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Performance Prediction and Analytics of Fuzzy, Reliability and Queuing Models

Theory and Applications

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Asset Analytics

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Busy Period Analysis of *GI/G/c* and *MAP/G/c* Queues

Srinivas R. Chakravarthy

Abstract The busy period analysis of queueing systems, in general, is very involved and complicated. Even for the simplest queueing model, namely $M/M/1$, the probability density function of the busy period is obtained in terms of modified Bessel function. A number of approaches using complex analysis, combinatorics, lattice path, and matrix-analytic methods have been applied to study some selected queueing models. While the steady-state analysis involving queue length and waiting times of queueing models, in general, has been receiving considerable and significant attention in the literature from both analytical and algorithmic points of view, the same cannot be said (relatively speaking) about busy period analysis. This is inherent in the nature of the busy period more than by choice. In this paper, after establishing the complexity involved in the study of the busy period, we record some interesting observations on the busy period under a wide variety of scenarios through simulation approach. The main purpose is to help researchers to look for novel theoretical and/or numerical approach to solving functional equations which naturally arise in the study of busy periods and use the simulated results here as one of the ways to confirm/validate their results.

Keywords Queueing · Busy period · Matrix-analytic method · Algorithmic probability · Simulation

1 Introduction and Notation

In this paper, we define the busy period (*BP*) to be the duration of the time interval that begins with an arrival of a customer to an empty system and ends with the system becoming empty again at the departure of a customer. This will be the case even for a multi-server queueing system. In the literature (see, e.g., [1, 2]), several authors recourse to full and partial busy periods when dealing with multiple-server system.

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Our definition here for multiple-server system is referred to as partial busy period. A full busy period is the one that starts with all servers becoming busy until at least one server becomes free. Note that in a single-server queueing system, the partial and full busy periods are the same.

The busy period analysis in queueing systems, in general, is very involved and complicated (see, e.g., [3–6]). Even for the simplest queueing model, namely $M/M/1$, the probability density function of the busy period is obtained in terms of modified Bessel function. A number of approaches using complex analysis, combinatorics, lattice path, and matrix-analytic methods have been applied to study some selected queueing models. While the steady-state analysis involving queue length and waiting times of queueing models, in general, has been receiving considerable and significant attention in the literature from both analytical and algorithmic points of view, the same cannot be said (relatively speaking) about busy period analysis. This is inherent in the nature of the busy period more than by choice. In fact, the busy period analysis got a new focus since the introduction of matrix-analytic methods by Neuts [7, 8] in the context of $M/G/1$ and $GI/M/1$ paradigms. In this paper, after establishing the complexity involved in the study of the busy period, we record some interesting observations on the busy period under a wide variety of scenarios through simulation approach.

The purpose of this paper is twofold. First one is to show the complexity involved in the study of the busy period. Secondly, we want to record some interesting observations on the busy period of queueing systems in general context through simulation approach. This will help researchers to look for novel theoretical and/or numerical approach to solving functional equations which naturally arise in the study of busy periods.

In the following, we will denote by $f(.)$ and $F(.)$, respectively, the probability density and probability distribution function of the inter-arrival times. Similarly, we define by $h(.)$ and $H(.)$ to be, respectively, the probability density and probability distribution function of the service times. We will also assume that means of $F(.)$ and $H(.)$ exist and are given by

$$\frac{1}{\lambda} = \int_0^\infty [1 - F(t)]dt \quad \text{and} \quad \frac{1}{\mu} = \int_0^\infty [1 - H(t)]dt, \tag{1}$$

so that λ denotes the rate of arrivals to the system and μ gives the rate of services.

Let Y denote the busy period of the queueing system under study, and let $\Phi(.)$ and $\phi(.)$ denote, respectively, the probability distribution and the density function of Y. We will denote by N_Y the number of customers served during the busy period, Y.

The Laplace–Stieltjes transforms (LST) of $F(.)$, $H(.)$, and $\Phi(.)$ are defined as

$$f^*(s) = \int_0^\infty e^{-st} dF(t),$$

$$h^*(s) = \int_0^\infty e^{-st} dH(t), \tag{2}$$

and

$$\phi^*(s) = \int_0^\infty e^{-st} d\Phi(t), \;\; Re(s) \;\geq 0.$$

The probability generating function, $N(z) = E[z^{N_Y}]$, is

$$N(z) = \sum_{n=1}^{\infty} z^n P(N_Y = n), \;\; |z| < 1. \tag{3}$$

The rest of the paper is organized as follows. In Sect. 2, we present known key results for the busy period for the classical $M/G/1$- and $M/G/1$-type queues. The corresponding known results for the classical $GI/M/1$- and $GI/M/1$-type queues are presented in Sect. 3. In Sect. 4, we look at $GI/G/1$ queues, and in Sect. 5, we look at multi-server queueing systems. A brief summary of some known papers dealing with algorithmic analysis of busy periods is presented in Sect. 6. Validation of our simulated results against some queueing models for which numerical results are reported is done in Sect. 7. Some interesting and illustrative simulated examples are discussed in Sect. 8. Finally, in Sect. 9, some concluding remarks are presented.

2 $M/G/1$-Type Queue

In this section, we will look at the classical (scalar) $M/G/1$- and the $M/G/1$-type queues.

2.1 Classical (Scalar) $M/G/1$ Queue

Here we look at the case of Poisson arrivals, and the service times following a general probability distribution function. That is,

$$F(t) = 1 - e^{-\lambda t}, t \geq 0, \text{ and } f^*(s) = \frac{\lambda}{\lambda + s}, Re(s) \;\geq 0. \tag{4}$$

In this case, Takacs [9] introduced a novel idea in obtaining the LST of the busy period (as it does not depend on the type of service discipline unlike the waiting time distribution) and showed (see also [6]) that $\phi^*(s)$ and $N(z)$ satisfy the following equations.

$$\phi^*(s) = h^*[s + \lambda - \lambda\phi^*(s)], \;\; Re(s) \;\geq 0, \;\; \text{and} \;\; N(z) = h^*[\lambda(1 - N(z)], \;\; |z| < 1. \tag{5}$$

Moments of busy period: Suppose that $\mu_Y^{(k)}$ denotes the kth moment of the busy period and σ_h^2 denotes the variance of the service time. Let $\rho = \frac{\lambda}{\mu}$. The following can be easily verified.

$$\mu_Y^{(1)} = -\frac{d\phi^*(s)}{ds}\Big]_{s=0} = \frac{1}{\mu(1-\rho)},$$

$$\mu_Y^{(2)} = -\frac{d^2\phi^*(s)}{ds^2}\Big]_{s=0} = \frac{\sigma_h^2 + \frac{1}{\mu^2}}{(1-\rho)^3}, \quad \sigma_Y^2 = \frac{\sigma_h^2 + \rho\frac{1}{\mu^2}}{(1-\rho)^3}. \tag{6}$$

The mean and the variance of the number of customers served during a busy period can be verified to be

$$\mu(N_Y) = \frac{1}{1-\rho} \text{ and } \sigma_{N_Y}^2 = \frac{\rho(1-\rho) + \lambda^2(\sigma_h^2 + \frac{1}{\mu^2})}{(1-\rho)^3}. \tag{7}$$

2.2 *M/M/1-Queue*

Here we look at the case of Poisson arrivals, and the service times are exponentially distributed. That is,

$$F(t) = 1 - e^{-\lambda t},\ t \geq 0, \quad H(t) = 1 - e^{-\mu t},\ t \geq 0,$$

$$f^*(s) = \frac{\lambda}{\lambda + s}, \quad h^*(s) = \frac{\mu}{\mu + s}, \ Re(s) \geq 0. \tag{8}$$

It is easy to verify that

(i) The LST of the busy period is given by [9]

$$\begin{aligned}\phi^*(s) &= \frac{\mu}{\mu + s + \lambda(1 - \phi^*(s))} \rightarrow \phi^*(s) \\ &= \frac{1}{2\lambda}\Big[(\lambda + \mu + s) - \sqrt{[(\lambda + \mu + s)^2 - 4\lambda\mu]}\Big].\end{aligned} \tag{9}$$

(ii) $N(z) = z\dfrac{\mu}{\mu + \lambda(1 - N(z))}$.

(iii) The density of the number served can be obtained explicitly as (see, e.g., [6, 9])

$$P(N_Y = n) = \frac{1}{n}\binom{2n-2}{n-1}\rho^{n-1}(1+\rho)^{1-2n},\ n \geq 1. \tag{10}$$

As part of transient analysis of this queueing system, Leguesdron et al. [10] obtained explicit expressions for the probability distribution function of Y and N_Y. Defining

$$\theta = \lambda + \mu,\ a = \frac{\mu}{\theta},\ b = 1 - a, \tag{11}$$

the probability distribution function of Y is obtained using Bessel function as

$$P(Y \le t) = \sum_{n=1}^{\infty} e^{-\theta t} \frac{(\theta t)^k}{k!} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{2k}{k} \frac{a^{k+1} b^k}{k+1},\ t \ge 0, \tag{12}$$

and

$$P(N_Y = n) = \binom{2n-2}{n-1} \frac{a^n b^{n-1}}{n},\ n \ge 1. \tag{13}$$

Note that (10) and (13) are identical, as it should be. Also, we refer to [3, 11–14] for different approaches to getting the LST of the busy period.

2.3 $M/G/1$-Type Queues

Understanding the important role of the busy period in the classical (scalar) $M/G/1$ queue, Neuts generalized it to the $M/G/1$-type queues using matrix formalism. We will briefly summarize the key results pertinent to our discussion here and refer the reader to [7, 8, 15–17] for full details. Consider a Markov renewal process (*MRP*) with transition probability matrix given by

$$Q(x) = \begin{bmatrix} B_0(x) & B_1(x) & B_2(x) & B_3(x) & \cdots \\ A_0(x) & A_1(x) & A_2(x) & A_3(x) & \cdots \\ & A_0(x) & A_1(x) & A_2(x) & \cdots \\ & & A_0(x) & A_1(x) & \cdots \\ & & & A_0(x) & \cdots \\ & & & & \ddots \end{bmatrix}, \tag{14}$$

where the entries are block matrices and govern transitions within each level. While the matrices $A_k(x),\ k \ge 0,\ x \ge 0$, representing possibly defective probability distributions on $[0, \infty)$ are square, the others are rectangular, and in some applications, they may also be square. The matrix, $A(x) = \sum_{i=0}^{\infty} A_i(x)$, is a stochastic semi-Markov matrix, and $A(\infty)$ is a stochastic matrix. Further, $Q(\infty)$ is a stochastic matrix. Note that the levels may represent the number of customers in the system and the auxiliary variable within the level may represent the phase of the service.

Due to simple boundary conditions (see [7, 8] for more complex boundary cases), the fundamental period (to be defined below) will be the busy period as defined in this paper. We will assume that the *MRP* is irreducible, which is the case in most applications.

Let $T(i+r, j; i, k)$ denote the first passage time from state $(i+r, j)$ to the state (i, k), for $i, r \geq 1$, $1 \leq j, k \leq m$. That is, $T(i+r, j; i, k)$ is the duration that the semi-Markov process $Q(.)$ starting in state $(i+r, j)$ visits level $\boldsymbol{i}$ for the first time by entering the state (i, k). Let $V(i+r, j; i, k)$ denote the number of transitions involved in the *MRP* during the first passage time $T(i+r, j; i, k)$.

The joint probability function of $T(i+r, j; i, k)$ and $V(i+r, j; i, k)$ plays a key role in $M/G/1$-type queues. Suppose that we define the matrix $G^{(r)}(n, x)$ such that its (j, k)th entry gives the joint probability as follows (note that these matrices do not depend on i due to the structure of $Q(x)$):

$$g_{j,k}^{(r)}(n, x) = P\{T(i+r, j; i, k) \leq x, V(i+r, j; i, k) = n\},\ r \geq 0, 1 \leq j, k \leq m,\ n \geq 0, \tag{15}$$

where we take

$$g_{j,k}^{(0)}(n, x) = \begin{cases} 1,\ j = k,\ n = 0,\ x = 0, \\ \qquad 0,\ \text{elsewhere}. \end{cases} \tag{16}$$

It can be verified that $G^{(r)}(n, x) = G^r(n, x)$. Also note that $G(n, x) = G^{(1)}(n, x)$, $n \geq 1, x \geq 0$, is such that $g_{j,k}(n, x)$ is the conditional probability that the first passage time from $(i+1, j)$ to (i, k), for $i \geq 1$, $1 \leq j, k \leq m$, occurs in exactly n transitions and no later than x.

Denoting $A_r^*(s)$ to be the LST of $A_r(.)$, the joint LST, $G(z, s)$, defined as

$$G(z, s) = \sum_{n=1}^{\infty} z^n \int_0^{\infty} e^{-sx} dG(n, x),\ |z| \leq 1, Re(s) \geq 0, \tag{17}$$

satisfies [8]

$$G(z, s) = z \sum_{r=0}^{\infty} A_r^*(s) G^r(z, s). \tag{18}$$

The substochastic matrix, $G(x) = \sum_{n=0}^{\infty} G(n, x)$, gives the matrix distribution for the fundamental period. That is, $g_{j,k}(x)$ is the conditional probability that the first passage time from $(i+1, j)$ to (i, k), for $1 \leq j, k \leq m$, occurs no later than x.

The sequence, $\{\widehat{G}(n) = G(n, \infty)\},\ n \geq 1$, of matrices gives the matrix-mass functions for the number of transitions during a fundamental period. That is, $\hat{g}_{j,k}(n)$, $1 \leq j, k \leq m, n \geq 1$, is the conditional probability that exactly n transitions occur during the first passage time from $(i+1, j)$ to (i, k).

The matrix $G = G(\infty) = \sum_{n=1}^{\infty} \widehat{G}(n)$ is such that its (j, k)th entry gives the conditional probability that the *MRP* eventually visits the state (i, k) for the first time starting in state $(i+1, j)$. This matrix plays an important role in $M/G/1$-type queues like R in $GI/M/1$-type queues (see [7, 8, 15]).

3 $GI/M/1$-Type Queues

In this section, we will look at the classical (scalar) $GI/M/1$- and the $GI/M/1$-type queues.

3.1 Classical (Scalar) $GI/M/1$ Queue

Here we look at the general independent and identically distributed inter-arrival times and exponential services. That is,

$$H(t) = 1 - e^{-\mu t},\ t \geq 0,\ h^*(s) = \frac{\mu}{\mu + s},\ Re(s)\ \geq 0. \tag{19}$$

Let $\{t_k : k \geq 0\}$ with $t_0 = 0$ denote the time points at which kth arrival occurs. Thus, for $k \geq 0$, $(t_{k+1} - t_k)$, denotes the duration of the time between kth and $(k+1)$st arrivals. Note that the nonnegative random variable $\tau_{k+1} = t_{k+1} - t_k$ has distribution function $F(.)$ that is independent of k. Let $N(t)$ denote the number of customers in the system at time t, and let $N_k = N(t_k - 0)$, $k \geq 0$, denote the number of customers in the system just before the kth arrival.

The process $\{(N_n, \tau_n) : n \geq 0\}$ is a Markov renewal process (*MRP*) with *TPM* given by

$$Q(x) = \begin{pmatrix} b_0(x) & a_0(x) & & & \cdots \\ b_1(x) & a_1(x) & a_0(x) & & \cdots \\ b_2(x) & a_2(x) & a_1(x) & a_0(x) & \cdots \\ b_3(x) & a_3(x) & a_2(x) & a_1(x) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{20}$$

where $a_k(x) = \int_0^x e^{-\mu t} \frac{(\mu t)^k}{k!} dF(t),\ k \geq 0$, gives the probability that k departures (or service completions) occur during an arrival that occurs at or before time x, and $b_k(x) = 1 - \sum_{i=0}^k a_i(x), k \geq 0,\ t \geq 0$.

Thus, we need to look at the imbedded Markov chain to study $GI/M/1$ queue.

For use in sequel, we define the taboo probability, ${}_i p_{i,i+1}^{(k)}(t)$, as the conditional probability that the *MRP* with *TPM* given in (20) starting in state i at time 0 makes k transitions by avoiding state i and visits state $i+1$ at the kth transition which occurs no later than time t. Note that due to the structure of the *TPM*, this conditional probability does not depend on i.

Define $r(t),\ t \geq 0$, as

$$r(t) = \sum_{k=1}^{\infty} {}_i p_{i,i+1}^{(k)}(t). \tag{21}$$

Note that $r(t)$ gives the expected number of visits to state $i+1$ during $\min\{t, \tau\}$ units of time before first return to state i given that the *MRP* started in state i. Here τ is the epoch at which the first return to state i occurs starting from state i.

Then it is easy to see (see, e.g., [9, 18]) that $\phi^*(s)$ satisfies

$$\phi^*(s) = \frac{\mu(1 - r^*(s))}{s + \mu(1 - r^*(s))}, \; Re(s) \; \geq 0, \tag{22}$$

where $r^*(s)$ is obtained as the solution to

$$r^*(s) = \sum_{n=0}^{\infty} (r^*(s))^n \int_0^{\infty} e^{-(s+\mu)t} \frac{(\mu t)^n}{n!} dF(t) = f^*[s + \mu(1 - r^*(s))], \; Re(s) \; \geq 0. \tag{23}$$

3.2 *M/M/1-Queue*

We will now revisit $M/M/1$ queue via $GI/M/1$ approach. Here we take $F(t) = 1 - e^{-\lambda t}], \; t \geq 0$. Then, we have

(i) $f^*(s) = \frac{\lambda}{s+\lambda}$
(ii) It is easy to verify that

$$\begin{aligned} r^*(s) &= \frac{\lambda}{\lambda + s + \mu(1 - r^*(s))} \rightarrow r^*(s) \\ &= \frac{1}{2\mu}\Big[(\lambda + \mu + s) - \sqrt{[(\lambda + \mu + s)^2 - 4\lambda\mu]}\Big], \end{aligned} \tag{24}$$

from which it can be seen that

$$\begin{aligned} \phi^*(s) &= \frac{\mu}{\lambda} r^*(s) \rightarrow \phi^*(s) \\ &= \frac{1}{2\lambda}\Big[(\lambda + \mu + s) - \sqrt{[(\lambda + \mu + s)^2 - 4\lambda\mu]}\Big], \end{aligned} \tag{25}$$

which (as it should be) is same as the one we got earlier [see (9)].

3.3 *GI/M/1-Type Queues*

Here we will look at the $GI/M/1$-type queues that were introduced and studied extensively by Neuts [7]. Consider a Markov renewal process (*MRP*) with transition probability matrix given by

$$Q(x) = \begin{bmatrix} B_0(x) & A_0(x) & & & \cdots \\ B_1(x) & A_1(x) & A_0(x) & & \cdots \\ B_2(x) & A_2(x) & A_1(x) & A_0(x) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{26}$$

where the matrices $A_n(x)$ and $B_n(x)$, for $n \geq 0$ and for $x \geq 0$, are square matrices of order m representing possibly defective probability distributions on [0, ∞). Further, $Q(\infty)$ is a stochastic matrix.

The *MRP* of the type given in (26) occurs naturally in many stochastic models including the well-known ones such as $GI/PH/1$ and $SM/M/1$. Also, the one given in (26) has a simple boundary condition and more complex boundary conditions also occur naturally, and we refer the reader to [7] for more details.

Before we proceed further, we will summarize some key concepts needed for discussion in the sequel.

Recall that if P is the *TPM* of a *DTMC*, then the (i, j)th entry of the matrix $\sum_{n=0}^{\infty} P^n$ gives the expected number of visits to state j starting in state i.

Suppose that H is a subset of the state space, Δ, of a *DTMC*. The taboo probability denoted by ${}_H p_{i,j}^{(n)}$ is the probability that starting in state i, the *DTMC* visits state j at time n without visiting any of the states in H. The taboo probabilities play an important in stochastic modeling, and Neuts used this concept extensively in the development of matrix-analytic methods.

Referring to the *TPM* given in (26), we define the level (or set) of states as follows.

$$\boldsymbol{i} = \{(i, j) : 1 \leq j \leq m\}. \tag{27}$$

Suppose we define the matrix $R^{(k)}(t) = \{(r_{j,l}^{(k)}(t))\},\ 1 \leq j, l \leq m, k \geq 0$, such that (by convention we take $R^{(0)}(t) = I$, an identity matrix of dimension m)

$$r_{j,l}^{(k)}(t) = \sum_{n=0}^{\infty} {}_{\boldsymbol{i}} p_{(i,j),(i+k,l)(t)}^{(n)}. \tag{28}$$

Note that ${}_{\boldsymbol{i}} p_{(i,j),(i+k,l)}^{(n)}(t)$ gives the conditional probability that the *MRP*, $Q(.)$, starting at time 0 in state (i, j) makes n transitions during $(0, t]$ by avoiding the level $\boldsymbol{i}$ and that the nth transition results in *MRP* being in state $(i + k, l)$. Observe from the structure of $Q(.)$, the taboo probabilities do not depend on i but rather depend only on the submatrix obtained from Q by deleting all rows and columns with indices $(l', j'), l' \leq i, 1 \leq j' \leq m$. Also, note that these submatrices are identical for all $i \geq 0$.

Also note that $r_{j,l}^{(k)}(t)$ gives the expected number of visits during the time interval $(0, \min(t, \tau)]$ (here τ is the epoch of the first return to level $\boldsymbol{i}$) to the state $(i + k, l)$ starting in state (i, j) before the first return to level $\boldsymbol{i}$.

We will denote $R^{(1)}(t)$ by $R(t)$. Let $R^*(s)$ denote the LST of $R(t)$. Suppose that $A_n^*(s)$ denote the LST of $A_n(x)$. Then, it is known (see [18]) that $R(s)$ is the minimal nonnegative solution (in the class of all such matrices whose spectral radius is at most one) to the matrix nonlinear equation:

$$R^*(s) = \sum_{n=0}^{\infty} (R^*(s))^n A_n^*(s), \ \ Re(s) \geq 0. \tag{29}$$

This matrix, $R^*(0) = R(\infty)$, is the well-known rate matrix (see [7]).

3.4 $GI/PH/1$ Queue

As seen in earlier sections, the busy period analysis tends to be very complicated. The first paper, to our knowledge, that dealt with non-exponential services is that of Conolly [19] who studied $GI/E_k/1$ queue. The methodology (based on difference equations) used in [19] is that of the one applied by the same author [20] in the context of $GI/M/1$ queue.

After more than three decades since the study of $GI/E_k/1$ queue, Ramaswami [18] used probabilistic arguments (developed and popularized by Neuts through his matrix-analytic methods) to develop transform-free methods for analyzing busy period for a large class of queues that possess matrix-geometric steady-state probability vector. We will briefly outline the key steps here and refer the reader to Ramaswami [18] for full details.

Let the service times be of phase type (PH) with representation $(\boldsymbol{\beta}, S)$ of order n (see [7]). Recall that the service rate is given by $\mu = [\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}]^{-1}$. In the sequel, we will assume that the queue is stable implying that $\lambda < \mu$. The probability density function, the distribution function, and the LST of service times are given by

$$h(t) = \boldsymbol{\beta} e^{St}\boldsymbol{S^0}, \ \ H(t) = 1 - \boldsymbol{\beta} e^{St}\boldsymbol{e}, t \geq 0, \ \ h^*(s) = \boldsymbol{\beta}(sI - S)^{-1}\boldsymbol{S^0}, \ Re(s) \geq 0. \tag{30}$$

Suppose that $\{N(t)\}$ denotes the counting process associated with a PH-renewal process. That is, $\{N(t)\}$ denotes the number of renewals if the times between renewals follow a PH-distribution with representation $(\boldsymbol{\beta}, S)$. Define the matrix $P(n, t)$, whose (i, j)th element is given by $p_{ij}(n, t) = P[N(t) = n, J(t) = j|N(0) = 0, J(0) = i]$. That matrix $P(n, t)$ satisfies (see, e.g., [7])

$$P'(0, t) = P(0, t)S, \ \ P'(n, t) = P(n, t)S + P(n - 1, t)\boldsymbol{S^0}\boldsymbol{\beta}, \ \ n \geq 1, \ t \geq 0, \tag{31}$$

with $P(n, 0) = \delta_{n0}I$, where δ_{n0} is the Kronecker delta.

The matrices, $A_k(t)$ and $B_k(t)$, in the $GI/PH/1$ case are given by

$$A_k(t) = \int_0^t P(k,u)dF(u),\ k \geq 0,\quad B_k(t) = \sum_{r=k+1}^{\infty} \int_0^t P(k,u)\boldsymbol{e}\boldsymbol{\beta} dF(u),\ k \geq 0. \tag{32}$$

Suppose that the $(i,\ j)$th element of the matrix $C_k(t)$ of dimension m denotes the conditional probability that the busy period starts at time 0 with an arrival which sees k customers already present in the system and the service phase at time 0 is i, the busy period ends by time t with exactly $k+1$ services, and that the next busy period starts in phase j. It can be verified (see, e.g., [18]) that

$$C_k(t) = \int_0^t P(k,u)\boldsymbol{e}\boldsymbol{\beta}[1 - F(u)]du,\ k \geq 0. \tag{33}$$

Let $C_n^*(s)$ denote the LST of $C_k(t)$. One can get $C_n^*(s)$ in terms of $A_k^*(s)$ as follows (see, e.g., [18])

$$C_0^*(s) = [I - A_0^*(s)](sI - S)^{-1}\boldsymbol{S^0}\boldsymbol{\beta},$$

$$C_k^*(s) = [C_{k-1}^*(s) - A_k^*(s)](sI - S)^{-1}\boldsymbol{S^0}\boldsymbol{\beta},\ k \geq 1. \tag{34}$$

Let $g_i(t)$, $1 \leq i \leq m$, denote the distribution function of the busy period that starts with an arrival of a customer whose service phase will be in state i. Defining $\boldsymbol{g(t)}$ to be the vector of dimension m whose ith component is given by $g_i(t)$ and $\boldsymbol{g^*(s)}$ to be the LST of $\boldsymbol{g(t)}$, we register the explicit expression for the (vector) LST of the busy period distribution as obtained by Ramaswami [18] as follows.

$$\boldsymbol{g^*(s)} = \sum_{n=0}^{\infty}[R^*(s)]^n C_n^*(s)\boldsymbol{e} = [I - R^*(s)][I - h^*(s)R^*(s)]^{-1}(sI - S)^{-1}\boldsymbol{S^0},\ Re(s) \geq 0. \tag{35}$$

We will look at the simplifications of Ramaswami's expression given in (35) for two special queueing models.

3.4.1 $GI/M/1$ Queue

In this case, the expression given in (35) reduces to (note that $g^*(s)$ and $R^*(s)$ are scalars)

$$g^*(s) = \frac{\mu(1 - R^*(s))}{s + \mu(1 - R^*(s))},\ \ Re(s)\ \geq 0, \tag{36}$$

and $R^*(s)$ is obtained as the solution to

$$R^*(s) = \sum_{n=0}^{\infty} (R^*(s))^n \int_0^\infty e^{-(s+\mu)t} \frac{(\mu t)^n}{n!} dF(t) = f^*[s + \mu(1 - R^*(s))], \; Re(s) \geq 0, \tag{37}$$

which agrees with (23).

3.4.2 $M/M/1$ Queue

In this case, verify that

(i) $f^*(s) = \frac{\lambda}{s+\lambda}$
(ii) $R^*(s)$ is obtained as the solution to

$$R^*(s) = \frac{\lambda}{\lambda + s + \mu(1 - R^*(s))} \rightarrow R^*(s) = \frac{1}{2\mu}\Big[(\lambda + \mu + s) - \sqrt{[(\lambda + \mu + s)^2 - 4\lambda\mu]}\Big], \tag{38}$$

from which it can be seen that

$$g^*(s) = \frac{\mu}{\lambda} R^*(s) \rightarrow g^*(s) = \frac{1}{2\lambda}\Big[(\lambda + \mu + s) - \sqrt{[(\lambda + \mu + s)^2 - 4\lambda\mu]}\Big], \tag{39}$$

which agrees with (9).

4 $GI/G/1$-Queue

The earliest known study on the busy period of $GI/G/1$ queue is by Finch [21] in which expressions for the LST of the busy period and the density of the number served during a busy period are obtained using combinatorial approach. The expressions are not only complicated but also numerically unstable due to alternating signs appearing in those expressions. For the sake of completeness, we will reproduce the results (just to show the complexity involved in the busy period analysis), and for details, we refer the reader to [21].

Suppose that $\xi_n(t), \; n \geq 1, \; t \geq 0$ denotes the joint probability of Y and N_Y such that

$$\xi_n(t) = P(Y \leq t, N_y = n), \; n \geq 1, \; t \geq 0, \tag{40}$$

and $\xi_n^*(s)$ denote the LST of $\xi_n(t)$. Denoting by $F^{(n)}(.)$ to be the n-fold convolution of $F(.)$ with itself, and $H^{(n)}(.)$ to be that of $H(.)$, and

$$a_n^*(s) = \int_0^\infty e^{-st}[1 - F^{(n)}(t)]dH^{(n)}(t), \; n \geq 1, \; Re(s) \geq 0, \tag{41}$$

Finch showed that the LST of the joint probability function, $\xi_n(t)$, to be

$$\xi_n^*(s) = \sum \frac{(-1)^{k_1+k_2+\cdots+\,k_n+1}}{k_1!k_2!\cdots k_n!}\left(a_1^*(s)\right)^{k_1}\left(\frac{a_2^*(s)}{2}\right)^{k_2}\cdots\left(\frac{a_n^*(s)}{n}\right)^{k_n}, \quad (42)$$

where $n \geq 1,\ Re(s) \geq 0$ and the summation is over all nonnegative integers k_r such that $\sum_{r=1}^{n} rk_r = n$.

From (42), it can be verified that, for $n \geq 1,\ Re(s) \geq 0$,

$$P(N_Y = n) = \xi_n^*(0) = \sum \frac{(-1)^{k_1+k_2+\cdots+\,k_n+1}}{k_1!k_2!\cdots k_n!}\left(a_1\right)^{k_1}\left(\frac{a_2}{2}\right)^{k_2}\cdots\left(\frac{a_n}{n}\right)^{k_n}, \quad (43)$$

where $a_n = a_n^*(0) = \int_0^\infty [1 - F^{(n)}(t)]dH^{(n)}(t),\ n \geq 1.$

Bertsimas et al. [22] analyzed the busy period by formulating it as Hilbert factorization problem using two-dimensional Lindley process. Subsequently, Bertismas and Nakazato [23] applied the method of stages by assuming the arrivals and services are of mixed generalized Erlang distributions and obtained the LST for the busy period.

It should be noted that Pakes [24], using duality results for $GI/G/1$ queue, derived expressions for the (a) probability of number served during a busy period of a $GI/G/1$ queue and (b) LST of $GI/M/1$ queue in which the first customer starting the busy period has a different service-time distribution compared to the other customers in that busy period.

In Baltrunas et al. [25, 26], the authors study the tail behavior of the busy period of a stable $GI/G/1$ queue with subexponential services.

5 Multi-server Queues

So far, we looked at single-server queueing systems under various scenarios. In this section, we will briefly summarize a few models dealing with multi-server case for which results are reported.

- Chae and Lim [27] derive the joint transform of the length of a busy period, the number of customers served during the busy period, and the remaining inter-arrival times at the instant the busy period ends for $GI/M/c$ queue with n-policy. By taking $n = 1$, their model reduces to the classical $GI/M/c$ queue. Some numerical results are reported.
- Natvig [28] derives the first- and second-order moments of the (partial) busy period as well as the distribution of the number of customers served by looking at a general birth-and-death queueing model with multiple servers.

– Omahen and Marathe [29] apply the technique of decomposition of busy periods to $M/M/c$ queueing system and derive recursive formulas for computing the LST of the (partial) busy period as well as the first two moments of the busy period.
– Artalejo and Lopez-Herrero [1] present an algorithmic analysis of the busy period in the context of $M/M/c$ queueing model. They obtain the LST as the solution of a finite system of linear equations. Further, they provide recurrent relations for computing the moments of the distribution of the length of the busy period as well as the number of customers served during a busy period. Some numerical examples are presented.
– Ghahramani and Wolff [30] provide a probabilistic proof for conditions that will guarantee (full) busy periods to have finite moments.

6 Algorithmic Analysis of Busy Periods

With a genuine concern for algorithmic feasibility of solutions of stochastic models to be useful in practice, Neuts [7] developed phase-type distribution, (batch) Markovian arrival processes, matrix-geometric methods, and later on matrix-analytic methods in stochastic modeling. In queueing theory, for what Neuts will be remembered (among many things) most in years to come, is the introduction and the development of the matrix-analytic methods (*MAMs*) for the solution of a wide variety of practical problems.

In the invited article published by European Journal of Operational Research [31], Neuts passionately says, "...The history of the matrix methods (so called for brevity) is short, but worth telling... I tackled a number of models involving embedded Markov renewal processes, evidently with some measure of success, since the papers were published in noted journals and some academic recognition came my way. It privately bothered me that, as the papers grew longer and the analysis more complex, the explicit or qualitative results in them became fewer and fewer." He continues further, "...In the history of mathematics, a similarity of formalism has always indicated similarity of structure and an ultimate level of understanding is that of unifying structure."

Stochastic modeling occurs naturally in many walks of life. The mathematical tools needed to solve a specific problem vary depending on the application of the stochastic model. Telecommunications area first adopted $MAMs$ soon after their introduction. Its well-known self-similarity property is seen in ethernet traffic, WWW traffic, signaling traffic, multi-media traffic, and other high-speed network traffic, and hence, Poisson/exponential distributions are not well suited for modeling such traffic. The benefit of using Markov-modulated Poisson process (which is a special class of Markovian arrival processes (MAP)) has been well documented.

Since the introduction of $MAMs$ by Neuts, the users of this methodology have grown from within queueing community to other areas due to solutions that are transparent, implementable, and probabilistic in nature. The practitioners benefiting from Neuts' contributions come from various fields such as health care, computer and communications engineering, production and manufacturing, industrial engineering, electrical engineering, actuarial science, transportation, and wireless sensor networks. Neuts' work has inspired many researchers from all over the world as can be seen from constant publications of research papers dealing with $MAMs$ in many diverse fields of applications.

While the steady-state analysis of many queueing models under a variety of scenarios can be performed both analytically and algorithmically using $MAMs$, only very few papers exist that address the busy period analysis in the same spirit. Again, this is inherent in the complexity of the busy period analysis as opposed to the choice of the methodology. As seen in previous sections, the busy period analysis is very complex. Even for the simplest queue in continuous time, namely $M/M/1$, the expression for the probability distribution function of the busy period is quite complex. Due to this complexity, many authors have resorted to deriving expressions, some of which are not algorithmically suited under a wide variety of parameters of the queueing model under study. Since most expressions are given in terms of LST, Abate and Whitt [32, 33] pioneered the numerical methods for computing LST expressions in the context of queueing models.

Some notable early papers, in addition to the ones mentioned earlier, dealing with computational aspects of the busy period include the following.

- Conolly [20] applied difference equations technique to derive probability distributions associated with the busy period for $GI/M/1$ queue and performed a few numerical comparisons in the case of $M/M/1$ and $D/M/1$ queues.
- Abate et.al. [34] show that the probability density of the busy period can be numerically inverted without the need to use iterative procedure for solving Kendall's functional equation. They apply their technique to $M/G/1$ queue with gamma service-time distribution.
- Garikiparthi et al. [35] derive the joint LST for the busy period and the number of customers served during a busy period for a finite QBD-process and propose algorithms for computing the moments of the busy period and the number served. Also, illustrative examples are presented.
- Artalejo and Gomez-Corral [36] using catastrophe method derived the LST of the busy period for an $M/G/1$ retrial queue with finite orbit size and discuss illustrative examples with the help of numerical inversion of transforms.
- Using lattice path approach, the authors in [37–40] study the busy period. They also discuss illustrative numerical examples for special cases under a wide range of values for the parameters of the model.
- In the context of $M/E_k/1$ queue, Baek et al. [41] give a closed form expression for the queue length within a busy period and discuss some illustrative numerical examples. It should be pointed out that the authors do not give expressions for the density of the busy period but rather the queue length during that busy period.
- Novak et al. [42, 43] provide analysis of the distribution of the number of arrivals in a subinterval of a busy period of an $M/D/1$ queue as well as for $M/G/1$ queue using Takacs' ballot theorem and its generalization. The authors compare simulated results to the actual density function.

– Assuming the inter-arrival and service times of a customer are correlated, Langaris [44] gives a closed form expression for the joint transform of the busy period and the number served during that period and discuss a few numerical results.

7 Validation of the Simulated Model

In the literature, to the best of our knowledge, there are very few exact and approximate results for the tail probabilities as well as for complementary distribution function for the busy period are available and that too for limited queueing models. We will use these results to validate/compare our simulated results.

We used ARENA [45] to get our simulated results for the busy period for various queueing systems. We simulated the model using 5 replications and for 1,000,000 units (which in our case is minutes) for each replicate.

7.1 *Abate and Whitt for $M/G/1$ Queueing Model*

In this section, we will compare the results given in Abate and Whitt [33] wherein the authors provide exact (through transform inversion) tail probabilities for various time points. These time points depend on the type of queueing model as well as the value of ρ. They fix the mean service time to be 1 in all cases and take the arrival rate so as to obtain a given value for ρ. Two queueing models, $M/E_4/1$ and $M/\Gamma(2, 0.5)/1$, are considered in [33] under three scenarios by varying ρ. We denote by $\Gamma(\alpha, \beta)$, a gamma distribution with shape parameter given by α and the scale parameter is β. Recall that the density function, $f(t)$, of $\Gamma(\alpha, \beta)$ is given by

$$f(t) = \frac{1}{\beta^\alpha \Gamma(\alpha)} t^{\alpha-1} e^{-t/\beta}, \ \alpha > 0, \beta > 0, \ t \geq 0,$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

In Table 1, we display the (absolute) error percentages of $P(Y > t)$ by comparing the results given in [33] with our simulated ones. The absolute error percentage, here and elsewhere, is defined as

$$100 \left| \frac{\text{reported} - \text{simulated}}{\text{reported}} \right| \%.$$

By looking at the entries in Table 1, we notice that, in general, our simulated results agree very well with the ones reported in [33]. The ones that have somewhat large error percentages (e.g., the largest one is 7.28%) we noticed that the underlying probabilities are very small and generally the values differ in the fifth or sixth decimal

Table 1 Absolute error percentages for $M/G/1$ queue

$\rho =$ 0.50	t	0.5	1	2	3	5	9	12	15	20	32
	$M/E_4/1$	0.51%	0.15%	0.1%	0%	0.15%	0.21%	1.19%	1.27%	0.55%	5.14%
	t	0.1	1	2	5	8	15	20	30	40	60
	$M/\Gamma(2, 0.5)/1$	6.46%	0.6%	1.77%	3.3%	3.28%	2.29%	1.42%	1.77%	3.65%	13.4%
$\rho =$ 0.75	t	0.5	1	2	6	10	15	30	40	80	120
	$M/E_4/1$	0.36%	0.09%	0.03%	0.06%	0.22%	0.41%	1.22%	0.25%	1.12%	6.75%
	t	0.1	1	5	8	15	30	60	80	120	250
	$M/\Gamma(2, 0.5)/1$	6.39%	1.76%	1.58%	1.98%	1.57%	1.35%	0.55%	1.07%	1.54%	6.39%
$\rho =$ 0.90	t	0.5	1	5	15	30	60	120	200	400	600
	$M/E_4/1$	0.36%	0.04%	0.04%	0.32%	0.05%	0.19%	1.52%	0.93%	2.72%	3.49%
	t	0.1	1	5	10	20	50	100	250	500	1000
	$M/\Gamma(2, 0.5)/1$	6.31%	1.58%	2.19%	2.23%	2.24%	1.55%	1.68%	2.13%	6.41%	7.28%

places. For example, corresponding to the scenario, $\rho = 0.9$, $M/\Gamma(2, 0.5)/1$, the simulated value for $P(Y > 32)$ is 0.00043577, whereas the reported value (see [33]) is 0.000470. So, the actual (absolute) difference is 0.00003423, which is a small number, but the percentage-wise it results in 7.28%. This is the case for other percentages too.

7.2 *Adan and Resing for* $M/M/1$ *Queueing Model*

In their book on queueing theory, Adan and Resing [46] report results on selected tail probabilities for $M/M/1$ queue by considering three values for the traffic intensities, $\rho = 0.8, 0.9, 0.95$, and fixing the mean service time to be 1. The tail probabilities are reported using two decimal places, and hence, we did the same so as to compare the results properly. In Table 2, we display the error percentages of our simulated results compared to the ones in [46]. Obviously, we notice that the results agree very much in all cases. In fact, only one error percentage is different from zero.

Table 2 Absolute error percentages for $M/M/1$

t	1	2	4	8	16	40	80
$\rho = 0.80$	0%	0%	0%	0%	0%	0%	0%
$\rho = 0.90$	0%	0%	0%	0%	0%	0%	0%
$\rho = 0.95$	0%	2.7%	0%	0%	0%	0%	0%

Table 3 Absolute error percentages for $M/M/5$

ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
μ_Y	0.02%	0.13%	0.10%	0.11%	0.14%	0.17%	0.02%	0.07%	0.31%
$\frac{\sigma_Y}{\mu_Y}$	0.17%	0.16%	0.06%	0.02%	0.07%	0.01%	0.23%	0.23%	0.08%
$E(N_Y)$	0.10%	0.08%	0.09%	0.11%	0.16%	0.16%	0.04%	0.07%	0.32%
$\frac{\sigma_{N_Y}}{E(N_Y)}$	0.21%	0.11%	0.05%	0.08%	0.05%	0.04%	0.27%	0.25%	0.06%

7.3 Artalejo and Lopez-Herrero for $M/M/5$ Queueing Model

Here we look at a multi-server system. Artalejo and Lopez-Herrero [1] consider $M/M/5$ queueing system by fixing the mean service time to be 1 and vary ρ from 0.1 through 0.9. Unlike the previous sections, the authors here report the various moments of the busy period and the number of customers served during a busy period. In Table 3, we display the (absolute) error percentages for the (a) mean busy period, μ_Y; (b) coefficient of variation of the busy period, $\frac{\sigma_Y}{\mu_Y}$; (c) mean number of customers served during a busy period, $E(N_Y)$; and (d) coefficient of variation of the number served during a busy period, $\frac{\sigma_{N_Y}}{E(N_Y)}$.

It is clear from the entries in the above table that simulated results agree very well with the ones reported in [1].

7.4 Blanc for $M/G/1$ and $GI/M/1$ Queueing Models

In [47], the author replaces the original contour integral involving implicitly known functions with alternative contour integrals with known transforms and numerically inverts the transforms. The author provides numerical examples for $GI/M/1$ queueing model by considering Erlang and gamma arrivals and $M/G/1$ queueing model by considering Erlang and gamma distributions for services. For all scenarios, the arrival rate is fixed at 0.8 and the service rate to be 1, so that the traffic intensity is set at $\rho = 0.8$. In Table 4, we display the (absolute) error percentages of our simulated results with the ones provided in [47].

A quick look at the entries in Table 4 indicates that generally our simulated results seem to be closer to the ones reported in [47] in all scenarios except for a few scenarios involving $M/\Gamma(8, 0.125)/1$ and $\Gamma(10, 0.125)/M/1$ queues. For example, for the $\Gamma(8, 0.125)$ services, we notice the error percentages to be very large for a few tail probabilities. We increased the simulation run for each replicate from 1,000,000 units to 10,000,000 units to see whether a longer run is warranted and thus possibly explain the higher values for the error percentages. That increase in the simulation run pretty much yielded the same results. In other comparisons (see Tables 1 and 5) involving service distributions having gamma distribution, we never noticed that

Table 4 Absolute error percentages for $M/G/1$ and $GI/M/1$ queues

	t	0.1	0.5	1	2	5	10	25	50	100
$M/G/1$	$M/E_8/1$	0.00%	0.06%	0.16%	0.11%	0.05%	0.00%	0.26%	0.88%	5.26%
	$M/E_2/1$	0.79%	0.39%	0.17%	0.13%	0.10%	0.44%	0.72%	1.36%	0.00%
	$M/M/1$	0.00%	0.02%	0.02%	0.00%	0.11%	0.46%	0.23%	0.00%	0.00%
	$M/\Gamma(2, 0.5)/1$	6.38%	3.82%	1.61%	0.07%	1.61%	2.03%	3.37%	3.90%	2.74%
	$M/\Gamma(8, 0.125)/1$	43.95%	27.62%	18.51%	9.84%	1.57%	2.63%	6.76%	8.02%	8.33%
$G/M/1$	$E_8/M/1$	0.02%	0.21%	0.10%	0.04%	0.40%	0.76%	1.36%	0.00%	0.00%
	$E_2/M/1$	0.02%	0.14%	0.04%	0.00%	0.00%	0.12%	0.95%	0.88%	0.00%
	$M/M/1$	0.00%	0.02%	0.02%	0.00%	0.11%	0.46%	0.23%	0.00%	0.00%
	$\Gamma(2.5, 0.5)/M/1$	0.48%	1.95%	2.38%	2.28%	1.20%	0.68%	0.15%	0.00%	2.78%
	$\Gamma(10, 0.125)/M/1$	2.18%	6.56%	9.08%	11.22%	11.64%	10.66%	8.62%	7.70%	8.11%

Table 5 Absolute error percentages for $\Gamma(5, 0.4)/M/c$

c	μ_Y	σ_Y	$E(N_Y)$	σ_{N_Y}
1	0.06%	0.21%	0.00%	0.34%
2	0.08%	0.17%	0.04%	0.03%
3	0.11%	0.09%	0.00%	0.00%

high error percentages making us believe that the results in [47] may need to be checked for possible numerical inversion problems.

7.5 Chae and Lim for $GI/M/c$ Queueing Model

Looking at $GI/M/c$ queue, Chae and Lim [27] present numerical results by considering inter-arrivals to follow a gamma distribution with shape parameter to be 5 and scale parameter to be 0.4 so that the mean time between arrivals is 2 and the mean service time to be 1. They vary c from 1 to 3 and present the mean and standard deviation of the (a) busy period and (b) number served during a busy period. In Table 5, we display the (absolute) error percentages of our simulated results with the ones reported in [27].

Once again, the entries in Table 5 reveal that the simulated results agree very well with the reported ones.

8 Illustrative Examples Based on Simulation

In this section, we will discuss a few illustrative and interesting examples obtained through simulation. We look at several classes of queueing systems of the type $GI/G/c$ and $MAP/G/c$. In all our simulation examples, we used 5 replicates and each replicate of length 1,000,000 units (in our case, minutes). Unless otherwise specified, we will fix the service mean to be 1. That is, we fix $\mu = 1$, and vary λ so that a given traffic intensity, $\rho = \frac{\lambda}{c\mu}$, is achieved.

8.1 $GI/G/c$

Here we look at four scenarios for arrival processes, Erlang, Poisson, hyperexponential, and two-parameter Weibull. For ease of reference, we list the type of arrival processes (TAP) and the type of service times (TS) with appropriate labels.

TAP 1: **Erlang** (*ERA*): Here we consider an Erlang distribution of order 2 with rate 2λ.
TAP 2: **Exponential** (*EXA*): This corresponds to the classical Poisson process with rate λ.
TAP 3: **Hyperexponential** (*HEA*): We look at a mixture of two exponentials with rates 1.9λ and 0.19λ, respectively, with probabilities 0.9 and 0.1.
TAP 4: **Weibull** (WEA): We consider a two-parameter Weibull whose CDF is given by

$$F_{WB}(x) = \begin{cases} 1 - e^{-(0.5\lambda x)^{0.5}}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

TS 1: **Erlang** (*ERS*): This is Erlang of order 2 with rate 2 in each stage.
TS 2: **Exponential** (*EXS*): This is an exponential distribution with mean 1.
TS 3: **Hyperexponential** (*HES*): Here we look at mixture of two exponentials with rates 1.9 and 0.19, respectively, with mixing probabilities 0.9 and 0.1.
TS 4: **Shifted exponential** (*SXP*): The shifted exponential with a shift of magnitude 0.2 one with *CDF* given by

$$F_{SE}(x) = \begin{cases} 1 - e^{-1.25(x-0.2)}, & x \geq 0.2, \\ 0, & x < 0.2. \end{cases}$$

TS 5: **Weibull** (*WES*): We consider a two-parameter Weibull whose *CDF* is given by

$$F_{WB}(x) = \begin{cases} 1 - e^{-(2x)^{0.5}}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

First note that the coefficient of variation of the four arrival processes labeled *ERA*, *EXA*, *HEA* and *WBA* are, respectively, 0.7071, 1, 2.2447, and 2.2361. Similarly, the coefficient of variation of the 5 service times labeled *ERS*, *EXS*, *HES*, *SXP*, and *WES* are, respectively, 0.7071, 1, 2.2447, 0.8, and 2.2361.

Recall from queueing literature (see, e.g., [3, 5, 6]) that the mean waiting time in the system is known to increase as the variability in the arrival (or services) increases (assuming that all other parameters are fixed). So, we decided to explore whether such a behavior is seen for the mean busy period.

In Table 6, we display the mean busy period under various scenarios. A quick look at the table reveals the following interesting observations.

- While for $c = 1$ and $c = 2$, we notice that the mean busy period appears to increase with increasing variability in the arrival processes, for $c = 5$, we notice a different trend. For example, by looking at the mean busy period when there are 5 servers in the system, a larger variability in the services such as *HES* appears to yield a smaller value as compared to ERS which has a smaller variability. This appears to be the case for $\rho = 0.80$ and $\rho = 0.95$.
- We notice the mean busy period to be very large when $c = 5$. This is not surprising since we use partial busy period in that the busy period starts when an arriving

Table 6 Mean busy period time for $GI/G/c$ queue

		$\rho = 0.80$				$\rho = 0.95$			
c	TS	ERA	EXA	HEA	WBA	ERA	EXA	HEA	WBA
1	ERS	3.644	4.978	14.795	14.261	14.037	20.117	68.954	64.184
	EXS	3.843	4.991	13.001	13.207	14.827	19.873	60.423	58.397
	HES	4.067	4.991	9.738	11.422	16.297	20.305	41.998	49.159
	SXP	3.679	5.016	14.337	14.084	14.267	20.168	66.311	64.772
	WBS	4.217	5.012	9.039	10.220	17.125	20.174	38.821	43.708
2	ERS	4.462	5.129	11.913	10.466	18.233	20.448	50.256	46.046
	EXS	4.313	5.004	10.606	10.014	16.922	19.839	45.208	44.887
	HES	3.889	4.614	8.512	9.051	14.904	18.040	36.172	38.325
	SXP	4.337	5.058	11.672	10.387	17.685	20.571	51.164	45.354
	WBS	4.147	4.717	7.753	8.486	16.125	18.478	32.714	36.034
5	ERS	31.685	19.607	12.939	12.483	191.903	100.306	52.459	53.448
	EXS	24.753	19.072	14.331	14.106	136.366	96.104	59.261	61.007
	HES	17.771	16.846	20.078	18.803	86.935	76.748	86.111	80.807
	SXP	29.623	19.225	13.806	13.009	178.140	98.166	55.653	56.053
	WBS	17.864	17.539	17.863	18.518	89.381	83.636	80.676	79.937

customer finds all servers to be idle and ends when a departure leaves all servers idle. However, if one looks at full busy period wherein the busy period starts with all servers becoming busy at an arrival and ends as soon as a server becomes idle at a departure point, one will see the difference. This will be explained later on.

– It is interesting to point out that for $c = 5$, the mean busy period as well as the ratio for Erlang arrivals are large compared to others for all types of service distribution. This is due to using the partial busy period as opposed to full busy period.

Now, we display the ratio, μ_Y/μ_W, of the mean busy period to the corresponding mean waiting time in the system under various scenarios in Table 7. First, note that as is to be expected this ratio is 1 for $M/M/1$ queue. For other scenarios, we notice the following interesting observations.

– When dealing with $GI/ERS/c$ and $GI/SXP/c$ queues, for all scenarios we notice that $\mu_Y > \mu_W$.
– In the case of $M/M/c$, for $c > 1$, we notice $\mu_Y > \mu_W$.
– When dealing with $GI/HES/c$ and $GI/WBS/c$ queues, we see an interesting pattern. For $c = 1$ and $c = 2$, $\mu_Y < \mu_W$; however, for $c = 5$, $\mu_Y > \mu_W$.
– Generally, we notice that ERS appears to have a larger ratio as compared to those of HES, indicating the less variability in the service times appears to have a larger busy period on the average.
– As the number of servers is increased, the $HEA/G/c$ queue appears to be insensitive to the type of services. However, this is not the case with other service types.

Table 7 Ratio of μ_Y/μ_W $GI/G/c$ queue

		$\rho = 0.80$					$\rho = 0.95$				
c	TAP	ERS	EXS	HES	SXP	WBS	ERS	EXS	HES	SXP	WBS
$c = 1$	ERA	1.28	1.00	0.34	1.17	0.36	1.37	1.00	0.30	1.23	0.31
	EXA	1.25	1.00	0.38	1.17	0.38	1.31	1.00	0.33	1.18	0.35
	HEA	1.26	0.99	0.44	1.19	0.42	1.26	0.97	0.47	1.23	0.41
	WBA	1.18	1.00	0.52	1.13	0.48	1.19	0.97	0.51	1.16	0.44
$c = 2$	ERA	2.46	1.93	0.70	2.25	0.74	3.28	2.21	0.57	2.80	0.64
	EXA	2.19	1.80	0.75	2.05	0.76	2.58	1.95	0.65	2.34	0.64
	HEA	1.91	1.59	0.82	1.84	0.75	1.87	1.56	0.80	1.84	0.69
	WBA	1.66	1.47	0.87	1.61	0.82	1.65	1.48	0.79	1.63	0.73
$c = 5$	ERA	25.50	18.07	7.78	23.24	7.67	70.52	38.28	8.30	60.36	8.35
	EXA	13.76	12.24	6.76	13.18	6.94	27.55	21.44	6.92	25.38	7.45
	HEA	4.50	4.71	4.99	4.76	4.40	4.68	4.85	4.44	4.91	4.15
	WBA	4.29	4.61	4.63	4.45	4.58	4.79	4.88	4.20	4.87	4.18

Table 8 Mean full busy period and the ratio of μ_{Y_F} to μ_W for $GI/G/5$ queue when $\rho = 0.95$

	Mean full busy period					μ_{Y_F}/μ_W				
TS	ERS	EXS	HES	SXP	WBS	ERS	EXS	HES	SXP	WBS
ERA	2.582	2.999	4.278	2.712	4.326	1.05	1.19	2.45	1.09	2.48
EXA	3.721	3.985	4.899	3.848	4.770	0.98	1.12	2.26	1.01	2.35
HEA	13.217	11.683	9.678	12.825	8.733	0.85	1.05	2.00	0.88	2.23
WBA	12.693	11.666	10.513	12.596	9.147	0.85	1.04	1.89	0.88	2.13

– As pointed out earlier, the partial busy period will be large for $c > 1$ as compared to the full busy period. Irrespective of whether one looks at partial or full busy period for any particular scenario, the mean waiting time in the system will remain the same. Hence, the ratio is large as c is increased.

In order to confirm that the partial busy period will be at least as large as the full busy period for any particular scenario (note that the busy periods will be identical only in the case of single-server queueing system), we ran our simulation for a few scenarios, and in Table 8, we display the mean busy period as well as the ratio of the mean full busy period to the corresponding mean waiting time in the system. It is clear from the entries in that table that there is a significant reduction in the entries as compared to those given in Tables 6 and 7 corresponding to $\rho = 0.95$ and $c = 5$.

It should be pointed that mean busy period of the system can be less than or greater than or equal to the mean waiting time in the system of a customer for a specific queueing model. As is known for $M/M/1$ queue, $\mu_Y = \mu_W = 1/(\mu - \lambda)$. However, there are other queueing models for which $\mu_Y < \mu_W$ or $\mu_Y > \mu_W$. While we noticed this in our simulated examples, one can intuitively explain this as follows.

Table 9 Complementary distribution of Y for $GI/G/c$ queue for $\rho = 0.95$, $c = 1, 2$

c	TAP	TS	$P(Y > 1)$	$P(Y > 2)$	$P(Y > 4)$	$P(Y > 8)$	$P(Y > 16)$	$P(Y > 40)$	$P(Y > 80)$
1	ERA	ERS	0.553	0.381	0.259	0.173	0.112	0.060	0.035
		EXS	0.468	0.329	0.226	0.154	0.101	0.056	0.034
		HES	0.297	0.183	0.119	0.084	0.060	0.036	0.024
		SXP	0.517	0.357	0.243	0.164	0.108	0.059	0.034
		WBS	0.281	0.200	0.140	0.096	0.066	0.039	0.025
	EXA	ERS	0.609	0.435	0.302	0.205	0.137	0.076	0.046
		EXS	0.519	0.376	0.265	0.181	0.122	0.069	0.043
		HES	0.345	0.221	0.146	0.101	0.071	0.043	0.028
		SXP	0.580	0.414	0.288	0.195	0.131	0.073	0.045
		WBS	0.309	0.224	0.158	0.110	0.075	0.045	0.029
	HEA	ERS	0.696	0.572	0.460	0.355	0.258	0.159	0.106
		EXS	0.589	0.486	0.389	0.297	0.216	0.132	0.088
		HES	0.420	0.310	0.229	0.170	0.123	0.077	0.052
		SXP	0.677	0.555	0.446	0.344	0.252	0.154	0.103
		WBS	0.345	0.273	0.211	0.158	0.114	0.070	0.047
	WBA	ERS	0.756	0.584	0.410	0.264	0.154	0.060	0.022
		EXS	0.665	0.523	0.375	0.245	0.145	0.058	0.023
		HES	0.493	0.347	0.243	0.171	0.111	0.052	0.025
		SXP	0.734	0.569	0.402	0.260	0.153	0.060	0.022
		WBS	0.416	0.326	0.242	0.167	0.106	0.050	0.023

Whenever the customers arrive in short intervals to, say, a single-server system, the waiting time in the system of these customers will be large while the busy period consisting of only the service times will be relatively small. Suppose the customers arrive pretty regularly such that their waiting time in the queue is very marginal but the busy period will be longer resulting in a larger mean for busy period as compared to the mean waiting time in the system.

We conclude this section by displaying the complementary distribution function of Y, namely $P(Y > t)$, for selected values of t and for $\rho = 0.95$ under different arrival and service times in Tables 9 and 10. Generally, we see the role of the variability either in the arrivals or in services in these probabilities.

8.2 $MAP/G/c$

Before we discuss simulated results for $MAP/G/c$ queueing systems, we briefly review the MAP in continuous time. Suppose that there is an underlying Markov chain which is irreducible with generator given by $Q^* = D_0 + D_1$, where $D_0 =$

Table 10 Complementary distribution of Y for $GI/G/c$ queue for $\rho = 0.95$, $c = 5$

c	TAP	TS	$P(Y > 1)$	$P(Y > 2)$	$P(Y > 4)$	$P(Y > 8)$	$P(Y > 16)$	$P(Y > 40)$	$P(Y > 80)$
2	ERA	ERS	0.671	0.529	0.397	0.282	0.188	0.097	0.052
		EXS	0.553	0.432	0.321	0.227	0.153	0.082	0.047
		HES	0.358	0.235	0.158	0.112	0.077	0.045	0.028
		SXP	0.638	0.496	0.368	0.259	0.172	0.090	0.049
		WBS	0.313	0.238	0.175	0.124	0.086	0.050	0.032
	EXA	ERS	0.692	0.537	0.395	0.276	0.184	0.098	0.055
		EXS	0.583	0.454	0.335	0.237	0.160	0.087	0.051
		HES	0.396	0.264	0.180	0.129	0.090	0.053	0.034
		SXP	0.666	0.515	0.377	0.266	0.177	0.095	0.054
		WBS	0.339	0.258	0.191	0.136	0.093	0.055	0.035
	HEA	ERS	0.779	0.660	0.522	0.386	0.272	0.158	0.100
		EXS	0.665	0.563	0.448	0.332	0.234	0.138	0.089
		HES	0.485	0.365	0.273	0.206	0.148	0.091	0.059
		SXP	0.766	0.644	0.509	0.376	0.265	0.156	0.100
		WBS	0.388	0.315	0.247	0.184	0.132	0.080	0.052
	WBA	ERS	0.741	0.599	0.460	0.340	0.244	0.151	0.100
		EXS	0.650	0.524	0.403	0.298	0.211	0.128	0.086
		HES	0.494	0.363	0.263	0.192	0.138	0.085	0.057
		SXP	0.725	0.581	0.445	0.328	0.236	0.144	0.095
		WBS	0.404	0.320	0.243	0.179	0.127	0.077	0.052
5	ERA	ERS	0.894	0.875	0.852	0.817	0.768	0.669	0.552
		EXS	0.757	0.728	0.700	0.665	0.613	0.517	0.411
		HES	0.582	0.509	0.445	0.391	0.343	0.270	0.206
		SXP	0.897	0.873	0.843	0.805	0.754	0.645	0.525
		WBS	0.432	0.390	0.357	0.328	0.294	0.240	0.192
	EXA	ERS	0.875	0.837	0.789	0.727	0.647	0.500	0.356
		EXS	0.762	0.722	0.678	0.624	0.555	0.433	0.314
		HES	0.589	0.504	0.432	0.377	0.326	0.250	0.187
		SXP	0.871	0.826	0.777	0.713	0.630	0.483	0.344
		WBS	0.448	0.404	0.368	0.333	0.294	0.236	0.184
	HEA	ERS	0.903	0.810	0.671	0.524	0.386	0.231	0.141
		EXS	0.806	0.734	0.630	0.513	0.395	0.248	0.157
		HES	0.672	0.571	0.480	0.412	0.345	0.253	0.182
		SXP	0.900	0.808	0.679	0.536	0.399	0.241	0.149
		WBS	0.516	0.466	0.412	0.358	0.305	0.226	0.167
	WBA	ERS	0.877	0.778	0.652	0.518	0.386	0.233	0.144
		EXS	0.796	0.719	0.621	0.512	0.394	0.250	0.159
		HES	0.655	0.546	0.456	0.390	0.327	0.239	0.173
		SXP	0.862	0.768	0.652	0.520	0.391	0.242	0.151
		WBS	0.412	0.378	0.349	0.323	0.293	0.246	0.201

$(d_{ij}^{(0)})$ and $D_1 = (d_{ij}^{(1)})$. Suppose that $d_{ii}^{(0)} = -\lambda_i, 1 \le i \le m, d_{ij}^{(0)} = \lambda_i p_{ij}^{(0)}$, for $j \ne i$ and $d_{ij}^{(1)} = \lambda_i p_{ij}^{(1)}, 1 \le i, j \le m$. At the end of a sojourn time in state i, that is exponentially distributed with parameter λ_i, one of the following two events could occur: With probability $p_{ij}^{(1)}$, the transition corresponds to an arrival and the underlying Markov chain is in state j with $1 \le i, j \le m$; with probability $p_{ij}^{(0)}$, the transition corresponds to no arrival and the state of the Markov chain is j, $j \ne i$. Note that the Markov chain can go from state i to state i only through an arrival. By assuming D_0 to be a nonsingular matrix, the inter-arrival times will be finite with probability one and the arrival process does not terminate. Hence, we see that D_0 is a stable matrix.

Thus, D_0 governs the transitions corresponding to no arrival and D_1 governs those corresponding to an arrival. It can be shown that MAP is equivalent to Neuts' versatile Markovian point process. The point process described by the MAP is a special class of semi-Markov processes. For further details on MAP and their usefulness in stochastic modeling, we refer to [8, 16, 48], and for a review and recent work on MAP, we refer the reader to [49–51].

Let $\boldsymbol{\eta}$ be the stationary probability vector of the Markov process with generator Q^*. That is, $\boldsymbol{\eta}$ is the unique (positive) probability vector satisfying $\boldsymbol{\eta} Q^* = \mathbf{0}$, $\boldsymbol{\eta}\boldsymbol{e} = 1$, where $\boldsymbol{e}$ is a column vector of 1's of appropriate dimension. We denote the average arrival rate, the average service rate, and the average retrial rate by, respectively, λ, μ, and θ. These are given by $\lambda = \boldsymbol{\eta} D_1 \boldsymbol{e},\ \mu = [\boldsymbol{\alpha}(-T)^{-1}\boldsymbol{e}]^{-1},\ \theta = [\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}]^{-1}$.

Here we look at two MAP which have correlated arrivals. The representation matrices D_0 and D_1 are given by

TAP 5: MAP with negative correlation ($MNCA$):

$$D_0 = \begin{pmatrix} -1.00222 & 1.00222 & 0 \\ 0 & -1.00222 & 0 \\ 0 & 0 & -225.75 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 & 0 \\ 0.01002 & 0 & 0.9922 \\ 223.4925 & 0 & 2.2575 \end{pmatrix}$$

TAP 6: MAP with positive correlation ($MPCA$):

$$D_0 = \begin{pmatrix} -1.00222 & 1.00222 & 0 \\ 0 & -1.00222 & 0 \\ 0 & 0 & -225.75 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 & 0 \\ 0.9922 & 0 & 0.01002 \\ 2.2575 & 0 & 223.4925 \end{pmatrix}.$$

Note that the above two MAP processes will be normalized so as to have a specified arrival rate. While the above two processes have identical means and identical standard deviations, they are qualitatively different in that MNA and MPA, respectively, have negative and positive correlation for two successive inter-arrival times with values -0.4889 and 0.4889.

In Table 11, we display the two measures, μ_Y and μ_Y/μ_W, under various scenarios by looking at the above two $MAPs$ as input processes. A quick look at these tables reveals some interesting observations.

Table 11 Mean busy period and the ratio of μ_Y to μ_W for $MAP/G/c$ queue

		Mean busy period				μ_Y/μ_W			
		$\rho = 0.80$		$\rho = 0.95$		$\rho = 0.80$		$\rho = 0.95$	
c	TS	MNA	MPA	MNA	MPA	MNA	MPA	MNA	MPA
1	ERS	7.028	7.840	27.402	32.586	1.65	0.03	1.75	0.03
	EXS	7.265	8.347	28.905	34.510	1.39	0.03	1.42	0.04
	HES	7.744	8.647	30.179	31.258	0.58	0.04	0.51	0.04
	SXP	7.073	8.060	27.791	29.487	1.56	0.03	1.63	0.03
	WBS	7.423	8.761	28.700	35.733	0.57	0.03	0.49	0.03
2	ERS	5.540	6.199	21.976	22.808	2.30	0.05	2.73	0.05
	EXS	6.003	6.255	23.933	24.351	2.12	0.05	2.30	0.05
	HES	6.113	6.204	23.437	27.106	0.98	0.05	0.82	0.05
	SXP	5.508	6.304	21.663	26.233	2.19	0.05	2.52	0.05
	WBS	6.473	6.790	25.389	24.334	1.05	0.05	0.90	0.06
5	ERS	16.146	11.285	80.786	56.840	11.22	0.22	21.93	0.26
	EXS	18.510	10.710	92.186	50.818	11.80	0.21	20.02	0.26
	HES	18.579	10.027	82.727	49.102	7.42	0.20	7.26	0.24
	SXP	15.658	10.993	76.737	52.807	10.66	0.22	19.57	0.26
	WBS	22.097	10.378	102.538	44.418	8.72	0.20	9.01	0.23

- While for $c = 1$ and $c = 2$, μ_Y is relatively large for positively correlated arrivals as opposed to the negatively correlated ones, we see for $c = 5$, it is the negatively correlated arrivals that yield a larger value for μ_Y compared to the positively correlated ones.
- Under all scenarios, we notice the ratio, μ_Y/μ_W, to be small for positively correlated arrivals, MPA.
- The ratio appears insensitive to the type of service times in the case of positively correlated arrivals, MPA.
- In the case of $MNA/HES/c$ queue with c small, we notice $\mu_Y < \mu_W$ whereas $\mu_Y > \mu_W$ when $c = 5$.
- Like we outlined earlier for $GI/G/c$ case, the partial busy period as compared to full busy period will go up as c is increased for MNA arrivals; however, for MPA, as noted in the literature (see, e.g., [51]), the mean waiting time in the system is still large enough to have the ratio $\frac{\mu_Y}{\mu_W}$ to be small.

The complementary distribution function, namely $P(Y > t)$, for selected values of t, for MNA and MPA, respectively, are displayed in Tables 12 and 13. Generally, we notice that MNA arrivals appear to have a larger value for this function compared to MPA. There are some scenarios for which this appears to be not the case but because the probabilities are small enough that they may be attributable to random (simulation) error.

Table 12 Complementary distribution of Y for $MNA/G/c$ queue

c	ρ	TS	$P(Y > 1)$	$P(Y > 2)$	$P(Y > 4)$	$P(Y > 8)$	$P(Y > 16)$	$P(Y > 40)$	$P(Y > 80)$
1	0.8	ERS	0.870	0.586	0.350	0.198	0.100	0.031	0.008
		EXS	0.754	0.520	0.325	0.191	0.103	0.036	0.012
		HES	0.553	0.322	0.196	0.128	0.082	0.040	0.020
		SXP	0.846	0.558	0.339	0.195	0.101	0.032	0.009
		WBS	0.457	0.320	0.213	0.136	0.083	0.039	0.019
	0.95	ERS	0.877	0.624	0.420	0.283	0.187	0.104	0.062
		EXS	0.764	0.551	0.380	0.258	0.173	0.099	0.062
		HES	0.565	0.348	0.225	0.155	0.109	0.066	0.043
		SXP	0.853	0.594	0.403	0.272	0.181	0.101	0.062
		WBS	0.464	0.333	0.233	0.161	0.110	0.065	0.042
2	0.8	ERS	0.751	0.519	0.329	0.182	0.081	0.016	0.002
		EXS	0.686	0.495	0.327	0.190	0.093	0.023	0.005
		HES	0.480	0.302	0.197	0.132	0.081	0.034	0.014
		SXP	0.711	0.499	0.319	0.178	0.082	0.017	0.003
		WBS	0.449	0.329	0.230	0.150	0.089	0.037	0.015
	0.95	ERS	0.774	0.585	0.426	0.297	0.198	0.105	0.059
		EXS	0.707	0.544	0.402	0.282	0.189	0.103	0.061
		HES	0.502	0.336	0.233	0.168	0.117	0.069	0.044
		SXP	0.739	0.559	0.409	0.285	0.189	0.101	0.058
		WBS	0.461	0.351	0.260	0.185	0.128	0.075	0.048
5	0.8	ERS	0.889	0.805	0.693	0.542	0.350	0.103	0.014
		EXS	0.817	0.740	0.650	0.530	0.373	0.144	0.031
		HES	0.636	0.510	0.407	0.332	0.259	0.149	0.068
		SXP	0.861	0.773	0.664	0.519	0.337	0.101	0.014
		WBS	0.552	0.480	0.417	0.354	0.285	0.175	0.089
	0.95	ERS	0.913	0.861	0.797	0.718	0.618	0.447	0.297
		EXS	0.847	0.796	0.739	0.671	0.584	0.440	0.310
		HES	0.681	0.575	0.486	0.421	0.362	0.275	0.204
		SXP	0.893	0.835	0.769	0.689	0.589	0.423	0.280
		WBS	0.577	0.517	0.467	0.422	0.373	0.298	0.229

9 Concluding Remarks

In this paper, after pointing out the complexity involved in the study of the busy period, we recorded some interesting observations on the busy period of queueing systems in general context through simulation. We also compared the mean busy period with the corresponding mean waiting time in the system for queues of the type $GI/G/c$ and $MAP/G/c$. The main purpose of this study through simulation is to help researchers to compare/validate their future novel theoretical and/or numer-

Table 13 Complementary distribution of Y for $MPA/G/c$ queue

c	ρ	TS	$P(Y > 1)$	$P(Y > 2)$	$P(Y > 4)$	$P(Y > 8)$	$P(Y > 16)$	$P(Y > 40)$	$P(Y > 80)$
1	0.8	ERS	0.453	0.191	0.063	0.021	0.013	0.011	0.009
		EXS	0.402	0.205	0.084	0.030	0.015	0.011	0.009
		HES	0.246	0.124	0.075	0.048	0.028	0.014	0.010
		SXP	0.415	0.190	0.070	0.024	0.013	0.011	0.009
		WBS	0.261	0.163	0.095	0.051	0.027	0.014	0.010
	0.95	ERS	0.467	0.218	0.084	0.029	0.015	0.013	0.011
		EXS	0.409	0.221	0.101	0.040	0.019	0.013	0.011
		HES	0.255	0.134	0.081	0.054	0.033	0.017	0.012
		SXP	0.428	0.212	0.087	0.032	0.016	0.012	0.011
		WBS	0.264	0.168	0.101	0.057	0.031	0.017	0.012
2	0.8	ERS	0.506	0.251	0.081	0.023	0.016	0.013	0.009
		EXS	0.436	0.247	0.103	0.033	0.017	0.013	0.010
		HES	0.269	0.140	0.083	0.052	0.028	0.014	0.010
		SXP	0.466	0.239	0.086	0.026	0.016	0.013	0.009
		WBS	0.273	0.179	0.107	0.057	0.029	0.015	0.010
	0.95	ERS	0.531	0.290	0.112	0.035	0.021	0.017	0.013
		EXS	0.453	0.274	0.129	0.046	0.022	0.017	0.013
		HES	0.283	0.154	0.093	0.059	0.034	0.018	0.013
		SXP	0.490	0.273	0.113	0.038	0.021	0.017	0.013
		WBS	0.282	0.190	0.119	0.067	0.036	0.018	0.014
5	0.8	ERS	0.686	0.531	0.338	0.158	0.069	0.037	0.023
		EXS	0.565	0.441	0.300	0.159	0.071	0.036	0.022
		HES	0.375	0.247	0.166	0.118	0.079	0.039	0.023
		SXP	0.656	0.500	0.322	0.155	0.068	0.036	0.022
		WBS	0.326	0.250	0.185	0.129	0.082	0.040	0.024
	0.95	ERS	0.733	0.609	0.442	0.254	0.126	0.067	0.046
		EXS	0.605	0.500	0.375	0.232	0.121	0.063	0.043
		HES	0.407	0.286	0.202	0.151	0.108	0.062	0.042
		SXP	0.707	0.578	0.419	0.247	0.122	0.066	0.045
		WBS	0.342	0.273	0.213	0.158	0.110	0.062	0.041

ical approach to solving functional equations involving Laplace transforms, which naturally arise in the study of busy periods, with our reported results here. The focus of this paper is on the classical queueing models, and hence, this can be generalized in a number of ways. These include (a) queueing models with group arrivals; (b) queueing models with finite capacity; (b) queues with balking; (c) infinite-server queueing system; (d) queues with catastrophes; (e) queues with priority; (f) retrial queues; and (g) queues with different types of policies such as N-policy and threshold policy. The results of these and other models will be presented elsewhere.

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Solving LP Models for Multi-objective Matrix Games with I-Fuzzy Goals

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Abstract The principal objective of this work is to obtain the optimal strategies for a multi-objective two-person zero-sum matrix game with intuitionistic fuzzy goals (MOMGIFG). In this problem, the fuzziness in aspiration levels of both players are characterized by intuitionistic fuzzy sets. The developed linear models are solved in maxmin–minmax way using linear membership function (*mf*) and non-membership function (*nmf*). A numerical example is incorporated to demonstrate the proposed solution procedure.

Keywords Matrix games · Intuitionistic fuzzy goals · Optimal strategies Intuitionistic fuzzy sets

1 Introduction

Multi-objective game theory optimizes those multi-objective problems that involve two or more than two decision makers. In fact, real game problems cannot be characterized precisely because of fuzzy information about their elements. Various studies about the zero-sum matrix game models with two players have been done so far, e.g., [6–8, 10, 16] and references therein, where fuzziness in payoffs and goals are characterized by fuzzy sets. But, a situation in which an element feels a hesitation to belong or not belong to a subset of universe cannot be represented by fuzzy sets. Intuitionistic fuzzy sets (I-fuzzy sets) [4] can give a suitable description of such kind vague information. Firstly, Atanassov [5] used I-fuzzy set in game models. Thereafter, many researchers studied single- and multi-objective two-person zero-sum matrix game in I-fuzzy environment [1, 2, 11–13, 17, 18] and references therein.

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The focus of this paper is introducing a solution approach for MOMGIFG. The notion for the proposed technique is inspired from max–min principle of classical game theory.

The outline of this research work is as follows: Sect. 2 introduces some preliminaries which are relevant to this work such as I-fuzzy set, maxmin–minmax solution and decision-making principle in I-fuzzy environment. In Sect. 3, a single-objective game model in matrix form with I-fuzzy goals is reviewed under some assumptions. A solution procedure for MOMGIFG with a set of assumptions is proposed in Sect. 4. In Sect. 5, an example is given to demonstrate the effectiveness of present work.

2 Preliminaries

Present section concerns some necessary definitions and one principle which are used throughout this paper.

Definition 1 (*I-fuzzy Set*) An I-fuzzy set $\widetilde{T}$ on space S is defined by two functions, μ_+ and μ_-, such that $\mu_+(s) \in [0, 1]$ represents the grade of membership of s in $\widetilde{T}$ and $\mu_-(s) \in [0, 1]$ represents the grade of non-membership of s in $\widetilde{T}$ with condition $0 \leq \mu_+(s) + \mu_-(s) \leq 1$. The expression $\mu^h(s) = 1 - \mu_+(s) - \mu_-(s)$ is called degree of hesitancy of s in $\widetilde{T}$. An I-fuzzy set $\widetilde{T}$ is denoted by

$$\widetilde{T} = \{\langle s,\ \ \mu_+(s),\ \ \mu_-(s)\rangle \mid s \in S\}\,.$$

In this paper, the goals for each player are viewed as I-fuzzy sets. The meaning of the value of $\mu_+(s)$ for an I-fuzzy goal is the grade of satisfaction of I-fuzzy goal for an expected payoff, whereas the value of $\mu_-(s)$ represents the degree of dissatisfaction of I-fuzzy goal. Recently, some I-fuzzy and fuzzy programming in term of goal programming have been found in [9, 14, 15].

A MOMGIFG is described by multi-payoff matrices $M^1, M^2, \ldots, M^r$. In this problem, Player I and II are denoted by P_1 and P_2, respectively. Suppose that I-fuzzy goal for kth payoff for P_1 and P_2 is denoted by $\tilde{g}^k_{P_1}$ and $\tilde{g}^k_{P_2}$, respectively. It is supposed that the r objectives of P_1 are also the objectives for P_2.

Definition 2 The maxmin–minmax value w. r. t. the grade of satisfaction of an aggregated I-fuzzy goal to P_1 is

$$\max_{p\,\in\,U^m}\ \min_{q\,\in\,U^n}\ \min_{k}\quad \{\mu_{\tilde{g}^k_{P_1}+}(p^T M^k q)\} \tag{1}$$

$$\min_{p\,\in\,U^m}\ \max_{q\,\in\,U^n}\ \max_{k}\quad \{\mu_{\tilde{g}^k_{P_1}-}(p^T M^k q)\} \tag{2}$$

where U^m/U^n is mixed strategy space to P_1/P_2. Such a strategy p^* is known as the maxmin–minmax solution of matrix game with aggregated I-fuzzy goal for P_1.

Similarly, the maxmin–minmax value w. r. t. the grade of satisfaction of an aggregated I-fuzzy goal to P_2 is

$$\max_{q\,\in\,U^n}\ \min_{p\,\in\,U^m}\ \min_{k}\ \{\mu_{\tilde{g}^k_{P_2}+}(p^T M^k q)\} \tag{3}$$

$$\min_{q\,\in\,U^n}\ \max_{p\,\in\,U^m}\ \max_{k}\ \{\mu_{\tilde{g}^k_{P_2}-}(p^T M^k q)\}. \tag{4}$$

Such a strategy q^* is known as the maxmin–minmax solution of matrix game with aggregated I-fuzzy goal for P_2.

Definition 3 (*Angelov's Decision-Making Principle*) Suppose that there are m goals $A_1, A_2, \ldots, A_m$ and n constraints $B_1, B_2, \ldots, B_n$ in a domain of alternatives Ω. All these goals ($A_i's$) and constraints ($B_j's$) are I-fuzzy sets on Ω. Angelov [3] proposed that an I-fuzzy decision which is evaluated by a suitable aggregation of the I-fuzzy sets $A_i (i = 1, 2, \ldots, m)$ and $B_j (j = 1, 2, \ldots, n)$. He used fuzzy intersection and fuzzy union as aggregation operators. Therefore, an I-fuzzy decision D which is an I-fuzzy set, defined by $\mu_{D+} : \Omega \to [0, 1]$ given by $\mu_{D+}(\omega) = \min_{i,j} \left(\mu_{A_i+}(\omega),\ \mu_{B_j+}(\omega)\right)$ and $\mu_{D-} : \Omega \to [0, 1]$ given by $\mu_{D-}(\omega) = \max_{i,j} \left(\mu_{A_i-}(\omega),\ \mu_{B_j-}(\omega)\right)$.

The optimal decision can be obtained as $\max_{\omega} \mu_{D+}(\omega)$ and $\min_{\omega} \mu_{D-}(\omega)$.

According to this principle, the crisp version of above I-fuzzy optimization problem in linear programming (LP) form can be formulated as follows:

$$\begin{aligned}
&\max \quad (\alpha_+ - \alpha_-)\\
&\text{s.t.,}\\
&\mu_{A_i+}(\omega) \geq \alpha_+,\\
&\mu_{A_i-}(\omega) \leq \alpha_-,\ \ (i = 1, 2, \ldots, m),\\
&\mu_{B_j+}(\omega) \geq \alpha_+,\\
&\mu_{B_j-}(\omega) \leq \alpha_-,\ \ (j = 1, 2, \ldots, n),\\
&\alpha_+ + \alpha_- \leq 1,\\
&\alpha_+ \geq \alpha_-,\ \alpha_- \geq 0,\ \omega \geq 0.
\end{aligned} \tag{5}$$

Here, the optimal solution of model (5) is denoted by $(\omega^*, {\alpha_+}^*, {\alpha_-}^*)$.

3 Single-Objective Matrix Game with I-Fuzzy Goal (SOMGIFG)

Present section demonstrates in what way a SOMGIFG can be solved through a pair of linear programming problem (LPP).

Let $M = [m_{ij}]_{m\times n}$ denote a payoff matrix of real constants for P_1. Since game is zero-sum, so $-M = [-m_{ij}]_{m\times n}$ is payoff matrix for P_2. Here, U^m/U^n represents a set of mixed strategies for P_1/P_2. The sets U^m and U^n are defined as:

$$U^m = \{p = (p_1, p_2, \ldots, p_m)^T | \sum_{i=1 \text{ to } m} p_i = 1, p_i \geq 0\},$$

and

$$U^n = \{q = (q_1, q_2, \ldots, q_n)^T | \sum_{j=1 \text{ to } n} q_j = 1, q_j \geq 0\}.$$

In this work, the goals of P_1 and P_2 are characterized by I-fuzzy sets. Suppose that $\bar{v}_a$ is the aspiration level for P_1 with tolerance error p_a and $\bar{v}_r$ is the rejection level for P_1 with tolerance error p_r. For P_2, let $\underline{v}_a$ be aspiration level with tolerance error q_a and $\underline{v}_r$ be rejection level with tolerance error q_r.

To solve two-person zero-sum SOMGIFG, the following conditions are assumed as:

(H_1) The I-fuzzy goals of both players P_1 and P_2 are represented by linear *mf* and *nmf*;
(H_2) For P_1, $\bar{v}_r - p_r \leq \bar{v}_a - p_a$ & $\bar{v}_r \leq \bar{v}_a$;
(H_3) For P_2, $\underline{v}_a + q_a \leq \underline{v}_r + q_r$ & $\underline{v}_a \leq \underline{v}_r$.

Using (H_1)–(H_2), the solution for optimization problem of P_1 will be produced as:

Theorem 1 [11] *The maxmin–minmax solution for P_1 is equivalent to the solution of a LPP which is described as*

$$\begin{aligned}
&\max \quad (\lambda_+ - \lambda_-) \\
&\text{s.t.,} \\
&\sum_{i=1 \text{ to } m} m_{ij} p_i + p_a - \bar{v}_a \geq p_a \lambda_+, \\
&\sum_{i=1 \text{ to } m} m_{ij} p_i - \bar{v}_r \geq -p_r \lambda_-, \quad (j = 1, 2, \ldots, n), \\
&\sum_{i=1 \text{ to } m} p_i = 1, 0 \leq \lambda_+, \lambda_- \leq 1, \\
&\lambda_+ + \lambda_- \leq 1, \lambda_+ \geq \lambda_-, p \geq 0.
\end{aligned} \tag{6}$$

Theorem 2 [11] *The maxmin–minmax solution for P_2 with assumptions (H_1) and (H_3) is equivalent to the solution of a LPP which is described as:*

$$
\begin{aligned}
&\max \quad (\eta_+ - \eta_-) \\
&\text{s.t.,} \\
&\sum_{j=1 \text{ to } n} m_{ij} q_j - \underline{v}_a - q_a \le -q_a \eta_+, \\
&\sum_{j=1 \text{ to } n} m_{ij} q_j - \underline{v}_r \le -q_r \eta_-, \quad (i = 1, 2, \ldots, m), \\
&\sum_{j=1 \text{ to } n} q_j = 1, 0 \le \eta_+, \eta_- \le 1, \\
&\eta_+ + \eta_- \le 1, \eta_+ \ge \eta_-, q \ge 0.
\end{aligned} \tag{7}
$$

4 Solution Procedure to MOMGIFG

In a multi-objective matrix game, each player has more than one objective and each objective is represented by a payoff matrix. Suppose that both players (P_1 and P_2) have same r objectives.

For this matrix game problem, following conditions are assumed as:

(H_4) The payoff values in each payoff matrix are real numbers;
(H_5) The fuzziness in aspiration level of each objective is represented by an I-fuzzy set; and
(H_6) *mf* and *nmf* for each I-fuzzy goal are linear.

Now, a methodology is proposed to obtain the models in LP form for strategic problem to P_1 and P_2, respectively, as follows:

Optimization problem for P_1

Suppose that *mf* and *nmf* of the I-fuzzy goal for kth objective of P_1 are denoted by $\mu_{\tilde{g}^k_{P_1}+}(p^T M^k q)$ and $\mu_{\tilde{g}^k_{P_1}-}(p^T M^k q)$, respectively. Using ($H_4$)–($H_6$), $\mu_{\tilde{g}^k_{P_1}+}(p^T M^k q)$ can be represented as

$$
\mu_{\tilde{g}^k_{P_1}+}(p^T M^k q) = \begin{cases} 0 & , \ p^T M^k q < \bar{v}^k_a - p^k_a, \\ 1 - \frac{\bar{v}^k_a - p^T M^k q}{p^k_a} & , \ \bar{v}^k_a - p^k_a \le p^T M^k q < \bar{v}^k_a, \\ 1 & , \ \bar{v}^k_a \le p^T M^k q, \end{cases} \tag{8}
$$

and *nmf* $\mu_{\tilde{g}^k_{P_1}-}(p^T M^k q)$ is

$$
\mu_{\tilde{g}^k_{P_1}-}(p^T M^k q) = \begin{cases} 1 & , \ p^T M^k q < \bar{v}^k_r - p^k_r, \\ 1 - \frac{p^T M^k q - (\bar{v}^k_r - p^k_r)}{p^k_r} & , \ \bar{v}^k_r - p^k_r \le p^T M^k q < \bar{v}^k_r, \\ 0 & , \ \bar{v}^k_r \le p^T M^k q, \end{cases} \tag{9}
$$

with conditions $\bar{v}^k_r - p^k_r \le \bar{v}^k_a - p^k_a$ and $\bar{v}^k_r \le \bar{v}^k_a$.

Using [3], *mf* and *nmf* for aggregated I-fuzzy goal to P_1 can be formed in respective order as:

$$\min_{k} \{\mu_{\tilde{g}^k_{P_1}+}(p^T M^k q)\} \tag{10}$$

and,

$$\max_{k} \{\mu_{\tilde{g}^k_{P_1}-}(p^T M^k q)\} \tag{11}$$

Assuming that

(H_7) The calculating *mf* in (10) and *nmf* in (11) are linear.

The maxmin–minmax value in terms of degree of acceptance of an aggregated I-fuzzy goal to P_1 is

$$\max_{p \in U^m} \min_{q \in U^n} \min_{k} \{\mu_{\tilde{g}^k_{P_1}+}(p^T M^k q)\},$$

$$\min_{p \in U^m} \max_{q \in U^n} \max_{k} \{\mu_{\tilde{g}^k_{P_1}-}(p^T M^k q)\}.$$

Theorem 3 *The maxmin–minmax solution for P_1 with assumption (H_7) is equivalent to the following LP model*

$$\max \quad (\lambda_+ - \lambda_-)$$

s.t.,

$$\begin{aligned}
\sum_{i=1 \text{ to } m} m^k_{ij} p_i + p^k_a - \bar{v}^k_a &\geq p^k_a \lambda_+, \\
\sum_{i=1 \text{ to } m} m^k_{ij} p_i - \bar{v}^k_r &\geq -p^k_r \lambda_-, \quad (j = 1, 2, \ldots, n), \\
\sum_{i=1 \text{ to } m} p_i &= 1, 0 \leq \lambda_+, \lambda_- \leq 1, \\
\lambda_+ + \lambda_- &\leq 1, \lambda_+ \geq \lambda_-, p \geq 0,
\end{aligned} \tag{12}$$

where $k = 1, 2, \ldots, r$.

Proof The maxmin–minmax problem for P_1 is

$$\max_{p \in U^m} \min_{q \in U^n} \min_{k} \{\mu_{\tilde{g}^k_{P_1}+}(p^T M^k q)\},$$

$$\min_{p \in U^m} \max_{q \in U^n} \max_{k} \{\mu_{\tilde{g}^k_{P_1}-}(p^T M^k q)\}.$$

For *mf*

$$\begin{aligned}
&\max_{p\,\in\,U^m}\ \min_{q\,\in\,U^n}\ \min_{k}\ \left(1-\frac{\bar{v}_a^k-p^TM^kq}{p_a^k}\right)\\
&=\frac{1}{p_a^k}\max_{p\,\in\,U^m}\ \min_{q\,\in\,U^n}\ \min_{k}\ \left(\sum_{i=1\text{ to }m}\ \sum_{j=1\text{ to }n} m_{ij}^k p_i q_j + c^k\right)\\
&=\frac{1}{p_a^k}\max_{p\,\in\,U^m}\ \min_{k}\ \min_{q\,\in\,U^n}\ \sum_{j=1\text{ to }n}\left(\sum_{i=1\text{ to }m} m_{ij}^k p_i + c^k\right) q_j\\
&=\frac{1}{p_a^k}\max_{p\,\in\,U^m}\ \min_{k}\ \min_{j\in J}\ \left(\sum_{i=1\text{ to }m} m_{ij}^k p_i + c^k\right).
\end{aligned}$$

Let $\min_{j\in J}\left(\sum_{i=1\text{ to }m} m_{ij}^k p_i + c^k\right) = \lambda_{k+}$ and further let $\min_{k}\ \lambda_{k+} = \lambda_{+}$. In similar way, for *nmf*, letting $\max_{k}\ \lambda_{k-} = \lambda_{-}$. The maxmin–minmax problem for P_1 reduces to LP model (12).

Optimization problem for P_2

Let *mf* and *nmf* of an I-fuzzy goal for k^{th} objective of P_2 be denoted by $\mu_{\tilde{g}_{P_2}^k+}(p^TM^kq)$ and $\mu_{\tilde{g}_{P_2}^k-}(p^TM^kq)$, respectively. Using (H_4)–(H_6), $\mu_{\tilde{g}_{P_2}^k+}(p^TM^kq)$ can be represented as

$$\mu_{\tilde{g}_{P_2}^k+}(p^TM^kq) = \begin{cases} 1 & ,\ p^TM^kq < \underline{v}_a^k,\\ 1-\frac{p^TM^kq-\underline{v}_a^k}{q_a^k} & ,\ \underline{v}_a^k \le p^TM^kq < \underline{v}_a^k + q_a^k,\\ 0 & ,\ \underline{v}_a^k + q_a^k \le p^TM^kq, \end{cases} \tag{13}$$

and $\mu_{\tilde{g}_{P_2}^k-}(p^TM^kq)$ is

$$\mu_{\tilde{g}_{P_2}^k-}(p^TM^kq) = \begin{cases} 0 & ,\ p^TM^kq < \underline{v}_r^k,\\ \frac{p^TM^kq-\underline{v}_r^k}{q_r^k} & ,\ \underline{v}_r^k \le p^TM^kq < \underline{v}_r^k + q_r^k,\\ 1 & ,\ \underline{v}_r^k + q_r^k \le p^TM^kq, \end{cases} \tag{14}$$

with conditions $\underline{v}_a^k + q_a^k \le \underline{v}_r^k + q_r^k$ and $\underline{v}_a^k \le \underline{v}_r^k$ for $k = 1, 2, \ldots, r$.

Using [3], *mf* and *nmf* for aggregated I-fuzzy goal can be calculated in respective order as

$$\min_{k}\ \{\mu_{\tilde{g}_{P_2}^k+}(p^TM^kq)\} \tag{15}$$

and,

$$\max_{k} \; \{\mu_{\tilde{g}^k_{P_2}-}(p^T M^k q)\} \tag{16}$$

In similar to problem of P_1, assuming that

(H_8) The calculating *mf* in (15) and *nmf* in (16) are linear.

The maxmin–minmax value in terms of the degree of acceptance of an aggregated I-fuzzy goal to P_2 is

$$\max_{p \in U^m} \min_{q \in U^n} \min_{k} \; \{\mu_{\tilde{g}^k_{P_2}+}(p^T M^k q)\},$$

$$\min_{p \in U^m} \max_{q \in U^n} \max_{k} \; \{\mu_{\tilde{g}^k_{P_2}-}(p^T M^k q)\}.$$

Theorem 4 *The maxmin–minmax solution for P_2 with assumption (H_8) is equivalent to the following LP model*

$$\begin{aligned}
&\max \quad (\eta_+ - \eta_-)\\
&\text{s.t.,}\\
&\sum_{j=1 \text{ to } n} m^k_{ij} q_j - \underline{v}^k_a - q^k_a \le - q^k_a \eta_+,\\
&\sum_{j=1 \text{ to } n} m^k_{ij} q_j - \underline{v}^k_r \le -q^k_r \eta_-, \quad (i = 1, 2, \ldots, m),\\
&\sum_{j=1 \text{ to } n} q_j = 1, 0 \le \eta_+, \eta_- \le 1,\\
&\eta_+ + \eta_- \le 1, \eta_+ \ge \eta_-, q \ge 0,
\end{aligned} \tag{17}$$

where $k = 1, 2, \ldots, r$.

Proof Proof is similar to Theorem 3.

5 Example

This section consists of an example of MOMGIFG which shows the validity of the proposed work.

The payoff matrices M^1, M^2 are separately indicated as:

$$M^1 = \begin{pmatrix} 4 & 2 & -1 \\ -2 & 0 & 1 \end{pmatrix}, M^2 = \begin{pmatrix} 10 & 24 & 9 \\ 7 & 15 & 11 \end{pmatrix}$$

Here, we assume that
$\bar{v}_a^1 = 3,\ p_a^1 = 4,\ \bar{v}_r^1 = 2,\ p_r^1 = 6$ and $\bar{v}_a^2 = 10,\ p_a^2 = 5,\ \bar{v}_r^2 = 7,\ p_r^2 = 4.$

Now, model (12) becomes,

$$\max \quad (\lambda_+ - \lambda_-)$$

s.t.,

$$\begin{aligned}
4p_1 - 2p_2 + 1 \geq 4\lambda_+ \ &,\ 2p_1 + 1 \geq 4\lambda_+,\\
-p_1 + p_2 \geq 4\lambda_+ \ &,\ 10p_1 + 7p_2 - 5 \geq 5\lambda_+,\\
24p_1 + 15p_2 - 5 \geq 5\lambda_+ \ &,\ 9p_1 + 11p_2 - 5 \geq 5\lambda_+,\\
4p_1 - 2p_2 - 2 \geq -6\lambda_- \ &,\ 2p_1 - 2 \geq -6\lambda_-,\\
-p_1 + p_2 - 2 \geq -6\lambda_- \ &,\ 10p_1 + 7p_2 - 7 \geq -4\lambda_-,\\
24p_1 + 15p_2 - 7 \geq -4\lambda_- \ &,\ 9p_1 + 11p_2 - 7 \geq -4\lambda_-,\\
p_1 + p_2 = 1 \ &,\ \lambda_+ + \lambda_- \leq 1,\\
p_1, p_2 \geq 0 \ &,\ \lambda_+ \geq \lambda_-, \lambda_- \geq 0.
\end{aligned} \tag{18}$$

The optimal solution for P_1 is obtained as;
$\left(p^* = (0.3750,\ 0.6250)^T,\ {\lambda_+}^* = 0.3125,\ {\lambda_-}^* = 0.2917\right).$
For P_2, we take $\underline{v}_a^1 = -2,\ q_a^1 = 5,\ \underline{v}_r^1 = 0,\ q_r^1 = 4$ and $\underline{v}_a^2 = 7,\ q_a^2 = 4,\ \underline{v}_r^2 = 10,\ q_r^2 = 5.$
Model (17) is reduced as follows,

$$\max \quad (\eta_+ - \eta_-)$$

s.t.,

$$\begin{aligned}
4q_1 + 2q_2 - q_3 - 3 \leq -5\eta_+ \ &,\ -2q_1 + q_3 - 3 \leq -5\eta_+,\\
10q_1 + 24q_2 + 9q_3 - 11 \leq -4\eta_+ \ &,\ 7q_1 + 15q_2 + 11q_3 - 11 \leq -4\eta_+,\\
4q_1 + 2q_2 - q_3 \leq -4\eta_- \ &,\ -2q_1 + q_3 \leq -4\eta_-,\\
10q_1 + 24q_2 + 9q_3 - 10 \leq -5\eta_- \ &,\ 7q_1 + 15q_2 + 11q_3 - 10 \leq -5\eta_-,\\
q_1 + q_2 + q_3 = 1 \ &,\ \eta_+ + \eta_- \leq 1,\\
q_1, q_2, q_3 \geq 0 \ &,\ \eta_+ \geq \eta_-, \eta_- \geq 0.
\end{aligned} \tag{19}$$

The optimal solution for P_2 is obtained as;
$\left(q^* = (0.25,\ 0,\ 0.75)^T,\ {\eta_+}^* = 0.25,\ {\eta_-}^* = 0.0625\right).$
These results are calculated by TORA software.

6 Conclusions

A solution procedure is introduced for MOMGIFG in this paper. This work shows that the strategic problems for both players are equivalent to two LPP. An example is given to show the existence of this theory. The author intends to study a case in which assumption (H_4) is violated, i.e., entries of payoff matrices having fuzziness in future.

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Fuzzy Integrated Super-Efficiency Slack Based Measure Model

Alka Arya and Shiv Prasad Yadav

Abstract Super-efficiency slack based measure (SESBM) model is a non radial approach proposed by Tone (J Oper Res 143:32–41, [19]) to rank the efficient DMUs. This model is extended to the additive SBM model by Du et al. (Eur J Oper Res 204(3):694–697, [12]). In additive SBM model, first find the efficient–inefficient DMUs and then apply the super-efficiency model to determine the super-efficiencies for efficient DMUs. This is time consuming model. Guo et al. (Omega 67:160–167, [14]) proposed an integrated super-efficiency SBM (ISESBM) model to find the super-efficiencies of the efficient DMUs. In this paper, we extend ISESBM model proposed by Guo et al. (Omega 67:160–167, [14]) to the fuzzy ISESBM using expected values of fuzzy numbers. Also, we propose a new approach to find the fuzzy input–output projections which help to make inefficient DMUs as efficient one in fuzzy environment.

Keywords Additive SBM model · Integrated super-efficiency SBM model
Fuzzy integrated super-efficiency SBM model · Fuzzy input–output projections

1 Introduction

Data envelopment analysis (DEA) is a linear programming based non-parametric technique for determining the relative efficiencies of decision making units (DMUs) which produce multiple outputs by making use of multiple inputs. Charnes et al. [5] proposed CCR DEA model which determines the performance efficiencies of DMUs. CCR DEA model is a radial efficiency model. In radial efficiency model, a DMU is weakly efficient if efficiency score is equal to one and at least one slack is zero and a DMU is strongly efficient if efficiency score is equal to one and all

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slacks are zero [5]. Charnes et al. [6] proposed an additive DEA model to determine the efficiencies of DMUs in terms of input excesses and output shortfalls directly. In additive DEA model, a DMU is efficient if slacks (excesses and shortfalls) are zero. Tone et al. [18] proposed a non-radial measure model based on slacks directly which is known as slack based measure (SBM) model. Andersen et al. [2] proposed a super-efficiency SBM (SESBM) model which is based on constant returns to scale (CRS), and the efficiencies of efficient DMUs in SESBM model are greater than or equal to one. The super-efficiency model may be infeasible if SESBM model is based on variable returns to scale (VRS) [8–10, 16]. Du et al. [12] extended the super-efficiency SBM model to additive super-efficiency SBM model. In this model, we first find the efficient DMUs and then apply additive super-efficiency SBM model to determine the super-efficiencies of efficient DMUs. Guo et al. [14] proposed a model to determine the efficiencies and super-efficiencies, and this model is known as integrated SESBM (ISESBM) model.

Real world problems have some input–output data which possess some degrees of imprecision or uncertainties. The imprecision can take the form of ordinal relations, intervals, fuzzy numbers, etc. [20, 21] There are some studies of fuzzy DEA (FDEA) in different areas [1, 4, 11, 17]. Hsiao et al. [15] proposed fuzzy SESBM (FSESBM) model. Due to uncertainty in data of real life problems, we extend crisp integrated SESBM (ISESBM) model to fuzzy integrated SESBM (FISESBM) model using fuzzy numbers in DEA. FISESBM model represents real world applications more realistically than the conventional ISESBM model. FISESBM model determines both the efficiencies and super-efficiencies of DMUs in one step. In this paper, we also extend crisp posterior super-efficiency (PSE) to fuzzy posterior super-efficiency (FPSE) by making the use of fuzzy numbers specially triangular fuzzy numbers (TFNs). Also, this study determines the fuzzy input projections and fuzzy output projection for inefficient DMUs.

The rest of the paper is organized as follows: Section 2 presents the preliminaries. Section 3 presents the proposed fuzzy integrated super-efficiency SBM model. Section 4 presents the illustrative example. Section 5 presents the conclusions of the study.

2 Preliminaries

Definition 1 (*Fuzzy Number (FN)*) An FN $\widetilde{M}$ [3] is defined as a convex fuzzy set $\widetilde{M}$ of the real line $\mathbb{R}$ such that

(1) there exists exactly one $x_0 \in \mathbb{R}$ with $\mu_{\widetilde{M}}(x_0) = 1$. x_0 is called the mean value of $\widetilde{M}$,

(2) $\mu_{\widetilde{M}} : \mathbb{R} \to [0, 1]$ is a piecewise continuous function, called the membership function of $\widetilde{M}$.

Definition 2 (*Triangular Fuzzy Number (TFN)*) The TFN $\widetilde{M}$ [3] is an FN denoted by $\widetilde{M} = (a, b, c)$ and is defined by the membership function $\mu_{\tilde{M}}$ given by

$$\mu_{\tilde{M}}(x) = \begin{cases} \frac{x-a}{b-a}, \ a < x \leq b, \\ \frac{c-x}{c-b}, \ b \leq x < c, \\ 0, \ elsewhere, \end{cases}$$

for all $x \in \mathbb{R}$, where b is the modal value of (a, b, c) and (a, c) is called the support of (a, b, c).

Definition 3 (*Arithmetic operations on TFNs*) Let $\widetilde{M}_1 = (a_1, b_1, c_1)$ and $\widetilde{M}_2 = (a_2, b_2, c_2)$ be two TFNs. Then, the arithmetic operations on TFNs [3] are given as follows:

- Addition: $\widetilde{M}_1 \oplus \widetilde{M}_1 = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$.
- Subtraction: $\widetilde{M}_1 \ominus \widetilde{M}_2 = (a_1 - c_2, b_1 - b_2, c_1 - a_2)$.
- Multiplication: $\widetilde{M}_1 \otimes \widetilde{M}_2 = (min(a_1a_2, a_1c_2, c_1a_2, c_1c_2), b_1b_2, max(a_1a_2, a_1c_2, c_1a_2, c_1c_2))$
- Scalar multiplication:

$$\lambda\tilde{M}_1 = \begin{cases} (\lambda a_1, \lambda b_1, \lambda c_1), \, for \, \lambda \geq 0 \\ (\lambda c_1, \lambda b_1, \lambda a_1), \, for \, \lambda < 0 \end{cases}$$

Definition 4 (*Expected values of FNs*) [13] The expected interval (EI) [3] of a TFN $\widetilde{M} = (a, b, c)$ defined as follows:
$EI(\widetilde{M}) = [E^L(\widetilde{M}), E^U(\widetilde{M})]$, where $E^L(\widetilde{M}) = \frac{a+b}{2}$ and $E^R(\widetilde{M}) = \frac{b+c}{2}$.
Expected value (EV) of a TFN $\widetilde{M} = (a, b, c)$ defined as follows:
$EV(\widetilde{M}) = \frac{1}{2}(E^L(\widetilde{M}) + E^U(\widetilde{M})) = \frac{a+2b+c}{4}$.

Definition 5 (*Ordering of TFNs*) [7] Let $\widetilde{M_1} = (a_1, b_1, c_1)$ and $\widetilde{M_2} = (a_2, b_2, c_2)$ be the two TFNs. Then
$\widetilde{M_1} \geq \widetilde{M_2} \iff EV(\widetilde{M_1}) \geq EV(\widetilde{M_2})$
$\widetilde{M_1} \leq \widetilde{M_2} \iff EV(\widetilde{M_1}) \leq EV(\widetilde{M_2})$
$\widetilde{M_1} = \widetilde{M_2} \iff EV(\widetilde{M_1}) = EV(\widetilde{M_2})$
$\widetilde{M_1} \leq \widetilde{M_2} \ or \, \widetilde{M_2} \geq \widetilde{M_1} \ if \ \min(\widetilde{M_1}, \widetilde{M_2}) = \widetilde{M_1}$
$\widetilde{M_1} \geq \widetilde{M_2} \ or \, \widetilde{M_2} \leq \widetilde{M_1} \ if \ \max(\widetilde{M_1}, \widetilde{M_2}) = \widetilde{M_1}$.

2.1 Slack-Based Measure (SBM) Model

Assume that the performance of a set of n homogeneous DMUs ($DMU_j, j = 1, 2, 3, \ldots, n$) be measured. The performance of DMU_j is described by a production process of m inputs x_{ij} (i=1,2,3,…,m) to produce s outputs y_{rj} (r=1,2,3,…,s) [3]. Let x_{ij} be the amount of the ith input used and y_{rj} be the amount of the rth output produced by DMU_j [3]. Assume that the input–output data are positive. The primal

Table 1 SBM and SESBM models

SBM model	SESBM model
$\min \rho_k = \dfrac{1-(1/m)\sum\limits_{i=1}^{m} w_{ik}^-/x_{ik}}{1+(1/s)\sum\limits_{r=1}^{s} w_{rk}^+/y_{rk}}$	$\min \tau_k = \dfrac{\frac{1}{m}\sum\limits_{i=1}^{m} x_i/x_{ik}}{\frac{1}{s}\sum\limits_{r=1}^{s} y_r/y_{rk}}$
$x_{ik} = \sum\limits_{j=1}^{n} x_{ij}\mu_{jk} + w_{ik}^- \ \forall i$	s.t. $x_i \geq \sum\limits_{j=1, \neq k}^{n} x_{ij}\mu_{jk} \ \forall i,$
$y_{rk} = \sum\limits_{j=1}^{n} y_{rj}\mu_{jk} - w_{rk}^+ \ \forall r$	$y_r \leq \sum\limits_{j=1, \neq k}^{n} y_{rj}\mu_{jk} \ \forall r,$
$\mu_{jk} \geq 0, \ \forall j = 1, 2, 3, \ldots, n,$ $\forall k = 1, 2, 3, \ldots, n,$	$\mu_{jk} \geq 0, \ \forall j = 1, 2, 3, \ldots, n,$ $\forall k = 1, 2, 3, \ldots, n$
$w_{ik}^- \geq 0 \ \forall i = 1, 2, 3, \ldots, m, \ w_{rk}^+ \geq 0$ $\forall r = 1, 2, 3, \ldots, s.$	$x_i \geq x_{ik} \ \forall i = 1, 2, 3, \ldots, m, \ 0 \leq y_r \leq$ $y_{rk} \ \forall r = 1, 2, 3, \ldots, s.$

SBM model [18] for DMU_k is given by the SBM model in Table 1. In Table 1, w_{ik}^- is ith input slack and w_{rk}^+ is the rth output slack of DMU_k.

Definition 6 ρ_k is called SBM efficiency (SBME) of DMU_k. Let ρ_k^* be optimal value of ρ_k. DMU_k is SBM efficient if $\rho_k^* = 1$ [18].

Theorem 1 $\rho_k^* = 1 \iff w_{ik}^{-*} = 0$ *and* $w_{rk}^{+*} = 0$, *i.e., no input excesses and no output shortfalls in optimal solution; otherwise,* DMU_k *is inefficient* [18].

SBM model determines the efficiencies of the DMUs and also determines the efficient and inefficient DMUs. If DMUs are efficient, then we apply super-efficiency SBM (SESBM) model which is given in Table 1.

SESBM efficiencies greater than or equal to one for efficient DMUs. SESBM model provides the efficiency equal to one for inefficient DMUs. Therefore, we cannot differentiate the efficient and inefficient DMUs by SESBM model if SESBM efficiency is equal to one.

2.2 Additive SBM (ASBM) and Additive SESBM (ASESBM)

Charnes et al. [6] developed an additive SBM (ASBM) model to determine the efficiencies of DMUs. ASBM model is given in Table 2. In Table 2, s_{ik}^- and s_{rk}^+ are the input excess and output shortage, respectively.

Du et al. [12] proposed an additive SESBM (ASESBM) model which is given in Table 2. In Table 2, t_{ik}^+ and t_{rk}^- are the input saving and output surplus, respectively.

The posterior additive efficiency [14] of ASBM model is defined as follows

$$\alpha_k^* = \frac{\frac{1}{m}\sum_{i=1}^{m} (x_{ik} - s_{ik}^{-*})/x_{ik}}{\frac{1}{s}\sum_{i=1}^{s} (y_{rk} + s_{rk}^{+*})/y_{rk}}$$

Table 2 ASBM and ASESBM models

ASBM model	ASESBM model
$\max a_k = \sum_{i=1}^{m} s_{ik}^- + \sum_{r=1}^{s} s_{rk}^+$	$\min b_k = \sum_{i=1}^{m} t_{ik}^+ + \sum_{r=1}^{s} t_{rk}^-$
s.t. $\sum_{j=1}^{n} x_{ij}\mu_{jk} = x_{ik} - s_{ik}^- \ \forall i,$	s.t. $\sum_{j=1,\neq k}^{n} x_{ij}\mu_{jk} \leq x_{ik} + \sum_{i=1}^{m} t_{ik}^+, \ \forall i,$
$\sum_{j=1}^{n} y_{rj}\mu_{jk} = y_{rk} + s_{rk}^+ \ \forall r,$	$\sum_{j=1,\neq k}^{n} y_{rj}\mu_{jk} \geq y_{rk} - \sum_{r=1}^{s} t_{rk}^- \ \forall r,$
$\mu_{jk} \geq 0 \ \forall j = 1, 2, 3, \ldots, n,$ $\forall k = 1, 2, 3, \ldots, n,$	$\mu_{jk} \geq 0, \ \forall j = 1, 2, 3, \ldots, n,$ $\forall k = 1, 2, 3, \ldots, n, \ j \neq k,$
$s_{ik}^- \geq 0 \ \forall i = 1, 2, 3, \ldots, m, \ s_{rk}^+ \geq 0$ $\forall r = 1, 2, 3, \ldots, s.$	$t_{ik}^+ \geq 0 \ \forall i = 1, 2, 3, \ldots, m, \ t_{rk}^- \geq 0$ $\forall r = 1, 2, 3, \ldots, s.$

where s_{ik}^{-*} and s_{rk}^{+*} are the optimal values of s_{ik}^- and s_{rk}^+ of ASBM model, respectively.

The posterior additive super-efficiency (ASE) [14] of ASESBM model is defined as follows

$$\bar{\alpha}_k^* = \frac{\frac{1}{m}\sum_{i=1}^{m}(x_{ik} + t_{ik}^{+*})/x_{ik}}{\frac{1}{s}\sum_{i=1}^{s}(y_{rk} - t_{rk}^{-*})/y_{rk}}$$

where t_{ik}^{+*} and t_{rk}^{-*} are the optimal values of t_{ik}^+ and t_{rk}^- of ASESBM model, respectively.

Similar to SBM and SESBM models, ASBM model is used to determine the efficiencies of DMUs and the efficient and inefficient DMUs are determined. Then ASESBM model is applied to measure the efficiency of efficient DMUs.

3 Proposed Fuzzy Integrated Super-Efficiency SBM Model

3.1 Integrated Super-Efficiency SBM (ISESBM) Model

Guo et al. [14] proposed an integrated SESBM (ISESBM) model. In Model 1, s_{ik}^- and s_{rk}^+ are the inefficiency slacks, and t_{ik}^+ and t_{rk}^- are the super-efficient slacks [14]. In this model, Guo et al. [14] determined the super-efficiency slacks first and then inefficiency slacks i.e., $t_{ik}^+ + t_{rk}^-$ is first minimized and then $s_{ik}^- + s_{rk}^+$ is maximized.

Model 1
$$\min \eta_k = \sum_{i=1}^{m} t_{ik}^+ + \sum_{r=1}^{s} t_{rk}^- - \varepsilon(\sum_{i=1}^{m} s_{ik}^- + \sum_{r=1}^{s} s_{rk}^+)$$
$$\text{s.t.} \sum_{j=1,\neq k}^{n} x_{ij}\mu_{jk} = x_{ik} + t_{ik}^+ - s_{ik}^-, \ \forall i,$$
$$\sum_{j=1,\neq k}^{n} y_{rj}\mu_{jk} = y_{rk} - t_{rk}^- + s_{rk}^+, \ \forall r,$$

$$\mu_{jk} \geq 0,\ \forall j = 1, 2, 3, \ldots, n,\ j \neq k,\ \ \forall k = 1, 2, 3, \ldots, n,$$
$$t_{rk}^{-} \geq 0,\ \ t_{ik}^{+} \geq 0,\ \ s_{ik}^{-} \geq 0,\ \ s_{rk}^{+} \geq 0\, \forall i = 1, 2, 3, \ldots, m,$$
$$\forall r = 1, 2, 3, \ldots, s.$$

Let $s_{ik}^{-*},\ s_{rk}^{+*},\ t_{ik}^{+*},\ t_{rk}^{-*}$ be the optimal values of $s_{ik}^{-},\ s_{rk}^{+},\ t_{ik}^{+},\ t_{rk}^{-}$. Then the posterior efficiency (PE) of the ISESBM model is denoted by δ_k^* and is defined by

$$\delta_k^* = \begin{cases} \dfrac{\frac{1}{m}\sum_{i=1}^{m}(x_{ik}-s_{ik}^{-*})/x_{ik}}{\frac{1}{s}\sum_{r=1}^{s}(y_{rk}+s_{rk}^{+*})/y_{rk}}, & \text{if} \quad (\sum_{i=1}^{m} t_{ik}^{+*} + \sum_{r=1}^{s} t_{rk}^{-*}) = 0, \\ \dfrac{\frac{1}{m}\sum_{i=1}^{m}(x_{ik}+t_{ik}^{+*})/x_{ik}}{\frac{1}{s}\sum_{r=1}^{s}(y_{rk}-t_{rk}^{-*})/y_{rk}}, & \text{elsewhere.} \end{cases}$$

Definition 7 *DMU_k* is said to be efficient if $\delta_k^* > 1$ and is said to be inefficient if $\delta_k^* \leq 1$.

3.2 Fuzzy Integrated Super-Efficiency SBM (FISESBM) Model

In conventional ISESBM model, the input data and output data are crisp values. But in the real world applications, these data may have fuzzy values [3]. Therefore, in this paper, we have taken fuzzy input–output data as TFNs. Let $\tilde{x}_{ij}$ be the fuzzy input and $\tilde{y}_{rj}$ be the fuzzy output for DMU_j. Model 1 reduces to Model 2 as given below:

Model 2 $$\min \tilde{\eta}_k = \sum_{i=1}^{m} t_{ik}^{+} + \sum_{r=1}^{s} t_{rk}^{-} - \varepsilon(\sum_{i=1}^{m} s_{ik}^{-} + \sum_{r=1}^{s} s_{rk}^{+})$$
$$\text{s.t.} \sum_{j=1,\neq k}^{n} \tilde{x}_{ij}\mu_{jk} = \tilde{x}_{ik} + t_{ik}^{+} - s_{ik}^{-},\ \forall i,$$
$$\sum_{j=1,\neq k}^{n} \tilde{y}_{rj}\mu_{jk} = \tilde{y}_{rk} - t_{rk}^{-} + s_{rk}^{+},\ \forall r,$$
$$\mu_{jk} \geq 0,\ \forall j = 1, 2, 3, \ldots, n,\ j \neq k,\ \ \forall k = 1, 2, 3, \ldots, n,$$
$$t_{rk}^{-} \geq 0\ \ t_{ik}^{+} \geq 0,\ \ s_{ik}^{-} \geq 0,\ \ s_{rk}^{+} \geq 0,\ \forall i = 1, 2, 3, \ldots, m,\ \forall$$
$$r = 1, 2, 3, \ldots, s.$$

Model 2 is known as FISESBM model. The posterior efficiency (PE) of the FISESBM model is denoted as $\tilde{\delta}_k^*$ and is defined as given below:

$$\tilde{\delta}_k^* = \begin{cases} \dfrac{\frac{1}{m}\sum_{i=1}^{m}(\tilde{x}_{ik}-s_{ik}^{-*})/\tilde{x}_{ik}}{\frac{1}{s}\sum_{r=1}^{s}(\tilde{y}_{rk}+s_{rk}^{+*})/\tilde{y}_{rk}}, & \text{if} \quad (\sum_{i=1}^{m} t_{ik}^{+*} + \sum_{r=1}^{s} t_{rk}^{-*}) = 0, \\ \dfrac{\frac{1}{m}\sum_{i=1}^{m}(\tilde{x}_{ik}+t_{ik}^{+*})/\tilde{x}_{ik}}{\frac{1}{s}\sum_{r=1}^{s}(\tilde{y}_{rk}-t_{rk}^{-*})/\tilde{y}_{rk}}, & \text{elsewhere.} \end{cases}$$

Taking $\tilde{x}_{ij} = (x^l_{ij}, x^m_{ij}, x^u_{ij})$ and $\tilde{y}_{rj} = (y^l_{rj}, y^m_{rj}, y^u_{rj})$, Model 2 is reduced to Model 3.

$$\text{Model 3}\quad \min(\eta_k) = \sum_{i=1}^{m} t^+_{ik} + \sum_{r=1}^{s} t^-_{rk} - \varepsilon(\sum_{i=1}^{m} s^-_{ik} + \sum_{r=1}^{s} s^+_{rk})$$
$$\text{s.t.} \sum_{j=1,\neq k}^{n} \mu_{jk}(x^l_{ij}, x^m_{ij}, x^u_{ij}) = (x^l_{ik}, x^m_{ik}, x^u_{ik}) + t^+_{ik} - s^-_{ik},\ \forall i,$$
$$\sum_{j=1,\neq k}^{n} \mu_{jk}(y^l_{rj}, y^m_{rj}, y^u_{rj}) = (y^l_{rk}, y^m_{rk}, y^u_{rk}) - t^-_{rk} + s^+_{rk},\ \forall r,$$
$$\mu_{jk} \geq 0,\ \forall j = 1, 2, 3, \ldots, n,\ j \neq k,\ \ \forall k = 1, 2, 3, \ldots, n,$$
$$t^-_{rk} \geq 0\ \ t^+_{ik} \geq 0,\ \ s^-_{ik} \geq 0,\ \ s^+_{rk} \geq 0,\ \forall i = 1, 2, 3, \ldots, m,$$
$$\forall r = 1, 2, 3, \ldots, s.$$

Model 3 is known as the proposed FISESBM model. Using expected values of TFNs, Model 3 is reduced to Model 4.

$$\text{Model 4}\quad \min \eta_k = \sum_{i=1}^{m} t^+_{ik} + \sum_{r=1}^{s} t^-_{rk} - \varepsilon(\sum_{i=1}^{m} s^-_{ik} + \sum_{r=1}^{s} s^+_{rk})$$
$$\text{s.t.} \sum_{j=1,\neq k}^{n} \mu_{jk}(x^l_{ij} + 2x^m_{ij} + x^u_{ij})/4 = (x^l_{ik} + 2x^m_{ik} + x^u_{ik})/4 + t^+_{ik} - s^-_{ik},\ \forall i,$$
$$\sum_{j=1,\neq k}^{n} \mu_{jk}(y^l_{rj} + 2y^m_{rj}, y^u_{rj})/4 = (y^l_{rk} + 2y^m_{rk} + y^u_{rk})/4 - t^-_{rk} + s^+_{rk},\ \forall r,$$
$$\mu_{jk} \geq 0,\ \forall j = 1, 2, 3, \ldots, n,\ j \neq k,\ \ \forall k = 1, 2, 3, \ldots, n,$$
$$t^-_{rk} \geq 0\ \ t^+_{ik} \geq 0,\ \ s^-_{ik} \geq 0,\ \ s^+_{rk} \geq 0,\ \forall i = 1, 2, 3, \ldots, m,$$
$$\forall r = 1, 2, 3, \ldots, s.$$

3.3 Proposed Posterior Super-Efficiency (PPSE)

The posterior super-efficiency of the Model 4 is denoted by χ^*_k and is defined by

$$\chi^*_k = \begin{cases} \dfrac{\frac{1}{m}\sum_{i=1}^{m}((x^l_{ik}+2x^m_{ik}+x^u_{ik})/4 - s^{-*}_{ik})/((x^l_{ik}+2x^m_{ik}+x^u_{ik})/4)}{\frac{1}{s}\sum_{r=1}^{s}(((y^l_{rk}+2y^m_{rk}+y^u_{rk})/4 + s^{+*}_{rk})/(y^l_{rk}+2y^m_{rk}+y^u_{rk})/4)}, & \text{if}\ \ \sum_{i=1}^{m} t^{+*}_{ik} + \sum_{r=1}^{s} t^{-*}_{rk} = 0, \\ \dfrac{\frac{1}{m}\sum_{i=1}^{m}((x^l_{ik}+2x^m_{ik}+x^u_{ik})/4 + t^{+*}_{ik})/(x^l_{ik}+2x^m_{ik}+x^u_{ik})/4}{\frac{1}{s}\sum_{r=1}^{s}((y^l_{rk}+2y^m_{rk}+y^u_{rk})/4 - t^{-*}_{rk})/(y^l_{rk}+2y^m_{rk}+y^u_{rk})/4}, & \text{elsewhere.} \end{cases}$$

Definition 8 Let χ^*_k be optimal value of χ_k. Then, DMU_k is said to be efficient if $\chi^*_k > 1$ and inefficient if $\chi^*_k \leq 1$.

3.4 Projection

The projection frontier of DMU_k is $(\bar{x}_{ik} = x_{ik} + t_{ik}^{+*} - s_{ik}^{-*}, \bar{y}_{rk} = y_{rk} - t_{rk}^{-*} + s_{rk}^{+*})$ for ISESBM model [14]. The projection frontier of DMU_k is $(\widetilde{X}_{ik} = \tilde{x}_{ik} + t_{ik}^{+*} - s_{ik}^{-*}, \widetilde{Y}_{rk} = \tilde{y}_{rk} - t_{rk}^{-*} + s_{rk}^{+*})$ for PFISESBM model.

4 Illustrative Example

In this section, we provide an example to illustrate the PFISESBM model. Let there be taken two fuzzy inputs: (i) $\tilde{x}_{1j}$, (ii) $\tilde{x}_{2j}$, and two fuzzy outputs: (i) $\tilde{y}_{1j}$, (ii) $\tilde{y}_{2j}$ as shown in Table 3, j = 1,2,3,…,10.

The inefficiency slacks and SE slacks of each DMU are determined using PFISESBM model (Model 4) as shown in Table 4. The PPSE, χ_k^*, for each DMU_k is determined and is shown in Table 4. In Table 4, DMUs 1 and 4 are inefficient other DMUs are efficient.

Finally, we determine the fuzzy input projections ($\widetilde{X}_{1j}$ and $\widetilde{X}_{2j}$) and fuzzy output projections ($\widetilde{Y}_{1j}$ and $\widetilde{Y}_{2j}$) discussed in Subsection 3.4 which are shown in Table 5. From fuzzy input projections, we conclude that for DMU 1, the first fuzzy input should be increased from (2.8, 3.6, 4.4) to (3.4, 4.2, 5) to become efficient. Similar conclusion can be drawn for other DMUs.

Table 3 Fuzzy input and fuzzy output data for 10 DMUs

DMUs	Fuzzy inputs		Fuzzy outputs	
	$\tilde{x}_{1j}$	$\tilde{x}_{2j}$	$\tilde{y}_{1j}$	$\tilde{y}_{2j}$
1	(2.8, 3.6, 4.4)	(5.8, 6.4, 7.3)	(6.9, 7.3, 7.7)	(3.8, 4.6, 4.9)
2	(1.8, 2.5, 3.2)	(3.5, 3.9, 4.5)	(2.8, 3.5, 4.0)	(5.2, 5.8, 6.6)
3	(4.2, 4.7, 5.2)	(3.1, 3.5, 3.9)	(2.1, 2.7, 3.3)	(4.9, 5.6, 6.2)
4	(2.2, 2.6, 3.0)	(3.5, 4.0, 4.5)	(3.7, 4.2, 4.7)	(7.9, 8.5, 9.1)
5	(5.5, 6.0, 6.7)	(4.6, 5.1, 5.6)	(5.2, 5.5, 5.7)	(6.8, 7.6, 8.4)
6	(3.1, 3.5, 3.9)	(4.5, 4.5, 4.5)	(3.6, 3.9, 4.2)	(7.1, 7.5, 7.9)
7	(4.6, 4.9, 5.2)	(6.2, 6.8, 7.2)	(1.9, 2.5, 3.1)	(8.1, 8.5, 8.9)
8	(4.3, 4.8, 5.3)	(6.1, 6.3, 6.5)	(1.3, 1.7, 2.1)	(3.1, 3.7, 4.3)
9	(5.9, 6.5, 7.4)	(5.1, 5.6, 6.4)	(2.6, 2.9, 3.2)	(7.1, 7.6, 8.1)
10	(7.6, 7.9, 8.2)	(4.6, 5.0, 5.4)	(2.7, 2.9, 3.1)	(3.2, 3.5, 3.8)

Table 4 Slacks and efficiencies

DMUs	Input slacks		Output slacks		Input slacks		Output slacks		PPSEs
	s_{1j}^-	s_{2j}^-	s_{1j}^+	s_{2j}^+	t_{1j}^+	t_{2j}^+	t_{1j}^-	t_{2j}^-	χ_j^*
1	0	0	0	9.284	0.6087	0	0.5012	0	1.5512
2	0	0.1038	0.588	2.32	0	0	0	0	0.444
3	0.0053	2.32	0	0.975	1.862	0	0	0	0.0053
4	2.5789	0	0	0	1.169	0.846	0	0.4230	2.5789
5	2.88	0	0	1.014	0	0	0	0	0.0378
6	0.575	0	0.825	2.062	0	0	0	0	0.343
7	0.5125	0	4.5875	5.843	0	0	0	0	0.1308
8	0.705	0	4.915	9.6875	0	0	0	0	0.0588
9	2.8862	0	3.058	4.459	0	0	0	0	0.285
10	4.65	0	2.35	7.125	0	0	0	0	0.3509

Table 5 Fuzzy input–output projections for 10 DMUs

DMUs	Fuzzy input projections		Fuzzy output projections	
	$\widetilde{X}_{1j}$	$\widetilde{X}_{2j}$	$\widetilde{Y}_{1j}$	$\widetilde{Y}_{2j}$
1	(3.4, 4.2, 5)	(5.8, 6.4, 7.3)	(6.39, 6.79, 7.19)	(13.1, 13.8, 14.2)
2	(1.8, 2.5, 3.2)	(3.39, 3.79, 4.39)	(3.38, 4.08, 4.58)	(7.5, 8.1, 8.9)
3	(6.05, 6.55, 7.05)	(0.781.18, 1.58)	(2.1, 2.7, 3.3)	(5.87, 6.5, 7.17)
4	(0.79, 1.19, 1.59)	(4.34, 4.84, 5.36)	(3.7, 4.2, 4.7)	(7.47, 8.07, 8.66)
5	(2.62, 3.12, 3.82)	(4.6, 5.1, 5.6)	(5.2, 5.5, 5.7)	(7.81, 8.61, 9.41)
6	(2.52, 2.92, 3.32)	(4.5, 4.5, 4.5)	(4.42, 4.72, 5.02)	(9.16, 9.56, 9.96)
7	(4.08, 4.38, 4.68)	(6.2, 6.8, 7.2)	(6.48, 7.08, 7.68)	(13.94, 14.34, 14.74)
8	(3.59, 4.09, 4.59)	(6.1, 6.3, 6.5)	(6.21, 6.61, 7.01)	(12.78, 13.38, 13.98)
9	(3.0, 3.6, 4.51)	(5.1, 5.6, 6.4)	(5.65, 5.95, 6.25)	(11.55, 12.05, 12.56)
10	(2.95, 3.25, 3.55)	(4.6, 5.0, 5.4)	(5.05, 5.25, 5.45)	(10.32, 10.62, 10.92)

5 Conclusion

The real world applications' data have some degree of uncertainties. To deal with such data, we have considered them as TFNs. In this paper, we extended the work of Guo et al. [14] and proposed fuzzy integrated super-efficiency SBM (FISESBM) model. Also, we have proposed a proposed posterior super-efficiency (PPSE) to determine the super-efficiencies of DMUs. To ensure the validity of the proposed models, we have considered the performance of 10 DMUs with two fuzzy inputs and two fuzzy outputs. We also determined the fuzzy input–output projections of DMUs (Table 5). The FISESBM model is more effective for real world applications.

The uncertainty in this paper is limited to fuzzy environment. In future, we plan to extend the present work to the intuitionistic fuzzy environment to determine the super-efficiencies of real world applications.

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Prioritizing Factors Affecting the Adoption of Mobile Financial Services in Emerging Markets—A Fuzzy AHP Approach

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Abstract Mobile financial services (MFSs) such as 'mobile banking' and 'mobile payments' have revolutionized the global banking and financial industry by bringing financial services closer to the consumers. Successful diffusion of various types of MFSs depends on their acceptance and adoption by the end-users (customers). Also, customers make trade-offs while choosing MFSs on the basis of various factors that are important to them. The present study attempts to find the relative importance of various factors that influence the customers' choice of MFSs. The study also prioritizes three MFSs, namely mobile banking, prepaid instruments (PPIs) and payments banks on the basis of multiple factors. The present problem is modelled as a multiple-criteria decision-making (MCDM) problem, wherein fuzzy analytic hierarchy process (FAHP) is used to rank the potential factors of MFS selection and to evaluate various MFSs. The findings of the study reveal that functional benefits and economic benefits dominate over trust and perceived risks in customers' decision-making regarding the selection of MFSs. With regard to the evaluation of the three MFSs, the findings indicate that payments bank is the superior choice as it offers best economic and functional benefits and involves minimum risks. The findings of the study may help MFS providers, to evaluate critical factors of adoption of MFSs. This may help them in achieving cost-effective implementation of MFSs by efficiently managing their resources.

Keywords Mobile financial services · Multiple-criteria decision-making
Fuzzy analytic hierarchy process · Mobile banking · Prepaid instruments
Payments banks

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1 Introduction

Over the past few years, the proliferation of mobile technology to use banking and financial services has been a major development in the financial services market. The use of mobile phones for providing financial services has changed the dynamics of the banking and financial industry, bringing financial services closer to the consumers. Mobile financial services (MFS) encompassing mobile banking and mobile payment have become increasing prevalent nowadays. The MFSs in emerging markets like India can be segmented into three categories: mobile banking, PPIs (prepaid instruments or mobile wallets) and payment banks. **Mobile banking** is offered by banks wherein customers can carry out most of their bank transactions by using their smartphones through a mobile application. These transactions may include transferring of funds, checking account activity, viewing balance, applying for chequebook, paying loan amount, opening a term deposit, recharging phone, or booking movie tickets. Examples of various mobile banking applications in emerging markets like India include—Kotak Bank App for Kotak Mahindra bank, iMobile for ICICI bank and SBI freedom app for State bank of India [1]. **PPI** or **Mobile wallet** (e.g. gift cards by Axis Bank, food card issued by HDFC Bank; Metro Card, Flipkart e-wallet, Oxigen, Paytm, Mobikwik) is a cashless payment service, provided by certain service providers, wherein people can load a certain amount of money that can be spent at online and offline merchants listed with the mobile wallet service provider [2]. PPIs can be used for paying bills, shopping, booking movie tickets, etc. PPIs can be issued by banks or Non-Banking Finance Companies (NBFCs) in the form of smart cards, magnetic stripe cards, mobile accounts, mobile wallets, paper vouchers and any such instrument which can be used to access the prepaid amount. **Payment Banks** is a recent initiative of the GoI which has been launched in 2014 on the lines of Kenya's payments bank 'M-Pesa'. Payment Banks are stripped-down type of banks, which can be promoted by existing nonbank PPI issuers, NBFCs, corporate Business Correspondents (BCs), mobile telephone companies, supermarket chains, etc. and can provide services like acceptance of demand deposits, issuance of ATM/debit cards, payments and remittance services through various channels, distribution of non-risk sharing simple financial products like mutual fund units and insurance products [3].

Successful diffusion of various types of MFSs depends on their acceptance and adoption by the end-users (customers). Technical issues, perceived risks, lack of trust and security concerns are generally found to be the major reasons behind customers' resistance to adopt these services [4–7], whereas value-added services and perceived monetary and non-monetary benefits are found to be the factors which motivate the users to use these services [8–10, 20]. Hence, it is pertinent to understand and address the needs and behavioural patterns of the customers so as to optimize their experience with MFSs. Moreover, with the availability of several options for MFSs, it is worth-seeing which options are more preferred by customers as different options have their pros and cons. MFS providers can increase their competitiveness if they are able to improve their performance in meeting the demands of customers. Consequently, it is important for them to understand the requirements of MFS users and the relative weights of the factors that determine the users' needs.

Though substantial research has been done towards determining the factors affecting the adoption of various MFSs, viz. mobile banking, mobile payments, most of the previous researches in this area have focused on the general factors related to MFS adoption using multiple regression analysis and structural equation modelling. The present study addresses the problem of MFS selection on the basis of multiple selection criteria (factors). The study attempts to prioritize three MFSs, namely mobile banking, PPIs and payments banks, on the basis of multiple factors which may be considered by the MFS users in India for selecting a particular MFS. Since assessing these factors includes multiple criteria, it can be modelled as a multiple-criteria decision-making (MCDM) problem. This study applies fuzzy analytic hierarchy process (FAHP), one of the MCDM methods, to evaluate the potential factors of MFS adoption/selection and to prioritize three different types of MFSs. Firstly, various factors are identified on the basis of the extant literature review, and their weights (priorities) are determined using AHP. Then, on the basis of these multiple criteria, the three types of MFSs are evaluated by determining their ranks. Practical implications that can be drawn from the findings of this study will assist service providers in understanding customers better and making appropriate decisions regarding delivery of services by allocating their limited resources to the most important factors. The rest of the paper is organized as follows: the next section gives an overview of the related researches. The proposed research methodology is discussed in Sect. 3. The data analysis is done in Sect. 4. Results and discussions are then presented in Sect. 5. Finally, the paper is concluded in Sect. 6.

2 Literature Review

The theoretical background to this study is derived from the literature areas of technology adoption and diffusion of innovation theory concerning consumer adoption of mobile banking and mobile payments. Previous researches have employed many such models such as the theory of planned behaviour (TPB) [11]; the diffusion of innovation theory (DOI) [12]; the technology acceptance model (TAM) [13]; the unified theory of acceptance and use of technology (UTAUT) [14], to study the factors that influence customers' decisions to adopt/use mobile banking and mobile payments. The constructs used in these models are defined in Table 1.

Apart from the models/theories discussed in Table 1, some researchers have used **Initial Trust Model (ITM)** [15] and **Theory of Innovation Resistance (TIR)** [16] to explain the customers' behaviour towards adoption of MFS. Trust comprises of three factors: institution-based trust factors (which include firm's characteristics such as size, capability, credibility, integrity, reputation, brand); environmental forces (which include structural assurances relevant to enhancing trustworthiness such as service guarantees); and personal trust propensity [15]. According to TIR, there are several

Table 1 Technology/innovation adoption models/theories

Model/Theory	Variables	Definition	Reference
TPB	Attitude	Attitude towards a behaviour is the degree to which performance of the behaviour is positively or negatively valued	Aizen [11]
	Subjective norms	The perceived social pressure to engage or not to engage in a behaviour	
	Perceived behavioural control	People's perceptions of their ability to perform a given behaviour	
DOI	Relative advantage	Degree to which an innovation is seen as being superior to its predecessor	Rogers [12]
	Complexity	Degree to which an innovation is seen by the potential adopter as being relatively difficult to use and understand	
	Compatibility	The degree to which an innovation is seen to be compatible with existing beliefs, values, experiences and needs of adopters	
	Trialability	Degree to which an idea can be experimented with, on a limited basis	
	Observability	Degree to which the results of an innovation are visible	
TAM	Perceived usefulness	Degree to which a person believes that using a particular system would enhance his or her performance	Davis et al. [13]
	Perceived ease of use	Degree to which a person believes that using a particular system would be free of effort	

(continued)

Table 1 (continued)

Model/Theory	Variables	Definition	Reference
UTUAT	Performance expectancy	Degree to which an individual believes that using a particular system would improve his or her performance	Venkatesh et al. [14]
	Effort expectancy	Degree of simplicity associated with the use of a particular system	
	Facilitating conditions	Degree to which an individual believes that technical infrastructure exists to support the use of a particular system	
	Social influence	Degree to which an individual perceives that others believe he or she should use a particular system	

barriers, viz. usage, value, risk, tradition and image barriers, due to which customers resist the use of an innovation [16].

Various researchers have employed the above-discussed models/theories and their extensions to study how customers formulate their perceptions, attitudes, intentions and behaviour towards MFSs. Many researchers have combined the constructs of the traditional models and theories (TPB, DOI, TAM and UTAUT) with trust and other value-based constructs, viz. perceived risks and perceived benefits to study the user's intentions to use MFSs. Table 2 summarizes various studies conducted in different contexts for studying the factors influencing customer's intentions to adopt MFSs (mobile banking or mobile payments).

It can be noticed from Table 2 that relative advantage [17–19]; effort expectancy and performance expectancy [20, 21]; and perceived usefulness and perceived ease of use [22–24] significantly influence customers' behavioural intentions to adopt MFSs. This indicates that customers are motivated to use a mobile financial service because they perceive it to be more beneficial over the traditional banking as it is easy to use and saves their time and effort. Facilitating conditions [20, 21] and compatibility [5, 17, 18] are also found to have positive effects on customers' intentions towards adopting MFSs. This implies that an individual who has the technical infrastructure and knowledge required for using mobile banking will be more likely to use the same. The society (friends, relatives, etc.) also plays an important role in influencing individuals for adopting the MFS, especially in the developing countries. Subjective norm is the most influential factor of mobile banking adoption in Thailand [25] and

Table 2 Mobile banking/payment adoption studies

Adoption model/theory	Studies	Country	Main Findings regarding factors influencing customer's intentions to adopt mobile banking/mobile payment
TPB	Tinga et al. [53]	Malaysia	Attitude, subjective norm and perceived behavioural control have positive effect on intention to use mobile payment
UTAUT2 and DOI	Oliveira et al. [54]	Portugal	Compatibility, perceived technology security, performance expectations, innovativeness and social influence have significant direct and indirect effects over the adoption of mobile payment
UTAUT2 and Trust	Alalwan et al. [20]	Jordan	Behavioural intention is significantly and positively influenced by performance expectancy, effort expectancy, facilitating conditions, hedonic motivation, price value and trust
Extended UTAUT	Alam [21]	Bangladesh	Individual intention to adopt mobile banking is significantly influenced by social influence, effort expectancy, performance expectancy, facilitating conditions and perceived financial cost
Extended TAM and Trust	Gu et al. [23]	Korea	Perceived usefulness is directly affected by perceived ease of use, trust and system quality. Perceived ease of use affects behavioural intention through perceived usefulness
Extended TAM	Amin et al. [22]	Malaysia	Human intentions to adopt mobile banking are significantly affected by perceived usefulness, perceived ease of use, perceived credibility, the amount of information and normative pressure
Extended TAM	Luarn and Lin [24]	Taiwan	Perceived self-efficacy, financial costs, credibility, ease of use and usefulness have positive effects on intention to adopt mobile banking
TAM, TPB and IDT	Riquelme and Rios [34]	Singapore	Usefulness, social norms and risk influence the intention to adopt mobile banking
TAM, DOI and Trust	Koenig-Lewis et al. [5]	Germany	Perceived usefulness, compatibility and risk have significant influence, while perceived costs, ease of use, credibility and trust do not have significant influence on consumer's intentions

(continued)

Table 2 (continued)

Adoption model/theory	Studies	Country	Main Findings regarding factors influencing customer's intentions to adopt mobile banking/mobile payment
DOI and TPB	Puschel et al. [19]	Brazil	Relative advantage, visibility, compatibility and perceived ease of use significantly affect attitude; attitude, subjective norm and perceived behavioural control ultimately enrich the customers' intention to adopt mobile banking
TAM and TPB	Sripalawat et al. [25]	Thailand	Subjective norm is the most influential factor, followed by perceived usefulness
Extended TAM	Dasgupta et al. [55]	India	Perceived usefulness, ease of use, image, perceived value, self-efficacy and credibility significantly affect intentions towards mobile banking usage
DOI and Decomposed TPB	Brown et al. [4]	South Africa	Relative advantage, trialability, number of banking services and risk significantly influence mobile banking adoption
DOI	Lee et al. [17]		Relative advantage, compatibility, trialability and complexity play a considerable role in forming customers' attitudes towards mobile banking adoption
DOI	Lin [18]		Relative advantage and compatibility are the key drivers of customers' attitudes towards mobile banking

Brazil [19]. Individual intention to adopt mobile banking is significantly influenced by social influence in Bangladesh [21] and by normative pressure in Malaysia [22].

Various researchers suggest that trust plays a vital role in influencing the customers' intentions to adopt MFS [20, 23]. Kim et al. [26] have explained initial trust in mobile banking by trust propensity, structural assurances and firm reputation. The authors argue that structural assurance in the form of guarantees of protection of information, assurance of transactional confidentiality, and contractual terms and conditions can build the initial trust and confidence in mobile banking services. Zhou [7] has empirically supported the considerable role of a bank's reputation, information quality, service quality and system quality in shaping the customers' initial trust in mobile banking.

On the basis of TIR, Laukkanen et al. [6] have studied five barriers, namely usage, value, risk, tradition and image barriers in adopting mobile banking. Through investigating 1525 usable respondents from a large Scandinavian bank, Laukkanen et al. [6] have identified that the value and usage barriers are the most intense barriers to mobile banking adoption, while tradition barriers (such as preferring to chat with the teller and patronizing the banking office) are not an obstacle. Perceived risk has been commonly observed as a negative factor which hinders the customers' willingness to adopt mobile financial services [4–6]. Brown et al. [4] have also highlighted that perceived risks associated with the use of mobile banking restrain customers from adopting the same. In spite of the numerous benefits of MFS, concerns of risks regarding privacy, security and financial issues are still imperative to the users.

Many studies have emphasized the role of perceived benefits in the adoption of mobile financial services. Brown et al. [4] opine that customers' intentions to adopt mobile banking are influenced by the number of banking services offered by banks through mobile banking. Hence, customers find the mobile banking advantageous if they get a wide range of services through mobile itself without physically going to the bank [4]. Kim et al. [27] opine that customers prefer mobile banking because it provides those benefits to them which traditional offline banking channels cannot provide. Many studies on MFS have concentrated on their functional or non-monetary values, e.g. mobility, convenience, accessibility, time value, usefulness, ease of use, performance expectancy [7, 8, 10], whereas others have focused on their monetary or price value [9, 20].

All the studies discussed above have used statistical techniques such as multiple regression analysis and structural equation modelling to empirically examine the general factors which influence customers' behavioural intentions to adopt MFSs. There are very few studies which have used the MCDM techniques for studying the factors influencing customers' decisions to select MFSs. Natarajan et al. [28] have used AHP to study customers' choices amongst self-service technology (SST) channels in retail banking. The authors conclude that purpose, perceived risks, benefits and requirements are the main criteria to influence customers to choose banking channels. Chou et al. [29] have employed AHP to understand the m-commerce payment systems. Komlan et al. [30] have employed AHP to evaluate and rank the different factors associated with perceived risks that contribute in the adoption of mobile banking in West Africa. Komlan et al. [31] have used AHP to evaluate mobile banking's perceived value on the basis of perceived benefits, perceived costs and perceived risks. Lin [32] have used with an extent analysis approach to develop a fuzzy evaluation model for prioritizing the relative weights of mobile banking quality factors. Recently, Osmani et al. [33] have used AHP method to evaluate e-payment system factors influencing mobile banking use in Iranian banks.

To summarize, we can conclude that traditional studies into customers' adoption of MFS focus on factors which affect their behavioural intentions such as effort expectancy, performance expectancy [20, 21]; perceived usefulness, perceived ease of use [22–24]; social influence [25, 34]; and facilitating conditions [20, 21], whereas studies that take a value-based approach involve assessment of factors that address the potential benefits [4, 8, 10], costs [20, 9], trust [23, 26] and the likely risks [5, 6]. However, certain factors in value-based approaches, viz. potential benefits, are derived on the basis of the constructs of traditional approaches, viz. performance expectancy, effort expectancy, relative advantage and ease of use.

3 Conceptual Framework

Since the purpose of the present study is to deal with the problem of MFS selection, the study focuses on the value-based approach, to identify the criteria which may be considered by the MFS users to select a particular MFS. The study particularly focuses on the benefits of the MFSs, perceived risks associated with the MFSs and the trust on the MFSs, to constitute the MFS selection criteria. On the basis of the theories and studies presented in the previous section, four main factors (main criteria) have been identified which are relevant to the selection of various MFSs in the context of India. These four criteria are 'economic benefits' (based on the 'price value' construct of UTAUT2, 'performance expectancy' construct of UTAUT and 'usefulness' construct of TAM); 'functional benefits' (based on 'relative advantage', complexity' and 'compatibility' constructs of DOI model, 'effort expectancy' construct of UTAUT and 'ease of use' construct of TAM); 'perceived risks' (based on TIR model) and trust (based on institution-based trust, environmental trust of ITM). The four main factors are further categorized into 12 sub-factors as discussed in Table 3.

The present problem of MFS selection has been modelled in a hierarchy consisting of four levels as depicted in Fig. 1. Level 1 is the general goal of the study, i.e. 'selection of MFS'. Level 2 comprises of the four main factors (main criteria) which are considered by the customers for selecting/adopting a particular MFS. Level 3 consists of the sub-factors (sub-criteria), and level 4 represents the three alternatives (MFSs), i.e. mobile banking, PPI and payment bank services.

4 Fuzzy AHP Methodology

The present study employs fuzzy AHP (FAHP) method to evaluate three MFSs (mobile banking, PPIs and payment banking services) on the basis of 12 criteria. The AHP is widely used to determine the relative importance of a set of activities in a multi-criteria decision-making problem (MCDM) [35]. When applying AHP, a hierarchical decision model is constructed by decomposing the MCDM into its decision criteria. The importance or preference of the decision criteria is compared by making pairwise comparisons with regard to the criterion preceding them in the hierarchy [35]. The use of such pairwise comparisons to collect data from the decision-makers is advantageous as it allows the decision-maker to focus on the comparison of just two objects (factors/criteria), thereby making the observation free from extraneous influences, as far as possible [36]. Despite its popularity and simplicity, AHP is often criticized for its inability to adequately handle the inherent imprecision and uncertainty associated with the mapping of decision-maker's perception to crisp values [37]. The use of fuzzy theory allows the decision-makers to incorporate unquantifiable, incomplete and non-obtainable information into a decision model [38]. As a result, fuzzy analytic hierarchy process (FAHP) is developed

Table 3 Conceptual framework

Main factors (main criteria)	Based on existing models/Theories	Sub-factors (sub-criteria)	Explanation	Literature
Economic benefits	UTAUT2 (price value)	Monetary benefits	Value for money received by using MFSs (e.g. interest on savings, less transaction charges)	Oliveira et al. [9, 20], Alam [21], Natarajan et al. [28], Komlan et al. [30, 31]
	UTAUT (performance expectancy); TAM (usefulness)	Time effectiveness	Saving of time as a result of using MFS	Taylor and Todd [10], Lee [8], Zhou [7], Natarajan et al. [28], Komlan et al. [30, 31]
Functional benefits	DOI (relative advantage, complexity, compatibility); UTAUT (effort expectancy); TAM (ease of use)	Simplicity	Effortlessness in using the MFS in terms of ease of use and minimum requirement of technical skills.	Nikou and Mezei [56], Taylor and Todd [10], Lee [8], Zhou [7]
		Convenience	Accessibility and suitability of using the MFS in terms of 24 × 7 availability, relaxation of KYC (know your customer) norms, minimum requirement of documents, etc.	Chou et al. [29], Luarn and Lin [24]
		Interoperability	Ability to operate with different handsets (mobile phones) without deteriorating the performance	Nava and Madhoushi [57], Osmani et al. [33], Puschel et al. [19]
		Variety of services	Wide range of banking and financial services that can be availed through MFS	Brown et al. [4], Amin et al. [22], Komlan et al. [30, 31]

(continued)

Table 3 (continued)

Main factors (main criteria)	Based on existing models/Theories	Sub-factors (sub-criteria)	Explanation	Literature
		Value additions	Additional services (other than banking and financial services) provided by the MFS provider (e.g. free talktime, mobile recharge, insurance)	Nikou and Mezei [56], Amin et al. [22]
Perceived risks	TIR (perceived risks)	Financial risk	Potential for monetary loss due to transaction errors or due to hidden costs	Lee [8], Featherman and Pavlov [58], Grewal et al. [59], Riquelme and Rios [34], [6], Natarajan et al. [28], Komlan et al. [30, 31]
		Privacy/Security risk	Possible loss of control over personal information	
		Performance risk	The possibility that the MFS will malfunction or not provide the expected services	
Trust	ITM (institution-based trust, environmental trust)	Trust in MFS provider	Trust in MFS provider's credibility, brand image, reputation, etc.	Amin et al. [22], Zhou [7], Kim et al. [26], Koenig-Lewis et al. [5]
		Structural Assurance	Trust in service quality in terms of effective delivery of services, guarantees of protection of information, assurance of transactional confidentiality, etc.	Gu et al. [23], Zhou [7], Kim et al. [26]

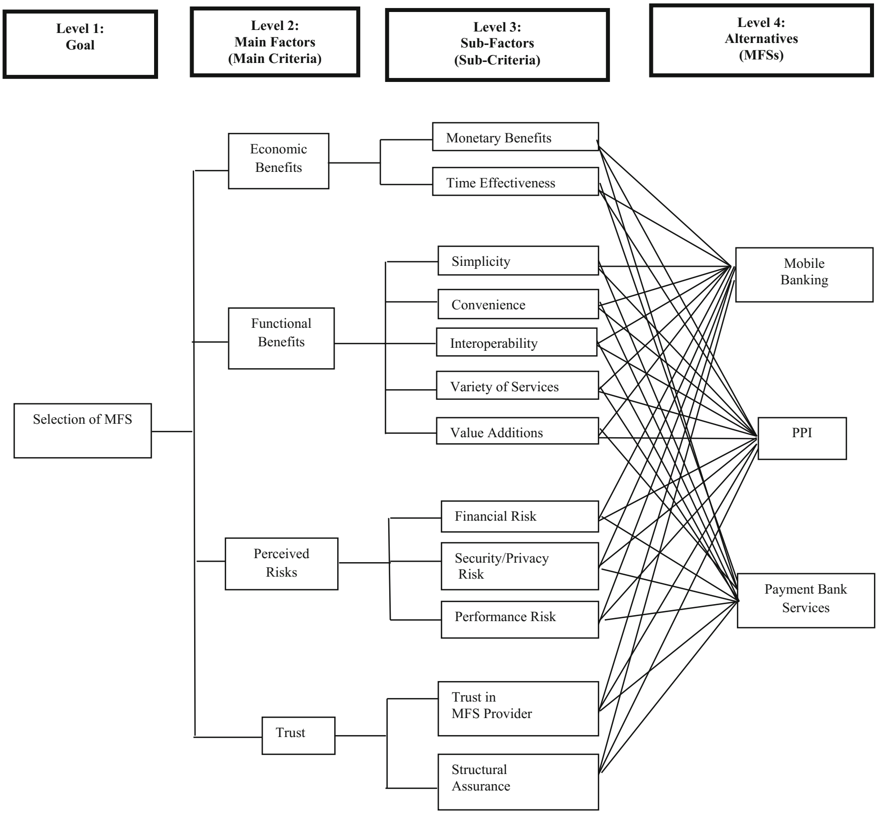

Fig. 1 AHP hierarchy

to solve alternative selection and justification problems, which is capable of capturing a human's appraisal of ambiguity especially when complex MCDM problems are considered. There are many FAHP methods proposed by various researchers. Laarhoven and Pedrycz [39] proposed triangular membership functions for comparing fuzzy ratios, while Buckley [40] compared the fuzzy comparison ratios by using trapezoidal membership functions. Mikhailov [42] proposed to obtain optimal crisp priorities from fuzzy pairwise comparison judgments based on a-cuts decomposition of the fuzzy judgments into a series of interval comparisons. Chang [43] introduced a new extent analysis approach for the synthetic extent values of the pairwise comparison for handling FAHP, wherein triangular fuzzy numbers (TFNs) are used as a pairwise comparison scale for deriving the priorities of factors and sub-factors. In the literature, FAHP has been used for solving many MCDMs such as evaluation of computer-integrated manufacturing alternatives [44], selection of the best location for a facility and in the evaluation of catering firms in Turkey [45], prioritizing customer requirements in quality function deployment [46] and selection of operating system [47]. Büyüközkan [48] used FAHP to prioritize mobile commerce user requirements, whereas Shieh et al. [49] and [56] applied FAHP for prioritizing the factors that affect the adoption of mobile services.

In this study, we use Chang's extent analysis as it is simple and easy to implement for prioritizing decision variables as compared with the other FAHP approaches [43]. The steps of the FAHP employed in the present study are discussed below:

Step I: Structuring the problem into a hierarchical framework

- Identify the factors (criteria), sub-factors (sub-criteria) and alternatives to be used in the model.
- Structure the model into a hierarchy such that the objective is in the first level, factors and sub-factors are in the second and third levels, respectively, and alternatives are in the fourth level.

Step II: Data Collection

- Design a questionnaire for pairwise comparing the relative importance of the factors (criteria) and sub-factors (sub-criteria) and comparing the preferences of the alternatives with respect to each sub-factor.
- Collect the responses from experts by using the linguistic scales given in Table 4.

Step III: Construction of fuzzy comparison matrices

- Use triangular fuzzy numbers (TFNs) given in Table 4, to construct fuzzy comparison matrices for the factors, sub-factors and alternatives.
- Each fuzzy comparison matrix is a square matrix of order n (n = number of attributes (factor/sub-factor/alternatives) being compared), which is constructed as follows:
  $\tilde{A} = \left[\tilde{a}_{ij}\right]_{nXn}$ where $\tilde{a}_{ij}$ is a fuzzy number constructed by attribute i and attribute j as follows:

$$\tilde{a}_{ij} = \begin{cases} 1 \text{ for } i = j \\ (l_{ij}, m_{ij}, u_{ij}) \text{ for } i < j \\ (\tilde{a}_{ij})^{-1} = (1/u_{ij}, 1/m_{ij}, 1/l_{ij}) \text{ for } i > j \end{cases} \quad (1)$$

Table 4 Linguistic scales for importance

Linguistic scale	Triangular fuzzy number	Triangular fuzzy reciprocal number
Just equal	(1, 1, 1)	(1, 1, 1)
Equally important	(1/2, 1, 3/2)	(2/3, 1, 2)
Weakly more important	(1, 3/2, 2)	(1/2, 2/3, 1)
Strongly more important	(3/2, 2, 5/2)	(2/5, 1/2, 2/3)
Very strongly more important	(2, 5/2, 3)	(1/3, 2/5, 1/2)
Absolutely more important	(5/2, 3, 7/2)	(2/7, 1/3, 2/5)

In a TFN (l, m, u), l and u, respectively, represent the minimum and maximum values of the triangular membership function, while m is the mean of the interval.

Step IV: Checking the consistencies of the fuzzy matrices

- Convert the fuzzy matrix $\tilde{A} = \left[\tilde{a}_{ij}\right]_{nXn}$ into the corresponding crisp matrix $A = \left[a_{ij}\right]_{nXn}$ by defuzzifying the TFNs using geometric means [40]. A fuzzy comparison matrix is consistent if the corresponding crisp matrix (defuzzified matrix) is consistent [41]. Check the consistency of the crisp matrix **A** as follows:

- A matrix **A** is said to be consistent if

$$AW = nW \tag{2}$$

Equation (2) is an eigenvalue problem. It is assumed that the largest eigenvalue $\lambda_{\max}$ is greater than or equal to n [35]. The closer $\lambda_{\max}$ is to n, the more consistent is the matrix **A**. $\lambda_{\max}$ is calculated by solving the following equation:

$$AW = \lambda_{\max} W \tag{3}$$

- Calculate consistency ratio (CR) by using the following formula:

$$\mathrm{CR} = \frac{\mathrm{CI}}{\mathrm{RI}} \tag{4}$$

where CI is the consistency index given by the following formula:

$$\mathrm{CI} = \frac{\lambda\max - n}{n - 1} \tag{5}$$

and RI is a random index. Different numbers of criteria (n) correspond to different values of RI (Table 5).

- If CR is $\leq$0.10, the level of inconsistency of comparison matrix A is considered acceptable.

Step V: Aggregation of fuzzy comparison matrices

- Aggregate the fuzzy comparison matrices of all the decision-makers by using geometric mean method [40] for all the factors and sub-factors and for the alternatives on the basis of each sub-factor.

Table 5 Table of random index [35]

n	1	2	3	4	5	6	7	8	9	10	11	12	13
RI	0	0	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49	1.51	1.58	1.56

- The aggregated fuzzy comparison matrix for a particular attribute is $\tilde{P} = \left[\tilde{p}_{ij}\right]_{nXn}$ where $\tilde{p}_{ij}$ is the aggregated TFN of the judgements of N decision-makers, calculated as follows:

$$\tilde{p}_{ij} = \left(\prod_{i=1}^{N} \tilde{a}_{ijk}\right)^{1/N} \tag{6}$$

Step VI: Computation of fuzzy synthetic extent

- Calculate the fuzzy synthetic extent S_i with respect to the ith attribute (factor/sub-factor/alternative) as follows:

$$S_i = \sum_{j=1}^{n} \tilde{p}_{ij} \otimes \left[\sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{p}_{ij}\right]^{-1}$$

where $\sum_{j=1}^{n} \tilde{p}_{ij} = \left(\sum_{j=1}^{n} l_j, \sum_{j=1}^{n} m_j, \sum_{j=1}^{n} u_j\right)$ and $\sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{p}_{ij} = \left(\sum_{j=1}^{n} l_i, \sum_{j=1}^{n} m_i, \sum_{j=1}^{n} u_i\right)$ (7)

The above calculations on the TFNs can be performed by using algebraic operations of addition ($\oplus$), multiplication ($\otimes$) and inverse $\left(^{-1}\right)$ of TFNs. For two TFNs $M_1 = (l_1, m_1, u_1)$ and $M_2 = (l_2, m_2, u_2)$,

$$M_1 \oplus M_2 = (l_1 + l_2, m_1 + m_2, u_1 + u_2) \tag{8}$$

$$M_1 \otimes M_2 = (l_1 \cdot l_2, m_1 \cdot m_2, u_1 \cdot u_2) \tag{9}$$

$$M_1^{-1} = (l_1, m_1, u_1)^{-1} = (1/u_1, 1/m_1, 1/l_1) \tag{10}$$

Step VII: Calculation of weights (priorities)

- Compute the degree of possibility that $S_i \geq S_j (i, j = 1, 2, \ldots, n)$ by using the following equation:

$$V\left(S_i \geq S_j\right) = \begin{cases} 1, \text{ if } m_i \geq m_j \\ 0 \text{ if } l_j \geq u_i \\ \frac{l_j - u_i}{(m_i - u_i) - \left(m_j - l_j\right)}, \text{ otherwise} \end{cases} \tag{11}$$

- Calculate the degree of possibility that $S_i \geq$ all other $(n - 1)$ fuzzy synthetic extents S_j as follows:

$$V\left(S_i \geq S_j | j = 1, 2, 3, \ldots, n; j \neq i\right) = \min V\left(S_i \geq S_j\right) \text{ for } j \in (1, \ldots, n); j \neq i \tag{12}$$

- Determine the non-fuzzy relative weight (priority) vector $W' = \left(w'_1, w'_2, \ldots, w'_n\right)^{\mathrm{T}}$ for the fuzzy comparison matrix $\tilde{A}$ where

$$w'_i = \min V\left(S_i \geq S_j\right) \text{ for } j \in (1, \ldots, n); j \neq i \tag{13}$$

- Determine the normalized weight vector $W = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ by normalizing the weights as follows:

$$w_i = \frac{w'_i}{\sum_{i=1}^{n} w'_i} \tag{14}$$

5 Results and Discussion

For the present study, the data has been collected through a structured questionnaire from 20 experts consisting of 5 researchers who have been involved in the research related to technology adoption and 15 practitioners who are involved in promoting mobile financial services. All the experts have more than 20 years of experience in their respective areas. Since AHP is not a statistics-based methodology, it does not always require a statistically significant sample size [50]. In AHP, it is not necessary to use a representative sample, because unit of analysis are the decisions made, and not who made the decisions [51]. Moreover, AHP is usually used to survey people who have knowledge about the topic under investigation and therefore a large number of samples are not needed [52]. Hence, the sample of 20 respondents may not be very large, but it is appropriate for the present study as all the respondents are quite experienced to provide the required information for the study. The questionnaire used for the data collection captured pairwise importance comparisons of all the three main factors and 12 sub-factors and pairwise preference comparisons of all the three alternatives (mobile banking, PPI, payment bank services) on the basis of each sub-factor, by using the linguistic scales given in Table 4.

After collecting the data, the aggregated fuzzy comparison matrices are constructed by using Eq. (2) for all the factors, sub-factors and alternatives. The consistency ratios (CR) for all the comparison matrices are calculated by using Eq. (4) for checking the consistencies of the judgments. The CR values for all the matrices have been found to be <0.1, indicating the acceptable level of consistency. The fuzzy synthetic extent values S_i are calculated by using Eq. (7), and the relative weights w'_i and normalized weights w_i are computed by using Eqs. (13) and (14), respectively.

Table 6 Fuzzy comparison matrix and the weight analysis of main factors

	Economic benefits	Functional benefits	Perceived risks	Trust	S_i	w_i'	w_i
Economic benefits	(1, 1, 1)	(0.74, 0.92, 1.17)	(1.06, 1.31, 1.62)	(1.11, 1.36, 1.66)	(0.20, 0.28, 0.38)	0.8112	0.3399
Functional benefits	(0.85, 1.09, 1.36)	(1, 1)	(1.29, 1.61, 1.94)	(1.21, 1.50, 1.80)	(0.23, 0.31, 0.43)	1.0000	0.4190
Perceived risks	(0.62, 0.76, 0.94)	(0.52, 0.62,0.77)	(1, 1, 1)	(0.80, 0.91, 1.04)	(0.15, 0.20, 0.26)	0.2417	0.1013
Trust	(0.6, 0.73, 0.90)	(0.56, 0.67, 0.82)	(0.96, 1.10, 1.25)	(1, 1, 1)	(0.16, 0.21, 0.28)	0.3339	0.1399

$\lambda_{max} = 4.00200$, CR = 0.000742

Table 7 Fuzzy comparison matrix and the weight analysis of sub-factors of economic benefits

	Monetary benefits	Time effectiveness	S_i	w_i'	w_i
Monetary benefits	(1, 1, 1)	(0.59, 0.76, 0.98)	(0.34, 0.43, 0.55)	0.4568	0.3136
Time effectiveness	(1.02, 1.32, 1.69)	(1, 1, 1)	(0.43, 0.57, 0.75)	1.0000	0.6864

$\lambda_{max} = 2.00$, CR = 0.00

5.1 *Determining Weights of Main Factors and Sub-factors*

This section discusses the weight analysis for the main factors and the sub-factors. The aggregated fuzzy comparison matrices and the values of S_i, w_i' and w_i for the main factors are presented in Table 6.

Tables 7, 8, 9 and 10 show the aggregated fuzzy comparison matrices, and the values of S_i, w_i' and w_i for the sub-factors: economic benefits (EB), functional benefits (FB), perceived risks (PR) and trust (TR), respectively.

Table 11 shows the local and global weights of the factors influencing the selection of MFSs. It can be observed that among the four main factors, functional benefits and economic benefits occupy the topmost rankings with weights 0.4190 and 0.3399, respectively. This is followed by trust (weight = 0.1399) and perceived risks (weight = 0.1013). These findings imply that the relative advantages in terms of various functional benefits and economic benefits play a predominant role in customers' decision-making regarding the selection of MFSs. On the other hand, trust and perceived risks play a relatively less important role in influencing customers' minds while selecting a particular MFS.

Table 8 Fuzzy comparison matrix and the weight analysis of sub-factors of functional benefits

	Simplicity	Convenience	Interoperability	Variety of services	Value additions	S_i	w_i'	w_i
Simplicity	(1, 1, 1)	(0.53, 0.66, 0.82)	(0.94, 1.08, 1.25)	(1.17, 1.38, 1.62)	(0.52, 0.64, 0.79)	(0.14, 0.18, 0.24)	0.4406	0.1490
Convenience	(1.22, 1.52, 1.88)	(1, 1, 1)	(1.66, 2.01, 2.35)	(0.73, 0.87, 1.04)	(0.97, 1.16, 1.38)	(0.18, 0.25, 0.33)	1.0000	0.3382
Interoperability	(0.80, 0.93, 1.07)	(0.43, 0.50, 0.60)	(1, 1, 1)	(0.47, 0.57, 0.68)	(0.68, 0.79, 0.91)	(0.11, 0.14, 0.18)	0.0104	0.0035
Variety of services	(0.62, 0.73, 0.85)	(0.96, 1.15, 1.37)	(1.47, 1.77, 2.11)	(1, 1, 1)	(1.19, 1.35, 1.51)	(0.17, 0.23, 0.30)	0.8369	0.2830
Value additions	(1.26, 1.56, 1.91)	(0.72, 0.86, 1.03)	(1.10, 1.27, 1.47)	(0.66, 0.74, 0.84)	(1, 1, 1)	(0.16, 0.20, 0.27)	0.6693	0.2263

$\lambda_{max} = 5.11600$, CR = 0.025894

Table 9 Fuzzy comparison matrix and the weight analysis of sub-factors of perceived risks

	Financial risk	Security/Privacy risk	Performance risk	S_i	w_i'	w_i
Financial risk	(1, 1, 1)	(0.94, 1.18, 1.49)	(1.17, 1.35, 1.54)	(0.30, 0.39, 0.49)	1.0000	0.4961
Security/Privacy risk	(0.67, 0.85, 1.07)	(1, 1, 1)	(0.98, 1.16, 1.36)	(0.26, 0.33, 0.42)	0.6773	0.3360
Performance risk	(0.65, 0.74, 0.85)	(0.73, 0.86, 1.02)	(1, 1, 1)	(0.23, 0.29, 0.35)	0.3386	0.1680

$\lambda_{max} = 3.00004$, CR = 0.0000328

Table 10 Fuzzy comparison matrix and the weight analysis of sub-factors of trust

	Trust in MFS provider	Structural assurance	S_i	w_i'	w_i
Trust in MFS provider	(1, 1, 1)	(0.94, 1.06, 1.21)	(0.45, 0.52, 0.59)	1.0000	0.5677
Structural assurance	(0.83, 0.94, 1.07)	(1, 1, 1)	(0.43, 0.48, 0.55)	0.7614	0.4323

$\lambda_{max} = 2.00$, CR = 0.00

Table 11 Local and global weights

Main factors (level 2)	Weights	Sub-factors (level 3)	Local weights	Global weights	Global rank
Economic benefits	0.3399	Monetary benefits	0.3136	0.1066	4
		Time effectiveness	0.6864	0.2876	1
Functional benefits	0.4190	Simplicity	0.1490	0.0624	7
		Convenience	0.3382	0.1417	2
		Interoperability	0.0035	0.0015	12
		Variety of services	0.2830	0.1186	3
		Value additions	0.2263	0.0948	5
Perceived risks	0.1013	Financial risk	0.4961	0.0502	9
		Privacy/security risk	0.3360	0.0340	10
		Performance risk	0.1680	0.0170	11
Trust	0.1399	Trust in MFS provider	0.5677	0.0794	6
		Structural assurance	0.4323	0.0605	8

Within the functional benefits (which is the most important factor), convenience (local weight = 0.3382) is found to be the most important sub-factor followed by variety of services (local weight = 0.2830), value additions (local weight = 0.2263), simplicity (local weight = 0.1490) and interoperability (local weight = 0.0035). The highest importance of convenience indicates that customers want the MFS to be hassle-free which can be used as per their convenience. This implies that customers prefer that MFS which is accessible anytime anywhere and requires minimum terms, conditions and documents. Availability of a wide range of banking and financial services along with other value-added benefits which can be additionally availed through the use of a MFS such as reward points, free talktime, free mobile recharge is also important for the customers while choosing MFSs. Simplicity and interoperability are found to be least important within functional benefits. This implies that difficulty in using MFSs is less of a concern for the customers as most of the MFSs have user-friendly interfaces. In addition, since use of smartphones is common nowadays, the issues related to interoperability are immaterial for the customers.

Economic benefits are found to be second important factor influencing the customers' decisions of selecting MFSs. Within economic benefits, time effectiveness (local weight = 0.6864) is found to be more important than monetary benefits (local weight = 0.3136). This indicates that time is more important than money, for the customers. The high importance of time effectiveness is due to the reason that nowadays, people are very busy with their hectic schedules and want to utilize their time effectively. Hence, they prefer those MFSs, which save their time, and carry out their financial transactions quickly and effectively.

Trust is the third important factor which influences the customers' choice of MFSs. Within trust, trust in MFS provider (local weight = 0.5677) is reported to be more important than structural assurance (local weight = 0.4323). This indicates that customers pay more importance to the credibility of the service provider than the structural assurance. This may be due to the reason that customers associate the structural assurance with the service provider only. They perceive that if the service provider is trustworthy, then the service quality and assurance will automatically be guaranteed.

Perceived risks are found to be the least important factor which may influence customers' decisions to select MFSs. The least importance of this factor may be due to the reason that nowadays customers are confident in using mobile interfaces and therefore the risks associated with MFSs do not restrain them from using these services. However, within perceived risks, financial risk (local weight = 0.4961) is found to be most important followed by privacy/security risk (local weight = 0.3360) with less emphasis on performance risk (local weight = 0.1680). The highest importance of financial risk indicates that customers are concerned about potential monetary losses which may occur due to transaction errors while using MFSs. Similarly, the importance of privacy/security risk indicates the customers' concerns about identity thefts and frauds. The least importance of performance risk implies that customers do not care much about the risk of malfunctioning of MFSs. Hence on the basis of risks, customers prefer that MFS which involves least financial and privacy/security risks.

Table 12 Weights of alternatives with respect to sub-factors of economic benefits

	Monetary benefits	Time effectiveness	Alternative Weight	Rank
Sub-factor weight	0.1066	0.2876		
Alternative				
Mobile banking	0.3149	0.2727	0.1120	3
PPI	0.3020	0.3180	0.1236	2
Payments bank services	0.3831	0.4093	0.1585	1

An examination of global ranks of the sub-factors indicates that 'time effectiveness', 'convenience', 'variety of services', 'monetary benefits' and 'value additions' are the top five factors that influence the customers' choice of MFSs. This implies that customers prefer that MFS which does not take much time for financial transactions, and is convenient to use, i.e. accessible round the clock and has minimum documentary requirements and conditions. Customers also look for availability of wide range of services in the MFS as they want to use the MFS for most of their banking and financial services. Monetary benefits and value additions also attract customers towards a particular MFS. Customers are inclined towards that MFS which is cost-effective, provides monetary advantages in terms of high interest rates and also provides additional benefits such as free talktime, mobile recharge, insurance.

5.2 Determining Weights of Alternatives

After achieving the normalized non-fuzzy weights for all the main factors and sub-factors, the same methodology has been applied to find the respective values for the three alternatives, wherein the alternatives are pairwise compared with respect to each sub-factor. That means, this analysis should be repeated for 12 more times for each sub-factor. However, it will be burdensome to show the fuzzy comparison matrices for each of the 12 sub-factors. Hence, the final normalized weights for the alternatives are presented in this section. Tables 12, 13, 14 and 15 show the normalized non-fuzzy weights for the three alternatives with respect to the sub-factors of economic benefits, functional benefits, perceived risks and trust.

The findings indicate that payments banks services are the most preferred ones with regard to all the factors, except for trust. This indicates that payments banks are better than PPI and mobile banking as they are more economical, provide more functional benefits and are least risky. As compared to PPIs and mobile banking, payments banks are more economical in terms of both time and money as they provide higher interest rates and faster transactions. Similarly, payments banks are more beneficial, in terms of convenience and value additions. As payments banks

Table 13 Weights of alternatives with respect to sub-factors of functional benefits

	Simplicity	Convenience	Interoperability	Variety of services	Value additions	Alternative weight	Rank
Sub-factor Weight	0.0624	0.1417	0.0015	0.1186	0.0948		
Alternative							
Mobile banking	0.2446	0.1816	0.4607	0.6240	0.2875	0.1429	2
PPI	0.4155	0.3596	0.2338	0.0122	0.2651	0.1038	3
Payments bank services	0.3399	0.4588	0.3055	0.3638	0.4474	0.1722	1

Table 14 Weights of alternatives with respect to sub-factors of perceived risks

	Financial risk	Security/Privacy risk	Performance risk	Alternative weight	Rank
Sub-factor Weight	0.0502	0.2354	0.0170		
Alternative					
Mobile banking	0.0993	0.3719	0.4734	0.1006	3
PPI	0.5027	0.3980	0.2786	0.1237	2
Payments bank services	0.3980	0.8835	0.2479	0.2322	1

Table 15 Weights of alternatives with respect to sub-factors of Trust

	Trust in MFS provider	Structural assurance	Alternative weight	Rank
Sub-factor Weight	0.0794	0.0605		
Alternative				
Mobile banking	0.8207	0.4103	0.090005	1
PPI	0.1395	0.3181	0.030315	2
Payments bank services	0.0398	0.2716	0.019585	3

offer a wide range of banking and financial services with bare minimum requirements from the customers in terms of documents and other terms and conditions, payments banks are preferred over PPIs and mobile banking. Along with providing convenient banking and financial services, payments banks also provide more value additions, which make them the first choice of the customers. Payments banks also involve least risks as compared to PPIs and mobile banking. Since payments banks do not involve the use of Internet, they are less prone to the financial and security/privacy risks, which may be associated with PPI and mobile banking which require Internet for financial transactions. However, payments banks are least preferred on the basis of trust. This can be attributed to the fact that payments banks are relatively new

Table 16 Weights of alternatives with respect to main factors

	Economic benefits	Functional benefits	Perceived risks	Trust	Alternative weight (overall)	Overall rank
Sub-factor weight	0.33987	0.41897	0.10125	0.13991		
Alternative						
Mobile banking	0.1120	0.1429	0.1006	0.090005	0.120724	2
PPI	0.1236	0.1038	0.1237	0.030315	0.102278	3
Payments bank services	0.1585	0.1722	0.2322	0.019585	0.152289	1

in the MFS market where PPI and mobile banking have already built their strong foundations. Hence, payments banks are yet to earn the trust of the customers.

The findings also indicate that PPIs are preferred over mobile banking with regard to economic benefits and perceived risks. PPIs are less time-consuming than mobile banking as they are relatively quicker at completing the financial transactions. Moreover, mobile banking is more risky as compared to PPIs as it is prone to data thefts and fraudulent transactions in case of mobile SIM swapping.

Further, it has been found that mobile banking is preferred over PPIs with regard to functional benefits. This is due to the reason that mobile banking covers the full range of banking and financial services as compared to PPIs, which majorly provide payment services only. Moreover, mobile banking also offers more value-added services to the customers as compared to PPIs. It can also be noticed that mobile banking is the most preferred MFS among the three MFSs with regard to trust. This is because mobile banking is offered by full-fledged banks which are more trustworthy than the service providers of PPIs and payments banks which include private players also.

The overall weights of the three alternatives with respect to the main factors are presented in Table 16. The results indicate that the overall preference of the three MFSs on the basis of all the factors taken together is as follows: payments banks (weight = 0.152289) followed by mobile banking (weight = 0.120724) and PPIs (weight = 0.102278). This implies that payments bank is the most preferred MFS, whereas PPI is the least preferred MFS.

6 Conclusion

The study has made an attempt to prioritize three MFSs, namely mobile banking, PPIs and payments banks on the basis of multiple factors including economic benefits, functional benefits, perceived risks and trust. The problem has been modelled as a multiple-criteria decision-making (MCDM) problem, wherein fuzzy analytic hierarchy process (FAHP) has been employed to rank the potential factors of MFS selection and to evaluate various MFSs. The findings of the study reveal that functional benefits and economic benefits dominate over trust and perceived risks in customers' decision-making regarding the selection of MFSs. With regard to the evaluation of the three MFSs, the findings indicate that payments bank is the superior choice because it offers best economic and functional benefits and is least risky.

This study represents a worthwhile direction by examining MFSs which, so far, have not been well evaluated in the Indian context. The hierarchy of influencing factors constructed in this study uses the factors that experts consider to be important for selection of MFSs. The assessment of the relative priorities of these factors using FAHP reveals the weightage of each factor. This approach differs from the previous examinations of MFSs, which have used statistical techniques like multiple regression analysis or structural equation modelling, for investigating the significant influencing factors.

The results of this study have provided clues for MFS providers about the important roles of 'functional benefits' and 'economic benefits'. Therefore, MFS providers should focus on providing time- and cost-effective services which are convenient to use and not only include a wide range of banking and financial services but also offer additional value-added services. Service providers should be sure about the ability of MFS to conduct financial transactions efficiently within less time along with the availability of information required by customers to successfully use the services. Additionally, service providers should offer monetary benefits in terms of less transaction handling fees and attractive interest rates on the savings, so that customers feel attracted towards using these services. Service providers can also provide value additions, viz. free talktime, mobile recharge, gift vouchers, to attract the customers. The evaluations of the three MFSs provide insights for the service providers in making appropriate decisions regarding delivery of services by allocating their limited resources to the relevant factors. As payments banks are found to be least trustworthy, the services providers of payments banks services should emphasize on establishing a relationship of trust with the customers through advertising and marketing campaigns. Similarly, PPI service providers should focus on providing more benefits to the customers by lowering their transaction handling charges, providing interest on savings and increasing the range of services. Finally, with regard to mobile banking, service providers should focus on reducing the associated financial and security risks in order to increase their customer base.

It is imperative to note that the MFS industry is a developing industry, and the factors affecting the adoption of various MFSs are changing constantly. Therefore, despite the fact that the researchers have tried to collect all the relevant influencing factors in the present study, it is possible that a more complete hierarchy of factors can be constructed for future study. Also, some factors selected for the model may have interrelationships, which are not explained by FAHP in the present study. In that case, analytic network process (ANP) can be a better option. Hence, this study can be further extended by considering some other factors influencing the choice of MFSs and applying ANP in the revised model.

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Design of Reliability Single Sampling Plan by Attributes Based on Exponentiated Exponential Distribution

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Abstract This paper considers designing of reliability single sampling plans for exponentiated exponential distribution assuming that the life test is conducted under hybrid censoring. Median of this distribution can be calculated easily using its closed form, whereas mean has to be computed using digamma functions. The computational difficulties can be overcome using appropriate computer programmes. This paper also attempts to study the efficiency of the reliability single sampling plans designed using median lifetime and mean lifetime of the products. Efficiency of these two kinds of sampling plans is analyzed with respect to the sample size and the sampling risks. Parameters of the sampling plans are determined using binomial probabilities with an objective of safeguarding producer and consumer simultaneously with specified risks.

Keywords Reliability sampling plan · Operating characteristic function
Hybrid censoring · Exponentiated exponential distribution

1 Introduction

Acceptance sampling provides statistical tools to carry out inspection of incoming raw materials and/or outgoing finished products. The items are sampled and inspected against the specified standards of quality characteristic(s). Lifetime is a quality characteristic for some products. Sampling inspection for such products is carried out by conducting suitable life tests. Sampling plans developed for conducting such sampling inspection may be called as reliability sampling plans. Censoring schemes may be employed while conducting the life test for various reasons including reduction in the time and cost of conducting the life test. Among different schemes, application of hybrid censoring method enables to reduce both test time and cost of life testing.

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Reliability sampling plan is designed using the lifetime distribution of the product considered for inspection. One of the approaches to handle this problem is finding the minimum sample size and the acceptance number ensuring the mean/median lifetime of the product for given producer's risk (PR) and consumer's risk (CR). Epstein [5], as a pioneering attempt, designed sampling plans for exponential distribution under hybrid censoring scheme. Under the same conditions, Balasooriya and Saw [4] made an attempt to determine the plan parameters for inspecting the products under progressive censoring scheme. Kantam et al. [8] and Rosaiah et al. [10] obtained the plan parameters under hybrid censoring scheme for log–logistic and exponentiated log–logistic distributions, respectively. Recently, Sriramachandran and Palanivel [14] determined sampling plans based on exponentiated inverse Rayleigh distribution. Attempts on designing reliability sampling plans can be found in the literature for generalized exponential distribution [1, 3, 12], Birnbaum–Saunders [2], Marshall–Olkin extended exponential [13], and Maxwell [9] distributions. All these works assumed that the sampling inspection is carried out under the hybrid censoring scheme. Also, the plan parameters were obtained considering the risk for consumer only. The risk of the producer due to rejection of the lots of good quality products is ignored.

The exponentiated exponential distribution was first introduced by Gupta and Kundu [6] and has been used as a lifetime distribution. This distribution is found to be more appropriate for some lifetime data than other frequently used lifetime distributions.

This paper aims to study the efficiency of the reliability single sampling plans designed using median lifetime and mean lifetime of the products with respect to the sample size and the sampling risks, when the lifetime distribution is the exponentiated exponential. Sampling plans are obtained using binomial probabilities considering the risks of both producer and consumer. Operating characteristic function of the reliability single sampling plan is derived in Sect. 2. Plan designing method is explained in Sect. 3. In Sect. 4, construction of tables of optimum sampling plans is discussed. Also, selection of the plan parameters for given requirements from the tables is explained. Performance of the two sampling plans is compared in Sect. 5. Results are summarized in Sect. 6.

2 Operating Characteristic Function of Reliability Single Sampling Plan under the Conditions of Exponentiated Exponential Distribution

A reliability single sampling plan under hybrid censoring scheme may be defined by a set of four parameters N (lot size), n (sample size), c (acceptance number), and t (test termination time). According to a given sampling plan, if the number of failures $X = x$ in a life test conducted for n sampled products exceeds c at time t or earlier, the lot is rejected. If $x \leq c$ at time t, the lot is accepted.

Let T denote the lifetime of the products, which has the exponentiated exponential distribution (EE (θ, λ)). The probability density function of this distribution is given by

$$f(t;\theta,\lambda)=\frac{\theta}{\lambda}\mathrm{e}^{-\frac{t}{\lambda}}\left(1-\mathrm{e}^{-\frac{t}{\lambda}}\right)^{\theta-1},t>0,\theta>0,\lambda>0 \tag{1}$$

The cumulative distribution function of this distribution is given by

$$F_T(t;\theta,\lambda)=\left(1-\mathrm{e}^{-\frac{t}{\lambda}}\right)^{\theta},t>0,\theta>0,\lambda>0 \tag{2}$$

Mean and variance of the EE (θ, λ) distribution are, respectively,

$$\mu=E(T)=\lambda[\psi(\theta+1)-\psi(1)] \tag{3}$$

and

$$\sigma^2=V(T)=\lambda^2[\psi'(1)-\psi'(\theta+1)] \tag{4}$$

where $\psi(\cdot)$ and $\psi'(\cdot)$ are, respectively, the digamma and polygamma functions, i.e.,

$$\psi(u)=\frac{d}{du}\Gamma(u),\ \ \psi'(u)=\frac{d}{du}\psi(u)$$

and $\Gamma(u)=\int_o^{\infty}y^{u-1}e^{-y}dy$.

Median of the EE (θ, λ) distribution is given by

$$m=\mathrm{Median}(T)=-\lambda\ \ln\left(1-\left(\tfrac{1}{2}\right)^{\frac{1}{\theta}}\right) \tag{5}$$

The lot fraction nonconforming, p, can be calculated corresponding to each value of t/μ and t/m, respectively, from

$$F_T(t/\mu)=p \tag{6}$$

$$F_T(t/m)=p \tag{7}$$

A sampling plan may be analyzed and its performance may be compared with other sampling plans using its operating characteristic (OC) function. The OC function of the reliability sampling plan based on the EE (θ, λ) distribution under hybrid censoring scheme is given by

$$P_a(p) = P(X \leq c) = \sum_{x=0}^{c} P(X = x)$$

Values of $P_a(p)$, for different levels of p, may be computed using hypergeometric, binomial and Poisson probabilities. As pointed out by Schilling and Neubauer [11], when N is large and $n/N \leq 0.10$, the distribution of X can be approximated by the Binomial (n, p) distribution. Under these circumstances, here, it is proposed to calculate the value of $P_a(p)$ using

$$P_a(p) = \sum_{x=0}^{c} \binom{n}{x} p^x (1-p)^{n-x} \tag{8}$$

3 Determination of Plan Parameters under the Conditions of the EE (θ, λ) Distribution

Usually, a sampling plan is determined considering two points, namely $(p_1, 1-\alpha)$ and (p_2, β) on the OC curve of the plan. Here, p_1 represents the acceptable quality level, α denotes the PR, p_2 represents the limiting quality level, and β denotes the CR. Such sampling plan protects the producer and consumer simultaneously at ensured levels. An optimum reliability single sampling plan can be obtained by satisfying the following two conditions,

$$P_a(p_1) \geq 1-\alpha$$
$$P_a(p_2) \leq \beta$$

Based on the binomial probabilities, these conditions may be rewritten as

$$\sum_{x=0}^{c} \binom{n}{x} p_1^x (1-p_1)^{n-x} \geq 1-\alpha \tag{9}$$

$$\sum_{x=0}^{c} \binom{n}{x} p_2^x (1-p_2)^{n-x} \leq \beta \tag{10}$$

It may be noted that the values of p_1 and p_2 are calculated corresponding to the specified lifetime of the product as expected by producer and consumer, respectively.

If γ_1 and γ_2 are the quantitative measurements of the lifetime of the product representing, respectively, the producer's quality (PQ) and consumer's quality (CQ), then the values of p_1 and p_2 can be calculated from

$$F_T(t/\gamma_1) = p_1 \tag{11}$$

$$F_T(t/\gamma_2) = p_2 \tag{12}$$

Optimum values of the plan parameters n and c may be obtained applying the following iterative procedure for each given set of values of θ, t, γ_1, γ_2, α and β with $\gamma_1 > \gamma_2$.

Step 1: Calculate λ_1 and λ_2 using $\lambda = h^*(\theta, \gamma)$.
Step 2: Determine p_1 and p_2 using (11) and (12).
Step 3: Set $c = 0$
Step 4: Find the largest n, say n_L, such that $P_a(p_1) \geq 1 - \alpha$
Step 5: Find the smallest n, say n_S, such that $P_a(p_2) \leq \beta$
Step 6: If $n_S \leq n_L$, then the optimum plan is (n_S, c); otherwise increase c by 1.
Step 7: Repeat Steps 4 through 6 until optimum values of n and c are obtained.

Suppose that $\gamma = h(\theta, \lambda)$ and $\lambda = h^*(\theta, \gamma)$ exist for some functions $h(\cdot)$ and $h^*(\cdot)$. If $\gamma = \mu$, then $\lambda = h^*(\theta, \gamma)$ can be determined from (3). Similarly, if $\gamma = m$, then $\lambda = h^*(\theta, \gamma)$ can be determined from (5). The sampling inspection may be carried out using the optimum values of n and c for a submitted lot under hybrid censoring scheme as described in Sect. 2.

4 Construction of Tables

Based on the procedure described in the preceding section, plan parameters of the optimum reliability sampling plans are obtained for some combinations of $\theta, t, \mu_1, \mu_2, \alpha$, and β. The PR and CR are considered as $\alpha = 0.05$ and $\beta = 0.10$ respectively. Various values taken for the mean lifetime of the PQ are $\mu_1 = 5000$, 6000, 7000, 8000, 9000 and 10,000 h. Values considered for t and θ are $t = 500$ h and $\theta = 2$. Eleven different values of μ_2 such as $\mu_2 = 1000$, 1200, 1400, 1600, 1800, 2000, 2200, 2400, 2600, 2800 and 3000 h are considered. The values of p_1 and p_2 are computed for each combination of t, μ_1 and μ_2 from (11) and (12), respectively, taking $\gamma_i = \mu_i$, $i = 1, 2$. Values of n and c are computed applying the iterative procedure discussed in Sect. 3 and are presented in Table 1.

Various levels of the median lifetime of the products as expected by the producer are taken as $m_1 = 5000$, 6000, 7000, 8000, 9000 and 10,000 h respectively. The median lifetime of the product as expected by the consumer are taken as $m_2 = 1000$, 1200, 1400, 1600, 1800, 2000, 2200, 2400, 2600, 2800 and 3000 h. Using the values of m_1, m_2 and t, the values of p_1 and p_2 are computed from (11) and (12), respectively, taking $\gamma_i = m_i$, $i = 1, 2$. Values of (n, c) are presented in Table 2. Procedure for selection of the plan parameters from these tables for given specifications is illustrated as follows.

Table 1 Parameters of reliability single sampling for the EE (2, λ) distribution based on mean lifetime with t = 500 h, α = 0.05, β = 0.10

		μ_1	5000	6000	7000	8000	9000	10,000
		t/μ_1	0.10000	0.08333	0.07143	0.06250	0.05556	0.05000
μ_2	t/μ_2	p_2	p_1					
			0.01940	0.01381	0.01032	0.00801	0.00639	0.00522
1000	0.5000	0.27840	(13,1)	(13,1)	(13,1)	(13,1)	(13,1)	(8,0)
1200	0.41667	0.21598	(17,1)	(17,1)	(17,1)	(17,1)	(17,1)	(17,1)
1400	0.35714	0.17202	(30,2)	(22,1)	(22,1)	(22,1)	(22,1)	(22,1)
1600	0.31250	0.14004	(37,2)	(37,2)	(27,1)	(27,1)	(27,1)	(27,1)
1800	0.27778	0.11612	(56,3)	(45,2)	(33,1)	(33,1)	(33,1)	(33,1)
2000	0.25000	0.09779	(67,3)	(53,2)	(53,2)	(39,1)	(39,1)	(39,1)
2200	0.22727	0.08345	(94,4)	(79,3)	(63,2)	(63,2)	(46,1)	(46,1)
2400	0.20833	0.07203	(127,5)	(91,3)	(73,2)	(73,2)	(53,1)	(53,1)
2600	0.19231	0.06279	(166,6)	(126,4)	(105,3)	(84,2)	(84,2)	(61,1)
2800	0.17857	0.05522	(233,8)	(143,4)	(120,3)	(95,2)	(95,2)	(95,2)
3000	0.16667	0.04893	(313,10)	(188,5)	(162,4)	(135,3)	(108,2)	(108,2)

Table 2 Parameters of reliability single sampling for the EE (2, λ) distribution based on median lifetime with t = 500 h, α = 0.05, β = 0.10

		m_1	5000	6000	7000	8000	9000	10,000
		t/m_1	0.10000	0.08333	0.07143	0.06250	0.05556	0.05000
m_2	t/m_2	p_2	p_1					
			0.01335	0.00946	0.00705	0.00546	0.00435	0.00355
1000	0.5000	0.21050	(17,1)	(17,1)	(17,1)	(17,1)	(10,0)	(10,0)
1200	0.41667	0.16039	(23,1)	(23,1)	(23,1)	(23,1)	(23,1)	(14,0)
1400	0.35714	0.12605	(41,2)	(30,1)	(30,1)	(30,1)	(30,1)	(30,1)
1600	0.31250	0.10156	(51,2)	(37,1)	(37,1)	(37,1)	(37,1)	(37,1)
1800	0.27778	0.08353	(79,3)	(63,2)	(46,1)	(46,1)	(46,1)	(46,1)
2000	0.25000	0.06987	(94,3)	(75,2)	(75,2)	(55,1)	(55,1)	(55,1)
2200	0.22727	0.05930	(133,4)	(111,3)	(89,2)	(65,1)	(65,1)	(65,1)
2400	0.20833	0.05095	(180,5)	(130,3)	(103,2)	(103,2)	(75,1)	(75,1)
2600	0.19231	0.04424	(236,6)	(179,4)	(150,3)	(119,2)	(119,2)	(87,1)
2800	0.17857	0.03877	(333,8)	(205,4)	(171,3)	(136,2)	(136,2)	(99,1)
3000	0.16667	0.03425	(448,10)	(269,5)	(194,3)	(194,3)	(154,2)	(154,2)

Illustration 1: Suppose that the probability distribution of the lifetime of products under inspection is EE($\theta = 2, \lambda$). The mean lifetime of the products corresponding to PQ and CQ are, respectively, $\mu_1 = 8000$ h and $\mu_2 = 2000$ h. If the test termination time is fixed as $t = 500$ h, then the values of p_1 and p_2 can be calculated, respectively, as 0.00801 and 0.09779. If α and β are prescribed as 0.05 and 0.10, respectively, then the plan parameters may be obtained from Table 1 as $n = 39$ and $c = 1$.

The sampling plan can be implemented as follows: 39 products may be selected randomly from the lot and life test may be conducted to all the sampled products. If the number of products failed till $t = 500$ h is one, the lot may be accepted. On the other hand, if the second failure occurs before $t = 500$ h, terminate the life test. The lot may be rejected.

Illustration 2: Let the lifetime distribution of the products be the EE (2, λ) distribution. Let the test termination time be prescribed as $t = 500$ h. If the median lifetime of the products corresponding to PQ and CQ are, respectively, $m_1 = 5000$ h and $m_2 = 1800$ h, then the values of p_1 and p_2 can be calculated using (11) and (12) respectively as $p_1 = 0.01335$ and $p_2 = 0.08353$. Now, (n, c) may be selected from Table 2 corresponding to $\alpha = 0.05$ and $\beta = 0.10$ as $n = 79$ and $c = 3$. The reliability sampling plan may be implemented as follows: Select 79 products randomly from the lot submitted for inspection and conduct the life test to all the 79 products. If, during 500 h, not more than three failures are observed, then the lot may be accepted. Otherwise, the lot may be rejected.

5 Comparison of Mean and Median Lifetime-based Sampling Plans

According to Gupta [7], population median can describe the population relatively better than the population mean, when the probability distribution of the lifetime is a positively skewed distribution. Implementation of mean-based reliability sampling plan may affect sampling risks. In view of elaborating this, true sampling risks of matched plans of both kinds are compared using the values of their OC functions.

Suppose that the lifetime expected by the producer is 6000 h, $\alpha = 0.05$, the lifetime expected by the consumer is 2400 h, $\beta = 0.10$ and test termination time is 500 h. Corresponding to this strength, the values of (n, c) for mean lifetime-based sampling plan and median lifetime-based sampling plan can be selected from Tables 1 and 2, respectively, as (91, 3) and (130, 3). The PR, CR, and the total sampling risk (TR) due to these two plans are computed. Values are presented in Table 3.

It may be noted from Table 3 that the PR due to the implementation of the mean lifetime-based and median lifetime-based sampling plans are, respectively, 3.77 and 3.57%. It indicates that mean lifetime-based sampling plan will yield relatively 0.20% additional risk to the producer. Similarly, observation can be made with respect to CR also. Median lifetime-based sampling plan reduces relatively 0.18% risk of consumer. In total, 0.39% of TR can be reduced by implementing median lifetime-based

Table 3 PR, CR, and TR

	(n,c)	p_1	p_2	$P_a(p_1)$	$P_a(p_2)$	PR	CR	TR
Mean-based sampling plan	(91,3)	0.01381	0.07203	0.96227	0.09946	0.037732	0.099461	0.137193
Median-based sampling plan	(130,3)	0.00946	0.05095	0.96434	0.09763	0.035661	0.097634	0.133295

sampling plan. Though median lifetime-based sampling plan requires inspection of additional number of products, it reduces the PR, CR, and TR.

6 Conclusion

Reliability single sampling plans are designed in this paper for carrying out life test-based sampling inspection under the hybrid censoring scheme assuming the exponentiated exponential distribution as the lifetime distribution. The plans will safeguard both producer and consumer at ensured levels. It is noted that implementation of the mean lifetime-based reliability sampling plan will increase the PR, CR, and the TR compared to the median lifetime-based sampling plan. Implementation of the median lifetime-based sampling plan may require relatively more number of products for inspection compared with the mean lifetime-based sampling plan. It may be noted further that if the hybrid censoring scheme is employed to carry out the life test, implementation of the corresponding sampling plans will reduce the time and cost of conducting the life test.

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Availability Prediction of Repairable Fault-Tolerant System with Imperfect Coverage, Reboot, and Common Cause Failure

Madhu Jain and Pankaj Kumar

Abstract In this article, we consider two Markov models with standbys and imperfect coverage for the performance prediction of a repairable redundant system. Model I deals with a redundant system comprising one operating unit and one standby unit. In model II, some realistic features such as common cause failure, reboot, and recovery are taken into account to analyze a two-unit system. The reliability and MTTF analyses have been carried out using Laplace transform approach for model I. The availability analysis of the system studied in model II has also been evaluated by implementing the recursive approach. The analytic expressions for predicting the availability and other performance measures of the systems are presented. Furthermore, the numerical results obtained from the analytical expressions are compared with the hybrid soft computing technique based on adaptive neuro-fuzzy inference system (ANFIS).

Keywords Reliability · MTTF · Availability · Standby support · Imperfect coverage, ANFIS

1 Introduction

Reliability modeling with redundancy is a common approach to study the reliability, availability, and maintainability (RAM) related issues of a computer system as well as complex engineering systems. Redundancy is a key element of real-time systems such as distributed networks and communication systems, manufacturing systems, banking and healthcare systems, cloud computing and data centers, telecommunications and power plants, etc. In several fault-tolerant systems, redundancy is employed to obtain high reliability and availability as well as safety of the systems operating

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under certain techno-economic constraints. The system redundancy via standby support is a common feature used in many real-time systems to tackle unexpected faults and failures. Redundancy techniques are commonly used to enhance the reliability and to achieve the desired goal of output/production. A detailed account of reliability analysis of repairable systems can be found in Osaki and Nakagwa [15]. With the provision of standbys, the repairable redundant system has been discussed by Kumar and Agarwal [10, 11], Gupta et al. [2], Goel and Shrivastava [1], and others. To make repairable redundant systems analysis more realistic, the feature of common cause failure (CCF) can be taken into account. In the last few decades, the concept of CCF has been discussed in different setup (cf. [6, 12, 16]. Recently, Jain et al. [7] discussed a Markovian multicomponent system by incorporating some features such as working vacation, unreliable server, and F-policy. In their study, they have evaluated the steady-state probabilities of various states by implementing the successive-over-relaxation (SOR) method. Furthermore, several indices to examine the reliability characteristics of the system have been established.

Sometimes, the failures are not detected or covered successfully in repairable redundant system; this is referred to as imperfect coverage. The notions of coverage probability, reboot, recovery, and common cause failure can be considered for the performance of real-time fault-tolerant systems (FTS). Initially, Pham [17] studied a high voltage system in which imperfect failure time was considered as a constant. Later work by Moustafa [14] enhanced Pham's approach for the reliability analysis of *k-out-of-n: G* configuration by incorporating imperfect failure. Wang and Chiu [20] studied an availability model by incorporating imperfect coverage and standby provisioning and also carried out a cost analysis. Wang et al. [19] studied the availability of two systems with different imperfect coverage. Recently, Jain [3] discussed the performance analysis of a redundant system with imperfect repair and facilitated the numerical simulation by using the Runge–Kutta method. Yuge et al. [21] studied the reliability indices of a *k-out-of-n: G* configuration with CCF. In this study, they focused on the Marshall–Olkin model by considering the failure governed by an exponential distribution. Jain and Meena [4] proposed the performance model of an FTS machining system by including the concept of working vacation, F-policy, standby support, and imperfect coverage. The steady-state probabilities of various states are determined to predict the system performance using successive-over-relaxation (SOR) method.

The soft computing technique has made a revolutionary contribution in the industrial scenarios to achieve tractability and robustness in an effective and efficient manner. The soft computing approach is also used to reduce the imperfection and uncertainty to enhance the reliability, availability, and maintainability of real-time systems. The composition of two approaches, namely artificial neural network and fuzzy logic, has been employed to develop a hybrid soft computing technique, viz. ANFIS. The contributions of ANFIS in different areas can be found in the literature (cf. [8, 9, 13, 18]. The comparative study of performance results determined using Runge–Kutta approach with ANFIS approach has been done by Jain and Meena [5] for a fault-tolerant system (FTS).

In this paper, to make a repairable redundant fault-tolerant system closer to realistic situations, imperfect fault coverage, common cause failure, and reboot are taken into consideration while developing two Markov models for a two-unit machining system. The rest of this paper is organized as follows. Section 2 presents the assumptions and notations of the model. The mathematical analysis of the reliability model is done in Sect. 3. Section 4 presents the numerical simulation by taking an illustration. The conclusion and future scope of the study done are presented in Sect. 5.

2 System Description

We consider two Markov models for a repairable redundant system comprising an active and a standby unit. The replacement of the failed unit by the standby unit is considered to be imperfect as such detection of the faulty unit is usually not perfect. The fault of the active (standby) unit is assumed to be not covered by the coverage probability $\overline{c}$ ($\overline{c_s}$). To formulate the Markov models for a two-unit redundant system, we use the following assumptions:

- The failure characteristics of the standby unit when in full operating mode are the same as that of the active unit. The active units are regularly monitored by a failure detection tool. Both the units may fail independently of the state of the other operating/standby units. The lifetimes of operating and standby units are exponentially distributed with rates λ and $\lambda_s (0 \le \lambda_s \le \lambda)$, respectively.
- Both units of the system are repairable, and if the active unit fails, then it is either sent for repair or replaced by standby with probability c as soon as possible. The repair time of the failed unit is exponentially distributed with mean time $1/\mu$. If the fault is not detected successfully with coverage probability c, then it is cleared by the reboot process which is exponentially distributed with parameter β. The reboot delay and recovery time of the system are exponentially distributed with mean time $1/\beta$ and $1/\sigma$, respectively.
- If the operating unit fails and is replaced by a standby unit, then the replaced unit begins to work in the same manner as operating unit. The system may also fail due to common cause failure (CCF) following an exponential distribution with rate λ_c. The common cause failure (CCF) duration as well as repair time of the units failed due to CCF is exponentially distributed with mean time $1/\lambda_c$ and $1/\mu_c$, respectively.

The other notations used to formulate the model are as follows:

MTTF: Mean time to failure of the repairable system

$A_V(\infty)$: Availability of the repairable system

$P_{i,j}(t)$: Probability that the system is residing in the (i, j)th state, where $i = 1, c;\ j = 0, 1, D$.

$P^*_{i,j}(s)$: Laplace transform of $P_{i,j}(t)$, i.e., $P^*_{i,j}(s) = \int_0^\infty e^{-st} P_{i,j}(t)\, dt$,

$$(i, j) = \{(1, 1), (1, 0), (1, D), (C, 1), (0, 0)\}$$

$P_{i,j}$ Steady-state probability of the (i, j) state, i.e.,

$$P_{i,j} = \lim_{t \to \infty} P_{i,j}(t)$$

3 Reliability Model

To predict the system behavior, we establish some performance indices of the repairable redundant system. The reliability indices such as MTTF and availability of the redundant system are obtained analytically.

3.1 Reliability and MTTF

The state transition diagram for model I for different system states is depicted in Fig. 1.

The differential difference equations of the system in terms of Laplace transforms of state probabilities are framed by using the birth–death process as follows:

$$sP^*_{1,1}(s) = 1 - (\lambda + \lambda_s)P^*_{1,1}(s) + \mu P^*_{1,0}(s) \tag{1}$$

$$sP^*_{1,0}(s) = -(\lambda + \mu)P^*_{1,0}(s) + (\lambda c + \lambda_s c_s)P^*_{1,1}(s) \tag{2}$$

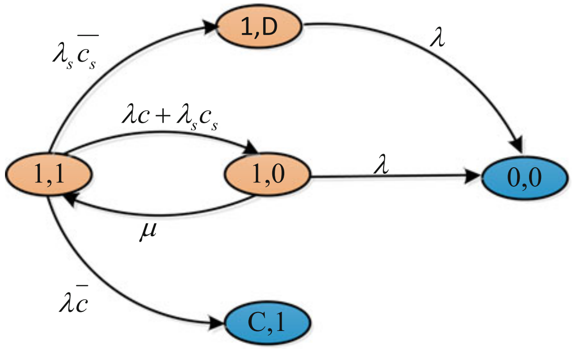
$$sP^*_{0,0}(s) = \lambda P^*_{1,0}(s) + \lambda P^*_{1,D}(s) \tag{3}$$

$$sP^*_{1,D}(s) = -\lambda P^*_{1,D}(s) + \lambda_s \overline{c_s} P^*_{1,1}(s) \tag{4}$$

$$sP^*_{C,1}(s) = \lambda \bar{c} P^*_{1,1}(s) \tag{5}$$

On solving the above system of linear Eqs. (1)–(5), we get

Fig. 1 State transition diagram for the reliability model I



$$P^*_{1,1}(s) = \frac{s+\lambda+\mu}{(s+\lambda+\lambda_s)(s+\lambda+\mu)-\mu(\lambda c+\lambda_s c_s)} \tag{6}$$

$$P^*_{1,0}(s) = \frac{\lambda c+\lambda_s c_s}{(s+\lambda+\lambda_s)(s+\lambda+\mu)-\mu(\lambda c+\lambda_s c_s)} \tag{7}$$

$$P^*_{0,0}(s) = \frac{\lambda[(\lambda c+\lambda_s c_s)(s+\lambda)+\lambda_s\mu\overline{c_s}]}{s(s+\lambda)[(s+\lambda+\lambda_s)(s+\lambda+\mu)-\mu(\lambda c+\lambda_s c_s)]} \tag{8}$$

$$P^*_{1,D}(s) = \frac{\lambda_s\overline{c_s}(s+\lambda+\mu)}{(s+\lambda)[(s+\lambda+\lambda_s)(s+\lambda+\mu)-\mu(\lambda c+\lambda_s c_s)]} \tag{9}$$

$$P^*_{C,1}(s) = \frac{\lambda\overline{c}(s+\lambda+\mu)}{s[(s+\lambda+\lambda_s)(s+\lambda+\mu)-\mu(\lambda c+\lambda_s c_s)]} \tag{10}$$

In this model, we assume that initially the system is in the operating state $(1, 1)$. So that $P_{1,1}(0) = 1,\ P_{1,0}(0) = 0,\ P_{0,0}(0) = 0,\ P_{1,D}(0) = 0$ and $P_{C,1}(0) = 0$. Let T be the mean time to failure of the system. The state probabilities $P_{0,0}(t)$ and $P_{C,1}(t)$ are the probabilities of failed states of the system at time t. Thus, the reliability function is obtained as follows:

$$R_T(t) = P_{1,1}(t) + P_{1,0}(t) + P_{1,D}(t) \tag{11}$$

The MTTF is obtained as

$$\text{MTTF} = \int_0^\infty R_T(t)\mathrm{d}t = \frac{\lambda^2(1+c)+\lambda(\mu+\lambda_s)+\mu\lambda_s\overline{c_s}}{\lambda[(\lambda+\lambda_s)(\lambda+\mu)-\mu(\lambda c+\lambda_s c_s)]} \tag{12}$$

3.2 The Availability Analysis of the System

The transition diagram of model II to describe the different states is displayed in Fig. 2. To explore the availability measures of the repairable redundant system with CCF, the steady-state difference equations are constructed as follows:

$$(\lambda+\lambda_s+\lambda_c)P_{1,1} = \mu P_{1,0} + \mu_c P_{0,0} \tag{13}$$

$$(\lambda+\mu)P_{1,0} = (\lambda c+\lambda_s c_s)P_{1,1} + \mu P_{0,0} + \sigma P_{1,D} + \beta P_{C,1} \tag{14}$$

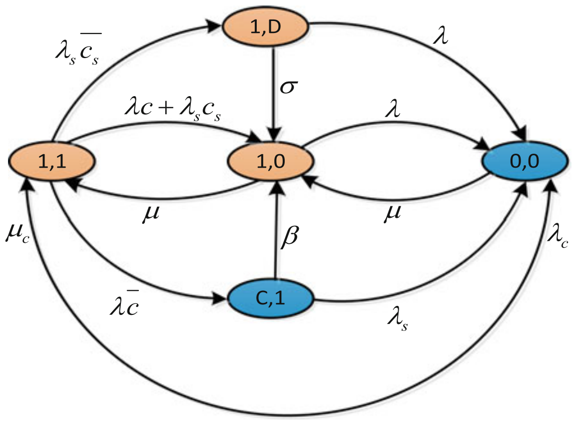
$$(\lambda_s+\beta)P_{C,1} = \lambda\overline{c}P_{1,1} \tag{15}$$

$$(\lambda+\sigma)P_{1,D} = \lambda_s\overline{c_s}P_{1,1} \tag{16}$$

$$(\mu+\mu_c)P_{0,0} = \lambda P_{1,0} + \lambda P_{1,D} + \lambda_s P_{C,1} + \lambda_c P_{1,1} \tag{17}$$

Equations (15)–(19) are solved to obtain the state probabilities in terms of $P_{1,1}$ as follows:

Fig. 2 State transition diagram for the availability model II



$$P_{1,0} = \frac{\mu_c}{\mu_c(\lambda+\mu)+\mu^2}\left[(\lambda c + \lambda_s c_s) + \frac{\mu(\lambda+\lambda_s+\lambda_c)}{\mu_c} + \frac{\sigma\lambda_s\overline{c_s}}{\lambda+\sigma} + \frac{\beta\lambda\bar{c}}{\lambda_s+\beta}\right]P_{1,1} \tag{18}$$

$$P_{1,D} = \frac{\lambda_s\overline{c_s}}{(\lambda+\sigma)}P_{1,1} \tag{19}$$

$$P_{C,1} = \frac{\lambda\bar{c}}{(\lambda_s+\beta)}P_{1,1} \tag{20}$$

$$P_{0,0} = \frac{\mu}{\mu_c(\lambda+\mu)+\mu^2}\left[\frac{\lambda(\lambda+\lambda_s+\lambda_c)}{\mu} + \frac{\lambda\lambda_s\overline{c_s}}{\lambda+\sigma} + \frac{\lambda\lambda_s\bar{c}}{\lambda_s+\beta} + \lambda_c\right]P_{1,1} \tag{21}$$

Using the normalizing condition

$$P_{1,1} + P_{1,0} + P_{1,D} + P_{C,1} + P_{0,0} = 1, \tag{22}$$

we get the probability $P_{1,1}$ as

$$P_{1,1} = \frac{1}{\left[1 + \frac{(\mu+\mu_c)(\lambda+\lambda_s+\lambda_c)}{\mu_c(\lambda+\mu)+\mu^2} + \frac{\lambda\bar{c}}{(\lambda_s+\beta)} + \frac{\lambda_s\overline{c_s}}{(\lambda+\sigma)}\right]} \tag{23}$$

Now, the availability of the system is obtained as

$$A = \frac{1 + \frac{\mu_c}{\mu_c(\lambda+\mu)+\mu^2}\left[(\lambda c + \lambda_s c_s) + \frac{\mu(\lambda+\lambda_s+\lambda_c)}{\mu_c} + \frac{\sigma\lambda_s\overline{c_s}}{\lambda+\sigma} + \frac{\beta\lambda\bar{c}}{\lambda_s+\beta}\right] + \frac{\lambda_s\overline{c_s}}{\lambda+\sigma}}{\left[1 + \frac{(\mu+\mu_c)(\lambda+\lambda_s+\lambda_c)}{\mu_c(\lambda+\mu)+\mu^2} + \frac{\lambda\bar{c}}{(\lambda_s+\beta)} + \frac{\lambda_s\overline{c_s}}{(\lambda+\sigma)}\right]} \tag{24}$$

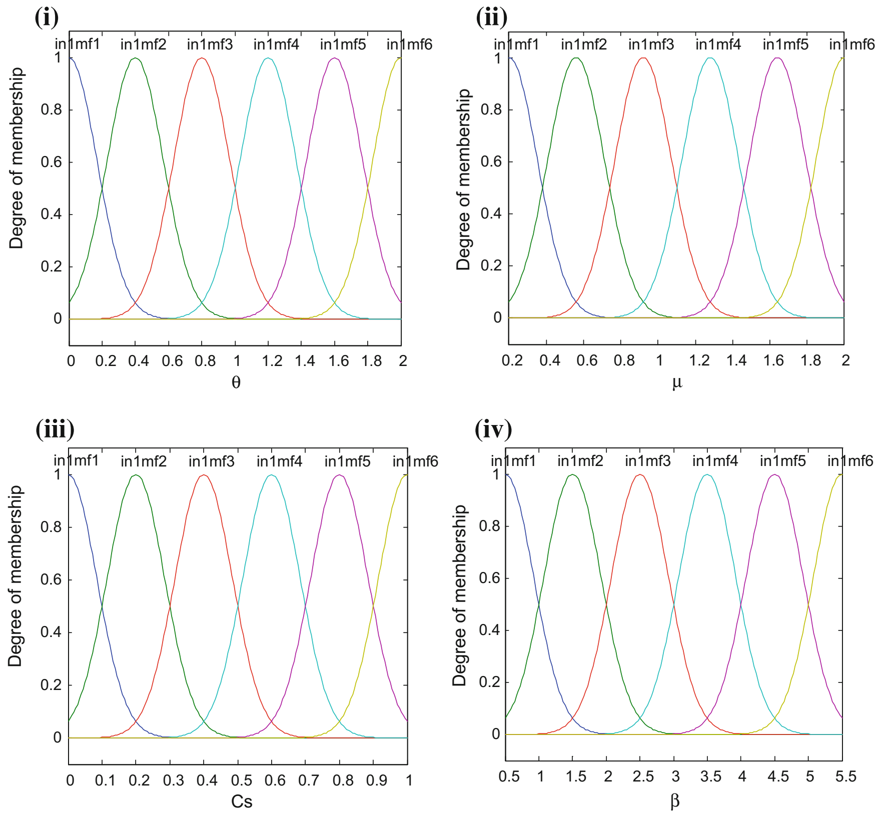


Fig. 3 Membership function for **i** $\theta = \lambda,\ \lambda_s$, **ii** μ, **iii** c_s, and **iv** β

3.3 Frequencies of Encountering of Individual States

The notion of frequency is very useful for the understanding of the system performance. The frequencies of encountering of all the individual states in the state space diagram for the availability model II (see Fig. 2) are obtained as follows:

$$f_1 = \frac{(\lambda + \lambda_s + \lambda_c)}{\left[1 + \frac{(\mu+\mu_c)(\lambda+\lambda_s+\lambda_c)}{\mu_c(\lambda+\mu)+\mu^2} + \frac{\lambda\overline{c}}{(\lambda_s+\beta)} + \frac{\lambda_s\overline{c_s}}{(\lambda+\sigma)}\right]} \tag{25.1}$$

$$f_2 = \frac{\frac{\mu_c}{\mu_c(\lambda+\mu)+\mu^2}\left[(\lambda c + \lambda_s c_s) + \frac{\mu(\lambda+\lambda_s+\lambda_c)}{\mu_c} + \frac{\sigma\lambda_s\overline{c_s}}{\lambda+\sigma} + \frac{\beta\lambda\overline{c}}{\lambda_s+\beta}\right](\lambda+\mu)}{\left[1 + \frac{(\mu+\mu_c)(\lambda+\lambda_s+\lambda_c)}{\mu_c(\lambda+\mu)+\mu^2} + \frac{\lambda\overline{c}}{(\lambda_s+\beta)} + \frac{\lambda_s\overline{c_s}}{(\lambda+\sigma)}\right]} \tag{25.2}$$

$$f_3 = \frac{\lambda_s\overline{c_s}}{\left[1 + \frac{(\mu+\mu_c)(\lambda+\lambda_s+\lambda_c)}{\mu_c(\lambda+\mu)+\mu^2} + \frac{\lambda\overline{c}}{(\lambda_s+\beta)} + \frac{\lambda_s\overline{c_s}}{(\lambda+\sigma)}\right]} \tag{25.3}$$

$$f_4 = \frac{\lambda\overline{c}}{\left[1 + \frac{(\mu+\mu_c)(\lambda+\lambda_s+\lambda_c)}{\mu_c(\lambda+\mu)+\mu^2} + \frac{\lambda\overline{c}}{(\lambda_s+\beta)} + \frac{\lambda_s\overline{c_s}}{(\lambda+\sigma)}\right]} \tag{25.4}$$

$$f_5 = \frac{\frac{\mu}{\mu_c(\lambda+\mu)+\mu^2}\left[\frac{\lambda(\lambda+\lambda_s+\lambda_c)}{\mu} + \frac{\lambda\lambda_s\overline{c_s}}{\lambda+\sigma} + \frac{\lambda\lambda_s\overline{c}}{\lambda_s+\beta} + \lambda_c\right](\mu+\mu_c)}{\left[1 + \frac{(\mu+\mu_c)(\lambda+\lambda_s+\lambda_c)}{\mu_c(\lambda+\mu)+\mu^2} + \frac{\lambda\overline{c}}{(\lambda_s+\beta)} + \frac{\lambda_s\overline{c_s}}{(\lambda+\sigma)}\right]} \tag{25.5}$$

4 Numerical Simulation

This section presents the computational results for various performance indices obtained analytically based on the expression from the previous section using MATLAB.

(i) **Effects of different parameters on MTTF**

Figure 4i shows that the MTTF decreases with an increase in the failure rate of the operating unit. It is noticed that initially, the MTTF decreases rapidly with an increase in λ and, later, when the failure rate further increases, MTTF decreases at a slower rate before becoming almost constant. Figure 4ii shows that expected MTTF increases with the increase in repair rate. For the lower value of the coverage factor, the effect of μ is not very significant. Also, the variation in MTTF is not very significant for small values of the coverage factor c. Figure 4iii shows that MTTF decreases rapidly initially with an increase in $\lambda_{s,}$ but the variation in MTTF diminishes for higher values of λ_s. In Fig. 4iv, we observe that MTTF increases very slowly by increasing the coverage factor of the standby unit.

(ii) **Effects of different parameters on availability**

In Fig. 5i, we notice that the availability decreases when the failure rate λ of the operating unit increases. Further, the availability increases as the coverage probability increases. In Fig. 5ii, we observe that the availability increases with an increase in the repair rate μ. From this figure, we notice that the availability increases sharply with a slight increase in the value of μ, but later on the impact diminishes. Further, the availability increases as the coverage factor increases. Figure 5iii shows that the availability can be improved by increasing the reboot rate (β). Also, availability is high for large values of c as can be seen from Fig. 5i–iii.

The ANFIS approach is also implemented to compute the system indices of the repairable redundant system. The membership function of input parameters $\lambda,\ \lambda_s,\ c_s,\ \mu$, and β are taken as a Gaussian function (see Fig. 3i–iv) by considering linguistic values as very small, small, average, high, and very high for parameters $\theta,\ \mu,\ c_s$ and β, respectively. The ANFIS results are computed by using the neuro-fuzzy tool in MATLAB software; the results obtained analytically are compared with the results of ANFIS. The results for the MTTF and availability by the ANFIS approach are depicted in Figs. 4i–iv and 5i–iii by tick marked over the curves drawn

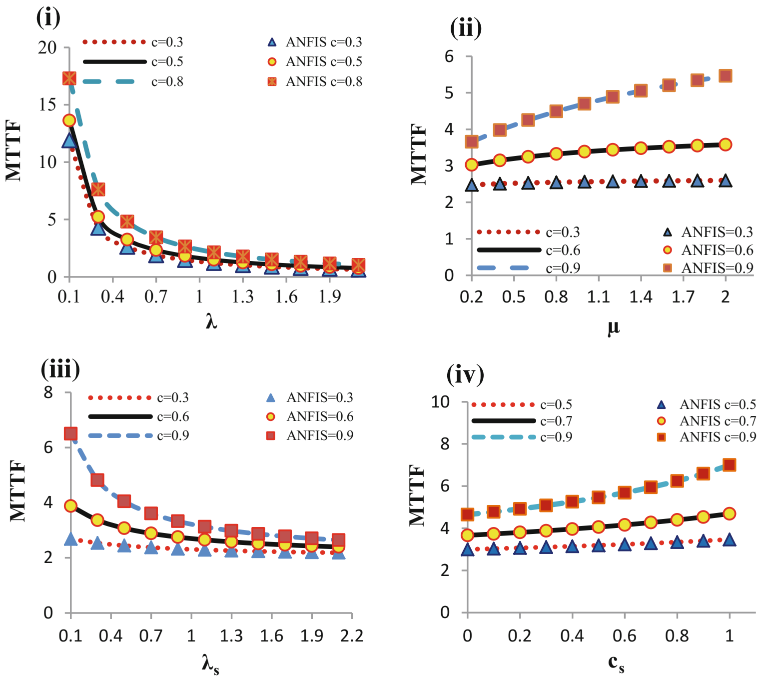


Fig. 4 Plot of MTTF versus **i** λ, **ii** μ, **iii** λ_s, and **iv** c_s for different value of c

based on analytical results. From these figures, we conclude that the analytical results and the ANFIS results are quite similar.

5 Conclusion

In this paper, we have derived explicit expressions for various indices for the performance prediction of a two-unit system by incorporating the features of redundancy and imperfect coverage. The explicit expressions for MTTF and availability derived for the system can be further used to design a suitable maintenance policy. Moreover, the results obtained analytically are found to be at par with the results obtained by ANFIS. Future work will extend this study for multicomponents redundant systems with dissimilar standby units.

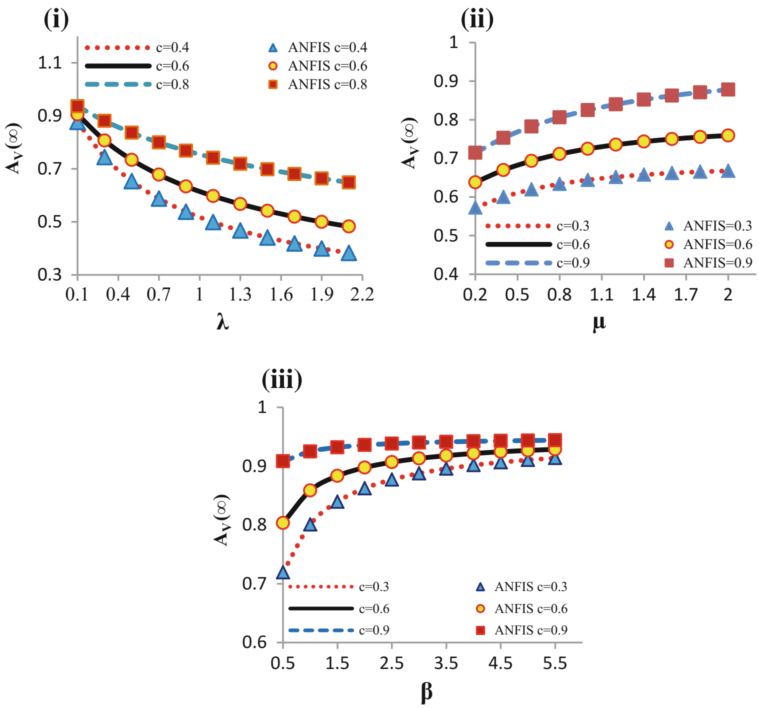


Fig. 5 System availability with variations in **i** λ, **ii** μ, and **iii** β for different value of c

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Software Reliability Growth Model in Distributed Environment Subject to Debugging Time Lag

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Abstract With the growing complexity of software in distributed computing environment, it is necessary to have knowledge of debugging process and testing coverage to make certain sure about achieved software reliability. This investigation is concerned with the reliability growth evaluation of the software system operating in the distributed development environment by considering the coverage factor and power function of testing time. The concept of delay effect factor is also taken into account which reveals the delay in removals of identified faults at any time. Based upon the cost and reliability criterion, the optimal policies for the software testing are suggested. Runge–Kutta technique is used to obtain the expected fault removals in a fixed time interval and others software reliability indices.

Keywords Fault removal process · Software reliability · Coverage factor · Fault dependency · Delay effect · Distributed software system

1 Introduction

With the remarkable advances in modern technology, software embedded systems have become an integral part of many industrial operations and day today human activities. With the wide applicability of Internet, the development and testing of the software for the distributed environment are one of the biggest challenges for the software developing enterprises. The main goals of software design in the dis-

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tributed environment are to pull off the high degree of quality which includes several attributes such as reliability, efficiency, security, transparency, scalability, portability, reusability.

The reliability improvement of the software is one of the key concerns for the developers and programmers. To fulfill this objective, from time to time, many analytical software reliability growth models (SRGMs) representing fault identification and removal phenomenon in different frameworks have been proposed by many researchers [1–6]. Fault identification and its removal processes require different strategies, resources, and efforts during testing of the software. In the literature, many researchers have categorized multiple types of faults according to their severity levels [7, 8]. A discrete software system consisting of some new, and reused components have been proposed by Khatri et al. [9]. They assumed the simple faults in reused components only, whereas new components contain hard and complex type of faults.

The occurrence of software failures may be due to many factors like imperfect debugging, error generation, debugging time lag. Recently, some researchers have also contributed in the area of testing resource allocations and models based on testing efforts. A testing effort-based SRGM has been suggested by Kapur et al. [10] by including the error generation and debugging criterion. In this direction, a Markovian model has also discussed by Jain et al. [11]. The fault identification and fault correction process reflect the failure occurrence rate. The fault reduction factor (FRF) plays an important feature to overcome the problem of failure occurrence by enhancing the growth of software reliability. Pachauri et al. [12] have developed the inflection *S*-shaped model by incorporating the fault reduction function. A new attempt in reliability improvement has been made by Dar et al. [13]. They considered a software system containing several types of faults and make provision of testing efforts which are required to remove them.

Testing coverage in software systems stands as a key factor for both developers and customers. The testing coverage can be measured in terms of block, branch, computation-use, and predicate-use. It reveals that in what manner testing efforts are required and how much testing efforts can be spent during software testing. This information is useful for the software developers and provides the confidence to the customers for using the software product. Malaiya et al. [14] discussed the test coverages to examine the software testing thoroughly. Testing coverage-dependent software reliability growth models have also been analyzed by taking block (or statement) measure [15]. Another interesting study on analyzing SRGM using test coverage and failure data simultaneously was given by Wang et al. [16]. They gave two different models: one is continuous based on testing time and another one is discrete based on test cases. Further, Wang et al. [17] extended their previous work and developed two models by considering executed test cases. Pham [18] also presented two new SRGMs under the assumption of testing coverage wherein fault identification rate follows log–log distribution. More recent developments on the same track were done by Anniprincy and Sridhar [19] by considering Cobb–Douglas production function for the removed faults. Pawar et al. [20] studied the improved coding structure of software via block-branch coverage.

The problem of optimal testing time to release the software is a major concern. Many researchers have proposed software release policies to resolve this problem in different discipline. In the past, the software release policies have been studied to minimize the maintenance cost subject to desired reliability objective [21–27]. As per requirement in global competition, Singh et al. [28] explored multiple release policies to preserve the effect of high levels of severe faults generated in the software.

The testing process of the software faults is not only liable for complete removal of faults, and it is also important to develop SRGMs that should include testing time and testing coverage effects together. A few researchers have come forward in developing SGRM with testing time and test coverage for the detection or removal processes of the faults by taking testing coverage factor dependent on faults detection/removal rate [29]. In the present investigation, we study the software reliability by assuming that the testing coverage of the faults is exponentially related with the testing time which is the new study in this direction. Furthermore, faults removal process complete in multistages as per increasing complexity in distributed software systems. The prime goal of present investigation is to develop SRGM for distributed software system. The fault removal process that incorporates coverage factor and testing time in power function form has been considered. The remaining contents of the paper are arranged section-wise. In next Sect. 2, the SRGM model is presented to elaborate the problem. The governing equations and analysis for the fault removal process have been provided in Sect. 3. The features of time lag and fault removal process are outlined in Sect. 4. Various performance indices are derived in Sect. 5. In Sect. 6, some optimal release policies are discussed. Numerical results and conclusions are given in Sects. 7 and 8, respectively.

2 Model Description

In this section, we consider distributed software system with reused as well as new components. It consists of total Y software components in which two components are reused and remaining are new components. For the modeling purpose, we assume that the removal of faults from the two reused components is completed in one-stage process and the faults which are present in newly developed components are removed in two, three, or more stages depending on the severity of the faults.

The basic assumptions concerned with the fault identification (observation/isolation) and removal processes in distributed software environment dependent to NHPP are given as:

- The number of fault removals depends upon the instructions executed in the S/w. The fault identification/removal rate is assumed as function of testing time in power function form.
- The detected faults which are dependent may not be removed immediately as well as perfectly, i.e., no other error is introduced during the debugging.

For the formulation of SRGM, the following notations are used.

a_{i_k}	Initial fault content per unit time in i_kth component of software.
b'_{i_k}	Failure observation/fault isolation rate per unit time in i_kth component of software.
i	The number of individual components of software system.
z	Constant parameter.
α	Any index representing the power of testing time ($\alpha \geq 0$).
k	The number of steps of fault removal process.
x	Mission time of the software
$C(t)$	Test coverage function.
$p(t)$	Function of instructions.
Δt	Delay effect factor.
C_0	Cost of setup for software.
C_1	Testing cost incurred in unit time.
C_2	Fault removal cost during testing incurred in unit time.
C_3	Cost incurred per unit on a fault removal during warranty period.
$C_4(T)$	Opportunity cost function.
$C_5(T)$	Cost associated with the S/w failure.
T	Software testing time.
T_w	Warrantyperiod.
δ	Discount rate in testing.
u_t	Expected time for a fault removal during the testing.
u_w	Expected time for a fault removal during the warranty/operation period.
v_0	The scale coefficient of delay in releasing the software.
v_1	The intercept value of delay in releasing the software.
v_2	The opportunity loss rate attributed to delay in releasing the S/w.
$m_{F_{i_k}}(t)$	Mean number of faults observed in i_kth component of the software in (0, t].
$m_{I_i,k_j}(t)$	Mean number of faults isolated in i_kth component of the software in (0, t], where $j = 2, 3, \ldots, k-2$.
$m_{R_{i_k}}(t)$	Mean number of faults removed in i_kth component of S/w in (0, t].
$R_s(x/t)$	Reliability function in (t, $t + x$].
R_0	Minimum required reliability.
EC(T)	Expected maintenance cost

3 Fault Removal Phenomenon for Software

Now we establish the relation between the expected number of fault removals and testing coverage function that also represents a function based on expected number of instructions executed during testing. Thus,

$$\frac{\mathrm{d}m_R(t)}{\mathrm{d}t} = \frac{\mathrm{d}m_R(t)}{\mathrm{d}p(t)} \frac{\mathrm{d}p(t)}{\mathrm{d}t} \tag{1}$$

The second part of Eq. (1) represents the instructions. It has assumed that the number of instruction of testing time in power function forms due to exponential learning of debuggers which is given as

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} = zt^{\alpha}, \quad \alpha \geq 0 \tag{2}$$

If we consider the effort to increase the coverage, then removal rate of the faults can be increased and the coverage of software depends on the test cases which are being used for testing of the software. Hence, we assume that coverage of software with respect to removal rate follows exponential distribution. The equation for the fault removal process is framed as

$$\frac{\mathrm{d}m_R(t)}{\mathrm{d}p(t)} = \frac{C'(t)}{1 - C(t)}[a - m_R(t)] \tag{3}$$

where $c(t) = [1 - e^{-bt}]$; $C'(t)$ reveals the rate of coverage and $1-C(t)$ represents percentage remaining uncovered part of the software faults.

Substituting the results of Eqs. (2) and (3) in (1), we yield

$$\frac{\mathrm{d}m_R(t)}{\mathrm{d}t} = b' t^{\alpha}[a - m_R(t)] \tag{4}$$

where $b' = zb$.

Now, we present the fault removal processes depending upon the number of stages of faults removal for a distributed software system.

3.1 One-Stage Removal Process

In this case, one type of faults in two reused components of software has considered. Here we consider that the faults' removal is completed in one-stage process. Further, reused components of the software are not affected by severity of the faults as such the faults are simple faults and they are immediately removed without requiring any time in observation/isolation processes. Thus,

$$\frac{\mathrm{d}m_{R_{i_1}}(t)}{\mathrm{d}t} = b'_{i_1} t^{\alpha}\left[a_{i_1} - m_{R_{i_1}}(t)\right] \tag{5}$$

3.2 Two-Stage Removal Process

In this case, the fault removal process for new components of the software that takes sometime for the fault observation in comparison with one-stage removal process.

The faults can be removed in two stages and categorized as hard faults. It can be modeled as given below:

$$\frac{\mathrm{d}m_{F_{i_2}}(t)}{\mathrm{d}t} = b'_{i_2} t^{\alpha}\left[a_{i_2} - m_{F_{i_2}}(t)\right] \tag{6}$$

$$\frac{\mathrm{d}m_{R_{i_2}}(t)}{\mathrm{d}t} = b'_{i_2} t^{\alpha}\left[m_{F_{i_2}}(t) - m_{R_{i_2}}(t)\right] \tag{7}$$

3.3 Three-Stage Removal Process

Sometimes removal process of the faults in new components may take time to observe the faults, and then after spending sometime to isolate the faults (due to complexity of faults), the faults' removal process is performed in three-stage process. In this case, the faults are defined as complex faults. Now

$$\frac{\mathrm{d}m_{F_{i_3}}(t)}{\mathrm{d}t} = b'_{i_3} t^{\alpha}\left[a_{i_3} - m_{F_{i_3}}(t)\right] \tag{8}$$

$$\frac{\mathrm{d}m_{I_{i_3}}(t)}{\mathrm{d}t} = b'_{i_3} t^{\alpha}\left[m_{F_{i_3}}(t) - m_{I_{i_3}}(t)\right] \tag{9}$$

$$\frac{\mathrm{d}m_{R_{i_3}}(t)}{\mathrm{d}t} = b'_{i_3} t^{\alpha}\left[m_{I_{i_3}}(t) - m_{R_{i_3}}(t)\right] \tag{10}$$

3.4 Multiple Stage Removal Process

The removal process for k-stages can be used due to high severity of faults in new components of software. There may be multiple types of faults which need more than three-stage removal processes. Since all faults are observed at one time but the isolation of faults is done in one or more stages, once all the faults are isolated, then they can be easily removed. The k-stage removal process for the newly developed components is framed as:

$$\frac{\mathrm{d}m_{F_{i_k}}(t)}{\mathrm{d}t} = b'_{i_k} t^{\alpha}\left[a_{i_k} - m_{F_{i_k}}(t)\right] \tag{11}$$

$$\frac{\mathrm{d}m_{I_{i,k_1}}(t)}{\mathrm{d}t} = b'_{i_k} t^{\alpha}\left[m_{F_{i_k}}(t) - m_{I_{i,k_1}}(t)\right] \tag{12}$$

$$\frac{\mathrm{d}m_{I_{i,k_j}}(t)}{\mathrm{d}t} = b'_{i_k} t^{\alpha}\left[m_{I_{i,k_{j-1}}}(t) - m_{I_{i,k_j}}(t)\right], \quad j = 2, 3, \ldots, k-2 \tag{13}$$

$$\frac{\mathrm{d}m_{R_{i_k}}(t)}{\mathrm{d}t} = b'_{i_k} t^{\alpha}\left[m_{Ii,k_{k-2}}(t) - m_{R_{i_k}}(t)\right] \tag{14}$$

The boundary conditions are

$$m_{F_{i_k}}(t=0) = m_{I_i,k_1}(t=0) = \cdots = m_{I_i,k_{k-2}}(t=0) = m_{R_{i_k}}(t=0) = 0 \quad (15)$$

The fault observation process is given in Eq. (11). Fault isolation processes are taken into consideration by Eqs. (12) and (13), whereas fault removal process is given in Eq. (14). Solving the above differential Eqs. (11)–(14) and using (15) and $y = \sum_{\tau=1}^{k} y_\tau$, we get

$$m_{F_{i_k}}(t) = a_{i_k}\left\{1 - \exp\left(-\frac{b'_{i_k}}{(s+1)} t^{(\alpha+1)}\right)\right\} \quad (16)$$

$$m_{I_i,k_1}(t) = a_{i_k}\left\{1 - \left(1 + b'_{i_k}\frac{t^{\alpha+1}}{(\alpha+1)}\right)\exp\left(-b'_{i_k}\frac{t^{(\alpha+1)}}{(\alpha+1)}\right)\right\} \quad (17)$$

$$m_{I_i,k_j}(t) = a_{i_k}\left\{1 - \left(1 + b'_{i_k}\frac{t^{(\alpha+1)}}{(\alpha+1)} + \sum_{\ell=2}^{j}\frac{b'^{\ell}_{i_k}}{\ell!}\left(\frac{t^{(\alpha+1)}}{(\alpha+1)}\right)^{(\ell)}\right)\exp\left(-b'_{i_k}\frac{t^{(\alpha+1)}}{(\alpha+1)}\right)\right\},$$

$$j = 2, 3, \ldots, k-2. \quad (18)$$

$$m_{R_{i_k}}(t) = a_{i_k}\left\{1 - \left(\sum_{q=0}^{k-2}\frac{\left[b'_{i_k}\frac{t^{(\alpha+1)}}{(\alpha+1)}\right]^q}{q!}\right)\exp\left(-b'_{i_k}\frac{t^{(\alpha+1)}}{(\alpha+1)}\right)\right\} \quad (19)$$

The total fault removal for the distributed software system can be given as:

$$\sum_{i=1}^{y} m_{R_{ik}}(t) = a_{1_1}\left\{1 - \exp\left(-\frac{b'_{1_1}}{(\alpha+1)}t^{(\alpha+1)}\right)\right\} + a_{2_1}\left\{1 - \exp\left(-\frac{b'_{2_1}}{(\alpha+1)}t^{(\alpha+1)}\right)\right\}$$
$$+ \sum_{k=3}^{Y}\eta_k\left[1 - \left(\sum_{q=0}^{k-2}\frac{\theta^q_{(q+2)}}{q!}\right)e^{-\theta_k}\right] \quad (20)$$

where $a_{k_{k-1}} = \eta_k, \theta_k = b'_{k_{k-1}}\frac{t^{(\alpha+1)}}{(\alpha+1)}$.

Also $b'_{1_1} = b'_{2_1}, b'_{3_2} = b'_{4_3} = \cdots = b'_{Y_k}, a = a_{1_1} + a_{2_1} + a_{3_2} + a_{4_3} + \cdots + a_{(Y_k)}$.

This mean value function for the proposed SRGM can be given in this form

$$m(t) = \sum_{i=1}^{Y} m_{F_{i_k}}(t) + \sum_{i=1}^{Y}\sum_{j=1}^{k-2} m_{I_i,k_j}(t) + \sum_{i=1}^{y} m_{R_{i_k}}(t) \quad (21)$$

4 Debugging Time Lag in Removal Process

In practical phenomenon, the detected faults cannot be removed immediately due to severity of the faults. Due to the complexity of faults, removal is a time-consuming process and shows the time delay in fault detection process. We denote the delay effect factor by Δt.

Now we consider the delay effect of the fault detection process for k-stages. Thus,

$$m_{R_{i_k}}(t) = m_{F_{i_k}}(t - \Delta t) \tag{22}$$

(i) If we take one-stage process and faults are immediately removed as they are identified, so in that case $\Delta t = 0$, then

$$m_{R_{i_1}}(t) = m_{F_{i_1}}(t) \tag{23}$$

(ii) For two-stage process, the time delay in the correction process of fault cannot be negligible. Thus,

$$t - \Delta t = \left\{ t^{\alpha+1} - \frac{(\alpha+1)}{b'_{i_k}} \log\left(1 + \frac{b'_{i_k}}{(\alpha+1)} t^{\alpha+1}\right) \right\}^{\frac{1}{(\alpha+1)}} \tag{24}$$

(iii) In three-stage process, the debugging time lag in faults detection and removal is given by

$$t - \Delta t = \left\{ t^{\alpha+1} - \frac{(\alpha+1)}{b'_{i_k}} \log\left(1 + \frac{b'_{i_k}}{(\alpha+1)} t^{\alpha+1} + \frac{b'^2_{i_k}}{2} \left(\frac{t^{\alpha+1}}{(\alpha+1)}\right)^2 \right) \right\}^{\frac{1}{(\alpha+1)}} \tag{25}$$

(iv) The debugging time lag for k-stages removal process reflects the ability of human learning. Thus,

$$t - \Delta t = \left\{ t^{\alpha+1} - \frac{(\alpha+1)}{b'_{i_k}} \log\left(\sum_{q=0}^{n} \frac{\left(b'_{i_k} \frac{t^{\alpha+1}}{(\alpha+1)}\right)^q}{q!} \right) \right\}^{\frac{1}{(\alpha+1)}} \tag{26}$$

5 Performance Indices

The effect of fault removal process on the reliability and maintenance cost is desirable factors to evaluate. In this section, we present some performance measures to judge the performance and improvement of concerned SRGM as follows:

(i) Failure intensity function is formulated by

$$\lambda(t) = \frac{\mathrm{d}m(t)}{\mathrm{d}t} \tag{27}$$

(ii) The reliability of the distributed software is computed by using

$$R_s(x/T) = \exp[-\{m(T + x) - m(T)\}] \tag{28}$$

(iii) The average total maintenance cost in incurred on the S/w is

$$\mathrm{EC}(T) = C_0 + C_1 T^{\delta} + C_2 m(T) u_t + C_3[m(T + T_w) - m(T)]u_w + C_4(T) + C_5(T) \tag{29}$$

where the risk cost due to software failure after releasing the software is computed using

$$C_5(T) = C_5\{1 - R_s(x/T)\} \tag{30}$$

and opportunity cost of software due to delaying the release time is computed by using

$$C_4(T) = v_0(v_1 + T)^{v_2}. \tag{31}$$

6 Optimal Release Policies

The delay in software releasing could not be tolerated by manufacturer as well as customer viewpoint. Moreover, if it is released to customer before complete removal of all types of faults, then maintenance cost may be very higher. Thus, it is very important to find an optimum cost under some desired reliability level so that the quality of the software should be controlled and maintained. To evaluate the optimal testing time, the total expenditure on the software will be minimized.

The cost optimization problem can be structured as

$$\begin{aligned} \text{Minimize } \mathrm{EC}(T) = C_0 + C_1 \int_T^{T+T_w} e^{\mu t}\mathrm{d}t + C_2\left[\int_0^T \lambda(t)e^{\mu t}\mathrm{d}t\right]u_t + C_3\left[\int_T^{T+T_w} \lambda(t)e^{\mu t}\mathrm{d}t\right]u_w \\ + C_5[1 - R_s(x/T)] + v_0(v_1 + T)^{v_2} \\ \text{subject to } R_s(x/(T + T_w)) > R_0, \end{aligned} \tag{32}$$

The optimum testing policies are stated as follows:

Optimal Testing Policy 1:

OPT 1.1: If $\lambda(0) > \lambda(T)$ then $T^* = T$
OPT 1.2: If $\lambda(0) \leq \lambda(T)$ then $T^* = 0$

Optimal Testing Policy 2:

Let T_r denote the optimal testing time when the reliability constraint is satisfied.
OPT 2.1: If $\lambda(0) > \lambda(T)$ and $R_s(x/0) < R_0$ then $T^* = \max\{T, T_r\}$
OPT 2.2: If $\lambda(0) > \lambda(T)$ and $R_s(x/0) \geq R_0$ then $T^* = T$
OPT 2.3: If $\lambda(0) \leq \lambda(T)$ and $R_s(x/0) < R_0$ then $T^* = T_r$
OPT 2.4: If $\lambda(0) \leq \lambda(T)$ and $R_s(x/0) \geq R_0$ then $T^* = 0$

7 Numerical Results

The analytical reliability indices and expected number of faults which are removed in (0, t] are computed by taking a numerical example. To facilitate the sensitivity analysis, the results are shown in Figs. 1, 2 and 3. The behavior of different parameters

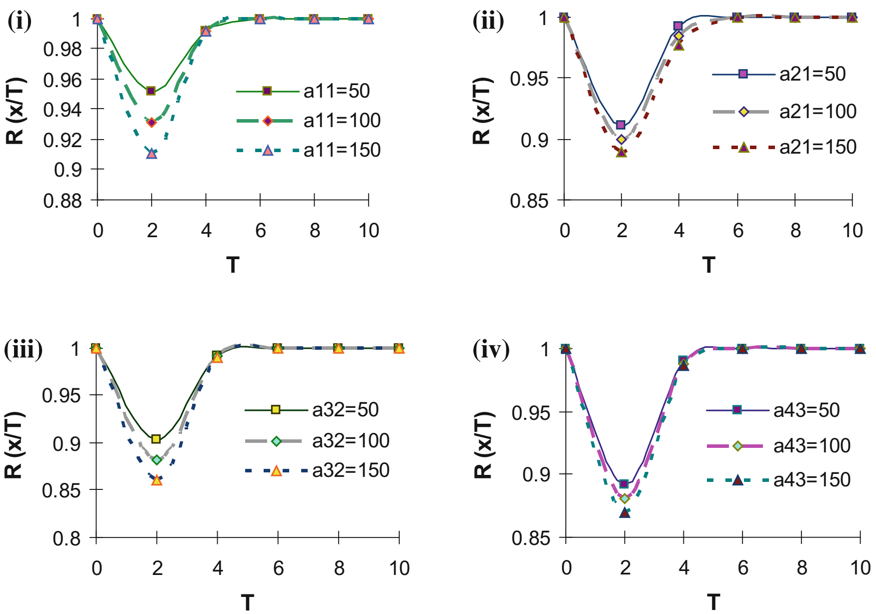

Fig. 1 **i** Reliability versus testing time by varying a_{11}. **ii** Reliability versus testing time by varying a_{21}. **iii** Reliability versus testing time by varying a_{32}. **iv** Reliability versus testing time by varying a_{43}

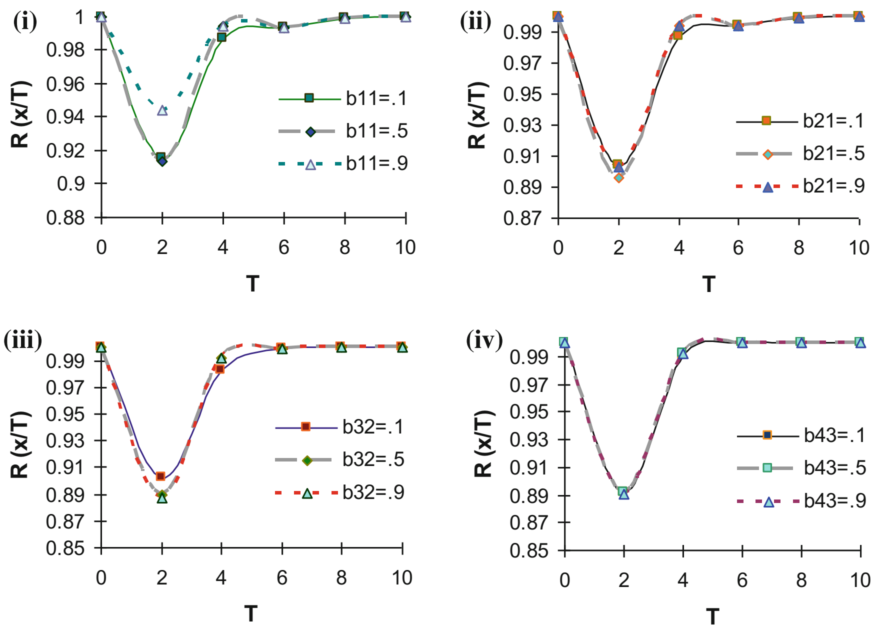

Fig. 2 **i** Reliability versus testing time by varying b_{11}. **ii** Reliability versus testing time by varying $b_{21.}$ **iii** Reliability versus testing time by varying $b_{32.}$ **iv** Reliability versus testing time by varying b_{43}

on the reliability is shown in Figs. 1 and 2 by varying testing time (T). Runge–Kutta fourth-order method is used to obtain MVFs which are displayed in Fig. 3.

Figures 1 and 2 exhibit reliability $R_s(x/T)$ by varying T for distinct initial number of faults (a_{11}, a_{21}, a_{32}, a_{43}) and error detection rates (b_{11}, b_{21}, b_{32}, b_{43}) in reused or new components of the software. The default parameters are taken as $a_{11} = 150$, $a_{21} = 50$, $a_{32} = 50$, $a_{43} = 100$, $b_{11} = 0.5$, $b_{21} = 0.1$, $b_{32} = 0.5$, $x = 0.0002$, $b_{43} = 0.5$, $s = 2$.

It is noticed that by increasing T, the reliability initially decreases sharply and then after increases steeply and it attains almost constant value for higher testing time T. In Fig. 1i–iv, we note the higher values of reliability for lower values of initial number of faults up to a certain value of T, then after it does not vary significantly. In Fig. 1ii, iv, for increasing the value of a_{21} and $a_{43,}$ respectively, $R_s(x/T)$ has lower value up to $T = 5$, after that there is no significant effect.

Figure 2i–iv depict the trend of S/w reliability $R_s(x/T)$ for different values of failure detection rates b_{11}, b_{21} (in reused components) and b_{32}, b_{43} (in new component). In the beginning reliability decreases sharply up to $T = 2$; then after as time goes on, it increases sharply up to its desired level. In Fig. 2i, S/w reliability growth seems to be enhanced for higher values of $b_{11.}$ In Fig. 2ii, iv, it is noticed that there is no significant effect of b_{21} and $b_{43,}$ respectively, on the reliability on increasing T. We

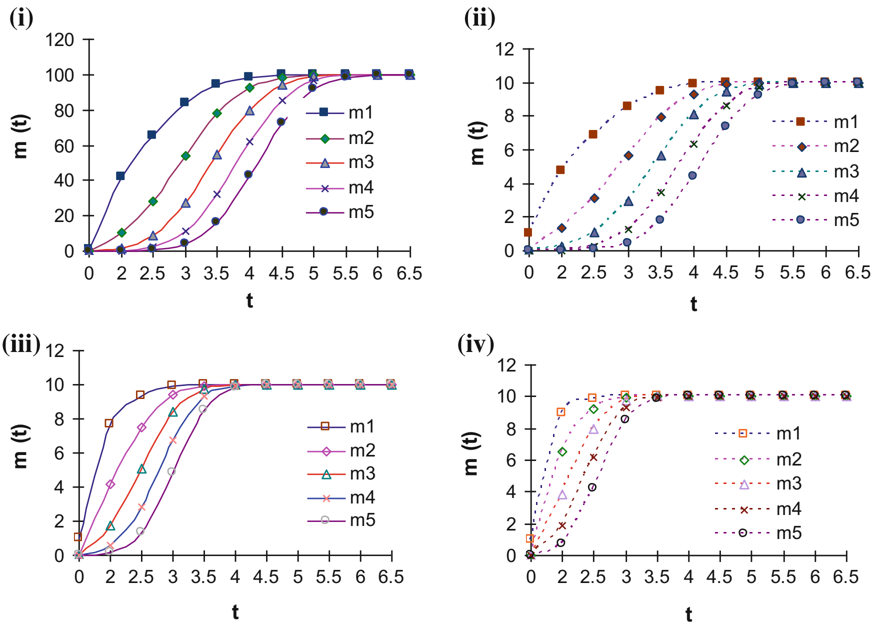

Fig. 3 MVF versus testing time by varying **i** $a = 100$ **ii** $a = 10$ **iii** $b = 0.5$ **iv** $b = 0.8$

also see in Fig. 2iii that initially reliability decreases slightly for higher b_{32} up to $T = 2$, and then after it increases moderately and then it shows asymptotic behavior.

The numerical results for the MVF depicting fault removals in $(0, t]$ from differential Eqs. (11)–(14) are obtained using Runge–Kutta approach. The default parameters are taken as $a = 10$, $b = 0.5$, and $s = 2$. In Fig. 3, $m(t)$ is shown by considering the removal process for the faults in five stages. We set the values of $a = 100$ in Fig. 3i, $a = 10$ in Fig. 3ii and $b = 0.5$ in Fig. 3iii while $b = 0.9$ in Fig. 3iv. It is observed that $m(t)$ increases for one-stage process.

Overall we conclude the findings of numerical experiment as follows:

- The reliability of the software first starts to decrease, then increase and finally it becomes almost constant with the increase in testing time.
- As we expect based on real-world experience, by increasing initial number of faults, $R_s(x/T)$ decreases but on increasing the error detection rates, reliability increases.
- The number of fault removals experienced in $(0, t]$, increases significantly time grows, but finally becomes constant.

8 Conclusion

The S/w reliability growth is investigated to determine the optimal value for testing time in distributed environment where faults come to surface during testing with high severity. The SRGM developed has incorporated the FRR based on testing coverage and testing time. Our study provides an insight to the software developer how the software reliability can be improved. The noble feature of our model is the realistic assumption of debugging time lag for fault removal. It is worth noting that for the prediction of measures of effectiveness of the concerned software system, the testing coverage facilitating power function of testing time is taken into account. The numerical illustration facilitated provides the valuable insight to determine optimal testing time. The proposed release policies of SRGM can be implemented in a wide range of distributed software to examine the different types of faults in the S/w design so as to resolve the issues related to faults before releasing the software in the market.

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Imperfect Software Reliability Growth Model Using Delay in Fault Correction

Bhoopendra Pachauri, Ajay Kumar and Sachin Raja

Abstract Software reliability growth models (SRGMs) are very useful tool to calculate the probability of software failure. A lot of mathematical models have been formulated to predict software reliability growth behavior. In the literature, most of the SRGMs is developed under consideration that the reliability growth of the software depends on testing time, and when a failure is occured, fault is immediately removed. In this article, an approach has been used which considers the testing effort and delay in removing the faults. Where testing time and testing effort are taken together. The combined effect of testing time and testing effort is considered using Cobb Douglas production function. Proposed model works in imperfect debugging environment where new faults may introduce in the fault detection and correction process. Time used by the testing team to remove any fault is also considered with some delay. The parameters are estimated using nonlinear regression. The developed model is validated on the real data sets. The performance of proposed study is compared based on mean square error (MSE) with existing models in the literature.

Keywords SRGM · NHPP · Optimal release policies · Cobb-douglus production function

1 Introduction

Software reliability growth models (SRGMs) can be specified in two types, parametric and nonparametric models. Parametric models are depend on the priori assumptions of the software failure nature, probability of failure occurrence and development

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environment, etc. However, the reliability metrics are predicted based on the failures history in nonparametric models. In any software, a failure means, the occurrence of an undesired result as an output for an input that is inward with specified conditions [19].

In parametric models, non-homogeneous Poisson process (NHPP)-based models are widely used. A lots of SRGMs have been discussed in existing literature considering different environmental conditions [9, 15–19, 24]. In 1979, Goel and Okumoto [2] proposed the NHPP-based -dependent failure rate model in perfect debugging environment. Inflection *S*-shaped model is given by Obha [12]. Musa studied the basic execution time model in [10]. After some time, log Poisson model is studied by Musa and Okumoto [11]. The *K*-Stage Erlangian model was proposed in [8]. Ohba [13] and Yamada et al. [25] attributed time delay effect between the fault detection and fault removal process. In this article, they showed that the skill level of the software testing team decreases in the later stages. Kapur et al. [5] studied an SRGM by considering three different types of faults that was modeled using delayed *S*-shaped, exponential, and three-stage Erlang distribution. Pham and Zhang [20] discussed a model in imperfect environment with time-dependent fault detection rate using delayed *S*-shaped function. Kapur et al. [6] discussed a integrated modeling framework to develop a SRGM in imperfect debugging environment. Pachauri et al. [14] studied the uncertainty of a imperfect debugging SRGM using fuzzy. At the same time period, some imperfect debugging models were also developed.

In the conventional SRGMs, it has been seen that one common assumption, fault that causes failure, is immediately removed. But it is not realistic because some time is needed to fix a fault by the programmer. Generally, there is a reasonable time taken to detect the fault and fix it. Basically, the fault fixing time depends on the difficulty of the fault, skill, and experience of the programmer, manpower, and so on. Therefore, the correction time cannot be neglected in fault correction processes [4]. On the other hand, most of the SRGMS have been discussed in one dimension that considers the reliability of a software depends only on testing time. In the literature, the same problem based on the NHPP has been studied in different way in [3, 26, 27]. They proposed some testing effort-dependent SRGMs where the testing time and used resources together is governed in the software reliability growth process. Therefore, two-dimensional SRGM is needed to capture the joint effect of testing time and used resources [7].

In this paper, two-dimensional SRGM using combined effect of testing time and used resources with debugging time lag is studied. Since, imperfect debugging reflects the realistic situation, the model is proposed in imperfect debugging environment. The rest of the manuscript is designed as follow: The concept of two-dimensional model is given in Sect. 2. Proposed models with debugging time lag and its result with discussion are shown in Sects. 3 and 4, respectively. Finally, conclusion and future scope is given in Sect. 5.

2 SRGM in Perfect Debugging Environment

In one-dimensional NHPP-based SRGMs, software failure/fault detection phenomenon is modeled as [19]:

$$P(N(t) = n) = \frac{(m(t))^n e^{-m(t)}}{n!}, \quad n = 0, 1, 2 \ldots, \tag{1}$$

where $N(t)$ is the random variable of faults/failure, $m(t)$ cumulative number of faults/failure at time t, and t represents the testing time. Since, two-dimensional SRGMs are used to govern the combined effect of testing time and used resources. Therefore, the combined effect of testing effort and used resources is modeled using Cobb–Douglas production function [7]. Mathematical expression of Cobb–Douglas production function is given as follows:

$$\tau \cong s^{\alpha} u^{1-\alpha}, \tag{2}$$

where $0 \leq \alpha \leq 1$.

Kapur et al. [7] studied a SRGM in two dimension with Cobb–Douglas production function in perfect debugging environment. Mathematical model shows the rate of changes in mean value function and given as follows:

$$m'(\tau) = \frac{b}{1 + ce^{-b\tau}}(a - m(\tau)), \tag{3}$$

The distribution function for the number of faults at time t after using u resources is given as follows:

$$m(s, u) = \frac{a(1 - e^{-bs^{\alpha}u^{1-\alpha}})}{1 + ce^{-bs^{\alpha}u^{1-\alpha}}}, \tag{4}$$

where a, b, α, s, and u represent total initial number of faults, inflection rate, output elasticities, testing time in weeks, and testing resources, respectively. Motivated from Kapur et al. [7] model, in the next section, this approach is extended considering time lag in fault debugging process in imperfect debugging environment.

3 Proposed Model

In this section, two-dimensional SRGM is discussed in imperfect debugging environment then same is extended considering the time lag in the fault removal process. For the proposed model, the assumptions based on the literature are as follows [1, 7, 19, 21, 22]:

(i) The fault removal phenomena follow NHPP.
(ii) The software system is subject to failures in operation caused by faults remaining in the system.
(iii) The failure rate is affected by the faults remaining in the system.
(iv) The fault detection rate is a non-decreasing time and resource-dependent function.
(v) The fault introduction rate is an exponential function of testing time.
(vi) To show the joint effect of testing time and testing resources, Cobb–Douglas production function is used

Under these assumptions, the mathematical model for the rate of change in the mean value function is given by

$$\frac{dm(\tau)}{d\tau} = b(\tau)(a(\tau) - m(\tau)), \tag{5}$$

$$a(\tau) = ae^{\beta t}, \tag{6}$$

$$b(\tau) = \frac{b}{1 + ce^{-b\tau}}. \tag{7}$$

where $a(\tau)$, $b(\tau)$, c, a, and b are the error introduction rate function, fault detection rate function, inflection factor, number of faults initially, and constant fault detection rate, respectively. The solution of the above differential equation is give as follows:

$$m(s, u) = \frac{ab}{b + \beta}\left(\frac{e^{(\beta+b)s^{\alpha}u^{1-\alpha}} - 1}{e^{bs^{\alpha}u^{1-\alpha}} + c}\right). \tag{8}$$

We introduced a delay factor $\varphi(s)$ in our original equation. Delay will be only function of time. After introduction of delay, the modified mean value function, $m(s, u)$, is given by

$$m(s, u) = m(s - \varphi(s), u) \tag{9}$$

Now we have considered different values of delay function. These values consider the three different cases with no delay, delayed S-shaped, and inflection S-shaped curves of delay function.

CASE 1: When there is no delay, value of $\varphi(s) = 0$ and equation will remain same as Eq. (8), but it is modeled in imperfect environment. Therefore, the results generated will be different from base model proposed [7].

$$m(s, u) = \frac{ab}{b + \beta}\left(\frac{e^{(\beta+b)s^{\alpha}u^{1-\alpha}} - 1}{e^{bs^{\alpha}u^{1-\alpha}} + c}\right) \tag{10}$$

CASE 2: The software fault detection process is modeled by an delayed S-shaped curve which can be regarded as a learning process because the tester's skills is directly proportional to time. Here, $\varphi(s) = \frac{1}{b}\log(1 + bs)$

$$m(s, u) = \frac{ab}{b + \beta}\left(\frac{e^{(\beta+b)\left(s-\frac{1}{b}\log(1+bs)\right)^{\alpha}u^{1-\alpha}} - 1}{e^{b\left(s-\frac{1}{b}\log(1+bs)\right)^{\alpha}u^{1-\alpha}} + c}\right) \tag{11}$$

CASE 3: Inflected S-shaped model curve takes the inflection factor as an parameter to vary the learning skills of tester's. Here, $\varphi(s) = \frac{1}{b}\log\left(\frac{1+\psi}{1+\psi e^{-bs}}\right)$ and our equation will be

$$m(s, u) = \frac{ab}{b + \beta}\left(\frac{e^{(\beta+b)\left(s-\frac{1}{b}\log\left(\frac{\psi+1}{1+\psi e^{-bs}}\right)\right)^{\alpha}u^{1-\alpha}} - 1}{e^{b\left(s-\frac{1}{b}\log\left(\frac{\psi+1}{1+\psi e^{-bs}}\right)\right)^{\alpha}u^{1-\alpha}} + c}\right) \tag{12}$$

4 Results and Discussion

To validate the model, real data set is used which is collected from [23]. The used resources in this data set are the CPU hours used in the testing, where the software release consists of 100 faults after 10,000 testing resources are used during 20 testing weeks. The detail description of data set is shown in Table 1.

Table 1 Testing data and resulted output

Time (in weeks)	Resource usages (CPU hours)	Actual defects	Predicted defects			
			Kapur's model	Case 1	Case 2	Case 3
1	519	16	–	11	11	10
2	968	24	–	19	19	19
3	1430	27	–	27	27	27
4	1893	33	–	34	34	34
5	2490	41	–	42	42	43
6	3058	49	–	49	49	49
7	3625	54	–	55	55	55
8	4422	58	–	63	63	63
9	5218	69	–	70	70	71
10	5823	75	98	76	75	76
11	6539	81	107	81	81	81
12	7083	86	116	85	85	85
13	7487	90	123	88	88	88
14	7846	93	129	91	90	91

(continued)

Table 1 (continued)

Time (in weeks)	Resource usages (CPU hours)	Actual defects	Predicted defects			
			Kapur's model	Case 1	Case 2	Case 3
15	8205	96	129	93	93	93
16	8564	98	134	95	95	95
17	8923	99	139	98	98	98
18	9282	100	138	100	99	100
19	9642	100	135	102	102	102
20	10000	100	133	104	105	104

Table 2 Results comparisons

Model	MVF, $a(\tau)$, $b(\tau)$	MLE's	MSE
Kapur et al. [7]	$m(s,u) = \frac{a(1-e^{-bs^{\alpha}u^{1-\alpha}})}{1+ce^{-bs^{\alpha}u^{1-\alpha}}}$	$\hat{a} = 114$	8.50
	$a(\tau) = a$	$\hat{b} = 0.046246$	
	$b(\tau) = \frac{b}{1+ce^{-b\tau}}$	$\hat{c} = 0.873236$	
		$\hat{\alpha} = 0.811683$	
Proposed model case 1	$m(s,u) = \frac{ab}{b+\beta}\left(\frac{e^{(\beta+b)s^{\alpha}u^{1-\alpha}}-1}{e^{bs^{\alpha}u^{1-\alpha}}+c}\right)$	$\hat{a} = 121$	6.7765
	$a(\tau) = ae^{\beta\tau}$	$\hat{b} = 7.215E-05$	
	$b(\tau) = \frac{b}{1+ce^{-b\tau}}$	$\hat{c} = -0.64423$	
	$\varphi(s) = 0$	$\hat{\alpha} = 0.0156$	
		$\hat{\beta} = 3.40E-05$	
Proposed model case 2	$m(s,u) = \frac{ab}{b+\beta}\left(\frac{e^{(\beta+b)\left(s-\frac{1}{b}\log(1+bs)\right)^{\alpha}u^{1-\alpha}}-1}{e^{b\left(s-\frac{1}{b}\log(1+bs)\right)^{\alpha}u^{1-\alpha}}+c}\right)$	$\hat{a} = 116$	7.5248
	$a(\tau) = ae^{\beta\tau}$	$\hat{b} = 7.45E-05$	
	$b(\tau) = \frac{b}{1+ce^{-b\tau}}$	$\hat{c} = -0.62134$	
	$\varphi(s) = \frac{1}{b}\log(1+bs)$	$\hat{\alpha} = 0.0025$	
		$\hat{\beta} = 3.42E-05$	
Proposed model case 3	$m(s,u) = \frac{ab}{b+\beta}\left(\frac{e^{(\beta+b)\left(s-\frac{1}{b}\log\left(\frac{\psi+1}{1+\psi e^{-bs}}\right)\right)^{\alpha}u^{1-\alpha}}-1}{e^{b\left(s-\frac{1}{b}\log\left(\frac{\psi+1}{1+\psi e^{-bs}}\right)\right)^{\alpha}u^{1-\alpha}}+c}\right)$	$\hat{a} = 117$	6.7558
	$a(\tau) = ae^{\beta\tau}$	$\hat{b} = 8.0E-05$	
	$b(\tau) = \frac{b}{1+ce^{-b\tau}}$	$\hat{c} = -0.59102$	
	$\varphi(s) = \frac{1}{b}\log\left(\frac{1+\psi}{1+\psi e^{-bs}}\right)$	$\hat{\alpha} = 0.0016$	
		$\hat{\beta} = 3.15E-05$	
		$\hat{\psi} = 0.6863$	

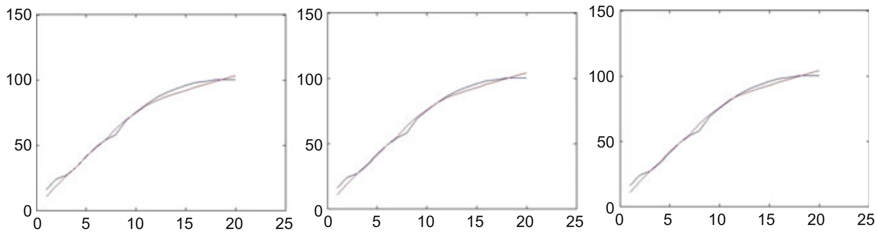

Fig. 1 Plots between faults and time of actual versus detected faults

The involved parameters in the model are estimated using nonlinear regression using SPSS. The estimated value of parameters of the model for data set is given in Table 2. To show quantitative comparisons with different fault prediction models, the mean square error (MSE) is used. From Table 2, it can be observed that the proposed model performs better than the others existing in the literature with minimum value of MSE in all three cases. To understand the performance of proposed work, graphical representation of cumulative fault versus actual faults with time are also shown in Figs. 1a, b.

5 Conclusions and Future Work

We extended the two-dimensional software reliability growth model proposed in [7] considering debugging time lag in imperfect debugging environment. The real data set taken from Tandem Computers is used to compare the results. Comparison criteria used is MSE calculated using actual and predicted defects. Based on MSE, it can be concluded that the proposed models are better in comparison to previous model. Moreover, model-3 is better among the other proposed models. Better results show that later model used with imperfect debugging and delay is more realistic model.

In this research work, we have considered the three types of time lag in the fault removal process. In future, other functions may be considered to model the time lag function. Other factors like testing coverage, change point concept, number of executed test cases, the effect of fault dependency may also be incorporated.

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F-Policy for M/M/1/K Retrial Queueing Model with State-Dependent Rates

Madhu Jain and Sudeep Singh Sanga

Abstract In this article, F-policy for the single-server finite capacity Markovian queueing model with retrial attempts is investigated. The system admits the customers to join the system till the system reaches its full capacity and then, the customers are restricted to join the system until the queue size reduces to threshold value 'F'. To deal with more realistic situations, the concepts of state-dependent arrivals and service process are incorporated while developing a Markov model. On the basis of birth–death process, Chapman–Kolmogorov equations governing the model are developed to analyze the queueing characteristics of the system. The steady-state queue size distributions are obtained by using recursive technique which are further used to establish numerous performance indices to predict the behavior of the studied model. A cost function is framed to compute the optimal service rate and corresponding minimum cost. To investigate the behavior of the system, numerical example, sensitivity analysis of the system, and descriptors for different indices are presented.

Keywords Finite capacity · Retrial queue · F-policy · Recursive method
Sensitivity analysis

1 Introduction

Now-a-days, retrial queueing models play a significant role in predicting the real-world congestion problems involved in the computer communication networks, business, industries, etc. To examine the practical applicability of retrial queues with finite capacity, consider an example of call center wherein the caller may try for a call to the center, and if the dialed number is busy, the caller gets a message of busy line.

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The caller may disconnect the call and remain in the orbit and may try after some time with the hope that the line becomes free to connect the call. In past years, many research works on retrial queues have appeared which dealt with Markov model. In a survey article, Yang and Templeton [1] presented various examples including reservation and packet switching networks. Some important works on Markovian retrial queues are reported in the survey articles by Artalejo and Falin [2] and Artalejo [3]. In recent years, Dudin et al. [4] and Taun [5] presented a noble work on single-server queue with retrial attempts by considering the group admission of the customers and setup time, respectively.

To control the arrivals in the queueing system, is a major issue. In order to maintain the quality of service rendered to the customers, the arrivals should be controlled and this can be done by introducing the suitable optimal control policy, namely F-policy. The control F-policy which restricts the arrivals of customers, when the capacity of the system is full, is applicable in many real-world situations like shopping malls, parking lot, transmission line, communication system, production system, and transport service. Gupta [6] developed the concept of F-policy to investigate Markovian single-server finite queue. F-policy can be used as optimal admission policy according to which as the system becomes full (say K), no more further arrivals are allowed to join the system until the queue size again drops to a threshold value 'F'. Later, Wang et al. [7, 8] extended the F-policy concept for the analysis of single-server Markovian queueing model to F-policy M/G/1/K and G/M/1/K queueing models. Admission control policy for M/M/1 queueing model was investigated by Ke et al. [9] by taking second optional services. Yang et al. [10] investigated Markovian queue with single working vacation that operates under F-policy. A finite-capacity queueing model was studied by Chang et al. [11] by considering two types of services that worked under control F-policy. <p-F>-policy for M/M/1/K queueing model with server breakdown and exponential startup time was considered by Chang and Ke [12] for obtaining the steady-state analytical solutions. In recent years, Jain et al. [13, 14] have developed Markovian model for control F-policy. Queueing characterization of machine repair problem with general retrial attempts by controlling the joining of failed machines operating under F-policy was also dealt by Jain and Sanga [15]. They obtained the steady-state probabilities of the system states and other indices by applying the recursive method.

In several queueing systems, the rates may be state-dependent; i.e, the arrival and service may be dependent on the number of customers present in the system. In some queueing situations, it is seen that the server may render service with faster rate as the queue size increases. On the contrary, sometimes it may happen that the server becomes slow due to stress. The important work on single-server retrial queue with state-dependent rates was done by Parthasarathy and Sudhesh [16]. Recently, Kumar and Darsana [17] analyzed an M/M/1 queueing model with state-dependent arrival rate and retention of reneged customers.

After the literature survey, it is noticed that no research paper has appeared on admission control based on F-policy for M/M/1/K retrial queueing model by considering state-dependent rates. This article is focused to analyze F-policy for the single-server finite capacity Markovian queueing model by considering the system-dependent arrival and service rates of the customers. After presenting some introductory and motivational issues, and concerned literature in the ongoing section, now we outline the contents of remaining sections of the present paper as follows. The model description is presented in Sect. 2. The steady-state Chapman–Kolmogorov equations for the governing model are constructed in Sect. 3. Section 4 provides some results for the system indices and the cost function. Numerical experiments and conclusion of the investigation done are provided in Sects. 5 and 6, respectively.

2 Model Description

Consider M/M/1/K retrial queueing model operating under F-policy and state-dependent rates. The service process follows the first-come-first-serve (FCFS) rule. The formulation of model is done based on some assumptions which are as follows:

(i) The arriving customer enters into the system by following Poisson fashion with rate λ.
(ii) If arriving customer sees the server free, he immediately gets served by the following exponential distribution with rate μ.
(iii) If the arriving customer sees that the server is engaged in serving some other customers, then he is forced to enter retrial pool. From the retrial pool, he retries for the service with exponential distributed retrial time with mean $1/\gamma$.
(iv) To stop the arrivals from entering the system when system capacity becomes full, a setup job is required according to exponential distributed with rate ε.
(v) As soon as the system becomes full, the arrivals are not acceptable until the enough customers are served and the system size further drops to prefixed level 'F'.

The mathematical formulation of the model is done as follows:

Let at time epoch $\tau, N(\tau)$, and $Y(\tau)$ be the random variables denoting the number of customers in the system and states of the server, respectively. The random variable $Y(\tau)$ is defined as follows:

$$Y(\tau) = \begin{cases} 0, & \text{the arriving customers are admitted to join the orbit when the server is engaged in rendering the service;} \\ 1(2), & \text{the admission of customers are permissible (not permissible) in the system when the server is busy} \end{cases}$$

At time τ for node (i, n), system states probabilities are defined by $P_{j,n}(\tau) = \text{Prob}\{Y(\tau) = j, N(\tau) = n\}$. It is noted that $\{Y(\tau), N(\tau) : \tau \geq 0\}$ is a bivariate Markov process which is discrete and continuous with respect to statespace and time, respectively. Markov model is analyzed at steady state, i.e., when $\tau \rightarrow \infty$ and the state probability is denoted by $P_{j,n} = \lim_{\tau \rightarrow \infty} P_{j,n}(\tau)$.

3 Governing Equations and Recursive Technique

To investigate the admission control for finite capacity retrial model with state-dependent rates, Chapman–Kolmogorov equations are formed based on birth–death process for three levels when $Y(\tau)$ takes values 0, 1, and 2, respectively. The state-dependent rates λ_n and μ_n are taken into account for generic formulation of the retrial model under F-policy. The in-flows and out-flows of different states of bivariate continuous time Markov chain (CTMC) are depicted in Fig. 1.

- **For** $j = 0$ and $0 \le n \le K - 1$.

$$-\lambda_0 P_{0,0} + \mu_1 P_{1,0} = 0 \tag{1}$$

$$-(\lambda_n + \gamma)P_{0,n} + \mu_{n+1} P_{1,n} = 0 \text{ for } 1 \le n \le K - 1. \tag{2}$$

- **For** $j = 1$ and $0 \le n \le K - 1$.

$$-(\lambda_1 + \mu_1)P_{1,0} + \lambda_0 P_{0,0} + \gamma P_{0,1} = 0 \tag{3}$$

$$-(\lambda_{n+1} + \mu_{n+1})P_{1,n} + \lambda_n P_{1,n-1} + \lambda_n P_{0,n} + \gamma P_{0,n+1} = 0 \text{ for } 1 \le n \le F - 2 \tag{4}$$

$$-(\lambda_F + \mu_F)P_{1,F-1} + \lambda_{F-1} P_{1,F-2} + \lambda_{F-1} P_{0,F-1} + \gamma P_{0,F} + \mu P_{2,F} = 0 \tag{5}$$

$$-(\lambda_{n+1} + \mu_{n+1})P_{1,n} + \lambda_n P_{1,n-1} + \lambda_n P_{0,n} + \gamma P_{0,n+1} = 0 \text{ for } F \le n \le K - 2 \tag{6}$$

$$-(\varepsilon + \mu_K)P_{1,K-1} + \lambda_{K-1} P_{1,K-2} + \lambda_{K-1} P_{0,K-1} = 0 \tag{7}$$

- **For** $j = 2$ and $F \le n \le K - 1$.

$$\mu P_{2,n} = \mu P_{2,n+1} = \varepsilon P_{1,K-1} \text{ for } F \le n \le K - 2 \tag{8}$$

For notational convenience, denote $\delta_i = \frac{\lambda_i + \gamma}{\gamma}$, $\rho_i = \frac{\lambda_i}{\mu_{i+1}}$, and $\Lambda_{0,n} = \prod_{j=0}^{n} \lambda_j$.

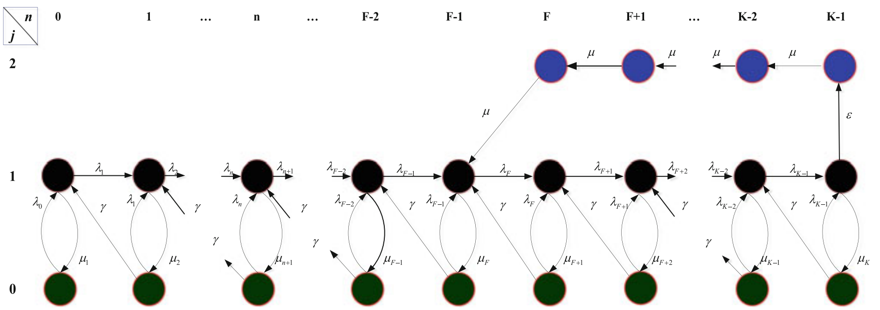

Fig. 1 Transition state diagram

Now, we solve Eqs. (1)–(8) using recursive approach as follows:
From (1),

$$P_{1,0} = \rho_0 P_{0,0} \tag{9}$$

Using result of (9) in (3), we obtain

$$P_{0,1} = \rho_0(\delta_1 - 1)P_{0,0} \tag{10}$$

Also, using value of $P_{0,1}$ from (10) in (2), we obtain $P_{1,1}$ as

$$P_{1,1} = \rho_0\rho_1\delta_1 P_{0,0} \tag{11}$$

Similarly, using (4) and (2), we obtain

$$P_{0,2} = \rho_0\rho_1\delta_1(\delta_2 - 1)P_{0,0} \tag{12}$$

$$\text{and} \quad P_{1,2} = \rho_0\rho_1\rho_2\delta_1\delta_2 P_{0,0} \tag{13}$$

Further solving recursively, in general, we find

$$P_{1,n} = \left(\prod_{i=0}^{n}\rho_i\right)\left(\prod_{j=1}^{n}\delta_j\right)P_{0,0} \text{ for } 1 \le n \le F-1 \tag{14}$$

$$P_{0,n} = (\delta_n - 1)\left(\prod_{i=0}^{n-1}\rho_i\right)\left(\prod_{j=1}^{n-1}\delta_j\right)P_{0,0} \text{ for } 1 \le n \le F-1 \tag{15}$$

Again using (5)–(7), we obtain

$$P_{1,n} = \frac{\Lambda_{0,n}S_n}{R_F}P_{0,0} \text{ for } F \le n \le K-2 \tag{16}$$

$$P_{0,n} = \frac{\Lambda_{0,n}(1 - 1/\delta_n)}{\rho_n}\frac{S_n}{R_F}P_{0,0} \text{ for } F \le n \le K-2 \tag{17}$$

$$P_{1,K-1} = \frac{\Lambda_{0,K-1}}{R_F}P_{0,0} \tag{18}$$

$$\text{and} \quad P_{0,K-1} = \frac{\Lambda_{0,K-1}(1 - 1/\delta_{K-1})}{\rho_{K-1}}\frac{1}{R_F}P_{0,0} \tag{19}$$

Substituting the value of $P_{1,K-1}$ from (18) in (8), we obtain

$$P_{2,n} = \frac{\Lambda_{0,K-1}\varepsilon}{\mu R_F}P_{0,0} \text{ for } F \le n \le K-1 \tag{20}$$

where

$$S_n = \varepsilon \Lambda_{n+2,K-1} + \varepsilon \sum_{i=n+3}^{K} (\Lambda_{i,K-1}\Lambda_{n+2,i-1}) + \prod_{j=n+2}^{K} \frac{\mu_j}{\delta_{j-1}}$$

$$\text{and} \quad R_F = \left[\mu_1 \prod_{j=1}^{F-1} \frac{\mu_{j+1}}{\delta_j}\right]\left[\varepsilon \Lambda_{F+1,K-1} + \frac{\mu_{F+1}}{\delta_F} S_F\right]$$

4 Performance Measures

To explore the performance characteristics of the system, we establish some indices, namely the average number of customers present in the system $E[N_S]$, in the queue $E[N_q]$, in the retrial orbit $E[N_R]$ and throughput TP. We also establish expressions for the probabilities of idle server (P_I), busy server (P_{SB}) corresponding to the server status at levels 0, 1 and 2, respectively. Now, in terms of steady-state probabilities $P_{j,n}$, the expressions for $E[N_S]$, $E[N_q]$, $E[N_R]$, TP, P_I, and P_{SB} are obtained as follows:

$$E[N_S] = \sum_{n=0}^{K-1} nP_{0,n} + \sum_{n=0}^{K-1} (n+1)P_{1,n} + \sum_{n=F}^{K-1} (n+1)P_{2,n} \tag{21}$$

$$E[N_q] = \sum_{n=0}^{K-1} nP_{1,n} + \sum_{n=F}^{K-1} nP_{2,n} \quad \text{and} \quad E[N_R] = \sum_{n=0}^{K-1} nP_{0,n} \tag{22a, b}$$

$$\text{TP} = \mu_n \sum_{n=0}^{K-1} P_{1,n} + \mu \sum_{n=F}^{K-1} P_{2,n} \tag{23}$$

$$P_I = \sum_{n=0}^{K-1} P_{0,n} \quad \text{and} \quad P_{SB} = \sum_{n=0}^{K-1} P_{1,n} + \sum_{n=F}^{K-1} P_{2,n} \tag{24a, b}$$

Cost Function

The system organizer always prefers to serve the customers with faster service rate and thereby reduces the waiting time, but cost increases in general if the service is improved. A cost function is formulated to determine the optimal service rate (μ^*) and minimum cost (TC(μ^*)) of the system. To formulate the cost function, the cost elements per unit time involved with several activities are considered.

By considering the service rate (μ) as a decision variable, the total cost per unit is formulated by:

$$TC(\mu) = C_I P_I + C_B P_{SB} + C_H E[N_q] + \mu C_F + \mu_n C_A + C_O E[N_R] \tag{25}$$

where cost elements per unit time used are

C_I Cost incurred on the system when the server is not rendering the service i.e., in idle state

C_B Cost incurred on the system when the server is busy in rendering the service

C_H Holding cost spend on each customer present in the system

$C_A(C_F)$ Cost for providing service to the each customers when the joining of the customers are permitted (not permitted)

C_O Cost spend on each customer in the retrial pool

5 Numerical Results

The effect of the system parameters on different system indices can be analyzed by taking a numerical illustration. *MATLAB* software is used to obtain the numerical results. The system parameters are set as $K = 7,\ F = 4,\ \varepsilon = 1,\ \mu = 10,\ \lambda = 2.9,\ \gamma = 1$. We consider the state-dependent rates as $\lambda_n = \frac{\lambda}{n+1}$ and $\mu_n = \frac{\mu}{n}$. Further, three cost sets are considered as given in Table 1 to compute μ^* and corresponding minimum system cost (TC(μ^*)).

Heuristic method is used to evaluate μ^* and corresponding average minimum system cost TC(μ^*). The graphs for cost versus 'μ' in Figs. 2 and 3 reveal that the cost function is unimodal and convex, and expected minimum cost is achieved. The effect of γ on (μ^*, TC(μ^*)) for different cost sets of elements C_I, C_B, C_H, C_F, C_A, and C_o is displayed in Table 2.

The sensitivity analysis is performed by varying the values of parameters λ, μ, γ. The numerical results for demonstrating the impact of different parameters on $E[N_q]$, $E[N_R]$, P_I, P_{SB}, and TC are depicted in Figs. 4, 5, 6, and 7 and Tables 3, 4, and 5.

Table 1 Different cost elements associated with cost set

Cost set	C_I	C_B	C_H	C_F	C_A	C_O
I	10	10	100	5	10	110
II	5	5	80	10	5	120
III	20	20	100	5	10	110

Table 2 Effect of γ on (μ^*, TC(μ^*))

Cost set	(μ^*, TC(μ^*))		
	$\gamma = 1$	$\gamma = 1.5$	$\gamma = 2$
I	(9.581, 248.05)	(8.629, 218.05)	(8.100, 201.41)
II	(9.284, 241.28)	(8.300, 209.70)	(7.755, 192.14)
III	(9.581, 258.05)	(8.629, 228.05)	(8.100, 211.41)

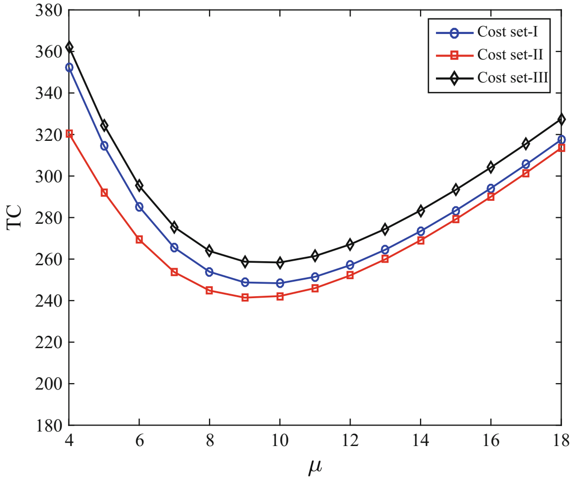

Fig. 2 TC versus μ for three different cost sets

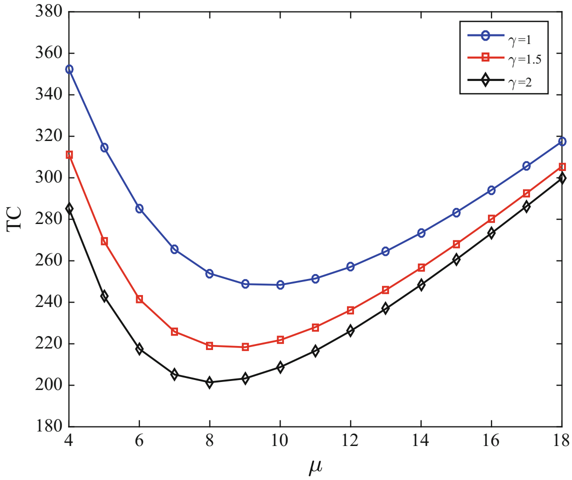

Fig. 3 TC versus μ for different value of γ

- **Effect of arrival rate** (λ)

From Figs. 4 and 6, it is clearly seen as λ goes up, $E[N_S]$ and TP increase. Table 3 displays that the queue length $E[N_q]$, $E[N_R]$, and probability of busy server (P_{SB}) build up as λ grows, but the probability of idle server decreases with the decrease in λ.

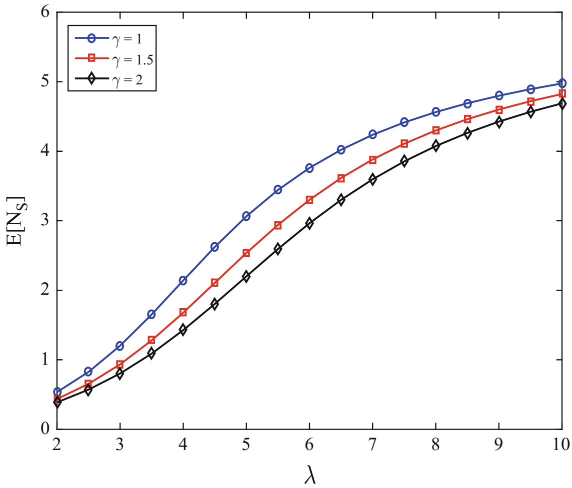

Fig. 4 $E[N_S]$ versus λ for different value of γ

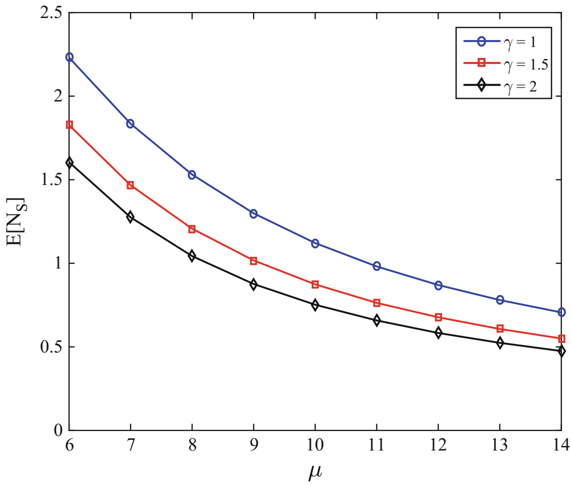

Fig. 5 $E[N_S]$ versus μ for different value of γ

- **Effect of service rate** (μ)

Table 4 indicates that $E[N_q]$, $E[N_R]$, and probability of busy server P_{SB} lower down as μ decreases. However, the probability of idle server P_I grows with the increment in service rate μ. Figure 5 demonstrates that $E[N_S]$ decreases as μ decreases. From Fig. 7, it is clear that as μ increases, throughput TP also increases.

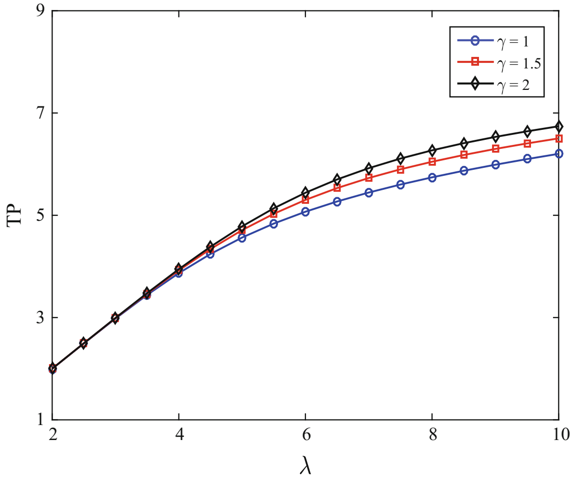

Fig. 6 TP versus λ for different value of γ

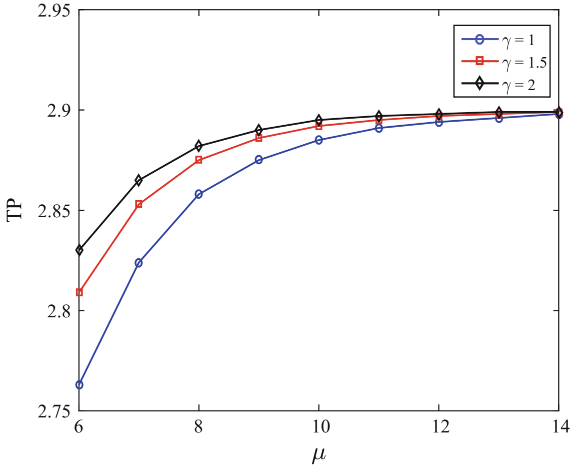

Fig. 7 TP versus μ for different value of γ

- **Effect of retrial rate** (γ)

Numerical results summarized in Table 5 reveal that $E[N_q]$, $E[N_R]$, and probability of idle server (busy server) decrease (increases) as γ increases.

Table 3 System indices for varying value of λ

λ	$E[N_q]$	$E[N_R]$	P_I	P_{SB}	TC
2	0.1066	0.2286	0.8001	0.1999	195.80
4	0.7834	0.9703	0.6131	0.3869	345.07
6	1.7264	1.5266	0.4930	0.5070	500.56
8	2.3404	1.6487	0.4258	0.5742	575.40
10	2.7372	1.6181	0.3798	0.6202	611.70

Table 4 System indices for varying value of μ

μ	$E[N_q]$	$E[N_R]$	P_I	P_{SB}	TC
6	0.9531	0.8174	0.5395	0.4605	285.22
8	0.5260	0.6479	0.6428	0.3572	253.87
10	0.3171	0.5152	0.7115	0.2885	248.37
12	0.2077	0.4209	0.7588	0.2412	257.07
14	0.1453	0.3536	0.7930	0.2070	273.43

Table 5 System indices for varying value of γ

γ	$E[N_q]$	$E[N_R]$	P_I	P_{SB}	TC
0.4	0.5876	1.2912	0.7172	0.2828	370.79
0.8	0.3674	0.6517	0.7121	0.2879	278.43
1.2	0.2831	0.4246	0.7111	0.2889	245.02
1.6	0.2405	0.3132	0.7107	0.2893	228.50
2	0.2152	0.2476	0.7105	0.2895	218.75

6 Conclusions

This article studies a single-server finite capacity retrial queueing model by adding the realistic features of admission control policy and state-dependent rates. The recursive method is applied to establish the steady-state queue size distributions of the system. The system indices established can be easily used for the computation purpose as validated by taking numerical illustration. Further, optimal service rate and corresponding minimal cost of the system determined by heuristic approach may be helpful to the system organizers and decision makers for improving the grade of service of the existing system. This model may further be extended by taking general retrial times and general service times.

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Time-Shared Queue with Nopassing Restriction for the Loss–Delay Customers and Additional Server

Madhu Jain, Shalini Shukla and Rakesh Kumar Meena

Abstract In the present chapter, we investigate the performance indices of a multi-server queueing system in order to examine the loss and delay behavior by incorporating the nopassing restrictions. The arriving customers who cannot wait for their service are lost forever from the system. The arriving customers who can wait for their turn in the queue to be served are referred as delay customers. The service crew has permanent multi-servers as well as one additional server; both types of servers work on the time-sharing basis. The additional repairman acts as a backup server and follows a threshold rule so as to reduce the total cost. The queueing indices including the waiting time are derived analytically using the product-type solution technique. Numerical simulation has been done to facilitate the sensitivity analysis of various parameters in the context of different performance results.

Keywords Time-shared queue · Loss and delay · Nopassing
Additional repairman · Controlled rates

1 Introduction

In the fast life of today world, both the time and money should be optimized by improving the efficiency of various queueing systems. Everyone is in a hurry to get their jobs to be done without any wastage of time and money. The provisioning of time-shared servers and additional removal server can play a significant role as far as

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a better grade of service is concerned. In many congestion scenarios, it is important to judge whether the customer would like to wait for the servers to be free for their service or they will prefer to leave the system forever on finding all the servers busy. Besides this, sometimes, the arriving customers are served in the same order in which they join the system; i.e., they are served under the nopassing restriction. The nopassing restriction according to which the customers are served in the same chronological order in which they join the system can be found in many practical congestion situations such as in military operation, toll check post on the highways, check-in process at airport, manufacturing industries, message transmission, etc.

The queueing systems having the facility of additional removable server provide better service in an economic way and are also helpful in reducing the heavy workloads. Some research works on the queueing systems with additional removable servers are available in the queueing literature [7–11, 18, 20, 28, 37]. In recent years, the queueing systems having permanent as well as additional server were also studied by Bieth et al. [2], Huang et al. [5], Li and Jiang [26]. Sharma and Sharma [30] proposed an optimal cost-minimization policy for a queueing system having the provision of additional removable server.

The time-sharing behavior of all the permanent as well as additional servers will be helpful in upgrading the performance of the system as a whole. It will also help in reducing the expenses and time which is a desirable trait. According to time sharing rule, more than one customer can approach a single server for the service simultaneously. Some very important investigations (cf. [14, 16, 23, 25, 34, 36]) have been done on queueing models with processor sharing servers in the past by many renowned investigators. A processor sharing queueing systems with variable service rates was studied by Litjens [27]. The discriminatory processor shared queues were studied by Altman et al. [1], Kim [24], Yu et al. [39], and Izagirre et al. [6] by including ergodic service time, impatient customers, and relative priorities, respectively. A machine repair problem has also been investigated by Jain et al. [17] in which there is a provision of permanent as well as additional servers working on time-sharing basis.

The loss–delay phenomenon can be observed in many real-time systems in which it is found that when the servers are busy, some customers will not like to enter the queue as such they are lost, whereas remaining customers called delay customers join the queue and wait for their turn for service. In the past, some research works have been done on queueing system having both loss and delay customers (cf. [3, 19, 22, 31, 32]). A queueing system with loss customers who arrive in batches was studied by Kim et al. [23]. Recently, Sharma [29] investigated a single-server queueing model with loss and delay customers and controllable arrival rates. The performance of an optical burst switched (OBS) combined node was studied by Hayat and Afzal [4] by developing a mixed loss delay queueing models.

The concept of nopassing was introduced by Washburn [38] for a queueing system with infinite capacity. Due to nopassing constraint, the customers have to follow their sequence of arrival while leaving the system even they do not require any kind of service. Since then, some more researchers have studied a variant of queueing systems with nopassing restrictions (cf. [12, 15, 21, 35]). In recent past, not much work has been done on the nopassing restriction. Srinivasan and Thiagarajan [33] investigated the loss and delay queueing system with controllable arrival rates by including the nopassing restriction on the departure of the customers. Later, Thiagarajan and Srinivasan [35] studied an M/M/c/K loss and delay interdependent queueing model with nopassing restrictions. Jain and Sharma [13] investigated a Markovian queue with heterogeneous servers by considering the nopassing restriction on the departure of the customers. In this chapter, we are aimed to analyze a time-sharing queueing system by considering some more realistic features, namely provision of both permanent and the additional server, loss of proportion of customers if the server is busy, balking behavior of the arriving customers, and departure of customers from the system by maintaining the chronological order of arrivals. The content-wise organization of the present paper is as follows: In Sect. 2, the mathematical problem of the concerned queueing system has been formulated which is followed by the queue size distribution in Sect. 3. Some performance measures have been derived in Sect. 4. Numerical illustration and sensitivity analysis have been given in Sect. 5. Finally in Sect. 6, the investigation done is summarized by highlighting the noble features and future scope of the present study.

2 The Model

Consider a finite capacity time-shared queueing system having the service facility consisting of two types of time-sharing servers, i.e., s permanent servers and one additional removable server which gets activated at the threshold N_0. The additional server is deployed in order to cope up with the increased workload. The additional server stops functioning or is removed as soon as the number of customers drops below N_0. On finding the permanent server busy, the arriving customers behave in two ways, i.e., either loss or delay customer. The nopassing restriction is imposed on the departing customers.

The state transition diagram of Markov model has been depicted in Fig. 1. For formulating the queueing model, the underlying assumptions are as follows. The queueing system has finite capacity L for the customers or jobs. In case when the server is busy, $(1-p)$ proportion of the total customers are lost and remaining p proportion of the total customers behave as delay customers. The lifetimes and service time of both the loss and delay customers follow the exponential distribution. The loss (delay) customers may arrive at a rate of λ_L (λ_D) and are served by following the FCFS rule on the time-sharing basis. The customers may also balk with the rate $\beta_0(\beta_1)$ when all permanent servers (all permanent servers along with the additional removable server) are busy. The loss customers require no service, whereas the delay customers require service which is governed by the exponential distribution.

The time-sharing rate of both types of servers depends on the number of customers present in the queue. Both the service rates and the arrival rates are controlled by a factor ε. The switching-over process of activation and deactivation of the servers is considered to be instantaneous.

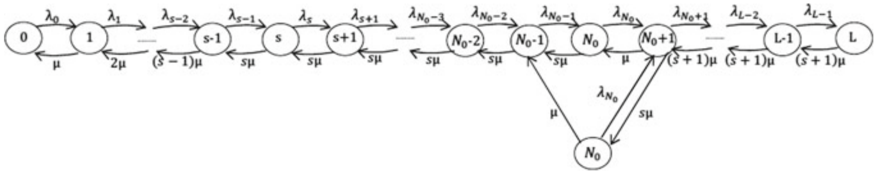


Fig. 1 State transition diagram

The state-dependent arrival rate λ_n and service rate μ_n are obtained as follows:

$$\lambda_n = \begin{cases} \{(\lambda_D + \lambda_L) - \varepsilon\}p; \; 0 \le n \le s \\ (\lambda_D - \varepsilon)\beta_0 p; \quad s < n < N_0 \\ (\lambda_D - \varepsilon)\beta_1 p; \quad N_0 \le n < L \end{cases} \text{ and } \mu_n = \begin{cases} n(\mu - \varepsilon)\phi(n); \quad 1 \le n < s \\ s(\mu - \varepsilon)\phi(n); \quad s \le n < N_0 \\ (s+1)(\mu - \varepsilon)\phi(n); \; N_0 \le n \le L \end{cases}$$

The steady-state probabilities of the system when there are 'n' customers in the system are defined as follows:

Q_n The steady-state probability that there are 'n' number of customers present in the system at any instant.

$Q_{N_0}(1)$ Probability that the first permanent server is providing the service to the customers at the threshold level N_0.

$Q_{N_0}(2)$ Probability that the second additional server is providing the service to the customers at the threshold level N_0.

Chapman–Kolmogrov equations at steady state can be constructed by balancing the inflows and outflows of each node depicted in Fig. 1. For different system states, we obtain

$$-\{(\lambda_D + \lambda_L) - \varepsilon\}pQ_0 + (\mu - \varepsilon)\varphi(1)Q_1 = 0 \tag{1}$$

$$\begin{aligned} &-[\{(\lambda_D + \lambda_L) - \varepsilon\}p + n(\mu - \varepsilon)\varphi(n)]Q_n + \{(\lambda_D + \lambda_L) - \varepsilon\}pQ_{n-1} \\ &+ (n+1)(\mu - \varepsilon)\varphi(n+1)Q_{n+1} = 0; \qquad 1 \le n \le s \end{aligned} \tag{2}$$

$$\begin{aligned} &-[(\lambda_D - \varepsilon)\beta_0 p + s(\mu - \varepsilon)\varphi(s+1)]Q_{s+1} + \{(\lambda_D + \lambda_L) - \varepsilon\}pQ_s \\ &+ s(\mu - \varepsilon)\varphi(s+2)Q_{s+2} = 0 \end{aligned} \tag{3}$$

$$\begin{aligned} &-[(\lambda_D - \varepsilon)\beta_0 p + s(\mu - \varepsilon)\varphi(n)]Q_n + \{(\lambda_D + \lambda_L) - \varepsilon\}pQ_{n-1} \\ &+ s(\mu - \varepsilon)\varphi(n+1)Q_{n+1}; \qquad s+2 \le n \le N_0 - 2 \end{aligned} \tag{4}$$

$$\begin{aligned} &-[(\lambda_D - \varepsilon)\beta_0 p + s(\mu - \varepsilon)\varphi(N_0 - 1)]Q_{1,N_0-1} + (\lambda_D - \varepsilon)\beta_0 pQ_{1,N_0-2} \\ &+ s(\mu - \varepsilon)\varphi(N_0)Q_{1,N_0} + (\mu - \varepsilon)\varphi(N_0)Q_{2,N_0} = 0 \end{aligned} \tag{5}$$

$$\begin{aligned} &-[(\lambda_D - \varepsilon)\beta_0 p + s(\mu - \varepsilon)\varphi(N_0)]Q_{1,N_0} + (\lambda_D - \varepsilon)\beta_0 pQ_{1,N_0-1} \\ &+ (s+1)(\mu - \varepsilon)\varphi(N_0+1)Q_{1,N_0+1} = 0 \end{aligned} \tag{6}$$

$$-[(\lambda_D - \varepsilon)\beta_0 p + (\mu - \varepsilon)\varphi(N_0)]Q_{2,N_0} + (s+1)(\mu - \varepsilon)\varphi(N_0+1)Q_{1,N_0+1} = 0 \tag{7}$$

$$- [(\lambda_D - \varepsilon)\beta_1 p + (s+1)(\mu - \varepsilon)\varphi(N_0+1)]Q_{1,N_0+1} + (\lambda_D - \varepsilon)\beta_0 p Q_{1,N_0} + (\lambda_D - \varepsilon)\beta_0 p Q_{2,N_0} + (s+1)(\mu - \varepsilon)\varphi(N_0+2)Q_{1,N_0+2} = 0 \tag{8}$$

$$- [(\lambda_D - \varepsilon)\beta_1 p + (s+1)(\mu - \varepsilon)\varphi(n)]Q_{1,n} + (\lambda_D - \varepsilon)\beta_1 p Q_{1,n-1} + (s+1)(\mu - \varepsilon)\varphi(n+1)Q_{1,n+1} = 0; \quad N_0 + 2 \le n \le L - 1 \tag{9}$$

$$-(s+1)(\mu - \varepsilon)\varphi(L)Q_{1,L} + (\lambda_D - \varepsilon)\beta_1 p Q_{1,L-1} = 0 \tag{10}$$

3 Queue Size Distribution

On solving the governing Eqs. (1)–(10), we can obtain the queue length for the steady-state probabilities, i.e., Q_n, $Q_{N_0}(1)$, and $Q_{N_0}(2)$, recursively.

Equation (1) yields

$$Q_1 = \frac{\lambda_0}{a_1} Q_0 \tag{11}$$

where $a_n = (\mu - \varepsilon)\phi(n)$.

Solving recursively for the nth nodes ($1 \le n \le s$), we get

$$Q_n = \frac{(\Lambda)^n}{n!\,\gamma_n} Q_0; \quad 1 \le n \le s \tag{12}$$

where $\gamma_n = \prod_{i=1}^{n} a_n$ and $\Lambda = \{(\lambda_D - \lambda_L) - \varepsilon\}p$.

Further, we get the results for nth node for the range $s < n \le N_0 - 1$ as

$$Q_n = \frac{\{(\lambda_D - \varepsilon)\beta_0 p\}^{n-s}(\Lambda)^s}{(s)^{n-s} \times s!\,\gamma_n} Q_0 \tag{13}$$

Further, solving Eqs. (5)–(7) recursively and denoting $A = \frac{\left\{\frac{(\lambda_D - \varepsilon)\beta_0 p}{s(\mu - \varepsilon)}\right\}^{N_0 - s}(\Lambda)^s}{s!}$, we get

$$Q_{N_0} = Q_{N_0(1)} + Q_{N_0(2)} \tag{14}$$

where

$$Q_{N_0(1)} = \frac{s(\lambda_D - \varepsilon)\beta_0 p + (2s)(\mu - \varepsilon)\phi(N_0)}{(s+1)(\lambda_D - \varepsilon)\beta_0 p + (2s)(\mu - \varepsilon)\phi(N_0)} \times \frac{A}{\gamma_{N_0}} Q_0 \tag{15}$$

and

$$Q_{N_0(2)} = \frac{s(\lambda_D - \varepsilon)\beta_0 p}{(s+1)(\lambda_D - \varepsilon)\beta_0 p + (2s)(\mu - \varepsilon)\phi(N_0)} \times \frac{A}{\gamma_{N_0}} Q_0 \tag{16}$$

Also

$$Q_{N_0+1} = \frac{\mathrm{CB}}{\gamma_{N_0+1}} Q_0 \tag{17}$$

where $B = \left[\left\{\frac{(\lambda_D-\varepsilon)\beta_0 p}{s(\mu-\varepsilon)}\right\} \times A\right]$ and $C = \frac{s(\lambda_D-\varepsilon)\beta_0 p+s(\mu-\varepsilon)\phi(N_0)}{(s+1)(\lambda_D-\varepsilon)\beta_0 p+(2s)(\mu-\varepsilon)\phi(N_0)} \times \frac{s}{s+1}$.

On solving Eqs. (8)–(10) recursively, we arrive at the following result:

$$Q_n = \frac{\left\{\frac{(\lambda_D-\varepsilon)\beta_1 p}{(1+s)(\mu-\varepsilon)}\right\}^{n-N_0-1}}{\gamma_n} \times CBQ_0; \quad N_0 + 2 \le n < L \tag{18}$$

Thus, the queue size distribution can be obtained from Eqs. (11)–(18). To obtain the probability Q_0, the normalizing condition $\sum_{n=0}^{L} Q_n = 1$ is used.

4 Performance Indices

In order to explore the performance of the system, the probabilities obtained in the previous section are used to establish some performance indices as follows:

- The expected number of customers in the system is

$$E(n) = \sum_{n=1}^{L} nQ_n \tag{19}$$

- The probability that only permanent servers are busy in servicing the customers is

$$P(\mathrm{PR}) = \sum_{n=0}^{N_0-1} Q_n \tag{20}$$

- The probability when both of permanent and additional servers are busy in providing the service of the customers is

$$P(\mathrm{BR}) = \sum_{n=N_0}^{L} Q_n \tag{21}$$

- The throughput of the queueing system is

$$\tau = \sum_{n=1}^{s-1} n(\mu - \varepsilon)\phi(n)Q_n + \sum_{n=s}^{N_0-1} s(\mu - \varepsilon)\phi(n)Q_n + \sum_{n=N_0}^{L} (s+1)(\mu - \varepsilon)\phi(n)Q_n \tag{22}$$

4.1 Expected Waiting Time

The expected waiting time of both types of customers can be obtained in the following manner. Let $E(W_1)$ and $E(W_2)$ denote the expected waiting times of delay and loss customers, respectively. Then, the expected waiting time for a particular customer will be obtained by using [12]:

$$E(W) = p \times E(W_D) + (1 - p) \times E(W_L) \tag{23}$$

Following Jain and Singh [12] for our model, we have

$$E(W_D) = \frac{1}{\mu}\left[\sum_{n=0}^{s} a_{n+1} Q_n + \sum_{n=s+1}^{N_0} \left\{\frac{n-s+1}{s} + a_s\right\} Q_n + \sum_{n=N_0+1}^{L} \left\{\frac{n-s}{s+1} + a_{s+1}\right\} Q_n\right] \tag{24}$$

$$E(W_L) = \frac{1}{\mu}\left[\sum_{n=0}^{s} a_n Q_n + \sum_{n=s+1}^{N_0-1} \left\{\frac{n-s+1}{s} + a_{s-1}\right\} Q_n + \sum_{n=N_0}^{L} \left\{\frac{n-s}{s+1} + a_{s+1}\right\} Q_n\right] \tag{25}$$

Using Eqs. (12), (13), and (16)–(18) in Eqs. (24) and (25), we get the expected waiting time for both types of customers as

$$\begin{aligned} E(W_D) = &\frac{1}{\mu}\left[\sum_{n=0}^{s} a_{n+1}\left\{\frac{(\Lambda)^n}{n!\,\gamma_n}\right\} + \sum_{n=s+1}^{N_0-1}\left\{\frac{n-s+1}{s} + a_s\right\} \times \left\{\frac{\{(\lambda_D - \varepsilon)\beta_0 p\}^{n-s}(\Lambda)^s}{(s)^{n-s} \times s!\,\gamma_n}\right\}\right. \\ &+ \left\{\frac{n-s+1}{s} + a_s\right\} \times \left\{\frac{2s(\lambda_D - \varepsilon)\beta_0 p + (2s)(\mu - \varepsilon)\phi(N_0)}{(s+1)(\lambda_D - \varepsilon)\beta_0 p + (2s)(\mu - \varepsilon)\phi(N_0)}\right\} \\ &\times \frac{A}{\gamma_{N_0}} + \left\{\frac{n-s}{s+1} + a_{s+1}\right\} \times \left\{\frac{CB}{\gamma_{N_0+1}}\right\} \\ &\left. + \sum_{n=N_0+2}^{L}\left\{\frac{n-s}{s+1} + a_{s+1}\right\} \frac{\left\{\frac{(\lambda_D-\varepsilon)\beta_1 p}{(1+s)(\mu-\varepsilon)}\right\}^{n-N_0-1}}{\gamma_n} \times \text{CB}\right] Q_0 \end{aligned} \tag{26}$$

$$\begin{aligned} E(W_L) =& \frac{1}{\mu}\left[\sum_{n=0}^{s} a_n \left\{\frac{(\Lambda)^n}{n!\,\gamma_n}\right\} + \sum_{n=s+1}^{N_0-1} \left\{\frac{n-s+1}{s} + a_{s-1}\right\} \times \left\{\frac{\{(\lambda_D-\varepsilon)\beta_0 p\}^{n-s}(\Lambda)^s}{(s)^{n-s}\times s!\,\gamma_n}\right\}\right. \\ & + \left\{\frac{n-s}{s+1} + a_s\right\} \times \left\{\frac{2s(\lambda_D-\varepsilon)\beta_0 p + (2s)(\mu-\varepsilon)\phi(N_0)}{(s+1)(\lambda_D-\varepsilon)\beta_0 p + (2s)(\mu-\varepsilon)\phi(N_0)}\right\} \\ & \times \frac{A}{\gamma_{N_0}} + \left\{\frac{n-s}{s+1} + a_{s+1}\right\} \times \left\{\frac{CB}{\gamma_{N_0+1}}\right\} \\ & \left. + \sum_{n=N_0+2}^{L} \left\{\frac{n-s}{s+1} + a_{s+1}\right\} \times \frac{\left\{\frac{(\lambda_D-\varepsilon)\beta_1 p}{(1+s)(\mu-\varepsilon)}\right\}^{n-N_0-1}}{\gamma_n} \times \mathrm{CB}\right] Q_0 \end{aligned} \tag{27}$$

Thus, the difference (D) between the mean waiting times of both types of customers can be obtained using

$$\begin{aligned} D =& \mu[E(W_D) - E(W_L)] \\ =& \mu\left[\sum_{n=0}^{s} (a_{n+1} - a_n)\left\{\frac{(\Lambda)^n}{n!\,\gamma_n}\right\} + \sum_{n=s+1}^{N_0-1} \{a_s - a_{s-1}\}\right. \\ & \times \left\{\frac{\{(\lambda_D-\varepsilon)\beta_0 p\}^{n-s}(\Lambda)^s}{(s)^{n-s}\times s!\,\gamma_n}\right\} + \left\{\frac{n-s+1}{s} - \frac{n-s}{s+1}\right\} \\ & \left.\times\left\{\frac{2s(\lambda_D-\varepsilon)\beta_0 p + (2s)(\mu-\varepsilon)\phi(N_0)}{(s+1)(\lambda_D-\varepsilon)\beta_0 p + (2s)(\mu-\varepsilon)\phi(N_0)}\right\} \times \frac{A}{\gamma_{N_0}}\right] Q_0 \end{aligned} \tag{28}$$

5 Numerical Simulation

In order to validate the applicability of the model, it is always better to evaluate the performance of the queueing model numerically. The numerical simulation will be of great help in making the study more practical and useful in real-life queueing scenarios. The effect of the arrival rate (λ_D) of the delay customers who are waiting to be served, the service rate (μ) of the server, balking rate (β_0) of customers when only first server is serving them on the various performance indices such as expected number of customers in the system $E(n)$, the throughput of the queueing system (τ), the probability that only permanent servers are servicing the customers P(PS), the probability when both servers are rendering the service of the customers P(BS), the expected waiting times of the delay customers $E(W_1)$, expected waiting times of loss customers $E(W_2)$ and the difference (Df) between the mean waiting times of both types of customer (Df) have been displayed in Tables 1, 2, 3 and Figs. 2 and 3. For the computation of numerical results, some parameters are fixed as $M = 10$, $m = 3$, $s = 2$, $N_0 = 5$, $\beta_1 = 0.2$, $\lambda_L = 0.6$, and $\varepsilon = 0.1$.

(i) ***Effect of the arrival rate of the delay customers (λ_D):*** In any queueing system, the arrival rate of the delay customers (λ_D) affects the performance of the whole

Table 1 Performance measures by varying λ_D for different values of p

p	λ_D	P(PS)	P(BS)	$E(W_1)$	$E(W_2)$	Df
1.5	0.5	0.077	0.923	6.61	5.51	0.28
	1.0	0.009	0.991	6.73	5.87	0.22
	1.5	0.002	0.998	6.68	5.96	0.18
2.5	0.5	0.016	0.984	6.75	5.80	0.24
	1.0	0.002	0.998	6.68	5.97	0.18
	1.5	0.000	1.000	6.61	6.05	0.14
3.5	0.5	0.005	0.995	6.74	5.88	0.21
	1.0	0.001	0.999	6.62	6.03	0.15
	1.5	0.000	1.000	6.55	6.10	0.11

Table 2 Performance measures by varying μ for different values of p

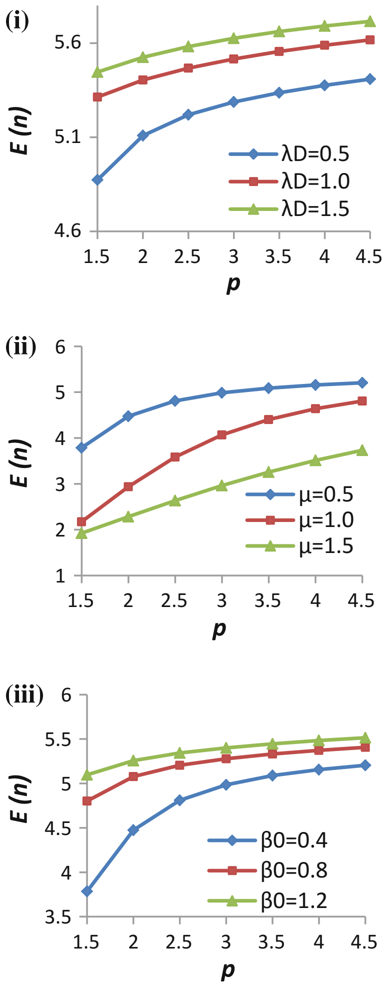
p	μ	P(PS)	P(BS)	$E(W_1)$	$E(W_2)$	Df
1.5	0.5	0.324	0.676	5.87	4.54	0.33
	1.0	0.761	0.239	2.49	1.54	0.95
	1.5	0.930	0.070	1.60	0.97	1.44
2.5	0.5	0.086	0.914	6.59	5.45	0.28
	1.0	0.424	0.576	2.82	2.18	0.64
	1.5	0.798	0.202	1.60	1.19	0.94
3.5	0.5	0.031	0.969	6.74	5.69	0.26
	1.0	0.227	0.773	3.09	2.54	0.55
	1.5	0.658	0.342	1.73	1.37	0.80

Table 3 Performance measures by varying β_0 for different values of p

p	β_0	P(PS)	P(BS)	$E(W_1)$	$E(W_2)$	Df
1.5	0.4	0.324	0.676	5.87	4.54	0.33
	0.8	0.094	0.906	6.57	5.44	0.28
	1.2	0.040	0.961	6.70	5.69	0.25
2.5	0.4	0.086	0.914	6.59	5.45	0.28
	0.8	0.019	0.981	6.74	5.78	0.24
	1.2	0.008	0.992	6.72	5.88	0.21
3.5	0.4	0.031	0.969	6.74	5.69	0.26
	0.8	0.006	0.994	6.74	5.88	0.21
	1.2	0.002	0.998	6.68	5.96	0.18

system to a great extent. The same is the case in the present model. It is clear from Table 1 that with the increase in λ_D, the probability that only permanent servers are servicing the customers P(PS), the expected waiting times of delay customers $E(W_1)$ and the difference (Df) between the mean waiting times of both types of customer (Df) and the probability when both types of servers are doing the service of the customers P(BS) decrease, whereas the expected waiting times of loss customers $E(W_2)$ increase continuously; however, the

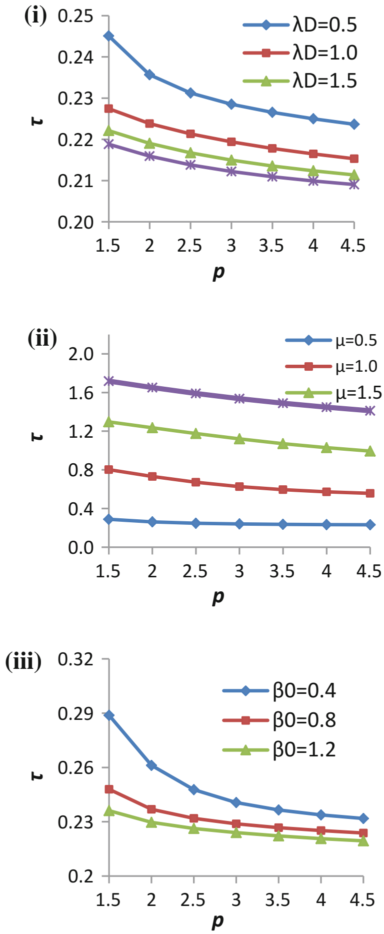
Fig. 2 $E(n)$ by varying p for different sets of **i** λ_D, **ii** μ, **iii** β_0



probability when both types of servers are doing the service of the customers $P(\text{BS})$ for the values of $p = 1.5$, increases first and then decreases. Figures 2i and 3i show that the expected number of customers in the system $E(n)$ increases, whereas the throughput of the queueing system (τ) decreases with the increment in the arrival rate λ_D.

(ii) ***Effect of the service rate of the server (μ):*** Table 2 reveals that with the increase in μ, the probability that only permanent server is servicing the customers $P(\text{PS})$ and the difference (Df) between the mean waiting times of both types of customer (Df) increase, whereas the expected waiting times of delay customers $E(W_1)$, the expected waiting times of loss customers $E(W_2)$, and the probability when both types of servers are doing the service of the customers

Fig. 3 Throughput by varying p for different sets of **i** λ_D, **ii** μ, **iii** β_0



$P(\mathrm{BS})$ decrease. Figures 2ii and 3ii depict that the expected number of customers in the system $E(n)$ decreases, whereas the throughput of the queueing system (τ) increases with the increase in the service rate μ.

(iii) ***Effect of the balking rate (β_0):*** From Table 3 and Figs. 2iii and 3iii, we notice that the increase in the balking rate of the customers when only permanent server is in service (β_0) affects the performance of the system significantly, which is quite realistic also. All the performance measures, namely $E(n)$, throughput (τ), the probabilities $P(\mathrm{PS})$, $P(\mathrm{BS})$, the expected waiting times $E(W_1)$ and $E(W_2)$ decrease, whereas the difference (Df) between the mean waiting times of both types of customers (Df) increases with the increase in the balking rate β_0.

(iv) ***Effect of the proportion (p):*** The proportion of the customers behaving as delay customers is also an important factor and affects the performance of the

Table 4 Performance measures by varying p

p	$E(n)$	P(PS)	P(BS)	τ	$E(W_1)$	$E(W_2)$	Df
1.5	3.79	0.324	0.676	0.29	5.87	4.54	0.33
2.0	4.48	0.161	0.839	0.26	6.36	5.16	0.30
2.5	4.81	0.086	0.914	0.25	6.59	5.45	0.28
3.0	4.99	0.050	0.950	0.24	6.69	5.61	0.27
3.5	5.09	0.031	0.969	0.24	6.74	5.69	0.26
4.0	5.16	0.021	0.979	0.23	6.76	5.75	0.25
4.5	5.21	0.014	0.986	0.23	6.77	5.79	0.25

system significantly. It is clear from Table 4 that indices $E(n)$, P(BS), $E(W_1)$, and $E(W_2)$ increase, whereas the throughput (τ), P(PS) and the difference (Df) decrease with the increase in the values of p, keeping all the other parameters fixed.

Overall, we can conclude that the arrival rate of the customers and the balking factor of the delay customers should be controlled, whereas the service rate should be kept high in order to enhance the performance of the system.

6 Discussion

The time-shared multi-component queueing system having controlled arrival rate of the customers has investigated by including the nopassing restrictions and balking behavior of the customers. To facilitate better grade of service, the service facility includes both permanent servers and a single additional removable server. The service of customers is provided on the time-sharing basis with rate dependent upon the number of customers present in the system. The performance results obtained are computationally tractable and provide valuable information for the improvement of day-to-day queueing scenarios encountered in time-shared system operating under nopassing constraints. The study done is relevant and helpful for the system analysts and decision makers in designing a reliable, economical, and practical service systems encountered in many real-life problems. The investigation can be further extended to bulk arrivals and multi-additional server system also.

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The Effect of Vacation Interruptions Policy on the Queueing System with Cost Optimization

Anupama, Anjana Solanki and Chandan Kumar

Abstract This paper analyzes a model of single-server, multiple working vacations with vacation interruption policy and general input process. In order to economize the cost of the system, we allow the server to take multiple vacations if there is less customer in the system waiting for the service, but it decreases the working speed and increases the customer's dissatisfaction in the system. This work also includes some performance measures that are queue length waiting for the service, waiting time and further numerical results have been given to show the effect of the measures on the system.

Keywords Multiple working vacations · Single server · Vacation interruption QFSM

1 Introduction

Queueing system with vacations makes the queueing system more flexible to use and apply in computer communication system, manufacturing and production system. In vacation queueing model, server can go on vacation after completing the service and utilizes the idle time by simply taking a break or doing another job or doing same job with different service rates. Extensive work in vacations may divide in two parts. In the first part, server that takes the vacation and stops serving the customer during the vacation period is known as classical vacation queueing theory, whereas in the second part, server that does not stop its service completely and serve the customer with a cheap rate is known as working vacation. Servi and Finn were the

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first to introduce multiple working vacations in 2002 [11]. Recently Goswami et al. [5] presented analysis of GI/M(n)/1/N queue with state dependent MWV (multiple working vacation) in 2013. Ke, J.C. [7] analyzed general input queue with N-policy in 2003. Doshi [4] survey gives more details about this theory. Baba [1] studied GI/M/1 model with working vacation. Banik et al. [2] worked on working vacation and gave many numerical results. Working vacation policy is practically applicable in optimal design of the system. Karaesmen and Gupta [6] investigated the same model with server vacations. Li et al. [9] analyzed the queue with memoryless working vacations with vacation interruptions. Kumar and Arivudainambi [8] surveyed the retrial queue with general retrial times. Further, Chao and Rahman [3] analyzed and presented a computer algorithm for queue. General input queue with working vacation and vacation interruption studied by Li and Tian [10].

In this paper, we analyze a model of single-server, multiple working vacations with vacation interruption policy and general arrival process. To economize the cost of the system, we allow the server to take multiple vacations, in case there is less customer in the system waiting for the service, but it also affects the working of the system. Along with the multiple working vacation policy in which customers served with cheap service rate, we also discuss the vacation interruption policy, where the server come back to the normal working rate immediately when number of customers or number of jobs increases to certain level. The vacation interruption determines when the server ends its vacation and comes back to normal service rate to the lower service rate.

1.1 Assumptions for the Model

- Time between two arrivals are (independently and identically distributed),
- Server takes working vacation at the end of busy period,
- Customers are served by the rule "first-in-first-out" queue discipline,
- The service times are exponentially distributed.

1.2 Notations for the Model

The expressions for μ, η, and γ which are mutually independent are given as below:

$$\mu = \sum_{n=1}^{K} \mu_n / K,$$

$$\eta = \sum_{n=1}^{K} \eta_n / K,$$

$$\gamma = \sum_{n=1}^{K} \gamma_n / K,$$

$$\rho = \lambda/\mu$$

- μ average service rate of regular busy period,
- η average service rate of MWV period,
- γ average vacation rates,
- $A(x)$ cumulative density function (c.d.f) of the inter-arrival times of successive arrival,
- $A^*(\Theta)$ Laplace Stieltjes transform (LST) of the inter-arrival times of successive arrival, $x \geq 0$,
- p probability of taking working vacation,
- q probability of vacation interruption,
- μ_n service rate during regular busy period, if there are n customers available in the queue before the beginning of a service $1 \leq n \leq K$,
- η_n service rate during multiple working vacations, if there are n customers available in the queue before the beginning of a service $1 \leq n \leq K$,
- γ_n rate of vacation times, if there are n customers available in the system $1 \leq n \leq K$,
- $N_s(t)$ the number of customers present in the system including the one who involves in service,
- $U(t)$ inter-arrival time remaining for the customer who just going to enter the system,

$$X(t) - \begin{cases} 0, & \text{the server is in WV period,} \\ 1, & \text{the server is in regular busy period.} \end{cases}$$

The joint probabilities are given as

$$J_{n,0}(x, t)\mathrm{d}x = P\{N_s(t) = n,\ x < U(t) \leq x + \mathrm{d}x,\ X(t) = 0\},\ x \geq 0,\ 0 \leq n \leq K,$$

$$J_{n,1}(x, t)\mathrm{d}x = P\{N_s(t) = n,\ x < U(t) \leq x + \mathrm{d}x,\ X(t) = 1\},\ x \geq 0,\ 1 \leq n \leq K,$$

2 Analysis of the Model

In the following section, analytic analysis of the model GI/M/1/K/MWV has been done using different analysis methods such as supplementary variable and recursive techniques.

Given that $J_{n,j}(x)$ are joint probabilities of n customers and the server in state of j at an arbitrary moment. Finding LST for above set of equations,

$$-\theta J^*_{0,0}(\theta) = \mu_1 J^*_{1,1}(\theta) + \eta_1 J^*_{1,0}(\theta) - J_{0,0}(0), \tag{1}$$

$$(\eta_n - \theta)J^*_{n,0}(\theta) = \eta_{n+1} J^*_{n+1,0}(\theta) + A^*(\theta)J_{n-1,0}(0) - J_{n,0}(0),\ 1 \le n \le N-1, \tag{2}$$

$$(\delta_n - \theta)J^*_{n,0}(\theta) = A^*(\theta)J_{n-1,0}(0) + p\eta_{n+1} J^*_{n+1,0}(\theta) - J_{n,0}(0),\ N \le n \le K-1, \tag{3}$$

$$(\delta_K - \theta)J^*_{K,0}(\theta) = A^*(\theta)\big(J_{K-1,0}(0) + J_{K,0}(0)\big) - J_{K,0}(0), \tag{4}$$

$$(\mu_1 - \theta)J^*_{1,1}(\theta) = \mu_2 J^*_{2,1}(\theta) - J_{1,1}(0), \tag{5}$$

$$(\mu_n - \theta)J^*_{n,1}(\theta) = \mu_{n+1} J^*_{n+1,1}(\theta) + A^*(\theta)J_{n-1,1}(0) - J_{n,1}(0),\ 2 \le n \le N-1 \tag{6}$$

$$\begin{aligned}(\mu_n - \theta)J^*_{n,1}(\theta) &= -J_{n,1}(0) + A^*(\theta)J_{n-1,1}(0) + \mu_{n+1} J^*_{n+1,1}(\theta) + \gamma_n J^*_{n,0}(\theta)\\ &+ q\eta_{n+1} J^*_{n+1,1}(\theta),\ \ N \le n \le K-1,\end{aligned} \tag{7}$$

$$(\mu_K - \theta)J^*_{K,1}(\theta) = -J_{K,1}(0) + A^*(\theta)\big(J_{K-1,1}(0) + J_{K,1}(0)\big) + \gamma_K J^*_{K,0}(\theta) \tag{8}$$

With the help of above equations, we find the following results:

Lemma 1

$$\sum_{n=0}^{K} J_{n,0}(0) + \sum_{n=1}^{K} J_{n,1}(0) = \lambda \tag{9}$$

Proof We obtain the above result by adding Eqs. (1)–(8), and applying the limit as $\theta \to 0$ and $\sum_{n=0}^{K} J_{n,0} + \sum_{n=1}^{K} J_{n,1} = 1$.

2.1 Recursive Method to Find the Probabilities

We find the probabilities $J_{n,j}(0)$ and $J^*_{n,j}(\theta)$ using Eqs. (1)–(8) as described below:

Substituting $\theta = \delta_K$ in (4), we obtain

$$\begin{aligned}0 &= A^*(\delta_K)\big(J_{K-1,0}(0) + J_{K,0}(0)\big) - J_{K,0}(0)\\ => J_{K-1,0}(0) &= \tfrac{J_{K,0}(0)(1-A^*(\delta_K))}{A^*(\delta_K)}\end{aligned} \tag{10}$$

From (4), we have

$$(\delta_K - \theta)J^*_{K,0}(\theta) = \ A^*(\theta)\left(\frac{J_{K,0}(0)(1 - A^*(\delta_K))}{A^*(\delta_K)} + J_{K,0}(0)\right) - J_{K,0}(0).$$

$$J^*_{K,0}(\theta) = \left(\frac{A^*(\theta) - A^*(\delta_K)}{A^*(\delta_K)((\delta_K - \theta))}\right)J_{K,0}(0) \tag{11}$$

In similar way, substituting $\theta = \delta_n$, where $n = K-1, \ldots, N$ in (3), we obtain

$$J_{n-1,0}(0) = \frac{J_{n,0}(0) - p\eta_{n+1} J^*_{n+1,0}(\delta_n)}{A^*(\delta_n)} \quad n = K-1, \ldots, N \tag{12}$$

From (3), we have

$$J^*_{n,0}(\theta) = \frac{A^*(\theta) J_{n-1,0}(0) + p\eta_{n+1} J^*_{n+1,0}(\theta) - J_{n,0}(0)}{(\delta_n - \theta)} \quad n = K-1, \ldots, N \tag{13}$$

Substituting $\theta = \eta_n$, where $n = N-1,\ldots 1$ in (2), we obtain

$$J_{n-1,0}(0) = \frac{J_{n,0}(0) - \eta_{n+1} J^*_{n+1,0}(\eta_n)}{A^*(\eta_n)} \quad n = N-1, \ldots 1 \tag{14}$$

From (2), we have

$$J^*_{n,0}(\theta) = \frac{A^*(\theta) J_{n-1,0}(0) + \eta_{n+1} J^*_{n+1,0}(\theta) - J_{n,0}(0)}{(\eta_n - \theta)} \quad n = N-1, \ldots 1 \tag{15}$$

Substituting $\theta = \mu_K$, in (8), we obtain

$$0 = -J_{K,1}(0) + A^*(\mu_K)\big(J_{K-1,1}(0) + J_{K,1}(0)\big) + \gamma_K J^*_{K,0}(\mu_K)$$

$$0 = -J_{K,1}(0) + A^*(\mu_K) J_{K-1,1}(0) + A^*(\mu_K) J_{K,1}(0) + \gamma_K J^*_{K,0}(\mu_K)$$

$$J_{K-1,1}(0) = \frac{1 - A^*(\mu_K)}{A^*(\mu_K)} J_{K,1}(0) - \frac{\gamma_K J^*_{K,0}(\mu_K)}{A^*(\mu_K)} \tag{16}$$

From (8),

$$J^*_{K,1}(\theta) = \frac{\gamma_K J^*_{K,0}(\theta) + A^*(\theta)\big(J_{K-1,1}(0) + J_{K,1}(0)\big) - J_{K,1}(0)}{(\mu_K - \theta)} \tag{17}$$

Substituting $\theta = \mu_n$, where $n = K-1,\ldots, N$ in (7), we obtain

$$J_{n-1,1}(0) = \frac{J_{n,1}(0) - \mu_{n+1} J^*_{n+1,1}(\mu_n) - \gamma_n J^*_{n,0}(\mu_n) - q\eta_{n+1} J^*_{n+1,0}(\mu_n)}{A^*(\mu_n)} \quad n = K-1, \ldots, N. \tag{18}$$

From (7),

$$J^*_{n,1}(\theta) = \frac{\gamma_n J^*_{n,0}(\theta) + \mu_{n+1} J^*_{n+1,1}(\theta) + q\eta_{n+1} J^*_{n+1,0}(\theta) + A^*(\theta) J_{n-1,1}(0) - J_{n,1}(0)}{(\mu_n - \theta)} \quad n = K-1, \ldots, N, \tag{19}$$

Substituting $\theta = \mu_n$, where $n = N - 1, \ldots, 2$ in (6), we obtain

$$J_{n-1,1}(0) = \frac{J_{n,1}(0) - \mu_{n+1} J^*_{n+1,1}(\mu_n)}{A^*(\mu_n)}, \quad n = N-1, \ldots 2. \tag{20}$$

From (6),

$$J^*_{n,1}(\theta) = \frac{\mu_{n+1} J^*_{n+1,1}(\theta) + A^*(\theta) J_{n-1,1}(0) - J_{n,1}(0)}{(\mu_n - \theta)} \quad n = N-1, \ldots 2. \tag{21}$$

$J^*_{1,1}(\theta)$ is obtained from (5)

$$J^*_{1,1}(\theta) = \frac{\mu_2 J^*_{2,1}(\theta) - J_{1,1}(0)}{\mu_1 - \theta} \tag{22}$$

Here, $J^*_{n,0}(\theta)(N \le n \le K)$ and $J^*_{n,0}(\theta)(1 \le n \le N-1)$ for $\theta = \delta_n$ and $\theta = \eta_n$ are, respectively, given by

$$J^*_{K,0}(\theta) = -A^{*(1)}(\theta)\big(J_{K-1,0}(0) + J_{K,0}(0)\big), \tag{23}$$

$$J^*_{n,0}(\theta) = -(A^{*(1)}(\theta) J_{n-1,0}(0) + p\eta_{n+1} J^{*(1)}_{n+1,0}(\theta)), \quad n = K-1, \ldots, N, \tag{24}$$

$$J^*_{n,0}(\theta) = -(A^{*(1)}(\theta) J_{n-1,0}(0) + \eta_{n+1} J^{*(1)}_{n+1,0}(\theta)), \quad n = N-1, \ldots, 1. \tag{25}$$

Again $J^*_{n,1}(\theta)$ for $\theta = \mu_n$ are given as

$$J^*_{K,1}(\theta) = -\Big((\gamma_K J^{*(1)}_{K,0}(\theta) + A^{*(1)}(\theta)\big(J_{K-1,1}(0) + J_{K,1}(0)\big)\Big), \tag{26}$$

$$J^*_{n,1}(\theta) = -\Big(\gamma_n J^{*(1)}_{n,0}(\theta) + \mu_{n+1} J^{*(1)}_{n+1,1}(\theta) + q\eta_{n+1} J^{*(1)}_{n+1,0}(\theta) + A^{*(1)}(\theta) J_{n-1,1}(0)\Big), \quad n = K-1, \ldots, N, \tag{27}$$

$$J^*_{n,1}(\theta) = -\Big(\mu_{n+1} J^{*(1)}_{n+1,1}(\theta) + A^{*(1)}(\theta) J_{n-1,1}(0)\Big), \quad n = N-1, \ldots, 2. \tag{28}$$

Hence, we can obtain $J_{n,j}(0), j = 0, 1; j \le n \le K$, using the above expressions.

3 Relationship of Arbitrary and Pre-arrival Instant

We assume that $J^{\wedge}_{n,j}, j = 0, 1; j \le n \le K$ represents the pre-arrival instant probabilities, i.e., a customer when enters the system finds n customers already exist and j state of the server. Using Bayes' theorem,

$$J^{\wedge}_{n,j} = \lim_{t\to\infty} \frac{P[N_s(t) = n, U(t) = 0, X(t) = j]}{P[U(t) = 0]}, \quad j = 0, 1; \; j \le n \le K$$

Using Eq. (9) in above expression, we obtain

$$J_{n,j}^{\wedge} = \frac{J_{n,j}(0)}{\lambda}, \quad j = 0,\ 1;\ j \le n \le K \tag{29}$$

where λ is given by Lemma.

Theorem 1 *The steady-state probabilities at arbitrary epochs are given by*

$$J_{K,0} = \left(\frac{\lambda}{\delta_K}\right) J_{K-1,0}^{\wedge}, \tag{30}$$

$$J_{n,0} = \frac{\lambda}{\delta_n}\left[J_{n-1,0}^{\wedge} + \left(\frac{q\eta_{n+1} - \gamma_{n+1}}{\delta_{n+1}}\right) J_{n,0}^{\wedge} + \sum_{j=n+1}^{K-1} \left(\frac{q\eta_{j+1} - \gamma_{j+1}}{\delta_{j+1}}\right) \times \prod_{s=n+1}^{j} \left(\frac{p\eta_s}{\delta_s}\right) J_{s,0}^{\wedge} \right]$$
$$n = K-1,\ K-2, \ldots,\ N, \tag{31}$$

$$J_{n,0} = \frac{\lambda}{\eta_n}\left[J_{n-1,0}^{\wedge} + \left(\frac{\gamma_N}{\delta_N}\right) J_{N-1,0}^{\wedge} + \left(\frac{q\eta_{N+1} - \gamma_{N+1}}{\delta_{N+1}}\right) J_{N,0}^{\wedge} + \sum_{j=N}^{K-1} \left(\frac{q\eta_{j+1} - \gamma_{j+1}}{\delta_{j+1}}\right) \times \prod_{s=N}^{j} \left(\frac{p\eta_s}{\delta_s}\right) J_{s,0}^{\wedge} \right]$$
$$n = N-1, \ldots, 1 \tag{32}$$

$$J_{K,1} = \frac{\lambda}{\mu_K}\left[\left(\frac{\gamma_K}{\delta_K}\right) J_{K-1,0}^{\wedge} + J_{K-1,1}^{\wedge} \right] \tag{33}$$

$$J_{n,1} = \frac{\lambda}{\mu_n}\left[J_{n-1,1}^{\wedge} + \left(\frac{\gamma_n}{\delta_n}\right) J_{n-1,0}^{\wedge} - \sum_{j=n}^{K-1} \left(\frac{q\eta_{j+1} - \gamma_{j+1}}{\delta_{j+1}}\right) \times \prod_{s=N}^{j} \left(\frac{p^{s-n}\eta_s}{\delta_s}\right) J_{s,0}^{\wedge} \right]$$
$$n = K-1,\ K-2,\ \ldots,\ N, \tag{34}$$

$$J_{n,1} = \frac{\lambda}{\mu_n}\left[J_{n-1,1}^{\wedge} + \left(\frac{\gamma_N}{\delta_N}\right) J_{N-1,0}^{\wedge} - \sum_{j=N}^{K-1} \left(\frac{q\eta_{j+1} - \gamma_{j+1}}{\delta_{j+1}}\right) \times \prod_{s=N}^{j} \left(\frac{p^{s-n}\eta_s}{\delta_s}\right) J_{s,0}^{\wedge} \right]$$
$$n = N-1,\ N-2,\ \ldots,\ 2 \tag{35}$$

$$J_{1,1} = \frac{\lambda}{\mu_1}\left[\left(\frac{\gamma_N}{\delta_N}\right) J_{N-1,0}^{\wedge} - \sum_{j=N}^{K-1} \left(\frac{q\eta_{j+1} - \gamma_{j+1}}{\delta_{j+1}}\right) \times \prod_{s=N}^{j} \left(\frac{p^{s-n}\eta_s}{\delta_s}\right) J_{s,0}^{\wedge} \right] \tag{36}$$

$$J_{0,0} = 1 - \sum_{n=1}^{K} \left(J_{n,0} + J_{n,1}\right) \tag{37}$$

Putting $\theta = 0$ in (2)–(8) and using (29), we find the relations (30)–(36) and using the normalization condition $J_{0,0}$ is obtained from (37).

4 System Measures

Since the steady-state probabilities at different instant are known, we can obtain the various system measures of the queue easily as below:

Average queue length in the system $(L_s) = \sum_{i=1}^{K} i\left(J_{i,0} + J_{i,1}\right)$
Average queue length in the system during service period $\left(L_q\right) = \sum_{i=1}^{K} (i-1)\left(J_{i,0} + J_{i,1}\right)$
Average queue length in the system during working vacation $(L_{\mathrm{wv}}) = \sum_{i=0}^{K} i\,J_{i,0}$
Average waiting time of a customer in the queue $\left(W_q\right) = \frac{Lq}{\lambda^\wedge}$
Average waiting time of a customer in the system $(W_s) = \frac{Ls}{\lambda^\wedge}$
The probability of loss or blocking $P_{\mathrm{loss}} = J_{\hat{K},0} + J_{\hat{K},1}$

where $\lambda^\wedge = \lambda(1 - P_{\mathrm{loss}})$ is the arrival rate.

5 Cost Model

We express an expected cost function per unit time in which the average service rate during working vacation (η) is the decision variable. Let

$C_{\mathrm{bp}} \equiv$ cost of regular busy period per unit time,
$C_{\mathrm{wv}} \equiv$ cost of WV period per unit time,
$C_q \equiv$ cost for a customer waiting in the queue per unit time,
$C_b \equiv$ cost when a customer is lost due to blocking per unit time

The total expected cost function per unit time according to the definitions of each cost element is given by

$$\textbf{Minimize}: \ \mathbf{C(\eta) = C_{bp}\mu + C_{wv}\eta + C_q L_q + C_b P_{loss}}.$$

The aim is to evaluate mean service rate of vacation (η^*) which minimizes the cost function $C(\eta)$. We applied **QFSM** to solve the above optimization problem.

6 Numerical Results

System parameters are taken as $K = 12$, $N = 6$, $\rho = 0.5$ and for $1 \leq n \leq K$, $\mu_n = \ln(n + 0.3)$, $\eta_n = \ln(n + 0.1)$, and $\gamma_n = \ln(n + 0.2)$ with means $\mu = 1.59681$, $\eta = 1.55021$, and $\gamma = 1.57430$, respectively. The cost elements are taken as $C_{\mathrm{bp}} = 15$, $C_{\mathrm{wv}} = 12$, $C_q = 20$, $C_b = 10$, and stopping tolerance $\in = 10^{-4}$.

Table 1 presents the observations that as q increases, the system characteristics increase and model of VI results good than the model with no VI as expected in real-world problems. Table 2 shows the effect of η on multiple working vacations with vacation interruption and multiple working vacations without vacation interruption. It is more clear through the visualization of the data by Fig. 1.

Table 1 Performance characteristics of $E_2/M(n)/1/12$ queue with $\lambda = 0.80862$

	$q = 0$	$q = 0.2$	$q = 0.4$	$q = 0.6$	$q = 0.8$	$q = 1$
L_q	1.22254	1.22261	1.22269	1.22281	1.22292	1.22306
L_s	2.15649	2.15658	2.15668	2.15679	2.15692	2.15708
P_{loss}	0.00002	0.00002	0.00002	0.00002	0.00002	0.00002
W_q	1.51194	1.51204	1.51214	1.51224	1.511236	1.51254
W_s	2.66696	2.66708	2.66721	2.667342	2.66749	2.66765

Table 2 Effect of η on MWV and MWV-VI

η	MWV-VI (L_q)	MWV (L_q)
1.551	1.475	1.481
1.565	1.469	1.476
1.578	1.463	1.469
1.591	1.458	1.462
1.605	1.453	1.556
1.62	1.4504	1.4504

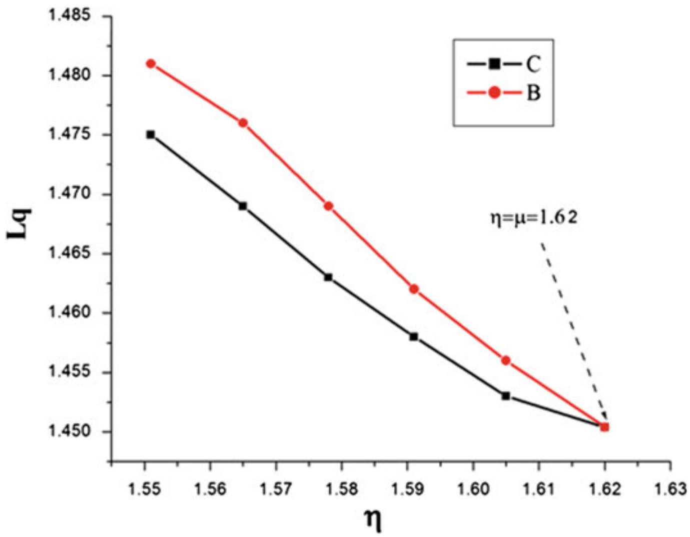

Fig. 1 Impact of η on L_q

Figure 1 depicts the effect of η on average queue length (L_q) in models of VI and with no VI for inter-arrival time distribution. Here, curve B represents multiple working vacations and curve C represents multiple working vacations with vacation interruption. We observe that as η increases L_q decreases and L_q converges to the same value as η approaches μ. Further, the model with vacation interruption yields lower queue lengths compared to model without vacation interruption.

Figure 2 shows the impact of threshold value (N) on W_q waiting time in models. It appears from the figure that W_q increases with the increase of N in both the models with and without vacation interruption. This trend is because as N increases more customers are required for vacation interruption resulting in increase of waiting time.

Table 2 presents the effect of η on the total expected cost function $f(\eta)$ for exponential inter-arrival time distribution, and constant service rate of regular busy period, and vacation period also $\mu = 2.4$, $\gamma = 0.6$, and $\lambda = 1.4$. Quadratic fit square method is taken into account to find the optimal η^* and the optimal cost function $f(\eta^*)$ (Table 3).

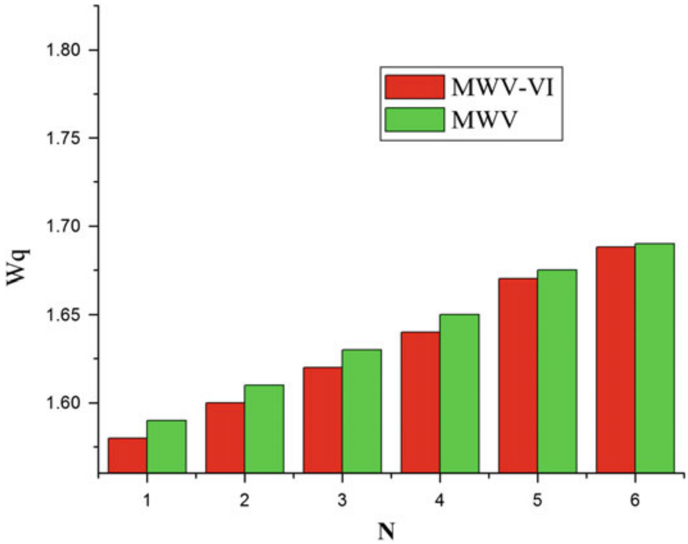

Fig. 2 Effect of threshold value

Table 3 Optimum service rate during working vacation period (η^*)

η^a	η^b	η^c	$f(\eta^a)$	$f(\eta^b)$	$f(\eta^c)$	η^q	$f(\eta^q)$
2.4	2.5	2.6	124.298	124.282	124.312	2.4234	124.270
2.4	2.4234	2.5	124.298	124.270	124.275	2.4221	124.270
2.4	2.4221	2.4234	124.298	124.270	124.270	2.4127	124.270
2.4	2.4127	2.4221	124.298	124.270	124.270	2.4219	124.270
2.4127	2.4219	2.4221	124.270	124.270	124.270	2.4219	124.270

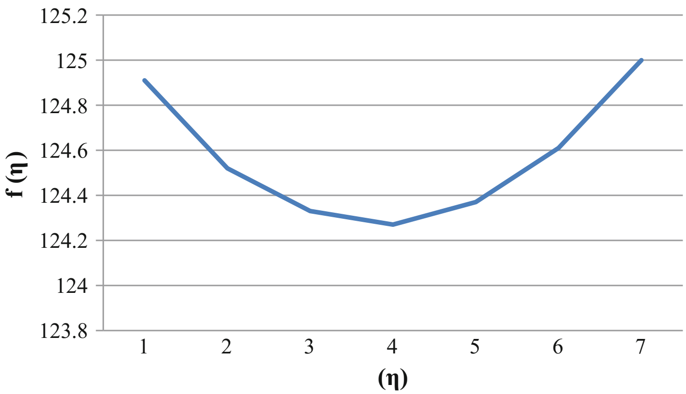

Fig. 3 Impact of η on $f(\eta)$

From Fig. 3, choosing the initial three-point pattern as $(\eta^a, \eta^b, \eta^c) = (2.4, 2.5, 2.6)$, QFSM is used for obtaining the optimal η. With the help of Table 2, one can observe that after five iterations, minimum expected cost per unit time is given as $f(\eta^*) = 124.270$ for $\eta^* = 2.4219$.

7 Conclusions

This chapter carried out an analysis of G1/M(n)/1/K model with multiple working vacations, vacation interruption with N-policy. Along with various performance measures, cost optimization problem is also done using QFSM. Numerical illustrations are reported to demonstrate how the various model parameters of the system influence the behavior of the system. Finally, we conclude that queue with vacation interruption and N-policy yield lower queue lengths and hence perform better due to the fact of the vacation interruption during working vacation.

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Balking Strategies for a Working Vacation Priority Queueing System with Two Classes of Customers

Anamika Jain and Madhu Jain

Abstract In this investigation, we develop a single unreliable server queueing model with working vacation and two classes of customers. The class P_1 customers have higher priority in comparison with class P_2 customers, each having its own respective line. Within a priority class, the service discipline is FCFS. To deal with realistic situation, some specific forms of balking behavior are considered. The lower priority customers observe the queue and decide whether join the queue or balk, depending upon the number of customers existing in the system. During the working vacation, the server continues the job with lower service rate rather than fully stopping the service. When the server is in working mode, at any instant breakdowns may occur randomly. The breakdown server is instantaneously sent for repair at the repair facility. The arrivals of priority and non-priority customers are independent Poisson processes. The service times and the working vacation times are exponentially distributed and indistinguishable from the respective priority classes. The matrix analytic method is implemented to obtain steady-state probability vectors. The expected waiting times of each class of customers in both queues are derived. For getting insight about the total number of customers in the system, numerical results are obtained which are also examined to facilitate the sensitivity analysis of system descriptions.

Keywords Preemptive priority · Working vacation · Balking · Unreliable server Matrix analytic method · Queue size

1 Introduction

In the classic priority queues, the customers are categorized based on their requirement or type of customers and then placed into different priority groups. Among each of priority class, the customers are scheduled in FIFO order. In a preemptive

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priority queueing system, the service of a customer is interrupted when a customer of higher priority class arrives. In preemptive resume priority system, the customer, whose service was interrupted, begins service from the point of interruption, as soon as all customers of higher priority have been served. There are many applications of the priority queueing models in computer and communication networks, i.e., in a packet switching network, the control packets that carry virtual instructions for the network operations are usually transmitted with a higher priority than that of data packets. Another example would be of the multimedia system in which the voice and data are carried in the same network; the voice packets may have accorded a higher priority than that of the data packets owing to real-time requirements. Heijden et al. [9] discussed Markovian multi-class queues with preemptive priorities associated with higher priority class customers. Derbala [2] studied an operating system by including the noble feature of priority queueing model. Dudin et al. [4] proposed the retrial priority queueing model in which they considered the reservation of servers with varying numbers.

It is seen that manufacturing/production system may not operate during the period of breakdown with full capacity and may lead to the loss in profit, production, and reputation. The queueing systems under random breakdown by developing priority models have also been studied by a few researchers in different frameworks. Grey et al. [6, 7] developed a queueing model with backup servers and service breakdowns. The method for the analysis of the queue length distribution was based on matrix geometric approach.

The customer's behavior is also very important in the study of queues. In the queueing system with discouragement, the customers either may not like to join the queue or may leave the system due to the impatience after joining it without getting service. In case of long queue, the balking behavior of the customers prevails due to which they cannot wait and thus do not join the queue. The behavior of the low priority customers can be treated to be discouraged as their queue becomes longer. In queueing literature, some researchers have investigated queueing models by incorporating the balking probability in their models. Drekic and Douglas [5] considered the balking behavior in a preemptive priority queue to determine the steady-state joint distribution of both types of customers in the system. Al-Seedy et al. [1] analyzed the multi-server queueing model with discouragement by considering both balking and reneging and obtained transient solution. A few authors have paid attention toward the working vacation queueing models in which the server is allowed to do some ancillary work in vacation period with different service rate; such problems can also realize in many industrial organizations including the manufacturing, production, transportation. Dimitriou [3] analyzed unreliable server queue with mixed priority, retrial attempts, and multiple vacations and obtained the stochastic decomposition results. Zhang et al. [13] proposed the work for Markovian queues with working vacations under equilibrium strategies for balking. Guha et al. [8] gave some balking strategies in working vacation queueing models with renewal input batch arrival by considering different cases for the equilibrium customer strategies. Yang et al. [12] analyzed working vacation queue with N-policy and server breakdowns and also suggested methodology to optimize the cost. Sun et al. [11] gave the different policies of equilibrium to study the optimal balking strategy of customers in Markovian queues where at least N-customers are accumulated in the queue after completion the vacation before starting of the service again.

In this present study, we consider a single unreliable server queueing system with balking and two classes of customers. The queueing model with priority, balking, and server breakdown can be considered as the extension of Drekic and Douglas (2005)'s model in which they have not taken the server's breakdown into consideration. The two categories of customers arriving in Poisson fashion from different arrival rates and having different general service distribution are considered. The type-1 customers have preemptive priority over type-2 customers and are served according to FCFS rule in their respective classes. The probabilities of the number of customers in the queue in equilibrium state and the stability condition are obtained using the matrix geometric analytical approach. The remaining of the paper is organized as follows. In Sect. 2, the mathematical model is described by describing the requisite assumptions and notations. For the queueing analysis of the proposed models, the global balance equations are constructed in Sect. 3. In Sect. 4, the steady-state probabilities are obtained by employing matrix geometric approach. Various performance measures are also established in terms of probabilities. A numerical example is given in Sect. 5. The sensitivity analysis is done in the next Sect. 6, in order to present the effect of different parameters on the system performance. Finally, Sect. 7 summarizes the investigation carried out in the paper and further draws the conclusions related to the works presented.

2 Model Description

In the present model, we consider two streams of customers, i.e., class p_1 (with higher priority) and the class p_2 (with lower priority). The service discipline among two classes is assumed to be governed by preemptive priority, i.e., higher priority class (p_1 type) customers can terminate the service of non-priority class (p_2 type) customers. However, inside the same class, the customers are provided service according to the first-come-first-served order. When the server returns back from vacation and if the queue is not empty, the first queued customers from the high priority class are selected to be served; otherwise, the low priority customers (i.e. p_2 type) are served in FCFS manner. In such a way, if there are no customers present in the class $p_{2,}$ then the server moves to a functioning (working) vacation mode for a random length V. During the working vacation, the server is available to work in operational mode with lower speed and renders the service if any arrival enters during the vacation period. In other way, if the system becomes empty (there are no class p_1 and class p_2 customer), then it goes for complete vacation.

The non-priority customers are assumed to follow the specific balking strategies and decide whether balk or join the queue based on information regarding the system size. The server is not reliable as such, may break down randomly during its busy period. When the server breakdown, it is sent for repair immediately. The life time and the repair time of the server are exponentially with parameter α and β, respectively. Class p_1 customers are serviced at the service rate μ_1, and the class p_2 customers are served at the rate $\mu_{2,\,i}$ (μ_{2V_i}) during the normal (working vacation) period with depending on the queue size 'i' of class p_2 customers. The class p_1 customers arrive in Poisson fashion with arrival rate λ_1 in normal operation period, during the vacation they arrive with rate λ_V, whereas class p_2 customers arrive in Poisson fashion with queue size-dependent rate due to the balking behavior of customers in both the states (busy and working vacation) and is given by

$$\boldsymbol{\lambda}_{2i} = \begin{cases} \boldsymbol{\lambda}_2\,\boldsymbol{\theta}_i;\ 0 \le i < N_s \\ 0; \qquad i \ge N_s \end{cases}, \quad \boldsymbol{\lambda}_{2\,Vi} = \begin{cases} \boldsymbol{\lambda}_{2V}\,\boldsymbol{\theta}_{li};\ 0 \le i < N_s \\ 0; \qquad i \ge N_s \end{cases}$$

Here N_s is the buffer size for class p_2 customers, where $N_s \in \mathbf{Z}^+$. Also, $\lambda_2 > 0$ is a given parameter; θ_i is a balking parameter and the server proceeds for the working vacation for exponential distributed duration with parameter η to class p_2. The service rate of class p_2 customers is given by

$$\boldsymbol{\mu}_{2i} = \begin{cases} \boldsymbol{\mu}; \qquad 0 \le i < \boldsymbol{\theta} \\ \boldsymbol{\mu}(1+\boldsymbol{\varepsilon});\ \boldsymbol{\theta} \le i \le N_s \end{cases}, \quad \boldsymbol{\mu}_{2\mathrm{V}_i} = \begin{cases} \boldsymbol{\mu}_{2V}; \qquad 0 \le i < \boldsymbol{\theta} \\ \boldsymbol{\mu}_{2V}(1+\boldsymbol{\varepsilon});\ \boldsymbol{\theta} \le i \le N_s \end{cases}$$

Let the total number of class p_1 and class p_2 customers in the system be denoted by N_{p_1} and N_{p_2}, respectively. We assume that $N_{p_1} \ge 0$, i.e., infinite buffer size for class p_1 customers and $0 \le N_{p_2} \le N_s$, θ is a threshold point and ε is fractional service rate. In this present work, following two specific balking functions are considered:

I. Fractional balking function: $\boldsymbol{\theta}_i = \frac{1}{i+1}$
II. For exponential balking function, $\boldsymbol{\theta}_i = \mathrm{e}^{-\delta i}$; $\delta > 0$

3 The Global Balance Equations

Let $\mathbf{P}^{\mathrm{T}} = \left[\mathbf{P}_0^{\mathrm{T}}, \mathbf{P}_1^{\mathrm{T}}, \mathbf{P}_2^{\mathrm{T}}, \ldots\ldots., \mathbf{P}_n^{\mathrm{T}}, \ldots\ldots\right]$ be the probability vector at steady state. Also, the probability vector for state 'n' is

$$\mathbf{P}_n^{\mathrm{T}} = \left[\mathrm{P}_{n,0,0}^{\mathrm{T}}, \mathrm{P}_{n,0,1}^{\mathrm{T}}, \mathrm{P}_{n,0,2}^{\mathrm{T}} \ldots, \mathrm{P}_{n,0,N_s}^{\mathrm{T}}, \mathrm{P}_{n,1,0}^{\mathrm{T}}, \mathrm{P}_{n,1,1}^{\mathrm{T}}, \mathrm{P}_{n,1,2}^{\mathrm{T}} \ldots, \right.$$
$$\left. \mathrm{P}_{n,1,N_s}^{\mathrm{T}}, \mathrm{P}_{n,2,0}^{\mathrm{T}}, \mathrm{P}_{n,2,1}^{\mathrm{T}}, \mathrm{P}_{n,2,2}^{\mathrm{T}} \ldots, \mathrm{P}_{n,2,N_s}^{\mathrm{T}}\right]; n \ge 0$$

The triplet (n, m, k) is used for indexing the probabilities, where n is the number of class p_1 customers; m defines the states of the server and takes values 0, 1, 2; k is used for number of class p_2 customers in the system. The server state at time 't' is defined as

$$G(t) = \begin{cases} 0; \text{ server in operating state} \\ 1; \text{ server in under repair state} \\ 2; \text{ server in working vacation mode} \end{cases}$$

The global balance equations can be written as:

$$\mathbf{P}_0^{\mathrm{T}}\,\mathrm{A}_{00} + \mathbf{P}_1^{\mathrm{T}}\,\mathrm{B}_{00} = 0^{\mathrm{T}} \tag{1}$$

$$\mathbf{P}_0^{\mathrm{T}}\,\mathrm{C}_{00} + \mathbf{P}_1^{\mathrm{T}}\,\mathrm{A}_0 + \mathbf{P}_2^{\mathrm{T}}\,\mathrm{B}_0 = \mathbf{0}^{\mathrm{T}} \tag{2}$$

$$\mathbf{P}_{\mathrm{n}-1}^{\mathrm{T}}\,\mathrm{B}_0 + \mathbf{P}_{\mathrm{n}}^{\mathrm{T}}\,\mathrm{A}_0 + \mathbf{P}_{\mathrm{n}+1}^{\mathrm{T}}\,\mathrm{C}_0 = 0^{\mathrm{T}}; \quad n > \mathbf{1} \tag{3}$$

where 0^{T} is a $(N_s + 1)$ order column vector of zeros. It can be verified that the system under deliberation is a quasi-birth death process with the infinitesimal generator

$$\mathbf{Q} = \begin{bmatrix} \mathbf{A_{00}} & \mathbf{B_{00}} & & & \\ \mathbf{C_{00}} & \mathbf{A_0} & \mathbf{B_0} & & \\ & \mathbf{C_0} & \mathbf{A_0} & \mathbf{B_0} & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot\,\cdot \\ & & & & \cdot\,\cdot \end{bmatrix} \tag{4}$$

Let the balance equations $\mathrm{C}_{00}\mathrm{U} + \mathrm{A}_0\mathrm{U} + \mathrm{B}_0\mathrm{U} = \mathrm{A}$, then $\mathrm{AI}_{Ns+1} = \underline{\mathbf{0}}$ and $\mathrm{A}_{00}\mathrm{U} + \mathrm{B}_{00}\mathrm{U} = \underline{\mathbf{0}}$ where U is the column vector defined as $\mathrm{U} = [1, 1, 1, \ldots\ldots]^{\mathrm{T}}$ and $\underline{\mathbf{0}}$ is the square matrices of order $(N_s + 1)$. For bravely, denote $\Lambda_i = (\lambda_1 + \lambda_{2i})$; $i \geq 0$, $\Delta_i = \left(\lambda_V + \lambda_{2V_i}\right)$; $i \geq 0$, $\nabla_i = \left(\mu_V + \mu_{2V_i}\right)$; $i \geq 0$. The sub-matrices are

$$\mathbf{B_{00}} = \mathrm{diag}\left[\Im_{N_s \times (N_s+1)},\ \underline{\mathbf{0}}_{N_s \times (N_s+1)},\ \boldsymbol{\lambda}_{\mathrm{V}}\mathrm{I}_{N_s+1}\right],\ \mathbf{C_{00}} = \begin{bmatrix} \boldsymbol{\varsigma}_{00} & \boldsymbol{\tau}_{00} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} \end{bmatrix}_{3(N_s+1)\times(3N_s+1)},$$

$$\boldsymbol{\omega}_{00} = \begin{bmatrix} \boldsymbol{\mu}_1 & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} \end{bmatrix}_{(N_s+1)\times(N_s+1)},\ \boldsymbol{\varsigma}_{00} = \mathrm{diag}\left[\Omega_{(N_s+1)\times N_s},\quad \underline{\mathbf{0}}_{(N_s+1)\times N_s}\right],$$

$$\Im = \left[0,\ \mathrm{diag}\,(\boldsymbol{\lambda}_1)\right]_{N_s \times (N_s+1)},\ \boldsymbol{\tau}_{00} = \begin{bmatrix} \boldsymbol{\omega}_{00} \\ \underline{\mathbf{0}} \end{bmatrix}_{2(N_s+1)\times(N_s+1)}$$

$$\mathbf{A_{00}} = \begin{bmatrix} \mathbf{D_{00}} & \mathbf{T_{00}} \\ \mathbf{M_{00}} & \mathbf{D_{01}} \end{bmatrix}_{(3N_s+1)\times(3N_s+1)},\ \mathbf{D_{00}} = \begin{bmatrix} \mathbf{D_{01}^B} & \mathbf{U_0^B} \\ \mathbf{L_0^B} & \mathbf{D_0^B} \end{bmatrix}_{2N_s \times 2N_s},$$

$$\mathbf{\Omega} = \begin{bmatrix} 0 & 0 & 0 \ldots \ldots & 0 \\ \mu_1 & 0 & 0 \ldots \ldots & 0 \\ 0 & \mu_1 & 0 \ldots \ldots & 0 \\ \ldots & \ldots & \ldots \ldots \ldots & \ldots \\ \ldots & \ldots & \ldots \ldots \ldots & \ldots \\ 0 & 0 & 0 \ldots \ldots & 0 \\ 0 & 0 & 0 \ldots \ldots & \mu_1 \end{bmatrix}_{(N_s+1)\times N_s}$$

$$\mathbf{D_{01}^B} = \begin{bmatrix} -(\Lambda_1+\mu_{21}+\alpha) & \lambda_{21} & 0 & \ldots\ldots & 0 & 0 \\ \mu_{22} & -(\Lambda_2+\mu_{22}+\alpha) & \lambda_{22} & \ldots\ldots & 0 & 0 \\ 0 & \mu_{23} & -(\Lambda_3+\mu_{23}+\alpha) & \ldots\ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots\ldots & -\left(\Lambda_{N-1}+\mu_{2(N_s-1)}+\alpha\right) & \lambda_{2(N_s-1)} \\ 0 & 0 & 0 & \ldots\ldots & \mu_{2N_s} & -\left(\Lambda_{N_s}+\mu_{2N_s}+\alpha\right) \end{bmatrix}_{N_s\times N_s}$$

$$\mathbf{D_0^B} = \begin{bmatrix} -(\lambda_{21}+\beta) & \lambda_{21} & 0 & \ldots\ldots & 0 & 0 \\ 0 & -(\lambda_{22}+\beta) & \lambda_{22} & \ldots\ldots & 0 & 0 \\ 0 & 0 & -(\lambda_{23}+\beta) & \ldots\ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots\ldots & -\left(\lambda_{2(N_s-1)}+\beta\right) & \lambda_{2(N_s-1)} \\ 0 & 0 & 0 & \ldots\ldots & 0 & -\left(\lambda_{2N_s}+\beta\right) \end{bmatrix}_{N_s\times N_s}$$

$$\mathbf{U_0^B} = \alpha \mathrm{I}_{N_s},\ \mathbf{L_0^B} = \beta \mathrm{I}_{N_s},\ \mathbf{T_{00}} = \begin{bmatrix} \mathbf{U_{00}} \\ \underline{\mathbf{0}} \end{bmatrix}_{2N_s\times(N_s+1)},\ \mathbf{M_{00}} = \begin{bmatrix} \mathbf{L_{00}} & \underline{\mathbf{0}} \end{bmatrix}_{(N_s+1)\times 2N_s},$$

$$\mathbf{U_{00}} = \begin{bmatrix} \mu_{21} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} \end{bmatrix}_{N_s\times(N_s+1)}$$

$$\mathbf{D_{01}} = \begin{bmatrix} -\Delta_0 & \lambda_{V_0} & 0 & \ldots\ldots & 0 & 0 \\ \mu_{2V_1} & -(\Delta_1+\mu_{2V_1}+\eta) & \lambda_{V_1} & \ldots\ldots & 0 & 0 \\ 0 & \mu_{2V_2} & -(\Delta_2+\mu_{2V_2}+\eta) & \ldots\ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots\ldots & -\left(\Delta_{N_s-1}+\mu_{2V_{N_s-1}}+\eta\right) & \lambda_{V_{N_s-1}} \\ 0 & 0 & 0 & \ldots\ldots & \mu_{2V_{N_s}} & -\left(\Delta_{N_s}+\mu_{2V_{N_s}}+\eta\right) \end{bmatrix}_{(N_s+1)\times(N_s+1)}$$

$$\mathbf{B_0} = \mathrm{diag}\left[\lambda_1 \mathrm{I}_{N_s+1},\ \underline{\mathbf{0}}_{N_s+1},\ \lambda_V \mathrm{I}_{N_s+1}\right], \mathbf{C_0} = \mathrm{diag}\left[\mu_1 \mathrm{I}_{N_s+1}, \underline{\mathbf{0}}_{N+1}, \underline{\mathbf{0}}_{N_s+1}\right],$$

$$\mathbf{A_0} = \begin{bmatrix} \mathbf{D_{10}} & \underline{\mathbf{0}} \\ \mathbf{M_{10}} & \mathbf{D_{20}} \end{bmatrix}_{(3N_s+3)\times(3N_s+3)}, \ \mathbf{D_{10}} = \begin{bmatrix} \mathbf{D^B_{10}} & \mathbf{U^B_{10}} \\ \mathbf{L^B_{10}} & \mathbf{D^B_{20}} \end{bmatrix}_{2(N_s+1)\times 2(N_s+1)}$$

$$\mathbf{U^B_{10}} = \alpha I_{N_s+1}, \ \mathbf{L^B_{10}} = \beta I_{N_s+1},$$

$$\mathbf{M_{10}} = \begin{bmatrix} \mathbf{L_{10}} & \underline{\mathbf{0}} \end{bmatrix}_{(N_s+1)\times 2(N_s+1)}, \ \mathbf{L_{10}} = \mathrm{diag}[\eta, \eta, \ldots\ldots, \eta]_{(N_s+1)\times(N_s+1)},$$

$$\mathbf{L_{00}} = \begin{bmatrix} 0 & 0 & 0 & .. & .. & 0 \\ \eta & 0 & 0 & .. & .. & 0 \\ 0 & \eta & 0 & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & .. & 0 \\ 0 & 0 & 0 & .. & .. & \eta \end{bmatrix}_{(N_s+1)\times N_s}$$

$$\mathbf{D^B_{10}} = \begin{bmatrix} -(\Lambda_0+\mu_1+\alpha) & \lambda_{20} & 0 & \ldots\ldots & 0 & 0 \\ 0 & -(\Lambda_1+\mu_1+\alpha) & \lambda_{21} & \ldots\ldots & 0 & 0 \\ 0 & 0 & -(\Lambda_2+\mu_1+\alpha) & \ldots\ldots & 0 & 0 \\ .. & .. & .. & \ldots\ldots & .. & .. \\ .. & .. & .. & \ldots\ldots & .. & .. \\ 0 & 0 & 0 & \ldots\ldots & -\left(\Lambda_{N_s-1}+\mu_1+\alpha\right) & \lambda_{2(N_s-1)} \\ 0 & 0 & 0 & \ldots\ldots & 0 & -\left(\Lambda_{N_s}+\mu_1+\alpha\right) \end{bmatrix}_{(N_s+1)\times(N_s+1)}$$

$$\mathbf{D^B_{20}} = \begin{bmatrix} -(\lambda_{20}+\beta) & \lambda_{20} & 0 & \ldots\ldots & 0 & 0 \\ 0 & -(\lambda_{21}+\beta) & \lambda_{21} & \ldots\ldots & 0 & 0 \\ 0 & 0 & -(\lambda_{22}+\beta) & \ldots\ldots & 0 & 0 \\ .. & .. & .. & \ldots\ldots & .. & .. \\ .. & .. & .. & \ldots\ldots & .. & .. \\ 0 & 0 & 0 & \ldots\ldots & -\left(\lambda_{2(N_s-1)}+\beta\right) & \lambda_{2(N_s-1)} \\ 0 & 0 & 0 & \ldots\ldots & 0 & -\left(\lambda_{2N_s}+\beta\right) \end{bmatrix}_{(N_s+1)\times(N_s+1)}$$

$$\mathbf{D_{20}} = \begin{bmatrix} -(\Delta_0+\mu_V+\eta) & \lambda_{V_0} & 0 & \ldots\ldots & 0 & 0 \\ \mu_{2V_1} & -(\Delta_1+\nabla_1+\eta) & \lambda_{V_1} & \ldots\ldots & 0 & 0 \\ 0 & \mu_{2V_2} & -(\Delta_2+\nabla_2+\eta) & \ldots\ldots & 0 & 0 \\ .. & .. & .. & \ldots\ldots & .. & .. \\ .. & .. & .. & \ldots\ldots & .. & .. \\ 0 & 0 & 0 & \ldots\ldots & -\left(\Delta_{N_s-1}+\nabla_{N_s-1}+\eta\right) & \lambda_{V_{N_s-1}} \\ 0 & 0 & 0 & \ldots\ldots & \mu_{2V_{N_s}} & -\left(\Delta_{N_s}+\nabla_{N_s}+\eta\right) \end{bmatrix}_{(N_s+1)\times(N_s+1)}$$

4 The Steady-State Probability Vector of Queue Length

To establish the analytical results under which the stationary probability exists, we outline the theorem of Neuts [10] as follows:

Theorem 1 *The continuous-time Markov chain described above is positive recurring if and only if the marginal nonnegative solution R to the matrix quadratic equation (cf. see Neuts [10], Theorem 3.1.1)*

$$R^2 C_0 + R A_0 + B_0 = 0, \tag{5}$$

has all its eigenvalues within the unit disk, i.e., sp(R) < 1, *and the finite system of equations*

$$\begin{cases} P_0^T (A_{00} + R B_{00}) \; = \; 0. \\ P_0^T (I - R)^{-1} U_{N_s+1} = 1. \end{cases} \tag{6}$$

has a (unique) positive solution P_0^T *where* U = [1, 1, ……,1]$^{\mathrm{T}}$. *The stationary probability vector is given by*

$$P_n^T \; = \; P_0^T R^n \;\; ; \, n \geq 0. \tag{7}$$

The $(N_s + 1) \times (N_s + 1)$ order matrix R is the key matrix which can be further used to determine the queue size distribution and other measures of system performance. We compute R by the following iterative procedure:

$$\begin{aligned} & R(0) = 0. \\ & R(n+1) = -B_0 A_0^{-1} - R^2(n) C_0 A_0^{-1} \, ; \qquad n \geq 0 \end{aligned} \tag{8}$$

Performance Indices:

Many performance measures and probabilities characterizing the system can be derived from the probability vectors evaluated and are given by

- The total number of customers in the system is

$$E(Q) = \sum_{i=0}^{N_s} \sum_{n=0}^{\infty} n \left(P_{n,0,i} \, + \, P_{n,1,i} \, + \, P_{n,2,i} \right) \tag{9}$$

- Throughput is obtained using

$$T(P) = \left(\sum_{i=1}^{N_s} \mu_{2,i} \; P_{0,0,i} + \sum_{i=1}^{N_s} \sum_{n=0}^{\infty} \mu_1 P_{n,0,i} + \sum_{i=1}^{N_s} \mu_{2,V_i} \; P_{0,2,i} + \sum_{i=1}^{N_s} \sum_{n=0}^{\infty} \mu_V P_{n,2,i} \right) \tag{10}$$

- Expected waiting time is

$$E(W) = \frac{E(Q)}{\lambda_{\text{eff}}} \tag{11}$$

where

$$\lambda_{\text{eff}} = \sum_{i=0}^{N_s}\sum_{n=0}^{\infty}(\lambda_{2i} + \lambda_1)P_{n,0,i} + \sum_{i=0}^{N_s}\sum_{n=0}^{\infty}\left(\lambda_{2V_i} + \lambda_V\right)P_{n,2,i} \tag{12}$$

- Average delay is computed using

$$D = \frac{E(Q)}{T(P)} \tag{13}$$

5 Numerical Illustration

Based on performance analysis detailed in Sects. 3 and 4, we validate the tractability of matrix geometric approach by taking a numerical example. MATLAB software has been used to develop the computer program for fixed parameter values chosen as

$$N_s = 5, \lambda_1 = 0.5, \lambda_2 = 2, \lambda_V = 0.5, \lambda_{2V} = 0.8, \mu_1 = 15, \mu = 20, \varepsilon = 1,$$
$$\mu_V = 2,\ \theta = 2, \mu_{2V} = 5,\ \delta = 1,\ \alpha = 0.5,\ \beta = 15,\ \eta = 0.5$$

The sub-matrices D_{00}, $D_{01,}$ D_0, and the nonnegative square matrix R are evaluated for default parameters as

$$D_{00} = \begin{bmatrix} -21.0996 & 0.0996 & 0 & 0 & 0 & 0.5000 & 0 & 0 & 0 & 0 \\ 20.0000 & -21.0050 & 0.0050 & 0 & 0 & 0 & 0.5000 & 0 & 0 & 0 \\ 0 & 20.0000 & -21.0002 & 0.0002 & 0 & 0 & 0 & 0.5000 & 0 & 0 \\ 0 & 0 & 20.0000 & -41.0000 & 0.0000 & 0 & 0 & 0 & 0.5000 & 0 \\ 0 & 0 & 0 & 40.0000 & -41.0000 & 0 & 0 & 0 & 0 & 0.5000 \\ 15.0000 & 0 & 0 & 0 & 0 & -15.0996 & 0.0996 & 0 & 0 & 0 \\ 0 & 15.0000 & 0 & 0 & 0 & 0 & -15.0050 & 0.0050 & 0 & 0 \\ 0 & 0 & 15.0000 & 0 & 0 & 0 & 0 & -15.0002 & 0.0002 & 0 \\ 0 & 0 & 0 & 15.0000 & 0 & 0 & 0 & 0 & -15.0000 & 0.0000 \\ 0 & 0 & 0 & 0 & 15.0000 & 0 & 0 & 0 & 0 & -15.0000 \end{bmatrix}$$

$$
D_{10} = \begin{bmatrix}
-16.0996 & 0.0996 & 0 & 0 & 0 & 0 & 0.5000 & 0 & 0 & 0 & 0 & 0 \\
0 & -16.0050 & 0.0050 & 0 & 0 & 0 & 0 & 0.5000 & 0 & 0 & 0 & 0 \\
0 & 0 & -16.0002 & 0.0002 & 0 & 0 & 0 & 0 & 0.5000 & 0 & 0 & 0 \\
0 & 0 & 0 & -16.0000 & 0.0000 & 0 & 0 & 0 & 0 & 0.5000 & 0 & 0 \\
0 & 0 & 0 & 0 & -16.0000 & 0.0000 & 0 & 0 & 0 & 0 & 0.5000 & 0 \\
0 & 0 & 0 & 0 & 0 & -15.5000 & 0 & 0 & 0 & 0 & 0 & 0.5000 \\
15.0000 & 0 & 0 & 0 & 0 & 0 & -15.0996 & 0.0996 & 0 & 0 & 0 & 0 \\
0 & 15.000 & 0 & 0 & 0 & 0 & 0 & -15.0050 & 0.0050 & 0 & 0 & 0 \\
0 & 0 & 15.0000 & 0 & 0 & 0 & 0 & 0 & -15.0002 & 0.0002 & 0 & 0 \\
0 & 0 & 0 & 15.0000 & 0 & 0 & 0 & 0 & 0 & -15.0000 & 0.0000 & 0 \\
0 & 0 & 0 & 0 & 15.0000 & 0 & 0 & 0 & 0 & 0 & -15.0000 & 0.0000 \\
0 & 0 & 0 & 0 & 0 & 15.0000 & 0 & 0 & 0 & 0 & 0 & -15.0000
\end{bmatrix}
$$

$$
D_{01} = \begin{bmatrix}
-1.3679 & 0.3679 & 0 & 0 & 0 & 0 \\
5.0000 & -6.6353 & 0.1353 & 0 & 0 & 0 \\
0 & 5.0000 & -6.5498 & 0.0498 & 0 & 0 \\
0 & 0 & 5.0000 & -6.5183 & 0.0183 & 0 \\
0 & 0 & 0 & 5.0000 & -6.5067 & 0.0067 \\
0 & 0 & 0 & 0 & 5.0000 & -5.5000
\end{bmatrix},
$$

$$
D_{20} = \begin{bmatrix}
-3.8679 & 0.3679 & 0 & 0 & 0 & 0 \\
5.0000 & -8.6353 & 0.1353 & 0 & 0 & 0 \\
0 & 5.0000 & -8.5498 & 0.0498 & 0 & 0 \\
0 & 0 & 5.0000 & -8.5183 & 0.0183 & 0 \\
0 & 0 & 0 & 5.0000 & -8.5067 & 0.0067 \\
0 & 0 & 0 & 0 & 5.0000 & -7.5000
\end{bmatrix}
$$

$$
R = \begin{bmatrix}
0.0331 & 0.0002 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0011 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.0333 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0 & 0.0011 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0333 & 0.0000 & 0.0000 & 0.0000 & 0 & 0 & 0.0011 & 0.0000 & 0.0000 & 0.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.0333 & 0.0000 & 0.0000 & 0 & 0 & 0 & 0.0011 & 0.0000 & 0.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.0333 & 0.0000 & 0 & 0 & 0 & 0 & 0.0011 & 0.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.0126 & 0.0007 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0004 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.2738 & 0.0118 & 0.0002 & 0.0000 & 0.0000 & 0.0000 \\
0.0083 & 0.0049 & 0.0001 & 0.0000 & 0.0000 & 0.0000 & 0.0003 & 0.0002 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.1600 & 0.1238 & 0.0020 & 0.0000 & 0.0000 & 0.0000 \\
0.0055 & 0.0033 & 0.0045 & 0.0000 & 0.0000 & 0.0000 & 0.0002 & 0.0001 & 0.0001 & 0.0000 & 0.0000 & 0.0000 & 0.0939 & 0.0726 & 0.1185 & 0.0007 & 0.0000 & 0.0000 \\
0.0037 & 0.0022 & 0.0030 & 0.0045 & 0.0000 & 0.0000 & 0.0001 & 0.0001 & 0.0001 & 0.0001 & 0.0000 & 0.0000 & 0.0552 & 0.0427 & 0.0697 & 0.1180 & 0.0003 & 0.0000 \\
0.0025 & 0.0015 & 0.0020 & 0.0030 & 0.0045 & 0.0000 & 0.0001 & 0.0000 & 0.0001 & 0.0001 & 0.0001 & 0.0000 & 0.0324 & 0.0251 & 0.0410 & 0.0694 & 0.1178 & 0.0001 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Further probabilities are obtained which are used to calculate various performance measures given in Eqs. (9)–(13) as

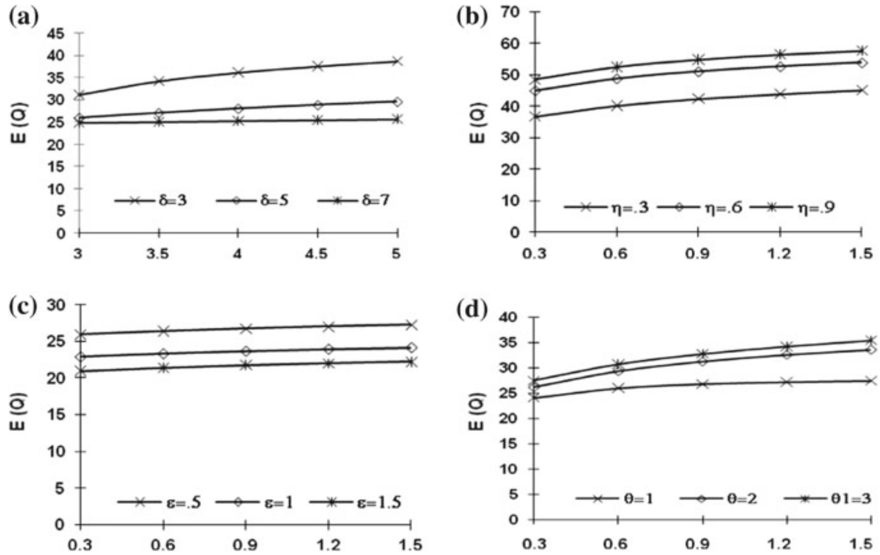

Fig. 1 $E(Q)$ versus λ_2 for exponential balking case with variation (**a**) δ (**b**) η (**c**) ε (**d**) θ

$$E(Q) = 34.9953,\ T(P) = 53.4308,\ E(W) = 2.224952,\ D = 0.6550.$$

6 Sensitivity Analysis

In order to reconnoiter the effect of different parameters on various system's performance measures, we facilitate the sensitivity analysis when the value of one or more parameters vary and the rest parameter values remain unchanged according to default values as fixed in numerical illustration given in the previous section.

For exponential balking case in Fig. 1a–d, the graphs of $E(Q)$ are plotted to see the effect of parameter λ_2 on the horizontal axis. It is observed that $E(Q)$ increases almost linearly for the value of λ_2. Figure 1a exhibits the decreasing trend with respect to δ, which is quite obvious. Figure 1b illustrates the variation of $E(Q)$ for different values of η and indicates the increasing pattern with respect to η. In Fig. 1c, we note that $E(Q)$ decreases with the increase in ε. Figure 1d depicts that $E(Q)$ increases as θ increases.

7 Discussion

Priority queues arise quite often in many real-life queueing applications. In classic priority queues, the customers are placed into different priority classes. In this paper, we have analyzed an unreliable server preemptive priority queueing system with working vacation by considering some balking strategies of the customers. The computational procedure involves a number of matrix inversions, but by noticing that the matrices to be inverted are tri-diagonal block matrix to determine the queue length distribution, the matrix geometric method is employed.

The incorporation of balking parameter seems to be fitted in many real-life congestion situations. Sensitivity analysis provided can be further utilized for the prediction of various systems performance indices so that some observations may be implemented to improve the grade of the service of the real time system.

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$M^X/G/1$ Queue with Optional Service and Server Breakdowns

Charan Jeet Singh and Sandeep Kaur

Abstract In the present study, a single server queueing system with batch arrivals of the units is considered. The provision of optional service after availing essential service is available in the system, and it is assumed that the server may break down during any stage of the service of the units and provides the repair facility immediately. The server may also avail the vacation under Bernoulli vacation policy after completion of the service of the units. The supplementary variable approach with probability generating functions is applied to analyze the system to find the system performance characteristics. The numerical illustration is considered to obtain the system state probabilities and queueing/reliability indices to study the effect of system parameters on the various performance measures.

Keywords Batch arrival · Optional service · Server breakdown
Bernoulli vacation · Reliability

1 Introduction

In mathematical analysis of queueing models, the effects of design, configuration, and implementation of the problems are encountered in many daily routine activities as well as industrial scenario including computer networks and other operating systems. The service phenomena of the system have significant place in queueing modeling, which includes various kinds of services such as single service, optional service, and phases service. Due to unpredictable breakdowns of the server, the service interruption can be experienced in various practical situations. In real practice, the server may break down and stops the service of the units of the queue during any

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stage of the service. The failed server needs to be repaired immediately and becomes ready to resume its work effectively.

The mechanism of the vacation policy of the server is realistic and can be observed in various real-time activities. The server may go for essential/optional vacation as per the certain vacation policy for the random period of time when no units are present in the system. On return of the vacation, if at least one unit is present in the queue, the server begins the service. If there is no unit in the queue, the server may avail the facility of a sequence of vacations for the period till the system is empty. The models of queueing problems with vacations provide the performance prediction of production and flexible manufacturing systems wherein the machines may undergo for the preventive maintenance with fixed period to achieve the optimal outputs. During this vacation time period, the units of the queue will have to wait.

An extensive survey on queueing problems with different variations can be searched in the literature. The detailed accounts of analysis of stochastic models are due to research works on optional service with unreliable server and vacation policy. In the recent past, several researchers have contributed significantly in this direction. Choudhury and Deka [1] have discussed a $M/G/1$ model in which they have assumed that the units arrive one by one with homogeneous arrival rate to get for two phases of essential service, and the server may break down at any instant of the service. Singh et al. [2] have developed the $M^X/G/1$ model of queueing system with state-dependent arrival rates under deterministic vacation policy and analyzed the queue size distribution and various performance indices of the system by using the supplementary variable approach. Chakravarthy [3] studied a single server queue with the essential/optional service and analyzed the model using supplementary variable approach. Jain and Bhagat [4] have described the model for retrial bulk input queueing system with k-essential phases of service provided to all customers by the server under the modified vacation policy with the assumption that the server may break down at any instant of the service. Wang [5] investigated the single arrival unreliable queueing system with general distributed service time under the randomized vacation policy. In this study, they have used the supplementary variable technique to obtain the performance indices of the system. In a survey on working vacation in many situations of queueing models, Chandrasekaran et al. [6] have presented the wide applications of different models with variations of behavior of the queueing systems. Recently, Lan and Tang [7] have investigated the queueing system under the assumption that service station may be subject to failures at random and optional services with D-policy.

2 Mathematical Model

Consider an unreliable single server queue with batch arrival of size (c_k) and arrival rates of the units $\lambda,\ \lambda_0 = \lambda b,\ \lambda_1 = \lambda b_1$ and $\lambda_2 = \lambda b_2$, in terms of respective joining probabilities $b,\ b_1$ and b_2 in busy, vacation, and repair states. The server provides the first essential services as well as one of the m optional services to each arriving unit, with respective distribution functions $B_0(x)$ and $B_i(x)$; $i = 1, 2, \ldots, m$. After all the unit served, the server may avail the vacation facility with distribution function $V(x)$. It is considerable that server may fail at any stage of the essential/optional service and undergoes for the repair immediately with distribution functions $G_0(x)$ and $G_i(x)$. The kth moment of repair time is assumed to be $g_0^{(k)}$ and $g_i^{(k)}$ when server fails during essential/optional service. The probabilities to opt ith optional service and optional vacation are described as r_i and p, respectively. It is also assumed that at time t, the queue size of the system is $N_q(t)$ with respective elapsed times of essential (ith optional service), vacation time, and repair time of failed units during essential (ith optional service) are identified as $B_0^0(t)(B_i^0(t))$, $V^0(t)$ and $G_0^0(t)(G_i^0(t))$. It is also noticed that $\bar{U}(\cdot)$ is considered as Laplace–Stieltjes transform (LST) of $U(*)$ with usual parameters. It is noted that after completion of any stage of service, if served unit is not satisfied, the unit may immediately join the tail of the original queue as a feedback customer for receiving another regular service with probability $\theta(0 \le \theta \le 1)$; or it may depart from the system with probability $(1 - \theta)$.

To analyze the model, the limiting probabilities are defined as follows:

$$P_0^{(0)}(t) = \text{Prob.}\{N_q(t) = 0,\ X(t) = 0\}; \tag{1.1}$$

$$P_n^{(0)}(x,t) = \text{Prob.}\{N_q(t) = n,\ X(t) = B_0^0(t);\ x \le B_0^0(t) \le x + \mathrm{d}x\};\ x > 0,\ n \ge 0, \tag{1.2}$$

$$\begin{aligned} P_n^{(i)}(x,t) &= \text{Prob.}\{N_q(t) = n,\ X(t) = B_i^0(t);\\ & x \le B_i^0(t) \le x + \mathrm{d}x\};\ x > 0,\ n \ge 0,\ 1 \le i \le m, \end{aligned} \tag{1.3}$$

$$V_n(y,t) = \text{Prob.}\{N_q(t) = n,\ X(t) = V^0(t);\ y \le V^0(t) \le y + \mathrm{d}y\};\ y > 0,\ n \ge 0, \tag{1.4}$$

$$\begin{aligned} R_n^{(0)}(x,y,t) &= \text{Prob.}\{N_q(t) = n,\ X(t) = R_0^0(t);\\ & y \le R_0^0(t) \le y + \mathrm{d}y / B_0^0(t) = x\};\ x > 0,\ n \ge 0, \end{aligned} \tag{1.5}$$

$$\begin{aligned} R_n^{(i)}(x,y,t) &= \text{Prob.}\{N_q(t) = n,\ X(t) = R_i^0(t);\\ & y \le R_i^0(t) \le y + \mathrm{d}y / B_i^0(t) = x\};\ x > 0,\ n \ge 0,\ 1 \le i \le m, \end{aligned} \tag{1.6}$$

The hazard functions and probability generating functions of the system at different service states are as follows:

$$\mu_i(x)\mathrm{d}x = \frac{\mathrm{d}B_i(x)}{1 - B_i(x)},\ \nu(y)\mathrm{d}y = \frac{\mathrm{d}V(y)}{1 - V(y)},\ g_i(y)\mathrm{d}y = \frac{\mathrm{d}G_i(y)}{1 - G_i(y)};\ 0 \le i \le m. \tag{2.1}$$

$$\begin{aligned} R^{(i)}(x,y,z) &= \sum_{n=0}^{\infty} z^n R_n^{(i)}(x,y);\ R^{(i)}(x,0,z) = \sum_{n=0}^{\infty} z^n R_n^{(i)}(x,0);\ P^{(i)}(x,z) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(x);\\ P^{(i)}(0,z) &= \sum_{n=0}^{\infty} z^n P_n^{(i)}(0);\ V(y,z) = \sum_{n=0}^{\infty} z^n V_n(y);\ V(0,z) = \sum_{n=0}^{\infty} z^n V_n(0). \end{aligned} \tag{2.2}$$

3 Analysis

Consider the stochastic model, wherein the arrival of the units follows the Poisson distribution and the service of the system is generally distributed. The supplementary variables technique is applied to study the behavior of the system and obtain its performance indices.

3.1 Governing Equations

The set of governing equations and boundary conditions by introducing the supplementary variable approach are constructed as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(i)}(x)+\left[\lambda b+\alpha_i+\mu_i(x)\right]P_n^{(i)}(x)=\lambda b\sum_{j=1}^{n}c_j\left(1-\delta_{n,0}\right)P_{n-j}^{(i)}(x)+\int_0^\infty g_i(y)R_n^{(i)}(x,y)dy;$$
$$x,y>0,\ n\geq 0;\ 0\leq i\leq m, \tag{3.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}V_n(y)+[\lambda b_1+\nu(y)]V_n(y)=\lambda b_1\sum_{j=1}^{n}c_j(1-\delta_{n,0})V_{n-j}(y);\ n\geq 0,y>0, \tag{3.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}R_n^{(i)}(x,y)+\left[\lambda b_2+g_i(y)\right]R_n^{(i)}(x,y)=\lambda b_2\sum_{j=1}^{n}c_j\left(1-\delta_{n,0}\right)R_{n-j}^{(i)}(x,y);$$
$$n\geq 0,x,\ y>0,\ 0\leq i\leq m, \tag{3.3}$$

$$\lambda P_0^{(0)}=q\left[(1-\theta)\left(r_0\int_0^\infty\mu_0(x)P_0^{(0)}(x)\mathrm{d}x+\sum_{i=1}^{m}\int_0^\infty\mu_I(x)P_0^{(I)}(x)\mathrm{d}x\right)\right]$$
$$+\int_0^\infty\nu(y)V_0(y)\mathrm{d}y, \tag{3.4}$$

3.2 Boundary Conditions

The set of equations are to be solved under the boundary condition at $x=0$:

$$P_n^{(0)}(0)=\lambda\sum_{j=1}^{n}c_j(1-\delta_{n,0})P_0^{(0)}+q\left[\theta\left(r_0\int_0^\infty\mu_0(x)P_n^{(0)}(x)\mathrm{d}x+\sum_{i=1}^{m}\int_0^\infty\mu_i(x)P_n^{(i)}(x)\mathrm{d}x\right)\right.$$
$$\left.+(1-\theta)\left(r_0\int_0^\infty\mu_0(x)P_{n+1}^{(0)}(x)\mathrm{d}x+\sum_{i=1}^{m}\int_0^\infty\mu_i(x)P_{n+1}^{(i)}(x)\mathrm{d}x\right)\right]$$
$$+\int_0^\infty\nu(y)V_{n+1}(y)\mathrm{d}y;\ n\geq 0, \tag{4.1}$$

$$P_n^{(i)}(0) = r_i \int_0^{\infty} \mu_0(x) P_n^{(0)}(x)\mathrm{d}x; \quad n \geq 0, \quad 1 \leq i \leq m. \tag{4.2}$$

The boundary conditions at $y = 0; i = 0, 1, 2, \ldots, m$ for fixed value of x are as follows:

$$V_n(0) = p\left[r_0 \int_0^{\infty} \mu_0(x) P_n^{(0)}(x)\mathrm{d}x + \sum_{i=1}^{m} \int_0^{\infty} \mu_i(x) P_n^{(i)}(x)\mathrm{d}x \right]; \quad n \geq 0 \tag{4.3}$$

$$R_n^{(i)}(x; 0) = \alpha_i P_n^{(i)}(x); \quad n \geq 0, \quad i = 0, 1, 2, \ldots, m \tag{4.4}$$

The normalizing condition of the system is defined as

$$P_0^{(0)} + \sum_{n=0}^{\infty} \sum_{i=0}^{m} \left[\int_0^{\infty} P_n^{(i)}(x)\mathrm{d}x + \int_0^{\infty} \int_0^{\infty} R_n^{(i)}(x, y)\mathrm{d}x\mathrm{d}y \right] + \sum_{n01}^{\infty} \int_0^{\infty} V_n(y)\mathrm{d}y = 1 \tag{4.5}$$

The stability condition for the existence of the solution at equilibrium is

$$\rho = \lambda_e E(X) \left\{ E(B_0)(1 + \alpha_0 g_0^{(1)}) + \sum_{i=1}^{m} r_i E(B_i) \left(1 + \alpha_i g_i^{(1)} \right) + pE(V) \right\} < 1;$$

$$\lambda_e = \frac{\lambda(1 - q\theta)}{1 - \chi_1 + \chi_2}$$

$$\chi_1 = q\theta + \lambda E(X) \left\{ E(B_0) \left(b + \alpha_0 g_0^{(1)} b_2 \right) + \sum_{i=1}^{m} r_i E(B_i) \left(b + \alpha_i g_i^{(1)} b_2 \right) + b_1 pE(V) \right\},$$

$$\chi_2 = \lambda E(X) \left\{ E(B_0) \left(1 + \alpha_0 g_0^{(1)} \right) + \sum_{i=1}^{m} r_i E(B_i) \left(1 + \alpha_i g_i^{(1)} \right) + pE(V) \right\}$$

3.3 Queue Size Distribution

(i) The partial probability generating functions of system state and queue size are

$$P^{(0)}(x, z) = \left[\phi_1(z)(1 - \chi_1)[1 - B_0(x)] \exp\{-\tau_0(z)x\} \right] [T]^{-1} \tag{5.1}$$

$$P^{(i)}(x, z) = [r_i \phi_1(z)(1 - \chi_1)\tilde{B}_0(\tau_0(z))[1 - B_i(x)] \exp\{-\tau_i(z)x\}][T]^{-1}; \ 1 \leq i \leq m \tag{5.2}$$

$$V(y,z) = [\phi_1(z)(1-\chi_1)\tilde{B}_0(\tau_0(z))p\left\{r_0 + \sum_{i=1}^{m} r_i \tilde{B}_i(\tau_i(z))\right\}$$
$$[1 - V(y)]\exp\{-\phi_3(z)y\}][T]^{-1} \quad (5.3)$$

$$R^{(0)}(x,y,z) = [\alpha_0\phi_1(z)(1-\chi_1)[1 - B_0(x)]\exp\{-\tau_0(z)x\}[1 - G_0(y)]$$
$$\exp\{-\phi_4(z)y\}][T]^{-1} \quad (5.4)$$

$$R^{(i)}(x,y,z) = [\alpha_i r_i \phi_1(z)(1-\chi_1)\tilde{B}_0(\tau_0(z))[1 - B_i(x)]\exp\{-\tau_i(z)x\}$$
$$[1 - G_i(y)]\exp\{-\phi_4(z)y\}][T]^{-1} \quad (5.5)$$

where $T = [S(z) - z](1 - \chi_1 + \chi_2)$

(ii) The marginal probability distribution of system state probability generating functions is

$$P^{(0)}(z) = [\phi_1(z)(1-\chi_1)[1 - \tilde{B}_0(\tau_0(z))]][T\tau_0(z)]^{-1} \quad (6.1)$$

$$P^{(i)}(z) = [r_i\phi_1(z)(1-\chi_1)\tilde{B}_0(\tau_0(z))[1 - \tilde{B}_i(\tau_i(z))]][T\tau_i(z)]^{-1},\ 1 \le i \le m \quad (6.2)$$

$$V(z) = [\phi_1(z)(1-\chi_1)\tilde{B}_0(\tau_0(z))p\left\{r_0 + \sum_{i=1}^{m} r_i \tilde{B}_i(\tau_i(z))\right\}[1 - \tilde{V}(\phi_3(z))]][T\phi_3(z)]^{-1} \quad (6.3)$$

$$R^{(0)}(z) = [\alpha_0\phi_1(z)(1-\chi_1)[1 - \tilde{B}_0(\tau_0(z))][1 - \tilde{G}_0(\phi_4(z))]][T\tau_0(z)\phi_4(z)]^{-1} \quad (6.4)$$

$$R^{(i)}(z) = [\alpha_i r_i\phi_1(z)(1-\chi_1)\tilde{B}_0(\tau_0(z))[1 - \tilde{B}_i(\tau_i(z))]$$
$$[1 - \tilde{G}_i(\phi_4(z))]][T\tau_i(z)\phi_4(z)]^{-1};\ 1 \le i \le m \quad (6.5)$$

with

$$S(z) = \{q(\theta z + 1 - \theta) + p\tilde{V}(\phi_3(z))\}\tilde{B}_0(\tau_0(z))\left\{r_0 + \sum_{i=1}^{m} r_i \tilde{B}_i(\tau_i(z))\right\}$$

$$\phi_1(z) = \lambda(1 - X(z)),\ \phi_2(z) = \lambda b(1 - X(z)),\ \phi_3(z) = \lambda b_1(1 - X(z)),$$
$$\phi_4(z) = \lambda b_2(1 - X(z))\tau_i(z) = \phi_2(z) + \alpha_i(1 - \tilde{G}_i(\phi_4(z))),\ 0 \le i \le m$$

(iii) The probability generating function of the stationary queue size at arbitrary epoch is

$$\pi_1(z) = \frac{(1-\chi_1)}{1-\chi_1+\chi_2}\left\{1 + \frac{\phi_1(z)}{[S(z)-z]}\left[\frac{[1-\tilde{B}_0(\tau_0(z))]}{\tau_0(z)}\left(\frac{\phi_4(z)+\alpha_0[1-\tilde{G}_0(\phi_4(z))]}{\phi_4(z)}\right)\right.\right.$$
$$+\sum_{i=1}^{m}\frac{r_i\tilde{B}_0(\tau_0(z))[1-\tilde{B}_i(\tau_i(z))]}{\tau_i(z)}\left(\frac{\phi_4(z)+\alpha_i[1-\tilde{G}_i(\phi_4(z))]}{\phi_4(z)}\right)$$
$$\left.\left.+\ \tilde{B}_0(\tau_0(z))p\left\{r_0+\sum_{i=1}^{m}r_i\tilde{B}_i(\tau_i(z))\right\}\frac{[1-\tilde{V}(\phi_3(z))]}{\phi_3(z)}\right]\right\} \tag{7}$$

(iv) The probability generating function of the stationary queue size at departure epoch is

$$\pi_2(z) = \frac{\phi_1(z)(1-\chi_1)\tilde{B}_0(\tau_0(z))\{r_0+\sum_{i=1}^{m}r_i\tilde{B}_i(\tau_i(z)\}(q(\theta z+1-\theta)+p\tilde{V}(\phi_3(z)))}{[S(z)-z]\lambda E(X)(\theta z+1-\theta)} \tag{8}$$

4 Performance Measures

In queueing system, the performance can be observed by the study of the behavior of queueing measures. In this section, we obtain the various performance characteristics such as system state probabilities, mean queue length, mean waiting time, and reliability indices.

4.1 System State Probabilities

The system state probabilities of the different server state are obtained and described as follows:

P_{B_0} (P_{B_i}) The probability that server is busy to provide essential and *i*th optional service.

$P_{R_0}(P_{R_i})$ The probability that server is under repair when it fails during essential service and *i*th optional service.

P_V The probability that server is under optional vacation.

P_I The probability when server is in idle state.

4.2 Mean Queue Length at Arbitrary Epoch

The mean queue length (L_q) at arbitrary epoch is obtained by using

$$L_q = \left.\frac{d\pi_1(z)}{dz}\right|_{z=1} = \frac{1}{(1-\chi_1+\chi_2)}\Bigg[\lambda E(X)\chi_3 + \left(\frac{\lambda E(X^2)}{2} + \frac{\lambda E(X)}{2}\chi_4\right) \left\{E(B_0)(1+\alpha_0 g_0^{(1)}) + \sum_{i=1}^{m} r_i E(B_i)(1+\alpha_i g_i^{(1)}) + pE(V)\right\}\Bigg] \quad (50)$$

with

$$\begin{aligned}\chi_3 = {} & \kappa_0 + \sum_{i=1}^{m} r_i\left[\kappa_i + \lambda E(X)E(B_0)(b+\alpha_0 g_0^{(1)} b_2)E(B_i)(1+\alpha_i g_i^{(1)})\right] \\ & + p\lambda E(X)E(B_0)(b+\alpha_0 g_0^{(1)} b_2)E(V) \\ & + p\sum_{i=1}^{m} r_i \lambda E(X)E(B_i)(b+\alpha_i g_i^{(1)} b_2)E(V) \\ & + p\left[\frac{E(V^2)(\lambda b_1 E(X))^2 + E(V)\lambda b_1 E(X^2)}{2\lambda b_1 E(X)} - \frac{\lambda b_1 E(X^2)}{2E(X)}E(V)\right]\end{aligned}$$

$$\begin{aligned}\chi_4 = {} & pE(V^2)\lambda(\lambda b_1 E(X))^2 + p(\lambda b_1)^2 E(X)E(X^2) \\ & + (2\theta q + pE(V)\lambda b_1 E(X))\lambda E(X)[E(B_0)(b+\alpha_0 g_0^{(1)} b_2) \\ & + \sum_{i=1}^{m} r_i E(B_i)(b+\alpha_i g_i^{(1)} b_2)] + E(B_0^{(2)})(\lambda E(X)(b+\alpha_0 g_0^{(1)} b_2))^2 \\ & + E(B_0)\lambda[bE(X^2) + \alpha_0(g_0^{(2)}\lambda(b_2 E(X))^2 \\ & + g_0^{(1)} b_2 E(X^2))] + 2E(B_0)(\lambda E(X))^2(b+\alpha_0 g_0^{(1)} b_2))\sum_{i=1}^{m} r_i E(B_i)(b+\alpha_i g_i^{(1)} b_2) \\ & + \sum_{i=1}^{m} r_i \Big(E(B_i^{(2)})(\lambda E(X)(b+\alpha_i g_i^{(1)} b_2))^2 \\ & \quad + E(B_i)\lambda[bE(X^2) + \alpha_i(g_i^{(2)}\lambda(b_2 E(X))^2 + g_i^{(1)} b_2 E(X^2))]\Big)\end{aligned}$$

$$\begin{aligned}\kappa_i = {} & \frac{E(B_i)\alpha_i g_i^{(2)}\lambda b_2 E(X)}{2} + (1+\alpha_i g_i^{(1)})\left\{\frac{E(B_i^{(2)})(\lambda E(X)(b+\alpha_i g_i^{(1)} b_2))}{2}\right. \\ & \left. - \frac{E(B_i)\lambda[bE(X^2) + \alpha_i(g_i^{(2)}\lambda(b_2 E(X))^2 + g_i^{(1)} b_2 E(X^2))}{2(\lambda E(X)(b+\alpha_i g_i^{(1)} b_2))}\right\}; \; 0 \le i \le m\end{aligned}$$

4.3 Mean Queue Length at Departure Epoch

The mean queue length (L_D) at departure epoch is

$$L_D = \lambda E(X)E(B_0)(b+\alpha_0 g_0^{(1)} b_2) + \sum_{i=1}^{m} r_i \lambda E(X)E(B_i)(b+\alpha_i g_i^{(1)} b_2) + \theta z + pE(V)\lambda b_1 E(X) - \theta + \frac{E(X^2)}{2E(X)} + \frac{1}{2}\chi_4$$

4.4 Mean Waiting Time

The mean waiting time (W_q) at arbitrary epoch is considered as

$$W_q = \frac{L_q}{\lambda_e E(X)}$$

4.5 Reliability Indices

Reliability indices are obtained in terms of steady state availability of the server and failure frequency of the system.

(a) The availability of the server is

$$A_v = P_0^0 + \sum_{i=0}^{m} \int_0^\infty P^{(i)}(x,1)\mathrm{d}x = P_0^{(0)} + \lim_{z\to 1}\left[\sum_{i=0}^{m} P^{(i)}(z)\right] = \left(1-\chi_1 + \lambda E(X)\left\{E(B_0) + \sum_{i=1}^{m} r_i E(B_i)\right\}\right)(1-\chi_1+\chi_2)^{-1} \quad (8.1)$$

(b) The steady state failure frequency is

$$F_f = \sum_{i=0}^{m} \alpha_i \int_0^\infty P^{(i)}(x,1) = \lim_{z\to 1}\left[\sum_{i=0}^{m} \alpha_i P^{(i)}(z)\right] = \lambda E(X)\{\alpha_0 E(B_0) + \sum_{i=1}^{m} \alpha_i r_i E(B_i)\}(1-\chi_1+\chi_2)^{-1} \quad (8.2)$$

Table 1 Effects of arrival rates and service rates

	$\mu = 3$				$\mu = 3.1$			
	$p = 0.3$		$p = 0.7$		$p = 0.3$		$p = 0.7$	
λ	L_q	W_q	L_q	W_q	L_q	W_q	L_q	W_q
1.5	4.30	2.39	9.40	5.57	4.03	2.22	9.03	5.30
1.6	5.24	2.81	11.28	6.45	4.93	2.61	10.83	6.13
1.7	6.30	3.26	13.38	7.41	5.93	3.03	12.85	7.03
1.8	7.48	3.75	15.72	8.45	7.05	3.49	15.09	8.02
1.9	8.79	4.28	18.32	9.57	8.28	3.98	17.59	9.08

Table 2 Effects of number of optional services and rates of feedback customers

		$m = 2$		$m = 1$		$m = 0$	
	p	L_q	W_q	L_q	W_q	L_q	W_q
$\theta = 0.01$	0.1	0.77	0.47	0.66	0.40	0.10	0.06
	0.2	1.40	0.87	1.28	0.79	0.60	0.35
	0.3	2.06	1.31	1.92	1.20	1.14	0.66
	0.4	2.75	1.77	2.60	1.65	1.70	1.00
	0.5	3.47	2.27	3.30	2.13	2.29	1.38
$\theta = 0.05$	0.1	0.79	0.49	0.69	0.42	0.11	0.06
	0.2	1.43	0.90	1.30	0.81	0.62	0.36
	0.3	2.09	1.33	1.94	1.23	1.15	0.67
	0.4	2.77	1.80	2.62	1.68	1.71	1.02
	0.5	3.49	2.30	3.31	2.16	2.3	1.39

5 Numerical Illustration

In the present section, the numerical illustration for the model is considered. It provides the validity and tractability of the investigation. The effects of parameters on applicability of the analytical results can be examined by using sensitivity analysis.

The following default parameters are taken for the computation purpose, and the effects of various parameters on the performance measures are displayed in Tables 1, 2, 3, and 4.

$$E(X) = 2, \mu_1 = \mu_2 = 2\mu_0, \alpha_0 = 0.01\,, \alpha_1 = 1.5\alpha_0, \alpha_2 = 2\alpha_0,$$
$$r_0 = r_1 = r_1 = 1/3,\ \ v = 5, g_0 = 10, g_1 = g_2 = 15,\ \ \lambda = 1.3, \mu_0 = 3.0,$$
$$b_1 = 0.6, b_2 = 0.5, b_3 = 0.3,\ \ \theta = 0.05$$

(i) **Effects on mean queue length and waiting time**

Table 1 displays the effects of parameters on mean queue length. From the table, we observe that mean queue length and waiting time increase (decrease) with the growth of arrival rates (service rates) of the units for the fixed values of probabilities

Table 3 Effects of failure rates and number of optional services

		$m = 2$		$m = 1$		$m = 0$	
	p	L_q	W_q	L_q	W_q	L_q	W_q
$\alpha_0 = 0.1$	0.1	1.16	0.68	1.03	0.60	0.28	0.15
	0.2	1.92	1.15	1.77	1.04	0.88	0.48
	0.3	2.72	1.66	2.54	1.53	1.52	0.84
	0.4	3.56	2.20	3.36	2.04	2.20	1.24
	0.5	4.43	2.77	4.20	2.60	2.91	1.67
$\alpha_0 = 0.05$	0.1	1.17	0.69	1.04	0.60	0.28	0.15
	0.2	1.94	1.16	1.78	1.05	0.89	0.48
	0.3	2.74	1.67	2.56	1.54	1.53	0.85
	0.4	3.58	2.21	3.37	2.06	2.20	1.24
	0.5	4.45	2.79	4.22	2.62	2.92	1.67

Table 4 Effects of service rates on system state probabilities

μ_0	P_I	P_{B_0}	P_{B_1}	P_{B_2}	P_V	P_{R_0}	P_{R_1}	P_{R_2}
3.1	0.0555	0.5336	0.1186	0.0889	0.1654	0.0005	0.0001	0.0374
3.2	0.0712	0.5218	0.1160	0.0869	0.1669	0.0005	0.0001	0.0365
3.3	0.0862	0.5105	0.1134	0.0851	0.1685	0.0005	0.0001	0.0357
3.4	0.1007	0.4996	0.1110	0.0833	0.1698	0.0005	0.0001	0.0349
3.5	0.1145	0.4892	0.1087	0.0815	0.1712	0.0005	0.0001	0.0343

Table 5 Effects of feedback customers on system state probabilities

θ	P_I	P_{B_0}	P_{B_1}	P_{B_2}	P_V	P_{R_0}	P_{R_1}	P_{R_2}
0.01	0.0509	0.5392	0.1198	0.0899	0.1618	0.0005	0.0001	0.0377
0.02	0.0480	0.5409	0.1202	0.0901	0.1623	0.0005	0.0001	0.0379
0.03	0.0450	0.5426	0.1206	0.0904	0.1628	0.0005	0.0001	0.0379
0.04	0.0420	0.5443	0.1209	0.0907	0.1633	0.0005	0.0001	0.03810
0.05	0.0390	0.5459	0.1213	0.0910	0.1638	0.0005	0.0001	0.03822

to opt optional service. From Table 2 (Table 3), the effects of the number of optional services available in the system and feedback probability of units (failure rates of the server) on mean queue length and waiting time are discussed. It is noticed that with the increments of these parameters, the mean queue length and waiting time increase.

(ii) **Effects on system state probabilities**

The values of system probabilities with the varying values of service rates (feedback probabilitiy) at different server states for fixed values of other parameters are obtained in Table 4 (Table 5).

Table 6 Effects of failure rates on reliability indices

α	$p = 0.1$		$p = 0.5$		$p = 0.9$	
	A_v	F_f	A_v	F_f	A_v	F_f
0.01	0.9636	0.0099	0.8354	0.0091	0.7258	0.0084
0.03	0.9619	0.0296	0.8340	0.0273	0.7246	0.0253
0.05	0.9603	0.0492	0.8326	0.0454	0.7234	0.0421
0.07	0.9586	0.0688	0.8312	0.0635	0.7223	0.0589
0.09	0.9570	0.0884	0.8299	0.0815	0.7211	0.0757

Fig. 1 L_q versus λ for *m*

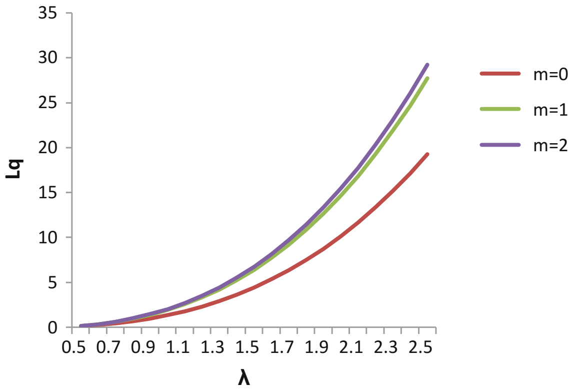
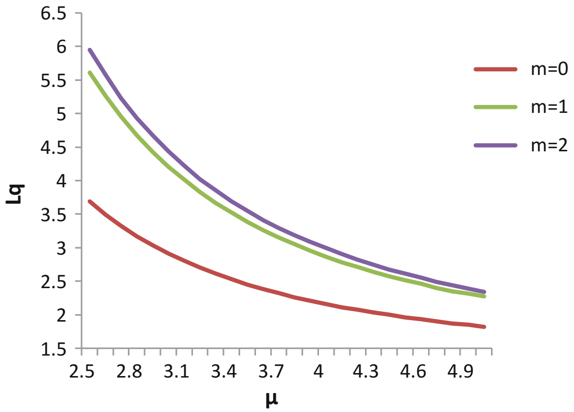


Fig. 2 L_q versus μ for *m*



(iii) **Effects on reliability indices**

From Table 6, it is noticed that the availability of the server and failure frequency decrease (increase) with the growth of failure rates of service system.

From Figs. 1 and 2, the effects of system parameters for different numbers of optional services on performance characteristics are observed.

6 Conclusion

The model investigated in the present study finds the many applications in every sphere on routine activities. It seems to be of enormous utility for the system designers and managements wherein their objective is to find the effects of the performance parameters of new study. Moreover, the concept of number of optional services available in the system and the unpredictable interruption in the service can be omitted by providing the standby components in terms of saving the time and cost. The investigation can be further extended by incorporating multi-server queue and N-policy.

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Performance Analysis of Series Queue with Customer's Blocking

Sreekanth Kolledath and Kamlesh Kumar

Abstract This investigation proposes a model for simple series queues with customer's blocking. We consider a series queue with three service stations as S_1, S_2, S_3 with single server at each station. No queue is allowed to form at any station. Since the model is a sequential model, all the customers require service at each station. The recursive method has been employed to obtain the steady-state probability distribution for the series queueing system analysis. For this purpose, various performance measures, namely the average numbers of customers in the queueing system, the proportion of customers entering the queueing system, average waiting time, have been obtained.

Keywords Series queue · Customers · Blocking · Recursive method
Performance analysis

1 Introduction

A group of stations with finite capacities through which jobs enter in order to gratify their service needs can be considered as a series queue with customer's blocking. It is because of the finite capacity of the individual station so that such phenomenon of blocking takes place. Blocking of customer happens when limitations are enforced upon the queue sizes (i.e., finite queues). It is complicated to solve series queue with blocking; generally speaking, the steady-state queue length distributions of series queue cannot be revealed to have product form solutions. Therefore, the majority of the methods used to analyze the series queue are in the form of approximations, simulation, and numerical methods. In the modeling of production lines, emergency healthcare units, and performance evaluation of computer systems, the series queues with customer's blocking have been of great use.

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Due to the widespread prevalence of series queues with customer's blocking and their importance in real life, many researchers have found a great deal of attention in the literature. Onvural [14], Perros [16], Hall and Sriskandarajah [10] have written significant survey works on series queues with blocking and showed great influence. Hunt [12] first time depicted the customer's blocking effects in a sequence of waiting lines. After that, [1] had studied the queueing system with arbitrary input and regular service times' parameters. Issues of series queues with blocking have been addressed by numerous research publications dating back to the late 1960s. Suzuki [18] considered Poisson arrivals in the case of two stations in series queue, arbitrarily distributed service times for both stations and also derived the expressions for the waiting time distribution. Avi-Itzhak and Yadin [5] have investigated the series queues without buffers and blocking of the customers.

The case of blocking generally refers to the set of rules that dictates when a node becomes blocked and unblocked. Several blocking mechanisms including blocking after service (BAS), blocking before service (BBS), repetitive service (RS) have been put forward in the literature in order to represent various queueing system. Generally, three types of blocking are regarded as **blocking after service**; it means that the customer gets blocked at station A even after completion of its service and it happens due to the unavailability of vacant space at station B. Customer can move from station A to station B only when station B is free for service. **Blocking before service** is the case in which the service of customer at station A will remain blocked unless there is an available space in station B. The customer will get service at station A only when the station B is free. Here for getting service, the customer at station A is completely dependent on the availability of station B. **Repetitive service** is the case in which the customer at station A gets service over and over again until the station B is available for service. In the literature, there are fundamentally only a few blocking mechanisms which have been studied by many authors such as [4, 6–8]. Onvural and Perros [15] have discussed these types of blocking of customers in the queueing system analysis. Akyildiz and von Brand [3] have considered a two station tandem queue with blocking after service and provided an accurate solution with correct stationary distribution.

Gomez-Corral [9] examined a series queue with blocking and repeated attempts by employing matrix geometric approximation method. Avrachenkov and Yechiali [2] considered a series queue with a common retrial queue. Based on mean value analysis and fixed point approach, they proposed an approximation procedure. Houdta and Alfa [11] have examined a series queues with blocking, Markovian arrivals, and phase-type service. They presented a novel approach to obtain the response time. Lekadir and Aissani [13] studied a series queue with blocking and non-preemptive priority. Shin and Moon [17] developed a multi-server series queueing model with blocking in which the service times are exponential. They also developed an approximation method for throughput of queueing system. Zhanga et al. [19] studied a series queue with blocking and evaluated several performance indices by using the rate iterative method embedded with generalized expansion method.

The remaining part of this paper is organized as follows. In Sect. 2, we describe the model in detail. Section 3 provides the steady-state solutions of the system governing equations by using recursive method. In Sect. 4, various performance indices are presented. Section 5 contains concluding remarks for series queue model with customer's blocking.

2 Model Description

A simple series queue is analyzed with three service stations as S_1, S_2, S_3 with the provision of single server at each station is taken into consideration. At each station, no queue is allowed to be formed. As this model is sequential, therefore a customer has to require service from each station. A customer is allowed to enter into the system only when station S_1 is empty irrespective of whether S_2 and S_3 are empty or not. After completing the service at S_1, the customer will go to S_2 and then go to S_3. Customer leaves the queueing system after getting the service at station S_3. A customer completing service at S_1 will go to S_2 if it is empty or will wait in S_1 until S_2, becomes empty; i.e, the station S_1 is blocked for a new customer. Likewise, a customer after completing service at S_2 will go to S_3 if it is empty or will wait in S_2 until S_3 becomes empty; i.e, the station S_2 is blocked for a new customer. The arrival of customers is turned away if a customer is in process at station S_1 or if S_2 is blocked.

It is assumed that the customers are arrived in the queueing system in accordance with a Poisson fashion with parameter λ and the service times at S_1, S_2, S_3 follow exponential distributions with parameters μ_1, μ_2, μ_3, respectively.

Notations

To formulate the series queueing model with customer's blocking, the following notations are used. The steady-state probabilities $P_{h,i,j}$ define that there are h number of customers ($h = 0$, or $h = 1$) at station S_1, i customer ($i = 0$, or $i = 1$) in S_2 and j customer ($j = 0$, or $j = 1$) in S_3. State of the model is denoted by triplet (h, i, j), and thus, the possible states of the queueing systems are

- $P_{0,0,0}$ No customer at any station S_1, S_2 and S_3.
- $P_{1,0,0}$ One customer is getting service at station S_1 and stations S_2 and S_3 are free.
- $P_{0,1,0}$ No customer at station S_1 and S_3, but one customer is getting service at station S_2.
- $P_{0,0,1}$ No customer at station S_1 and S_2, but one customer is getting service at station S_3.
- $P_{1,1,0}$ One customer at station S_1 and one customer at station S_2 are getting service and no customer at station S_3.
- $P_{1,0,1}$ One customer is getting service at station S_1 and no customer at station S_2 and also one customer getting service at station S_3.

$P_{0,1,1}$ No customer at station S_1, but one customer at station S_2 and one customer at station S_3 are getting service.

$P_{1,1,1}$ One customer at station S_1, one customer at station S_2 and one customer at S_3 are getting service (All the three stations are busy).

$P_{b,1,0}$ One customer after completing service blocked at station S_1, but one customer at station S_2 is getting service and no customer at station S_3.

$P_{0,b,1}$ No customer at station S_1, one customer at station S_2 completed service and blocked and one customer is getting service at station S_3.

$P_{b,1,1}$ One customer after completing service blocked at station S_1, but one customer at station S_2 and one customer at station S_3 are getting service.

$P_{b,b,1}$ One customer at station S_1 and one customer at station S_2 are blocked, but one customer at station S_3 is getting service.

$P_{1,b,1}$ One customer at station S_1 and one customer at station S_3 are getting service, but after completing service one customer is blocked at station S_3.

3 Governing Equations

By using the transition state diagram depicted in Fig. 1, the steady-state balance equations can be written as

$$\lambda P_{0,0,0} = \mu_3 P_{0,0,1} \tag{1}$$

$$\mu_1 P_{1,0,0} = \lambda P_{0,0,0} + \mu_3 P_{1,0,1} \tag{2}$$

$$(\lambda + \mu_2) P_{0,1,0} = \mu_1 P_{1,0,0} + \mu_3 P_{0,1,1} \tag{3}$$

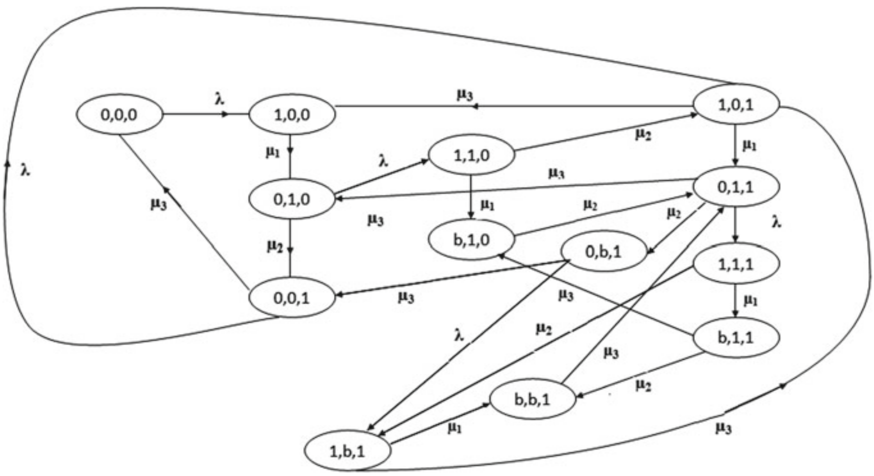

Fig. 1 State transition diagram

$$(\lambda + \mu_3)P_{0,0,1} = \mu_2 P_{0,1,0} + \mu_3 P_{0,b,1} \tag{4}$$

$$(\mu_1 + \mu_2)P_{1,1,0} = \lambda P_{0,1,0} \tag{5}$$

$$\mu_2 P_{b,1,0} = \mu_1 P_{1,1,0} + \mu_3 P_{b,1,1} \tag{6}$$

$$(\mu_1 + \mu_3)P_{1,0,1} = \mu_2 P_{1,1,0} + \mu_3 P_{1,b,1} + \lambda P_{0,0,1} \tag{7}$$

$$(\lambda + \mu_2 + \mu_3)P_{0,1,1} = \mu_1 P_{1,0,1} + \mu_2 P_{b,1,0} + \mu_3 P_{b,b,1} \tag{8}$$

$$(\mu_1 + \mu_2)P_{1,1,1} = \lambda P_{0,1,1} \tag{9}$$

$$(\mu_2 + \mu_3)P_{b,1,1} = \mu_1 P_{1,1,1} \tag{10}$$

$$\mu_3 P_{b,b,1} = \mu_1 P_{1,b,1} + \mu_2 P_{b,1,1} \tag{11}$$

$$(\mu_1 + \mu_3)P_{1,b,1} = \lambda P_{0,b,1} + \mu_2 P_{1,1,1} \tag{12}$$

$$(\lambda + \mu_3)P_{0,b,1} = \mu_2 P_{0,1,1} \tag{13}$$

From (1), we get

$$P_{0,0,1} = \frac{\lambda}{\mu_3} P_{0,0,0} \tag{14}$$

By substituting $P_{0,0,1}$ in (4), and using (13)

$$P_{0,1,0} = \frac{\lambda(\lambda + \mu_3)}{\mu_2 \mu_3} P_{0,0,0} - \frac{\mu_3}{\lambda + \mu_3} P_{0,1,1} \tag{15}$$

Using (5) and (15), we obtain

$$P_{1,1,0} = \frac{\lambda^2(\lambda + \mu_3)}{\mu_2 \mu_3 (\mu_1 + \mu_2)} P_{0,0,0} - \frac{\lambda \mu_3}{(\lambda + \mu_3)(\mu_1 + \mu_2)} P_{0,1,1} \tag{16}$$

Also from (9)

$$P_{1,1,1} = \frac{\lambda}{\mu_1 + \mu_2} P_{0,1,1} \tag{17}$$

Substitute the value of $P_{1,1,1}$ from (17) in (10), we get

$$P_{b,1,1} = \frac{\lambda \mu_1}{(\mu_1 + \mu_2)(\mu_2 + \mu_3)} P_{0,1,1} \tag{18}$$

By substituting $P_{0,1,0}$ in (3), we get

$$P_{1,0,0} = \frac{\lambda(\lambda + \mu_2)(\lambda + \mu_3)}{\mu_1 \mu_2 \mu_3} P_{0,0,0} - \mu_3 \left[\frac{(2\lambda + \mu_2 + \mu_3)}{\mu_1} \right] P_{0,1,1} \tag{19}$$

By using the above $P_{1,0,0}$ in (2),we get

$$P_{1,0,1} = \frac{\lambda(\lambda^2 + \lambda\mu_2 + \mu_3)}{\mu_2\mu_3^2}P_{0,0,0} - \frac{(2\lambda + \mu_2 + \mu_3)}{\mu_1}P_{0,1,1} \tag{20}$$

Substituting $P_{b,1,1}$ and $P_{1,1,0}$ in (6), we get

$$P_{b,1,0} = \frac{\lambda\mu_1\mu_3(\lambda - \mu_2)}{\mu_2(\mu_1 + \mu_2)(\mu_2 + \mu_3)(\lambda + \mu_3)}P_{0,1,1} - \frac{\lambda^2\mu_1(\lambda + \mu_3)}{\mu_2^2\mu_3(\mu_1 + \mu_2)}P_{0,0,0} \tag{21}$$

From (13), we have

$$P_{0,b,1} = \frac{\mu_2}{\lambda + \mu_3}P_{0,1,1} \tag{22}$$

Using $P_{0,b,1}$ and $P_{1,1,1}$ in (12), we get

$$P_{1,b,1} = \lambda\mu_2\left[\frac{(\lambda + \mu_1 + \mu_2 + \mu_3)}{(\mu_1 + \mu_2)(\mu_1 + \mu_3)(\lambda + \mu_3)}\right]P_{0,1,1} \tag{23}$$

In (7) by using $P_{0,1,1}$, $P_{1,0,1}$, $P_{1,1,0}$, $P_{1,b,1}$, we get

$$P_{0,1,1} = \left[\frac{(\mu_1 + \mu_3)(\lambda + \mu_3)(\mu_1 + \mu_2)\{(\mu_1 + \mu_3)(\lambda^3 + \lambda^2\mu_2 + \lambda\mu_3) - 1\}}{\lambda\mu_1\mu_2^2\mu_3^3(\lambda + \mu_2) + [(\mu_2\mu_3^2)(\mu_1 + \mu_2)(\mu_1 + \mu_3)^2(\lambda + \mu_3)(2\lambda + \mu_2 + \mu_3)]}\right.$$
$$\left. - \frac{(\mu_1 + \mu_3)(\lambda + \mu_3)\lambda^2\mu_1\mu_2\mu_3(\lambda + \mu_3)}{\lambda\mu_1\mu_2^2\mu_3^3(\lambda + \mu_2) + [(\mu_2\mu_3^2)(\mu_1 + \mu_2)(\mu_1 + \mu_3)^2(\lambda + \mu_3)(2\lambda + \mu_2 + \mu_3)]}\right]P_{0,0,0}$$

$$P_{0,1,1} = \delta P_{0,0,0} \tag{24}$$

where

$$\delta = \frac{(\mu_1 + \mu_3)(\lambda + \mu_3)[(\mu_1 + \mu_2)\{(\mu_1 + \mu_3)(\lambda^3 + \lambda^2\mu_2 + \lambda\mu_3) - 1\} - \lambda^2\mu_1\mu_2\mu_3(\lambda + \mu_3)]}{\lambda\mu_1\mu_2^2\mu_3^3(\lambda + \mu_2) + [(\mu_2\mu_3^2)(\mu_1 + \mu_2)(\mu_1 + \mu_3)^2(\lambda + \mu_3)(2\lambda + \mu_2 + \mu_3)]}$$

By substituting the value of $P_{0,1,1}$ in Eqs. (16)–(23), respectively, we get the following probabilities.

$$P_{0,b,1} = \left[\frac{\mu_2\delta}{\lambda + \mu_3}\right]P_{0,0,0}$$
$$P_{0,1,0} = \left[\frac{\lambda(\lambda + \mu_3)^2 - \mu_2\mu_3^2\delta}{\mu_2\mu_3(\lambda + \mu_3)}\right]P_{0,0,0}$$
$$P_{1,1,0} = \left[\frac{\lambda(\lambda + \mu_3)^2 - \mu_2\mu_3^2\delta}{\mu_2\mu_3(\mu_1 + \mu_2)(\lambda + \mu_3)}\right]\lambda P_{0,0,0}$$

$$P_{1,1,1}=\left[\frac{\lambda\delta}{(\mu_1+\mu_2)}\right]P_{0,0,0}$$
$$P_{b,1,1}=\left[\frac{\lambda\mu_1\delta}{(\mu_1+\mu_2)(\mu_2+\mu_3)}\right]P_{0,0,0}$$
$$P_{1,0,0}=\left[\frac{\lambda(\lambda+\mu_2)(\lambda+\mu_3)-(2\lambda+\mu_2+\mu_3)\mu_2\mu_3^2\delta}{\mu_1\mu_2\mu_3}\right]P_{0,0,0}$$
$$P_{1,0,1}=\left[\frac{\mu_1\lambda(\lambda^2+\lambda\mu_2+\mu_3)-(2\lambda+\mu_2+\mu_3)\mu_2\mu_3^2\delta}{\mu_1\mu_2\mu_3}\right]P_{0,0,0}$$
$$P_{b,1,0}=\left[\frac{\lambda(\lambda-\mu_2)\mu_1\mu_2\mu_3^2\delta-\lambda^2\mu_1(\lambda+\mu_3)^2(\mu_1+\mu_3)}{\mu_2^2\mu_3(\mu_1+\mu_2)(\mu_2+\mu_3)(\lambda+\mu_3)}\right]P_{0,0,0}$$
$$P_{0,b,1}=\left[\frac{\mu_2\delta}{(\lambda+\mu_3)}\right]P_{0,0,0}$$
$$P_{b,b,1}=\left[\frac{(\mu_2+\mu_3)(\mu_1+\mu_2+\mu_3+\lambda)+(\mu_1+\mu_3)(\lambda+\mu_3)}{\mu_3(\mu_1+\mu_2)(\mu_1+\mu_3)(\mu_2+\mu_3)(\lambda+\mu_3)}\right]\lambda\mu_1\mu_2\delta P_{0,0,0}$$
$$P_{1,b,1}=\left[\frac{\lambda\mu_2\delta(\mu_1+\mu_2+\mu_3+\lambda)}{(\mu_1+\mu_2)(\mu_1+\mu_3)(\lambda+\mu_3)}\right]P_{0,0,0}$$

Using the normalized condition, we get

$$\begin{aligned}
&P_{0,0,0}+P_{1,0,0}+P_{0,1,0}+P_{0,0,1}+P_{1,0,1}+P_{0,1,1}+P_{1,1,1}+P_{b,1,1}\\
&\quad+P_{1,1,0}+P_{b,1,0}+P_{0,b,1}+P_{b,b,1}+P_{1,b,1}=1\\
&\quad\Rightarrow P_{0,0,0}+\left[\frac{\lambda(\lambda+\mu_2)(\lambda+\mu_3)-(2\lambda+\mu_2+\mu_3)\mu_2\mu_3^2\delta}{\mu_1\mu_2\mu_3}\right]P_{0,0,0}\\
&\quad+\left[\frac{\lambda(\lambda+\mu_3)^2-\mu_2\mu_3^2\delta}{\mu_2\mu_3(\lambda+\mu_3)}\right]P_{0,0,0}\\
&\quad+\frac{\lambda}{\mu_3}P_{0,0,0}+\left[\frac{\mu_1\lambda(\lambda^2+\lambda\mu_2+\mu_3)-(2\lambda+\mu_2+\mu_3)\mu_2\mu_3^2\delta}{\mu_1\mu_2\mu_3}\right]P_{0,0,0}\\
&\quad+\delta P_{0,0,0}+\left[\frac{\lambda\delta}{(\mu_1+\mu_2)}\right]P_{0,0,0}+\left[\frac{\lambda\mu_1\delta}{(\mu_1+\mu_2)(\mu_2+\mu_3)}\right]P_{0,0,0}\\
&\quad+\left[\frac{\lambda(\lambda+\mu_3)^2-\mu_2\mu_3^2\delta}{\mu_2\mu_3(\mu_1+\mu_2)(\lambda+\mu_3)}\right]\lambda P_{0,0,0}\\
&\quad+\left[\frac{\lambda(\lambda-\mu_2)\mu_1\mu_2\mu_3^2\delta-\lambda^2\mu_1(\lambda+\mu_3)^2(\mu_1+\mu_3)}{\mu_2^2\mu_3(\mu_1+\mu_2)(\mu_2+\mu_3)(\lambda+\mu_3)}\right]P_{0,0,0}+\left[\frac{\mu_2\delta}{(\lambda+\mu_3)}\right]P_{0,0,0}\\
&\quad+\left[\frac{(\mu_2+\mu_3)(\mu_1+\mu_2+\mu_3+\lambda)+(\mu_1+\mu_3)(\lambda+\mu_3)}{\mu_3(\mu_1+\mu_2)(\mu_1+\mu_3)(\mu_2+\mu_3)(\lambda+\mu_3)}\right]\lambda\mu_1\mu_2\delta P_{0,0,0}\\
&\quad+\left[\frac{\lambda\mu_2\delta(\mu_1+\mu_2+\mu_3+\lambda)}{(\mu_1+\mu_2)(\mu_1+\mu_3)(\lambda+\mu_3)}\right]P_{0,0,0}=1
\end{aligned}$$

$$\Rightarrow P_{0,0,0} = \left[1 + \frac{\lambda}{\mu_3} + \frac{1}{\mu_2\mu_3}\alpha + \beta\right]^{-1} \tag{25}$$

where

$$\begin{aligned}\alpha = &\frac{\lambda(\lambda+\mu_2)(\lambda+\mu_3) - (3\lambda+\mu_2+2\mu_3)\mu_2\mu_3^2\delta}{\mu_1} + \frac{\lambda(\lambda+\mu_3)^2 - \mu_2\mu_3^2\delta}{\lambda+\mu_3} \\ &+ \frac{\mu_1\lambda(\lambda^2+\lambda\mu_2+\mu_3) - (2\lambda+\mu_2+\mu_3)\mu_2\mu_3^2\delta}{\mu_1\mu_3} + \frac{\lambda(\lambda(\lambda+\mu_3)^2 - \mu_2\mu_3^2\delta)}{(\mu_1+\mu_2)(\lambda+\mu_3)} \\ &+ \frac{\lambda(\lambda-\mu_2)\mu_1\mu_3^2\delta - \lambda^2\mu_1(\lambda+\mu_3)^2(\mu_1+\mu_3)}{\mu_2(\mu_1+\mu_2)(\mu_1+\mu_3)(\lambda+\mu_3)}\end{aligned}$$

and

$$\begin{aligned}\beta = &\frac{\mu_3(\mu_1+\mu_3)(\lambda+\mu_3)[\lambda(\mu_1+\mu_2+\mu_3) + (\mu_1+\mu_2)(\mu_2+\mu_3)(1+\mu_2)]}{\mu_3(\mu_1+\mu_2)(\mu_1+\mu_3)(\mu_2+\mu_3)(\lambda+\mu_3)} \\ &+ \frac{(\lambda+\mu_1)\lambda\mu_1\mu_2 + \lambda\mu_2(\mu_1+\mu_2+\mu_3+\lambda)[\mu_1(\mu_2+\mu_3)+\mu_3]}{\mu_3(\mu_1+\mu_2)(\mu_1+\mu_3)(\mu_2+\mu_3)(\lambda+\mu_3)}\end{aligned}$$

4 Performance Measures

For examining the series queueing system behavior, various performance indices are constructed using steady-state probabilities as:

(i) The proportion of customer entering the system $Q_s = P_{0,0,0} + P_{0,1,0} + P_{0,0,1}$

$$Q_s = \left[1 + \frac{\lambda(\lambda+\mu_3)^2 - \mu_2\mu_3^2\delta}{\mu_2\mu_3(\lambda+\mu_3)} + \frac{\lambda}{\mu_2}\right]\left[1 + \frac{\lambda}{\mu_3} + \frac{1}{\mu_2\mu_3}\alpha + \beta\right]^{-1} \tag{26}$$

(ii) The average number of customers in the system:

(a) If only one customer is available in the system, then its probability is $P_{1,0,0} + P_{0,1,0} + P_{0,0,1}$
(b) If only two customers are available in the system, then its probability is $P_{1,0,1} + P_{0,1,1} + P_{1,1,0} + P_{b,1,0} + P_{0,b,1}$
(c) If only three customers are available in the system, then its probability is $P_{1,1,1} + P_{b,1,1} + P_{b,b,1} + P_{1,b,1}$

Hence, the average number of customer in the system is

$$\begin{aligned}L_s = &1(P_{1,0,0} + P_{0,1,0} + P_{0,0,1}) + 2(P_{1,0,1} + P_{0,1,1} + P_{1,1,0} + P_{b,1,0} + P_{0,b,1}) \\ &+3(P_{1,1,1} + P_{b,1,1} + P_{b,b,1} + P_{1,b,1})\end{aligned}$$

$$= \Bigg\{ \left[\frac{\lambda(\lambda+\mu_2)(\lambda+\mu_3) - 3(2\lambda+\mu_2+\mu_3)\mu_2\mu_3^2\delta}{\mu_1\mu_2\mu_3} \right] + \Bigg[2 + \frac{2\mu_2}{\lambda+\mu_2}$$
$$+ \frac{3\lambda}{\mu_1+\mu_2} + \frac{3\lambda\mu_1}{(\mu_1+\mu_2)(\mu_1+\mu+3)} \Bigg] \delta + \left[\frac{\lambda(\lambda+\mu_3)^2 - \mu_2\mu_3^2\delta}{\mu_2\mu_3(\lambda+\mu_3)} \right]$$
$$+ \frac{\lambda}{\mu_3} + 2 \left[\frac{\lambda(\lambda+\mu_3)^2 - \mu_2\mu_3^2\delta}{\mu_2\mu_3(\mu_1+\mu_2)(\lambda+\mu_3)} \right] \lambda$$
$$+2 \left[\frac{\lambda(\lambda-\mu_2)\mu_1\mu_2\mu_3^2\delta - \lambda^2\mu_1(\lambda+\mu_3)^2(\mu_1+\mu_3)}{\mu_2^2\mu_3(\mu_1+\mu_2)(\mu_2+\mu_3)(\lambda+\mu_3)} \right]$$
$$+3 \left[\frac{(\mu_2+\mu_3)(\mu_1+\mu_2+\mu_3+\lambda) + (\mu_1+\mu_3)(\lambda+\mu_3)}{\mu_3(\mu_1+\mu_2)(\mu_1+\mu_3)(\mu_2+\mu_3)(\lambda+\mu_3)} \right] \lambda\mu_1\mu_2\delta$$
$$+3 \left[\frac{\lambda\mu_2\delta(\mu_1+\mu_2+\mu_3+\lambda)}{(\mu_1+\mu_2)(\mu_1+\mu_3)(\lambda+\mu_3)} \right] \Bigg\} \left[1 + \frac{\lambda}{\mu_3} + \frac{1}{\mu_2\mu_3}\alpha + \beta \right]^{-1} \tag{27}$$

(iii) Average amount of time spent by the customer in the system

$$W_s = \frac{L_s}{\lambda Q_s}$$
$$= \frac{L_s}{\lambda \left[1 + \frac{\lambda(\lambda+\mu_3)^2 - \mu_2\mu_3^2\delta}{\mu_2\mu_3(\lambda+\mu_3)} + \frac{\lambda}{\mu_2} \right] \left[1 + \frac{\lambda}{\mu_3} + \frac{1}{\mu_2\mu_3}\alpha + \beta \right]^{-1}} \tag{28}$$

(iv) The probability that all the stations are free

$$P_{0,0,0} = \left[1 + \frac{\lambda}{\mu_3} + \frac{1}{\mu_2\mu_3}\alpha + \beta \right]^{-1} \tag{29}$$

(v) The probability that station S_1 is busy and S_2, and S_3 are free

$$P_{1,0,0} = \left[\frac{\lambda(\lambda+\mu_2)(\lambda+\mu_3) - (2\lambda+\mu_2+\mu_3)\mu_2\mu_3^2\delta}{\mu_1\mu_2\mu_3} \right] P_{0,0,0} \tag{30}$$

(vi) The probability that no customer at S_1, and S_2 and S_3 are busy

$$P_{0,0,1} = \frac{\lambda}{\mu_3} P_{0,0,0} \tag{31}$$

(vii) The probability that all the stations S_1, S_2, S_3 are busy

$$P_{1,1,1} = \left[\frac{\lambda\delta}{(\mu_1+\mu_2)} \right] P_{0,0,0} \tag{32}$$

(viii) The probability that S_1 is free, but a customer is waiting in S_1 and S_2, S_3 are busy

$$P_{b,1,1} = \left[\frac{\lambda \mu_1 \delta}{(\mu_1 + \mu_2)(\mu_2 + \mu_3)} \right] P_{0,0,0} \tag{33}$$

5 Conclusions

In this paper, the analysis of a series queue with three service stations and customer's blocking has been done. By using the recursive method, we have established the formulae for various performance measures. We can further extend this paper by increasing the number of service stations for the customers to study the series queue with customer's blocking. Also, it may be noticed that as the number of stations increases, it becomes complicated to solve the governing equations of the series queueing model by using recursive method. Therefore, in order to solve the system governing equations, the numerical techniques or approximation methods are more easy to use rather than applying the recursive method approach. It would be more complicated to get the problem solutions for the series queue with more than three service stations.

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Markovian Multi-server Queue with Reneging and Provision of Additional Removable Servers

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Abstract Many queuing situations with discouragement are encountered in real life in which the customers do not wait for a longer time and may renege from the system after waiting for some time. To avoid this reneging behavior of the customers, the provision of additional removable servers may be helpful. From economic viewpoint, the additional servers may be installed or removed from the system according to a threshold policy on the basis of number of clients present in the system. In this paper, a steady state solution of a finite M/M/R+r Markov model with discouragement and operating under threshold policy for inducting the additional removable r servers along with R permanent servers is presented. The first-come first-served discipline is followed in providing services to the customers. The queue size distribution at the equilibrium is obtained using a recursive approach. The formulations of various performance measures, namely expected number of customers in the system, throughput, have been done in terms of system state probabilities. By taking numerical illustration, the system behavior via sensitivity analysis and cost optimization has been examined.

Keywords Markov finite queue · Reneging · Additional servers · Queue size
Cost optimization

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1 Introduction

Due to impatient nature of the customers, the arrivals in the system may or may not like to join the long queues. To overcome this problem, the facility of additional servers in the system can be made according to a threshold policy based on the number of customers present in the system. In case of additional provisioning of servers according to threshold rule, as soon as the number of customers in the system becomes greater than the threshold value of the customers, the additional servers are added into the system. Also, the additional servers are removed from the system as the number of customers present in the system decreases below the threshold value. In this investigation, Markov queueing model with crew of R permanent and r additional servers and reneging has been investigated.

Sometimes, the impatient customers standing in the long queues leave the system due to their impatient behavior. The impatient behavior of the customers plays a vital role while dealing with realistic situations of congestion problems encountered in day-to-day as well as industrial environment. It has been observed that when the queue size becomes larger, the arriving customers may be discouraged and do not like to wait. The queueing situation with reneging for Markovian system was first investigated by Haight [1]. Another important early past contribution on M/M/1/N queuing problem with balking and reneging was presented by Ancker and Gafarian [2]. Significant work on Markov queueing system has been done by Courtois and Georges [3] by considering the state-dependent arrival and service process. Natvig [4] and Robert [5] have also contributed significantly in the direction of the development of queuing models with reneging.

The queueing modeling and performance measures for an M/M/1 and M/M/c/K queues with balking and reneging have been presented by Gross and Harris [6] and Abou-el-ata et al. [7]. Cost analysis for the finite M/M/R queuing system with balking, reneging, and server breakdowns was given by Wang and Chang [8]. In queueing literature, Yue et al. [9] and many others have dealt with finite capacity multi-server queue with reneging. Perel and Yechiali [10] obtained the performance indices for the M/M/c queueing system with impatient customers by considering the noble idea of the provision of slow servers. The performance prediction of M/M/1/N queue with the provision of retention of reneged customers was done by Kumar and Sharma [11]. The queueing analysis of finite capacity queueing system with reneging was also presented by Ying-Yi et al. [12].

In the present investigation, the performance indices and cost analysis are provided to analyze the behavior of reneged customers in the queueing system with the provision of installing and removing additional servers in the system according to a threshold policy. Section 2 outlines the assumptions and notations that would be used further for the model description. In Sect. 3, queue size distribution has been obtained. In Sect. 4, performance measures have been established. In Sect. 5, numerical illustrations are given by taking an example. All the numerical results derived in previous sections are validated for the finite capacity model. Finally, Sect. 6 highlights the noble features of the present study and future scope.

2 Model Description

The performance analysis of M/M/R + r Markovian queue model is done by including the concept of reneging behavior of the customers. The following assumptions have been made for developing Markov model:

- The customers join the system in Poisson fashion, and the arrival rate for the finite capacity model is denoted by λ.
- The service provided to the customers follows an exponential distribution, having the following density function (Fig. 1)

$$Z(t) = \mu \ \exp(-\mu t); \, t \geq 0, \ \mu \geq 0$$

- If there are k customers such that $k < R$, then only k servers provide service with a rate μ.
- If the number of customers in the system are k such that $K_j < k \leq K_{j+1}$, then all R servers and j additional servers will provide the service.
- The customer while waiting in the queue may get impatient and departs from the system following exponential process with rate υ.

The service rate for the finite capacity Markov model with reneging is state-dependent and can be defined based on number of available servers and number of customers present in the system. To define effective service rate, the notations used are as follows:

$$\mu^{(j)} = \sum_{i=1}^{j} \mu_i, \quad \psi_j = R\mu + \mu^{(j)}, \quad \psi_r = R\mu + \mu^{(r)}$$

$$\eta_{0,k} = R\mu + (k - R)\nu, \quad \eta_{j,k} = R\mu + (k - (R + j))\nu_j + \mu^j$$

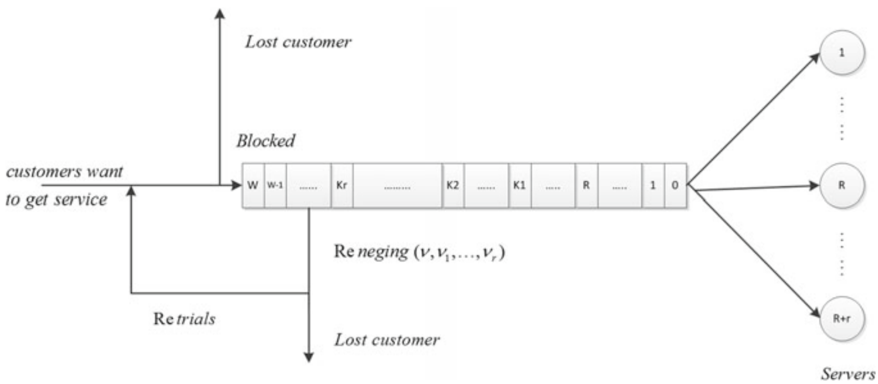

Fig. 1 Additional server queue with reneging and retrial attempts

If the number of customers in the system are $k\,(< R)$, then the customers depart from the system after service with a rate $k\mu$. If the number of customers is k such that $R < k \leq K_1$, then all R permanent servers will be busy and $k - R$ customers will wait in the queue. In this case, the customer may renege with the rate v resulting in a effective service rate $R_\mu + (k - R)v$. Similarly, we define the service rate when additional servers are busy. Thus, the overall effective service rate is defined by-

$$\mu_k = \begin{cases} k\mu; \ \ 1 \leq k \leq R \\ \eta_{0,k}; \ R < k \leq K_1 \\ \eta_{j,k}; \ K_j < k \leq K_{j+1}; \ \ 1 \leq j \leq r-1 \\ \eta_{r,k}; \ \ K_r < k \leq W \end{cases}$$

3 Queue Size Distribution

The formulation of the queue size distribution at the equilibrium and the expected number of customers in the system have been done for the finite capacity model by considering the reneging factor into effect. A product type solution is obtained by using the appropriate arrival and service rates. Let p_k and p_0 denote the probability that there are k number of customers present in the system and the probability that there are no customers in the system, respectively. Thus, to establish the queue size distribution, we use the product type solution given by (Fig. 2)

$$p_k = p_0 \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}} \tag{1}$$

where the values of p_0 can be calculated as follows:

$$p_0 = \left[1 + \sum_{R=1}^{W} \left(\prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}} \right) \right]^{-1} \tag{2}$$

The steady state probabilities p_k which represent the probability of k customers in the system have been obtained for finite capacity model by using birth and death

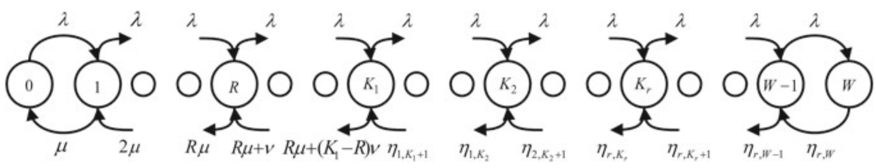

Fig. 2 Transition diagram

rates. The birth rate λ_k accounts for the transition of state k into a $k+1$ state. If the system is in k state and a customer departs the system after receiving the service or due to his impatient behavior, this results in the transition from k state to $k-1$ with a rate μ_k. Thus, we obtain

$$p_k = \begin{cases} \frac{\lambda^k}{k!\mu^k} p_0 & ; k \le R \\ \frac{\lambda^k}{R!\mu^R \left(\eta_{0,k}\right)^{k-R}} p_0 & ; R < k \le K_1 \\ \frac{\lambda^k}{R!\mu^R \left(\eta_{0,k}\right)^{K_1-R} \left(\eta_{j,k}\right)^{k-K_j} \prod\limits_{i=1}^{j-1} \left(\eta_{i,k}\right)^{K_{i+1}-K_i}} p_0 & ; K_j < k \le K_{j+1};\ 1 \le j \le r-1 \\ \frac{\lambda^k}{R!\mu^R \left(\eta_{0,k}\right)^{K_1-R} \left(\eta_{r,k}\right)^{k-K_r} \prod\limits_{i=1}^{r-1} \left(\eta_{i,k}\right)^{K_{i+1}-K_i}} p_0 & ; K_r < k \le W \end{cases} . \tag{3}$$

4 Performance Measures

The performance metrics, namely carried load, throughput, expected number of customers in the system as well as in the queue, have been established. The formulations of the expected waiting time of the customers in system and in the queue are done in this section. Furthermore, the probability of reneging and probability that permanent and additional servers being busy are provided. Finally, a cost function and the optimum value of the function have been computed.

4.1 Carried Load (λ_{eff})

The effective arrival rate (λ_{eff}) is obtained as

$$\begin{aligned} (\lambda_{\text{eff}}) &= \sum_{k=0}^{W-1} \lambda_k p_k = \lambda \sum_{k=0}^{W-1} p_k = \lambda(1 - p_R) \\ &= \lambda \left[1 - \frac{\lambda^w}{R!\mu^R \left(\eta_{0,k}\right)^{K_1-R} \left(\eta_{r,k}\right)^{W-K_r} \prod\limits_{i-1}^{r-1} \left(\eta_{i,k}\right)^{K_{i+1}-K_i}} P_0 \right] \end{aligned} \tag{4}$$

4.2 Throughput

The service rates at different instances in a model are represented by a single service rate, namely μ_{eff}. For finite capacity model considering reneging, the throughput is the same as μ_{eff} and has been computed by

$$\begin{aligned}
\mu_{\text{eff}} &= \sum_{k=1}^{W} \mu_k p_k \\
&= p_0 \left[\sum_{k=1}^{R} k\mu \frac{\lambda^k}{k!\mu^k} + \sum_{k=R+1}^{K_1} (\eta_{0,k}) \frac{\lambda^k}{R!\mu^R (\eta_{0,k})^{k-R}} \right. \\
&\quad + \sum_{j=1}^{r-1} \sum_{k=K_j+1}^{K_{j+1}} (\eta_{j,k}) \frac{\lambda^k}{R!\mu^R (\eta_{0,k})^{K_1-R} (\eta_{j,k})^{k-K_j} \prod_{i=1}^{j-1} (\eta_{i,k})^{K_{i+1}-K_i}} \\
&\quad \left. + \sum_{k=K_r+1}^{W} (\eta_{r,k}) \frac{\lambda^k}{R!\mu^R (\eta_{0,k})^{K_1-R} (\eta_{r,k})^{k-K_r} \prod_{i=1}^{r-1} (\eta_{i,k})^{K_{i+1}-K_i}} \right]
\end{aligned} \tag{5}$$

4.3 Expected Number of Customers in the System

The expected number of customers in the system (L_S) is

$$\begin{aligned}
L_S &= \sum_{k=0}^{W} k p_k \\
&= p_0 \left[\sum_{k=0}^{R} \frac{\lambda^k}{k!\,\mu^k} + \sum_{k=R+1}^{K_1} \frac{\lambda^k}{R!\,\mu^R (\eta_{0,k})^{k-R}} \right. \\
&\quad + \sum_{j=1}^{r-1} \sum_{k=K_j+1}^{K_j+1} \frac{\lambda^k}{R!\,\mu^R (\eta_{0,k})^{K_1-R} (\eta_{j,k})^{k-K_j} \prod_{i=1}^{j-1} (\eta_{i,k})^{K_{i+1}-K_i}} \\
&\quad \left. + \sum_{k=K_r+1}^{W} \frac{\lambda^k}{R!\,\mu^R (\eta_{0,k})^{K_1-R} (\eta_{r,k})^{k-K_r} \prod_{i=1}^{r-1} (\eta_{i,k})^{K_{i+1}-K_i}} \right].
\end{aligned} \tag{6}$$

4.4 Expected Waiting Time of the Customers in System

The expected time that a customer has to wait in the system (W_S) is obtained using

$$W_S = \frac{L_S}{\lambda_{\text{eff}}} \tag{7}$$

where λ_{eff} and L_S are determined using Eqs. (4) and (6)., respectively.

4.5 Expected Waiting Time of the Customers in Queue

The time a customer is expected to wait in the queue (W_q) is

$$W_q = W_s - \frac{1}{\mu_{\text{eff}}} = \frac{L_s}{\lambda_{\text{eff}}} - \frac{1}{\mu_{\text{eff}}} \tag{8}$$

4.6 Expected Length of the Queue

The expected number of customers present in the queue is obtained using

$$L_q = \lambda_{\text{eff}} W_q = L_S - \frac{\lambda_{\text{eff}}}{\mu_{\text{eff}}} \tag{9}$$

where λ_{eff}, μ_{eff} and L_S are given in Eqs. (4), (5) and (6), respectively.

4.7 Probability of Reneging

In case when all servers are being busy, the customers may wait for some time and then after may leave or renege from the system due to impatience. The probability that a customer may renege is obtained as follows:

$$
\begin{aligned}
P(\text{Reneging}) &= \sum_{k=1}^{W} (1-\nu_k) p_k \\
&= p_0 \Bigg[\sum_{k=R+1}^{K_1} (1-\nu) \frac{\lambda^k}{R!\,\mu^R \big(\eta_{0,k}\big)^{k-R}} \\
&\quad + \sum_{j=1}^{r-1} \sum_{k=K_j+1}^{K_{j+1}} \big(1-\nu_j\big) \frac{\lambda^k}{R!\,\mu^R \big(\eta_{0,k}\big)^{K_1-R} \big(\eta_{j,k}\big)^{k-K_j} \prod_{i=1}^{j-1} \big(\eta_{i,k}\big)^{K_{i+1}-K_i}} \\
&\quad + \sum_{k=K_r+1}^{W} (1-\nu_r) \frac{\lambda^k}{R!\,\mu^R \big(\eta_{0,k}\big)^{K_1-R} \big(\eta_{r,k}\big)^{k-K_r} \prod_{i=1}^{r-1} \big(\eta_{i,k}\big)^{K_{i+1}-K_i}} \Bigg]
\end{aligned} \tag{10}
$$

4.8 Probability that Permanent Server is Busy

The probability that mth permanent server being busy is obtained as

$$
\begin{aligned}
P_{B_{P_m}} &= \sum_{k=m}^{W} k p_k \\
&= p_0 \Bigg[\sum_{k=m}^{R} \frac{\lambda^k}{k!\mu^k} + \sum_{k=R+1}^{K_1} \frac{\lambda^k}{R!\mu^R \big(\eta_{0,k}\big)^{k-R}} \\
&\quad + \sum_{j=1}^{r-1} \sum_{k=K_j+1}^{K_j+1} \frac{\lambda^k}{R!\mu^R \big(\eta_{0,k}\big)^{K_1-R} \big(\eta_{j,k}\big)^{k-K_j} \prod_{i=1}^{j-1} \big(\eta_{i,k}\big)^{K_{i+1}-K_i}} \\
&\quad + \sum_{k=K_r+1}^{W} \frac{\lambda^k}{R!\mu^R \big(\eta_{0,k}\big)^{K_1-R} \big(\eta_{r,k}\big)^{k-K_r} \prod_{i=1}^{r-1} \big(\eta_{i,k}\big)^{K_{i+1}-K_i}} \Bigg] \quad 1 \le m \le R
\end{aligned} \tag{11}
$$

4.9 Probability that Additional Server is Busy

Let P_{B_m} denote the probability that mth additional server is busy. Then $P_{B_m} = p(K_m < k \le W)$. Thus, we obtain

$$P_{B_m} = p_0 \left[\sum_{j=m}^{r-1} \sum_{k=K_j+1}^{K_{j+1}} \frac{\lambda^k}{R!\mu^R \left(\eta_{0,k}\right)^{K_1-R} \left(\eta_{j,k}\right)^{k-K_j} \prod_{i=1}^{j-1} \left(\eta_{i,k}\right)^{K_{i+1}-K_i}} + \sum_{k=K_r+1}^{W} \frac{\lambda^k}{R!\mu^R \left(\eta_{0,k}\right)^{K_1-R} \left(\eta_{r,k}\right)^{k-K_r} \prod_{i=1}^{r-1} \left(\eta_{i,k}\right)^{K_{i+1}-K_i}} \right] \quad 1 \le m \le R. \tag{12}$$

4.10 Total Cost

Total cost function (TC) is constructed in terms of various cost elements defined as

C_H holding cost per unit time for each customer present in the system
C_P cost incurred per unit time on each permanent server when rending service
C_I cost incurred per unit time on each permanent server when it is idle
C_m cost incurred per unit time on mth ($m = 1,\ldots, r$) additional server.

Now, the cost function TC is given by

$$\mathrm{TC} = C_H L_S + C_P \left(\sum_{k=1}^{R} k\mu \frac{\lambda^k}{k!\mu^k} p_0 + R\mu \left(1 - \sum_{k=0}^{R} \frac{\lambda^k}{k!\mu^k} p_0 \right) \right) + C_I \sum_{k=1}^{R} (R-k) \frac{\lambda^k}{k!\mu^k} p_0 + \sum_{m=1}^{r} \left[C_m P_{B_m} \mu_{k_m+1} \right]. \tag{13}$$

5 Numerical Illustrations

This section presents the numerical simulation and sensitivity analysis by obtaining the numerical results of the performance measures established in Sect. 5. The software "MATLAB" is used to develop computer programs. The default parameters used for computation purpose are as follows:

$$\mu = 0.4,\ r = 3,\ R = 3,\ K_1 = R + 10,\ K_2 = R + 20,\ K_3 = R + 30,\ \mu_1 = 0.2\mu,$$
$$\mu_2 = 0.3\mu,\ \mu_3 = 0.4\mu,\ \nu = 0.35\mu,\ \nu_1 = 1.1\nu,\ \nu_2 = 1.2\nu,\ \nu_3 = 1.3\nu,\ W = 40.$$

The default parameters have been set, and the variations of cost have explored. For computing the cost, the following default parameters have been set for finite capacity model:

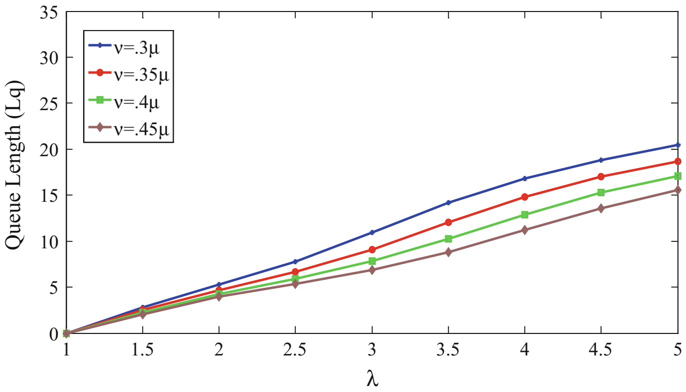

Fig. 3 L_q versus λ by varying ν

$$\lambda = 10,\ \mu = 2.5,\ r = 2,\ R = 2,\ K_1 = 5R,\ K_2 = 6R,\ \mu_1 = 0.02\mu,$$
$$\mu_2 = 0.003\mu,\ \nu = 0.035\mu,\ \nu_1 = 1.01\nu,\ \nu_2 = 1.02\nu,\ W = 40,$$
$$C_H = 5,\ C_p = 20,\ C_I = 10,\ C_1 = 20,\ C_2 = 50.$$

To illustrate the application of the finite capacity Markov model by considering the realistic assumptions of additional removable servers and reneging, we consider the example of a bank where two permanent servers provide service to the customers. The customers are assumed to arrive in the bank following Poisson fashion with an arrival rate λ.The service provided to the customers follows an exponential distribution with a service rate μ. If both the permanent servers are busy, the customers may be discouraged and depart from the system after waiting for some time in the system and without getting service. Considering this reneging scenario into account, there is provision of two additional removable servers in the system. The additional *j*th (*j* = 1, 2) server is added into the system as soon as there are K_j customers present in the system. Due to constraints on the finite waiting space (W), further arriving of customers is not allowed, when the number of customers reaches to its full capacity.

The trends of average number of customers with respect to arrival rates for varying values of different parameters are shown in Figs. 3, 4, 5, and 6. In Fig. 3, the variation of queue length versus arrival rate with the different values of reneging parameter (υ) for finite capacity model is displayed. It is noticed that by increasing the reneging rate, the queue length decreases significantly.

In Figs. 3, 4, 5, and 6, we see that by increasing the arrival rate, there is remarkable increment in the queue length. Figure 4 depicts the variation of the queue length versus arrival length with the variation of service rate (μ). As expected, by increasing the service rate, the queue length seems to decrease for a given arrival rate. Figure 5 reveals the variation of queue length for increasing arrival rate and with the variation of number of permanent servers (*R*). It is noticed that as the number of permanent servers increases, the queue length seems to decrease due to fact that the effective

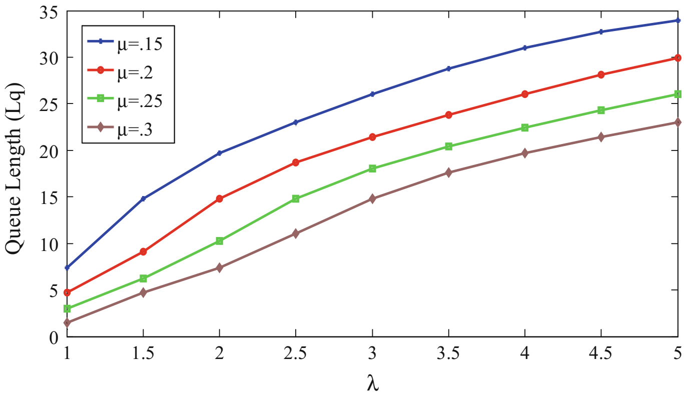

Fig. 4 L_q versus λ by varying μ

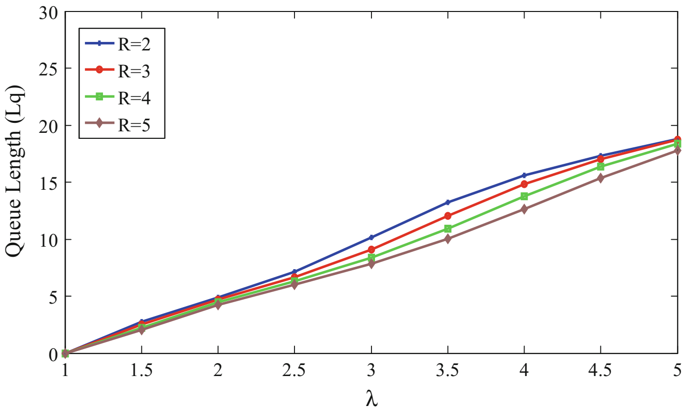

Fig. 5 L_q versus λ by varying R

service is improving by adding more permanent servers. Figure 6 displays the trend of queue length for the variation of capacity (W) of the system. There is increment in the queue length as W increases which is what we expect in real-time system.

Figures 7, 8, 9, and 10 exhibit the variation of total cost with service rate by varying other parameters viz. υ, K_1, R, and W, respectively. The cost function seems to be convex with respect to service rate, so the optimal rate can be obtained for the set default cost factors. One can see from the graph plotted in Fig. 7 that the total cost decreases with an increase in reneging rate. In Fig. 8, we notice that the cost increases with the increase in the value of K_1, which demonstrates that an additional server should be installed at an early stage. The optimal values in this figure are $\mathrm{TC}^* = 197.21,\ \mu^* = 2.510,\ K_{L^*} = 3$.

Figure 9 demonstrates the variation of total cost by the increasing the number of permanent servers as well as service rate. It can be seen from the graph that the total

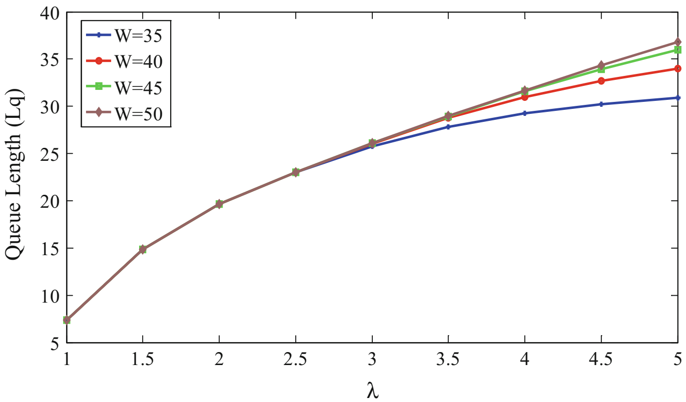

Fig. 6 L_q versus λ by varying W

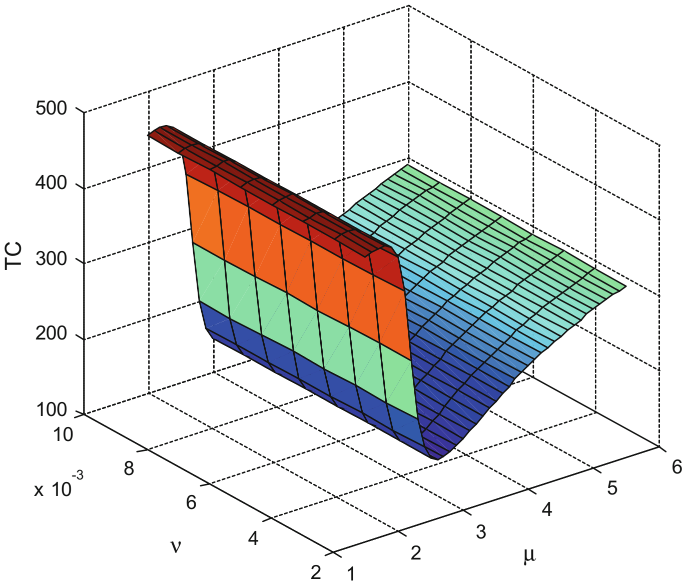

Fig. 7 Trend of TC by varying (μ, υ)

cost increases with an increase in the number of permanent servers which can also realized in real-time system too. Here we find the minimum cost and optimal service rate and number of servers as $\text{TC}^* = 177.21, \ \mu^* = 2.608, \ R^* = 2$. Figure 10 displays the variation of total cost with respect to capacity of the system while varying the service rate. It can be seen from the graph that the total cost increases with the increase in the capacity of the system. We find the minimum cost and optimal parameter values as $\text{TC}^* = 176.42, \ \mu^* = 2.583, \ W^* = 30$.

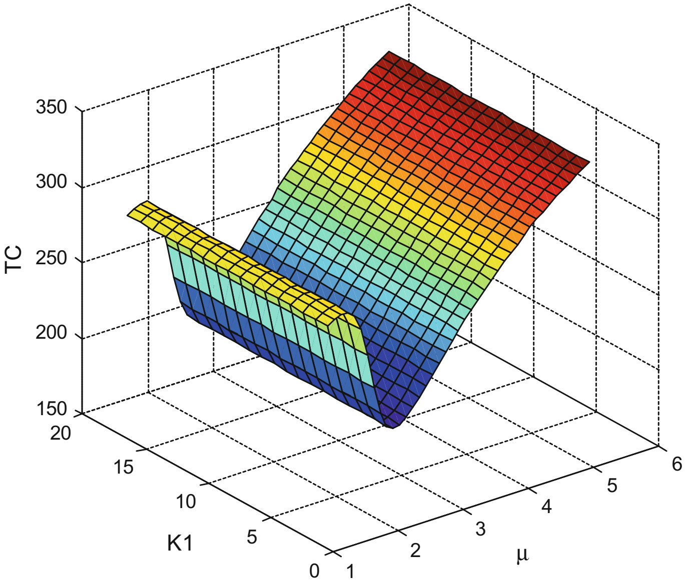

Fig. 8 Trend of TC by varying (μ, K_1)

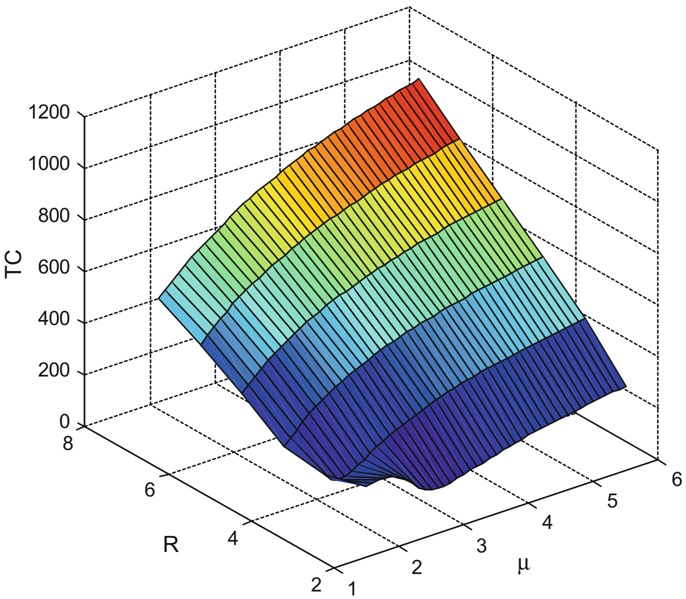

Fig. 9 Trend of TC by varying (μ, R)

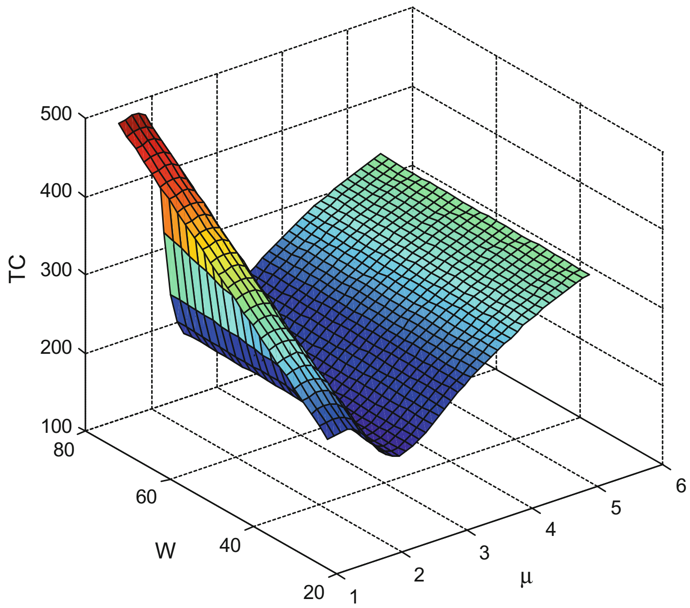

Fig. 10 Trend of TC by varying (μ, W)

6 Conclusions

The cost analysis by establishing various system metrics for the finite capacity Markovian queuing model has been presented by considering the reneging behavior of the customers. To reduce the adverse impact of reneging on the system efficiency, the trade-off between the service cost and waiting cost can play an important role. The provision of additional servers that can turn on based on workload of the system according to a threshold policy has been analyzed. The facility of removable extra servers along with permanent servers may be helpful to the system organizers and decision-makers for the economic design of the concerned system. Due to high setup cost and other factors, more number of permanent servers cannot be incorporated in the system as such additional removable servers can be installed as and when required.

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Analysis of Queues with Impatient Clients: An Application to Online Shopping

Yogesh Shukla, Nasir Khan and Sonia Shivhare

Abstract Queuing models have been largely used in the literature for obtaining routine actions and developing employment policies. However, the majority of this work has started a pure probabilistic point of view and has not addressed issues of statistical conclusion. In this article, we have tried to study bounce rate of shopping of clients with particular emphasis on online shopping operations and discuss the various reasons for high bounce rates for five leading online shopping sites in the Indian context. We have prepared a table and graph on the queue of the people who visit the sites and bounce back without placing any orders. We have tried to point out the actual arrival, service, and leaving pattern from online shopping. We have tried to establish a relationship between various variables among this data and also the presentation of all the sites and its effect on bounce rate.

Keywords Queuing model · Probabilistic point · Bounce rate

Ams subject classification 60K25 · 60K37 · 90B50

1 Introduction

Queuing theory has been worn in the before period to assess such things as staff plans, working condition, effectiveness, customer holding up time, and customer holding up condition. In online Web site, lining supposition can be utilized to evaluate a monstrous measure of variables, for example, treatment fill-time, customer holding up time, customer advising time. The function of queuing theory may be of meticulous profit in hospitals with high-volume outpatient workloads and/or those that offer different point of examine. Lining hypothesis uses scientific models and execution strategies to judge and hopefully improve the stream of customer through a lining framework. Lining hypothesis has innumerable applications and has been worn

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generally by the administration ventures. The capacity of lining hypothesis might be of fastidious preferred standpoint in receptionists with high-volume outpatient workloads and additionally people that supply a few purposes of analyze, for example, people in the Department of Veterans Affairs (VA), Department of Defense (DoD), college well-being frameworks, and oversaw mind associations. Still we breathe in an era who is wearing the eyeglasses of silver screen but wants to get things in an hour or less. A epigrammatic legend told from the patient's outlook will be of assistance to promote demonstrate this position [8]. Presently, the retail business in India is quickening. It is not extensively settled as the way it is conventional in Asian partner. India is vivified to create to be a most basic rival in the retail showcase. In view of the fact that India is a developing nation, it is tranquil not organized in favor of it. And they are parting no stone unturned to develop into the best in retail industry. Also India amid a high on cross-culture factor, it allows unlike companies bringing in diversity of products targeting unusual shopper segments. Indian market is not conquered by organized dramatis personae; however, there is planned in the territory of retail organization also. Big organization, for example Big Bazaar, wants to secure the advancement of its organization even in the rustic zones. There has been an increase in the Indian working class individuals because of rapid financial growth. Despite the fact that the number of inhabitants in utilizing Internet in India is low as far as all things considered level of aggregate populace in any case, in absolute figures it is high. This gives huge open doors for different online business locales to connect with this area. The populace getting to in India is the age gather from 1845. Independent of this reality web-based distributing shapes a small 0.08% of the whole Indian retail advertises. Market players must be more positive, reforming and spearheading in their approach and offering to make genuine advances. In the present market, main part of online deals is in a scope of things. This market in India needs to jump to the following level [1].

2 Literature Review

The progress of online shopping is also encouraged by the growing number of Internet users. "E-commerce proceed has grown from $7.4 billion in the middle of 2000 to $34.7 billion in the third sector of 2007. On-line shopping is one of key business behavior existing over the Internet," stated an investigation with reference to the conduct of clients in Poland [2]. This investigation confirmed the inclination observed in America. "Among various business instruments universally available on-line since 1990s, like auctions and banking and secure payment," mentioned in [5, 3, 10, 4], shopping [7], electronic libraries [6], etc."

"The number of on-line users either buying or searching for goods on-line since 2000 has generally doubled. While in 2000, 22% of Americans (46% of on-line users) had some incident with buying products in fundamental shops, the ratio grew to 39% in 2003 and reached 49% (66% of on-line users) in 2007," found an investigation about American Internet users performance available by Pew Internet & American

Life Project in February 2008 [9]. This investigation states, "The inhabitants of on-line clients grows rapidly and systematically from year to year on-line vend residue the service existing by the highest amount of providers."

"A rising digit of on-line shops and improved accessibility to clients world-wide due to the use of credit card payment in on-line transactions are key attributes that strength strong competition on the market, keep prices low (when compared to off-line shopping) and make more and more clients interested in on-line purchasing," as stated in [4, 10, 11].

"However, a wide choice of on-line shops makes it not easy to physically put side by side all the offers and select most favorable providers for the obligatory set of yield. This approach is normally accepted by clients and, according to Alexa Rank, popular price evaluation services feel right to the group of 1000 most viewed sites world-wide: shopping.com: 518-th place, nextag.com: 533-th place, bizrate.com: 600-th place, shoplocal.com: 932-th place," site popularity results registered in October 2008, www.alexa.com. A clarification of this dilemma has been supported by software agents purported price evaluation sites. "The thought of a worth comparator is built on the proposal of collecting offers of many on-line shops and build a price level on a client's request," stated their study.

"It is significant noting that price level built on-line on a client's appeal articulated in a text inquiry (product description) is a explanation to a specific case of shopping, in which a client wants to buy a single product. Multiple item shopping is not supported by price comparators existing in our time. As a result, price evaluation sites play the responsibility of recommender systems which tend to become aware of a client's preferences and interests in order to recommend goods to buy. A part outcome of the troubles mentioned over is the loss of client assurance," as observed by Satzger [9].

Besides, price evaluation sites, being marketable projects, have a propensity to optimize their incomes from directing clients to meticulous online shops.

3 Numerical Illustrations of Bounce Rate Using Online Shopping Customer Data

Here, we analysis different–different sites for reduce the queue and improve the service of online Web site. For this, here we fix some factor for analysis this problem like number of Web site, month of a fixed year.

Table 1 shows the percentage of people who surf different online shopping sites, namely Amazon.in, Flipkart.com, Snapdeal.com, EBay. in, and Jabong.com. The ranking of each Web site in India has also been mentioned on the basis of their overall goodwill in the Indian online market. The total of clients who visit any Web site have been classified into two categories: those who surf the Web site on desktops or laptops and those who surf the Web site on their smartphones, for instance, of the total 100% of people who visit Amazon.in, 48.27 visit it on their desktop systems or laptops and the remaining 51.73% visit it on their smart phones. The entire amount of

Table 1 Overview of particulars

Parameter ↓	Name of online Web site				
	Amazon.in	Flipkart.com	Snapdeal.com	EBay.in	Jabong.com
Country rank	7	10	19	28	54
Desktop users (%)	48.27	55.02	52.24	48.07	60.69
Mobile users (%)	51.73	44.98	47.76	51.93	39.31
Total visit (Aug 2016)	188.6M	106.2M	55.7M	36.7M	24.7M
Average daily visits	6.739M	3.793M	1.989M	1.310M	0.865M
Average visit duration	0:07:22	0:06:45	0:05:29	0:05:23	0:03:53
Pages per visit	8.54	7.6	4.91	5.4	4.02
Bounce rate (%)	32.06	29.94	39.99	42.90	52.84

Table 2 Analysis of visiting users and bounce users

Parameter ↓	Name of online Web site				
	Amazon.in (%)	Flipkart.com (%)	Snapdeal.com (%)	EBay.in (%)	Jabong.com (%)
Mobile users	51.73	44.98	47.76	51.93	39.31
Desktop users	48.27	55.02	52.24	48.07	60.69
Bounce rate	32.06	29.94	39.99	42.90	52.84

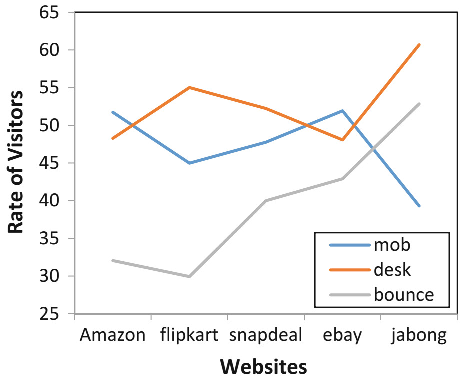
visits, average number of daily visits, and average visit durations on these websites during August' 16 have also been mentioned in Table 1 along with the average number of pages surfed in a single visit.

Table 2 has been drawn to emphasize the percentage of users preferring smartphones for online shopping and the remaining that prefer desktops or laptops for the same. The bounce rates of all the online shopping Web sites have been emphasized upon in Table 2 (Fig. 1).

Now, by analyzing the graph, we see that the percentage of mobile users for Ebay.in is the highest (51.93%), very closely followed by Amazon.in with 51.73% mobile application users. Jabong.com faces the minimum proportion (39.31%) of mobile app users. In contrast, Jabong faces the highest proportion (60.69%) of desktop users who surf Jabong.com, while Amazon.in and Ebay.in face the lowest proportions of people surfing the Web sites on their desktops with 48.27% and 48.07%, respectively.

Bounce rate is the percentage of visits that end up without any orders getting placed. The bounce rate profile for the Web sites shows that Jabong.com has the highest bounce rate of 52.84%, while Flipkart.com and Amazon.in have the lowest bounce rates. In other words, Jabong.com is less capable of turning its visitors into

Fig. 1 Graph depicts the data in Table 2



clients when compared to Flipkart.com and Amazon.in. The reason for this may be cited as the lower percentage of mobile app users of Jabong.com in comparison to Amazon.in and Flipkart.com. It can be suggested that Jabong.com should go for the reintroduction of their mobile application toward a more user-friendly mobile application which people tend to prefer more than they prefer the desktop.

While Ebay.in and Snapdeal.com face almost equal visitors in terms of both desktop visitors and mobile app users, they face a mediocre bounce rate when compared to Jabong.com (with the highest bounce rate) and Flipkart.com and Amazon.in (with the lowest bounce rates).

It can be stated in general that the average bounce rate faced by the online visitors of these Web sites is substantially high. The reasons for this may be

- Lower high-speed Internet penetration in the country;
- Impatience of the Indian clients due to low speed while browsing the sites, both on mobile apps and desktops;
- Less techno-savvy population of the country, for example, some people surf the various products, but are not able to place orders;
- Less user-friendly application of offers available on the sites, etc.

4 Conclusion

These bounce rates can be reduced by introducing more user-friendly online Web sites for desktop users and more user-friendly mobile apps for smart phone users. Also, it is worth mentioning that the best offers should be availed automatically on these online shopping platforms, as the impatience in the Indian clients makes it hard for them to give time to search and avail offers. Also, a higher penetration of high-speed Internet will be substantially helpful in eradicating the problem of impatience of the Indian clients that eventually end up increasing the bounce rate.

Also, the products should be clearly classified in categories of offers; i.e., there should be separate pages for all the categories of offers and for the products without any offers. This would help the client in directly browsing the page with the required offer that the client intends to. For instance, someone who only wishes to browse products with offers should be able to clearly go for a page which has the products with offers.

Also, to gain more and more clients, points should be credited to any registered viewer who visits the site even once. Any credit point may be equated to a desired money value, for example, Rs. 0.5. This would help in retaining a visitor on the site and would automatically lead to an increased market share and sale.

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Designing Bulk Arrival Queue Model to an Interdependent Communication System with Fuzzy Parameters

Reeta Bhardwaj, T. P. Singh and Vijay Kumar

Abstract This study attempts to investigate and fuzzify the bulk queuing model of a communication system with statistical multiplexers containing the packet voice source and the data. In this study, we assumed that the incoming messages and transmitted messages are correlated at the node of network that make the system interdependent. The uncertain parameters are considered in fuzzy numbers. A set of parametric values are developed to evaluate the bounds of system characteristic functions at possibility level α. Numerical examples with graphical insight are also illustrated to check the validity of the proposed approach.

Keywords Bulk arrival queue · Membership function
Trapezoidal fuzzy numbers · Statistical multiplexing voice packetized · DCE

1 Introduction

Bulk arrival queue models have applications in various systems such as telecommunication, manufacturing, and computer networking. Traffic in communication system has significantly changed with the invention of fax and Internet. In an expanded system of data processor, computers talk to each other over public phone lines that are analog in nature. A less degradation of analog signals over long phone lines has been observed compared to data stream over identical lines due to high harmonic content of the digital signals and limited frequency response of the lines. The modems are

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the data communication equipment (DCE). The sequence of events for the control line can be summarized as:

a. Data terminal ready comes on to the modem.
b. The modem responds with data set ready on, which signals that both devices are turned on.
c. When the terminal is ready to send the message, it will signal the modem with a request to send activated.
d. If the modem is ready, it will reply with a clear to send.

The modern communication system can better be analyzed in light of the stochastic modeling of queue networks. The queue models play an important role in modeling voice calls also. In communication system, one can consider the messages as customers/units, the buffer (region of physical memory storage used to temporarily hold data) as waiting lines and the concerned activity in transmission of messages as services. The control register supplies the information to the receive control circuit, which counts the data bits, examines the parity and compares it to the control parity bit and looks for only one stop bit. This fact justifies the resemblance between queue system and communication system.

The function of packet switching is to divide data message into small bundles of information and then transmit through communication network to their destination using computer control switches. Nowadays, communication network sources are managed by statistical multiplexing which reduces the delay in packet switching.

In the study of communication systems, Jenq [1], Hayashida [2] assumed that the arrival and service processes are independent but later on other researchers observed that these system can be better analyzed by assuming dependence between arriving and transmitted messages. Srinivas Rao [3, 4] and Singh [5] presented stochastic queue model to interdependent communication network in which arrival messages were distributed geometrically and transmission at the node of network are correlated following a bi-variate Poisson process. Using Chapman Kolmogorov equation, Singh [5] calculated the average delay transmission and variance of number of packets in buffer. The work was further extended by Singh and Kusum [6] and Singh and Arti [7] who explored the interdependent queuing model to a communication system with voice packets and data under fuzzy environment.

In many real-world problems, the arrival and service nodes are more correctly expressed by linguistic terms (slow, moderate, and fast) as compared to probability distribution. Due to fluctuation and variation in demands at transmission lines, congestion occurs in communication system; therefore, the concerned activities in transmission of messages, service patterns, etc., have been assumed uncertain in nature. The covariance between the composite input and output transmission completion has also been considered in fuzzy parameters. Our study is more realistic than the work done of earlier authors. This interdependent network can reduce the mean buffer length and variability of buffer contents when the environment is uncertain and complex.

Singh et al. [6, 7] explored the application of interdependent queuing model to a communication system with voice packets and data in fuzzy environment using Yager's and Chanes's defuzzification formula, but the results are not so encouraging. Recently, Bhardwaj [8] extended the work of Singh et al. [7] by considering the system parameters and covariance through fuzzy triangular numbers. This paper is further an extension of the work done by Bhardwaj [8] in the sense that the parametric values have been considered as trapezoidal fuzzy numbers and the model have been developed on the basis of lower and upper bounds of system performance measure at possibility level α. The validity of proposed approach has been checked through numerical examples.

2 Preliminaries

2.1 Fuzzy Set and Fuzzy Logic

Fuzzy logic (many valued logic) has various application in networks and subsidized to development in network efficiency. It deals with reasoning which is inaccurate and uncertain rather than fixed and precised. Zadeh [9] first of all introduced the fuzzy concept which can be demonstrated by a simple set inclusion operator, but there is a degree of membership. The Boolean logic has just two values either true (numerically represented by 1) and false (numerically represented by 0). The fuzzy logic extends the values between 0 and 1 using concept of degree of membership. It makes use of fuzzy interference tool.

In the universe of discourse X, a fuzzy subset $\widetilde{A}$ on X is defined by membership function $\mu_{\widetilde{A}}(X)$ such that $\mu_{\widetilde{A}}(X) : X \to [0, 1]$.

2.2 α-Cut

Let $\widetilde{A}$ be a fuzzy set on R and $0 \leq \alpha \leq 1$. Then, **α-cut** of the fuzzy set $\widetilde{A}$ is the crisp set A that contains all the elements of universe of discourse X, whose membership grade in A is greater than or equal to specified value α i.e., $A_\alpha = \left\{x/\mu_{\widetilde{A}}(x) \geq \alpha, x \in X\right\} = \left\{L_{\widetilde{A}}(\alpha), U_{\widetilde{A}}(\alpha)\right\}$, where $L_{\widetilde{A}}(\alpha)$ and $U_{\widetilde{A}}(\alpha)$ represent the lower bound and upper bound of the α cut of $\underset{\sim}{A}$, respectively.

2.3 Trapezoidal Fuzzy Numbers

$$\widetilde{A} = \begin{cases} \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & b \le x \le c \\ \dfrac{d-x}{d-c} & c \le x \le d \\ 0 & \text{otherwise} \end{cases}$$

3 Model Description and Notation

In this model, we assume that the incoming and transmitted packets are correlated and succeeding a bi-variate Poisson's law. We also assume that the buffer having infinite capacity and the number of incoming packets are considered as a random variable x.

3.1 Notation

λ_x	Arrival rate of message containing x packets
$\in_x$	Covariance between packet arrival and number of completed transmission
μ	Average transmission rate
$\lambda = \sum_x \lambda_x$	Composite arrival rate of packets
$\in = \sum_x \in_x$	Covariance between composite arrivals and completed transmission.

The bit dropping of flow control mechanism generates the covariance which induces the dependence between incoming and transmitted messages. Flow control plays an important role because the sending computer may transmit information at a faster rate than the destination computer can receive and process them. The function of congestion control is to control the flow of data. This type of situation arises only if the receiving computers have a heavy traffic load or has less processing power than the sending computer. The network diagram is as shown (Fig. 1).

3.2 Assumptions

1. The probability that there is no arrival and no service in a small interval of time Δt is $1-(\lambda+\mu-2\in)\Delta t + o(\Delta t)$.

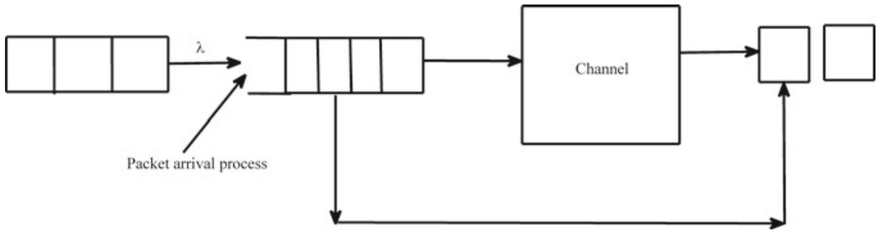

Fig. 1 Bit dropping or flow control

2. The probability of one arrival and no service in a small interval of time Δt is $(\lambda - \in)\Delta t + o(\Delta t)$.
3. The probability of no arrival and one service completion during a small interval of time Δt is $(\mu - \in)\Delta t + o(\Delta t)$.

4 Mathematical Study

The model differential difference equations in steady state can be written as:

$$0 = -(\lambda + \mu + 2\in)P_n + (\mu - \in)P_{n+1} + (\lambda - \in)\sum_{r=1}^{n} P_{n-r}C_r \quad n \geq 1 \tag{1}$$

$$0 = -(\lambda - \in)P_0 + (\mu - \in)P_1; \quad n = 0 \tag{2}$$

In order to solve, applying generating function technique

$$P(z) = \sum_{n=0}^{\infty} P_n Z^n, \ \ |Z| \leq 1$$

$$C(z) = \sum_{n=0}^{\infty} C_n Z^n, \ \ |Z| \leq 1$$

Solving on the basis of Singh [5], we get

$P(z) = \frac{(\mu-\in)(1-az)P_0}{(\mu-\in)(1-az)-(\lambda-\in)z}$ for $(\lambda - \in) < (\mu - \in)(1-a)$, $0 < a < 1$

5 Performance Measure

Using initial condition

$P(1) = 1,\ P(0) = 1 - \frac{\lambda-\in}{(\mu-\in)(1-a)} = 1 - \rho_0$ where $\rho_0 = \frac{\lambda-\in}{(\mu-\in)(1-a)}$

we ultimate find the following performance measures for the stated model:

(i) The expected number of packets in the system is
$L = \frac{\rho_0}{(1-a)(1-\rho_0)}$ where $\rho_0 = \frac{\lambda-\in}{(\mu-\in)(1-a)}$

(ii) The expected number of packets in buffer

$$L_q = \frac{\rho_0}{(1-a)(1-\rho_0)} - \rho_0$$

(iii) Variance of the number of packets in the system

$$\text{Var} = \frac{a\rho_0(1-\rho_0) + \rho_0}{(1-a)^2(1-\rho_0)^2}.$$

6 Fuzzified Model

In many practical situations, the parameters are characterized subjectively; i.e., arrival and service pattern (mess in communication system) are described in linguistic variables such as fast, slow, medium. The role of fuzzy arithmetic is very important here, or we can say that the fuzzified model is potentially more practical than stochastic model. To explain the model, we discussed the trapezoidal fuzzy system through $\alpha-$ cut.

6.1 The Numerical Illustration

Mean Queue Length

Consider $\widetilde{\lambda} = (2, 3, 4, 5)$, $\widetilde{\mu} = (9, 10, 11, 12)$, $\widetilde{\varepsilon} = (0.01, 0.02, 0.03, 0.04)$, $a = 0.3$

Using $\alpha-$ cut

$$\lambda_\alpha = \left\{\lambda_\alpha^{\text{lower}}, \lambda_\alpha^{\text{upper}}\right\} = \{\alpha + 2,\ 5 - \alpha\}$$

$$\mu_\alpha = \left\{\mu_\alpha^{\text{lower}}, \mu_\alpha^{\text{upper}}\right\} = \{9 + \alpha,\ 12 - \alpha\}$$

$$\varepsilon_\alpha = \left\{\varepsilon_\alpha^{\text{lower}}, \varepsilon_\alpha^{\text{upper}}\right\} = \{0.01\alpha + 0.01,\ 0.04 - 0.01\alpha\}$$

$$(\rho_0)_\alpha = \left\{\rho_{0\alpha}^{\text{lower}}, \rho_{0\alpha}^{\text{upper}}\right\} = \left\{\frac{1.01\alpha + 1.96}{8.393 - 0.707\alpha}, \frac{4.99 - 1.01\alpha}{0.707\alpha + 6.272}\right\}$$

Table 1 Mean queue length versus α

α	$L_{\alpha}^{\text{lower}}$	$L_{\alpha}^{\text{upper}}$
0	0.4352	5.5600
0.3	0.5462	3.7258
0.5	0.6317	2.9930
0.7	0.7283	2.4630
1.0	0.8996	1.8958

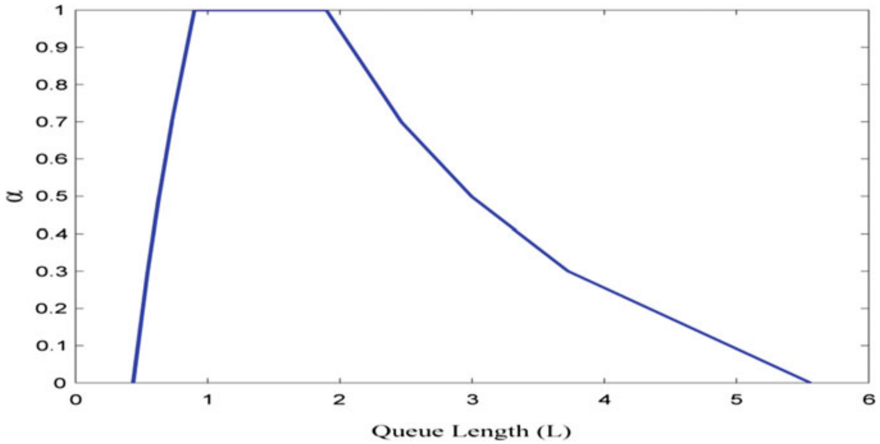

Fig. 2 Mean queue length versus α

$$L_{\alpha} = \left\{L_{\alpha}^{\text{lower}}, L_{\alpha}^{\text{upper}}\right\} = \left\{\frac{\rho_{0\alpha}^{\text{lower}}}{(0.7)\left(1-\rho_{0\alpha}^{\text{lower}}\right)}, \frac{\rho_{0\alpha}^{\text{upper}}}{(0.7)\left(1-\rho_{0\alpha}^{\text{upper}}\right)}\right\}$$
$$= \left\{\frac{1.01\alpha + 1.96}{4.5301 - 1.2019\alpha}, \frac{4.99 - 1.01\alpha}{1.2019\alpha + 0.8974}\right\}$$

Table 1 presents the mean fuzzy queue length of the system for the different parametric values of α (Fig. 2).

Variance

Consider $\widetilde{\lambda} = (1, 2, 3, 4)$, $\widetilde{\mu} = (15, 16, 17, 18)$, $\widetilde{\varepsilon} = (0.01, 0.02, 0.03, 0.04)$, $a = 0.5$

Using $\alpha-$ cut

$$\lambda_{\alpha} = \left\{\lambda_{\alpha}^{\text{lower}}, \lambda_{\alpha}^{\text{upper}}\right\} = \{\alpha + 1,\ 4 - \alpha\}$$

$$\mu_{\alpha} = \left\{\mu_{\alpha}^{\text{lower}}, \mu_{\alpha}^{\text{upper}}\right\} = \{15 + \alpha,\ 18 - \alpha\}$$

Table 2 Variance versus α

α	V_α^{lower}	V_α^{upper}
0	0.7739	12.087
0.2	0.9920	9.981
0.4	1.236	8.318
0.6	1.511	6.982
0.8	1.823	5.892
1.0	2.178	4.990

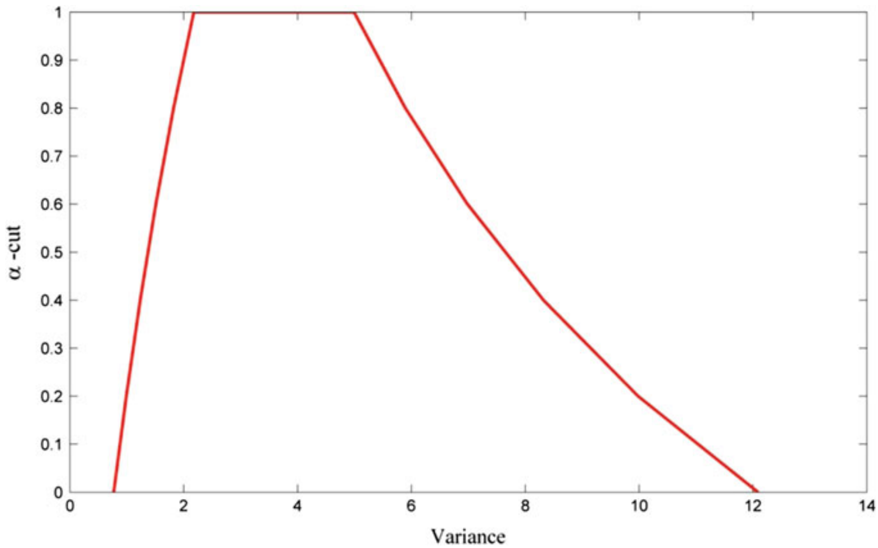

Fig. 3 Variance versus α

$$\varepsilon_\alpha = \left\{\varepsilon_\alpha^{\text{lower}}, \varepsilon_\alpha^{\text{upper}}\right\} = \{0.01\alpha + 0.01,\ 0.04 - 0.01\alpha\}$$

$$\text{Var} = V_\alpha = \left\{V_\alpha^{\text{lower}}, V_\alpha^{\text{upper}}\right\} = \left\{\frac{a\rho_{0\alpha}^{\text{lower}}\left(1-\rho_{0\alpha}^{\text{lower}}\right)+\rho_{0\alpha}^{\text{lower}}}{(1-a)^2\left(1-\rho_{0\alpha}^{\text{lower}}\right)^2}, \frac{a\rho_{0\alpha}^{\text{upper}}\left(1-\rho_{0\alpha}^{\text{upper}}\right)+\rho_{0\alpha}^{\text{upper}}}{(1-a)^2\left(1-\rho_{0\alpha}^{\text{upper}}\right)^2}\right\}$$

$$= \left\{\begin{array}{c}\dfrac{(0.505\alpha+0.48)(8.035-1.515\alpha)+(8.995-0.505\alpha)(1.01\alpha+0.96)}{0.25(8.035-1.515\alpha)^2}, \\ \dfrac{(1.995-0.505\alpha)(1.515\alpha+3.49)+(0.505\alpha+7.48)(3.99-1.01\alpha)}{0.25(1.515\alpha+3.49)^2}\end{array}\right\}$$

Table 2 presents the fuzzy variance of the system for different value of α.

7 Conclusion

Authors observed that when the parameters (input and output transmission rate) are considered as fuzzy, the performance measures of system can also be expressed by fuzzy parameters which show a complete conservation of input/output information. Since the system characteristics are represented by membership function rather than by a crisp value, it preserves the fuzziness of input and the outcomes which can analyze the fuzzy bulk system more accurately. As the value of α increases, lower bound of queue length L increases and upper bound correspondingly decreases. Authors found that uncertainty decreases. Again from the table of variance, it is found that the variability of number of packets in buffer decreases as the dependence between arrival of packets and transmission increases positively (Fig. 3).

The results tally with the work done by Singh and Arti [7] but in different way analytically with more clear arguments. When $\alpha \to 0,\ \rho_n = (1-\rho_0)\rho_0^n\ ,\ n > 0$ gives the interdependent communication network without bulk arrivals, the results tally with the study of Srinivas Rao et al. [3].

8 Future Scope

In the given system, the parametric values (arrival rate, service rate, and covariance coefficient) are considered as trapezoidal fuzzy numbers. The proposed approach is not only restricted to these parameters. Either case with arrival rate and service rate being convex fuzzy sets can also be applicable, or the work can be extended. The proposed work can be continued to the cases of fuzzy batch sizes which need further research. To avoid a sudden drop in graph, a nonlinear threshold can be suggested in which the membership function goes down to zero as slower pace. The performance measures can also be converted into linguistic variable (short, average, medium, and large) which needs further research.

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Transient Analysis of Markov Feedback Queue with Working Vacation and Discouragement

Madhu Jain, Shobha Rani and Mayank Singh

Abstract This article is concerned with the performance prediction of feedback Markovian queueing model with working vacations. During the period of vacation, the server continues its job with slower rate. In case when the server is on normal busy state, the customers may be discouraged to join the queue and depart from the system. After the service completion, the unsatisfied customer may also feedback to the system when operating in normal busy as well as working vacation mode. The transient solution of the system size probability distribution is obtained in terms of modified Bessel's function by using the probability generating function and continued fractions. The time-dependent results for various performance indices including cost function are derived. By taking illustration, numerical simulation and sensitivity of performance indices with respect to different parameters are provided.

Keywords Transient probabilities · Markov process · Working vacations
Feedback queue · Balking · Continued fractions · Modified Bessel's function

1 Introduction

Recent researches in the queueing theory show the emergence of an important and interesting topic of queueing system with working vacations which in turn has discernible effect on the queueing applications. To investigate the system behavior over a time, we need time-dependent analysis for the queue under study. The transient state measures are important to examine the functioning of the system. It has been observed that a very few research works have been done on the time-dependent

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analysis of Markovian queueing system including either vacation or working vacations. Such types of queueing models have many applications in real life, especially in globalized computer networks, manufacturing, and production systems, etc. The main objective in this article is to study the transient behavior of Markovian queueing systems with working vacation and feedback.

Sometimes, it may happen that the server may not be available at the service counter in the absence of customers. That is the case when the server goes for the vacation for a random length of time and remains idle. But, sometime the vacationing server may go passively engaged in some activity. This concept of 'working vacation' which actually works as a semi-vacation scheme was initiated by Servi and Finn [11]. They generalized $M/M/1$ classical vacation model where the server continues its service to the customer with a lower speed in place of stopping the service. In the standard working vacation queueing problems, during the period of vacation, the server may not maintain the original speed of their service. In this case, impatience factor can also take place in the system; if the waiting line is too long, then the customer may balks from the service area without joining the queue. The queueing systems in which unsatisfied customers can again get a chance to be served is treated as a feedback system.

Feedback queues are common in many congestion scenarios due to the fact that after getting service, sometimes the unsatisfied customers demand for more service time instead of departing from the system. Such situation of feedback can be seen at many queueing systems including the doctors' clinics, supermarkets, banking services, medical stores.

A lot of research and survey have been done on the steady and transient state analysis of Markovian queueing systems that are designed in different context to serve the customers. Baba [6] presented the performance modeling and analysis of $G/M/1$ queueing model by including the concept of multiple vacations. It was an extension work of $M/M/1$/WV queueing model discussed by Servi and Finn [11]. Zhang et al. [15] presented a paper on finite capacity Markovian queue model with reneging, balking, and server vacation and acquired the steady-state solution in matrix form. Altman et al. [2] have described a comprehensive analysis of $M/M/1$, $M/M/c$, and $M/G/1$ queueing models with the emergence of impatient customers and server vacation. Jain and Agarwal [8] studied an $M/E_k/1$ model with vacation scheme under the features of server breakdown and state-dependent arrival. To develop an $M/M/1$ single server working vacation queueing model, Tian et al. [12] have done the analysis and derived the distribution of the total number of the customers present in the service area by using the birth and death process. Seddy et al. [1] gave the analysis of transient behavior of an $M/M/C$ Markov queue model by considering the discouraging aspects of both balking and reneging to establish the expression of distribution of queue length and average number of the customers in the system at the stationary level. By applying the matrix-geometric solution technique, Xiu et al. [14] discussed the performance of $M/M/1$ vacation queue model with waiting server and impatience customers. Jain [9] studied the queueing problem under the multiple breakdown servers and working vacation to obtain the queue size distribution by using the matrix analysis approach. Baba [7] developed the approximation formula of the queue length distribution of $M^X/M/1$ queueing system with multiple vacations and unreliable server by using the entropy principle. Arivudainambi et al. [5] have done the analysis of a steady-state distribution and the total jobs in the orbit of single server vacationing retrial queue. Vijayashree et al. [13] studied an $M/M/C$ queue model with multiple types of working vacations by evaluating the transient solution of the total customers in the system. Ammar [3] discussed a single server vacation model with an impatient factor, to derive the time-dependent probabilities in explicit form. Ammar [4] also studied the transient study of an $M/M/1$ vacation queue model with waiting server and impatience customer to obtain the transient solution in explicit form. In recent times, Panda et al. [10] have investigated the Markovian queue with working vacation and impatient customers to discuss its equilibrium behavior and social optimization.

A very few papers have presented in the queueing literature on transient analysis of Markovian queueing problems. This article attempts to figure out the transient solution of an $M/M/1$ Markov queue by considering the feedback customers along with working vacation. It has been observed that no related research work has been done to analyze the transient probabilities for the feedback queueing problem under the assumption of state-dependent rates, balking behavior, and working vacation, altogether. For the solution purpose, generating function and continued fraction techniques are used. The closed-form formulae of the queue size distribution and other performance indices to examine the transient behavior of the system are provided. The numerical illustration for the model is also discussed to look the sensitivity outcomes of the system descriptors on several indices. The contents of the paper have been presented as follows. In Sect. 2 we discuss the concerned model and governing equations with suitable notations. The closed-form expressions for the generating function of queue length distribution and system's state probabilities have derived in Sect. 3. The stationary distribution is also obtained by using the results of transient state probabilities in Sect. 4. Some other performance measures are obtained in Sect. 5. Numerical simulation and sensitivity analysis have been done to demonstrate the effects of parameters for the performance measures in Sect. 6. Finally, the conclusion for the investigation done has been presented in Sect. 7.

2 Model Description

To interpret the transient behavior of $M/M/1$/WV feedback model with discouragement factor, we develop the queueing model by considering the birth–death process. We assume that if there is no customer to serve in the system, then the server goes to vacation state for random length of time but continues his job with slower rate instead of ceasing service completely. At the state of the busy period of the server, the unsatisfied customer may or may not join again the waiting line. After getting the unsuccessful service, some of the customers join the queue again for service completion; such type of queue is known as feedback queue. The impatient customers considered here show the balking behavior, i.e., the arriving customers do not join the

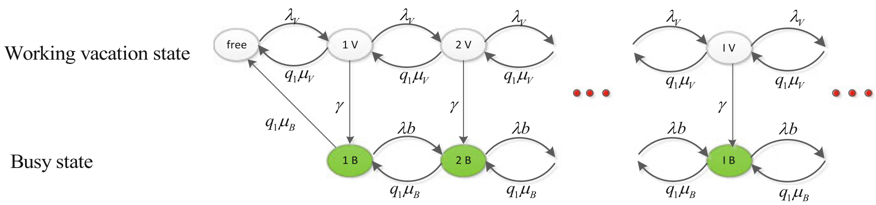

Fig. 1 Transition state diagram for the $M/M/1/WV$ queue with feedback and balking

queue if observe that the waiting line is too long. The customers are served according the FIFO rule by the single server in case of both normal busy period and working vacation period. The other assumptions for model formulation are as follows (Fig. 1):

(a) The customers join the system in Poisson fashion with rate λ and λ_V when it works at normal operating mode and at working vacation mode.
(b) The working vacation time follows the exponential distribution with rate γ.
(c) Both the service times during normal busy and on working vacation mode of the server are distributed exponentially with rates μ_B and μ_V, respectively.
(d) During the working vacation after completion of the service, the feedback customer, i.e., the unsatisfied customer may again join the queue with probability q_0 or departs from the service area with probability $1 - q_0$. During the busy period when the server is rendering service in normal mode, after the service completion, the unsatisfied customer may join the end of queue with probability q_1 or may not join the queue with probability $1 - q_1$.
(e) When the server is on normal busy state, the arriving customer may discourage and enters the queue with some probability b.
(f) The arrival and service processes are mutually independent.

Let $M_V(t)$ be the total number of customers present in the system at time t and

$$S(t) = \begin{cases} V, \text{ working vacation state of the system at time } t, \\ B, \text{ busy state of the system at time } t. \end{cases}$$

Bivariate stochastic process $\{M_V(t), S(t)\}$ is governed by

$$\Omega = \{0, 0\} \cup \{(i.j) : i \geq 1, j = V \text{ or } B\}$$

Let $P_{i,j}(t)$ be the probability of i customers in the system and $j \in \{V \text{ or } B\}$; if $j = V$ it means that the server is on working vacation and if $j = B$, then the server is in normal busy mode.

2.1 Governing Equations

Ongoing to the forward direction, Kolmogorov differential–difference equations for the concerned study are formed as follows:

For Working Vacation State:

$$P'_{00}(t) = -\lambda_V P_{00}(t) + q_0\mu_V P_{1V}(t) + q_1\mu_B P_{1B}(t) \quad (1)$$

$$P'_{1V}(t) = -(\lambda_V + q_0\mu_V + \gamma)P_{1V}(t) + \lambda_V P_{00}(t) + q_0\mu_V P_{2V}(t) \quad (2)$$

$$P'_{i,V}(t) = -(\lambda_V + \gamma + q_0\mu_V)P_{i,V}(t) + \lambda_V P_{i-1,V}(t) + q_0\mu_V P_{i+1,V}(t); \quad \text{for } i \geq 2 \quad (3)$$

For busy state:

$$P'_{1,B}(t) = -(\lambda b + q_1\mu_B)P_{1,B}(t) + \gamma P_{1,V}(t) + q_1\mu_B P_{2,B}(t) \quad (4)$$

$$P'_{2,B}(t) = -(\lambda b + q_1\mu_B)P_{2,B}(t) + \gamma P_{2,V}(t) + q_1\mu_B P_{3,B}(t) + \lambda b P_{1,B}(t)$$

$$P'_{i,B}(t) = -(\lambda b + q_1\mu_B)P_{i,B}(t) + q_1\mu_B P_{i+1,B}(t) + \gamma P_{i,v}(t) + \lambda b P_{i-1,B}(t), \quad \text{for } i \geq 3 \quad (5)$$

The initial conditions are $P_{00}(t) = 1$; $P_{i,j} = 0, i \neq 0, j \neq 0$.

3 Transient Probabilities

3.1 Evaluation of $P_{i,B}(t)$

Here we obtain the value of $P_{i,B}(t)$ in terms of $P_{i,V}(t)$, for $i \geq 1$.

Define probability generating function of busy state by

$P_B(z, t) = \sum_{i=1}^{\infty} z^i P_{i,B}(t)$ and $P_B(z, 0) = 0$.

Solving (4) and (5), we get

$$\frac{\partial P_B(z,t)}{\partial t} = \sum_{i=1}^{\infty} \left[-(\lambda b + q_1\mu_B)P_{i,B}(t) + \gamma P_{i,V}(t) + q_1\mu_B P_{i+1,B}(t) + \lambda b P_{i-1,B}(t)\right] z^i$$

which after some algebraic manipulation yields

$$\frac{\partial P_B(z,t)}{\partial t} = \left\{-(\lambda b + q_1\mu_B) + z\lambda b + \frac{q_1\mu_B}{z}\right\} P_B(z,t) + \gamma \sum_{i=1}^{\infty} z^i P_{i,V}(t) - q_1\mu_B P_{1,B}(t) \quad (6)$$

Further solving Eq. (6), we get

$$P_B(z,t) = \gamma \int_0^t \left[\sum_{m=1}^{\infty} z^m P_{m,V}(y) \right] \mathrm{e}^{-(\lambda b + q_1\mu_B)(t-y)} \mathrm{e}^{\left(\lambda bz + \frac{q_1\mu_B}{z}\right)(t-y)} \mathrm{d}y$$

$$- q_1\mu_B \int_0^t P_{1,B}(y)\, \mathrm{e}^{-(\lambda b + q_1\mu_B)(t-y)}\, \mathrm{e}^{\left(\lambda bz + \frac{q_1\mu_B}{z}\right)(t-y)} \mathrm{d}y \quad (7)$$

Let $I_i(t)$ be the modified Bessel's function of first kind of order i.
If $\alpha_B = 2\sqrt{\lambda b q_1 \mu_B}$, $\beta_B = \sqrt{\frac{\lambda b}{q_1\mu_B}}$, then we find

$$\exp\left[\left(\lambda z b + \frac{q_1\mu_B}{z}\right)t\right] = \sum_{i=-\infty}^{\infty} \left(\sqrt{\beta_B z}\right)^i I_i(\alpha_B t) \quad (8)$$

Using (7), and then comparing the coefficients of z^i on both sides for $i \geq 1$, we obtain

$$P_{i,B}(t) = \gamma \int_0^t \left[\sum_{m=0}^{i-1} P_{i-m,V}(y)\beta_B^m I_m(\alpha_B(t-y)) + \sum_{m=1}^{\infty} P_{i+m,V}(y)\beta_B^{-m} I_m(\alpha_B(t-y)) \right]$$

$$\times\, \mathrm{e}^{-(\lambda b + q_1\mu_B)(t-y)} \mathrm{d}y - q_1\mu_B \int_0^t P_{1,B}(y)\beta_B^i I_i(\alpha_B(t-y))\, \mathrm{e}^{-(\lambda b + q_1\mu_B)(t-y)} \mathrm{d}y \quad (9)$$

It is well known that Eq. (9) holds for $i \leq -1$ with the left side probabilities replaced by the zero value. Also, identity $I_{-i}(x) = I_i(x)$ holds for $i \geq 1$. Thus, we get

$$0 = \gamma \int_0^t \left[\sum_{m=1}^{\infty} P_{m,V}(y)\beta_B^{-i-m} I_{i+m}(\alpha_B(t-y)) \right] \mathrm{e}^{-(\lambda b + q_1\mu_B)(t-y)} \mathrm{d}y$$

$$- q_1\mu_B \int_0^t P_{1,B}(y)\beta_B^{-i} I_i(\alpha_B(t-y)) \mathrm{e}^{-(\lambda b + q_1\mu_B)(t-y)} \mathrm{d}y \quad (10)$$

Equations (8) and (9) yield

$$P_{i,B}(t) = \gamma \int_0^t \left[\sum_{m=0}^{i-1} P_{i-m,V}(y)\beta_B^m I_m(\alpha_B(t-y)) + \sum_{m=1}^{\infty} P_{i+m,V}(y)\beta_B^{-m} I_m(\alpha_B(t-y)) - \sum_{m=1}^{\infty} P_{m,V}(y)\beta_B^{i-m} I_{i+m}(\alpha_B(t-y)) \right] \mathrm{e}^{-(\lambda b+q_1\mu_B)(t-y)}\mathrm{d}y \tag{11}$$

Finally for $i = 1, 2, 3, \ldots$ we get the closed form of $P_{i,B}(t)$ in terms of $P_{i,V}(t)$ as

$$P_{i,B}(t) = \gamma \int_0^t \mathrm{e}^{-(\lambda b+q_1\mu_B)(t-y)} \sum_{m=1}^{\infty} \beta_B^{i-m} P_{m,V}(y)\big[I_{i-m}(\alpha_B(t-y)) - I_{i+m}(\alpha_B(t-y))\big]\mathrm{d}y \tag{12}$$

3.2 Evaluation of $P_{i,V}(t)$

For the evaluation of $P_{i,V}(t)$, the continued fraction method is used. Now we will express $P_{i,V}(t)$ in terms of $P_{00}(t)$.

Denoting Laplace transformation of $f(t)$ by $\tilde{f}(s)$ and taking Laplace transformation of (1) and (2), we have

$$\tilde{f}_{00}(s) = \frac{1}{s + \lambda_v - q_0\mu_v \frac{f_{1V}(s)}{f_{00}(s)} - q_1\mu_B \frac{f_{1B}(s)}{f_{00}(s)}}, \tag{13}$$

and

$$\frac{\tilde{f}_{1V}(s)}{\tilde{f}_{00}(s)} = \frac{\lambda}{s + \lambda_v + \gamma + q_0\mu_v - q_0\mu_v \frac{f_{2V}(s)}{f_{1V}(s)}}. \tag{14}$$

Using (3) for $i \geq 2$, we have

$$\frac{\tilde{f}_{i,V}(s)}{\tilde{f}_{i-1,V}(s)} = \frac{\lambda_V}{s + \lambda_V + q_0\mu_V + \gamma - q_0\mu_V \frac{f_{i+1,V}(s)}{f_{i,V}(s)}}.$$

On iteration sequel, we get

$$\frac{\tilde{f}_{i,V}(s)}{\tilde{f}_{i-1,V}(s)} = \frac{\lambda_V}{s + \lambda_V + \gamma + q_0\mu_V - \frac{q_0\mu_V\lambda_V}{s+\lambda_V+q_0\mu_V+\gamma-\cdots}}. \tag{15}$$

From brevity, denote $p_V = s + \lambda_V + \gamma + \mu_V, \alpha_V = \sqrt{4\lambda_V q_0\mu_V},$ and $\beta_V = \sqrt{\frac{\lambda_V}{q_0\mu_V}}$.

Solving (13) and (14), we obtain

$$\tilde{f}_{i,V}(s) = \beta_V^i \left[\frac{p_V - \sqrt{p_V^2 - \alpha_V^2}}{\alpha_V} \right]^i \tilde{f}_{00}(s). \tag{16}$$

On taking inverse Laplace transform of (15), we have

$$P_{i,V}(t) = \lambda_V \beta_V^{i-1} \int_0^t \mathrm{e}^{-(\lambda_V + q_0\mu_V + \gamma)} \left[I_{i-1}(\alpha_V y) - I_{i+1}(\alpha_V y) \right] P_{00}(t-y)\mathrm{d}y \tag{17}$$

Using (11) and (15), for $i = 1$, we find

$$\frac{\tilde{f}_{1,V}(s)}{\tilde{f}_{00}(s)} = \beta_V E_V(s) \tag{18}$$

$$\text{and} \quad \frac{\tilde{f}_{1,B}(s)}{\tilde{f}_{00}(s)} = \frac{\gamma}{\mu_B} \sum_{m=1}^{\infty} [\eta]^m \left[E_V(s) E_B^{-1} \right]^m, \tag{19}$$

$$E_V(s) = \frac{p_V - \sqrt{p_V^2 - \alpha_V^2}}{\alpha_V}; \quad E_B(s) = \frac{p_B - \sqrt{p_B^2 - \alpha_B^2}}{\alpha_B}, \quad \eta = \frac{\beta_v}{\beta_B}$$

and $p_B = s + \lambda b + q_1\mu_B$.

Using (17) in (12), and after some algebraic manipulation, we obtain

$$\begin{aligned} \tilde{f}_{00}(s) = \frac{2}{\alpha_V} \sum_{m=0}^{\infty} (\eta)^m \sum_{k_1+k_2+k_3=m} \frac{m!}{k_1!\,k_2!\,k_3!} \left(\frac{\beta_B(q_0\mu_V + \gamma)}{\lambda_V} \right)^{k_2} \left(\frac{q_0\mu_V}{\sqrt{\lambda_V q_0 \mu_V}} \right)^{k_3} \\ \times \left[A_V(s)^{m+k_3+1} A_B(s)^{k_1+k_3} - \frac{\beta_V}{\beta_B} A_V(s)^{m+k_3+2} A_B(s)^{k_1+k_3+1} \right]. \end{aligned} \tag{20}$$

The inverse Laplace transformation of Eq. (19) gives,

$$\begin{aligned} P_{00}(t) = &\frac{2}{\alpha_V} \sum_{m=0}^{\infty} (\eta)^m \sum_{k_1+k_2+k_3=m} \frac{m!}{k_1!\,k_2!\,k_3!} \left(\frac{\beta_B(q_0\mu_V + \gamma)}{\lambda_V} \right)^{k_2} \left(\frac{q_0\mu_V}{\sqrt{\lambda_V q_0 \mu_V}} \right)^{k_3} \\ &\times \left[\frac{\alpha_V \alpha_B}{4} \left\{ I_{m+k_3}(\alpha_V t) - I_{m+k_3+2}(\alpha_V t) \right\} \mathrm{e}^{-(\lambda_V + q_0\mu_V + \gamma)t} \right. \\ &* \left\{ I_{k_1+k_3-1}(\alpha_B t) - I_{k_1+k_3+1}(\alpha_B t) \right\} \mathrm{e}^{-(\lambda b + q_1\mu_B)t} \\ &- \lambda b q_1 \mu_B \left\{ I_{m+k_3+1}(\alpha_v t) - I_{m+k_3+3}(\alpha_v t) \right\} \mathrm{e}^{-(\lambda_v + q_0\mu_v + \gamma)t} \\ &\left. * \left\{ I_{k_1+k_3}(\alpha_B t) - I_{k_1+k_3+2}(\alpha_B t) \right\} \mathrm{e}^{-(\lambda b + q_1\mu_B)t} \right]. \end{aligned} \tag{21}$$

4 Stationary Distribution

The system size probabilities at the stationary level of $M/M/1$ Markov model with working vacation and balking can be deduced from the transient probabilities presented in the last section. The stationary distribution denoted by $\lim_{t\to\infty} P_{i,j}(t) = \pi_{i,j}, i \geq 0; j \in \{0, V, B\}$ can be evaluated by using the standard properties of the Laplace transformation given by $\pi_{i,j} = \lim_{s\to 0} \tilde{f}_{i,j}(s)$, where $i \geq 0; j \in \{0, V, B\}$.

For the steady state, we have

$$\rho_V = \frac{\lambda_V}{q_0 \mu_V} < 1 \quad \text{and } \rho_B = \frac{\lambda}{q_1 \mu_B} < 1,$$

From (16), the stationary distribution for vacation state is obtained as

$$\pi_{i,V} = \lim_{s\to 0} s\,\tilde{f}(s) = \frac{1}{2^i}\left[\varsigma - (\varsigma^2 - 4\rho_V)^{1/2}\right]^i \pi_{0,0}, \quad i = 1, 2, 3, \ldots \tag{22}$$

Laplace transformation of (11) gives

$$\tilde{f}_{i,B}(s) = \frac{\gamma\, f_{0,0}(s)}{\sqrt{P_B^2 - \alpha_B^2}} \sum_{m=1}^{\infty} \beta_B^{i-m} \left(\frac{P_V - \sqrt{P_V^2 - \alpha_V^2}}{2\mu_V}\right)^m \times \left[\left(\frac{P_B - \sqrt{P_B^2 - \alpha_B^2}}{\alpha_B}\right)^{|i-m|} - \left(\frac{P_B - \sqrt{P_B^2 - \alpha_B^2}}{\alpha_B}\right)^{i+m}\right]. \tag{23}$$

From relation (23) for $i = 1, 2, 3, \ldots$, we get

$$\pi_{i,B} = \lim_{s\to 0} s\,\tilde{f}_{i,B}(s) \frac{2\gamma \pi_{0,0}}{\mu_B\left[\varsigma + \sqrt{\varsigma^2 - 4\rho_V}\right]} \sum_{m=1}^{i} \rho_B^{i-m} \frac{1}{2^m}\left[\varsigma - (\varsigma^2 - 4\rho_V)^{1/2}\right]^m \tag{24}$$

Further, normalizing condition $\sum_{i=0}^{\infty} \pi_{i,V} + \sum_{i=1}^{\pi} \pi_{i,B} = 1$, yields

$$\pi_{0,0} = \frac{(1-\rho_B)(1-A_V)}{(1-\rho_B) - \gamma K_V} \tag{25}$$

$$\text{where} \quad K_V = \frac{A_V}{\mu_{\mathbf{B}}(1-A_V)},\ A_V = \frac{\varsigma - \sqrt{\varsigma^2 - 4\rho_V}}{2} \quad \text{and} \quad \delta_V = \frac{\gamma}{\rho_V}.$$

5 Some Performance Measures

Now we evaluate the expected values related to the time-dependent measures of the queue size distribution as follows:

I. Let, $N(t)$ indicate the total number of the customers at time t. So the expected value of the total customers in the system at time t is provided by

$$u(t) = E(N(t)) = \sum_{i=1}^{\infty} i(P_{i,B}(t) + P_{i,V}(t)) \tag{26}$$

$$\text{Also} \quad u(0) = \sum_{i=1}^{\infty} i(P_{i,B}(t) + P_{i,V}(t)) = 0 \tag{26a}$$

$$\text{and} \quad u'(t) = \sum_{i=1}^{\infty} i(P'_{i,B}(t) + P'_{i,V}(t)) \tag{26b}$$

$$u(t) = \sum_{i=1}^{\infty} i \int_{0}^{t} (P_{i,B}(t') + P_{i,V}(t'))\mathrm{d}t' \tag{26c}$$

The transient probabilities $p_{i,B}(t)$ and $p_{i,V}(t)$ used in above equations are given in (11) and (17).

II. Let $Y(t)$ denote the number of the customers in the system at time 't'. We obtain the variance for the total number of customers in the system at time t as follows:

$$\text{Var}\,(Y(t)) = E(Y^2(t)) - (E(Y(t))^2 \tag{27}$$

$$\text{Var}\,(Y(t)) = r(t) - (u(t))^2$$

where

$$r(t) = E(Y^2(t)) = \sum_{i=1}^{\infty} i^2(P_{i,B}(t) + P_{i,V}(t)),$$

$$\text{and} \quad r(0) = E(Y^2(0)) = \sum_{i=1}^{\infty} i^2(P_{i,B}(0) + P_{i,V}(0)) \tag{27a}$$

$$\text{Also} \quad r'(t) = E(Y^2(t)) = \sum_{i=1}^{\infty} i^2(P'_{i,B}(t) + P'_{i,V}(t)). \tag{27b}$$

III. Let TP(t) be the throughput of the system. To explore the system utility, we formulate TP(t) in terms of transient probabilities as

$$\mathrm{TP}(t) = \sum_{i=0}^{\infty} q_0 \mu_V P_{i,V} + \sum_{i=0}^{\infty} q_1 \mu_B P_{i,B} \tag{28}$$

IV. The probabilities of the states of the server, when it is in normal busy mode and when it is in working vacation mode are denoted by $p_B(t)$ and $p_V(t)$, respectively. Then

$$\mathrm{PB}(t) = \sum_{i=1}^{\infty} P_{iB}(t), \mathrm{PWV}(t) = \sum_{i=0}^{\infty} P_{iV}(t) \tag{29}$$

V. Here we formulate the expression of the cost function for the system using the following cost elements:

C_H Holding cost of a customer per unit time in the system.
C_B Cost spent/unit time in the system.
C_{WV} Cost spent/unit time in the system.
C_1 Cost/unit time spent on the service counter in busy state.
C_2 Cost per unit time spent on the service counter in working vacation mode.

So, the cost function is obtained as

$$\mathrm{TC}(t) = C_H \mathrm{EN}(t) + C_B \mathrm{PB}(t) + C_{\mathrm{WV}} \mathrm{PWV}(t) + C_1 \mu_B + C_2 \mu_V \tag{30}$$

6 Numerical Simulation

Here we are presenting some numerical results to look into the sensitiveness of the studied model for several performance measures and system cost. To demonstrate the system behavior with respect to various system descriptors, the numerical outcomes are displayed clearly in Tables 1 and 2. The numerical results are calculated by taking the default parameters fixed as

$$\lambda = 0.5, \lambda_V = 0.4, b = 0.1, q_0 = 0.15, q_1 = 0.15\, \gamma = 1\, \mu_V = 5 \quad \text{and} \quad \mu_B = 7.$$

In Table 1, we notice that as arrival rate goes on increasing, the throughput TP(t), probability of normal busy state PB(t), and the total cost of the system TC(t) increases. This trend is expected because high arrival rate is directly increase the number of customers as such there may be increment in TP(t), PB(t) & TC(t). It is seen that the working vacation state probability PWV(t) decreases by increasing the arrival rate λ. In Table 2, the results for throughput, busy state probability, and working state probability show the increasing trend as μ_B increases. However as time passes, PWV(t) decreases but TP(t) and PB(t) increase. The cost function TC(t) reveals the increment for the higher value of μ_B and time both.

Table 1 Various system indices for different value of λ with respect to t

γ	λ	TP(t)	PB(t)	PWV(t)	TC(t)
2	0.5	6.0747	0.074662	0.925338	381.4559
8		6.187	0.187031	0.812969	400.9074
14		6.1876	0.187632	0.812368	400.9951
20		6.1876	0.187634	0.812366	400.9952
2	3.5	6.0763	0.076313	0.923687	382.3431
8		6.2364	0.236425	0.763575	415.4825
14		6.2457	0.245667	0.754333	417.8772
20		6.2466	0.246642	0.753358	418.1388
2	6.5	6.0778	0.077838	0.922162	383.2484
8		6.2944	0.29438	0.70562	437.3875
14		6.3293	0.329305	0.670695	448.8049
20		6.3378	0.337774	0.662226	451.6178

Table 2 Various system indices for different values of μ_B with respect to t

γ	μ_B	TP(t)	PB(t)	PWV(t)	TC(t)
2	5	4.08756	0.08756	0.91244	294.8988
8		4.276375	0.276375	0.723625	326.9953
14		4.279463	0.279463	0.720537	327.4882
20		4.279519	0.279519	0.720481	327.4962
2	6	4.167396	0.083698	0.916302	339.3876
8		4.481368	0.240684	0.759316	366.9213
14		4.484275	0.242138	0.757862	367.1372
20		4.484285	0.242142	0.757858	367.137
2	7	4.240284	0.080095	0.919905	383.9108
8		4.638584	0.212861	0.787139	408.0148
14		4.640753	0.213584	0.786416	408.1142
20		4.640746	0.213582	0.786418	408.1133

For display of graphs for EN(t) and PB(t), the default parameters are chosen as $\lambda = 0.5, \lambda_V = 0.4, b = 0.1, q_0 = 0.15, q_1 = 0.15, \gamma = 1, \mu_V = 1$ and $\mu_B = 1.5$. The numerical results for EN(t) and PB(t) are exhibited in Figs. 2, 3, 4, 5, 6 and 7, respectively.

From Figs. 2 and 4, it is clear that at the initial stage, the mean value of the total customers present in the system increases sharply as time increases. However as time grows higher, the increasing trend diminishes and finally becomes almost constant. The increasing trends of EN(t) for increasing value of λ can be justified due to the fact that there will be more customers in the system with the higher arrival rate λ at any time. Figure 3 shows that the trends of expected number of customers for different arrival rate of working vacation λ_V. The increasing trends of $EN(t)$ for the higher values of λ_v is clearly seen. From Figs. 4 and 5, it can be noticed clearly

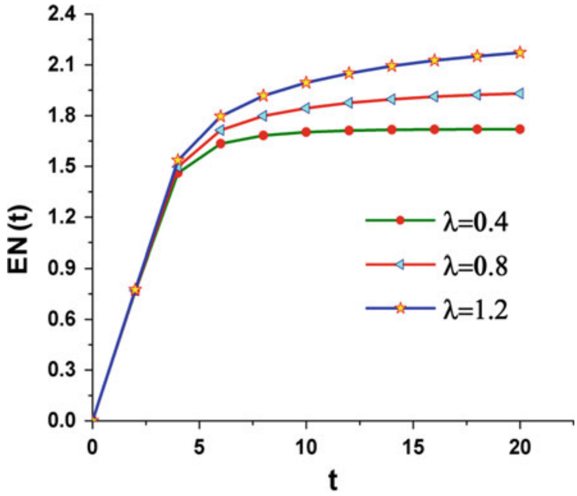

Fig. 2 Variation EN(t) with respect to time for different values of λ

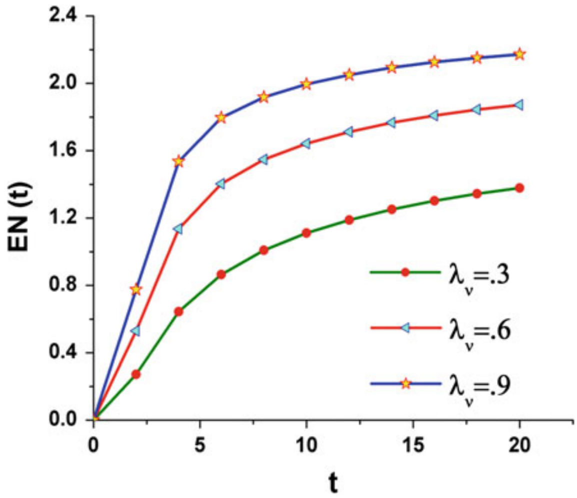

Fig. 3 Variation EN(t) with respect to time for different values of λ_v

that as service rates μ_B and μ_V increase, the queue length $EN(t)$ decreases. The decreasing trends of $EN(t)$ for the increment in service rate as shown in Figs. 4 and 5 are also as per our expectation due to the reason that an improvement in the service can lower down the queue size. The larger value of service rates in both normal busy as well as working vacation modes of the server results in lower value of $EN(t)$. The graphs of normal busy and working vacation probabilities for different arrival rate are depicted in Figs. 6 and 7, respectively. It is clearly noticed from these figures that for large values of arrival rate in busy operating mode, the probability PB(t) is high

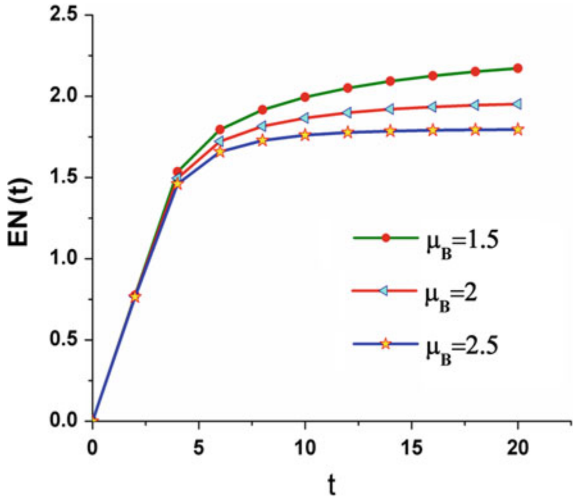

Fig. 4 Variation EN(t) with respect to time for different values of μ_B

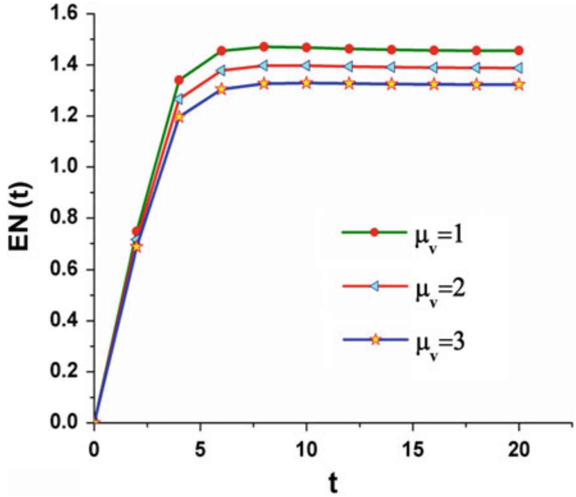

Fig. 5 Variation EN(t) with respect to time for different values of μ_v

but in working vacation state, the greater the value of arrival rate, lesser is the value of probability PWV(t). It is seen that the effects of λ and λ_V become significant as time increases.

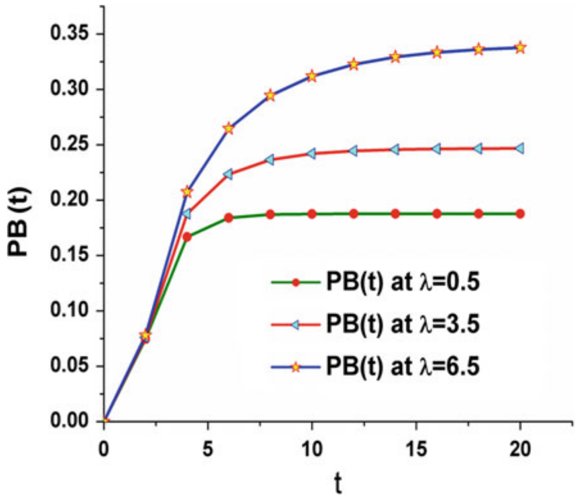

Fig. 6 Variation PB(t) with respect to time for different values of λ

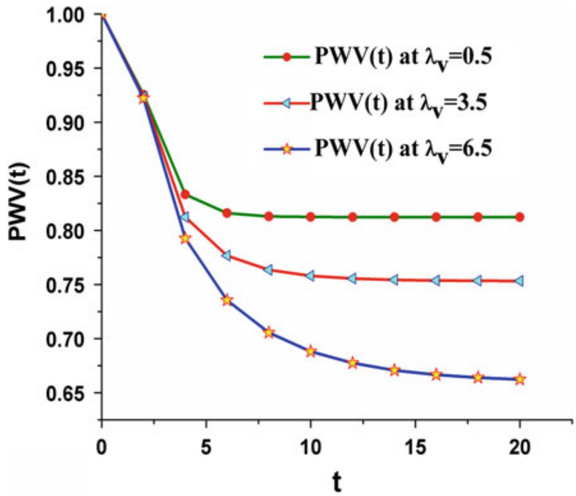

Fig. 7 Variation PB(t) with respect to time for different values of λ_v

7 Conclusion

In this article, an analytic approach is suggested to investigate the transient probabilities for the total number of customers present in the system at time epoch 't'. Transient Markov process is studied by involving the continued fraction method and generating function technique to derive the analytical expressions of transient probabilities in the explicit form for $M/M/1$ feedback model operating under the working vacation and discouragement. The queueing model is studied by involving realistic assumptions and has many applications in industrial scenario, namely production and manufacturing systems, computer and communication networks, etc. By evaluating the performance measures such as mean and variance for the system size, the optimal parameters can be determined which can be further used to improve the system capacity/efficiency. Furthermore, the numerical simulation done by examining the variations in different parameters facilitates the insight for the optimal control design of the existing/new systems.

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Transient and Steady-State Behavior of a Two-Heterogeneous Servers' Queuing System with Balking and Retention of Reneging Customers

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Abstract This paper studies a two-heterogeneous servers' queuing model with customers' impatience and retention. The time-dependent and steady-state analyses of the model are performed. Some measures of effectiveness like mean number of customers in the system, mean reneging rate, and mean rate of retention are obtained. Finally, some particular cases are discussed.

Keywords Customers' impatience · Heterogeneous servers · Mean retention rate · Probability generating function

1 Introduction

The studies on queuing systems with multi-servers generally expect the servers to serve the customers at the same rate. But such scenario occurs only if the service system is automatic in nature; otherwise, the servers work with different rates. We cannot expect that the work will be carried out at same rate in a queuing system with human servers. In our daily life, we usually face situations of this kind, e.g., at check-out counters in retail stores, in hospitals. Heterogeneity in service systems is first presented by Morse [1]. Then, this problem is further extended by Saaty [2]. Singh [3] analyzes a heterogeneous queuing system with balking. He performs comparative studies. The steady-state probabilities of system size of a limited capacity queuing model with heterogeneous servers are obtained by Sharma and Dass [4]. Yue et al. [5] extend Singh [3]'s work by considering the server breakdown in two-heterogeneous servers' queuing system. They derive various measures of performance by employing matrix-geometric method. Kumar and Madheswari [6] obtain the stationary proba-

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bilities of a heterogeneous queueing systems having multiple vacations. Dharmaraja [7] obtained the time-dependent probabilities of two-processor heterogeneous system using probability generating function technique. Sridhar and Pitchai [8] derive the stationary probabilities of heterogeneous two-server queuing model. Ammar and Alharbi [9] obtain the transient solution of heterogeneous queuing model with time-dependent arrival and service rates.

A number of authors study customers' impatience in queuing models with many servers. Haight [10, 11] is the first to work on queues with impatience. Subba Rao [12] studies a non-Markovian single-server queuing model with impatient customers and service interruptions. Baccelli et al. [13] obtain the results for $GI/GI/1 + GI$ queuing system with impatient customers in terms of Volterra equation. Kumar and Raja [14] obtain various system performance measures of a multi-server retrial queue with feedback using matrix-geometric method. Yechiali [15] derives the stationary probabilities of various queueing systems with catastrophes and customers' impatience. Baek et al. [16] compute the steady-state probabilities of an $M/M/1$ queuing system with heterogeneous impatient customers and consumable additional items. Vijaya Laxmi and Jyothsna [17] study a heterogeneous servers' queue with multiple vacations and customers' impatience.

The concept of retaining a reneging customers is given by Kumar and Sharma [18]. They introduce this concept into a finite capacity, single-server Markovian queuing model with reneging phenomenon and derive its stationary system size probabilities. Further, they extend this model by incorporating balking in it [19]. Thiagarajan and Premalatha [20] derived the steady-state solution of a finite waiting space Markovian single-server system with controllable arrival rates and retention.

After reviewing the literature, we can notice that the time-dependent solution of this model is never carried out. Therefore, we study a queuing model with two servers having different rates of service. The transient analysis of the model is performed.

The remaining paper is written as follows: Sect. 2 describes the system model. Section 3 presents the mathematical model. In Sect. 4, the time-dependent behavior is studied. Stationary probabilities of the model are provided in Sect. 5. Particular cases are given in Sect. 6. Section 7 is related to expected system size and variance. Measures of effectiveness are presented in Sect. 8. Finally, the numerical analysis is done in Sect. 9.

2 System Model

The arrivals occur one by one in a Poisson stream with intensity λ. The system has two servers who serve the customers with different service rates μ_1 and μ_2. The service time distribution is negative exponential. If both the servers are free, arriving customers always go to server S_1 for service else he joins the server who is idle. If an arriving customer finds at least two customers in the system, he either balks with probability 1-β or enters the queue with complementary probability. Upon joining the queuing system and waiting for some time, a customer may become impatient and abundance the queue with probability p or gets retained with complementary probability. The probability distribution for reneging times is assumed negative exponential with rate ξ.

3 Mathematical Model

First of all, we define some notations:

- $Q_{0,0}(t)$ the probability that at time t the system is empty.
- $Q_{0,1}(t)$ the probability that at time t the first server is idle and the second server is occupied with no waiting line.
- $Q_{1,0}(t)$ the probability that at time t the first server is occupied and the second server is idle with no waiting line.
- $Q_n(t)$ the probability that at time t the system size is n.

The queuing model is based on following equations:

$$\frac{dQ_{0,0}(t)}{dt} = -\lambda Q_{0,0}(t) + \mu_1 Q_{1,0}(t) + \mu_2 Q_{0,1}(t) \tag{1}$$

$$\frac{dQ_{1,0}(t)}{dt} = -(\lambda + \mu_1) Q_{1,0}(t) + \lambda Q_{0,0}(t) + \mu_2 Q_2(t), \tag{2}$$

$$\frac{dQ_{0,1}(t)}{dt} = -(\lambda + \mu_2) Q_{0,1}(t) + \mu_1 Q_2(t), \tag{3}$$

$$\frac{dQ_2(t)}{dt} = -(\beta\lambda + \mu) Q_2(t) + \lambda Q_{0,1}(t) + \lambda Q_{1,0}(t) + (\mu + \xi p) Q_3(t), \tag{4}$$

$$\begin{aligned}\frac{dQ_n(t)}{dt} = &-(\beta\lambda + \mu + (n-2)\xi p) Q_n(t) + \beta\lambda Q_{n-1}(t) \\ &+(\mu + (n-1)\xi p) Q_{n+1}(t), \quad n = 3, 4 \ldots\end{aligned} \tag{5}$$

where $\mu = \mu_1 + \mu_2$.
Initial condition: $Q_{0,0}(0) = 1$.

4 Time-Dependent Behavior of the Model

Define p.g.f. $Q(z, t)$ of probabilities $Q_n(t)$ by

$$Q(z, t) = R(t) + \sum_{n=0}^{\infty} Q_{n+3}(t) z^{n+1}; \; Q(z, 0) = 1 \tag{6}$$

with

$$R(t) = Q_{0,0}(t) + Q_{0,1}(t) + Q_{1,0}(t) + Q_2(t) \tag{7}$$

Equations (1)–(5) give

$$\frac{\partial Q(z,t)}{\partial t} = \xi p(1-z)\frac{\partial Q(z,t)}{\partial z} + [(\mu-\xi p)(z^{-1}-1)+\beta\lambda(z-1)] \times [Q(z,t)-R(t)] + \beta\lambda(z-1)Q_2(t) \tag{8}$$

On solving (8), we obtain

$$Q(z,t) = \exp\left\{[(\mu-\xi p)(z^{-1}-1)+\beta\lambda(z-1)]t\right\} + \int_0^t [\beta\lambda(z-1)Q_2(u) - ((\mu-\xi p)(z^{-1}-1)+\beta\lambda(z-1))R(u)] \times \exp\left\{[(\mu-\xi p)(z^{-1}-1) + \beta\lambda(z-1)](t-u)\right\} \mathrm{d}u \tag{9}$$

Taking $a = 2\sqrt{\beta\lambda(\mu-\xi p)}$ and $b = \sqrt{\frac{\beta\lambda}{\mu-\xi p}}$ and using the modified Bessel function of first kind $I_n(.)$, we get

$$\exp\left\{\left(\beta\lambda z + \frac{\mu-\xi p}{z}\right)t\right\} = \sum_{n=-\infty}^{\infty}(bz)^n I_n(at) \tag{10}$$

4.1 Determination of $Q_{n+2}(t)$

After comparison of coefficients of z^n on both sides of (9), we get

$$Q_{n+2}(t) = \exp\{-(\beta\lambda+\mu-\xi p)t\}b^n I_n(at) + \beta\lambda\int_0^t \exp\{-(\beta\lambda+\mu-\xi p)(t-u)\} \times \left[I_{n-1}(a(t-u))b^{n-1} - I_n(a(t-u))b^n\right] Q_2(u)\,\mathrm{d}u - \int_0^t \exp\{-(\beta\lambda+\mu-\xi p)(t-u)\}R(u)[\beta\lambda I_{n-1}(a(t-u))b^{n-1} - (\beta\lambda+\mu-\xi p)I_n(a(t-u))b^n + (\mu-\xi p)I_{n+1}(a(t-u)b^{n+1}]\,\mathrm{d}u; \quad n = 1,2,\ldots \tag{11}$$

By replacing the left-hand side by zero, Eq. (11) holds for $n = -1, -2, -3, \ldots$. For obtaining $n = 1, 2, 3, \ldots$, we use $I_{-n}(t) = I_n(t)$,

$$\int_0^t \exp\{-(\beta\lambda+\mu-\xi p)(t-u)\}R(u)[\beta\lambda I_{n+1}(a(t-u))b^{n-1} - (\beta\lambda+\mu-\xi p) I_n(a(t-u))b^n + (\mu-\xi p)I_{n-1}(a(t-u))b^{n+1}]\,\mathrm{d}u$$

$$= \exp\{-(\beta\lambda + \mu - \xi p)t\} I_n(at) b^n + \beta\lambda \int_0^t \exp\{-(\beta\lambda + \mu - \xi p)(t-u)\} Q_2(u) \times [I_{n+1}(a(t-u)) b^{n-1} - I_n(a(t-u)) b^n] \, \mathrm{d}u \tag{12}$$

Using (12) in (11), we get

$$Q_{n+2}(t) = nb^n \int_0^t \exp\{-(\beta\lambda + \mu - \xi p)(t-u)\} \frac{I_n(a(t-u))}{(t-u)} Q_2(u) \, \mathrm{d}u; \quad n = 1, 2, \ldots \tag{13}$$

4.2 Determination of $Q_2(t)$

Equations (1)–(3) in matrix notation can be written as follows:

$$\frac{\mathbf{dQ(t)}}{dt} = V\mathbf{Q(t)} + \mu_2 Q_2(t)\mathbf{e_1} + \mu_1 Q_2(t)\mathbf{e_2} \tag{14}$$

where the matrix V is given as:

$$V = \begin{pmatrix} -\lambda & \mu_1 & \mu_2 \\ \lambda & -(\lambda + \mu_1) & 0 \\ 0 & 0 & -(\lambda + \mu_2) \end{pmatrix}$$

$\mathbf{Q(t)} = (Q_{0,0}(t) \;\; Q_{1,0}(t) \;\; P_{Q,1}(t))^T$, $\mathbf{e_1} = (0 \;\; 1 \;\; 0)^T$ and $\mathbf{e_2} = (0 \;\; 0 \;\; 1)^T$

Let $Q_n^*(s)$ denotes the Laplace transform (L.T.) of $Q_n(t)$. Taking the L.T. of Eq. (14), we get

$$\mathbf{Q^*(s)} = (sI - V)^{-1}[\mu_2 Q_2^*(s)\mathbf{e_1} + \mu_1 Q_2^*(s)\mathbf{e_2} + \mathbf{Q(0)}] \tag{15}$$

with $\mathbf{Q(0)} = (1 \;\; 0 \;\; 0)^T$. To find $Q_2^*(s)$, we have

$$\mathbf{e}^T\mathbf{Q^*(s)} + Q_2^*(s) = R^*(s) \tag{16}$$

where $\mathbf{e} = (1 \;\; 1 \;\; 1)^T$.

Taking

$$f(s) = \left[(s + \beta\lambda + (\mu - \xi p)) - \sqrt{(s + \beta\lambda + (\mu - \xi p))^2 - a^2}\right]$$

When $n = 0$, Eq. (9) yields

$$R(t) = \exp\{-(\beta\lambda + \mu - \xi p)(t)\}I_0(at) + \beta\lambda \int_0^t Q_2(u) \exp\{-(\beta\lambda + \mu - \xi p)(t-u)\}[I_{-1}(a(t-u))b^{-1} - I_0(a(t-u))]\,\mathrm{d}u + \int_0^t R(u)\exp\{-(\beta\lambda + \mu - \xi p)(t-u)\}[(\beta\lambda + \mu - \xi p)I_0(a(t-u)) - \{\beta\lambda b^{-1}I_{-1}(a(t-u)) + (\mu - \xi p)bI_1(a(t-u))\}]\,\mathrm{d}u \tag{17}$$

The L.T. of (17) gives

$$sR^*(s) = 1 + Q_2^*(s)\left[\frac{\beta\lambda}{ab}\{f(s)\} - \beta\lambda\right] \tag{18}$$

Using Eq. (18) in (16), we obtain

$$Q_2^*(s) = \frac{1 - s\mathbf{e}^{\mathbf{T}}(sI - V)^{-1}\mathbf{Q(0)}}{\{(s+\beta\lambda) - \frac{1}{2}[f(s)] + s\mathbf{e}^{\mathbf{T}}(sI - V)^{-1}[\mu_2\mathbf{e_1} + \mu_1\mathbf{e_2}]\}} \tag{19}$$

Let

$$(sI - V)^{-1} = (m_{ij}^*(s))_{3\times 3}$$

The inverse of matrix $(sI - V)$ is given by

$$\frac{1}{\mid D(s)\mid}\begin{pmatrix} (s+\lambda+\mu_1)(s+\lambda+\mu_2) & \mu_1(s+\lambda+\mu_2) & \mu_2(s+\lambda+\mu_1) \\ \lambda(s+\lambda+\mu_2) & (s+\lambda)(s+\lambda+\mu_2) & \lambda\mu_2 \\ 0 & 0 & (s+\lambda)(s+\lambda+\mu_1) - \lambda\mu_1 \end{pmatrix} \tag{20}$$

where $\mid D(s) \mid = s^3 + (3\lambda + \mu)s^2 + (3\lambda^2 + \lambda(\mu_1 + \mu_2) + \mu_l\mu_2)s + \lambda^2(\lambda + \mu_2)$.

To find the characteristic roots of matrix V

$$\mid D(s) \mid = 0 \tag{21}$$

Let $a_k, k = 1, 2, 3$ denote the characteristic roots of (20). Then, $a_1 = -(\lambda + \mu_2)$, $a_2, a_3 = \frac{-(2\lambda+\mu_1)\pm\sqrt{4\lambda\mu_1+\mu_1^2}}{2}$.

The use of (20) gives

$$s\mathbf{e}^T(sI - V)^{-1}Q(0) = s\sum_{j=1}^{3} m_{j1}^*(s) = r_1^*(s) \tag{22}$$

$$s\mathbf{e}^T(sI - V)^{-1}[\mu_2\mathbf{e}_1 + \mu_1\mathbf{e}_2] = s\left[\mu_2\sum_{j=1}^{3} m_{j2}^*(s) + \mu_1\sum_{j=1}^{3} m_{j3}^*(s)\right]$$
$$= r_2^*(s) \tag{23}$$

Substituting (22) and (23) in (19), we get

$$Q_2^*(s) = \frac{1 - r_1^*(s)}{(s+\beta\lambda) - \frac{1}{2}[f(s)] + r_2^*(s)} \tag{24}$$

Hence, (24) simplifies into

$$Q_2^*(s) = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{(-1)^l}{\Psi}\left(\frac{a}{2\lambda\beta}\right)^{n+1}\left(\frac{1}{\Psi}\right)^l\binom{n}{l}$$
$$\left[\frac{[(s+\beta\lambda+\mu-\xi p) - \sqrt{(s+\beta\lambda+\mu-\xi p)^2 - a^2}]^{n+1}}{a^{n+1}}\right.$$
$$(1 - r_1^*(s))(1 - r_2^*(s))^l\,] \tag{25}$$

where $\Psi = (\mu - \xi p)$.

The Laplace inverse of (25) is

$$Q_2(t) = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{(-1)^l}{\Psi}\left(\frac{a}{2\lambda\beta}\right)^{n+1}\left(\frac{1}{\Psi}\right)^l\binom{n}{l}\times\int_0^t r_2^{C(l)}(u-v)$$
$$\times\left[\exp\{-(\beta\lambda+\mu-\xi p)u\}\{I_n(au) - I_{n+2}(au)\} - \int_0^u r_1(u-v)\right.$$
$$\times\exp\{-(\beta\lambda+\mu-\xi p)v\}\{I_n(av) - I_{n+2}(av)\}dv\,]\,\mathrm{d}u \tag{26}$$

where $r_2^{C(l)}(t)$ is $l-fold$ convolution of $r_2(t)$ with itself with $r_2^{C(0)} = \delta(t)$, $a = 2\sqrt{\beta\lambda(\mu-\xi p)}$, $b = \sqrt{\frac{\beta\lambda}{\mu-\xi p}}$, and $\Psi = (\mu - \xi p)$.

4.3 Determination of $Q_{0,0}(t)$, $Q_{0,1}(t)$ and $Q_{1,0}(t)$

Using (20) in (15), we get

$$Q_{00}^*(s) = \frac{1}{\mid D(s)\mid}\left[(s+\lambda+\mu_1)(s+\lambda+\mu_2) + \mu_1\mu_2(s+\lambda+\mu_2)Q_2^*(s)\right.$$
$$+\,\mu_1\mu_2(s+\lambda+\mu_1)Q_2^*(s)\,] \tag{27}$$

$$Q_{10}^*(s) = \frac{1}{|\,D(s)\,|}\left[\lambda(s+\lambda+\mu_1) + \mu_2(s+\lambda)(s+\lambda+\mu_2)Q_2^*(s)\right.$$
$$\left. + \mu_1\mu_2\lambda Q_2^*(s)\,\right] \tag{28}$$

$$Q_{01}^*(s) = \frac{1}{|\,D(s)\,|}\left[(\mu_1(s+\lambda)(s+\lambda+\mu_1) - \lambda\mu_1)Q_2^*(s)\right] \tag{29}$$

By taking the inverse of Eqs. (27), (28), and (29), we get

$$Q_{00}(t) = m_{11}(t) + \int_0^t [\mu_1 m_{12}(u) + \mu_2 m_{13}(u)]Q_2(t-u)\,\mathrm{d}u \tag{30}$$

$$Q_{10}(t) = m_{21}(t) + \int_0^t [\mu_2 m_{22}(u) + \mu_1 m_{23}(u)]Q_2(t-u)\,\mathrm{d}u \tag{31}$$

$$Q_{01}(t) = m_{31}(t) + \int_0^t [\mu_1 m_{33}(u)]Q_2(t-u)\,\mathrm{d}u \tag{32}$$

Thus, Eqs. (13), (26), (30), (31), and (32) constitute the transient solution to the model.

5 Stationary Probabilities

Case 1: If arrival rate is not equal to service rate
In Eq. (25), we use Tauberian theorem and get

$$Q_2 = \frac{1}{\frac{1}{2}[(\beta\lambda - \mu + \xi p) + \sqrt{(\beta\lambda - \mu + \xi p)^2 - a^2}]} \tag{33}$$

By taking Laplace transform of (13), we have

$$Q_{n+2}^*(s) = n\left(\frac{b}{a}\right)^n [f(s)]^n Q_2^*(s); \quad n \geq 1 \tag{34}$$

By applying the Tauberian theorem again, we obtain

$$Q_{n+2} = \left(\frac{b}{a}\right)^n [\beta\lambda + \mu - \xi p + \sqrt{(\beta\lambda + \mu - \xi p)^2 - a^2}]^n Q_2; n \geq 1 \tag{35}$$

In the similar manner from Eqs. (30), (31), and (32), we get

$$Q_{00} = \frac{\mu_1\mu_2(2\lambda + \mu)}{\lambda^2(\lambda + \mu_2)} Q_2 \tag{36}$$

$$Q_{10} = \frac{\mu_2(\lambda+\mu)}{\lambda(\lambda+\mu_2)} Q_2 \tag{37}$$

and

$$Q_{01} = \frac{\mu_1}{\lambda+\mu_2} Q_2 \tag{38}$$

Case 2: If arrival rate is equal to service rate
In (33), (35), (36), (37), and (38), we put $\lambda = \mu$; then,

$$Q_2 = \frac{1}{\frac{1}{2}[(\beta\lambda-\mu+\xi p)+\sqrt{(\beta\lambda-\mu+\xi p)^2-a^2}]} \tag{39}$$

$$Q_{n+2} = \left(\frac{b}{a}\right)^n [\beta\lambda+\mu-\xi p+\sqrt{(\beta\lambda+\mu-\xi p)^2-a^2}]^n Q_2 \tag{40}$$

$$Q_{00} = \frac{3\mu_1\mu_2}{\mu(\mu+\mu_2)} Q_2 \tag{41}$$

$$Q_{10} = \frac{2\mu_2}{\mu+\mu_2} Q_2 \tag{42}$$

and

$$Q_{01} = \frac{\mu_1}{\mu+\mu_2} Q_2. \tag{43}$$

6 Particular Cases

Case 1 If there is no balking, no reneging, and no retention i.e., ($\beta = 0$, $\xi = 0$, $p = 1$), then the results of our model coincide with [21] with $\pi_1 = \pi_2 = 1$ in his model.
Case 2 If $p = 1$, then the results of our model coincide with [22].
Case 3 If there is no retention and no reneging, i.e., ($p = 1$ and $\xi = 0$), then the results of our model coincide with [3].

7 Expected System Size and Variance

7.1 Expected System Size

At time t, the expected number of customers in the system is given by

$$L_s(t) = E\{X(t)\} = Q_{1,0}(t) + Q_{0,1}(t) + \sum_{n=0}^{\infty}(n+2)Q_{n+2}(t)$$

On solving, we obtain

$$L_s(t) = E\{X(t)\} = 2Q_2(t) + \int_0^t [\mu_2 m_{22}(t-u) + \mu_1(m_{23}(t-u) + m_{33}(t-u))]Q_2(u)\,\mathrm{d}u \\ + m_{21}(t) + \sum_{n=1}^{\infty} n(n+2)b^n \int_0^t \exp\{-(\beta\lambda + \mu - \xi p)(t-u)\} \\ \times \frac{I_n(a(t-u))}{(t-u)} Q_2(u)\,\mathrm{d}u \tag{44}$$

where $Q_2(t)$ is given in Eq. (26).

7.2 Variance

At time t, the variance number of customers in the system is given by

$$V\{X(t)\} = E\{X^2(t)\} - [E\{X(t)\}]^2 \tag{45}$$

where

$$E\{X^2(t)\} = 4Q_2(t) + \int_0^t [\mu_2 m_{22}(t-u) + \mu_1(m_{23}(t-u) + m_{33}(t-u))]Q_2(u)\,\mathrm{d}u \\ + m_{21}(t) + \sum_{n=1}^{\infty} n^2(n+2)b^n \int_0^t \exp\{-(\beta\lambda + \mu - \xi p)(t-u)\} \\ \times \frac{I_n(a(t-u))}{(t-u)} Q_2(u)\,\mathrm{d}u \tag{46}$$

On putting the above equation in (47), we obtain

$$V\{X(t)\} = 4Q_2(t) + \int_0^t [\mu_2 m_{22}(t-u) + \mu_1(m_{23}(t-u) + m_{33}(t-u))]Q_2(u)\,\mathrm{d}u \\ + m_{21}(t) + \sum_{n=1}^{\infty} n^2(n+2)b^n \int_0^t \exp\{-(\beta\lambda + \mu - \xi p)(t-u)\} \\ \times \frac{I_n(a(t-u))}{(t-u)} Q_2(u)\,\mathrm{d}u - [E\{X(t)\}]^2 \tag{47}$$

where $Q_2(t)$ and $E\{X(t)\}$ are provided in Eqs. (26) and (46), respectively.

8 Measures of Effectiveness

8.1 Mean Reneging Rate

$$R_r(t) = \sum_{n=2}^{\infty}(n-2)\xi p Q_n(t)$$

8.2 Mean Retention Rate

$$R_R(t) = \sum_{n=2}^{\infty}(n-2)\xi q Q_n(t)$$

where $Q_n(t)$ is given in Eqs. (13) and (26).

9 Numerical Analysis

The numerical analysis of the model with reference to various measures of effectiveness like the mean reneging rate ($R_r(t)$) and mean retention rate ($R_R(t)$) is performed in this section.

To compute the numerical results, we use MATLAB software. Figures 1, 2, 3, and 4 presented all the numerical results. From these figures, the following observations can be made:

1. The transient state probabilities of system size are shown in Fig. 1.
2. Figure 2 shows that as the mean retention rate increases, the probability of retaining a reneging customer (q) also increases.
3. In Fig. 3, the effect of the probability of retaining customer on the expected system size is presented in transient state. We can see that the expected system size increases as the probability of retention increases.
4. Figure 4 shows the effect of mean reneging rate with respect to the change in probability of retention. We can observe that the mean reneging rate decreases as the probability of retention increases.

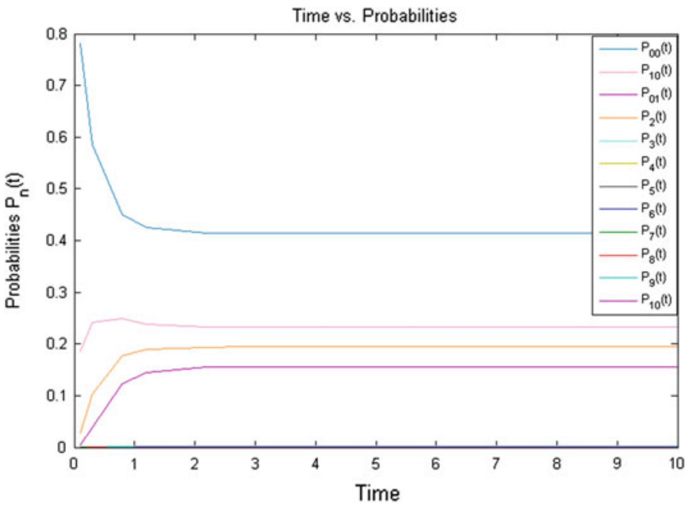

Fig. 1 The transient state probabilities are plotted

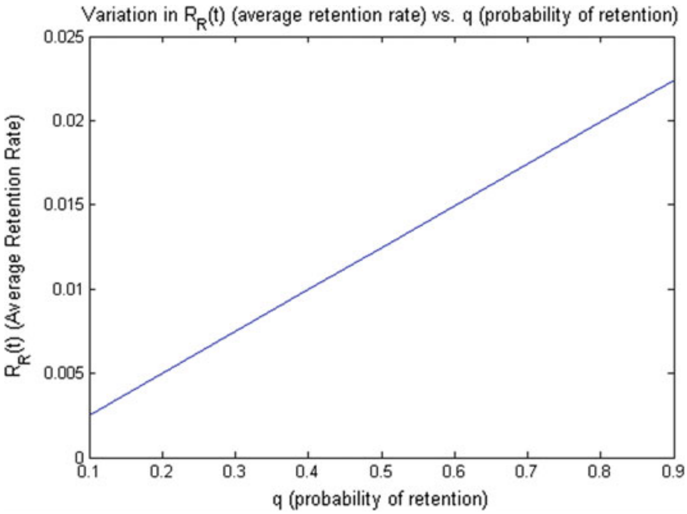

Fig. 2 Variation in probability of retention with respect to mean retention rate

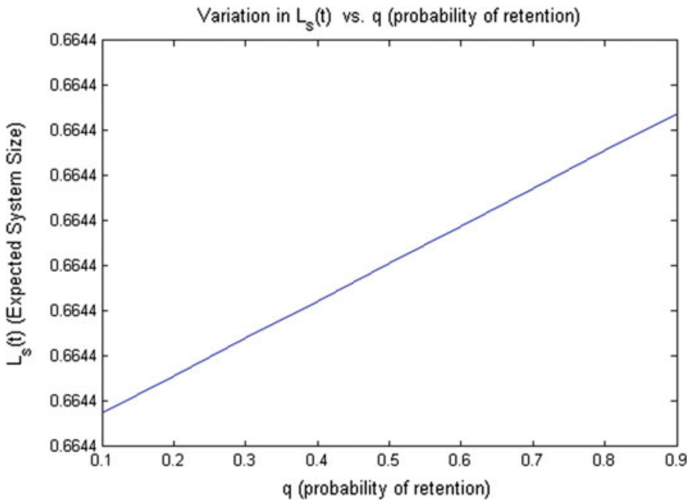

Fig. 3 Probability of retention (q) versus expected system size

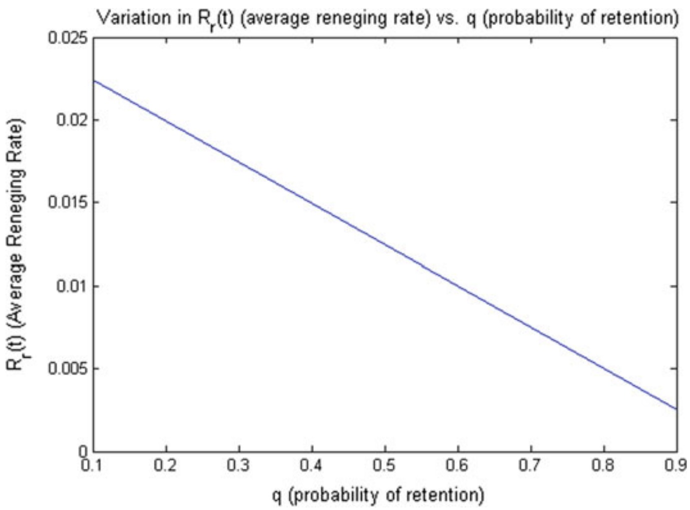

Fig. 4 Variation in probability of retention with respect to mean reneging rate

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Mehar Methods to Solve Intuitionistic Fuzzy Linear Programming Problems with Trapezoidal Intuitionistic Fuzzy Numbers

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Abstract While solving real-life linear programming problems, there may exist the situation of uncertainty and hesitancy due to several factors. To deal with such situations, intuitionistic fuzzy data representation is widely used. Many authors have investigated and proposed methods to find the solution of intuitionistic fuzzy linear programming (IFLP) problems in the last decade. In this paper, the limitations of the existing methods are pointed out and new methods (named as Mehar methods) are proposed to overcome these limitations. Further, the proposed method is illustrated with an example.

Keywords Intuitionistic fuzzy linear programming problem · Trapezoidal intuitionistic fuzzy numbers · Intuitionistic fuzzy optimal solution

1 Introduction

Fuzzy set theory has been extensively used in the area where the information/data is vague or imprecise. Zadeh [16] was the first who introduced the concept of fuzzy sets. Atanassov [1] generalized the concept of fuzzy set to intuitionistic fuzzy set. In intuitionistic fuzzy set [1], the degree of non-membership indicating the non-association to a set is also included to the degree of membership of association to a set. In fuzzy set, the sum of membership degree and non-membership degree is always equal to one, whereas in intuitionistic fuzzy set, sum of both membership and non-membership degree should not exceed one.

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Nehi [8] introduced the concept of trapezoidal/triangular intuitionistic fuzzy numbers as well as the proposed method for ranking of intuitionistic fuzzy numbers based on the characteristic values of membership and non-membership functions. Dubey and Mehra [3] proposed a more general definition of triangular intuitionistic fuzzy numbers as compared to [4] and defined a ranking function based on value and ambiguity indexes.

Parvathi and Malathi [10] introduced symmetric trapezoidal intuitionistic fuzzy (STIF) numbers and the arithmetic operations of symmetric trapezoidal intuitionistic fuzzy numbers based on $\alpha-$cuts.

By using $\alpha-$cuts, the operations of division for triangular intuitionistic fuzzy numbers are defined by Nagoorgani and Ponnalagu [7] and they also described scoring and accuracy function to rank triangular intuitionistic fuzzy number.

Parvathi and Malathi [11] proposed intuitionistic fuzzy simplex method to obtain the solution of such fuzzy linear programming problems in which parameters are represented by STIF numbers.

Suresh et al. [14] presented the ranking of triangular intuitionistic fuzzy numbers by means of magnitude and obtained the solution of IFLP problems using this ranking. Sidhu and Kumar [12] pointed out error in ranking function [14] and proposed the new ranking function.

Sidhu [13] proposed a new approach to find the solution of IFLP problems in which the variables/parameters are represented by STIF number. Nguyen [9] proposed new entropy measure for intuitionistic fuzzy sets and applied it to multiple attribute group decision making. Xu and Liao [15] presented a comprehensive study on decision making with intuitionistic fuzzy preference relations.

In this paper, the limitations of the existing method [11] are pointed out and new methods (named as Mehar methods) are proposed to solve IFLP problems with trapezoidal intuitionistic fuzzy (TIF) numbers. The same methods can be applied to solve IFLP problems with STIF numbers.

The paper is organized systematically as follows: Sect. 2 comprises of preliminaries related to intuitionistic fuzzy numbers. An existing method to solve IFLP problems is described in Sect. 3. In Sect. 4, a numerical example solved by the existing method is presented. The linearity property of the existing ranking function is presented in Sect. 5. The limitations of the existing methods are pointed out in Sect. 6. Errors in the existing results are showed in Sect. 7. In Sect. 8, Mehar methods are proposed to solve IFLP problems with TIF numbers with nonnegative coefficients as well as unrestricted coefficients. The exact solution of the existing problem is presented in Sect. 9. Finally, the concluding remarks are given in Sect. 10.

2 Preliminaries

In this section, some basic definitions, arithmetic operations, and comparison of TIF numbers are presented.

2.1 Some Basic Definitions

In this section, some basic definitions related to intuitionistic fuzzy numbers are presented.

Definition 1 ([2]) An intuitionistic fuzzy set $\tilde{A}$ in X is defined as a set of the form $\tilde{A} = \{< x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) >: x \in X\}$ where the functions $\mu_{\tilde{A}} : X \to [0, 1]$ and $\nu_{\tilde{A}} : X \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in X$, respectively, and for every $x \in X$ in $\tilde{A}$, $0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1$ holds.

Definition 2 ([2]) The intuitionistic fuzzy index of x in $\tilde{A}$ is defined as $\pi_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x)$. It is also known as degree of uncertainty or degree of hesitancy of the element x in $\tilde{A}$. So, for every $x \in X$, $0 \leq \pi_{\tilde{A}}(x) \leq 1$.

Definition 3 ([2]) An intuitionistic fuzzy set $\tilde{A} = \{< x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) >: x \in X\}$ is said to be intuitionistic fuzzy normal if there exist at least two points $x_0, x_1 \in X$ such that $\mu_{\tilde{A}}(x_0)=1$, $\nu_{\tilde{A}}(x_1)=1$.

Definition 4 ([6]) An intuitionistic fuzzy set $\tilde{A}$ is said to be intuitionistic fuzzy number $\tilde{A}^I$ if it is

(a) Intuitionistic fuzzy normal.
(b) Convex for the membership function $\mu_{\tilde{A}^I}(x)$, i.e., $\mu_{\tilde{A}^I}(\lambda x_1 + (1-\lambda)x_2) \geq min(\mu_{\tilde{A}^I}(x_1), \mu_{\tilde{A}^I}(x_2))$ for every $x_1, x_2 \in R$, $\lambda \in [0, 1]$.
(c) Concave for the non-membership function $\nu_{\tilde{A}^I}(x)$, i.e., $\nu_{\tilde{A}^I}(\lambda x_1 + (1-\lambda)x_2) \leq max(\nu_{\tilde{A}^I}(x_1), \nu_{\tilde{A}^I}(x_2))$ for every $x_1, x_2 \in R$, $\lambda \in [0, 1]$.

Definition 5 ([5]) An intuitionistic fuzzy number is said to be a TIF number if it has the following membership function $\mu_{\tilde{A}^I}(x)$ and non-membership function $\nu_{\tilde{A}^I}(x)$:

$$\mu_{\tilde{A}^I}(x) = \begin{cases} \frac{x-(a_1-\alpha)}{\alpha}, & x \in [a_1-\alpha, a_1); \\ 1, & x \in [a_1, a_2]; \\ \frac{a_2+\beta-x}{\beta}, & x \in (a_2, a_2+\beta]; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\nu_{\tilde{A}^I}(x) = \begin{cases} \frac{a_1-x}{\alpha'}, & x \in [a_1-\alpha', a_1); \\ 0, & x \in [a_1, a_2]; \\ \frac{x-a_2}{\beta'}, & x \in (a_2, a_2+\beta']; \\ 1, & \text{otherwise.} \end{cases}$$

where $\alpha, \beta, \alpha', \beta' > 0$.
The TIF number is denoted by $\tilde{A}^I = [a_1, a_2, \alpha, \beta; a_1, a_2, \alpha', \beta']$.

Definition 6 ([10]) A TIF number is called a STIF number if $\alpha = \beta$ (say h) and $\alpha^{'} = \beta^{'}$ (say $h^{'}$); i.e., if there exist real numbers $a_1, a_2, h, h^{'}$ where $a_1 \leq a_2, h \leq h^{'}$ and $h, h^{'} > 0$ such that the membership and non-membership functions are as below:

$$\mu_{\tilde{A}^I}(x) = \begin{cases} \frac{x-(a_1-h)}{h}, & x \in [a_1 - h, a_1); \\ 1, & x \in [a_1, a_2]; \\ \frac{a_2+h-x}{h}, & x \in (a_2, a_2 + h]; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\nu_{\tilde{A}^I}(x) = \begin{cases} \frac{a_1-x}{h^{'}}, & x \in [a_1 - h^{'}, a_1); \\ 0, & x \in [a_1, a_2]; \\ \frac{x-a_2}{h^{'}}, & x \in (a_2, a_2 + h']; \\ 1, & \text{otherwise.} \end{cases}$$

The STIF number is denoted by $\tilde{A}^I = [a_1, a_2, h, h; a_1, a_2, h^{'}, h^{'}]$.

2.2 Arithmetic Operations on TIF Numbers

In this section, the arithmetic operations on TIF numbers are illustrated [10].

If $\tilde{A}^I = [a_1, a_2, h_1, h_2; a_1, a_2, h_1^{'}, h_2^{'}]$ and $\tilde{B}^I = [b_1, b_2, k_1, k_2; b_1, b_2, k_1^{'}, k_2^{'}]$ are two TIF numbers. Then,

(i) $\tilde{A}^I + \tilde{B}^I = [a_1 + b_1, a_2 + b_2, h_1 + k_1, h_2 + k_2; a_1 + b_1, a_2 + b_2, h_1^{'} + k_1^{'}, h_2^{'} + k_2^{'}]$.

(ii) $\tilde{A}^I - \tilde{B}^I = [a_1 - b_2, a_2 - b_1, h_1 + k_2, h_2 + k_1; a_1 - b_2, a_2 - b_1, h_1^{'} + k_2^{'}, h_2^{'} + k_1^{'}]$.

(iii) $\lambda\tilde{A}^I = \begin{cases} ([\lambda a_1, \lambda a_2, \lambda h_1, \lambda h_2; \lambda a_1, \lambda a_2, \lambda h_1^{'}, \lambda h_2^{'}]), & \text{if } \lambda \geq 0; \\ ([\lambda a_2, \lambda a_1, -\lambda h_2, -\lambda h_1; \lambda a_2, \lambda a_1, -\lambda h_2^{'}, -\lambda h_1^{'}]), & \text{if } \lambda < 0. \end{cases}$

2.3 Comparison of STIF Numbers

To obtain the intuitionistic fuzzy optimal solution of the IFLP problem, there is need to compare intuitionistic fuzzy numbers. In this section, the method, used by the Parvathi and Malathi [11], for comparing intuitionistic fuzzy numbers is presented.

If $\tilde{A}^I = [a_1, a_2, h, h; a_1, a_2, h', h']$ and $\tilde{B}^I = [b_1, b_2, k, k; b_1, b_2, k', k']$ are two STIF numbers, then $\tilde{A}^I \succeq \tilde{B}^I$ if and only if $\Re(\tilde{A}^I) \geq \Re(\tilde{B}^I)$, $\tilde{A}^I \succ \tilde{B}^I$ if and only if $\Re(\tilde{A}^I) > \Re(\tilde{B}^I)$, $\tilde{A}^I \approx \tilde{B}^I$ if and only if $\Re(\tilde{A}^I) = \Re(\tilde{B}^I)$, where $\Re(\tilde{A}^I) = a_1 + a_2 + \frac{1}{2}(h' - h)$ and $\Re(\tilde{B}^I) = b_1 + b_2 + \frac{1}{2}(k' - k)$.

Remark 1 If $\tilde{A}^I = [a_1, a_2, h_1, h_2; a_1, a_2, h'_1, h'_2]$ and $\tilde{B}^I = [b_1, b_2, k_1, k_2; b_1, b_2, k'_1, k'_2]$ are non-symmetric trapezoidal intuitionistic fuzzy numbers, then $\Re(\tilde{A}^I) = (a_1 + a_2) + \frac{1}{4}(h'_1 + h'_2 - h_1 - h_2)$ and $\Re(\tilde{B}^I) = (b_1 + b_2) + \frac{1}{4}(k'_1 + k'_2 - k_1 - k_2)$.

3 Solution of IFLP Problem by the Existing Method

Parvathi and Malathi [11] proposed intuitionistic fuzzy simplex method to solve the IFLP problems with STIF numbers (P_1).

$$\begin{aligned}&\text{Maximize/Minimize } \left[\tilde{z}^I \approx \sum_{j=1}^{n} c_j \tilde{x}_j^I\right]\\&\text{Subject to}\\&\sum_{j=1}^{n} a_{ij}\tilde{x}_j^I \preceq, \approx, \succeq \tilde{b}_i^I,\ i = 1, 2, \ldots, m,\\&\tilde{x}_j^I \succeq 0,\ j = 1, 2, \ldots, n.\end{aligned} \qquad (P_1)$$

The steps of the existing method [11] are as below:

Step 1 Using Sect. 2.3, the problem (P_1) can be modified as problem (P_2).

$$\begin{aligned}&\text{Maximize/Minimize } \left[\Re(\tilde{z}^I) = \Re\left(\sum_{j=1}^{n} c_j \tilde{x}_j^I\right)\right]\\&\text{Subject to}\\&\Re\left[\sum_{j=1}^{n} a_{ij}\tilde{x}_j^I\right] \leq, =, \geq \Re(\tilde{b}_i^I),\ i = 1, 2, \ldots, m,\\&\Re(\tilde{x}_j^I) \geq 0,\ j = 1, 2, \ldots, n.\end{aligned} \qquad (P_2)$$

Step 2 Using the property $\Re\left(\sum_{i=1}^{n} \lambda_i \tilde{A}_i^I\right) = \sum_{i=1}^{n} \lambda_i \Re(\tilde{A}_i^I)$, where λ is a real number, the problem (P_2) can be modified as problem (P_3).

$$\begin{aligned}&\text{Maximize/Minimize } \left[\Re(\tilde{z}^I) = \sum_{j=1}^{n} c_j \Re(\tilde{x}_j^I)\right]\\&\text{Subject to}\\&\left[\sum_{j=1}^{n} a_{ij}\Re(\tilde{x}_j^I)\right] \leq, =, \geq \Re(\tilde{b}_i^I),\ i = 1, 2, \ldots, m,\\&\Re(\tilde{x}_j^I) \geq 0,\ j = 1, 2, \ldots, n.\end{aligned} \qquad (P_3)$$

Step 3 Use any appropriate existing method to find the fuzzy optimal solution $\{\Re(\tilde{x}_j^I)\}$ of the problem (P_3).

4 Intuitionistic Fuzzy Optimal Solution of the Existing Problem by the Existing Method

In this section, the IFLP problem (P_4), considered by Parvathi and Malathi [11, Sect. 5.5, pp. 45], is solved by the existing method [11].

Example 1 ([11, Sect. 5.5, pp. 45])

$$\begin{array}{l}\text{Maximize } \left[\tilde{z}^I \approx 5\tilde{x}_1^I + 4\tilde{x}_2^I\right]\\ \text{Subject to}\\ 6\tilde{x}_1^I + 4\tilde{x}_2^I \preceq [23, 25, 1, 1; 23, 25, 3, 3],\\ \tilde{x}_1^I + 2\tilde{x}_2^I \preceq [5, 7, 2, 2; 5, 7, 4, 4],\\ -\tilde{x}_1^I + \tilde{x}_2^I \preceq [3, 5, 4, 4; 3, 5, 6, 6],\\ \tilde{x}_2^I \preceq [1, 3, 2, 2; 1, 3, 4, 4],\\ \tilde{x}_1^I, \tilde{x}_2^I \succeq 0,\end{array} \qquad (P_4)$$

where $\tilde{x}_1^I$ and $\tilde{x}_2^I$ are STIF numbers.

Using the existing method [11], the intuitionistic fuzzy optimal solution of problem (P_4) can be obtained as below:

Step 1 Using Step 1 of Sect. 3, the problem (P_4) can be modified as problem (P_5).

$$\begin{array}{l}\text{Maximize } \left[\Re(\tilde{z}^I) = \Re(5\tilde{x}_1^I + 4\tilde{x}_2^I)\right]\\ \text{Subject to}\\ \Re(6\tilde{x}_1^I + 4\tilde{x}_2^I) \leq \Re[23, 25, 1, 1; 23, 25, 3, 3],\\ \Re(\tilde{x}_1^I + 2\tilde{x}_2^I) \leq \Re[5, 7, 2, 2; 5, 7, 4, 4],\\ \Re(-\tilde{x}_1^I + \tilde{x}_2^I) \leq \Re[3, 5, 4, 4; 3, 5, 6, 6],\\ \Re(\tilde{x}_2^I) \leq \Re[1, 3, 2, 2; 1, 3, 4, 4],\\ \Re(\tilde{x}_1^I), \Re(\tilde{x}_2^I) \geq 0.\end{array} \qquad (P_5)$$

Step 2 Using Step 2 of Sect. 3, the problem (P_5) can be modified as problem (P_6).

$$\begin{array}{l}\text{Maximize } \left[\Re(\tilde{z}^I) = 5\Re(\tilde{x}_1^I) + 4\Re(\tilde{x}_2^I)\right]\\ \text{Subject to}\\ 6\Re(\tilde{x}_1^I) + 4\Re(\tilde{x}_2^I) \leq \Re[23, 25, 1, 1; 23, 25, 3, 3],\\ \Re(\tilde{x}_1^I) + 2\Re(\tilde{x}_2^I) \leq \Re[5, 7, 2, 2; 5, 7, 4, 4],\\ -\Re(\tilde{x}_1^I) + \Re(\tilde{x}_2^I) \leq \Re[3, 5, 4, 4; 3, 5, 6, 6],\\ \Re(\tilde{x}_2^I) \leq \Re[1, 3, 2, 2; 1, 3, 4, 4],\\ \Re(\tilde{x}_1^I), \Re(\tilde{x}_2^I) \geq 0.\end{array} \qquad (P_6)$$

Step 3 Adding slack variables $\Re(\tilde{x}_3^I), \Re(\tilde{x}_4^I), \Re(\tilde{x}_5^I)$, and $\Re(\tilde{x}_6^I)$ into first, second, third, and fourth constraints of problem (P_6), respectively, the problem (P_6) can be modified as problem (P_7).

Table 1 The initial simplex table

Basis ($\tilde{x}_B^I$)	$\Re(\tilde{x}_1^I)$	$\Re(\tilde{x}_2^I)$	$\Re(\tilde{x}_3^I)$	$\Re(\tilde{x}_4^I)$	$\Re(\tilde{x}_5^I)$	$\Re(\tilde{x}_6^I)$	Solution
$\Re(\tilde{x}_3^I)$	6	4	1	0	0	0	$\Re[23, 25, 1, 1; 23, 25, 3, 3] = 49$
$\Re(\tilde{x}_4^I)$	1	2	0	1	0	0	$\Re[5, 7, 2, 2; 5, 7, 4, 4] = 13$
$\Re(\tilde{x}_5^I)$	−1	1	0	0	1	0	$\Re[3, 5, 4, 4; 3, 5, 6, 6] = 9$
$\Re(\tilde{x}_6^I)$	0	1	0	0	0	1	$\Re[1, 3, 2, 2; 1, 3, 4, 4] = 5$
$z_j - c_j$	−5	−4	0	0	0	0	$\Re(\tilde{0}^I) = 0$

$$\begin{aligned}
&\text{Maximize } \left[\Re(\tilde{z}^I) = 5\Re(\tilde{x}_1^I) + 4\Re(\tilde{x}_2^I)\right]\\
&\text{Subject to}\\
&6\Re(\tilde{x}_1^I) + 4\Re(\tilde{x}_2^I) + \Re(\tilde{x}_3^I) = \Re[23, 25, 1, 1; 23, 25, 3, 3],\\
&\Re(\tilde{x}_1^I) + 2\Re(\tilde{x}_2^I) + \Re(\tilde{x}_4^I) = \Re[5, 7, 2, 2; 5, 7, 4, 4],\\
&-\Re(\tilde{x}_1^I) + \Re(\tilde{x}_2^I) + \Re(\tilde{x}_5^I) = \Re[3, 5, 4, 4; 3, 5, 6, 6],\\
&\Re(\tilde{x}_2^I) + \Re(\tilde{x}_6^I) = \Re[1, 3, 2, 2; 1, 3, 4, 4],\\
&\Re(\tilde{x}_1^I), \Re(\tilde{x}_2^I), \Re(\tilde{x}_3^I), \Re(\tilde{x}_4^I), \Re(\tilde{x}_5^I), \Re(\tilde{x}_6^I) \geq 0.
\end{aligned} \tag{P_7}$$

Table 1 is the initial simplex table of the problem (P_7).

Since minimum $\{-5, -4\} = -5$, $\Re(\tilde{x}_1^I)$ is entering variable. Also, minimum $\{\frac{49}{6}, \frac{13}{1}\} = \frac{49}{6}$ corresponding to $\Re(\tilde{x}_3^I)$. So, $\Re(\tilde{x}_3^I)$ is a leaving variable. Now, after applying the required row operations, Table 2 is obtained.

Now, $-\frac{2}{3}$ is only negative entry in $z_j - c_j$. So, $\Re(\tilde{x}_2^I)$ is an entering variable and minimum $\left\{\frac{\frac{49}{6}}{\frac{2}{3}}, \frac{\frac{31}{6}}{\frac{4}{3}}, \frac{\frac{103}{6}}{\frac{5}{3}}, \frac{5}{1}\right\} = \frac{\frac{31}{6}}{\frac{4}{3}} = \frac{31}{8}$ corresponding to $\Re(\tilde{x}_4^I)$. So, $\Re(\tilde{x}_4^I)$ is a leaving variable. The next updated table is Table 3.

Since all the values of $z_j - c_j \geq 0$, the obtained fuzzy solution is a fuzzy optimal solution. The obtained fuzzy optimal solution is $\tilde{x}_1^I \approx [\frac{9}{4}, \frac{15}{4}, \frac{5}{4}, \frac{5}{4}; \frac{9}{4}, \frac{15}{4}, \frac{11}{4}, \frac{11}{4}]$ with $\Re(\tilde{x}_1^I) = \Re[\frac{9}{4}, \frac{15}{4}, \frac{5}{4}, \frac{5}{4}; \frac{9}{4}, \frac{15}{4}, \frac{11}{4}, \frac{11}{4}] = \frac{27}{4} = 6.75$; $\tilde{x}_2^I \approx [\frac{5}{8}, \frac{19}{8}, \frac{13}{8}, \frac{13}{8}; \frac{5}{8}, \frac{19}{8}, \frac{27}{8}, \frac{27}{8}]$ with $\Re(\tilde{x}_2^I) = \Re[\frac{5}{8}, \frac{19}{8}, \frac{13}{8}, \frac{13}{8}; \frac{5}{8}, \frac{19}{8}, \frac{27}{8}, \frac{27}{8}] = \frac{31}{8} = 3.875$, and the obtained fuzzy optimal value is $\tilde{z}^I \approx [\frac{55}{4}, \frac{113}{4}, \frac{51}{4}, \frac{51}{4}; \frac{55}{4}, \frac{113}{4}, \frac{109}{4}, \frac{109}{4}]$ with $\Re(\tilde{z}^I) = \Re[\frac{55}{4}, \frac{113}{4}, \frac{51}{4}, \frac{51}{4}; \frac{55}{4}, \frac{113}{4}, \frac{109}{4}, \frac{109}{4}] = \frac{197}{4} = 49.25$.

5 Linearity Property of the Existing Ranking Function

In this section, it is shown that for ranking function $\Re$, used by Parvathi and Malathi [11], the property $\Re\left(\sum_{i=1}^{m} \lambda_i \tilde{A}_i^I\right) = \sum_{i=1}^{m} \lambda_i \Re(\tilde{A}_i^I)$ will be satisfied only if $\lambda_i \geq 0$. But, if $\lambda_i < 0$, then this property is not satisfied.

Table 2 The first iteration table

Basis ($\tilde{x}_B^I$)	$\Re(\tilde{x}_1^I)$	$\Re(\tilde{x}_2^I)$	$\Re(\tilde{x}_3^I)$	$\Re(\tilde{x}_4^I)$	$\Re(\tilde{x}_5^I)$	$\Re(\tilde{x}_6^I)$	Solution
$\Re(\tilde{x}_1^I)$	1	$\frac{2}{3}$	$\frac{1}{6}$	0	0	0	$\Re[\frac{23}{6}, \frac{25}{6}, \frac{1}{6}, \frac{1}{6}; \frac{23}{6}, \frac{25}{6}, \frac{3}{6}, \frac{3}{6}] = \frac{49}{6}$
$\Re(\tilde{x}_4^I)$	0	$\frac{4}{3}$	$-\frac{1}{6}$	1	0	0	$\Re[\frac{5}{6}, \frac{19}{6}, \frac{13}{6}, \frac{13}{6}; \frac{5}{6}, \frac{19}{6}, \frac{27}{6}, \frac{27}{6}] = \frac{31}{6}$
$\Re(\tilde{x}_5^I)$	0	$\frac{5}{3}$	$\frac{1}{6}$	0	1	0	$\Re[\frac{41}{6}, \frac{55}{6}, \frac{25}{6}, \frac{25}{6}; \frac{41}{6}, \frac{55}{6}, \frac{39}{6}, \frac{39}{6}] = \frac{103}{6}$
$\Re(\tilde{x}_6^I)$	0	1	0	0	0	1	$\Re[1, 3, 2, 2; 1, 3, 4, 4] = 5$
$z_j - c_j$	0	$-\frac{2}{3}$	$\frac{5}{6}$	0	0	0	$\Re[\frac{115}{6}, \frac{125}{6}, \frac{5}{6}, \frac{5}{6}; \frac{115}{6}, \frac{125}{6}, \frac{15}{6}, \frac{15}{6}] = \frac{245}{6}$

Table 3 The final iteration table

Basis ($\tilde{x}_B^I$)	$\Re(\tilde{x}_1^I)$	$\Re(\tilde{x}_2^I)$	$\Re(\tilde{x}_3^I)$	$\Re(\tilde{x}_4^I)$	$\Re(\tilde{x}_5^I)$	$\Re(\tilde{x}_6^I)$	Solution
$\Re(\tilde{x}_1^I)$	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	$\Re[\frac{9}{4}, \frac{15}{4}, \frac{5}{4}, \frac{5}{4}; \frac{9}{4}, \frac{15}{4}, \frac{11}{4}, \frac{11}{4}] = \frac{27}{4}$
$\Re(\tilde{x}_2^I)$	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\Re[\frac{5}{8}, \frac{19}{8}, \frac{13}{8}, \frac{13}{8}; \frac{5}{8}, \frac{19}{8}, \frac{27}{8}, \frac{27}{8}] = \frac{31}{8}$
$\Re(\tilde{x}_5^I)$	0	0	$\frac{9}{24}$	$-\frac{5}{4}$	1	0	$\Re[\frac{23}{8}, \frac{65}{8}, \frac{55}{8}, \frac{55}{8}; \frac{23}{8}, \frac{65}{8}, \frac{97}{8}, \frac{97}{8}] = \frac{109}{8}$
$\Re(\tilde{x}_6^I)$	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\Re[-\frac{11}{8}, \frac{19}{8}, \frac{29}{8}, \frac{29}{8}; \frac{11}{8}, \frac{19}{8}, \frac{59}{8}, \frac{59}{8}] = \frac{23}{8}$
$z_j - c_j$	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	$\Re[\frac{55}{4}, \frac{113}{4}, \frac{51}{4}, \frac{51}{4}; \frac{55}{4}, \frac{113}{4}, \frac{109}{4}, \frac{109}{4}] = \frac{197}{4}$

Let $\tilde{A}_i^I = [x_i, y_i, h, h; x_i, y_i, h', h']$, $i = 1, 2, \ldots, m$ be a STIF number.
Case 1 If $\lambda_i \geq 0$.
Then

$$\begin{aligned}
\Re\left(\sum_{i=1}^{m}\lambda_i\tilde{A}_i^I\right) &= \Re\left(\sum_{i=1}^{m}\lambda_i[x_i, y_i, h, h; x_i, y_i, h^{'}, h^{'}]\right)\\
&= \Re\left(\sum_{i=1}^{m}[\lambda_i x_i, \lambda_i y_i, \lambda_i h, \lambda_i h; \lambda_i x_i, \lambda_i y_i, \lambda_i h^{'}, \lambda_i h^{'}]\right)\\
&= \Re\left[\sum_{i=1}^{m}\lambda_i x_i, \sum_{i=1}^{m}\lambda_i y_i, \sum_{i=1}^{m}\lambda_i h, \sum_{i=1}^{m}\lambda_i h; \sum_{i=1}^{m}\lambda_i x_i, \sum_{i=1}^{m}\lambda_i y_i, \sum_{i=1}^{m}\lambda_i h', \sum_{i=1}^{m}\lambda_i h'\right]\\
&= \sum_{i=1}^{m}\lambda_i x_i + \sum_{i=1}^{m}\lambda_i y_i + \frac{1}{2}\sum_{i=1}^{m}\lambda_i(h' - h)\\
&= \sum_{i=1}^{m}\lambda_i(x_i + y_i + \frac{1}{2}(h' - h)) = \sum_{i=1}^{m}\lambda_i\Re(\tilde{A}_i^I),
\end{aligned}$$

Hence, $\Re\left(\sum_{i=1}^{m}\lambda_i\tilde{A}_i^I\right) = \sum_{i=1}^{m}\lambda_i\Re(\tilde{A}_i^I)$.
Case 2 If $\lambda_i < 0$ say $\lambda_i = -k_i, k_i > 0$.
Then

$$\begin{aligned}
\Re\left(\sum_{i=1}^{m}\lambda_i\tilde{A}_i^I\right) &= \Re\left(\sum_{i=1}^{m}(-k_i)[x_i, y_i, h, h; x_i, y_i, h^{'}, h^{'}]\right)\\
&= \Re\left(\sum_{i=1}^{m}[(-k_i)y_i, (-k_i)x_i, k_i h, k_i h; (-k_i)y_i, (-k_i)x_i, k_i h^{'}, k_i h^{'}]\right)\\
&= \Re\left[\sum_{i=1}^{m}(-k_i)y_i, \sum_{i=1}^{m}(-k_i)x_i, \sum_{i=1}^{m}k_i h, \sum_{i=1}^{m}k_i h;\right.\\
&\left.\sum_{i=1}^{m}(-k_i)y_i, \sum_{i=1}^{m}(-k_i)x_i, \sum_{i=1}^{m}k_i h', \sum_{i=1}^{m}k_i h'\right]\\
&= \sum_{i=1}^{m}(-k_i)y_i + \sum_{i=1}^{m}(-k_i)x_i + \frac{1}{2}\sum_{i=1}^{m}k_i(h' - h)\\
&= \sum_{i=1}^{m}(-k_i)(x_i + y_i - \frac{1}{2}(h' - h)) \neq \sum_{i=1}^{m}\lambda_i\Re(\tilde{A}_i^I).
\end{aligned}$$

Hence, $\Re\left(\sum_{i=1}^{m}\lambda_i\tilde{A}_i^I\right) \neq \sum_{i=1}^{m}\lambda_i\Re(\tilde{A}_i^I)$.

6 Limitations of the Existing Method

In this section, the limitations of the existing method [11] are presented.

1. Parvathi and Malathi [11] have used the linearity property $\Re\left(\sum_{i=1}^{m}\lambda_i\tilde{A}_i^I\right) = \sum_{i=1}^{m}\lambda_i\Re(\tilde{A}_i^I)$ in Step 2 of their proposed method. However, as proved in Sect. 5, this property is valid only if all the coefficients c_j and a_{ij} in problem (P_1) are nonnegative real numbers. Hence, if any of the coefficients c_j or a_{ij} is negative real number, then the existing method [11] cannot be used for solving problem (P_1).
2. The method, proposed by Parvathi and Malathi [11], is applicable only if variables and right-hand side vector are represented by STIF numbers. But, this method [11] is not applicable to solve IFLP problems (P_8) in which variables and right-hand side vector are represented by non-symmetric trapezoidal intuitionistic fuzzy(NSTIF) numbers.

$$\begin{aligned}&\text{Maximize/Minimize } \left[\tilde{z}^I \approx \sum_{j=1}^{n} c_j\tilde{x}_j^I\right]\\&\text{Subject to}\\&\sum_{j=1}^{n} a_{ij}\tilde{x}_j^I \preceq, \approx, \succeq \tilde{b}_i^I,\ i=1,2,\ldots,m;\\&\tilde{x}_j^I \succeq 0,\ j=1,2,\ldots,n.\end{aligned} \qquad (P_8)$$

7 Error in the Existing Results

In Sect. 5, it is proved that if λ_i is a negative real number, then $\Re\left(\sum_{i=1}^{m}\lambda_i\tilde{A}_i^I\right) \neq \sum_{i=1}^{m}\lambda_i\Re(\tilde{A}_i^I)$. It is obvious that the coefficients of $\tilde{x}_1^I$ in third constraint of the problem (P_4) are the negative real number. So, $\Re(-\tilde{x}_1^I + \tilde{x}_2^I) \neq -\Re(\tilde{x}_1^I) + \Re(\tilde{x}_2^I)$.

However, it is obvious from Step 2 of Sect. 4 that Parvathi and Malathi [11] have used the property $\Re(-\tilde{x}_1^I + \tilde{x}_2^I) = -\Re(\tilde{x}_1^I) + \Re(\tilde{x}_2^I)$ to transform the problem (P_5) into problem (P_6). Due to the same reason, the optimal solution of the problem (P_4), obtained by Parvathi and Malathi [11], is not satisfying the first and second constraints of problem (P_4). This is shown as given below:

(i) Putting the values of $\tilde{x}_1^I$ and $\tilde{x}_2^I$ in left-hand side of first constraint, we have $6[\frac{9}{4}, \frac{15}{4}, \frac{5}{4}, \frac{5}{4}; \frac{9}{4}, \frac{15}{4}, \frac{11}{4}, \frac{11}{4}] + 4[\frac{5}{8}, \frac{19}{8}, \frac{13}{8}, \frac{13}{8}; \frac{5}{8}, \frac{19}{8}, \frac{27}{8}, \frac{27}{8}] = [\frac{27}{2}, \frac{45}{2}, \frac{15}{2}, \frac{15}{2}; \frac{27}{2}, \frac{45}{2}, \frac{33}{2}, \frac{33}{2}] + [\frac{5}{2}, \frac{19}{2}, \frac{13}{2}, \frac{13}{2}; \frac{5}{2}, \frac{19}{2}, \frac{27}{2}, \frac{27}{2}] = [16, 32, 14, 14; 16, 32, 30, 30]$ and $\Re[16, 32, 14, 14; 16, 32, 30, 30] = 56$. Right-hand side of first constraint is $[23, 25, 1, 1; 23, 25, 3, 3]$ and $\Re[23, 25, 1, 1; 23, 25, 3, 3] = 49$. It is obvious that $[16, 32, 14, 14; 16, 32, 30, 30] \npreceq [23, 25, 1, 1; 23, 25, 3, 3]$.

(ii) Putting the values of $\tilde{x}_1^I$ and $\tilde{x}_2^I$ in left-hand side of second constraint, we have $[\frac{9}{4}, \frac{15}{4}, \frac{5}{4}, \frac{5}{4}; \frac{9}{4}, \frac{15}{4}, \frac{11}{4}, \frac{11}{4}] + 2[\frac{5}{8}, \frac{19}{8}, \frac{13}{8}, \frac{13}{8}; \frac{5}{8}, \frac{19}{8}, \frac{27}{8}, \frac{27}{8}] = [\frac{9}{4}, \frac{15}{4}, \frac{5}{4}, \frac{5}{4}; \frac{9}{4}, \frac{15}{4}, \frac{11}{4}, \frac{11}{4}] + [\frac{5}{4}, \frac{19}{4}, \frac{13}{4}, \frac{13}{4}; \frac{5}{4}, \frac{19}{4}, \frac{27}{4}, \frac{27}{4}] = [\frac{7}{2}, \frac{17}{2}, \frac{9}{2}, \frac{9}{2}; \frac{7}{2}, \frac{17}{2}, \frac{19}{2}, \frac{19}{2}]$ and $\Re[\frac{7}{2}, \frac{17}{2}, \frac{9}{2}, \frac{9}{2}; \frac{7}{2}, \frac{17}{2}, \frac{19}{2}, \frac{19}{2}] = \frac{29}{2}$. Right-hand side of second constraint is [5, 7, 2, 2; 5, 7, 4, 4] and $\Re[5, 7, 2, 2; 5, 7, 4, 4] = 13$. It is obvious that $[\frac{7}{2}, \frac{17}{2}, \frac{9}{2}, \frac{9}{2}; \frac{7}{2}, \frac{17}{2}, \frac{19}{2}, \frac{19}{2}] \npreceq [5, 7, 2, 2; 5, 7, 4, 4]$.
Hence, the fuzzy optimal solution, obtained by Parvathi and Malathi [11], is not valid.

8 Proposed Mehar Methods to Find the Solution of IFLP Problems with TIF Numbers

To overcome the limitations of the existing method [11], discussed in Sect. 6, new methods (named as Mehar methods) are proposed to find the intuitionistic fuzzy optimal solution of (P_8).

8.1 Proposed Mehar Method to Find the Solution of IFLP Problems with TIF Numbers with Nonnegative Coefficients

In this section, a new method is proposed to solve IFLP problem (P_8) with TIF numbers in which all the coefficients of the variables are nonnegative real numbers. The same method can be applied to solve IFLP problems with STIF numbers in which all the coefficients of the variables are nonnegative real numbers.

The following are the steps of proposed Mehar method.

Step 1 Using Sect. 2.3, the problem (P_8) can be modified as problem (P_9).

$$\begin{aligned}
&\text{Maximize/Minimize } \left[\Re(\tilde{z}^I) = \Re\left(\sum_{j=1}^{n} c_j \tilde{x}_j^I\right)\right] \\
&\text{Subject to} \\
&\Re\left[\sum_{j=1}^{n} a_{ij}\tilde{x}_j^I\right] \le, =, \ge \Re(\tilde{b}_i^I),\ i = 1, 2, \ldots, m; \\
&\Re(\tilde{x}_j^I) \ge 0,\ j = 1, 2, \ldots, n.
\end{aligned} \qquad (P_9)$$

Step 2 Using the property $\Re\left(\sum_{i=1}^{n}\lambda_i\tilde{A}_i^I\right)=\sum_{i=1}^{n}\lambda_i\Re(\tilde{A}_i^I)$, where λ is a nonnegative real number, the problem (P_9) can be modified as problem (P_{10}).

$$\begin{aligned}&\text{Maximize/Minimize }\left[\Re(\tilde{z}^I)=\sum_{j=1}^{n}c_j\Re(\tilde{x}_j^I)\right]\\&\text{Subject to}\\&\left[\sum_{j=1}^{n}a_{ij}\Re(\tilde{x}_j^I)\right]\le,=,\ge\Re(\tilde{b}_i^I),\ i=1,2,\ldots,m;\\&\Re(\tilde{x}_j^I)\ge 0,\ j=1,2,\ldots,n.\end{aligned}\qquad (P_{10})$$

Step 3 Since the rank of an intuitionistic fuzzy number is a real number, assuming $\Re(\tilde{z}^I)=z$, $\Re(\tilde{x}_j^I)=x_j$ and $\Re(\tilde{b}_i^I)=b_i$, the problem (P_{10}) can be modified as (P_{11}).

$$\begin{aligned}&\text{Maximize/Minimize }\left[z=\sum_{j=1}^{n}(c_jx_j)\right]\\&\text{Subject to}\\&\left[\sum_{j=1}^{n}a_{ij}x_j\right]\le,=,\ge b_i,\ i=1,2,\ldots,m;\\&x_j\ge 0,\ j=1,2,\ldots,n.\end{aligned}\qquad (P_{11})$$

Step 4 Find the optimal solution of the problem (P_{11}) by using a suitable existing method.

Step 5 As there exist infinite many intuitionistic fuzzy numbers which have the same rank, if $x_1=a_1, x_2=a_2,\ldots,x_n=a_n$ is an optimal solution of the problem (P_{11}), then all that TIF numbers $\tilde{x}_1^I,\tilde{x}_2^I,\ldots,\tilde{x}_n^I$ such that $\Re(\tilde{x}_1^I)=a_1$, $\Re(\tilde{x}_2^I)=a_2$,…,$\Re(\tilde{x}_n^I)=a_n$ will be the fuzzy optimal solution of the problem (P_8).

8.2 Proposed Mehar Method to Find the Solution of IFLP Problems with TIF Numbers with Unrestricted Coefficients

In this section, a new method is proposed to solve IFLP problems (P_8). The same method can be applied to solve IFLP problems (P_1), by replacing $h_{1j}=h_{2j}=h_j$ and $h'_{1j}=h'_{2j}=h'_j$.

The following are the steps of the proposed Mehar method.

Step 1 Substituting $\tilde{x}_j^I=[x_j,y_j,h_{1j},h_{2j};x_j,y_j,h'_{1j},h'_{2j}]$ and $\tilde{b}_i^I=[b_i,g_i,k_{1i},k_{2i};b_i,g_i,k'_{1i},k'_{2i}]$ into the problem (P_8), the problem (P_8) can be modified as problem (P_{12}).

$$\begin{aligned}&\text{Maximize/Minimize } \left[\tilde{z}^I \approx \sum_{j=1}^{n} c_j[x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}]\right]\\&\text{Subject to}\\&\sum_{j=1}^{n} a_{ij}[x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}] \preceq, \approx, \succeq [b_i, g_i, k_{1i}, k_{2i}; b_i, g_i, k'_{1i}, k'_{2i}],\\&[x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}] \succeq 0.\end{aligned} \tag{P_{12}}$$

Step 2 Assuming $c_j[x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}] = [p_j, q_j, r_{1j}, r_{2j}; p_j, q_j, r'_{1j}, r'_{2j}]$ and $a_{ij}[x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}] = [d_j, e_j, f_{1j}, f_{2j}; d_j, e_j, f'_{1j}, f'_{2j}]$, the problem ($P_{12}$) can be modified as problem (P_{13}).

$$\begin{aligned}&\text{Maximize/Minimize } \left[\tilde{z}^I \approx \sum_{j=1}^{n}[p_j, q_j, r_{1j}, r_{2j}; p_j, q_j, r'_{1j}, r'_{2j}]\right]\\&\text{Subject to}\\&\sum_{j=1}^{n}[d_j, e_j, f_{1j}, f_{2j}; d_j, e_j, f'_{1j}, f'_{2j}] \preceq, \approx, \succeq [b_i, g_i, k_{1i}, k_{2i}; b_i, g_i, k'_{1i}, k'_{2i}],\\&[x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}] \succeq 0.\end{aligned} \tag{P_{13}}$$

Step 3 The problem (P_{13}) can be modified as problem (P_{14}).

$$\begin{aligned}&\text{Maximize/Minimize}\\&\quad\left[\tilde{z}^I \approx [\sum_{j=1}^{n} p_j, \sum_{j=1}^{n} q_j, \sum_{j=1}^{n} r_{1j}, \sum_{j=1}^{n} r_{2j}; \sum_{j=1}^{n} p_j, \sum_{j=1}^{n} q_j, \sum_{j=1}^{n} r'_{1j}, \sum_{j=1}^{n} r'_{2j}]\right]\\&\text{Subject to}\\&\left[\sum_{j=1}^{n} d_j, \sum_{j=1}^{n} e_j, \sum_{j=1}^{n} f_{1j}, \sum_{j=1}^{n} f_{2j}; \sum_{j=1}^{n} d_j, \sum_{j=1}^{n} e_j, \sum_{j=1}^{n} f'_{1j}, \sum_{j=1}^{n} f'_{2j}\right] \preceq,\\&\quad \approx, \succeq [b_i, g_i, k_{1i}, k_{2i}; b_i, g_i, k'_{1i}, k'_{2i}],\\&[x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}] \succeq 0.\end{aligned} \tag{P_{14}}$$

Step 4 The problem (P_{14}) can be modified as problem (P_{15}).

$$\begin{aligned}&\text{Maximize/Minimize}\\&\quad\left[\tilde{z}^I = \Re\left[\sum_{j=1}^{n} p_j, \sum_{j=1}^{n} q_j, \sum_{j=1}^{n} r_{1j}, \sum_{j=1}^{n} r_{2j}; \sum_{j=1}^{n} p_j, \sum_{j=1}^{n} q_j, \sum_{j=1}^{n} r'_{1j}, \sum_{j=1}^{n} r'_{2j}\right]\right]\\&\text{Subject to}\\&\Re\left[\sum_{j=1}^{n} d_j, \sum_{j=1}^{n} e_j, \sum_{j=1}^{n} f_{1j}, \sum_{j=1}^{n} f_{2j}; \sum_{j=1}^{n} d_j, \sum_{j=1}^{n} e_j, \sum_{j=1}^{n} f'_{1j}, \sum_{j=1}^{n} f'_{2j}\right] \leq, =, \geq\\&\Re[b_i, g_i, k_{1i}, k_{2i}; b_i, g_i, k'_{1i}, k'_{2i}],\\&\Re[x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}] \geq 0.\end{aligned} \tag{P_{15}}$$

Step 5 Using Sect. 2.3, the problem (P_{15}) can be modified as problem (P_{16}).

$$\begin{array}{l}\text{Maximize/Minimize } \left[\tilde{z}^I = \sum_{j=1}^{n} p_j + \sum_{j=1}^{n} q_j + \frac{1}{4}\left(\sum_{j=1}^{n} r'_{1j} + \sum_{j=1}^{n} r'_{2j} - \sum_{j=1}^{n} r_{1j} - \sum_{j=1}^{n} r_{2j}\right)\right] \\ \text{Subject to} \\ \left(\sum_{j=1}^{n} d_j + \sum_{j=1}^{n} e_j + \frac{1}{4}\left(\sum_{j=1}^{n} f'_{1j} + \sum_{j=1}^{n} f'_{2j} - \sum_{j=1}^{n} f_{1j} - \sum_{j=1}^{n} f_{2j}\right)\right) \leq, =, \geq \\ \left(b_i + g_i + \frac{1}{4}(k'_{1i} + k'_{2i} - k_{1i} - k_{2i})\right), \\ x_j + y_j + \frac{1}{4}(h'_{1j} + h'_{2j} - h_{1j} - h_{2j}) \geq 0, \\ x_j \leq y_j, h_{1j} \leq h'_{1j}, h_{2j} \leq h'_{2j}, \\ x_j, y_j \text{ are unrestricted and } h_{1j}, h_{2j}, h'_{1j}, h'_{2j} \geq 0. \end{array} \quad (P_{16})$$

Step 6 Solve the problem (P_{16}) by using any suitable technique/method to find the values of $x_j, y_j, h_{1j}, h_{2j}, h'_{1j}, h'_{2j}$ and put these values in $\tilde{x}_j^I = [x_j, y_j, h_{1j}, h_{2j}; x_j, y_j, h'_{1j}, h'_{2j}]$ to obtain the intuitionistic fuzzy optimal solution.
Step 7 Find the intuitionistic fuzzy optimal value $\tilde{z}^I$ by substituting the values of $\tilde{x}_j^I$ in $\sum_{j=1}^{n}(c_j\tilde{x}_j^I)$.

9 Exact Solution of the Existing Problem

In this section, the exact solution of the problem (P_4) is obtained by using the proposed Mehar method.
Step 1 Substituting the values of $\tilde{x}_1^I = \left[x_1, y_1, h_1, h_1; x_1, y_1, h'_1, h'_1\right]$ and $\tilde{x}_2^I = \left[x_2, y_2, h_2, h_2; x_2, y_2, h'_2, h'_2\right]$ in the problem (P_4), it can be converted into problem (P_{17}).

$$\begin{array}{l}\text{Maximize } \left[\tilde{z}^I \approx 5\left[x_1, y_1, h_1, h_1; x_1, y_1, h'_1, h'_1\right] + 4\left[x_2, y_2, h_2, h_2; x_2, y_2, h'_2, h'_2\right]\right] \\ \text{Subject to} \\ 6\left[x_1, y_1, h_1, h_1; x_1, y_1, h'_1, h'_1\right] + 4\left[x_2, y_2, h_2, h_2; x_2, y_2, h'_2, h'_2\right] \\ \quad \preceq [23, 25, 1, 1; 23, 25, 3, 3], \\ \left[x_1, y_1, h_1, h_1; x_1, y_1, h'_1, h'_1\right] + 2\left[x_2, y_2, h_2, h_2; x_2, y_2, h'_2, h'_2\right] \preceq [5, 7, 2, 2; 5, 7, 4, 4], \\ -\left[x_1, y_1, h_1, h_1; x_1, y_1, h'_1, h'_1\right] + \left[x_2, y_2, h_2, h_2; x_2, y_2, h'_2, h'_2\right] \preceq [3, 5, 4, 4; 3, 5, 6, 6], \\ \left[x_2, y_2, h_2, h_2; x_2, y_2, h'_2, h'_2\right] \preceq [1, 3, 2, 2; 1, 3, 4, 4], \\ \left[x_1, y_1, h_1, h_1; x_1, y_1, h'_1, h'_1\right] \succeq 0, \\ \left[x_2, y_2, h_2, h_2; x_2, y_2, h'_2, h'_2\right] \succeq 0. \end{array} \quad (P_{17})$$

Step 2 Using Step 2 of Sect. 8.2, the problem (P_{17}) can be modified as problem (P_{18}).

$$\begin{array}{l}
\text{Maximize } \left[\tilde{z}^I = \left[5x_1, 5y_1, 5h_1, 5h_1; 5x_1, 5y_1, 5h'_1, 5h'_1\right]\right. \\
\quad \left. + \left[4x_2, 4y_2, 4h_2, 4h_2; 4x_2, 4y_2, 4h'_2, 4h'_2\right]\right] \\
\text{Subject to} \\
\left[6x_1, 6y_1, 6h_1, 6h_1; 6x_1, 6y_1, 6h'_1, 6h'_1\right] + \left[4x_2, 4y_2, 4h_2, 4h_2; 4x_2, 4y_2, 4h'_2, 4h'_2\right] \\
\quad \preceq [23, 25, 1, 1; 23, 25, 3, 3], \\
\left[x_1, y_1, h_1, h_1; x_1, y_1, h'_1, h'_1\right] + \left[2x_2, 2y_2, 2h_2, 2h_2; 2x_2, 2y_2, 2h'_2, 2h'_2\right] \\
\quad \preceq [5, 7, 2, 2; 5, 7, 4, 4], \\
\left[-y_1, -x_1, h_1, h_1; -y_1, -x_1, h'_1, h'_1\right] + \left[x_2, y_2, h_2, h_2; x_2, y_2, h'_2, h'_2\right] \\
\quad \preceq [3, 5, 4, 4; 3, 5, 6, 6], \\
\left[x_2, y_2, h_2, h_2; x_2, y_2, h'_2, h'_2\right] \preceq [1, 3, 2, 2; 1, 3, 4, 4], \\
\left[x_1, y_1, h_1, h_1; x_1, y_1, h'_1, h'_1\right] \succeq 0, \\
\left[x_2, y_2, h_2, h_2; x_2, y_2, h'_2, h'_2\right] \succeq 0.
\end{array} \tag{P_{18}}$$

Step 3 Using Step 3 of Sect. 8.2, the problem (P_{18}) can be modified as problem (P_{19}).

$$\begin{array}{l}
\text{Maximize} \\
\left[\tilde{z}^I = \left[5x_1 + 4x_2, 5y_1 + 4y_2, 5h_1 + 4h_2, 5h_1 + 4h_2; 5x_1 + 4x_2, 5y_1 + 4y_2,\right.\right. \\
\quad \left.\left. 5h'_1 + 4h'_2, 5h'_1 + 4h'_2\right]\right] \\
\text{Subject to} \\
\left[6x_1 + 4x_2, 6y_1 + 4y_2, 6h_1 + 4h_2, 6h_1 + 4h_2; 6x_1 + 4x_2, 6y_1 + 4y_2,\right. \\
\quad \left. 6h'_1 + 4h'_2, 6h'_1 + 4h'_2\right] \preceq \\
[23, 25, 1, 1; 23, 25, 3, 3], \\
\left[x_1 + 2x_2, y_1 + 2y_2, h_1 + 2h_2, h_1 + 2h_2; x_1 + 2x_2, y_1 + 2y_2, h'_1 + 2h'_2, h'_1 + 2h'_2\right] \\
\preceq [5, 7, 2, 2; 5, 7, 4, 4], \\
\left[-y_1 + x_2, -x_1 + y_2, h_1 + h_2, h_1 + h_2; -y_1 + x_2, -x_1 + y_2, h'_1 + h'_2, h'_1 + h'_2\right] \\
\quad \preceq [3, 5, 4, 4; 3, 5, 6, 6], \\
\left[x_2, y_2, h_2, h_2; x_2, y_2, h'_2, h'_2\right] \preceq [1, 3, 2, 2; 1, 3, 4, 4], \\
\left[x_1, y_1, h_1, h_1; x_1, y_1, h'_1, h'_1\right] \succeq 0, \\
\left[x_2, y_2, h_2, h_2; x_2, y_2, h'_2, h'_2\right] \succeq 0.
\end{array} \tag{P_{19}}$$

Step 4 Using Step 4 of Sect. 8.2, the problem (P_{19}) can be modified as problem (P_{20}).

$$\begin{aligned}
&\text{Maximize}\\
&\left[\tilde{z}^I = \Re\left[5x_1 + 4x_2, 5y_1 + 4y_2, 5h_1 + 4h_2, 5h_1 + 4h_2; 5x_1 + 4x_2, 5y_1 + 4y_2,\right.\right.\\
&\quad\left.\left. 5h'_1 + 4h'_2, 5h'_1 + 4h'_2\right]\right]\\
&\text{Subject to}\\
&\Re\left[6x_1 + 4x_2, 6y_1 + 4y_2, 6h_1 + 4h_2, 6h_1 + 4h_2; 6x_1 + 4x_2,\right.\\
&\quad\left. 6y_1 + 4y_2, 6h'_1 + 4h'_2, 6h'_1 + 4h'_2\right] \le\\
&\Re\,[23, 25, 1, 1; 23, 25, 3, 3]\,,\\
&\Re\left[x_1 + 2x_2, y_1 + 2y_2, h_1 + 2h_2, h_1 + 2h_2; x_1 + 2x_2, y_1 + 2y_2, h'_1 + 2h'_2, h'_1 + 2h'_2\right] \le\\
&\Re[5, 7, 2, 2; 5, 7, 4, 4],\\
&\Re\left[-y_1 + x_2, -x_1 + y_2, h_1 + h_2, h_1 + h_2; -y_1 + x_2, -x_1 + y_2, h'_1 + h'_2, h'_1 + h'_2\right] \le\\
&\Re[3, 5, 4, 4; 3, 5, 6, 6],\\
&\Re\left[x_2, y_2, h_2, h_2; x_2, y_2, h'_2, h'_2\right] \le \Re[1, 3, 2, 2; 1, 3, 4, 4],\\
&\Re\left[x_1, y_1, h_1, h_1; x_1, y_1, h'_1, h'_1\right] \ge 0,\\
&\Re\left[x_2, y_2, h_2, h_2; x_2, y_2, h'_2, h'_2\right] \ge 0.
\end{aligned} \tag{P_{20}}$$

Step 5 Using Step 5 of Sect. 8.2, the problem (P_{20}) can be modified as problem (P_{21}).

$$\begin{aligned}
&\text{Maximize } \left[\tilde{z}^I = \left[5x_1 + 4x_2 + 5y_1 + 4y_2 + \tfrac{1}{2}[5h'_1 + 4h'_2 - 5h_1 - 4h_2]\right]\right]\\
&\text{Subject to}\\
&6x_1 + 4x_2 + 6y_1 + 4y_2 + \tfrac{1}{2}[6h'_1 + 4h'_2 - 6h_1 - 4h_2] \le 49,\\
&x_1 + 2x_2 + y_1 + 2y_2 + \tfrac{1}{2}[h'_1 + 2h'_2 - h_1 - 2h_2] \le 13,\\
&-y_1 + x_2 - x_1 + y_2 + \tfrac{1}{2}[h'_1 + h'_2 - h_1 - h_2] \le 9,\\
&x_2 + y_2 + \tfrac{1}{2}[h'_2 - h_2] \le 5,\\
&x_1 + y_1 + \tfrac{1}{2}[h'_1 - h_1] \ge 0,\\
&x_2 + y_2 + \tfrac{1}{2}[h'_2 - h_2] \ge 0,\\
&x_1 \le y_1, x_2 \le y_2, h_1 \le h'_1, h_2 \le h'_2,\\
&\text{where } x_1, x_2, y_1, y_2 \text{ are unrestricted and } h_1, h'_1, h_2, h'_2 \ge 0.
\end{aligned} \tag{P_{21}}$$

Step 6 Solving the problem (P_{21}), the obtained values of x_1, y_1, h_1, h'_1 are $\frac{3}{32}$, $\frac{3}{32}$, 0, $\frac{89}{8}$ and x_2, y_2, h_2, h'_2 are 0, 0, 0, $\frac{29}{4}$, respectively. Hence, the intuitionistic fuzzy optimal solution is $\tilde{x}_1^I = \left[\frac{3}{32}, \frac{3}{32}, 0, 0; \frac{3}{32}, \frac{3}{32}, \frac{89}{8}, \frac{89}{8}\right]$ and $\tilde{x}_2^I = \left[0, 0, 0, 0; 0, 0, \frac{29}{4}, \frac{29}{4}\right]$.
Step 7 Substituting the values of $\tilde{x}_1^I$ and $\tilde{x}_2^I$ in $5\tilde{x}_1^I + 4\tilde{x}_2^I$, the intuitionistic fuzzy optimal value is $\left[\frac{15}{32}, \frac{15}{32}, 0, 0; \frac{15}{32}, \frac{15}{32}, \frac{677}{8}, \frac{677}{8}\right]$.

10 Conclusions and Future Scope

In this paper, new methods are proposed to find the solution of intuitionistic fuzzy linear programming problems with trapezoidal intuitionistic fuzzy numbers with nonnegative coefficients as well as with unrestricted coefficients.

To find a new method for intuitionistic fuzzy multiobjective linear programming problems would be the future scope of work.

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