Chapter 6
General Relativity

The theory of special relativity discussed in Chaps. 2–4 is based on the Einstein Principle of Relativity and requires flat spacetime and inertial reference frames. The aim of this chapter is to discuss the extension of such a theoretical framework in order to include gravity and non-inertial reference frames.

From simple considerations, it is clear that Newtonian gravity needs a profound revision. As was already pointed out in Sect. 2.1, Newton's Law of Universal Gravitation is inconsistent with the postulate that there exists a maximum velocity for the propagation of interactions. However, this is not all. In Newton's Law of Universal Gravitation, there is the distance between the two bodies, but distances depend on the reference frame. Moreover, we learned that in special relativity we can transform mass into energy and vice versa. Thus, we have to expect that massless particles also feel and generate gravitational fields, and we need a framework to include them. Lastly, if gravity couples to energy, it should couple to the gravitational energy itself. We should thus expect that the theory is non-linear, which is not the case in Newtonian gravity, where the gravitational field of a multi-body system is simply the sum of the gravitational fields of the single bodies.

6.1 General Covariance

The theory of *general relativity* is based on the *General Principle of Relativity*.

> **General Principle of Relativity**. The laws of physics are the same in all reference frames.

C. Bambi, *Introduction to General Relativity*, Undergraduate Lecture Notes in Physics, https://doi.org/10.1007/978-981-13-1090-4_6

An alternative formulation of the idea that the laws of physics are independent of the choice of reference frame is the *Principle of General Covariance*.

Principle of General Covariance. The form of the laws of physics is invariant under arbitrary differentiable coordinate transformations.

Both the General Principle of Relativity and the Principle of General Covariance are principles: they cannot be proved by theoretical arguments but only tested by experiments. If the latter confirm the validity of these principles, then we can take them as our postulates and formulate physical theories that are consistent with these assumptions.

A *general covariant transformation* is a transformation between two arbitrary reference frames (i.e. not necessarily inertial).[1] An equation is *manifestly covariant* if it is written in terms of tensors only. Indeed we know how tensors change under a coordinate transformation and therefore, if an equation is written in a manifestly covariant form, it is easy to write it in any coordinate system.

Note that in this textbook we distinguish the theory of general relativity from Einstein's gravity. As we have already pointed out, here we call the theory of general relativity the theoretical framework based on the General Principle of Relativity or, equivalently, on the Principle of General Covariance. With the term *Einstein's gravity* we refer instead to the general covariant theory of gravity based on the Einstein–Hilbert action (which will be discussed in the next chapter). Note, however, that different authors/textbooks can use a different terminology and call general relativity the theory of gravity based on the Einstein–Hilbert action.

With the theory of general relativity we can treat non-inertial reference frames and phenomena in gravitational fields. As we have seen at the beginning of the previous chapter, the fact that the motion of a body in a gravitational field is independent of its internal structure and composition permits us to absorb the gravitational field into the metric tensor of the spacetime. The same is true in the case of a non-inertial reference frame. If we are in an inertial reference frame, all free particles move at a constant speed along a straight line. If we consider a non-inertial reference frame, free particles may not move at a constant speed along a straight line any longer, but still they move in the same way independent of their internal structure and composition. We can thus think of absorbing the effects related to the non-inertial reference frame into the spacetime metric.

An example can clarify this point. We are in an elevator with two bodies, as shown in Fig. 6.1. In case A (left picture), the elevator is not moving, but it is in an external gravitational field with acceleration g. The two bodies feel the acceleration g. In case B (central picture), there is no gravitational field, but the elevator has acceleration $a = -g$ as shown in Fig. 6.1. This time the two bodies in the elevator feel an acceleration $-a = g$. If we are inside the elevator, we cannot distinguish

[1] Note that in special relativity we talked about "covariance" or "manifestly Lorentz-invariance". "General covariance" is their extension to arbitrary reference frames.

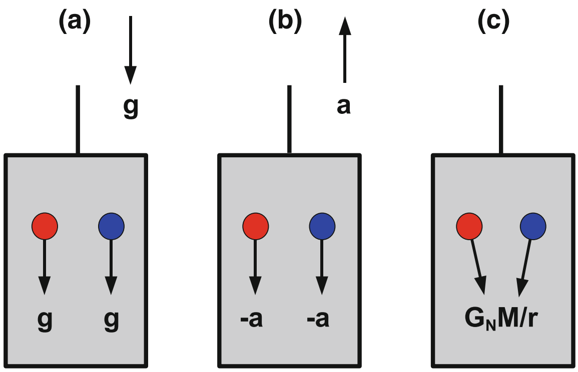

Fig. 6.1 The elevator experiment. See the text for the details

cases A and B. There is no experiment that can do so. This suggests that we should be able to find a common framework to describe physical phenomena in gravitational fields and in non-inertial reference frames.

However, such an analogy holds only locally. If we can perform our experiment for a "sufficiently" long time, where here "sufficiently" depends on the accuracy/precision of our measurements, we can distinguish the cases of a gravitational field and of a non-inertial reference frame. This is illustrated by case C (right picture in Fig. 6.1). The gravitational field generated by a massive body is not perfectly homogeneous. If the gravitational field is generated by a body of finite size, the spacetime should indeed tend to the Minkowski one sufficiently far from the object. Mathematically, this is related to the fact that it is always possible to change reference frame from a non-inertial to an inertial one and recover the physics of special relativity in all spacetime, but it is never possible to remove the gravitational field in *all* spacetime simply by changing coordinate system. Otherwise, the gravitational field would not be real! We can at most consider a coordinate transformation to move to a locally inertial frame (see Sect. 6.4).

An example similar to that introduced at the beginning of the previous chapter can better explain this point. Let us consider a flat spacetime. In the inertial reference frame with the Cartesian coordinates (ct, x, y, z) the line element reads

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \,. \tag{6.1}$$

Then we consider a rotating Cartesian coordinate system (ct', x', y', z') related to (ct, x, y, z) by

$$\begin{aligned} x' &= x \cos \Omega t + y \sin \Omega t \,, \\ y' &= -x \sin \Omega t + y \cos \Omega t \,, \\ z' &= z \,, \end{aligned} \tag{6.2}$$

where Ω is the angular velocity of the rotating system. The line element becomes

$$ds^2 = -\left(c^2 - \Omega^2 x'^2 - \Omega^2 y'^2\right) dt^2 - 2\Omega y' \, dt dx' + 2\Omega x' \, dt dy' + dx'^2 + dy'^2 + dz'^2 \, . \quad (6.3)$$

Whatever the transformation of the time coordinate is, we see that the line element is not that of the Minkowski spacetime.

We already know how tensors of any type change under a coordinate transformation $x^\mu \to x'^\mu$. This was given in Eq. (1.30). The difference is that now we can consider arbitrary coordinate transformations $x'^\mu = x'^\mu(x)$, even those to move to non-inertial reference frames. This also shows that the spacetime coordinates cannot be the components of a vector; they only transform as the components of a vector for linear coordinate transformations.

6.2 Einstein Equivalence Principle

The observational fact that the ratio between the inertial and the gravitational masses, m_i/m_g, is a constant independent of the body is encoded in the so-called Weak Equivalence Principle.

Weak Equivalence Principle. The trajectory of a freely-falling test-particle is independent of its internal structure and composition.

Here "freely-falling" means that the particle is in a gravitational field, but there are no other forces acting on it. "Test-particle" means that the particle is too small to be affected by tidal gravitational forces and to alter the gravitational field with its presence (i.e. the so-called "back-reaction" is negligible).

The Einstein Equivalence Principle is the fundamental pillar of the theory of general relativity.

Einstein Equivalence Principle.
1. The Weak Equivalence Principle holds.
2. The outcome of any local non-gravitational experiment is independent of the velocity of the freely-falling reference frame in which it is performed (*Local Lorentz Invariance*).
3. The outcome of any local non-gravitational experiment is independent of where and when it is performed (*Local Position Invariance*).

The Local Lorentz Invariance and the Local Position Invariance replace the Einstein Principle of Relativity in the case of non-inertial reference frames. The Local Lorentz Invariance implies that – locally – we can always find a quasi-inertial reference frame. How this can be implemented will be shown in the next sections of this chapter. The Local Position Invariance requires that the non-gravitational laws of physics are the same at all points of the spacetime; for example, this forbids variation of fundamental constants.

The theories of gravity that satisfy the Einstein Equivalence Principle are the so-called *metric theories of gravity*, which are defined as follows:

Metric theories of gravity.
1. The spacetime is equipped with a symmetric metric.
2. The trajectories of freely-falling test-particles are the geodesics of the metric.
3. In local freely-falling reference frames, the non-gravitational laws of physics are the same as in special relativity.

6.3 Connection to the Newtonian Potential

We have already seen in Sect. 5.1 how to connect the gravitational field of Newtonian gravity, Φ, with the spacetime metric $g_{\mu\nu}$. In this section, we propose an alternative derivation.

We employ Cartesian coordinates and require the following conditions in order to recover Newton's gravity: (i) the gravitational field is "weak", so we can write $g_{\mu\nu}$ as the Minkowski metric plus a small correction, (ii) the gravitational field is stationary, i.e. independent of time, and (iii) the motion of particles is non-relativistic, i.e. the particle speed is much smaller than the speed of light. These three conditions can be written as

$$(i) \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad |h_{\mu\nu}| \ll 1 \,, \tag{6.4}$$

$$(ii) \quad \frac{\partial g_{\mu\nu}}{\partial t} = 0 \,, \tag{6.5}$$

$$(iii) \quad \frac{dx^i}{dt} \ll c \,. \tag{6.6}$$

From the condition in (6.6), the geodesic equations become

$$\ddot{x}^\mu + \Gamma^\mu_{tt} c^2 \dot{t}^2 = 0 \,, \tag{6.7}$$

because $\dot{x}^i = (dx^i/dt)\dot{t} \ll c\dot{t}$. Note that here the dot $\dot{}$ indicates the derivative with respect to the proper time τ. In the Christoffel symbols Γ^μ_{tt}s, we only consider the linear terms in h and we ignore those of higher order

Table 6.1 Mass, radius, and value of $|2\Phi/c^2|$ at their radius for the Sun, Earth, and a proton. As we can see, $|2\Phi/c^2| \ll 1$

Object	Mass	Radius	$\|2\Phi/c^2\|$
Sun	$1.99 \cdot 10^{33}$ g	$6.96 \cdot 10^5$ km	$4.24 \cdot 10^{-6}$
Earth	$5.97 \cdot 10^{27}$ g	$6.38 \cdot 10^3$ km	$1.39 \cdot 10^{-9}$
Proton	$1.67 \cdot 10^{-24}$ g	0.8 fm	$3 \cdot 10^{-39}$

$$\Gamma^t_{tt} = \frac{1}{2} g^{t\nu} \left(\frac{1}{c}\frac{\partial g_{\nu t}}{\partial t} + \frac{1}{c}\frac{\partial g_{t\nu}}{\partial t} - \frac{\partial g_{tt}}{\partial x^\nu} \right) = O\left(h^2\right) ,$$
$$\Gamma^i_{tt} = \frac{1}{2} g^{i\nu} \left(\frac{1}{c}\frac{\partial g_{\nu t}}{\partial t} + \frac{1}{c}\frac{\partial g_{t\nu}}{\partial t} - \frac{\partial g_{tt}}{\partial x^\nu} \right) = -\frac{1}{2}\eta^{ij}\frac{\partial h_{tt}}{\partial x^j} + O\left(h^2\right) . \tag{6.8}$$

The geodesic equations reduce to

$$\ddot{t} = 0 , \tag{6.9}$$
$$\ddot{x}^i = \frac{1}{2}\frac{\partial h_{tt}}{\partial x^i} c^2 \dot{t}^2 . \tag{6.10}$$

$\dot{x}^i = (dx^i/dt)\dot{t}$ and, from Eq. (6.9), $\ddot{x}^i = (d^2x^i/dt^2)\dot{t}^2$. Equation (6.10) becomes

$$\frac{d^2x^i}{dt^2} = \frac{c^2}{2}\frac{\partial h_{tt}}{\partial x^i} . \tag{6.11}$$

In Newtonian gravity in Cartesian coordinates, Newton's Second Law reads

$$\frac{d^2x^i}{dt^2} = -\frac{\partial \Phi}{\partial x^i} . \tag{6.12}$$

From the comparison of Eqs. (6.11) and (6.12), we see that

$$h_{tt} = -\frac{2\Phi}{c^2} + C , \tag{6.13}$$

where C is a constant that should be zero because at large radii $h_{tt} \to 0$ in order to recover the Minkowski metric. We thus obtain again the result

$$g_{tt} = -\left(1 + \frac{2\Phi}{c^2}\right) . \tag{6.14}$$

Table 6.1 shows the value of $|2\Phi/c^2|$ at their surface for the Sun, the Earth, and a proton. This quantity is always very small and a posteriori justifies our assumption (6.4).

6.4 Locally Inertial Frames

6.4.1 Locally Minkowski Reference Frames

Since the metric is a symmetric tensor, in an n-dimensional spacetime $g_{\mu\nu}$ has $n(n+1)/2$ distinct components. If $n = 4$, there are 10 distinct components. In a curved spacetime, it is not possible to reduce $g_{\mu\nu}$ to the Minkowski metric $\eta_{\mu\nu}$ over the whole spacetime with a coordinate transformation. If it were possible, the spacetime would be flat and not curved by definition. From the mathematical point of view, the fact that such a transformation is not possible in general is because one should solve six differential equations for the off-diagonal components of the metric tensor

$$g_{\mu\nu} \to g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} = 0 \qquad \mu \neq \nu\,, \tag{6.15}$$

for the four functions $x'^\mu = x'^\mu(x)$.

On the contrary, it is always possible to find a coordinate transformation that reduces the metric tensor to $\eta_{\mu\nu}$ at a point of the spacetime. In such a case we have to make diagonal a symmetric matrix with constant coefficients (the metric tensor in a certain reference frame and evaluated at a certain point of the spacetime) and then rescale the coordinates to reduce to ± 1 the diagonal elements. Formally, we perform the following coordinate transformation

$$dx^\mu \to d\hat{x}^{(\alpha)} = E^{(\alpha)}_\mu dx^\mu\,, \tag{6.16}$$

such that the new metric tensor is given by the Minkowski metric diag$(-1, 1, 1, 1)$

$$g_{\mu\nu} \to \eta_{(\alpha)(\beta)} = E^\mu_{(\alpha)} E^\nu_{(\beta)} g_{\mu\nu}\,. \tag{6.17}$$

A similar coordinate system is called *locally Minkowski reference frame*.

$E^\mu_{(\alpha)}$s are the inverse of $E^{(\alpha)}_\mu$s and we can write

$$E^{(\alpha)}_\mu E^\nu_{(\alpha)} = \delta^\nu_\mu\,, \quad E^{(\alpha)}_\mu E^\mu_{(\beta)} = \delta^{(\alpha)}_{(\beta)}\,. \tag{6.18}$$

The coefficients $E^\mu_{(\alpha)}$s are called the *vierbeins* if the spacetime has dimension $n = 4$ and *vielbeins* for n arbitrary.

If a vector (dual vector) has components V^μ (V_μ) in the coordinate system $\{x^\mu\}$, the components of the vector and of the dual vector in the locally Minkowski reference frame are

$$V^{(\alpha)} = E^{(\alpha)}_\mu V^\mu\,, \quad V_{(\alpha)} = E^\mu_{(\alpha)} V_\mu\,. \tag{6.19}$$

From the definition of vierbeins, it is straightforward to see that

$$V^\mu = E^\mu_{(\alpha)} V^{(\alpha)}\,, \quad V_\mu = E^{(\alpha)}_\mu V_{(\alpha)}\,. \tag{6.20}$$

6.4.2 Locally Inertial Reference Frames

A *locally inertial reference frame* represents the best local approximation to a Minkowski spacetime that we can always find in a generic curved spacetime. In the case of a locally Minkowski spacetime, the metric at a point is given by the Minkowski metric, but we have not yet locally removed the gravitational field.

Let us consider an arbitrary coordinate system $\{x^\mu\}$. We expand the metric tensor around the origin

$$g_{\mu\nu}(x) = g_{\mu\nu}(0) + \left.\frac{\partial g_{\mu\nu}}{\partial x^\rho}\right|_0 x^\rho + \frac{1}{2}\left.\frac{\partial^2 g_{\mu\nu}}{\partial x^\rho \partial x^\sigma}\right|_0 x^\rho x^\sigma + \cdots . \tag{6.21}$$

We consider the following coordinate transformation $x^\mu \to x'^\mu$

$$x^\mu \to x'^\mu = x^\mu + \frac{1}{2}\Gamma^\mu_{\rho\sigma}(0) x^\rho x^\sigma + \cdots , \tag{6.22}$$

with inverse

$$x^\mu = x'^\mu - \frac{1}{2}\Gamma^\mu_{\rho\sigma}(0) x'^\rho x'^\sigma + \cdots . \tag{6.23}$$

Since

$$\frac{\partial x^\alpha}{\partial x'^\mu} = \delta^\alpha_\mu - \Gamma^\alpha_{\mu\nu}(0) x'^\nu + \cdots , \tag{6.24}$$

the metric tensor in the new coordinate system $\{x'^\mu\}$ is

$$\begin{aligned} g'_{\mu\nu} &= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} \\ &= g_{\alpha\beta}\left(\delta^\alpha_\mu - \Gamma^\alpha_{\mu\rho}(0) x'^\rho + \cdots\right)\left(\delta^\beta_\nu - \Gamma^\beta_{\nu\sigma}(0) x'^\sigma + \cdots\right) \\ &= g_{\mu\nu} - g_{\mu\beta}\Gamma^\beta_{\nu\sigma}(0) x'^\sigma - g_{\alpha\nu}\Gamma^\alpha_{\mu\rho}(0) x'^\rho + \cdots . \end{aligned} \tag{6.25}$$

For simplicity, in what follows we omit ... at the end of the expressions to indicate that we are only interested in the leading order terms.

The partial derivative of the metric tensor evaluated at the origin is

$$\left.\frac{\partial g'_{\mu\nu}}{\partial x'^\rho}\right|_0 = \left.\frac{\partial g_{\mu\nu}}{\partial x^\rho}\right|_0 - g_{\mu\beta}(0)\Gamma^\beta_{\nu\rho}(0) - g_{\alpha\nu}(0)\Gamma^\alpha_{\mu\rho}(0) . \tag{6.26}$$

The second term on the right hand side in Eq. (6.26) is

$$\Gamma^{\beta}_{\nu\rho} = \frac{1}{2} g^{\beta\gamma} \left(\frac{\partial g_{\gamma\rho}}{\partial x^{\nu}} + \frac{\partial g_{\nu\gamma}}{\partial x^{\rho}} - \frac{\partial g_{\nu\rho}}{\partial x^{\gamma}} \right) ,$$
$$g_{\mu\beta} \Gamma^{\beta}_{\nu\rho} = \frac{1}{2} \left(\frac{\partial g_{\mu\rho}}{\partial x^{\nu}} + \frac{\partial g_{\nu\mu}}{\partial x^{\rho}} - \frac{\partial g_{\nu\rho}}{\partial x^{\mu}} \right) . \tag{6.27}$$

The third term on the right hand side in Eq. (6.26) is

$$g_{\alpha\nu} \Gamma^{\alpha}_{\mu\rho} = \frac{1}{2} \left(\frac{\partial g_{\nu\rho}}{\partial x^{\mu}} + \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial g_{\mu\rho}}{\partial x^{\nu}} \right) . \tag{6.28}$$

We sum the expressions in Eqs. (6.27) and (6.28)

$$g_{\mu\beta} \Gamma^{\beta}_{\nu\rho} + g_{\alpha\nu} \Gamma^{\alpha}_{\mu\rho} = \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} , \tag{6.29}$$

and we thus see that the expression in Eq. (6.26) vanishes. The metric tensor $g'_{\mu\nu}$ around the origin is

$$g'_{\mu\nu}(x') = g'_{\mu\nu}(0) + \frac{1}{2} \frac{\partial^2 g'_{\mu\nu}}{\partial x'^{\rho} \partial x'^{\sigma}} \bigg|_0 x'^{\rho} x'^{\sigma} , \tag{6.30}$$

because all its first partial derivatives vanish. The geodesic equations around the origin are

$$\frac{d^2 x'^{\mu}}{d\lambda^2} = 0 . \tag{6.31}$$

The motion of a test-particle around the origin in this reference frame is the same as in special relativity in Cartesian coordinates and we can thus say that we have locally removed the gravitational field.

6.5 Measurements of Time Intervals

In general relativity, the choice of the coordinate system is arbitrary and we can move from one reference frame to another one with a coordinate transformation. In general, the value of the coordinates of a certain reference frame has no physical meaning; the coordinate system is just a tool to describe the points of the spacetime. When we want to compare theoretical predictions with observations, we need to consider the reference frame associated with the observer performing the experiment and compute the theoretical predictions of the physical phenomenon under consideration there.

In Sect. 2.4, we discussed the relations between time intervals measured by clocks in different reference frames. Here we want to extend that discussion in the presence of gravitational fields.

Let us consider an observer in a gravitational field described by the metric tensor $g_{\mu\nu}$. The observer is equipped with a locally Minkowski reference frame. His/her proper time is the time measured by a clock at the origin of his/her locally Minkowski reference frame and is given by

$$-c^2 d\tau^2 = ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,, \tag{6.32}$$

where ds is the line element of his/her trajectory (which is an invariant) and $\{x^\mu\}$ is the coordinate system associated to the metric $g_{\mu\nu}$.

A special case is represented by an observer at rest in the coordinate system $\{x^\mu\}$; that is, the observer's spatial coordinates are constant in the time coordinate: $dx^i = 0$ in Eq. (6.32). In such a situation we have the following relation between the proper time of the observer, τ, and the temporal coordinate of the coordinate system, t,

$$d\tau^2 = -g_{tt} dt^2 \,. \tag{6.33}$$

In the case of a weak gravitational field, g_{tt} is given by Eq. (6.14), and we find

$$d\tau^2 = \left(1 + \frac{2\Phi}{c^2}\right) dt^2 \,. \tag{6.34}$$

If the gravitational field is generated by a spherically symmetric source of mass M, $\Phi = -G_{\rm N} M/r$ and

$$d\tau = \sqrt{1 - \frac{2G_{\rm N}M}{c^2 r}} dt < dt \,. \tag{6.35}$$

In this specific example, the temporal coordinate t coincides with the proper time τ for $r \to \infty$, which means that the reference frame corresponds to the coordinate system of an observer at infinity. In general, $\Delta\tau < \Delta t$, and the clock of the observer in the gravitational field is slower than the clock of an observer at infinity.

6.6 Example: GPS Satellites

The Global Position System (GPS) is the most famous global navigation satellite system. It consists of a constellation of satellites orbiting at an altitude of about 20,200 km from the ground and with an orbital speed of about 14,000 km/h. With a typical GPS receiver, one can quickly determine his/her position on Earth with an accuracy of 5–10 m.

GPS satellites carry very stable atomic clocks and continually broadcast a signal that includes their current time and position. From the signal of at least four satellites, a GPS receiver can determine its position on Earth. Even if the gravitational field of Earth is weak and the speed of the satellites is relatively low in comparison to the speed of light, special and general relativistic effects are important and cannot be ignored.

A GPS receiver on Earth is a quasi-inertial observer, while the GPS satellites are not and move with a speed of about 14,000 km/h with respect to the ground, so we have

$$\frac{v}{c} = 1.3 \cdot 10^{-5}\,, \quad \gamma \approx 1 + \frac{1}{2}\frac{v^2}{c^2} = 1 + 8.4 \cdot 10^{-11}\,. \tag{6.36}$$

If $\Delta\tau$ is the measurement of a certain time interval by the GPS receiver and $\Delta\tau'$ is the measurement of the same time interval by the GPS satellites, we have that $\Delta\tau = \gamma\Delta\tau'$. After 24 h, the clocks of the GPS satellites would have a delay of 7 µs due to the orbital motion of the satellites.

Assuming that the Earth is a spherically symmetric body of mass M, the Newtonian gravitational potential is

$$\Phi = -\frac{G_N M}{r}\,, \tag{6.37}$$

where r is the distance from the center of Earth. The GPS receiver is on the Earth surface, so $r = 6,400$ km. For the GPS satellites, the distance from the center of Earth is $r = 26,600$ km. From Eq. (6.35), we can write the relation between the proper time interval of the GPS receiver $\Delta\tau$ and the proper time interval of the GPS satellites $\Delta\tau'$

$$\frac{\Delta\tau}{\sqrt{1 + 2\Phi_{\rm rec}/c^2}} = \frac{\Delta\tau'}{\sqrt{1 + 2\Phi_{\rm sat}/c^2}}\,, \tag{6.38}$$

where $\Phi_{\rm rec}$ and $\Phi_{\rm sat}$ are, respectively, the Newtonian gravitational potential at the position of the GPS receiver on the surface of Earth and at the positions of the GPS satellites at an altitude of about 20,200 km from the ground. We have

$$\Delta\tau = \sqrt{\frac{1 + 2\Phi_{\rm rec}/c^2}{1 + 2\Phi_{\rm sat}/c^2}}\Delta\tau' \approx \left(1 + \frac{\Phi_{\rm rec}}{c^2} - \frac{\Phi_{\rm sat}}{c^2}\right)\Delta\tau'\,. \tag{6.39}$$

$\Phi_{\rm rec}/c^2 = -6.9 \cdot 10^{-10}$, $\Phi_{\rm sat}/c^2 = -1.7 \cdot 10^{-10}$, and, after 24 h, the clock of the GPS receiver would have a delay of 45 µs with respect to the clocks of the GPS satellites due to the Earth's gravitational field.

If we combine the effect of the orbital motion of the GPS satellites with the effect of the Earth's gravitational field, we find that, after 24 h, the difference between the time of the clock of the GPS receiver and that of the clocks of the GPS satellites is

$$\begin{aligned} \delta t = \Delta\tau - \Delta\tau' &\approx \left(\frac{\Phi_{\rm rec}}{c^2} - \frac{\Phi_{\rm sat}}{c^2} + \frac{1}{2}\frac{v^2}{c^2}\right)\Delta\tau' \\ &= -45\,\mu{\rm s} + 7\,\mu{\rm s} = -38\,\mu{\rm s}\,. \end{aligned} \tag{6.40}$$

Since the communication between GPS satellites and GPS receiver is through electromagnetic signals moving at the speed of light, an error of 38μs is equivalent to an error of $c\delta t \approx 10\,\mathrm{km}$ in space position, which would make GPS navigation systems in cars and smartphones completely useless.

6.7 Non-gravitational Phenomena in Curved Spacetimes

Now we want to figure out how to write the laws of physics in the presence of a gravitational field. In other words, we want to find a "recipe" to apply the Principle of General Covariance: we know the mathematical expression describing a certain non-gravitational phenomenon in the Minkowski spacetime in Cartesian coordinates and we want to write the same physical law in the presence of a gravitational field in arbitrary coordinates. The case of non-inertial reference frames in flat spacetime will be automatically included.

As it will be more clear from the examples below, there are some ambiguities when we want to translate the laws of physics from flat to curved spacetimes. This means that purely theoretical arguments are not enough to do it and experiments have to confirm if the new equations are right or not.

As the first step to solve this problem, we can start considering how to write a physical law valid in flat spacetime and Cartesian coordinates for the case of flat spacetime and non-Cartesian coordinates. This has been partially discussed in the previous chapters. We start from manifestly Lorentz-invariant expressions and we make the following substitutions:

1. we replace the Minkowski metric in Cartesian coordinates with the metric in the new coordinates: $\eta_{\mu\nu} \to g_{\mu\nu}$;
2. partial derivatives become covariant derivatives: $\partial_\mu \to \nabla_\mu$;
3. if we have an integral over the whole spacetime (e.g. an action), we have to integrate over the correct volume element $d^4\Omega$: $d^4x \to |J| d^4x$, where J is the Jacobian. We will see below that $J = \sqrt{-g}$, where g is the determinant of the metric tensor.

The new equations are written in terms of tensors and therefore they are manifestly covariant. We could apply the same recipe to write the physical laws in curved spacetimes. For the time being, experiments confirm that we obtain the right equations.

For the point 3 above, we know that, if we move from an inertial reference frame with the Minkowski metric $\eta_{\mu\nu}$ and the Cartesian coordinates $\{x^\mu\}$ to another reference frame with the coordinates $\{x'^\mu\}$, the new metric is

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \eta_{\alpha\beta} , \tag{6.41}$$

and therefore the determinant is

$$\det \left| g'_{\mu\nu} \right| = \det \left| \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \eta_{\alpha\beta} \right| = \det \left| \frac{\partial x^\alpha}{\partial x'^\mu} \right| \det \left| \frac{\partial x^\beta}{\partial x'^\nu} \right| \det \left| \eta_{\alpha\beta} \right| . \tag{6.42}$$

Note that[2]

$$\det \left| g'_{\mu\nu} \right| = -g \, , \quad \det \left| \frac{\partial x^\alpha}{\partial x'^\mu} \right| = J \, , \quad \det \left| \eta_{\alpha\beta} \right| = 1 \, , \tag{6.43}$$

where J is the Jacobian of the coordinate transformation $x'^\mu \to x^\mu$. From Eq. (6.42), we see that $J = \sqrt{-g}$, and therefore we find that we need the following substitution when we have an integral and we move to an arbitrary reference frame

$$\int d^4x \to \int \sqrt{-g} \, d^4x \, . \tag{6.44}$$

Let us consider the electromagnetic field. The fundamental variable is the 4-potential A_μ. In the Faraday tensor $F_{\mu\nu}$, we replace the partial derivatives with the covariant ones, but there is no difference

$$\begin{aligned} F_{\mu\nu} &= \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma^\sigma_{\mu\nu} A_\sigma - \partial_\nu A_\mu + \Gamma^\sigma_{\nu\mu} A_\sigma \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu \, . \end{aligned} \tag{6.45}$$

The Maxwell equations in the manifestly Lorentz-invariant form, in flat spacetime, and in Cartesian coordinates are given in Eqs. (4.40) and (4.48). We replace the partial derivatives with the covariant ones and we have

$$\nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} = 0 \, , \tag{6.46}$$

$$\nabla_\mu F^{\mu\nu} = -\frac{4\pi}{c} J^\nu \, . \tag{6.47}$$

Let us now talk about possible ambiguities of our recipe. First, we have to replace partial derivatives with covariant ones. However, partial derivatives commute while covariant derivatives do not. This may be a problem, but in practical examples it is not because it is possible to find a way to figure out the right order.

Second, in general we may expect that tensors that vanish in flat spacetime may affect the physical phenomenon when gravity is turned on. For example, the Lagrangian density of a real scalar field in Minkowski spacetime in Cartesian coordinates is

$$\mathscr{L} = -\frac{\hbar}{2} \eta^{\mu\nu} \left(\partial_\mu \phi \right) \left(\partial_\nu \phi \right) - \frac{1}{2} \frac{m^2 c^2}{\hbar} \phi^2 \, , \tag{6.48}$$

[2] We indicate with g the determinant of the metric $g'_{\mu\nu}$. Since $g < 0$ in a (3+1)-dimensional spacetime, the absolute value of the determinant of $g'_{\mu\nu}$ is $\det \left| g'_{\mu\nu} \right| = -g$.

and the action is

$$S = \frac{1}{c} \int \mathscr{L} d^4x \, . \tag{6.49}$$

If we apply our recipe, the Lagrangian density describing our real scalar field in curved spacetime should be (we remind that, for scalars, ∇_ν reduces to ∂_ν)

$$\mathscr{L} = -\frac{\hbar}{2} g^{\mu\nu} \left(\partial_\mu \phi\right) \left(\partial_\nu \phi\right) - \frac{1}{2} \frac{m^2 c^2}{\hbar} \phi^2 \, . \tag{6.50}$$

The action is

$$S = \frac{1}{c} \int \mathscr{L} \sqrt{-g} \, d^4x \, . \tag{6.51}$$

We say that such a scalar field is *minimally coupled* because we have applied our recipe that holds for both non-inertial reference frames and gravitational fields. However, we cannot exclude that the Lagrangian density in curved spacetime is

$$\mathscr{L} = -\frac{\hbar}{2} g^{\mu\nu} \left(\partial_\mu \phi\right) \left(\partial_\nu \phi\right) - \frac{1}{2} \frac{m^2 c^2}{\hbar} \phi^2 + \xi R \phi^2 \, , \tag{6.52}$$

where ξ is a coupling constant. In this case, we say that the scalar field is *non-minimally coupled* because we have a term coupling the scalar field with the gravitational field. The Lagrangian density in Eq. (6.52) reduces to Eq. (6.48) when gravity is turned off. There are no theoretical reasons (such as the violation of some fundamental principle) to rule out the expression in Eq. (6.52), which is instead more general than the Lagrangian density in (6.50) and therefore more "natural" – in theoretical physics, it is common to follow the principle according to which whatever is not forbidden is allowed. For the time being, there are no experiments capable of testing the presence of the term $\xi R \phi^2$, in the sense that it is only possible to constrain the parameter ξ to be below some huge unnatural value.

Up to now, there are no experiments that require that some physical law requires a non-minimal coupling when we move from flat to curved spacetimes. However, we have to consider that the effects of gravitational fields via non-minimal coupling are extremely weak in comparison to non-gravitational interactions. This is because the Planck mass $M_{\rm Pl} = \sqrt{\hbar c / G_{\rm N}} = 1.2 \cdot 10^{19}$ GeV is huge in comparison with the energy scales in particle physics and there are no environments in the Universe today where gravity is "strong" in terms of the Planck mass.

Problems

6.1 In the Schwarzschild metric, the line element reads

$$ds^2 = -f(r)c^2dt^2 + \frac{dr^2}{f(r)} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\,, \tag{6.53}$$

where

$$f(r) = 1 - \frac{r_{\rm Sch}}{r}\,, \tag{6.54}$$

and $r_{\rm Sch} = 2G_{\rm N}M/c^2$. The Schwarzschild metric describes the spacetime around a static and spherically symmetric object of mass M. Write the vierbeins to move to a locally Minkowski reference frame.

6.2 In the Kerr metric, the line element reads

$$\begin{aligned} ds^2 = &-\left(1 - \frac{r_{\rm Sch}r}{\Sigma}\right)c^2dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \\ &+\left(r^2 + a^2 + \frac{r_{\rm Sch}ra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 - \frac{2r_{\rm Sch}ra\sin^2\theta}{\Sigma}cdtd\phi\,, \end{aligned} \tag{6.55}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta\,, \quad \Delta = r^2 - r_{\rm Sch}r + a^2\,, \tag{6.56}$$

$r_{\rm Sch} = 2G_{\rm N}M/c^2$, and $a = J/(Mc)$. The Kerr metric describes the spacetime around a stationary and axisymmetric black hole of mass M and spin angular momentum J. Write the relation between the proper time of an observer with constant spatial coordinates and the time coordinate t.

6.3 In Sect. 4.3, we met the equation of the conservation of the electric current in flat spacetime in Cartesian coordinates: $\partial_\mu J^\mu = 0$. Rewrite this equation in: (i) a general reference frame, and (ii) flat spacetime in spherical coordinates.

6.4 The field equation for the scalar field ϕ that we obtain from the Lagrangian density (6.48) in Cartesian coordinates is the Klein–Gordon equation

$$\left(\partial_\mu\partial^\mu - \frac{m^2c^2}{\hbar^2}\right)\phi = 0\,. \tag{6.57}$$

Write the Klein–Gordon equation in curved spacetime, first assuming minimal coupling and then non-minimal coupling.

6.5 Write the Lagrangian in (6.52) and the associated Klein–Gordon equation in the case of a metric with signature $(+ - - -)$.

6.6 The energy-momentum tensor of a perfect fluid in flat spacetime and Cartesian coordinates is given in Eq. (3.104). Write its expression for a general reference frame.