Chapter 15
Supporting Young Children to Develop Combinatorial Reasoning

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Abstract The goal of this chapter is to discuss young children's approaches to dealing with combinatorial tasks and to present some teachers' strategies to support children's combinatorial reasoning. The discussions are based on clinical interviews with young children (ages 6–8) who were asked to solve a combinatorial task centered on the process of combinatorial counting. Children were interviewed in a private setting and were given some manipulative to help them visualize, explore, model, and solve the combinatorial task. The results revealed by the clinical interviews were contrasted with those disclosed by the literature on children's combinatorial development. Such a contrast suggests that some strategies could be used to support children's combinatorial reasoning. One of the important contributions of this chapter is that it reveals the close relation between young children's combinatorial reasoning and multiplicative reasoning. Consequently, teachers' strategies to support young children's combinatorial reasoning need to be grounded on the development of multiplicative reasoning and to support exploration of combinatorial counting processes. The chapter closes by presenting and discussing some strategies for teachers to support young children in their combinatorial reasoning.

15.1 Statement of Problem

Nowadays, probability is a topic integrated in most elementary school curricula in different countries. Probability includes, among other topics, combinatorial counting which is considered a very fundamental topic in the development of mathematical ideas, and which is based on additive and multiplicative reasoning[1] (Shin & Steffe, 2009). Although combinatorics seems to be a high-level topic for elementary school

[1] Additive reasoning is related to children's first organized attempt to understand and operate with adults' number system and it is mainly based on addition and subtraction while multiplicative reasoning recognizes and uses grouping to manage the number system.

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curricula, literature has shown that the evolution of combinatorial counting is essential in the establishment of the ideas of chance and probability. Piaget and Inhelder, for example, stated that "the child constructs his notions of probability by his ability to subordinate the disjunctions effected within mixed sets to all the possible combinations, using a multiplicative and not simply an additive mode" (p. 161). Some authors consider that the development of the combinatorial counting is important because it is the basis for more complex subjects (Ura, Stein-Barana, & Munhoz, 2011); others defend that the nature of formal reasoning is based on the combinatorial capability of the learner (Fischbein & Grossman, 1997).

Research literature on teaching and learning statistics has shown that students of all ages struggle with different types of combinatorial counting problems (Lockwood, 2011; Batanero, Navarro-Pelayo, & Godino, 1997) mainly because there is not an upfront way to solve them using procedural reasoning and because they require deep mathematical thinking. In spite of the fact that combinatorics is in the curriculum, there are few resources available for teachers to help them support young children in polishing their combinatorial reasoning. Discussing the path young children go through while exploring, approaching, modeling, and solving combinatorial counting situations could be an important source of reflection for teachers' practice as well as a valuable resource for researchers and statistics educators.

A crucial issue today is that the resources available to support teachers in teaching combinatorics to young children are separated from the daily life of students. Informal knowledge of students is infrequently taken into account when building new knowledge. Consequently, school mathematics is disconnected from the way young children solve problems and do mathematics in their daily lives (Bosch, 2012). Research results have shown, for example, that in the mathematical curriculum of certain educational systems, it is common to find statistics instruction as a set of procedures and algorithms that need to be memorized and applied without any contact with practical situations of the world (Zapata-Cardona & González Gómez, 2017). This form of instruction assumes and accepts statistics as a disarticulated science with no relation to the world experienced by the student. To transform this ingrained practice, daily situations have to be the basis of teaching in elementary education.

Teachers need to understand how young children think and do combinatorial counting tasks. When teachers understand children's ways of thinking and offer them opportunities to construct their own knowledge, it is easy to see how much students are able to learn. In a similar way, learning is meaningful when tasks are connected to students' lives.

Piaget and Inhelder's (1975) work studying the origin of the idea of chance in children has been very influential for those interested in children's combinatorial reasoning. However, in spite of its recognized impact and usefulness, it has raised strong criticism because of the characteristics of the tasks used in the clinical interviews with children. According to some critics, Piaget's research used material that was not very familiar for the children interviewed and this might have affected children's performance in the tasks (English, 1991). Piaget (as cited in Batanero, 2013) indicated that combinatorial reasoning is fully developed during the stage of formal operations (ages 11–15). However, there are some other researchers that stated that some teaching strategies could challenge young children to develop their combinatorial reasoning before coming to the stage of formal operations (Cañadas & Figueiras, 2010; Itzcovich, Ressia de Moreno, Novembre, & Becerril, 2009).

The intent of this chapter is to address the tensions found in the literature relating to the unfamiliarity of the tasks used in research with young children, the lack of detail in the strategies young children use to solve combinatorial counting task, and the limited resources teachers have to support and challenge young children's combinatorial reasoning. The chapter presents a description of the combinatorial counting strategies young children activate when they solve a familiar task. The purpose is not to judge their performance but to illustrate the kind of questions and strategies that researchers and teachers could use to challenge young children's combinatorial reasoning beyond their actual state. This chapter presents the reflections after interviewing three young children when solving a combinatorial counting task related to the multiplication principle.

15.2 Theoretical Framework

Combinatorial reasoning can be defined as the activation of resources (mental or physical) to complete a combinatorial task. Combinatorial counting is essential for the study of discrete mathematics and is the basis for other branches of mathematics. It is fundamental in the study of biology, economics, transportation, agriculture, and others related areas. For some authors, combinatorial reasoning is an important prerequisite for the dynamic and creative power of logical reasoning (Fernández Millán, 2013). Frequently, the study of combinatorics appears in the secondary education curriculum. However, there is an important aspect of combinatorics that can be introduced and successfully worked on with young children in the elementary school mathematics curriculum if it is carried out in conjunction with the strategies to develop multiplicative reasoning.

Some authors (Roa, Batanero, & Godino, 2001) have suggested that the difficulties that even advanced students of mathematics have solving combinatorial problems are related to the way combinatorics is taught. Every so often, teaching is focused on the formula, definition, and combinatorial operation. These authors suggest that in teaching combinatorics, the teacher should privilege problem-solving, systematic enumeration, and tree diagrams. They also indicate that combinatorial reasoning is developed in the stage of formal operations and recognize the strong influence of the environment and personal capacities of the individual. Despite localizing combinatorics at the end of the development period, the multiplication principle can be promoted early on in schooling if teachers support their teaching by using different typologies of problems to develop the multiplicative schemes. In this regard, other authors (Pessoa & Borba, 2012) have shown that combinatorial reasoning is not exclusively a characteristic of the stage of formal operations—as mentioned in Piagetian studies. Pessoa and Borba (2012) provide empirical evidence that children as young as preschool are able to solve, by manipulation of figures, combinatorial problems of different types: arrangement (from a larger set some elements are chosen whose ordering generates different possibilities), permutation (all elements of the set are used, only the order of the presentation varies), combination (these are similar to arrangements in terms of choice of elements, with the difference that the order of the elements does not generate distinct possibilities), and Cartesian product (the total number of all ordered k-tuples from multiple sets).

Scholarly literature in education and psychology reports several studies interested in the way children—from prekindergarten up to high school levels—explore, deal, and come up with solutions for combinatorial situations. The most influential study is that of Piaget and Inhelder (1975) who studied the genesis of the notion of chance by means of clinical interviews with children of a wide range of ages. Piaget and Inhelder's work has been the inspiration for a number of subsequent studies. One of these studies was carried out by English (1991) who explored 50 children's (4–9 years old) strategies to solve combinatorial problems by using seven different forms of the same problem: "find all the possible outfits for toy bears." English found that children could go from trial and error to very sophisticated and efficient algorithmic actions with the potential to generate all possible combinations. One of the important findings of this study was that children in the concrete operational stage, under proper learning conditions, are able to independently develop a method for the multiplication principle prior to formal instruction.

Another study explored the combinatorial abilities of 720 secondary school students (14–15-year-old pupils) finding that children make several mistakes in the combinatorial procedures (Batanero et al., 1997). The authors presented a list of 14 different types of errors participants made during the study. However, one of the most important results arises from comparing the performance of students who received direct instruction with those who did not, and the frequency of errors was reduced in the instruction group. One more study carried out by Cañadas and Figueiras (2010) investigated how students (11–12 years old) solve combinatorial problems using manipulative and how they make a generalization. In the cited study, one of the important results was the different interpretations students gave to multiplication. Students studied multiplication as Cartesian products going beyond the tradition of primary school mathematics instruction that promotes multiplication mainly as repeated addition. Another study carried out by Fuentes and Roa (2014) required 54 compulsory secondary students (12 and 13 years old) to solve a task making all the possible outfits from some shirts, pants, and hats. Fuentes and Roa found that participants were successful 78% of the time and used different strategies like multiplication (59.2% of the time), addition, seriation, and tree diagrams. In another study, Lockwood (2011) examined how students transfer some knowledge developed in solving some combinatorial problems to other types of problems. The study concluded that to help students with the reported difficulties in the literature for combinatorial counting problems, teachers need to pay closer attention to the connections students naturally make.

Studies exploring children's combinatorial abilities have been abundant and they have taken different approaches. Some of them have focused on children's mistakes and some on children's strategies. But what is important to highlight is that some of them have shown that children as young as prekindergarten and elementary school have available the operational structures necessary for dealing successfully with combinatorial counting tasks. This chapter takes into account the reflections and results that the available literature has suggested but the intention is to offer a little more depth and detail in the strategies that young children use when solving combinatorial counting tasks. It also attempts to reveal the type of child–adult interactions that support young children in moving a little beyond their actual capacities.

15.2.1 Multiplicative Reasoning

Usually, a multiplicative structure is constructed prior to operating. Such structure allows the child to shorten a counting activity and later on to internalize conducted actions and operations and use them a priori for the construction of more abstracted combinatorial reasoning. This is also called recursive multiplicative reasoning.

Multiplication in primary school is encouraged under its common intuitive meanings: (1) repeated addition (Cañadas & Figueiras, 2010; Steffe, 1994), (2) ratio and proportion—if a package of cookies has four cookies, how many cookies are there in five packages?—(Cañadas & Figueiras, 2010; Itzcovich et al., 2009), (3) rectangular arrangements—you need to set up a carpet on a surface of 3 by 5 m, how much carpet do you need?—(Itzcovich et al., 2009), and (4) Cartesian product—how many different outfits are you able to create from four shirts and three pants—(Cañadas & Figueiras, 2010; Itzcovich et al., 2009). Nonetheless, there is a disparity in the way multiplicative tasks are stimulated in curriculum materials. Usually, tasks that bring to mind repeated addition, ratio and proportion (formation of groups) or rectangular arrangements are stimulated but those that evoke Cartesian products are left out, having devastating implications for the development of combinatorial reasoning in young children.

15.2.2 Combinatorics and Combinatorial Counting Problems

Combinatorics is defined as "a principle of calculation involving the selection and arrangement of objects in a finite set" (English, 2005, p. 121). It includes areas like combinatorial counting, computations, and probability. In combinatorial counting problems, children are asked to count the number of ways that certain patterns can be formed. However, these are different from simple counting problems such as "how many color pencils do you have?" Counting is understood by Steffe (1983) as "the production of a sequence of number words, such that each number word is accompanied by the production of a unit item" (p. 111). Combinatorial counting problems involve more than children's basic counting schemes and require several connected actions. Initially, children need to deal with units of the indeterminate quantity to be counted in the problem situation; then, they need to properly combine elements from the different sets to create the new counting units; finally, they need to check the counting activity to decide when to stop counting (Shin & Steffe, 2009). In a simple counting situation, the child is asked to count single elements like “how many coins do you have in your pocket.” In a combinatorial counting situation, the counting unit is a combination of single units that the child properly creates. Combinatorial counting problems facilitate the development of enumeration processes, conjectures, generalizations, and systematic thinking. Combinatorial activities also help with the development of important concepts such as relations, equivalence classes, mapping, and functions (Batanero et al., 1997; English, 2005).

Within combinatorial counting, problems are those that involve the multiplication principle. This principle declares that if one event can occur in n ways and another event in m ways, then the two events together can occur in $n \times m$ ways. It is important that young children understand and properly use this elemental principle since it is the basis for more complex subjects like combinatorics, probability, and statistics (Ura et al., 2011).

15.3 Methodology

The goal of this chapter is to make evident children’s strategies to solve enumerative combinatorial counting situations so that the reflection on such strategies can orient teachers’ actions in the classroom when teaching combinatorics to young children. This chapter also pays close attention to how certain questions, actions, and suggestions indicated by the researchers challenge young children’s strategies and make young children go beyond their initial strategies. To address this goal, three young children (ages 6–8) were the participants who inspired the reflections discussed here. It will be too ambitious to call this experience a formal study since the participants do not represent all primary school students. The sample was a convenience sample of three girls. They were interviewed in a home setting while they solved the following combinatorial task:

> You have a doll and have four shirts and three pants. If you were to dress the doll in these clothes in how many different ways, could you combine those tops with those bottoms?

The task was presented in verbal form and some manipulative (silhouettes of tops and pants) were given to help visualize, explore, model, and solve the combinatorial task (like the ones shown in Fig. 15.1). The researcher did not reinforce correct choices and avoided referring to the quality of the participants’ decisions (as it is recommended by Falk, Yudilevich-Assouline, & Elstein, 2012).

The participants had not had formal academic training in combinatorics during their schooling, which was an advantage in that it made it easier to induce children to express their informal ideas during the interviews. Each child’s performance was videotaped, with the camera positioned to capture eye, head, and hand movements, and the use of manipulative.

Fig. 15.1 Silhouettes of tops and pants given to children to model the task

There are different reasons that support the use of attractive manipulative materials. First, images are important in helping children to communicate scientific ideas and support conceptualization. Second, manipulative stimulate children's minds and help them to explore different solutions without giving them the exact way to solve it. Third, children are able to develop concepts related to multiplication and combinations based on their own concrete experience (Ura et al., 2011).

In this study, attractive manipulative materials and an attractive task were used to explore young children's counting combinatorial strategies. By using attractive manipulative materials, teachers and researchers can increase the willingness for children to explore and attempt a solution using their informal knowledge. When children are exposed to tasks that are attractive to them, they increase the possibilities of exhibiting sophisticated solutions (English, 1993; Falk et al., 2012; Ura et al., 2011).

The analysis of data occurred at multiple levels. The researcher reviewed the videos several times to construct a content log. Special attention was paid to young children's strategies and how they reacted to challenging questions from the researcher. The interviews were transcribed verbatim, translated from Spanish into English, and reviewed to refine the understanding and descriptions of key aspects of the children's combinatorial reasoning.

15.4 Results

In this section, some segments of the interviews with the three young children are presented. The order in which the segments are displayed is related to the level of sophistication the young children displayed in the interview. The rudimentary strategies are presented first and then the more elaborate ones. The goal is to look beyond young children' strategies to focus on the potential support they could get from adults (teachers or researchers) to refine their combinatorial reasoning.

The first child is Valery, a seven-year-old girl who was in second grade of elementary school.

Researcher:	Let us suppose you have a doll with different clothes: four shirts and three pants. If you were to dress the doll in these clothes in how many different ways, could you combine those tops with those bottoms?
Valery:	Three ways
Researcher:	How did you do it?
Valery:	I have three outfits. I have three pants that I can dress the doll with and every day I put one on
Researcher:	Show me the three outfits you say
Valery:	[*She pairs up one top with one bottom*] this way
Researcher:	One way. Show me another way
Valery:	This way and this way [*She pairs up two more tops with two bottoms, but one top is left aside as it is shown in Fig.* 15.2]
Researcher:	And with that one [*the top left aside*], what are you going to do with it?
Valery:	If I have another doll out there, I can put it on [*the top left aside*] to it
Researcher:	I see. Thank you so much

Fig. 15.2 Combinations done by Valery

Valery's counting strategy was very straightforward. She paired up bottoms with tops, and once she ran out of pants, she stopped counting. She left one top aside without using it and when was asked what to do with it she recycled it to use it with another doll. The child used a very simple counting scheme and did not even intend to combine the elements of the sets to create the new counting units. The researcher did not ask further questions in this situation. This is a common strategy used by young children in solving combinatorial counting tasks.

The next child is Eileen. She was a six-year-old girl who was in first grade of elementary school.

Researcher:	You have a doll, four tops and three bottoms. How many ways do you have to dress your doll?
Eileen:	[*She pairs up a top and a bottom*] this one
Researcher:	Do you have any other way to dress the doll?
Eileen:	And these ones [*She pairs up two tops with two pants leaving one top apart*]
Researcher:	What are you going to do with that top? [*Pointing to the top left aside*]
Eileen:	I am going to put it here [*she puts the silhouette of the top on her own chest*]
Researchers:	To whom?
Eileen:	To the doll [*she exchanges the top on her chest with one of the tops that was already paired up with one of the bottoms. She ends up with a different top on her hand*]
Researcher:	So, what are you going to do with this one? [*The one on her hands*]
Eileen:	[*She exchanges the top again with another top already paired up with a bottom. She does that several times completing seven different ways and ends up with one top on her hands*]
Researcher:	So, what are you going to do with this one [*The one on her hands*]?
Eileen:	I will throw it away

In Eileen's interview, she paired up each bottom with each top and she stopped the combinatorial counting when she did not have any more bottoms for the tops. Initially, she formed three different ways and only after being asked what she was going to do with the remaining top, she came up with four more ways. In total, she created seven different ways by using random (unstructured) strategies. She did not use any systematic way to list the combinations or to keep track of the possibilities. Eileen was able to create four more ways because of the researcher intervention. The researcher pushed her to think about what to do with the remaining top and she was able to react to the query by coming up with an action. Eileen's action did not allow her to find all the different ways but at least allowed her to further extend her initial strategy. Comparing Eileen's with Valery's performance, it is evident that Eileen was able to go beyond her initial strategy. Even though both children were asked the same question about what to do with the single remaining top, the question presented the necessary motivation for Eileen to explore more ways to combine clothes. In this sense, the same question challenged only one child.

The third child is Sandy, an eight-year-old girl who was in second grade of elementary school.

Researcher:	You have a doll, three bottoms and four tops. If you were going to combine bottoms and tops in how many different ways, could you dress the doll?
Sandy:	I can dress my doll with this dress [*top*] and with the orange one [*bottom*]. This pink one [*bottom*] with the red one [*top*], and the yellow one [*bottom*] with this one [*top*]
Researcher:	In how many ways could you dress the doll?
Sandy:	Three ways
Researcher:	[*Pointing out to the top that was left without bottom*] And this one, what is going to happen with this one?
Sandy:	That one does not have a pant
Researcher:	So, would you put it on to the doll?
Sandy:	No
Researcher:	And what do you think could happen if we do this? [*Pairing up one of the pants with the shirt that has been left alone*]. One day you dress the doll with this pant and this shirt, and the next day you dress the doll with this other shirt?
Sandy:	Or you could also do this. This one with this one [*she moves the pants around and leaves the tops fixed*] and this yellow one [*pant*] can be also worn with this one
Researcher:	How many outfits do you have then?
Sandy:	Four
Researcher:	Show me the four outfits
Sandy:	I have this one [*she makes some exchanges with the pants*] and also this one. This one with this one
Researcher:	Then, it seems you have found more than four ways
Sandy:	Five then
Researcher:	Show me the five ways
Sandy:	[*she puts together five outfits*] This one with this one, this one with this one, this one too
Researcher:	Do you have more ways?
Sandy:	I have one more.
Researcher:	Which one?
Sandy:	[*she moves two bottoms again getting two new ways*]
Researcher:	Now you have seven. Do you think you have more ways?
Sandy:	Yes [*she moves two pants getting two more ways but one of them is already repeated*], eight and nine
Researcher:	Do you have more forms?
Sandy:	And ten [*she puts together another repeated outfit*]
Researcher:	Do you think that you have repeated some outfits?
Sandy:	This one and this one [*she points out two outfits, one of them was not repeated*]
Researcher:	Do you have more outfits?
Sandy:	No
Researcher:	Then, how many forms in total do you have to dress your doll?
Sandy:	Ten
Researcher:	Thank you so much

In Sandy's interview, she initially paired up bottoms with tops paying attention primarily to the proper combination of colors. Once she ran out of bottoms, she stopped making combinations. She did not consider dressing the doll with the fourth top (the one left out). Sandy paired up one top with one bottom and stopped when she exhausted the elements of the smaller set. The transformation of Sandy's strategy, at the end of the interview, was due to the researcher's stimulating question "And what do you think could happen if we do this? [*Pairing up one of the pants with the shirt that has been left alone*]. One day you dress the doll with this pant and this shirt, and the next day you dress the doll with this other shirt?" After this question, Sandy started randomly (unstructured) matching shirts with pants without being systematic in her approach. In doing so, Sandy found ten combinations but not all of them were different. She repeated two counting units but she was not fully aware of this. This was mainly in part because she did not follow any systematic strategy to keep track of the repeated counting units.

15.5 Discussion

The three young children participating in this experience, at first, used the same strategy to combine shirts and pants in order to find out the different combinations of outfits for the doll. All the children started by pairing up pants with shirts and left one shirt out. They stopped the combinations when they did not have more pants left to combine with the shirts. Similar results were found by Piaget and Inhelder (1975) and later on by English (1991, 1993) who stated that young children initially tend to approach combinatorial problems using very simple counting schemes and empirical approaches.

Valery, as well as Eileen, initially found the same number of outfits by combining shirts with pants using the same rudimentary strategy. They paired the elements of one set with the elements of the other set until they ran out of elements from the smaller set. However, when they were asked what to do with the shirt left aside, the answers were very different. The question the researcher asked did not have any effect on Valery's actions and her task ended there, whereas the same question allowed Eileen to explore other options slightly modifying her strategy and consequently the results. Eileen got four counting units more than in her initial attempt. This is a very interesting result because it shows that the same researcher strategy had different effects on children's actions. These differences in children's performance might be attributed to the different resources children come with to the interview (influence of family, schooling, or culture). Children before being interviewed have previous knowledge that cannot be separated from their essence and constitutes what they are and what they do. In this chapter, knowledge is conceived in a sense similar to Radford: "knowledge […] is considered to be constituted of forms of human action that have become historically and culturally synthesized" (2016, p. 199). Despite this interesting hypothesis, this experience does not offer sufficient data to support this claim. This is just a hypothesis that could be explored in future studies. What is important to highlight is that the researcher's intervention was essential for one of these children. The researcher's question challenged the child to go further in her combination strategies, and even though she did not use a systematic approach, she was able to find four more outfits. This could be explained using the zone of proximal development in which the learner is able to do something unaided but their capacities are potentiated with the help of an adult or a teacher. In other words, "children 'appropriate' knowledge and skills from more expert members of their society" (Fernández, Wegerif, Mercer, & Rojas-Drummond, 2015, p. 55) and "the child develops through participating in the solution of problems with more experienced members of his or her cultural group" (p. 55).

Sandy's initial strategy was very similar to Valery's and Eileen's strategies. Sandy paired up bottoms with tops and she stopped when she ran out of bottoms. It gives the impression that children see an implicit one-to-one correspondence between the shirts' set and the pants' set, and those single elements (without their respective pair) that cause difficulties in such a correspondence are just left out. The three children in this experience, initially, did not consider interchanging the tops to create more counting units. Apparently, young children's enumerative combinatorial counting strategies are very concrete, probably resembling the same counting strategy they use when counting a simple list of discrete elements as it has been mentioned by Shin and Steffe (2009). In enumerative combinatorial counting situations, the counting units are beyond concrete. The child needs to create those new combinatorial counting units, which usually are a challenge for young children. It is worth noticing Sandy's strategy transformation at the end of the interview. Although she was not able to generate all the new counting units from the combinatorial counting situation, she was able to increase the number of combinations compared to her efforts in her first attempt. This increment in the number of counting units was due to the researcher's intervention through the use of stimulating questions. The researcher did not ask leading questions but those asked made the child either think twice about her decisions or conceive the situation from a different approach. This interaction with a more experienced individual contributes to child development and knowledge in the sense stated by Fernández and colleagues:

> the child develops through participating in the solution of problems with more experienced members of his or her cultural group. […] the development of the child towards more able ways of participation in society is carried out through a process of 'guided participation,' which may or may not include explicit teaching. (2015, p. 55)

That young child–teacher interaction could be oriented, taking into account some aspects of the combinatorial counting. According to the level of the child, the teacher might monitor that there are not elements left out in the new counting units; that there are clear intentions for combining all the elements from one collection with all the elements from the other collection; that there is an explicit use of tools to organize the combinatorial counting units like lists, draws, tables, flow diagrams; that there are clear indications of approaches to keep track of the possibilities to avoid repetitions of the combinatorial counting units. In all these situations, the teacher might ask probing questions or explanations. That does not mean that the child will be successful but at least will be challenged without being explicitly taught. Those "interactions give to each child the opportunity to participate in activities and goals that would be very difficult for them to achieve alone" (Fernández et al., 2015, p. 56).

In terms of the contributions for developing multiplicative reasoning in young children, there is a need to incorporate a wide variety of multiplicative situations in instruction. Most multiplicative situations proposed in the school for young children when learning multiplication have the form of repeated addition, direct proportionality, or rectangular arrangements (arrays). However, on very few occasions, are multiplicative situations that resemble the Cartesian product—like the one discussed in this chapter—used in elementary school to orient the work with multiplication. Some authors have stated that in the proportionality situations or rectangular arrangements, the conception of multiplication as a repeated addition is clear; however, this repeated addition is not as clear in situations that require combinatorial counting to reach a solution (Itzcovich et al., 2009). As a result, in order to contribute to the development of multiplicative reasoning early in elementary education, teachers need to propose a variety of situations in which young children could explore different ways to approach multiplication. Multiplicative reasoning cannot be fully developed using primarily (or exclusively) direct proportionality situations that are very straightforward for most young children. Children's multiplicative reasoning needs to be challenged with multiplicative situations that require deep exploration like Cartesian product tasks. This statement holds firm, taking into account the fact that the literature has shown that young children with no instruction in multiplication are able to solve direct proportion multiplicative situations using their previous knowledge (English, 1991; Park & Nunes, 2001). This suggests that teachers and schools have to do something else. Teachers and schooling must challenge young children to go beyond what they can do using their own resources.

In terms of the familiarity, young children have with the tasks, it is crucial to mention that most combinatorial tasks used in research are too formal and abstract for young children to connect with their daily life. In the experience reported here, having a familiar situation was essential for young children to understand and engage in the solution of the task. Proposing tasks that have some familiarity for young children could potentially activate children's informal knowledge to build new knowledge. In this regard, there are scholars who state that children's abilities are better revealed when the proposed tasks are motivating and meaningful (Falk, et al. 2012).

Providing young children with manipulative to support the exploration of the task was vital to keep track of their approaches to get a solution. By using the manipulative provided, the young children were able to visualize, explore, and model different strategies, and the researcher was able to figure out and follow young children's reasoning while they explored the task.

15.6 Conclusions and Implications

This experience shows that young children's combinatorial reasoning could be stimulated from the moment children begin to work with multiplication. It is not necessary to wait until formal combinatorial instruction that usually takes places in secondary education since the formation of the ideas of probability depends on the evolution of combinatorial counting. Teachers could expose young children to exploring and solving different formats of multiplicative situations, focusing not only on those that follow the structure of either direct proportionality or rectangular arrangements but also on those that follow the structure of Cartesian product. Frequently, in elementary school education, the tasks for developing multiplicative reasoning are based on straightforward strategies without the possibility of exploration. Young children need to be challenged with interesting and familiar situations to enhance their capacities.

In this experience, young children sense of fashion came out in the interviews. Young children wanted to combine tops with bottoms attending to the proper coordination of color. This aspect rises up two contradictory reflections. First, the silhouettes used to help young children visualize, explore, model, and solve the combinatorial task seemed to distract children from creating all the combinatorial units. The researcher could be tempted to simplify the material or the task by taking out the context to warrant young children do not get distracted with fashion issues. However, this could take us back to the criticism received in Piagetian tasks that were too unfamiliar for students. Second, the fashion issue is intrinsic to the task proposed in this experience. Most situations students find in their daily life are not clear and cut. Generally, they incarnate the characteristics of a particular context that in some cases could be considered potential distractions. The researcher could keep the task as it is but emphasizing the probing questions on the creation of the counting units more than in the fashion aspect of it. Either decision the researcher makes will leave something crucial out. This experience also reveals that young children's implicit knowledge can be strengthened by creating hands-on tasks that allow them to deal with combinatorial counting situations early on in schooling in a playful, attractive, and familiar way. Since young children are still concrete thinkers, the use of manipulative is always a welcome support in the modeling and exploration of combinatorial situations. To carry out a simple counting activity, the units to be counted are tangible to the child. However, the combinatorial counting activity requires the child to create the new counting units, which is not a simple task. To help young children with this challenge, teachers could complement the combinatorial task with attractive manipulative that help them in the exploration and modeling.

The results from this experience show that whereas young children explore combinatorial tasks, teachers' questions are essential to focus children's attention and to challenge their reasoning. Teachers' questions could have different purposes: close questions (How many ways do you have to dress the doll?), probing questions (How did you do it? Can you show me those outfits? How many outfits do you have then?), or challenging questions that require the children go beyond their actual state (Do you have any other way to dress the doll? What are you going to do with this piece [*the* *top left aside*]? What do you think could happen if we do this [*Pairing up one of the pants with the shirt that has been left alone*]?). This young child–teacher interaction is fundamental for child development. After all, learning is the result of interaction with more experienced members of the cultural group.

In this experience, young children did not use structured strategies to find all the different counting units; however, this does not mean that young children were not ready to engage in combinatorial reasoning. The fragments of young child–adult interaction shown here had the intention to illustrate different ways to challenge young children, but literature has revealed previously that young children could develop efficient and sophisticated strategies with the potential to generate all the possible counting units (English, 1991).

Combinatorial reasoning, although developed slowly, can be favored by simple enumeration combinatorial counting activities. Teacher should promote combinatorial tasks early on in schooling to encourage reflection and problem-solving skills that contribute to the development of combinatorial reasoning. Combinatorial tasks, when accompanied with challenging questions from more experienced members of the cultural group (teachers, researchers, parents), could help young children to confront their primary intuitions and polish their reasoning.

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