*, p. 1141).
68. *Mṛtyuñjaya-yantra* (*MM*, XX . 38–39, p. 183; *YCD*, No. 6, p. 22).
69. *Ṇamokāra-yantra* (Varni, p. 353, see Fig. 3 in Sec. 2).
70. *Navakoṇātmaka-yantra* (see Sec. 6 of this paper).
71. *Nigada mocana-yantra* (*YCD*, No. 78, p. 57–58).
72. *Nṛsiṃha-yantra* (*PC*, p. 1141; *MM*, XIV 7–8, p. 121).
73. *Paraśurāma-yantra* (*PC*, p. 1142).
74. *Pavitrayajana-yantra* (*MM*, XXIII. 51–54, p. 214–215).
75. *Rāma Pūjana-yantra* (*PC*, p. 1142).
76. *Rāma Dhāraṇa-yantra* (*PC*, p. 1142–1144).
77. *Rudra-yantra* (*MM*, XVI 78–79, p. 143). See Sec. 6 in paper.
78. *Śānti-yantras* (*MM*, XX. 105–111, p. 187; Varni. pp. 361–363).
79. *Śarabha-yantra* (*PC*, plate 14 at the end).
80. *Sarasvatī-yantra* (*PC*, p. 1157; cf. Mishra, p. 161).
81. *Sarvatobhadra-yantra* (See Section 6 of present paper).
82. *Ṣaṭkūṭā Bhairavī-yantra* (*PC*, p. 1155).
83. *Siddhilakṣmī-yantra* (*PC*, p. 1151).
84. *Śītalā-yantram* (*PC*, p. 1139 and plate 10).
85. *Śiva-yantras* (*PC*, p. 1145) (from *Prapañcasāra* etc).
86. *Smara* (cupid)-*yantra* (*PC*, p. 1147).
87. *Śmaśānakālī-yantra* (*PC*, p. 1150; Mishra, p. 122).
88. *Śrīyantra* (See Section 5 of the present paper).
89. *Sumukhī pūjāyantra* (*MM*, III. 56, p. 26).
90. *Sūrya-yantra* (*PC*, p. 1140–41; *MM*, x. 28, p. 131).
91. *Svapnavārāhī Pūjā-yantra* (*MM*, X. 41, p. 80; *PC*, p. 1158).
92. *Svayaṃvarakalā-yantra* (*MM*, VI. 60–61, p. 47; See. 6 above).
93. *Tārā-yantra* (*PC*, p. 1151; *MM*, IV. 87, p. 34).
94. *Tripura Bhairavī-yantra* (*PC*, p. 1154–1155; Mishra, p. 100).
95. *Vāmana-yantra* (*PC*, p. 1141).
96. *Varāha-yantra* (from *Prapañcasāra*) (*PC*, p. 1141).
97. *Vardhamāna-yantra* (Varni, p. 359).
98. *Vārtālī Pūjana-yantra* (*MM*, X. 76–78, p. 82–83).
99. *Vidyārājñī-yantra* (*MM*, V. 32–34, p. 39).
100. *Viṣṇu-yantra* (*PC*, p. 1141).

Appendix II: Select Glossary

For details of references, see at the end, e.g. *M M (= Mantra-mahodadhi)* in ref. no. 32.

1. *Adhara* (lip): Number 2 (used in *Kālacakra-tantrarāja*).[83]
2. *Aditya* (sun): number 1 and 12 (see *Ekādisaṃkhyākośa*. Jodhpur, 1964).

3. *Agni*: A Hindu god; vedic *citi* (altar); number 3; a metaphysical element-*bhūta* q.v.; consonant *r*.
4. *Agni-bīja* : raṃ (*MM*, p. 2 gives *vahni-bījam* = *raṃ*).
5. *Agni-priya*: *svāhā*.
6. *Agni-trikoṇa* : a triangle with apex upwards.
7. *Ākāśa* : number 0; consonant *h*; a *bhūta* q.v.
8. *Antya* : *kṣa* (last consonant in *tantras*, see *MM*, p. 63, 2391.
9. *Anugraha* : vowel *au*.
10. *Ara, āra* : corner, angle, spoke, petal.
11. *Āṣādhī* : consonant *t*.
12. *Aṣṭa-dala (or patra)* : 8-petalled lotus.
13. *Balaḥ* : *vaḥ*.
14. *Bhaga* : vowel *e* (*MM*, pp. 31, 45, 237).
15. *Bhṛgu* : consonant *s*.*Bhū* (earth) : number 1; a gross element *(bhūta)*; *la*.
16. *Bhūpura* : a decorated square with 4 gates (see Fig. 10).
17. *Bhūta* : a gross or meta physical element, see *Pañca-mahābhūta* for 5 such elements.
18. *Bīja* (seed) : mystic root syllable (of a *mantra* etc.).
19. *Cakra* : astrological diagram; mystic diagram (*yantra*); a mystical nerve plexus, wheel weapon of Viṣṇu.
20. *Candra* (moon) : number 1; vowel *aṃ* or *anusvāra bindu*; consonant *s* (*MM*, p. 239).
21. *Candra-bījam* : *ṭhaṃ*.[84]
22. *Damodara* : vowel *ai* (*MM*, pp. 58, 237).
23. *Daṇḍi* : consonant *th* (*MM*, pp. 53 and 238).
24. *Daśa-mahā-vidyās* : ten *tāntrika* goddesses.[85]
25. *Dhruvam* : the syllable *om* (*MM*, p. 38 and 237).
26. *Dik* : (direction-cardinal): number 8 (*MM*, p. 121) or 10 (usually).
27. *Gadi* : consonant *kh* (*MM*, pp. 35 and 237).
28. *Gagana* : synonym of *ākāsa* q.v.
29. *Gajapūrva* : number 7 (used in *Śrutabodha*, see ref. 2, p. 643).
30. *Gaṇanāyaka* (Gaṇeśa) : letter *ga* or *bīja gaṃ*.
31. *Govinda* : vowel *i* (*MM*, pp. 27 and 237).
32. *Gupta* : number 7 (used in *Mānasāra*).[86]
33. *Hali* : consonant *c* (1st in *cavarga*).
34. *Haṃsa* : consonant *s* (*MM*, pp. 5, 33, and 239).
35. *Harabīja* : mercury (chemical element).
36. *Hariḥ* : *tah* (*MM*, pp. 27, 52, and 238).
37. *Indra* : letter *la* (*MM*, pp. 42 and 239).
38. *Indu* : synonym of *candra* q.v.
39. *Jala* : letter *va*; a gross element (see *Pañca-mahābhūta*).
40. *Jhintiśa* : vowel *e* (*MM*, pp. 17 and 237).
41. *Kaḥ* : Brahmā of the Hindu Trinity.
42. *Kala* (time) : number 3 (see Ref. no. 83, Appendix I).
43. *kālibīja* : *kriṃ* (*MM*, p. 25 and ref. 85, p. 40).

44. *Kamikā* letter *ta* (*MM*, pp. 14, 37 and 238).
45. *karṇa* (ear) : diagonal: hypotenuse; vowel *u* or *ū* (*u* is right ear and *ū* is left ear, *MM*, p. 237).
46. *Kesari* (lion) : number 24 (used by *Pūjyapada*).[87]
47. *Keśava* : vowel *a* (*MM*, pp. 50 and 237).
48. *kham* : synonym of *ākāśa* q.v.
49. *Koṇa* : corner; angle; planet Saturn number 4 (used in *Mohacūdottara*, see Ref. 3, p. 2080).
50. *Kriyā* : letter *la* (*MM*, pp. 23 and 239).
51. *Krodhabīja* : *huṃ* (*MM*, p. 32).
52. *Kṣiti* (earth) : synonym of *bhū* q.v.; letter *la*.
53. *Kūṭabīja* : the phoneme *kṣa* (ref. 85, p. 46).
54. *Laksmībīja* : *śrīṃ*.
55. *Laṃgalī* : letter *tha* (*MM*, p. 238; ref. 16, p. 1148).
56. *Lotus* : Its botanical name is *Nelumbo nucifera, Gaertn,* and the red, pink, blue, and white flowers are called *kamala, padma, utpala, puṇḍarīka*.
57. *Madana* (cupid) : number 13 (ref. 83, Appendix I).
58. *Mahābhūta* (gross elements) : see *Pañca-mahā bhūta*.
59. *Mahāśūnya* (great vacuity) : a mental condition of *yogin*.
60. *Maṇḍala* : mystical or symbolic diagram.
61. *Manu* : *mantra*; number 14; etc. (see Sec. 4 of the paper).
62. *Mātṛkās* : alphabet; *varṇas a* to *kṣa* (*MM*, p. 5).
63. *Māyābīja* : *hrīṃ* (Ibid.).
64. *Mṛtyuḥ* (death) : Letter *śaḥ* (*MM*, pp. 31 and 239)
65. *Nabha* : synonym of *ākāśa* q.v.
66. *Nāḍītrayam* : *iḍā, piṅgalā,* and *suśumnā*.
67. *Nandaja* : letter *ṭha* (*MM*, pp. 26 and 238).
68. *Netra* (eye) : number 2; vowel *i* (right eye) or (left eye).
69. *Pañca-mahābhūta* : 5 gross or metaphysical elements viz. *bhū* (earth), *jala* (water), *agni* (fire), *vāyu* (air), *ākāśa* (sky or ether).
70. *Pañca-makāra* : *madhya, māṃsa, mīna, mūdrā* and *maithuna*.
71. *Pavana* : synonym of *vāyu* q.v.
72. *Pradakṣiṇā* : going round (clock wise) a deity etc.
73. *Pṛthivī or Pṛthvī* : synonym of *bhū* q.v.; letter *la*.
74. *Sahasrara* : 1000-petalled lotus supposed to exist in the head.
75. *Śaktibīja* : *hriṃ* (*MM*, p. 3).
76. *Śakti-trikoṇa* : triangle with apex downwards (ref. 16, p. 1149).
77. *Śānti* (Peace): vowel *i* (*MM*, p. 25, 27, 37 and 237).
78. *Saptamātṛikā* : 7 universal mothers, see ref. 69, p. 30 for names.
79. *Ṣaṭkoṇa* : (six-angled): hexagram (see Fig. 2).
80. *Śiva-trikoṇa* : triangle with apex upwards.
81. *Surpatlocana* : number 1000 (from Indra's eyes).[88]
82. *Tāraḥ* : the sacred syllable *om* (*MM*, pp. 5 and 237).
83. *Tattva* (element) : number 5 (cf. *bhūta*), or 24 (in *Mahābhārata* of *MM*, p. 167), or 25 (usual in *sāṃkhya*).

84. *Ṭha* : number 0 (according to *Ekākṣaranāma-koṣa*).
85. *Ṭhadvayam* : *svāhā* (*MM*, pp. 9 and 32).
86. *Trika* : trinity of Brahmā, Viṣṇu and Maheśa or of Śiva, Śakti and Nara; etc.
87. *Tri-pancāra yantra* : a special mystic diagram (ref. 69, p. 126).
88. *Trirekhāpuṭam* : triangle (see *Rāmāpurvatapini Upaniṣad*): (on page 237 of *MM*, *trikoṇaka* means *e*!).
89. *Vahni* : synonym of *agni* q.v.
90. *Varāha* (boar) : Letter *ha* (*MM*, pp. 23 and 239).
91. *Vasu* :letter *ra* (*MM*, p. 69).
92. *Vāyu* (air) : letter *ya* (*MM*, p. 239); a *bhūta* q.v.
93. *Vedādi* (origin of *Veda*) : sacred syllable *om* (*MM*, p. 237).
94. *Viyat* : synonym of *ākāśa* q.v.
95. *Yantra-gāyatrī* : *Gāyatrī mantra* for *yantras*.[89]
96. *Yoni-trikoṇa* : same as *śakti-trikoṇa* q.v.; triangle.
97. *Yoni-yugma* : same as *ṣatkoṇa* q.v. (ref. 85, p. 105).[90]

References and Notes

1. V. S. Apte, *The Student's Sanskrit-English Dictionary*, Reprinted, Delhi, 1965; p. 454.
2. M. Monier-Williams, *A Sanskrit-English Dictionary*, Reprinted, Delhi, 1972; p. 845.
3. Taranatha Bhattacharya, *Vācaspatyam* (1884), Reprinted in six volumes, Varanasi, 1990; vol. VI, p. 4771.
4. M. Sripathi Sastri (Compiler), *Sanskrit Dhātusāgara Taraṇiḥ*, Madras, 1968; pp. 113–114.
5. Apte, *op. cit.* (see reference no. 1 above), p. 454; Ishvarchandra Vidyasagar's *Subodha Sanskrit Vyākaraṇa Kaumudī*, edited by Ram Sundar Sharma, Ranchi, 1964; p. 251, *yantra* word comes from *yam* via '*tran*' *pratyaya*.
6. S. K. Ramachandra Rao, *The Yantras*, Sri Garib Dass Oriental Series No. 48, Delhi, 1988; p. 10.
7. Ph.D. Thesis, Lucknow University, 1990.
8. For details of all these *yantras*, see Ohashi's Thesis (ref. 7 above), pp.157 and 344 etc.
9. *Ibid*, p. 344.
10. *Ibid*, pp. 366–367.
11. *Ibid,* pp. 370–371. Also see Bapudeva Shastri's *Mānamandira Observatory of Kāśī* edited and translated by S.D. Sharma, Kurali, 1982; p. xiv.
12. For details of these *yantras*, see Satya Prakash, *Prācīna Bhārat me Rasāyan kā Vikās* (in Hindi), Lucknow, 1960, pp. 500–518.
13. Rao, *op. cit* (ref. 6 above), pp. 10–11.
14. References to the use of many mechanical *yantras* are found in ancient classical works such as *Mahābhārata* (see *Sabhāparva*, 5.10, and *Śāntiparva*, 58.65).
15. R. P. Anuruddha, *An Introduction into Lamaism*, V.V.R.I., Hoshiarpur, 1975; p. 142.
16. See *Puraścaryārṇava* (of Pratapa Simha Sahadeva, fl. 1775 AD) ed. by Murlidhara Jha, Chowkhamba Pratisthan, Delhi, 1985; p. 1158.
17. For details of these *yantras* of Sun, Moon, and other ancient planets, see R. C. Gupta, "Mystical Mathematics of Ancient Planets." *IJHS*, 40.1 (2005) 31–53.
18. For example see the *Yantracintāmaṇiḥ* of Damodara edited by Hans-Georg Turstig, Franz Steiner, Stuttgart, 1988; *yantra* Nos. 4 and 23.
19. *Webster's Seventh New Collegiate Dictionary*, Indian edition, Calcutta, 1971; p. 831.
20. Rao, ref. 6 above, p. 46.

21. Anuruddha, ref. 15, pp. 103–105.
22. *Ibid.,* p. 108.
23. Jinedra Varni, *Jainendra Siddhānta Kośa*, Part III, Delhi 1997, p. 353.
24. *Bṛhaddaivañja-rañjanam* of Rāmadīna (1897 AD) published from Venkatesvara Press, Bombay, 1987; pp. 96–97.
25. But the quoted verses are not found in the above mentioned (ref. 18) *Yantra-cintāmaṇi.*
26. See R. C. Gupta, "Early Pandiagonal Magic squares in India," *Bulletin of Kerala Mathematics Association*, 2.2 (2005), 25–44 for full details.
27. T. Hayashi, "Varāhamihira's Pandiagonal Magic Square of Order Four", *Historia Mathematica*, 14 (1987) 159–166.
28. S. Cammann, "Islamic and Indian Magic Squares, Part II", *History of Religion*, 8 (1969) 271–299; p. 272.
29. See the *Nārada-purāṇa* ed. by Shriram Sharma, Bareilly, 1971; Part II, p. 318. In the second verse of the chapter, the *Jyotiṣa* science is said to contain 4 lakh verses.
30. See *Jyautiṣa-śabdakoṣah* by Mukund Sharma, Amola (Garhwal), 1967; p. 112 where the quoted *mantra* is given.
31. *Gaṇita-kaumudī* ed. by Padmakara Dvivedi, Part II, Benares, 1942, p. 353.
32. *Mantra-mahodadhi* with commentary, Venkateśvara Press, Bombay, 1988; p. 180.
33. For details, see the recent paper of R. C. Gupta mentioned above (ref. 26), pp. 27–32.
34. See *Puraścaryārṇava* (ref. 16), p. 524.
35. R. C. Gupta, "*Agni-Kuṇḍas*—A neglected area of study in the history of ancient Indian Mathematics", *IJHS*, 38.1 (2003) 1–15; p. 6.
36. See *Puraścaryārṇava* (ref. 16), p. 525.
37. According to Rao (ref. 6), p. 31.
38. See the *Sūrya-siddhānta* (with commentary of Raṅganātha) ed. by B. P. Mishra, Bombay, 1956, pp. 300–301.
39. See the *Kalyāṇa* (Hindi Monthly), vol. 42, No. 1 (Upāsanā-Number), 1968, p. 355.
40. For details of the various Indian systems, see, e.g. B. Datta and A. N. Singh, *History of Hindu Mathematics,* Single Vol. edition, Bombay, 1962, Part I, pp. 53–85.
41. *Mantra-mahodadhi* (ref. 32 above), p. 82. Here *rudra* = 11.
42. See Apte's *Dictionary* (ref. 1), p. 1 where *'a'* is said to denote *Viṣṇu* and *'u'* stands for *Maheśvara*.
43. See the *Mātṛkā-nighaṇṭu (Kośa)* at the end of *Mantra-mahodadhi* (ref. 32), pp. 237–239.
44. *Ibid.,* p. 27 (of ref. 32).
45. *Ibid.,* p. 42.
46. *Puraścaryārṇava* (ref. 16), p. 1149.
47. See *Jinendra-siddānta-kośa* (ref. 23), p. 365.
48. See *Gaṇita-kaumudī* (ref. 31), p. 397. For construction details and arrangement of numbers see *Gaṇita Bhāratī*, vol. 24 (2002) 86–88.
49. See Rao (ref. 6 above), p. 14.
50. N. J. Bolton and D. N. G. Macleod, "The Geometry of the *Śrīyantra*", *Religion*, vol. VII (1) (1977) 65–85, pp. 68–69. also Vaishampayam, "Sri-Chakra", *Dilip*, Vol. 1, no. 5 (1974) 6–7.
51. P. Ramakrishnan, *Indian Mathematics Related to Architecture,* Ph.D. Thesis, Cochin University of Science and Technology, 1998; pp. 120–123.
52. For other details see R. C.Gupta, "A Little Known Text and Version of Śrīyantra", *Gaṇita-Bhāratī*, 25 (2003) 22–28.
53. See Bolton and Macleod (ref. 50 above), p. 68.
54. *Ibid.,* pp. 76–76 contain the details.
55. *Kalyāṇa*, Special Issue on *Śakti Upāsanā*, Gorakhpur, 1987, pp. 255–256.
56. See *Puraścaryārṇava* (ref. 16), p. 1152; *Vācaspatyam* (ref. 3), vol. VI, p. 5154; also the next reference.
57. See *Kalyāṇa Śakti Upāsana* Issue (ref. 55 above), p. 255.
58. See Rao (ref. 6 above) p. 53 for a Sanskrit verse and the figure of the complex.
59. *Mantra-mahodadhi*, XI, 53b–54a (ref. 32, p. 93).

60. A. P. Kulaichev, "*Śrīyantra* and its Mathematical Properties", *IJHS* 19.3 (1984) 279–292.
61. Bolton and macleod (ref. 50 above), pp. 71–79.
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63. Anuruddha (ref. 15 above), p. 108.
64. *Puraścaryārṇava* (ref. 16), p. 525.
65. See Kulaichev (ref. 60 above), p. 279.
66. See *Saundarya-laharī* ed. by V. K. Subramanian, Delhi, 1990, pp. 6–7. It may be pointed out that a commentator of the above work attributes it to Prabarasena, see *Tāntrika Sāhitya* (in Hindi) by Gopinath Kaviraj. Lucknow, 1972, p. 712.
67. See *IJHS*, 33 (1998) 227.
68. Same journal, Vol. 24 (1989) 137–149.
69. A recent good book is the *Camatkārī Fifty-five Pūja yantra* (in Hindi) by Kulapati Mishra, Randhir Prakashan, Hardwar, 2004.
70. See above book, pp. 35–38, and P.C, p. 1140. However, the Jaina *Vināyaka-yantra* is different (ref. 23, p. 360).
71. See Mitali Dev (editor), *Kuṇḍaratnāvalī of Rāmacandra-dīkṣita* (1868), Varanasi, 2003, p. 116.
72. See K. Mishra, *op. cit.* above (ref. 69), pp. 100 and 193. There is an error in drawing the top line of the third triangle on p. 100 (line PQ is not drawn along KL).
73. *Ibid.,* p. 79.
74. *Ibid.,* p. 76; *Kalyāṇa–Śakti Upāsanā* Issue (ref. 55 above), plate facing p. 38.
75. MM commentary (p. 167) takes *'tattva'* equal to 24. Also cf. (*Upāsanā-Number* (ref. 39), p. 405.
76. See K. Mishra's book (ref. 69), pp. 79–81. He calls Fig. 21 as *nava-koṇa cakra* but does not quote any text.
77. C. A. Pickover, *The Zen of Magic Squares, Circles, and Stars,* Universities Press, Hyderabad, 2002, pp. 339–340. Also see plate xvii in the *Album of Yantras* (in Hindi) by R. Mishra, Delhi, *s.a.*
78. See the *Grahaśānti Prayogaḥ* edited with Daulata Ram Gaud's commentary, Varanasi, 2001, p. 155 where *bhūpura* is shown as triple-lined and some other variants of *Rudra yantra* are mentioned.
79. For details see R. C. Gupta's paper (ref. 17 above) especially pp. 36–38 and 45–48.
80. The new translation has been suggested by Dr. Takao Hayashi in a personal letter.
81. See R. C. Gupta (ref.17), pp. 46–48.
82. *Vācaspatyam* (see ref. 3 above), vol. VI, p. 5258–5259.
83. See *Śrī Kālacakratantra-raja* ed. by Biswanath Banerjee, Asiatic Society, Kolkata, 1985, Appendix I.
84. See *Vācaspatyam* (see ref. 3 above), vol. VI, p. 4686.
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86. LXI. 32–33, see P.K. Acharya, *Hindu Architecture*, p. 279.
87. See Ganitanand (=R. C. Gupta), Hindu Gods on the Gateway of a Mosque and Some World-Numerals, *Gaṇita-Bhāratī*, 23 (2001), 120–121, for details and more such nos.
88. *Ibid.* The expression *surpatlocana* (God Indra's eye) is found in a mosque inscription of 1587 AD.
89. Rao (ref. 6, p. 30) gives it as follows: *Yantra-rājāya vidmahe; mahāyantraya dhimahi; tanno yantraḥ pracodayāt.*
90. In India geometry forms and figures (e.g. *yonikuṇḍa*) which resemble a leaf of the *pippal* tree of fig family (*Ficus Religiosa*) are also usually called *yoni* figures.

Part VI

Interpolations and Combinatorics

Second-Order Interpolation in Indian Mathematics up to the Fifteenth Century

The computational abilities of ancient Indian mathematicians are well known. The paper deals with the second-order interpolation schemes found in a few astronomical works of India. The earliest one is the rule of Brahmagupta (c. AD 625) for equal intervals, which resembles the modern Newton–Stirling interpolation formula up to the second-order. Later on (AD 665), Brahmagupta also gave a modified form of his rule to cover the case of unequal intervals. Then we come across a peculiar set of rules for second-order interpolation in a work of Govindasvāmin (c. AD 800–850). The famous Bhāskara II (c. AD 1150) gave an empirical derivation of Brahmagupta's rule for equal knots. Next are described the Indian forms of the second-order Taylor series approximations which are attributed to Mādhava (AD 1350–1410). Finally are given the forms of various rules quoted by Parameśvara (c. first quarter of the fifteenth century AD).

Symbols

a	the argument, circular arc measured in angular units; anomaly.
a_1, a_2, etc.	successive unequidistant values of a.
h	equal (common) arcual interval; elemental arc.
h_1, h_2, etc.	unequal arcual intervals (*gatis*); $h_1 = a_1$; $h_2 = a_2 - a_1$; $h_3 = a_3 - a_2$, etc.
R	*sinus totus* (radius).
$R \sin a$, $R \cos a$, R versin a	Indian sine, cosine and versed sine of the arc a
$f(a)$	the functional value of sine, versed sine or certain astronomical function called 'equation' (*phala*).
p, q	positive integers; $x = p \cdot h$ or a_p; arc passed over, such that $f(x)$ is known.

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θ	residual arc such that $f(x+\theta)$ is required to be interpolated, θ being positive and less than h or h_{p+1}.
n	$\frac{\theta}{h}$.
D_1, D_2, etc.	tabulated functional differences; $D_1 = f(a_1)$ or $f(h)$; $D_2 = f(a_2) - f(a_1)$ or $f(2h) - f(h)$; $D_3 = f(a_3) - f(a_2)$ or $f(3h) - f(2h)$, etc.
Δ	first-order forward difference operator; $\Delta f(a) = f(a+h) - f(a)$; $\Delta f(a_q) = f(a_{q+1}) - f(a_q)$; $\Delta f(x) = D_{p+1}$.
Δ^2	second-order difference operator.
h_p, h_{p+1}	argumental intervals just passed over (last or *bhukta-gati*) and yet to be passed over (current or *bhogya-gati*), respectively.
D_p, D_{p+1}	the corresponding tabulated functional differences passed over (*bhukta-khaṇḍa* or *gatiphala*) and to be passed over (*bhogya-khaṇḍa* or *gatiphala*), respectively.
D_t	the envisaged true (*sphuṭa*) value of the functional difference to be passed over.
Z_p	'adjusted' value of the functional difference passed over in case of unequal intervals.

1 Introduction

Tabular values of the trigonometric functions R sin a, R versin a or their differences and of certain astronomical functions are found almost in every work on astronomy of ancient and medieval India. Various numerical values of R and h were taken by the Indians. For computing the functional values corresponding to the intervening values of the argument, the ordinary method used was that of linear proportion, i.e. first-order interpolation. For better results, more elegant techniques using second-order interpolation schemes are also found in few Hindu works. Below we give the methods described by few Indian astronomer–mathematicians starting with Brahmagupta (seventh century) who taught, 'for the first time in the History of Mathematics, the improved rules for interpolation by using the second differences'.[1]

2 Brahmagupta's Rule for Equal Intervals

It is well known[2] that Brahmagupta composed *Brāhmasphuṭa-siddhānta* in AD 628 and *Khaṇḍakhādyaka* in AD 665. The famous couplet containing Brahmagupta's interpolation rule for equal intervals is found in the *uttara* (supplementary) part of

Khaṇḍakhādyaka. However, an earlier reference is worth noting. The same couplet occurs in Brahmagupta's earliest known work, the *Dhyāna-graha-adhikāra* or *Dhyāna-graha-adhāya* or the *Dhyānagrahopadeśādhyāya*. That the *Dhyānagraha* was written earlier than the *Brāhmasphuṭa-siddhānta* is concluded on the ground that the latter (XXIV, 9) quotes the former.[3] Hence, the invention of the second-order interpolation formula given by Brahmagupta should be placed near the beginning of the second quarter of the seventh century AD, if not earlier.

Now we quote the couplet:

> गतभोग्यखण्डकान्तरदलविकलवधात् शतैर्नवभिराप्तैः ।
> तद्युतिदलं युतोनं भोग्यादूनाधिकं भोग्यम् ॥

(*Dhyāna-graha-Upadeśa-adhyāya*, 17;
Khaṇḍakhādyaka, IX, 8, etc.)[4]

> Multiply half the difference of the tabular differences crossed over and to be crossed over by the residual arc and divide by 900' (= *h*) by the result (so obtained) increase or decrease half the sum of the same (two) differences, according as this (semi-sum) is less or greater than the difference to be crossed over. We get the true functional differences to be crossed over.

That is

$$D_t = \frac{1}{2}(D_p + D_{p+1}) \pm \frac{1}{2}(D_p \sim D_{p+1})\frac{\theta}{h}, \tag{1}$$

the upper or lower sign is to be taken according as $\frac{1}{2}(D_p + D_{p+1})$ is less than or greater than D_{p+1}, i.e. according as D_p is less or greater than D_{p+1}. Then we have

$$f(x+\theta) = f(x) + \frac{\theta}{h} \cdot D_t. \tag{2}$$

Combining (1) and (2) and using $D_{p+1} = \Delta f(x)$, we easily obtain

$$f(x+nh) = f(x) + \frac{n}{2}\{\Delta f(x-h) + \Delta f(x)\} + \frac{n^2}{2}\{\Delta f(x) - \Delta f(x-h)\}$$

which may be regarded as the modern form of Brahmagupta's rule and is a particular case (up to second-orders) of the more general Newton–Stirling interpolation formula of modern Mathematics.[5] From the context in the work *Dhyāna-graha-Upadeśa-adhyāya*, it is clear that Brahmagupta gave the rule for the interpolation of sine ($D_p > D_{p+1}$) and the versed sine ($D_p < D_{p+1}$). However, in the statement of the rule itself, there is no such limitation on the scope of its use and the rule may be applied to other functions tabulated at equal intervals. In fact, the commentators Pṛthūdaka (AD 864) and Āmarāja (1180) both explained its use in finding the equation of centre (*manda-phala*).[6] Brahmagupta's rule is also found in the *Vaṭeśvara-siddhānta* (II, i, 62) of AD 904.

3 Bhāskara II's Form of Brahmagupta's Formula and Its Rationale

In the *Grahagaṇita* part of his *Siddhānta-śiromaṇi*, Bhāskara II (AD 1150) gives the rule as

यातैष्ययोः खण्डकयोर्विशेषः शेषांशनिघ्नो नखहृत् तदूनम् ।
युतं गतैष्यैक्यदलं स्फुटं स्यात् क्रमोत्क्रमज्या करणेऽत्र भोग्यम् ॥ १६ ॥

(*Siddhānta-śiromaṇi*, II (*Spaṣṭādhikāra*), 16)[7]

Multiply the difference of the tabular differences passed over and to be passed over by the residual arc and divide by 20. By the result decrease or increase half the sum of (the differences) passed over and to be passed over to get the true difference to be passed over for interpolating sine and versed sine respectively.

That is,

$$D_t = \frac{1}{2}(D_p + D_{p+1}) \pm \frac{\theta}{20} \times (D_p \sim D_{p+1}),$$

where we have to take the negative and positive signs, respectively, for computing sine and versed sine.

Thus, we see that the formula is same as that of Brahmagupta except that Bhāskara II takes $h = 10°$, instead of $h = 900'(= 15°)$. The rationale (*upapatti*) of the rule given by Bhāskara II is as follows.[8]

D_{p+1} and $\frac{1}{2}(D_p + D_{p+1})$ are the tabular differences at the end and beginning of the current interval, respectively. Take proportional part of their difference (to get the necessary correction to be made for finding the true difference corresponding to the intervening point).

By the rule of three, this correction (change)

$$= \left[\frac{1}{2}(D_p + D_{p+1}) \sim D_{p+1}\right] \frac{\theta}{10}$$
$$= \frac{\theta}{20} \times (D_p \sim D_{p+1}).$$

This combined with $\frac{1}{2}(D_p + D_{p+1})$ (which has been taken above as the tabular difference at the beginning of the current interval) will give the required D_t. We have to add or subtract the correction term since tabular differences decrease (*apacaya*) for sine and increase (*upacaya*) in case of versed sine. Thus, it is proved.

In the above proof of Bhāskara II, the assumption that $\frac{1}{2}(D_p + D_{p+1})$ is the tabular difference at the beginning of the interval considered is wrong since it is actually D_p there[*] Kamalākara (AD 1658) attacked Bhāskara II on this point in his

[*]However, these arguments are not adequate, since the true functional difference, corresponding to the residual arc, is $\frac{\theta}{h} \cdot D_t$ (and not simply D_t) which will be zero (as it ought to be) when $\theta = 0$, whether D_t is taken D_p or $\frac{1}{2}(D_p + D_{p+1})$ there.

Siddhānta-tattva-viveka (II, 179).[9] Whether Brahmagupta argued in the same manner as Bhāskara II to arrive at his rule or had some other approach for it is difficult to say in the absence of specific evidence.

In case of sine function, the following derivation of the rule will not be without interest here. We have

$$\begin{aligned} \frac{1}{2}(D_p + D_{p+1}) &= \frac{R}{2}\{\sin x - \sin(x-h) + \sin(x+h) - \sin x\} \\ &= R \cos x \sin h \end{aligned} \tag{3}$$

and

$$\begin{aligned} \frac{1}{2}(D_p - D_{p+1}) &= \frac{R}{2}\{\sin x - \sin(x-h) - \sin(x+h) + \sin x\} \\ &= R \sin x(1 - \cos h). \end{aligned} \tag{4}$$

If θ and h are small, we may assume

$$\frac{\sin\theta}{\sin h} = \frac{\sin\left(\frac{1}{2}\theta\right)}{\sin\left(\frac{1}{2}h\right)} = \frac{\theta}{h}. \tag{5}$$

Now the 'true' *bhogyaphala*, $\frac{\theta}{h} \cdot D_t$

$$\begin{aligned} &= R \sin (x+\theta) - R \sin x \\ &= R \cos x \sin \theta - R \sin x(1 - \cos \theta) \\ &= R \cos x \sin \theta - R \sin x \cdot 2\sin^2\left(\frac{1}{2}\theta\right) \\ &= R \cos x \cdot \frac{\theta}{h} \cdot \sin h - R \sin x \cdot 2\frac{\theta^2}{h^2} \sin^2\left(\frac{1}{2}h\right) \qquad \text{by (5)} \\ &= R\frac{\theta}{h} \cdot \cos x \cdot \sin h - R \cdot \frac{\theta^2}{h^2} \cdot \sin x(1 - \cos h) \\ &= \frac{\theta}{h} \cdot \frac{1}{2}(D_p + D_{p+1}) - \frac{\theta}{h} \cdot \frac{\theta}{h} \cdot \frac{1}{2}(D_p - D_{p+1}) \qquad \text{by (3) and (4)} \end{aligned}$$

hence we get the required expression for the 'true' *bhogya-khaṇḍa*, D_t.

4 Brahmagupta's Rule for Unequal Intervals

This rule is given by Brahmagupta in *Khaṇḍakhādyaka* (AD 665) in connection with computing the *gatiphala* (change in the equation) corresponding to any given *gati* (change in anomaly) by using the tabulated values of the *gatiphala* at unequal intervals. The relevant Sanskrit passage is

भुक्तगतिफलांशगुणाभोग्यगतिः भुक्तगतिहृता लब्धम् ।
भुक्तगतेः फलभागास्तद्भोग्यफलान्तरार्धहतम् ॥
विकलं भोग्यगतिहृतं लब्धेनोनाधिकं फलैक्यार्धम् ।
भोग्यफलादधिकोनं तद्भोग्यफलं स्फुटं भवति ॥

(*Khaṇḍakhādyaka*, II, 12–13, etc.)[10]

Multiply the last *gatiphala* (in degree) by the current *gati* and divide by the last *gati*; the result is the "adjusted" last *gatiphala* (in degrees). Multiply half the difference of the "adjusted" last *gatiphala* and the current *gatiphala* by the residual arc and divide by the current *gati*. By the new result decrease or increase half the sum of the "adjusted" last *gatiphala* and the current *gatiphala* when this half sum is more or less than the current *gatiphala*. The final result is the true current *gatiphala*, i.e. true functional difference to be passed over.

In symbols

$$Z_p = D_p \cdot \frac{h_{p+1}}{h_p}, \qquad \text{and}$$

$$D_t = \frac{1}{2}(Z_p + D_{p+1}) \pm \frac{\theta}{h_{p+1}} \frac{1}{2}(Z_p \sim D_{p+1})$$

the upper or lower sign is to be taken according as $\frac{1}{2}(Z_p + D_{p+1})$ is less or greater than D_{p+1}, i.e. according as Z_p is less or greater than D_{p+1}.

The desired result is given by

$$\begin{aligned} f(x+\theta) &= f(x) + \frac{\theta}{h_{p+1}} \cdot D_t, \\ \text{where} \qquad x &= a_p \\ &= h_1 + h_2 + \cdots + h_p \\ \text{and} \qquad f(x) &= D_1 + D_2 + \cdots + D_p. \end{aligned}$$

The numerical illustration of this rule given by Sengupta[11] in his paper is wrong. Instead of $D_p (= 12^\circ$ in his example), he put θ $(= 14^\circ)$ in finding Z_p. However, this error was avoided by him while illustrating the rule in his translation of the *Khaṇḍakhādyaka*.[12]

In case the intervals are equal, i.e. $h_p = h_{p+1}$, we will have $Z_p = D_p$ itself, and the rule will reduce, as should be expected, to his earlier rule for equal intervals.

5 Govindasvāmin's Rule for Second-Order Interpolation

About two centuries after Brahmagupta, we come across a set of peculiar rules of making second-order interpolation to compute the intermediary functional values. These rules are found in Govindasvāmin's (c. AD 800–850)[13] commentary on

Mahābhāskarīya of Bhāskara I (seventh century AD). The peculiar thing to note is that different formulae are laid down for different argumental intervals. The relevant text is as follows:

गच्छद्यातगुणान्तराहतवपुर्यातैष्यदिष्वासन-
च्छेदाभ्याससमूहकार्मुककृतिप्राप्तात्, त्रिभिस्ताडितात् ।
वेदैः षड्भिरवाप्तमन्त्यगुणजे राश्योः क्रमाद्, अन्त्यभे
गन्तव्याहतवर्तमानगुणजाच्चापाप्तमेकादिभिः ॥
अन्त्यादुत्क्रमतः क्रमेण विषमैः संख्याविशेषैः क्षिपेत् *
भक्त्वाप्तं, यदि मौर्विकाविधिरयं मख्याः क्रमाद् वर्तते ।
शोध्यं व्युत्क्रमतस्तथाकृतफलं, ...

(Govindasvāmin's comm. on *Mahābhāskarīya*)[14]

Multiply the difference of the last and the current sine differences by the two parts of the elemental arc (made by any intermediary point on it) and divide by the square of the elemental arc and further multiply by three. Now divide the result so obtained by four, in the first *rāśi*, or by six, in the second *rāśi*. The final result thus obtained should be added to the portion of the current sine difference (got by linear proportion).

In the last (third) *rāśi*, multiply the linearly proportional part of the current sine difference by the remaining part of the elemental arc and divide by the elemental arc. Now divide the result (so obtained) by the odd numbers 1,[3,5], etc., according as the current sine difference is first, (second, third), etc., when counted from the end in the reversed order. Add the final result thus obtained to the portion of the current sine difference (got by ordinary proportion).

These are the rules of computing true sine difference for (direct) sines. In case of versed sines apply the rules in the reversed order* and the above corrections are to be subtracted from the respective differences (got by linear interpolation).

Let the true sine difference (*bhogyaphala*) desired,

$$R\sin(x+\theta) - R\sin\ x = \frac{\theta}{h} \cdot D_{p+1} + E, \quad \text{approximately,}$$

where $\frac{\theta}{h} \cdot D_{p+1}$ is the portion of the current sine difference, D_{p+1}, obtained by the ordinary first-order linear interpolation, and E is the term got by second-order interpolation. Then, according to the above rules, we have (assuming 24 equal divisions of the first quadrant).

$$E = \frac{1}{4} \times \frac{3\theta(h-\theta)}{h^2}(D_p - D_{p+1}), \qquad \text{when } p = 1 \text{ to } 7$$
$$E = \frac{1}{6} \times \frac{3\theta(h-\theta)}{h^2}(D_p - D_{p+1}), \qquad \text{when } p = 8 \text{ to } 15$$
$$E = \frac{(h-\theta)}{h} \times \frac{\theta}{h} D_{p+1} \times \frac{1}{(47-2p)}, \quad \text{when } p = 16 \text{ to } 23.$$

*In the published article, the reading was: 'विषमैः विशेषैः क्षिपेद्'. This has been refined in consultation with the source work. (–*ed.*).

*That is, the first, second, third *rāśi* rules of sines are to be used, respectively, for third, second, first *rāśi* in case of versed sines.

Using the general functional notation and finite difference operator, the rule for the second *rāśi* (30° to 60°) may be put as

$$f(x+nh) = f(x) + n\Delta\, f(x) + \frac{n(n-1)}{2}\{\Delta\, f(x) - \Delta\, f(x-h)\},$$

which is the modern form of Govindasvāmin's rule and is a particular case (up to the second-order) of the general Newton–Gauss interpolation formula.[15] Mathematically, this rule of Govindasvāmin is equivalent to that of Brahmagupta for equal knots. But, unlike Brahmagupta, Govindasvāmin gives different rules for the three *rāśis* of the first quadrant.[†]

6 Mādhava's Taylor Series Approximation

Nīlakaṇṭha (1443–1545)[16] in his commentary on *Āryabhaṭīya* quotes the text of rules for computing sine and cosine functions, which are equivalent to modern Taylor series approximations up to the second-order of small quantities. Nīlakaṇṭha attributes these rules to Mādhava (1350–1410),[17] who antedates Taylor by more than 300 years.[18] The same verses containing the same rules (but without mentioning Mādhava) are also included by Nīlakaṇṭha in his *Tantrasaṅgraha* (AD 1500), a 'compendium' on astronomy. The text is

तत्राह माधवः

इष्टदोःकोटिधनुषोः स्वसमीपसमिरिते ।
ज्ये द्वे सावयवेऽन्यस्य कुर्यादूनाधिकं धनुः ।
द्विघ्नतल्लिप्तिकाप्तैकशरशैलशिखीन्दवः ।
न्यस्याच्छेदाय च मिथः तत्संस्कारविधित्सया ।
छित्वैकां प्राक् क्षिपेज्झह्यात् तद्धनुष्यधिकोनके ।
अन्यस्यामथ तां द्विघ्ना तथा स्यामिति संस्कृतिः ।
इति ते कृतसंस्कारे स्वगुणौ धनुषोस्तयोः ।

(*Āryabhaṭīya-bhāṣya* (II, 12);[19]
Tantrasaṅgraha (II, 10–13))[20]

Thus spake Mādhava:
Placing the (sine and cosine) chords nearest to the arc whose sine and cosine chords are required get the arc difference to be subtracted or added. For making the correction, 13751 should be divided by twice the arc difference in minutes and the quotient is to be placed as the divisor. Divide the one (say sine) by this (divisor) and add to or subtract from the other (cosine) according as the arc difference is to be added or subtracted. Double this (result) and do as before (i.e., divide by the divisor). Add or subtract the result (so obtained) to or from the first sine and cosine to get the desired sine or cosine chords.

[†]The author of the present paper proposes to publish a separate article about Govindasvāmin's computations of Indian sines.

That is,

$$\text{`divisor'} = \frac{13751}{2\theta} = D, \qquad \text{say.}$$

$$\text{Then} \qquad \sin(x+\theta) = \sin\ x + \left(\cos\ x - \frac{\sin\ x}{D}\right)\frac{2}{D}$$

$$\cos(x+\theta) = \cos\ x - \left(\sin\ x + \frac{\cos\ x}{D}\right)\frac{2}{D}.$$

The *sinus totus* in this case is

$$\begin{aligned} R &= \frac{21600}{2\pi} = 3437.75 \quad \text{very nearly} \\ &= \frac{13751}{4} \\ \therefore \qquad D &= \frac{2R}{\theta}. \end{aligned}$$

Using this, the above approximations can be easily put as

$$\sin(x+\theta) = \sin\ x + \frac{\theta}{R}\cdot\cos\ x - \frac{\theta^2}{2R^2}\cdot\sin\ x \qquad \text{and}$$

$$\cos(x+\theta) = \cos\ x - \frac{\theta}{R}\cdot\sin\ x - \frac{\theta^2}{2R^2}\cdot\cos\ x$$

which are particular cases of the well-known Taylor series,

$$f(x+\theta) = f(x) + \theta f'(x) + \frac{\theta^2}{2!}f''(x) + \ldots,$$

for sine and cosine, respectively, up to second power of small quantity (in radians when using the Taylor series).

Shukla[21] interprets the text to yield the rule in the following mathematical form:

$$\sin(x+\theta) = \sin\ x + \frac{\theta}{2R}\{\cos\ x + \cos(x+h)\}.$$

But this interpretation does not seem to conform to the text closely. Our interpretation is justified by the commentator Śaṅkara Vāriar[22] (AD 1556) as well as by the exposition of the text in *Yukti-bhāṣā,*[23] a work attributed to Jyeṣṭhadeva (c. 1475–1575) by K. V. Sarma.[24]

7 Forms of Various Rules Found in the Works of Parameśvara

In Parameśvara's commentary (AD 1408)[25] on *Laghubhāskarīya* is found a second-order interpolation rule described in the following text:

गतैष्यचापांशकयोः संवर्गेण समाहतम् ।
पूर्वापरोत्थखण्डज्या विवरस्य दलं हरेत् ॥
चापवर्गेण तत्राप्तिमिष्टज्यासु विनिक्षिपेत् ।
यत्राधिका पराखण्डजीवा तत्र तु शोधयेत्[*] ॥

(Parameśvara's comm. on *Laghubhāskarīya*)[26]

By the product of the two parts of the elemental arc (made by any intermediary point on it) multiply half the difference of the last and the current sine differences and divide by the square of the elemental arc. By the quotient (so obtained) increase or decrease the sine desired for correction accordingly as the current sine difference is less or greater than the last sine difference.

That is,

$$E = \pm\frac{\theta(h-\theta)}{h^2} \times \frac{1}{2}(D_p \sim D_{p+1}).$$

Thus, we see that this formula is same as that of Govindasvāmin for the argumental interval from 30° to 60°. However, unlike Govinasvāmin, Parameśvara recommends the use of this single rule for the whole of the first quadrant.

Exactly the same rule but described in different words is found quoted in Parameśvara's supercommentary (called *Siddhāntadīpikā*) on Govindasvāmin's commentary on the *Mahābhāskarīya*.[27] But here Parameśvara accepts that the rule is not his own, for the statement of the rule is found to be preceded by the words

केचिदाहुः or केचिदेवमाहुः
Thus is said by others.

thereby clearly ascribing the rule to other persons.

Finally, in the *Siddhāntadīpikā* mentioned above are also found the rules, again ascribed to others, for computing sines and cosines using central values but ultimately amounting to the use of Taylor series approximations up to the second-order. The text quoted is

चापखण्डस्य मध्योत्था या कोटिज्या तया हतात् ।[†]
चापखण्डात् त्रिज्ययाप्तं तत्खण्डे दोर्गुणो भवेत् ॥ ७ ॥
चापखण्डस्य मध्योत्थभुजज्यानिहतात् तथा ।
चापखण्डात् त्रिज्ययाप्तं तत्खण्डे कोटिका भवेत् ॥ ८ ॥
चापखण्डार्धसंभूतदोःकोट्योर्विधिरुच्यते ।
दोर्ज्यां चापान्तजां चापखण्डार्धेन समाहताम् ॥ ९ ॥

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[*]In the published article the reading was: 'विशोधयेत्'. This has been refined in consultation with the source work. (–*ed.*).

[†]We have changed the printed हृतात् to हतात् for an obvious reason. (–*ed.*).

व्यासार्धेन विभज्याप्तं यत् स्यात्तेन विवर्जिता ।
चापान्तोत्था कोटिजीवा चापखण्डार्धजा भवेत् ॥ १० ॥
तया कोट्या हताच्चापखण्डार्धात् त्रिज्यया हृतम् ।
यत् स्यात् तेन तु संयुक्ता दोर्ज्या चापान्तसंभवा ॥ ११ ॥
चापखण्डार्धजा सा स्याद्, भूयोऽप्येवं गुणद्वयम् ।
साध्यं परस्परं चापखण्डमध्योत्थजीवया † ॥ १२ ॥
अविशिष्टं तु तद् द्वन्द्वं ज्याखण्डाप्तौ स्फुटं भवेत् ।

(*Siddhāntadīpikā* of Parameśvara)[28]

7. Multiply the cosine at the middle of the residual arc by the residual arc and divide by the radius. That becomes the sine-difference for the residual arc.
8. Multiply the sine at the middle of the residual arc by the residual arc and divide by the radius. That becomes the cosine-difference for the residual arc.
9–10. (Now) is described the method of finding the sine and cosine at the middle of the residual arc (needed above). Multiply the sine of the arc passed over by half the residual arc, and divide by the radius. By the result, so obtained, subtract the cosine of the arc passed over. This gives the cosine at the middle of the residual arc.
11. Multiply the cosine of the arc passed over by half the residual arc and divide by the radius. The result, so obtained, is to be added to the sine of the arc passed over.
12. That becomes the sine at the middle of the residual arc. Thus compute the sine and cosine differences (of the residual arc) mutually from the cosine and sine at the middle of the residual arc.
13. Combine the sine and cosine of the arc passed over respectively with the sine and cosine differences of the residual arc (got above), we get the true desired sine and cosine for any arc....

In symbols, we can write the rules as follows:

$$R\sin(x+\theta) - R\sin\, x = R\cos\left(x+\frac{\theta}{2}\right)\cdot\frac{\theta}{R} \tag{6}$$

$$R\cos(x+\theta) \sim R\cos\, x = R\sin\left(x+\frac{\theta}{2}\right)\cdot\frac{\theta}{R} \tag{7}$$

$$R\cos\left(x+\frac{1}{2}\theta\right) = R\cos\, x - R\sin\, x\cdot\frac{\theta}{2R} \tag{8}$$

$$R\sin\left(x+\frac{1}{2}\theta\right) = R\sin\, x + R\cos\, x\cdot\frac{\theta}{2R} \tag{9}$$

Combining (6) with (8) and (7) with (9), we get

$$R\sin(x+\theta) = R\sin\, x + \frac{\theta}{R}\cdot R\cos\, x - \frac{\theta^2}{2R^2}\cdot R\sin\, x$$

$$R\cos(x+\theta) = R\cos\, x - \frac{\theta}{R}\cdot R\sin\, x - \frac{\theta^2}{2R^2}\cdot R\cos\, x$$

which are the modern Taylor series approximations up to the second-order and which are already ascribed to Mādhava by Nīlakaṇṭha as has been already pointed out.

†In the published article the reading was: `चापखण्डोत्थजीवया'. This has been refined in consultation with the source work. (–*ed.*).

Also, as noted above, Parameśvara himself attributes the rules to others although he does not mention any specific names in this connection. But since Parameśvara (c. AD 1360–1460) is stated[29] to have studied under Mādhava (c. AD 1350–1410) in his young age, it is very likely that Mādhava was the source for Parameśvara for these rules involving values at the centre of the residual arc.

8 Concluding Remarks

The difference of the first-order finite differences, which is the second-order differences, was used as early as the fifth century AD by the Indian astronomer Āryabhaṭa I. His *Āryabhaṭīya* (II, 12)[30] contains a rule equivalent to the relation

$$\Delta^2 \sin\, x = -k \sin\, x$$

which was apparently used in finding tabular sine differences. But the use of the second-order differences for interpolating intervening functional values appears in India in the early part of the seventh century in the works of Brahmagupta. In modern language, Brahmagupta's technique of interpolating the functional value between a pair of tabular entries amounts to passing a parabola through the two functional values at the end points of the interval and the next preceding tabular value.

It is stated[31] that Al-Bīrūnī (eleventh century AD) in his work *Canon Masudicus* employs a parabola through the same end points. The scheme found in the work *Zij-i-Khāqānī* of Al-Kāshī (d. AD 1429) using second-order differences is about interpolating planet's longitudinal speed denoted by the Persian-Arabic word '*buht*' which, as pointed out by Kennedy,[32] is from the Sanskrit word '*bhukti*'. It is not difficult to understand the Indian influence in general when we remember that both the important works of Brahmagupta, viz. *Brāhmasphuṭa-siddhānta* and *Khaṇḍakhādyaka*, were translated into Arabic at Baghdad under the titles '*Sind-Hind*' and '*Al-arkand*' as early as eighth century of our era.[33]

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$$D_{p+1} - D_p = -\frac{1}{225}\sum_{r=1}^{p} D_r$$

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Munīśvara's Modification of Brahmagupta's Rule for Second-Order Interpolation

When the values of a function are tabulated for some discrete values of the argument, the functional values corresponding to intermediary argumental values are obtained ordinarily by linear interpolation. For greater accuracy, higher order technique is necessary. It is known that the famous Indian mathematician Brahmagupta (seventh century AD) gave a rule for second-order interpolation. This yields results equivalent to what one will get by using the Newton–Stirling formula upto the same order.

Munīśvara (seventeenth century) has given a modification, which consists of applying a process of iteration and leads to better results in some cases. The paper presents a discussion on Brahmagupta's original rule, its modification by Munīśvara, and the example which has been worked out by the latter.

Symbols

a	– the argument, circular arc measured in angular units
$a_0, a_1, a_2 \ldots,$	– successive and equidistant values of a with $a_0 = 0$.
$D_1, D_2, D_3 \ldots$	– tabulated functional differences $D_1 = f(a_1) - f(a_0)$, $D_2 = f(a_2) - f(a_1)$, etc.
D_p, D_{p+1}	– tabulated functional difference just crossed over *(bhukta-khaṇḍa)* and the current tabulated functional difference *(bhogya-khaṇḍa)*. $D_m = \frac{1}{2}\left(D_p + D_{p+1}\right)$.
D_t	– the true *(sphuṭa)* value of the current functional difference as given by Brahmagupta.

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$f(a)$ – functional value corresponding to the argumental value a. The function considered here is either the Indian Sine ($= R\,sin\,a$), or the Indian Versed Sine ($R\,vers\,a$). So that we have $f(a_0) = 0$.

p – positive integer.

h – equal (or common) arcual interval

$h = a_1 - a_0 = a_2 - a_1$ and so on.

$n = \frac{\theta}{h}$.

$x = p.h$, the arc crossed over.

R – *Sinus totus,* radius of the circle of reference defining Sine and Versed Sine.

T – mathematically exact value of the current functional difference, so that $f(x + \theta) = f(x) + \theta \,.\, \frac{T}{h}$.

T_1, T_2, etc., are successive approximations to T

$T_\infty-$ the theoretically ultimate or limiting value

Δ – first-order forward finite difference operator

$\Delta f(a) = f(a + h) - f(a)$

$\Delta f(x) = D_{p+1}$.

∇ – first-order backward finite difference operator

$\nabla f(a) = f(a) - f(a - h)$

$\nabla f(x) = D_p$.

θ – residual arc such that $f(x + \theta)$ is required to be found out or interpolated, θ being positive and less than h.

1 Introduction and Brahmagupta's Rule

Tabular values of the trigonometric functions $R \sin a$ and R vers a or of their differences are found in several astronomical works of ancient and medieval India. For computing the functional values corresponding to the intervening values of the argument the ordinary method used was that of linear proportion. This usual method of first-order interpolation can be expressed as

$$\begin{aligned} f(x+\theta) &= f(x) + \left(\frac{\theta}{h}\right) . [f(x+h) - f(x)] \\ &= f(x) + \left(\frac{\theta}{h}\right) . D_{p+1}. \end{aligned} \tag{1}$$

For better results, more elegant techniques using second-order interpolation schemes also found in some of the Indian works of the earlier period.

One such rule is found in the works of Brahmagupta (seventh century AD). What he gives is equivalent to the following expression called *sphuṭa* (true) functional difference to be crossed over[1]

$$D_t = \left(\frac{1}{2}\right) . \left(D_p + D_{p+1}\right) - \left(\frac{1}{2}\right) . \left(D_p - D_{p+1}\right) . \left(\frac{\theta}{h}\right) \tag{2}$$

Then the required result is obtained by using the relation

$$f(x+\theta) = f(x) + \left(\frac{\theta}{h}\right) . D_t. \tag{3}$$

Combining (2) and (3) and, using the notation of finite difference operators, we easily get the following formulas

$$f(x+nh) = f(x) + \left(\frac{n}{2}\right) .[\Delta f(x-h) + \Delta f(x)] + \left(\frac{n^2}{2}\right) . \Delta^2 f(x-h) \tag{4}$$

and

$$f(x+nh) = f(x) + \left(\frac{n}{2}\right) .[\nabla f(x) + \nabla f(x+h)] + \left(\frac{n^2}{2}\right) . \nabla^2 f(x+h), \tag{5}$$

which can be regarded as the modern forms of Brahmagupta's second-order interpolation rule using forward and backward differences, respectively.

The formula (4) is a particular case (upto second-order) of the more general Newton-Stirling Interpolation Formula of modern calculus of finite differences.[2]

Brahmagupta's rule is found subsequently in the works of Govindasvāmin (ninth century); Vaṭeśvara (tenth century). Bhāskara II (twelfth century) and Parameśvara (fifteenth century).[3] The general form of Brahmagupta's expression (2) will be

$$D_t = \left(\frac{1}{2}\right) .[f(x+h) - f(x-h)] - \left(\frac{1}{2}\right) .[2f(x) - f(x-h) - f(x+h)]. \left(\frac{\theta}{h}\right)$$

which, on using Taylor Series expansions, will become

$$\begin{aligned} D_t = h \Bigg[f'(x) + \left(\frac{\theta}{2}\right) . f''(x) + \left(\frac{h^2}{6}\right) . f'''(x) \\ + \left(\frac{\theta h^2}{24}\right) . f^{iv}(x) + \left(\frac{h^4}{120}\right) . f^{v}(x) + \cdots \Bigg] \end{aligned} \tag{6}$$

Now, the mathematically exact value of the current functional difference will be given by

$$\begin{aligned} T &= \left(\frac{h}{\theta}\right) . [f(x+\theta) - f(x)] \\ &= h \Bigg[f'(x) + \left(\frac{\theta}{2}\right) . f''(x) + \left(\frac{\theta^2}{6}\right) . f'''(x) \\ &\quad + \left(\frac{\theta^3}{24}\right) . f^{iv}(x) + \left(\frac{\theta^4}{120}\right) ., f^{v}(x) + \cdots \Bigg] \end{aligned} \tag{7}$$

Thus we have

$$T - D_t = -h\frac{(h^2 - \theta^2)}{6} f'''(x) - \frac{\theta h(h^2 - \theta^2)}{24} f^{iv}(x) + \cdots \tag{8}$$

Since h is small and θ still smaller, we may leave the subsequent terms involving higher powers of these small quantities in order to consider the sign of the R.H.S. of (8). Since the third and fourth derivatives of the Versed Sine function are both negative, the R.H.S. of (8), presumed to be dominated by the first two terms, will be positive. And hence T will be greater than D_t.

In the case of the Sine function, we have

$$T - D_t = \left(\frac{h}{6}\right) . (h^2 - \theta^2) . \left[\cos x - \left(\frac{\theta}{4}\right) . \sin x\right] \tag{9}$$

neglecting subsequent terms. So that T will be greater than D_t provided that

$$\cot x > \frac{\theta}{4} \qquad \text{or, a } \textit{fortiori}^{\dagger} \text{ if,} \qquad \tan h > \frac{h}{4}$$

which is always true under the conditions. Therefore, Brahmagupta's 'true' functional difference D_t may be taken to be less than the *really true* (or exact) functional difference.

Thus we see that, if one wants to improve Brahmagupta's expression (2), it should be modified in such a way as to yield an expression which is greater in magnitude. One such modification, found in a commentary (*circa* 1635) by Munīśvara, is discussed below.

2 Munīśvara's Modification of the Rule

Brahmagupta's rule (adopting it for a tabular interval of 10°, instead of 15°) has been given by Bhāskara II (1150 AD) in the *Graha-gaṇita* part (Chapter II, stanza 16) of his *Siddhānta-śiromaṇi* and the scholiast Munīśvara (1635) in his commentary *Marīci* (=*MC*) on it, gives not only an exposition of the subject but also a modification of the rule.[4] This modification, which is meant for achieving greater accuracy *(sūkṣmatā)*, consists of applying a process of iteration *(asakṛt-karma)*. The theory of the process, as gathered or based on the numerical example worked out in the *MC* (p. 134), may be outlined as follows.

We successively find the values of T_1, T_2, ... by using (2) which can be written as:

$$D_t = D_m - \left(\frac{\theta}{2h}\right) . D_p + \left(\frac{\theta}{2h}\right) . D_{p+1} \tag{10}$$

[†] Since, in the first quadrant, cotangent decreases and the greatest values of θ and x are h and $90° - h$, respectively.

The initial value is taken as $T_1 = D_t$ and the subsequent values are computed by the iteration formula

$$T_{n+1} = D_m - \left(\frac{\theta}{2h}\right) . D_p + \left(\frac{\theta}{2h}\right) . T_n \tag{11}$$

obtained from (10). The limiting value will be obtained by making $n \to \infty$. Thus we get,

$$T_\infty = D_m - \left(\frac{\theta}{2h}\right) . D_p + \left(\frac{\theta}{2h}\right) . T_\infty \tag{12}$$

giving

$$T_\infty = \frac{[(h-\theta).D_p + h.D_{p+1}]}{2h-\theta}. \tag{13}$$

3 Example from the *Marīci*

By applying the iteration process represented by (11), the *MC* (p. 134) works out an example of computing the Sine of 24° (which was the Indian value for the obliquity of the ecliptic) from the following (here partially reproduced) Table belonging to the *Siddhānta-Śiromaṇi, Graha-gaṇita,* II, 13 (*MC,* p 127) (Table 1).

Table 1 ($R = 120$)

a	$R \sin a$	Functional difference
10°	21′	$21 = D_1$
20°	41′	$20 = D_2$
30°	60′	$19 = D_3$
40°	77′	$17 = D_4$

Here, $h = 10°$, $\theta = 4°$, $x = 20$, and $p = 2$. And, $D_p = D_2 = 20$; $D_{p+1} = D_3 = 19$. Thus from (11) we have,

$$T_{n+1} = 15'30'' + \left(\frac{1}{5}\right) . T_n, \tag{14}$$

which is the required iteration formula for finding T_n to any desired degree. However, we have noticed some calculation and printing mistakes in the *MC* values while doing the computation work ourselves. The results are shown in Table 2.

Using (13), the limiting value will be $T_\infty = 19; 22, 30$. With this value used for T, we have $\sin 24° = 41 + 7; 45 = 48; 45$.

Brahmagupta's rule (2) would give $\sin 24° = 41 + 7; 43, 12 = 48; 43, 12$, while the modern value is about 48; 48, 30.*

*Linear interpolation yields 48; 36.

Table 2 The values of T_{n+1} and $R \sin 24°$

n		Actual value of $T_{n+1} = 15;30 + \left(\frac{1}{5}\right).T_n$	Printed text value of T_{n+1} (*MC*, p. 134)	Value of T_{n+1} as calculated (by us) by using the printed value of T_n *each time*
0	T_1	19;18	19;18	–
1	T_2	19;21,36	19;21,36	19;21,36
2	T_3	19;22,19,12	19;22,22,12	19;22,19,12
3	T_4	19;22,27,50,24	19;22,28,26,24	19;22,28,26,24
4	T_5	19;22,29,34,4,48	19;22,29,41,16,47	19;22,29,41,16,48
5	T_6	19;22,29,54,48,57,36	19;22,29,56,15,21,36	19;22,29,56,15,21,24
6	T_7	19;22,29,58,57,47,31,12	19;22,29,59,15,4,19,12	19;22,29,59,15,4,19,12
7	T_8	19;22,29,59,47,33,30,14,24	19;22,29,59,51,0,51,50,24	19;22,29,59,51,0,51,50,24
8	T_9	19;22,29,59,57,30,42,2,52,48	19;22,29,59,58,12,10,22,4,48	19;22,29,59,58,12,10,22,4,48
9	T_{10}	19;22,29,59,59,30,8,24,34,33,36	19;22,29,59,59,38,20,34,14,57,36	19;22,29,59,59,38,26,4,24,57,36
sin	24°	48;44,59,59,59,48,3,21,49,49,26,24	48;44,59,59,59,51,20,13,45,59,2,24	48;44,59,59,51,20,13,41,59,2,24
		(got by using the above value of T_{10})	(as *printed* in the text)	(got by using the *printed* value of T_{10})

Although the *MC* (p. 134) states that the technique can be used for the Versed Sine also, but its author has not worked out there any example to illustrate the process for the Versed Sine. On the other hand, we found that the process does not give satisfactory results. So I leave the matter for further discussion and investigation.

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Varāhamihira's Calculation of ${}^nC_r$ and the Discovery of Pascal's Triangle

In ancient time, the Jaina School of Indian mathematics took great interest in the subject of permutations and combinations as is clear from their canonical and other literature. The *Bhagavatī-sūtra* (dated about 300 BC) is said to have mentioned combinations of n objects taken one at a time *(eka-saṃyoga)*, two at a time *(dvika-saṃyoga)*, three at a time *(trika-saṃyoga)*, or more at a time.[1] The Jainas had correctly found the values of ${}^nC_1$, ${}^nC_2$., and ${}^nC_3$ by rules which are particular cases of the formula that we now write as

$$ {}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1\times 2\times 3\times\cdots\times r}. \qquad (1)$$

This formula for finding ${}^nC_r$ or the number of ways in which r things can be selected out of n is more specifically indicated by Śrīdhara (about AD 750) in his *Pāṭīgaṇita*, rule 72, and several subsequent works.[2] In fact, it represents the current modern method.

However, Varāhamihira (AD sixth century) had given a different method for numerically finding ${}^nC_r$ for various values of n and r. This method is based on what is called the *loṣṭa-prastāra*. The relevant rule has been mentioned very briefly in a condensed form in his *Bṛhat-saṃhitā*, Chap. 76, verse 22. The original Sanskrit text of the rule is[3]

पूर्वेण पूर्वेण गतेन युक्तं स्थानं विनान्त्यं प्रवदन्ति सङ्ख्याम् ।

It may be translated thus:

> Leaving out the last (from the bottom in a column) and by adding any earlier number to still earlier ones, are obtained the said numbers (${}^nC_{r+1}$ from the column containing ${}^nC_r$).

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Following the example and exposition in the accompanying Sanskrit commentary by Bhaṭṭotpala (also called Utpala simply) on the rule, the details may be explained as follows.

Let n be 16. We write the numbers 1 to 16 one above the other in a column starting from the bottom. These are shown in column 1 in Table 1. Of course these numbers are also the values of $^nC_1$ for n equal to 1, 2, 3, etc., up to $n = 16$.

Table 1 Calculation of ^nC_r for $n = 16$

16	–	–	–
15	120	–	–
14	105	560	–
13	91	455	1820
12	78	364	1365
11	66	286	1001
10	55	220	715
9	45	165	495
8	36	120	330
7	28	84	210
6	21	56	126
5	15	35	70
4	10	20	35
3	6	10	15
2	3	4	5
1	1	1	1

The entries (from bottom) of column 2 in Table 1 are obtained by Varāhamihira's above rule thus:

1st entry is same as in column 1. For the next entry , the number 3 is obtained by adding 2 of column 1 to its earlier member which is 1. That is,

$$1 + 2 = 3, \qquad \text{the 2nd entry of column 1.}$$

Similarly, by adding 3 of column 1 to its earlier members (or entries) we get 6 for the third entry of column 2. That is,

$$1 + 2 + 3 = 6.$$

In the same manner, the next number 10 of column 2 is obtained by adding 4 to 3, 2, and 1 in column 1. That is,

$$1 + 2 + 3 + 4 = 10$$

This operation is continued on all the entries in column 1 except on the last (namely 16) which is left out as such. By this process, we will get the entries or numbers of column 2 in Table I. These numbers (starting from the bottom) will be the values of $^nC_2$ for $n = 2, 3, 4, \ldots 16$.

The numbers in column 3 of Table 1 are obtained from those in column 2 by the same procedure as was followed to get the numbers of column 2 from those in column 1. Thus, we have

$$\begin{aligned} &1 = 1, \text{ the first entry of column 3.} \\ &1 + 3 = 4, \text{ the second entry;} \\ &1 + 3 + 6 = 10, \text{ the third entry} \\ \text{till } &1 + 3 + 6 + \ldots + 105 = 560, \text{ the 14th entry.} \end{aligned}$$

The last number 120 of column 2 is to be left out as such according to the rule. These numbers of column 3 will give the values of

$$^nC_3 \text{ for } n = 3 \text{ to } 16.$$

The 4th column of Table 1 is obtained from column 3 by following the same procedure. The numbers of column 4 give the values of

$$^nC_4 \text{ for } n = 4 \text{ to } 16.$$

In this way, we find that the last entries (or rather the first entries from the top) in the first four columns of Table 1 represent all the values of

$$^{16}C_r, \text{ for } n = 1, 2, 3, 4.$$

It is clear that by this method, we can find ^nC_r for any n and r. Although somewhat lengthy, the use of this method will be beneficial if we wish to tabulate all possible numerical values of ^nC_r upto a particular n and for all r which are less than or equal to n.

The tabulation of the full set of these values of ^nC_r $(r \leq n)$ by Varāhamihira's rule is what seems to be called the *loṣṭa* or *loṣṭaka-prastāra*. Verse No. 30 of the same chapter gives the value of $^9C_3$ $(= 84)$ and Utpala's commentary therein explains this by finding

$$^9C_1, \quad ^9C_2, \quad ^9C_3$$

and adds that for showing all this we should have the *loṣṭaka-prastāra*.

Varāhamihira stated his rule for forming the *loṣṭaka-prastāra* in conformity with the ancient Indian traditional practice of operating with a column of numbers. According to this practice, any repetitive type of a prescribed operation (like the one followed above) is to be started from the lower end or bottom of the column, and then to be carried out upwards in the laid down manner. Other examples of this bottom-to-top-type operation are

(i) the *'adhauparigunitam-antyayuk'* rule of Āryabhaṭa I (born AD 476) for solving the problem of two divisions.[4]
(ii) Mādhava's power-series method of computing the sine.[5]

However, for getting the *loṣṭa-prastāra*, the numbers can be written in an increasing order, and the prescribed operation can be done by starting from the top. We take the case $n = 11$ to illustrate this. The result is shown in Table 2 in which an extra trivial column has been added in the beginning.

Table 2 Binomial coefficients

1	1	1	1	1	1	1	1	1	1	1	1
$\nearrow$ 1	2	3	4	5	6	7	8	9	10	11	–
$\nearrow$ 1	3	6	10	15	21	28	36	45	55	–	–
$\nearrow$ 1	4	10	20	35	56	84	120	165	–	–	–
$\nearrow$ 1	5	15	35	70	126	210	330	–	–	–	–
$\nearrow$ 1	6	21	56	126	252	462	–	–	–	–	–
$\nearrow$ 1	7	28	84	210	462	–	–	–	–	–	–
$\nearrow$ 1	8	36	120	330	–	–	–	–	–	–	–
$\nearrow$ 1	9	45	165	–	–	–	–	–	–	–	–
$\nearrow$ 1	10	55	–	–	–	–	–	–	–	–	–
$\nearrow$ 1	11	–	–	–	–	–	–	–	–	–	–
$\nearrow$ 1	–	–	–	–	–	–	–	–	–	–	–

When the ancient Indian *loṣṭa-prastāra* is depicted in this way, the so-called Pascal's Triangle is at once seen formed in it. The binomial coefficients are seen contained in it in the *diagonal* direction as viewed in the marked directions indicated by arrows.

We have thus the successive sets:

$$1 = {}^0C_0?;\ \text{expansion of}\ (a+b)^0?$$
$$1, 1 = {}^1C_0, {}^1C_1;\ \text{coefficients in}\ (a+b)^1$$
$$1, 2, 1 = {}^2C_0, {}^2C_1, {}^2C_2;\ \text{coefficients in}\ (a+b)^2$$
$$1, 3, 3, 1 = {}^3C_0, {}^3C_1, {}^3C_2, {}^3C_3;\ \text{coefficients in}\ (a+b)^3$$

and so on.

It must be pointed out that the original form of the Arithmetical Triangle as given by Blaise Pascal in his Traité du triangle arithmétique (AD 1665) was in fact same as Table 2 upto $n = 9$ with only very minor difference.[6] It should also be noted that Girolamo Cardano had also given exactly the same Table 2 in his Opus novum of 1570 AD[7].

In India, the 'Pascal's Triangle' was known much earlier and was called *meru* or *meru-prastāra*. Each row of the *meru* was called *sūcī* or *sūcī-prastāra*. Thus the *sūcī* corresponding to $n = 6$ will be

$$1, 6, 15, 20, 15, 6, 1$$

that is,

$${}^6C_{0'}\ {}^6C_{1'}\ {}^6C_{2'}\ {}^6C_{3'}\ {}^6C_{4'}\ {}^6C_{5'}\ {}^6C_{6'}.$$

More refined rules for finding the combinatorial and binomial coefficients, and for forming the *sūcī* and *meru* are found in the works of Janāśraya (about AD 600 or before), Virahāṅka (between 600 and 800), Jayadeva, the prosodist (between 600 and 900), Halāyudha (tenth century), and others.[8] It seems that what was perhaps missed by Varāhamihira was noted by others.

References and Notes

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6. A. W. F. Edwards, *Pascal's Arithmetical Triangle,* Charles Griffin, London and Oxford University Press, New York, 1987, p. ii.
7. *Ibid.,* p. 44. On p. 51, Edwards has reproduced the "Pascal's Triangle" as found in Chu Shih-chieh's Chinese work *Precious Mirror of the Four Elements* (AD 1303). Recently Jordanus de Nemore (about AD 1225) has been assigned its construction and use; see B. Hughes' article on the subject in *Historia Mathematica,* Vol. 16 (1989), pp. 213–223.
8. Parmanand Singh, "Binomial and Multinomial Theorems in India and Central Asia", in the Indo-Soviet Joint Monograph called *Interaction between Indian and Central Asian Science and Technology in Medieval Times* (INSA, New Delhi, 1990), Vol. I, pp. 297–311.

The Last Combinatorial Problem in Bhāskara's *Līlāvatī*

There is no doubt that the most prominent name in the history of ancient and medieval Indian mathematics is that of Bhāskarācārya (AD twelfth century). He is usually designated as Bhāskara II in order to differentiate him for his earlier namesake the Āryabhaṭan scholiast Bhāskara I who wrote a commentary in AD 629 on the *Āryabhaṭīya* of Āryabhaṭa I (born AD 476).[1]

According to Bhāskara II's own statement,[2] he was born in *Śaka* 1036 or AD 1114. His father Maheśvara was also his teacher as well. Bhāskara II composed a number of works (all in Sanskrit) three of which are very famous:

1. *Līlāvatī* (on arithmetic, geometry, mensuration, etc.) which is the most popular work on ancient Indian or Hindu mathematics.
2. *Bījagaṇita* (on algebra including indeterminate analysis) which is a standard Hindu work on algebra.
3. *Siddhānta-śiromaṇi*[3] which is standard work on Hindu astronomy, and which was composed in 1150.

The fame and usefulness of the above three works are also illustrated by their one or more Persian translations. The *Līlāvatī* was translated into Persian by Abū al-Fayḍ Fayḍī in 1587, and the *Bījagaṇita* by 'Aṭa' Allāh Rushdī in 1635.[4] The *Zij-i-Sarūmanī* (1797) by Ṣafdar 'Alī Khān is presumably the Persian translation of the *Siddhānta-Śiromaṇi*.[5]

The other works of Bhāskara II include the *Karaṇa-kutūhala* (also called *Brahmatulya*), a commentary on Lalla's astronomical work.[6] But his authorship of *Bījopanaya* has been refuted by T. S. Kuppanna Sastry.[7] It is interesting to note that Bhāskara's grandson Caṅgadeva (who was the chief astronomer in the court of king Siṅghaṇa) had established, in 1207, a residential institution for the study of the works of his grandfather.[8]

Gaṇita Bhāratī, Vol. 18, Nos. 1–4 (1996), pp. 14–20.

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The popularity of *Līlāvatī* is shown by the fact that it is still used as a textbook in the Sanskrit-medium schools and colleges throughout India. Ever since its composition, the *Līlāvatī* ("the beautiful") has been commented by a large number of scholars. Some of the Sanskrit commentators were Gaṅgādhara (c. 1420), Parameśvara (c. 1430), Lakṣmīdāsa (c. 1500), Sūryadāsa (c. 1540), Gaṇeśa (1545), Mahīdhara (c. 1590), Munīśvara (c. 1635), Rāmakṛṣṇa (1687), etc. The *Kriyākramakarī* (c. 1534) commentary is a joint work of Śaṅkara Vāriyar and Mahiṣamaṅgala Nārāyaṇa (who completed it after the demise of Śaṅkara). This commentary is a rich source for traditional Indian mathematics.[9]

It was through H. T. Colebrooke's translation of 1817 that the West became quite familiar with *Līlāvatī* although this was not the first English translation.[10] Of course, there exists a large number of other translations in Indian as well as foreign languages.[11]

The last chapter of *Līlāvatī* is called *Aṅkapāśa* ("Net of Numbers") and is devoted to combinatorics or the theory of permutations and combinations. In it the last problem deals with the formula for finding the number of n-digit numbers with a given digital sum. Take the nine digits 1–9 (0 is excluded here). Then the total number of n-digit numbers formed using these digits will by 9^n, because each digital positional place can be occupied by any of the said nine digits. But in Bhāskara's problem, we have to find the total number of those n-digit numbers in each of which the sum of the digits is a given fixed quantity, say S.

Let an n-digit number be represented in the usual decimal number form as

$$d_1\, d_2\, d_3\ \ldots\ldots d_n, \tag{1}$$

where each digit d satisfies the relation

$$1 \leq d_i \leq 9, \quad i = 1, 2, \ldots\ldots n. \tag{2}$$

Then Bhāskara's problem is to find the total number of numbers of the form (1) where

$$d_1 + d_2 + d_3 + \ldots\cdots + d_n = S. \tag{3}$$

For example, if $S = 13$, and $n = 5$, the problem is to find the total number of numbers of the type

$$11119,\ 11146,\ 44221,\ 71131,\ \text{etc.}$$

in each of which the sum of the digits is 13.

Bhāskara's rule (*sūtra*) for solving such problems reads verbally as[12]

> निरेकमङ्कैक्यमिदं निरेकस्थानान्तमेकापचितं विभक्तम् ।
> रूपादिभिस्तन्निहतैः समास्स्युः संख्याविभेदा नियतेऽङ्कयोगे ॥
> नवान्वितस्थानकसंख्यकाया ऊनेऽङ्कयोगे कथितं तु वेद्यम् ।
>
> When the digital sum (S) is fixed, subtract one from it (to get $S-1$). Again subtract one, and continue this subtraction one-less times than the number of digital places (i.e., $n-1$ times). Divide the results (so obtained) by one etc. and multiply together the quotients obtained. The product will be equal to the number of number-variations. Here the given digital sum is understood to be less than the number of digital places plus nine.

That is, we have to first form the quantities

$$\frac{(S-1)}{1}, \frac{(S-2)}{2}, \ldots\ldots \frac{(S-n+1)}{(n-1)},$$

where n is the number of digital places. Then the required number of the n-digit numbers will be

$$\frac{(S-1)}{1} \times \frac{(S-2)}{2}, \ldots \times \frac{(S-n+1)}{(n-1)}$$

$$= {}^{S-1}C_{(n-1)} \tag{4}$$

$$\text{provided} \quad S < (n+9) \tag{5}$$

To illustrate his rule, Bhāskara took the example in which $n = 5$ and $S = 13$ (as mentioned above) and got the answer 495 which is correct. However, as is usual with Hindu mathematical texts, he has not given any proof of his solution, although he may have known one as is indicated by the restriction (5). A simple proof of Bhāskara's rule follows from the following lemma.

LEMMA: The number of the ways in which r similar balls can be placed in n bags or compartments (empty cases allowed) is equal to

$$^{(n+r-1)}C_r \tag{6}$$

This lemma is easily proved by considering the n compartments formed by placing $(n-1)$ dividing boundaries, each of which is represented by the capital letter I; for example, one case in which 13 balls are distributed over 5 compartments (formed by 4 I's) may look like

$$\circ\circ\circ\circ \,\mathrm{I}\, \circ\circ\circ \,\mathrm{I}\, \circ \,\mathrm{I}\quad \mathrm{I}\, \circ\circ\circ\circ\circ$$

Then the required number of ways of placing the r balls ($\circ$) variously in the n compartments will be the same as the total number of arrangements of $(n-1+r)$ letters, of which $(n-1)$ are of one kind (I) and the rest of the other kind ($\circ$). This is a standard exercise, and the solution was also known to Bhāskara.[13] The answer will be

$$\frac{(n-1+r)!}{(n-1)!\,r!} = {}^{(n+r-1)}C_r \tag{7}$$

which proves the lemma.

For Bhāskara's problem, the n compartments may be taken to be the n positional places to be filled by digits instead of balls. But unlike the balls, the digits are not similar. So we can imagine the digits or the single digital numbers 1 to 9 to be represented by the corresponding number of vertical strokes as was done by the Egyptians or the Indus Valley people about 4000 years ago! For example, the 5-digit number 23125 will be represented as

|| I ||| I | I || I |||||.

The only difference is that here empty space is not allowed (since 0 has been excluded). This is equivalent to having only $(S-n)$ similar 'strokes' (instead of balls) to be distributed over the n places or compartments, so that $r = S - n$. Thus, the required number of arrangements will be, by the Lemma formula (6), equal to

$${}^{(n+S-n-1)}C_{S-n} = {}^{(S-1)}C_{S-n} = {}^{(S-1)}C_{n-1}$$

as given by Bhāskara in his rule (4). The restriction or condition (5) or $(S-n) < 9$ is obvious here as no digit (in the used decimal base) can exceed 9, and hence the number of additional 'strokes' which can be placed in any compartment is necessarily less than 9. Bhāskara was equipped with the tool and method of proof given here, and he most probably followed the above line of arguments or some very similar to it.[14] The restriction (5) in the above proof is due to employment of the commonly used decimal place-value system of numerals which the Indians have been possessing since about two millennia.

Now Bhāskara's problem will be worked out without the restriction (5). For this we take another approach. We know that the total number of ways in which a score of S points can be depicted in a throw of n ordinary dice (each having six faces marked by dots or numbers 1–6) is equal to the coefficient of x^S in the expansion of

$$(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^n.$$

Similarly, if we take dice with nine faces each marked by numbers 1 to 9, the total number of ways in which a score of S points will appear (when n such dice are thrown) is the coefficient of x^S in

$$(x^1 + x^2 + x^3 + \cdots + x^9)^n. \tag{8}$$

Here the score S is the sum of the numbers shown on the n faces of the dice thrown. These faces may be treated as cells or positional places in which any of the digits may appear. For example, when 5 dice (each having nine faces) are rolled, the score 13 may appear as

[2] [3] [1] [2] [5].

With such an analogy, the total n-digit numbers (formed from the digits 1 to 9) in which the digital sum is S will be the coefficient of x^S in (8), or the coefficient of $x^{(S-n)}$ in

$$(1 + x + x^2 + \cdots + x^8)^n.$$

With some simplifications, the required number will be the coefficient of $x^{(S-n)}$ in

$$(1 - x^9)^n (1 - x)^{-n},$$

or in

$$\left[\sum_{r=0}^{n} {}^nC_r \, (-1)^r \cdot x^{9r}\right]\left[\sum_{t=0}^{\infty} {}^{(n+t-1)}C_{(n-1)} \cdot x^t\right] \tag{9}$$

where we have used the result

$$\frac{n(n+1)(n+2)\ldots(n+t-1)}{t!} = {}^{(n+t-1)}\,C_{n-1}.$$

The coefficient of $x^{(S-n)}$ can now be easily collected in the above product of the series (9). Corresponding to $r = 0, 1, 2, 3, \ldots$, the value of t will be

$$(S-n), (S-n-9), (S-n-18), \ldots$$

Thus, the required number n-digit numbers with fixed digital sum S will be given by

$$\begin{aligned} {}^nC_0 \cdot {}^{(S-1)}C_{n-1} - {}^n\, C_1 \cdot {}^{(S-10)}C_{n-1} + {}^n\, C_2 \cdot {}^{(S-19)}C_{n-1} - \\ \ldots (-1)^r \cdot {}^nC_r \cdot {}^{(S-9r-1)}C_{n-1} \end{aligned} \tag{10}$$

where

$$\begin{aligned} (n-1) &\le S - 9r - 1 \\ \text{or} \quad r &\le \frac{(S-n)}{9}. \end{aligned} \tag{11}$$

COROLLARY: Under Bhāskara's restriction (5), the value of r, by (11), will be less than unity. Hence, only the first term in (10) will be taken in this case, and the required answer will be the same as Bhāskara's solution (4).

Example 1: Let $S = 20$, and $n = 5$. Hence we have by (11), $r \leq \frac{5}{3}$, so that we can take $r = 0, 1$ only. The answer will be by (10),

$$
\begin{aligned}
&= 1 \cdot {}^{19}C_4 - 5 \cdot {}^{10}C_4 \\
&= 2826
\end{aligned}
$$

Example 2: Let $S = 45$, $n = 5$. In this case, we have, by (11), $r \leq \frac{40}{9}$, so that $r = 0, 1, 2, 3, 4$. The required answer will be, by (10),

$$
\begin{aligned}
&{}^{44}C_4 - 5 \cdot {}^{35}C_4 + 10 \cdot {}^{26}C_4 - 10 \cdot {}^{17}C_4 + 5 \cdot {}^{8}C_4 \\
&= 135751 - 261800 + 149500 - 23800 + 350 \\
&= 1
\end{aligned}
$$

which is correct, since the only 5-digit number with digital sum 45 will be 99999.

Example 3: Let $S = 46$ and $n = 5$. Here, by (11), we have $r \leq \frac{41}{9}$, so that we can take $r = 0, 1, 2, 3, 4$. Then by (10) the required answer will be

$$
\begin{aligned}
&= {}^{45}C_4 - 5 \cdot {}^{36}C_4 + 10 \cdot {}^{27}C_4 - 10 \cdot {}^{18}C_4 + 5 \cdot {}^{9}C_4 \\
&= 148995 - 294525 + 175500 - 30600 + 630 \\
&= 0.
\end{aligned}
$$

This answer is correct, since no 5-digit number in decimal digital representation can have a digital sum 46 (cf. Example 2).

Historically, the name of A. De Moivre (1730) is associated with the problem of finding the ways of getting scores S when n dice (each with f faces) are thrown together.[15] Here our purpose is not to go into more historical details related to above problems. But it is worthwhile to quote the remarks of Sharon Kunoff:[16]

> "Many of the permutation and combination formulas attributed to Cardan, Tartaglia, and Pascal were known to the Hindus". And that "the algebra of combinatorics in the twelfth century and perhaps earlier was considerably more advanced in Hindu mathematics than in Western circles."

References and Notes

1. Bhāskara I also composed earlier, the *Mahābhāskarīya* and *Laghubhāskarīya*.
2. Towards the end of the *Golādhyāya* part of the *Siddhānta-śiromaṇi* occurs the verse
   रसगुणपूर्णमही १०३६ समशकनृपसमयेऽभवन्ममोत्पत्तिः।
   रसगुण ३६ वर्षेण मया सिद्धान्तशिरोमणी रचितः॥
3. See the above verse. Often *Līlāvatī, Bījagaṇita, Grahagaṇita* and *Golādhyāya* are taken to be the four parts of *Siddhānta-śiromaṇi* itself.
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13. *Līlāvatī*, Poona edition, 1937, Part II, p. 278, verse 264; and Colebrooke's translation. Allahabad, 1967, p. 168, Rule 270.
14. C. N. Srinivasiengar, *The History of Ancient Indian Mathematics*, World Press, Calcutta, 1967, pp. 83–84.
15. See H. S. Hall and S. R. Knight, *Higher Algebra* (1st edition 1887); Metric Edition, Macmillan, London, 1964; pp. 388–389.
16. S. Kunoff, "A curious counting/summation formula from the ancient Hindus." Paper contributed to the 16th Annual Conference (1990) of the CSHPM; *see Proceedings* (ed. by F. F. Abeles *et al.*), Toronto, 1991, pp. 101–107.

Early Pandiagonal Magic Squares in India

1 Introduction

In ancient India, arts and sciences were hand-maiden of religions. In fact religion has been the dominant feature of Indian culture through the ages. Almost all sciences have been attributed a divine origin. For instance, the exposition of the 54th chapter (on astronomy and mathematics) of the *Nārada-purāṇa* commences with the line.[1]

ज्यौतिषाङ्गं प्रवक्ष्यामि यदुक्तं ब्रह्मणा पुरा।

(Sanandana says) I shall now set out the *Jyotiṣa* portion which was enunciated in antiquity by (god) Brahma.

Nārāyaṇa Paṇḍita begins Chap. 14 (on magic squares and other magic figures) of his *Gaṇita-kaumudī* (AD 1356)[2] by stating that the subject was taught to Maṇibhadra by Lord Śiva, the Master of the three worlds.

Actually magic squares (*aṅka-yantras*) are particular type of the more general figures or diagrams called *yantras* (mystic diagrams). According to Mahīdhara,[3] the *yantras* were enunciated by Lord Śiva.

Another characteristic of religious domination of Indian history and culture is to trace the beginning of everything to *Vedas*. The six ancient broad sciences namely *śikṣā* (phonetics), *kalpa* (ritual), *vyākaraṇa* (grammar), *nirukta* (etymology), *chandaḥ-śāstra* (prosody) and *jyotiṣa* are considered only *vedāṅgas* or limbs of *Vedas*. The sciences of *āyurveda* (medical sciences), *śilpa* (architecture and fine arts), and of music and war etc. are also included in the vedic fold as *Upavedas*. This attitude of assigning divine and vedic origin to Indian sciences automatically attaches a hoary past and authority to them. Manu even claims.[4]

भूतं भव्यं भविष्यं च सर्वं वेदात्प्रसिद्ध्यति।

All that was, is, and will be (in future) can be derived from the Vedas.

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Magic square in seed form has been traced to *Ṛgveda* (see below).

A magic square of order n is an arrangement of n^2 different numbers (usually positive integers) in n rows and n columns such that the sum of the numbers along each row, each column, and along each main diagonal is same. This constant sum is called the magical sum or magic constant of the magic square. In a simple way, a magic square of order n may be written in the form of a square array of its n^2 numbers which may be called its elements. Usually the square array is depicted in a square consisting of n^2 sub-squares or cells each of which contains an element (Fig. 1a).

(a)

a_{11}	a_{12}	...	...	a_{1n}
a_{21}	a_{22}	...	...	a_{2n}
...	...	...	...	...
...	...	...	...	...
a_{n1}	a_{n2}	...	...	a_{nn}

(b)

a_1	b_1	c_1	d_1
a_2	b_2	c_2	d_2
a_3	b_3	c_3	d_3
a_4	b_4	c_4	d_4

Fig. 1 General magic squares

In addition to the constancy of the sum along every row, column, and main diagonal, if the sum of elements along each broken diagonal is also equal to the same magic constant, then the array is called a diabolic or pandiagonal magic square. Here we are mainly concerned with 4th order magic squares. In Fig. 1b, the main diagonals are represented by a_1, b_2, c_3, d_4 (forward), and d_1, c_2, b_3, a_4 (back-ward). The six broken diagonals are formed by (b_1, c_2, d_3, a_4); (a_2, b_3, c_4, d_1); (c_1, d_2, a_3, b_4); etc.

2 The Vedic Method

For introducing a condensed notation, let p_i and q_i denote the pairs of elements or numbers of magic square in Fig. 1b as follows:

$$p_i = (a_i, b_i);\ \ q_i = (c_i, d_i) \tag{1}$$

where i takes the values 1–4. In terms of these new pairs, the magic square of Fig. 1b can be written as Fig. 2a.

Similarly by pairing the elements vertically, Fig. 1b becomes Fig. 2b where,

$$r_1 = (a_1, a_2),\ \ r_2 = (b_1, b_2),\ \ \text{etc.}$$

and

$$s_1 = (a_3, a_4),\ \ s_2 = (b_3, b_4),\ \ \text{etc.}$$

(a)

p_1	q_1
p_2	q_2
p_3	q_3
p_4	q_4

(b)

r_1	r_2	r_3	r_4
s_1	s_2	s_3	s_4

(c)

6	1	8
7	5	3
2	9	4

Fig. 2 Condensed notations

Now let us have a look at the simple ancient magic square of Fig. 2c. It will be clear that the constancy of the magical sum (= 15) is due to arrangements of the pairs.

$$(1, 9), (2, 8), (3, 7), (4, 6) \tag{2}$$

around the central number 5. The important point to note is that the sum of numbers in each pair of (2) is 10 (which is partial magical sum). Perhaps taking idea or hint from this, a technique emerged to form magic squares of order 4 by taking two sets of pairs of numbers suitably.

To illustrate the method, we take the first sixteen natural numbers 1–16. From the first eight we form the four pairs.

$$p, q, r, s \equiv (1, 8), (2, 7), (3, 6), (4, 5). \tag{3}$$

The sum of numbers in each pair is 9. It should be noted that the first members in (3) are from the set 1, 2, 3, 4 and the second members from the set 5, 6, 7, 8 taken in the reversed order to achieve constancy of sum. The next four pairs are to be formed similarly by taking numbers respectively from the sets

$$9, 10, 11, 12; \text{ and } 16, 15, 14, 13. \tag{4}$$

This time we start from the end and take

$$P, Q, R, S \equiv (13, 12), (14, 11), (15, 10), (16, 9) \tag{5}$$

(a)

p	P
q	Q
r	R
s	S

(b)

p	P
Q	q
s	S
R	r

(c)

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

Fig. 3 Magic squares by condensed notation

We see that the sum in each pair here is 25 which with earlier partial sum of 9, will give us the total 34 which is magic constant of the 4th order magic square formed from first sixteen natural numbers.[5]

Using the condensed notation of Fig. 2a, if we set the pairs from (3) and (5) as shown in Fig. 3a, we will get an array in which the sum along the rows will be same (namely 34) but the sum along the columns will not be so. But by making diagonal moves in the sets p, q, s, r and P, Q, S, R the symmetric scheme of Fig. 3b was discovered.

Putting of numerical values from (3) and (5) in Fig. 3b leads us to the magic square of Fig. 3c (according to the condensed notation of Fig. 2a). Surprisingly, it is a pandiagonal magic square!

The magic square of Fig. 3c is found in some Sanskrit works including the *Gaṇita Kaumudī* where it is frequently used.[6] In the light of the above method, a rule given in this work for constructing 4th order magic square *(catur-bhadra)* from numbers of an arithmetical series *(śreḍhī)* (here 1–16) may be interpreted as follows.[7]

The diagonal move from p to q (Fig. 3b) implies moves from 1 to 2 and from 8 to 7 (Fig. 3c). Each of these two moves is similar to a knight's move in chess *(caturaṅga)*. So it is called movement of numbers taking in pairs or two at a time *(dvau dvau)* according to *(turaga-gati)* ("horse-move"). The move from p to q is towards right (*savya*) and that from P to Q to left *(asavya)*. Also the pair of pairs p, q is taken in the direct order *(krama)* but the next pair of pairs r, s is taken in the reverse order *(utkrama)* namely s to r (and S to R). Thus the method amounts to moving numbers in pairs (p, q, s, r and P, Q, S, R) in pairs of cells *(koṣṭhaikya)* diagonally from left to right and right to left alternately *(ekāntareṇa)*.

It should be noted that although the above numerical illustration is simple, the method is quite general and can be applied to any 16 different numbers of an arithmetical progression. If the series is

$$f, f+e, f+2e, \ldots, f+15e \tag{6}$$

then we take

$$p, q, r, s = (f, f+7e), (f+e, f+6e), (f+2e, f+5e), (f+3e, f+4e) \tag{7}$$

and

$$P, Q, R, S = (f+12e.f+11e), (f+13e, f+10e), (f+14e, f+9e), (f+15e, f+8e) \tag{8}$$

respectively. The Vedic method represented by the condensed notation of Fig. 3b will easily lead[8] to the magic square of Fig. 4a. Indeed it is a pandiagonal magic square of magic constant $(4f+30e)$.

Instead of given series of 16 numbers, if we are to construct a magic square of given constant $K(=2m$ say) we can proceed as follows. The set p, q, r, s is taken

(a)

f	f+7e	f+12e	f+11e
f+13e	f+10e	f+e	f+6e
f+3e	f+4e	f+15e	f+8e
f+14e	f+9e	f+2e	f+5e

(b)

1	8	m-4	m-5
m-3	m-6	2	7
4	5	m-1	m-8
m-2	m-7	3	6

Fig. 4 Vedic magic squares

to be as given by (3). For the second set, we take the series $a, a+d, a+2d, a+3d, a+4d, a+5d, a+6d, a+7d$. Like (5) we form P, Q, R, S respectively as

$$(a+4d, a+3d), (a+5d, a+2d), (a+6d, a+d), (a+7d, a) \tag{9}$$

If we use values from (3) and (9) in the scheme of Fig. 3b, the sum of each row in the magic square formed will be $(2a+7d+9)$ which should be $2m$. Assuming $d=1$, we get, $a=m-8$.

Putting this and $d=1$ in (9), we have (with $m>16$)

$$P, Q, R, S = (m-4, m-5), (m-3, m-6), (m-2, m-7), (m-1, m-8). \tag{10}$$

By the scheme of Fig. 3b, the required magic square is (Fig. 4b).

Considering the *Vedas* (from the root *vid*, to know) to be the source of all knowledge, the Vedic words and phrases are thought to be multi-intentional as well as polysemous. Often many meanings are derived from passages and verses of the *Vedas*. It is frequently demonstrated that the Tantric works follow the line of canonical Vedic literature. Tantric *yantras* (mystic diagrams) include magic squares (*aṅka-yantras).*

Nīlakaṇṭha Caturdhara has claimed that the practice of generating magic squares is hinted in certain *Ṛg*-vedic verses. He is famous for his commentary on the *Mahābhārata* but wrote several other works. These include his commentary on the *Śivatāṇḍava-tantra* which deals with magic squares. This commentary was composed in AD 1680 and contains his rather unexpected interpretation of *Ṛg-veda*, X, 114, 6–7. The new meaning and analysis claim that the *Ṛg-vedic* verses contain, in the coded seed form, the hint to obtain the basic pairs (3) and other related information.[9]

The presence of the initial pair 1, 8 has been identified by Nīlakaṇṭha by applying the *Kaṭapayādi Nyāya* to the word *yajña* in a peculiar way. No doubt *ya* stands for 1. But *jña* as *ña* will give zero which is meaningless *(anarthakaḥ)*, so it indicates *ja*, that is 8. The indicated sum $(1 + 8)$ or 9 is then to be measured in different forms (*vividha rūpeṇa*) to get the set (3) etc.

According to Nīlakaṇṭha, *Ṛg-veda* text word *caturaḥ* (four) indicated the order of the magic square and the word *ṣaṭtriṃśān* indicates its magic constant.[10] After filling the pairs (3) in Fig. 4b, the remaining entries are to be filled through intelligence *(manīṣā)*. Nīlakaṇṭha rightly points out that the number $(m - x)$ is to be put in the cell reached by the camel-step *(uṣṭra-pada) ie*. by a diagonal jump from the number x etc.[11]

3 Varāhamihira's Method

Let the arithmetical numbers 1–8 be put in two groups.

$$1, 2, 3, 4 \tag{11}$$

and

$$5, 6, 7, 8 \tag{12}$$

From (11) we form pairs, as usual, with equal sums in direct and reverse order, namely

$$(1, 4), (3, 2) = p, q \text{ say} \tag{13}$$

We do same with (12) by starting from the end as in (5), to get

$$(8, 5), (6, 7) = P, Q \text{ say} \tag{14}$$

(a)

p	Q
Q	p
q	P
P	q

(b)

1	4	6	7
6	7	1	4
3	2	8	5
8	5	3	2

Fig. 5 First steps for 4th order square

By the scheme shown in Fig. 5a (which is comparable to Fig. 3b), we form the square shown in Fig. 5b. This square represents a preliminary step in the method but it is not a magic square. However, except for the repetition of numbers, it satisfies all the conditions of a pandiagonal square, the sum of each row, column, and diagonal being 18. To get a magic square of constant $K = 2m$ from it, we have to add half of $(2m - 18)$ i.e. $(m - 9)$ to half the numbers in Fig. 5b staggerly in an intelligent manner (using *manīṣā*). This can be done by keeping in mind the camel move *(uṣṭra-pada)* or Diagonal Jump method of Fig. 4b. That is, starting with the first number 1 in the first cell in Fig. 5b, $(m - 1)$ should appear in place of 8 in the 11th cell (here $m - 1$ comes from $m - 9$ added to 8). From the first cell we now go respectively to the next numbers 2, 4 and 3 obtained by *aśvagati* (knight's move). Then, of course, the numbers $(m - 2)$, $(m - 4)$, and $(m - 3)$ should appear in the cells located by the usual camel move.

The result at this stage is shown in Fig. 6a for convenience after dropping the duplicated numbers 1, 2, 3, 4 (which have already been attended from other cells of Fig. 5b following knight's move in the order 1, 2, 4, 3). The final pandiagonal magic square of Fig. 6b is now easily obtained from Fig. 6a by filling the empty entries through the usual camel moves. Figures 4b and 6b are comparable.

(a)

1		6	$m-2$
m-3	7		4
	2	m-1	5
8	m-4	3	

(b)

1	m-5	6	m-2
m-3	7	m-8	4
m-6	2	m-1	5
8	m-4	3	m-7

Fig. 6 Forming pandiagonal magic square

(a)

2	3	5	8
5	8	2	3
4	1	7	6
7	6	4	1

(b)

m-7	3	m-4	8
5	m-1	2	m-6
4	m-8	7	m-3
m-2	6	m-5	1

Fig. 7 Varāhamihira's *kacchapuṭa*

Varāhamihira's *Bṛhat-saṃhitā* (sixth century AD) is a historically important encyclopedic work. In its Chap. 76, verses 23–26, he gives the method of preparing the *sarvatobhadra* perfumes.[12] As explained by his ancient commentator *Bhaṭṭotpala*, the method clearly involved the use of the following 16 celled square which Varāhamihira calls *Ṣodaśa-kacchapuṭa* (Fig. 7a).

Figure 7a is practically same as Fig. 5b whose rows are now written from bottom to top and the numbers in each row are reversed. The Pandiagonal magic square corresponding to Fig. 7a is shown in Fig. 7b (as obtained by above method). With the minimum value $m = 17$, we get the magic square of Fig. 8a.

In addition to Fig. 8a, Takao Hayashi[13] reconstructed three more magic squares from Varāhamihira's *kacchapuṭa* (Fig. 7a) by adding the (constant) same number to

(a)

10	3	13	8
5	16	2	11
4	9	7	14
15	6	12	1

(b)

2	11	5	16
13	8	10	3
12	1	15	6
7	14	4	9

Fig. 8 Pandiagonal magic squares

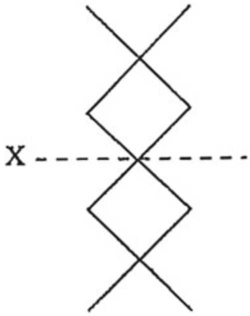

Fig. 9 *Matsyadvaya* (fish-pair)

exactly two terms of each row, each column, and each of the two main diagonals. But only one of these three is found to be pandiagonal (Fig. 8b).

It may be pointed out that the diagrammatic patterns implied in the schemes represented by Figs. 3b and 5a are same. The resulting geometrical diagram may be depicted as shown in Fig. 9 and is called *matsyadvaye* (fish-pair) in ancient terminology.[14] The diagram is formed by two zigzag lines joining the centres of the cells of the columns of the Fig. 3b (or Fig. 5a) alternately. The mouths of the two fishes in Fig. 9 meet at the centre line XY. It may also be noted that the order p, q, s, r (instead of p, q, r, s) in Fig. 3b is comparable to the order 1, 2, 4, 3 (instead of 1, 2, 3, 4) used in forming Fig. 6a.

(a)

0	1	0	8
0	9	0	2
6	0	3	0
4	0	7	0

(b)

$m-3$	1	$m-6$	8
$m-7$	9	$m-4$	2
6	$m-8$	3	$m-1$
4	$m-2$	7	$m-9$

Fig. 10 Nāgārjuna's code and magic square

4 Nāgārjuna's Pandiagonal Magic Squares

There were at least two famous ancient Indian scholars bearing the name Nāgārjuna. The well known Buddhist philosopher and logician Nāgārjuna lived in the first or second century AD Subsequent Nāgārjunas are often confused and wrongly identified with him. Here we are concerned with the Tantric Nāgārjuna who was the author of a work called *Kakṣapuṭa*. His date is variously given from seventh to tenth century AD In this work he gives briefly a rule for the construction of pandiagonal magic square of order four. His mnemonic Sanskrit line is [15]

$$\underset{(01)}{\underline{\textit{arka}}}\ \underset{(0809)}{\underline{\textit{indunidhā}}}\ \underset{(02)}{\underline{\textit{nārī}}}\ \underset{(60)}{\underline{\textit{tena}}}\ \underset{(30)}{\underline{\textit{lagna}}}\ \underset{(4070)}{\underline{\textit{vināsanam}}}$$

Nāgārjuna has used the popular *Kaṭapayādi* alphabetic system in his mnemonic rule. The square of Fig. 10a is obtained when the decoded sixteen numbers are filled in the cell in order. Actually Fig. 10a represents the preliminary steps and the zeros in it are to be considered in the sense of *śūnyas* ("empty spaces") to be filled suitably to get a required magic square. If we want a pandiagonal magic square of constant $2m$, then the blank cell lying next plus one cell diagonally to any number x (among the eight positive numbers of Fig. 10a) is to be filled by $m - x$ according to Nāgārjuna (the diagonal direction may be upwards or downwards, right or left[16]. The method is same as Nīlakaṇṭha's camel-step i.e. the diagonal jump method used in getting Fig. 4b in Sect. 2.

(a)

30	16	18	36
10	44	22	24
32	14	20	34
28	26	40	6

(b)

15	8	9	18
5	22	11	12
16	7	10	17
14	13	20	3

Fig. 11 Nāgārjuna's other magic squares

A more interesting case is that of the magic square (given by Nāgārjuna) shown in Fig. 11a which has been similarly expressed by a Sanskrit verse or mnemonic formula using the *kaṭapaya* system.[17]

The pandiagonal magic square of Fig. 11a is called *Nāgārjunīya* (because probably he himself constructed it). Its magic sum is 100. But it cannot be obtained from Fig. 10b by putting $m = 50$. How Nāgārjuna got it, is not satisfactorily known. We use the Vedic method to construct it or its equivalent form (Fig. 11b) obtained by halving the elements.

First a general pandiagonal magic square is constructed by using four subsets *(caraṇas)* of arithmetical numbers namely

$$a_1, a_1 + d, a_1 + 2d, a_1 + 3d \tag{15}$$

$$a_2, a_2 + d, a_2 + 2d, a_2 + 3d \tag{16}$$

$$a_3, a_3 + d, a_3 + 2d, a_3 + 3d \tag{17}$$

$$a_4, a_4 + d, a_4 + 2d, a_4 + 3d \tag{18}$$

where an essential condition is that

$$a_1 + a_4 = a_2 + a_3. \tag{19}$$

(a)

a_1	$a_2 + 3d$	a_4	$a_3 + 3d$
$a_4 + d$	$a_3 + 2d$	$a_1 + d$	$a_2 + 2d$
$a_1 + 3d$	a_2	$a_4 + 3d$	a_3
$a_4 + 2d$	$a_3 + d$	$a_1 + 2d$	$a_2 + d$

(b)

3	14	16	17
18	15	5	12
9	8	22	11
20	13	7	10

Fig. 12 General pandiagonal magic square (with example)

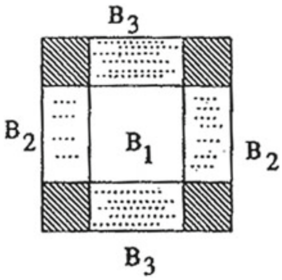

Fig. 13 Forming blocks in a magic square

We can form p, q, r, s and P, Q, R, S and proceed according to scheme of Fig. 3b of Sect. 2. But since we already know the Fig. 3c, the required Vedic square can be easily obtained by replacing the numbers 1–16 (of Fig. 3c) by the sixteen terms of the sets (15)–(18) respectively in order. The resulting pandiagonal square will be (Fig. 12a).

By taking the values $a_1 = 3, a_2 = 8, a_3 = 11, a_4 = 16$, and $d = 2$, we get the Fig. 12b (the values have to be chosen avoiding repetition of numbers subject to condition (19)). It can be noted that the numbers in Fig. 12b are same as in *Nāgārjunīya* square, reduced form of Fig. 11b (compare especially the leading diagonals). A suitable transformation will change Fig. 12b exactly to Fig. 11b.

A simple transformation is based on the constancy of magic sum in blocks of four cells selected in several ways. Figure 13 shows four such ways namely by combining

(i) 4 central squares (B_1);
(ii) Two middle cells from the first and the last columns (to form B_2);
(iii) Two middle cells from the top and bottom rows (to form B_3); and
(iv) 4 corner cells to form the block B_4 (see Fig. 13).

If the formation of these four combinations is applied to the square of Fig. 12b, the four numerical blocks of Fig. 14 will be obtained.

15	5
8	22

18	12
9	11

14	16
13	7

3	17
20	10

Fig. 14 The Four Blocks

(a)

15	5	18	12
8	22	9	11
14	16	3	17
13	7	20	10

(b)

15	8	9	18
5	22	11	12
16	7	10	17
14	13	20	3

Fig. 15 Block diagram and resulting square

From the blocks of Fig. 14, we get the block diagram (Fig. 15a) which is an intermediary step. The final magic square (Fig. 15b) is then obtained by resetting the numbers of Fig. 15a such that the FIRST number of EACH block should be placed in the respective CORNER cell of the magic square being formed and rows of EACH block be changed to columns from top to bottom or bottom to top (starting from the corner-filled number as shown by arrows in Fig. 15b). Indeed we thus get the *Nāgārjunīya* square of Fig. 11b. The transformation may be called *Nāgārjunīya*.

If we perform this transformation on the square of Fig. 15b, we get the square of Fig. 16a which itself will transform to Fig. 16b if the process is repeated. However, a further application of the process will yield the original square to Fig. 12b (Cyclic shifting of leading diagonal elements may be noted). In this way[18] three new pandiagonal magic squares are easily obtained from the general square of Fig. 12a.

(a)

22	7	16	5
11	10	17	12
9	20	3	18
8	13	14	15

(b)

10	20	9	11
17	3	18	12
16	14	15	5
7	13	8	22

Fig. 16 Magic squares by transformation

(a)

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

(b)

16	9	4	5
3	6	15	10
13	12	1	8
2	7	14	11

Fig. 17 The Dudhai and Gwalior magic squares

5 Epilogue

Discussion of material from some tantric works like *Bhairava Tantra* and *Śivatāṇḍava Tantra* is not included in this paper because of their uncertain dates.[19] In fact, it is difficult to assign definite historical value to them due to their *apauruṣeya* (non-human or divine) nature and authorship.

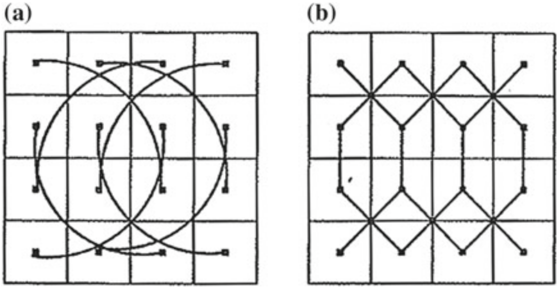

Fig. 18 Geometrical designs in magic squares

However, some early archaeological sources may be mentioned. Earliest of these seems to be the case of the Jhansi Magic Square (Fig. 17a) which was found carved on the underside of a lintel from the collapsed doorway of a shrine known as the Chota Surang (situated at the Dudhai village of Jhansi district and supposed to have been built in the first half of the eleventh century AD)[20]. Exactly the same square (Fig. 17a) is now found preserved in an inscription of a temple at the famous Khajuraho (situated about a hundred miles east of Jhansi town) and dated about AD 1200. The Gwalior

Magic Square (Fig. 17b) was discovered in a ruined temple in the Gwalior Fort and is dated[21] *Saṃvat* 1540 (= AD 1483). It is same as the Vedic square (of Fig. 3c) with its corner blocks interchanged diagonally. It seems that pandiagonal magic squares of order four were quite popular in India.

Often artistic significance is attached to the beautiful symmetrical designs which are generated from the geometrical patterns and other properties of numbers arranged in the form of magic squares. If we join centres of the cells (taken two at a time) which have numbers whose sum is equal to half the magic constant 17 in Fig. 3c (or in Fig. 17a or b), then we will obtain the geometrical pattern shown in Fig. 18a. Of course, the same method can be applied to other squares (dealt in this paper) to obtain the said pattern. The implied property is an extension of the constancy of sum of blocks of four cells described and used in Sect. 4 above. The geometrical design of Fig. 18a also represent the so called camel-step moves *(uṣṭra-pada)* of each joined-pair of dots mutually towards each other.

Another symmetrical design can be obtained by joining the centres of the cells containing odd and even numbers separately in squares of Figs. 3c or 17a or b. The resulting symmetry is shown in Fig. 18b. The joining of odd and even number cells in Varāhamihira's pandiagonal magic squares (Fig. 8a and b) will yield the symmetry of Fig. 18b as rotated through 90 degrees. Other designs may be obtained by considering other properties e.g. cells which have constant partial magical sum in pairs.

In the Islamic world, pandiagonal magic squares are first found in the twelfth or thirteenth century AD and their most famous such square is same as that of Varāhamihira (Fig. 8a) rotated through 90 degrees.[22] The popular Albert Dürer's magic square (AD 1514) is not a pandiagonal or diabolic although it has some interesting features.[23]

References and Notes

1. See the *Nāradapurāṇa* edited by Shriram Sharma, Bareilly, 1971, Part II, p. 318. In the second verse of Chap. 54, the science of *Jyotiṣa* (astronomy, astrology, and mathematics) is said to have been expounded in 4 lakh verses.
2. *Gaṇita-kaumudī* edited by Padmakara Dvivedi, Part II, p. 353 (Benares, 1942.)
3. See his *Mantra-mahodadhi* (AD 1588) XX, 1 and its autocommentary (Mumbai, 1988 ed., p. 180).
4. *Manusmṛti*, XII, 97; ed. by H. G. Mishra, Varanasi, 1952, p. 681. Also see R. C. Gupta, "Six Type of Vedic Mathematics", *Gaṇita Bhāratī*, 16 (1994) 5–15 and 23, p. 7.
5. It can be easily seen that the magic constant of the magic square formed from first n^2 positive integers is $\frac{n(n^2+1)}{2}$.
6. According to Takanori Kusuba, *Combinatorics and Magic Squares in India* etc., Doctoral Dissertation, Brown University, 1993, p. 51.
7. See *Gaṇita-kaumudī, loc. cit* (under Ref. 2 above), XIV, 10–11, pp. 358–359.
8. Of course, if the magic square of Fig. 3c is already formed or known, then the Fig. 4a can be directly derived from Fig. 3c by writing the terms of (6) in place of number elements 1–16 respectively.

9. For Nīlakaṇṭha's text and interpretations, we mostly rely on the forthcoming paper on "Nīlakaṇṭha and Magic Squares in the *Ṛg-veda*" by Christopher Minkowski who kindly sent a preprint.
10. *Ṛgveda*, X, 114, 6 begins with *ṣaṭtriṃśāṃśca caturaḥ kalpayantraś-chandāṃsi* (see Bareilly ed., 1962, p. 1808).
11. In Indian chess, the camel *(uṣṭra)* moves diagonally. Thus if Fig. 4b, move from 1 to $(m - 1)$ is a diagonal jump. Nīlakaṇṭha is wrong to consider $2m = 32$ as a possible magic constant of 4th order squares with positive integers.
12. *Bṛhat-saṃhitā*, Part II, edited, with the commentary of Bhaṭṭotpala, by A. V. Tripathi, Varanasi, 1968, pp. 846–848.
13. T. Hayashi, "Varāhamihira's Pandiagonal Magic Square of Order Four", *Historia Mathematica*, Vol. 14 (1987), pp. 159–166, p. 163.
14. See R. C. Gupta, "Mystical Mathematics of Ancient Planets", *Indian Journal of History of Science,* Vol. 40 (2005), 31–53, pp. 46–48.
15. See "Magic Squares in India" by B. Datta and A. N. Singh (revised by K. S. Shukla), *I.J.H.S.*, 27 (1992), 51–120; pp. 52–53. The authors of the paper have confused the author of *Kakṣapuṭa* with the famous Buddhist Scholar (c. 100 AD). The earlier source seems to be the article of W. Goonetillke, "The American Puzzle", *Indian Antiquary*, 11 (1882), pp. 83–84.
16. Datta and Singh, *op.cit* above, p. 53.
17. *Ibid.,* pp. 56 and 118.
18. The change due to *Nāgārjunīya* transformation is not so trivial as caused by change of rows into columns and columns into rows, and by reflections or rotations of the square etc.
19. See *Gaṇita-kaumudī edition* (ref. 2 above), Introduction, p. XVI. Also see S. Y. Wakankar and S. D. Khadilkar. "Magic Squares of Sanskrit Origin", *Journal of Oriental Institute Baroda,* 30 (1981), 269–271.
20. See S. Cammann, "Islamic and Indian Magic Squares, Part II", *History of Religion*, 8 (1969), 271–299, p. 272, quoting an *ASI Annual Progress Report* (Lahore, 1916), p. 3.
21. R. Shortreede, "On An Ancient Magic Square cut in a Temple at Gwalior", *Journal of the Asiatic Society of Bengal* (N.S.), 11 (1842), 292–293.
22. See Hayashi, *op. cit* (ref. 13), pp. 164–165.
23. See C. A. Pickover, *The Zen of Magic Squares, Circles, and Stars,* Universities Press, Hyderabad, 2004, pp. 19–20. Dürer square has symmetry of Fig. 18b, but not of Fig. 18a.

Part VII

Trigonometry and Spherical Trigonometry

Bhāskara I's Approximation to Sine

The *Mahābhāskarīya* of Bhāskara I (c. AD 600) contains a simple but elegant algebraic formula for approximating the trigonometric sine function. It may be expressed as

$$\sin\alpha = \frac{4\alpha(180-\alpha)}{40500-\alpha(180-\alpha)},$$

where the angular arc α is in degrees. Equivalent forms of the formula have been given by almost all subsequent Indian astronomers and mathematicians. To illustrate this, relevant passages from the works of Brahmagupta (AD 628), Vaṭeśvara (AD 904), Śrīpati (AD 1039), Bhāskara II (twelfth century), Nārāyaṇa (AD 1356) and Gaṇeśa (AD 1520) are quoted.

Accuracy of the rule is discussed and comparison with the actual values of sine is made and also depicted in a diagram. In addition to the two proofs given earlier by M. G. Inamdar (*The Mathematics Student*, Vol. XVIII, 1950, p. 10) and K. S. Shukla, three more derivations are included by the present author. We are not aware of the process by which Bhāskara I himself arrived at the formula which reflects a high standard of practical Mathematics in India as early as seventh century AD.

1 Introduction

Indians were the first to use the trigonometric sine function represented by half the chord of any arc of a circle. Hipparchus (second century BC) who has been called the 'Father of Trigonometry' dealt only with chords and not the half-chords as done by Hindus. So also Ptolemy (second century AD), who is much indebted to Hipparchus and has summarized all important features of Greek Trigonometry in his famous *Almagest*, used chords only. The history[1] of the word 'sine' will itself tell the story as to how the Indian Trigonometric functions sine, cosine and inverse sine were introduced into the Western World through the Arabs.

Among the Indians, the usual method of finding the sine of any arc was as follows: First a table of sine-chords (i.e. half-chords), or their differences, was prepared on the basis of some rough rule, apparently derived geometrically, such as given in *Āryabhaṭīya* (AD 499) or by using elementary trigonometric identities, as seems to

Indian Journal of History of Science, Vol. 2, No. 2. (1967) pp. 121–136.

K. Ramasubramanian (ed.), *Gaṇitānanda*,
https://doi.org/10.1007/978-981-13-1229-8_32

be done by Varāhamihira (early sixth century). To avoid fractions the *Sinus Totus* was taken large and the figures were rounded off. Most of the tables prepared gave such values of 24 sine-chords in the first quadrant at an equal interval of $3\frac{3}{4}$ degrees assuming the whole circumference to be represented by 360 degrees. For getting sine corresponding to any other arc, simple forms of interpolation were used. In Bhāskara I we come across an entirely different method for computing sine of any arc approximately. He gave a simple but elegant algebraic formula with the help of which any sine can be calculated directly and with a fair degree of accuracy.

2 The Rule

The rule stating the approximate expression for the trigonometric sine function is given by Bhāskara I in his first work now called *Mahābhāskarīya*. The relevant Sanskrit text is:

मख्यादिरहितं कर्म वक्ष्यते तत्समासतः ।
चक्रार्धांशकसमूहाद्विशोध्या ये भुजांशकाः ॥ १७ ॥
तत्छेषगुणिता द्विष्ठा शोध्याः खाभ्रेषुखाब्धितः ।
चतुर्थांशेन शेषस्य द्विष्ठमन्त्यफलं हतम् ॥ १८ ॥
बाहुकोट्योः फलं कृत्स्नं क्रमोत्क्रमगुणस्य वा ।
लभ्यते चन्द्रतीक्ष्णांश्वोस्ताराणां वापि तत्त्वतः ॥ १९ ॥

(*Mahābhāskarīya*, VII, 17–19)[2]

Dr. Shukla[3] translates the text as follows:

(Now) I briefly state the rule (for finding the *bhujaphala* and the *koṭiphala*, etc.) without making use of the Rsine-differences, 225, etc. Subtract the degrees of the *bhuja* (or *koṭi*) from the degrees of half a circle (i.e. 180 degrees). Then multiply the remainder by degrees of the *bhuja* (or *koṭi*) and put down the result at two places. At one place subtract the result from 40500. By one-fourth of the remainder (thus obtained) divide the result at the other place as multiplied by the *antyaphala* (i.e. the epicyclic radius). Thus is obtained the entire *bāhuphala* (or *koṭiphala*) for the sun, moon or the star-planets. So also are obtained the direct and inverse Rsines.

In current mathematical symbols the rule implied can be put as

$$R\sin\phi = \frac{R\phi(180-\phi)}{\frac{1}{4}\{40500-\phi(180-\phi)\}} \tag{1}$$

i.e.

$$\sin\phi = \frac{4\phi(180-\phi)}{40500-\phi(180-\phi)} \tag{2}$$

where ϕ is in degrees.

3 Equivalent Forms of the Rule from Subsequent Works

Many subsequent authors who dealt with the subject of finding sine without using tabular sines have given the rule more or less equivalent to that of Bhāskara I, who seems to be the first to give such rule. Below we give few instances of the same.

(i) Brāhmasphuṭa-Siddhānta

The fourteenth chapter has the couplets:

भुजकोट्यंशोनगुणा भार्धांशास्तच्चतुर्थभागोनैः ।
पञ्चद्वीन्दुखचन्द्रैर्विभाजिता व्यासदलगुणिता ॥ २३ ॥
तज्ज्ये परमफलज्या संगुणितास्तत्फले विना ज्याभिः ।
इष्टोच्चनीचवृत्तव्यासार्धं परमफलजीवा ॥ २४ ॥

(*Brāhmasphuṭa-siddhānta*, XIV, 23–24)[4]

Multiply the degrees of the *bhuja* or *koṭi* by degrees of half a circle diminished by the same (the product so obtained) be divided by 10125 lessened by the fourth part of that same product. The whole multiplied by the semi-diameter gives the sine …

i.e.

$$R \sin \phi = \frac{R\phi(180 - \phi)}{10125 - \frac{1}{4}\phi(180 - \phi)}$$

which is equivalent to Bhāskara's rule.

The *Brāhmasphuṭa-siddhānta* was composed by Brahmagupta in the year AD 628 according to what the author himself says in the work at XXIV, 7–8.[5]

Dr. Shukla[6] points out that Bhāskara I's commentary on the *Āryabhaṭīya* was written in AD 629 and his *Mahābhāskarīya* was written earlier than this date. Thus Bhāskara I seems to be a senior contemporary of Brahmagupta. Kuppanna Sastri[7] even asserts that the statements of Pṛthūdaka svāmī (AD 860), the commentator of *Brāhmasphuṭa-siddhānta*, imply that the Bhāskara I's works must have been known to Brahmagupta.

(ii) Vaṭeśvara-Siddhānta

In the fourth *adhyāya* of the *Spaṣṭādhikāra* of *Vaṭeśvara-siddhānta* the rule occurs in two forms as follows:

चक्रार्धांशा भुजांशैर्विरहितनिहतास्तद्विहीनैर्विभक्ताः
खव्योमेष्वभ्रवेदैः सलिलनिहताः पिण्डराशिः प्रदिष्टः ।
षड्भांशघ्ना भुजांशा निजकृतिरहितास्तत्तुरीयांशहीनैः
र्भक्ताः स्यात्पिण्डराशिर्विशिखनयनभूव्योमशीतांशुभिर्वा ॥

(*Vaṭeśvara-siddhānta, Spaṣṭādhikāra*, IV, 2)[8]

Multiply degrees to half the circle less the degrees of *bhuja* (by the degrees of *bhuja*). Divide (the product so obtained) by 40500 less that product. Multiplied by four is obtained the required sine. *Or* the *bhuja* in degrees be multiplied by 180 degrees and (the result) be lessened by its own square. Fourth part of the quantity (so obtained) be subtracted from 10125, and by this (new) result the first quantity be divided. The sine is obtained.

Thus we have the two forms as

$$\sin\phi = \frac{\phi(180-\phi)\times 4}{40500-\phi(180-\phi)}$$

and

$$\sin\phi = \frac{\phi\cdot 180-\phi^2}{10125-\frac{1}{4}(\phi\cdot 180-\phi^2)}$$

both being equivalent to Bhāskara I's rule.

According to a verse[9] of the work itself, the *Vaṭeśvara-siddhānta* was composed in AD 904.

(iii) Siddhānta-śekhara

In this astronomical treatise the rule is given as:

दोःकोटिभागरहिताभिहताः खनागचन्द्रास्तदीयचरणोनशरार्कदिग्भिः ।
ते व्यासखण्डगुणिता विहृताः फले तु ज्याभिर्विनैव भवतो भुजकोटिजीवे ॥ १७ ॥

(*Siddhānta-śekhara* III, 17)[10]

The degrees of *doh* or *koṭi* multiplied to 180 less degrees of *doh* or *koṭi*. Semi-diameter times the product (so obtained) divided by 10125 less fourth part of that product becomes the *bhuja* or *koṭīphala*.

i.e.

$$R\sin\phi = \frac{R\phi(180-\phi)}{10125-\frac{1}{4}\cdot\phi\cdot(180-\phi)}$$

This form is exactly the same as that of Brahmagupta. Sengupta[11] gives AD 1039 as the date of *Siddhānta-śekhara* which was composed by Śrīpati.

(iv) Līlāvatī

The concerned stanza is:

चापोननिघ्नपरिधिः प्रथमाह्वयः स्यात् पञ्चाहतः परिधिवर्गचतुर्थभागः ।
आद्योनितेन खलु तेन भजेच्चतुर्घ्नव्यासाहतं प्रथममाप्तमिह ज्यका स्यात् ॥

(*Līlāvatī, Kṣetravyavahāra,* No. 48)

Circumference less (a given) arc multiplied by that arc is *prathama*. Multiply square of the circumference by five and take its fourth part. By the quantity so obtained, but lessened by *prathama*, divide the *prathama* multiplied by four times the diameter. The result will be chord (i.e. *pūrṇa jyā* or double-sine) of the (given) arc.

i.e.

$$\begin{aligned}(360-2\phi)\cdot 2\phi &= prathama\\ 2R\sin\phi &= \frac{4\times 2R\times(prathama)}{\frac{1}{4}\times 5\times 360^2-(prathama)}\\ &= \frac{8R(360-2\phi)\cdot 2\phi}{\frac{1}{4}\times 5\times 360^2-(360-2\phi)\cdot 2\phi}\end{aligned}$$

giving

$$\sin\phi = \frac{4\phi(180-\phi)}{40500-\phi(180-\phi)}$$

which is mathematically equivalent to the rule of Bhāskara I. *Līlavatī* was composed by Bhāskara II in the first half of the twelfth century.

It should be noted that Bhāskara II himself accepted the rule to be very approximate. He says in his *jyotpatti*[12] (an appendix to *Golādhyāya*):

स्थूलं ज्यानयनं पाट्यामिह तन्नोदितं मया

The rough (crude) rule for finding sine given in my arithmetic (*Līlāvatī*) is not discussed here.

(v) *Gaṇita-Kaumudī*

As in case of *Vaṭeśvara-siddhānta*, two forms of the rule occur here in the chapter called *Kṣetravyavahāra* as follows:

वृत्त्यर्धं धनुरूनितं स्वगुणितं तेनोनयुक्ते क्रमाद्
वृत्त्यर्धं च वृतिश्चते स्वगुणिते तौ गुण्यहाराह्वयौ ।
व्यासे गुण्यहते हराङ्घ्रिविहृते ज्यास्यादथाद्यज्यया-
ऽऽसन्ना ज्यारहिता ग्रहाख्यगणिते स्युर्व्यासखण्डानि च ॥

अथवा

वृत्तेधनूरहितनिघ्नवृतिर्द्धिधा तां व्यासाहतां च विभजेदितराङ्घ्रिहीनैः ।
वृत्त्यङ्घ्रिवर्गगणितैर्विषयैश्च जीवा स्यात् खेचराख्यगणितेऽप्युपयोग एषः ।

(*Gaṇita-kaumudī, Kṣetravyavahāra,* 69–70)[13]

Multiply to itself half the circumference less (a given) arc. The quantity so obtained when respectively subtracted from and added to the squares of half the circumference and circumference respectively gives the Numerator and Denominator. Multiply the diameter by the Numerator and divide by one-fourth of the Denominator to get the chord. …

Alternately,

Multiply the circumference less the given arc by the arc and put the result in two places. In one place multiply it by the diameter and divide the result by five times the square of the quarter circumference less the quarter of the result in the other place. The final result is chord. …

In symbols these can be expressed as follows. Let c stand for circumference,

First form:

$$\text{Numerator, } N = \left(\frac{1}{2}c\right)^2 - \left(\frac{1}{2}c - 2\phi\right)^2$$

$$\text{Denominator, } D = c^2 + \left(\frac{1}{2}c - 2\phi\right)^2$$

then

$$\text{Chord} = \frac{N \times 2R}{\frac{1}{4} \cdot D}$$

i.e.

$$2R\sin\phi = \frac{2R\{180^2 - (180 - 2\phi)^2\}}{\frac{1}{4}\{360^2 + (180 - 2\phi)^2\}}$$

Second form:

$$\text{Chord} = \frac{(c - 2\phi)2\phi \cdot 2R}{5\left(\frac{1}{4}c\right)^2 - \frac{1}{4} \cdot 2\phi(c - 2\phi)}$$

or

$$2R\sin\phi = \frac{2R(360^2 - 2\phi)2\phi}{40500 - \phi(180 - \phi)}.$$

On simplification both the forms reduce to the rule of Bhāskara I.

It is stated by Padmākara Dvivedī[14] that *Gaṇitakaumudī* was composed by Nārāyaṇa Paṇḍita in AD 1356. *See* colophonic verse No. 5 of the work in the edition referred.

(vi) Grahalāghava

In this work the rule is given in various modified forms adopted for particular cases. The relevant text from *Ravicandra-spaṣṭādhikāra* for one such case is:

विधोः केन्द्रदोर्भागषष्ठोननिघ्नाः खरामाः पृथक् तन्नखांशोनितैश्च ।
रसाक्षैर्हृतास्ते लवाद्यं फलं स्यात् रवीन्दू स्फुटौ संस्कृतौ स्तश्च ताभ्याम् ॥ ३ ॥

(*Grahalāghava*, II, 3)[15]

> Subtract the sixth part of the degrees of the *bhuja* of the moon from 30 and multiply the result by the same sixth part. Put the product in two places. By 56 minus the twentieth part of the product in one place divide the product of the other place. The result is the *Mandaphala* ....

$$R \sin\phi = \frac{\left(30 - \frac{\phi}{6}\right) \times \frac{\phi}{6}}{56 - \frac{1}{20}\left(30 - \frac{\phi}{6}\right)\frac{\phi}{6}}$$

$$\text{or} \qquad \sin\phi = \frac{20\phi(180-\phi)}{R\{40320 - \phi(180-\phi)\}}$$

$$= \frac{4 \cdot \phi(180-\phi)}{40320 - \phi(180-\phi)}$$

since the value of the maximum *mandaphala* for moon, i.e. the value of R for moon, is 5 degrees approximately.[16] Thus we see that the form of the rule given here is slightly modified. In place of the figure 40500 of Bhāskara I we have 40320 here. Another modified form is given in the stanza preceding the one we have quoted above.

Grahalāghava was written by Gaṇeśa Daivajña in the year 1520. He dealt with the whole of astronomy contained in the work without using *Jyāgaṇita* (i.e. sine-chords). For sine, whenever needed, he used the rule equivalent to that of Bhāskara I after duly modifying it and rounding off the fractions.

Table 1 Accuracy of Bhāskara's rule

ϕ	$\sin\phi$ by Bhāskara's rule	Actual value of $\sin\phi$
0	0.00000	0.00000
10	0.17525	0.17365
20	0.34317	0.34202
30	0.50000	0.50000
40	0.64183	0.64279
50	0.76471	0.76604
60	0.86486	0.86603
70	0.93903	0.93969
80	0.98461	0.98481
90	1.00000	1.00000

4 Discussion of the Rule

The rule of Bhāskara I, mathematically expressed by relation (2), is a representation of the transcendental function $\sin\phi$, by means of a rational function, i.e. the quotient of two polynomials. This algebraic approximation is not only simple but surprisingly accurate remembering that it was given more than thirteen hundred years ago. The

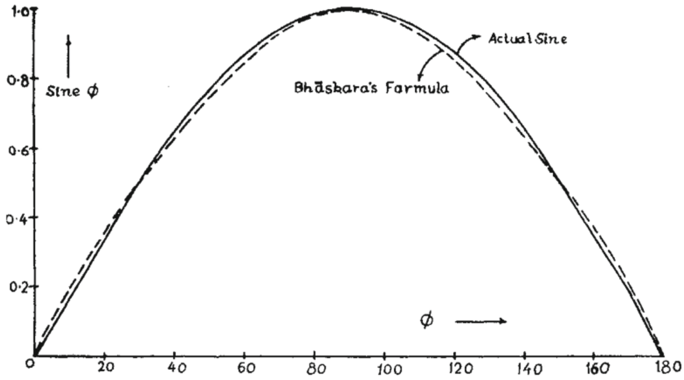

Fig. 1 Geometrical representation

relative accuracy of the formula can be easily judged from Table 1 which gives the comparison of the values of sines obtained by using Bhāskara's rule with the correct values. Calculations have been made by using Castle's Five Figure Logarithmic and Other Tables (1959 ed.) from which the actual values of $\sin\phi$ are also taken.

A glance at Table 1 will show that Bhāskara's approximation affects the third place of decimal by one or two. The Maximum deviation in the table occurs at 10 degrees and is

$$= +0.00160$$

The corresponding percentage error can be seen to be less than one per cent.

Figure 1 shows how nicely the curve represented by Bhāskara's formula approximates the actual sine curve. The deviations have been exaggerated so that the two curves may be clearly distinguished; otherwise, they agree so well that, on the size of the scale used, it was not practically possible to draw them distinctly.

The proper range of validity of the rule is from $\phi = 0$ to $\phi = 180$. It cannot be used directly for getting sine between 180 and 360. However with slight modification[17] it can yield $\sin\phi$ for any value of ϕ.

The last word *tattvataḥ* ('truly' or 'really') of the text, quoted for Bhāskara I's rule, indicates that the rule is to be taken as 'accurate.' Dr. Singh[18] used the word 'grossly' in his translation of the passage. As already pointed out earlier (*see* under (iv) *Līlāvatī*), Bhāskara II clearly stated his equivalent rule as *sthūla* ('rough' or 'gross'). But whether Bhāskara I also meant his rule to be so is doubtful.

5 Derivation of the Formula

The procedure through which Bhāskara I arrived at the rule is not given in *Mahābhāskarīya* which contains the rule. But this seems to be the general feature in case of most of the results in ancient Indian mathematics. This may be partly due to the fact that these mathematical rules are found mostly in works which do not deal exclusively with mathematics as they are treatises on astronomy and not on mathematics. Below we give few methods of arriving at the approximate formula.[†]

(i) First Approach

Following[*] Shukla,[19] let CA be the diameter of a circle of radius R, where the arc AB is equal to ϕ degrees, and (Fig. 2)

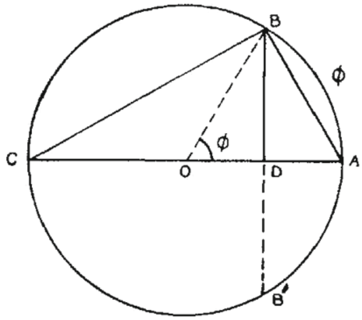

Fig. 2 Derivation of the rule

$$BD = R \sin \phi$$

Now

$$\text{Area of the } \Delta ABC = \frac{1}{2} AB \cdot BC$$
$$\text{and also} = \frac{1}{2} AC \cdot BD$$

Therefore

$$\frac{1}{BD} = \frac{AC}{AB \cdot BC}$$

so that

$$\frac{1}{BD} > \frac{AC}{(\text{arc} AB) \cdot (\text{arc} BC)} \tag{3}$$

[†]Now also see *Gaṇita Bhāratī*, Vol. 8 (1986), pp. 39–41.

[*]*See* M. G. Inamdar's paper in *The Mathematics Student*, Vol. XVIII, 1950.

After this Shukla assumes

$$\begin{aligned}\frac{1}{BD} &= \frac{X.AC}{(\text{arc}AB)\cdot(\text{arc}BC)} + Y \\ &= \frac{2XR}{\phi(180-\phi)} + Y\end{aligned} \tag{4}$$

so that

$$R\sin\phi = \frac{\phi(180-\phi)}{2XR + Y\phi(180-\phi)} \tag{5}$$

By taking two particular values, viz., 30 and 90, for ϕ we get two equation in X and Y from (5). Solving these it is easily seen that $Y = -\frac{1}{4R}$ and $2XR = \frac{40500}{4R}$. Putting these in (5), Bhāskara's formula is readily obtained.

If X is greater than one and Y is positive the assumption (4) is justified by virtue of the inequality (3). But as noted above, Y comes out to be negative. Hence some more investigation to justify (4) is needed which I give below.

Let a, b, p and q be four positive quantities all different from each other. If $a > b$, then we can write

$$a = pb - q. \tag{6}$$

That is

$$b = \frac{a+q}{p}$$

provided

$$a > \frac{a+q}{p}, \qquad \text{since} \qquad a > b.$$

That is to say we can write (6) if

$$\begin{aligned} & p > \frac{a+q}{a} \\ \text{that is} \qquad & p > 1 + \frac{q}{a}\end{aligned} \tag{7}$$

In the above case

$$\begin{aligned} a &= \frac{1}{R\sin\phi} \\ b &= \frac{AC}{(\text{arc}AB)\cdot(\text{arc}BC)} \\ p &= X = \frac{40500}{8R^2}\end{aligned} \tag{8}$$

and

$$q = -Y = \frac{1}{4R}$$

therefore

$$1 + \frac{q}{a} = 1 + \frac{R \sin\phi}{4R}$$

Now greatest value of this

$$= 1 + \frac{1}{4} = \frac{5}{4}$$

Also, since $R = \frac{360}{2\pi}$, from (8) we get

$$p = \frac{405 \cdot \pi^2}{2592} > \frac{5}{4}$$

So that the condition (7) is satisfied and hence Shukla's assumption (4) is justified.

(ii) Second Approach

Let[20]

$$\sin\lambda = \frac{a + b\lambda + c\lambda^2}{A + B\lambda + C\lambda^2} \tag{9}$$

where λ is in radians and corresponds to ϕ degrees.

Out of the six unknown coefficients a, b, c, A, B, C, only five are independent. Those can be found by using the following five mathematical facts:

$$\begin{aligned}
\lambda = 0, &\quad \sin\lambda = 0 \\
\lambda = \pi, &\quad \sin\lambda = 0 \\
\sin\lambda &= \sin(\pi - \lambda) \\
\lambda = \frac{1}{6}\pi, &\quad \sin\lambda = \frac{1}{2} \\
\lambda = \frac{1}{2}\pi, &\quad \sin\lambda = 1
\end{aligned}$$

Utilizing these five conditions we easily arrive at Bhāskara's formula. Perhaps the justification for supposition (9) is that the very form of Bhāskara's rule is of type (9).

(iii) Third Approach

Shifting the origin through 90 degrees by putting $\phi = 90 + X$ (2) becomes

$$\cos X = \frac{4(90+X)(90-X)}{40500-(90+X)(90-X)}$$

or

$$\cos X = \frac{4(8100-X^2)}{32400+X^2} \tag{10}$$

By elementary empirical arguments we shall derive formula (10) which resembles the first form given by Nārāyaṇa Paṇḍita (*vide* under (v) *Gaṇita-kaumudī*).

Since the sine function, in the interval 0 to 180, is symmetrical about $\phi = 90$, it follows that the cosine function will be symmetrical about $X = 0$ in the interval -90 to $+90$. In other words, $\cos X$ is an even function of X, say

$$\cos X = f(X^2)$$

Now $\cos X$ decreases as X increases from 0 to 90. The simplest way of effecting this is to take

$$\cos X \propto \frac{1}{X^2}$$

But $\cos X$ also remains finite at $X = 0$; hence, we should assume, instead of above,

$$\cos X \propto \frac{1}{X^2+a}, \qquad \text{where } a \neq 0$$

Finally, remembering that $\cos X$ vanishes for finite values of X, we take

$$\cos X = \frac{C}{X^2+a} - k$$

k being positive.

It should be noted that the possibility of taking simply

$$\cos X = \frac{C}{X^2+a}$$

is ruled out since by this assumption we cannot make $\cos X$ to vanish for any finite value of X while we know $\cos 90 = 0$. The three unknowns a, c and k can be found by taking particular known simple rational values, e.g.

$$\cos 0 = 1; \qquad \cos 60 = \frac{1}{2}; \qquad \cos 90 = 0.$$

Utilizing these we get (10) without difficulty.

(iv) Fourth Approach

Now we take the method of approximation by continued fractions. The general formula which we shall use is[21]

$$f(X) = a_0 + \cfrac{X - X_0}{a_1 + \cfrac{X - X_1}{a_2 + \cfrac{X - X_2}{a_3 + \cdots}}} \tag{11}$$

where

$$a_k = \phi_k[X_0, X_1, X_2 \ldots X_{k-1}, X_k]$$

The quantities ϕ_k's are called the 'inverted differences' and are defined as follows:

$$\phi_0[X] = f(X)$$

$$\phi_1[X_0, X] = \frac{X - X_0}{\phi_0[X] - \phi_0[X_0]} = \frac{X - X_0}{f(X) - f(X_0)}$$

$$\phi_2[X_0, X_1, X] = \frac{X - X_1}{\phi_1[X_0, X] - \phi_1[X_0, X_1]}$$

and so on.

Now we form the table of 'inverted differences' for the function $f(X) = \sin X$ by taking some simple rational values of the sines (*see* Table 2).

Table 2 Inverted differences

X in degrees	$f(X) = \phi_0 = \sin X$	ϕ_1	ϕ_2	ϕ_3	ϕ_4
$X_0 = 0$	$\underline{0 = a_0}$	—	—	—	—
$X_1 = 30$	$\frac{1}{2}$	$\underline{60 = a_1}$	—	—	—
$X_2 = 90$	1	90	$\underline{2 = a_2}$	—	—
$X_3 = 150$	$\frac{1}{2}$	300	$\frac{1}{2}$	$\underline{-40 = a_3}$	—
$X_4 = 180$	0	$\propto$	0	-45	$\underline{-6 = a_4}$

Using (11) the successive convergents are:

$$\text{First,} \qquad = a_0 = 0$$

$$\text{Second,} \qquad = a_0 + \frac{X - X_0}{a_1} = \frac{X}{60}$$

$$\text{Third,} \qquad = a_0 + \cfrac{X - X_0}{a_1 + \cfrac{X - X_2}{a_2}}$$

$$= \frac{2X}{X + 90}$$

$$\text{Fourth,} \qquad = a_0 + \cfrac{X - X_0}{a_1 + \cfrac{X - X_1}{a_2 + \cfrac{X - X_2}{a_3}}}$$

$$= \frac{X(170 - X)}{9000 - 20X}$$

$$\text{Fifth,} \qquad = a_0 + \cfrac{X - X_0}{a_1 + \cfrac{X - X_1}{a_2 + \cfrac{X - X_2}{a_3 + \cfrac{X - X_3}{a_4}}}}$$

$$= \frac{4X(180 - X)}{40500 - X(180 - X)}$$

which is the rule of Bhāskara I.

Inverted differences or rather the related quantities called 'reciprocal differences' were introduced by T. N. Thiele[22] in AD 1909. We do not know if any ancient Indian mathematician ever used the inverted or reciprocal differences, although Brahmagupta (AD 665) has been credited for being the first to use 'direct' differences up to second-order.[23]

(v) Fifth Approach

It will be noted that the quantity

$$\phi(180 - \phi) = p, \qquad \text{say}$$

is an important function of ϕ in connection with $\sin\phi$. In fact it is a quarter of what Bhāskara II has called *prathama* (*see* under (iv) *Līlāvatī*). Now p can be written as

$$p = 8100 - (\phi - 90)^2.$$

It follows that the maximum value of p will be 8100 when $\phi = 90$. Also the function p vanishes for $\phi = 0$ and 180 and is symmetrical about $\phi = 90$. Thus the nature of p and $\sin\phi$ resembles in certain points. Since the maximum value of $\sin\phi$ is 1, we may take as a crude approximation

$$\sin\phi = \frac{p}{8100} = P, \qquad \text{say.}$$

However Bhāskara I's formula is far better than the above simplest (linear) but very rough approximation. We now attempt to find a better relation between $\sin\phi$ and P. Since $0 \times 0 = 0$ and $1 \times 1 = 1$, the product function $P \times \sin\phi$ will also have the same nature as P or $\sin\phi$. The simplest relation between $P, \sin\phi$ and $P\sin\phi$ will be a linear one. Therefore we assume

$$lP\sin\phi + mP + n\sin\phi = 0 \tag{12}$$

which is the general form of a linear relation. Taking $\phi = 90$ and $\phi = 30$ we get respectively

$$l + m + n = 0$$

and

$$\frac{5}{18}l + \frac{5}{9}m + \frac{1}{2}n = 0.$$

Solving the above two equations we get

$$l = -\frac{1}{5}n$$
$$m = -\frac{4}{5}n$$

Using these, the linear relation (12) becomes, after simplification (i.e. solving for $\sin\phi$),

$$\sin\phi = \frac{4P}{5-P} \tag{13}$$

which is the formula of Bhāskara I in disguise. Form (2) of the rule can be obtained by substitution of the value of P, viz.

$$P = \frac{\phi(180-\phi)}{8100}$$

Relation (13) may be regarded as the simplest form of Bhāskara's rule and may be derived in many ways.

6 Conclusion

To Bhāskara I (early seventh century AD) goes the credit of being the first to give a surprisingly simple algebraic approximation to the trigonometric sine function. In its simplest form his rule may be expressed by the mathematical formula

$$\sin\phi = \frac{4P}{5-P},$$

where P may be called as the 'modified *prathama*' (accepting the definition of Bhāskara II). If ϕ is measured in terms of right angles (quadrants) then P will be given by

$$P = \phi(2-\phi).$$

The formula is fairly accurate for all practical purposes. A better formula, which will have the simplicity and practicability of Bhāskara I's formula, can hardly be given without introducing bigger rational numbers or irrational numbers and higher degree polynomials of ϕ. No such mathematical formula approximating algebraically a transcendental function seems to be given by other nations of antiquity. The formula as such, or its modified form, has been used by almost all the subsequent authors, a few instances of which are given in this paper. Remembering that it was given more than one thousand and three hundred years ago, it reflects a high standard of mathematics prevalent at that time in India. 'How Indians arrived at the rule' may be taken as an open question.

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Fractional Parts of Āryabhaṭa's Sines and Certain Rules found in Govindasvāmin's *Bhāṣya* on the *Mahābhāskarīya*

The commentary of Govindasvāmin (*circa* AD 800–850) on the *Mahābhāskarīya* contains the sexagesimal fractional parts of the 24 tabular Sine-differences given by Āryabhaṭa I (born AD 476). These lead to a more accurate table of Sines for the interval of 225 minutes. Thus the last tabular Sine becomes.

$$3437 + \frac{44}{60} + \frac{19}{60^2},$$

instead of Āryabhaṭa's 3438.

Besides this improvement of Āryabhaṭa's sine table, the paper also deals with some empirical rules given by Govindasvāmin for computing tabular Sine-differences in the argumental range of 60 to 90 degrees. The most important of these rules may be expressed as

$$D_{(24-p)} = \left[D_{24} - \frac{(1+2+\ldots+p)\cdot c}{60^2}\right]\cdot(2p+1),$$

where

$$p = 1,2,\ldots,7;$$

and $D_{17}, D_{18}, \ldots D_{24}$ are the tabular Sine-differences with D_{24} being given, in the usual mixed sexagesimal notation, as

$$a + \frac{b}{60} + \frac{c}{60^2}.$$

Symbols

$a; b, c$	The usual notation for writing a number with whole part 'a' (say, in minutes) separated from its sexagesimal fractional parts (of various orders, 'b' (in second), 'c' (in thirds),..., by a semicolon.
$D_1, D_2 \ldots$	Tabular Sine-differences such that $D_n = R\sin nh - R\sin(n-1)h$; $n = 1,2,\ldots$
h	Uniform tabular interval.

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$L(h)$	Last tabular Sine-difference when the tabular interval is h, so that $L(h) = R - R\cos h$.
m, n, p	Positive integers.
R	Radius, *Sinus Totus,* norm.

1 Introduction

It is well known[1] that the *Āryabhaṭīya* of Āryabhaṭa I (born AD 476) contains a set of 24 tabular Sine-differences. In the modern language we can say that the work tabulates, to the nearest whole number, the values of

$$D_n = R\sin nh - R\sin(n-1)h$$

for

$$n = 1, 2, \ldots\ldots, 24;$$

where the uniform tabular interval h is equal to 225 minutes and the norm R is defined by

$$R = \frac{21600}{2\pi} \tag{1}$$

Āryabhaṭīya, II, 10 gives[2]

$$\pi = 3.1416, \text{ approximately.}$$

Using this approximation of π, the definition (1) gives

$$\begin{aligned} R &= 3437.73872, \text{ nearly} \\ &= 3437;\ 44, 19 \text{ to the nearest third.} \end{aligned}$$

Thus, to the nearest minute, the 24th tabular Sine (the *Sinus Totus* or the radius) will be given by

$$R = R\sin 90^\circ = 3438.$$

By employing his own peculiar alphabetic system[3] of expressing numbers, Āryabhaṭa could express the 24 tabular Sine-differences just in one couplet which runs as follows: (*Āryabhaṭīya* I, 10; pp. 16–17)

225	224	222	219	215	210	205	199	(191)	183	174	164
मखि	भखि	फखि	धखि	णखि	ञखि	ङखि	हस्झ	स्ककि	किष्ग	श्घकि	किघ्व ।
154	143	131	119	106	93	79	65	51	37	22	7
घ्लकि	किग्र	हक्य	धाहा	स्त	स्ग	श्झ	ङ्व	ल्क	प्त	फ	छ कलार्धज्याः ॥

In Kern's edition (Leiden 1874), which is used here, the text and commentary both give the reading *svaki*, 250 (a wrong value), for the ninth tabular Sine-difference. It is stated by Sen[4] that Fleet pointed out the mistake as early as 1911. However, it must be noted that although the commentary reading is *svaki,* the translation or explanation given by the commentator (Parameśvara, *circa* AD 1430) is '*candrāṅkaikaḥ*', 191, which is correct. This shows that *svaki* was not the original reading.

In fact, Śaṅkaranārāyaṇa (AD 869) in his commentary[5] on *Laghu-bhāskarīya* quotes the above couplet in full with the reading *skaki,* 191 (which is correct), instead of the wrong reading *svaki,* 250. That in the original text of *Āryabhaṭīya* the reading was *skaki* has also been confirmed by consulting the manuscripts[6] of the commentaries of Bhāskara I (AD 629)[7] and Sūryadeva Yajva (born AD 1191). Hence it is certain that the original reading was *skaki* which is adopted here as well as by other translators.[*]

These tabular Sine-differences are shown in Table 1.

Table 1 Tabular Sine-differences

n	Actual value of $R \sin nh$ ($R = \frac{10800}{3.1416}$, and $h = 225$ min.)	Actual Sine-diff. D_n	Ārya-bhaṭa's Sine-diff.	Govinda-svāmin's fractional parts	Āryabhaṭa's Sine-diff. improved by Govindasvāmin
1	224; 50, 19, 56	224; 50, 19, 56	225	−9, 37	224; 50, 23
2	448; 42, 53, 48	223; 52, 33, 52	224	−7, 30	223; 52, 30
3	670; 40, 10, 24	221; 57, 16, 36	222	−2, 42	221; 57, 18
4	889; 45, 08, 06	219; 04, 57, 42	219	+4, 57	219; 04, 57
5	1105; 01, 29, 37	215; 16, 21, 31	215	+16, 22	215; 16, 22
6	1315; 33, 56, 21	210; 32, 26, 44	210	+32, 26	210; 32, 26
7	1520; 28, 22, 38	204; 54, 26, 17	205	−5, 34	204; 54, 26
8	1718; 52, 09, 42	198; 23, 47, 04	199	−36, 12	198; 23, 48
9	1909; 54, 19, 05	191; 02, 09, 23	191	+2, 09	192; 02, 09
10	2092; 45, 45, 51	182; 51, 26, 46	183	−8, 33	182; 51, 27
11	2266; 39, 31, 06	173; 53, 45, 15	174	−7, 02	173; 52, 58
12	2430; 50, 54, 06	164; 11, 23, 00	164	+12, 10	164; 12, 10
13	2584; 37, 43, 44	153; 46, 49, 38	154	−13, 11	153; 46, 49
14	2727; 20, 29, 23	142; 42, 45, 39	143	−17, 14	142; 42, 46
15	2858; 22, 31, 00	131; 02, 01, 37	131	+2, 02	131; 02, 02
16	2977; 10, 08, 37	118; 47, 37, 37	119	−12, 22	118; 47, 38
17	3083; 12, 50, 56	106; 02, 42, 19	106	+2, 42	106; 02, 42
18	3176; 03, 23, 11	092; 50, 32, 15	93	−9, 28	92; 50, 32
19	3255; 17, 54, 08	079; 14, 30, 57	79	+14, 31	79; 14, 31
20	3320; 36, 02, 12	065; 18, 08, 04	65	+18, 08	65; 18, 08
21	3371; 41, 00, 43	051; 04, 58, 31	51	+4, 59	51; 04, 59
22	3408; 19, 42, 12	036; 38, 41, 29	37	−21, 19	36; 38, 41
23	3430; 22, 41, 43	022; 02, 59, 31	22	+3, 00	22; 03, 00
24	3437; 44, 19, 23	007; 21, 37, 40	07	+21, 37	07; 21, 37

[*] It is now evident that the reading in the commentary by Parameśvara has also been *skaki* originally and not *svaki* as appears in the printed edition.

Instead of tabulating the Sine-differences to the nearest whole minutes, if they are tabulated up to the second-order sexagesimal fraction, then the tabular values should be given in minutes, seconds, and thirds. The sexagesimal fractional parts (seconds and thirds), in defect or in excess, of the Āryabhaṭa's Sine-differences are found stated in the commentary (gloss) of Govindasvāmin (*circa* AD 800–850)[8] on the *Mahābhāskarīya* of Bhāskara I (early seventh century AD), both belonging to the Āryabhaṭa School of Indian Astronomy. These fractional parts *(avayavāḥ)* are described below in section two of the paper. Certain other rules concerning the computations of Sine-differences, as found in the same commentary, are discussed in the subsequent sections of the paper.

2 Fractional Parts of Āryabhaṭa's Sine-Differences

Described in the usual Indian word-numerals (Bhūtasaṅkhyās), the seconds and thirds (in defect or in excess) of all the 24 Āryabhaṭa's Sine-differences appear on page 200 of the printed edition (Madras, 1957) of Govindasvāmin's commentary on the *Mahābhāskarīya*. They are as follows (the first two digits in each figure-group of the *text* denote the thirds):

9,37	7,30	2,42	4,57
सप्ताग्निरन्ध्राणि,	वियद्गुणागं,	नेत्राब्धिनेत्रं,	मुनिपञ्चवेदाः ।
16,22	32,26	5,34	36,12
द्व्यक्ष्यष्टयः,	षण्णयनद्विरामा,	वेदाग्निभूतं,	रविषड्कृशानुः ॥
2,09	8,33	7,02	12,10
रन्ध्राभ्रपक्षं,	गुणपावकाष्टौ,	चक्षुर्वियत्सप्त,	खचन्द्रसूर्याः ।
13,11	17,14	2,02	12,22
रुद्राग्निचन्द्रा,	मनुसप्तसोमा,	दस्राभ्रनेत्रं,	नयनं द्विसूर्यम् ॥
2,42	9,28	14,31	18,08
अक्ष्यब्धिपक्षं	वसुनेत्ररन्ध्रं,	चन्द्राग्निविद्या,	वसुखाष्टचन्द्रम् ।
4,59	21,19	3,00	21,37
रन्ध्रेषुवेदं,	नवरूपमिध्मं,	खाभ्राग्नयस्,	सप्तगुणेध्मसंख्यम् ॥

(Govindasvāmin's commentary on the *Mahābhāskarīya* under IV, 22).

After describing these values the commentary says (p. 201):

इत्युक्तास्तत्पराद्याः स्युरेते हीनाधिकांशकाः ।
गुणानां ते ततः शोध्या मख्यादौ योजिता अपि ॥
त्रि-त्रि-द्वि-रूप-नेत्रै-क-द्वि-चन्द्रै-के-न्दु-संख्यया ।
एक-त्रि-रूप-नेत्रैश्च ज्याविद्भिर्गणकैः क्रमात् ॥

These are the fractional parts, thirds first, in defect or in excess, of the Sine-differences. They are subtracted from, and added to, (the Āryabhaṭa's Sine-differences) *makhi,* etc., by the calculators expert in Sines (taking) 3, 3, 2, 1, 2, 1, 2, 1, 1, 1, 1, 3, 1, 2, in succession (from the set).

These fractional parts with their proper signs are tabulated in Table 1. The resulting tabular Sine-differences are also given in the table along with the actual values for the purpose of comparison.

3 An Approximate Rule Concerning The Last Tabular Sine-Difference

For finding an approximate value of the last Sine-difference with tabular interval $\frac{h}{2}$, from the last Sine-difference when the tabular interval is h, the commentary (p. 199) of Govindasvāmin on the *Mahābhāskarīya* gives a simple rule as follows:

> अन्त्यगुणस्य तावत् चतुर्भागः, तदर्धकाष्ठान्त्यज्या
>
> The fourth part of the last (tabular) Sine-difference (corresponding to a tabular interval of arc h) is the last (tabular) Sine-difference corresponding to half of the (given tabular) arc.

That is,

$$\left(\frac{1}{4}\right)\cdot\ L(h) = L\left(\frac{h}{2}\right)$$

The work gives the following illustrations of the rule:

$$\left(\frac{1}{4}\right)\cdot L(450) = L(225),$$
$$\left(\frac{1}{4}\right)\cdot L(225) = L(112;30)$$
$$\left(\frac{1}{4}\right)\cdot L(112;30) = L(56;15)$$

'In this way', says the author, 'the last tabular Sine-difference corresponding to any tabular arc (of the type $\frac{h}{2^n}$) should be obtained'. Thus we have the rule

$$L\left(\frac{h}{2^n}\right) = \frac{L(h)}{4^n}.$$

Rationale: We have

$$L\left(\frac{h}{2}\right) = R - R\cos\frac{(h)}{2} = 2R\sin^2\left(\frac{h}{4}\right).$$

Now

$$
\begin{aligned}
L(h) = R - R\cosh &= 2R\sin^2\left(\frac{h}{2}\right) \\
&= 8R\sin^2\left(\frac{h}{4}\right)\cdot\cos^2\left(\frac{h}{4}\right) \\
&= 4\cdot L\left(\frac{h}{2}\right)\cdot\cos^2\left(\frac{h}{4}\right), \text{ by the above.}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
L\left(\frac{h}{2}\right) &= \left(\frac{1}{4}\right)\cdot\ L(h)\ \sec^2\left(\frac{h}{4}\right) \\
&= \left(\frac{1}{4}\right)\cdot L(h) + \left(\frac{1}{4}\right)\cdot\ L(h)\cdot\tan^2\left(\frac{h}{4}\right)
\end{aligned}
$$

From this the rule follows, since (when h is small)

$$
\begin{aligned}
&\left(\frac{1}{4}\right)\cdot\ L(h), \tan^2\left(\frac{h}{4}\right) \\
&= \left(\frac{1}{4}\right)\cdot\ 2R\sin^2\left(\frac{h}{2}\right)\cdot\tan^2\left(\frac{h}{4}\right) \\
&= \frac{h^4}{128R^3}, \text{ approximately,}
\end{aligned}
$$

which is negligible. For an alternative rationale *see* Sect. 4.

4 A Crude Rule for Computing Tabular Sine-Differences in the Third Sign (60° to 90°)

After giving the method of finding the last tabular Sine-difference D_n (described in the last section), the commentary (p. 199) of Govindasvāmi on *Mahābhāskarīya* gives the following crude rule for obtaining the other tabular Sine-differences (lying in the third sign only) from D_n.

> सा पुनस्त्र्यादिविषमसंख्यागुणिता तदधःप्रभृत्युत्क्रमतस्तद्भागज्या ।
> एवं तृतीयराशिज्याकल्पना ।
>
> That (that is, the last tabular Sine-difference) severally multiplied by the odd numbers 3, etc., become the Sine-difference below that (that is, the last-but-one), etc. (that is, the other Sine-differences), counted in the reversed order. This is the method of getting Sine-differences in the third sign.

That is, from

$$L(h) = D_n,$$

we get

$$\begin{aligned}
3 \times L(h) &= D_{n-1},\\
5 \times L(h) &= D_{n-2},\\
&\dots\dots\dots\dots\\
(2p+1)L(h) &= D_{n-p};\ p = 0, 1, 2, \dots
\end{aligned}$$

Rationale: We have

$$\begin{aligned}
D_{n-p} &= R\sin\,(n-p)h - R\sin\,(n-p-1)h\\
&= R\cos\ ph - R\cos(p+1)h,\ \text{as } nh = 90°,\\
&= 2R\sin\left(\frac{h}{2}\right).\ \sin\left(ph + \frac{h}{2}\right)\\
&= 2R\sin^2\left(\frac{h}{2}\right).\ \frac{\sin\left(ph+\frac{h}{2}\right)}{\sin\left(\frac{h}{2}\right)}\\
&= D_n.\ \frac{\sin\left\{\frac{(2p+1)h}{2}\right\}}{\sin\left(\frac{h}{2}\right)}\\
&= (2p+1).\ D_n,\ \text{roughly,}
\end{aligned}$$

since $h\left(=\frac{90}{n}\text{ degrees}\right)$ is small and $\left(ph+\frac{h}{2}\right)$ is less than 30 degrees in the third sign. Thus follows the above crude rule.

From this rule it is clear that

$$\begin{aligned}
D_{n-1} &= 3D_n\\
D_{n-2} &= 5D_n\\
D_{n-3} &= 7D_n, \text{etc.}
\end{aligned}$$

Now

$$\begin{aligned}
L(h) &= D_n\\
L(2h) &= D_n + D_{n-1} = (1+3)D_n\\
&= 4L(h)\\
L(4h) &= D_n + D_{n-1} + D_{n-2} + D_{n-3}\\
&= (1+3+5+7)D_n\\
&= 4^2 L(h).
\end{aligned}$$

Thus, in general, we have

$$L(2^n h) = 4^n L(h),$$

or

$$L(h) = \frac{L(2^n h)}{4^n}$$

which is equivalent to the rule described in Sect. 3.

It can be easily seen that the rule, although simple, is very gross. The D_{24}, of Table 1, when multiplied by 3, 5, 7,…,15, will not give results equal to D_{23}, D_{22}, $D_{21}, \ldots, D_{17}$, respectively. 'This is no fault, as the manipulation is not complete', says Govindasvāmin. He, therefore, gives a modification of this rule which we describe now.

5 Govindasvāmin's Modified Rule for Computing Tabular Sine-Differences in the Third Sign

In the commentary (p. 201) of Govindasvāmin on the *Mahābhāskarīya* is found an excellent rule for computing, from a given last tabular Sine-difference D_n, the other Sine-differences lying in the third sign (60 degrees to 90 degrees). The text says:

अन्त्यज्या तावदेकादिसंकलितगुणिततत्पराहीना
त्र्यादिविषमगुणिता फादितुल्याऽन्त्यभवने भवेदिति ।

Diminish the last (tabular) Sine-difference by its thirds multiplied (severally) by the sums of (the natural numbers) 1, etc. The results (so obtained) multiplied by the odd number 3, etc., become the (tabular) Sine-differences, in the third sign, starting from *"pha"* (that is, the last-but-one Sine-difference).

That is, taking the last Sine-difference

$$\begin{aligned} D_n &= a + \frac{b}{60} + \frac{c}{60^2} \text{ minutes} \\ &= a;\ b, c, \text{ say,} \end{aligned}$$

we have

$$\begin{aligned} D_{n-1} &= \left[D_n - 1 \times \frac{c}{60^2} \right] \times 3 \\ D_{n-2} &= \left[D_n - \frac{(1+2)c}{60^2} \right] \times 5 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ D_{n-p} &= \left[D_n - \frac{(1+2+\ldots+p)c}{60^2} \right] . (2p+1), \\ p &= 0, 1, 2, \ldots \end{aligned}$$

Illustration: We take, for the last Sine-difference, the value

$$D_{24} = 7;\ 21,37$$

as found in the work itself (*see* Table 1). Applying the above rule, we get

$$\begin{aligned} D_{23} &= \left(D_{24} - 1 \times \frac{37}{60^2} \right) \times 3 \\ &= 22;3,0. \\ D_{22} &= \left[D_{24} - (1+2) \times \frac{37}{60^2} \right] \times 5 \\ &= 36;38,50. \end{aligned}$$

Similarly all the differences up to D_{17} may be worked out. These are shown in Table 2 and may be compared with the set of values given in the work itself.

Table 2 Sine-difference in third sign

p	D_{n-p} ($n = 24$)	Sine-diff. by the Rule applied to $D_n = 7;21,37$	Sine-diff. as given in the work	Actual value	By the Rule applied to $D_n = 7;21,38$
0	D_{24}	7; 21, 37	7; 21, 37	7; 21, 38	7; 21, 38
1	D_{23}	22; 03, 00	22; 03, 00	22; 03, 00	22; 03, 00
2	D_{22}	36; 38, 50	36; 38, 41	36; 38, 41	36; 38, 40
3	D_{21}	51; 5, 25	51; 04, 59	51; 04, 59	51; 04, 50
4	D_{20}	65; 19, 03	65; 18, 08	65; 18, 08	65; 17, 42
5	D_{19}	79; 16, 02	79; 14, 31	79; 14, 31	79; 13, 28
6	D_{18}	92; 52, 40	92; 50, 32	92; 50, 32	92; 48, 20
7	D_{17}	106; 05, 15	106; 02, 42	106; 02, 42	105; 58, 30

Rationale: We have already shown (*see* Sect. 4)that

$$D_{n-p} = \frac{D_n \cdot \left[\sin \left\{ \frac{(2p+1)h}{2} \right\} \right]}{\sin \left(\frac{h}{2} \right)}.$$

Now it is known that[9]

$$\sin m\theta = m \sin\theta - \frac{m(m^2-1^2)}{3!} \sin^3\theta + \frac{m(m^2-1^2)(m^2-3^2)}{5!} \sin^5\theta - \ldots$$

Taking in this,

$$m = 2p + 1,\ \text{and}\ \theta = \frac{h}{2}$$

we get

$$\frac{\left[\sin\left\{\frac{(2p+1)h}{2}\right\}\right]}{\sin\left(\frac{h}{2}\right)} = (2p+1) - \left(\frac{2}{3}\right) p(p+1)(2p+1)\sin^2\left(\frac{h}{2}\right) + f(p)\sin^4\left(\frac{h}{2}\right) - \dots$$

Using this we get

$$\begin{aligned} D_{n-p} &= (2p+1)D_n - D_n.\left(\frac{4}{3}\right)(2p+1)(1+2+\dots+p)\sin^2\left(\frac{h}{2}\right) + \dots \\ &= \left[D_n - \left(\frac{4}{3}\right)(1+2+\dots+p)D_n.\sin^2\left(\frac{h}{2}\right)\right].(2p+1), \end{aligned}$$

neglecting higher terms which are comparatively small. This we can write as

$$D_{n-p} = [D_n - (1+2+\dots+p)k]\cdot(2p+1),$$

where

$$\begin{aligned} k &= \left(\frac{4}{3}\right)\sin^2\left(\frac{h}{2}\right)\cdot D_n \\ &= \left(\frac{2}{3R}\right)D_n^2,\ \text{or}\ \left(\frac{8R}{3}\right)\sin^4\left(\frac{h}{2}\right). \end{aligned}$$

since

$$D_n = R(1-\cos h) = 2R\sin^2\left(\frac{h}{2}\right).$$

Now, in our case,

$$\begin{aligned} h &= 225\ \text{minutes}, \\ R &= \frac{10800}{3.1416}. \end{aligned}$$

Hence we easily get

$$k = \frac{1}{95.2},\ \text{nearly}.$$

The numerical value implied in the rule given by Govindasvāmin is

$$\begin{aligned} &= \frac{37}{60^2} \\ &= \frac{1}{97.3},\ \text{nearly}. \end{aligned}$$

This is quite comparable to the actual value calculated above.

Acknowledgements I am grateful to Dr. T. A. Sarasvati for checking the English rendering of the Sanskrit passages.

References and Notes

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2. The *Āryabhaṭīya* with the commentary *Bhaṭadīpikā* of Paramādīśvara (Parameśvara); edited by H. Kern, Leiden, 1874, p. 25. In our paper the page-references to *Āryabhatīya* and Parameśvara's commentary on it are according to this printed edition.
3. For an exposition of his alphabetic system of numerals *see,* for example, *History of Hindu Mathematics: A Source Book* by B. Datta and A. N. Singh; Asia publishing House, Bombay, 1962; pp. 64–49 of part I.
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9. *Higher Trigonometry* by A. R. Majumdar and P. L. Ganguli; Bharti Bhawan, Patna, 1963; p. 128.

Early Indians on Second-Order Sine-Differences

The well-known property that the second-order differences of sines are proportional to the sines themselves was known even to Āryabhaṭa I (born AD 476) whose *Āryabhaṭīya* is the earliest extant historical work (of the dated type) containing a sine table. The paper describes the various forms of the proportionality factor involved in the mathematical formula expressing the above property. Relevant references and rules are given from the Indian astronomical works such as *Āryabhaṭīya, Sūrya-siddhānta, Golasāra* and *Tantrasaṅgraha* (AD 1500).

The commentary of Nīlakaṇṭha Somayāji (born AD 1443) on the *Āryabhaṭīya* discusses the property in detail and contains an ingenious geometrical proof of it. The paper gives a brief description of this proof which is merely based on the similarity of triangles.

The Indian mathematical method based on the implied differential process is found, in the words of Delambre, "neither amongst the Greeks nor amongst the Arabs."

1 Introduction

Let (n being a positive integer)

$$S_n = R \sin nh \tag{1}$$

$$D_1 = S_1$$

$$D_{n+1} = S_{n+1} - S_n \tag{2}$$

It is easily seen that

$$D_n - D_{n+1} = F \cdot S_n \tag{3}$$

where the proportionality factor F (independent of n) is given by

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$$F = 2(1 - \cos h). \tag{4}$$

Relation (3) represents the fact that in a set of equidistant tabulated Indian Sines defined by (1), the differences of the first Sine-differences $(S_{n+1} - S_n)$, that is, the second Sine-differences $(D_n - D_{n+1})$ are proportional to the sines S_n themselves. This fact seems to be recognized in India almost since the very beginning of Indian trigonometry. In Sect. 2 below we shall describe some of the forms of the rule (3) along with various forms of the factor F as found in important Indian works. In Sect. 3 we shall outline an Indian proof of the rule as found in Nīlakaṇṭha Somayāji's *Āryabhaṭīya-Bhāṣya* (= *NAB*) which was written in the early part of the sixteenth century of our era.

2 Forms of the Rule

It is easy to see that

$$F = \frac{(D_1 - D_2)}{D_1}. \tag{5}$$

When the norm (radius or *sinus lotus*) R is equal to 3438 min and the uniform tabular interval h is equal to 225 min (as is the case with the usual Indian sine tables), we have

$$\begin{aligned} D_1 &= 3438 \sin 225' = 224.86 \text{ nearly}, \\ D_2 &= 3438 \sin 450' - 3438 \sin 225' = 223.89 \text{ nearly}, \\ D_1 - D_2 &= 0.97 = 1 \text{ approximately}. \end{aligned}$$

Using this value and (5), we can put (3) as

$$D_{n+1} = D_n - \frac{S_n}{D_1}. \tag{6}$$

A rule which is equivalent to (6) is found[1] in the *Āryabhaṭīya* II, 12 of Āryabhaṭa I (born 476 AD) which is the earliest extant historical work of the dated type containing a sine table. The rule found[2] in the *Sūrya-siddhānta* II, 15–16 is also equivalent to (6) according to the interpretations of the commentators Mallikārjuna (1178 AD) and Rāmakṛṣṇa (1472 AD). The *NAB* also accepts that the *Sūrya-siddhānta* rule is same as above and further gives an exact form of the rule (3) which can be expressed in our notation as follows[3]

$$D_{n+1} = D_n - \frac{S_n(D_1 - D_2)}{D_1}$$

or,

$$D_{n+1} = D_n - \frac{(D_1 + D_2 + \cdots + D_n).(D_1 - D_2)}{D_1}$$

The *Golasāra* III, 13–14 gives a rule equivalent to[4]

$$S_{n-1} = S_n - \left[\left(\frac{2}{R}\right)\{R\sin 90^\circ - R\sin(90^\circ - h)\}S_n + D_{n+1}\right]$$

which implies (3) with

$$F = \frac{2(R - R\cos h)}{R}.$$

The *NAB* (part I, p. 53) quotes the *Golasāra*-rule and further adds that we equivalently have

$$F = \frac{2(R\,\mathrm{vers}\,h)}{R}.$$

The actual value of F (independent of R) is given by

$$F = (2\sin 112.5')^2 = \frac{1}{233.53}\text{very nearly.}$$

The *Tantrasaṅgraha* (= *TS*) II, 4 gives[5] the value of the reciprocal of F as 233.5 and the commentator thereof even gives it as

$$233 + \frac{32}{60}$$

which is almost equal to the true value.

A rule equivalent to (3) occurs in the *TS* II, 8–9 (p. 18), which was written in AD 1500, as follows:

अन्त्योपान्त्यान्तरं द्विघ्नं गुणो व्यासदलं हरः।
आद्यज्यायास्तथापि स्यात् खण्डज्यान्तरमादितः॥ ८॥
ताभ्यां `तु गुणहाराभ्यां' द्वितीयादेरपि क्रमात्।
उत्तरोत्तरखण्डज्याभेदाः पिण्डगुणार्धतः॥ ९॥

Twice the difference between the last and the last-but-one (Sines) is the multiplier; the semi-diameter is the divisor. The first Sine then (that is, when operated by the multiplier and divisor defined above) becomes the difference of the initial Sine-differences. With those very multiplier and divisor (operated upon) the tabular Sines starting from the second, (we get) the successive differences of Sine-differences respectively.

That is, $2[R\sin 90^\circ - R\sin(90^\circ - h)] =$ Multiplier, M; Semi-diameter or radius R = Divisor D. Then

$$\left(\frac{M}{D}\right) S_1 = D_1 - D_2$$

$$\left(\frac{M}{D}\right) S_n = D_n - D_{n+1} \qquad n = 2, 3 \ldots$$

So that we have

$$D_n - D_{n+1} = 2(1 - \cos\ h) S_n,$$

which, is equivalent to (3).

Finally, we also have

$$F = \frac{(\text{crd}\ h)^2}{R^2} \tag{A}$$

where crd h denotes the full chord of the arc h in a circle of radius R. With (A) as the value of the proportionality factor, the *NAB* (part I, p. 52) gives the verbal statement of the rule (3) as follows (*NAB* was composed after *TS*):

चापसन्धिगतभुजाज्याः समस्तज्यावर्गो गुणकारः,
त्रिज्यावर्गो भागहारः। फलं खण्डज्यान्तरम्।

For the Sine at any arc-junction (that is, at any point where two adjacent elemental arcs meet) the square of the full chord is the multiplier; the square of the radius is the divisor. The result (of operating the Sine by multiplier and divisor) is the difference of the (two adjacent) Sine-differences.

That is,

$$D_n - D_{n+1} = \frac{S_n (\text{crd}\ h)^2}{R^2} \tag{7}$$

From this, the *NAB* rightly concludes that

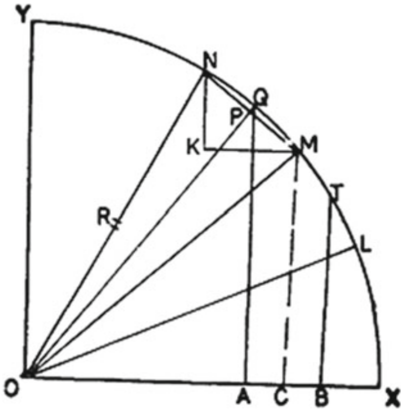
भुजाज्यानुसारिण्येव ज्याखण्डानां वृद्धिः।

The (numerical) increase of the Sine-differences is proportional to the very Sines.

3 Proof of the Rule

An Indian proof of the rule (7) as found in the *NAB* (part I, pp. 48–52) may be briefly outlined in the modern language as follows:

Make the reference circle on a level ground and draw the reference lines XOX' and YOY' (*see* the accompanying figure where only a quadrant is shown). Mark the parts of the arc on the circumference (by points, such as L, M, N, which are at the arcual interval h).



Sine-chord differences

Take a rod OQ equal in length to the radius R and fix firmly and crossly (and symmetrically) another rod MN whose length is equal to the full chord of the (elemental) arc h at the point P which is at a distance equal to the Versed Sine of half the elemental arc h from the end Q of the first rod.

The sides of the similar triangles NKM and OAQ are proportional. Therefore, by the Rule of Three we have

$$NK = \frac{OA \cdot MN}{OQ}$$
$$MK = \frac{QA \cdot MN}{OQ}$$

In other words we have*

Lemma I The difference of Sines, corresponding to the end-points of any elemental arc, is proportional to the Cosine at the middle of the arc;

Lemma II The difference of Cosines, corresponding to the end-points of any elements arc, is proportional to the Sine at the middle of the arc; the proportionality factor in both cases being $= \dfrac{\text{(chord of the arc)}}{\text{Radius}} = \dfrac{(\text{crd } h)}{R}$

Thus, in our symbols we have (when arc $MX = nh$)

$$D_{n+1} = \frac{(\text{crd } h) \cdot OA}{R}$$

*The Sanskrit text (एकचापसमस्तज्या ......खण्डज्ये ज्ञेयता ययोः), as quoted in the *NAB*, states the Lemmas as two Rules of Three. *See* Gupta R.C., Some Important Indian Mathematical Methods as conceived in Sanskrit Language, paper presented at the International Sanskrit Conference, New Delhi, March 1972, p. 3. For a nice statement of the Lemmas, *see* Gupta R. C., Second-Order Interpolation in Indian Mathematics etc., *I.J.H.S.*, Vol.4 (1969), p. 95, verses 7–8.

and, similarly

$$D_n = \frac{(\text{crd } h) \cdot OB}{R}.$$

Therefore,

$$D_n - D_{n+1} = \frac{(\text{crd } h) \cdot (OB - OA)}{R}. \tag{8}$$

Now the *second half* (TM) of the first (lower) arc LTM and the *first half* (MQ) of the second (upper) arc MQN together form the arc TMQ whose length is equal to that of an elemental arc h. Thus we can place the above frame of two rods such that the radial rod coincides with OM and the cross radial rod (therefore) coincides with the full chord of the arc TQ and consider the proportionality of sides as before.

In other words we use *Lemma II* for the arc TQ. This will mean that the difference of the Cosines, OB and OA, corresponding to the end-points T and Q, will be proportional to the Sine, MC, at the middle point M of the arc TQ. That is, we have

$$\begin{aligned} OB - OA &= \frac{(\text{crd } h) \cdot MC}{R} \\ &= \frac{(\text{crd } h) \cdot (R \sin nh)}{R}. \end{aligned}$$

Hence by (8)

$$D_n - D_{n+1} = \frac{(\text{crd } h)^2 \cdot (R \sin nh)}{R^2}$$

which is equivalent to (7).

4 Concluding Remarks

An Indian method of computing tabular Sines by using a process given basically by the rule expressed by (7) has been regarded curious by Delambre whom Datta[6] quotes as remarking thus:

"This differential process has not upto now been employed except by Briggs (c.1615 AD) who himself did not know that the constant factor was the square of the chord or the interval (taking unit radius), and who could not obtain it, except by comparing the second differences obtained in a different manner. The Indians also had probably done the same; they obtain the method of differences only from a table calculated previously by a geometrical process. Here then is a method which the Indians possessed and which is found neither amongst the Greeks nor amongst the Arabs."

Like Delambre, Burgess[7] also thinks that the property, that the second differences of Sines are proportional to Sines themselves, 'was known to the Hindus only by observation. Had their trigonometry sufficed to demonstrate it, they might easily have constructed much more complete and accurate table of Sines.'

Datta *(op. cit.)*, however, sees no reason to suspect that Indians obtained the above formula (6) by inspection after having calculated the table by a different method; "there is no doubt that the early Hindus were in possession of necessary resources to derive the formula," he adds.

Finally it may be stated that various geometrical proofs of the rule have been given[8] by modern scholars like Newton, Krishnaswami Ayyangar, Naraharayya and Srinivasiengar. However, it may be pointed out that the rule given by (7) is exact, and not approximate as assumed by some of the above scholars. The exposition and the limiting forms of the rules and results from the *NAB* and *Yukti-bhāṣā* (17th century AD) as given by Saraswathi[9] should also be noted. Many other modern proofs have been given.[10]

References and Notes

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An Indian Form of Third-Order Taylor Series Approximation of the Sine

The paper describes an approximation formula for sine $(x + h)$ that differs from the first four terms of the Taylor expansion only by having 4 in place of 6 in the denominator of the fourth term. It appears in Sanskrit stanzas quoted in a work of about the fifteenth century and given here with translation and explanation.

लेख में इष्टचाप की ज्या निकालने की उस भारतीय विधि का वर्णन किया गया है जो कि इष्ट ज्या के टेलर श्रेणी के चार पदों तक के विस्तार से मिलती-जुलती है, केवल चतुर्थ पद के हर में ६ के स्थान पर ४ है। तत्सम्बन्धी गणितीय नियम संस्कृत के उन तीन श्लोकों में निहित है जो कि ईसवीं की लगभग १५ वीं शताब्दी में लिखे एक ग्रन्थ में उद्धृत हैं।

Approximation formulas for the sine of $x + h$ equivalent to the first two terms of the Taylor series expansion and to the first three terms were known from at least as early as the tenth century AD and the fourteenth century AD to Indian mathematicians [Sengupta 1932, 5; Gupta 1969, 92–94].

An approximation formula equivalent to the first four terms, except that the denominator in the fourth term is 4 instead of 6, appears in a literary form in three stanzas quoted by Parameśvara (circa 1360–1455) in his super-commentary, called *Siddhānta-dīpikā*, on Govindasvāmin's gloss (circa 800–850) on the *Mahābhāskarīya* (about early seventh century) [Kuppanna Sastri 1957, 205].

Below we give the text in Sanskrit from the *Siddhānta-dīpikā* and an almost literal translation of the Sanskrit stanzas. Then follows an explanation of the various steps in the rule laid down in the text. (the Indian sine, or Sine, of an angular arc ϕ is equal to $R \sin \phi$, where $\sin \phi$ is the modern sine of the angle ϕ and R is the radius of the circle of reference).

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दोश्चापखण्ड-संभक्तं व्यासार्धं भाजको भवेत् ।
दोर्ज्यामतीतचापान्ते कोटिज्यां च न्यसेत् पुनः ॥ १४ ॥
कोटिज्यातो भाजकेन लब्धस्यार्धेन संयुतात् ।
दोर्गुणाद् भाजकाप्तार्धं हित्वा कोटिगुणात् पुनः ॥ १५ ॥
तस्मादाप्तं भाजकेन दोर्ज्याखण्डः स्फुटो भवेत् ।
चापान्तदोर्ज्या तद्युक्ता स्यादिष्टज्या भुजोद्भवा ॥ १६ ॥

The semi-diameter divided by the residual arc becomes the divisor. Put down the Sine and again the Cosine at the end of the arc traversed. (14)

From the Cosine, subtract half the quotient obtained from the divisor-divided Sine [which is] increased by half the quotient obtained from the Cosine by the divisor. Again, (15)

[The quotient] obtained from that [above difference] by dividing by the divisor becomes the true Sine difference. The Sine at the end of the arc traversed increased by that [true Sine-difference] becomes the desired Sine for a [given] arc. (16)

1 Explanation

Suppose the given arc lies between the tabulated argument α and the next tabulated argumental value. Let the given arc be $\alpha + \theta$, where α is called the arc traversed and θ is the residual arc which is necessarily less than the current tabular interval. We have to find $R\sin(\alpha + \theta)$. The quantities $R\sin\alpha$ and $R\cos\alpha$ are the Sine and the Cosine at the end of the arc traversed. $R\sin\alpha$ can be read directly from a sine table. Thus $R\sin\alpha$ and $R\cos\alpha$ are taken to be known.

The rule starts with the introduction of a quantity D, called "divisor", defined by $D = \frac{R}{\theta}$. The second half of the first stanza (14) simply asks us to write down $R\sin\alpha$ and $R\cos\alpha$ for the operations which are given in the two subsequent stanzas.

The contents of the next stanza (15) may be put as follows:

1. Divide the Cosine at the end of the arc traversed by the divisor, so that we get $\frac{(R\cos\alpha)}{D}$.
2. Half the above quotient is added to the Sine at the end of the arc traversed. Thus we get $R\sin\alpha + \frac{(R\cos\alpha)}{2D}$.
3. Divide the above result by the divisor, so that we get $\frac{\left[R\sin\alpha + \frac{(R\cos\alpha)}{2D}\right]}{D}$.
4. Half the above quotient is to be subtracted from the Cosine at the end of the arc traversed. Hence we now get $R\cos\alpha - \frac{\left[R\sin\alpha + \frac{(R\cos\alpha)}{2D}\right]}{2D}$.

The first half of the last stanza (16) again asks us to divide the results obtained above by the divisor.

Thus we finally get

$$\frac{\left[R\cos\alpha - \frac{\left[R\sin\alpha + \frac{(R\cos\alpha)}{2D}\right]}{2D}\right]}{D}$$

which is stated to be the true Sine-difference needed. That is, the above final result is taken to be the value of $R\sin(\alpha+\theta) - R\sin\alpha$. The second half of the last stanza asks us to add the true Sine difference, obtained above, to the Sine at the end of the arc traversed to get the desired Sine corresponding to a given arc.

Thus the rule, expressed mathematically, is equivalent to the formula

$$R\sin(\alpha\ +\ \theta)\ =\ R\sin\alpha + \frac{\left[R\cos\alpha - \frac{\left(R\sin\alpha + \frac{(R\cos\alpha)}{2D}\right)}{2D}\right]}{D}.$$

Substitution of $D = \frac{R}{\theta}$ and simplification yield

$$R\sin(\alpha+\theta) = R\sin\alpha + \left(\frac{\theta}{R}\right)(R\cos\alpha) - \left(\frac{\theta}{R}\right)^2 \frac{(R\sin\alpha)}{2} - \left(\frac{\theta}{R}\right)^3 \frac{(R\cos\alpha)}{4}.$$

When $R = 1$, this becomes the third-order Taylor series approximation except for the 4 in place of 6 in the last term.

It is interesting that a four-term approximation formula for the Sine function so close to the Taylor series approximation was known in India more than two centuries before the Taylor expansion was discovered by Gregory about 1668 [Boyer 1968, 422].

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Solution of the Astronomical Triangle as Found in the *Tantrasaṅgraha* (AD 1500)

The spherical triangle formed on the celestial sphere by the positions of the Sun, north pole and the zenith on it is called an astronomical triangle. The three sides of this triangle are the co-latitude of the place of observation, the co-altitude of the Sun and its co-declination at the time of observation. The internal angles formed at the zenith and the north pole are the azimuth and the hour angle respectively. If any three of the above named five elements are known, the remaining two can be found out. This gives rise to ten cases to be considered.

A complete solution of the astronomical triangle dealing systematically with all the ten cases is found in the Sanskrit work *Tantrasaṅgraha* which was composed by Nīlakaṇṭha Somayājī's in the year AD 1500. However, some material of the work belongs to an earlier period of Indian astronomy.

The present paper contains the translations (for the first time?) of the various rules given in the above work for solving the astronomical triangle in the ten different cases. When these rules are expressed in modern forms (as done in the present paper), it is seen that they give results which are same as those obtained by using the current standard formulas of modern spherical trigonometry, such as the Sine, Cosine and Cotangent Rules, and applying the theory of quadratic equation in some cases. For example, in Case I, where latitude, declination and azimuth are given, the rule given in the *Tantrasaṅgraha* for finding out the altitude of the Sun, yields a result which is same as that obtained from the Cosine Rule

$$\sin\delta = (\sin\phi)\cdot(\sin\alpha) + (\cos\phi)\cdot(\cos\alpha)\cdot(\cos A)$$

when this relation is converted into a quadratic equation in $\sin\alpha$ and solved.

The said work, however, does not contain the rationales of the rules, and the present paper does not make any attempt to investigate as to how the rules were arrived at.

Symbols and Select Glossary

A Azimuth measured from the north.

B *Bhā-bhuja* ('Shadow-arm') which is the distance of the Sun's projection on the plane of the celestial horizon from the east–west line.

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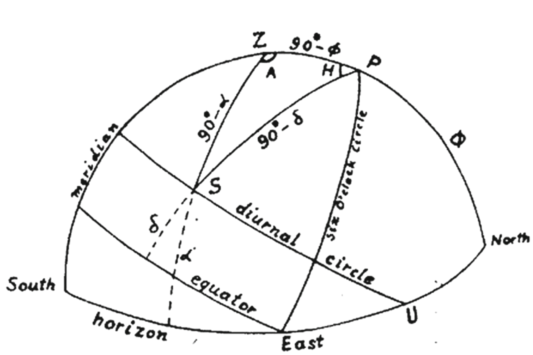
C	Cosine of the local hour angle; $\sqrt{R^2 - J^2}$.
D	Certain divisor (s).
'Day-sine'	Radius of the Sun's diurnal circle; $R\cos\delta$.
'Gnomon'	Sine of the altitude of the Sun.
H	Hour angle measured eastward.
J	*Svanata-jyā*, the Sine of the local hour angle defined by $J = \frac{(R\sin H).(R\cos\phi)}{R}$.
K	*Bhā-koṭi* ('Shadow-upright') which is the distance of the Sun's projection on the plane of the celestial horizon from the north–south line.
R	Radius, norm, *trijyā* or *sinus totus*.
'Shadow'	Cosine of the altitude of the Sun.
α	Altitude of the Sun or its co-zenith distance.
γ	*Digagrā* (directional amplitude), the (Indian) azimuth measured from the east–west line; so that we have $A = 90^\circ \pm \gamma$.
δ	Declination of the Sun.
ϕ	Terrestrial latitude.

1 Introduction

Let S be the position of the Sun on the celestial sphere, P the position of the north pole and Z the zenith. Then the astronomical triangle SPZ is a spherical triangle in which the arcual sides PZ, ZS and SP are equal to the co-latitude, co-altitude and co-declination, respectively. The angles SPZ and PZS are the hour angle and azimuth, respectively. Knowing any three out of the above five elements of the triangle, the remaining two can be found. A nice exposition of the subject of determining the remaining two elements, when any three of the above named five elements are given, is found in the Sanskrit work *Tantrasaṅgraha* (= TS) which was composed in AD 1500 by Nīlakaṇtha Somayājī (1444–1545)[1]. TS is an important work



The astronomical triangle

of the late Āryabhaṭa I School of the Indian astronomy. The title of the work signifies that it is a "Compendium" or "Collection" of astronomical rules and no doubt it includes some earlier material on Indian astronomy. The work has been published[2] with the commentary (= TSC) *Laghuvivṛti* written in AD 1556 by Śaṅkara Vāriar (circa 1500–1560) who was a disciple of the author of TS.[3]

The above-mentioned almost complete solution of the astronomical triangle has been dealt in the third chapter of the work. TS, III, 60 (p. 64) says:

इह शङ्कुनतक्रान्तिदिगग्राक्षेषु पञ्चसु ।
द्वयोर्द्वयोरानयनं दशधा स्यात् परैस्त्रिभिः ॥ ६० ॥

The determination of any two elements at a time out of the five elements, altitude, hour angle, declination, (Indian) azimuth and latitude, from the other three (being given) is of ten types (that is, there are ten cases).*

The ten cases have been dealt systematically and one by one in TS, III, 62–87 (pp. 65–88) and each case is followed by numerical exercises (*uddeśakas*). In the following pages we shall describe these ten cases one by one giving each time the method of solution as stated in the TS. Most of the TS rules under these cases are included in the *Trigonometry in Ancient and Medieval India* by the author of the present paper (Table 1).[4]

Table 1 Elements given in the ten cases

Case	Given Elements	Elements to be found out	Reference
I	δ, A, ϕ	α, H	TS. III, 62–67
II	H, A, ϕ	α, δ	TS. III, 68–73
III	H, δ, ϕ	α, A	TS. III, 74–75
IV	H, δ, A	α, ϕ	TS. III, 75–78
V	α, A, ϕ	H, δ	TS. III, 78–79
VI	α, δ, ϕ	H, A	TS. III, 80–81
VII	α, δ, A	H, ϕ	TS. III, 81–83
VIII	α, H, ϕ	δ, A	TS. III, 83–85
IX	α, H, A	δ, ϕ	TS. III, 86–87
X	α, H, δ	A, ϕ	TS. III, 86–87

2 Case I: Given δ, A, ϕ

In order to find the Sun's altitude, the *TS*, III, 62–65 (p. 65) states:

आशाग्रा लम्बकाभ्यस्ता त्रिज्याभक्ता च कोटिका ॥ ६२ ॥
भुजाक्षज्या तयोर्वर्गयोगमूलं श्रुतिर्हरः ।
क्रान्त्यक्षवर्गौ तद्वर्गात् त्यक्त्वा कोट्यौ तयोः पदे ॥ ६३ ॥
कुर्यात् क्रान्त्यक्षयोर्घातं कोट्योर्घातं तथा परम् ।

*Angle ZSP has not been considered here.

> सौम्ये गोले तयोर्योगात् भेदाद्याम्ये तु घातयोः ॥ ६४ ॥
> आद्यघातेऽधिके सौम्ये योगभेदद्वयादपि ।
> त्रिज्याघ्नाद्धारवर्गाप्तः शङ्कुरिष्टदिगुद्भवः ॥ ६५ ॥
>
> ....The Sine of the directional amplitude multiplied by the Cosine of the latitude and divided by the radius is the upright (a small side of some right-angled plane triangle). The Sine of the latitude is the base (the other small side of the same triangle). The square root of the sum of their squares is the hypotenuse which is the Divisor.
>
> The square roots of the quantities got by subtracting (separately) the squares of the Sines of the declination and the latitude from the square of that (Divisor) are the two *koṭis*.
>
> Obtain the product of the Sines of the declination and the latitude and (also) the product of the two *koṭis*. Take their (of the above two products) sum (when the Sun's declination is) in north and difference (when it is) in south and both the sum as well as the difference (when it is) in north and the first product is greater (than the second product), multiply (the sum and/or the difference) by the radius and divide by the square of the Divisor. (The result is) the Gnomon (Sine of the altitude) in the desired direction.[*]

That is,

$$\text{Divisor} = \sqrt{(R\sin\phi)^2 + \left\{\frac{(R\sin\gamma \cdot R\cos\phi)}{R}\right\}^2} = D, \qquad \text{say.}$$

Then[†]

$$R\sin\alpha = \left(\frac{R}{D^2}\right)\cdot\left[R\sin\delta\cdot R\sin\phi \pm \sqrt{D^2-(R\sin\delta)^2}\cdot\sqrt{D^2-(R\sin\phi)^2}\right]$$

This solution can be easily seen to be equivalent to the result obtained by solving the quadratic equation

$$\sin^2\phi + \cos^2\phi\cdot\cos^2 A)\sin^2\alpha - 2\sin\phi\cdot\sin\delta\cdot\sin\alpha + (\sin^2\delta - \cos^2\phi\cdot\cos^2 A) = 0$$

which is derived from the relation $\sin\delta = \sin\phi\cdot\sin\alpha + \cos\phi\cdot\cos\alpha\cdot\cos A$.

This last relation is written down by applying the modern cosine rule to the spherical triangle ZSP.

Then TS, III, 66 (p. 65) includes a rule which can be expressed in modern symbols as follows:

$$R\sin H = \frac{(R\cos\alpha\cdot R\cos\gamma)}{R\cos\delta}.$$

This relation for finding out the hour angle is equivalent to the sine formula for the spherical triangle ZSP.

[*] *TS*, III, 67 (p. 65) says that, in the case of the southern declination, the desired altitude of the Sun will not be attained if the first product is greater than the second (numerically); so also if $R\sin\delta$ is greater than the Divisor (whatever be the direction of δ).

[†] In the equation below and all the other subsequent equations throughout the article wherever '±' appears the '−' sign denotes the positive difference of the quantities.

3 Case II: Given H, A, ϕ

For finding the altitude, the *TS*, III. 68–71 (p. 70) says:

नतलम्बकयोर्घातात् त्रिज्याप्तं तत्स्वदेशजम् ।
स्वदेशनतकोट्याप्तं नताक्षज्यावधात्तु यत् ॥ ६८ ॥
तदाशाग्रावधे कोट्योस्तयोर्घातं क्षिपेदुदक् ।
शोधयेद् दक्षिणाग्रायां त्रिज्यया च ततो हरेत् ॥ ६९ ॥
लब्धात् स्वनतकोटिघ्नात् पृथक् त्रिज्याप्तवर्गितम् ।
युतं स्वनतवर्गेण तन्मूलेन हृतं फलम् ॥ ७० ॥
पृथक्कृताद् भवेच्छङ्कुः .... ।
.... ॥ ७१ ॥

The product of the Sine of the hour angle and the Cosine of the latitude divided by the radius is the Sine of the local hour angle. Divide the product of the Sines of the hour angle and the latitude by the Cosine of the local hour angle and multiply by the Sine of the directional amplitude (the azimuthal angle measured from the east). The result should be added to, in case the directional amplitude is towards north, or subtracted from, in case the directional amplitude is towards south, the product of their *'koṭis'* (that is, their uprights when they are taken as bases and the radius is taken as the hypotenuse in each case). The result (now obtained) be divided by the radius and the quotient (thus obtained) multiplied by the Cosine of the local hour angle be put separately (at two places).

At one place divide (the quantity) by the radius and add the square (of the result) to the square of the Sine of the local hour angle. By the square root of that (the sum of the squares just now obtained) divide the quantity (placed) separately. (The final result) becomes the Gnomon (the Sine of the altitude)...........

That is,

Sine of the local hour angle

$$= \frac{(R\sin H \cdot R\cos\phi)}{R} = J, \qquad \text{say.}$$

Cosine of the local hour angle

$$C = \sqrt{R^2 - J^2}.$$

Then we form the quantity

$$\left(\frac{C}{R}\right)\left[R\cos\gamma \cdot \sqrt{R^2 - \left\{\frac{(R\sin H \cdot R\sin\phi)}{C}\right\}^2} \pm \frac{R\sin\gamma \cdot (R\sin H \cdot R\sin\phi)}{C}\right] = Q, \text{ say.}$$

The rule then gives

$$R\sin\alpha = Q\sqrt{J^2 + \left(\frac{Q}{R}\right)^2}.$$

By combining the various above steps, the solution given in the TS can be seen to be equivalent to the relation.

$$\sin\alpha = \frac{\sin A \cdot \cos H + \cos A \cdot \sin H \cdot \sin\phi}{\sqrt{(\sin H \cdot \cos\phi)^2 + (\sin A \cdot cos H + \cos A \cdot \sin H \cdot \sin\phi)^2}}$$

which is transformed form of the following result obtained by using the modern cotangent formula of the spherical trigonometry[5]

$$\tan\alpha \cdot \cos\phi = \sin A \cdot \cos H + \cos A \cdot \sin\phi$$

After finding the altitude, TS, III, 71 (second half) gives the equivalent of

$$R\cos\delta = \frac{R\cos\alpha \cdot R\cos\gamma}{R\sin H}$$

which completes the desired computations in the present case.

4 Case III: Given H, δ, ϕ

For finding the altitude, TS, III, 74–75 (p. 74) states:

नतकोट्या हता द्युज्या विभक्ता त्रिभजीवया ।
सौम्ययाम्यदिशोर्भूज्यायुतोना लम्बकाहता ॥ ७४ ॥
त्रिज्याप्ता शङ्कुः .... ॥ ७५ ॥

Multiply the Cosine of the hour angle by the Day-sine (Cosine of the declination) and divide by the radius. (The quotient obtained be) increased or diminished, according as the direction (of the declination) is north or south, by the Earth-sine (the distance between the rising-setting line and the line joining the points of intersection of the diurnal circle and the six O'clock circle). (The result now obtained) multiplied by the Cosine of the latitude and divided by the radius is the Sine of the altitude.

That is,

$$R\sin\alpha = \left[\frac{(R\cos H \cdot R\cos\delta)}{R} \pm (\text{Earth-sine})\right] \cdot \frac{R\cos\phi}{R}$$

Now we know that (see TS, III, 59)[6]

$$\text{Earth-sine} = \frac{R\sin\delta \cdot R\sin\phi}{R\cos\phi}$$

So that the rule is equivalent to the relation

$$\sin\alpha = \sin\delta \cdot \sin\phi + \cos\delta \cdot \cos\phi \cdot \cos H$$

which can be directly written by using the cosine formula for a spherical triangle.

Just after giving the above rule, TS text says that the Cosine of the directional amplitude should be found as before and the TSC (p. 74) gives the following usual rule for the purpose:

$$R\cos\gamma = \frac{(R\sin H)\cdot(R\cos\delta)}{R\cos\alpha}.$$

5 Case IV: Given H, δ, A

TS, III, 75–78 (p. 76) states:

छायां नीत्वाथ तत्कोटिद्युज्यावार्गन्तरात् पदम् ॥ ७५ ॥
तच्छायाबाहुघातो यश्शङ्कुक्रान्त्योर्वधोऽपि यः ।
क्रान्त्यग्रयोस्तुल्यदिशोस्तयोर्भेदोऽन्यथा युतिः ॥ ७६ ॥
उन्मण्डलक्षितिजयोरन्तरेऽर्के च तद्युतिः ।
तद्धतां विभजेत् त्रिज्यां तच्छायाकोटिवर्गयोः ॥ ७७ ॥
अन्तरेण भवेदक्षः .... ॥ ७८ ॥

After getting the Shadow (Cosine of the altitude), take the square root of the difference of the squares of its upright (east-west component) and the Day-sine (Cosine of the declination).

The product of that (square root) and the Shadow-arm and also the product of the Sine of the altitude and the Sine of the declination (are formed). Take their (of the above two products) difference, if the declination and the directional amplitude are in the same direction, otherwise sum, and (also take their) sum when the Sun is between six O'clock circle and the horizon. That (the sum when the Sun is between the six O'clock circle and the horizon. That (the sum or difference) multiplied by the radius and divided by the difference of the squares of the radius and the Shadow-upright becomes the Sine of the latitude.......

Explanation : For finding the latitude by the above rule, we should first determine the 'Shadow', Shadow-upright, and the Shadow-arm needed in the rule. For this purpose the TSC (p. 76) on the text of the rule gives the equivalent of the following formulas:

$$\begin{aligned}
\text{Shadow} &= \frac{(R\sin H\cdot R\cos\delta)}{R\cos\gamma} = R\cos\alpha;\\
\text{Shadow-upright} &= \frac{(R\sin H\cdot R\cos\delta)}{R},\\
\text{that is,}\quad K &= \frac{(R\cos\alpha\cdot R\cos\gamma)}{R};\\
\text{and}\qquad \text{Shadow-arm} &= \sqrt{(R\cos\alpha)^2 - K^2}\\
\text{or}\quad B &= \frac{(R\cos\alpha\cdot R\sin\gamma)}{R}.
\end{aligned}$$

Then the above rule gives

$$R\sin\phi = \frac{R\cdot\left[(R\sin\alpha)\cdot(R\sin\delta)\pm B\cdot\sqrt{(R\cos\delta)^2-K^2}\right]}{R^2-K^2}.$$

This solution can be seen to be equivalent to the roots of the equation

$$(1-\cos^2\alpha\cdot\sin^2 A)\sin^2\phi - \sin\alpha\sin\delta\cdot\sin\phi + (\sin^2\delta - \cos^2\alpha\cdot\cos^2 A) = 0$$

which is derived from the relation

$$\sin\delta = \sin\alpha\cdot\sin\phi + \cos\alpha\cdot\cos\phi\cdot\cos A$$

already mentioned in case I.

6 Case V: Given α, A, ϕ

For finding the declination of the Sun, the TS, III, 78–79 (p. 79) states:

> अक्षशङ्क्वोर्वधो यश्च यश्च भाबाहुलम्बयोः ॥ ७८ ॥
> सौम्ययाम्यस्थिते भानौ तयोर्योगान्तरात्ततः ।
> क्रान्तिस्त्रिज्याहृता .... ॥ ७९ ॥
>
> …Take the product of the Sine of the latitude and the Sine of the altitude and also that of the Shadow-arm and the Cosine of the latitude. Their (of the two products) sum or difference, according as the Sun's position is north or south (of the prime vertical), divided by the radius is the Sine of the declination ……

That is,

$$R\sin\delta = \frac{[(R\sin\phi\cdot R\sin\alpha)\pm B\cdot Rcos\phi]}{R}.$$

$$\text{Since}\qquad B = \frac{(R\cos\alpha)\cdot(R\cos A)}{R},$$

the above rule gives the same result as obtained by using the Cosine formula which has already been mentioned in cases I and IV.

Just after stating the above rule, the TS says that the hour angle should be obtained as before. The TSC (p. 80) gives two methods for this. One of these is same as that contained in TS. III, 66 (p. 65) and which we have already mentioned under case I.

7 Case VI: Given α, δ, ϕ

To find the azimuth or the directional amplitude, TS, III, 80–81 (p. 81) says:

त्रिज्यापक्रमघातो यो यश्च शङ्कक्षयोर्वधः ।
तयोर्योगान्तरं यत्तु गोलयोर्याम्यसौम्ययोः ॥ ८० ॥
भाबाहुर्लम्बकाप्तोऽस्मात्त्रिज्याघ्नाद् भाहृतेष्टदिक् ।
.... ॥ ८१ ॥

Take the product of the radius and the Sine of the declination and also the product of the Sine of the altitude and the Sine of the latitude. Their (of the two products) sum or difference, (according as Sun's declination is) in southward or northward, divided by the Cosine of the latitude is the Shadow-arm. This multiplied by the radius and divided by the Cosine of the altitude is the desired Sine of the directional amplitude................

That is,

$$\frac{[(R \cdot R\sin\delta) \pm (R\sin\alpha \cdot R\sin\phi)]}{R\cos\phi} = B$$

Then

$$\frac{B \cdot R}{R\cos\alpha} = R\sin\gamma$$

giving the Sine of the (Indian) azimuth which is measured from the east. The above result may be seen to be same as that obtained by solving the following cosine relation for getting A:

$$\sin\delta = (\sin\alpha \cdot \sin\phi) + (\cos\alpha \cdot \cos\phi \cdot \cos A)$$

As explained in the TSC on the text of the above rule, we can then get the hour angle by using the relation

$$R\sin H = \frac{K \cdot R}{(R\cos\delta)}$$

where

$$K = \sqrt{(R\cos\alpha)^2 - B^2}$$

8 Case VII: Given α, δ, A

For finding the latitude, the TS, III, 81–83 (p. 83) says:

वर्गान्तरपदं यत्स्याच्छायाकोटिद्युजीवयोः ॥ ८१ ॥
तच्छायाबाहुयोगो यः शङ्कुक्रान्त्यैक्यवर्गतः ।
तेनाप्तं यत् फलं तस्मिन्नेव तत् स्वमृणं पृथक् ॥ ८२ ॥
तयोरल्पहता त्रिज्या महताप्ताक्षमौर्विका ।
.... ॥ ८३ ॥

........Take the square root of the difference of the squares of the Shadow-upright and the Cosine of the declination and add it to the Shadow-arm. By the quantity so obtained, divide the square of the sum of the Sines of the altitude and the declination. The quotient should be separately added to or subtracted from that very quantity. When the radius is multiplied by the smaller result (of the above subtraction) and divided by the greater result (of the last addition), we get the sine of the latitude…

That is,

$$B + \sqrt{(R\cos\delta)^2 - K^2} = D, \qquad \text{say}$$

Then form the two quantities

$$D + \frac{(R\sin\alpha + R\sin\delta)^2}{D} = Q_1, \qquad \text{say}$$

and

$$D \sim \frac{(R\sin\alpha + R\sin\delta)^2}{D} = Q_2, \qquad \text{say.}$$

Finally we get

$$R\sin\phi = \frac{R \cdot Q_2}{Q_1}$$

so that we have

$$R\sin\phi = R \cdot \frac{\left\{D^2 \sim (R\sin\alpha + R\sin\delta)^2\right\}}{\left\{D^2 + (R\sin\alpha + R\sin\delta)^2\right\}}$$

This rule may be compared with that given under case IV for finding the latitude.

For finding the hour angle, the *TSC* (P. 83) gives the equivalent of the following rule before commenting on the above rule proper:

$$R\sin H = \frac{(R\cos\alpha).(R\cos\gamma)}{R\cos\delta}$$

However, the same occurs in the text of the *TS* also and we have already given it (see under case I).

9 Case VIII: Given α, H, ϕ

For finding the declination, *TS*, III, 83–85 (p. 85) says:

त्रिज्याहताक्षशङ्कू स्वनतकोट्युद्धृतौ पृथक् ॥ ८३ ॥
ये तत्कोट्यौ च तत्त्रिज्यावर्गभेदपदीकृतौ ।
मिथः कोटिघ्नयोर्योगाद्याम्ये सौम्येऽन्तरात्तयोः ॥ ८४ ॥
त्रिज्यया विहृता द्युज्या .... ॥ ८५ ॥

Multiply the Sine of the latitude and the Sine of the altitude (each) by the radius and divide (the results) separately by the Cosine of the local hour angle. The corresponding uprights (with the above two quotients as bases) are the square roots of the radius-square minus (each of) the quotients.

Multiply the (above) quotients crossly by their uprights. Their (of the two products just obtained) sum (when the Sun is) in the south (of the six O'clock circle), or difference in the north, divided by the radius is the Day-sine, (Cosine of the declination) ....

That is.

$$R\cos\delta=\left(\frac{1}{R}\right)\left[(R\sin\phi)\left(\frac{R}{C}\right)\sqrt{R^2-\left\{(R\sin\alpha)\left(\frac{R}{C}\right)\right\}^2}\pm(R\sin\alpha)\left(\frac{R}{C}\right)\sqrt{R^2-\left\{(R\sin\phi)\left(\frac{R}{C}\right)^2\right\}}\right]$$

On simplification this will become

$$\cos\delta=\frac{\sin\phi\cdot\sqrt{\sin^2\phi+\cos^2\phi\cdot\cos^2 H-\sin^2\alpha}\pm\cos\phi\cdot\sin\alpha\cdot\cos H}{(\sin^2\phi+\cos^2\phi\cdot\cos^2 H)}$$

This solution is same as that obtained by solving the quadratic equation

$$(\sin^2\phi+\cos^2\phi\cdot\cos^2 H)\cos^2\delta\pm 2\cos\phi\cdot\sin\alpha\cdot\cos H\cdot\cos\delta+(\sin^2\alpha-\sin^2\phi)=0$$

which itself is derived from the already mentioned (see case III) cosine relation, namely,

$$\sin\alpha=\sin\phi\cdot\sin\delta+\cos\phi\cdot\cos\delta\cdot\cos H$$

After giving the above rule the TS asks us to find the directional amplitude as before and the TSC (p. 86) lays down the equivalent of the following procedure for finding the azimuthal angle.

$$\frac{(R\cos\delta)\cdot(R\sin H)}{R}=K,$$

$$\sqrt{(R\cos\alpha)^2-K^2}=B.$$

Finally,

$$R\sin\gamma=B\cdot\frac{R}{(R\cos\alpha)}$$

10 Case IX: Given α, A, H

For finding the declination, TS, III, 86 (p. 88) says:

> दिगग्रायास्तु तत्कोटिस्तच्छायाघाततो हृता ।
> नतज्यया भवेद् द्युज्या................. ॥ ८६ ॥
>
> From the (Sine of) the directional amplitude get its Cosine. The product of that (the above Cosine) and the Cosine of the altitude divided by the Sine of the hour angle becomes the Day-sine (the Cosine of the declination)..............

That is

$$R\cos\delta=\frac{(R\cos\gamma)\cdot(R\cos\alpha)}{R\sin H}$$

a rule which has already been given earlier in TS, III, 71 (see under case II above) and which is equivalent to the sine formula for the spherical triangle SZP.

For finding the latitude see under case X below.

11 Case X: Given α, H, δ

> द्युज्यानतज्ययोर्घातादग्राकोटिः प्रभाहृता ॥
> .... ॥ ८७ ॥
>
> The product of the Day-sine (Cosine of the declination) and the Sine of the hour angle divided by the Shadow (Cosine of the altitude) becomes the Cosine of the directional amplitude...............

That is,

$$R\cos\gamma = \frac{(R\cos\delta)\cdot(R\sin H)}{R\cos\alpha}$$

which is another form of the rule given under case IX above.

The result of finding one unknown element in each of the above two cases IX and X is that we know now the four elements α, H, A and δ in either or the cases. The problem in both these cases is therefore same, namely, to find out the remaining fifth element ϕ. For this the TS simply says.

'Latitude (should be found) as before'.

The TSC (p. 88) at this point asks us to use either the rule given under case IV or the rule given under case VII for determining the latitude.

12 Concluding Remarks

Just after giving the said solutions in the ten cases. TS, III, 87 (p. 88) says

> (Here) ends the description of the answers to the ten problems.

However, the work contains several other rules which provide alternate methods of solution in some of the above general cases or their particular ones. For example, TS, III, 88–91 (pp. 89–90) gives an alternate rule for finding the 'Shadow' (Cosine of the altitude) from given azimuth, declination and latitude (Cf. Case I dealt above).

From the present study, the readers must not conclude that Indian solution of the astronomical triangle in each case was given for the first time in the TS. In fact, solution in many general and particular cases were known in India much earlier than the date of the TS. It is outside the scope of the present paper to give the history and development of the Indian solutions in the various cases.

As is usual with most of the ancient Indian original texts, the TS does not state explicitly the methods through which the rules were arrived at. However, many of the ancient ways of deriving these rules can be known from the material found in the commentaries on various astronomical works. It is believed that most of the Indian rules were derived by working 'inside' the armillary sphere rather than 'on its surface'. For this purpose the Indians also employed the so-called latitudinal and declinational triangles. In this connection the following remark of Nīlakaṇṭha Somayājī is noteworthy.[7]

> The whole of the planetary-mathematics is pervaded by the two theorems (namely) the *Bhujā-koṭi-karṇa-nyāya* (the so-called Pythagoras Theorem) and the Rule of Three (the proportionality of sides in similar triangles)

Some other methods, like those based on the theory of successive approximations or of quadratic equations, were also employed by the Indians.

References and Notes

1. Sarma, K. V. : *History of the Kerala School of Hindu Astronomy (in perspective)*, Vishveshvarananda Institute, Hoshiarpur, 1972, pp. 55–56.
2. The *Tantrasaṅgraha* by Nīlakaṇṭha Somasutvan, Edited by S. K. Pillai, Trivandrum Sanskrit Series No. 188, University of Kerala, Trivandrum, 1958. All references to TS and TSC in the article are according to this edition.
3. Sarma, K. V. : op, cit., p. 58.
4. Gupta, R. C. : *Trigonometry in Ancient and Medieval India*. Doctoral Thesis, Ranchi University, Ranchi, 1970 (not yet published), Chapter VII, Part two (Solution of Triangles), pp. 238–259.
5. Todhunter, I. : *Spherical Trigonometry*, Revised by G. Prasad, Pothishala, Allahabad, 1965, pp. 17–18.
6. The expression for the Earth-sine was well known to the Indian astronomers. It is found even in early works like *Āryabhaṭīya* (IV. 26) and *Mahā-bhāskarīya* (III, 6) etc.
7. See his commentary on the *Āryabhaṭīya*, edited by K. Sambasiva Sastri, Trivandrum, 1930, Part I, p. 100.

Addition and Subtraction Theorems for the Sine and the Cosine in Medieval India

The paper deals with the rules of finding the sines and the cosines of the sum and difference of two angles when those of the two angles are known separately. The rules, as found in the important medieval Indian works, are equivalent to the correct modern mathematical results. Indians of the said period also knew several proofs of the formulas. These proofs are based on simple algebraic and geometrical reasoning, including proportionality of sides of similar triangles and the Ptolemy's theorem. The enunciations and derivations of the formulas presented in the paper are taken from the works of the famous authors of the period, namely, Bhāskara II (AD 1150), Nīlakaṇṭha Somayājī (1500), Jyeṣṭhadeva (sixteenth century), Munīśvara (1st half of the seventeenth century) and Kamalākara (2nd half of the seventeenth century).

1 Introduction

According to Carl B. Boyer[1], the introduction of the sine function represents the chief contribution of the *Siddhāntas* (Indian astronomical works) to the history of mathematics. The Indian Sine (usually written with a capital S to distinguish it from the modern sine) of any arc in a circle is defined as the length of half the chord of double the arc. Thus the (Indian) Sine of any arc is equal to $R \sin A$, where R is the radius (norm or *Sinus totus*) of the circle of reference and $\sin A$ is the modern sine of the angle, A, subtended at the centre by the arc. Likewise, the (Indian) Cosine function is equivalent to $R \cos A$ and similarly for the Versed Sine and its complement. The *Āryabhaṭīya* of Āryabhaṭa I (born 476 AD) is the earliest extant historical work of the dated type in which the Indian trigonometry is definitely used.

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The modern forms of the Addition and the Subtraction Theorems for the sine and the cosine functions are:

$$\sin(A + B) = (\sin A) \cdot (\cos B) + (\cos A) \cdot (\sin B) \tag{1}$$

$$\sin(A - B) = (\sin A) \cdot (\cos B) - (\cos A) \cdot (\sin B) \tag{2}$$

$$\cos(A + B) = (\cos A) \cdot (\cos B) - (\sin A) \cdot (\sin B) \tag{3}$$

$$\cos(A - B) = (\cos A) \cdot (\cos B) + (\sin A) \cdot (\sin B) \tag{4}$$

The present paper concerns the equivalent forms of the above four Theorems for the corresponding Indian trigonometric functions. Statements as well as derivations of these formulas, as found in important Indian works, are described in it.

2 Statement of the Theorems

Bhāskara II (AD 1150) in his *Jyotpatti*, which is given at the end of the *Golādhyāya* part of his famous astronomical work called *Siddānta-śiromaṇi,* states[2]

चापयोरिष्टयोर्दोर्ज्ये मिथः कोटिज्यकाहते ।
त्रिज्याभक्ते तयोरैक्यं स्याच्चापैक्यस्य दोर्ज्यका ॥ २१ ॥
चापान्तरस्य जीवा स्यात् तयोरन्तरसंमिता ॥ २१$\frac{१}{२}$ ॥

The Sines of the two given arcs are crossly multiplied by (their) Cosines and (the products are) divided by the radius. Their (that is, of the quotients obtained) sum is the Sine of the sum of the arcs; their difference is the Sine of the difference of the arcs.

$$R \sin(A \pm B) = \frac{(R \sin A) \cdot (R \cos B)}{R} \pm \frac{(R \cos A) \cdot (R \sin B)}{R} \tag{5}$$

Thus we get the Addition Theorem (called the *Samāsa-Bhāvanā* by Bhāskara II) and the Subtraction Theorem (called the *Antara-Bhāvanā*) for the Sine.

We have some reason (see below and also Sect. 3) to believe that Bhāskara II was aware of the corresponding Theorems for the Cosine. According to the *Marīci* commentary (=*MC*) by Munīśvara (1638) on the *Jyotpatti*, a reason for Bhāskara's omission *(upekṣā)* of the Cosine formulas was that the following alternately shorter procedure, after having obtained $R \sin(A \pm B)$, was known[3]

$$R \cos(A \pm B) = \sqrt{R^2 - \{R \sin(A \pm B)\}^2} \tag{6}$$

Kamalākara (1658) also mentions (or quotes *MC*) in his commentary on his own *Siddhānta-tattva-viveka* (=*STV*) that the *Ācārya* (Bhāskara) has not followed or given the Cosine Theorems because of the exactly same reason as stated in the *MC*.[4]

In the late Āryabhaṭa School the Addition-Subtraction Theorem for the Sine was known as the *Jīveparaspara-Nyāya* and is attributed to the famous Mādhava of

Saṅgamagrāma (*circa* 1340–1425) who is also referred as *Golavid* (Master of spherics).[5] The *Tantrasaṅgraha* (=*TS*), composed by Nīlakaṇṭha Somayājī (AD 1500), gives Mādhava's rule in Chap. 2 as[6]

जीवे परस्परनिजेतरमौर्विकाभ्यामभ्यस्य विस्तृतिदलेन विभज्यमाने ।
अन्योन्ययोगविरहानुगुणे भवेतां यद्वा स्वलम्बकृतिभेदपदीकृते द्वे ॥ १६ ॥

> The Sines (of two arcs) reciprocally multiplied by the Cosines and divided by the radius, when added to and subtracted from each other, become the Sines of the sum and difference of the arcs (respectively). Or (we get the same results when the mutual addition and subtraction is performed with) the two (positive) square-roots of the (two) differences of their own (that is, of the two Sines themselves) and *lamba* squares.[7]

So that the first part of the rule gives the formula (5), while the second part contains the alternate formula

$$R\sin(A \pm B) = \sqrt{(R\sin A)^2 - (lamba)^2} \pm \sqrt{(R\sin B)^2 - (lamba)^2} \qquad (7)$$

Nīlakaṇṭha in the *Āryabhaṭīya-bhāṣya* (=*NAB*) and Śaṇkara Vāriar (AD 1556) in his commentary (=*TSC*)[8] on the above rule explains that the *lamba* involved is to be calculated from the relation

$$lamba = \frac{(R\sin A)\cdot(R\sin B)}{R} \qquad (8)$$

Thus it will be noticed that the form (7) is mathematically equivalent to the formula (5).

An important point to note is that the *TSC* (pp. 22–23) makes it clear that the word *jīve* (which we have translated as Sines) can be also taken to mean Cosines. But in such a case the phrase *nijetaramaurvikās* ('the other chords') should be taken to mean, as the *TSC* points out, the corresponding Sines. In other words we can get the same Addition and Subtraction Theorems for the Sine, if we interchange "sin" and "cos" with each other in the right-hand sides of (5). Following this interpretation the form (8) can also be expressed as

$$R\sin(A \pm B) = \sqrt{(R\cos A)^2 - (lamba)^2} \pm \sqrt{(R\cos B)^2 - (lamba)^2} \qquad (9)$$

where the *lamba* will now be given by

$$lamba = \frac{(R\cos A)\cdot(R\cos B)}{R} \qquad (10)$$

This interpretation also leads to the same Addition and Subtraction Theorems for the Sine.

For a geometrical interpretation of the quantity *lamba*, see Sect. 4.

Almost the same Sanskrit text of Mādhava's rule is also found quoted in the *NAB* (part I, p. 58) where it is explicitly mentioned to be *'Mādhava-nirmitaṃ padyam'*, that is a stanza composed by Mādhava. In this connection the *NAB* (part I, p. 60) also mentions the variant readings:

विस्तृतिगुणेन and विस्तृतिदलेन

For the Addition and Subtraction Theorems of the Cosine function, we may quote the *Siddhānta-sārvabhauma* (= *SSB*, 1646 AD), II, 57 which says[9]

यदंशज्ययोर्घातहीनाऽधिका च स्वकोटिज्ययोराहतिस्त्रिजकाप्ता ।
तदंशैक्यविश्लेषहीनाभ्रनन्दांशयोर्ज्ये स्तः ॥ ५७ ॥

The product of the Sines of the degrees (of two arcs) subtracted from or added to the product of their Cosines, and (the results) divided by the radius, become the Sines of the sum or difference of the degrees diminished from the ninety degrees.

That is,

$$R\sin(90^\circ - \overline{A \pm B}) = \frac{(R\cos A \cdot R\cos B \mp R\sin A \cdot R\sin B)}{R} \tag{11}$$

which are equivalent to the Theorems (3) and (4).

The *STV* (AD 1658), III, 68–69 (p. III) puts all the four Theorems side by side in the following words clearly.

मिथः कोटिज्यकानिघ्न्यौ त्रिज्याप्ते चापयोर्ज्यके ।
तयोर्योगान्तरे स्यातां चापयोगान्तरज्यके ॥ ६८ ॥

दोर्ज्ययोः कोटिमौर्व्योश्च घातौ त्रिज्योद्धृतौ तयोः ।
वियोगयोगौ जीवे स्तश्चापैक्यान्तरकोटिजे ॥ ६९ ॥

Multiply the Sines of the two arcs crossly by the Cosines and divide (separately) by the radius. Their (that is, of the two quotients obtained) sum and difference are the Sines of the sum and difference of the arcs (respectively). The products of the Sines and the Cosines are (each) divided by the radius. Their (that is, of the two quotients just obtained) difference and sum are (respectively) the Cosines of the sum and difference of the arcs.

That is,

$$R\sin(A \pm B) = \frac{(R\sin A)\cdot(R\cos B)}{R} \pm \frac{(R\sin B)\cdot(R\cos A)}{R}$$

$$R\cos(A \pm B) = \frac{(R\cos A)\cdot(R\cos B)}{R} \mp \frac{(R\sin A)\cdot(R\sin B)}{R}$$

Immediately after the statement of the above Theorems, the author, Kamalākara, in the next two verses (*STV*, III, 70–71), says

एवमानयनं चक्रे पूर्वं स्वीयशिरोमणौ ।
भावनाभ्यामतिस्पष्टं सम्यगार्योऽपि भास्करः ॥ ७० ॥
तस्य चानयनस्यार्यैः सिद्धान्तज्ञैः पुरोदिता ।
वासना बहुभिः स्वस्वबुद्धिवैचित्र्यतः स्फुटा ॥ ७१ ॥

Such a computation, which is quite evident from the two *bhāvanās* (see the next Section), was given earlier also by the highly respected Bhāskara in his (*Siddhānta-śiromaṇi*).† And many accurate proofs of that computation have been given previously by the respected astronomers according to the manifoldness of their intelligence.

Below we outline the various derivations as found in some Indian works and which indicate the ways through which Indians understood the rationales of the Theorems.

3 Method Based on the Theory of Indeterminate Analysis

The second-degree indeterminate equation

$$Nx^2 + k = y^2 \tag{12}$$

is called *varga-prakṛti* (square-nature) in the Sanskrit works. In connection with its solution the following two Lemmas, referred to as Brahmagupta's (AD 628) Lemmas by Datta and Singh,[10] have been quite popular in Indian mathematics since the early days.

Lemma I: If $x_1,\ y_1$ is a solution of (12) and $x_2,\ y_2$ that of

$$Nx^2 + g = y^2 \tag{13}$$

then (*Samāsa-bhāvanā*)

$$x = x_1y_2 + y_1x_2, \quad y = y_1y_2 + Nx_1x_2$$

is a solution of the equation

$$Nx^2 + kg = y^2 \tag{14}$$

Lemma II: (*Antara-bhāvanā*)

$$x = x_1y_2 - y_1x_2, \quad y = y_1y_2 - Nx_1x_2$$

is also a solution of (14).

An elaborate discussion of the subject including references to Sanskrit works, translations, proofs, and terminology is available and need not to be reproduced here.[11] Dr. Shukla's paper on Jayadeva (not later than 1073) is an additional noteworthy publication in this connection.[12]

Now, as indicated by the terminology used by Bhāskara II and clearly explained by his great mathematical commentator, Munīśvara, it is evident that Bhāskara II

†Alternately, the first verse may be translated thus: 'This was very clearly computed earlier by the respected Bhāskara also through the *bhāvanās* in his *Siddhānta-śiromaṇi*'.

arrived at the truth of the Addition and Subtraction Theorems for the Sine (and Cosine) possibly by applying the above Lemmas as follows.

As explained in the *MC* (pp. 150–151) on *Jyotpatti* 21–25, if we compare the Eq. (12) with the relation

$$-(R\sin Q)^2 + R^2 = (R\cos Q)^2 \tag{15}$$

we see that we can take

$$\begin{aligned} \textit{guṇaka}(\text{multiplier}) \quad N &= -1 \\ \textit{kṣepaka}(\text{interpolator}) \quad k &= R^2 \\ x &= R\sin Q \\ y &= R\cos Q \end{aligned}$$

Hence, on identifying

$$\begin{aligned} k &= g = R^2 \\ x_1 &= R\sin A,\, y_1 = R\cos A \\ x_2 &= R\sin B,\, y_2 = R\cos B \end{aligned}$$

we at once see, from Lemma I, that

$$x = (R\sin A)\cdot(R\cos B) + (R\cos A)\cdot(R\sin B) \tag{16}$$

and

$$y = (R\cos A)\cdot(R\cos B) - (R\sin A)\cdot(R\sin B) \tag{17}$$

is a solution of the equation

$$-x^2 + R^4 = y^2$$

that is, $\left(\dfrac{x}{R}\right)$ and $\left(\dfrac{y}{R}\right)$ will be a solution of (12), since

$$-\left(\frac{x}{R}\right)^2 + R^2 = \left(\frac{y}{R}\right)^2 \tag{18}$$

Thus comparing (15) and (18), we see, from (16) and (17), that

$$\frac{(R\sin A)\cdot(R\cos B) + (R\cos A)\cdot(R\sin B)}{R}$$

and

$$\frac{(R\cos A)\cdot(R\cos B) - (R\sin A)\cdot(R\sin B)}{R}$$

will represent some sort of additive solutions for the Sine and Cosine functions, respectively. The above were taken to represent $R\sin(A+B)$ and $R\cos(A+B)$, respectively. From mathematical point of view there is a lacuna in such an identification without further justification.[13]

Similarly, by using Lemma II. the expansions of $R\sin(A-B)$ and $R\cos(A-B)$ were identified.

Such a derivation undoubtedly supports the view that Bhāskara must have been aware of the Addition and Subtraction Theorems for the Cosine, although he did not state them.

The *SSB*, II, 58–59 (pp. 144–145). whose author is same as that of *MC*, also gives the same derivation of the Theorems (also see *STVC*, pp. 112–113).

4 Geometrical Derivation as Given in the *NAB*

While explaining Mādhava's Sanskrit stanza (*Jīve-paraspara* etc.) and the implied rules, which we have already mentioned above, the *NAB* (part I, p. 59) says:

तत्राद्यपादत्रयात्मकमेकं वाक्यम्, चरमः पादो वाक्यान्तरमिति विभागः। तत्राद्ये वाक्ये त्रैराशिकेन तदानयनं प्रदर्श्यते। अन्यस्मिन् भुजाकोटिकर्णद्वारा वर्गमूलपरिकल्पनया

The first three lines (of the stanza) form one rule (or method). The last line represents another rule. This is the break-up. We demonstrate the derivation of the first rule by (applying) the Rule of Three (that is, the proportionality of sides in the similar triangles). The other (rule) will follow from the relation between the base, upright and hypotenuse (or Sine, Cosine and radius) by extracting the square-root.

The geometrical demonstration given in the *NAB* (part I. pp. 58–61) may be substantially outlined as follows:

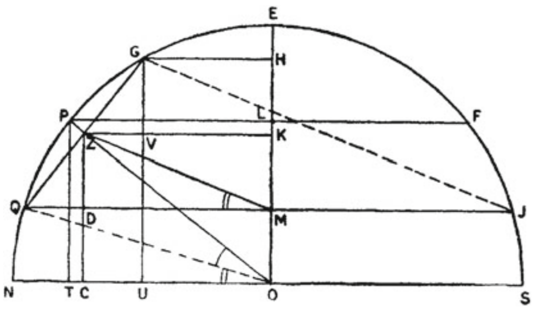

Fig. 1 Geometrical demonstration of *jīvepaspara-nyāya* as outlined by Nīlakaṇṭha

In Fig. 1 (East direction is upwards).

$$\text{arc } EP = A$$
$$\text{arc } PQ = \text{arc } PG = B \text{ (being less than } A).$$

So that

$$\text{arc } EQ = A + B$$
$$\text{arc } EG = A - B.$$

Here OP and QG intersect at Z and

$$PL = R\sin A = TO$$
$$OL = R\cos A$$
$$QZ = R\sin B = ZG$$
$$OZ = R\cos B = OP - PZ$$
$$QM = R\sin(A + B), \text{ which is required to be found out.}$$

In order to find QM, we determine its two portions QD and DM, made by the line ZC (drawn westwards from Z), separately and add them. Now to find the southern portion DM (which is equal to KZ) we have, from the similar right triangles OZK and OPL,

$$\frac{ZK}{OZ} = \frac{PL}{OP}$$

or

$$\frac{DM}{(R\cos B)} = \frac{(R\sin A)}{R}$$

giving

$$DM = \frac{(R\sin A)\cdot(R\cos B)}{R}. \tag{19}$$

Again, to find the northern portion DQ, we have, from the similar right triangles DQZ and OLP

$$\frac{DQ}{ZQ} = \frac{OL}{OP}$$

or

$$\frac{DQ}{(R\sin B)} = \frac{R\cos A}{R}$$

giving

$$DQ = \frac{(R\cos A)\cdot(R\sin B)}{R}. \tag{20}$$

By adding (19) and (20) we get QM which represents $R\sin(A+B)$. Thus is proved the Addition Theorem for the Sine.

For proving the Subtraction Theorem, drop perpendicular GU from G on ON. It divides ZK, which is equal to DM given by (19), into two portions VZ and VK. The northern portion VZ is equal to DQ given by (20) because the hypotenuse ZG is equal to the hypotenuse ZQ. Hence the southern portion

$$VK = ZK - DQ$$

or

$$GH = DM - DQ.$$

That is,

$$R\sin(A-B) = \frac{(R\sin A)\cdot(R\cos B)}{R} - \frac{(R\cos A)\cdot(R\sin B)}{R}$$

the required Subtraction Theorem.

Again, since

$$\begin{aligned}\frac{(R\sin A)\cdot(R\cos B)}{R} &= \frac{(R\sin A)\cdot\{\sqrt{R^2-(R\sin B)^2}\}}{R}\\ &= \sqrt{(R\sin A)^2 - \left\{\frac{(R\sin A)\cdot(R\sin B)}{R}\right\}^2}\\ &= \sqrt{(R\sin A)^2 - (lamba)^2}\end{aligned}$$

we can easily get the form (7) from the form (5) mathematically (see *NAB*, part I, pp. 86–87).

The *NAB* (part I, pp. 87–88) has also given some further geometrical interpretations and computations which we now indicate. In Fig. 1, $R\ \sin\ (A + B)$, that is, QM, is the base of the triangle ZQM. The second (or smaller) Sine, $R\sin B$, that is, QZ is the left side. The greater Sine, $R\sin A$, is the right side ZM (How?).

The foot of the perpendicular *(lamba)*, D, divides the base into two segments (*ābādhās*) DQ and DM which have been already found out. So that the *lamba*, given by (8), can be easily identified with the length ZD, the altitude of the triangle ZQM (this follows from $ZD^2 = ZQ^2 - DQ^2$). Then, from (see *NAB* part I, p. 88)

$$\sqrt{DM^2 + ZD^2} = ZM$$

we get, using (8) and (19),

$$ZM = R\sin A.$$

5 A Proof Based on Ptolemy's Theorem

Jyeṣṭhadeva (circa 1500–1610)[14] wrote *Yuktibhāṣā* (=*YB*) in Malayalam. Part I of the work presents an elaborate and systematic exposition of the rationale of the mathematical formulas.[15]

YB, (pp. 206–208 and 212–213) explains Mādhava's rules concerning the Addition and Subtraction Theorems for the Sine more or less on the same lines as given in the *NAB*. However, the *YB* (pp. 237–238) also indicates a proof of the Addition Theorem for the Sine by applying the so-called Ptolemy's theorem, namely:

'In a cyclic quadrilateral the sum of the products of the opposite sides is equal to the product of the diagonals'.

Of course, before indicating this use of the Ptolemy's theorem, the *YB* (pp. 228–236) has given a proof of it. According to Kaye,[16] a proof of the Ptolemy's theorem was also given by a commentator (Pṛthūdaka?, ninth century) of Brahmagupta (AD 628), the famous Indian mathematician who knew the correct expressions (which immediately yield the Ptolemy's theorem on multiplication) for the diagonals of a cyclic quadrilateral.[17]

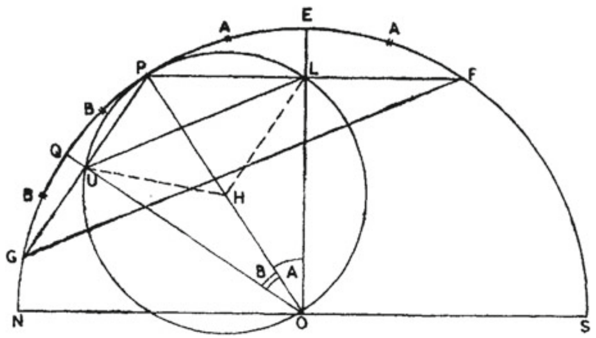

Fig. 2 Geometrical demonstration for *jīvepaspara-nyāya* as outlined in *Yuktibhāṣā*

The proof indicated in the *YB* and as explained by its editors (pp. 237–239) may be outlined as follows:
In Fig. 2

$$\text{arc } PE = A$$
$$\text{arc } QP = \text{arc } QG = B.$$

The radius OQ intersects PG in U. Thus PL and OL are the Sine and the Cosine of A and PU and OU those of B. From the cyclic quadrilateral $LPUO$, we have, by applying the rule of *bhujā-pratibhujā* etc. (that is, the Ptolemy's theorem),

$$PL.\ OU + OL \cdot\ PU = LU \cdot\ OP$$

or

$$(R\sin A)\cdot(R\cos B)+(R\cos A)\cdot(R\sin B)=LU\cdot R \tag{21}$$

The relation (21) will establish the Addition theorem for the Sine provided we are able to identify that LU represents the Sine of $(A+B)$. For seeing this, it may be noted that LU is the full chord of the arc $(LP+PU)$ in the circle which circumscribes the quadrilateral in question and whose radius is $\frac{R}{2}$ (as the centre of this smaller circle will be at H. The middle points of the radius OP equal to R). Thus

$$\begin{aligned} LU &= 2\left(\frac{R}{2}\right)\sin\left\{\frac{(2A+2B)}{2}\right\} \\ &= R\sin(A+B) \end{aligned}$$

We can also prove this by observing that LU is parallel to and half of the side FG in the triangle PFG. But FG itself is the full chord of the arc GPF in the bigger circle, so that

$$FG = 2R\sin\left\{\frac{(2A+2B)}{2}\right\}.$$

6 A Geometrical Proof Quoted in the *MC* (1638)

The *MC* (pp. 154–155) contains a geometrical proof, ascribed to others (*kecid*), which is only slightly different from that found in the *NAB* (see Sect. 4). It may be outlined as follows:

Firstly, the *MC* asks us to draw a figure similar to Fig. 1 which may be referred now. In the triangle ZQM, the base QM is the desired Sine of the combined arc $(A+B)$. The smaller side QZ is $R\sin B$. The distance between Z and M, that is, the larger (lateral) side ZM is equal to $R\sin A$ evidently (*pratyakṣa-pramāṇāvagatā?*) In order to know the base QM, its two segments QD and DM should be found out.

Now the *MC* finds QD exactly in the same manner as *NAB* (see the derivation of the relation (20)). Similarly, from the similar right triangles DMZ and OZQ, we have

$$\frac{DM}{MZ}=\frac{OZ}{OQ}$$

or

$$\frac{DM}{(R\sin A)}=\frac{(R\cos B)}{R}$$

which gives the bigger segment DM and hence their sum $(QD+DM)$ proves the Addition Theorem for the Sine.

After this, the *MC* also indicates the method for proving the Subtraction Theorem for the Sine.

We note that, in proving the Addition Theorem above, the *MC* does not give any theoretical details to demonstrate that the length ZM is equal to $R\sin A$. One way of proving this could be by noting that ZM is parallel to and half of the side GJ in the triangle QGJ; and GJ is itself the full chord of the arc GEJ which is easily seen to be equal to $2A$, so that $GJ = 2R\sin A$.

Alternately, we can see that a circle, of radius $\frac{R}{2}$, drawn on OQ as the diameter will pass through the points Q, Z, M and O and ZM will be a full chord (subtending angle $2A$ at the centre) of this smaller circle. So that we have

$$ZM = 2\left(\frac{R}{2}\right)\sin A.$$

Once the flank sides of the triangle ZQM are thus identified, the perpendicular ZD could also be obtained directly by using a well-known geometrical rule equivalent to[18]

$$\text{perp.} = \frac{\text{product of flank sides}}{\text{twice the circum-radius}}$$

giving

$$\begin{aligned} ZD &= \frac{ZQ \cdot ZM}{2 \cdot \left(\frac{R}{2}\right)} \\ &= \frac{(R\sin B)\cdot(R\sin A)}{R}. \end{aligned}$$

Thus, knowing ZQ, ZM and ZD, we can easily get the segments QD and DM and hence the required length QM. This provides an alternate and independent rationale of the Addition Theorem for the Sine in the form (7).

7 Proofs Found in *STVC*

We have already mentioned the observation of *STV*, III, 71 that several proofs of these Theorems were given by the previous writers. One set of derivations as given in the *STVC* (pp. 125–129) may be briefly outlined as follows:

In Fig. 3 arcs EP and EQ are equal to A and B, respectively. Other constructions are obvious from the figure. It can be easily seen that

$$\begin{aligned} PK &= PL + MQ = R\sin A + R\sin B \\ QK &= OM - OL = R\cos B - R\cos A \end{aligned}$$

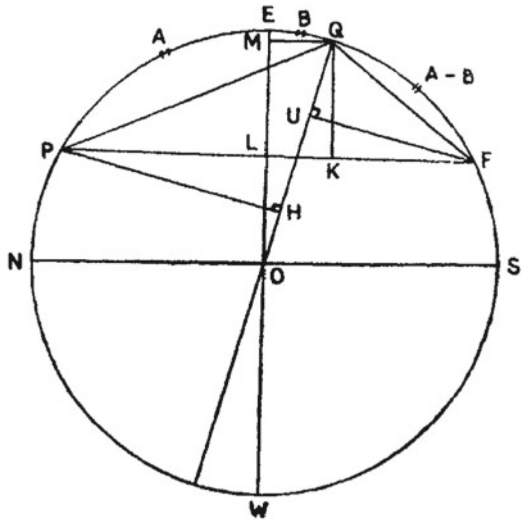

Fig. 3 Proof given in *Siddhānta-tattvaviveka* of Kamalākara

Therefore,

$$\begin{aligned} PQ^2 &= (R\sin A + R\sin B)^2 + (R\cos B - R\cos A)^2 \\ &= 2R^2 + 2R\sin A \cdot R\sin B - 2R\cos A \cdot R\cos B \end{aligned} \tag{22}$$

But PQ is the full chord of the arc $(A+B)$, so that

$$\frac{PQ}{2} = R\sin\left\{\frac{(A+B)}{2}\right\} \tag{23}$$

Now from a rule given in the *Jyotpatti*, 10 (p. 282), which the *STVC* (p. 126) quotes, we have

$$R\sin\frac{(A+B)}{2} = \sqrt{\left(\frac{R}{2}\right)\cdot R \text{ vers } (A+B)} \tag{24}$$

That is,

$$\begin{aligned} R \text{ vers } (A+B) &= \left(\frac{2}{R}\right)\cdot \left\{\frac{R\sin(A+B)}{2}\right\}^2 \\ &= \left(\frac{1}{2R}\right)\cdot PQ^2, \qquad \text{by (23)} \end{aligned}$$

Using (22), we easily get

$$R \text{ vers } (A+B) = R + \frac{(R\sin A \cdot R\sin B - R\cos A \cdot R\cos B)}{R}$$

from which the required expression for $R\cos(A+B)$ follows, since

$$R\cos\,(A+B) = R - R \text{ vers } (A+B).$$

Here it may be pointed out that the *STVC* (p. 126) also states that PQ, which we have found above from the triangle PQK, is also the hypotenuse for the right-angled triangle PQH (PH being perpendicular to the radius OQ). Incidentally, this gives an alternate procedure for proving the Addition Theorem for the Cosine. For, we have

$$PK^2 + QK^2 = PQ^2 = PH^2 + QH^2$$

or

$$(R\sin A + R\sin B)^2 + (R\cos B - R\cos A)^2 = \{R\sin(A+B)\}^2 + \{R - R\cos(A+B)\}^2$$

or

$$2R^2 + 2R\sin A \cdot R\sin B - 2R\cos A \cdot R\cos B = 2R^2 - 2R \cdot\ R\cos(A+B)$$

giving the required expansion of $R\cos(A+B)$.

Anyway, after getting the expression for $R\cos(A+B)$, the *STVC* (pp. 127–128) derives the corresponding expression for $R\sin(A+B)$ by using the relation

$$\{R\sin(A+B)\}^2 = R^2 - \{R\cos(A+B)\}^2.$$

Again, in the same figure the arc QF represents $(A-B)$. Also we have

$$FQ^2 = FK^2 + QK^2 = (PL - QM)^2 + (OM - OL)^2$$

or

$$\left\{2R\sin\frac{(A-B)}{2}\right\}^2 = (R\sin A - R\sin B)^2 + (R\cos B - R\cos A)^2 \qquad (25)$$

If we proceed as we did in the case of proving $R\cos(A+B)$ above, we easily get the desired expression for $R\cos(A-B)$. Alternately we get the same expansion by starting with the relation

$$FK^2 + QK^2 = FU^2 + QU^2$$

and proceeding as before.

Finally, the corresponding Subtraction Theorem for the Sine can be derived from that for the Cosine.

It is interesting to note that an equivalent of the identity (25) already occurs in the *jyotpatti*, 13, (p. 282). Thus Bhāskara II's familiarity with the relation (25) and that implied in (24) (where $A - B$ should be used for $A + B$) was enough to derive the Subtraction Theorems by this method (if he wanted to do so).

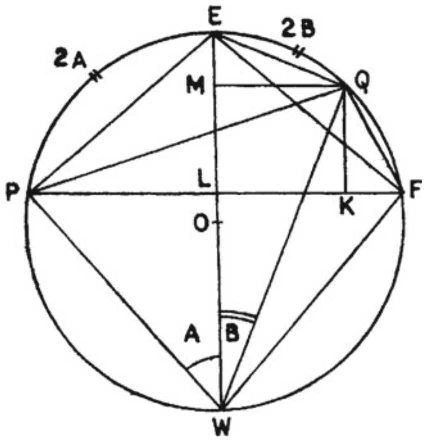

Fig. 4 Another proof given in *Siddhānta-tattvaviveka*

Another proof given in the *STVC* (pp. 130–135) may be briefly outlined as follows: In Fig. 4

$$\text{arc } EP = \text{arc } EF = 2A$$
$$\text{arc } EQ = 2B$$

It is important to note that the *STVC* says that the radius of the circle drawn is $\frac{R}{2}$ where R is the *Sinus totus*, so that the full chords EP, EQ etc., will themselves behave as the Sines. That is, we have

$$EP = 2\left(\frac{R}{2}\right)\sin A = R\sin A$$
$$EQ = R\sin B$$
$$PW = R\cos A$$
$$QW = R\cos B$$

etc., and, of course

$$EW = R$$

Now, by the methods of finding the altitude and segments of the base in a triangle, we have

$$\begin{aligned}
\text{segment } EM &= \frac{(R\sin B)^2}{R} \\
\text{segment } WM &= \frac{(R\cos B)^2}{R} \\
\text{perpendicular } QM &= \frac{(R\sin B)\cdot(R\cos B)}{R} \\
\text{segment } EL &= \frac{(R\sin A)^2}{R} \\
\text{segment } WL &= \frac{(R\cos A)^2}{R} \\
\text{perp } PL = \text{perp } LF &= \frac{(R\sin A)\cdot(R\cos A)}{R}.
\end{aligned}$$

(Of course, all these results also follow from similar right triangles in the figure.) Now we have

$$\begin{aligned}
PQ^2 &= PK^2 + QK^2 \\
&= (PL + QM)^2 + (EL - EM)^2
\end{aligned}$$

On substituting from the above expressions, simplifying, and taking the square-roots we easily get the required expression for $R\sin(A+B)$ represented by PQ.

Again we have

$$\begin{aligned}
FQ^2 &= FK^2 + KQ^2 \\
&= (PL - QM)^2 + (EL - EM)^2
\end{aligned}$$

Thus, following the same procedure, we get the required expression for R sin $(A - B)$ represented by QF.

However, before closing this article, it may not be out of place to mention that by using the Ptolemy's theorem in Fig. 4, we get the expressions for PQ and QF almost in one step. For, Ptolemy's theorem applied to the quadrilateral $EPWQ$ yields

$$EP \cdot QW + PW \cdot EQ = PQ \cdot EW$$

which gives the desired PQ; and Ptolemy's theorem applied to the quadrilateral $EQFW$ yields

$$EQ \cdot FW + QF \cdot EW = EF \cdot QW$$

which gives the desired QF.

Acknowledgements I am grateful to Prof. M. B. Gopalakrishnan for reading the relevant passages from the *YB* for me and to Dr. T. A. Sarasvati Amma for some discussion on the subject.

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11. *Ibid.,* pp. 141–172.
12. Shukla, K.S., "Ācārya Jayadeva, the mathematician," *Gaṇita* 5, No. 1 (June 1954), pp. 1–20.
13. If we compare Eq. (12) with the identity

$$\tan^2 Q + 1 = \mathrm{see}^2 Q,$$

we see that ($\tan A$, $\sec A$) and ($\tan B$, $\sec B$) are solutions of the equation

$$x^2 + 1 = y^2,$$

whose *samāsa-bhāvanā* solution. therefore, will be given by

$$x = \tan A \cdot \sec B + \sec A \cdot \tan B$$
$$y = \sec A \cdot \sec B + \tan A \cdot \tan B.$$

But here x and y do not represent $\tan(A + B)$ and $\sec(A + B)$.
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Parameśvara's Rule for the Circum-radius of a Cyclic Quadrilateral

The expression for the circum-radius of a cyclic quadrilateral in terms of its sides, usually attributed to L'Huilier in 1782, was known in India to Parameśvara (circa 1430). The present paper contains the original Sanskrit text of the rule, its English translation, and a discussion of its derivation as given by Saṅkara Vāriar in his *Kriyākramakarī* (sixteenth century) along with relevant historical remarks.

चक्रीय चतुर्भुज के भुजमानों से उसके परिगत वृत्त की त्रिज्या के ज्ञान एवं प्रकाशन (१७८२ ई.) का श्रेय एक यूरोपीय लेखक को दिया गया है। लेकिन तत्संबंधी नियम भारत में परमेश्वर (लगभग १४३० ई.) को पहले से ही ज्ञात था। निम्न लेख में उस नियम का मूल संस्कृत पाठ, आंग्ल अनुवाद, तथा शंकर कृत क्रियाक्रमकरी (१६ वीं शताब्दी) में वर्णित उपपत्ति का विवेचन सुसंगत ऐतिहासिक टिप्पणियों के साथ दिये हैं।

Let $ABCD$ be a cyclic quadrilateral with sides AB, BC, CD, and DA; equal to a, b, c, and d, respectively. Smith [1958, II, 287] stated that S. A. J. L'Huilier discovered and published in 1782 a formula which reduces

$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}} \tag{1}$$

where R is the radius of the circle circumscribing the quadrilateral and s is its semi-perimeter. The formula (1) was already known about 350 years earlier in India and is given verbally by Parameśvara (circa 1360–1455) in his commentary (before 1432) on the *Līlāvatī* (circa 1150) of Bhāskara II [Saraswathi 1969, 69; Sarma 1972, 19].

The purpose of the present paper is to bring to the notice of scholars the Sanskrit verse and the Indian derivation of the rules as found in another commentary,

Historia Mathematica 4 (1977), pp. 67–74.

K. Ramasubramanian (ed.), *Gaṇitānanda*,
https://doi.org/10.1007/978-981-13-1229-8_38

called *Kriyākramakarī* (sixteenth century), on the *Līlāvatī* recently published [Sarma (editor) 1975].

The composition of the *Kriyākramakarī* (=*KKK*) commentary was started by Saṅkara Vāriar (c. 1500–1560) and, after his death, finished by Mahiṣamaṅgalam Nārāyaṇa (1540–1610). The rule and its rationale are found in the portion (c. 1534) which was written by Saṅkara [Sarma (ed.) 1975, xxii].

The original Sanskrit text of the rule as found in the *KKK* (p. 363), and which is almost the same as that given by Parameśvara, is

दोष्णां द्वयोर्द्वयोर्घातयुतीनां तिसृणां वधे ।
एकैकोनेतरत्र्यैक्यचतुष्केण विभाजिते ॥
लब्धमूलेन यद् वृत्तं विष्कम्भार्धेन निर्मितम् ।
सर्वं चतुर्भुजं क्षेत्रं तस्मिन्नेवावतिष्ठते ॥

This may be translated almost literally thus:

> The three sums of the products of the sides taken two at a time are to be multiplied together and divided by the tetrad formed by diminishing one (of the sides) at a time from the sum of the other three. If a circle is drawn with the square-root of the quotient (just obtained) as semi-diameter, the whole quadrilateral figure will be located therein.

That is,

$$R = \sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(b+c+d-a)(c+d+a-b)(d+a+b-c)(a+b+c-d)}} \tag{2}$$

which is equivalent to (1). After explaining the rule, the *KKK* gives (pp. 364–365) its rationale *(upapatti)*, using the following three results.

Lemma I The product of the flank sides of any triangle divided by the diameter of its circumscribed circle is equal to the altitude of the triangle.

Lemma II The area of the cyclic quadrilateral is given by

$$S = \sqrt{(s-a)(s-b)(s-c)(s-d)} \tag{3}$$

Lemma III (cf. Ptolemy's theorem): Let $ABCD'$ be the quadrilateral formed from $ABCD$ by interchanging the sides AD and CD, that is, by taking $AD' = CD = c$ and $CD' = AD = d$. If x, y, z denote the three diagonals AC, BD, and BD', respectively, then $yz = ab + cd$, $zx = bc + da$, $xy = ca + bd$.

Lemma I was known to Indians for about a thousand years before the date of the KKK. In the equivalent form circum-radius $= \dfrac{\text{(product of flank sides)}}{\text{(twice the altitude),}}$ ...... it is implied in a rule given by Brahmagupta (AD 628) in his *Brāhmasphuṭa-siddhānta,* XII, 27 [Sharma (ed.) 1966, III, 834; Gupta 1974b, 173]. (The *KKK* itself proves it separately (pp. 365–366). It is also used and proved in another Indian work called

Yuktibhāṣā (=*YB*) which is attributed to Jyeṣthadeva (c. 1500–1610) [Sarma 1972, 59–60; Thampuran and Aiyar 1948, 231, 243–246].

Lemma II has been very popular in India since it was first stated by Brahmagupta in his *Brāhmasphuṭa-siddhānta* (=BSS), XII, 21 [Sharma (ed.) 1966, III, 816; Gupta 1974a, 34–35]. According to Dr. K.S Shukla (a great authority on Hindu astronomy and mathematics), Brahmagupta and other early Indian Mathematicians have committed an error in declaring the formula (3) "as applicable to all quadrilaterals (with unequal altitudes), when in fact it is applicable to cyclic quadrilaterals only" [Shukla (ed.) 1959, p. 90 (translation)]. However, a recent scholar has been "unable to accept that Brahmagupta could have imagined that his rules would apply to all quadrilaterals whatsoever" [Pottage 1974, 354]. The whole difficulty arises out of the fact that Brahmagupta himself has neither explicitly specified the correct range of application of his rule (3) nor given any derivation for it. But this state of affairs was not an unusual feature of ancient Indian mathematical texts.

Gaṇeśa in his commentary (AD 1545) on the *Līlāvatī* [Apte (ed.) 1937, 156–157] attempted to prove the rule (3) but the demonstration is incorrect [Inamdar 1946, 36–42].

A detailed proof of Lemma II is found in the *YB* (pp. 247–257). When the product of the two diagonals is needed in the course of this proof, it is derived by making use of the following (the so-called Brahmagupta's expressions for diagonals of a cyclic quadrilateral):

$$x = \sqrt{\frac{(ac + bd)(ad + bc)}{(ab + cd)}} \tag{4}$$

$$y = \sqrt{\frac{(ac + bd)(ab + cd)}{(ad + bc)}} \tag{5}$$

These results are given by Brahmagupta in his *BSS*, XII, 28 [Sharma (ed.) 1966, III 836] and are considered to be the "most remarkable in Hindu Geometry and solitary in its excellence" by a recent historian of mathematics [Eves 1969, 187]. The formula (5) is stated to be rediscovered in Europe by W. Snell who gave it in his edition (1619) of Van Ceulen's work [Smith 1958, 287]. In fact the expressions (4) and (5) are separately derived in the *YB* (p. 233) from Lemma III which we now consider.

The Indian discussion of Lemma III is quite interesting because of the concept of the third diagonal of a cyclic quadrilateral. Bhāskara II had shown that the interchange of two adjacent sides of a (cyclic) quadrilateral alters the length of one of the diagonals (thereby getting a third diagonal), and this area and perimeter preserving construction appears in his *Līlāvatī* [Apte (ed.) 1937, II, 187; Colebrooke (tr.) 1967, 110; Pottage 1974, 306].

The geometry of the three diagonals of a cyclic quadrilateral is discussed in greater detail by Nārāyaṇa Paṇḍita (not to be confused with Mahiṣamaṅgalam Nārāyaṇa mentioned above) in his *Gaṇita-kaumudī* (c. 1356). For instance, Rule 52 from the *Kśetravyavahāra* portion of the work runs as follows [Dvivedi (ed.) 1942, 59]:

द्विगुणव्यासविभक्ते त्रिकर्णघातेऽथवा गणितम् ।
त्रिभुजे चतुर्भुजे वा व्यासस्य दलं प्रजायते हृदयम् ॥ ५२ ॥

> The product of the three diagonals divided by twice the diameter (of the circumscribed circle) is the area of a triangle or quadrilateral; half of the diameter becomes the *hṛdayam* (circum-radius).

That is, Area $S = \frac{xyz}{4R}$ for a cyclic quadrilateral as well as a triangle (in which case the three sides themselves will be its three diagonals).

Rule $137\frac{1}{2}$ from the same portion of the work gives the above relation in the form $R = \frac{xyz}{4}$ (area). It is interesting to note that, after stating this rule, Nārāyaṇa criticized Brahmagupta's rule for the circum-radius [*BSS*, XII. 26; Pottage 1974, 334–335] as being *avyāpaka* ('not universal') and further said that Lalla (c. 748 AD) and Śrīpati (c. 1039) blindly followed Brahmagupta in this respect [Dvivedi (ed.) 1942, 175].

The discussion of the three diagonals as found in the *KKK* is more subtle. Firstly, it shows that in a cyclic quadrilateral more than three diagonals are not possible. The arguments given are substantially as follows (p. 351):

Let $\alpha, \beta, \gamma, \delta$ be the angular measures of the arcs corresponding to the sides a, b, c, d (respectively) of a cyclic quadrilateral. Now a sum of any two arcs can be made to define a diagonal. Hence there can be six cases. But because $\alpha + \beta + \gamma + \delta = 360°$ there will be only three final possibilities (e.g. if $\alpha + \beta$ defines one diagonal, $\gamma + \delta$ will define the same diagonal). Hence only three diagonals are possible (our x, y, z will be found to correspond to $\alpha + \beta, \beta + \gamma, \gamma + \alpha$ respectively).

The complete proof of Lemma III as given in the *KKK* (pp. 349–351) may be briefly mentioned in terms of modern symbols as follows.

Simple geometrical proofs of the following two preliminary results are given

$$ch^2\theta - ch^2\phi = ch(\theta + \phi) \cdot ch(\theta - \phi) \tag{6}$$

$$ch\lambda \cdot ch\mu = ch^2 \left\{\frac{(\lambda + \mu)}{2}\right\} + ch^2 \left\{\frac{(\lambda - \mu)}{2}\right\}, \tag{7}$$

where *ch* stands for the chord of the arc (for the sine function also, similar results hold good). Now we have, with reference to the accompanying figure,

$$\begin{aligned} ab + cd &= ch\ \alpha \cdot ch\ \beta + ch\ \gamma \cdot ch\ \delta \\ &= ch^2 \left\{\frac{(\alpha + \beta)}{2}\right\} - ch^2 \left\{\frac{(\beta - \alpha)}{2}\right\} + ch^2 \left\{\frac{(\gamma + \delta)}{2}\right\} - ch^2 \left\{\frac{(\gamma - \delta)}{2}\right\}. \end{aligned} \tag{8}$$

by one of the above results.

If E and W be the mid-points of the arcs ABC and ADC, respectively, then

$$ch\left\{\frac{(\alpha+\beta)}{2}\right\} = AE, \text{ and } ch\left\{\frac{(\gamma+\delta)}{2}\right\} = AW.$$

Also AEW is a right-angled triangle with hypotenuse EW equal to the diameter of the circle. Hence (8) gives

$$\begin{aligned} ab+cd &= (2R)^2 - ch^2\left\{\frac{(\beta-\alpha)}{2}\right\} - ch^2\left\{\frac{(\gamma-\delta)}{2}\right\} \\ &= ch^2\left(180° - \frac{\beta-\alpha}{2}\right) - ch^2\left\{\frac{(\gamma-\delta)}{2}\right\} \\ &= ch\left(180° - \frac{\beta-\alpha}{2} + \frac{\gamma-\delta}{2}\right)\cdot ch\left(180° - \frac{\beta-\alpha}{2} - \frac{\gamma-\delta}{2}\right) \end{aligned}$$

by the formula (6). Because $\alpha+\beta+\gamma+\delta = 360°$, we finally get

$$\begin{aligned} ab+cd &= ch\ (\alpha+\gamma)\cdot ch\ (\alpha+\delta) \\ &= (\text{chord of the arc } BAD')\cdot(\text{chord of the arc } BAD) \\ &= BD'\cdot BD = z\cdot y \end{aligned}$$

which is the first equation of Lemma III. The other equations can be derived similarly, and the proof of the Lemma III is thus completed. The proof given in the *YB* (pp. 228–233) is somewhat similar to this.

These Indian proofs of the so-called Ptolemy's theorem are radically different from that given about 1500 years earlier by Ptolemy in his *Almagest* [Taliaferro 1952, 16–17].

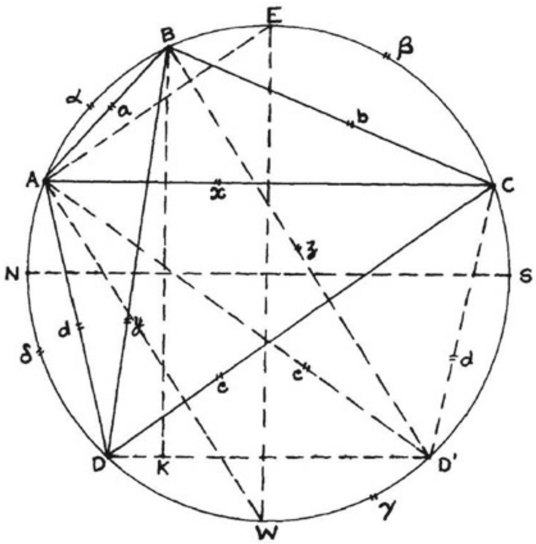
After proving Lemma III, the *KKK* (p. 351) derives the expressions for the squares of the two diagonals (x and y) from it, that is, from the equation in Lemma III. These expressions are equivalent to the famous Indian formulas (4) and (5). Finally, a similar expression for the third diagonal is also derived but "it is not given here (that is, in the original text) because of its non-utility (*anupayoga*)", the KKK says. Almost the same discussion is found in the *YB* (p. 233). These Indian derivations may be contrasted with the conjectural Brahmaguptan proofs as suggested by Pottage [1974, 344–349].

The derivation of the main result (1) may now be presented briefly.

The *KKK* starts (p. 364) by asking us to draw a diagram similar to the accompanying figure. In it EW and NS are east-west and north-south lines (east was represented upwards by Indians). BK is drawn perpendicular to DD' (which is parallel to AC). Other details are self-evident in the figure.

By Lemma I, applied to the triangle BDD', we get

$$\text{perp. } BK = \frac{yz}{2R}\dots \tag{9}$$



Three diagonals of a cyclic quadrilateral

This perpendicular *BK* will be the sum of the altitudes of the two triangles *BAC* and *DAC* into which the quadrilateral *ABCD* is divided by the diagonal *AC* (which becomes their common base). Thus, the area of the quadrilateral

$$s = \left(\frac{1}{2}\right) BK \cdot x \ldots \tag{10}$$

Therefore, by (9) and (10), we get

$$R = \frac{xyz}{4S} \ldots . \tag{11}$$

As stated above, this result was already known to Nārāyaṇa Paṇḍita (c. 1356).

Parameśvara's rule (1) now immediately follows from (11) by using Lemma II and Lemma III, that is, by multiplying the equations in Lemma III to get *xyz* as needed in (11).

Just after completing the proof, the *KKK* (p. 365) adds an intelligent remark which renders unnecessary the alternate reading (involving the work *vadha* or *ghāta,* that is, 'product') of the original Sanskrit stanza, as mentioned by the editor and found quoted elsewhere [Saraswathi 1969, 69; Sarma 1972, 19].

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Indian Values of the Sinus Totus

Unlike the modern trigonometric sine of an angle which is defined as the ratio of the side (facing that angle) to the hypotenuse in a right-angled triangle, the ancient Sine of an arc was defined (apparently in India for the first time) as half the chord of double the arc in a circle of reference. The radius of this circle thus became the *Trijyā* (the Sine of the three signs) or the *sinus totus* (the total or complete Sine).

It is curious as well as interesting to know that Indians, through the ages, used a variety of values for the *sinus totus* such as, 45, 60, 120, 150, 200, 300, 500, 1000, 3270 and 3600 beside those typically Indian values which were based on the relation

$$R = \frac{21600}{2\pi} \quad \text{minutes.}$$

The value 3438 has been the most popular for Indian standard tables of the Sines and 120 was frequently used for shorter tables.

Detailed discussions of the various values are presented in the paper along with full references. Terminology and some instances of transmission are also described. The value 150 which was used in India by Brahmagupta (seventh century AD) and Lalla (eighth century) has been found to be used later on in several foreign works obviously under Indian influences.

1 Introduction

The predecessor of the modern trigonometric function known as the sine of an angle was born, apparently, in India.[1] The Greek trigonometry had been based on the functional relationship between the chords of a circle and the central angles they subtend. The Indians, on the other hand, used half of a chord of a circle as their basic trigonometric function. The Indian (or Hindu) Sine (usually written with a capital letter to distinguish it from the modern Sine) of an arc in a circle is defined as half the length of the chord of double the arc. Thus the (Indian) Sine of an arc α is equal

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to $R \sin\theta$ where R is the radius of the circle of reference and $\sin\theta$ is the modern sine of the angle θ subtended at the centre by the arc α.

The relations between (Indian) Sine, modern sine and the Greek chord (=crd) functions may be expressed as

$$\text{Sin}\,\alpha = R\sin\theta = \frac{1}{2}\,\text{crd}\,2\alpha \tag{1}$$

$$\text{crd}\,\alpha = 2R\sin\left(\frac{\theta}{2}\right) = 2\,\text{Sin}\left(\frac{\alpha}{2}\right). \tag{2}$$

In these relations the angular measure α of the arc is exactly equal to the angle θ.

The *Āryabhaṭīya* (*AB*) of Āryabhaṭa I (born 476 AD) is the earliest extant Indian work of the historical or dated type in which the Indian Sine is definitely used. However, there seems to be some evidence for earlier and possible use of Sine in India in some of the old *Siddhāntic* works which have been summarized later on by Varāhamihira (c. 550 AD) in his *Pañcasiddhāntikā* (= *PS*).[2] Hereafter such abbreviations will be used for standard Sanskrit works; all of them are listed in Appendix 1.

From the definition of the Sine, it is clear that its greatest value will be equal to R when the arc is equal to 90 degrees. That is why the norm R is called *Sinus Totus*[*] ('total or complete Sine'), the Sines corresponding to other arcs being regarded as parts or fractions of this.

The ancient length-definition (even with R equal to one) has thus at least one advantage over the modern definition of the Sine, as the ratio of perpendicular to the hypotenuse in a right-angled triangle, because in the case of 90° the former definition presents no difficulty, while the latter can yield the sine of 90° only by considering it as a limiting case.

For the parametric norm R, a variety of values were used by the Indians during the ancient and medieval periods of their trigonometry. The purpose of this paper is to present and discuss those values along with some other related aspects.

2 Values of the Sinus Totus

The constancy of the ratio of the circumference of any circle to its diameter was known in the ancient world. So that when circumference C is known, the diameter D (=2R) can be written down, their ratio being π.

The most typically Indian values of the *Sinus Totus* R were obtained from the relation

[*]The term sinus totus was introduced for the first time by Gerhard of Cremona (1114–1187) in his translation of *Al-Zarqālī*'s astronomical tables. (information given in the reference).

$$R = \frac{C}{2\pi} \tag{3}$$

after having first chosen the value of C. Now C is a linear quantity and should be specified in linear units. But the Indians took a different attitude. They took the angular measure of the circumference (equivalent to the measure of 360° angle subtended by it at the centre) itself to represent its linear measure.

However, this attitude is in itself not sufficient to fix the size of the circle because we cannot associate any absolute length to an angular unit, say a degree or a minute. For example, two arcs, one of a smaller and other of a bigger circle, can both be said to be of equal length (say 1800 min) in angular units when each of them subtends the same angle (30° in said case) at their centres although their linear sizes, that is lengths, are different.

Another difficulty created by first specifying C in ordinary angular units and then calculating R by (3) is that the latter cannot be found exactly since the number π is transcendental. Even if we use an approximate value of π, we may be compelled to involve another approximation in deciding the value of R from (3) in a nice form, for example, in whole number of minutes, seconds, depending on the accuracy desired.

Even with all these choices when we determine and accept a particular value of R specified in the angular units, we cannot still draw the circle of a definite size because the value of R is not in the absolute units of length (What is the size of one degree or one minute length?).

Of course, this difficulty of drawing the circle will come even if R is specified directly in angular units (say minutes) or as an absolute number instead of determining it from (3). To overcome this theoretical difficulty, an angular unit was taken to be equivalent to some known unit of length (This practice is similar to what we do when we have to draw, for example, a force diagram where we decide that so many units of force are represented by so many units of length).

Thus in connection with the description of a 'shadow-instrument' (*chāyā-yantra*, the lost work of) Āryabhaṭa I asks us to draw a circle of radius 57 *aṅgulas* (finger-breadth) to represent the 57° of the *Sinus Totus* (when the circumference is taken to be equal to 360°).[3]

Again, while giving the details of the graphical method of finding the Sines as described in the *Brāhmasphuṭa-siddhānta* (=*BSS*) of Brahmagupta (628 AD), the commentator Pṛthudaka (c. 860) asks us to draw, by a pair of compass (*karkaṭa*), a circle of radius 3270 *aṅgulas* on a level ground, the number being the *Sinus Totus* used in the above work.[4] If the *aṅgula* mentioned be taken to be equivalent to about three-fourths of an inch, then the radius will measure more than 200 feet. We may have a level ground to draw the said circle but wherefrom such a big compass (of more than 100 ft arm) is to be obtained. Or we have to give some different explanations. Bhāskara II (1150) also, in a similar context, talks of drawing a circle with desired radius in *aṅgulas*.[5]

Anyway, whether be the practical difficulties or conventions in drawing the circle of reference, the values of the *Sinus Totus* (and so of all other Sines) were not represented in absolute units of length. The same may be said of Ptolemy who took a diameter of 120 parts for his table of chords. All this shows that the ancient Sines were defined as lengths but not as absolute lengths.

In spite of all these defects, the Indians have been praised for their practice of taking the circumference and radius in the same angular units. Thus Otto Neugebauer remarked[6]

> ....The Hindus took the reasonable attitude that the radial distances should be measured in the same units in which the length of the circumference is measured, an approach which would have led to the modern concept of radians. Had they not retained the Babylonian sexagesimal division of a circle into 360 parts.

Now according to *AB*, II, 10 the circle of diameter 20,000 is nearly equal to 62,832 units.[7] This implies the approximation

$$\pi = \frac{62{,}832}{20{,}000} = 3.1416. \tag{4}$$

Using this and taking 360° (or 21,600 min) as the measure of C, the relation (3) gives

$$\begin{aligned} R &= \frac{75000}{1309} = 57 + \frac{387}{1309} \quad \text{degrees} && (5)\\ &= \frac{4500000}{1309} = 3437 + \frac{967}{1309} \quad \text{minutes} && (6)\\ &= 206264 + \frac{424}{1309} \quad \text{seconds} && (7)\\ &= 12375859 + \frac{569}{1309} \quad \text{thirds} && (8)\\ &= 57°17'44''19''' + \frac{569}{1309} && (9)\\ &= 57.2956, 4553 \quad \text{nearly} && (10) \end{aligned}$$

The value (6), inclusive of the fractional part as such, is mentioned or quoted in the Utpala's commentary (tenth century) on the *Bṛhat-saṃhitā*.[8] Also we see that, depending on the degree of accuracy desired, the value of the *Sinus Totus* can be taken to nearest degree, minute, second, third, etc. We discuss these individually.

(I) From (5) we get (to the nearest degree)

$$R = 57 \quad \text{degrees} \tag{11}$$

We have already pointed out that the lost work, called *Āryabhaṭa-siddhānta* (=*AS*), of Āryabhaṭa I talks of drawing a circle with *trijyā* (*sinus totus*) or radius 57 units which represent the 57° (in a radian). However, we have not come across any evidence to show that this lost *AS* had used a sine table with *R* equal to 57°.

According to the *Hayata* (1764), a very late Sanskrit work on Arabic astronomy, some *Sūrya-siddhānta* (= *SS*) also used a radius of 57 units.[9] Which *SS* is referred to here is not clear, but it may be pointed out that the lost *AS* (which used the same radius) was based on the old *SS*.[10]

Moreover it may also be pointed out that the *Hayata* (p. 16) roughly derives the radius 57 by using the approximation

$$\pi = \frac{22}{7} \tag{12}$$

instead of (4). Although this does not matter much, it should be noted that the value (12) is the first fractional approximation of the value (4) when the latter is expressed in continued fraction.[11]

The *PS*, IV, 1 (see Sect. 3) gives the rule

$$\text{diameter} = \sqrt{\frac{360^2}{10}}$$

which implies the radius (11) to the nearest degree of (22) below.

(II) The relation (6) shows that we shall have (to the nearest minute)

$$R = 3438\,\text{minutes} \tag{13}$$

By far this is the most commonly used value of the *sinus totus* in Indian trigonometry. Sine-differences[12] stated in *AB*, I, 10 (pp. 16–17) imply it. These tabular differences have been referred[13] and used by Bhāskara I in his works (early seventh century). The resulting 24 tabular Sines are given[14] by Lalla (c. 748) who rightly calls them *Bhaṭoditā* (i.e. as computed by Āryabhaṭa), the last Sine being equal to the *Sinus Totus* given by (13). Moreover, Lalla has also given the value[15]

$$R^2 = 1181,9844 \quad \text{(square minutes)} \tag{14}$$

which is obviously derived from (13).

The tabular Sines as given[16] in the extant *SS*, II, 17–22, also imply the value (13). The same is the case with the *Soma-siddhānta*, II, 4–8 (p. 7)[17a] Sumati of Nepal (before 950 AD) also used the value (13) in the *Sumati Mahātantra* as well as in his *Sumati-Karaṇa*.[17b]

Āryabhaṭa II (950) has employed the value (13) in his *Mahā-siddhānta* (*MS*)[18], but one of his tabular Sine is different from the corresponding value found in Lalla or *SS*. Bhāskara II has followed Āryabhaṭa II for his standard table of the Great Sines[19] although author's own accompanying commentary states that the same values are given in the *SS* (?) and the *Āryabhaṭa Tantra* (=*AB?*).

The use of the value (13) for a radius by Puliśa is also attested.[20]

(III) The relation (7) shows that, to the nearest second of the angular arc, we have

$$\begin{aligned} R &= 206264 \text{ seconds} \\ &= 3437'44'' \end{aligned} \tag{15}$$

The value (15) for the *sinus totus* (to the nearest second) is given by Vaṭeśvara (early tenth century) in his *Vaṭeśvara-siddhānta* (=*VS*)[21] just after stating the minutes and seconds of his tabular Sines and Versed Sines. Immediately after stating the value (15), the *VS* gives

$$R^2 = 1181,\, 8047'35''. \tag{16}$$

Now (15) will give

$$\begin{aligned} R^2 &= \left(3437 + \frac{44}{60}\right)^2 \\ &= 1181,\, 8010 + \frac{28}{60} + \left(\frac{16}{60^2}\right), \end{aligned} \tag{17}$$

which does not agree with (16) to the nearest sixtieth part. However, the original relation (6) gives

$$\begin{aligned} R^2 &= \frac{4,\,500,\,000^2}{1309^2} \\ &= 2025 \times \frac{10^{10}}{1,\,713,\,481} \\ &= 1181,\, 8047 + \frac{(35.3)}{60} \text{ nearly} \end{aligned} \tag{18}$$

which agrees with the *VS* value (16) to the nearest second. Thus Rai[22] need not remark that there is an error in the *VS* value of the square of the radius nor his suggested emendation of the printed reading *jala* (=4) to *jalada* (=0), thereby getting

$$R^2 = 1181,\, 8007'35''$$

in place of the correct value (16), is necessary.

Parameśvara[23] in his commentary (c. 1408) on the *Laghu-bhāskarīya* (=*LB*) gives a table of Sines in which the last value (representing the Sine of 90°) is same as (15).

(IV) From the relation (8) and (9) we have (to the nearest third)

$$R = 3437'44''19''' \tag{19}$$

$$= 1237,5859 \text{ thirds} \tag{20}$$

Govindasvāmin (c. 800–850) in his commentary on the *MB*, IV, 22 (pp. 199–201) has given fractional parts meant for improving the *AB* Sine-differences and the resulting sine table[24] imply the radius (19).

For the value of the *Sinus Totus* in the form (20), a set of tabular Sines and their differences is found in the *Sundarī* commentary[25] by Udaya Divākara (c. 1073) on the *LB*.

It is well-known that, for finding the circumference of a circle, the Indians often used the rough formula

$$C = \sqrt{10D^2} \tag{21}$$

which gives

$$D = \sqrt{\frac{C^2}{10}}. \tag{22}$$

By comparing this with (3), we may say that (22) implies

$$\pi = \sqrt{10}. \tag{23}$$

Instead of (4) with C equal to 21600 min the relation (22) gives

$$D = \sqrt{4665,6000} = 6830.52 \text{ nearly}. \tag{24}$$

Thus (to the nearest minute)

$$R = 3415. \tag{25}$$

The value (25) has been worked out by Pṛthūdaka in his commentary on the *BSS*, XXI, 15 (Vol. IV, p. 1626) which contains the rule (22). For the square-root in (24), he gives 6830 which is correct if we disregard the fractional part though greater than half.

The sine table given in the *Laghu Vasiṣṭha Siddhānta*[26] implies the radius (25). Same is the case with the *Siddhānta-śekhara* of Śripati (c. 1040) who adds that his *sinus totus* is the radius of the circle whose circumference is equal to minutes in one revolution and also gives[27]

$$R^2 = 1166,225 \tag{26}$$

which is exactly the square of the value (25).

The famous Mādhava's (c. 1340–1425) sine table which is quoted by Nīlakaṇṭha Somasutvan (c. 1500) in his commentary (*NAB*) on the *AB* imply the radius[28]

$$R = 3437'44''48'''. \tag{27}$$

The *NAB* (Part I, p. 55) states that the value (27) was obtained by Mādhava by using a (very accurate) relation, between the circumference and diameter of a circle, which is expressed by the rule *vibudhanetra*, etc. The full Sanskrit stanza[29] giving this rule of Mādhava is quoted by *NAB* at another place (Part I, p. 42) and imply a value of π correct to 11 decimals. Thus we get (27) by using (3) with C equal to 21600 min and a more accurate value of π than (4).

The value (27) when rounded off to the nearest second becomes

$$\begin{aligned} R &= 3437'45'' = 3437.75 \\ &= \frac{13751}{4} \end{aligned} \tag{28}$$

which is implied in another rule (*NAB,* part I, pp. 54–55), given by Mādhava, that contains the Taylor series expansions of the Sine and the Cosine upto the second-order.[30]

We note that, even with adoption of submultiple units of seconds and thirds, the various values of the *Sinus Totus* considered above were obtained by neglecting smaller fractional parts or by rounding off. That is why Brahmagupta in his *BSS*, XXI, 16 (Vol. IV, p. 1626) states that by taking the circumference equal to 21,600 min, we do not get the radius fully represented in terms of minutes or its (sexagesimal) parts; hence the corresponding computed tabular Sines will not be accurate (or exact), and consequently he took a different radius. For his standard table of Sines he took, *BSS*, II, 9 (Vol. II, p. 141.)

$$R = 3270. \tag{29}$$

However, Brahmagupta has not explained as to how he selected the number 3270 for the radius. Whether this number was picked up at random or whether there was some basis for the choice seems to be a difficult problem. Anyway, I may submit the following facts in this connection:

Brahmagupta's *BSS*, XII, 40 (Vol. III, p. 857) gives the rule (2) and *BSS*, XXII, 15 (Vol. IV, p. 1626) gives the rule (22). If we use the very crude approximation

$$\sqrt{10} = 3.3 \tag{30}$$

for this implied value of π, then we get from (3),

$$R = \frac{21600}{6.6} = \frac{36000}{11} = 3273 \quad \text{nearly} \tag{31}$$

which is conveniently near (29). The crude value (30) can be obtained, roughly, by using the approximation

$$\sqrt{N} = \sqrt{a^2 + x} = a + \left(\frac{x}{a}\right) \tag{32}$$

so that

$$\sqrt{10} = \sqrt{3^2 + 1} = 3 + \left(\frac{1}{3}\right)$$
$$= 3.3 \quad \text{roughly.}$$

The unusual gross rule (32) occurs as an intermediate step in the Babylonian Hernian algorithm[31] according to which, if a_1, equal to a, be the first approximation (in defect) to the square-root of N, then

$$b_1 = \frac{N}{a_1} = a + \left(\frac{x}{a}\right)$$

will be the second approximation (in excess), etc.

The gross approximation (32) seems also to be implied in a verbal rule found stated in the *Arithmetic in Nine Sections*, an ancient Chinese work, for finding the fractional part ($\frac{x}{a}$) of the root.[32]

However, these facts and our derivation of the value (31) are not meant here as any possible explanation for the true reason, if any, for Brahmagupta's supposition or imagination (*kalpanā*, according to Pṛthūdaka, his commentator, *BSS*, Vol, IV, p. 1626) of the value (29).

Mahendra Sūri (c. 1370), a court *Paṇḍita* of Firoz Shah Tugalaq, wrote the *Yantrarāja* (=*YR*) which is based on foreign material and is commented upon by the author's own pupil. This work contains a sine table based on the radius[33]

$$R = 3600 = 60^2. \tag{33}$$

Although this value can be easily derived from the relation (3) by using the simplest approximation

$$\pi = 3, \tag{34}$$

it is better to consider the choice of (33) as based on the convenience it provides for calculations with sexagesimal fractions which have been in common use throughout.

A short sine table for the radius

$$R = 1000, \tag{35}$$

which is more suitable for the decimal rather than the sexagesimal system is found[34] in the *Vṛddha-vaśiṣṭha-siddhānta* (=*VVS*), III, 9–10, whose date cannot be given with certainty.

Another conveniently or arbitrarily chosen value, namely,

$$R = 500 \tag{36}$$

is implied in the tabular Sines and their differences which are found in the *Karaṇakaustubha* composed about 1650 by Kṛṣṇa Daivajña.[35]

Still another such value, namely

$$R = 700 \tag{37}$$

was used in *Karaṇa Vaiṣṇava* of Śaṅkara (eighteenth century).[36]

3 Smaller and Miscellaneous Values

For convenience, the Indians also used some smaller values of *Sinus totus* which we take up now.

(I) The following radius was used as early as the sixth century AD

$$R = 120 \tag{38}$$

For this purpose we quote the *PS*, IV-1 which states[37]

षष्टिशतत्रयपरिधेर्वर्गदशांशात्पदं स विष्कम्भः ।
तदिहांशचतुष्कं सम्प्रकल्प्य राश्यष्टभागज्या ॥ १ ॥

> Take the square-root of the tenth part of the square of the circumference (which is) three hundred and sixty (degrees); it is the diameter. Here (in this work), assuming that (that is, the diameter) to be four degrees, the Sines are given at the (interval of) eighth part of a sign.

The first part of the rule gives the relation (22) with C equal to 360°, and the second part implies a radius of 2° or the value (38) in minutes.

The referred tabular Sines are given *PS*, IV, 6–15. But due to a wrong amendation by Thibaut and Dvivedi of the otherwise correct original Sanskrit reading, these two scholars were led to the erroneous value[38]

$$R \sin 90^\circ = 120'1'' \tag{39}$$

instead of (38). The Sine differences given by Brahmadeva (1092) in his *Karaṇaprakāśa*[39] imply the radius (38).

The same radius is used by Bhāskara II for his shorter table of Sines whose differences he has given in his *SSGG*, II, 13 (p. 41) and also in his *Karaṇakutūhala*[40] which was composed about 1183 AD The printed *Marīci* commentary (1638) in the *SSGG* gives a better sine table for the same radius.[41]

The *Yantra-śiromaṇi*[42] of Viśrāma (1615) contains a set of Sines, directly in a tabular form, for the radius (38). A similar table also appears in the *Marīci* commentary (p. 140) on the *SSGG*.

(II) Instead of (38), Brahmagupta took

$$R = 150 \tag{40}$$

for his short sine table. The Sanskrit stanza containing the related tabular differences is given by him in his *Dhyānagrahopadeśa* (verse 16)[43] as well as in his *Khaṇḍa-khādyaka* (=*KK*) III, 6 which[44] was composed about 37 years after his *BSS* (wherein is quoted the first of the above two works).[45]

Another short sine table for the radius (40) is given by Lalla in his *SVGG*, XIII, 2–3, (p. 48.)

(III) The use of the following two values for the *sinus totus* has come to light now.

$$R = 200 \tag{41}$$

$$R = 300 \tag{42}$$

According to Al-Bīrūnī (died 1048)[46] the first value was used by Vijayanandin in his *Karaṇatilaka* and the second by Vitteśvara (=Vaṭeśvara?) in his *Karaṇasāra*.

(IV) It is surprising that the use of the simplest sexagesimal value

$$R = 60 \tag{43}$$

which is used by the Greek astronomer Ptolemy (second century AD) for his table of chords and frequently by medieval Arab authors is found quite late in India. It is used by Kamalākara in his *Siddhānta-tattva-viveka* (=*STV*)[47] which contains sexagesimally five-figured sine table (p. 168).

The work *Samrāt-siddhānta* (1st half of the eighteenth century)[48] is a Sanskrit translation of Ptolemy's *Almagest* made by Jagannātha from an Arabic version. In addition to Ptolemy's table of chords (Vol. I, pp. 30–40), the work contains a Sine table (pp. 55–57) for the same radius (43). The printed edition of the work contains some additional material on trigonometry (apparently by Jagannātha) which is also based on the same value of the radius.

(V) Of the various miscellaneous values, we first take

$$R = 8°8' = 8 + \frac{2}{15} = 488'. \tag{44}$$

This *Sinus totus* is arrived at by interpreting a rule[49] given by Muñjāla (c. 932) in his *Laghumānasam*, II, 12. However, according to another interpretation found[50] stated by Mukhopadhyaya, we have

$$R = 8°11' = 491' \tag{45}$$

instead of (44).

The set of six tabular Sine differences found in the *Vākya-karaṇa* (c. 1300)[51] III, 2–3, implies the value

$$R = 43 \quad \text{parts} \tag{46}$$

which, according to the editors of the work (p. xxi), is obtained from (13) by dividing it by 80 for convenience.

Another peculiar Indian value of the *sinus totus* is

$$R = 191. \tag{47}$$

This was used by Gaṅgādhara (c. 1434) in his *Candramāna*.[52] The two sets of tabular Sines given by Muniśvara in his *Siddhānta-sārva-bhauma* (1946) are also based on the same radius.[53] Although awkward, the value (47) might have been obtained from (13) by removing the simple factors or divisors 2, 3 and 3.

Lastly, we mention that the value

$$R = 24 \tag{48}$$

is stated to be used in the *Karaṇa-Vaiṣṇava* of Śaṅkara (eighteenth century).[54]

We present the various Indian values of the *Sinus totus* in a consolidated form in the accompanying table.

Sl. No.	Sinus totus	Reference
1	2 parts or degrees	*Pañca-siddhāntikā* (c. 550), IV, 1; cf. No. 7 below
2	$8°8'(=488')$	*Laghu-mānasa* of Muñjāla (932)
3	24	*Karaṇa-vaiṣṇava* of Śaṅkara (1766)
4	43	*Vākya-karaṇa* (c. 1300)
5	57°	Old *Sūrya-siddhānta* (?); Āryabhaṭa I's lost work (c. 500); Some *Sūrya-siddhānta* according to Hayata (1764)
6	60	*Siddhānta-tattva-viveka* of Kamalākara (1658); *Samrāt-siddhānta* of Jagananātha (c. 1730). *Siddhānta-rāja* of Nityānanda (1639)
7	120 (min)	*Pañca-siddhāntikā* of Varāhamihira (c. 550); *Karaṇa-prakāśa* of Brahmadeva (1092); *Siddhānta-śiromaṇi* (1150) and *Karaṇakutūhala* (1183) of Bhāskara II; *Yantraśiromaṇi* of Viśrama (1615); *Karaṇendu-śekhara*
8	150	*Dhyāna-grahopadeśa,* (c. 625) and *Khaṇḍa-khādyaka* (665) of Brahmgupta; *Śiṣya-dhīvṛddhida* of Lalla (c. 748)
9	191	*Cāndramāna* of Gaṅgādhara (1434); *Siddhānta-sārva-bhauma* of Munīśvara (1646)
10	200	*Karaṇa-tilaka* of Vijayanandi (before c. 1000)
11	300	*Karaṇa-sāra* of Vitteśvara (c. 900). *Karaṇaratna* of Deva (c. 689 AD)
12	491	*Laghu-mānasa* (cf. No. 2 above) (according to D. N. Mukhopadhyaya)
13	500	*Karaṇa-kaustubha* of Kṛṣṇa-Daivajña (c. 1650)
14	700	*Karaṇa-vaiṣnava* of Śañkara (1766)
15	1000	*Vṛddha-vaśiṣṭha-siddhānta* (undated ?)
16	3270	*Brāhmasphuṭa-siddhānta* of Brahmagupta (628)
17	3415	*Laghu-vasiṣṭha-siddhānta* (undated ?); *Siddhānta- śekhara* of Śripati (c. 1039)
18	$3437'44''$	*Vaṭeśvara-siddhānta* of Vaṭeśvara (904); Parameśvara's commentary (1408) on the *Laghu-bhāskarīya*
19	$3437'44''19'''$	Govindasvāmin's commentary (c. 800–850) on the *Mahābhāskarīya*; cf. No. 25 below
20	$3437+\frac{967}{1309}$	Utpala's commentary (c. 966) on the *Bṛhat-saṃhitā.*
21	$3437'44''48'''$	Sine table Mādhava (c. 1400) quoted by Nīlakaṇṭha (c. 1500) and Śaṅkara Vāriar (1556)
22	$3437'45''$	Implied in a rule of Mādhava which is quoted by Nīlakaṇṭha in his commentary on the *Āryabhaṭīya* (II, 12) and also in his *Tantrasaṅgraha* (II, 10–13)
23	3438	*Āryabhaṭīya* of Āryabhaṭa I (born 476); extant *Sūrya-siddhānta*; *Mahābhāskarīya* of Bhāskara I (c. 625); Works of Sumati (before 950); *Mahā-siddhānta* of Āryabhaṭa II (950?); *Śiṣya-dhīvṛddhida* of Lalla (c. 748); *Siddhānta-śiromaṇi* of Bhāskara II (1150); Puliśa or Pauliśa (?)
24	3600	*Yantra-rāja* of Mahendrasūri (c. 1370). Malayendu in his commentary on *YR*
25	21600	Madanapāla in his commentary of SS (fourteenth century)
26	$12375859'''$	Udayadivākara's commentary (1073) on the *Laghu-bhāskarīya* (cf. No. 19 above)

4 Terminology

From definition it is clear that the Sine of three signs or of 90° arc will be equal to the radius of the circle of reference. Therefore this value of the radius is commonly called *trijyā* which is a short form of the terms like *tri-rāśi-jyā* meaning 'Sine of three signs' literally. Most of the sanskrit terms are based in this interpretation. Few terms which literally mean *sinus totus* (total or complete Sine) and 'greatest Sine' are also used for obvious reasons. All these terms are listed along with at least one reference of their use in Appendix 2.

Beside the listed one, many terms which mean radius or semi-diameter geometrically have been used synonymous to *trijyā*. Conversely *trijyā* has been frequently used for radius of any circle without confirming its use in the sense of 'Sine of three signs'. However, expressions like

त्रिज्याव्यासार्धेन वृत्तं कृत्वा

(– Poona edition of *Marīci on Jyotpatti*, part I, p. 154.)

clearly bring out the distinction between *trijyā* '(Sine of three signs)' and *vyāsārdha* ('semi-diameter' or radius).

The use of such a large number of synonymous terms was partly necessitated by the fact that mathematical rules were to be given in verses which involved definite number of syllables. Of course, the richness of the Sanskrit language easily provided them.

5 Transmission of the Indian Values of the *Sinus Totus*

Below we give a few cases of the use of some Indian values of *Sinus totus* in the works of foreign writers.

For example Yaqūb Ibn Tariq (2nd half of the eighth century) gives the following rules.[55] Radius of the diurnal circle

$$R \cos\delta = 3438 - Vers\ \delta$$

and Sine of ascensional difference

$$= 3438\ \frac{e\ (\sin\delta)}{g(\cos\delta)}$$

where e is the equinoctial noon-day shadow and g is the length of the gnomon. These clearly imply a *sinus totus* which was used in India since, at least, about AD 500. Indian table of 24 Sines for $R = 3438$ min was reproduced in the Chinese *Chiu Chih li* calendar (A. D. 718).[56]

For the smaller *Sinus totus* 150, which was used in India by Brahmagupta (seventh century) and Lalla (eighth century), the following instances may be noted:

1. A radius of 150 min is associated with al-Fazārī (c. 750) by Bīrūnī,[57a] and also with Yaqūb ibn Tāriq (eighth century) and Abū Ma'shar (c. 850).[57b]
2. The same radius is stated to be associated with the Shāh-Zij (c. 790) according to passage given by Bīrūnī.[58]
3. It is stated[59] that the original *Zij* of al-Khwārizmī (c. 840) had a sine table for R equal to 150.
4. The Arab Az-Zarqālī (Arzachel of the Latins), a celebrated astronomer of the eleventh century Spain, also took a radius equal to 150 min.[60a]
5. $R = 150$ is also used in an anonymous Byzantine treatise of (eleventh century).[60b]
6. An anonymous thirteenth century Latin manuscript also assumes the radius of 150 min for Sines.[61]
7. The same radius also appears in a fifteenth century Newminister (England) manuscript.[62]

On the other hand, it may be pointed out that the Greek value 60 for the radius, which was used by Ptolemy (150 AD), is found in India in the *STV* (1658) whose author was familiar with foreign material.

Similarly, the radius 3600, used in the *YR* (c. 1370), is obviously due to Islamic influence.

Appendix 1

The following abbreviations used in the paper for some works.

AB	: *Āryabhaṭīya*, see Ref. 7
AS	: *Āryabhaṭa-siddhānta* (lost). see Ref. 3
BSS	: *Brāhmasphuṭa-siddhānta*, see Ref. 4
KK	: *Khaṇḍa-khādyaka*, see Ref. 44
LB	: *Laghu-bhāskarīya*, see Ref. 23
MB	: *Mahā-bhāskarīya*, see Ref. 13
MS	: *Mahā-siddhānta*, see Ref. 18
NAB	: Nīlakaṇṭha's commentary of the *AB*, see Ref. 28
PS	: *Pañca-siddhāntikā*, see Ref. 37
SS	: *Sūrya-siddhānta*, see Ref. 16
SSGG	: *Siddhānta-śiromaṇi Graha-gaṇita* , see Refs. 5 and 19
SSK	: *Siddhānta-śekhara*, see Ref. 27
STV	: *Siddhānta-tattva-viveka*, see Ref. 47
SVGG TS	: *Tantra-saṅgraha*, see Ref. 28
VS	: *Vaṭeśvara-siddhānta*, see Ref. 21
VVS	: *Vṛddha-vaṣiṣṭha-siddhānta*, see Ref. 23
YR	: *Yantra-rāja*, see Ref. 33

Appendix 2

List of Sanskrit Terms for sinus totus

Antyā or antya-jyā (last Sine)	*Soma-siddhānta*, II, 3; *YR*, I, 42 and its commentary (p. 30)
gagṛha-maurvī	*MS*, III 36. (Note *ga* = 3)
gajyā	*MS*, III, 1 etc.
gabha-maurvī	*MS*, IV, 21
triguṇa	*VVS*, II, 16
trigṛha-guṇa	*MB*, III, 27
trigṛha-jyā	*BSS*, XVI, 11, *KK*, III, 9
trigṛha-maurvī	*Śiṣyadhī-vṛddhida, Golā*, IX, 37
trigṛha-śiñjinī	*SSK*, III, 50
trijaka	*Siddhānta-sārvabhauma*. II, 57
trijīvā	*SS*, II, 28.
trijyā	the most common and popular term
tribhaguṇa or *bhatraya-guṇa*	*SVGG*, IV, 5 and II, 37
tribha-jīvā	*VS*, II, iii, 2
tribha-jyā	*PS, IV*, 5
tribha-maurvī	*VS*, II, i, 69
tribhavana-guṇa	*VS* II, iv, 4
tribhavana-jyā or *bhavanatraya-jyā*	*SVGG*, II, 18 and 20 ; *Karaṇa-prakāśa*, III, 9 (p. 30) uses *bhavana-tritayottha-jīvā*
tribhavanasya-guṇapratānam	*MB*, III, 5
tribhavanasya-jīvā	*MB*, III, 19
tribha-śiñjinī	*SSK*, III, 50
tri-maurvī	*MB*, III, 39 etc., *LB*, III. 12 etc.
trirāśi-guṇa	*SSK*, IV, 119
trirāśi-jīvā	*LB*, III, 29
trirāśi-jyā	*MB*, III, 16
tri-śiñjinī	*SSK*, III, 14
padasamuttha-jīvā (Sine for one quadrant)	*SSK*, III, 63
parama-jyā (great Sine)	*VVS*, II, 8 and II, 41, etc.
paramaśiñjinī	*MS*, III, 2
bhatraya-guṇa	*see* serial No. 13 above
bhavana-traya-jyā	*see* serial No. 18 above
vyāsa (=*trijyā*)	*MB*, III, 20 and 38
vyāsa-khaṇḍa	*MB*, III, 7
vyāsa-khaṇḍa-nicaya	*MB*, III, 20
sakala-guṇa (total Sine or *sinus totus*)	*MB*, II, 10 and III, 27

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20. See Sachau, E. C. (translator), *Alberuni's India.* Two volumes in one, S. Chand, Delhi 1964; Vol. II pp. 69 and 74. Also see Al-Bīrūnī *on Transits.* (Ref. 46 below), pp. 37 and 150.
21. The *VS, spaṣṭādhikāra*, I, 46 (p, 312). Vol. I edited by R. S. Sharma and M. Mishra, Indian Inst. of Astronomical and Sanskrit Research, New Delhi, 1962.
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25. Lucknow University Manuscript (Transcript) No. 46338, pp. 37–44.
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27. The *Siddhānta-śekhara*, III, 11 (p. 140), Edited by Babuaji Misra, Part I, Calcutta, 1932.
28. The *NAB*, Part I, p. 55. Edited by K. Sambasiva Sastri Trivandrum, 1930. Mādhava's sine table is also quoted by Śaṅkara Vāriar (c. 1556) in his commentary on the *Tantrasaṅgraha* (*-TS*), p. 19 ; edited by S. K. Pillai, Trivandrum, 1958. Also *See* Rai, R. N. Ref. 22, p. 11.

29. The same stanza is also quoted in the *Kriyākramakarī* (sixteenth century), edited by K. V. Sarma, Hoshiarpur. 1975. p. 377. Also *see* Sarma, *A History of the Kerala School of Hindu Astronomy*, pp. 25–26 (Hoshiarpur, 1972).
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35. The *Karaṇakaustubha* edited by D. V. Apte, pp. 6–8; Poona, 1927 (Ananda asrama Sanskrit Series No. 96.)
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38. For a correction, see A. K. Bag *Indian J. Hist. Science*, 7, pp. 71–74.
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41. The *SSGG* with *Marīci* edited by K. D. Joshi, Part II, p. 129. B. H. U., Varanasi, 1964.
42. The *Yantra-śiromaṇi*, Ref. 33, pp. 85–87.
43. See *BSS.* Vol. IV, p. 1540.
44. The *KK*, Vol. II. p. 94. Edited and published by Bina Chaterjee, New Delhi, 1970: The statement of Dr. D. A. Somayaji, *A Critical Study of Hindu Astronomy,* p. 4 (Dharwar. 1971), that Brahmagupta took the radius to be 120 is wrong.
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57. [a] Ref. 46, pp. 29, 36, 138.

57. [b] Kennedy, E. S. *J. Near Eastern Studies* 27, (1908), p. 120.
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South Indian Achievements in Medieval Mathematics

1 Introduction

The development of Hindu mathematics did not come to a standstill after the famous Bhāskarācārya or Bhāskara II (circa 1150 AD) although many scholars believed and still believe that.

This is mainly because the historians failed to notice or take cognizance of the findings published by C.M. Whish (as early as 1835),[1] C.T. Rajagopal and others.[2]

Thus for quite some time the view that there was no progress in Indian mathematics and astronomy after Bhāskara II continued to prevail and still prevails in some quarters due to ignorance. Of course one reason for this is that the important works (some of which are in Malayalam and other regional languages) of the post-Bhāskara II period have not been translated into English.

However, nice editions of several important south Indian works (original texts as well as commentaries) are now available. They are *Vyavahāra-gaṇitam* (in Kannada of Rājāditya (c. 1190), a native of Puvina Bage (in North Karnataka); the commentary by Sūryadeva Yajvan (born 1191) of Gangaikonda Colapuram on the *Āryabhaṭīya (= AB);* some works of Mādhava of Saṅgamagrāma near Cochin (c. 1340–1425), and of Parameśvara (between 1360 and 1460) of Alattur village in Kerala; the *Tantrasaṅgraha (= TS)* and *Āryabhaṭīya-bhāṣya (= NAB)* by Nīlakaṇṭha Somayāji (c. 1500) of Kundapura (near Tirur, South Malabar); the Malayalam *Yukti-bhāṣā (= YB)* of Jyeṣṭhadeva (c. 1500–1610); the *Kriyākramakarī (= KKK)* which is an elaborate commentary on Bhāskara's *Līlāvatī* and composed by Śankara Vāriyar (1500–1560) and, after his demise, by Mahiṣamaṅgalam Nārāyaṇa (c. 1530–1610), both of Kerala; and a host of others.

The purpose of the present paper is to give a topic-wise survey of the south Indian achievements in medieval mathematics (twelfth to seventeenth century) based mostly on primary sources.

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2 Decimal Place Value Names

Śrīdhara in his *Pāṭīgaṇita (= PG)* (c. 750 AD), 7–8, gives names of 18 decimal notational places,[3] the last being *Parārdha* which stood for 10^{17}. These names (sometimes with slight variations) are later on given by Śrīpati (c. 1040), Bhāskara II, and Nārāyaṇa Paṇḍita (1356). This shows that the practice of 18 notational place-names became more or less standard (especially in north India).

In south India, however, we find bigger lists of names of notational places. Thus the *Gaṇitasārāsaṅgraha (= GSS)*, 1, 63–68 of Mahāvīra (850) written under the Rāṣṭrakūṭa king Amoghavarṣa (who ruled in the Kanarese area of south), extends the list to 24 places ending with[4]

महाक्षोभं चतुर्नयम् ।

The twenty-fourth (place) is *Mahākṣobha* (= 10^{23}).

Pāvaḷūri Mallikārjuna in his *Gaṇita-śāstra* gives[5] the list up to 36th place which is called *mahābhūri*. The author seems to be same as Pāvaḷūri Mallana (c. 1100), son of Sivanna, who wrote a Telugu version of the *GSS*[6]. But there was another south Indian author called Mallikārjuna Sūri (c. 1178) who composed a Telugu commentary on the *Sūrya-siddhāntā (= SS)* and some other works.[7]

Yallaya (c. 1480) of Skanda-somesvara (S.E. of Srisaila in Kurnool District of Andhra Pradesh) restricts his list to 29 places. He gave the list in his commentary on the *AB*, II, 2 where list of only 10 names occurs.[8]

The biggest south Indian list of notational places is found in the Kannada work, *Vyavahāra-gaṇitam* of the Jaina author Rājāditya (twelfth century). It extends to 40 places and the names are[9]

1 *ekam*	21 *kṣiti*
2 *dāham*	22 *mahākṣiti*
3 *śatam*	23 *kṣobha*
4 *sābira*	24 *mahākṣoba*
5 *dāsābira*	25 *nadī*
6 *lakṣa*	26 *mahānadī*
7 *dālakṣa*	27 *naga*
8 *koṭi*	28 *mahānaga*
9 *dākoṭi*	29 *ratha*
10 *śatakoṭi*	30 *mahāratha*
11 *arbuda*	31 *hari*
12 *nyarbuda*	32 *mahāhari*
13 *kharva*	33 *phaṇi*
14 *mahākharva*	34 *mahāphaṇi*
15 *padma*	35 *kratu*
16 *mahāpadma*	36 *mahākratu*
17 *kṣoṇī*	37 *sāgara*
18 *mahākṣoṇī*	38 *mahāsāgara*

19 *śaṅkha* 39 *parimita*
20 *mahāśankha* 40 *mahāparimita*$= 10^{39}$

3 Geometrico Algebra

Inheriting the tradition of the *Śulba-sūtras,* the full bloom of the elementary geometric algebra is found developed in the works of the late Āryabhaṭa School in South India.

The *Yuktibhāṣā (= YB)*[10] demonstrates geometrically the multiplication identity

$$ab = (a+c)\left[b - \frac{b.c}{(a+c)}\right]$$

for

$$a = 12,\ b = 20,\ \text{and}\ c = 4.$$

If one side of a rectangle is increased arbitrarily, in what proportion the other side should be decreased in order to preserve the area unchanged. The above identity provides the answer to the question. Such questions were relevant to construction of altars.

An algebraic rule from the *Līlāvatī* says that $a^2 \pm b^2 - 1$ will be a perfect square if $a = 8c^4 + 1$, and $b = 8c^3, c$ being arbitrary.

The *Kriyākramakarī (=KKK)* (c. 1534) commentary on this rule presents an interesting graphical demonstration which may be outlined as follows.[11]

The quantity $(8c^4)^2$ can be represented as a square $PQRS$, and $b^2 = (8c^3)^2 = (8c^4)\cdot(8c^2)$ as a rectangle $ABCD$ (see the accompanying Fig. 1) of sides $8c^4$ and $8c^2$.

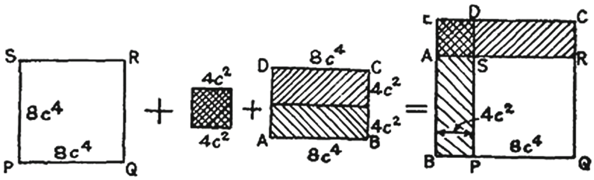

Fig. 1 Geometrical demonstration (case I)

$ABCD$ is divided into two equal rectangular strips each of sides $8c^4$ and $4c^2$. Applying these strips to the square $PQRS$ so as to form a square of side $(8c^4 + 4c^2)$ but with a square of side $4c^2$ remaining unfilled at one corner. However,

$$a^2 = (8c^4+1)^2 = (8c^4)^2 + 16c^4 + 1$$

Hence $(16c^4 + 1)$ is still to be used. The part $16c^4$ can be used to fill the empty corner square of side $4c^2$ needed above. Hence when unity is subtracted from the sum of their squares, the result is a perfect square. Or

अतस्तयोर्वर्गयोगो, व्येको मूलीकर्तव्य एव

as the *KKK* (p. 139) puts it. That is,

$$(8c^4 + 1)^2 + (8c^3)^2 - 1 = (8c^4 + 4c^2)^2.$$

A similar demonstration will hold for the other case (see Fig. 2).

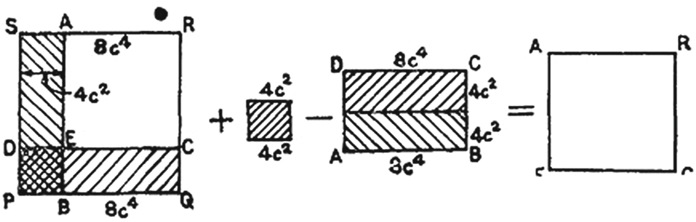

Fig. 2 Geometrical demonstration (case II)

The geometrical demonstration of the formula

$$S = \frac{a(r^n - 1)}{(r - 1)}$$

for the sum of a geometrical progression is more interesting. The *KKK* (pp. 263–264) gives it for the case when r is equal to 4 somewhat as follows.

Let a long rectangular strip $ABCD$ (see the Fig. 3) be taken to represent the $(n+1)$th term, T_{n+1}, of the series. Let it be divided into 3 (i.e. $r - 1$) equal parts.

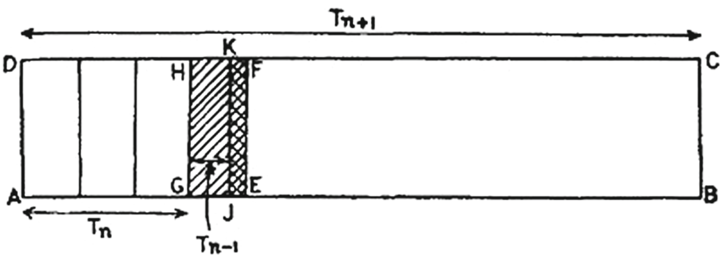

Fig. 3 Sum of a G.P.

One of these parts is $AEFD$ which is further divided into 4 (i.e. r) equal parts. Three (i.e. $r-1$) of these smaller divisions will be equal to

$$\left(\frac{3}{4}\right)\cdot\left(\frac{1}{3}\right)\cdot T_{n+1}=\left(\frac{1}{r}\right)\cdot T_{n+1}=T_n$$

Thus, the rectangle $AGHD$ represents the nth term of the series and the rectangle $GEFH$ is one-third of $AGHD$, just as the rectangle $AEFD$ was one-third of $ABCD$ which represented T_{n+1}.

So we repeat the above process of dividing $GEFH$ into 4 equal parts and choose 3 of them to get the rectangle $GJKH$ to represent T_{n-1}.

The remaining rectangle $JEFK$ can be similarly treated. This process is repeated till first term is reached. The part ultimately left will be thus one-third of T_1, Hence

$$\begin{aligned}\left(\frac{1}{3}\right)T_{n+1} &= \text{rectangle } AEFD\\ &= T_n+T_{n-1}+\ldots+T_1+\left(\frac{1}{3}\right)\cdot T_1\\ &= (\text{sum of } n \text{ terms})+\left(\frac{1}{3}\right).T_1\end{aligned}$$

which gives the required S. With the general common ration r, the above implies

$$\frac{(ar^n)}{(r-1)}=S+\frac{a}{(r-1)}$$

giving the general formula for S.

The enunciation of the general formula for the sum of an infinite geometrically progressing convergent series and its proof in particular cases seems to be first (?) given by Nīlakaṇṭha Somayāji (c. 1500) of Kerala.[12]

4 Solution of Algebraic Equations

Citrabhānu (1475–1550) of the village Covvaram (or Sivapuram) near Trichur in Central Kerala has not yet found place in books on Indian mathematics. His work on solution of equations is quoted in that portion of the KKK which was composed by Śaṅkara Vāriar who not only provides detailed explanations of the rules but speaks as if he had received instructions from the former.

Citrabhānu gave method of solving each of a set of 21 pairs of simultaneous equations in two unknowns, say x and y. The 21 pairs arise by taking, at a time, any two of the following seven quantities as known:[13]

(i) $x+y=a$, say;
(ii) $x-y=b$

(iii) $xy = p$
(iv) $x^2 + y^2 = m$
(v) $x^2 - y^2 = n$
(vi) $x^3 + y^3 = r$
(vii) $x^3 - y^3 = s$

Thus there will be ${}^7C_2 = 21$ cases all of which are dealt by Citrabhānu in his work called *Ekaviṃśati-praśnottara* ("Twenty-one Question-Answers"). They are

1. $x + y = a, \quad x - y = b$
2. $x + y = a, \quad xy = p$
3. $x + y = a, \quad x^2 + y^2 = m$
4. $x + y = a, \quad x^2 - y^2 = n$
5. $x + y = a, \quad x^3 + y^3 = r$
6. $x + y = a, \quad x^3 - y^3 = s$
7. $x - y = b, \quad xy = p$
8. $x - y = b, \quad x^2 + y^2 = m$
9. $x - y = b, \quad x^2 - y^2 = n$
10. $x - y = b, \quad x^3 + y^3 = r$
11. $x - y = b, \quad x^3 - y^3 = s$
12. $xy = p, \quad x^2 + y^2 = m$
13. $xy = p, \quad x^2 - y^2 = n$
14. $xy = p, \quad x^3 + y^3 = r$
15. $xy = p, \quad x^3 - y^3 = s$
16. $x^2 + y^2 = m, \quad x^2 - y^2 = n$
17. $x^2 + y^2 = m, \quad x^3 + y^3 = r$
18. $x^2 + y^2 = m, \quad x^3 - y^3 = s$
19. $x^2 - y^2 = n, \quad x^3 + y^3 = r$
20. $x^2 - y^2 = n, \quad x^3 - y^3 = s$
21. $x^3 + y^3 = r, \quad x^3 - y^3 = s$

The exact solution in 15 of the above 21 cases is more elementary. As a sample, Citrabhānu's rule for solution in the 5th case starts with (KKK, p. 112) the stanza

योगघनाद् घनयोगे त्यक्ते त्रिगुणेन राशियोगेन । शिष्टे हृतेऽथ राश्योर्द्वयोर्भवेदिष्टयोर्घातः ॥

From the cube of the sum, subtract the sum of the cubes and divide the remainder by three times the sum of the quantities. The result is the product of the two quantities.

That is,
$$\frac{\left[(x+y)^3 - (x^3 + y^3)\right]}{3(x+y)} = xy.$$

In this way the problem reduces to case 2 and the solution is easily obtained.

To take a sample of non-elementary type, the second equation in case 6 can be written as
$$(x-y)\left[3(x+y)^2 + (x-y)^2\right] = 4s$$

or, with $x - y = u$, as

$$u(3a^2 + u^2) = 4s$$

The exact solution, therefore, depends on the method of solving a general cubic which does not seem to be discussed by ancient or medieval Indians. Citrabhānu's procedure (*KKK*, p. 112) may be translated as:

> The (given) difference of the cubes (of the unknowns), multiplied by four, be divided by three times the square of the (given) sum (of the unknowns). The quotient (thus determined) is the (estimated) difference of the (desired) quantities, if the remainder (left in the above division) is capable of being cancelled by the cube of the (estimated) difference.

This rule is easily seen to be based on the identity

$$\begin{aligned}\frac{4(x^3 - y^3)}{3(x+y)^2} &= (x-y) + \frac{(x-y)^3}{3(x+y)^2}\\ &= Q + \frac{R}{3(x+y)^2}, \qquad \text{say.}\end{aligned}$$

We shall get the exact result, if, as the commentary states, the remainder R is equal to the cube of the quotient Q. This is illustrated by taking an example in which $a = 25$ and $s = 2375$. We get $(x - y) = 5$ exactly, and so $x = 15$ and $y = 10$.

It can be easily seen that the method will also be useful for the integral values of the involved quantities if

$$(x-y)^3 \text{ is less than } 3(x+y)^2.$$

In other situations, this empirical method will be rather impractical, e.g. when one tries to solve

$$x + y = 11,\ x^3 - y^3 = 999.$$

We have to follow trial-and-error method to get correct solution. Anyway, the effort to discuss all the 21 possible cases indicates south Indian interest in pure mathematics in medieval times.

5 Values of π

The best ancient Indian approximation, as given in the *AB*, II, 10, is[14]

$$\pi = \frac{C}{D} = \frac{62832}{20000}$$

This is later on put by the Andhra Pradesh commentator Yallaya (fifteenth century) in the popular south Indian alphabetic system, called Kaṭapayādi Nyāya, as[15]

रङ्गेहरिस्तुपरिधेर्विष्कम्भोज्ञानिनोनरः ।

The Chinese value $\frac{355}{113}$ for π, which was given by Tsu Chhung-chih (fifth century AD), occurs in India explicitly in the *Tantra-samuccaya* of Nārāyaṇa (fifteenth century),[16] a priest of Travancore for temple construction. The same approximation is given by Nīlakaṇṭha (c. 1500) in his *TS*, II, 7; as well as in his *Golasāra*, III, 12 where the rule is[17]

विश्वैकसमोव्यासः परिधेः प्रायोऽर्थबाणगुणभागः ।

(When) the diameter equals 113, the circumference has 355 parts nearly.

It is interesting to note that the *KKK* (p. 377) quotes a variant reading of a rule from Bhāskara II in order to credit him with the above value.

A very good value is given by Mādhava (c. 1340–1425) in the verse[18]

विबुधनेत्रगजाहिहुताशनत्रिगुणवेदभवारणबाहवः ।
नवनिखर्वमिते वृतिविस्तरे परिधिमानमिदं जगदुर्बुधाः ॥

In a circle of diameter nine *nikharva* (that is, 9×10^{11}), the measure of the circumference is taken to be 2827, 4333, 88233 by the learned.

That is,

$$\pi = \frac{(2827433388233)}{9\times10^{11}}$$

which yields a value correct to 11 decimal places (after rounding off).

When expressed as a continued fraction, Mādhava's value yields the Chinese approximation $\frac{355}{113}$ as its fourth convergent. The six convergent comes

$$\frac{104348}{33215}$$

for which a Sanskrit stanza is quoted in the *KKK* (p. 377) but with a wrong remark that this value is a closer approximation than Mādhava's (*ato'api sukṣmatama*).

The *Karaṇapaddhati (=KP),* VI, 7 composed by Putumana Somayāji (c. 1600–1740) of Sivapuram (Trichur) gives a value of π which is correct to 10 decimal places.[19] It might have been obtained from that of Mādhava.

A value correct to 17 decimal places is found in a late south Indian work, namely, the *Sadratnamālā* of Śaṅkara Varmā (1800–1838) who belonged to a royal house of north Malabar.[20]

Before going on to next topic it may be worthwhile to quote the *NAB* on the incommensurability of π as follows:[21]

> ...इत्येकेनैवमानेन मीयमानयोरुभयोः क्वापि न निरवयवत्वं स्यात् ।
> महान्तमध्वानं गत्वापि अल्पावयवत्वमेव लभ्यम् । निरवयवत्वं तु क्वापि न लभ्यम् ।...
>
> …Hence the two (that is, the diameter and the circumference of a circle) measured by the same unit will never be without remainder. By carrying out the process further, smallness in the remainder may be obtained. But remainderlessness can never be achieved.…

Thus irrationality of π is clearly stated. Series for π are dealt below.

6 Cyclic Quadrilaterals

The cyclic quadrilateral has significant place in the history of Indian mathematics. The concept of the third diagonal is quite interesting and reflects a sort of abstract mathematical thinking. When two adjacent sides of a (cyclic) quadrilateral are interchanged, the length of one diagonal is altered (thereby we get the third diagonal). This area and perimeter preserving property is mentioned by Bhāskara II, and the geometry of three diagonals of a cyclic quadrilateral is discussed in greater details by Nārāyaṇa Paṇḍita (fourteenth century)[22].

The discussion of the three diagonals as found in the KKK is more subtle. It shows that in a cyclic quadrilateral, more than three diagonals are not possible. The arguments given are substantially as follows (KKK, p. 351).

Let $\alpha, \beta, \gamma, \delta$ be the angular measures of the arcs corresponding to the sides a, b, c, d (respectively). Now a sum of any two arcs can be made to define a diagonal. Thus there will be six cases. But because of

$$\alpha + \beta + \gamma + \delta = 360^\circ$$

there will be only three final possibilities (for example, if $\alpha + \beta$ defines one diagonal, $\gamma + \delta$ will define the very same diagonal). Hence only three diagonals are possible.

The KKK also proves the following beautiful generalization of the so-called Ptolemy's theorem:

THEOREM: Let $ABCD'$ be the quadrilateral formed from the cyclic quadrilateral $ABCD$ by interchanging the sides AD and CD. That is, by taking $AD' = CD = c$, and $CD' = AD = d$, and $AB = a$, $BC = b$. If x, y, z denote the three diagonals AC, BD and BD', respectively, then

$$\begin{aligned} yz &= ab + cd \\ zx &= bc + da \\ \text{and} \qquad xy &= ca + bd \end{aligned}$$

We have already published[23] the KKK (pp. 349–351) proof in modern language and notation and pointed out that it is different from Ptolemy's proof.

The KKK (p. 351) then obtains the expressions for the three diagonals from the above theorem. The resulting expressions for the diagonals AC and BD were already known to Brahmagupta (628 AD) who[24] gave them in his *Brāhmasphuṭa-siddhānta (= BSS),* XII, 28. These two expressions are considered to be the "most remarkable to Hindu geometry and solitary in its excellence" by a modern historian and one of them was rediscovered in Europe by W. Snell (seventeenth century).[25]

The usual expressions for the area of a cyclic quadrilateral in terms of its sides, namely,

$$S = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

were first stated in BSS, XII, 21 by Brahmagupta who did not, as usual, prove it.[26] And the proof attempted by Gaṇeśa (c. 1545) of Nandigram (Nanded, Gujarat) is found to be incomplete.[27]

A detailed and complete proof is given in the Malayalam YB (pp. 247–257) which may be briefly outlined as follows:

By adding the areas of the triangles BAC and DAC (see Fig. 4) and squaring, we easily get

$$\begin{aligned} S^2 &= \frac{(AC)^2(BE+DF)^2}{4} \\ &= \frac{(AC)^2(BD^2-EF^2)}{4} \end{aligned}$$

Now from the individual triangles BAC and ADC we can show that

$$EM = \frac{(b^2-a^2)}{2 \cdot AC}$$
$$FM = \frac{(c^2-d^2)}{2 \cdot AC}$$

where M is the mid-point of AC. Thus, the *lambanipātāntara* is given by

$$EF = \frac{\left[(a^2+c^2)-(b^2+d^2)\right]}{2.AC}$$

Hence we get

$$S^2 = \left(\frac{1}{4}\right) \cdot (AC)^2 \cdot (BD)^2 - \frac{\left[(a^2+c^2)-(b^2+d^2)\right]^2}{16}$$

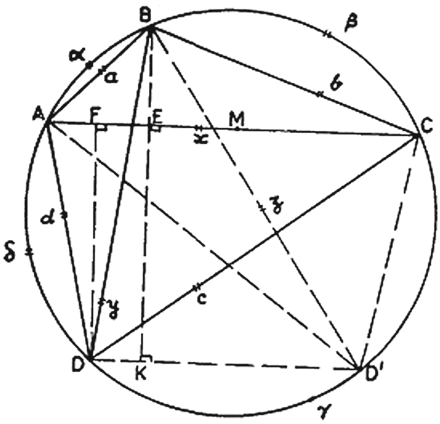

Fig. 4 Area of a cyclic quadrilateral

Using Brahmagupta's expressions for the diagonals AC and BD (or using the expression for xy from the above Theorem) and simplifying, we get the required result in terms of sides only.

Another glorious achievement of medieval south Indian mathematics is the expression for the circum-radius of the cyclic quadrilateral. This is given by Parameśvara in his commentary (before 1432) on the *Lilāvatī* and the rule is also quoted in the KKK (p. 363). The rule is[28]

दोष्णां द्वयोर्द्वयोर्घातयुतीनां तिसृणां वधे । एकैकोनेतरत्र्यैक्यचतुष्केण विभाजिते ॥
लब्धमूलेन यद् वृत्तं विष्कम्भार्धेन निर्मितम् । सर्वं चतुर्भुजं क्षेत्रं तस्मिन्नेवावतिष्ठते ॥

The three sums of the products of the sides taken two at a time are to be multiplied together and divided by the tetrad formed by diminishing one (of the sides) at a time form the sum of the other three. If a circle is drawn with the square-root of the quotient (just obtained) as semi-diameter, the whole quadrilateral figure will be located therein.

That is

$$R = \sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(b+c+d-a)(c+d+a-b)(d+a+b-c)(a+b+c-d)}}$$

This formula for the circum-radius was rediscovered in Europe about 350 years later on by S. A. J. L'Huilier (c. 1780).[29] The KKK (pp. 364–365) has an elegant derivation. Applying a well-known result to triangle BDD' (see Fig. 4), it gets

$$\text{perpendicular } BK = \frac{yz}{2R}$$

Also

$$S = \left(\frac{1}{2}\right) BK.x$$

Hence

$$R = \frac{xyz}{4S}$$

which was already known to Nārāyaṇa Paṇḍita (c. 1356) and from which the required result follows.[30]

7 Sine Tables

The AB, I, 10 contains differences corresponding to a set of 24 tabular Sines ($h = 225$ minutes) for the radius[31]

$$\begin{aligned} R &= \frac{21600}{2\pi} \\ &= 3438 \text{ to nearest minute,} \end{aligned}$$

which can be derived by using the value of π given in the AB itself (see above). The fractional parts beyond minutes are either left out or rounded off. The fractional parts consisting of (sexagesimal) seconds and thirds are given by a south Indian, Govindasvāmin (c. ninth century), who lived in a place which had a latitude of about 10° (in his gloss on the *Māhā-Bhāskarīya*).[32] Parameśvara in his commentary (1408) on the *Laghubhāskarīya* quotes a sine table in which the *sinus totus* is 3437; 44 minutes.[33]

Employing a better value of π, Mādhava (c. 1400) obtained, from the above relation, the value

$$R = 3437;44,48 \text{ (to nearest second).}$$

Mādhava's tabular Sines given in the *Kaṭapayādi* system are contained just in six couplets which are quoted in NAB (Part I, p. 55) and in a commentary on TS.[34] We reproduce the Sines in the accompanying Table.

Table: Mādhava's Mahājyā $\left(R = \dfrac{21600}{2\pi} \text{ and } h = 225'\right)$

(*Note:* Here ळ, ळा, that is, *ḷa, ḷā,* denote 9.)

n	in Devanāgarī	in Transliteration	Implied $R\sin nh$
1	श्रेष्ठं नाम वरिष्ठानाम्	*śreṣṭhaṃ nāma variṣṭhānām*	0224; 50, 22
2	हिमाद्रिर्वेदभावनः	*himādrirveda bhāvanaḥ*	0448; 42, 58
3	तपनो भानुसूक्तज्ञो	*tapano bhānusūktajño*	0670; 40, 16
4	मध्यमं विद्धि दोहनम्	*madhyamaṃ viddhi dohanam*	0889; 45, 15
5	धिगाज्यो नाशनं कष्टम्	*dhigājyo nāśanaṃ kaṣṭam*	1105; 01, 39
6	छन्नभोगाशयाम्बिका	*channabhogāśayāmbikā*	1315; 34, 07
7	मृगाहारो नरेशोऽयम्	*mṛgāhāro nareśo'yam*	1520; 28, 35
8	वीरो रणजयोत्सुकः	*vīro raṇajayotsukaḥ*	1718; 52, 24
9	मूलं विशुद्धं नाळस्य	*mūlaṃ viśuddhaṃ nāḷasya*	1909; 54, 35
10	गानेषु विरळा नराः	*gāneṣu viraḷā narāḥ*	2092; 46, 03
11	अशुद्धिगुप्ता चोरश्रीः	*aśuddhiguptā coraśrīḥ*	2266; 39, 50
12	शङ्कुकर्णो नगेश्वरः	*śaṅkukarṇo nageśvaraḥ*	2430; 51, 15
13	तनुजो गर्भजो मित्रम्	*tanujo garbhajo mitram*	2584; 38, 06
14	श्रीमानत्र सुखी सखे	*śrīmānatra sukhī sakhe*	2727; 20, 52
15	शशी रात्रौ हिमाहारो	*śaśī rātrau himāhāro*	2858; 22, 55
16	वेगज्ञः पथि सिन्धुरः	*vegajñaḥ pathi sindhuraḥ*	2977; 10, 34
17	छायालयो गजो नीलो	*chāyālayo gajo nīlo*	3083; 13, 17
18	निर्मलो नास्ति सत्कुले	*nirmalo nāsti satkule*	3176; 03, 50
19	रात्रौ दर्पणमभ्राङ्गम्	*rātrau darpaṇamabhrāṅgam*	3255; 18, 22
20	नगस्तुङ्गनखो बली	*nagastuṅganakho balī*	3320; 36, 30
21	धीरो युवा कथालोलः	*dhīro yuvā kathālolaḥ*	3371; 41, 29
22	पूज्यो नारीजनैर्भगः	*pūjyo nārījanairbhagaḥ*	3408; 20, 11
23	कन्यागारे नागवल्ली	*kanyāgāre nāgavallī*	3430; 23, 11
24	देवो विश्वस्थली भृगुः	*devo viśvasthalī bhṛguḥ*	3437; 44, 48

8 Second-Order Differences of Sines

It can be easily seen that

$$D_n - D_{n+1} = F.S_n \tag{1}$$

where (for positive integral values of n)

$$\begin{aligned} S_n &= R\sin nh \\ D_1 &= S_1 \\ D_{n+1} &= S_{n+1} - S_n \end{aligned}$$

and F is the proportionality factor, a constant independent of R and n. The relation (1) expresses the simple property that second-order differences of sines are proportional to sines themselves, a fact which was known to Indians from almost the beginning of their trigonometry, e.g. see the AB, II, 12 and the SS, II, 15–16 (according to the interpretations of Mallikārjuna Sūri and Rāmakṛṣṇa), *Golasāra*, III, 13–14, and NAB (Part I, p. 53).[35] The TS, II, 4 gives the value of the reciprocal of F as 233.5 and the commentator thereof even gives this as

$$233 + \frac{32}{60}$$

which is almost equal to the actual value.[36]

More details of the rule (1) are given in NAB (part I, 48–53) which contains a beautiful simple geometrical proof of it. We have already published the proof in modern language and notation[37] but a condensed mathematical version may be given as follows:

From the similar triangles OAQ and NKM (see Fig. 5), we have

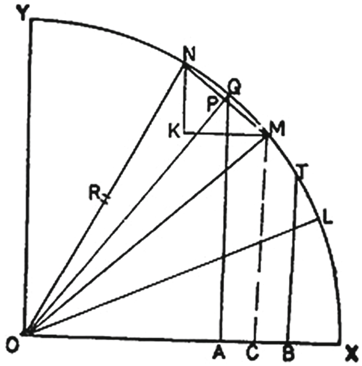

Fig. 5 Second order Sine-difference

$$NK = \frac{OA \cdot MN}{OQ} \tag{2}$$

and

$$MK = \frac{QA \cdot MN}{OQ} \tag{3}$$

That is,

$$D_{n+1} = R\cos\left(nh + \frac{h}{2}\right) \cdot \frac{(\text{crd } h)}{R} \tag{4}$$

and

$$E_{n+1} = R \sin\left(nh + \frac{h}{2}\right) \cdot \frac{(\text{crd } h)}{R} \tag{5}$$

where

$$C_n = R \cos nh$$
$$E_{n+1} = C_n - C_{n+1}$$

and

$$E_1 = C_1; \text{ arc } MX = nh; \text{ arc } MN = h;$$
$$\text{crd } h = \text{ chord of arc } h$$

Now, using (2),

$$D_n - D_{n+1} = \frac{OB \cdot (\text{crd } h)}{R} - \frac{OA \cdot (\text{crd } h)}{R}$$
$$= \frac{(\text{crd } h)(OB - OA)}{R}$$

But by (5) applied to the elemental arc TQ, we get

$$OB - OA = \frac{(\text{Sine at the middle of } TQ) \cdot (\text{crd } h)}{R}$$
$$= \frac{MC \cdot (\text{crd } h)}{R}$$

Hence

$$D_n - D_{n+1} = \frac{(\text{crd } h)^2 MC}{R^2}$$
$$= \frac{(\text{crd } h)^2 \cdot (S_n)}{R^2}$$

which proves the required property (1).

9 Addition and Subtraction Theorems for Sine

Although these were already known to Bhāskara II, in the Āryabhaṭa School of South India these theorems, called *Jīveparaspara-nyāya*, were attributed to the famous Mādhava (c. 1400) who gave two forms, namely[38]

$$R\sin(A \pm B) = \frac{(R\sin A)\cdot(R\cos B)}{R} \pm \frac{(R\cos A)\cdot(R\sin B)}{R} \tag{6}$$

and

$$R\sin(A \pm B) = \sqrt{(R\sin A)^2 - (lamba)^2} \pm \sqrt{(R\sin B)^2 - (lamba)^2}$$

where

$$lamba = \frac{(R\sin A)(R\sin B)}{R}$$

Geometrical proofs of the theorems are found in the NAB (Part I, pp. 58–61) and the Malayalam YB (pp. 206–208, 212–213, 237–238). We have already brought out these in modern notation in a paper[39] from which one proof (YB, 237–238) may be briefly extracted as follows:

Referred to Fig. 6, arc $PE = A$; arc $PQ = B$

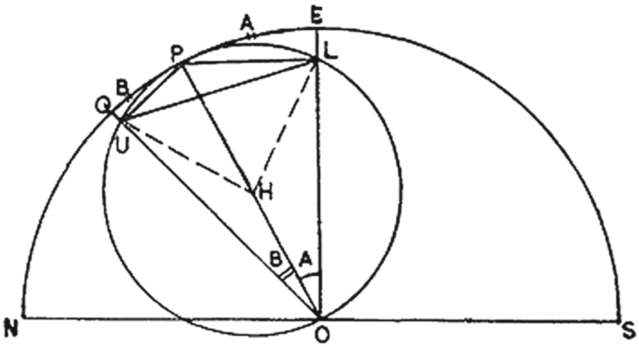

Fig. 6 Proof of addition theorem for Sine

Thus PL and OL are Sine and Cosine of A, and PU and OU those of B. By applying the *bhujā-pratibhujā* rule (i.e. Ptolemy's theorem) to the cyclic quadrilateral $LPUO$, we get

$$PL\cdot OU + OL\cdot PU = LU\cdot OP$$

or

$$(R\sin A)\cdot(R\cos B) + (R\cos A)\cdot(R\sin B) = LU\cdot R.$$

Now LU is the full chord of the arc $(LP + PU)$ in the circle which circumscribes the quadrilateral $LPUO$ whose radius is $\frac{R}{2}$. Thus

$$\begin{aligned} LU &= 2\cdot\left(\frac{R}{2}\right)\cdot\sin\left\{\frac{(2A+2B)}{2}\right\} \\ &= R\sin(A+B) \end{aligned}$$

Hence the theorem follows from the above relation.

10 Solution of Spherical Triangles

Consider the spherical triangle SPZ on the celestial sphere, where S, P, Z are the positions of the sun, north pole and the zenith. It has following five useful elements:

(i) side PZ, the co-latitude, $90^\circ - \phi$
(ii) side ZS, the co-altitude, $90^\circ - \alpha$
(iii) side SP, the co-declination, $90^\circ - \delta$
(iv) angle SPZ, the hour angle, H
(v) angle PZS, the azimuth (measured from the north), $A = 90^\circ \pm \gamma$.

When any three out of the above five elements are given, the remaining two can be found out. This leads to ten different cases all of which are discussed systematically in the TS (Chapter III).[40] The rules are fully explained and the examples worked out in the commentary of TS.[41]

The solutions given are equivalent to what one will get by using modern spherical trigonometry. We have already shown this in our detailed paper.[42] Here we take, as a sample, the case II, namely, given H, A and ϕ, to find α and δ. For finding the altitude TS, III, 68–71, says:

> The product of the Sine of the hour angle and Cosine of the latitude divided by the radius is the Sine of the local hour angle. Divide the product of the Sines of the hour angle and the latitude by the Cosine of the local hour angle and multiply by the Sine of the directional amplitude (azimuthal angle measured from the east). The result should be added to, in case the directional amplitude is towards north, or subtracted from, in case the directional amplitude is towards south, the product of their *'koṭis'* (i.e. their uprights when they are taken as bases and the radius is taken as the hypotenuse in each case). The result (now obtained) be divided by the radius and the quotient (thus obtained) multiplied by the Cosine of the local hour angle be put separately (at two places).
>
> At one place divide (the quantity) by the radius and add the square (of the result) to the square of the sine of the local hour angle. By the square root of that (the sum of the squares just obtained) divide the quantity (placed) separately. (The final result) becomes the Gnomon (the Sine of the altitude)....

That is

$$\text{Sine of the local hour angle} = \frac{(R\sin H)\cdot(R\cos\phi)}{R} = J, \text{ say}$$

$$\text{Cosine of the local hour angle} = \sqrt{R^2 - J^2} = C, \text{ say}$$

Then we form the quantity

$$\left(\frac{C}{R}\right)\cdot\left[R\cos\gamma\cdot\sqrt{R^2 - \left\{\frac{(R\sin H)\cdot(R\sin\phi)}{C}\right\}^2} \pm \frac{R\sin\gamma\cdot(R\sin H)\cdot(R\sin\phi)}{C}\right] = Q, \text{say}$$

The rule then gives

$$R\sin\alpha = \frac{Q}{\sqrt{J^2 + \left(\frac{Q}{R}\right)^2}}.$$

By combining the various steps, the solution given can be seen to be equivalent to

$$\sin\alpha = \frac{\sin A \cdot \cos H + \cos A \cdot \sin H \cdot \sin\phi}{\sqrt{(\sin H \cdot \cos\phi)^2 + (\sin A \cdot \cos H + \cos A \cdot \sin H \cdot \sin\phi)^2}}$$

which is a transformed form of the following result obtained by using the modern cotangent formula of spherical trigonometry:

$$\tan\alpha \cdot \cos\phi = \sin A \cdot \cot H + \cos A \cdot \sin\phi.$$

The complicated Indian solution does not make use of tangent and cotangent functions. Another complication results by deriving the rules from working 'inside' the celestial sphere instead of 'on its surface'.

We have brought out these complicated rules into modern forms but the more challenging task of publishing, in modern forms, their ancient rationales from the south Indian elaborate commentary of TS still remains to be done.[43]

11 Series for π

Several such series were known to medieval south Indians. We shall give a few of them here. One verse, which is found in the *Gaṇita-Yuktibhāṣā (= GYB),* the *Yuktidīpikā* commentary or *Vyākhyā* on TS $(= TSV)$, and in the KP, says:[44]

व्यासाद्वारिधिनिहतात् पृथगाप्तं त्र्याद्ययुग्विमूलघनैः ।
त्रिघ्नव्यासे स्वमृणं क्रमशः कृत्वापि परिधिरानेयः ॥

The circumference is likewise obtained when four times the diameter is (successively) divided by the cubes of the odd numbers, beginning with 3, diminished by these numbers themselves, and the (respective) quotients are alternately added to, and subtracted from, the diameter multiplied by three.

That is,

$$C = \pi D = 3D + \frac{4D}{3^3 - 3} - \frac{4D}{5^3 - 5} + \frac{4D}{7^3 - 7} - \dots$$

Another Sanskrit text says[45]

The fifth powers of odd numbers are increased by 4 times themselves; 16 times the diameter is successively divided by all such numbers (so obtained); the results (of division) of odd rank are added and those of even rank are subtracted. The circumference corresponding to the diameter is (thereby) obtained.

That is

$$\pi D = \frac{16D}{1^5 + 4 \cdot 1} - \frac{16D}{3^5 + 4 \cdot 3} + \frac{16D}{5^5 + 4 \cdot 5} - \dots$$

Now to give a sample of finite series, we have Sanskrit stanzas which contain series[46]

$$\pi D \approx 4D - \frac{4D}{3} + \frac{4D}{5} - \ldots \mp \frac{4D}{n} \pm 4D \cdot f(n)$$

where

$$f(n) = \frac{\frac{(n+1)}{2}}{(n+1)^2 + 1}$$

or, more accurately,

$$f(n) = \frac{[\frac{(n+1)}{2}]^2 + 1}{[(n+1)^2 + 5]\frac{(n+1)}{2}}.$$

These two closer approximations are also stated in the sixteenth century Malayalam work YB which is said to have given a third form of the closing term but as an intermediate step (see $AHES$, Vol 18, pp. 89–90; cf. ref. 44). All these can be put in an alternative modern form as

$$\frac{\pi}{4} \approx 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \pm \frac{1}{n} \mp f_i(n+1)$$

where

$$f_1(n) = \frac{1}{(2n)}$$

$$f_2(n) = \frac{\left(\frac{n}{2}\right)}{(n^2+1)}$$

and $$f_3(n) = \frac{\left(\frac{n^2}{4}\right) + 1}{\left(\frac{n}{2}\right) \cdot (n^2+5)}$$

It is said that Prince Rāma Varma of Cochin had verified, by taking $n = 55$, that π obtained by f_1, f_2 and f_3 is correct to 1, 6 and 10 decimal places, respectively.

Rounding off $f_i(n)$ enormously improves slowness of convergence of the series used. It is therefore rightly remarked that apparently "we have lost some wider theory of accelerating the convergence of singly infinite series by applying such corrections as have now come down to us in medieval Hindu texts the TS and YB" (*AHES*, Vol. 35, p. 99).

12 Series for Sine and Cosine

For the Sine series we quote the following stanzas (nos. 440–441) from the TSV of Saṅkara Vāriar (1st half of sixteenth century)[47].

निहत्य चापवर्गेण चापं तत्तत्फलानि च ।
हरेत् समूलयुग्वर्गैस्त्रिज्यावर्गहतैः क्रमात् ॥
चापं फलानि चाधोऽधोऽन्यस्योपर्युपरि त्यजेत् ।
जीवाप्त्यै संग्रहोऽस्यैवं 'विद्वान्' इत्यादिनाकृतः ॥

The arc is to be repeatedly multiplied by the square of itself and is to be divided (in order) by the square of each even number increased by itself and multiplied by the square of the radius. The arc and the terms obtained by these repeated operations are to be placed in sequence in a column and any last term is to be subtracted from the next above, the remainder from the term then next above, and so on, to obtain the Sine of the arc. It was this procedure which was briefly mentioned in the verse starting with *'Vidvān'*.

That is,

$$\sin s = s - s \cdot \frac{s^2}{(2^2+2)r^2} + s \cdot \frac{s^2}{(2^2+2)\cdot r^2} \cdot \frac{s^2}{(4^2+4)\cdot r^2} - \ldots$$

The interesting reference to another set of rules starting with *'Vidvān'* points out to a pair of famous stanzas which contain the rule[48]

$$\sin s = s - t^3[Q_5 - t^2[Q_4 - t^2\{Q_3 - t^2(Q_2 - t^2 Q_1)\}]]$$

where

$$\begin{aligned} t &= \frac{s}{h} \\ h &= 5400;0,0 \\ R &= 3437;44,48 \\ Q_1 &= 0;00,44 \\ Q_2 &= 0;33,06 \\ Q_3 &= 16;05,41 \\ Q_4 &= 273;57,47 \\ Q_5 &= 2220;39,40. \end{aligned}$$

The important point to note is that the *'Vidvān'* stanzas are attributed to the famous Mādhava (c. 1400) of Saṅgamagrāma (near Cochin) in the *NAB* (Part I, p. 113). This shows that the power series expansion of sine was known already in India more than 200 years before it was known in Europe.

There are corresponding stanzas[49] for the cosine series which are given side by side with the above rules.

South Indian medieval derivations of these series have been published and to reproduce them here will take us too far.[50]

13 The Mādhava-Gregory Series

Three and a half Sanskrit couplets containing series for inverse tangent function have been translated as:[51]

> The product of the given Sine and the radius divided by the Cosine is the first result. From the first, (and then, second, third), etc., results, obtain (successively) a sequence of results by taking repeatedly the square of the Sine as the multiplier and the square of the Cosine as the divisor. Divide (the above results) in order by the odd numbers one, three, etc., (to get the full sequence of terms). From the sum of the odd terms, subtract the sum of the even terms. (The result) becomes the arc. In this connection, it is laid down that the (Sine of the) arc or (that of) its complement, whichever is smaller, should be taken here (as the 'given Sine'); otherwise, the terms, obtained by the (above) repeated process will not tend to the vanishing-magnitude.

That is,

$$R\theta = \frac{R\cdot(R\sin\theta)^1}{1\cdot(R\cos\theta)^1} - \frac{R\cdot(R\sin\theta)^3}{3\cdot(R\cos\theta)^3} + \frac{R\cdot(R\sin\theta)^5}{5\cdot(R\cos\theta)^5} - \dots$$

or

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots.$$

The Sanskrit text translated is found in the KKK as well as TSV both of which belong to the first half of sixteenth century while James Gregory knew the series about 1670 in Europe.

But there are reasons to believe that the series was known still one more century earlier to Mādhava (c. 1340–1425). First, the KKK (p. 379) expressly attributes to him certain Sanskrit stanzas which contain the series[52]

$$\pi D = 4D - \frac{4D}{3} + \frac{4D}{5} - \dots \mp \frac{4D}{n} \pm 4D\cdot f(n)$$

where (see Sect. 11)

$$f(n) = \frac{\frac{(n+1)}{2}}{(n+1)^2+1}.$$

This shows that Mādhava knew the Gregory series for $\theta = \frac{\pi}{4}$. Also the YB (pp. 113–116) proves both these (the general and the particular cases) without making any distinction between them by essentially same approach.[53]

This south Indian proof starts with a geometrical derivation of a rule which is basically equivalent to

$$d\theta = \frac{d(\tan\theta)}{1+\tan^2\theta}.$$

Then it consists of steps which are equivalent to expansion and term-by-term integration in modern analysis. Since the details of the proof (from the Malayalam YB) which belongs to pre-calculus period have been already brought out in modern notation, it is not necessary to reproduce it here.[54] The Indian proofs of $\sin x, \cos x$, and $\tan^{-1} x$ show that the seeds of modern analysis were first sown in India.

14 Taylor Series for the Sine and Miscellaneous Matters

Rules equivalent to second-order Taylor series expansions of the sine and the cosine were known to Mādhava whose Sanskrit stanzas are given in the *TS*, II, 10–13 (p. 112) and quoted in the *NAB* (Part I, pp. 54–55).[55]

A sort of third-order Taylor series expansion of the sine is implied in the following verses which are quoted anonymously by Parameśvara of Kerala (c. 1360–1460) in his supercommentary on the *Mahābhāskarīya*.[56]

दोश्चापखण्ड-संभक्तं व्यासार्धं भाजको भवेत् ।
दोर्ज्यामतीतचापान्ते कोटिज्यां च न्यसेत् पुनः ॥
कोटिज्यातो भाजकेन लब्धस्यार्धेन संयुतात् ।
दोर्गुणाद्, भाजकाप्तार्धं हित्वा कोटिगुणात् पुनः ॥
तस्मादाप्तं भाजकेन दोर्ज्याखण्डः स्फुटो भवेत् ।
चापान्तदोर्ज्या तद्युक्ता स्यादिष्टज्या भुजोद्भवा ॥

The semi-diameter divided by the residual arc becomes the divisor. Put down the Sine and again the Cosine at the end of the arc traversed. From the Cosine, subtract half the quotient obtained from the divisor-divided Sine (which is) increased by half the quotient obtained from the Cosine by the divisor. Again (the quotient) obtained from that (above difference) by dividing by the divisor becomes the true Sine-difference. The Sine at the end of the arc traversed increased by that (true Sine-difference) becomes the desired Sine for a (given) arc.

That is, let

$$\text{divisor} = \frac{R}{\theta} = D, \text{ say;}$$
$$\text{true Sine-difference} = R\sin(\alpha+\theta) - R\sin\alpha.$$

Then the above rule gives[57]

$$R\sin(\alpha+\theta) = R\sin\alpha + \left[R\cos\alpha - \left\{R\sin\alpha + \frac{R\cos\alpha}{2D}\right\}\frac{1}{2D}\right]\frac{1}{D}$$

which on simplification becomes

$$R\sin(\alpha+\theta) = R\sin\alpha + \left(\frac{\theta}{R}\right)\cdot R\cos\alpha - \left(\frac{\theta}{R}\right)^2 \cdot \frac{(R\sin\alpha)}{2} - \left(\frac{\theta}{R}\right)^3 \cdot \frac{(R\cos\alpha)}{4}.$$

Putting $R = 1$, this becomes the third-order Taylor Series approximation of the sine function except that we have 4 instead of 6 in the last term.

No doubt the Indian form is far from being the true Taylor series expansion of 4 terms; the fact that it was given more than two centuries before Taylor expansion was discovered by Gregory (about 1668) is interesting.[58] Parameśvara's supercommentary, called *Siddhānta-dīpikā*, contains several other rules for computations of trigonometric functions.

The TSV verses 455–456 (p. 119) contain the series for the square of the sine function equivalent to

$$\sin^2 x = x^2 - \frac{x^4}{\left(2^2 - \frac{2}{2}\right)} + \frac{x^6}{\left(2^2 - \frac{2}{2}\right)\left(3^2 - \frac{3}{2}\right)} - \frac{x^8}{\left(2^2 - \frac{2}{2}\right)\left(3^2 - \frac{3}{2}\right)\left(4^2 - \frac{4}{2}\right)} + \dots$$

This topic of *jyā-vargānayanam* is also found treated in the sixteenth century Malayalam work YB (Part I, pp. 203–206) as has been mentioned elsewhere.[59]

Lastly, it may be pointed out that the possibility of a principle to frame the transcendental number e (the base of natural logarithms) in the medieval Indian mathematics has been discussed. More than a decade ago scholars have exemplified the mechanism of the motivational kinetics by examining the possibility that the basic constant e of mathematical analysis would have appeared as a definite object at the horizon of mathematical thinking of medieval India.[60]

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Munīśvara's Traditional Sine Table

1 Introduction

Munīśvara's another name was Viśvarūpa. He was born in 1603 AD and his father was Raṅganātha (a commentator of *Sūryasiddhānta*). He resided at Varanasi and wrote the following works (in Sanskrit) related to Indian astronomy and mathematics (Census of Exact Sciences in Sanskrit, Vol. 4 of Series A, pp. 436–441; Philadelphia, 1981):

1. A commentary called *Marīci* on the *Siddhāntaśiromaṇi* in 1635/1638. It has been published in the edition of various scholars from various institutions from 1908 to 1988.
2. *Siddhānta-sārva-bhauma* (1646) with auto-commentary (1650). The *pūrvārdha* of this has been published in 3 volumes, Varanasi, 1932, 1935, and 1978.
3. A commentary on the famous *Līlāvatī*.
4. *Pāṭīsāra* on traditional arithmetic, algebra, and geometry.
5. *Gaṇitaprakāśa*.
6. Commentary on *Cābukayantra* of Gaṇeśa.
7. *Ekanātha-mukha-bhañjana* on *krāntipātāryātraya* from the *Siddhāntaśiromaṇi* (separate from *Marīci*).

The *Siddhānta-sārva-bhauma* (= SSB) is composed in the style and pattern of other traditional siddhāntic works of India. It has some new things and features. The work is comparable to the *Siddhānta-tattva-viveka* (1657) composed by his rival Kamalākara in some respect.

2 The Sine Table of Munīśvara

The second chapter (called *spaṣṭādhikāra*) of the SSB contains a sine table, with the *Sinus Totus* $R = 191$ and tabular interval of one degree, given verbally in 16 Sanskrit verses (II, 3–18). The classical value of the *Sinus Totus* is $3438 = 2 \times 3 \times 3 \times 191$,

Gaṇita Bhāratī, Vol. 25, Nos. 1–4 (2003), pp. 119–123.

K. Ramasubramanian (ed.), *Gaṇitānanda*,
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and Munīśvara's R comes by omitting the simple factors 2, 3, 3. Another connection may be the value $\pi = \frac{600}{191}$ quoted by him (I. 134) from ancient tradition (Vasiṣṭha, etc.). Described by profusely using word-numerals, the 90 tabular Sines are expressed in a peculiar mixture of the whole parts with their positive or negative fractions (both vulgar and sexagesimal). In effect each verbal value can be expressed in an integral number (*rūpaḥ*) and his sexagesimal part (*adhaḥ-avayava*) called *kalā, liptā,* etc. For example, the 4th and the 8th Sines are:

सरामांशगुणेन्दवः $= 13 + \dfrac{1}{3} = 13;20$ (usual notation);

and तत्त्वकलोनभानि $= 27 - 25' = 26;35$

$= 191 \sin 8^\circ$ (= Sin 8°).

Munīśvara was aware of the inaccuracy of his traditionally described verbal table of 90 Sines. (II. 20). So he also gave a set of 90 Sines in a *direct tabular form* in which each numerical value is conveniently expressed in figures (not words) upto the 3rd sexagesimal part. We described his both sets of Sines in the accompanying table. In expressing numerical values, the usual notation for sexagesimal system (fractions) is followed here, i.e. $a, b, c, d,$ ......stands for

$$a + \frac{b}{(60)} + \frac{c}{(60)^2} + \frac{d}{(60)^3} + \ldots$$

We have made a few minor changes and corrections in the text and table found in the printed edition of the *SSB* (Vol. I, Varanasi, 1932, pp. 119–124). He also gave a *direct table* (*Ibid.,* 125–126) of *Utkramajyās* (Versed Sines) for the same R and interval (h = 1°). This table can be derived from the corresponding table of Sines since

$$\text{Vers}\, n^\circ = R - \text{Cos}\, n^\circ = 191 - \text{Sin}\,(90 - n)^\circ.$$

This relation helped in checking some tabular entries.

Arc in deg.	Verbal value of the Sine	Symbolic form	Sexagesimal form	From his 4-fig sine table
1	सत्र्यंशरामाः *satryaṃśarāmāḥ*	$3 + \frac{1}{3}$	3; 20	3; 20,00,17
2	त्रिलवोनितागाः *trilavonitāgāḥ*	$7 - \frac{1}{3}$	6; 40	6; 39,56,54
3	दिशः *diśaḥ*	10	10; 0	9; 59,46,12
4	सरामांशगुणेन्दवः *sarāmāṃśaguṇendavaḥ*	$13 + \frac{1}{3}$	13; 20	13; 19,24,33

5	वित्र्यंशमेघाः *vitryaṃśameghāḥ*	$17-\frac{1}{3}$	16; 40	16; 38,48,17
6	खयमाः *khayamāḥ*	20	20; 0	19; 57, 53,46
7	स्वपादगुणाश्विनः *svapādaguṇāśvinaḥ*	$23+\frac{1}{4}$	23; 15	23; 16,37,22
8	तत्त्वकलोनभानि *tattvakalonabhāni*	$27-25'$	26; 35	26; 34,55,25
9	त्रिंशन्नगाधोऽवयवोनिताः *triṃśannagādho'vayavonitāḥ*	$30-7'$	29; 53	29; 52, 44,20
10	सषडंशदेवाः *saṣaḍaṃśadevāḥ*	$33+\frac{1}{6}$	33; 10	33; 10,00,29
11	सभा अङ्गगुणाः[1] *sabhā aṅgaguṇāḥ*	$36+27'$	36; 27	36; 26,40,16
12	मेघलिप्तोनशून्याब्धिमिताः *meghaliptonaśūnyābdhimitāḥ*	$40-17'$	39; 43	39; 42,40,5
13	त्रिवेदाः *trivedāḥ*	43	43; 0	42; 57,56,21
14	सपञ्चांशतर्काब्धयः *sapañcāṃśatarkābdhayaḥ*	$46+\frac{1}{5}$	46; 12	46; 12,25,30
15	षड्विलिप्तायुताङ्काब्धयः[2] *saḍdviliptāyutāṅkābdhayaḥ*	$49+26'$	49; 26	49; 26,3,58
16	त्र्यंशहीनाग्निबाणाः *tryaṃśahīnāgnibāṇāḥ*	$53-\frac{1}{3}$	52; 40	52; 38,48,5
17	वियद्भागषड्पञ्च *viyadbhāgaṣaḍpañca*	$56+0$	56; 0	55; 50,34,15
18	गोऽक्षाः *go'kṣāḥ*	59	59; 0	59; 1,20,6
19	सषड्भागपक्षारयः *saṣaḍbhāgapakṣārayaḥ*	$62+\frac{1}{6}$	62; 10	62; 11,00,40

[1] The reading found in the printed version of the paper is *sabhānyaṅgaguṇāḥ*. According to this reading, the word has to be split as *sabhāni + aṅgaguṇāḥ*. This is grammatically incorrect, as the gender of the qualifier is neuter, whereas that of the qualified in masculine. Hence the reading has been altered as above. It may also be added here that the printed version of the text *SSB* edited by Gopinatha Kaviraja, and published Benares—as a part of Saraswati Bhavana Texts, No. 41, in 1935 (p. 119)—presents the reading as सभात्र्यङ्गगुणाः, which is obviously incorrect. – *Editor*.

[2] Here again the reading षड्विलिप्ता युताङ्काब्धयः presented in the printed version of the text *SSB* (cited in the previous footnote), is incorrect. The Sanskrit phrase quoted by Prof. RC Gupta in his paper, as well as the numerical values decoded by him contained inaccuracies. These inaccuracies have been rectified above. – *Editor*.

20	त्र्यंशयुक्तेषुतर्काः *tryaṃśayukteṣutarkāḥ*	$65+\frac{1}{3}$	65; 20	65; 19,33,4
21	सभा अष्टरसाः *sabhā aṣṭarasāḥ*	$68+27'$	68; 27	68; 26,53,48
22	सदेवकलाक्वगाः *sadevakalākvagāḥ*	$71+33'$	71; 33	71; 32,59,30
23	सार्द्धकलाब्धिसप्त *sārddhakalābdhisapta*	$74+\frac{1}{2}(74')$	$74; 37'$	74; 37,14,53
24	वित्र्यंशनागाद्रिमिताः *vitryaṃśanāgādrimitāḥ*	$78-\frac{1}{3}$	77; 40	77; 41,12,7
25	घनोनकलाङ्कवर्गः *ghanonakalāṅkavargaḥ*	$9^2-17'$	80; 43	80; 43,12,19
26	वितदब्धिनागाः *vitadabdhināgāḥ*	$84-17'$	83; 43	83; 43,44,00
27	मेघोनलिप्ताद्रिगजाः *meghonaliptādrigajāḥ*	$87-17'$	86; 43	86; 42,43,52
28	विरामभागाभ्रनन्दाः *virāmabhāgābhranandāḥ*	$90-\frac{1}{3}$	89; 40	89; 40,8,39
29	जिनलिप्तिकोनाः त्र्यङ्काः *jinaliptikonāḥ tryaṅkāḥ*	$93-24'$	92; 36	92; 35,55,6
30	शराङ्काः सदलाः *śarāṅkāḥ sadalāḥ*	$95+\frac{1}{2}$	95; 30	95; 30,00,00
31	सजातिलिप्ताष्टनन्दाः *sajātiliptāṣṭanandāḥ*	$98+22'$	98; 22	98; 22,20,11
32	सशरांशभूदिक् *saśarāṃśabhūdik*	$101+\frac{1}{5}$	101; 12	101; 12,52,29
33	वेदाभ्रचन्द्राः *vedābhracandrāḥ*	104	104; 0	104; 1,33,48
34	विशरांशसप्तदिशः *viśarāṃśasaptadiśaḥ*	$107-\frac{1}{5}$	106; 48	106; 48,21,2
35	तलत्र्याग्नियुताङ्कदिक्काः *talatryāgniyutāṅkadikkāḥ*	$109+33'$	109; 33	109; 33,11,9
36	साङ्घ्र्यर्कचन्द्राः *sāṅghryarkacandrāḥ*	$112+\frac{1}{4}$	112; 15	112; 16,1,8
37	विनखांशपञ्चेशाः *vinakhāṃśapañceśāḥ*	$115-\frac{1}{20}$	114; 57	114; 56,48,1
38	सप्तरुद्राः सशराग्निलिप्ताः *saptarudrāḥ saśarāgniliptāḥ*	$117+35'$	117; 35	117; 35,28,50

39	सपञ्चांशखार्काः *sapañcāṃśakhārkāḥ*	$120+\frac{1}{5}$	120; 12	120; 12,00,41
40	विपञ्चांशरामाद्विचन्द्राः *vipañcāṃśarāmādvicandrāḥ*	$123-\frac{1}{5}$	122; 48	122; 46,20,46
41	सरामांशपञ्चद्विचन्द्राः *sarāmāṃśapañcadvicandrāḥ*	$125+\frac{1}{3}$	125; 20	125; 18,26,11
42	विपञ्चांशनागार्कसंख्याः *vipañcāṃśanāgārkasaṃkhyāḥ*	$128-\frac{1}{5}$	127; 48	127; 48,14,12
43	सपादखरामेन्दवः *sapādakharāmendavaḥ*	$130+\frac{1}{4}$	130; 15	130; 15,42,4
44	त्र्यंशहीनाग्निविश्वे *tryaṃśahīnāgniviśve*	$133-\frac{1}{3}$	132; 40	132; 40,47,6
45	सनखांशाक्षविश्वे *sanakhāṃśākṣaviśve*	$135+\frac{1}{20}$	135; 03	135; 3,26,37
46	सजिनलिप्तागविश्वकाः *sajinaliptāgaviśvakāḥ*	$137+24'$	137; 24	137; 23,38,3
47	वित्र्यंशखेन्द्राः *vitryaṃśakhendrāḥ*	$140-\frac{1}{3}$	139; 40	139; 41,18,48
48	तिथ्यंशहीनद्वीन्द्राः *tithyaṃśahīnadvīndrāḥ*	$142-\frac{1}{15}$	141; 56	141; 56,26,23
49	कृतेन्द्रकाः साङ्गांशाः *kṛtendrakāḥ sāṅgāṃśāḥ*	$144+\frac{1}{6}$	144; 10	144; 8,58,18
50	सत्रिभागाङ्गशक्राः *satribhāgāṅgaśakrāḥ*	$146+\frac{1}{3}$	146; 20	146; 18,52,10
51	तत्त्वकलाधिकाः गजेन्द्राः *tattvakalādhikāḥ gajendrāḥ*	$148+25'$	148; 25	148; 26,5,34
52	सार्द्धखाक्षक्ष्माः *sārddhakhākṣakṣmāḥ*	$150+\frac{1}{2}$	150; 30	150; 30,36,12
53	द्वितिथ्योऽधो रदैर्युताः *dvitithyo'dho radairyutāḥ*	$152+32'$	152; 32	152; 32,21,47
54	सार्द्धाब्धितिथ्यः *sārddhābdhitithyaḥ*	$154+\frac{1}{2}$	154; 30	154; 31,20,5
55	सभलिप्तिकाङ्गतिथ्यः *sabhaliptikāṅgatithyaḥ*	$156+27'$	156; 27	156; 27,28,57
56	सरामांशगजाक्षचन्द्राः *sarāmāṃśagajākṣacandrāḥ*	$158+\frac{1}{3}$	158; 20	158; 20, 46,14
57	पञ्चांशयुक्ताभ्रनृपाः *pañcāṃśayuktābhranṛpāḥ*	$160+\frac{1}{5}$	160; 12	160; 11,7,53

58	द्विभूपाः *dvibhūpāḥ*	162	162; 0	161; 58,37,52
59	घनोनलिप्ता युगतर्कचन्द्राः *ghanonaliptā yugatarkacandrāḥ*	$164-17'$	163; 43	163; 43,8,14
60	सतत्त्वलिप्ताक्षनृपाः *satattvaliptākṣanṛpāḥ*	$165+25'$	165; 25	165; 24,58,10
61	नखांशयुक्तागभूपाः *nakhāṃśayuktāgabhūpāḥ*	$167+\frac{1}{20}$	167; 03	167; 3,8,31
62	त्रिलवेन हीनाः नवाङ्गचन्द्राः *trilavena hīnāḥ navāṅgacandrāḥ*	$169-\frac{1}{3}$	168; 40	168; 38,34,46
63	सशरांशखात्यष्टयः *saśarāṃśakhātyaṣṭayaḥ*	$170+\frac{1}{5}$	170; 12	170; 10,56,5
64	त्रिभागोनयमागचन्द्राः *tribhāgonayamāgacandrāḥ*	$172-\frac{1}{3}$	171; 40	171; 40,10,47
65	दिगंशयुक्ताग्निघनाः *digaṃśayuktāgnighanāḥ*	$173+\frac{1}{10}$	173; 06	173; 6,14,3
66	दलाढ्याब्ध्यगेन्दवः *dalāḍhyābdhyagendavaḥ*	$174+\frac{1}{2}$	174; 30	174; 29,13,51
67	व्यङ्गलवाङ्गमेघाः *vyaṅgalavāṅgameghāḥ*	$176-\frac{1}{6}$	175; 50	175; 48,59,8
68	दिगंशयुक्तागघनाः *digaṃśayuktāgaghanāḥ*	$177+\frac{1}{10}$	177; 06	177; 5,31,37
69	सरामांशाष्टागचन्द्राः *sarāmāṃśāṣṭāgacandrāḥ*	$178+\frac{1}{3}$	178; 20	178; 18,49,54
70	सदलाङ्कमेघाः *sadalāṅkameghāḥ*	$179+\frac{1}{2}$	179; 30	179; 28,53,42
71	जिनोनलिप्तेन्दुधृतिः *jinonaliptendudhṛtiḥ*	$181-24'$	180; 36	180; 35,38,14
72	त्रिभागहीनाश्व्यहिक्ष्माः *tribhāgahīnāśvyahikṣmāḥ*	$182-\frac{1}{3}$	181; 40	181; 39,6,28
73	त्रिलवेन हीनाः त्र्यष्टेन्दवः *trilavena hīnāḥ tryaṣṭendavaḥ*	$183-\frac{1}{3}$	182; 40	182; 39,15,9
74	सिद्धकलोनवेदाष्टक्ष्माः *siddhakalonavedāṣṭakṣmāḥ*	$184-24'$	183; 36	183; 36,3,33
75	दलाढ्याब्धिगजेन्दवः *dalāḍhyābdhigajendavaḥ*	$184+\frac{1}{2}$	184; 30	184; 29,30,35
76	सत्र्यंशपञ्चाष्टभुवः *satryaṃśapañcāṣṭabhuvaḥ*	$185+\frac{1}{3}$	185; 20	185; 19,35,20

77	दिगंशाढ्याङ्गाष्टचन्द्राः *digaṃśāḍhyāṅgāṣṭacandrāḥ*	$186 + \frac{1}{10}$	186; 06	186; 6,16,51
78	रसभागहीनाः सप्ताष्टचन्द्राः *rasabhāgahīnāḥ saptāṣṭacandrāḥ*	$187 - \frac{1}{6}$	186; 50	186; 49,34,17
79	सदलागनागचन्द्राः *sadalāganāgacandrāḥ*	$187 + \frac{1}{2}$	187; 30	187; 29,26,51
80	दिगंशाढ्यगजाष्टचन्द्राः *digaṃśāḍhyagajāṣṭacandrāḥ*	$188 + \frac{1}{10}$	188; 06	188; 5,53,49
81	त्र्यंशोननन्दधृतयः *tryaṃśonanandadhṛtayaḥ*	$189 - \frac{1}{3}$	188; 40	188; 38,54,30
82	अष्टकलाढ्यगोऽष्टचन्द्राः *aṣṭakalāḍhyago'ṣṭacandrāḥ*	$189 + 8'$	189; 08	189; 8,20,29
83	वितत्त्वकलिकाभ्रनवेन्दुसंख्याः *vitattvakalikābhranavendusaṃkhyāḥ*	$190 - 25'$	189; 35	189; 34,34,44
84	खाश्व्यंशहीनखनवेन्दुमिताः *khāśvyaṃśahīnakhanavendumitāḥ*	$190 - \frac{1}{20}$	189; 57	189; 57,13,15
85	सपादखाङ्केन्दवः *sapādakhāṅkendavaḥ*	$190 + \frac{1}{4}$	190; 15	190; 16,23,28
86	रदकलाढ्यनभोऽङ्कचन्द्राः *radakalāḍhyanabho'ṅkacandrāḥ*	$190 + 32'$	190; 32	190; 32,5,2
87	पादोनभूनवभुवः *pādonabhūnavabhuvaḥ*	$191 - \frac{1}{4}$	190; 45	190; 44,17,40
88	विदशांशभूगोचन्द्राः *vidaśāṃśabhūgocandrāḥ*	$191 - \frac{1}{10}$	190; 54	190; 53,1,8
89	यमोनकलिकावनिनन्दचन्द्राः *yamonakalikāvaninandacandrāḥ*	$191 - 2'$	190; 58	190; 58,15,17
90	भूनन्दभूपरिमिताः *bhūnandabhūparīmitāḥ*	191	191; 0	191; 00,00,00 $= R$

Part VIII

Bio-bibliographical Sketches of Some Historians of Mathematics

Prabodh Chandra Sengupta (1876–1962): Historian of Indian Astronomy and Mathematics

1 Introduction

Prabodh Chandra Sengupta, the younger son of Ram Chandra Sengupta, was born in a village near Tangail in Mymensingh district (now in Bangladesh) on 21 June 1876. He had his early education in the Santosh Jahnavi H. E. School and passed the Entrance (Matric) examination with sufficient merit to obtain a scholarship. Subsequently he studied in Calcutta passing the First Arts (Intermediate) examination from the Presidency College, the B. A. examination with first class honours in Mathematics from the General Assembly's Institution, and the M. A. examination in Mathematics from the Presidency College in 1901.

Professor Prabodh Chandra Sengupta (1876–1962)

Gaṇita Bhāratī, Vol. 1, (1979), pp. 31–35.

K. Ramasubramanian (ed.), *Gaṇitānanda*,
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Professor Sengupta entered the educational service under the Government of Bengal in July 1902 and worked as a teacher in various government schools till 1914. Several renowned scholars like M. N. Saha and R. C. Majumdar were his students during their school days in Dacca.

Shortly after passing the B. T. examination, Prof. Sengupta was appointed as a Lecturer in Mathematics at the Chittagong College in 1914. Later on in 1916, he joined the Bethune College, Calcutta, which he served till his retirement from the government service in January 1934. He was made Professor of Mathematics in 1921 under Bengal Educational Service.

2 Research Contributions

Professor Sengupta is best known for his researches and publications in the field of Indian astronomy and chronology which date from 1916 and lasted for a long period of 40 years. He also delivered lectures in Indian Mathematics and Astronomy at the Calcutta University. The arrangement of teaching Indian Astronomy (2 papers) and Indian Mathematics (2 papers) in M. A. Course (Group IV) was there under the University Department of Ancient Indian, History and Culture (see *Journal of Ancient Indian History,* Vol. II, 1968–1969, p. 3).

Besides translating (1927a) the *Āryabhaṭīyam* of Āryabhaṭa I (born 476 AD). Professor Sengupta gave us his famous translation (1934) and edition (1941) of Brahmagupta's *Khaṇḍakhādyaka* (665 AD). These two parts were dedicated to Sir Ashutosh Mukherjee (1864–1924), the Founder of Research Studies in the University of Calcutta, who had nicely utilized the handsome donation from Maharaja Manindra Chandra Nandy of Cossimbazar for the promotion of researches in the domain of ancient Indian Mathematics and Astronomy. Professor Sengupta got inspiration also from others like Professor Ganesh Prasad (1876–1935), Hardinge Professor of Pure Mathematics, Calcutta University, whose two students, B. Datta (1888–1958) and A. N. Singh (1901–1954), turned out be famous historians of Indian Mathematics. Professor Sengupta's numerous papers on various aspects of ancient Indian Mathematics and Astronomy including comparison with Greek methods are the result of his deep research and labour. His introductions attached to his translation of *Khaṇḍakhādyaka*, to the Calcutta edition (1935) of Burgess's translation of the *Suryasiddhānta* and to B. Misra's edition of the *Siddhāntaśekhara* (see [1944/47]) are equally valuable.

By applying the so-called 'astronomical method', Professor Sengupta determined the dates of a number of events and works related to Indian history, culture and civilization and published several papers on the subject. At the suggestion of Professor M. N. Saha, FRS, Professor Sengupta submitted a scheme of research work to the Calcutta University which was duly approved. Mr. Nirmal Chandra Lahiri worked as a research assistant in the scheme which was carried out from 1939 to 1941. The result is the famous work *Ancient Indian Chronology* (Calcutta 1947) which reflects profound knowledge of Astronomy, Mathematics and Sanskrit.

Professor Sengupta was the President of the Technical Sciences Section of the XIIth All-India Oriental Conference (Benares, 1944). His publications continued to come out when he was well over 80 years. He died in Calcutta on 6 August 1962 leaving his widow, five sons and three daughters and grandchildren to mourn his loss. In his passing, India lost a pioneer worker in the field of ancient Indian exact sciences. A very good way to cherish his work and memory will be to bring out in a book form a collection of his numerous papers on Indian Mathematics and Astronomy.

3 Bibliography of P. C. Sengupta

The following abbreviations are used:

BCMS = *Bulletin of the Calcutta Math. Society*.
JASP[L] = *Journal of the Asiatic Society of Bengal (Letters).* Was called *Journal of the Royal Asiatic Society of Bengal earlier.*
JDL/JDS = *Journal of the Department of Letters/Science* (University of Calcutta).
YB = *Year Book of the Asiatic Society of Bengal*.

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Note: Prof. Sengupta also authored numerous contributions on Indian chronology in Bengali which appeared in Bengali periodicals like *Sri Bharati, Bharatavarsa*, etc.

Acknowledgements I am grateful to Dr. P. C. Sengupta, M. B., D. Phil, son of Professor P. C. Sengupta, for supplying valuable information and material for the present article.

Bibhutibhusan Datta (1888–1958): Historian of Indian Mathematics

Born to a poor Bengali family, Bibhutibhusan Datta (1888–1958) was indifferent to worldly pleasures and gains. He never married. His doctoral thesis was on hydrodynamics, but he is best known for his work on the history of mathematics. He retired voluntarily from the University of Calcutta at the age of 45 and in 1938 took *sanyāsa* (literally, renunciation) to become known as Swami Vidyāraṇya. He also wrote on Indian religion and philosophy.

Bibhutibhusan Datta (1888–1958), Sohn einer armen Bengali-Familie, war an weltlichen Vergnügungen und Reichtümern nicht interessiert und blieb unverheiratet. Er promovierte auf dem Gebiet der Hydrodynamik, doch erwarb er sich vor allen Dingen durch sein werk über die Geschichte der indischen Mathematik grosse Verdienste Im Alter von 45 Jahren zog er sich aus freier Entscheidung von der Universität Calcutta zurück, nahm 1938 sanyāsa an und wurde unter dem Namen Swami Vidyāraṇya bekannt. Er schrieb auch über indische Religion und Philosophie.

– বিভূতি ভূষণ দত্ত (১৮৮৮ – ১৯৫৮) এক
দরিদ্র বঙ্গ পরিবারে জন্মগ্রহণ করিয়া-
ছিলেন। পার্থিব সুখ বা লাভের প্রতি
তাঁহার আসক্তি ছিলনা। তিনি চিরকুমার
ছিলেন। হাইড্রোডিনামিকসে ডক্টরেট
ডিগ্রী করিলেও ভারতীয় গণিতের ইতিহাসে
উল্লেখনীয় অবদানের জন্য তিনি বিখ্যাত।
তিনি ৪৫ বৎসর বয়সে কলিকাতা বিশ্ব-
বিদ্যালয় হইতে স্বেচ্ছায় অবসর গ্রহণ
করেন। পরে ১৯৩৮ খৃষ্টাব্দে সন্ন্যাস গ্রহণ
করিয়া স্বামী বিদ্যারণ্য নামে পরিচিত হন।
তিনি ভারতীয় ধর্ম ও দর্শন সম্বন্ধেও লিখিয়াছেন।

Historia Mathematica 7 (1980), pp. 126–133. A Bengali translation appeared in *Ganit Charcha* 8(1–2) (1989), pp. 73–78.

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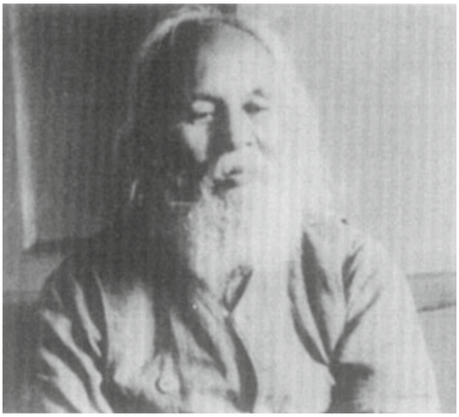

Dr. Bibhutibhusan Datta D.Sc. (Swami Vidyāraṇya), 1888–1958

Bibhutibhusan Datta was born on Thursday, June 28, 1888, in Kanungoyapara village of Chittagong (now in Bangladesh) [1]. His father, Rasikacandra Datta (1854–1926), was a poor but honest and religious man who worked in the office of the subjudge. His mother, Muktakeśī Datta (1861–1958), was a kind person, rendering help to her neighbours during the famine of 1943. Bibhutibhusan was their third son (of 11) and, like his brothers, was handsome [2].

From the beginning Bibhutibhusan exhibited saint-like qualities. Even as an adolescent, he told his parents that he would never marry and would become a *sanyāsī* (one who renounces the world and worldly pleasures in order to realize *Brahman*, the Infinite Self). During his school days, he wore nothing more than *kopīna* (loincloth). He studied religious books and was deeply influenced by the writings of Ramakrishna and Vivekananda. He had an analytical view of philosophy, even from his youth.

In 1907, B. Datta passed the matriculation examination of Calcutta University and was awarded a scholarship. He received a Bachelor of Science degree at the Presidency College (under Calcutta University) in 1911–1912. In November of 1913 (a few months before his master's examination), he left home (very likely he intended to become *sanyāsī*) and was reported missing. His eldest brother found him in Haridwar (near the Himalayas) and brought him home, taunting him that he had run away for fear of failing the examination. But Datta said he would receive a first class, which in fact he did in 1914 when he passed the master's examination in Mixed Mathematics (a course which involved pure as well as applied mathematics).

Thereafter he was awarded a scholarship to do mathematical research at Calcutta University. (He had already written a paper before the results of the master's examination were known.) He was appointed lecturer in Mixed Mathematics at the University Science College, where he taught planetary theory. Later he was awarded the Premchand Roychand Scholarship and the Elliot Prize. After defending his thesis on hydrodynamics (circa 1920), he received the degree of Doctor of Science.

Datta always retained his religious beliefs and saint-like nature. He completed critical studies and reviews of the *upaniṣads* and other philosophical works, always remaining aloof from worldly pursuits. So unconcerned was he for personal gain that when the Rashbehari Ghosh Professorship of the Science College fell vacant and was offered to him, he rejected the honour, saying: “After a couple of days I shall become *sanyāsī* (and so) I have no need for the promotion.” However, since no other suitably qualified person was available for the post, he took the assignment and carried out the job successfully for three years without accepting any additional allowance (Dutt 1963, 6).

Throughout his life, Datta was a follower of Shankaracharya, the Vedantin, and believed in the *Viśuddha-advaitavāda* (pure non-dualism). He was initiated in 1920, and his *guru* was Swami Viṣṇutīrtha Mahārāja. Datta never ate meat in his life. His serenity was not disturbed by worldly pains and sorrows, not even by the death of his father in 1926.

In 1923, Ganesh Prasad (then Principal of the Central Hindu College, Benares) was appointed Hardinge Professor of Higher Mathematics at Calcutta University [3] who deserves credit for creating interest in the history of mathematics in India. He was himself a historian of mathematics, and two of his students, A. N. Singh and B. Datta, became pioneers in the history of Hindu mathematics.

Datta’s writings in the history of mathematics date from the mid-1920s. He delivered an address, “Contribution of the Ancient Hindus to Mathematics,” to the Allahabad University Mathematical Association on December 20, 1927. This was published in Volumes I and II of the Association’s Bulletin and later became the nucleus of Datta’s major work (with A. N. Singh), *History of Hindu Mathematics* (1935–38).

By now Datta had become even more uninterested in the routine work of a professional mathematician, and in 1929 he resigned from Calcutta University (Jones 1976, 77). However, his retirement was brief. In 1931, by special invitation, he delivered the readership lectures at Calcutta University. Datta states (1932a, viii) that he delivered these lectures on “The Science of the *Śulba*” in deference to the wish of his teacher (Professor Ganesh Prasad). In April of 1932, he wrote a short review of the edition of *Siddhānta-śekhara* edited by Babua Misra (1932, 521).

But Datta was not anxious to remain at the University, and in the Preface (dated July 28, 1932) to *The Science of the Śulba* (1932a, viii) he writes, “I tender grateful thanks to Mr. A. C. Ghatak, Superintendent, and to the staff of Calcutta University Press for kindly expediting publication of the book in order to help me to go back to my retirement earlier.” In 1933, at the age of 45, he finally retired from the University (Dutt 1972, 8). He then became an itinerant wanderer, moving here and there, living on the charity of others, and was seen at the Dharmasindhu Ashram during March of 1934 (Jones 1976, 77–78). During this period he continued to write on the history of mathematics.

In 1938, Datta took *sanyāsa* (Dutt 1934, 14). As a *sanyāsin* he became known as Swami Vidyāraṇya. In the same year Part II of his *History of Hindu Mathematics* appeared [4]. In the last years of his life, Swami Vidyāraṇya lived mainly at Puṣkara (in Rajasthan). On March 24, 1958, his mother died, and his own death followed shortly thereafter, on October 6.

1 Bibliography of Bibhutibhusan Datta

The work done by B. Datta falls into three distinct categories according to subject matter. These three categories—namely Applied Mathematics, History of Mathematics, and Religion and Philosophy—also mark the three phases or periods of his work.

1.1 Applied Mathematics

Datta's papers in this category are devoted to hydrodynamics, which was his area of research as a university scholar. As a sample we cite his paper, "On the Periods of Vibrations of a Straight Vortex Pair," (1921), *Proceedings of the Benares Mathematical Society* 3, 13–24.

These writings, completed before 1925, represent the first phase of his work. They are little known, and since they contain no historical treatment, we will not list them here. For a list of seven published and a dozen unpublished papers on Applied Mathematics, see *Pioneer Mathematicians of Calcutta University* (Kolkata, 2014), pp. 40–42.

1.2 History of Mathematics

This is the most well-known category of Datta's writings. His reputation rests mainly on original research in this area. These publications date from 1925 to 1947. They are listed below in chronological order, with the following abbreviations used:

AMM	*American Mathematical Monthly.*
BCMS	*Bulletin of the Calcutta Mathematical Society.*
B. S.	*Bengali San* (i.e. Bengali Year).
BSPP	*Baṅgīya Sāhitya Pariṣad Patrikā* (Calcutta).
IHQ	*Indian Historical Quarterly* (now defunct).
JA	*Jaina Antiquary* (Arrah).
JASB	*Journal of the Asiatic Society of Bengal* (Calcutta).
PBMS	*Proceedings of the Benares Mathematical Society.*

[1925–26] "Al-Bīrūnī and the origin of Arabic numerals," *PBMS* 7–8, 9–23.
1926a "A note on Hindu-Arabic numerals," *AMM* 33, 220–221.
1926b "Two Āryabhaṭas of Al-Bīrūnī," *BCMS* 17, 59–74.
1926c "Hindu (non-Jaina) values of π," *JASB* (n. s.) 22, 25–42.
1926–31 "Early literary evidence of the use of zero in India," *AMM* 33, 449–454, and 38, 566–572.
1927a "On *mūla*, the Hindu term for 'root," *AMM* 34, 420–423.
1927b "On the origin and development of the idea of percent," *AMM* 34, 530–531.
1927c 'Āryabhaṭa, the author of *Gaṇita*," *BCMS* 18, 5–18.
1927d "Early history of the arithmetic of zero and infinity in India," *BCMS* 18, 165–176.
1927e "The present mode of expressing numbers," *IHQ* 3, 530–540.
1927f "The Hindu method of testing arithmetical operations," *JASB* (n. s.) 23, 261–267.
1927–29 "Hindu contributions to mathematics," *Bulletin of the Allahabad University Mathematical Association,* 1, 49–73, 2, 1–36. Part one reprinted, 12, 1–36.
1928a "The science of calculation by the board," *AMM* 35, 520–529.
1928b "The Hindu solution of the general Pellian equation," *BCMS* 19, 87–94.
1928–29a "On Mahāvīra's solution of rational triangles and quadrilaterals," *BCMS* 20, 267–294 (1930).
1928–29b "*Śabda-saṃkhyā-praṇālī*" (The word numeral system) (in Bengali), *BSPP* for B. S. 1335, 8–30.
1929a "The Bakshali mathematics," *BCMS* 21, 1–60.
1929b "The Jaina School of mathematics," *BCMS* 21, 115–145.
1929c "The scope and development of Hindu Gaṇita," *IHQ* 5, 479–512.
1929d "A short review of G. R. Kaye, *The Bakhshali Manuscript—A study in Mediaeval mathematics* (Calcutta, 1927)," *Bulletin of the American Mathematical Society* 35, 579–580.
1929–30 "*Akṣara-saṃkhyā-praṇālī*" (Alphabetic numeral system) (in Bengali), *BSPP* for B. S. 1336, 22–50.
1929–31a "Origin and history of the Hindu names for geometry," *Quellen und Studien zur Geschichte der Mathematik*, B I, 113–119 (1930).
1929–31b "Geometry in Jaina cosmography," *Quellan und Studien zur Geschichte der Mathematik*, B I, 245–254.
1930a "On the supposed indebtedness of Brahmagupta to *Chiu-chang Suan-shu,*" *BCMS* 22, 39–51.
1930b "The two Bhāskaras," *IHQ* 6, 727–736.
1930c "On the Hindu names for rectilinear geometrical figures," *JASB* (n.s) 26, 283–290.
1930–31a "*Nāma-saṃkhyā*" (Nominal numerals) (in Bengali), *BSPP* for B. S. 1337, 7–27.
1930–31b "*Jaina-sāhitye nāma-saṃkhyā*" (Nominal numerals in Jaina literature) (in Bengali), *BSPP* for B. S. 1337, 28–29.
1930–31c "*Aṇkānāṃ vāmato gatiḥ*" (in Bengali), *BSPP* for B. S. 1337, 70–80.
1931a "On the origin of the Hindu terms for 'root," *AMM* 38, 371–376.
1931b "The origin of Hindu indeterminate analysis," *Archeion* 13, 401–407.
1931c "Nārāyaṇa's method for finding approximate value of a surd," *BCMS* 23, 187–194.
1931d "Early history of the principle of place value," *Scientia* 50, 1–12.
1932a *The Science of the Śulba,* Calcutta.
1932b "Elder Āryabhaṭa's rule for the solution of indeterminate equations of the first degree," *BCMS* 24, 19–36.
1932c "Testimony of early Arab writers on the origin of our numerals," *BCMS* 24, 193–218.
1932d "On the relation of Mahāvīra to Śrīdhara," *ISIS* 17, 25–33.
1932e "Introduction of Arabic and Persian mathematics into Sanskrit literature," *PBMS* 14, 7–21.

1933	"The algebra of Nārāyaṇa." *Isis* 19, 472–485.
1933a	"Ancient Bengali Astronomer: Mallikārjuna Sūri", *BSPP* for B. S. 1340, 83–94; Reprinted in Proceedings of the Radhanagar Seminar on History of Mathematics, Radhanagar, 2013, 18–29.
1933–34	"Ācārya Āryabhaṭa and his disciples and followers" (in Bengali), *BSPP* for B. S. 1340, 129–158.
1935	"Mathematics of Nemicandra," *JA* 1, 25–44.
1935–36	"Āryabhaṭa and the theory of the motion of the earth" (in Bengali), *BSPP* for B. S. 1340, 167–183.
1935–38	*History of Hindu Mathematics. A Source Book* (with A. N. Singh). Parts I, II, Lahore, 1935, 1938. Reprinted as two parts in one, Bombay, 1962. Part I translated into Hindi by K. S. Shukla, Lucknow 1956 (2nd ed. 1963). For part III, see note [4] below.
1936	"A lost Jaina treatise on arithmetic," JA 2, 38–41.
1937	"Vedic Mathematics," *In Cultural Heritage of India,* Vol. III, pp. 378–401, Calcutta.
1938–39	"Application of indeterminate analysis to astronomical problems," *Archeion* 21, 28–34.
1941	"Chronology of the history of science in India during sixteenth century," *Archeion* 23, 78–83.
1946–47	"Some instruments of ancient India and their working," *Journal of the Ganganatha Jha Research Institute* 4, 249–270.
1983–1987	*Prācīna Hindu Jyotisī* (in Bengali). Published serially in *Ganit Charcha* from Vol. 2, No.1 (March 1983) to Vol. 6, No. 4 (Dec.1987).

1.3 Philosophy and Religion

Datta's writings on the historical aspects of Indian philosophical and religious system form the third and the last phase of his work. They are not widely known (doubtless because they are all in Bengali). Two works published posthumously are:

1963–66	*Bhāgavata-dharmer prācīna itihāsa* (Ancient History of Bhagavata Religion) (in Bengali). 4 vols., Calcutta.
1972	*Advaitavāder prācīna kahānī* (Ancient Story of Advaita Philosophy) (in Bengali). Vol. I, Calcutta. Other parts not published.

Datta left a work on Jaina philosophy and another on Buddhist philosophy in addition to several incomplete essays (Dutt 1963, 11).

2 Notes

1. The date of birth as recorded in the local system is Thursday, the 15th of *Āṣāḍha* in Śaka 1810 (Dutt 1963, 5), which is taken to be the 28th of June, 1888. This Gregorian date of birth is also recorded in the list of members of the *comité Internationale d' Histoire des Sciences* of May 15, 1931 (Jones 1976, 78). The

citation on a plate included in some of the copies of (1963–66) of June 29, a Friday, as Datta's birth date is incompatible with the officially recorded date.

2. The names of the ten brothers, in order of seniority, are: Rebati Raman Dutt, Bhupati Mohan Dutt, Nirode Lal Dutt, Binode Behari Dutt, (Ex-Registrar and Controller of Examinations, University of Calcutta), Harihar Dutt, Pramatha Ranjan Dutt, Subimal Dutt (former Ambassador to USSR), Sukomal Dutt, Parimal Dutt, and Ranjit Dutt. These names have been kindly communicated by Professor Bhupal K. Dutt (son of Rébati Raman). Bibhutibhusan wrote the family name as Datta, which is the proper transliteration.
3. This was the second time he joined the Calcutta University. From 1914 to 1917 he had been the Rashbehari Ghosh Professor of Applied Mathematics [Narayan 1939, 108].
4. Part III (Geometry, Trigonometry, Calculus, etc.) of the *History of Hindu Mathematics* by Datta and A. N. Singh (died 1954) has never been published although more than 40 years have passed since the appearance of Part II. The information given by the late Binode Bihari Dutt in a personal communication dated September 11, 1966 [also see his (1963, 12)], that Part III has been lost, turned out to be wrong. Manuscripts of Part III exist at Lucknow with Dr. K. S. Shukla (1976, 52) and with the writer (R. C. G.) of the present article who received it from (and due to kindness of) Dr. S. N. Singh (son of A. N. Singh). It is unfortunate that the authors (particularly A. N. S.) could not ensure the publication of Part III (Sinvhal 1954, iii), although they lived long enough after the appearance of Part II to have perhaps done so. It is also unfortunate that when Parts I and II were reprinted, no attempt was made to bring the work up-to-date. Part III is expected to appear shortly, in serialized form, in the *Indian Journal of the History of Science.*

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6. Narayan, L., 1939. Ganesh Prasad. *PBMS* (n. s.) I, 107–114.
7. Shukla, K. S., 1976. (ed.) *Aryabhaṭīya* of Aryabhaṭa. Critically edited with translation. New Delhi: Indian National Science Academy. (In collaboration with K. V. Sarma.).
8. Sinvhal, S. D., 1954, Dr. Avadhesh Narain [sic] Singh (a life sketch). *Gaṇita* 5 (2), i–vii.

Review of Pingree's Census of Exact Sciences in Sanskrit (1992)

[Series A, Volume 4, by David Pingree; American Philosophical Society, Philadelphia, 1981 (*Memoirs of the A.P.S.,* Vol. 146); Pages 447, List Price U.S. $ 30.00.]

While reviewing C. N. Srinivasiengar's *The History of Ancient Indian Mathematics* (Calcutta 1967), the late E. B. Allen of New York has remarked that "an adequate history of mathematics of India can be written only after a comprehensive survey has been made of manuscripts available in Sanskrit and vernacular languages, the dates and authors of manuscripts, and the studies bearing on their place in the history of mathematics" (*Math. Reviews,* Vol. 36, p. 269). S. N. Sen's *A Bibliography of Sanskrit Works on Astronomy and Mathematics, Part-I* (New Delhi 1966), according to Allen (*ibid*)., "attempts to fulfil these requirements but does not do so" (other parts of Sen's work never appeared).

It is therefore a very happy affair to find that Dr. David B. Pingree (born 1933), now Professor of the History of Mathematics at the Brown University (USA), undertook single-handed a giant project of surveying all the Sanskrit works and all the authors that can be identified in the field of *jyotiṣa-śāstra* and allied disciplines. The usefulness of the project is to lie, as Pingree himself points out, "in providing a preliminary exploration and organisation of the vast mass of Sanskrit and Sanskrit-influenced literature devoted to the exact sciences (including astronomy, mathematics, astrology, and divination), and in detailing under each item not only what preceding work has been done, but what manuscript material is known to be available for future investigations" (Introduction to Vol. I).

To achieve the desired utility of his project, the author has been collecting material since 1955 from libraries in America, Europe and India (which he visited in 1965 under a grant from the A.P.S.). Due to the tremendous labour done in this way, the

Gaṇita Bhāratī, Vol. 4, Nos. 3–4 (1982), pp. 157–161.

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cooperation and help received by the searching author, and also due to his own high scholarly calibre, the author has been successful in providing the above laid down utility of the project. As a result, we have the *Census of the Exact Sciences in Sanskrit* (= *CESS*) which may very well be said to be author's magnum opus. The plan is to issue the *CESS* in two series (of 6 volumes each). Series A has articles on authors arranged in Sanskrit alphabetic order, and Series B is to have articles on works arranged similarly. An extra volume containing tables of astronomical parameters, genealogy of authors and scribes, and indexes of scribes and proper names, is also promised.

Up to now we have four volumes of Series A as follow:

Vol. I: Philadelphia, 1970 (*MAPS* 81) devoted to authors whose names begin with a vowel.
Vol. II: 1971 (*MAPS* 86) for names which begin with a guttural *(kavarga)*.
Vol. III: 1976 (Introduction is dated 1974) (*MAPS* 111) for names which begin with a cerebral *(cavarga)* or reflexive *(ṭavarga)* or dental (*tavarga*).
Vol. IV: 1981 (Introduction dated 1979) *MAPS* 146 for names which begin with a labial *(pavaraga p, ph, b, bh and m)* which is being reviewed here.*

However, it must be noted that, starting with Vol. II, each volume has a substantial amount of material which is supplemental to all the preceding volumes. This fine scheme enables the entries to become up-to-date (though not consolidated).

In series A, the bio-bibliographical information on each author is presented in an orderly sequence. These include dates of birth and death, names of ancestors, parents and other relatives (including teachers and taughts), places of birth, work, and death, religious and social status. Then follows information about the author's works on exact sciences. Under each work is given its short review, list of commentators (whose entries themselves may be consulted for more details), manuscripts, editions and translations and references to its studies already done. Frequently the chapter-headings of listed works and extracts containing information on the author are also given.

Full details of the Studies and Discussions referred are found in the comprehensive bibliography given in Vol. I (pp. 3–25) and supplemented in others. This bibliography of about 3000 items (books, articles, translations, etc.) in various languages is the most comprehensive in the field available so far. The main basis of information about works of each author is the catalogues of Sanskrit manuscripts and books which are listed in Vol. I (pp. 26–32) and supplemented in others.

In comprehension the general purpose and value of the *CESS* in the field of exact sciences may be compared to those of the *New Catalogus Catalogorum* (=NCC) in the field of Sanskrit literature in the wider sense (hence including exact sciences also). In fact, Prof. Pingree himself has indicated (see preface to Vol. II) this comparison and acknowledged his debt to Prof. V. Raghavan (died 1979) whose name is invariably associated with the *NCC* (being published by the University of Madras). Ironically, there is another similarity between the *CESS* and the *NCC*. It is the slowness of their

*Vol. V appeared in 1994.

printing. Vol. I of the *NCC* (to be completed in 40 Volumes) was published in 1949 (revised edition, 1969), and Vol. X (the last so far?) was brought out in 1978. The entries in *NCC* cover both authors and works in Sanskrit alphabetic order. Of course, it must be noted that the task of either of the projects is very difficult and encyclopedic. The number of Sanskrit authors on exact sciences is quite large (about 2500 so far). Mentioning the Indian Concept of an ocean of knowledge, Pingree has compared his *CESS* to a "raft to rescue those in danger of drowning in it" (Introduction to *Vol. II*).

To give more details of Vol. IV, it comprises of Introduction (p. 1), additional abbreviations of journals and serials (p. 2), additional Bibliography (pp. 3–7), additional List of Catalogues and Sanskrit Manuscripts and Books (p. 8), Census entries supplemental to Vol. I (11–31), to Vol. II (32–88), to Vol. III (88–164) and the authors of Vol. IV (164–447) from Paūmaṇandi (= Padmanandin, tenth century AD), author of the Prakrit *Jambūdīva-paṇṇati-saṃgaho* to Mhālugi (son of Vāsudeva) who wrote a *Jātaka-paddhati* (whose one manuscript was copied in Śaka 1350 = AD 1424). For additions to the Bibliography, the author is "deeply indebted to R. C. Gupta of Ranchi and to A. Volodarsky of Moscow" (p. 1).

About 1000 authors are covered in the present volume. These include some very famous ancient and medieval astronomers and mathematicians like Parameśvara of Vaṭaśrenī (c. 1380/1460) (pp. 187–192), Brahmagupta (b. 598 AD) (254–257), Bhāskara I (fl.629 AD) (297–299), Bhāskara II (b. 1114) (299–326), Mahāvīra (fl. ca. 850 AD) (388–389) Mādhava of Saṅgamagrāma (fl. ca. 1380/1420) (414–415), Muñjāla (fl. 932 AD) (435–436), Munīśvara alias Viśvarūpa b. 1603 AD) (436–441). The modern authors treated include Bāpūdeva Sāstrin (b. 1821) who was "invited by Lancelot Wilkinson to study with Sevārāma at the Saṃskṛta Pāṭhaśālā at Sihora". Bāpūdeva is stated to be "especially influential in spreading the knowledge of European mathematics in India" (p. 241).

The following points mentioned in *Vol. IV* may be interesting to note. The Līlāvatī (of Bhāskara II), the most popular work of Hindu mathematics, was translated into Persian by Fayḍī (1587 AD), by Medinīmalla (1663/64) and by Muḥammad Amīn (1678) (p. 300). It was translated into Hindi/Rajasthani by Amīcandra (c. 1842) who also translated Mahāvīra's *Gaṇita-sāra-saṅgraha* into Rajasthani (p. 23). The Manuscript No. 45 (100 folios) entitled *Jīca Ulugbegī*, in Mahārāja's Museum Library at Jaipur is the Sanskrit translation of Ulugh Beg's famous Persian *Zij-i Ulugh Beg* (fifteenth century) (p. 31), although the Indian version may not be complete *(cf. J. Hist. Arabic Science*, Vol. 4, No. 1, p. 84). The anonymous Sanskrit *Hayata-grantha (c. 1700)* "appears" to be the translation of the Persian *Risālah dar hay'at* (also called *Fārsī hay'at*) by al-Kūshjī (or al-Qūshjī) who died at Istanbul in 1474 (p. 57). Pingree considers al-Bīrūnī's (b. 973 AD) claims of translating Euclid's Elements and Ptolemy's *Almagest* into Sanskrit to be "implausible" (p. 248).

The area covered by *CESS* is really very vast and scope of its utility tremendous. Besides dealing with the three branches (*skandhas*) of *Jyotiṣa-śāstra*, namely *gaṇita* (mathematics and mathematical astronomy), *hora* (horoscopic astrology), and *saṃhitā* (of encyclopedic nature), the *CESS* treats several related fields such as cosmology, chronology, geography, and rituals (which need properly determined times). The scope may further increase, e.g. by including relevant tantric literature.

This oceanic coverage changed the tone from the intention "to include all of the works and all of the authors" (Introduction to *Vol. I*) to the admission that such a work "can never be complete" (Introduction to *Vol. IV*). Pingree also grossly underestimated the task when he stated the successive volumes of the *CESS* will appear at the interval of about one year (*Vol. I,* Introd.).

To be more specific, there are several libraries which have not been catalogued at all, and hundreds of family-collections of manuscripts which have not seen the light of the day since long. In fact, some of the owners of private collections are very hesitant to even show the manuscripts which, therefore, cannot reach *CESS*, nor the *CESS* project can reach them!

Moreover, the scope of the *CESS* is to cover entries even about living Sanskrit authors of exact sciences and this is a never-ending process. Regular supplements will keep the information complete and up-to-date but only to a certain degree.

Another point is that when more detailed surveys of manuscripts are done on regional basis, some still new authors and works come to light. For instance this will be clear from the *A History of the Kerala School of Hindu Astronomy* by K. V. Sarma (Hoshiarpur 1972) which escaped Pingree's notice earlier. In fact while reviewing the first three volumes of the CESS in *Vishveshvaranand Indological Jour.*, 17 (1979), 362–364, Sarma has already mentioned about 25 authors not found in those volumes of the CESS. Fortunately, Pingree promptly rectified this lacuna in the present volume. In fact he is too good to miss any such opportunity. Herein lies his greatness. And the monumental CESS will ever remain not only a masterpiece of Pingree's scholarship but also that of American scholarship. In fact there is no surprise to find that more serious studies of Indian mathematics and astronomy are being done outside than in India itself.

The American Philosophical Society is to be congratulated for bringing out the CESS. The printing and other errors are minimum for such a boring work. The price is relatively low to enable even the individual scholars to have personal copies. No serious work on Indian exact sciences can be done without the CESS which symbolizes both—the labour of love and love of knowledge.

Homage to Professor Abraham Seidenberg (1916–1988)

Professor Abraham Seidenberg died on May 3, 1988. He belonged to the Department of Mathematics, University of California, Berkeley, USA, and was a member of the Editorial Board of the *Gaṇita Bhāratī*. He was a great scholar and was actively involved in research work on history of ancient mathematics. A few months before his death, he had sent his manuscript "On the Volume of a Sphere" for publication in the *Archive for History of Exact Sciences (= AHES)*. He could not correct the proofs of the article which was published posthumously in December 1988. Perhaps this may be his last paper.

Professor Abraham Seidenberg (1916–1988)

Gaṇita Bhāratī, Vol. 11, Nos. 1–4 (1989), pp. 57–59.

K. Ramasubramanian (ed.), *Gaṇitānanda*,
https://doi.org/10.1007/978-981-13-1229-8_45

Professor Seidenberg was a pioneer in the field of real Vedic mathematics, i.e. the one which is found in the genuine Vedic literature or was known to the Vedic seers. He has successfully shown that the mathematics as found in the *Śulba-sūtras* "already existed before 1700 BC" He has published a number of significant papers on the subject.

Professor Seidenberg has given us not only new findings but also some great ideas as applied to history of ancient mathematics. One of them is the hypothesis of diffusion, i.e. the view that great and important ideas arise only once and spread from a centre. In other words, when significant theorems and formulas are found in several culture areas, it is sound to assume a theory of dependence through some links.

1 Major Contribution of Seidenberg

A more original hypothesis of Seidenberg is that of 'ritual origin' of mathematics. For example, the ancient texts that describe geometrical constructions clearly specify ritual purpose such as the construction of altars of specified form and size for fulfilling specific ends.

If we combine diffusion theory with the hypothesis of ritual origin, we are led somewhat to the assumption that many geometrical and religious ideas must have had a common origin and source. Similar stand for counting practices is found to be valid. In fact, all such considerations imply a real historical common origin for mathematics. Convinced about Seidenberg's views, Professor B. L. van der Waerden has put forward his bold hypothesis of the common origin of mathematics in the Indo-European people before their dispersal (about 3500 BC to 2500 BC).

The passing away of Professor Seidenberg is a great loss to the historians of mathematics. But he will ever be remembered for his new and deep researches in the field. The Indian historians of science should take lessons from his comparative methodology, ritualistic analysis and diffusion theory and carry on the work further.

2 Selected Publications of Abraham Seidenberg

1. "Peg and Cord in Ancient Greek Geometry", *Scripta Mathematica*, 24 (1959), 107–122.
2. "On the Eastern Bantu Root for Six", *African Studies*, 18 (1959), 28–34, and 22 (1963), 116–117.
3. *The Diffusion of Counting Practices*, Univ. of California Publication in Math.,[*] Vol. 3, 1960.
4. "The Ritual Origin of Geometry", *AHES*, 1(5) (1962), 488–527.

[*] His papers are preserved in the Bancroft Library at the University of California, Berkeley, U.S.A.

5. "The Ritual Origin of Counting", *AHES*, 2(1) (1962), 1–40.
6. "The Sixty System of Summer", *AHES*, 2(5), (1965), 436–440.
7. "Remarks on Nicomedes' Duplication", *AHES*, 3(2) (1966), 97–101.
8. "On the Area of a Semi-Circle", *AHES*, 9(3) (1972), 171–211.
9. "Did Euclid's *Elements, Book I*, Develop Geometry Axiomatically?", *AHES*, 14(4) (1975), 263–295.
10. "*km*, a widespread root for ten", *AHES*, 16(1) (1976), 1–16.
11. "Pappus implies Desargues", *Amer. Math. Monthly*, 83 (1976), 190–192.
12. "The Origin of Mathematics", *AHES*, 18(4) (1978), 301–342 (This has an Appendix on "Vedic Mathematics").
13. (With J. Casey), "The Ritual Origin of the Balance", *AHES*, 23(3) (1980), 179–226.
14. "The Ritual Origin of the Circle and Square", *AHES*, 25(4) (1981), 269–327.
15. "Ratio of Arc to Chord", In *Selected Studies: Physics, Mathematics, History of Science* (Amsterdam, 1982), pp. 293–305.
16. "On Ancient Mathematical Terminology", *AHES*, 31(1) (1984), 1–13.
17. "The Geometry of the Vedic Rituals", In *Agni, the Vedic Rituals of Fire* (edited by F. Staal), Delhi, 1984.
18. "Essay Review" of B. L. van der Waerden's *Geometry and Algebra in Ancient Civilizations* (Berlin, 1983), *Gaṇita Bhāratī*, 7 (1985), 31–39.
19. "The Zero in the Mayan Numerical Notation", In *Native American Mathematics* (edited by M. P. Closs), Austin, 1986, pp. 371–386.
20. "Comments on Knorr's Review of van der Waerden's book, Geometry and Algebra in Ancient Civilizations", *Gaṇita Bhāratī*, 9 (1987), 49–50.
21. "On the Volume of a Sphere", *AHES*, 39(2) (1988), 97–119.

Sudhākara Dvivedī (1855–1910): Historian of Indian Astronomy and Mathematics

1 Introduction

Once a king wanted to know as to why the Moon (candā) is addressed चन्दामामा (*candāmāmā*) or "Moon, the Maternal Uncle" in India. Most of the people were puzzled at the question. But one man could find an answer. He said that goddess Lakṣmī (the consort of Lord Viṣṇu) is universally accepted as Mother by all, and the Moon was her brother, since both were born out of the sea when it was churned in remote antiquity (So says the story of *sāgara-manthana* or Sea-Churning, according to Hindu mythology). The reply was quite witty.

MM. Sudhākara Dvivedī (1888–1910)

Gaṇita Bhāratī, Vol. 12, Nos. 3–4 (1990), pp. 83–96.

K. Ramasubramanian (ed.), *Gaṇitānanda*,
https://doi.org/10.1007/978-981-13-1229-8_46

The man who frequently showed such wit was no other than Sudhākara Dvivedī. He was an intelligent person and a reputed scholar of his time well known for his exposition of ancient as well as modern astronomy and mathematics, both in Hindi and Sanskrit. He was a great educationist and an eminent writer. His prolific works provided not only a synthesis of Eastern (Indian) and Western (European) sciences but also played a significant role in introducing modern astronomy and mathematics in India through the national media of Sanskrit and Hindi. The British Government awarded the title 'Mahāmahopādhyāya' (abbreviated as M. M.) for his distinguished scholarship and educational service. Judged by any standard, he was a man of extraordinary talent and capacity.

2 Birth and Education

Sudhākara Dvivedī was born in *Vikrama Saṃvat* 1912 (or AD 1855) in the village Khajurī near Vārānasī. Dvivedī (also written as Dube or Dvivedīn) was his family name. It is said that the news of the child's birth was given to his uncle (father was away from home) at the time when a local newspaper called "Sudhākara" was being delivered to him. So he eventually named the child also as Sudhākara which literally means "maker or store of nectar". There were some disruptions in the family in those days at the time of Sudhākara's birth, his father Kṛpāludatta was also not at home. Due attention was not paid to record accurately the event of the child's birth. According to Joshi,[1] the date of Sudhākara's birth was back-calculated at the time of his marriage, and was fixed at 4th day of the light half of the month of *Caitra* of *Saṃvat* 1912 (or *Śaka* 1777). The same date of birth (recorded according to the Indian calendar) is also found—given, 20 years earlier than Joshi's work, in another source which seems to be quite reliable.[2]

However, Dikshit in his famous work on history of Indian astronomy has given[3] Sudhākara's birth date as Monday the 4th day of the light half of *Caitra* in *Śaka* 1782 (or AD 1860). So there is a confusion as to whether the year of birth is *Śaka* 1777 (AD 1855) or 1782 (AD 1860). In this connection it may be pointed out that Sudhākara wrote a Sanskrit work called *Pratibhābodhakam* in *Śaka* 1795. Dikshit knew this,[4] and if his statement of Sudhākara's birth year *Śaka* 1782 is correct, then it would mean that the above-mentioned work was composed at the age of 13. Since Sudhākara started getting education late when he was 8 years old, his authoring the *Pratibhābodhakam* at the age of 13 seems unlikely (although not impossible). Anyway, the year AD 1860 for Sudhākara's birth has been given, apparently on the authority of Dikshit, by G. Prasad,[5] R. C. Jha,[6] B. Mohan,[7] S. Shukla,[8] and Krishnamurthy.[9] The writer of the present article has come across some other dates also, but the year AD 1855 for Sudhākara's birth seems to be somewhat authentic and reliable. Of course, it should be corroborated and checked from other sources.

Sudhākara's mother, Lācī, left for heaven quite early leaving the child to be looked after by others who showed more than enough affection to him. They always tried to keep the child within the range of their sight. Also his father's place of work was away from home, and he came only occasionally. As a result, the boy's early education was delayed. Sudhākara was quite intelligent and had sharp memory. So, although he started education at the age of 8, he soon learnt reading and writing. Later on he studied Sanskrit grammar under Paṇḍit Devakṛṣṇa. These subjects were enough to make him fit to work as a priest (*purohita)* and astrologer (*jyotiṣī*) in his native and nearby villages. He was married at the age of 14.

Actually Sudhākara was more interested in the study of mathematics which attracted him. He made serious studies of the subject along with astronomy under Paṇḍit Devakṛṣṇa (born 1818 AD) who was the Professor of *Jyoiṣa* at the Benares Sanskrit College and a worthy disciple of Paṇḍit Lajjāśaṅkara (who had earlier served the College in the same capacity). During those days there was another famous Indian astronomer, mathematician and educationist, Prof. Bāpudeva Śāstrī (born 1821 AD) who was teaching mathematics (especially *Rekhāgaṇita* or geometry) in the same institute. Sudhākara had the benefit of his knowledge and experience as well, although they developed some differences later on.

3 Professional Career

As already mentioned, Sudhākara Dvivedī had deep interest in mathematics and mathematical astronomy. For more satisfaction and greater insight, he also studied the then available modern exposition of mathematical topics through European textbooks. Being inspired by higher educational objective he devoted more and more time to gain and disseminate knowledge of the exact sciences and neglected the profession of priesthood. That created some temporary financial difficulty, but he gained eminences as a great scholar and teacher especially through scores of students whom he taught free.[10] It is stated that his spreading reputation impressed King Lakṣmīśvara Siṃha of Darbhanga who recognized his talent and appointed him as a teacher of *Jyotiṣa-śāstra* in a school in Varanasi.[11] Thus the financial constraints were eased and he could devote more efficiently to his pet subjects. Learning and teaching as well as writing and publishing went on side by side. Even the European scholars such as George Thibaut were greatly impressed by his work and knowledge.

In 1883, Sudhākara Dvivedī was appointed Chief Librarian *Pustakālaya-adhyakṣa* of the Govt. Sanskrit College Library, Benares. This library, called Sarasvatī Bhavan, was very rich especially in manuscripts of Sanskrit works. It provided easy opportunity to S. Dvivedī to further widen his knowledge and scope of writing as well as editing (by collation of manuscripts). And, in fact, he did all that sincerely. For instance, he soon edited and published (1884/85) the *Siddhānta-tattva-viveka* of Kamalākara (AD 1658). His famous treatise on differential calculus in Hindi appeared in 1886, and the very next year the title 'Mahāmahopādhyāya' was conferred on him. At the retirement of Bāpudeva Śāstrī in 1889, Dvivedī was appointed to succeed him as Professor of Mathematics and Astronomy at the Govt. Sanskrit College, Benares. During the same year appeared his Sanskrit commentary on the *Pañcasiddhāntikā* of Varāhamihira (sixth century AD). He also taught Mathematics and Indian Astronomy to the postgraduate students. It is stated[12] that earlier these classes were taken by Mr. M. N. Datta who had to leave the assignment due to his appointment as District Inspector of Schools. It should be noted that although Dvivedī did not himself have any formal postgraduate degree, he did the job quite successfully.

Benares (now called Varanasi) was a famous centre for traditional learning especially for north Indian scholars. Hence the pupils of Dvivedī were soon found scattered in the United Province (now called Uttar Pradesh), Bihar and Bengal. Among them some became quite famous such as Baladeva Pāṭhaka, Baladeva Miśra, Genālāla Caudharī, Buddhinātha Jhā, Dayānātha Jhā and Muralīdhara Jhā. A student named Śaśipāla Jhā translated Euclid's *Elements*, Book XV into Sanskrit and another named Gaurī Śaṅkara Prasāda donated money to institute a medal in Dvivedī's name (see below).

4 Service to Hindi

As a classical and sacred language, Sanskrit had been always enjoying a respectful place among learned people in India. Urdu was already being used as an official language in north India in those days (nineteenth century). But this was not the case with Hindi. Dvivedī did a lot to propagate the cause of Hindi language whose fuller recognized form and status were being slowly evolved. In this regard the Kashi Nagari Pracharini Sabha (founded in 1893) was a leading organization at Varanasi. Dvivedī played his due role in it as an official as well as the editor of the Nagari Pracharini Granthamālā (Book-Series) from 1905 to 1908. He was its President from 1902 to 1910. Dvivedī composed a work on Hindi grammar and several other works in that language including *Rāmakahānī* and *Rādhākṛṣṇa-rāsalīlā*. He edited the *Rāmāyaṇa* of Rudrasiṃha, and also edited (in collaboration with G. A. Grierson) the *Padmāvata* of Malika Muhammada Jāyasī along with his own commentary.[13]

However, it must be noted that Dvivedī was not in favour of 'Sanskritization' of Hindi. He used the commonly spoken language employing even domestic terms. His style was simple and lively. He was very fond of composing easy Hindi couplets (*dohās*). As a sample we quote the following two colophonic from his *History of Mathematics* (in Hindi), Part I (1910):

ऐहि अंग्रेजी-राज-बल सब देशन की रीति ।
समुझि बूझ लखि मनन करि भिइन पर कर प्रीति ॥ 1 ॥
अंकगणित की कछु कथा लिखी सुधाकर धीर ॥
ताहि बांचि पुरबहु कसर निज बुधि-बल लखि हीर ॥ 2 ॥

In the field of scientific education, his most significant service to Hindi was the writing of text-books on modern mathematics which we now take.

5 Mathematics Text-Books in Sanskrit and Hindi

S. Dvivedī believed in propagating correct and current astronomical theories based on modern mathematical treatment. Through his expository works, he wished to incite and encourage his own countrymen "towards the cultivation of Western science" as he put it.[14] As a sort of synthesis of Eastern and Western science, Bāpudeva Śāstrī had already started putting Indian traditional knowledge and material from classical Sanskrit works (such as those of the twelfth century Bhāskara II) in modern style. Thus text-books on elementary arithmetic, geometry, and trigonometry were available in new mathematical treatment and notation. It may be recalled that Euclidean geometry was already there in India, e.g. in the eighteenth century Sanskrit translation called *Rekhāgaṇita* which was made by Jagannātha (1718) from an Arabic version. Dvivedī not only followed similar modern exposition but made original contributions by extending the treatment to cover higher topics of mathematics and mathematical astronomy based on new theories. These included conic sections, theory of equations, differential and integral calculus. His earliest expository work was *Pratibhābodhakam* (in Sanskrit) which was composed in 1873. The central subject matter of the tract is the determination of the shape and size of the various sections (*pratibhā-s*) of a general conical surface.

A better known and typical Sanskrit work of Dvivedī is the *Dīrghavṛttalakṣaṇam* which is a treatise on the properties of an ellipse (*dīrghavṛtta*). Properties have been expressed verbally but symbolic mathematical language has been used in giving the derivations. Without using coordinates and analytical geometry, this was a heroic venture. The work was published in 1881. In a year's time appeared his another similar tract called *Bhābhramarekhā-nirūpaṇa* in which the path of the tip of the shadow of a gnomon was discussed. More praiseworthy are Dvivedī's text-books on higher mathematics in Hindi. These include the *Calanakalana* on differential calculus (1886), *Calarāśikalana* on integral calculus (1895) and *Samīkaraṇa-mīmāṃsā* on the theory of equations (1897). These new attempts can be regarded as commendable successes especially keeping in view the difficulty of technical terminology to be used in Indian languages. It was just not available.

The differential calculus book covered all topics of a classical degree level course including Maclaurin and Taylor series expansions, indeterminate forms (*luptabhinna*), functions of several variables and the theory of maxima and minima. The surprising thing is that there is no mention of limit or any concept of it in the whole work. The integral calculus book covered even such topics as change of order of integration (*kramaparivartana*), calculus of variation (*vaiśeṣika-kalana*) and differential equations (*calana-samīkaraṇa*). The author has cited the works of Todhunter, Williamson, Hymers, Cox and De Morgan. But he has also given some methods of his own.

Of course, we cannot expect very high rigorousness of treatment in these new native attempts. Thus, while finding the derivative of $\sin x$, the expression

$$\cos\left(x+\frac{h}{2}\right)\cdot\frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

is coolly equated to $\cos x$ without saying a word about

$$\frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

which could not be explained (since concept of limit was avoided). However, it will not be fair to judge those books by modern standards. Dvivedī's algebraic book discussed lot of topics including cubics, quartics, determinants *kaniṣṭhaphala* and elimination. The Newton–Raphson method is shown by Dvivedī, to be an extension of Kamalākara's method (p. 287).

6 Editions of Ancient Works on Astronomy and Mathematics

A great contribution of S. Dvivedī was to make available authentic editions of several important Sanskrit works on astronomy and mathematics. These provided reliable primary sources for studying and writing the history of mathematics and mathematical astronomy in India during ancient and medieval periods. Frequently, the edited works were accompanied by commentaries (by author, by other ancient writers or by editor), notes and rationales. These explanations and expositions themselves contributed to valuable interpretations of ancient theories and methods which were often hidden obscurely in ancient *sūtras*, verses and passages.

It was but natural that the first work which attracted him was *Līlāvatī*, the most popular work of ancient Indian mathematics and the standard text-book on the subject (for Sanskrit medium courses) through the ages and throughout India ever since it was composed by Bhāskara II in AD 1150. Dvivedī's edition (1878) contained his derivations (*upapatti*) of the rules. A few years later he edited, with his own commentary, the *Karaṇakutūhala*, an astronomical manual of the same author.

The *Siddhānta-tattva-viveka* (1658) was composed by Kamalākara when the famous Newton was just a boy of 16. Dvivedī edited it in 1884 along with his valuable notes. He considered the work to be the best among the Indian *siddhāntas* (astronomical treatises) although Kamalākara still adhered, due to orthodoxy, to the approximation $\pi = \sqrt{10}$. A couple of years later, Dvivedī brought out his edition of Lalla's *Śiṣyadhī-vṛddhida-tantra* (eighth century AD). There is no commentary, possibly because the text itself was simple and expository, or more probably because the editor was lacking peace of mind due to the sad loss of his father which he mentioned in the words[15]

यातेदिवं पितरि तद्विरहज्वरेण सन्तापतप्तहृदयेन सुधाकरेण । संशोधितम् ....

After normalization, Dvivedī took up more important works. His edition of Bhāskara's *Bījagaṇita* ("Algebra") was supplied with his derivations (1888). He also completed his valuable Sanskrit commentary *Prakāśikā* on the *Pañcasiddhāntikā* of Varāhamihira. This was a difficult task in the absence of any ancient commentary on the *Pañcasiddhāntikā* (c. 550 AD). Nevertheless, this work is a very important primary source for history of ancient astronomy not only in India but in the whole world. The text along with the Sanskrit commentary and an English translation became available to scholars in 1889 due to joint efforts of Dvivedī and G. Thibaut. A few years later, Dvivedī brought out his edition of Varāhamihira's another work the *Bṛhat-saṃhitā* which is equally important for history of astrology. In fact, it is an encyclopedic work and Dvivedī's edition contained the equally important ancient commentary of Bhaṭṭotpala (tenth century). The versatility of Dvivedī is further illustrated by the fact that in the 12 years period from 1899 to 1910, he edited several other important works after supplying with his own detailed commentary and notes. These include *Karaṇaprakāśa* (1899), *Triśatikā* (1899), *Brāhmasphuṭa-siddhānta* (1901/1902), *Grahalāghava* (1904), *Sūrya-siddhānta* (1906), *Vedāṅga-Jyotiṣa* (1907/1908) and *Mahā-siddhānta* (1910). During the last days of his life, he wrote a commentary on the *Gaṇita-kaumudī* of Nārāyaṇa Paṇḍita (1356 AD).[16] More information about above works is given in the chronological bibliography at the end of this article.

Besides the two categories of works (text-books and editions) which we have discussed under the present and the last section, there were many others which may be considered under different categories. For instance, works which are stated to contain mathematical and astronomical tables include[17]

(i) *Laghurikta-sāriṇī* (Logarithmic Tables).
(ii) *Candrasāriṇī*, etc. (Tables for the Moon and Planets) (translated from French).
(iii) *Sūrya-siddhānta-sāriṇī* (Tables, based on the Sūrya-sidhānta).

Dvivedī's scores of students often helped him in working out these tables. Then there was a book by him on magic squares (*yantras*) in Hindi, some others on *pañcāṅga* (Almanac) and still others on minor topics like rotation of the earth and construction of a regular polygon of 17 sides in a circle, etc. (see bibliography at the end).

7 Contribution to History and Historiography of Mathematics

The valuable contribution of S. Dvivedī to history and historiography of Mathematics and Astronomy in the form of a large number of primary sources has been already discussed above. The material in the form of his interpretations and explanations of ancient Indian mathematical and astronomical methods is equally significant. In *Vikrama Saṃvat* 1947 (or AD 1890), he completed his *Gaṇaka-taraṅgiṇī* (in Sanskrit) which is on the lives and works of important Indian astronomers and mathematicians (*gaṇakas*) from antiquity to his own time presented chronologically. Thomas Carlyle (1795–1881) had said that "the history of the worlds is but the biography of great men". If that is so, we can regard *Gaṇaka-taraṅgiṇī* as a history of Indian astronomy and mathematics written through biographies of those who contributed significantly to the development of the twin-disciplines. The work shows author's vast knowledge of the history and sources of Indian exact sciences. Although a century old, much of its material will be found to be still useful.

Dvivedī seems to have fully realized the value and usefulness of history of mathematics in teaching and learning to mathematics. He tried to incorporate relevant historical information in his text-books to make them more interesting and to bring out the human side of mathematics. For example, one of his calculus book contains a footnote which states that in 1711 Leibnitz filed a case with the Royal Society of London against Dr. Keil who had charged Leibnitz of plagiarizing Newton's work, etc. (read the book of history of mathematics to know as to what happened subsequently). Dvivedī accepted mathematics as a universal subject in which contributions were made by different countries, nations and cultural groups from time to time. In fact in 1910, he published his *A History of Mathematics,* Part I (Arithmetic), in Hindi, in which he dealt with the history of the world arithmetic paying special attention to numbers and numerals. It was a unique work of its kind in India, and he deserves all praise for the rich material it contains. According to the Preface, he had desired to write three more parts of the work devoted respectively to Algebra, Geometry and Mensuration, and Trigonometry and Astronomy. But perhaps he could not do this due to his early death.

It is to be noted that in his view the history of mathematics (or of science in general) is to be studied honestly and to bring out the facts to light impartially.

8 Epilogue

According to Jha,[18] S. Dvivedī left for heaven in January 1910. But this conflicts with the date 29-10-1910 on which Dvivedī apparently wrote the *Bhūmikā* to his *History of Mathematics Part I* (In Hindi). However, the year 1910 (or 1911) for Dvivedī's expiry seems to be correct, being corroborated by *Saṃvat* 1967 found in an earlier source.[19] Nevertheless, to worsen the confusion another year, namely, AD 1922, is also found mentioned in some works.[20] However, this seems to be wrong.

Dvivedī had a daughter named Sītā Priyājī (who had already died young) and three sons of whom Padmākara followed the footsteps of his father. He was a scholar in the same field and edited some of the works of his father. He was also Professor of Mathematics and Astronomy in the Government Sanskrit College, Benares.

So we are coming to the end of this short sketch of the life and work of M. M. Sudhākara Dvivedī. He was a unique man and a great scholar. He often demonstrated that Hindi can be written rapidly in a good hand. He took part in a movement which ultimately resulted in the introduction of Nāgarī script in the official work of U. P. Government (AD 1900).[21] In the field of education Dvivedī had a burning zeal to see that his countrymen and students learn modern exact sciences quickly. So he wrote expository books on higher mathematics in Sanskrit and Hindi. Being gifted with talent and sense of devotion, he could devise new terminology which was simple and elegant. Because of his text-books it was possible to raise the standard of mathematics students graduating through the medium of Sanskrit at least in north India.

He had proposed to write similar books on even advanced topics such as analytic geometry and quaternions.[22] However, subsequent Hindi and Sanskrit scholars and teachers of mathematics could not carry on the task with similar enthusiasm. In this respect Dvivedī was much ahead of his times. It was only towards the last days of his life that he realized the need to replace *Līlāvatī* by the better and more advanced *Gaṇita-kaumudī* (of Nārāyaṇa Paṇḍita) as a text-book for mathematics courses through Sanskrit.[23] But the many years needed to achieve this were not granted to him by God. At present some educational institutions are named after Dvivedī. The Nagari Pracharini Sabha awards a medal in his name.[24] It is called '*Sudhākara-padaka*'. Recently a new series of monographs, called M. M. Sudhākara Dvivedī Granthamālā, has been started by the Sampurnanand Sanskrit University to commemorate his memory.

May this brief sketch inspire others to take up the writing of a fuller and more authentic biography of this remarkable man:

सुधाकर सुधावर्षि वाग्भूषणविभूषिते । भारते गणितज्ञानामादर्शत्वम् उपस्थितम् ॥

A Selected Scientific Bibliography of Sudhākara Dvivedī

1873 *Pratibhābodhakam* (in Sanskrit):

(i) According to Dikshit (p. 421), a work of this name was published as Dvivedī's commentary (dated 1873) on *Yantrarāja* in 1882 (see below).

(ii) Edited with his own commentary by Gaṅgādhara Miśra, Varanasi, 1942. This edition also contains author's later additions to the work. The year of composition of the original work appears on page 35 as *'śarāṅkasaptenduśake'* or *Śaka* 1795 (=AD 1873).

1878 (editor) *Līlāvatī* with his own Notes, Benares, *Śaka* 1800 (= AD 1878). Also posthumously published as Benares Sanskrit Series No. 39 (Benares, 1912).

1878/1881 *Dīrghavṛtta-lakṣaṇam* (in Sanskrit):

(i) Stated to be composed in *Śaka* 1800 or AD 1878 (Dikshit, p. 420) and published in 1881 (Joshi, p. 72).

(ii) A second edition was brought out by Baladeva Miśra in 1943 from Varanasi.

(iii) Now edited by K. C. Dvivedī, Varanasi, 1981. The colophonic verse no. 4 says that it was composed by Sudhākara at the age of 18 which means either in 1873 or in 1878.

1879 *Vicitra-praśna* (in Sanskrit): Composed in *Śaka* 1801 or AD 1879, this work contains 20 difficult mathematical problems with solutions (Dikshit, p. 420).

1880 *Vāstava-candra-śṛṅgonnati-sādhanam* (in Sanskrit): Edited by Gaṅgādhara Miśra with his own Commentary and proof, Allahabad 1923. Verse 91 (p. 81) states that it was completed in *Vikrama* year *'Nagalokāṅkabhū'* (1937) or AD 1880.

1881 (editor) *Karaṇa-kutūhala* (of Bhāskara II) with his own commentary, Benares, 1881.

1882 *Dyucaracāra* (in Sanskrit): It is work on planetary orbits according to European astronomy.

1882 *Bhābhramarekhā-nirūpaṇam* (in Sanskrit): Recent edition by K. C. Dvivedī, Varanasi, 1981. The data of composition is given to be *Vikrama* year *'Navarāmanavendu'* (i.e. 1939) or AD 1882. There was an earlier edition by Padmākara Dvivedī in 1933.

1882/1883 (editor) *Yantrarāja* (of Mahendra Sūri, AD 1370) with the commentary of Malayendu and with his own commentary, Benares, *Śaka* 1804 *vedanabho'ṣṭarūpa*.

1883/1884 बाबू हरिश्चन्द्रजी की जन्मपत्री or *The Horoscope of Babu Harishchandra* (in Hindi). Composed in 1883 and printed in 1884 (Medical Hall Press, Benares).

1884/1885 (editor) *Chādaka-nirṇaya* (of Kṛṣṇa): Benares, *Śaka* 1806. It is on eclipses in the form of a dialogue between a couple.

1885 (editor) *Sidhānta-tattva-viveka* (of Kamalākara, AD 1658), with his own notes, Benares, 1885. The edition was completed in *Śaka* 1806 *'rasanabhogajabhū'*. A revised edition by Muralidhara Jha and Muralidhara Thakkura was published at Benares, 5 fasciculi, 1924–1935. It includes author's *Śeṣavāsanā*.

1886 (editor) *Śiṣyadhīvṛddhida-tantra* (of Lalla), Benares, 1886. The *Bhūmikā* (Preface) is dated *Saṃvat* 1943.

1886 चलनकलन (*Calana-kalana*) (Differential Calculus, in Hindi), Benares, 1886. There is an edition of Chapters I to VIII by Padmākara Dvivedī, Benares, 1941.

1888 (editor) *Bījagaṇita* (Algebra) of Bhāskara II with his own notes, Benares, 1888. A posthumous edition was brought out by Muralidhara Jha, Benares, 1927.

1888/1889 (with G. Thibaut) (editor and translator): *Pañcasiddhāntikā* of Varāhamihira (sixth century) edited with his own Sanskrit commentary called *Prakāśikā* (*Śaka* 1810) and English translation, Benares 1889 (Thibaut's preface is dated 1888). Reported subsequent reprints are Lahore, 1930 and Varanasi, 1968.

1890/1892 *Gaṇaka-taraṅgiṇī* (in Sanskrit), on the lives and works of Indian astronomers and mathematicians. Originally composed in *Saṃvat* 1947 (*'nagasāgararatnabhū'*) or AD 1890, serially it first appeared in the monthly journal *The Paṇḍit* in 1892 and then published in a book form, Benares, 1892. Posthumous editions include one by Padmākara Dvivedī (Benares, 1933) and another by Sadananda Shukla (Varanasi, 1986).

1895 चलराशिकलन (*Calarāśi-kalana*) (Integral Calculus, in Hindi), Benares, 1895. Posthumous editions by Baladeva Miśra, Benares, 1941 (Part I) and 1943 (Part II).

1895/1897 (editor) *Bṛhat-saṃhitā* of Varāhamihira with the commentary *vivṛti* of Bhaṭṭotpala (tenth century), 2 Vols., Benares, 1895, 1897. Posthumously edited by Avadhavihari Tripathi. Varanasi, 1968.

1897 *Samīkaraṇa-mīmāṃsā* (in Hindi) completed in *Saṃvat* 1954 (or AD 1895). Posthumously edited by Padmākara Dvivedī, 2 Vols., Allahabad (undated).

1898/1899 *Diṅmīmāṃsā* (in Sanskrit), Benares, 1899. It was completed in *Saṃvat* 1955 (Joshi, p. 63) or AD 1898.

1899 (editor) *Karaṇa-prakāśa* (of Brahmadeva, 1192 AD) with his own commentary (called *Sadvāsanā*) and supplementary material on *Kṣepasādhanam, Āsannamānam* (Approximations) and Prime Numbers *Dṛḍhāṅka*. Benares, 1899.

1899 (editor) *Triśatikā* (of Śrīdhara, about 750 AD), Benares, 1899.

1901/1902 (editor) *Brāhmasphuṭa-siddhānta* (of Brahmagupta, AD 628) with his own Sanskrit commentary (called *Tilaka*), Benares, 1902. First the work was published serially in *The Paṇḍit*, Vol. 23 (1901) and Vol. 24 (1902).

1904 (editor) *Grahalāghava* (of Gaṇeśa, 1520), with the commentaries of Mallāri and Viśvanātha, and of his own, Benares, 1904. Reprinted, Bombay, 1925.

1906 (editor) *Sūryasidhānta* with his own commentary called *Sudhāvarṣiṇī* (completed in 1906), Calcutta, two parts, 1909, 1911 (Second edition, Calcutta, 1925 ?). Recently edited by K. C. Dvivedī, Varanasi, 1987. The colophonic verse No. 3 (p. 256) states that the commentary was completed in *Vikrama* year 1963 *'lokāṅganandavidhu'* or AD 1906.

1906/1908 (editor) *Vedāṅga-jyotiṣa* (of Lagadha) with his own commentary, Benares, 1906. Another edition with the commentary of Somākara (on Yājuṣa version) and with his own commentary (on Yājuṣa as well as Ārca versions) etc., Benares, 1908.

1910 (editor) *Mahā-siddhānta* of Āryabhaṭa II with his own commentary, 3 fasciculi, Benares, 1910.

1910 *A History of Mathematics*, Part I (Arithmetic) (in Hindi), Benares, 1910. No other part has come to light.

There is a work in Sanskrit called *Dharābhramaḥ* which was written by S. Dvivedī. It has been edited by his son P. Dvivedī with his own commentary (Benares, 1940) but the date of original composition could not be found. It is also called *Bhūbhramaṇa* and is on the two views regarding the rotation of the Earth (प्राचीननवीनयोर्विवादः). There is a recent edition by S. Shukla.

Many other mathematical works by S. Dvivedī are found reported in various sources but details are not known. These include poetical Sanskrit translation of Euclid's *Elements*, Books VI, XI and XII *Spherical Geometry* (in Sanskrit), and *Grahaṇakaraṇa* (in Sanskrit) on eclipses (Dikshit, pp. 420–421). His book on magic squares is called *Vargakoṣṭha-pūrṇarīti* (in Hindi) and that on the heptadecagon as *Vṛttāntargata-samasaptadaśabhujakṣetra-racanāprakāra* (Jha, pp. 69–70).

Titles of the reported works of S. Dvivedī on calendar or almanac are *Pañcāṅga-Prapañca* (in Hindi, Joshi, p. 16) and *Pañcāṅga-Vicāra* (Jha, p. 69). An edition of Śatānanda's *Bhāsvatī* by Dvivedī is mentioned by S. N. Sen in his *Bibliography of Sanskrit Works*, etc. (New Delhi, 1966, p. 194) but I have not come across corroborative information elsewhere. Dvivedī's writings dealing with tables have been already mentioned in the main paper (Sect. 6).

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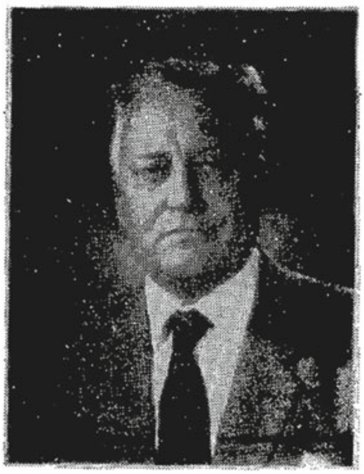
2. Vedavrata Śāstrī, *The Half-Century History of Kāśī Nāgarī Pracāriṇī Sabhā* (in Hindi), Varanasi, *Saṃvat* 2000 (= AD 1943), p. 163. This is quoted by Joshi, *op. cit.,* p. 83.
3. S. B. Dikshit, *Bhāratīya-jyotiṣa* ("Indian Astronomy"), translated into Hindi by S. N. Jharkhandi, Second Edition, Lucknow, 1963, p. 420. Also translated into English by R. V. Vaidya, Parts, I and II, Delhi, 1969 and 1981. Dikshit's original work was in Marathi and was published in 1896 (reprinted, Poona, 1931).
4. *Ibid*., p. 421.
5. G. Prasad, *History of Indian Astronomy* (in Hindi), Lucknow, 1959, p. 244.
6. R. C. Jha, *Vidvadvibhūti* (in Hindi), Varanasi, 1959, p. 65.
7. B. Mohan, *History of Mathematics* (in Hindi), Lucknow, 1965, pp. 337–338.
8. S. Shukla (editor), *Gaṇakataraṅgiṇī* (of S. Dvivedī), Varanasi 1986, introductory page 17.
9. K. R. Krishnamurthy, "Mahamahopadhyaya Sudhakara Dvivedī", *Bhavan's Journal*, 37(9) (Dec. 15, 1990), pp. 60–61.
10. Joshi, *op.cit.* (ref. 1), p. 14.
11. Jha, *op.cit.* (ref. 6), p. 66.
12. Joshi, ref. 1, pp. 14–16.
13. *Ibid.,* pp. 84 and 88.
14. See his *Calarāśi-kalana*, Part I, edited by Baladeva Mishra, Benares, 1941, Preface, p. 5.
15. See *Bhūmikā* of the edited work (Varanasi, 1886). This was written just about a month after his father's death on the 7th day of dark half of *Vaiśākha* in Saṃvat 1943 at the age of 60 (see Joshi, ref. 1, p. 6).
16. Jha, ref. 6, p. 68.
17. Joshi, ref. 1. pp. 15 and 32.
18. Jha, ref. 6, p. 67. It could be January 1911 (see the next note no. 19).
19. Vedavrata, ref. 6, p. 67. It could be noted that *Saṃvat* 1967 actually corresponds to the period AD 1910 (roughly March/April) to 1911 (roughly March/April).
20. For example, see Prasad, ref. 5, p. 244, and Mohan, ref. 6, pp. 337–338. Joshi, ref. 1, p. 102, gives *Saṃvat* 1967.
21. Joshi, ref. 1, pp. 79–80, and Vedavrata, ref. 2, pp. 129–130.
22. See his *Calarāśi-kalana* (ref. 14), Preface, p. 3.

23. Jha, ref. 6, p. 68.
24. Vedavrata, ref. 2, pp. 163 and 258.

Clas-Olof Selenius (1922–1991): An Expert in Indian Cyclic Method

"The cyclic method (*cakravāla*) was in fact a very natural, effective and labour-saving method with deep-seated mathematical properties. It anticipated the European methods by more than a thousand years and surpassed all other oriental performances. Since it did not occur in China at all, it must be regarded as a purely Indian creation. The cyclic method is the absolute climax of the Indian mathematics. In my opinion, no European performance at the time of Bhāskara (AD twelfth Century), nor much later, came up to this marvellous height of mathematical complexity."

–C.-O. Selenius (1971)



Dr. Clas-Olof Selenius (1922–1991)

A great scholar as he was, Clas-Olof Selenius could successfully illustrate the use of modern mathematics in highlighting the significance of the ancient Indian master-

Gaṇita Bhāratī, Vol. 15, Nos. 1–4 (1993), pp. 74–78.

K. Ramasubramanian (ed.), *Gaṇitānanda*,
https://doi.org/10.1007/978-981-13-1229-8_47

piece called *Cakravāla* (चक्रवाल) or cyclic method in solving certain indeterminate equations. As a historian of mathematics, he corrected some misconceptions about the method and gave better interpretations. As a researcher, he developed a theory of ideal continued fractions, being motivated by the investigation into that method. In fact, his work shows that a study of history of mathematics can enrich modern stock of mathematics by inducing new researches.

Selenius was born in Helsinki, Finland, on September 28, 1922, and had his early schooling there. He obtained his M.Sc. degree from the University of Helsinki. His further studies were continued at the Åbo Academy, the Swedish University at Turku on the Baltic Sea in S. W. Finland. The Academy awarded him a doctorate degree in mathematics in 1961 for his thesis on "Konstruktion und Theorie half-regelmässiger Kethenbrüche mit idealer relativer Approximation" (1960). In this work he made a thorough study of the various types of semi-regular continued fractions. In particular he investigated the special ideal continued fraction expansions (of real numbers) originating in the cyclic method. A couple of years later he gave a table of the ideal expansions of all the quadratic surds from $\sqrt{2}$ to $\sqrt{1000}$ and thus provided all the *cakravāla* cycles leading to the solution of the corresponding *varga-prakṛti* indeterminate equations.

Selenius had started his teaching career in Helsinki (1945–1949) and then continued in Ekenäs (1949–1960), about 60 miles west of the former. He was also the town councillor of Ekenäs from 1957 to 1960. Later on he was awarded a medal for his services to this town.

In 1963, he became docent at the Åbo Academy. Three years later he was appointed a lecturer at Uppsala University, Sweden. During 1963, he visited the Cambridge University, UK, and worked there for some time with Prof. J. W. S. Cassels. He continued to lecture at the Uppsala University till his retirement in 1983, but he also held professorship of the Åbo Academy from 1975 to 1979 simultaneously. After retirement, he continued lecturing at the Academy as docent till 1990. The subject of his lecturing included history of mathematics.

To mention his profession activities, he was one of the editors of the journal of *Nordisk Matematisk Tidskrift* for several years, and in 1975, he became a member of the International Commission on History of Mathematics. He participated in many international conferences on history of mathematics, especially those held at the Mathematical Research Institute, Oberwolfach, Germany.

The most important contribution of Dr. Selenius to history of mathematics is his deep study of the Indian cyclic method for solving the *varga-prakṛti* equation (in intergers, N being non-square)

$$Nx^2 + 1 = y^2$$

He brought to light many hidden aspects of the method. In his extensive paper on the subject (1963), he showed that the cyclic process produces quantities all of which have their simple counterparts in the ideal continued fraction algorithm. He fully demonstrated that the Indian *cakravāla* method

(i) represents one of the shortest possible algorithms,
(ii) always produces the least positive integral solution,
(iii) involves rules which are quite sophisticated,
(iv) can be extended to cover other equations like

$$Nx^2 + 1 = xy + y^2 \text{ and } Nx^2 + 1 = y^3$$

In fact, Selenius was able to highlight the ingenious core of the method. His exposition of its secrets is far illuminating than given by previous scholars both Indian as well as foreign.

Besides being a sincere scientific worker, Prof. Selenius had many other remarkable qualities. He not only liked music and poetry but had talent to compose them. In 1971 he won the first prize in a national poetry competition. He was a thorough gentleman and an unforgettable human being. When the author of the present article first contacted him through a letter, he paid great attention to it and sent a long reply. Since it will be of interest to historians of mathematics, his reply dated 7 February 1973 is being reproduced here (with only minor corrections).

Uppsala 07.02.73

Professor R. C. Gupta, Ph.D.
Birla Institute of Technology
P. O. Mesra, Ranchi.

Dear Colleague,

Many thanks for your kind letter. I am delighted at your interest in ancient and medieval Indian mathematics. Therefore I have sent you my foremost paper (1963) about the cyclic method and also my doctoral thesis (1960). All my papers written in German, but I assume you read German well.

The subject of my historical research was at first the cyclic method of Bhāskara to solve the so-called Pellian equation. I found two very remarkable facts: (1) no one had interpreted (defined) the continued fraction expression that corresponds to the completely continued-fraction-like cyclic method (the expression in question is not an usual c.f. expression, but a halvregular expression), (2) a type of such continued fractions (ideal c.f.) constructed in my doctoral thesis was in fact equivalent with the cyclic method.

In October last year, I was invited to give a lecture at Mathematisches Frschungsinstitut in Oberwolfach. There I lectured about the two facts mentioned, and (3) about the fact that I later (not yet published but announced in Moscou by me, 1971) proved: Bhāskara's method surprisingly could be generalized to (a) the case $x^3 - Dy^3 = 1$, and (b) case $x^2 - Dy^2 = 1$, $D \,\varepsilon Z(i)$. In the planned new journal Historia Mathematica I will publish these results.

If you are interested in these problems, demonstrating the high value of the old Indian mathematics and culture (my paper in 1963 I dedicated to the Indian mathematics) I should like to inform you about these theories, connections, interpretations, results etc.

Naturally, I, as an European far away from the centre of my subject, have had great difficulties to find the right sources, texts, communications, contacts, libraries, etc., and to publish papers there. I have not had opportunity to travel to India and explore there these circumstances.

I am very interested in getting your papers, especially your doctoral thesis but also all other papers. I was very thankful for your information about the latest (recent) editions of the works of Āryabhaṭa, Brahmagupta and Bhāskara (and other). Also, I have got only sparse information about papers of Indian (or non-Indian) authors in Indian journals.

I wish to express my appreciation of the achieved contact between us. My admiration for Indian (Asian) mathematics is very deep. As I said in Moscou, "no European performance at the time of Bhāskara, nor much later, came up to this marvellous height of mathematical complexity".

Yours sincerely,
(signed)

Clas-Olof Selenius

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75428 Uppsala (Sweden)

In spite of great difficulties (which Selenius mentions), it is to be appreciated that he could consult a lot of Indian publications relevant to his work including those of A. A. Krishnaswami Ayyangar. This shows his sincere and determined efforts. In fact the bibliography in his 1975 paper is quite rich and shows the thoroughness and up-to-dateness of his knowledge. This should be contrasted with the writings of many Indians who are ignorant (often intentionally) of the current researches and publications. For instance, an article in the recent issue of *Mathematical Education* (Vol. 8, 1991, 23–27) does not mention the contributions of Jayadeva (who already knew *cakravāla* a century before Bhāskara or even earlier), K. S. Shukla (1954), or Selenius, and repeats older writings.

The work of Selenius is of paramount importance to historians of mathematics because the cyclic method is, in the words of H. Hankel (1874), "ohne Zweifel der Glanzpunkt" of ancient Indian exact science.

Acknowledgements The author is grateful to Mm. Singe Selenius, wife of Prof. C. -O. Selenius, for sending material including his photographs.

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16. "Rationale of the *Cakravāla* Process of Jayadeva and Bhāskara II", *HM*, 2 (1975), 167–184.
17. "Umwertung der alten orientalischen Mathematik", talk at the 23th Oberwolfach Conference (1980). See *HM* (1981), p. 471 for abstract.
18. "Warurn gibt es für Mathematik Keinen Nobelpreis'?" (Why is there no Nobel prize for mathematics?). In *Mathemara* (ed. by M. Folkerts and U. Lindgreen), Stuttgart, 1985, pp. 613–624.
19. "Sof'ja Kovalevskaja und der Bordinpreis", contributed to 18th I.C.H.S. (Hamburg-Munich, 1989). See abstract F4 in the Congress Abstracts (but wrongly listed as P4 in the Index).
20. About Selenius, see "In Memorium: Clas-Olof Selenius (1922–1991)", by C. J. Scriba, *HM,* 19 (1992), 325–327.

Kripa Shankar Shukla (1918–2007): Veteran Historian of Hindu Astronomy and Mathematics

Kripa Shankar Shukla's birth took place at Lucknow on July 10, 1918. From the very early years, he was a brilliant student of Mathematics and Sanskrit. He passed the High School Examination of U.P. Board in 1934 in First Division with Distinction in Mathematics and Sanskrit and the Intermediate Examination of that Board again in First Division with Distinction in Mathematics.

He had his higher education at Allahabad, passing the B. A. examination in the second division from Allahabad University in 1938. From the same University, he obtained his Master of Arts degree in Mathematics in the first division in 1941. During his M. A. studies in Allahabad, Paṇḍit Devi Datta Shukla (editor of the Hindi monthly *Sarasvati*) greatly helped K. S. Shukla like his own son and taught him the full *pūja-paddhati* (ritual worship) of Śrī Bālā Devī.

Professor Kripa Shankar Shukla (1918–2007)

Dr. Avadhesh Narain (or Narayan) Singh (1905–1954), a student of Prof. Ganesh Prasad, was quite enthusiastic about the study of history of mathematics and was associated with Dr. B. B. Datta (1888–1958) in that field. The *History of Hindu Mathematics*, part II, by Datta and Singh, was published in 1938 from Lahore (then

Gaṇita Bhāratī, Vol. 20, Nos. 1–4 (1998), pp. 1–7.

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in India). Dr. Singh, although still a Lecturer in the Department of Mathematics and Astronomy, Lucknow University, was very sincerely interested in promoting the study of history of Indian mathematics. In 1939 he started a Scheme of Research in Hindu Mathematics in the Department. Dr. Oudh (i.e. Avadha) Upadhyaya (1894–1941) who had just returned from France with a D.Sc. (Math.) was appointed in the Scheme (see P. D. Shukla's note on Upadhyaya in *Proc. Benaras Math. Soc, N.S.,* III, 95–98).

Dr. K. S. Shukla joined the Department and the Scheme in 1941, and his whole-hearted devotion in the field of study and research in ancient Indian astronomy and mathematics proved very fruitful. His very first research paper on "The Eviction and Deficit of Moon's Equation of Centre" (1945) showed his talent. He concentrated more in studying the works of Bhāskara I, a follower (but not a direct pupil) of Āryabhaṭa I (born AD 476). As early as in 1950, Dr. Shukla studied Bhāskara I's commentary (AD 629) on the *Āryabhaṭīya* and prepared a full Hindi translation of it (see Introduction, p. cxiii, in Shukla's 1976 edition of the commentary).

Dr. Shukla investigated thoroughly the works of Bhāskara I and studied other relevant primary and secondary materials. Under the supervision of Dr. A. N. Singh, Shukla prepared a thesis on "Astronomy in the Seventh Century India: Bhāskara I and His Works". But Dr. Singh died before the Lucknow University awarded the D.Litt. degree on the thesis to Dr. Shukla in 1955. Perhaps by divine plan Singh's death occurred on July 10 which is the date of Shukla's birth in Gregorian Calendar.

Shukla's doctoral thesis was in four parts:

(i) Introduction;
(ii) Edition and Translation of the *Mahābhāskarīya*;
(iii) Edition and Translation of the *Laghu-Bhāskarīya*; and
(iv) Bhāskara I's commentary on the *Āryabhaṭīya* with English Translation of *Āryabhaṭīya*.

The significance of the Thesis lies not only in providing a genuine additional source for the history of early Indian exact sciences but also in bringing to light many new historical and methodological facts. By now most of the material from the thesis has been published in various forms.

In fact, Dr. Shukla proved to be a worthy successor in carrying on the study and research in the field of Hindu astronomy and mathematics. With the help of research assistants like Markandeya Mishra, Dr. Shukla brought out the editions of several Sanskrit texts which were published under the "Hindu Astronomical and Mathematical Texts Series" (= *HAMTS*) of the Department of Mathematics and Astronomy of Lucknow University. Dr. Shukla supervised the research work of a number of theses. Under his guidance the following scholars got their doctoral degree.

(i) Usha Asthana, *Ācarya Śrīdhara and His Triśatikā* (Lucknow University, 1960) (She started her research under A.N. Singh's guidance).
(ii) Mukut Bihari Lal Agrawal, *Contribution of Jaina Ācaryas in the development of mathematics and astronomy (in Hindi)* (Agra Univ. 1973).

(iii) Paramanand Singh, *A Critical Study of the Contributions of Nārāyaṇa Paṇḍita to Hindu Mathematics* (Bihar Univ. 1978).
(iv) Loknath Sharma, *A study of Vedāṅga-jyotiṣa* (L. N. Mithila Univ. 1984).
(v) Yukio Ohashi, *A History of Astronomical Instruments in India* (Lucknow Univ. 1992).

After serving the Lucknow University department with distinction for 38 years, Professor Shukla retired formally under rules on June 30, 1979. But he continued his outstanding and creative works actively in his cherished field for many more years, and scholars still continue to get ideas, suggestions and encouragement from him. One of the tasks he completed after retirement was to bring out a revised edition of the manuscript of Part III of Datta and Singh's *History of Hindu Mathematics*. The manuscript was lying with Dr. Shukla since long (see *Gaṇita Bhāratī*, Vol. 10, 1988, pp. 8–9) but now he found time to publish it in the form of a series of eight articles on Geometry, Trigonometry, Calculus, Magic Squares, Permutations and Combinations, Series, Surds and Approximate Values of Surds in the *IJHS*, Vols. 15 (1980), 121–188; 18 (1983), 39–108; 19 (1984), 95–104; 27 (1992), 51–120; 231–249; and 28 (1993), 103–129; 253–264; 265–275, respectively. It is unfortunate that parts I and II of *HHM* were reprinted (Bombay, 1962) without any revision. Anyway, there is an urgent national need to bring out a consolidated edition of all the three parts possibly after making them up to date, and also to take up the writing of a national history of mathematics in India as team work.

Working wholeheartedly with single minded devotion for more than half a century, Dr. Shukla's contribution in the field of history of ancient and medieval Indian mathematics forms a pioneer work which will continue to motivate future research and investigations. He gave new interpretations of many obscure Sanskrit passages and corrected misinterpretations and other errors committed by others. He has worked diligently and is proud of India's scientific heritage. He has been working silently without caring for publicity. Yet he is greatly reputed for his in depth research among the scholars, and the merit of his work is widely recognised as shown by various citations.

Dr. Shukla was awarded the Banerji Research Prize of the Lucknow University. He was associated with the editorial work of the Journal *Gaṇita* of the Bhārata Gaṇita Pariṣad (formerly the Benaras Mathematical Society) for many years. He was elected Fellow of the National Academy of History of Science, Paris, in 1988. He Served as a member of several national and international committees.

As a student of the Lucknow University, the writer of the present article (*RCG*) attended B.Sc. and M.Sc. courses in the Department of Mathematics and Astronomy during 1953–1957; and Dr. Shukla taught him the subject of a paper in M.Sc. Part I. But there was no course available in History of Mathematics or Hindu Mathematics then (and even now). It is a tragedy that our educational set-up is deficient in this respect. A course in the history (in wide sense) of any subject should form a part of postgraduate curriculum to justify the award of "Master's" title in that subject. It is also hoped that the glorious tradition of study and research in the field of ancient

Indian Mathematics and Astronomy will be maintained in the concerned Lucknow University Department.

A Preliminary note on Dr. Shukla's work appeared in "Two Great Scholars", *Gaṇita Bhāratī*, 12 (1990), 39–44 and Dr. Yukio Ohashi discussed "Prof. Shukla's contribution to the study of history of Hindu astronomy", in the same journal, Vol. 17 (1997, 29–44). The present article is a humble tribute and felicitation on the occasion of the 80th birth-anniversary of respected Shuklaji. May God grant him best health, happiness and long life.

Dr. K. S. Shukla's publications

(I) Edited, Translated and Other Books:

1. *Hindu Gaṇita-Śāstra kā Itihāsa* being a Hindi translation of B. B. Datta and A. N. Singh's *History of Hindu Mathematics Part I* (Lahore 1935), Hindi Samiti, Lucknow, 1956. Reprinted many times.
2. *The Sūrya-siddhānta with the commentary of Parameśvara* (1431). Edited with an introduction in English. *HAMTS* No. 1 Lucknow, 1957.
3. *Pāṭīgaṇita* of Śrīdharācārya edited with an ancient commentary, introduction, and English translation. *HAMTS* No. 2, Lucknow, 1959.
4. *Mahābhāskarīya* (of Bhāskara I) edited with introduction and tranlsation. *HAMTS* No. 3, Lucknow, 1960.
5. *Laghubhāskarīya* (of Bhāskara I) edited with introduction and translation. *HAMTS* No. 4, Lucknow, 1963.
6. *Dhīkoṫida-karaṇa* (of Śrīpati) edited with introduction and translation. Akhila Bharatiya Sanskrit Parishad, Lucknow, 1969.
7. *Bījagaṇitāvataṃsa* (of Nārāyaṇa Paṇḍita) edited with introduction. Akhila Bharatiya Sanskrit Parishad, Lucknow, 1970.
8. *Āryabhaṭa, Indian Astronomer and Mathematician* (fifth century). INSA, New Delhi, 1976.
9. *Āryabhaṭīya* of Āryabhaṭa I edited (in collaboration with K. V. Sarma) with introduction and translation. INSA, New Delhi, 1976.
10. *The Āryabhaṭīya* with the commentary of Bhāskara I (629 AD) and Someśvara, edited with introduction and appendices. INSA, New Delhi, 1976.
11. *Karaṇa-ratna of Devācārya* (689 AD) edited with introduction and translation, *HAMTS* No. 5, Lucknow, 1979.
12. Late Bina Chatterjee's edition and translation of *Lalla's Śiṣyadhī-vṛddhida-tantra* completed and edited. Two volumes, INSA, New Delhi, 1981 (Chatterjee's edition contains the commentary of Mallikārjuna Sūri in Vol. 1 and 17 appendices after the translation in Vol. 2).
13. *Vaṫeśvara-siddhānta and Gola* edited with introduction and translation. Part I (text) and Part II (translation), INSA, New Delhi, 1985–1986.

14. *History of Astronomy in India* edited by S. N. Sen and K. S. Shukla, INSA, New Delhi, 1985 (also issued as *IJHS* Vol. 20).
15. *History of Oriental Astronomy* edited by G. Swarup, A. K. Bag and K. S. Shukla, Cambridge Univ. Press, Cambridge, 1987 (The book constitute Proceeding of IAU Colloquium No. 91, New Delhi, 1985.
16. *A Critical Study of Laghumānasa of Mañjula* (with edition and translation of the text). INSA, New Delhi, 1990. (It was issued as supplement to IJHS, Vol. 25).
17. *A Text book on Algebra* (for B.A. and B.Sc.) by K. S. Shukla and R. P. Agarwal, Kanpur, 1959.
18. [*]*A Text book on Trigonometry* (for B.A. and B.Sc.) by Shukla and R. S. Verma, Allahabad, 1951.
19. *Avakalan Gaṇita* (in Hindi) by M. D. Upadhyay, revised by Shukla, Hindi Sansthan, Lucknow, 1980.

(II) Research Papers and Other Articles:

1. "The evection and the deficit of the equation of the centre of the Moon in Hindu Astronomy". *Proc. Benaras Math. Soc. (N. S.),* 7(2) (1945), 9–28.
2. "On Śrīdhara's rational solution of $Nx^2 + 1 = y^2$". *Gaṇita*, I(2) (1950), 1–12.
3. "Chronology of Hindu Achievements in Astronomy". *Proc. National Inst. Sci. India,* 18(4) (July–Aug. 1952), 336–337 (Summary of a 1950 symposium paper).
4. "The *Pāṭīgaṇita* of Śrīdharācārya" (in Hindi). *Jñānaśikhā* (Lucknow), 2(1) (Oct. 1951), 21–38.
5. "Ācārya Jayadeva, the mathematician", *Gaṇita*, 5(1) (1954), 1–20.
6. "On the three stanzas from the *Pañca-siddhāntikā* of Varāhamihira," *Gaṇita*, 5(2) (1954), 129–136.
7. "A note on the *Rājamṛgāṅka* of Bhoja published by the Adyar Library," *Ibid.,* 149–151.
8. "Indian Geometry" (in Hindi). *Svatantra-Bhārata* (Lucknow), dated 24 Nov. 1957, pp. 1 and 11.
9. "Hindu methods of finding factors or divisors of number". *Gaṇita*, 17(2) (1966), 109–117.
10. "Ācārya Āryabhaṭa's *Ārdharātrika-Tantra*" (in Hindi) *C. B. Gupta Abhinandana Grantha*, New Delhi, 1966, 483–494.
11. "Āryabhaṭa I's astronomy with midnight day reckoning." *Gaṇita*, 18(1) (1967), 83–105.
12. "Early Hindu methods in spherical astronomy." *Ganita*, 19(2) (1968), 49–72.
13. "Astronomy in Ancient and Medieval India." *IJHS*, 4 (1969), 99–106. (cf. no. 15 below).
14. "Hindu mathematics in the seventh century AD as found in Bhāskara I's commentary on the *Āryabhaṭīya*." *Gaṇita*, 22(1) 1971, 115–130; 22(2) (1971), 61–78; 23(1) (1972), 57–79; and 23(2), 41–50.

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15. “Ancient and Medieval Hindu Astronomy” (in Hindi) *Jyotish-Kalp* (Lucknow), 3(6) (March 1972), 32–37. (cf. no. 13).
16. “Characteristic features of the six Indian seasons as described by astronomer Vaṭeśvara”. *Jyotish-Kalp*, 3(11) (Aug. 1972) 65–74.
17. “Hindu astronomer Vaṭeśvara and his works”. *Ganita*, 23(2) (1972), 65–74.
18. “Use of hypotenuse in the computations of the equation of the centre under the epicyclic theory in the school of Āryabhaṭa”. *IJHS*, 8 (1973), 43–57.
19. “The *Pañca-siddhāntikā* of Varāhamihira (I)”. *Gaṇita*, 24(1) (1973), 59–73; also same in *IJHS*, 8 (1974), 62–76. (cf. no. 22 below).
20. “Āryabhaṭa”. In *Cultural Leaders of India: Scientists* (edited by V. Raghavan), Ministry of Information and Broadcasting, Delhi, 1976, reprinted 1981, pp. 83–99.
21. “Astronomy in India before Āryabhaṭa”. Paper read at the Symposium on Hindu Astronomy, Lucknow, 1976, 11 pages (cyclostyled).
22. “*The Pañca-siddhāntikā* of Varāhamihira (II)”. *Gaṇita*, 28 (1977), 99–116. (cf. no. 19).
23. “Glimpses from the *Āryabhaṭa-siddhānta*”. *IJHS*, 12 (1977), 181–186.
24. “Series with Fractional Number of Terms” *Bhāratī Bhavanam* (K. V. Sarma Felicitation Volume) = *Vishveshvaranand Indolog. Jour.* 18 (1980), 475–481.
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26. “A note on R. P. Mercier’s review of *Karaṇaratna of Devācārya*.” *Gaṇita Bhāratī*, 6 (1984), 25–28.
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28. “Main characteristics and Achievements of Ancient Indian Astronomy in Historical Perspective” In *History of Oriental Astronomy* (edited by G. Swarup *et al*), Cambridge 1987, 7–22.
29. “The Yuga of the *Yavana-jātaka*: David Pingree’s Text and Translation Reviewed”. *IJHS*. 24 (1989), 211–223.
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32. “Graphic Methods and Astronomical Instruments” being translation (with notes) of Chapter XIV of the *Pañcasiddhāntikā* of Varāhamihira. Pages 261–281 in K. V. Sarma’s edition of *Pañcasiddhāntikā with Translation of T. S. Kuppanna Sastry*, Madras 1993.

(III) Book Reviews:

1. Review of the *Pañcasiddhāntikā* of Varāhamihira (ed. by O. Neugebauer and D. Pingree, Two Parts, Copenhagen), 1970–1971 *Journal of the American Oriental Society* Vo. 93 (?), 1973, pp.?
2. Review of *Census of Exact Sciences in Sanskriti Series A*, Vol. 3. (by D. Pingree, Philadelphia, 1976) *IJHS*, 13, (1978), 72–73.

3. Review of *Candracchayagaṇitam* of Nīlakaṇṭha Somayājī, (edited by K. V. Sarma, Hosiarpur, 1976) *IJHS*, 13, (1978), p. 73.
4. *Review of Siddhānta-darpaṇam* of Nīlakaṇṭha Somayājī, (edited by K. V. Sarma, Hosiarpur, 1976), *IJHS*, 13 (1978), p. 73–74.
5. Review of *Rāśigolasphuṭanītiḥ* of Acyuta Piṣaraṭi, (in Hindi) ed. by K. V. Sarma, Hosiarpur, 1977, *IJHS*, p. 74.
6. Review of A. K. Bag, *Mathematics in Ancient and Medieval India* (Varanasi, 1979), *Gaṇita Bhāratī*, 3 (1981), 107–108.
7. Review of R. C. Pandeya (editor), *Grahalāghavaṃ Karaṇam* (Parts 1 and 2, Jammu, 1976 and 1977), *Ibid*. 108–109.
8. Review of *Census of Exact Sciences in Sanskrit Series*, A. Vol. 4, (by D. Pingree, Philadelphia, 1981). *Jour. Hist. Astron.* Vol-13 (1982), 225–226. Also *IJHS*, 18 (1983), 221–222.
9. Review of *'Prāchīn Bhārat Mein Vijñān'* (in Hindi) (by S. L. Dhani, Panchkula, 1982). *IJHS*, 19 (1984) 86–87.
10. Review of Rahman, A. et. al, *Science and Technology in Medieval India—A Bibliography of Source Materials in Sanskrit. Arabic and Persian* (INSA, New Delhi, 1982). *IJHS*, 19 (1984), 412–413.

Obituary—T. A. Sarasvati Amma

Dr. T. A. Sarasvati Amma was born as the second daughter of her mother Kuttimalu Amma and father Marath Achutha Menon. The year of her birth was apparently 1094 of the Kollam (Kolamba) era which is prevalent in Kerala and which corresponds to AD 1918–1919. The initial letters in her name indicate the place of her birth which was Tekkath Amayankoth Kalam (Cherpulassery) in the Palakkad district of Kerala. In correspondence, Dr. Sarasvati always signed her letters as T. A. Saraswathi which is spelled in the usual south Indian style and which appear in some of her papers.

T.A. Sarasvati Amma (1918–2000)

Dr. Sarasvati graduated from the University of Madras with first class in Part II (Sanskrit) and Part III (Physics and Mathematics). She obtained M.A. degree in

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Sanskrit from the Banares Hindu University in 1st class and secured 2nd rank. Later on she took M.A. degree in English Literature from Bihar University.

Smt. Sarasvati worked as a Government of India (Ministry of Education) scholar in the Sanskrit Department of the Madras University for three years (apparently between 1957 and 1960). She worked under the great Sanskrit scholar Dr. V. Raghavan who asked her to specialise in the field of Indian contribution to mathematics.

For brief periods, Sarasvati worked as a teacher in the Sree Kerala Varma College, Thrissur, and the Maharaja College, Ernakulam. She was appointed Lecturer in Sanskrit in 1961 for the Ranchi Women's College which was a constituent college of Ranchi University. She served in that capacity for about a dozen years.

She submitted her doctoral thesis (prepared under the guidance of Dr. Raghavan) on "Geometry in Ancient and Medieval India" (about 300 pages) to the Ranchi University in January 1963. It was examined by an eminent mathematician of north India (Dr. R. S. Mishra at Allahabad) and another of south India (Dr. A. Narasinga Rao).

Viva voce was held in Madras in February 1964 and it was approved for the award of Ph.D. degree by the Ranchi University soon. While in Ranchi, Dr. Sarasvati also supervised the doctoral thesis of R. C. Gupta (who was then serving B.I.T. Mesra, Ranchi) on "Trigonometry in Ancient and Medieval India" (Ranchi University, 1970–71).

Dr. Sarasvati was the Principal of the Shree Shree Lakshmi Narain Trust Mahila Mahavidyalaya, Dhanbad, Bihar, from 1973 to about 1980. This administrative assignment did not allow her any time for research which she enjoyed. In her letter of April 16, 1973, to R. C. Gupta, she wrote:

> I do not do any useful work now-a-days, immersed as I am in the squabbles and problems of an affiliated college accustomed to tactics to which I am not accustomed.

Dr. Sarasvati tried to publish her doctoral thesis privately at Ranchi. In fact, the whole thesis was printed (240 pages) at the G. E. L. Church Press, Ranchi. But due to presence of a very large number of printing errors (which escaped proof-reading), the whole lot was abandoned (R. C. G. has a copy of this).

Luckily, the thesis was published later on by the famous Motilal Banarsidass (Delhi, 1979; Revised edition, 1999). The delay in publishing was caused because the Ranchi University took a long time in releasing the financial aid it had sanctioned for the purpose.

The Delhi print of Dr. Sarasvati's *Geometry in Ancient and Medieval India* was a great welcome. It was praised by scholars and reviewers. One of them says that the book "is an almost exhaustive survey of geometry in Sanskrit and Prakrit literature right from the Vedic times down to the early part of the seventeenth century AD" (Deccan Herald dated 21 October, 1979). Dr. Michio Yano of Japan writes that "Sarasvati's discussion of the cyclic quadrilaterals treated by Brahmagupta (AD 628) reveals her remarkable competence in dealing with mathematical Sanskrit texts" (*Historia Mathematica*, Vol. 10, p. 469). Another reviewer remarks that "an admirable feature of the book is the impartial scholarly attitude to the study and a complete absence of parochialism" (*Annals of the Bhandarkar Oriental Research Institute*, Vol. 69, 1988).

After retirement from the Principal's post at Dhanbad, Dr. Sarasvati went back to her home town Ernakulam in Kerala. She wanted to continue her study and research but could not do much due to domestic and family work (her ailing aged mother needed care and attention). In 1986 she moved to a smaller house in Ottappalam.

Dr. Sarasvati breathed her last on August 15, 2000. Her only son (an engineering graduate) is living with his family in Australia (Dr. Sarasvati was separated from her husband soon after the birth of the only child). Her younger sister T. A. Rajalakshmi was a famous story writer and novelist but committed suicide in 1965.

Dr. Sarasvati was a simple lady but a great scholar. Her book on Geometry, in the words of Dr. Yano, "has established a firm foundation for the study of Indian geometry". It will continue to stimulate and inspire students of history of mathematics. The Kerala Mathematical Association has started a regular Prof. T. A. Sarasvati Amma Memorial Lecture in its annual conference to honour her memory (the 1st lecture was delivered by P. Rajasekhar in March 2002 on the "*Golayantra* according to Nīlakaṇṭha").

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3. "The Mathematics of the First Four Mahādhikāras of the Trilokaprajñapti", *J. of the Ganganath Jha Research Inst.,* 18 (1961–62), 27–51.
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6. "Development of Mathematical Ideas in India", *Indian Journal of History of Science,* 4 (1969), 59–78.
7. "Sanskrit and Mathematics", Paper read at the First International Sanskrit Conference, New Delhi, 1972; see Summary of Papers, Vol. III, 15–16. Published in the Proceedings (of the Conference), Vol. III, Part I, 196–200 (New Delhi, 1980). Also reprinted in the *Souvenir of the World Sanskrit Conference,* New Delhi, 2001, 63–78.
8. "The Treatment of Geometrical Progressions in India", Paper sent to Indian Science Congress for Symposium on History of Mathematics (Delhi, 1975). For a Summary, See *Advance Notes on Symposia,* pp. 7–8.
9. "Indian Methods of Calculating the Volume of the Frustum of a Pyramid", *In Sanskrit and Indology: Dr. V. Raghavan Felicitation Vol.,* Delhi, 1975, 335–339.
10. "Bhāskarācārya", in *Cultural Leaders of Indian Scientists* (ed. by Raghavan), New Delhi, 1976, 100–106.
11. Reviews of *Candracchāyāgaṇita, Siddhāntadarpaṇa and Sphuṭanirṇaya-tantra. Vishveshvaranand Indological J.,* 15 (1977), 173–176.

12. *Geometry in Ancient and Medieval India,* Delhi, 1979; pages xii+280, Revised Second edition, Delhi 1999. This book was reviewed by S. Balachandra Rao in *Deccan Herald* Magazine dated October 21, 1979; by A. K. Bag in *Gaṇita Bhāratī* Vol. 3 (1981), 53–54; by Michio Yano in *Historia Mathematica* 10 (1983), 467–470; by A. I. Volodarsky in *Mathematical Reviews* 84 (1984), 2516–2517; by D. G. Dhavale in the *Annals of Bhandarkar Oriental Research Institute* Vol. 69 (Pune, 1988); and by J. N. Kapur in *IJHS* 24 (1989), 93–94.
13. Review (by T. A. Sarasvati) of *Geometry according to Śulbasūtra* (authored by R. P. Kulkarni, Pune, 1983), *Gaṇita Bhāratī*, 8 (1986), 64–65. See Vol. 11 (1989), 60–62 for Kulkarni's reply to the review and etc.
14. "T. A. Sarasvati Amma (c. 1920–2000): A Great Scholar of Indian Geometry" (by R. C. Gupta), *Gaṇita Bhāratī*, 23 (2001), 125–127.
15. "Dr. T. A. Sarasvati Amma (1920–2000)" (by Academic Secy, KMA), *Proceedings of the Annual Conference of KMA*, Payyanur, 2004, p. 102.

The India-Born First President of the London Mathematical Society and His Discovery of Ramachandra

Augustus De Morgan (1806–1871), the founder President of the London Mathematical Society was born in Madurai of Madras Presidency (South India) on Friday, the 27th June, 1806. His father was associated with the East India company. His great-grand father James Dodson (died 1757) was the author of *Anti-Logarithmic Canon* (1742) and *Mathematical Repository* (1755).

Professor Augustus De Morgan (1806–1871)

Due to an early infection, Augustus De Morgan lost the sight of his right eye from the very beginning of his life. This deficiency, however, could not prevent him

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The Mathematics Teacher (India), Vol. 41 (1–2) (2005), pp. 100–115; This article is a humble homage to De Morgan on the eve of the bicentenary year of his birth.

K. Ramasubramanian (ed.), *Gaṇitānanda*,
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from attaining excellence in intellectual activities and achievements. His birth in the spiritual land of India (famous for religious tolerance and intellectual freedom) perhaps provided the *saṃskāras* to infuse in him pious qualities such as free and secular thinking. Universal catholicity of ideas and morality in character.

1 Education and Professional Life

The early education of Augustus De Morgan took place in England, and in February 1823 he entered the famous Trinity College, Cambridge. There, he was introduced to continental analytic mathematics from the very start. He was a brilliant student and, under the influence of good tutors such as George Peacock (algebraist), George Biddel Airy (astronomer) and William Whewell (philosopher and historian of science), his talents were further developed.

De Morgan graduated from the Trinity College in 1827 as a wrangler. He was a free thinker and his conscience made him reluctant to blindly accept everything of orthodox religion. He was often hesitant to act according to the doctrines of the established church, especially in academic matters. This prevented him from proceeding to the M.A. degree. Earlier, in 1788, his father-in-law (see below) was deprived of tutorship (also the Jews were debarred from University education!).

In 1837, De Morgan was married to Sophia Elizabeth, daughter of William Frend (1757–1841) who, himself, was a mathematician. Frend's *The Principles of Algebra* was published from London in two parts (1796 and 1799). The married life of the De Morgan couple was fruitful, both socially and academically. Their first son was born in 1839. Sophia Elizabeth posthumously compiled and edited her husband's *A Budget of Paradoxes* (London, 1872) a collection of anecdotes, reviews and humorous writings which De Morgan had published in the journal *Atheneum* from time to time. Her Memoir of Augustus De Morgan by His Wife with a selection from His Letters (London, 1882) contains valuable information and useful bibliography.

A. De Morgan was a man of principles, and far above the narrow thinking of sectarian religions. He cared more for his principles than for superfluous orthodox doctrines. To him the spirit of liberality appealed more than the rigid articles of the Anglican episcopal church. Thus he could not hold a fellowship at Cambridge (or Oxford) because he declined to undergo the prescribed religious test for the sake of that academic gain.

Nevertheless, due to his excellent merit, De Morgan was unanimously selected as founder Professor of Mathematics in February 1828 at the newly founded 'The London University' which was subsequently called the University College of London. It may be mentioned that at the time of selection, he was the youngest of the thirty-one aspiring candidates and had no teaching experience! The University College offered good intellectual freedom so necessary for academic development. This secular nature (of "godless institution") may be contrasted with that of the King's College, London (established in 1829), where attendance in lectures on theology was compulsory.

De Morgan held the position (up to 1866) for most of his remaining life except for a short period of five years (1831–1835) when he had resigned in protest against the unfair dismissal of one of his colleagues as a matter of principle.

De Morgan was a dedicated teacher and discharged his duties conscientiously. His extensive course trained students from elementary arithmetic to calculus of variation. He was so confident in making his teaching of mathematics interesting to students that they needed no stimulus "beyond their own pleasure in learning", he believed. He was a brilliant mathematician and made new discoveries in the field of advanced algebra, series, logic, etc. Yet he spent considerable time in devising better ways of teaching mathematics. He encouraged students to carry out long arithmetical computations for the sake of acquiring the art, power and skill of rapid and accurate computation (in those pre-computer days). But he discouraged cramming for examinations. Examination questions should be for eliciting the thinking power of the examinees and for testing real understanding of the subject (and not merely for showing the mathematical gymnastics of the setters). Many students of De Morgan became famous mathematicians, such as J. J. Sylvester (1814–1897), Isaac Todhunter (1820–1884), and Francis Guthrie (1831–1899) (originator of the famous four colour problem in 1852).

The idea of forming the LMS or London Mathematical Society ("to which all discoveries in mathematics could be reported") first came during a talk between two ex-students of the University College of London—Arthur Cowper Ranyard (1845–1894) who was elected Fellow of the Royal Astronomical Society in 1863 and George Campbell De Morgan (1841–1867), who was a University of London Gold Medalist of 1863, and a son (not the eldest) of Augustus De Morgan. A meeting was convened on November 7, 1864 with Prof. A. De Morgan in chair. It was quite natural and befitting that Prof. De Morgan became its first President. His speech at the first regular meeting of the LMS (held on January 16, 1865) was published in the *Proceedings of the LMS,* Vol. 1, 1866, pp. 1–9.

Under the presidentship of Prof. De Morgan, the LMS was "very high in the newest developments", and there was "no penny fine for reticence or occult science". It may be pointed out that the old Spitalfields Mathematical Society (SMS which flourished from 1717 to 1845) held that "it is the duty of every member, if he be asked any mathematical or philosophical question by another member to instruct him in the plainest and the easiest manner he is able". Another point to note is that while smoking and drinking were permitted at the meetings of the old SMS, "not a drop of liquor is seen at our (LMS) meetings" (claimed De Morgan).

De Morgan had hoped that the LMS would cultivate and support every branch of mathematics (and its application) including the then neglected areas of Logical Mathematics (i.e. connection between logic and mathematics) and History of Mathematics. He was re-elected President of the LMS at its Annual Meeting of January 1866.

In November 1866, De Morgan resigned from professorship, again on a matter of principle, because the policy of religious secularism (equality or neutrality) was betrayed by the council in appointing a candidate. The last years of De Morgan were not happy. The deaths of his son George (1867) and daughter Helen (1870) gave him further shocks. His health was badly affected. He breathed his last on March 18, 1871.

2 Contribution to Logic, Mathematics and Its History

It is said that in the nineteenth century, professors were appointed and paid solely for teaching. Research was not taken to be included in their formal duties. Thus academics with interest in research only found it difficult to support by pursuit of research alone. Of course, it is natural that a professor sincerely devoted to the exposition of subtleties of a subject will have ideas which can lead to its advancement.

Augustus De Morgan contributed to the development of mathematics as well as of logic. He believed that the two disciplines are connected closely, but complained that their followers are not paying adequate mutual attention to the twin disciplines. He said: "We know that mathematicians care no more for logic than logicians for mathematics. The two eyes of exact science are mathematics and logic: the mathematical sect puts out the logical eye, the logical sect puts out the mathematical eye, each believing that it sees better with one eye than with two (eyes)".

De Morgan's famous formula related to duality may be stated as follows:

> If A and B are subsets of a set S, then the complement of the union of A and B is the intersection of the complements of A and B; and the complement of the intersection of A and B is the union of the complements of A and B.

In the modern Boolean symbology, De Morgan's (above) Laws can be expressed as follows:

$$(A \cup B)' = A' \cap B'$$

and

$$(A \cap B)' = A' \cup B'.$$

The truth of these laws can be easily verified by drawing the Venn diagrams of the sets A and B, their union $A \cup B$, their intersection $A \cap B$, also then drawing diagrams of their complements A', B', etc.

De Morgan was an outstanding mathematician and an inspiring teacher. He had a particularly important role to play in the revival of mathematics in Britain. Florian Cajori writes (*History of Elementary Mathematics*, p. 208):

> Think of the pains taken by Augustus De Morgan to reform elementary mathematical instruction. The man who could write a brilliant work on Calculus, who could make new discoveries in advanced algebra, series, and in logic, was the man who translated Bourdon's arithmetic from French, composed an arithmetic and elementary algebra for younger students and endeavoured to simplify, without loss of rigour, Euclidean geometry.

To secure rightful place for algebra in liberal education was also a task of De Morgan. George Peacock (1791–1858) was among the earliest mathematicians to recognize fully the purely abstract nature and symbolic character of algebra. In his Algebra (1830) he defined symbolic algebra as "the science which treats of the combinations of arbitrary signs and symbols by means of defined, though arbitrary, laws".

De Morgan clearly saw that the laws of algebra could be created without using those of arithmetic. He believed that an abstract algebraic system could be created with arbitrary symbols and a set of laws under which these symbols are operated on. Only afterwards would one need to provide interpretations of these laws. Thus he asserted the freedom to create algebraic axioms for symbols and even realized that the symbols could represent things other than quantities, magnitudes or numbers. Soon systems were created which obeyed laws different from those obeyed by numbers in arithmetic (e.g. non-commutative algebras).

De Morgan's work greatly advanced that of Peacock in the area of foundation of algebra and encouraged W. R. Hamilton's on quaternions and George Boole's on algebraic logic and Ernst Schroeder's on algebraic logic and lattices. He established the logarithmic criteria for testing the convergence of series although the subject was discussed by others also.

A scholar of any science cannot be a full authority of the discipline without a knowledge of its history. According to De Morgan "the history of most of the sciences resembles a river which sinks underground at a certain part of its course, and emerges again at a distant, spot, swelled by certain tributaries, which have joined it in the tunnel" (Introduction to *Arithmetic Books*). He was a great lover of History of Mathematics and studied it with zeal and scholarly attitude. He acquired a profound knowledge of the subject and made significant contribution in the area. He said: "The early history of the mind of men with regard to mathematics leads us to point out our own errors; and in this respect it is well to pay attention to history of mathematics".

In fact, a study of history of mathematics is not only instructive and enlightening but is charming in itself. J. W. L. Glaisher even said that "no subject loses more than mathematics by any attempt to dissociate it from its history". According to the opinion of the famous historian of mathematics, Florian Cajori, "Few contemporaries were as profoundly read in history of mathematics as was De Morgan" (*History of Mathematics*, p. 331).

The *Arithmetical Books from the invention of Printing to the Present Time* (London, 1847) of De Morgan is a chronologically arranged descriptive catalogue of a very large number of works made from "actual inspection". It is a comprehensive and unique bibliographical work which is useful also for the study of the spread of Indian decimal place-value notation and the arithmetic based on the system. His celebrated *Budget of Paradoxes* (1872) is a mine of historical information.

Regarding the infamous calculus priority dispute between Newton and Leibnitz, De Morgan felt it his duty to examine various documents and publish his new findings. The result was the rehabilitation of Leibnitz among the British historians of mathematics. De Morgan paid attention to historical writings of earlier mathematicians such as Wallis (1657), Dechales (1690), Heibronner (1742), etc.

De Morgan was a lover of mathematical puzzles and conundrums. For his birth year (= y, say) he declared: “I was x years old in the year x^2”. So that we have the relation

$$y + x = x^2.$$

Now, in the seventeenth to nineteenth centuries, the possible values of x^2 are 1681 (= 41^2), 1764 (= 42^2) and 1849 (= 43^2). Of these, the last 43^2–43 gives the birth-year $y = 1806$ of De Morgan. In printing mathematical symbols, De Morgan proposed the use of the slant line or ‘solidus’ for printing fractions in the text.

3 De Morgan’s Writings and Publications

Augustus De Morgan had wide knowledge of mathematics, logic, philosophy and history of mathematics. He was a prolific writer throughout his career. He wrote about a score of books and about two hundred articles on various topics. Some of his more important books may be mentioned here. His writings show a balanced attention to professor’s twin duty of teaching (dissipation of knowledge) and research (advancement of knowledge).

De Morgan had begun his career, as an author, by a translation of part of Bourdon’s work on algebra (*Arithmetic Books*, p. 91). The first edition of his *The Elements of Arithmetic* appeared in 1830. His *On the Study and Difficulties of Mathematics* (London, 1831) is devoted to pedagogy. Then there is his *The Elements of Algebra Preliminary to Differential Calculus*, etc., (London, 1835). A year later he published *The Connexion of Numbers and Magnitudes*: An attempt to Explain the Fifth Book of Euclid which is in dialogue form.

De Morgan’s famous *Treatise on the Differential Calculus* appeared in 1842. It was a work of great ability, especially in clarifying the concept of limit and in the treatment of infinite series which were considered “very perplexing” by the students. His equally famous *Arithmetical Books* (1847) has been mentioned above. His *Formal Logic or Calculus of Inference* (1847) sets forth the algebra of logic or algebra of sets.

De Morgan’s *Trigonometry and Double Algebra* (London, 1849) has a peculiar title. In 1851, he published *The Book of Almanacs* in which were provided 35 rigorously constructed charts (“almanacs” he called them) for any year upto 2000 AD These charts were based on the tables of Louis Benjamin Francoeur who published them in 1842 from Paris (see *Gaṇita Bhāratī*, Vol. 17, p. 63).

De Morgan’s *Contents of the Correspondence of Scientific Men of the Seventeenth Century* (Oxford University Press, 1862) is useful for the historical study of the crucial century. His remarkable *A Budget of Paradoxes* (1872) a posthumous compilation by his wife is already mentioned. It is his collection of eccentrics (including circle-squarers) which were featured in a magazine. The second edition (by D. E. Smith) of this work was published in two volumes (Chicago, 1915), of which the first volume has been reprinted as *The Encyclopedia of Eccentrics* (Open Court, La Salle, 1974).

De Morgan's variety of articles on the history of mathematics were published in *Penny Cyclopedia, English Cyclopedia, Companion to (British) Almanac, Philosophical Magazine*, etc. Profuse use of his historical writings was made by W. W. Rouse Ball, Florian Cajori and others in their historical writings. Large space will be needed to enlist his articles which have still historical, educational and mathematical significance and relevance. Some of these are

1. "On the Foundation of Algebra", *Transac. Cambr. Philos. Soc.*, 7, 1839–43, pp. 287–300.
2. "Table", *Penny Cyclopedia*, 23 (1842), pp. 496–501, and supplement 2, (1846), pp. 595–605.
3. "On the almost total disappearance of the earliest trigonometrical canon", *Philos. Magazine, Series*, 3, Vol. 26 (1845), pp. 517–526.
4. "On the Early History of Infinitesimal Calculus in England", *Philos, Magazine Ser.* IV, Vol. 4 (1852), pp. 321–330.

4 De Morgan's Discovery of Ramachandra

Ramachandra (1821–1880) was a science teacher at the Delhi College (now called Zakir Hussain College). He was born at Panipat in a Hindu *Kayastha* family. His father Rai Sunder Lal Mathur (died 1831) worked in the East Indian Company's revenue department. The early formal education of Ramachandra took place in an English government school (1833–1839) where he showed his talent as a bright student of mathematics. Khushal Rai, a reputed rich man (*rais*), succeeded in arranging the marriage of his deaf and dumb daughter Sita with Ramachandra even before the latter attained teenage (*pandits* and *purohits* had been lobbying for Rai since 1832, and enquiries about the girl were considered taboo in those days of arranged child marriages).

While teaching in the Delhi College (from 1843 onwards), Ramachandra was also involved in translating European scientific works into Urdu as undertaken by the Vernacular Translation Society. These included Hutton's Trigonometry, Boucharlet's Conic Sections, Simon's Analytic Geometry and some books of I. Todhunter. His Urdu mathematical primer *Sari-ul-Fahm* was published from Delhi in 1849. It was said to be written in 1845 in which year his two other Urdu books, namely *Asool-i-Jabr-O-Muqabal* (Principles of Algebra) and *Asool-i-Ilm-i-Hisab-Juziat-O-Kuliyat* (Principles of Maxima and Minima), were published from the same place.

Above all, Ramachandra could also find time to complete his mathematical investigations and writing of his famous book Treatise on Problems of Maxima and Minima Solved by Algebra.[1] It was published from Calcutta in 1850 and created stir in mathematical circles. It made him a reputed mathematician but some native educationists had rebuked the author's 'temerity in publishing the book in English' (instead of

[1] AMTI has now published this book as edited by Prof. M. S. Rangachari with his comments.

vernacular language). It was meant to advance the educational standard and to serve cause of science.

Ramachandra was a believer in rationalism and in the neutrality of science. He was against the blind religious practices (cf. reformists like Raja Rammohan Roy and Swami Dayananda Saraswati). In fact, it was the time of encounter between science and religion, between eastern and western, and also between traditional or old and new ideas and systems.

Anyway, Ramachandra's perception of the then prevailing Hindu religion and culture, his administration of the emerging scientific West and, possibly, prospect of a brighter future career were factors which caused a great change in his faith. He was baptized in 1852 and thus became known as Yesudas Ramachandra.

It is said that Drinkwater Bethune, Chairman of the Education Commission, Calcutta, forwarded a copy of Ramachandra's *Treatise on Problems of Maxima and Minima* to Augustus De Morgan, the Professor of Mathematics at the University College, London, for comments. In this way De Morgan discovered Ramachandra (Just as Hardy was to discover the famous Ramanujan later on).

De Morgan arranged the re-publication of Ramachandra's book in England with an Introduction for distribution in Europe. He had said: "I would point out how to bring Ramachandra under the notice of scientific men in Europe". Thus the second edition of the book was published from London in 1859. It was done so "by the order of the honourable court of directors of the East India Company … in acknowledgement of the merit of the author (Ramachandra)". The book showed to English men of science that "the Hindu mind masters problems without the aid of Differential Calculus".

In 1863, Ramachandra's *A Specimen of a New Method of Differential Calculus called the Method of Constant Ratios* was published from Calcutta. Towards the end of his life, he wrote on some religious topics including *Aitaraz-i-Quran* (Delhi, 1876) which is critical of Islam.

5 Miscellany

In Latin, the word 'augustus' means 'grand'. Indeed Augustus De Morgan was a grand mathematician of his time. He was an outstanding teacher, an eminent scholar, a prominent historian and a veteran writer in mathematics. Recently Dr. Adrian Rice reported to have successfully completed his doctoral thesis on "De Morgan and the Development of University Level Mathematics in London During the nineteenth century" (Middlesex University, 1997).

To cherish the loving memory of the first and founder professor of mathematics, the alumni of the Department of Mathematics, University College of London, call their association as the 'De Morgan Association'. Similarly, to commemorate its founder president, the London Mathematical Society awards the De Morgan Medal every three years for outstanding achievements in mathematics.

Professor De Morgan had great love especially for rare and ancient books and in their history. He had collected more than 3000 books. After his death, the collection was purchased by Lord Overstone (1796–1883), who presented it to the University Library where it is called the De Morgan Collection. De Morgan had a habit of pasting letters (he received) and pictures (cut out from magazines) in suitable places in the books he possessed. The numerous De Morgan's manuscripts, nearly 200 notebooks, items of correspondence, etc., were given to the University Library by his eldest son.

A man of principle, De Morgan also depicted his eccentric nature. He never voted at an election, declined the offer of an honorary L. L. D. degree and even refused to be proposed for fellowship of the Royal Society although he was a Fellow of the Cambridge Philosophical Society (Astronomer T. I. M. Forster who discovered a comet in 1819 also refused the offer of FRS because he "disapproved of certain of its rules").

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M. Rangacharya and His Century Old Translation of the *Gaṇita-sāra-saṅgraha*

Lots of Achievements in ancient Indian Mathematics as reflected in the works of Āryabhaṭa I (born 476 AD), Brahmagupta (seventh century AD) and Bhāskara II (twelfth century) were freshly made known in modern form to the Western world during the nineteenth century. Leading role in this regard was displayed by Western scholars such as R. Barrow, H. T. Colebrooke, S. Davies, C. Hutton, H. Kern, L. Rodet, E. Strachey and John Taylor.

Often some Indian scholars (e.g. Bapudeva Sastri and Sudhakara Dvivedi) were also involved and associated in this academic and educational propagation. However, according to the then well-known historian of mathematics, D. E. Smith, "native scholars under the English supremacy have done so little to bring to light ancient mathematical material known to exist and to make it known to the Western world". Nevertheless, he had soon found a sort of exception in Prof. M. Rangacharya whom he met in Madras (about 1905). He came to know about latter's edition and translation of the *Gaṇita-sāra-saṅgraha (=GSS)* then contemplated to be published for the first time. With Smith's introduction, the fruitful work appeared a century back (Madras 1912).

Prof. M Rangacharya (1861–1916)

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Rangacharya was born in 1861 in Melkote† (of the then Mysore State) where he also received his early education. In 1881, he passed the B.A. and worked in Madras Christian College. Soon, the Government College, Kumbakonam offered him a lecturership in science. On completing M.A. in Physical Sciences, he served the Govt. College, Rajahmundry, the Presidency College, Madras (1890), and the Maharaja College, Trivandrum after which he returned to the Presidency College. Here, he became professor of Sanskrit and Comparative Philosophy in 1901. It was the result of the great impression, influence and impact which he had already made by his scholarly lectures and publications. He also became Curator of the Govt. Oriental Manuscripts Library which was a literary 'Laboratory' for him. In 1903, the title "Rao Bahadur" was awarded to him. He died in 1916 and thus a brilliant career ended.

It is interesting to note that Rangacharya's interest in *GSS* was initiated by the then Director of Public Instruction, G. H. Stuart who had asked him to find out if the G. O. M. Library has "any work of value capable of throwing new light on the history of Hindu mathematics, and publish it, if found with English translation etc." A search yielded three incomplete manuscripts of *GSS* in the Library. On the advise of Stuart, some other *mss* (luckily found to be complete) were procured. The tough job of editing and translating could be taken to final stage due to Rangacharya's labour of love. Unfortunately, Stuart did not live long enough to enjoy the delight of seeing the final form.

Rangacharya was a remarkable scholar, scientist and educator. He enriched his formal routine knowledge further by self study of various branches of learning. As a result, he could successfully teach a variety of subjects such as biology, chemistry, physics, mathematics, history, philosophy, Sanskrit and Indology. The variety and depth of his scholarship is well reflected in his studies, researches and publications which cover scientific as well as humanity topics. We take an example.

The cyclic division of time is peculiar feature of ancient Indian Chronology. But the four *yugas* (called *kṛta, tretā, dvāpara and kali*) have also been given a variety of interpretation in Indology. For instance, the *Aitareya Brāhmaṇa* is said to have stated that "one who sleeps is *kali*, one who gets up is *dvāpara*, one who stands up is *tretā* and one who moves is *kṛta*" (Rajneesh, p. 18). Rangacharya interpreted the four *yugas* in terms of historical periods. According to him, the *kṛta* ("deed") *yuga* refers to the period when the Āryans performed heroic deeds in conquering lands and establishing their supremacy in the Indian subcontinent, the *tretā* i.e. the ("three" i.e. the three *nitya-agnis* dealt so frequently in *Śulba-sūtras*), *yuga* refers to the period when the Aryans were concentrating on the vedic sacrifices and priesthood, etc. (1909). He mentioned the Sanskrit *Sūrya-siddhānta* for giving the life of Brahmā as 31104×10^{10} solar years which are taken to form the life of universe according to Hindu theory of cycles of creation and *pralaya* (destruction). But some *Purāṇas* consider even Brahmā's whole life just as twinkling of the eye of Lord Kṛṣṇa or Śiva (Gangooly, p. 12)!

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†For obvious reasons, the reading: 'Malkota' in the published article has been changed as above.

The *GSS* is a very significant work not only of the Jaina School but of ancient Indian Mathematics in general. However, due to religious bias, it did not find the mention it deserved in the works of Hindu mathematical writers belonging to ancient and medieval periods. Although entitled *saṅgraha* ("collection") it contains many original contributions of the author Mahāvīra who was a Digambara Jaina belonging to the ninth century.

The translation of *GSS* into English is Rangacharya's greatest contribution to history of mathematics and by doing so he has made, in the words of Smith, "the mathematical world his perpetual debtor". The interest shown by Smith in *GSS* also helped in its quick worldwide publicity. Rangacharya's work has useful appendices. During the last 100 years, scholars in the field have been benefited by his clear edition of the Sanskrit text, faithful translation and accompanying notes. Scores of research papers, essays and popular articles on *GSS* as well as on its author have been published (see a brief bibliography at the end). There is a need to make a fresh and deep critical study of the *GSS*. It will not be out of place to mention a few things.

The *GSS* 1.49 gives the wrong rule $\frac{N}{0} = N$

possibly because division by zero was looked upon as of no effect (cf. distribution of N things among zero persons). Mahīdhara's commentary (1587) on *Līlāvatī* contains the above rule numerically with $N = 9$ (Ganitanand, p. 139). A simple and practical algorithm to express a given fraction $\frac{p}{q}$ into unit fractions is the Mahavira–Fibonacci method (Gupta 2010, pp. 87–88). A typically Jaina formula for finding the arcual length s of a circular segment of chord c and height h is their very ancient empirical rule (*Ibid.*, pp. 66–68)

$$s = \sqrt{c^2 + 6h^2}.$$

Its history, rationale and related forms are interesting. GSS (VII. 63) appears to have used it for accurate rectification of the 'elongated circle' or ellipse (Gupta 1974).

Takao Hayashi has discussed several mathematical formulas and aspects of *GSS*. He [1987] gives a new interpretation of the Quiver Problem and [1992] deals with the Conch-like plane figure. Links between Mahāvīra and the non-Jaina Nārāyaṇa Paṇḍita (1356) are clearly reflected in many ways. For finding the area of the curved surface of spherical segment, *GSS* (VII. 25) prescribes.

$$A = \frac{p \cdot w}{4}$$

where p is the perimeter of the base circle and w is the curvilinear width of the bulged surface (Gupta 1989). This empirical rule easily leads to the expressions

$$S = \frac{C^2}{4} = \left(\frac{C}{2}\right)^2$$

for the full surface S of a sphere where C is the circumference of any great circle. Interestingly such a rule for finding S (in form of above expressions) was known to

Ṭhakkura Pherū (c. 1300) in India and to the seventeenth-century Japanese mathematicians as old method (Gupta 2011). Surprisingly, $S = (\frac{c}{2})^2$ also appears in some Italian manuscripts of the fourteenth and fifteenth centuries (Simi and Rigatelli., p. 469). *GSS* rules for volume of a sphere and frustum like solids have been given a new look (Gupta 1986 and 2011).

Rangacharya's endeavour could procure only a few manuscripts of *GSS* for consultation. Now D. Pingree's *Census*, Vol. 4 (1981), pp. 388–389, lists about 50 manuscripts. Daivajña Vallabha's *Kanarese* commentary on GSS is known and he is also claimed to be the author of a Telugu commentary on the same work. Other Commentators of *GSS* include Varadarāja and Sumatikīrti who belong to the sixteenth century (?). In the eleventh century, *GSS* was translated into Telugu by Pāvuḷūri Mallana, son of Sivanna. But he also made some changes and additions and the whole work is popularly called Pāvuḷūri Gaṇitamu. His grandfather was also named Mallana whom some scholars regard the real author of the Telugu work (Arunachalam p. 149).

It so happened that, about two centuries, a keen scholar-officer named Benjamin Heyne studied the Pāvuḷurī Gaṇitamu and translated its *kṣetragaṇita* chapter into English. This was published as "A free translation of the Chetri Ganitam or Field Measuring of the Hindoos" in *Tracts of India* (London 1814) [Gupta 2002].

According to Pingree (p. 388), *GSS* was translated into Rajasthani by Amīcandra in 1842. L. C. Jain's edition with Hindi translation (Sholapur, 1963) is based on Rangacharya's version. B. B. Bagi's Introduction says that "a new edition with English translation by an experienced mathematician who knows Sanskrit well is an urgent need". In 2000, Sri Hombuja Jain Math published the *GSS* with Rangacharya's translation along with a Kannada.

The relation and relative chronology of Mahāvīra (c. 850) and Śrīdhara (eighth century) has been often discussed by scholars. Although K. S. Shukla's introduction to Śrīdhara's *Pāṭīgaṇita* (Lucknow, 1959) placed Śrīdhara after Mahāvīra, later on he accepted the usual dates (see *Gaṇita Bhāratī*, Vol. 9, 1987, pp. 54–56, and Vol. 25, 2003, 146–149).

A critical edition of *GSS* based on more mss along with some ancient commentary will be a tribute to Rangacharya on the occasion of his coming death centenary.

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Part IX

Transmission of Mathematics and Astronomy Between India and Other Civilizations

Indian Astronomy and Mathematics in the Eleventh-Century Spain

According to Ibn al-Ādamī (c. 950 AD) as quoted by Qāḍī Ṣā'id al Andalūsī (d. 1070), Caliph al-Manṣūr of Baghdad (755–775 AD) ordered a Sanskrit astronomical work to be translated into Arabic.[1] This translation was made by al-Fazārī with the help of the Indian astronomer who had brought the said Sanskrit astronomical (i.e. *Siddhānta*) work to the Abbasid Court (in 771 or 773), and the Arabic version was called *Zij al-Sindhind* from which descended a long tradition within Islamic astronomy extending upto Spain for several centuries.[2]

Al-Fazārī also composed (c. 780) the *ZIj al-Sindhind al-Kabīr* (The Great Sindhind) which was based on the *Zij al-Sindhind*, and in which three Indian values, namely 3270, 3438 and 150, were used for the *Sinus Totus (i.e. trijyā)* or radius. A similar Arabic work called *Zīj maḥlūl fī al-Sindhind li daraja daraja* ("Astronomical Tables in the Sindhind Resolved for every Degree") was composed by Ya'qūb ibn Ṭāriq who had collaborated personally with the Indian astronomer who went to Baghdad in 771 or 773 (as mentioned above).[3]

Al-Khwārizmī who flourished under the region of Caliph al-Ma'mūn (813–833), made extensive use of the *Zīj al-Sindhind* and its derivative works in composing his *Zīj* (Astronomical Tables) which became famous through out the Islamic world upto Spain and in Europe through subsequent Latin translations. It is said for his astronomical work, al-Khwārizmī was fār more heavily indebted to Indian work than to other sources.[4] His *Astronomical Tables* were redacted by Masalama al-Majrīṭī who flourished in Spain and died there about 1007 (or later).[5] This was one of the channels through which Indian astronomy and mathematics penetrated Spain and the influence of Indian astronomy represented by the tradition of *Sindhind* continued there even after Ptolemy's *Almagest* (on Greek astronomy) came to be known.[6]

In the *Astronomical Tables* of al-Khwārizmī, the corrections for the planets and the reckoning of time are made with reference to the central place of the earth,

Gaṇita Bhāratī, Vol. 2, Nos. 3–4 (1980), pp. 53–57; Paper presented at the Symposium on History and Philosophy of Mathematical Sciences held at the Osmania University, Hyderabad, on January 2, 1979, under the joint auspices of the Indian Society for Histroy of Mathematics and the Indian Association for the History and Philosophy of Science.

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called Arin, i.e. Ujjain which is the zero meridian of Hindu astronomy.[7] However, it was but natural for al-Majrīṭī to work out his adaptation of these Tables for the longitude of Córdoba (Spain) where he flourished.[8] Thus the crescent visibility table was computed on the basis of Indian visibility theory for the latitude of northern Spain.[9]

Al-Majrīṭī had several disciples who made his work known throughout the peninsula and through them exercised considerable influence on the work of later scientists.[10] For instance, one of the disciples was Ibn al-Samḥ (d. 1035) of Granada (Spain) who followed Indian astronomical techniques and about 1010 AD wrote a *zīj* (not extant) which is reported to be based on the methods of the famous *Sindhind*.[11]

Such was the impact of Indian scientific achievements in Spain that Qaḍī Sā'id (d. Toledo, Spain, 1070) included Indians among the nations which have cultivated the sciences in his *Ṭabaqāt al-Umam* ("Category of Nations") which he wrote there in 1062. He says:[12]

> The first nation (that has cultivated the sciences) is the people of India who form a nation vast in numbers, powerful, within great dominations. All former kings and past generations have acknowledged their wisdom and admitted their pre-eminence in various branches of learning....
>
> The king of India was called King of Wisdom because of the concerns of the Indians for the sciences and their distinction in all branches of knowledge....
>
> Among all nations, during the course of centuries and throughout the passage of time, India was known as the mine of wisdom,... and the Indians were credited with excellent intellects, exalted ideas, universal maxims, rare inventions and wonderful inventions.
>
> .... they (the Indians) have studied arithmetic and geometry. They have also acquired copious and abundant knowledge of the movements of the stars, the secrets of the celestial sphere and all other kinds of mathematical sciences.

About the systems of astronomy followed in India Qāḍī Sā'id says:[13]

> Among the Indian systems of astronomy, there are three famous schools, i.e. those of Sindhind, Arjabhar and Arkand. Exact information has reached us only about the school of Sindhind (*Siddhānta*) which has been adopted by a group of Muslim scholars who have used it for the compilation of astronomical tables ....
>
> The followers of Sindhind state that the apogee *(awjāt)* and nodes *(Jawzahrāt)* of seven planets are all assembled at the head of Aries once in 4320,000,000 solar years, and they name this period as 'world-period' *[*cf. kalpa*]*. . . . As for the followers of Arjabhar [Āryabhaṭa], they are in agreement with the followers of Sindhind except with regard to the length of the 'world-period' [which is taken to be 4320,000 solar years in this system]. . . . As regards the followers of Arkand school, they differ from the former schools in respect of movements of planets and the 'world-period' but exact nature of this difference is not known to us.

Al-Zarqālī or Al-Zarqāll (Azarchiel of the Latins), the most celebrated astronomer of his time, lived in Toledo and Córdova (both in Spain) where he died in 1100 AD His name is associated with the famous *Toledo Tables* which enjoyed an enormous circulation. They were extraordinarily successful in the Latin world, and by the twelfth century they were used throughout Europe.[14]

The *Toledo Tables* have lot of material on Indian astronomy. It has a set of tables, of Indian origin, for computing oblique ascensions in terms of right ascensions.[15] Its table of Sines (with $R = 150'$) and the table of Solar declinations (with $e = 24°$, the Hindu value of the obliquity of the ecliptic) are identical with the corresponding tables in the Sanskrit *Khaṇḍa-khādyaka* (665 AD) of Brahmagupta.[16]

Like Indian, al-Zarqāli assumed circumference equal to 360° and diameter equal to $300'$, and defined *kardaga* as arc of 15° (cf. *gṛhārdha* or half sign which is used as a tabular interval by Brahmagupta).[17] He defined Sinus Rectus and Sinus Versus like the Indian *Kramajyā* and *Utkramajyā,* and gave the usual Indian methods for computing the tabular Sines.[18] Using the standard Hindu gnomon of 12 units and the Indian radius of 150 minutes, al-Zarqālī found the shadow of the sun from its altitude and vice versa. For π, he gave the approximation $\frac{22}{7}$, $\sqrt{10}$ and $\frac{62832}{20000}$, the last two of which are obtained from Indian sources.[19] The value $\sqrt{10}$ is found in old Jaina Canonical works, and $\frac{62832}{20000}$ is exactly the approximation given by Āryabhaṭa I (born 476 AD).[20]

In 1089, al-Zarqālī elaborated the *Almanac* of Ammonius in which the trigonometrical portion presents the mingling of the Indian material with that from other sources.[21]

Regarding the knowledge of Indian arithmetic in the eleventh-century Spain, we again quote the words of Qāḍī Sa'id who says:[22]

> In the domain of numerical sciences, we have their (i.e. of Indians) *hisāb al-ghubār* which was explained by *al-Khwārizmī*. It is a very compendious and quick system of calculation, easy to understand, simple to adopt, and remarkable in its composition, bearing testimony to the sharp intelligence, creative power and remarkable faculty of invention of the Indians.

Unfortunately, like his *Zīj*, the Arabic original of al-Khwārizmī's work on Indian arithmetic is lost, one of its suggested titles is *Kitāb Ḥisāb al-'Adad al-Hindi* ("Treatise on Calculation with Hindu Numerals").[23] However, its Latin version entitled *Algoritmi De Numero Indorum* is well known, and this played a very important role in introducing the Indian decimal place-value system of numerals and the corresponding computational methods in Europe.[24]

Similarly, the first serious Latin work on astronomy was a translation (via Arabic) of a redaction of *Sindhind* which was a translation of an Indian astronomical work in Sanskrit.[25]

We have already mentioned al-Majrīṭī's redaction (made in Spain) of al-Khwārizmī's *Zīj* which had strong influence of Sindhind.[26] Ibn al-Muthannā wrote a commentary (lost) on the original *Zīj* of al-Khwārizmī, but a Hebrew translation (of this lost commentary) by Ibn Ezra (born in Toledo, ca. 1090 and died at Calahorra, Spain, c. 1165) is extant. In some of the matters, e.g. table of ascensional difference which is missing in al-Majrīṭī's version, Ibn al-Muthannā's version shows further Indian influence by the use of Indian Sinus Totus of 150' and an interval of a *kardaga* of 15°.[27]

Ibn Ezra himself composed *Sefer ha-Mispar* ("Book of the Number") which describes the decimal system of numerals with zero showing a deep influence, although he often used the letters of the Hebrew alphabet as numeral-signs.[28] Just a few years before and after Ibn Ezra put the material, containing Indian techniques and parameters, into Hebrew, a host of other scholars, like Adelared of Bath (fl. 1116–1142), Plato of Tivoli (fl. 1132–1146) and Gerard of Cremona (d. Toledo, 1187), translated Graeco-Arabic and Indo-Arabic scientific literature into Latin. There were several centres where this translation work was done but Spain had major share in the activity. And it was through these Latin translations that astronomy and mathematics flowed wider into Europe causing a step towards renaissance.

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Ibn Ezra also knew Āryabhaṭa I's value of π (attributing it to "Indian sages") although there is some mistake or scribal error in quoting the value (see *Historia Mathematica,* I, 1974, 25).

Indian Astronomy in West Asia

1 Upto the Middle of Eighth Century AD

It is difficult to gather trustworthy knowledge of astronomy in Iran before the reign of Ardashīr I (AD 226–240) and Shāpūr I (241–272)[1] who encouraged the spread of Indian Science in the region. The ninth century Pahlavi (Middle Persian) Dēnkart informs us authoritatively that the two kings had Indian and Greek works translated into Pahlavi and that they were revised under Khusro Anūshirwān (sixth century) (Pingree, 1964–66, p. 119).

According to Abū Sahl ibn Nawbakht who worked in the library of Hārūn al-Rashīd (786–809), there was made then a translation of a Sanskrit work composed by an Indian whose name was Farmāsb which has been taken equivalent of Parameśvara (Pingree 1976a, p. 146).

In Iran, about the middle of the fifth century, an official royal handbook on astronomy was compiled. It is generally called *Zīj al-Shāh* (Royal Astronomical Table). It used Indian parameters from the Sanskrit *Paitāmaha Siddhānta* of the *Viṣṇudharmottara Purāṇa* (Haddad, p. 213). The tenth century Cairo astronomer Ibn Yūnis says that, about 450 AD, the solar apogee was placed by Persians in Gemini 17; 35° which is precisely the longitude it had at that date in the above Sanskrit work (Pingree 1963–64, pp. 3–4; and 1967–68 for the work).

Māshā ʻallāh (fl. 750–815), a Persian Jew from Baṣra has been quoted by ʻAli ibn Sulaymān al-Hāshimī in his *Kitab fiʻīlal al-zījāt* (The Book of the Reasons Behind Astronomical Tables). He says:

[1] All dates are in AD unless otherwise stated.

Revised version of a paper published in the *Vishveshvaranand Indological Journal* Vol. XX, 1982, pp. 219–236. Originally this was author's lecture delivered on 19th September 1979 at the University of Jodhpur under the *INSA* Programme in History of Science.

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> Khusro Anūshirwān, when he beheld the difference between the Arkand and what Ptolemy asserted, he gathered together the people learned in computation and in (astrological) judgements, and he looked over these two books. He found the Arkand (based on Indian material) to be the most accurate by observation and eyesight, and judgements based on its planets more accurate. So he worked out a *Zīj* called *The Shāh* (Haddad, p. 95).

Thus a new version of the Royal Astronomical Table was written which is called the *Zīj Shahriyārān* of AD 556. Another version of the *Zīj al-shāh* was written during the reign of Yazdigird III (632–652) but the source was still Arkand (Haddad, p. 212–213).

The midnight system of Āryabhaṭa I (b. 476) is not extant. Its parameters are same as found in the Old *Sūryasiddhānta* (in the redaction of Lāṭadeva) which is summarized by Varāhamihira in his *Pañcasiddhāntikā* and whose epoch is AD 505. Kennedy has been able to demonstrate that *Zīj al-Shāh* contains the same parameters (Kennedy 1956, p. 130 and Pingree 1963, p. 242).

The standard Arabic alphabet has no characters to render the guttural sound of the Sanskrit letter *g* (ग, as in get) which is therefore usually transliterated in Arabic by guttural letter *k* (kāf) as done by al-Birūnī when he wrote *ankula* in Arabic for *aṅgula* in Sanskrit (Kennedy 1976, II, p. 27). Thus it is generally accepted that the Arabic word *arkand* comes from the Sanskrit word *ahargaṇa* (heap of days).

For calculating planetary positions, the concept of *ahargaṇa* is basic in the Indian astronomy. It has been suggested that the midnight system (of about AD 500) as expounded by Āryabhaṭa I, "could easily have been translated into Pahlavi as the *Zīj al-Arkand* by about 550" (Haddad p. 212). A better surmise will be that the translation could have been done of the Old *Sūryasiddhānta* which had the same system.

The last version (seventh century) of the *Zīj al-Shāh* was translated into Arabic in about 790 perhaps by Abū-al-Ḥasan 'Alī bin Ziyād al-Tamimī from a Pahlavi (Haddad, p. 216). Bīrūnī states (Sachau, II, p. 7) that the *Khaṇḍakhādyaka* (AD 665) of Brahmagupta is known among the Muslims as *Al-Arkand*. Of course, both belong to the same astronomical tradition, the old midnight Indian system.

Bīrūnī had come across an Arabic *al-Arkand* (of about 735 AD) which was a bad translation of the *Khaṇḍakhādyaka* (Haddad, p. 207). In the true spirit of scholarship, he himself made a corrected version of the Arabic translation in which the Sanskrit technical terms were properly translated. It is called *Tahdhib Zīj al-Arkand* (MAIC, p. 153) and (Haddad, p. 211).

According to Bīrūnī, the maximum equations for the Sun ($2°14'$) and moon ($4°56'$) in the *Zīj al-Shāh* are derived from the Hindus (Saffouri and Ifram, p. 28). Elsewhere he gives other informations about the same *Zīj* such as (Kennedy 1956, p. 130, Pingree 1970a, p. 118):

1. Use of midnight epoch in contrast to the general practice of using noon.
2. Use of Hindu methods and parameters at least with regard to planetary equations.
3. Use of the standard Hindu gnomonic height of 12 digits in connections with shadow problems.

Now we briefly discuss about the origin of an Arabic word frequently used in *Zījes*. It is *kardaja* which is the interval employed in the column of arguments in tables of sines and etc. (Haddad, p. 214). Scholars (D. Pingree etc.) think that the above word comes from the Sanskrit *Kramajya* (*Ibid*). An objection to this is that this Sanskrit word stands for sine, which is the function and not the argument (arc or degree). In my view *kardaja* comes from the Sanskrit *gṛhārdha* ("half *rāśi*" *or* 15°) by the basic change of Sanskrit *g* to Arabic *k*, etc. In fact, al-Hāshmī says "each *kardaja* is 15° which is 900 minutes" (Haddad p. 153). Even in Latin works it is so. Regiomontanus (ed. by D. Santbech, Basle 1561) explains "Kardaja portio arcus 15 gr. appellatur" i.e. *Kardaja* is arc of 15°.

Of course, as it happens usually, slowly *kardaja* was used for any argumental measure or interval of argument (not simply 15°). No doubt it acquired even wider and more general meanings but originally it sprang from *gṛhārdha*.

2 Arabs and the Second Half of the Eighth Century

Before the rise of Islam, no scientific literature existed in Arabia beyond a few magical, meteorological and medical formulas (Hitti, pp. 91 and 307). After the historic hegira flight (AD 622), Prophet Muḣammad united all Arab tribes under the flag of Islam to launch *jihād*. About 637 AD, the might of Persians was broken and the Sasanian empire ended. Arab *Chronicles* estimate the booty and treasures captured to billions of *dirhams* (silver coins).

An anecdote from them says that when an Arab warrior was blamed for selling a noble man's daughter who fell as his share of booty, for only 1000 dirhams, his reply was that he "never thought there was a number above ten hundred" (Hitti, pp. 156-7). Caliph ʻAli ibn Abī Ṭalib (656–661) who himself was a mathematician and astronomer, was influenced by mathematicians of the Pre-Islamic Iranian Centre at Fars (Gundishpur) (MAIC, p. 13).

In 662 AD, the Syrian Christian bishop Severus Sēbōkht (died 667) composed a book which contains information about Indian decimal place-value numeration and arithmetic (*Ibid.*).

Better knowledge of Hindu astronomy spread among the Arabs in the eighth century when direct contact with India took place. An Indian astronomer visited Baghdad as a member of an embassy from Sind. The story was told by Ibn al-Adamī (Baghdad, about 920) and is quoted by Qāḍī Ṣāʻid al-Andalusī (d. 1078) some what as follows (Pingree 1970a, pp. 105–106):

> …Ibn al Adamī in his large *Zīj* called *Nazm al-ʻiqd* says that in the (Hijri) year 156 (=AD 772–773) there came to the Caliph al-Manṣūr (755–775) a man from India, an expert in the calculation (ḣisāb) called *al-Sindhind* concerning the motions of planets … Al-Manṣūr ordered the translations of this book into Arabic, and that there should be written from it a book which the Arabs might use as a basis for the motions of the planets. For this al-Fazārī was made incharge. He made of it a book which is called *al-Sindhind al-kabīr*.

According to Pingree (*DSB* IV, p. 555), the Sanskrit text translated belonged to the *Brahmapakṣa* (school of Indian astronomy) whose most immediate cognates were the *Paitāmahā Siddhānta* of the *Viṣṇudharmottara Purāṇa* and the *Brāhmasphuṭa Siddhānta* of Brahmagupta (AD 628). The name of the visiting astronomer is given as Kanaka (or kankah) by Abraham ibn Ezra (c. 1090–1167) in the preface of his translation of the Ibn Muthannā's *Fi'ilal Zīj al-kwārizmī* and his *Liber de Rationibus Tabularum* (DSB VII, 223). In the latter work he says (Pingree 1968, p. 215):

> The Indians said that the (maximum) declination of the Sun is 24° as Iacob Abentaric (=Ya'qūb ibn Ṭāriq) transmitted from the words of chenche (= Kanaka) the most learned of the Indians.

Al-Bīrūnī in his *Chronology* states that Kanakah was an astrologer at the court of Hārūn al-Rashid (786–809) while Abū Ma'shār in his *Kitāb al-ulūf* (Book of Thousands) (c. 850) says that Kanaka was an authority in Indian astronomy "in ancient times".

In my opinion the name Kanaka (kankah) among the Arabs in this context came from the Sanskrit *gaṇaka* (*gaṇakaḥ*) and stood for an astronomer in general. Different *gaṇakas* visited Baghdad at different times. Nevertheless an Indian teacher named Kanakācārya is quoted by Kalyaṇa Varman (800 AD or so) in his *Sārāvalī* (Jha, p. 426).

3 Al-Fazārī and Ya'qūb

We have already mentioned that al-Fazārī was asked to work with the Indian astronomer on an Arabic translation of a Sanskrit astronomical text brought by the latter to Baghdad in 772/773. Another Muslim scholar who collaborated in the project at the Abbasid court was Ya'qūb ibn Ṭāriq. These Arab scholars played the significant role of introducing a large body of Indian parameters and computational techniques to Islamic scientists. The Arabic translation was entitled *Zīj al-Sindhind* from which descended a long tradition in Islamic astronomy that survived in the 'East' until the tenth century and the 'West' (i.e. Spain) till the twelfth century. Kennedy (1956, p. 129) has listed about a dozen *zījes* which were computed by the method of the *Sindhind* or strongly affected by it. Unfortunately most of these have not come to light.

The first work derived from the *Sindhind* was evidently the *Zīj al-Sindhind al-kabīr* (The Great Sindhind) of al-Fazārī himself. In this, the system of the *kalpa*, the mean motions of the planets, their apogees and their nodes were all according to the *Brahmapakṣa*. Two Indian values of the *sinus totus* namely 3438 and 3270 are found in the work (Pingree 1971, p. 555).

Probably about 790, al-Fazārī composed his *Zīj 'alā sinī al-'Arab* ("Astronomical Tables according to the years of the Arabs") in which he apparently tabulated the mean motions of planets from 1 to 60 *saura* days and added tables for converting *kalpa ahargaṇas* to Hijra dates (*Ibid.*)

Ya'qūb ibn Ṭariq composed the *Zīj maḥlūl fī al-sindhind li daraja daraja* ("Astronomical Tables of the *Sindhind* Resolved for Every Degree) in which the basic parameters were very similar to those of *Zīj al-sindhind al-kabīr* of al-Fazāri (DSB XIV, p. 546). In his *Tarkīb al-aflak* (c. 777), Ya'qūb drew upon the *Zīj al Sindhind* and the *Zīj al-Arkand* (based on the midnight system) as well as on his conversation with the Indian astronomer who visited Baghdad in his time. In the same work, he gives the following (Sachau, I, 316, 353; II, 35):

1. Latitude of Ujjain, given as 4.4 digits which is equinoctial noon shadow length for the Indian gnomonic height of 12 digits. Another value is 4.6 digits.
2. The 4 kinds of measure namely *sauramāna, sāvanamāna, cāndramāna* and *nākṣatramāna*.
3. Hindu method of finding the *adhimāsa* (intercalary month).

Bīrūni says (Sachau I, 169 and II, 67–68) that Yā'qūb in the same work gave the Hindu planetary distances and the circumference of the zodiac which he had drawn from the well-known scholar who accompanied with the embassy to Baghdad in the (Hijri) year 161 (=AD 777/778) (was this another delegate?). Ya'qūb's diameter of the earth (given in the same book) is precisely that of Āryabhaṭa I (=1050 *yojanas* or 2100 *farsakh*) (Pingree 1968, p. 109). Ibn al-Nadīm (c.987) in his Fihrist mentions Ya'qūb's *Kitāb taqṭī kardajat al-jayb* which deals with table of sines (*Ibid*. p. 98). The Arabic *jayb* or *jyb* (for Sine) comes from the Sanskrit *jīvā* (*jībā*) (Plofker, 257).

Another important work of Ya'qūb is the *Kitāb al-'ilal* (book of Reasons) which explained the rationale for mathematical procedures followed by astronomers. Unfortunately this book is not extant but fragments are found quoted by Bīrūnī who, for example, quotes in his book *On Shadows* Yaqūb's rule regarding the equation of daylight etc. (Kennedy 1976, Vol. I, pp. 175–76). The verbal rule may be expressed in modern way as follows:

$$\text{Day-radius} = 3438 - \text{Vers}\,\delta,$$

$$\text{and}\qquad Car\bar{a}rdhajy\bar{a} = \frac{(\sin\delta.\,e.\,3438)}{(g.\ \cos\delta)}$$

where δ is Sun's declination, e is equinoctial noon shadow, and g in gnomonic height. Bīrūnī mentions Arjabhar (or Āryabhaṭa) for using 3438 as *sinus totus*. It is a common Indian value and is also found in the *Sūryasiddhānta* which in fact also contains (II. 59–60) the above rule (Shukla 1957, pp. 36–37).

4 Al-Khwārizmī (Ninth Century AD)

Muḥammad ibn Mūsa al-Khwārizmī (780–c.850) was one of the greatest scientific minds of Islam. He influenced mathematical thought to a greater extent than any other medieval writer (Hitti, p. 379). Based on the famous *Sindhind* (Arabic translation of a Sanskrit work), al-Khwārizmī composed about 820 AD his work which was appropriately called *Zīj al-Sindhind* according to the *Fihrist* of al-Nadīm (c.987) (Toomer, p. 360). Most of the basic parameters in his *Tables (=Zīj al-Sindhind)* are derived from Hindu astronomy. For all seven bodies, the mean motions, mean positions at epoch and the positions of the apogee and node all agree with what can be derived from Brahmagupta's *Brāhmasphutasiddhānta*.

Chapter 29 in *Zīj al-Sindhind* is on *elbuht* (true motion) which comes from the Sanskrit *bhukti*. The maximum equations for the sun (2; 14°) and moon (4; 56°) although stated to be derived from *Zīj al-Shāh*, they ultimately are Hindu values. Bīrūnī says that these numbers "passed from India to the Persians" (Toomer, p. 361; Kennedy 1956, p. 148 & 1963, p. 326).

Al-Khwārizmi's maximum latitude of the moon is 4; 30° as in *Sūryasiddhānta* (Shukla, p. 22) (Ptolemy's value being 5°). His rules yield a sidereal year of 365;15, 30, 22, 30 days which is same as in a work of Brahmagupta (Haddad, p. 218).

It is a long established fact that al-Khwārizmī's planetary theory is based on Hindu procedures. According to Toomer (p. 363), his method by 'halving the equation' is purely a Hindu procedure. It has been shown that the crescent visibility table found in al-Khwārizmī's *Zīj* was computed on the basis of Indian visibility theory according to which the crescent will be visible when the difference in setting time between Sun and Moon is 12° or more (Kennedy 1965, p. 73). Similarly a set of planetary latitude table found in the same work corresponds completely to the demands of Indian model (Kennedy 1969, p. 86).

The rule given by al-Khwārizmī in his *Zīj* for finding the apparent diameter of the solar disc (in connection with eclipse) occurs in *Khaṇḍakhādyaka I*.31 (Sengupta p. 32) and that for finding the radius of the shadow at Moon's place is same as in *Khaṇḍakhādyaka IV*.2 (*Ibid.* p. 83) (Neugebauer, pp. 58–59, and 107). Bīrūnī adds that the same rules are also found in Indian *Karaṇasāra* (Sachau II, 79), Same thing can be said about al-Khwārizmī's table of parallax in latitude in which case the theory and parameters are found in Hindu sources (Neugebauer p. 122 and Kennady 1956, p. 150).

Al-Khwārizmī's *Zīj al-Sindhind* was used as a classroom text book in the ninth and tenth centuries and continued to be used, studied and commented on. In fact, it was the first such work to reach the West in the Latin translation of Adelard of Bath (twelfth century). In addition to astronomical *Tables* (= above *Zīj*), al-Khwārizmī also wrote on arithmetic and algebra. These writings were also popular. He helped in popularizing Indian decimal place-value system of numerals as well as Indian arithmetic in Europe through his *Kitāb al-ḥisab al-hindi* (MAIC, p. 22).

According to al-Hāshimī, during the days of (Ja'far) al-Mutawakkil (2nd half of ninth century) a delegation presented itself from India and informed about practice and parameters of astronomy in India (Haddad, p. 96). Ḥabash al-Ḥāsib al-Marwazī (d. 864/74) worked at Baghdad under the Abbasid Caliphs and took astronomical observations from 825 to 835. His works include a reworking of the famous *al-Sindhind* (Tekeli, p. 612).

5 Al-Bīrūnī (973–1048 AD)

Abū'l-Rayhan al-Bīrūnī is the most famous scholar of Science of Medieval Islam. He was an encyclopedist and knew many languages including Arabic, Persian and Sanskrit. His interest in Indian civilization is due to his being part of an empire that had extended then to India. But his respect and admiration for the people of India which kept high traditions of learning, must have been an additional incentive for him to visit India (1017 to 1030) (Kennedy, DSB II, p. 150). During his stay in India, he not only acquired knowledge of Indian Sciences but also recorded them in his works. His *Kitāb fi Taḥrīr mā li'l Hind*...(= *India* in short) was completed in 1030 and contains mine of information about Indian astronomy. A few of his works related to Indian astronomy are (MAIC, pp. 152–153):

1. *Jawāmī al-mawjūd li-khowāṭir al-Hunud fī al-ḥisāb al-tanjīm* (Collection of Ideas of Indians on astronomical calculations). It is a book on Indian *siddhāntas* exposed in the "Sindhind".
2. *Tahdhīb Zīj al-Arkand* (Correction of Zīj al-Arkand). Already mentioned (see Section I above).
3. *Khayāl al-Kusufayn inda'l-Hind* (Representation of Both [Kinds of] eclipses by the Indians).
4. *Al-Jawābāt an al-masa'l al-wārids min munajjimī al-Hind* (Answers to Questions asked by Indian Astronomers).
5. Arabic translation of Vijayanandin's Sanskrit *Karaṇatilaka*. The Arabic title is *Ghurra al-zījāt*.*

On the other hand he was also keen of translating into Sanskrit some foreign works which included Euclid's *Elements* and Ptolemy's *Almagest* (MAIC, pp. 147, 154). Sachau (p. II, 303) states Bīrūnī was translating Brahmagupta's *Brāhmasphuṭa Siddhānta* into Arabic about 1030 AD. It is doubtful whether he could complete this tough task.

Through his Arabic writings on and about India and through his Arabic translations of original Sanskrit works, Bīrūnī can be credited for spreading the knowledge of Indian astronomy in West Asia. But for his researches, translations, and studies of Indian works, the Islamic world (especially the Arabic knowing people of the time) would have remained ignorant of the achievements of India in the field of astronomy etc. (Ahmad, pp. 7–9).

In his treatise *On Shadows*, he gives several rules for finding equation of daylight, rising times of the signs etc. from the works of Brahmagupta, Vaṭeśvara, Vijayanandin and one Yaltabān whom we cannot identify (Kennedy 1976, I, pp. 173–83 and II, pp. 98–113).

Bīrūnī in his *On Transits* says that the Hindus originated the maximum equation of the Sun and Moon and that these parameters passed from India to the Persians whence to others (Saffouri, p. 28). In the same work (*Ibid*, pp. 30–32), he correctly mentions the *sinus totus* used by Āryabhaṭa I as 3438 and by Brahmagupta as 3270.

*The title occurs as *Ghurrat uz-zījāt* also.

He further says that the *sinus totus* used by Vaṭeśvara in his *Karaṇasāra* is 300 and that by Vijayanandin in his *Karaṇatilaka* in 200. The latter work is not extant in original Sanskrit and so we should thank Bīrūnī for his Arabic translation of it. This Arabic translation of a lost Sanskrit work has been studied, edited, translated into English, and even revised by Saiyid Samad Husain Rizvi in various publications from 1963 to 1979 (MAIC, pp. 152 and 672).

In his book *On Coordinates* (completed in 1025), Bīrūnī says that Indians possessed a book in which determination of distance between two global places is dealt. He calls the book as *Kitāb taḥdid al-arḍ wal-falak* whose author is not named. But the mentioned rule may be compared with that found in *Mahābhāskarīya* (II. 3–4 of Bhāskara I) (c. 625 AD) (Ali, p. 193).

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Spread and Triumph of Indian Numerals

According to Menninger,[1] it is quite probable that due to active commercial relations with India, the first Indian numerals became known in Alexandria sometime in the fifth century AD and from there they might have penetrated farther westward.

Menninger says that the Indian numerals did not arrive in Egypt as a scientific treasure, but rather like the numerals of alien peoples that become known in the harbours and ports. However, we may point out that some Brahmins (who are not supposed to be traders) who visited Alexandria in AD 470 were the guests of Consul Severus.[2]

Hindu numerals are found in several manuscripts of the Geometry of Boethius (c. 500), and if the relevant portions of the manuscripts are regarded as genuine, it will show that Indian numerals had reached southern Europe about the close of the fifth century.[3]

The Mayan vigesimal abstract place-value notation contains the oldest zero in the New-World. Although the Mayan system occurred in apparent isolation, Menninger (*CHN*, 405) suspects a possible borrowing from India, the Mayan culture (now extinct) being at its height during the period from the sixth to eleventh centuries AD

According to Werner,[4] the Chinese adopted the Indian decimal system and notation introduced by the Buddhists and changed their custom of writing figures from top to bottom for the Indian custom from left to right.

Wei Chih's (died 643) *Sui-shu* (Records of the Sui Dynasty, 581–618 AD) mentions the Chinese translations of Indian works like[5]

1. *Brahman Suan-fa* (Brahman Arithmetical Rules) in one book.
2. *Brahman Suan-ching* (Brahman Arithmetical Classic) in 3 books.

Indian Journal of History of Science, 18(1), (1983), pp. 23–38; Lecture delivered on 17th September, 1979 at the University of Jodhpur under the I.N.S.A Programme in History of Science.

K. Ramasubramanian (ed.), *Gaṇitānanda*,
https://doi.org/10.1007/978-981-13-1229-8_54

This shows that Indian calculation methods and numerals (?) were known in China already about the end of the sixth century. It is unfortunate that these Chinese translations are lost.

That the fame of the Indian numerals had already reached the banks of Euphrates in the seventh century, is shown by passage in a work of the Syrian monk Severus Sebokht (AD 662) who lived in a monastery at Kenneshre (*HHM*, I, 95–96). He[6] refers to the Hindus and "their valuable methods of calculation; and their computing that surpasses description. I wish only to say that this computation is done by means of nine signs."

This clearly shows that the Syrian scholars understood the full significance of the Indian numerals although, like so many others, he mentions only the nine Indian numbers signs which, of course, without a zero symbol would not have been considered at all remarkable. Karpinski[7] is also of the opinion that the numerals referred by the Syrian Bishop are those "which we now use."

The Khmere inscription at Sambor (683 AD) and the Malay inscription (684 AD) at Palembang (Indonesia, Sumatra) give their dates (in *Śaka* years) by employing Indian decimal place system of numerals with zero.[8]

The Malay inscription at Kotakapur (716 AD) also gives its *Śaka* year 608 in the same system.[9]

In the Khai-yuan Period (713–741), Chhüthan Hsita (Gautama Siddhārtha) was appointed Royal Astronomer of China. Under imperial order he made the Chinese translation, called *Chiu-chih li* (Navagraha Calendar), of an Indian work. The Chinese version contained the Indian numerals following decimal place value notation and using a dot (instead of a circle) for zero, and further remarked[10] that "with these numerals, calculation is easy to the eyes." Of course, it is.

The Dinaya Sanskrit inscription (760 AD) at Java gives the *Śaka* year both in Indian place value notation and in Indian word numerals.[11]

During the reign of Caliph al-Manṣūr (755–775), works on Indian mathematics and astronomy (including those of Brahmagupta of seventh century) were translated into Arabic at Baghdad (*HHM*, I, 89). It is believed that it was at that time that the Hindu numerals were definitely introduced amongst the Arabs and the Baghdad scholars greatly appreciated the Indian system.[12]

An inscription (813 AD) at Po-nagar, Champa, gives the *Śaka* year in two slightly different forms employing the Indian system of positional numerals.[13]

The famous Abū Jafar Muḥammad al-Khwārizmī (c. 800–850) wrote about 820 a work on Indian numerals. The original in Arabic is not extant but we have its Latin version, entitled *Liber Algorismi de numero Indorum* (The Book of al-Khawārizmī on Indian numerals), possibly by Adelard of Bath (c. 1120)[14] or by Robert of Chester (*CHN*, 411). Menninger thinks (*CHN*, 411) that al-Khwārizmī had probably learned the numerals himself from the Indian writings (several of which were available in Arabic translations.)

According to Ibn al-Qifṭī's indication, the title of the Arabic original may have been like *Kitāb hisāb al-adad al-ḥindī* (Treatise on calculation with Hindu Numerals) but J. Ruska[15] conjectures its title equivalent to "*Book of Addition and Subtraction by the Method of Calculation of the Hindus.*"

The treatise of al-Khwārizmī, as we have it, expounds the use of the Hindu (or as they are misnamed "Arabic") numerals 1–9 and 0 and the place-value system and then explains various applications. It is an elementary arithmetic treatise using Indian numerals.

Arabs were already using alphabetic numerals system (as shown by an eighth-century Arabic Papyrus from Egypt) similar to Greek. The Indian decimal place value system was also already known, but al-Khwārizmī's work was the first to expound it systematically. Unfortunately, even with this important introduction of useful symbols into more general use in Islamic lands, there was delay in adopting them quickly in all spheres of life. And the treatise achieved greatest success only when introduced to the West through Latin translations in early twelfth century.[16]

As regards the forms of the number symbols, al-Khwārizmī stated that the new numerals, particularly 5, 6, 7 and 8 were written differently by different peoples but that this circumstance was no obstacle to their use as a place-value notation (*CHN*, 413).

Another example of the use of Indian numerals in S. E. Asia is provided by the inscription at Bakul (839 AD) in which the corresponding *Śaka* year is mentioned in decimal place value system.[17]

In his work, *Ancient Alphabets and Hieroglyphic Characters Explained etc.* the Syrian Ibn Wahshḥīya (c. 855) gives three forms of Hindu numerals as three species of Hindu alphabets which shows that the forms were well known in his time in Arabia (*HHM*, I, 96–97).

The Indian numerals are also mentioned by the Arab philosopher al-Jāḥiẓ (d. 868–869), who calls them 'figure of Hind' and observes that with these numerals large numbers can be represented with great facility (*HHM*, I, 97).

A concrete example of the use of the decimal place value system is provided by an Egyptian Papyrus (written in the year 873) in which the year 260 is expressed in Indian numerals (*CHN*, 414).

Abū Yūṣuf al-Kindī (died c. 873) and al-Dīnawarī (died c. 895) each wrote a tract on Indian Computation (Hiṣābul Hindī).[18] Al-Dināwarī was a lawyer and attempted to introduce Hindu methods in business.

Abū Sahl Ibn Tamim (d. 950), a native of Kairwān a village in Tunis in the north of Africa (*HHM*, I, 98), wrote in the *Sefer Yezirah* that he had used the Indian nine signs in his work on Hindu calculation, *Ḥisāb al-ghubār*.[19]

Abul-Ḥasan-Masaūdī (d. 956–957) visited India about 915 and later on mentioned the Indian numerals in his work (c. 943) with the remark that "a congress of sages at the command of the Creator Brahmā invented the nine figures" which shows that no inventor was known in India even at that time (*HHM*, I, 97).

Abul-Ḥasan al-Uqlīdisī wrote his *Kitāb al-Fuṣūl fī al-Ḥisāb al-Ḥindī* (Book on Principles of Hindu Computation) in Arabic at Damascus in 952–953 AD.[20] It is said to be the earliest extant Arabic book that presents Indian system.

In the introduction al-Uqlīdisī states that he has travelled extensively and read all books on Indian arithmetic that he found. Hindu numerals and place-value notation is discussed in the first part of the work.

He has made several interesting suggestions such as[21]

1. Modifications of Indian schemes whereby the (dust) abacus can be dispensed with, and ink and paper used instead. (This was the first step in discarding abacus slowly).
2. Greek letters might replace the nine Indian numerals. (This was, in fact, done sometimes).
3. The Indian numerals with superimposed dots might form a new Arabic alphabet.

Gerbert (c. 940–1003) visited Spanish border country around 967 and enriched his mathematical knowledge. It was probably here that he first became acquainted with the Indian numerals, for the Arabs had been in Spain since 713 (*CHN*, 322).

He carried the *ghubār* forms of Indian numerals (learned in Spain) back to his home place (Auvergne, France) and inscribed them on the counters of the monastic abacus in the form of *apices*. In this form the Indian numerals (without zero, as columns with no digits were simply left vacant) made their first definite excursion into the West towards the end of the tenth century.

But Europe was not ready for them; neither their nature nor their advantages were appreciated, and they soon retreated into the cells of learned monks as they failed to survive in ordinary use (*CHN*, 417).

However, it must be pointed out that the full set of the *ghubār* forms of Indian numerals (including the zero symbol) is found in an Arabic manuscript dated 970 AD.[22]

Although Indian numerals were known and appreciated by the Baghdad scholars as early as the eighth century, yet when Abūl Wafā (died 997–998 at Baghdad) wrote his Arabic text book on practical arithmetic, called *Book on What Is Necessary From the Science of Arithmetic for Scribes and Businessmen* (written between 961 and 976), he avoided the use of numerals by writing the numbers in words.[23]

Some historians (such as M. Cantor and H. Zeuthen) explain the lack of Indian numerals by presuming the existence of two opposing schools among the Arabic mathematicians one following Greek models and the other Indian models. However M. I. Medovy[24] shows that such a hypothesis is not supported by facts (as some writers/scholars are found to use both the systems).

It is more probable that the use of Indian numerals simply spread very slowly among the businessmen, scribes and general public whose needs were heeded by the text-book writers. Whatever be the reason, the victory of Indian numerals, though delayed, was unavoidable.

A good example of Indian numerals is found in an European manuscript written in Spain in 976 AD.[25]

Alī Ibn Aḥmad al-Mujtabā, who lived in Baghdad and died in 987, wrote the *Kitāb al-takht al-kabīr fī al-ḥisāb al Hindī* (The Great Book of the Board on Hindu Arithmetic); and his contemporary, al-Kalwādānī, (living at Baghdad) also wrote a similar work, *The Book of the Board on the Hindu Arithmetic*.[26] Both the works employ the dust or *ghubār* form of Indian numerals and are among the several Arabic books written by the Eastern Muslim scholars on the subject in the tenth century.

The Indian numerals are mentioned by al-Nadīm (d. 995) in his Kitāb al-Fihrist (c. 987) and are called *hindisah* (*HHM*, I, 98). Ibn Nadīm reports of custom of writing the zeros beneath the figures.[27]

The Indian numerals of the *ghubār* type (but without zero) are given, as an addition (992 AD) in a Spanish copy of the *Origines* by Isidorus of Seville (d. 636).[28]

The Indian numerals (with dot for zero) are found in the philosophical treatises of the brothers Ikwān as-Ṣafā (c. 1000) along with their Arabic names and the old Arabic alphabetic numerals following the *abjād* system.[29]

A similar set of Indian numerals (but with different form of 5) is found in a treatise on Hindu arithmetic written about 1000 AD by the famous Arabic scientist al-Bīrūnī.[30]

In his *Āthār al-Baqiyah* (Vestiges of the Past), written in 1000 AD, Bīrūnī calls the then modern numerals as 'al-arqam al-hind', i.e. 'the Indian Ciphers' distinguishing them from other systems (*HHM*, I, 99).

Kūshyār Ibn Labbān (c. 1010) wrote his *Kitāb fī Uṣul Ḥisāb al-Hind* (Book of Principles of Hindu Reckoning) in Arabic and is based on Indian system of numerals including zero which is represented by a circle.[31] His importance lies in his having written the work to introduce the Hindu methods into astronomical calculations.

Abu Bakr al-Karkhī (d. 1029) wrote (Probably between 1010 and 1016) his *kāfi fī al-Ḥisāb* (Sufficient of the Computation) which was largely based on Hindu sources.[32] His other work using Indian numerals is the *Kitāb fī al-Ḥisāb al-Hindī* (Book of Indian Computation) which is cited in his Algebra.[33]

Al-Bīrūnī (whom we have already mentioned) visited India and studied Indian sciences between 1017 and 1030 and so he was more qualified than his predecessors to speak with authority about Indian numerals (*HHM*, I, 98). Two of his works, namely *Kitāb al-arqam* (Book of Ciphers) and another called *A Treatise on Arithmetic and the system of Counting with the Ciphers of Sindh and India* are quite relevant in the matter (*Ibid*).

His knowledge and opinion about Indian numerals are expressed in the following words:[34]

> As in different parts of India the letters have different shape, the numeral signs too, which are called *aṅka*, differ. The numeral signs which we use are derived from the finest forms of Hindu signs.

For some nice examples of the actual use of Indian numerals by al-Bīrūnī, reference may be made to facsimile of pages from his original Arabic work as reproduced in L. C. Karpinaski's *The History of Arithmetic* (New York, 1965), pp. 47 and 51 (*vide* ref. No. 7). These show the use of Indian form of several numbers up to 1000.

Abu'l-Ḥasan al-Nasawī, who lived in Baghdad (1029–1044), wrote his *Al-Muqni' fī al-Ḥisāb al-Hindī* (An Account of Indian Computation) which employs the numerical symbols obtained from the Indians. Introduction of the book shows that al-Nasawī wrote in Persian a book on Indian arithmetic for presentation to Magd al-Dawla, the Buwayhid ruler (who was dethroned in 1029–1030). Later on it was presented to Sharaf al-Mulūk, vizier of Jalā al-Dawala, ruler of Baghdad. But the vizier ordered al-Nasawī to write it in Arabic and the result was the above work *al-Muqni*.[35]

In the Islamic astronomical literature, sexagesimal digits were written from right to left in Arabic alphabetic numerals. But al-Nasawī placed successive digits in a vertical column and used Indian numerals only.

A nice detail about the transmission of Indian numerals to the Islamic world is accidentally preserved in the autobiography of Ibn Sīnā or Avicenna (c. 980–1037). When he was about 10 years old, missionaries of an Islamic sect, called Ismaelites, came to his native place, Bukhara (then under the Iranian Dynasty of the Samanids) from Egypt. Through the teachings of these missionaries, Ibn Sīnā learned the Hindu method of computing. Without this explicit bit of information, no body would have dreamt that Indian influence (and numerals) reached southern Russia via Egypt.[36]

It has been stated by Ali bin Abil-Regal Abul-Hasan, called Abenragel (1048), in the preface to his treatise on astronomy, that the invention of reckoning with nine ciphers is due to Hindu philosophers (*HHM*, **I**, 99).

Abu Jafar al-Ṭabarī, who lived in the town of Āmul (south-east of the Caspian) in the last half of the eleventh century, wrote the *Shumār-nāme* (Reckoning Book). It is a text-book on Hindu computation and is said to be earliest extant book on the subject in Persian.[37]

We have already mentioned that al-Khwārizmī's book on Indian computation was translated into Latin in the early twelfth century by Adelard of Bath or by Robert of Chester. In fact the book quickly spawned a number of adaptations and off-shoots such as the *Liber alghoarismi* of John of Seville (c. 1135), the *Alghorismus* of John of Sacrobosco (thirteenth century) and the twelfth century work *Ysagogarum Alchorizmi*.[38] Other twelfth century epitomes exist in manuscripts form in the Royal Library, Vienna and the University Library, Heidelberg (*CHN*, p. 411). (Also see *Math. Reviews* 44, 481–482.)

Two more such Latin 'algorisms' are reported to exist in the British Museum, the one is the Royal MS. 15B. IX and the other is the Egerton MS. 2261. The Royal Manuscript begins:[39]

> The intention of al-Khwārizmī in this work is to present the teaching of numeration, addition, subtraction, duplication and mediation, multiplication, and division by the ten characters of the Hindus (per X karacteres indorum).

In fact al-Khwārizmī's name became so closely associated with the 'new arithmetic' using the Hindu numerals that the Latin form of his name, algorismus, was given to any treatise on that topic. Hence by a devious path, is derived the modern word 'algorism' (corrupted by false etymology to algorithm').[40]

The oldest year-date to appear in Europe in the new Indian numerals occurs on a Sicilian coin of the Norman King Roger II (*CHN*, 439). The year marked is 533 A. H. (= 1138 AD).

Out of several works on number written by Abraham Ibn Ezra (d. 1167), the most important is his *Sefer ha-Mispar* (Book of the Number). It is based on the Indian system of positional numerals but uses the first nine Hebrew letters for the figures 1–9 and the zero as in algorism.[41] The zero symbol is given as *galgal* ('wheel' or 'circle').

Saraf Eddin (c. 1172) of Mecca wrote a treatise entitled *Fi al-handasa wa al-arqam al-hindi* (On Geometry and the Indian Ciphers) (*HHM*, I, 99).

Al-Samaw'al (died c. 1180) was a native of Baghdad and studied the Hindu computational methods. In writing polynomials, he assigned to each power of x a place in a table in which the polynomial was represented by the sequence of its coefficients, written in Indian numerals. Only by employing the new numerals, he could easily handle large number of equations. His techniques helped development of symbolism necessary for the progress of algebra.[42]

Leonardo of Pisa or Fibonacci (c. 1170–1250) wrote his great work, *The Liber Abaci* (Book of Computation), in 1202 fully based on Indian numerals. The work prepared the ground for the widespread adoption of the Indian numerals in the West (*CHN*, 425).

As a young man he travelled about the Mediterranean visiting Egypt, Syria, Greece, Sicily and southern France, meeting the scholars and becoming acquainted with the various computational systems in use among the merchants of different lands. But he reports that all systems appeared to him in error as compared to the Indian mode ("quasi errorem computavi respectu modi indorum").[43]

He introduced the new numerals in the following words (*CHN*, 425):

> The nine Indian numerals (figure *indorum*) are 9, 8, 7, 6, 5, 4, 3, 2, 1. With them and with the sign 0, which in Arabic is called *zephirum* (cipher), any desired number can be written.

Fibonacci's works did pioneering service in bringing Indian numerals into ordinary use. With him a new epoch in Western mathematics began. Although all his ideas were not taken up immediately, great influence was exerted by those portions of his work that served to introduce Indian numerals and methods.[44]

It is unfortunate that two of his works, namely *Dimmor Guisa* (A book on Commercial Arithmetic) and a tract on book X of *Eculid's Elements* (in which he promised a numerical treatment of irrationals instead of Euclid's geometrical presentation), are lost.

We have already mentioned the name of John of Sacrobosco who was educated at Oxford and later on taught mathematics in Paris where he died in 1244 or 1256. His *Algorismus* or *Tractatus de Arte Numerandi* was the first arithmetic, based on Indian numerals, written by an Englishman. His work was widely used all over western Europe for centuries and thus he did much to spread the Indian numerals and computation.[45]

But the most interesting among the computational works based on Indian numerals is the *Carmen de Algorismo* (Song of Algorisums) by the French monk Alexandre De Ville Dieu who taught in Paris about 1240. In his version, an Indian king named Algor figures as the inventor of the new art which itself is called algorismus (*CHN*, 412).

The opening lines from a thirteenth century manuscript (at Darmstadt) of his work may be translated thus (CHN, 412):

> Here begins the alogorismus. This present art is called algorismus, in which we use twice the five figures of the Indians (*bis quinque figuris indorum*).

These lines and the myth of king Algor again appear in the first English arithmetic (c. 1300), the anonymous *The Crafte of Nombryng* whose manuscript is in the British Museum (Egerton MS. 2622).[46]

We find that the nine Indian numerals were called *figure* in the thirteenth century and the name was retained in English and French. Thus zero, the 'figure of nothing' was no numeral or no figure at all, *nulla figura* in Latin whence came the name 'null' for zero (*CHN*, 403).

In the eighth century, when Indian numerals were definitely introduced in China, a dot was used for zero (see above). A small circle as the symbol for zero is first found in print in the Chinese work *Su Shu Chiu Chang* (Mathematical Treatise in Nine Sections) of Chhin Chiu-Shao (1247. AD) but many believe in its use in the earlier period of the Sung Dynasty (950–1280) after its arrival from India.[47]

It is stated that *The Comprehensive Work on Computation with Board and Dust* (in Arabic) by Naṣir al-Dīn al-Tūṣī (1201–1274) marks an important stage in the development of the Indian numerals.[48]

A thirteenth century monastic manuscript (State Library, Munich) contains the Indian numerals along with Roman (*CHN*, 282).

Towards the end of the thirteenth century, an enemy suddenly appeared from an unexpected direction. As numbers began to be written in the new Indian numerals by some Italian trading houses, the City Council of Florence in 1299 issued an ordinance which forbade to enter the amounts of money in the accounts book in Indian numerals. (*CHN*, 426).

The argument was that the new numerals were more easily forged or changed than Roman numerals. People were still too insecure about the new numerals. It was not only their forms that were unfamiliar but also the method of writing them. It is not therefore surprising that the local chambers of commerce in Italy resisted the adoption of Indian numerals.

Thus, although computations with Indian numerals were known to commercial and trading establishments in the thirteenth century, book-keeping continued in old manner. This, of course, was a serious obstacle to the spread of the Indian numerals. However, the teachers and students of universities at Paris, Oxford, Padua and Naples kept alive the knowledge of Indian numerals.

Gregory Chioniades, who studied astronomy in Tabriz (in Azerbaijan) around 1290, used the Eastern Arabic forms of Indian numerals while he was in Byzantium (from 1298 to 1302).[49] These forms of Indian numerals may have been learned from him by Planudes (see below) who also used them.

Like Abraham Ibn Ezra (twelfth century), Levi ben Gershon used, in his *Sefer Maasei Hoshev* (Book of the Calculator) (completed in 1321), the Indian place-value numeration but employs the first nine letters of the Hebrew alphabet for numerals 1–9 (and a circle for zero).[50]

About 1330 (?), the Byzantine scholar Maximus Plandudes (a Greek monk and Constantinople ambassador to Venice in 1327?) wrote the *Psēphophoria kat' Indous e Legomenē Megalē* (Computation According to the Indians, Which is Great) based on Indian numerals. It sets forth the system of notation by the "nine figures received from the Hindus" together with the zero, and is the first of the Greek works to give any attention to Indian methods.[51]

T. L. Heath (*Hist. of Greek Math.* Oxford, 1965; II, 547) quotes Planudes more fully as follows:

> (The symbols were) invented by certain distinguished astronomers for the most convenient and accurate expression of numbers. There are nine of these symbols (our 1, 2, 3, 4, 5, 6, 7, 8, 9), to which is added another called *zifra* (cypher), written 0 and denoting zero. The nine signs as well as this are Indian.

Heath mentions an earlier Greek work, with similar title, written in 1252 (extant as Paris MS. Suppl. Gr. 387) and believes that Plaundes may have raided it. But the forms of numerals are stated to be different in the two works. Planudes is placed earlier by Heath and still earlier by Pingree (ref. 49).

Al-Umawī taught arithmetic in Damascus in fourteenth century. He wrote about 1373 the *Marāsīm al-intisāb fī'ilm al-ḥisāb* which represents a trend of Arabic arithmetic in which the Indian dust board calculations had began to be modified to suit paper and ink.[52] In a table of sequences he used the Western Arabic forms of Indian numerals.

Indian numerals appear also on several manuscripts such as:[53]

1. Latin manuscripts (c. 1294) of Boethius arithmetic.
2. Latin manuscripts (c. 1294) of Euclid.
3. Italian manuscripts (c. 1339) of the *Trattao d' Abbaco, etc.* by Paolo Dagomari (d. 1373/1374.)
4. French manuscript (fourteenth century) of *Algorismus Proportionum* by Nicole Oresme (c. 1323–82).

In spite of widespread use of Indian numerals, a class of arithmetical works, called *Computi* (which were treatises on Church Calendar), was mostly confined to Roman numerals. Was it due to orthodoxy or prejudice? However, Indian numerals were known to the authors who occasionally used them—sometimes in a peculiar way. For instance a Latin manuscript (dated AD 1384) of an anonymous *computus* gives its date as (*Rara*, 443):

anno dnj 1000.300.80.$\mathring{4}$

As the Indian place-value notation penetrated deeper and deeper in the West, it gradually displaced computations with alphabetic numerals which, like Roman numerals, were deeply rooted. In the beginning there was a sort of "equilibrium" (or compromise) between the two systems. The Greek alphabetic numerals for the units (α to θ including the now obsolete 'vau', 'stigma' or 'digamma' which stood for six) were used as "Indian numerals". Of course a zero symbol had to be adopted for there was no such thing in the alphabetic system. Thus a fifteenth century Greek manuscript of a text-book on arithmetic contains the following (*CHN*, 274):

$\alpha\epsilon$ for 15

$\delta\gamma\bullet$ for 430

$\gamma\beta\theta\bullet$ for 1290

It may be recalled that such a compromise with the Hebrew alphabetic numerals was already employed by Abraham Ibn Ezra (twelfth century) and Levi Ben Gershon (fourteenth century). The oldest German coin which gives the date of the year in Indian numerals is a silver medallion struck by the town of St. Gall in 1424 (*CHN*, 439).

The progress in the use of Indian numerals in the accounts books of the imperial free city of Augsburg (West Germany) may be summarised as follows (*CHN*, 289–293):

Sl. no.	Year	Use of numerals
(i)	1410	iiij^{M} lb vij^{c} lxxxx viij lb for 4798 lb
(ii)	1430	$iiij^{C}$ for amount 400; but year in Indian numerals
(iii)	1470	amount as iij^{C} and lxiij which is repeated in Indian numerals as 363
(iv)	1500	amounts in both systems but their total in Indian numerals only
(v)	1533	all entries in Indian numerals only

Before their regular appearance in printed arithmetical books, the Indian numerals were extensively used in several computational treatises of which the following fifteenth century manuscripts may be noted (*Rara*, pp. 443–465):

1. Italian Ms. (1422) of *Trattato di aritmetica* by Giovanni son of Luca da Firenze.
2. Latin Ms. (1424) of *Scientia de namero ac virtue numeri* by Rollandus (c. 1425).
3. Italian Ms. (c. 1430) of an anonymous work on Florentine commercial arithmetic.
4. Latiin Ms. (c. 1442) of a work on *algorismus* by John of Sacrobosco (thirteenth century).
5. Italian Ms. (c. 1456) of anonymous work on business arithmetic.
6. Italian Ms. (c. 1460) of a work on mathematics possibly by Raffaele Canacci (of Florence).
7. Italian Ms. (c. 1460) of a work on mercantile arithmetic by Benedetto da Firenze.

The work on *Etymologies* (also called *Origines*) by Isidorus of Seville (d. 636 AD) was printed at Augsburg in 1472. The subject of arithmetic is treated in its book III (*Rara*, 8). It is not known whether this printed version contains the Indian numerals which were added to chapter one of book III in some tenth century copies of the work (*HHM*, I, 102). Indian numerals are profusely used in Regiomontanus's *Calendar des Magister* which was printed in Nuremberg in 1473 AD.[54]

The first truly dated computational work (using Indian numerals) to appear in print in the West is called *Treviso Arithmetic* (Treviso, 1478) from its place of printing in Italy, the author being unknown. The numerical 1 was printed as *i* generally (*Rara*, 3–7).

Wide penetration of Indian numerals and methods can be ascertained from the fact that in Italy the very first computation text-books has no traces of counting board (*CHN*, 441). In England, Indian numerals appeared on an illustration in an English work printed about 1480 (see below). In Germany, the first printed arithmetic text-book employing Indian numerals is the Bamberg arithmetic of 1483 by the Nuremberg rechenmeister Ulrich Wagner (*CHN*, 335 and 434).

Pietro Borghi's *Arithmetic* (Venice, 1484) is more elaborate than Treviso arithmetic and had far greater influence on education. Borghi first treats of the Indian place value notation, carrying his numbers as high as 'numero de million de million de million', and making no mention whatever of the Roman numerals (*Rara*, 16–19).

Abū'l-Ḥasan al-Qalaṣādī (1412–1486) is the last known Muslim mathematician of Spain who wrote several books on arithmetic. His *Kashf al-asrār 'an waḍ 'ḥurūf al-ghubār* (Unfolding the Secrets of the Use of Dust letters, i.e. Indian Numerals) was a text-book in school of North Africa.[55]

Even at this juncture when Indian numerals were marching to victorious triumph, some setbacks did exist. For instance, the Frankfurt Mayor's Book of 1494 ordered the rechenmeister to abstain from calculating with Indian numerals (*CHN*, 427).

Great were the success which the Indian numerals achieved. Greater was the revolution which they were creating. Opposition to them attracted more attention.

With the introduction of printing in the fifteenth century, the contest between the old counting board and the few Indian place-value numerals in Europe becomes visible in various ways. Thus the old and the new are symbolically represented in the *Margarita Philosophica* of Gregor Reisch (1503 AD). Next to Pythagoras with his sorrowful face working at a counting board sits a cheerful and serene Boethius contemplating his computations in Indian numerals. Arithmetic (personified as a female figure) hovers with her books between them, looking at the computer with digits and indicates her approval of him by two geometric series in Indian numerals on her garment (*CHN*, 350 and 431).

In another illustration, a woodcut by the Nuremberg artist Hans Sebald Beham (d. 1550), Winged Arithmetic is shown to turn her back on the counting board and point emphatically to the tablet with the new Indian numerals (*CHN*, 431).

Roman numerals were so deep rooted in Europe that it made exceedingly difficult for the Indian numerals to replace the old numerals even from those situations where the latter deserved no place. For some time they boiled in the same pot leading to a sort of confusion and multiplicity as illustrated by the following examples (*CHN*, 287).

$$\begin{aligned}\text{M.CCCC.8II} &\ \text{ for } 1482\\ 15\times 5 &\ for\ 1515\\ \text{I.O.VIII.IX} &\ \text{ for } 1089\\ \text{ICCOO or I.II. } \tau\,\tau &\ for\ 1200\\ (\tau\ \ for\,zero\,from\,Greek\,word\ \tau\zeta\iota\phi\rho\alpha,\ i.e.\,tzifra\,or\,cifra)\end{aligned}$$

As mentioned above, the Indian numerals appeared in England in an English work printed as early as 1480 (*Rara*, 10) or 1481 by the Caxton Press.[56] This was the *Mirrour of the World* in which arithmetic is briefly discussed but the author is not known and the numeral forms appear only on an illustration.

The *De Arte Supputandi* (The Art of Computation) by Cuthbert Tonstall (London, 1522) is the first book wholly on arithmetic (using Indian numerals) that was printed in England (*Rara*, 132–135). But the English treatise which was most influential in popularizing the Indian numerals was Robert Recorde's *The Grounde of Artes*. It appeared first about 1542 and in 27 further editions up to 1699.[57]

In Germany the work of Jakob Köbel (died 1533) played similar role. He wrote several books on arithmetic out of which his *Rechenbiechlin* first appeared at Augsburg in 1514. This has a table for learning the Indian numerals (*Rara*, 104) which were still considered difficult. (In this table, the number 89 is wrongly shown as LXXXXI).

Like Ibn Ezra (d. 1167), Elias Misrachi (d. 1526) used the Hebrew letters and O in his arithmetic which was based on former's work and bore same title (*Rara*, 521).

We have already mentioned the interesting illustration, contained in Reisch's *Margarita Philosophica* (1503 AD) in which Boethius is shown as the representative of Indian numerals. In fact the Medieval Europe (among some circles at least) believed erroneously that Boethius (c. 500 AD) was the inventor of 'Indian' computation (*CHN*, 350). This belief may be partly due to the fact that Indian numerals are found in the manuscripts of his *Geometry* as early as tenth century (*HHM*, I, 92) or may be due to hero-worship.

Similarly some Latin writers, in their desire to exalt the classical (Greek) learning, assigned the Indian numerals to the Pythagoreans. For instance, Valentin Nabod did so in his *De Calculatoria Numerorum* (Cologne, 1556) which was written for the classical schools of Germany (*Rara*, 281). But could such writers succeed in misleading their readers and discredit the Indians?

Anyway, Indian numerals and computations continued to spread and hundreds of books appeared in print in the sixteenth-century Europe on the subject.

The first arithmetical work (based in Indian numerals) printed in America appeared in Mexico in 1556. This was the *Sumario Compendioso* of Juan Diez Freyle.[58]

Towards the end of the sixteenth century, a book on arithmetic based on Indian numerals was submitted to a deacon of the cathedral at Antwerp for his approval. The decision was (*CHN*, 427):

> These rules and procedures for computation and for finding the answers to problems are admittedly useful for merchants, and for their sake permission is granted for them to be printed; but they (the merchants) must see to it that they avoid usury and other illicit transactions and exchanges.

That is, the new numerals are not to be used for dealings that are not approved of. Prejudice and suspicion continued to exist about the Indian numerals and the long struggle between the 'abacists' and the 'algorithmicists' extended even beyond the sixteenth century. Orthodoxy was hard but cases of prosecution (like that of Galileo for his astronomical theories) have not come to light.

The duality of computation on the counting board and writing the numbers in Roman numerals were finally (in seventeenth century) replaced by single-step procedure of written computations with Indian numerals.

This was, no doubt, the victory of Indian culture; but it was also a victory for the mind of man, who finally, in the long history of written numerals, arrived at a mature, abstract decimal place-value notation.

The Indian numerals are now used all over the civilized world. In vain Charles XII, King of Sweden (1682–1718), tried to abolish the Indian decimal system in favour of duo-decimal.[59] The decimal system is not the best but God favoured it by giving us 10 fingers.

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Indian Mathematical Sciences Abroad During Pre-modern Times

1 Introduction

Views regarding the origin of mathematics have been changing fast during the last 50 years. The publication of old Babylonian texts in the thirties has not only upset the theory of the Greek origin of mathematics but rather gave rise to the view that the Greek mathematics itself was a derivative of the Babylonian mathematics. According to a tradition, Thales visited Egypt and Pythagoras is said to have even visited India.

The researches of A. Seidenberg during the last 25 years have been boosting the thesis of a single origin for mathematics in ritual such as the construction of Vedic altars in India. Recently he has shown that the Vedic mathematics cannot all be derived from the Babylonian lot what to say of the Greeks.

These considerations combined further with the traditions of Chinese and Egyptians and with European megalithic constructions led B. L. van der Waerden to argue that the original mathematics had its source among the Indo-Europeans (Aryans) before their dispersion (c. 3500 to 2500 BC).[1] However, a very recent study of Chinese right-angled triangles (*AHES*, 30, 111, 1984) raises objection to van der Waerden's hypothesis.

As regards early transmissions to and from India are concerned, the problem of dating early Indian texts and traditions accurately is a very serious difficulty. There are quite divergent views. Relative chronology is useful but has its limitation. Exaggerated claims by Indians are not unusual but cases of biased views of seeing foreign origin in every Indian achievement are also not lacking; for instance, the statement that the Indian value[2]

$$\sqrt{2} = 1 + \left(\frac{1}{3}\right) + \left(\frac{1}{3 \cdot 4}\right) - \left(\frac{1}{3 \cdot 4 \cdot 34}\right)$$

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was derived from the Babylonian value

$$\sqrt{2} = 1 + \left(\frac{24}{60}\right) + \left(\frac{51}{60^2}\right) - \left(\frac{10}{60^3}\right).$$

Some scholars are ready to twist the text and even prepared to give undue credit to Indians just to show borrowing from the Greeks.[3] So we take known periods.

2 Indian Mathematics in China

Following the introduction of Buddhism in China, a number of texts were translated into Chinese of which the *Mātaṅgāvadāna* (third century AD) has a list of 28 *nakṣatras* and lengths of shadows for the Hindu gnomonic height of 12 units. Similar is the case with the Sanskrit work *Śārdūla-karṇāvadāna*.[4]

In the sixth century, Indian astronomical works were translated into Chinese by Bodhiruchi, Paramārtha, and by Upaśūnya whose translation of the *Mahāratnakūṭa-sūtra as Ta Pao Chi Ching* (541 AD) had an Indian system of numeration. The *Sui-shu* (636–656 AD) mentions the Chinese translations of at least seven Indian works on mathematics and astronomy such as *Po-lo-mên Suan Fa* (Brahmin Arithmetical Rules) in one book and *Po-lo-mên Suan Ching* (Brahmin Arithmetical Classic) in three books. These works were in the Royal Library (c. 1150) but are lost now.

In the glorious Thang Period (618 to 907 AD), Indian mathematical astronomy was taught in Chang-Nan, and Indians were employed in the Chinese Astronomical Board with Gautama Siddhārtha (Hsia-Ta) becoming even its President, and Director of the Royal Observatory. He translated an Indian calendar as the famous *Chiu Chi li* (718 AD) which also had sections on Indian numerals[5] (with a thick dot symbol for zero) and Indian trigonometry (with 24 Hindu sines for $R = 3438$).

I-hsing, one of the greatest astronomers of China, constructed a tangent table which "was clearly derived[6] from Indian sine tables". He, along with Nan Kung Yueh, made full use of Indian trigonometrical tables in conducting an official survey in 725 AD.

According to K. Yabuuti,[7] "there can be no doubt that the Indian astronomy was highly valued in Thang times for the superior results it achieved in the prediction of solar and lunar eclipses". Even the then contemporaries such as Sulaymāna al-Tājir (ninth century) had recorded the impression that Indian astronomy was more advanced than Chinese.

3 Transmission and Triumph of Indian Numerals

Whatever be the origins of the initial and separate concepts of place-value notion and zero, it is well known that the present positional decimal system of numerals started in India from where it spread to other parts of the world.[8]

According to Werner, the Chinese adopted the Indian decimal system and notation introduced by the Buddhists. The mention of several Indian arithmetical books in the *Sui-shu* (c. 600 AD) imply that Indian numerals were possibly known in China quite early. Chinese Buddhists, such as Hui Shun (c. 500 AD), also reached North America and may have influenced the Mayan system. Menninger also suspects a borrowing from India for the Mayan system. According to him, the Indian numerals reached Alexandria some time in the fifth century AD.

That the fame of the Indian numerals had already reached the banks of Euphrates in the seventh century is shown by a passage in a work of the Syrian monk Severus Sebokht (662 AD) who has praised the Indian system. Several inscriptions from Southeast Asia, such as the Khmere inscription at Sambor (683 AD), the Malay inscriptions at Palembang (684 AD) and Kotakapur (716 AD), show that Indian system was in use there in the seventh century. The Chinese translation (c. 718) called *Chiu-chih li* (Navagraha Calendar) of an Indian work contains the Indian system of numerals using a thick dot (instead of a circle) for zero. The Dinaya Sanskrit inscription (760 AD) in Java gives the *Śaka* year both in Indian place-value notation and in Indian word numerals.

It is believed that Indian numerals were formally and definitely introduced among the Arabs during the reign of Caliph al-Manṣūr (755–775) when Indian works were translated at Baghdad. Al-Khwarizmī (c. 820) wrote an Arabic work on Indian numerals and calculations, which is extant in the Latin version as *Liber-Algorismi de Numero Indorum* (twelfth century) which quickly spawned a number of adaptations and offshoots such as the *Liber alghoarismi* of John of Seville (c. 1135), the *Algorismus* of John of Sacrobosca (thirteenth century), etc. In fact, following (and improving) al-Khwārizmī, several works on arithmetical computations using Indian numerals were written by Arabic, Persian, Latin and other European scholars such as

(i) Abū Yūsuf al-Kindī (died c. 873): *Ḥisābul Hindī*
(ii) Al-Dinawarī (d. 895) who attempted to introduce Indian numerals in business.
(iii) Abū Sahl Ibn Tamim (d. 950); *Ḥisāb al-ghubār.*
(iv) Abū'l Ḥasan al-Uqlidisī : *Kitāb al-Fuṣūl fī al-Ḥisāb al-Hindī* (952/953).
(v) Alī ibn Aḥmad al-Mujtabā (d. 987): *Kitāb al-takht al-kabīr fī al-ḥisāb al-Hindī.*
(vi) Kūshyār ibn Labbān : *Kitāb fī Uṣūl Ḥisāb al-Hindī* (c. 1000 AD).
(vii) Abu Bakr al-Karkhī (d. 1029) : *Kitāb fī al-Ḥisāb al-Hindī* .
(viii) Abu'l-Hasan al-Nasawī : *Al-Muqni fī al-Ḥiṣāb al-Hindī* (both in Arabic and Persian) (c. 1030/1040 AD)
(ix) Abraham Ibn Ezra (d. 1167) : *Sefer ha-Mispar.*
(x) Saraf Eddin (c. 1172) : *Fi al-handasa wa al-arqam al-Hindī.*

Earlier in his *Āthar al-Baqiyah* (Vestiges of the Past) of about 1000 AD, al-Bīrūnī had called the then modern numerals as "al-arqm al-hind" ('the Indian ciphers') distinguishing them from other systems. Later on, Fibonacci's *Liber Abaci* (1202 AD) was fully based on Indian numerals and it helped widespread adoption of the system. About a century later, Maximus Planudes wrote his Greek work called *Computation According to the Indians, Which is Great.* Similar and more advanced works were written in various parts of Europe thus furthering the cause of Indian numerals. The first truly dated computational work using Indian numerals to appear in print is the *Treviso Arithmetic* (1478 AD), and the first printed such work in America was *Sumario-compendioso* (1556 AD).

4 Indian Mathematics and Astronomy Among the Arabs

According to Ibn al-Adamī (Baghdad, c. 920) as quoted by Qāḍi Ṣā'id al-Andalusī (d. 1078) and by al-Qifṭī (d. 1248/49), Caliph al-Manṣūr (755–775) ordered a Sanskrit astronomical work (brought by an expert from India) to be translated into Arabic. This translation was called *Sindhind* (from Sanskrit *Siddhānta*) from which descended a long tradition within Islamic astronomy extending up to Spain for several centuries. E. S. Kennedy has listed about a dozen *Zījes* which were computed by the method of *Sindhind* or strongly affected by it (*TAPS*, N.S., 46, Part 2, 1956).[9]

Based on the Indian *Sindhind*, al-Fazārī (c. 775) composed his *Zij al-Sindhind al-kabīr* in which he used three Indian values of the *sinus totus*, namely 3438, 3270 and 150. He also used the Hindu gnomonic length of 12 units. A similar work was written by Ya'qub ibn Ṭāriq. His diameter of earth as mentioned in his *Tarkīb al-aflāk* is precisely the same as that of Āryabhaṭa I (1050 *yojanas* or 2100 *farasakh*) and the circumference of earth is 3298 plus $\frac{17}{25}$ *yojanas* which precisely implies *Āryabhaṭīya's* value of π $(= 3.1416)$.

In Abū Ma'shar's (787–886) *Zīj al-hazārāt*, the mean motions of the planets are computed by the Indian method of the *Yuga* and by using Indian parameters.

Al-Khwārizmī composed his *Zīj al-Sindhind* about 820 AD in which most of the parameters were derived from Hindu astronomy, e.g. maximum latitude for moon as $4°30'$ (Ptolemy's value being $5°$). His rules for finding true longitude of a planet, apparent diameter of solar disc, radius of shadow at moon's place, crescent visibility, parallax in latitude, etc. are all based on Indian procedures.

Ḥabash al- Ḥāsib (d. 864/884), al-Sarakhsī and al-Sizjī also used Indian methods and parameters. Al-Bīrūnī (b. 973) visited India and acquired knowledge of Indian sciences. Through his works this knowledge further spread among the Arabs.

Al-Khwārizmī's *Zīj al-Sindhind* which is full of Indian material was redacted in Spain by Masalama al-Majrīṭī (d.c. 1000 AD). This was one of the channels through which Indian astronomy and mathematics penetrated Spain and the influence of Indian astronomy represented by the tradition of *Sindhind* continued there even after Ptolemy's *Almagest* (on Greek astronomy) came to be known.

Such was the impact of Indian scientific achievements in Spain that Qāḍī Sā'id of Toledo included Indians among the "first" nation which cultivated sciences in his *Ṭabaqat al-Umam* (1062 AD). He says further:

> ...they (the Indians) have studied arithmetic and geometry. They have also acquired copious and abundant knowledge of the movements of the stars, the secrets of the celestial sphere and all other kinds of mathematical sciences.....

> In the domain of numerical sciences, we have their (i.e. of Indians) *ḥisāb al-ghubār* which was explained by al-Khwārizmī. It is very compendious and a quick system of calculation, easy to understand, simple to adopt, and remarkable in its composition, bearing testimony to the sharp intelligence, creative power and remarkable faculty of invention of the Indians.

Unfortunately, the original Arabic versions of al-Khwārizmī's *Zīj al-Sindhind* as well as his work on Indian arithmetic are both lost but we have the Latin versions of both available and they played important role in spreading Indian astronomy and mathematics in the medieval Western world.

The *Toledo Tables*, associated with the name of al-Zarqālī (d. 1100 AD), enjoyed enormous circulation and were used throughout Europe by the twelfth century. They have a lot of material of Indian origin, e.g. tables of sines ($R = 150$), solar declination ($\epsilon = 24°$) and oblique ascensions. Al-Zarqālī gave $\pi = \frac{22}{7}$, $\sqrt{10}$, and $\frac{62832}{20000}$, the last two of which are Indian and used Hindu gnomonic length of 12 units. Ibn Muthannā wrote a commentary on al-Khwārizmī's *Zīj al-Sindhind*, and Abraham Ibn Ezra (d. c. 1165) translated that commentary into Hebrew and this version also shows Indian influence.

In the twelfth century, a host of European scholars translated Graeco-Arabic and Indo-Arabic scientific works into Latin. Through these translations oriental astronomy and mathematics flowed wider into Europe causing a helpful step towards renaissance.

5 Indian Astronomy and Mathematics in Late Foreign Works

According to W. Petri[10], Tibetan astronomy was virtually Indian astronomy preserved by Buddhist monks. One of the books of the *Tibetan Tripitaka* is ascribed to the Indian astronomer Garga and contains a list of 28 lunar mansions mostly agreeing with the Hindu tradition. *Kālacakrāvatāra* is an important Indo-Tibetan text based on the *kālacakra*, the Indian *yuga* system of periodicity, the basic text for which is the *Kālacakra-tantra* (c. 1000 AD).

The *mKhas pa dga'byed* (1356 AD) by Bu-ston is a Tibetan treatise on mathematical astronomy based on the above *tantra*. Its recent study by Y. Ohashi[11] shows that its constants are very similar to those of Indian treatises, *Sūrya-siddhānta* and *Pañca-siddhāntikā*. Similar works were written in 1447 and 1683 AD *(Baiḍūrya dkar-po)*.

This influence of Indian astronomy extended even beyond Tibet. In medieval Uigur (Turkish) fragments from Turfan (Central Asia), the lunar mansions are listed by their Sanskrit names. A study of a treatise of 1712 AD by L. S. Baranovskaya revealed that Indo-Tibetan astronomical ideas and names were alive in Mongolia until modern times (*Vistas of Astronomy*, Vol. 9, 164).

As an instance of Indian influence in Maghrib countries in late period, we may mention the paper of Kennedy and King entitled "Indian Astronomy in fourteenth century Fez (Morocco) : the Versified *Zij* of al-Qusunṭīnī" (*JHAS*, 6, 3–45, 1982).

It is said that this *Zij* "is the only known document extant in Arabic in which the planetary theory is essentially Indian" (p. 4).

Similarly, a glimpse of Indian influence in the fifth-century England can be obtained from the paper of Neugebauer and Schmidt on "Hindu Astronomy at New minster in 1428" (*AS* 8, 221–228, 1952). It is shown that methods used in the Latin manuscript are closely related to *Sūrya-siddhānta*. The parameters used are mostly Indian, e.g. $R = 150$ and $g = 12$.

Astronomical calculations based on the Indian gnomonic height of 12 units are also mentioned in the Altimetry portion of the twelfth-century Latin geometry *Artis Cuiuslibet Consummatio* (1193), which also mentions some other Indian astronomical notions. According to S. K. Victor, who has edited and translated the work (Philadelphia, 1979), the method of finding oblique ascension in it is an Indian procedure (pp. 257–259). It is stated that *Pratika de Geometric* is a thirteenth-century adaptation of the above work in old French and is considered to be "the oldest known treatise on geometry in the French language" (*Ibid.,* p. 27).

Cammann (*HR*, 1969, p. 188) suspects that yang Hui's magic squares of order 8 and 10 were probably borrowed from India. He found that the magic square of order 5 on a fifth-century Latin manuscript is the exact Hindu form (pp. 291–292). He says that Vincent learnt Hindu continuous method of making magic squares and taught it to the French Loubere in 1688.

According to Imai (see *Mathematical Reviews*, 58, 3154, 1979) a seventeenth-century Japanese formula for the rectification of the arc of the circular segment was due to an old Indian mathematical rule. A Japanese bonze, Yentsu[12], published the *Bukkoku Reki sho-hen* in 1815 on the "Discussion on the Astronomy of Buddha's Country" (India). A computation for a possible eclipse based on the *Sindhind* tradition is said to be made even in the nineteenth-century Egypt.[13]

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Sino-Indian Interaction and the Great Chinese Buddhist Astronomer-Mathematician I-Hsing (AD 683–727)

1 Buddhism, the Medium of Interaction

The rock edicts of king Aśoka (third century BC) show that he had already paved the way for the expansion of Buddhism outside India.[1] Subsequently Buddhist missionaries took Buddhism to Central Asia, China, Korea, Japan and Tibet in the North, and to Burma, Ceylon, Thailand, Cambodia and other countries in the South. This helped in spreading Indian culture to these countries. It is well said that "Buddhism was, in fact, a spring wind blowing from one end of the garden of Asia to the other and causing to bloom not only the lotus of India, but the rose of Persia, the temple flower of Ceylon, the zebina of Tibet, the chrysanthemum of China and the cherry of Japan. It is also said that Asian culture is, as a whole, Buddhist culture".[2] Moreover, some of these countries received with Buddhism not only their religion but practically the whole of their civilization and culture.

The generally accepted view is that China received Buddhism from the nomadic tribes of Eastern Turkestan towards the end of the first century BC, although there is evidence to show that Indians had gone there earlier to propagate the faith.[3] The Chinese tradition narrates that the Han emperor Ming-Ti (first century AD) had sent an embassy to India to bring back Buddhist priests and scriptures.[4] Consequently two Indian monks Kia-yeh Mo-than (Kāśyapa Mātaṅga) and Chu-fa-lan (probably Dharmaratna or Gobharaṇa) reached the Han capital Loyang. They learnt Chinese and translated Buddhist books the 1st of which was *Fo-shuo-ssu-shih-erh-cheng-ching* (the Sūtra of Forty-two Sections Spoken by Buddha).[5] With the arrival of more monks, both from India and Central Asia, the Loyang monastery became a centre of Indian culture. A large number of Indian books were translated, and people began to adopt Buddhist monastic rituals. Buddhism prevailed so extensively that by sixth century, the number of monasteries rose to 30000 and the number of monks and nuns to 2 million.[6]

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The tradition of Buddhist education system gave birth to large-scale monastic universities. Some of these famous universities were Nālandā, Valabhī, Vikramśilā, Jagaddala and Odantapurī. They attracted students and scholars from all parts of Asia. Of these, the Nālandā University was most famous with about 10000 students and 1500 teachers. The range of studies covered both sacred and secular subjects of Buddhist as well as Brahminical learning. The monks eagerly studied, besides Buddhist works (including *Abhidharmakośa*, the Vedas, medicine, arithmetic, occult sciences and other popular subjects.[7] There was a special provision for the study of astronomy, and an astronomical observatory is said to be a part thereof.[8]

According to the findings of a modern Chinese historian (Liang Chi-Chao), more than 160 Chinese pilgrims and scholars came to India from fifth to eighth centuries.[9] Of these Fa-Hien (fifth century) Yuan Chwang (seventh century), and I-tsing (eighth century) are the most famous. Some of them stayed and studied in India for several years. While going back, they took loads of Pali and Sanskrit works to China where hundreds of these works were translated into Chinese.

2 Indian Astronomy and Mathematics in Ancient China

We have seen that Buddhism was the medium for mutual intercourse between India and China, and provided opportunities for exchange of ideas. Buddhism exerted great influence in various fields in China and was the main vehicle for transmission of Indian scientific ideals to that land. The influence was so high that even scientists embraced the faith, e.g. astronomer Han Chai and mathematician Wang Fan (about AD 200) both became Buddhists (Mikami, p. 57). Lot of Indian astronomy and mathematics became known in China through the translation of Indian works and through the visits of Indian scholars. We shall briefly outline the broad facts in this section now.

The *Mātaṅga-avadāna* was translated (or re-translated) into Chinese about the third century AD although the original is believed to date earlier.[10] It gives the lengths of monthly shadows of a 12-inch gnomon which is the standard parameter of Indian astronomy. The work also mentions the 28 Indian *nakṣatras*.

Śārdūlakarṇāvadāna was translated into Chinese several times starting with the second century. This work contains the usual Sanskrit names of the 28 *nakṣatras* starting with *kṛttikā*, but the number of *grahas* mentioned are only 7, thereby excluding Rāhu and Ketu which were often added in the manuscripts and translations.[11] The measures of shadows for various parts of the day mentioned in the work (pp. 54–55) are same as in the *Atharva Vedāṅga Jyotiṣa,* verses 6 to 11.

Lalitavistara is another work which was translated into Chinese several times from first century onwards. It is in this work that the famous Buddhist centesimal scale counting occurs during dialogue between Prince Gautama and mathematician Arjuna. The Ist series of count ends with *tallakṣaṇa* ($= 10^{53}$) beyond which 8 more *gaṇanā* series are mentioned.[12] Atomic scale counting is also there (there being 7^{10} *paramaṇus* in one *aṅgulaparva*) (p. 104).

Vasubandhu (fourth century) was so much honoured for his work that he was known as the Second Buddha. His *Abhidharma-kośa* on which he wrote his own commentary is an encyclopedic work and played a great role in propagating Buddhist philosophy and thought in Asia. It was translated into Chinese and also in Tibetan. It contains early Buddhist ideas in Cosmography (Jambūdvīpa being given the form of a *śakaṭa*) and astronomy (sun and moon revolving round the Meru).[13] It is through this work that we know that Buddhist school used 60 decuple terms in decimal counting.[14]

The *Mahāprajña-pāramitā Śāstra* (of Nāgarjuna, second century) was translated into Chinese by Kumārajīva in early fifth century.[15] The astronomical parameters mentioned in this translation are comparable to those given in the *Vedāṅga-jyotiṣa*.[16]

Bodhiruci I arrived in China (from Central India) in AD 508 and said to have translated quite a few Indian astronomical books into Chinese.[17]

An Indian system of numeration appeared in the Chinese work *Tapao Chi Ching (Mahāratnakūṭa-sūtra)* translated by *Upaśūnya* (in AD 541).[18] Paramārtha (Po-lo-mo-tho), a native of Ujjain, arrived in China in AD 548 and translated about 70 works including the *Abhidarmakośa (vyākhyā)-śāstra* and the *Lokasthiti-abhidhrmaśāstra* (which is astronomical content).

There was a short setback to Indian activities in China when Wu-Ti came to power in AD 557. But they were resumed during the role of Sui Dynasty (581–618). The Indian paṇḍita Narendrayaśas was called back from exile in 582. Among the works he translated was the *Mahāvaipulya Mahāsannipāta-sūtra* from Sanskrit. It contains *nakṣatras,* zodiacal cycle, calendrical material and other Indian astronomical theories.[19]

The Chinese translations of the following works are mentioned in the *Sui Shu* or *Official History of the Sui Dynasty* (seventh century):[20]

1. *Po-lo-mên Thien Wên Ching* (Brahminical Astronomical Classic) in 21 books.
2. *Po-lo-mên Chieh-Chhieh Hsien-jen Thien Wên Shuo* (Astronomical Theories of Brāhmaṇa Chieh-Chhieh Hsienjen) in 30 books.
3. *Po-lo-mên Thien Ching* (Brahminical Heavenly Theory) in I book.
4. *Mo-têng-Chia Ching Huang-thu* (Map of Heavens in the Mātāṅgī-sūtra) in I book.
5. *Po-lo-mên Suan Ching* (Brahminical Arithmetical classic) in 3 books.
6. *Po-lo-mên Suan Fa* (Brahminical Arithmetical Rules) in I book.
7. *Po-lo-mên Ying Yong Suan Ching* (Brahminical Method of Calculating Time) in I book.

Although these translations are lost, they were also mentioned in other sources.

More vigorous contacts and activities took place during the glorious period of Thang dynasty (618–907). In return to an envoy sent by the Indian king Harṣavardhana in 641 to China, two missions came to India from there. Hiuen Tsang (or Yuan Chwang) needed 22 horses to carry the works which he took from India to China in 645. He translated 75 of these including *Abhidharmakośa*.

The great influence which the Indian astronomy had at that time can be seen by the presence of a number of Indian astronomers in the Chinese Capital Chang-Nan where there was a school in which Indian *siddhāntas* were taught.[21] In fact there were three clans of Indian astronomers, namely Kāśyapa, Gotama and Kumāra. These Indians were employed in the Chinese National Astronomical Bureau and helped in improving local calendar.

The greatest of these was Gotama Siddha (or Gautama Siddhārtha). He became the president of the Chinese Astronomical Board and Director of the Royal Observatory. Under the imperial order (from Hsuan-tsung) he translated in AD 718 the famous *Chiu Chih Li* ("Navagraha-karaṇa") from Indian astronomical material. A few years later, he compiled the *Khai-Yuan Chan Ching* (The Khai Yuan Treatise on Astronomy and Astrology) in 120 volumes of which the 104th is the *Chiu Chih Li*. It includes Indian sine table ($R = 3438$, $h = 225$ minutes) and Indian methods of calculations with nine numerals and zero (denoted by thick dot $\bullet$). The astronomy was based on 9 planets including *Lo-hou and Chi-tu* (which are Chinese forms of the Sanskrit names Rāhu and Ketu).[22]

3 Earlier Chinese Parallels of Indian Mathematical Pieces

Before talking up the question of mutual transmission further, we shall first mention the close resemblances which exist between some mathematical problems, rules and formulas as found in China and India.

1 The Broken Bamboo Problem (वंशभङ्गोद्देशकः):

In China this is found in the famous *Chiu Chang Suan Shu* (Nine Chapters on the Mathematical Art) whose present text is placed in the first century AD. Its ninth chapter, entitled "kou ku" (Right-Angled Triangles), contains the following problem.[23]

Problem 13: A bamboo is 1 chang (= 10 *chhih*) tall. It is broken, and the top touches the ground 3 *chhih* from the root. What is the height of the break?

The verbal solution given is equivalent to (see Fig. 1)

$$y = \frac{\left(h - \frac{x^2}{h}\right)}{2} = 4\frac{11}{20} \quad chhih$$

It is understood that the solution is based on the Pythagorean property, so that

$$y + z = h, \text{ and } z^2 - y^2 = x^2$$

One of the two similar examples given by Bhāskara I (AD 629) reads[24]

अष्टादशकोच्छ्रायो वंशो वातेन पातितो मूलात् ।
षड्गत्वाऽसौ पतितस्त्रिभुजं कृत्वा क्व भग्नः स्यात् ॥

A bamboo of height 18 is felled by the wind. It falls at (a distance of) 6 from the root (thus) forming a triangle. Where is the break?

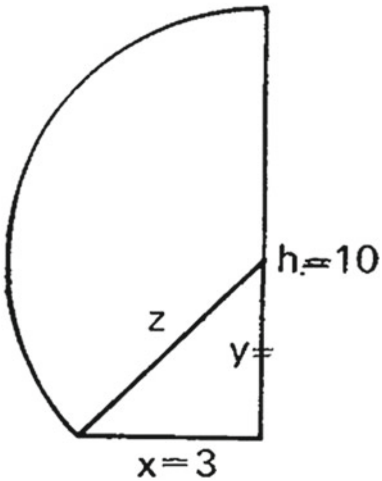

Fig. 1 Broken bamboo problem

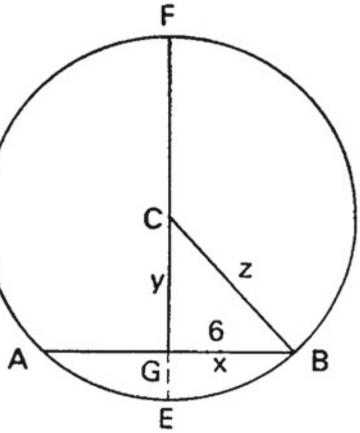

Fig. 2 Bhāskara's solution

Bhāskara's solution is based on applying the relation (see Fig. 2)

$$GF \cdot GE = GB^2$$

which is given in *Āryabhaṭīya* II, 17 (second half) on which he is commenting. He gets

$$GE = \frac{x^2}{h} = 2 = z - y$$

The doing *saṅkramaṇa* with $z + y = 18$, he found z and y to be 10 and 8.

2 Problem of Reed in a Pond (कमलोद्देशकः):

This is problem no. 6 in the 9th chapter of the *Chiu Chang Suan Shu*:[25]

> There is a pond whose section is square of side I *chang (= 10 chhih)*. A reed grows in its centre and extends 1 *chhih* above water. If the reed is pulled to the side (of the pond), it reaches the bank precisely. What are the depth of the water and the length of the reed?

Solution given[26] is $x = \frac{(z^2 - e^2)}{2e}$, where z is half the side of the pond, and $y = x + e$ (see Fig. 3).

Bhāskara I's first similar example (out of two) reads[27]

> कमलं जलात् प्रदृश्यं विकसितमष्टाङ्गुलं निवातेन ।
> नीतं मज्जति हस्ते, शीघ्रं कमलाम्भसी वाच्ये ॥
> A full-blown lotus of 8 *aṅgulas* as visible (just) above the water. When carried away by the wind, it submerges just as the distance of *1 hasta (=24 aṇgulas)*. Tell quickly (the height of) the lotus plant and (the depth of) the water.

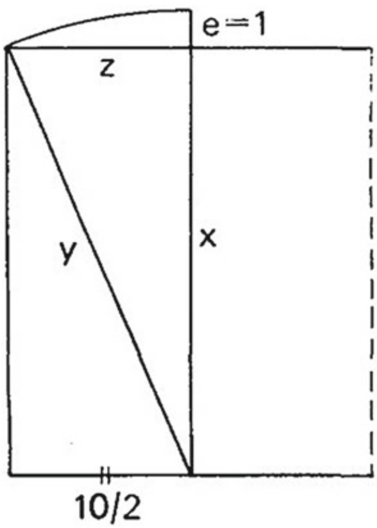

Fig. 3 Reed problem

His solution is again based on the same property of chords, namely (see Fig. 4)

$$BC = \frac{BM^2}{AB} = \frac{z^2}{e}$$

And, then applying *saṃkramaṇa* to $y + x = \frac{z^2}{e}$, and $y - x = e$, he gets lotus-measure, y and water-measure, x as 40 and 32 (*aṅgulas*). On Simplification, Bhāskara's solution

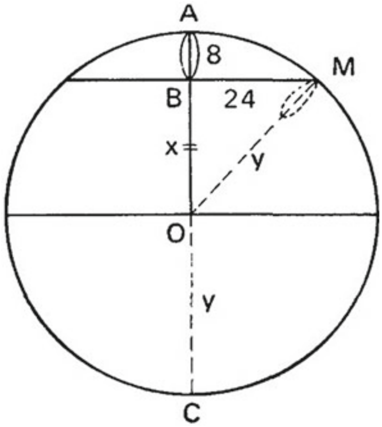

Fig. 4 Solution of reed problem

$$x = \frac{\left(\frac{z^2}{e} - e\right)}{2}$$

becomes same as the Chinese solution.

3 Approximate Volumes of a Sphere

The *Chiu Chang Suan Shu* (first century AD) uses the approximate rule

$$V = \left(\frac{9}{2}\right) r^3 \tag{1}$$

For calculating the diameter of a sphere when its volume V is known.[28] In India Bhāskara I quotes a rule which gives (1) directly[29]

> व्यासार्धघनं भित्त्वा नवगुणितमयो गुडस्य घनगणितम् ।
>
> The product of 9 and half the cube of the radius is ball's volume.

Two centuries later Mahāvīra (about AD 850) gave the same and regarded it, like Bhāskara, only a *vyāvahārika* or practical (not exact) rule.[30] The same is also found in other Jaina works such as *Tiloya-sāra* (*gāthā* 19) of Nemicandra (about 975 AD) and the *Gaṇita-sāra* (V. 25) of Ṭhakkura Pheru (about 1300 AD). This shows a Jaina tradition for (1).

There is another interesting thing. In China, Liu Hui (third century) interpreted (1) wrongly as equivalent to[31]

$$V = \left(\frac{\pi^2}{2}\right) r^3 \tag{2}$$

In India also, Māhāvīra seems to have thought that (1) was based on (2) with the practical value $\pi = 3$, and he further derived a better formula by taking $\pi = \sqrt{10}$ which he considered to be *sūkṣma*.[32]

4 The Problem of 100 Fowls

In China, the earliest statement of this problem is found in the *Chang Chhiu-Chien Suan Ching* (Arithmetical Classic of Chang Chhiu Chien) which is generally placed in the 2nd half of the fifth century. It runs as follows:[33]

> A cock costs 5 pieces (*wên*) of money, a hen 3 pieces, and 3 chickens 1 piece. If we buy, with 100 pieces, 100 fowls, what will be their respective numbers (answers: 4+18+78; 8 + 11+81; 12+4+84).

A century later Chen Luan gave two similar problems with costs 5, 4, $\frac{1}{4}$ (answer: $15 + 1 + 84$) and 4, 3, $\frac{1}{3}$ (answer: $8 + 14 + 78$).[34]

In India, such problems appear in the *Bakshālī Manuscript* (whose exact date is uncertain or controversial). One problem relates to buying a total of 20 animals (monkeys, horses and deer) for a total of 20 *paṇas* at costs $\frac{1}{4}$ (say), 4 and $\frac{1}{2}$ (answer: $2 + 3 + 15$).[35] Another similar example relates to prices or earnings of men, women and *śūdras* or children at rates 3, $\frac{3}{2}$ and $\frac{1}{2}$ (answer: $2 + 5 + 13$).[36]

Example of buying 100 birds (pigeons, cranes, swans and peacocks) with 100 *rūpas* (or paṇas) with rates $\frac{3}{5}$, $\frac{5}{7}$, $\frac{7}{9}$, $\frac{9}{3}$ occurs in the *pāṭīgaṇita* (Exam. 78–79) of Śrīdhara (eighth century) as well as in the *Gaṇita-sāra-saṅgraha* (VI, 152–153) of Mahāvīra (ninth century).[37] This problem was quite popular in India, and one of the many solutions is 15 pigeons, 28 cranes, 45 swans, 12 peacocks.[38] Similar problems were also popular in other parts of the world as shown by works of various authors starting with Alcuin (ninth century).[39]

In simple matters, like the use of $\pi = 3$, we may accept independent discoveries or inventions by different culture groups. But when specifically characteristic rules and problems, such as (I) to (IV) considered above, are found to occur in different culture areas, we have to favour a theory of diffusion. Of course an older common source may have been there from which material was possibly transmitted to the various culture areas. B. L. van der Waerden (p. 66) considers a pre-Babylonian common source of Chinese and Babylonian algebra. In fact he has formulated the thesis of a common Indo-European origin of mathematics which flowed to China, India, Babylonia, Greece and Egypt (pp. 67–69). We have evidence that some peculiar rules such as the "surveyor's rule" for the area of a quadrilateral,[40] or the use of $\frac{h(c+h)}{2}$ (or its other derived forms) for the area of a segment of a circle, are found widely diffused.

Regarding pieces (I) to (IV) discussed above, we have not come across specific earlier instances where these are found as such. It is, therefore, to be presumed that there was some interaction which ultimately led to transmission between China and India. We have already noted above that even Chinese mathematicians such as Wag Fan (about 200 AD) became Buddhist (Mikami, ref, 28, 57). Needham[41] mentions monk Than Ying (about 440) who could be a teacher of *Chiu Chang Suan Shu* and its commentary by Liu Hui.

References to Buddhism and Buddhist works are found even in the mathematical treatises of China such as the *Sun Tzu Suan Ching* or Arithmetical Manual of Master Sun, which is placed[42] between AD 280 and 473. Master Sun's work is important for early indeterminate analysis, and he was an ardent believer in Buddhism. He profoundly read the Buddhist works and mentioned them in his writings.

At least some of the Indian scholars who visited China, must have become familiar to some extent, with the local mathematical traditions especially the more popular common and recreational type of problems. A few of these Indians had often come back to India (maybe even temporarily). Also Chinese pilgrims, scholars, and envoys (including diplomats) who visited India, may have taken some Chinese mathematical classics, such as the famous *Chiu Chang Suan Shu,* with them. Books may have been part of gifts which may have been presented to the kings or universities. All such things indicate a strong possibility of mathematical interaction between China and India. But while things were documented in Chinese sources, there is no similar positive literary or other documentary evidence known from Indian sources which specify clearly the arrival of any Chinese mathematical material in India.[43]

4 I-Hsing (683–727), the Great Chinese Astronomer-Mathematician

By the end of the seventh century AD, a lot of Indian mathematics and mathematical astronomy were known in China. The compilation of *Chiu Chih Li* in Chinese by Gotama Siddha from Sanskrit sources represents the culmination of such transmissions in AD 718. Through this work the Indian methods of computations based on the decimal place-value system (with a zero symbol) and the Indian trigonometry (based on Sines) were formally introduced in China. The analysis of the contents of *Chiu Chih Li* by Yabuuti (ref. 22 at the end) shows that the mathematical astronomy as found in *Sūrya-siddhānta* and in the works of Varāhamihira (sixth century AD) and Brahmagupta (seventh century) was known in China in the beginning of the eighth century.

At this time appears I-Hsing on the Chinese scene. He was an able mathematician deeply learned in astronomy and was well versed in Sanskrit (Mikami, ref. 28, p. 60). He combined in himself the traditions of Chinese as well as Indian mathematical sciences. He became a Buddhist monk, attended conglomerations of monks and *śramaṇas* and travelled widely to acquire knowledge (Needham, ref. 17, p. 38).

I-Hsing won a great reputation for his combinatory calculations. Due to his Buddhist training, he could easily handle large numbers such as 3^{361} or 10^{172}. His methods were capable of enumerating all possible changes and transformations occurring on go board or chess board (Needham, *Ibid.,* p. 139). He could also handle indeterminate problems involving large numbers (*Ibid.,* p. 119–120). In India similar problems had already been solved by Bhāskara I (early seventh century). Some scholars have confused him with I-tsing, the pilgrim.[44]

Between 721 and 727, I-Hsing prepared, by imperial order, a calendar known as *Ta Yen Li* (Needham, ref. 17, p. 37) in which he had applied higher mathematics. Out of the 23 different systems of calendars known by that time, I-Hsing's was found to be accurate and stood the test of time (Mikami, ref. 28, p. 60).

Gautama Chuan (of Kumāra clan) probably knew that one of his Indian colleagues has taught I-Hsing method (say as given in the *Sūrya-siddhānta*) for relating gnomon shadows and solar zenith distance (or altitude) by means of sine table of *Chiu Chih Li*.[45] I-Hsing fully used this knowledge.

Much influenced by Indian astronomy, I-Hsing made measurements in ecliptic coordinates which played little role before (Needam, ref. 17, p. 202). He was associated in training the officials and observers for the great meridian survey of 724 AD[46] The observed data was also analysed by him. He developed a tangent table which is the earliest of its kind in the world. This development was based on Indian information about the use and values of sines from which his tangent table was derived.[47] He used methods of finite differences, fitting of polynomials and interpolations.[48]

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14. Bapat (ref. 1) p. 115.
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16. Mukherjee (ref. 4), 38.
17. J. Needham, *Science and Civilization in China,* Vol. III, Cambridge, U.K. 1959, p. 88.
18. See Bapat (ref. 1), p. 214; Mukherjee (ref. 4), p. 34; and Needhan (ref. 17), p. 707 where the Chinese title of the second work appears as *Li Shih A-Pi-Than Lun* (Philosophical Treatise on the Preservation of the World).
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20. R. C. Gupta "Indian Astronomy in China During Ancient Times", *Vishveshvaranand Indological Journal,* XIX (1981), 266–276, p. 270.
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27. Shukla (editor), *op. cit,* (ef. 24), pp. 100–102. Shukla's remark (p. 299) that the Chinese and Hindu solutions are "quite different" is not justified as both are ultimately based on Pythagorean property. The relation $BC = y + x = \frac{z^2}{e}$ follow from property of Chords (which itself is based on Pythagorean property) or from $y^2 - x^2 = z^2$ and $y - x = e$. The slight difference in methods is not significant.
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33. Mikami (ref. 28), p. 43, On p. 39 he says that the work "probably belongs to latter half of sixth century"
34. *Ibid.,* p. 44.
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38. Shukla (ref. 37) has given all the 16 solutions. Also see Hayashi (ref. 35), p. 650 for more reference.
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43. There exist similarities in many other mathematical pieces which we have not discussed here. Some of these are treated by B. Datta in his paper, "On the supposed Indebtedness of Brahmagupta to *Chiu Chang Suan Shu*", *Bulletin of the Calcutta Math. Soc.,* Vol. XXII (1930), pp. 39–51. Datta does not mention Bhāskara I. Also see Waerden (ref. 23), pp. 196–208, for $\pi = 3.1416$, and L. C. Jain, "Jaina School of Mathematics (A study in Chinese Influences and Transmissions)". In *Contribution of Jainism to Indian Culture* (ed. by R. C. Dwivedi), Delhi, 1975, 206–220.
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48. See Cullen's paper (ref. 45) and *Historia Mathematica,* Vol. 11 (1984), pp. 45–46, where it is stated that Liu Ch'uo (about 600 AD) knew formula for interpolation for equal intervals and Liu Ch'un-feng (AD 665) had studied finite differences. In India, Brahmagupta (seventh century) had used finite differences up to second-order, and interpolations for equal as well as for unequal intervals. See R. C. Gupta, "Second-Order Interpolation in Indian Mathematics etc.", *Indian J. Hist. Sci.,* 4 (1969). 86–98.

Indian Mathematical Sciences in Ancient and Medieval China

1 Buddhism, the Vehicle of Transmission

The great Indian emperor Aśoka (usually called *Devānāṃpriya*, 'Beloved of Gods') ruled from about 272–232 BC His missionary activities are recorded in Rock Edict No. XIII as follows (Sircar, p. 54, and P. Thomas, p. 15).[1]

> So what is conquest through *Dharma* is now considered to be the best conquest by the Beloved of Gods. And such a conquest has been achieved by the Beloved of Gods not only here in his own dominions but also in the territories bordering his dominions as far away as the distance of 600 *yojanas* where the *yavana* king Antiyoka (Antiochus) is ruling; further where four other kings named Turamāya (Tulamāya, Ptolemy), Antikini (Antigonas), Makā (Magā, Magas), and Alikasundara (Alexander) are ruling ...

Subsequently, Buddhism was taken to various countries of Asia and this helped in spreading Indian culture there. It is well said that "Buddhism was, in fact, a spring wind blowing from one end of the garden of Asia to the other and causing to bloom not only the lotus of India, but the rose of Persia, the temple flower of Ceylon, the zebina of Tibet, the chrysanthemum of China, and the cherry of Japan. Asian culture is, as a whole, Buddhist culture" (Bapat, p. 397). During the full first millennium of the present era "all the countries of Asia from Persia to Japan, from Mongolia to Ceylon formed one cultural commonwealth with India as its centre and fountainhead" (P. Thomas, p. 2).

It was in the reign of Ming Ti (first century AD) that Buddhism was formally introduced in China with his imperial sanction (Chou, p. 66). According to Chihpang's *Records* of the *Lineage of Buddha and Patriarchs* (c. 1200 AD), Ming Ti had sent envoys to bring Buddhist scriptures and priests to China (*Ibid*., 66–67). Consequently two Indian monks Kia-yeh-Mo-than (Kaśyapa Mātaṅga) and Chu-fa-lan (Dharmarakṣa or Dharmaratna, etc.)[2] reached the Chinese capital Loyang (in AD 64) where a monastery was built soon. They learnt Chinese and translated

Gaṇita Bhāratī, Vol. 27, Nos. 1–4 (2005), pp. 26–63; Revised version from an unpublished report which was prepared by the author for his INSA project.

K. Ramasubramanian (ed.), *Gaṇitānanda*,
https://doi.org/10.1007/978-981-13-1229-8_57

Buddhist books the first of which was the *Fo-shuo-ssu-shih-erh-cheng-ching* (Sūtra of Forty-Two Sections Spoken by Buddha) (*Ibid.,* p. 67 and Mukherjee, pp. 2–3).

Buddhism spread in China steadily. More and more monks went there from India and Central Asia and a large number of monasteries started working. By the end of fourth century there were over 17,000 Buddhist institution in China (HCIP, Vol. III, p. 615, and Varma, p. 137). Quoting from *Records of Buddhism and Taoism of the Book of the Wei Dynasty*, Chou (p. 106), has given a table according which there were more than 3000 monasteries and about two million monks and nuns in China in AD sixth century. In fact by that time Buddhism had become China's own religion (Varma, p. 137), and that, according to *T'ung Chien* (Mirror of History), almost every household had been converted (Chou, p. 106).

During prehistoric times, the Peking Man (whose bones were found in 1929) is said to have lived about 5,00,000 year ago (*Ibid.,* 14). During the historical period of over 5000 years, the Chinese civilization and culture have been quite high and more or less continuous. The remarkable history of China is the story of a large number of dynasties which ruled variously as shown in the Table 1.[3]

During the last 2000 years there have been active cultural and civilizational interaction and vibrations (and even war) between India and China. The whole historical period has been divided conveniently into five phases:

1. (Before 64 AD) Thoughts of Confucius (sixth century BC) prevailed. His *Spring and Autumn Annals* is the political bible of China (Chou., p. 52).
2. (64–644 AD) Movement of Buddhism into China.
3. (645–1160) Building up of a Buddhist socio-political and cultural infrastructure in China with Indian monks.
4. (1220–1765) Diplomatic and Commercial activities.
5. (After 1765) Matured political diplomacy involving even international considerations.

Thus during the intermediary ancient and medieval periods Buddhism was the medium first of transmission and then of interaction with a sort of regular exchange of visitors. According to the findings of Liang Chi-chao (a modern Chinese historian), more than 160 Chinese pilgrims and scholars came to India during fifth to eighth centuries AD (Bapat, 163–164). The author of the Chinese book *She-Kia-Fang-Che* claims to have "studied all the biographies of monks (who went to the country of Buddha) and heard many things personally." But he adds that those who went and came back in Sui and Tang periods "are too many to be mentioned" (Bagchi, p. 129). Life sketches of 226 eminent Indian monks travelling to China and 118 Chinese pilgrims travelling to India may be found in the India and China: *20 Centuries of Civilizational Interaction and Vibrations* by Tan Chung et al., Delhi, 2005 (MLBD Newsletter, Vol. 27, No. 6, p. 2). Indian king Śrīgupta built a special temple for the Buddhist priests who came from China (*HCIP*, Vol. 2, p. 649).

Many of the Chinese pilgrims had scholarly taste and studied in Buddhist universities in India. Such monastic universities (*Vihāras*) were at Nālandā, Valabhī,

Table 1 Chronology of Chinese history

Name of dynasty period		Time	Remark
HSIA (Xiā)		About 2200–1600 BC	(2357–1767 BC)
SHĀNG		About 1600–1100 BC	(1767–1122 BC)
Chou (Zhou):	Western Chou	1122–771 BC	
	Eastern Chou	770–256 BC	
	Spring and autumn period	770–476 BC	
	Warring states period	475–221 BC	
Ch'in (Chhin or Qin)		221–201 BC	(256–206 BC)
HAN (Han.):	Western Han	206 BC–24 AD	
	Eastern Han	25–220 AD	
Three kingdoms:	Wei	220–265 AD	
	Shu Han	221–263 AD	
	Wu	222–280 AD	
Chin (Tsin or Jin):	Western Chin	265–316 AD	
	Eastern Chin	317–420 AD	
Southern:	Liu Sung	420–479 AD	
	Ch'i (Chhi or Qi)	479–502 AD	
	Liang	502–557 AD	
	Chen (or Chhen)	557–589 AD	
Northern:	Northern Wei	386–534 AD	
	Eastern Wei	534–550 AD	
	Western Wei	535–556 AD	
	Northern Ch'i	550–577 AD	
	Northern Chou	557–581 AD	
Sui		581–618 AD	
Tang (Thang)		618–907 AD	
Five dynasties:	(Later Liang, Tang, Chin, Han and Chou)	907–960 AD	
Sung (Song):	Northern Sung	960–1279 AD	(960–1126 AD)
	Southern Sung	1227–1279 AD	(1127–1279 AD)
Also:	Liao	916–1125 AD	
	Hsia	1032–1227 AD	
	and Chin	1115–1234 AD	
Yuan (Mongol)		1271–1368 AD	
Ming		1368–1644 AD	
Ching (Chhing or Qing Manchu)		1644–1911 AD	
The Republic of China		1912–1949 AD	
Peoples Republic of China		1949 onwards	

Vikramaśilā, Jagaddala and Odantapurī. They attracted students from various parts of Asia (Bapat, 156). Of these the most famous and magnificent (and architecturally the grandest) was the Nālandā Mahāvihāra with 1500 teachers and about 10000 students (*Ibid.,* 164). Subjects of study included even Brahmanical philosophy, mathematics, astronomy and medicine (*HCIP*, Vol. 3, p. 618).

Several of the Chinese pilgrims were keen to personally see and know details of Buddha's country and toured various parts and kingdoms of India. Of such pilgrim-scholars the names of Fa-Hien (fifth century AD), Hiuen Tsang or Yuan Chwang (seventh century), and I-tsing (c. 700) are well-known. They wrote vivid accounts of their travels. Their descriptions throw valuable light on the history of ancient India.

Some of the visiting Chinese scholars collected a large number of Buddhist and other works especially copying those texts which were unknown in China. Thus they took loads of Indian works to China for study and translation.

In fact the number of Indian works which were translated into Chinese by various Indian and Chinese scholars (often jointly) was very enormous. For instance, Hiuen Tsang, after going back to China, along with his associates, translated 600 Sanskrit works into Chinese (Bapat, 241). Catalogues of the translated works were prepared by ancient and medieval historians and scholars from time to time. A sixth century catalogue lists texts numbering about 5400 volumes and the Sui catalogues list about 8000 Buddhist works (Mukherjee, 46–47). The famous modern B. Nanjio Catalogue (1883) records 1962 Chinese translations of Indian texts.

Paradoxically, it often turns out that some works are now available only in foreign translations (originals being lost!). More ironic is the fact that while India has practically no substantial records about her cultural conquests abroad, foreign literary sources do tell about them. Early political and commercial relations between Indian and China were also there.

2 *Mātaṅgāvadāna*, *Lalitavistara* and *Śārdūla-karṇāvadāna*

When the Buddhist monk Kāśyapa Mātaṅga arrived in China in AD 64, the Chinese king honoured him. Mātaṅga (Mo-Than in Chinese) translated a number of Indian works into Chinese and he soon became a legendary figure in China and his name carried a great authority. In fact his name is associated with a large number of Chinese translations of Indian works although some of these Chinese translations belong to later periods (but a few may be revised versions of translations started by Mātaṅga and his associates). Interestingly a French translation of the Mongol version of Mātaṅga's *Sūtra of the forty-two sections* appeared in 1848 (Bapat, 354).

According to Yabuuti (1954, p. 585), the *Mātaṅgāvadāna* was translated into Chinese about the third century AD although the original is believed to date a century earlier (and attributed to Mātaṅga). It gives the lengths of monthly shadows of a 12-inch gnomon which is the standard parameter (12 *aṅgulas*) of Indian astronomy.

According to Shinjo Shinzo, the data implies a place of observation at 43° north latitude which may indicate the region from which Buddhism entered China (*Ibid.*). Of course Mātaṅga went to China from Central Asia (Bapat, 110, and Chou, p. 67).

Needham (p. 710) states that the Chinese work *Mo-téng-Chia Ching* (*Mātaṅgī-Sūtra* or *The Book of Mātaṅga*), supposed to be translated by Chu L-Yen from Sanskrit, contains a list of *hsiu (nakṣatra)* with the number of stars in each *nakṣatra*. It is ascribed to the period of Three Kingdoms or San Kuo (AD 220–265) although Needham places it in the eighth century AD.

A work called *Mo-téng-Chia-Ching Huang-Thu* or 'Map of Heavens in *Mātaṅgī-sūtra*' (one book) is mentioned in the *Official History of the Sui Dynasty* (seventh century) (see Sect. 4).

Lalitavistara, a Buddhist Sanskrit work of first century BC (*HHM*, I, 10), contains a biographical account of Buddha (Bapat, 124–127). It belongs to a class of nine sacred texts called *Vaipulya-sūtra* or Nine Dharmas. It calls itself as *Mahānidāna* as well as *Purāṇa* (Vaidya, p. ix).

Lalitavistara was translated into Chinese several times. The Indian priest Chu-fa-lan (first century AD) translated a few books into Chinese but these translations are lost (Mukherjee, 1–3). The Chinese title of one of them is Fo-pan-hsin-chin which, according to B. Nanjio and S. Julien, was a translation of the *Lalitavistara* although there are other views (*Ibid.,* p. 3). Two other translations were also lost by the year 730 when *Khai-yuen-lu-Catalogue* was compiled. Both translations are called *Phu-Yao-Ching* (*'Samantaprabhava-sūtra'*). The name of 1st translator (third century AD) is lost but the other was done jointly by Pao-yun and Chu-yen in the fifth century (*Ibid.,* 62–63).

Two extant Chinese translations of *Lalitavistara* are recorded in *Nanjio's Catalogue* (Nos. 159 and 160). One of these, called *Pou-yao-king* (=Phu-Yao-Ching), was made by Chu-Fa-hu or Dharmarakṣa (AD 305/308), and the other, called *Fang-kwang-ta, chwang* (*Vaipulya-mahāvyūha-sūtra*), etc., by Śramaṇa Divākara of Central India about AD 685 (Vaidya, xi; Mukherjee, 9–10, 62–63).

The Sanskrit text of the *Lalitavistara* was edited by R. L. Mitra (Calcutta, 1877) and then by S. Lefmann (Two Parts, Halle, 1902, 1908). These were used by P. L. Vaidya for his recent edition (Darbhanga, 1958) which is used here. There is a French translation by P. E. Foucaux (Paris, 1847–48; 1887–92) with the text of the Tibetan translation of the ninth century scholars Jinamitra, Dānaśīla, Munivarma and Ye-śes-sde (*Ibid.*). Śāntibhikṣu Śāstrī has translated the work into Hindi (Lucknow, 1985). Dharmarakṣa's translation is reported to have only 8 chapters while present text has 27 chapters.

Some scientific ideas are found embedded in Buddhist views on evolution and nature of the world, life and matter. Concepts of atomism were developed by Buddhist and metrological systems for measuring distance, capacity, weight and time are found in their works. The microscopic linear system of units found in the *Lalitavistara* (p. 104) is as follows:

7 *paramāṇus*	1 *aṇu*
7 *aṇus*	1 *truti (truṭi)*
7 *truṭis*	1 *vātāyana-raja*
7 *vātāyana-rajas*	1 *śaśaraja*
7 *śaśarajas*	1 *eḍakaraja*
7 *eḍakarajas*	1 *goraja*
7 *gorajas*	1 *likṣā*
7 *likṣās*	1 *sarṣapa*
7 *sarṣapas*	1 *yava*
7 *yavas*	1 *aṅgulī-parva (aṅgula)*

Thus one *aṅgula* (finger-breadth or digit) was subdivided into 7^{10} *paramāṇus* or atomic-measures. The Chinese subunits are said to follow a decimal scheme in which (Needham, 85; Mikami, 26).

1 *Chhih* ('foot') = 10^6 *hu* ('diameter of silk string'.)

On the macroscopic side, the *Lalitavistara* (p. 103) contains a numeration system in centesimal scale (beyond *koṭi* = 10^7) going up to very very large numbers. The names of various denominations in the first series of counts are *ayuta* (= 10^9), *niyuta* (= 10^{11}), *kaṅkara, vivara, akṣobhya,* etc., up to *tallakṣaṇa* (= 10^{53}), the number of terms after *ayuta* being 23. Beyond this *Tallakaṣaṇa-gaṇanā*, the names of eight more (similar) series of counts are mentioned starting with *Dhvajāgravatī-Gaṇanā*. Thus (as explained by Karl Menninger, p. 137) the last number in the last series of count (called Agrasārā) will stand for

$$10^{(53+8\times 46)} = 10^{421}$$

a monstrous number.

The *Lalitavistara* terms, *koṭi, ayuta* and *niyuta* were transliterated phonetically into Chinese as *juzhi, ayuduo* and *nayouta*, respectively (Martzloff, p. 97). The last one seems directly from the Buddhist form *Nayuta* (pali, *nahuta*) of *niyuta* (Hayashi, 111).[4] The *Lalitavistara* (p. 103) mentions that the *Dhvajāgravatī* count of the above numeration scale can be used to count even the number of sand particles of the river *Gaṅgā*. Now if we use two series (each of 23 terms) of the above centesimal numeration, we get

10^{99} when we start with usual *koṭi* $=10^7$
10^{97} when we start with Chinese *koṭi* $=10^5$ (Martzloff, 97)
10^{96} if we use the Chinese higher base *wan* $=10^4$.

And it is interesting to note that Chu (= Zhu Shijie) in his system of large numbers calls 10^{96} as *heng ho sha* which literally means "sands of the River Ganges" (Lam, pp. 8–9). Another expression used in the *Lalitavistara* (p. 104) in connection with counting of very large numbers is *trisāhasra mahāsāhasra lokadhātu* ('a major universe of 3000 great chiliocosms') which is semantically or phonetically transliterated in Chinese as *dagian* ('great thousands') in *Shushu jiyi* (Martzloff, 96).

A more important work which was translated into Chinese is the *Śārdūlakarṇāvadāna* (first century AD or earlier). Mukhopadhyaya (pp. xii–xiii) mentions the following four translations of the work:

(i) Oldest Chinese translation is by the Parthian prince Ān Shi-kāo who, after renouncing his kingdom, went to China in AD 148 and worked at the rendering till AD 170 (*Nanjio Catalogue*, No. 643).

(ii) The most elaborate translation was done between AD 223 and 253 jointly by Chu Lüh-yen (an Indian śramaṇa) and Che K'ien (as *Upāsaka* or householder of Tukhar).

(iii) An elaborate Chinese translation was made by Chu-Fā-hu (Dharmarakṣa) between AD 266 and 317. He was a native of Tunhuang and, after being educated in India (*HCIP*, II, 647), went to China in AD 266. According to Needham (p. 712), the title of the Chinese translation was *Shê-Thou-Chien Thai-Tzu Erh-shih-pa Hsiu Ching*.

(iv) Another translation was done under the Tsin or Chin Dynasty (between 265 and 420) but the name of the translator is not known.

The work was also translated into Tibetan by Ajitaśrībhadra and Śākyaprabhā about 864 AD.

The importance of *Śārdūlakarṇavadāna* (which is said to be a part of *Divyadāna*) lies in the fact that through its various translations, Indian scientific ideas reached China quite early. Following astronomical and mathematical features/data from the work may be briefly mentioned.

(1) *List of Indian Nakṣatras* (pp. 46–52): Needham (p. 712) has already pointed out the transmission in this regard. The 28 *nakṣatras (hsiu)* listed are the usual Sanskrit names which are known in India since Vedic times. Here the list starts with *Kṛttikā, Rohiṇī* and ends with *Aśvinī, Bharaṇī*. The number of stars in each *nakṣatra* is also given along with some other details.

(2) *Names of Grahas* (pp. 53 and 104): The text mentions only 7 *grahas* (planets in ancient or astrological sense) in two places in two different orders, one of which is Venus, Jupiter, Saturn, Mercury, Mars, Sun, Moon. The exclusion of *Rāhu* and *Ketu* (of Indian *Navagrahas*) shows that these two were not accepted properly at that remote time. Often these were added in the Chinese translations and manuscripts.

(3) *Linear Units* (p. 58): A table of microscopic measures from *paramāṇu* to *aṅgula*, similar to that of *Lalitavistara* (see above), is given here with slight variations in some names but with the omission of one subunit, namely *truṭi* (between *aṇu* and *vātāyanaraja*).

(4) *Units of Time* (pp. 54–57): For subdividing the *muhūrta* ($= \frac{1}{30}$ of *ahorātra* or day and night), two systems are mentioned. One of them is as follows:

$$\begin{aligned} 1 \, muh\bar{u}rta &= 30 \, lavas \\ &= 30 \times 60 \, k\d{s}a\d{n}as = 30 \times 60 \times 120 \, tatk\d{s}a\d{n}as. \end{aligned}$$

It appears that in early ancient India, the semi-sexagesimal division played a role. According to the *Manusmṛti*, I, 64, a month of 30 days can be divided as[5]

$$1 \text{ month} = 30 \text{ days} = 30 \times 30 \, muh\bar{u}rta = 30 \times 30 \times 30 \, kal\bar{a}s$$
$$= 30 \times 30 \times 30 \times 30 \, k\bar{a}\d{s}\d{t}h\bar{a}s$$

(5) *Duration of Daylight* (pp. 53, and 100–108): The longest and shortest durations of the daylight are mentioned to be of 18 and 12 *muhūrtas*. This is exactly same as in the *Vedāṅga Jyotiṣa*. The months of *Kārtika* and *Vaiśākha* are said to be of equal day and night (each = 15 *muhūrtas*).

(6) *Shadows at various parts of a Day* (pp. 54–55): After sunrise, the *puruṣī* shadows are given to be of measure 96, 60, 12, 6, 5, 4 and 3 during the first seven *muhūrtas*, while during the 8th *muhūrta* (middle of day), it is stated to be stationary (constant). Since the duration of the day is given to be 15 *muhūrtas*, the data may be taken to refer to an equinoctial day. Exactly same data is found in the *Atharva Vedāṅga Jyotiṣa* (verses 6–10) except for the change of measure unit from *aṅgula* to *puruṣa* here. S. S. Lishk and S. D. Sharma tried to establish the observational origin or basis of the data but without success (*IJHS,* Vol. 15, pp. 193–200).

(7) *Noon Shadows for the Whole Year* (pp. 100–103): For a gnomon of height 16 *aṅgulas*, there is a table of 12 monthly shadows as follows:

$$\frac{1}{2}, 2, 4, 6, 8, 10, 12, 10, 8, 6, 4, 2 \, a\dot{n}gulas.$$

The shortest shadow is for the month *Śrāvaṇa* when the day is longest (of 18 *muhūrtas*), and the longest is for *Māgha* month when day is shortest (of 12 *muhūrtas*).

3 Other Transmissions to China Upto About 600 AD

Certain empirical rules used by Zhang Heng (c. 130 AD) imply the value $\pi = \sqrt{10}$ which was used in India centuries earlier (Gupta 1992, pp. 2–3). Nāgārjuna, a great dialectian (c. 200), propounded the *Śūnyavāda* philosophy. His main work, the *Mādhyamika-śāstra* (*Chun-lun* in Chinese), displays a rare insight into the science of logic. His other works were also translated into Chinese, and Kumārajīva (fifth century) rendered his biography into it (Bapat, 194–195). Mou-tseu (third century) tried to establish the superiority of Buddhism over local religions prevalent in China (*Ibid.,* 59).

Written expressions for very large numbers in different numeration system (in powers often) are found the *Shushu Jiyi* ('Memoir on Some Traditions of Mathematical Art') attributed to Xu Hue (c. 200 AD) who is believed to be a pupil of the great astronomer Liu Hung (Mikami, 44). The numeration systems indicate Indian influence especially of the Buddhist idea of infinite cycles of reincarnations (*Ibid.,* 57–58; Martzloff, 99 and 141), and of *kalpas* (Needham, 30 and 87). The influence was so high that even the astronomer Han Chai (c. 220) and his contemporary mathematician Wang Fan both became Buddhist (Mikami, 57) and possibly then the problem of circle measurement in China was 'transplanted from Indian soil.'

Dharmakāla (Than-mo-chiao-lo) of central India reached China in 222 and translated the *Vinaya-Piṭaka* which mentions three classes of *Gaṇita*, namely *mudrā, gaṇanā* and *saṃkhyāna* (*HHM,* I, p. 7). According to Muroga and Unno (p. 56), the Indian theory of dividing the world into four parts was introduced as early as the third century AD. The date of Sun Tzu or Sunzi is controversial. He is now usually placed in the fourth century (often between 280 and 473). His *Sunzi Suanjing* mentions a Buddhist *sūtra* and employs word-numerals resembling Indian concept of *bhūta-saṃkhyās* (Martzloff, 136–138). The calendarist He Chengtian (370–447) tried to find out about Indian astronomy from the famous monk Hui Yan (*Ibid.,* p. 96). The great Chinese scholar Tao-ngan (fourth century) wrote a book on India (*HCIP*, III, p. 613).

The *Abhidharma-kośa* of the celebrated teacher Vasubandhu (fourth century) is an encyclopedic work which proved valuable for the propagation of Buddhism in Asia. It was especially taught at the Nālandā University where of Chinese priest Tao-Lun (or Śilaprabha as he was called in India) and Wou-Lung studied it (Mukherjee, 78–79). Vasubandhu's other works include his two treatises on logic, namely *Tarkaśāstra* and *Vādavidhi* (Bapat, 197). He wrote his own *bhāṣya* on the *Abhidharma-kośa*. A French translation of *Abhidharma-kośa* based on the Chinese and Tibetan versions was published by Louis de la Vallé Poussin in 5 vols. (1923–1925). Here the recent edition (with auto-commentary etc.) by D. Śāstrī is used. The scientific contents from the work of Vasubandhu and his commentary on it may be briefly mentioned now.

(1) The auto-commentary under III, 94 (p. 544) quotes the *muktaka-sūtra* ('free secular aphorism').

षष्टिः स्थानान्तराण्यसंख्येयम्

from which we know that 60 decuple terms in decimal counting scale were already in use in India in those days. However, only 52 names are given, and for the remaining ones the auto-commentary remarks that they are forgotten (*vismṛtam*)! The eight names are lost from the middle (*madhyāt*) and the super-commentator Yaśomitra advised to restore full table with suitable names. So the present author (R. C. Gupta) reconstructed the table with the help of names found in *Lalitavistara* mostly (*GB*, Vol. 23, p. 85). The reconstructed Buddhist set of 60 decuple terms is:

eka, *daśa (daśaka)*, *śata* (100), *sahasra* ($= 10^3$), *prameda*, *lakṣa*, *atilakṣa*, *koṭi*, *madhya*, *ayuta*, *mahā-yuta*, ($= 10^{10}$), *kaṅkara*, *mahā* ..., *balāksa* ($= 10^{57}$), *mahā*, *asaṃkhya* ($= 10^{59}$).

These large numbers are mentioned in connection with measures of *kalpas* in relation to birth of various Buddhas. The same idea influenced Xu Hue of China (c. 200) as mentioned above.

(2) The *Abhidharma-kośa*, III, 55–86 (p. 536) contains, like *Lalitavistara*, a table of linear measures starting with *paramāṇu* ('atomic particle') and ending with[6] *aṅgulī-parva* ('finger-breadth'), the increase each time being seven fold. But there are some changes, and the table here may be summarized as follows:

$$\begin{aligned} \text{One } a\dot{n}gula\ parva &= 7 y\bar{u}ka = 7^2 lik\d{s}\bar{a} = 7^3 chidraraja \\ &= 7^4\ goraja = 7^5 aviraja = 7^6 \acute{s}a\acute{s}araja = 7^7 aparaja \\ &= 7^8 loharaja = 7^9 a\d{n}us = 7^{10} param\bar{a}\d{n}us. \end{aligned}$$

There is an interesting difference regarding higher units of length or distance. The *Lalitavistara* (p. 104) and *Śārdūla-karṇāvadāna* (p. 58) both give

$$\begin{aligned} 1\ yojana &= 4\ kro\acute{s}a\ (m\bar{a}gadh - kro\acute{s}a) \\ &= 4 \times 1000\ dhanus \end{aligned}$$

But the *Abhidharma-kośa* (p. 536) (III, 87–88) gives

$$1\ yojana = 8\ kro\acute{s}a = 8 \times 500\ dhanus.$$

However, the units of time from this work tally with those in the *Śārdūla-karṇāvadāna* (from *tatkṣaṇa* to year).

(3) *Cosmography*: Somewhat similar to other Indian schools, the Buddhist cosmography, as described in the *Abhidharma-kośa* (pp. 507–511), consists of the Sumeru mountain in the centre surrounded alternately by seven annular seas and seven annular mountains all concentric with the Sumeru (= Meru). The height of Meru above water is 80,000 *yojanas* and that of the surrounding closest *yugandhara* mountain is 40,000 *yojanas*. The heights of the successive mountains from a G. P. with common ratio $\frac{1}{2}$.

The 7th surrounding mountain Nimindhara (whose height is 625 *yojanas*) is itself surrounded by the annular salty large sea (*Mahodadhi*) of width 322000 *yojanas*.

In this large sea (or ocean) are situated four *dvīpas* (islands) in the four cardinal directions. Their names, form and dimensions are as follows:

(i) *Jambūdvīpa* (*yen-fu-t'i* in Chinese) in the south in the form of a *śakaṭa* or isosceles (inverted) trapezium of sides 2000, 2000, 2000 and 3.5 *yojanas* (the last dimension represents the short base of the inverted trapezium like the southern tip of India). Some scholars have interpreted the relevant text (III, 53–54, pp. 511–512) to take *śakaṭa* as a triangle of sides 2000, 2000 and 3500 or 3050 (A. Jain, p. 20). In the *Kāla Cakra Tantrarāja* I, 17 also the shape is a triangle.

(ii) Godānīya Island in the west is in the form of a circle of perimeter 7500 *yojanas* and diameter 2500 *yojanas* (III, 54; p. 512.) implying the use of the simple approximation $\pi = 3$.

(iii) Kuru Island in the north is in the form of a square of perimeter 8000 *yojanas*.

(iv) Videha Island in the East is said to be of the form like more or less the half-moon (*ardha-candra-vat*) whose *pārśva-trayam* ("three sides"?) are like those (of Jambūdvīpa) and one of length 350 *yojanas* (cf. A. Jain, p. 20 with text reference III, 54).

It may be mentioned that the measures of possible perimeters of the above four Islands are given to be apparently as 7000, 8000, 9000 (Videha's) and 10000 (Kuru's) *yojanas* in *Lalitavistara* (p. 104).

(4) Value of π: We have already mentioned above a case in which $\pi = 3$ is used. *Abhidharma-kośa*, III, 48 (p. 507) contains a sort of *sūtra 'samantatastu triguṇam'* which means, according to auto-commentary, that circumference is three times the diameter. It is applied here to D=1203450 to get 3610350 = C (p. 507).

Named after the Indo-Greek king Menander (first century BC). The Pali book *Milinda-pañha* ("Questions of Milinda") was translated into Chinese between AD 317 and 420 under the title *Nāgasena-sūtra* (Bapat, 182). The book states that all *vidyās* were taught in Jambūdvīpa and scientific subjects were discussed (A. Jain, 21). In early days Jambūdvīpa was understood in China to be synonymous with India (Muro and Unno, 51) like *Vinayapiṭaka* (mentioned above). The three types of *gaṇita*, namely *mudrā, gaṇanā* and *saṃkhyāna* are enumerated in the *milinda-pañha* (*HHM*, I, 7).

Arithmetic (*gaṇanā* or *saṃkhyāna*) is regarded noblest of arts in *Majjhima Nikāya* (*Ibid.,* 4) whose Chinese translation by Dharmanandi (c. 390 AD) is lost but that by Gautama Saṅghadeva (c. 400) was made from the Sanskrit version called *Madhyamāgama* (Mukherjee, 13). The *Dīghanikāya* is the 1st book of *Sutta-piṭaka* (whose 2nd book is *Majjhima-nikāya*). Its Sanskrit version, the *Dīrghāgama* was rendered into Chinese by the Indian monk Buddhayaśa with the help of Chu Fo-nien in 412–413 (*Ibid.,* 23). It enumerates the three types of *gaṇita* mentioned above. Like *Abhidharma-kośa*, it explains the structure of the universe with Sumeru (Xumixan in Chinese) at its centre (Unno, 1980a, 57–58).

Kumārajīva (Ciu-mo-lo-shi) is a great name in Chinese Buddhism. He was born of an Indian father (Kumārāyana) and Jīvā of the royal family of Kuci (Eastern Turkistan). He was educated in Kashmira under Ācārya Buddhadatta and, in AD 401, went to China where he was greatly respected, although some controversy exists about the status (Bapat, 211; Puri, 329). He had thoroughly studied Buddhist works and was well-versed in *Vedas*, five sciences,[7] astronomy (e.g. *nakṣatravidyā*) and *Pyotiṣa* (Puri, 46 and 76). His Chinese translations of about a hundred Buddhist works covered almost all branches of learning (Mukherjee, 17). He also revised the earlier translations, and 50 extant translations are still ascribed to him (*Ibid.*).

One of the most important works translated by Kumārajīva was the *Mahāprajñā-pāramita Śāstra* which is an encyclopedic commentary by Nāgārjuna (c. 100 AD) on the *Pañcaviṃśati-śatasāhasrikā Prajña pāramitā*, the Chinese version being called *Ta-ci-tu-lum* (Nanjio No. 1169; Bapat, 115; Mukherjee, 22). A French translation of the great commentary, after the Chinese version, with notes was published by E. Lamotte in several volumes the first of which appeared in 1944 (Bapat, 356).

The astronomical parameters mentioned in the *Mahāprajñā-pāramita Śāstra* translation are comparable to those given in the *Vedāṅga Jyotiṣa*, e.g. solar month of $30\frac{1}{2}$ days (making year of 366 days) (Chin Keh-mu; 784, Gupta 1989, 40). The work or *Chih-tu lun* is said to state that "if a stone falls from the heaven called Rūpadhātu, it takes 18383 years to reach the earth" (Bagchi, p. 1). The height of Heaven is given to be 78940 *li* in the *Shūgaishō* (Unno 1980a, 69), and elsewhere Brahmaloka (Heaven) is said to be at the height of 1465×10^4 *yojanas* (Hayashi, 549). Fall of a ball from[8] Indraloka for 6 months defines *Rajju* (*Jainendra-kośa*). A model of mount Sumeru was constructed in China in AD 405 while in Japan it was built in 612 (Unno 1980a, 65).

Dharmarakṣa was a great Indian Translator of Buddhist works into Chinese during early fifth century. He translated more than a score books which include the *Mahāvaipulya-Mahāsannipāta-sūtra* (Mukherjee, 24). It contains material on astronomy and calendrical science (Chin Keh-mu, 784; Needham, 716). At that time the fame of Kashmir as a centre for learning was at zenith in the Buddhist world and several of its scholars visited China. Guṇavardhana went to Java via Ceylon and from there to China in a ship owned by an Indian, reaching Nankin in AD 431. He was welcomed by the Chinese king himself (Puri, 330). Chinese mathematician Sunzi (dated between 280 and 473 AD) has been mentioned above. A problem from his *Sunzi Suanjing* (Master Sun's Arithmetical Manual) reads thus (Mikami, 26)

> There is a Buddhist work consisting of 29 stanzas, each of which contains 63 ideographs. It is required to find how many ideographs will be contained in all?

The *Hsiahou Yang Suan Ching* of Hsiahou (= Xiahou) Yang has names for certain fractions such as 'average half' or 'dead on half' ($\frac{1}{2}$), 'lesser half' ($\frac{1}{3}$), 'greater half' ($\frac{2}{3}$) and 'weak half' ($\frac{1}{4}$). Van Hée suspected Indian origin for these names (Needham, 34, Martzloff, 124, 193). In AD 472, Chi-Chia-yé and Than-Yao translated the history of succession of 23 patriaches from Mahākāśyapa to Bhikṣu Siṃha (Mukherjee, 39).

In the history of Buddhism in China there were two famous scholars of the name Bodhiruci ('chia-ai' in Chinese). One of them belonged to the Northern Wei dynasty period and the other to the Tang (Bapat, 219; Mukherjee, 38 and 68). Bodhiruci I arrived in China in AD 508 from India. With the help of some Indian and Chinese monks, he translated a few Indian astronomical books into Chinese and this translation ran into more than 200 chapters (Mukherjee, p. 38, giving reference of P. N. Bose's *Indian Teachers in China*). Around 518 the agents of the Chinese dowager empress (Tai-Hau) procured about 200 volumes from India, all standard works (*Ibid.;* and Beal, 55).

Chinese king Wu-ti of Liang Dynasty (502–557) embraced Buddhism, collected 5400 volumes of canonical books and himself wrote a book on Buddhist ritual (Mukherjee, p. 31). A lost catalogue indicates that, between 67 and 518 AD, 1432 distinct works under 20 classes were translated from Sanskrit into Chinese (*Ibid.,* 31–32). Monk Hui-Chiao's Kao Sêng Chuan ('Biographies of Outstanding Monks') compiled in the sixth century, covers learned scholars including Indians (Needham, 705). Around 520, the Chinese priest Sang Yien compiled *Chu-San-tang-tsi* or a collection of the records of translations of the *Tripiṭakas* (Mukherjee, p. 32).

Paramārtha (Po-lo-mo-tho) is a great name in Chinese Buddhism. Known by some other names (including Guṇaratna) also, he was born and educated in Ujjain, later on went to Magadha and probably settled in Pāṭalīputra the capital of Gupta empire (Bapat, 212). In response to a Chinese royal mission to Magadha, he was sent to China where he arrived in 548 with a large number of Indian books.

During the next 20 years Paramārtha translated about 70 Indian works into Chinese and died in 569 at the age of 71 (*Ibid.,* 213). Among the works he translated are the *Abhidharma-kośa-śāstra*, the *Aṣṭādaśa-śūnyatā-śāstra* and the *Sāṅkhyakārikā* (of Iśvarakṛṣṇa), a non-Buddhist work (Bapat, 214; Mukherjee, 34). He also translated the *Lokasthiti-abhidharma-śāstra* in 558 under the Chinese title *Li Shih A-pi-Than-Lun* (Philosophical Treatise on the preservation of the world) which is astronomical in content (Needham, 707). The subject of the 1st chapter is the motion of earth, and that of the 19th chapter is that of the sun and the moon (Mukherjee, 34). The main scientific contents of the *Abhidharma-kośa-vyākhyā-śāstra* (*O-phi-ta-mo-ku-sho-shih-lun* in Chinese, Nanjio No. 1269) have been already briefly mentioned above, and "Paramārtha rendered a great service to China by translating it" (*Ibid.,* and Bapat, 214).

An Indian system of numeration appeared in the Chinese work *Ta Pao Chi Ching* ("Mahāratna-kūṭa-sūtra") which was translated from Sanskrit by Upaśūnya (Yueh-Po-Shu-Na in Chinese) of India in 541 (Needham, 88; Mukherjee, 36). Another Chinese translation of *Mahāratna-kūṭa-sūtra* (Nanjio, No. 23) was made c. 700 by Bodhiruci II (Bapat, 220; Mukherjee, 68). A parallel to the story of counting as told in the legend of Nala and Rituparṇa in the great epic *Mahābhārata* (*Vanaparva*, Chapter 72) is found in the story of Chhiwu Huai-Wen (flourished between 530 and 570) as found in his biography (Needham, 78). According to Chin Keh-mu (p. 783), the *Tang Dynasty Lives of Eminent Monks* records that "between 566 and 571, an Indian monk named Dharmaruci translated twenty books of Brahman astronomy." (Unfortunately, 'all these are lost', Chin adds). It may be relevant to point out here that 'Dharmaruci' was also the earlier name of Bodhiruci II (571–727) who is said to have a long life of 156 years (Bapat, 219). He was expert in sciences of astronomy, medicine, geography and divinity. His work of translating Indian works into Chinese is discussed in a subsequent section.

The Chinese astronomer and mathematician Chen (or Zhen) Luan (c. 570) became a Buddhist. He made calendar for the Chou dynasty. He was profoundly read in books on Buddhism and had expressly referred to Buddhist *sūtras* in Chinese translations (Mikami, 30 and 58). He wrote several commentaries also, including those on the *Wujing suanshu* and the *Shushu Jiyi* both of which deal with large numbers and have Buddhist influence (Martzloff. 140–141). Mikami (p. 58) also suspects Indian origin for some of his problems.

During the Northern Chhi Dynasty (550–557), Narendrayaśas and Dharmajñāna (both from India) translated several Indian works into Chinese including *Candraprabhā-vaipulya, Sumeru-garbha* and *Abhidharma-hṛdaya-śāstra* (Mukherjee, 41). In the various Indian schools (Brahmanic, Buddhist and Jaina), Sumeru is the celestial axis around which sun and moon revolve, cosmographical structure is erected, etc. Even the number 33 (= no. of Hindu gods) is frequently explained in Buddhist works in astronomical terms (Beal, 81).

Jñānabhadra and Jinayaśa translated a work *Pañcavidyā* into Chinese but it was lost by 750 AD (Mukherjee, 42). According to Chin Keh-mu (p. 787), *pañcavidyā* ('five sciences') stood for grammar, medicine, technology, philosophy and logic.

4 During and From the Records of the Sui Dynasty, 581–618 AD

There was a short setback in Indian activities in China when emperor Wu-Ti of the Northern Chou Dynasty came to power in AD 557. He put a ban on Buddhism and Taoism, and Narendrayaśas and other monks fled; but 5 years later Wu-Ti's son withdrew his father's edict (Mukherjee, p. 42).

In AD 581, the Chou Dynasty's rule came to an end and the Sui Dynasty began to rule. Narendrayaśas (Na-Lien-Thi-Yeh-Shê in Chinese), the Indian Paṇḍita, was called back from exile in 582, and he resumed the translation work with the help of others. Among the new works taken up were the *Sūryagarbha, Śrīgupta* and *Mañjuśrī-vikrīḍita-sūtras* (*Ibid*., 44). According to Needham (p. 716), he also translated the *Mahāvaipulya-mahāsannipāta-sūtra* (*Ta Fang Têng Ta Chi Ching*) from Sanskrit into Chinese. It contains calendrical material, zodiacal cycle, association of planets with *hsiu (nakṣatra)*, etc. (*Ibid.,* Chin Keh-mu, 784). Of course, this work had been already translated earlier by Dharmarakṣa (c. fifth century) as mentioned above.

By special invitation, the Sui emperor also recalled Jinagupta from exile and made him the President of the Board of Translation. About 40 works were translated into Chinese between 585 and 592 (Mukherjee, 45). One of his assistant was Dharmagupta, an Indian who was educated at Kannauj (Puri, 332) and who also wrote a book giving geographical description of his journey to China where he reached in 590 (*Ibid.*).

Chi-k'ai (died 597) founded the T'ien-t'ai Buddhist school in China (earlier he followed the teachings of the school founded by Bodhidharma). The main texts of the new school included the famous *Mahāprajñāpāramitā-sūtra* (Nanjio, No. 1) whose Chinese titles was Ta-pan-jo-po-lo-mi-to-cin (Bapat, 115 and 218) and which was partly translated by Kumārajīva in the fifth century AD (Mukherjee, 53).

The Sui rulers of China were great patrons of Buddhist learning. The *Sui Shu* ('Sui History') in its digest of books mentions 1950 distinct Buddhist works including the *Po-lo-mên-shu* (or Poluomen Shu) or "Brahmanical Writings" on the Indian system of writing with alphabetic symbols (*Ibid.,* 43). The Chinese writing was based on ideographs, and scientific alphabetic system of India was a welcome. Due to royal encouragement, it made it possible to translate about 60 works into Chinese in a short rule of 37 years of Sui Dynasty. Also several catalogues of Indian works translated into Chinese were compiled during the Sui reign. (Mukherjee, 46–47): The first catalogue, complied in 594 by Fa-Ching and others mentions 2257 distinct works in 5310 fasciculi. The next, *Li-tai-san-pao-chi* ("Record Concerning the Triratna under Successive Dynasties"), complied by Fa-chan-fang in 597, has a list of 1076 works. The third catalogue, *Sui-chung-ching* (603 AD) lists 2109 books. The 4th catalogue compiled by Shaman Chi-kuo (during 605–616) was based on the imperial collection in the palace premises. There was also a 5th catalogue in which 1962 works were arranged in 77 classes. The last two catalogues (out of the five mentioned above) seems to be lost.

The *Sui Shu* (Records or official History of the *Sui* Dynasty) was written by Wei Chih (also called Wei Chang or Cheng) in 636 or soon after that (Needham, 715; Martzloff, 137). In it, the section on classical and other works mentions the following Indian astronomical and mathematical works in Chinese translation (Mikami, 58; Sen 1963, 22, and Gupta 1981, 270).

1. *Po-lo-mên Thien Wên Ching* (Brahmanical Astronomical Classic) in 21 books (or volumes, *juan*).
2. *Po-lo-mên Chieh-Chhieh Hsien-jen Thien Wen Shuo* (Astronomical Theories of Brāhmaṇa Chie-Chhieh Hsien-jen) in 30 books.
3. *Po-lo-mên-Thien Ching* (Brahmanical Heavenly Theory) in 1 book.
4. *Mo-têng-Chia Ching Huang-Thu* (Map of Heavens in the Mātaṅgī-sūtra) in 1 book.
5. *Po-lo-mên Suan Ching* (Brahmanical Arithmetical or Computational Classic) in 3 books or *Juan*.
6. *Po-lo-mên Suan Fa* (Brahmanical Arithmetical Rules or Methods) in 1 book or 3 *juan* (Martzloff, 96). According to Needham (p. 37) *this work may be on jyotiṣa instead of mathematics*.
7. *Po-lo-mên Ying Yang Suna Ching* (Brahmanical method of calculating time) in 1 book.

Most of these works are also mentioned in the *Hsin Thang Shu* (New History of Thang Dynasty) which was compiled by Ouyang Hsiu and Sung Chhi in 1061 (Cullen, 19; and Needham, 703). Unfortunately all these translations are irremediably lost. But they were also mentioned in Cheng Chhiao's *Thung Chih Lueh* (Historical Collections) in which the catalogue was based on the imperial library collection (Needham, 207).

There had existed, no doubt, various other translations of similar works in China (Mikami, 58). We cannot say whether these Indian works (which were translated into Chinese) contained only the pre-siddhāntic astronomy of India or the *Siddhāntic* astronomy as well which was definitely introduced in China during the seventh century (see next section).

Nevertheless, the rendering of Indian books into Chinese shows that a knowledge of Indian astronomy and mathematics was prevalent in China at that time. Also some scientific knowledge must have been transmitted orally as a large number of Indian scholars went and were there.

5 The Glorious Period of Tang Dynasty, 618–907

The fame of the Nālandā University (Mahāvihāra) was at its peak during the seventh century AD as a full-fledged institutions and as an international centre of learning. The famous scholar Hiuen Tsang (also called Yuan Chwang) studied there from about 635 to 640. He specialized in Vijñānavāda philosophy under the guidance of Śīlabhadra, the Head of the Mahāvihāra, and, later on, introduced that philosophy in China. Yuan Chwang has left a description (of Nālandā) which has been further supplemented by his disciple and biographer, Hwui-Li. According to it, 10000 students studied the Mahāyāna and the works of 18 sects (Bapat, 165). *Vedas* and books on *hetuvidyā* (logic), *Śabdavidyā* (Grammar etc.,) *Cikitsāvidyā* (medicine), *Sāṅkhya* (philosophy), etc., were also studied (*Ibid.,*). Thus Buddhist, Brahmanical and secular subjects (mathematics, astronomy, occult sciences, etc.) were all taught there (*Ibid.,* 239; *HCIP*, III, 618) Hiuen Tsang's *Si-yu-ki* ("Buddhist Records of the[9] western world") contains an account of Indian calendar and of the Sumeru, the cosmographical axis (Mukherjee, 50–52).

Hiuen Tsang returned to China in 645 with a large number of Indian works (which required 22 horses to carry them !). He opened a new era of translation in China and himself translated 75 Indian Treatises in 1335 fasciculi before his death in 664 (Mukherjee, 53). The most stupendous work he took was the *Mahā-prajñā-pāramitā-sūtra* (Sanskrit text of 200,000 *ślokas*) and translated 120 volumes entire "in all their wearisome re-iteration of metaphysical paradoxes" (*Ibid.*). A preface by the Chinese priest (Hiuen Tson) was added to the translation of each of the 16 *sūtras* of the above work (whose earlier translation by Paramārtha was partial). Yuan Chwang also translated the *Abhidharma-kośa* of Vasubandhu and its auto-commentary into Chinese, as well as some books on Indian logic which had great influence in China (*Ibid.,* 57–60).

Hiuen-Chiu (c. 638–665) had studied Sanskrit literature. He was a Chinese monk and took the Indian name Prakāśamati. Crossing Tibet, he came to Magadha and studied various books for 4 years. (Mukherjee, 75). Hwui-Lun (a native of Corea) took the Indian name Prajñāvarman, visited holy places (including Nālandā) and learnt Sanskrit language (*Ibid*., 76). Tao-Lun (or Śīlabhadra) and Wou-Lung both studied the famous *Abhidharma-kośa* at Nālandā where, according to Hwui-Lun, there was a special arrangement for Chinese (*Ibid.,* 76–77).

There was considerable flow of Indian arts and sciences into China in the seventh century through the Buddhist Channels, and some Indian scientists were active there. Wang Hsiao-Thung who wrote the *Chhi Ku Suan Ching* ("Continuation of Ancient Mathematics") or *Jigu suanjing* in the seventh century may have known some of these scholars (Needham, 37). Close mutual contacts between India and China are shown by the fact that in return to an envoy sent by Harṣavardhana to China in 641, two missions came to India from there (Mukherjee, 48). In 648 monk Hsüan Chuang wrote the *Ta Thang Hsi Yü Chi* or "Records of the Western Countries in the Time of Thang" (Needham, 716). Chinese translations were made of *Nyāyapraveśa-tarkaśāstra* in 647 and Candra's *Daśapadārthaśāstra* (in 648) which is extant only in that translation now (*HCIP*, III, 390 and 301). About that time the Chinese monk Hsuan-Chhao spent a few years in India to study Sanskrit and Buddhism and helped in procuring Indian scientists for China (Sen 1963, 24).

In 664 Tao Suen complied the *Ta-thang-mu-tien-lu* or Catalogue of Buddhist Books. It has biographical note on each author and the number of works covered is 2487 in 8476 fasciculi (Mukherjee, 72). In the same year Tsing Mai compiled another catalogue of all translated works from the time of Kāśyapa Mātaṅga to that of Hiuen Tsang (*Ibid.*). Next year Tao Suen (or Tao-süan) wrote *Shih-chia-shih-pu* in which he referred to the Yen-fu-t'u (literally, the map of Jambūdvīpa) (Muroga and Unno, 50). He had helped Hiuen Tsang in translating Indian works into Chinese (*Ibid.*) About 666, the Buddhist monk Shou-Chen wrote the *Thien Ti Jui Hsiang Chih* or "Record of Auspicious Phenomena in the Heavens and the Earth". Two years later a Buddhist encyclopaedia entitled *Fayuan Chu Lin* ("Forest Pearls in the Garden of the Law") was produced by Tao-Shih (Needham, 698). An *Encyclopedia of Abhidharma* was edited critically by *Kātyāyanī-putra*. There were six supplements (called pādas) which are said to have the same relation to the main work as *Vedāṅgas* have to the Veda (Mukherjee, 54–55). The pādas were Saṅgīti-paryāya (*Chi-i-men-tsu-lun* in Chinese), *Prakaraṇa-pāda, Vijñāna-kāya-pāda, Dhātu-kāya, Dharma-skandha* and *Prajñapti-pāda Śāstra*, all of which were translated into Chinese, mostly by Hiuen Tsang (*Ibid.,* 56–61) who died in 664 (at the age of 64) "honoured by all and mourned by all".

I-tsing came to India by sea route via Śrīvijaya (now called Palembang) which was then a great centre of Indian culture (Mukherjee, 66). He landed at Tāmralipti (Sri Lanka) in 673 and then went to Nālandā where he spent 10 years (Bapat, 242). After studying and touring in India for about 20 years, he returned to China in 695 taking about 400 Indians books with him. He translated 56 works including Vinaya texts and Diṅnāga's *Nyāya-praveśa* on Indian logic (Mukherjee, 66–67). He wrote the book *Ch'iu-fa-ko-sang-chuan* which contains biographies of Chinese Buddhist priests who visited India during the early T'ang (= Thang) period (*Ibid.,* 74). His *Ta-t'ang-si-yii-ch'iu-fa-kao-sêng-ch'uan* (which seems same as above) has details about India which called Jambūdvīpa by Chinese commonly (Muroga and Unno, 51). His contribution to diffuse the knowledge about India and Indian literature is significant. He died in 713 at the age of 78.

In 676 Śramaṇa Divākara of central India went to China. His Chinese translation of *Lalitavistara* has been already mentioned above (see Sect. 2). The Chinese title literally means *"Vaipulya-Mahāvyūha-sūtra"* (Mukherjee, 62). He also translated a few other works, and the Chinese Thang empress wrote preface to some of these whose rendering was accomplished with the help of Chinese assistants (*Ibid.,*). Under the order of the empress, Wu Tso-thien (684–705), an official catalogue was complied by Ming-Chuen about 695 with the help of others. The total number of works listed in it was 3616 (*Ibid.,* 72).

Bodhiruci II was another Indian Śramaṇa who worked in China. His original name Dharmaruci (or Fa-hsi, i.e. 'law-loving' literally) and Ta-mo-liu-cim in Chinese was changed to Bodhiruci ('intelligence-loving') and Ciaoai in Chinese, by the empress Wu (Bapat, 219). He had knowledge of such sciences as astronomy, medicine, geography, etc. (*Ibid.*). Between 693 and 713, he translated more than 50 works into Chinese but the most stupendous work was his Chinese edition of the *Mahāratnakūṭa-sūtra* (*Ta-pao-tsi-cin* in Chinese) with an introduction by Su-no, and a royal preface (Bapat, 220; Mukherjee, 68). Bodhiruci settled in China, and when he died in 713 it was 156th year of his age !

The T'ang or Thang period was very favourable and fertile for the interaction between India and China especially for transmission of Indian ideas to the Asian neighbour. Many aspects of Chinese culture were profoundly influenced by India.

According to Liang Chi'i-ch'ao, "the academies which flourished since the Tang dynasty cannot be other than Buddhist in origin" (Mair, 65–66). Indian cultural influence in arts (music, dance, drama, painting, sculpture, etc.) and architecture, language and literature, religion and philosophy is there (*Ibid.,* 78–79). In science and technology, Needham's *Science and Civilization in China* does deal with exchanges.

In the field of mathematical sciences of the heavens, the main subject of Chinese astronomy throughout its history was calendrical computation. According to the *Jiu Tang shu* (The Old History of Tang Dynasty), the royal observatory at that time employed high officials and other astronomical workers in large number in its four departments as follows:

1. Calendar making (63 persons)
2. Astronomical Observations (147 persons)
3. Time-keeping (90 persons)
4. Time-service or announcing time by bell and drum (200 persons) (Xi Zegong, p. 38). Astronomical records were kept.

The great influence which the Indian astronomy had exercised in China during the Tang period can be seen by the presence of Indian astronomers in the Chinese Capital Chang-Nan which had a school where Indian *Siddhāntas* (mathematical astronomy) were taught (Sen, 1963, 22). Actually, there were three clans of Indian astronomers who were active there during the period (Needham, 202; Yano, 130).

1. Chiayeh (Jiyaye) or Kāśyapa.
2. Chhüthan (Qutan) or Gudon, i.e. Gotama or Gautama.
3. Chümole (Jumoluo) or Chü-mo-jo, i.e. Kumāra.

They were employed in the National Astronomical Bureau of China and collaborated with the Chinese Imperial astronomers (Martzloff, 100).

Kāśyapa Hsiao-Wei (c. 650 AD) was occupied with the improvement of the Chinese calendar and helped Li Shun-Fêng in the preparation of the Lin-Tê calendar of 664–665 (Needham, 202). His contribution is mentioned in the calendrical volume (Li-chih) of *Jiu Tang Shu* mentioned above (Yabuuti 1979, 8). The *Tang shu* (Records or History of the Tang Dynasty) mentions four astronomers of the Gautama clan who were officials in the Astronomical Board (Mikami, 58). The first was Gautama Lo who was President of the Kuang-chai Calendar (*Ibid.,* 59; Needham, 202).[10] He was director of the Royal observatory in 698 and compiled the Kuang-chai li by the order of the emperor Kao-tasung. It is believed to be based on Indian astronomy (Yabuuti 1954, 586 and 1979, p. 8).

The greatest Indian calendar expert in China was Gautama Siddhārtha (Qutan Xida) or Gotama Siddha (Hsita) who became the President of the Astronomical Board and Director of the Royal Observatory (Yabuuti 1954, 586). As an 'astronomer royal', he was asked to translate Indian astronomical work into Chinese. Adapting various Indian works, he translated some astronomical material based on the *navagraha* ('nine planets')-system under the Chinese title *Chiu-chih li* or *Jiuzhi li* (Navagraha Calendar). In this '*navagraha karaṇa*', the 9 planets stand for the sun, moon, the 5 star-planets together with *Lo-hou* (Rāhu) and *Chi-tu* (Ketu) (Yabuuti 1979, p. 9).

The Chinese translation was done in AD 718 under order from Hsüan-tsung (Yabuuti 1954, p. 586).

A few years later Gautama Siddhārtha complied the *Khai-yüan chan Ching* (Khai-yuan Treatise on Astronomy and Astrology) in 120 volumes of which the 104th volume was the *Chiu-chih li*. The full work is the "greatest collection of ancient and medieval Chinese fragments on the subject" (Needham, 707) compiled in Khai Yuan period (713–741). Piety and some sort of secrecy seems to be attached to the work.[11]

The *Chiu-chih li* (*Navagraha-karaṇa*) includes a section on Indian methods of calculation based on nine numerals and a symbol for zero which in this work is thick dot (•). Such a zero symbol had already appeared in India, say in the *Bakhshālī Manuscript* (seventh century) and in the Cambodia inscriptions under Indian influence (Martzloff, 97 and 207; Bag, 250–251). In India, a small circle has been used, since ancient times, as a symbol for zero. A common Sanskrit name for it is *bindu* whose etymological meaning is 'a drop' (from root *bhid*) according to the *Nirukta* (c. 500 BC); see L. Sarup's edition of the text (p. 44) and translation (p. 21), Delhi, 1984. Interestingly, the written Chinese character *ling*, which is used for zero from the Ming, also means 'dewdrop' (Martzloff, 208).[12]

The text of the Chinese *Chiu-chih li* was translated by Kiyosi Yabuuti in 1963 and he revised this in 1979. In the original work, the methods are said to be originated by Brahma. They were received and handed down by Wut'ung Hsienjen, "the excellent scholar of full understanding of the five"; perhaps this refers to five *Siddhāntas* summarised by Varāhamihira (c. 550 AD) in his *Pañcasiddhāntikā* or to the five components of *pañcāṅga*, the Indian calendar and its science. In fact the *Chiu-chih li* is taken to be based on the *Pañcasiddhāntikā* with which it has several parallel passages. A dozen such passages have been listed in the Introduction (p. 16) of recent edition of the latter (by Neugebauer and Pingree, 1970/1971) and reproduced by Sen (1985, 99–100). However, the use of R = 3438 for *sinus totus* in connection with table of 24 Sines (h = 225 minutes) shows that other Indian sources were also utilized by Gautama Siddhārtha in preparing his Chinese work *Chiu-chih li*. The Sanskrit word *jīvā* (or *jyā*) for sine was literally translated as *ming* in Chinese (Yabuuti 1979, 36).

M. Yano points out (*Ibid.,* 10) that the main idea of the *Chiu-chih li* was the midnight reckoning system which was represented by Āryabhaṭa's lost work, Lāṭadeva's revision of *Sūrya-siddhānta* and Brahmagupta's *Khaṇḍakhādyaka* (AD 665). Close relation of *Chiu-chih li* is found with the last work in matters of solar and lunar equations for finding true longitudes of the sun and moon and daily motion of moon. Further parallelism with *Pañcasidhāntikā* exists in matters of latitude of moon, length of daylight, and magnitude and duration of lunar eclipse, etc.

Gautama Chuan is another Indian astronomer mentioned in the *Tang Shu*, who served the Chinese Astronomical Board (*Ibid.,* 8). He belonged to the Khai Yuan period (713–741) although Mikami (p. 59) placed him a century earlier in 618 (Needham, 203). Possibly, he composed a calendar (Sen 1970, 334). Other Indian astronomers of the same clan were Gautama Chhien who served the Astronomical Board during the first half of the eighth century, and possibly Gautama Yen who seems to have contributed to the calendrical methods (Needham, 201).

I-Hsing or Yi Xing (c. 682–727) was a great Chinese astronomer-mathematician of Tang period. An account of his life is found in Chêng Chhu-Hui's *Ming Huang Tsa Lu* (Miscellaneous Records of the Brightness of Imperial Court) of AD 855 (Needham, 38). Due to his outstanding contributions, the *Chhou Jen Chuan* (Biographies of Mathematicians and Astronomers) of Juan Yuan (1799) devotes three full chapters on him (*Ibid.,* 37). Yi Xing was a Tantrik Buddhist monk versed in Sanskrit. He was taught by an old monk and travelled widely to attend conglomerations of *śramaṇas* (Mikami, 60; Needham 38). Under a royal order, he investigated the chronological and computational ideas introduced into China from India by Gautama Siddhārtha (Sen 1963, 22).

Only a few of Yi Xing's works are extant. The Buddhist *Tripiṭaka* still contains his *Hsiu Yao I Kuei* (The Tracks of the Hsiu and Planets) and the *Pei Tou Chhi Hsing Nein sung I Kuei* (Mnemonic of Seven Stars of the Great Bear and Their Tracks) (Needham, 202). The *Chhi Yao Hsing Chhen Pieh* (The Different Influences of the Seven Luminaries and the Constellations) is ascribed to him and lists *hsiu (nakṣatras)* and their stars, etc. (*Ibid.,* 696). Another work ascribed to him is the *Fan Thien Huo-Lo Chiu Yao* (Horā of Brahmā and seven or nine Luminaries) but Needham (p. 698). thinks it to be of later times (874).

Due to his Buddhist training I-Hsing (= Yi Xing) could easily handle large numbers, e.g. 3^{361} or 10^{172}. He was expert in combinatory calculations and was capable of enumerating all possible transformations occurring on go board or chess board (*Ibid.*, 139). He was much influenced by Indian astronomy (*Ibid.,* 202). His *'thai yen'* method is quite comparable to that of Bhāskara I (c. 625) in solving astronomical problems by using indeterminate analysis (*Ibid.,* 119–120; and K. S. Shukla's edition of *Āryabhaṭīya* with the commentary of Bhāskara I).

Under royal order and patronage, I-Hsing was asked to compose a calendrical system in 721. He applied his contrived arithmetical method of *'thai yen'* (based on indeterminate analysis) and produced the system called *Ta Yen Li* (727). An Indian astronomer of Chümolo (Kumāra) clan contributed a method of computation of solar eclipse to the calendrical system. I-Hsing's *Ta Yen Li Shu* (which contains the above calendar) was edited by Chang Yueh and Chhen Hsuan-Ching shortly after his death. Final draft was promulgated officially in 729 (Cullen, 2). But four years later Gautama Chuan and Chhen declared that the *Ta Yen* calendar was a plagiarism of the *Chiu-chih li* with added mistakes (Needham, 203). However, the charge was found untrue. In fact, out of the 23 different systems of calendars of that time, the *Ta Yen Li* was most accurate, stood the test of time and was used for long time (Mikami, 60).

Much influenced by Indian astronomy, I-Hsing made measurements in ecliptic coordinates which played little role before (Needham, 202). He trained the officials and observers for the great meridian survey which was conducted in 724 under the direction of Nan-Kung Yueh, director of Astronomical Bureau (Beer et al., 14; Cullen, 1). The observed data was also analysed by I-Hsing. Details of the survey experiments are found Chinese sources (Cullen, p. 1 gives references).

The ten places of observations in the meridian survey extended from Lin-i or Indrapura (latitude 17.4 *tu*, in Champa) to Thich-lo (latitude 52 *tu*). The old belief was that the Sun's shadow lengths change one Chinese inch (= $\frac{1}{10}$ *Chhih*) over a distance of 1000 *li* in north–south direction on ground but the survey experiment's result was 4 inches per 1000 *li* (Beer et al., 14 and 25). I-Hsing established the true relation between terrestrial distance and the change in polar altitude. His main conclusion was (*Ibid.,* 15).

$$1^\circ = 351\ li\ 80\ pu, \ \text{where } 1\ li = 300\ pu$$

In arriving at the various result, his knowledge of Indian astronomical achievements may have helped him (*Ibid.,* 25). Possibly, he was taught (by an Indian astronomer) the Hindu method (such as given in the *Sūrya-siddhānta*) for relating gnomon shadows and solar altitude by means of sine table transmitted in *Chiu-Chih li* (Cullen, 32). I-Hsing was fully capable of adapting calculations for different localities. Even Nan-Kung Yueh knew that the mathematical techniques then used in China were derived from India (*Ibid.*).

About 725 I-Hsing developed a table of functions equivalent to a tangent table. According Cullen (p. 32) it was "clearly derived from Indian sine tables" although Duan (p. 111) says that he made it by the Chinese method of differences. The table gives the values of $g \tan\theta$ where $g = 8$ *Chhih* (Chinese foot) and $\theta = 1$–79 *tu* (Chinese degree where 1 $tu = \frac{360}{365.25^\circ}$). This table is the earliest of its kind in the world. Cullen (pp. 8–10) has published it in modern form with translation of relevant part of *Ta Yen Li*. I-Hsing used methods finite differences, fitting of polynomials and interpolations. Methods in these areas appear simultaneously (Martzloff, 339) in India, e.g. in Brahmagupta (628/665) and China (*HM,* Vol. II, 1984, p. 45) in Liu Ch'uo (c. 600) and Li Chiun-feng (665) (Gupta 1989, p. 49 and *IJHS*, Vol. 4, 1969, 86–98).

It is already mentioned (see Sect. 2) that the Chinese translation of the *Mātaṅgī-sūtra* must, according to Needham (p. 710), belong to the eighth century AD. It has a list of Indian *nakṣatras* (*hsiu,* in Chinese) with their stars and other details. Needham (p. 698) also mentions the *Fo Shuo Pei Tou Chhi Hsing Yen Ming Ching* (Sūtra Spoken by a Bodhisattva on Delaying of Destiny according to the Seven Stars of the Great Bear) belonging to the Tang period. It contains western zodiacal cycle but the translator is not known. In 730, Chi-Shang compiled the *Khai-Yuen Lu*, a catalogue of Chinese *Tripiṭakas* (Mukherjee, 72). It lists 2278 works ascribed to 176, Indian and Chinese translators and 741 books by unknown translators. According to a narration of 749, there were three Brāhmaṇa-Vihāras in Canton (Puri, 337).

The antiquity of the Buddhist *Tāntrika* system is shown by the first century work *Mañjuśrī-mūlakalpa* which contains not only the *mantras* and *dhāraṇīs* but numerous *maṇḍalas* or mystic diagrams, etc. (Bapat, 316). According to Tibetan chronicles, the first Tantrik who went to China from India was Sthavira Śrīmitra (Mukherjee, 68) who translated *Mahāyāna* and other *Dharaṇīs* into Chinese during the period 307–312 AD.

In 719, Vajrabodhi, along with his disciple Amoghavajra, arrived in China (under reign of Hsuan Tsung), instructed two Chinese monks in Tantric mysticism and translated 11 works (Mukherjee, 69). After the death of his guru in 732, Amoghavajra (Pu-Khung) visited India in 741. Five years later he went back to China with a large number of Indian texts and translated more than 70 works before his death in 774 (*Ibid.*). The Ming dynasty catalogue ascribed 108 works to him showing his great contribution to Chinese literature (*Ibid.,* 70). He was greatly honoured by Tang emperors and, according to Tibetan sources, he performed the 'Vajrabodha Maṇḍala' ceremony for king's benefit (*Ibid.,* 69).

In 759, Amoghavajra wrote in Chinese (or translated from Sanskrit) a work on Indian *Jyotiṣa* whose full title can be rendered as "Sūtra on Auspicious and Inauspicious Times and Days, and on the Good and Evil Nakṣatras (Lunar Mansions) and Planets Promulgated by Bodhisattva Mañjuśrī and other sages" (Needham, 720 gives full Chinese title also). Mañjuśrī is the legendary promulagator of the Tantric lore. A shorter title of the work is *Hsia Yao Ching (= HYC)* or "Nakṣatra and Planet Sūtra".

According to Yano (p. 126), Amoghavajra's original (or translation) work was recorded by his Chinese pupil Shih-yao, but a revised translation was made in 764 by another Chinese scholar Yang Ching-fêng under direct supervision of Amoghavajra. Both these recensions are found in the *Tripiṭaka* (whether Chinese, Korean, or Japanese) *HYC* as contained in Vol. 21 of the *Taisho Tripiṭaka* of Japan which is based on the Korean *Tripiṭaka* along with the variant readings from the Chinese *Tripiṭaka* of Ming Dynasty (*Ibid.,* 125). While preparing the revised version of *HYC*, Yang made additions which include:

(i) A chapter on the method of computing the seven planetary days (*i.e.* weekdays) from *Chiu-chih lī* of Gautama Siddhānta (AD 718). Yang also changed the epoch of AD 714–788 BC.
(ii) A multilingual list of names of planetary weekdays (see below). This was added by Shih-yao (*Ibid.,* 131).
(iii) Perhaps the 28th *nakṣatra*, Abhijit (Niu in Chinese) to the original list of 27 as given by Amoghavajra.

Table 2 (Cf. Yano, p. 131)

	Weekday	Planet (Old)	Lord (Sanskrit)	Sogdian (Kang)	Persian (Sambhih)
1	Sunday	Sun	Āditya	myr	ēw
2	Monday	Moon	Soma	m'x	dō
3	Tuesday	Mars	Aṅgāraka	wnx'n	sē
4	Wednesday	Mercury	Budha	ṭyr	cahār
5	Thursday	Jupiter	Bṛhaspati	wrmgṭ	panj̄
6	Friday	Venus	Śukra	n'xys	šaš
7	Saturday	Saturn	Śanaiścara	kyw'n	haft

Further additions were made by monk Kakusho who published the 'best' text of *HYC* in Japan (1736). Contents of *HYC* are briefly as follows (Yano, 129–132).

Chapter I is on classification of *nakṣatras* and zodiacal signs. Equating the first point of *Aśvinī* to that of *Meṣa* (Aries), the distribution of the 27 *nakṣatras* to the 12 *rāśis* (signs) is exactly same as in ancient texts on *jyotiṣa*. The sizes of the 7 planets (named after weekdays), the topic being comparable to that in *Abhidharmakośa* III, 60 and its commentary, Vol. I, p. 518.

The configuration of each *nakṣatra* and the number of stars comprising it are described in Chapter II with some other details all of which are standard features in Indian *Jyotiṣa* subject. In his notes on Chapter III, Yang says that Indian method was to be used for computing the positions of the 7 luminaries (*tārāgrahas*). He mentions the Gautama, Kāśyapa and Kumāra clans of Indian astronomers who were active in China during Tang period. He says that he himself used the work of Gautama (Siddhārtha).

Chapter IV is "On the Rule of the Seven Luminaries." These are said to govern each day of the week in turn. It is here that Shih-yao, pupil of the Indian Amoghavajra, added the Sogdian and Persian names of the weekdays along with the Sanskrit names (see the Table 2).[13]

Chapter V deals with miscellaneous matters. The next chapter is on the Indian division of a lunar month into *śukla-pakṣa* ('po-pochha') and *kṛṣṇa-pakṣa* (hei-pochha). The last chapter is not by Amoghavajra. It was added by Yang from, as already mentioned above, *Chiu-chih li* whose epoch was also altered.

A work which was probably translated from the Sogdian (Needham, 715) by the priest Ching-Ching (Adam, the Nestorian) is the *Ssu Men Ching* (Manual of the Four Gates) of about 780. It deals with the distribution of the *hsiu (nakṣatras)*. It may have Indian influence because Adam worked closely with the Indian scholar Prajñā (c. 781) in preparing a translation of the *Ṣat-Pāramitā-sūtra*. However, the Chinese emperor disapproved such collaboration saying that Adam should devote to doctrines of Meshia leaving Buddhists to propgate the teachings of Buddha (Mukherjee, 70).

The *Futian* Calendar was compiled in China in the Jianzhong reign period (780–783) and was taken to Japan in 957 by Buddhist monk (Nakayama, 135). It was imported from Indian under Buddhist influence. According to Wang Yinglin, a Song Dynasty Chinese scholar, it was "originally an Indian method of astronomical calculation" taking its epoch like *Chiu-chih li* (*Ibid.*). It relates the equation of centre y to mean solar anomaly x by

$$y = kP, \qquad k = \frac{1}{3300}$$

where P represents the parabolic function

$$P = x(182 - x)$$

In India such techniques were already known to the early seventh century, e.g. Bhāskara I's formula approximates

$$\sin A^\circ = \frac{4Q}{(40500 - Q)}$$

where $Q = A(180 - A)$ is like P in form (Gupta 1967 and 1986).

About AD 800, Chhü-Kung translated, from Sogdian, the *Tu-Li-Yü-Sss-Ching*, and astronomical manual, and a few years later Chin Chü-chha wrote the *Chhi Yao Jang Tsai Chüeh* which gives planetary ephemerides from AD 794, etc. (Needham, 696 and 719). The Chinese version of the *Horā of Brahmā* and the *Nine Planets* dated 874 by Needham (p. 698) has been already mentioned under I-Hsing to whom it is usually ascribed.

The Tang Shu refers to various works called *Rāśi-sūtras* which are taken to be mathematical in content because, according to Chin Keh-mu (p. 786), *rāśi* was an ancient Indian word for mathematics. The Jaina canon *Śthānāṅga-sūtra* (c. 300 BC) includes *rāśi* among topics of mathematics and *rāśi* is taken here in the context of Rules of Three *trai-rāśika*, Rule of five, etc., (*HHM*, I, 8 and 104) and even in the sense of conglomeration, i.e. set theory! It may also be recalled that the ancient Indian sage Nārada included *rāśi-vidyā* (arithmetic) and *nakṣatra-vidyā* (astronomy) among the scientific subjects he studied (*Ibid.,* I, 4). Usually in Indian astronomy *rāśi* is zodiacal sign and *nakṣatra* is lunar mansion or lunar division of circle ($=\frac{40}{3}$ degree, just as *rāśi* is also a measure of 30°).

Whatever be that and other things, it has been made clear in this section that good amount of Indian scientific and mathematical astronomy was definitely transmitted to China before the end of the Tang period. Contemporary world tourists such as Sulaiman al-Tajir̥ (ninth century) had recorded the impression that Indian astronomy was more advanced than Chinese (Needham, 203).

A Buddhist monk named Shou-wen even devised an ‘alphabet’ of 36 letters at the end of the Tang period, and although Sung (or Song, next dynasty) “phonologists adopted its principles in their analyses, it is unfortunate that full-scale alphabetization of Chinese languages failed to materialize” (Mair, 73). Some Chinese books of Sui period had treated the mode of writing by alphabetic symbols of India. It is called *Si-yo-hu-shu* or “Foreign Writing of the Western Conutries” and also *Po-lo-men-shu* or “Brahmanical Writing” (Mukherjee, p. 43).

6 After the Tang (Thang) Period

Due to changed political conditions, resulting from the entry of the Arabs in West and Central Asia, the harmonious relations between Indian and China got a set back, and mutual contacts starting fading away about AD 900. In 907, the Tang Dynasty, itself collapsed and the brisk age-old Indo-Chinese cultural and scientific programmes got a shock. During the trouble time of half a century, five short dynasties (including those of Turkish race) rose and fell. An Indian monk Samant along with his companies arrived in China in 951 (Mukherjee, 83).

Political stability was restored only when unification under the Sung Dynasty (960–1279) was achieved and then some activities in Indo-Chinese cultural relationship started. Taistu, who united the Chinese people, encouraged activities regarding Buddhism, although he himself was not Buddhist (*Ibid.,* 82). In 965, the Chinese priest Tan-Yuen returned to China after collecting Sanskrit works from India (*Ibid.,* 83). In 971, Taitsu ordered the Buddhist canon to be written in gold and silver paints and next year the first printed edition (using wooden blocks) of the *Tripiṭaka* was published. Round about this time 44 Indian monks went to China (Puri, 338). Also between 964 and 976, about 300 Chinese śramaṇas travelled to India. Of these 157 came together in a batch and collected lot of Indian books (Mukherjee, 83–84).

Indian scholars also continued to visit China. In 973, Dharmadeva, who studied at Nālandā, went there and was received royally. He did translation work, and in 982 he was honoured by the emperor, changing his name to Fa-Hsein. (*Ibid.,* 84; and Puri, 338). The Ming catalogue ascribes 118 works to Dharmadeva of which 78 were translations of new books, one of which was the *Mañjuśrī-Bodhisattvaśrīgāthā* (Mukherjee, 87). According to Needham (710–711), the book *Manual of the chih Cycle spoken by Nan-Chi-shih-Lo-Thien* was translated by him about 985 AD. He died in China in 1001 (Puri, 338).

According to Chin-Keh-mu (p. 785), the *Sung Dynasty History* mentions a "Calendar of Seven Heavenly Bodies" based on the Western (i.e. Indian) astronomy. It also speaks of *Rāśi Rhyming Tables (Ibid.* 786) which seems to contain material related to scientific correlation. The *Annals of Sung Period* mentions a number of Indian monks who went to China and Chinese monks who visited India (Mukherjee, 86). This mutual link is also proved by several eleventh century Bodhgaya inscriptions, one of which mentions the name Chi-i associated with distribution of 3000 books in Charity (Mukherjee, 86). Puri (p. 338) also says that between 970 and 1026 AD, there were large scale mutual visits of Indian and Chinese monks and that about 200 works were translated into Chinese by a Board of Translation which included three Indians.

Among the values of π used by Chen Huo (died 1075) and other subsequent Chinese mathematicians is $\sqrt{10}$ which is a well-known ancient Indian approximation of Jaina School (Gupta, 1992, 1–5). The *Meng Chhi Pi Than* or *Menggi bitan* (c. 1086) of Shen Kua (=Gua) has the interesting title "Dream pool essays" and contains notes on various scientific topics (Needham 38–39).[14] His "notation is perhaps of Indian origin" (Martzloff, 98). As late as in the twelfth century, it is stated by Shen Tso-Che that even children learn mathematics in China from printed Buddhist text books (Psu-Sa-Suan Fa) (Needham, 88). We have already noted above (see Sect. 4) the half a dozen Indian books on astronomy and mathematics which are mentioned in Cheng Chiao's Historical Collection of about 1150 AD (*Ibid.,* 206–207).

The thirteenth century AD is called the golden era of Chinese mathematics. In 1221, the work of I-Hsing's survey was completed by the Taoist Chhiu-Chhung and his party (Beer et al., p. 14). According to Martzloff (p. 105), "magic squares of order greater than three are first found in China in the *Yang Hui Suanfa* (1275 AD)," and that they were introduced from outside possibly. Camman (p. 188) suspected that Yang Hui's magic squares of orders 8 and 10 were probably borrowed from India.

Towards the end of the Sung period, the monk Chih-Phan wrote the *Fo Tsu Thung Chi* or "Records of the Lineage of Buddha and the Patriarchs" (1270) (Needham, 698). It contains cosmographical ideas and descriptions as found in the Indian work *Abhidharma-kośa* and its auto-commentary (Unno 1980a, 58–59). It also contains the oldest map of India based on the *Si-yü-ki* of Hiuen Tsang (seventh century) (Muroga and Unno: 55) Chih-Phan states that India was claimed to be the centre of the world in the Holy Buddhist Writings (*Ibid*).

Lists of mathematical terms borrowed from Sanskrit and denoting large numbers and decimal fractions are found in China in Zhu-Shijie's *Suanxue qimeng* (1299) (Martzloff, 98), and 10^{96} is called *heng ho sha* ("*Sands of the river Ganges*"), and 10^{128} is called *wu liang shu*, i.e. *asaṃkheya* (Lam, 8).[15]

During the Ming Dynasty (1368–1644), some Islamic works on astronomy and astrology were translated into Chinese. One of them has the title *Ming-i Tien-wen-shu* (Astronomical Book Translated During the Ming Dynasty), the original of which was written by K'uo-shih-ya-erh who has been identified with Kūshyār ibn Labbān (Yabuuti 1987, pp. 551–552). It should be noted that Kūshyār is the tenth century Islamic author of the famous on *Principles of Hindu Reckoning*.

Another work of Islamic astronomy translated into Chinese is the *Ch'i-cheng-t'ui-pu* (1385) in which a lunar node is called *chih-tu*, and this is clearly derived from Sanskrit word *ketu*, (*Ibid.,* 557).

As a significant event of the Ming period, the manuscript of the *Khai-yuan Chan Ching* (which was compiled by the Indian scholar Gautama Siddhārtha in the eighth century China) was discovered by Chhêng Ming-shan (c. 1600) eventually in a Buddha statue (Yabuuti 1979, p. 9). It was then duly published in its 1st 104th Chapter on *Chiu-chih-li* (Navagraha Karaṇa on Indian astronomy) was studied by Ku-Kuan-Kuang (1799–1861), and by Haü Yu-jên (1800–1860).

In Jên-ch'ao's *Fa-chieh-an-li-t'u* (1602), the Jambūdvīpa is represented as the India centric continent (as was done in Holy Buddhist Writings) (Muroga and Unno, p. 55). That very time Hsieh Chao-chih wrote his *Wu tsa-tsu* (Miscellanea in Five Parts) (Mair, p. 79). The *Suan-fa tangzong* (1592) (General Source of Computational Methods) of Cheng Dawei (1533–1606) was a popular book (containing versified rules) which was published many times. It gives system of very large number whose terminology was borrowed from Sanskrit similar to what is found in *Suanxue qimeng* of 1299 (Martzloff, 98). The translation technique involved when Matteo Ricci (1552–1610) translated Euclid's *Elements* into Chinese is said to follow the same method as was used earlier in translating the Buddhist texts (Martzloff, 21).

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References and Notes

1. Full details of the foreign kings mentioned in the Edict can be found in Sircar (Appendix No. 1, pp. 78–84). For instance, Alikasundara (Alexander) was the Greek king of Epirus (272–255 BC) or of Corinth (252–244 BC), and a contemporary of Aśoka.
2. The (Indian) name of Chu-fa-lan is lost but translated as Dharmarakṣa by modern scholars and Gobharaṇa or Bharaṇa by Tibetan historians (Mukherjee 1). Dharmaratna seems better translation, as Dharmarakṣa is given a rendering of Chu-fa-hu (*Ibid.,* p. 10; also cf. F. W. Thomas, 66).
3. Capitals of each kingdom can be found in Chou (last page) and number of mathematicians (of note) in each dynasty is given in Vanhee (1926 Isis article, pp. 104–105. Alternative names of dynasties) in mostly current popular Pingyin system are given in the parentheses. Due to consultation of a large number of sources (both old and new) uniformity was not possible.

4. Indian influence of terminology is also suspected in systems of notation for large numbers in many early Chinese mathematical works of third to sixth century AD (Martzloff, 97–98).
5. See *Manusmṛti* ed. with Hindi translation by Haragovinda (Miśra) Śāstrī, Benares, 1952, p. 18.
6. *Abhidharma-kośa* auto-commentary (p. 536) adds that 3 (*aṅgulī) parvas* make one *aṅguli* (or *aṅgurī*) suggesting that *aṅguli* was possibly length of a finger (if not simply a finger without measure-use). Also see the *Śārdūla-karṇavadāna*, p. 58 f.n. for same.
7. According to Chin-Keh-mu (p. 787), the *pañcavidyā* (or five sciences) comprised of grammar, medicine, technology, philosophy, and logic.
8. See *Jainendra Siddhānta Kośa* by Jinendra Varni, Vol. III, p. 401; and L. C. Jain, *Exact Sciences from Jaina Sources*, Vol. II, 1983, pp. 16–19.
9. 'Western World' of those days for Chinese scholars included all Central Asia and India.
10. The date 684 AD as the time of adoption of the Kuang-chai Calendar, as given by Li-ung-bing in his *Outlines of Chinese History* (Shanghai, 1914) (Mukherjee, 71) seems to be wrong.
11. According to M. Yano (see Selin, p. 1059), The Indian zodiacal signs were transmitted to China in the eighth century by Buddhist astrology.
12. Hindi words *būṃda* and *bindī* are from Sanskrit *Vindu*. Also see Wang Yusheng's paper (2003) now for many details in this regard.
13. It will be noted that the Persian names for week-days in the table are simply words for natural numbers.
14. Sher Kua (1030–1094) in his *Collated Dream Brook Essays* (1086/1091) says: "Now on the lintel of the Tower of Leisure in P'u, this is some horizontal writing [in contrast to Chinese Vertical] in a Devanāgarī-like script by person of tang" (Mair, p. 74.)
15. The Chinese terms (in the same work). For 10^{120} is *na-yu-t'a* which clearly seems to be same as Sanskrit word *nayuta*. A few other terms are also said to have Buddhist origin and influence (Lam, 8–9).

Indian Influence on Early Arabic and Persian Writers of Mathematical Sciences

India has given to the world outstanding gifts in sacred spiritual as well as in secular scientific fields. In spiritualism, India gave the great Buddhism through which all the Asian countries during the first millennium of our era "formed one fountain-head". Then there is the gift of the supreme philosophy of Vedanta which through the effective speeches of Swami Vivekananda, "invaded" America and Europe. The Advaita Vedanta philosophy also deeply influenced the Sufi thought which is widely and universally appreciated in the Islamic world of Middle East.

In the scientific field, the grand gifts from the simple but unique decimal place-value system of numeration to the advance marvellous achievements of Srinivasa Ramanujan in modern mathematics are the distinctly shining examples. In the present article, we will highlight the diffusion and enrichment of Indian mathematics through the Arabic and Persian works during ancient times. Several works based on or translated from Sanskrit sources will be mentioned with some details. The spread, penetration and triumph of the Indian system of numerals will also find a sort of concise documentation in the following pages.

Ali al-Masudi (died 956 AD), the encyclopaedist scholar of Baghdad who was in India from 912 AD to 916 AD, wrote that a congress of sages at the command of the Creator Brahma invented the "nine figures" (that is, the decimal place-value system along with zero), astronomy and other sciences. Such a statement is not surprising because in ancient India, all arts and sciences were assigned a divine origin. But, it may very well contain factual information that the decimal place-value system emerged out of a deliberate discussion in a conference of learned scholars. In this connection, it should be noted that the Sanskrit alphabet which is so scientifically designed may also have been the result of similar group discussion in a systematic manner (cf. English, Greek, or Arabic alphabet). Any way, the positional decimal system of numerals was already in use in India in the beginning of the present era.

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According to Parsi tradition the Sasanian king Shapur I (241–272 AD), “caused to be included, among the holy books, secular works on medicine, astronomy and metaphysics found in India, Greece and other countries”. The ninth-century Pahlavi (Middle Persian) Denkart informs us authoritatively that Ardashir (226–240 AD) and he had Indian works translated into Pahlavi and that they were revised under the rule of Khusru I Anushirvan (531–579 AD). Ibn al-Nadim (d.987 AD) quotes the Persian scholar Abu Sahl ibn Naubakht (flourished in 800 AD) as saying that an Indian text on Jyotisha (ascribed to Farmasb) was translated as well. Barzouhyeh, a subject of Khusru I, visited India to acquire proficiency in Indian sciences.

“Strangely enough, although Ptolemy’s Almagest (famous Greek work on mathematical astronomy) was known to the Sasanians, Indian doctrines seem to have been preferred” (al-Abhath, vol. 23, p. 341). Thus, the Iranian official work Zij-I Shah contained lot of Indian astronomy. The work was revised under Khuusru I during whose rule the game of chess was also imported from India. The Arabic–Persian word *shatoranj* comes from the Sanskrit *caturaṅga*. The chess-related popular problem about the series

$$1 + 2 + 4 + 8 + - - - - + 2^{63}$$

is also attributed to Indians by Masudi in his Meadows of Gold and Mines of Jewels.

Before the rise of Islam, no scientific literature existed in Arabia beyond a few magical, meteorological and medicinal formulas. The famous flight of Prophet Muhammad took place in 622 AD. With astounding rapidity, the sons of the Arabian desert conquered most of the then civilized world in a century following the death of their Prophet in 632 AD. On the Indian front Qandahar was reduced and the statue of Lord Buddha found there was demolished. Still the Arabs had very little science and philosophy of their own and they carried no scientific study.

In 762 AD, Caliph al-Mansur laid the foundation of his new capital at Baghdad. Soon it grew into an international centre of trade, commerce and intellectual activities. Many oriental ideas and thoughts flowed in and a new era of cultivation of science and scholarly pursuits began. In the early stage the channel of inflow of scientific knowledge was Persia.

Direct infusion of Indian material on mathematical sciences into Islam took place when an embassy from Sind visited the court of Caliph al-Mansur in Baghdad in Hijiri year 156 (i.e. 772/773 AD). This visit of Indian embassy is mentioned by Muhammad ibn al-Adami in his Great *Zij* which is also called Threading the Pearl Necklace (Nazm al-Iqd) and which was completed about 920 AD (posthumously) by his pupil Qasim. The story is quoted by many other scholars.

Ibn al-Adami says that the embassy included an Indian Scholar who was an expert in astronomy and had a work on the subject (*Siddhānta*) with him. The Caliph ordered that the work be studied and an Arabic treatise be made out of it. The work was translated into Arabic by al-Fazari (in collaboration with others)—The Arabic version of the Sanskrit *Siddhānta* astronomy was called Zij al-Sindhind or Zij al-Sindhind al-Kabir, i.e. The Grand Sindhind from which descended a long tradition of other Arabic astronomical works.

Although not explicitly mentioned, the main original Sanskrit work involved in Arabic translation seems to be Brahmagupta's Brāhmasphuta-siddhānta (628 AD). This specification is supported by several ancient (e.g. al-Biruni) as well as modern scholars. Following the tradition of the Sindhind, Yaqub ibn Tariq (d. about 796 AD) who also worked under Caliph al-Mansur wrote his Zij Extracted from Sindhind Degree by Degree. Yaqub also wrote another work called Sine Division of Kardajas (the word *kardaja* is said to be from the Sanskrit *kramajyā*). Mention may also be made of an Astronomical Table (Zij) "composed according to Indian sources" by Siman ibn Sayyar of Kabul about the same time.

We might say that the visit of the Indian embassy (mentioned above) to Baghdad court was the formal occasion when Indian decimal place-value system of numerals was transmitted to the Arabs through official royal channel. Otherwise it is known that Indian numerals have been already spread westward much earlier to Persia, etc. and even to Alexandria which was a great international centre where not only commercial products but scientific and philosophical ideas were exchanged.

The first definitely known evidence of the knowledge of Indian decimal place-value system among the Arabs is provided by the praise it got from Severus Sebokht (d. 667 AD), the Syrian scholar and Christian bishop in the convent of Kenneshre on the bank of Euphrates. In his Syrian work Reasoning on Priority of Syrians over Greeks in Mathematics and Astronomy (662 AD), he stated that the Indian place-value arithmetic "surpasses description". He refers to the Indian computation as carried out by "nine figures" which along with zero formed the system (the Sanskrit word *aṅka*, as a bhūta-saṃkhyā, denotes 9.) the well-known Arabic alchemist Jabir (known in Europe as "Geber") also used the Indian decimal system with zero in his Book on Poisons (eighth century) before the famous al-Khwārizmī did so.

Of course, after the translation of some Indian astronomical works into Arabic in the form of Sindhind, there developed a category of Arabic writings called "fi al-hisab al-Hindi" (denumero lndorum in Latin). The Indian system of arithmetic (based on decimal place-value notation of numerals with zero) was called "al-hisab al-Hindi" and often also as "hisab al-takht" (cf. pāṫigaṇita) or as "hisb al-ghubar", etc. because it spread in Arabic world with dust board as tool.

The Indian numerals themselves were called as huruf al-Hind or huruf al-ghubar (i.e. Hindu or dust letters) because Arabs then demoted numbers by letters (huruf is plural of Arabic word harf "letter" or literae in Latin).

The known earliest text of the category (or genre) mentioned above is the Kitab al-Hisab al-Hindi (Book on Hindu Reckoning) by the famous al-Khwārizmī (c.780–c.850) whose works pioneered the spread of Indian mathematical sciences among Arabs and in Latin world. Unfortunately, the original Arabic work is not extant. However, Latin translation made by Adelard of Bath (c. 1120) in Spain and some other Latin versions are available. The Latin title is "Algoritmi de Numero Indorum" as published by B. Boncompagni in 1857. Cambridge University Library Latin text "Dixit Algorismi" of the work has been translated into English by P. J. Witz as part of her M.A. 348 Project (1980).

The Indian astronomical tradition of the Sindhind was especially used by al-Khwārizmī in his Zij al-Sindhind composed during the reign of Caliph al-Mamun (813–833 AD). The Arabic original of this work is also not extant. But its tenth century Spain revision by al-Majriti and lbn al-Saffar became quite popular and was translated into Latin by Adelard in 1126 AD.

Al-Khwārizmī's abbreviated Book on the Reckoning of Algebra and Almucabala was the first book that included the term 'algebra'. Chapter 15 of this book is on mensuration of plane and solid figures and includes the Indian value of π, namely $\frac{62832}{20000}$ (of Āryabhaṭa). Interestingly the modern mathematical term 'algorithm' comes from his medieval European name Algorismus or Algorithmus.

The name of the famous Indian scientist Āryabhaṭa (born 476 AD) appears as Arjabhar in Arabic sources. His Āryabhaṭīya is said to be translated into Arabic by Abu al-Hasan (or Husayn) al-Ahwazi who belonged to the ninth century or so. To the same century belonged al-Quyrawani of Africa who wrote a Book of Indian Calculation (see *HPM* Newsletter No. 37). About the same time al-Battani dealt with the novelties of trigonometry (based on the sine function introduced from India) in his Arabic work which was translated into Latin as De Scientia Stellarum.

Yaqub al-Kindi (d. about 873 AD) was a prolific writer and was known as "Philosopher of Arabs" and "Alkindus" in Medieval Europe. His Treatise on the Use of Hindu Arithmetic deals with Medieval Europe. His Treatise on the Use of Hindu Arithmetic deals with integers which are called adad al-hindi. Another Arabic writer was al-Dinawari (d. 895 AD) whose Book of Board on Hindu Reckoning had good reputation. These two books are mentioned in Fihrist of al-Nadim but are not extant now.

The earliest extant Arabic text on Indian arithmetic is the Book of Sections on Hindu Arithmetic written by Ahmed al-Uqlidisi in Damuscus in Hijri year 341 (952/953 AD). A. S. Saidan has brought out a fine English translation of the work (Dordrecht 1978) which includes good material on the history of arithmetic among the Arabs.

Several other Arabic writers of the tenth century took keen interest in Indian arithmetic Al-Karabisi wrote his Kitab al-hisab al-hindi while the work on the subject by Sinan ibn al-Fath as well as by al-Kalwadhani had the same title namely Book of Board on Hindu Arithmetic. On the other hand, the work of Ali al-Antaki (d. 987 AD) was titled Great Book of Board for Hindu Reckoning. This work also mentioned by al-Nasawi, a pupil of Kushyar (see below). The famous Book of Bibliography of Sciences (Fihrist in short) by Muhammad ibin al-Nadim contains good information on Arabic authors of Indian arithmetic. Although Abu al-Wafa (d. 998 AD) in his famous "A Book About What is Necessary for Scribes, Dealers and Others from the Science of Arithmetic" avoided Hindu numerals, but he did include Indian schemes of multiplication and division in the work.

The popular astronomer and mathematician Kushyar ibn Labban (971–1029 AD) wrote two works related to Indian mathematics, namely "Principles of Hindu Arithmetic" and the "Sources of the Principles in Hindu Arithmetic". The Arabic text of the first work along with English translation has been brought out by M. Levey and M. Petruck (Madison 1965). A Persian appendix on Indian fractions was added in

1283, and a Hebrew translation and commentary were written by Shalom ben Jeseph Anabi who lived in Constatinople in the fifteenth century.

Quite a few significant works on new arithmetic were written by al-Karaj (d. about 1025 AD) whose name is also written as al-Karkhi. One title is Book on Hindu Arithmetic. In his important work Sufficient on the Science of Arithmetic, he avoided Hindu numerals like Abu al-Wafa, "but even in their attempt to turn their back on Hindu devices, they prove to have borrowed from them" (Saidam, p. 8).

The penetration and popularity of Indian arithmetic are interestingly illustrated by a statement of the then contemporary writer. Referring to the Ismaili missionaries, Ibn Sina (or 'Avicenna' of Medieval Age) (980–1037.AD) says:

> Presently they began to invite me to join the movement, rolling in their thoungue talk about philosophy, geometry, and Indian arithmetic; and my father sent me to certain vegetable seller who used Indian arithmetic so that I might learn from him. (Neugebauer, p. 24; Rahman, p. viii).

Some astronomical works of the time related to Indian context include the Zij according to the Indian Method by Asbagh al-Gharnati (d. 1035 AD), the concise Zij according to the Model of Sindhind by Ahmad al-Ghafigi (d. 1035 AD), and the Book on the Cause of Mediation of Equation of Sindhind by Abu Nasr (d. 1036 AD). In book of completion on the science of Arithmetic Abd al-Qahir al-Baghdadi (d. 1038 AD), the first two chapters are devoted to 'Hindu Arithmetic' of integers and fractions, respectively.

Muhammad al-Shanni (tenth–eleventh century) in his book on the Measurement of a Triangle and of the Quadrangle Inscribed in a Circle gives a proof of the famous Brahmagupta's expression (s is semi-perimeter):

$$\text{Area} = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

During the same period, Muhammad ibn al-Haytham wrote his Book on Hindu Reckoning (Maqate fi al-Hisab al-Hindi) in Arabic, which he mentions in his autobiography. The more famous al-Hasan (Alhazen of medieval Europe) ibn al-Haytham (965–1041 AD) was a prolific writer. He wrote more than 50 books on mathematics alone one of which is Book of Defects of Indian Arithmetic.

Abu al-Rayhan al-Biruni (973–1048 AD) is doubtlessly the most famous scientist of Medieval Islam. He was a very prolific writer and encyclopaedist. He was in India for a few years and wrote the famous book containing explanations of doctrines of Indians, both acceptable by reason or rejectable (in Arabic). Titles of his works on Indian sciences are:

(i) Memorandum on Arithmetic and Reckoning by Means of Hindu Figures.
(ii) Modes of Indian Records in Learning Arithmetic.
(iii) Book on Indian *Rashikas*. It deals with the Rule of Three (*trairāśika*) and other higher rules.
(iv) Numerical Sankalitas. It deals with summation of series.

(v) Translation of astronomical work based on (now lost) "Karaṇatilaka" of Vijayanandin. The Arabic title is "Ghurrat al-Zijat" (see English translation by Rizvi).
(vi) Collection of Ideas of Indians on Astronomical Calculations.
(vii) Representation of Both Kinds of Eclipses by Indians.
(viii) Corrected and Revised Arabic translation of Brahmagupta's *Khaṇḍakhādyaka* (see Sachau, Alberuni's India, Vol. II, p. 339).
(ix) Answers to Questions Asked by Indian Astronomers.
(x) Answers to Ten Questions Asked by the people of Kashmir (Rosenfeld and Ihsanoglu, p. 153).

Sachau (India, p. 303) states that al-Biruni was translating *Brāhmasphuta-siddhānta* into Arabic about 1030 AD. It was doubtful whether he could complete this tough task.

Ali al-Nasawi (d. about 1070 AD) was a pupil of Kushyar ibn Labban and wrote the expository work Sufficient on Hindu Reckoning. It was first written in Persian for Majd al-Dawla and later in Arabic for Mahmud Ghaznawi (c. 1030). The famous Toledo Tables by al-Zarqal (d. 1099 AD) had followed the Sindhind tradition in Spain through the Zij of al-Khwārizmī mentioned above. The Mushkilat al-Hisale (Problems of Arithmetic) by Umar Khayyam (d. 1131 AD) is described by the author himself as "a treatise on the proof of Indian methods of extraction of square and cube roots, etc."

Some other works of the early twelfth century include the Book on Indian Multiplication by Ishaq al-Sardafi (d. about 1105 AD) and the "Hisab al-Hindi" by Asad al-Bayhaqi. The Book on Hindu Reckoning for a Qiwan al-Din was composed by al-Samawal al-Andalusi (d. about 1175 AD). Sharaf al-Din al-Amuni of Mecca wrote his "On Geometry and Indian Figures" in Arabic in 1172 AD.

Ibn al-Yasmin (d. 1204 AD) composed his work on Correction of Opinions on the Science of Arithmetic by Means of Figures (al-ghubar) while Muhanumad al-Hassar's work on Indian arithmetic is entitled Book of al-Hassar on the Science of Ghubar in which old and new techniques have been unified. Al-Hassar belonged to the last part of the twelfth century. Of an unknown date is the work Comments on an Indian Book on arithmetic by Isa ibn Ahmad ibn Yusuf in Arabic.

After the twelfth century, the use of Indian place-value system of numerals became more common and most of the arithmetical books used the decimal system with positional numerals although other older systems also continued.

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A Bibliography of R. C. Gupta

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Being much impressed with the contributions of Professor R. C. Gupta, on the occasion of his 60th birthday, in 1996, Professor Takao Hayashi brought out a bibliography of his writings that was published in *Historia Scientiarum*. He also prepared an updated version of it in 2011, perhaps to commemorate his 75th birthday, which was published in *Gaṇita Bhāratī*. The third edition of the bibliography that is being presented to the readers, is an up-to-date version and covers his writings over a span of 60 years (1958–2017). This has heavily borrowed on the earlier version prepared by Hayashi and hence we are deeply indebted to him for his pioneering efforts in this direction.

The rich variety of entries we find in the bibliography clearly mirrors the passionate dedication of Professor Gupta to create pathways to approach the history of mathematics and astronomy of India at multiple levels. The author strives to bring to light the achievements of Indian mathematicians both to the academic community and the generally informed lay person. He dedicates himself to painstakingly collect theses from various universities of India and collate them, and puts it in the form of short articles for his readers. He has also created bibliographies facilitating the search for source material.

In his capacity as the editor of *Gaṇita Bhāratī* he has been meticulously gathering information regarding conferences, meetings, symposiums and seminars on topics related to history of mathematics, held at different places in India (and abroad), in order to help the fellow academicians to keep abreast of research done in this field. Being well aware that this is a much neglected field, he strives hard to disseminate this material to an international readership, by way of sending short reports to various reputed journals of distinguished societies of history of mathematics and astronomy in various parts of the world, and thereby to stimulate curiosity and interest in fellow mathematicians towards promoting studies in history of mathematics.

While assiduously addressing the academicians he does not lose track of the common reader in towns of India, such as Jhansi and Ranchi where he lived and worked. Through his writings in the local newspapers about great Indian mathematicians on their birth and death anniversaries, on science days, he makes the common readers and students aware of India's contribution to the world heritage of science. His research is not confined to ancient mathematicians alone. The sev-

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eral bio-bibliographies of historians of mathematics and astronomy of recent times, such as P. C. Sengupta, Sudhakara Dvivedi, etc., that Professor Gupta has brought out reveal his admiration for his immediate predecessors as well as contemporary historians.

A mention may be made here of this version of bibliography providing glimpses of Professor Gupta's work translated into Hindi, Gujarati, Kannada, Bangla, Tamil and Malayalam, in a separate section. While most of the articles in Hindi are authored by Gupta himself, all the Kannada ones have been prepared by Venugopal D. Heroor. It is to the credit of Gupta that he draws the uninitiated readers into the folds of scientific scholarship through encouraging translations of his own and other writings in the regional languages of India. Towards the end of this bibliography we find a list of all of his works that have been translated into other Indian languages. This is followed by a list of articles that have been written about R. C. Gupta.

Besides articles meant for technical audience, the list presented in the bibliography also includes his writings in the form of very short notes on ancient Indian mathematicians for encyclopedias of science. It may also be noted that such notes as well as a few of his articles have reappeared in different publications. Given the fact that India's contribution to science does not find its way to the informed academic community in India as well as in other parts of the world due to a lack of awareness, and also with the mission to disseminate knowledge, he has not shied away from rewriting or translating into Hindi some of the articles that have already been published elsewhere. In order to indicate readers, the equivalence of one article with the other (in the case that they reappear elsewhere) is indicated at the end of those entries. In doing so, we have essentially followed Hayashi's style that was found novel and useful. Professor Gupta has also been writing under his pen name 'Ganitanand' for some time. Those publications which appeared with this name have been indicated by including the word 'Ganitanand' in parentheses at the end of those entries.

Finally, it may be stated that this bibliography all in all has more than 500 entries. This by no stretch of the imagination can be considered a small accomplishment, particularly considering the fact that by profession Professor Gupta has been a teacher of modern mathematics—which unfortunately has been completely severed from the history of mathematics—all through this career (which in itself would have been demanding a significant portion of his time) and also given the fact that all his publications are single authored!

Note Apparently 'missing' items such as (1973c) and many more both detected and undetected (eg. 1998b; 2000c, d, e) may be found in the section on various Indian languages.

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राधाचरणगुप्तप्रशस्तिः

राधायाश्चरणाख्यगुप्तकुलजो वैदुष्यरत्नप्रभः
छोटेलालबिनोबयीति महतोः जातो हि झान्सीपुरे ।
श्रेष्ठे श्रावणपूर्णचन्द्रदिवसे शाके समाहात्म्यके
सावित्रीसहितो विराजत इह स्वाभारवेर्ज्योतिषा ॥ १ ॥

आरब्धा तव जीविका हि गणिते प्रायोगिके विश्रुते
पश्चादादृतमैतिहासिककृतेः संशोधनं जाटिलम् ।
दृष्ट्वा दत्तकृतेर्विमर्शमधमं खिन्नाय तुभ्यं द्रुतं
अम्बा सा हि सरस्वती निरदिशत् मार्गं तु संशोधने ॥ २ ॥

ख्याता ‘गाणितभारती’ तु विदुषां मोदाय नेयेत्यतः
वोढ्वायासमहर्निशं विरचिता नैका प्रबन्धावलिः ।
अज्ञातं विदितं कृतं परिहृतिः संशीतिभावस्य च
विज्ञातेऽपि च रन्ध्रपूरणकृतौ श्लाघ्यं भवत्कौशलम् ॥ ३ ॥

उत्साहं तव कर्मयोगमतुलं ज्ञात्वा मुदामन्त्रितः
`केन्नेता'ख्यमहोदयेन रचिते वोढुं पदं वैश्विके।
आयोगे बहुराष्ट्रके गणितगे चैतिह्यसंशोधके
स्थानं प्राप्य सुभारतीवरसुतो देदीप्यतेऽयं मुदा ॥ ४ ॥

ऐतिह्यं गणितस्य भारतभवं ज्ञाप्यं तदर्थं त्वया
स्वल्पादेव सदर्जितात् श्रमभवात् कोशात्स्वकीयान्मुदा ।
`नासी'त्यादिषु भूयसीषु हि निधिर्व्याख्याकृते स्थापितः
सुज्ञानां हि चरित्रदीपनविधौ संपोषकः प्रेरकः ॥ ५ ॥

आचार्यार्यभटादि-विज्ञकृतिभां संदर्शयन् सन्ततं
प्राप्तोत्कृष्टपदस्स भारतसुतो विज्ञानवृत्तान्तधीः ।
`केन्नत्तोमयि' नामकेन पदकेनायं सुधीरादृतो
गुप्ताचार्यवरो धिया विलसितो विश्वस्तरे राजते ॥ ६ ॥

लक्ष्यैकाग्रमतिस्तु मत्पतिवरो मग्नो महाम्भोनिधौ
इत्येवं बहुधा विचिन्त्य समये संभोजयन्तीं पतिम् ।
सावित्री हि सती तु सत्यमनुगा साध्ये च संप्रेरिका
त्यागेनामृतता भवेदिति दिशन्त्येषा गुणै राजते ॥ ७ ॥

येनाचार्यवरेण तीक्ष्णमतिना पीयूषधारार्पिता
विज्ञानस्य महर्षिभिर्विरचितस्यैतिह्यविज्ञापने ।
संनीता शरदामशीतिगणना नीत्या विनीत्या युता
रराजेत स मार्गदर्शनसुधीः क्रान्त्वा समानां शतम् ॥ ८ ॥

Translation of the *Praśasti*

1. Rādhācharaṇa, effulgent by the gem of his scholarship, was born in Gupta lineage in Jhansi to a noble couple, Chotelal and Binobai in the Śālivāhana *Śaka* year 1857 (समाहात्म्य) on the beautiful *Śrāvaṇa-Pūrṇimā* day. Accompanied by Sāvitrī, he shines with the light (Jyoti) of the Sun (Ravi) of his own brilliant knowledge (Ābhā).[1]
2. You started your (academic) career in the field of applied mathematics. Later, you were keen to do research on challenging topics about historical texts [of mathematics]. [This choice was made] when you were pained by seeing the dismissive review of the work of Datta [and Singh]. Then indeed the mother[2] Sarasvatī guided you [to tread] in the path of research.
3. In order to make the journal *Gaṇita Bhāratī* a source of delight to scholars, you wrote a number of articles by painstakingly working day and night. [Through these articles,] unknown was made known, doubts were dispelled, and the gaps wherever amongst the known things were filled. Your talents are indeed praiseworthy.
4. Having recognized your unparalleled commitment to the work and your zeal, Kenneth invited you to become a member in his International Commission on History of Mathematics. Through the satisfaction derived by this recognition [as first Indian], you, the excellent son of Mother *Bhāratī* (India) continued to glow further.
5. In order to make the History of Indian mathematics known widely, with the sense of great contentment you established several endowments for the conduct of the lectures [annually] from your own meagre, hard and rightfully earned money, in organizations like NASI.[3] That [gesture] indeed is greatly encouraging and supporting the cause of illuminating (recounting) the contributions of renowned scholars [of history of science].
6. You, the son of India, are an eminent historian of science, who by continually exploring the brilliance of the works of the savants like Āryabhaṭa have gained a great prominence [among historians of mathematics]. For this, [you] the great Guptācārya, were honoured with the Kenneth O. May medal, and thereby you shine [among the top historians] in the world.
7. Noticing that her own husband who is steadfast on the goal is completely immersed in the ocean [of research], your wife Savitri multiply thinking (whether to disturb or not to) has kindly been serving food to you at appropriate times. This virtuous lady, by upholding the eternal principles (*śraddhā and bhakti*), has been [greatly] assisting you in achieving the goals. Glowing with her [divine] qualities, she demonstrates that immortality can be achieved through sacrifice.

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[1] Pun is intended here, as Ravīndra, Ābhā and Jyoti are the names of his son and two daughters.

[2] Here too pun is intended as Sarasvatī is the name of the goddess of knowledge as well as his doctoral thesis advisor.

[3] National Academy of Sciences, India, Allahabad.

8. May the sharp-witted *ācārya*—by whom eighty years [of fruitful life] has been led with morality and modesty, offering the nectar in the form of scientific expositions of the historical facts in the writings of the great seers—remain the torch bearer crossing the milestone of hundred years!