Second-Order Evolution Problems with Time-Dependent Maximal Monotone Operator and Applications

C. Castaing, M. D. P. Monteiro Marques, and P. Raynaud de Fitte

Abstract We consider at first the existence and uniqueness of solution for a general second-order evolution inclusion in a separable Hilbert space of the form

$$0 \in \ddot{u}(t) + A(t)\dot{u}(t) + f(t, u(t)),\ t \in [0, T]$$

where $A(t)$ is a time dependent with Lipschitz variation maximal monotone operator and the perturbation $f(t, .)$ is boundedly Lipschitz. Several new results are presented in the sense that these second-order evolution inclusions deal with time-dependent maximal monotone operators by contrast with the classical case dealing with some special fixed operators. In particular, the existence and uniqueness of solution to

$$0 = \ddot{u}(t) + A(t)\dot{u}(t) + \nabla\varphi(u(t)),\ t \in [0, T]$$

where $A(t)$ is a time dependent with Lipschitz variation single-valued maximal monotone operator and $\nabla\varphi$ is the gradient of a smooth Lipschitz function φ are stated. Some more general inclusion of the form

$$0 \in \ddot{u}(t) + A(t)\dot{u}(t) + \partial\Phi(u(t)),\ t \in [0, T]$$

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C. Castaing (✉)
IMAG, Université de Montpellier, Montpellier, France

M. D. P. Monteiro Marques
CMAF-CIO, Departamento de Matemática, Faculdade de Ciências da Universidade de Lisboa, Lisbon, Portugal
e-mail: mdmarques@fc.ul.pt

P. Raynaud de Fitte
Laboratoire de Mathématiques Raphaël Salem, Normandie Université, Rouen, France
e-mail: prf@univ-rouen.fr

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where $\partial\Phi(u(t))$ denotes the subdifferential of a proper lower semicontinuous convex function Φ at the point $u(t)$ is provided via a variational approach. Further results in second-order problems involving both absolutely continuous in variation maximal monotone operator and bounded in variation maximal monotone operator, $A(t)$, with perturbation $f : [0, T] \times H \times H$ are stated. Second- order evolution inclusion with perturbation f and Young measure control ν_t

$$\begin{cases} 0 \in \ddot{u}_{x,y,\nu}(t) + A(t)\dot{u}_{x,y,\nu}(t) + f(t, u_{x,y,\nu}(t)) + \text{bar}(\nu_t),\ t \in [0, T] \\ u_{x,y,\nu}(0) = x, \dot{u}_{x,y,\nu}(0) = y \in D(A(0)) \end{cases}$$

where $\text{bar}(\nu_t)$ denotes the barycenter of the Young measure ν_t is considered, and applications to optimal control are presented. Some variational limit theorems related to convex sweeping process are provided.

Keywords Bolza control problem · Lipschitz mapping · Maximal monotone operators · Pseudo-distance · Subdifferential · Viscosity · Young measures

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1 Introduction

Let H be a separable Hilbert space. In this paper, we are mainly interested in the study of the perturbed evolution problem

$$0 \in \ddot{u}(t) + A(t)\dot{u}(t) + \partial\Phi(u(t)),\ t \in [0, T]$$

where $\partial\Phi(u(t))$ denotes the subdifferential of a proper lower semicontinuous convex function Φ at the point $u(t)$, $A(t) : D(A(t)) \to 2^H$ is a maximal monotone operator in the Hilbert space H for every $t \in [0, T]$, and the dependence $t \mapsto A(t)$ has *Lipschitz variation*, in the sense that there exists $\alpha \geq 0$ such that

$$\text{dis}(A(t), A(s)) \leq \alpha(t - s),\ \ \forall s, t \in [0, T]\ (s \leq t)$$

dis(., .) being the *pseudo-distance* between maximal monotone operators (m.m.o.) defined by A. A. Vladimirov [53] as

$$\text{dis}(A, B) = \sup\left\{\frac{\langle y - \hat{y}, \hat{x} - x\rangle}{1 + ||y|| + ||\hat{y}||} : x \in D(A), y \in Ax, \hat{x} \in D(B), \hat{y} \in B\hat{x}\right\}$$

for m.m.o. A and B with domains $D(A)$ and $D(B)$, respectively; the dependence $t \mapsto A(t)$ has *absolutely continuous variation*, in the sense that there exists $\beta \in W^{1,1}([0, T])$ such that

$$\mathrm{dis}(A(t), A(s)) \le |\beta(t) - \beta(s)|, \ \forall t, s \in [0, T],$$

the dependence $t \mapsto A(t)$ has *bounded variation* in the sense that there exists a function $r : [0, T] \to [0, +\infty[$ which is continuous on $[0, T[$ and nondecreasing with $r(T) < +\infty$ such that

$$\mathrm{dis}(A(t), A(s)) \le dr(]s, t]) = r(t) - r(s) \ \text{ for } \ 0 \le s \le t \le T$$

The paper is organized as follows. Section 2 contains some definitions, notation and preliminary results. In Sect. 3, we recall and summarize (Theorem 3.2) the existence and uniqueness of solution for a general second-order evolution inclusion in a separable Hilbert space of the form

$$0 \in \ddot{u}(t) + A(t)\dot{u}(t) + f(t, u(t)), \ t \in [0, T]$$

where $A(t)$ is a time dependent with Lipschitz variation maximal monotone operator and the perturbation $f(t, .)$ is *dt-boundedly Lipschitz* (short for *dt-integrably Lipschitz on bounded sets*). At this point, Theorem 3.2 and its corollaries are new results in the sense that these second-order evolution inclusions deal with time-dependent maximal monotone operators by contrast with the classical case dealing with some special fixed operators; cf. Attouch et al. [4], Paoli [43], and Schatzman [48]. In particular, the existence and uniqueness of solution, based on Corollary 3.2, to

$$0 = \ddot{u}(t) + A(t)\dot{u}(t) + \nabla\varphi(u(t)), \ t \in [0, T]$$

where $A(t)$ is a time dependent with Lipschitz variation single-valued maximal monotone operator and $\nabla\varphi$ is the gradient of a smooth Lipschitz function φ, have some importance in mechanics [40], which may require a more general evolution inclusion of the form

$$0 \in \ddot{u}(t) + A(t)\dot{u}(t) + \partial\Phi(u(t)), \ t \in [0, T]$$

where $\partial\Phi(u(t))$ denotes the subdifferential of a proper lower semicontinuous convex function Φ at the point $u(t)$.

We provide (Proposition 3.1) the existence of a generalized $W^{1,1}_{BV}([0, T], H)$ solution to the second-order inclusion $0 \in \ddot{u}(t) + A(t)\dot{u}(t) + \partial\Phi(u(t))$ which enjoys several regularity properties. The result is similar to that of Attouch et al. [4], Paoli [43], and Schatzman [48] with different hypotheses and a different method that is essentially based on Corollary 3.2 and the tools given in [22, 23, 27] involving the Young measures and biting convergence [9, 22, 32]. By $W^{1,1}_{BV}([0,T],H)$, we denote the space of all absolutely continuous mappings $y:[0,T]\to H$ such that $\dot{y}$ are BV. Further results on second-order problems involving both the absolutely continuous in variation maximal monotone operators and the bounded in variation maximal monotone operator $A(t)$ with perturbation $f:[0,T]\times H\times H$ are stated.

Finally, in Sect. 4, we present several applications in optimal control in a new setting such as Bolza relaxation problem, dynamic programming principle, viscosity in evolution inclusion driven by a Lipschitz variation maximal monotone operator $A(t)$ with Lipschitz perturbation f, and Young measure control ν_t

$$\begin{cases} 0\in \ddot{u}_{x,y,\nu}(t)+A(t)\dot{u}_{x,y,\nu}(t)+f(t,u_{x,y,\nu}(t))+\mathrm{bar}(\nu_t),\ t\in[0,T] \\ u_{x,y,\nu}(0)=x,\dot{u}_{x,y,\nu}(0)=y\in D(A(0)) \end{cases}$$

where $\mathrm{bar}(\nu_t)$ denotes the barycenter of the Young measure ν_t in the same vein as in Castaing-Marques-Raynaud de Fitte [25] dealing with the sweeping process. At this point, the above second-order evolution inclusion contains the evolution problem associated with the sweeping process by a closed convex Lipschitzian mapping $C:[0,T]\to \mathrm{cc}(H)$

$$\begin{cases} 0\in \ddot{u}(t)+N_{C(t)}(\dot{u}(t))+f(t,u(t))+\mathrm{bar}(\nu_t),\ t\in[0,T] \\ u(0)=u_0,\dot{u}(0)=\dot{u}_0\in C(0) \end{cases}$$

(where $\mathrm{cc}(H)$ denotes the set of closed convex subsets of H) by taking $A(t)=\partial\Psi_{C(t)}$ and noting that if $C(t)$ is a closed convex moving set in H, then the subdifferential of its indicator function is $A(t)=\partial\Psi_{C(t)}=N_{C(t)}$, the outward normal cone operator. Since for all $s,t\in[0,T]$

$$\mathrm{dis}\big(A(t),A(s)\big)=\mathscr{H}\big(C(t),C(s)\big),$$

where $\mathscr{H}$ denotes the Hausdorff distance; it follows that our study of these time-dependent maximal monotone operators includes as special cases some related results for evolution problems governed by sweeping process of the form

$$0\in \ddot{u}(t)+N_{C(t)}(\dot{u}(t))+f(t,u(t)),\ t\in[0,T].$$

Since now sweeping process has found applications in several fields in particular to economics [29, 31, 35], we present also some variational limit theorems related to convex sweeping process; see [1, 3, 34] and the references therein.

There is a vast literature on evolution inclusions driven by the sweeping process and the subdifferential operators. See [2, 5, 6, 10, 17, 18, 20, 21, 25, 26, 28, 30, 37, 39–41, 45, 47, 49–52] and the references therein. We refer to [9, 12, 13, 54] for the study of maximal monotone operators.

2 Notation and Preliminaries

In the whole paper, $I := [0, T]$ ($T > 0$) is an interval of $\mathbb{R}$, and H is a real Hilbert space whose scalar product will be denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\| \cdot \|$. $\mathscr{L}([0, T])$ is the Lebesgue σ-algebra on $[0, T]$, and $\mathscr{B}(H)$ is the σ-algebra of Borel subsets of H. We will denote by $\overline{\mathbf{B}}_H(x_0, r)$ the closed ball of H of center x_0 and radius $r > 0$ and by $\overline{\mathbf{B}}_H$ its closed unit ball. $\mathrm{C}(I, H)$ denotes the Banach space of all continuous mappings $u : I \to H$ equipped with the norm $\|u\|_C = \max_{t \in I} \|u(t)\|$. For $q \in [1, +\infty[$, $L^q_H([0, T], dt)$ is the space of (classes of) measurable $u : [0, T] \to H$, with the norm $\|u(\cdot)\|_q = (\int_0^T \|u(t)\|^q dt)^{\frac{1}{q}}$, and $L^\infty_H([0, T], dt)$ is the space of (classes of) measurable essentially bounded $u : [0, T] \to H$ equipped with $\|.\|_\infty$.

If E is a Banach space and E^* its topological dual, we denote by $\sigma(E, E^*)$ the weak topology on E and by $\sigma(E^*, E)$ the weak star topology on E^*. For any $C \subset E$, we denote by $\delta^*(., C)$ the support function of C, i.e.

$$\delta^*(x^*, C) = \sup_{x \in C} \langle x^*, x \rangle , \forall x^* \in E^*.$$

A set-valued map $A : D(A) \subset H \to 2^H$ is monotone if $\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0$ whenever $x_i \in D(A)$ and $y_i \in A(x_i)$, $i = 1, 2$. A monotone operator A is maximal if A is not contained properly in any other monotone operator, that is, for all $\lambda > 0$, $R(I_H + \lambda A) = H$, with $R(A) = \bigcup \{Ax, \ x \in D(A)\}$ the range of A and I_H the identity mapping of H. In the whole paper, $I := [0, T]$ ($T > 0$) is an interval of $\mathbb{R}$, and H is a real Hilbert space whose scalar product will be denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\| \cdot \|$. Let $A : D(A) \subset H \to 2^H$ be a set-valued map. We say that A is monotone, if $\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0$ whenever $x_i \in \mathscr{D}(A)$ and $y_i \in A(x_i)$, $i = 1, 2$. If $\langle y_1 - y_2, x_1 - x_2 \rangle = 0$ implies that $x_1 = x_2$, we say that A is strictly monotone. A monotone operator A is said to be maximal if A could not be contained properly in any other monotone operator.

If A is a maximal monotone operator, then, for every $x \in D(A)$, $A(x)$ is nonempty closed and convex. So the set $A(x)$ contains an element of minimum norm (the projection of the origin on the set $A(x)$). This unique element is denoted by $A^0(x)$. Therefore $A^0(x) \in A(x)$ and $\|A^0(x)\| = \inf_{y \in A(x)} \|y\|$. Moreover the set $\overline{D(A)}$ is convex.

For $\lambda > 0$, we define the following well-known operators:

$$J^A_\lambda = (I + \lambda A)^{-1} \text{ (the resolvent of } A),$$

$$A_\lambda = \frac{1}{\lambda}(I - J^A_\lambda) \text{(the Yosida approximation of } A).$$

The operators J^A_λ and A_λ are defined on all of H. For the terminology of maximal monotone operators and more details, we refer the reader to [9, 13], and [54].

Let $A : D(A) \subset H \to 2^H$ and $B : D(B) \subset H \to 2^H$ be two maximal monotone operators, and then we denote by $\operatorname{dis}(A, B)$ the pseudo-distance between

A and B defined by A. A. Vladimirov [53] as

$$\mathrm{dis}(A, B) = \sup\left\{\frac{\langle y - y', x' - x\rangle}{1 + \|y\| + \|y'\|} : \ x \in D(A),\ y \in Ax,\ x' \in D(B),\ y' \in Bx'\right\}.$$

Our main results are established under the following hypotheses on the operator A:

$(H1)$ The mapping $t \mapsto A(t)$ has Lipschitz variation, in the sense that there exists $\alpha \geq 0$ such that

$$\mathrm{dis}(A(t), A(s)) \leq \alpha(t - s),\ \ \forall s, t \in [0, T]\ (s \leq t).$$

$(H2)$ There exists a nonnegative real number c such that

$$\|A^0(t, x)\| \leq c(1 + \|x\|)\ \text{ for }\ t \in [0, T],\ x \in D(A(t)).$$

We recall some elementary lemmas, and we refer to [38] for the proofs.

Lemma 2.1 *Let A and B be maximal monotone operators. Then*

(1) $\mathrm{dis}(A, B) \in [0, +\infty]$, $\mathrm{dis}(A, B) = \mathrm{dis}(B, A)$ *and* $\mathrm{dis}(A, B) = 0$ *iff* $A = B$.
(2) $\|x - Proj(x, \overline{D(B)}\| \leq \mathrm{dis}(A, B)$ *for* $x \in \overline{D(A)}$.
(3) $\mathcal{H}(D(A), D(B)) \leq \mathrm{dis}(A, B)$.

Lemma 2.2 *Let A be a maximal monotone operator. If $x, y \in H$ are such that*

$$\langle A^0(z) - y, z - x\rangle \geq 0\ \ \forall z \in D(A),$$

then $x \in D(A)$ and $y \in A(x)$.

Lemma 2.3 *Let A_n $(n \in \mathbb{N})$ and A be maximal monotone operators such that* $\mathrm{dis}(A_n, A) \to 0$. *Suppose also that $x_n \in D(A_n)$ with $x_n \to x$ and $y_n \in A_n(x_n)$ with $y_n \to y$ weakly for some $x, y \in H$. Then $x \in D(A)$ and $y \in A(x)$.*

Lemma 2.4 *Let A and B be maximal monotone operators. Then*

(1) *for* $\lambda > 0$ *and* $x \in D(A)$

$$\|x - J_\lambda^B(x)\| \leq \lambda\|A^0(x)\| + \mathrm{dis}(A, B) + \sqrt{\lambda\big(1 + \|A^0(x)\|\big)\,\mathrm{dis}(A, B)}.$$

(2) *For* $\lambda > 0$ *and* $x, x' \in H$

$$\|J_\lambda^A(x) - J_\lambda^B(x')\|^2 \leq \|x - x'\|^2 + 2\lambda\big(1 + \|A_\lambda(x)\| + \|B_\lambda(x')\|\big)\,\mathrm{dis}(A, B).$$

(3) *For* $\lambda > 0$ *and* $x, x' \in H$

$$\|A_\lambda(x) - B_\lambda(x')\|^2 \leq \frac{1}{\lambda^2}\|x - x'\|^2 + \frac{2}{\lambda}\big(1 + \|A_\lambda(x)\| + \|B_\lambda(x')\|\big)\,\mathrm{dis}(A, B).$$

3 Second-Order Evolution Problems Involving Time-Dependent Maximal Monotone Operators

In the sequel, H is a separable Hilbert space. For the sake of completeness, we summarize and state the following result. We say that a function $f = f(t, x)$ is *dt-boundedly Lipschitz* (short for *dt-integrably Lipschitz on bounded sets*) if, for every $R > 0$, there is a nonnegative dt-integrable function $\lambda_R \in L^1([0, T], \mathbb{R}; dt)$ such that, for all $t \in [0, T]$

$$\|f(t, x) - f(t, y)\| \leq \lambda_R(t)||x - y||, \ \forall x, y \in \overline{\mathbf{B}}(0, R).$$

Theorem 3.1 *Let for every $t \in [0, T]$, $A(t) : D(A(t)) \subset H \to 2^H$ be a maximal monotone operator satisfying*

$(H1)$ *there exists a real constant $\alpha \geq 0$ such that*

$$\text{dis}(A(t), A(s)) \leq \alpha(t - s) \ \textit{for} \ \ 0 \leq s \leq t \leq T.$$

$(H2)$ *there exists a nonnegative real number c such that*

$$\|A^0(t, x)\| \leq c(1 + \|x\|), t \in [0, T], x \in D(A(t))$$

Let $f : [0, T] \times H \to H$ satisfying the linear growth condition

$(H3)$ *there exists a nonnegative real number M such that*

$$\|f(t, x)\| \leq M(1 + \|x\|) \ \textit{for} \ \ t \in [0, T], \ x \in H.$$

and assume that $f(., x)$ is dt-integrable for every $x \in H$. Assume also that f is dt-boundedly Lipschitz, as above.

Then for all $u_0 \in D(A(0))$, the problem

$$-\frac{du}{dt}(t) \in A(t)u(t) + f(t, u(t)) \ dt - \text{a.e.}\, t \in [0, T], \quad u(0) = u_0$$

has a unique Lipschitz solution with the property: $||u(t) - u(\tau)|| \leq K \max\{1, \alpha\}|t - \tau|$ for all $t, \tau \in [0, T]$ for some constant $K \in]0, \infty[$.

Proof See [7, Theorem 3.1 and Theorem 3.3].

Theorem 3.2 *Let for every $t \in [0, T]$, $A(t) : D(A(t)) \subset H \to 2^H$ be a maximal monotone operator satisfying*

$(H1)$ *there exists a real constant $\alpha \geq 0$ such that*

$$\text{dis}(A(t), A(s)) \leq \alpha(t - s) \ \textit{for} \ \ 0 \leq s \leq t \leq T.$$

(*H*2) *there exists a nonnegative real number c such that*

$$\|A^0(t,x)\| \le c(1+\|x\|), t \in [0,T], x \in D(A(t))$$

Let $f : [0,T] \times H \to H$ *satisfying the linear growth condition:*
(*H*3) *there exists a nonnegative real number M such that*

$$\|f(t,x)\| \le M(1+\|x\|) \text{ for } t \in [0,T],\ x \in H.$$

and assume that $f(.,x)$ *is* dt*-integrable for every* $x \in H$*. Assume also that* f *is* dt*-boundedly Lipschitz.*

Then the second-order evolution inclusion

$$(\mathscr{S}_1) \begin{cases} 0 \in \ddot{u}(t) + A(t)\dot{u}(t) + f(t,u(t)),\ t \in [0,T] \\ u(0) = u_0, \dot{u}(0) = \dot{u}_0 \in D(A(0)) \end{cases}$$

admits a unique solution $u \in W_H^{2,\infty}([0,T], dt)$.

Proof The proof is a careful application of Theorem 3.1. In the new variables $X = (x, \dot{x})$, let us set for all $t \in I$

$$B(t)X = \{0\} \times A(t)\dot{x},\ g(t,X) = (-\dot{x}, f(t,x)).$$

For any $u \in W^{2,\infty}(I,H;dt)$, define $X(t) = (u(t), \frac{du}{dt}(t))$ and $\dot{X}(t) = \frac{dX}{dt}(t)$. Then the evolution inclusion $(\mathscr{S}_1)$ can be written as a first-order evolution inclusion associated with the Lipschitz maximal monotone operator $B(t)$ and the locally Lipschitz perturbation g:

$$\begin{cases} 0 \in \dot{X}(t) + B(t)X(t) + g(t,X(t)),\ t \in [0,T] \\ X(0) = (u_0, \dot{u}_0) \in H \times D(A(0)). \end{cases}$$

So the existence and uniqueness solution to the second-order evolution inclusion under consideration follows from Theorem 3.1.

There are some useful corollaries to Theorem 3.2.

Corollary 3.1 *Assume that for every* $t \in [0,T]$, $A(t) : H \to H$ *is a single-valued maximal monotone operator satisfying* (*H*1) *and* (*H*2)*. Let* $f : [0,T] \times H \to H$ *be as in Theorem 3.2. Then the second-order evolution equation*

$$\begin{cases} 0 = \ddot{u}(t) + A(t)\dot{u}(t) + f(t,u(t)),\ t \in [0,T] \\ u(0) = u_0, \dot{u}(0) = \dot{u}_0 \end{cases}$$

admits a unique solution $u \in W_H^{2,\infty}([0,T])$.

Corollary 3.2 *Assume that for every $t \in [0, T]$, $A(t) : H \to H$ is a single-valued maximal monotone operator satisfying $(H1)$ and $(H2)$. Assume further that $A(t)$ satisfies*

(i) *$(t, x) \mapsto A(t)x$ is a Caratheodory mapping, that is, $t \mapsto A(t)x$ is Lebesgue measurable on $[0, T]$ for each fixed $x \in H$, and $x \mapsto A(t)x$ is continuous on H for each fixed $t \in [0, T]$,*
(ii) *$\langle A(t)x, x\rangle \geq \gamma ||x||^2$, for all $(t, x) \in [0, T] \times H$, for some $\gamma > 0$.*

Let $\varphi \in C^1(H, \mathbb{R})$ be Lipschitz and such that $\nabla\varphi$ is locally Lipschitz. Then the evolution equation

$$(\mathscr{S}_2) \begin{cases} 0 = \ddot{u}(t) + A(t)\dot{u}(t) + \nabla\varphi(u(t)), \ t \in [0, T] \\ u(0) = u_0, \dot{u}(0) = \dot{u}_0 \end{cases}$$

admits a unique solution $u \in W^{2,\infty}([0, T], H; dt)$; moreover, u satisfies the energy estimate

$$\varphi(u(t)) - \frac{1}{2}||\dot{u}(t)||^2 \leq \varphi(u(0)) - \frac{1}{2}||\dot{u}(t)||^2 - \gamma \int_0^t ||\dot{u}(s)||^2 ds, \ t \in [0, T].$$

Proof Existence and uniqueness of solution follows from Theorem 3.2 or Corollary 3.1. The energy estimate is quite standard. Multiplying the equation by $\dot{u}(t)$ and applying the usual chain rule formula gives for all $t \in [0, T]$

$$\frac{d}{dt}\left(\varphi(u(t)) + \frac{1}{2}||\dot{u}(t)||^2\right) = -\big\langle A(t)\dot{u}(t), \dot{u}(t)\big\rangle.$$

By (i) and (ii) and by integrating on $[0, t]$, we get the required inequality

$$\begin{aligned} \varphi(u(t)) + \frac{1}{2}||\dot{u}(t)||^2 &= \varphi(u(0)) + \frac{1}{2}||\dot{u}(0)||^2 - \int_0^t \big\langle A(s)\dot{u}(s), \dot{u}(s)\big\rangle ds \\ &\leq \varphi(u(0)) + \frac{1}{2}||\dot{u}(0)||^2 - \gamma \int_0^t ||\dot{u}(s)||^2 ds, \ t \in [0, T], \end{aligned}$$

which completes the proof.

It is worth mentioning that the uniqueness of the solution to the equation $(\mathscr{S}_1)$ is quite important in applications, such as models in mechanics, since it contains the classical inclusion of the form

$$0 \in \ddot{u}(t) + \partial\Phi(\dot{u}(t)) + \nabla g(u(t))$$

where $\partial\Phi$ is the subdifferential of the proper lower semicontinuous convex function Φ and g is of class C^1 and ∇g is Lipschitz continuous on bounded sets. We also note that the uniqueness of the solution to the equation $(\mathscr{S}_2)$ and its energy estimate allow to recover a classical result in the literature dealing with finite dimensional space H and $A(t) = \gamma I_H,\ t \in [0, T]$, where I_H is the identity mapping in H. See Attouch et al. [4]. The energy estimate for the solution of

$$\begin{cases} 0 = \ddot{u}(t) + \gamma \dot{u}(t) + \nabla\varphi(u(t)),\ t \in I \\ u(0) = u_0, \dot{u}(0) = \dot{u}_0 \end{cases}$$

is then

$$\varphi(u(t)) + \frac{1}{2}||\dot{u}(t)||^2 = \varphi(u_0) + \frac{1}{2}||\dot{u}_0||^2 - \gamma \int_0^t ||\dot{u}(s)||^2 ds.$$

Actually the dynamical system $(\mathscr{S}_1)$ given in Theorem 3.2 has been intensively studied by many authors in particular cases. See Attouch et al. [4] dealing with the inclusion

$$0 \in \ddot{u}(t) + \gamma \dot{u}(t) + \partial\varphi(u(t))$$

and Paoli [43] and Schatzman [48] dealing with the second-order dynamical systems of the form

$$0 \in \ddot{u}(t) + \partial\varphi(u(t))$$

and

$$0 \in \ddot{u}(t) + A\dot{u}(t) + \partial\varphi(u(t))$$

where A is a positive autoadjoint operator. The existence and uniqueness of solutions in $(\mathscr{S}_2)$ are of some importance since they allow to obtain the existence of at least a $W^{1,1}_{BV}([0, T], H)$ solution with conservation of energy (see Proposition 3.1 below) for a second-order evolution inclusion of the form

$$(\mathscr{S}_3) \begin{cases} 0 \in \ddot{u}(t) + A(t)\dot{u}(t) + \partial\Phi(u(t),\ t \in I \\ u(0) = u_0 \in \operatorname{dom}\Phi, \dot{u}(0) = \dot{u}_0 \in D(A(0)) \end{cases}$$

where $\partial\Phi$ is the subdifferential of a proper convex lower semicontinuous function; the energy estimate is given by

$$\Phi(u(t)) + \frac{1}{2}||\dot{u}(t)||^2 = \Phi(u(0)) + \frac{1}{2}||\dot{u}(0)||^2 - \int_0^t \langle A(s)\dot{u}(s), \dot{u}(s)\rangle ds.$$

Taking into account these considerations, we will provide the existence of a generalized solution to the second-order inclusion of the form

$$0 \in \ddot{u}(t) + A(t)\dot{u}(t) + \partial\phi(u(t))$$

which enjoy several regular properties. The result is similar to that of Attouch et al. [4], Paoli [43], and Schatzman [48] with different hypotheses and a different method that is essentially based on Corollary 3.2 and the tools given in [22, 23, 27] involving the Young measures [9, 32] and biting convergence.

Let us recall a useful Gronwall-type lemma [21].

Lemma 3.5 (A Gronwall-like inequality.) *Let $p, q, r : [0, T] \to [0, \infty[$ be three nonnegative Lebesgue integrable functions such that for almost all $t \in [0, T]$*

$$r(t) \le p(t) + q(t) \int_0^t r(s)\, ds.$$

Then

$$r(t) \le p(t) + q(t) \int_0^t \left[p(s) \exp\left(\int_s^t q(\tau)\, d\tau \right) \right] ds$$

for all $t \in [0, T]$.

Proposition 3.1 *Assume that $H = \mathbb{R}^d$ and that, for every $t \in [0, T]$, $A(t) : H \to H$ is single-valued maximal monotone satisfying*

$(H1)$ *there exists $\alpha > 0$ such that*

$$\mathrm{dis}(A(t), A(s)) \le \alpha(t - s) \ \ \textit{for} \ \ 0 \le s \le t \le T,$$

$(H2)$ *there exists a nonnegative real number c such that*

$$\|A(t, x)\| \le c(1 + \|x\|) \ \ \textit{for} \ \ t \in [0, T], \ x \in H.$$

Assume further that $A(t)$ satisfies

A-1. *$(t, x) \to A(t)x$ is a Caratheodory mapping, that is, $t \mapsto A(t)x$ is Lebesgue-measurable on $[0, T]$ for each fixed $x \in H$, and $x \mapsto A(t)x$ is continuous on H for each fixed $t \in [0, T]$,*

A-2. *$\langle A(t)x, x\rangle \ge \gamma ||x||^2$, for all $(t, x) \in [0, T] \times H$, for some $\gamma > 0$.*

Let $n \in \mathbb{N}$ and $\varphi_n : H \to \mathbb{R}^+$ be a C^1, convex, Lipschitz function and such that $\nabla\varphi_n$ is locally Lipschitz, and let φ_∞ be a nonnegative l.s.c proper function defined on H with $\varphi_n(x) \le \varphi_\infty(x), \forall x \in H$. For each $n \in \mathbb{N}$, let u^n be the unique $W_H^{2,\infty}([0, T])$ solution to the problem

$$\begin{cases} 0 = \ddot{u}^n(t) + A(t)\dot{u}^n(t) + \nabla\varphi_n(u^n(t)), t \in [0, T] \\ u^n(0) = u_0^n, \ \dot{u}^n(0) = \dot{u}_0^n \end{cases}$$

Assume that

(i) *φ_n epiconverges to φ_∞,*

(ii) $u^n(0) \to u_0^\infty \in \operatorname{dom}\varphi_\infty$ *and* $\lim_n \varphi_n(u^n(0)) = \varphi_\infty(u_0^\infty)$,

(iii) $\sup_{v\in \overline{B}_{L_H^\infty([0,T])}} \int_0^T \varphi_\infty(v(t))dt < +\infty$, *where* $\overline{B}_{L_H^\infty([0,T])}$ *is the closed unit ball in* $L_H^\infty([0,T])$.

(a) *Then up to extracted subsequences,* (u^n) *converges uniformly to a* $W_{BV}^{1,1}([0,T],\mathbb{R}^d)$*-function* u^∞ *with* $u^\infty(0) \in \operatorname{dom}\varphi_\infty$*, and* $(\dot u^n)$ *pointwisely converges to a BV function* v^∞ *with* $v^\infty = \dot u^\infty$*, and* $(\ddot u^n)$ *biting converges to a function* $\zeta^\infty \in L^1_{\mathbb{R}^d}([0,T])$ *so that the limit function* u^∞, $\dot u^\infty$ *and the biting limit* ζ^∞ *satisfy the variational inclusion*

$$-A(.)\dot u^\infty - \zeta^\infty \in \partial I_{\varphi_\infty}(u^\infty)$$

where $\partial I_{\varphi_\infty}$ *denotes the subdifferential of the convex lower semicontinuous integral functional* I_{φ_∞} *defined on* $L^\infty_{\mathbb{R}^d}([0,T])$

$$I_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\,dt,\ \ \forall u \in L^\infty_{\mathbb{R}^d}([0,T]).$$

(b) $(\ddot u^n)$ *weakly converges to a vector measure* $m \in \mathscr{M}_H^b([0,T])$ *so that the limit functions* $u^\infty(.)$ *and the limit measure* m *satisfy the following variational inequality:*

$$\int_0^T \varphi_\infty(v(t))\,dt \geq \int_0^T \varphi_\infty(u^\infty(t))\,dt + \int_0^T \langle -A(t)\dot u^\infty(t), v(t)-u^\infty(t)\rangle\,dt$$
$$+ \langle -m, v-u^\infty\rangle_{(\mathscr{M}^b_{\mathbb{R}^d}([0,T]),\mathscr{C}_E([0,T]))}.$$

(c) *Furthermore* $\lim_n \int_0^T \varphi_n(u^n(t))dt = \int_0^T \varphi_\infty(u^\infty(t))dt$*. Subsequently the energy estimate*

$$\varphi_\infty(u^\infty(t))+\frac{1}{2}||\dot u^\infty(t)||^2 = \varphi_\infty(u_0^\infty)+\frac{1}{2}||\dot u_0^\infty||^2+\int_0^t \langle -A(s)\dot u^\infty(s), u^\infty(s)\rangle ds$$

holds a.e.

(d) *There is a filter* $\mathscr{U}$ *finer than the Fréchet filter* $l \in L^\infty_{\mathbb{R}^d}([0,T])'$ *such that*

$$\mathscr{U} - \lim_n [-A(.)\dot u^n - \ddot u^n] = l \in L^\infty_{\mathbb{R}^d}([0,T])'_{\text{weak}}$$

where $L^\infty_{\mathbb{R}^d}([0,T])'_{\text{weak}}$ *is the second dual of* $L^1_{\mathbb{R}^d}([0,T])$ *endowed with the topology* $\sigma(L^\infty_{\mathbb{R}^d}([0,T])', L^\infty_{\mathbb{R}^d}([0,T]))$*, and* $\mathbf{n} \in \mathscr{C}_{\mathbb{R}^d}([0,T])'_{\text{weak}}$ *such that*

$$\lim_n [-A(.)\dot u^n - \ddot u^n] = \mathbf{n} \in \mathscr{C}_{\mathbb{R}^d}([0,T])'_{\text{weak}}$$

where $\mathscr{C}_{\mathbb{R}^d}([0,T])'_{\text{weak}}$ *denotes the space* $\mathscr{C}_{\mathbb{R}^d}([0,T])'$ *endowed with the weak topology* $\sigma(\mathscr{C}_{\mathbb{R}^d}([0,T])', \mathscr{C}_{\mathbb{R}^d}([0,T]))$. *Let* l_a *be the density of the absolutely continuous part* l_a *of* l *in the decomposition* $l = l_a + l_s$ *in absolutely continuous part* l_a *and singular part* l_s. *Then*

$$l_a(f) = \int_0^T \langle f(t), -A(t)\dot{u}^\infty(t) - \zeta^\infty(t)\rangle dt$$

for all $f \in L^\infty_{\mathbb{R}^d}([0,T])$ *so that*

$$I^*_{\varphi_\infty}(l) = I_{\varphi^*_\infty}(-A(.)\dot{u}^\infty - \zeta^\infty) + \delta^*(l_s, \operatorname{dom} I_{\varphi_\infty})$$

where φ^*_∞ *is the conjugate of* φ_∞, $I_{\varphi^*_\infty}$ *the integral functional defined on* $L^1_{\mathbb{R}^d}([0,T])$ *associated with* φ^*_∞, $I^*_{\varphi_\infty}$ *the conjugate of the integral functional* I_{φ_∞}, $\operatorname{dom} I_{\varphi_\infty} := \{u \in L^\infty_{\mathbb{R}^d}([0,1]) : I_{\varphi_\infty}(u) < \infty\}$, *and*

$$\langle \mathbf{n}, f\rangle = \int_0^T \langle -A(t)\dot{u}^\infty(t) - \zeta^\infty(t), f(t)\rangle dt + \langle \mathbf{n}_s, f\rangle, \quad \forall f \in \mathscr{C}_{\mathbb{R}^d}([0,T]).$$

with $\langle \mathbf{n}_s, f\rangle = l_s(f)$, $\forall f \in \mathscr{C}_{\mathbb{R}^d}([0,T])$. *Further* $\mathbf{n}$ *belongs to the subdifferential* $\partial J_{\varphi_\infty}(u^\infty)$ *of the convex lower semicontinuous integral functional* J_{φ_∞} *defined on* $\mathscr{C}_{\mathbb{R}^d}([0,T])$

$$J_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\, dt, \;\; \forall u \in \mathscr{C}_{\mathbb{R}^d}([0,T]).$$

Consequently the density $-A(.)\dot{u}^\infty - \zeta^\infty$ *of the absolutely continuous part* $\mathbf{n}_a$

$$\mathbf{n}_a(f) := \int_0^T \langle -A(t)\dot{u}^\infty(t) - \zeta^\infty(t), f(t)\rangle dt, \quad \forall f \in \mathscr{C}_{\mathbb{R}^d}([0,T])$$

satisfies the inclusion

$$-A(t)\dot{u}^\infty(t) - \zeta^\infty(t) \in \partial\varphi_\infty(u^\infty(t)), \quad \text{a.e.}$$

and for any nonnegative measure θ *on* $[0,T]$ *with respect to which* $\mathbf{n}_s$ *is absolutely continuous*

$$\int_0^T r_{\varphi^*_\infty}\left(\frac{d\mathbf{n}_s}{d\theta}(t)\right) d\theta(t) = \int_0^T \left\langle u^\infty(t), \frac{d\mathbf{n}_s}{d\theta}(t)\right\rangle d\theta(t)$$

where $r_{\varphi^*_\infty}$ *denotes the recession function of* φ^*_∞.

Proof The proof is long and based on the existence and uniqueness of $W^{2,\infty}_H([0,T])$ solution to the approximating equation (cf. Corollary 3.2)

$$\begin{cases} 0 = \ddot{u}^n(t) + A(t)\dot{u}^n(t) + \nabla\varphi_n(u^n(t)), t \in [0,T] \\ u^n(0) = u^n_0,\ \dot{u}^n(0) = \dot{u}^n_0 \end{cases}$$

and the techniques developed in [22, 23, 27]. Nevertheless we will produce the proof with full details, since the techniques employed can be applied to further related results.

Step 1. Multiplying scalarly the equation

$$-A(t)\dot{u}^n(t) - \ddot{u}^n(t) = \nabla\varphi_n(u^n(t))$$

by $\dot{u}^n(t)$ and applying the chain rule theorem [42, Theorem 2] yields

$$-\langle \dot{u}^n(t), A(t)\dot{u}^n(t)\rangle - \langle \dot{u}^n(t), \ddot{u}^n(t)\rangle = \frac{d}{dt}[\varphi_n(u_n(t))],$$

that is,

$$-\langle \dot{u}^n(t), A(t)\dot{u}^n(t)\rangle = \frac{d}{dt}\left[\varphi_n(u^n(t)) + \frac{1}{2}||\dot{u}^n(t)||^2\right].$$

By integrating on $[0,t]$ this equality and using the condition (ii), we get

$$\begin{aligned}\varphi_n(u^n(t)) + \frac{1}{2}||\dot{u}^n(t)||^2 &= \varphi_n(u^n(0)) + \frac{1}{2}||\dot{u}^n(0)||^2 - \int_0^t \langle \dot{u}^n(s), A(s)\dot{u}^n(s)\rangle ds \\ &\le \varphi_n(u^n(0)) + \frac{1}{2}||\dot{u}^n(0)||^2 + \gamma \int_0^t ||\dot{u}^n(s||^2 ds.\end{aligned}$$

Then, from our assumption, $\varphi_n(u^n(0)) \le$ positive constant $< +\infty$ and $\frac{1}{2}||\dot{u}^n(0)||^2 \le$ positive constant $< +\infty$ so that

$$\varphi_n(u^n(t)) + \frac{1}{2}||\dot{u}^n(t)||^2 \le p + \gamma \int_0^t ||\dot{u}^n(s||^2 ds,\ t \in [0,T]$$

where p is a generic positive constant. So by the preceding estimate and the Gronwall inequality [21, Lemma 3.1] , it is immediate that

$$\sup_{n\ge 1}\sup_{t\in[0,T]} ||\dot{u}^n(t)|| < +\infty \quad \text{and} \quad \sup_{n\ge 1}\sup_{t\in[0,T]} \varphi_n(u^n(t)) < +\infty. \tag{1}$$

Step 2. Estimation of $||\ddot{u}^n(.)||$. For simplicity, let us set $z^n(t) = -A(t)\dot{u}^n(t) - \ddot{u}^n(t)$, $\forall t \in [0,T]$. As

$$z^n(t) := -A(t)\dot{u}^n(t) - \ddot{u}^n(t) = \nabla\varphi_n(u^n(t))$$

by the subdifferential inequality for convex lower semicontinuous functions, we have

$$\varphi_n(x) \geq \varphi_n(u^n(t)) + \langle x - u^n(t), z^n(t)\rangle$$

for all $x \in \mathbb{R}^d$. Now let $v \in \overline{B}_{L^\infty_{\mathbb{R}^d}([0,T])}$, the closed unit ball of $L^\infty_{\mathbb{R}^d}[0, T])$. By taking $x = v(t)$ in the preceding inequality, we get

$$\varphi_n(v(t)) \geq \varphi_n(u^n(t)) + \langle v(t) - u^n(t), z^n(t)\rangle.$$

Integrating the preceding inequality gives

$$\int_0^T \langle v(t) - u^n(t), z^n(t)\rangle dt \leq \int_0^T \varphi_n(v(t))dt - \int_0^T \varphi_n(u^n(t))dt.$$

Whence follows

$$\int_0^T \langle v(t), z^n(t)\rangle dt$$
$$\leq \int_0^T \varphi_n(v(t))dt - \int_0^T \varphi_n(u^n(t))dt + \int_0^1 \langle u^n(t), z^n(t)\rangle dt. \quad (2)$$

We compute the last integral in the preceding inequality. By integration and taking account of (1), we have

$$\begin{aligned}
&\int_0^T \langle u^n(t), z^n(t)\rangle dt \\
&= \int_0^T \langle u^n(t), -A(t)\dot{u}^n(t) - \ddot{u}^n(t)\rangle dt \\
&= - [\langle u^n(t), \dot{u}^n(t)]_0^T + \int_0^T \langle \dot{u}^n(t), \dot{u}^n(t)\rangle dt - \int_0^T \langle u^n(t), A(t)\dot{u}^n(t)\rangle dt \\
&= - \langle u^n(T), \dot{u}^n(T)\rangle + \langle u^n(0), \dot{u}^n(0)\rangle \\
&\qquad + \int_0^T ||\dot{u}^n(t)||^2 dt - \int_0^T \langle u^n(t), A(t)\dot{u}^n(t)\rangle dt. \quad (3)
\end{aligned}$$

As $||A(t)\dot{u}^n(t)|| \leq c(1 + ||\dot{u}_n(t)||)$ by (H_2), so that by (1) it is immediate that $\int_0^T \langle u^n(t), A(t)\dot{u}^n(t)\rangle dt$ is uniformly bounded so that by (1), (2), and (3), we get

$$\int_0^T \langle v(t), z^n(t)\rangle dt \leq \int_0^T \varphi_n(v(t))dt + L$$
$$\leq \sup_{v \in \overline{B}_{L^\infty_{\mathbb{R}^d}([0,T])}} \int_0^T \varphi_\infty(v(t))dt + L < \infty \quad (4)$$

for all $v \in \overline{B}_{L^\infty_{\mathbb{R}^d}([0,T])}$. Here L is a generic positive constant independent of $n \in \mathbb{N}$. By (4), we conclude that $(z^n = -A(.)\dot{u}^n - \ddot{u}^n)$ is bounded in $L^1_{\mathbb{R}^d}([0,T])$, and then so is $(\ddot{u}^n)$. It turns out that the sequence $(\dot{u}^n)$ of absolutely continuous functions is uniformly bounded by (1) and bounded in variation and by Helly's theorem; we may assume that $(\dot{u}^n)$ pointwisely converges to a BV function $v^\infty : [0,T] \to \mathbb{R}^d$ and the sequence (u^n) converges uniformly to an absolutely continuous function u^∞ with $\dot{u}^\infty = v^\infty$ a.e. At this point, it is clear that $A(t)\dot{u}^n(t) \to A(t)v^\infty(t)$ so that $A(t)\dot{u}^n(t) \to A(t)\dot{u}^\infty(t)$ a.e. and $A(.)\dot{u}^n(.)$ converges in $L^1_{\mathbb{R}^d}([0,T])$ to $A(.)\dot{u}^\infty(.)$, using (1) and the dominated convergence theorem.

Step 3. Young measure limit and biting limit of $\ddot{u}_n$. As $(\ddot{u}_n)$ is bounded in $L^1_{\mathbb{R}^d}([0,T])$, we may assume that $(\ddot{u}^n)$ stably converges to a Young measure $\nu \in \mathscr{Y}([0,T]); \mathbb{R}^d)$ with $\mathrm{bar}(\nu) : t \mapsto \mathrm{bar}(\nu_t) \in L^1_{\mathbb{R}^d}([0,T])$ (here $\mathrm{bar}(\nu_t)$ denotes the barycenter of ν_t). Further, we may assume that $(\ddot{u}^n)$ biting converges to a function $\zeta^\infty : t \mapsto \mathrm{bar}(\nu_t)$, that is, there exists a decreasing sequence of Lebesgue-measurable sets (B_p) with $\lim_p \lambda(B_p) = 0$ such that the restriction of $(\ddot{u}_n)$ on each B_p^c converges weakly in $L^1_{\mathbb{R}^d}([0,T])$ to ζ^∞. Noting that $(A(.)\dot{u}^n)$ converges in $L^1_{\mathbb{R}^d}([0,T])$ to $A(.)\dot{u}^\infty$. It follows that the restriction of $z^n = -A(.)\dot{u}^n - \ddot{u}^n$ to each B_p^c weakly converges in $L^1_{\mathbb{R}^d}([0,T])$ to $z^\infty := -A(.)\dot{u}^\infty - \zeta^\infty$, because $(-A(.)\dot{u}^n)$ converges in $L^1_{\mathbb{R}^d}([0,T])$ to $A(.)\dot{u}^\infty$ and $(\ddot{u}^n)$ biting converges to $\zeta^\infty \in L^1_{\mathbb{R}^d}([0,T])$. It follows that

$$\lim_n \int_B \langle -A(.)\dot{u}^n - \ddot{u}^n, w(t) - u^n(t)\rangle = \int_B \langle -A(.)\dot{u}^\infty - \mathrm{bar}(\nu_t), w(t) - u(t)\rangle dt \tag{5}$$

for every $B \in B_p^c \cap \mathscr{L}([0,T])$ and for every $w \in L^\infty_{\mathbb{R}^d}([0,T])$. Indeed, we note that $(w(t) - u^n(t))$ is a bounded sequence in $L^\infty_{\mathbb{R}^d}([0,T])$ which pointwisely converges to $w(t) - u^\infty(t)$, so it converges uniformly on every uniformly integrable subset of $L^1_{\mathbb{R}^d}([0,T])$ by virtue of a Grothendieck Lemma [33], recalling here that the restriction of $-A(.)\dot{u}^n - \ddot{u}^n$ on each B_p^c is uniformly integrable. Now, since φ_n lower epiconverges to φ_∞, for every Lebesgue-measurable set A in $[0,T]$, by virtue of [23, Corollary 4.7], we have

$$+\infty > \liminf_n \int_A \varphi_n(u^n(t))dt \geq \int_A \varphi_\infty(u^\infty(t))dt. \tag{6}$$

Combining (1), (2), (3), (4), (5), and (6) and using the subdifferential inequality

$$\varphi_n(w(t)) \geq \varphi_n(u^n(t)) + \langle -A(.)\dot{u}^n - \ddot{u}^n(t), w(t) - u^n(t)\rangle,$$

we get

$$\int_B \varphi_\infty(w(t))\, dt \geq \int_B \varphi_\infty(u^\infty(t))\, dt + \int_B \langle -A(.)\dot{u}^\infty - \mathrm{bar}(\nu_t), w(t) - u^\infty(t)\rangle\, dt.$$

This shows that $t \mapsto -A(.)\dot{u}^\infty - \text{bar}(\nu_t)$ is a subgradient at the point u^∞ of the convex integral functional I_{φ_∞} restricted to $L^\infty_{\mathbb{R}^d}(B_p^c)$, consequently,

$$-A(.)\dot{u}^\infty - \text{bar}(\nu_t) \in \partial\varphi_\infty(u^\infty(t)), \text{ a.e. on } B_p^c.$$

As this inclusion is true on each B_p^c and $B_p^c \uparrow [0,T]$, we conclude that

$$-A(.)\dot{u}^\infty - \text{bar}(\nu_t) \in \partial\varphi_\infty(u^\infty(t)), \text{ a.e. on } [0,T].$$

Step 4. Measure limit in $\mathscr{M}^b_{\mathbb{R}^d}([0,T])$ *of* $\ddot{u}^n$. As $(\ddot{u}_n)$ is bounded in $L^1_{\mathbb{R}^d}([0,T])$, we may assume that $(\ddot{u}^n)$ weakly converges to a vector measure $m \in \mathscr{M}^b_{\mathbb{R}^d}([0,T])$ so that the limit functions $u^\infty(.)$ and the limit measure m satisfy the following variational inequality:

$$\int_0^T \varphi_\infty(v(t))\,dt \geq \int_0^T \varphi_\infty(u^\infty(t))\,dt + \int_0^T \langle -A(t)\dot{u}^\infty(t), v(t) - u^\infty(t)\rangle\,dt$$
$$+ \langle -m, v - u^\infty\rangle_{(\mathscr{M}^b_E([0,T]),\mathscr{C}_{\mathbb{R}^d}([0,T]))}.$$

In other words, the vector measure $-m - A(t)\dot{u}^\infty(t)dt$ belongs to the subdifferential $\partial J_{\varphi_\infty}(u^\infty)$ of the convex functional integral J_{φ_∞} defined on $\mathscr{C}_{\mathbb{R}^d}([0,T])$ by $J_{\varphi_\infty}(v) = \int_0^T \varphi_\infty(v(t))\ dt$, $\forall v \in \mathscr{C}_{\mathbb{R}^d}([0,T])$. Indeed, let $w \in \mathscr{C}_{\mathbb{R}^d}([0,T])$. Integrating the subdifferential inequality

$$\varphi_n(w(t)) \geq \varphi_n(u^n(t)) + \langle -A(t)\dot{u}^n(t) - \ddot{u}^n(t), w(t) - u^n(t)\rangle$$

and noting that $\varphi_\infty(w(t)) \geq \varphi_n(w(t))$ gives immediately

$$\int_0^T \varphi_\infty(w(t))dt \geq \int_0^T \varphi_n(w(t))dt$$
$$\geq \int_0^T \varphi_n(u^n(t))dt + \langle -A(t)\dot{u}^n(t) - \ddot{u}^n(t), w(t) - u^n(t)\rangle dt.$$

We note that

$$\lim_n \int_0^T \langle -A(t)\dot{u}^n(t), w(t) - u^n(t)\rangle dt = \int_0^T \langle A(t)\dot{u}^\infty(t), w(t) - u^\infty(t)\rangle dt$$

because $(-A(.)\dot{u}^n)$ is uniformly integrable and converges in $L^1_H([0,T])$ to $A(.)\dot{u}^\infty$ and the sequence in $(w - u^n)$ converges uniformly to $w - u^\infty$. Whence follows

$$\int_0^T \varphi_\infty(w(t))dt \geq \int_0^T \varphi_\infty(u^\infty(t))dt + \int_0^T \langle -A(t)\dot{u}^\infty(t), w(t) - u^\infty(t)\rangle dt$$
$$+ \langle -m, w - u^\infty\rangle_{(\mathscr{M}^b_{\mathbb{R}^d}([0,T]),\mathscr{C}_{\mathbb{R}^d}([0,T]))},$$

which shows that the vector measure $-m - A(.)\dot{u}^{\infty}dt$ is a subgradient at the point u^{∞} of the of the convex integral functional J_{φ_∞} defined on $\mathscr{C}_{\mathbb{R}^d}([0,T]))$ by $J_{\varphi_\infty}(v) := \int_0^T \varphi_\infty(v(t))dt, \forall v \in \mathscr{C}_{\mathbb{R}^d}([0,T])$.

Step 5. Claim $\lim_n \varphi_n(u^n(t)) = \varphi_\infty(u^\infty(t)) < \infty$ a.e. and $\lim_n \int_0^T \varphi_n(u^n(t))dt = \int_0^T \varphi_\infty(u^\infty(t))dt < \infty$, and subsequently, the energy estimate holds for a.e. $t \in [0,T]$:

$$\varphi_\infty(u^\infty(t)) + \frac{1}{2}||\dot{u}^\infty(t)||^2 = \varphi_\infty(u_0^\infty) + \frac{1}{2}||\dot{u}_0^\infty||^2 - \int_0^t \langle A(s)(\dot{u}^\infty(s), \dot{u}^\infty(s)\rangle ds.$$

With the above stated results and notations, applying the subdifferential inequality

$$\varphi_n(w(t)) \geq \varphi_n(u^n(t)) + \langle -A(t)\dot{u}^n(t) - \ddot{u}^n(t), w(t) - u^n(t)\rangle$$

with $w = u^\infty$, integrating on $B \in B_p^c \cap \mathscr{L}([0,T])$, and passing to the limit when n goes to ∞, gives the inequality

$$\int_B \varphi_\infty(u^\infty(t))dt \geq \liminf_n \int_B \varphi_n(u^n(t))dt$$
$$\geq \int_B \varphi_\infty(u^\infty(t))dt \geq \limsup_n \int_B \varphi_n(u^n(t))dt$$

so that

$$\lim_n \int_B \varphi_n(u^n(t))dt = \int_B \varphi_\infty(u^\infty(t))dt \tag{7}$$

on $B \in B_p^c \cap \mathscr{L}([0,T])$. Now, from the chain rule theorem given in Step 1, recall that

$$-\langle \dot{u}^n(t), A(t)\dot{u}^n(t)\rangle - \langle \dot{u}^n(t), \ddot{u}^n(t)\rangle = \frac{d}{dt}[\varphi_n(u_n(t))],$$

that is,

$$\langle \dot{u}^n(t), z^n(t)\rangle = \frac{d}{dt}[\varphi_n(u_n(t))].$$

By the estimate (1) and the boundedness in $L^1_{\mathbb{R}^d}([0,T])$ of (z^n), it is immediate that $(\frac{d}{dt}[\varphi_n(u_n(t))])$ is bounded in $L^1_{\mathbb{R}}([0,T])$ so that $(\varphi_n(u_n(.))$ is bounded in variation. By Helly's theorem, we may assume that $(\varphi_n(u_n(.))$ pointwisely converges to a BV function ψ. By (1), $(\varphi_n(u_n(.))$ converges in $L^1_{\mathbb{R}}([0,T])$ to ψ. In particular, for every $k \in L^\infty_{\mathbb{R}^+}([0,T])$, we have

$$\lim_{n\to\infty} \int_0^T k(t)\varphi_n(u_n(t))dt = \int_0^T k(t)\psi(t)dt. \tag{8}$$

Combining with (7) and (8) yields

$$\int_B \psi(t)\, dt = \lim_{n\to\infty} \int_B \varphi_n(u^n(t))\, dt = \int_B \varphi_\infty(u^\infty(t))\, dt$$

for all $\in B_p^c \cap \mathscr{L}([0, T])$. As this inclusion is true on each B_p^c and $B_p^c \uparrow [0, T]$, we conclude that

$$\psi(t) = \lim_n \varphi_n(u_n(t)) = \varphi_\infty(u^\infty(t)) \text{ a.e.}$$

Subsequently, using (iii), the passage to the limit when n goes to ∞ in the equation

$$\varphi_n(u^n(t)) + \frac{1}{2}||\dot{u}^n(t)||^2 = \varphi_n(u^n(0)) + \frac{1}{2}||\dot{u}^n(0)||^2 - \int_0^t \langle A(s)\dot{u}^n(s), \dot{u}^n(s)\rangle ds$$

yields for a.e. $t \in [0, T]$

$$\varphi_\infty(u^\infty(t)) + \frac{1}{2}||\dot{u}^\infty(t)||^2 = \varphi_\infty(u_0^\infty) + \frac{1}{2}||\dot{u}_0^\infty)||^2 - \int_0^t \langle A(s)\dot{u}^\infty(s), \dot{u}^\infty(s)\rangle ds.$$

Step 6. Localization of further limits and final step.
As $(z^n = -A(.)\dot{u}^n - \ddot{u}^n)$ is bounded in $L^1_{\mathbb{R}^d}([0, T])$ in view of Step 3, it is relatively compact in the second dual $L^\infty_{\mathbb{R}^d}([0, T])'$ of $L^1_{\mathbb{R}^d}([0, T])$ endowed with the weak topology $\sigma(L^\infty_{\mathbb{R}^d}([0, T])', L^\infty_{\mathbb{R}^d}([0, T]))$. Furthermore, (z^n) can be viewed as a bounded sequence in $\mathscr{C}_{\mathbb{R}^d}([0, T])'$. Hence there is a filter $\mathscr{U}$ finer than the Fréchet filter $l \in L^\infty_{\mathbb{R}^d}([0, T])'$ and $\mathbf{n} \in \mathscr{C}_{\mathbb{R}^d}([0, T])'$ such that

$$\mathscr{U} - \lim_n z^n = l \in L^\infty_{\mathbb{R}^d}([0, T])'_{\text{weak}} \tag{9}$$

and

$$\lim_n z^n = \mathbf{n} \in \mathscr{C}_{\mathbb{R}^d}([0, T])'_{\text{weak}} \tag{10}$$

where $L^\infty_{\mathbb{R}^d}([0, T])'_{\text{weak}}$ is the second dual of $L^1_{\mathbb{R}^d}([0, T])$ endowed with the topology $\sigma(L^\infty_{\mathbb{R}^d}([0, T])', L^\infty_{\mathbb{R}^d}([0, T]))$ and $\mathscr{C}_{\mathbb{R}^d}([0, T])'_{\text{weak}}$ denotes the space $\mathscr{C}_{\mathbb{R}^d}([0, T])'$ endowed with the weak topology $\sigma(\mathscr{C}_{\mathbb{R}^d}([0, T])', \mathscr{C}_{\mathbb{R}^d}([0, T]))$, because $\mathscr{C}_{\mathbb{R}^d}([0, T])$ is a separable Banach space for the norm sup, so that we may assume by extracting subsequences that (z^n) weakly converges to $\mathbf{n} \in \mathscr{C}_{\mathbb{R}^d}([0, T])'$. Let l_a be the density of the absolutely continuous part l_a of l in the decomposition $l = l_a + l_s$ in absolutely continuous part l_a and singular part l_s, in the sense there is a decreasing sequence (A_n) of Lebesgue-measurable sets in $[0, T]$ with $A_n \downarrow \emptyset$ such that $l_s(f) = l_s(1_{A_n} f)$ for all $h \in L^\infty_{\mathbb{R}^d}([0, T])$ and for all $n \geq 1$. As $(z^n = -A(.)\dot{u}^n - \ddot{u}^n)$ biting converges to $z^\infty = -A(.)\dot{u}^\infty - \zeta^\infty$ in Step 4, it is already known [22] that

$$l_a(f) = \int_0^T \langle f(t), -A(t)\dot{u}^\infty(t) - \zeta^\infty(t)\rangle dt$$

for all $f \in L^\infty_{\mathbb{R}^d}([0,T])$, shortly $z^\infty = -A(t)\dot{u}^\infty(t) - \zeta^\infty(t)$ coincides a.e. with the density of the absolutely continuous part l_a. By [19, 46], we have

$$I^*_{\varphi_\infty}(l) = I_{\varphi^*_\infty}(-A(.)\dot{u}^\infty - \zeta^\infty) + \delta^*(l_s, \mathrm{dom}\, I_{\varphi_\infty})$$

where φ^*_∞ is the conjugate of φ_∞, $I_{\varphi^*_\infty}$ is the integral functional defined on $L^1_{\mathbb{R}^d}([0,T])$ associated with φ^*_∞, $I^*_{\varphi_\infty}$ is the conjugate of the integral functional I_{φ_∞}, and

$$\mathrm{dom}\, I_{\varphi_\infty} := \{u \in L^\infty_{\mathbb{R}^d}([0,T]) : I_{\varphi_\infty}(u) < \infty\}.$$

Using the inclusion

$$z^\infty = -A(.)\dot{u}^\infty - \zeta^\infty \in \partial I_{\varphi_\infty}(u^\infty),$$

that is,

$$I_{\varphi^*_\infty}(-A(.)\dot{u}^\infty - \zeta^\infty) = \langle -A(.)\dot{u}^\infty - \zeta^\infty, u^\infty\rangle - I_{\varphi_\infty}(u^\infty),$$

we see that

$$I^*_{\varphi_\infty}(l) = \langle -A(.)\dot{u}^\infty - \zeta^\infty, u^\infty\rangle - I_{\varphi_\infty}(u^\infty) + \delta^*(l_s, \mathrm{dom}\, I_{\varphi_\infty}).$$

Coming back to $z^n(t) = \nabla\varphi_n(u^n(t))$, we have

$$\varphi_n(x) \geq \varphi_n(u^n(t)) + \langle x - u^n(t), z^n(t)\rangle$$

for all $x \in \mathbb{R}^d$. Substituting x by $h(t)$ in this inequality, where $h \in \mathscr{C}_{\mathbb{R}^d}([0,T])$, and integrating, we get

$$\int_0^T \varphi_n(h(t))\, dt \geq \int_0^T \varphi_n(u^n(t))\, dt + \int_0^T \langle h(t) - u^n(t), z^n(t)\rangle\, dt.$$

Arguing as in Step 4 by passing to the limit in the preceding inequality, involving the epiliminf property for integral functionals (cf. (6)), it is easy to see that

$$\int_0^T \varphi_\infty(h(t))\, dt \geq \int_0^T \varphi_\infty(u^\infty(t))\, dt + \langle h - u^\infty, \mathbf{n}\rangle.$$

Whence **n** belongs to the subdifferential $\partial J_{\varphi_\infty}(u^\infty)$ of the convex lower semicontinuous integral functional J_{φ_∞} defined on $\mathscr{C}_{\mathbb{R}^d}([0,T])$ by

$$J_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\,dt, \ \ \forall u \in \mathscr{C}_{\mathbb{R}^d}([0,T]).$$

Now let $B : \mathscr{C}_{\mathbb{R}^d}([0,T]) \to L^\infty_{\mathbb{R}^d}([0,T])$ be the continuous injection, and let $B^* : L^\infty_{\mathbb{R}^d}([0,T])' \to \mathscr{C}_{\mathbb{R}^d}([0,T])'$ be the adjoint of B given by

$$\langle B^* l, f\rangle = \langle l, Bf\rangle = \langle l, f\rangle, \quad \forall l \in L^\infty_{\mathbb{R}^d}([0,T])', \quad \forall f \in \mathscr{C}_{\mathbb{R}^d}([0,T]).$$

Then we have $B^*l = B^*l_a + B^*l_s$, $l \in L^\infty_{\mathbb{R}^d}([0,T])'$ being the limit of z_n under the filter $\mathscr{U}$ given in Sect. 4 and $l = l_a + l_s$ being the decomposition of l in absolutely continuous part l_a and singular part l_s. It follows that

$$\langle B^* l, f\rangle = \langle B^* l_a, f\rangle + \langle B^* l_s, f\rangle = \langle l_a, f\rangle + \langle l_s, f\rangle$$

for all $f \in \mathscr{C}_{\mathbb{R}^d}([0,T])$. But it is already seen that

$$\begin{aligned}\langle l_a, f\rangle &= \langle -A(.)\dot{u}^\infty - \zeta^\infty, f\rangle \\ &= \int_0^T \langle -A(.)\dot{u}^\infty(t) - \zeta^\infty(t), f(t)\rangle dt, \quad \forall f \in L^\infty_{\mathbb{R}^d}([0,T])\end{aligned}$$

so that the measure B^*l_a is absolutely continuous

$$\langle B^* l_a, h\rangle = \int_0^T \langle -A(.)\dot{u}^\infty(t) - \zeta^\infty(t), f(t)\rangle dt, \quad \forall f \in \mathscr{C}_{\mathbb{R}^d}([0,T])$$

and its density $-A(.)\dot{u}^\infty - \zeta^\infty$ satisfies the inclusion

$$-A(t)\dot{u}^\infty(t) - \zeta^\infty(t) \in \partial\varphi_\infty(u^\infty(t)), \quad \text{a.e.}$$

and the singular part B^*l_s satisfies the equation

$$\langle B^* l_s, f\rangle = \langle l_s, h\rangle, \quad \forall f \in \mathscr{C}_{\mathbb{R}^d}([0,T]).$$

As $B^*l = \mathbf{n}$, using (9) and (10), it turns out that **n** is the sum of the absolutely continuous measure $\mathbf{n}_a$ with

$$\langle \mathbf{n}_a, f\rangle = \int_0^T \langle -A(t)\dot{u}^\infty(t) - \zeta^\infty(t), f(t)\rangle dt, \quad \forall f \in \mathscr{C}_{\mathbb{R}^d}([0,T])$$

and the singular part $\mathbf{n}_s$ given by

$$\langle \mathbf{n}_s, f\rangle = \langle l_s, f\rangle, \quad \forall f \in \mathscr{C}_{\mathbb{R}^d}([0,T]).$$

which satisfies the property: for any nonnegative measure θ on $[0, T]$ with respect to which $\mathbf{n}_s$ is absolutely continuous

$$\int_0^T r_{\varphi_\infty^*}\left(\frac{d\mathbf{n}_s}{d\theta}(t)\right) d\theta(t) = \int_0^T \left\langle u^\infty(t), \frac{d\mathbf{n}_s}{d\theta}(t)\right\rangle d\theta(t)$$

where $r_{\varphi_\infty^*}$ denotes the recession function of φ_∞^*. Indeed, as $\mathbf{n}$ belongs to $\partial J_{\varphi_\infty}(u^\infty)$ by applying [46, Theorem 5], we have

$$J_{\varphi_\infty}^*(n) = I_{\varphi_\infty^*}\left(\frac{d\mathbf{n}_a}{dt}\right) + \int_0^T r_{\varphi_\infty^*}\left(\frac{d\mathbf{n}_s}{d\theta}(t)\right) d\theta(t), \tag{11}$$

with

$$I_{\varphi_\infty^*}(v) := \int_0^T \varphi_\infty^*(v(t))dt, \forall v \in L^1_{\mathbb{R}^d}([0, T]).$$

Recall that

$$\frac{d\mathbf{n}_a}{dt} = -A(.)\dot{u}^\infty - \zeta^\infty \in \partial I_{\varphi_\infty}(u^\infty),$$

that is,

$$I_{\varphi_\infty^*}\left(\frac{d\mathbf{n}_a}{dt}\right) = \langle -A(.)\dot{u}^\infty - \zeta^\infty, u^\infty\rangle_{\langle L^1_{\mathbb{R}^d}([0,T]), L^\infty_{\mathbb{R}^d}([0,T])\rangle} - I_{\varphi_\infty}(u^\infty). \tag{12}$$

From (12), we deduce

$$\begin{aligned}
J_{\varphi_\infty}^*(n) &= \langle u^\infty, \mathbf{n}\rangle_{\langle \mathscr{C}_{\mathbb{R}^d}([0,T]), \mathscr{C}_{\mathbb{R}^d}([0,T])'\rangle} - J_{\varphi_\infty}(u^\infty) \\
&= \langle u^\infty, \mathbf{n}\rangle_{\langle \mathscr{C}_{\mathbb{R}^d}([0,T]), \mathscr{C}_{\mathbb{R}^d}([0,T])'\rangle} - I_{\varphi_\infty}(u^\infty) \\
&= \int_0^T \langle u^\infty(t), -A(.)\dot{u}^\infty - \zeta^\infty(t)\rangle dt \\
&\quad + \int_0^T \left\langle u^\infty(t), \frac{d\mathbf{n}_s}{d\theta}(t)\right\rangle d\theta(t) - I_{\varphi_\infty}(u^\infty) \\
&= I_{\varphi_\infty^*}\left(\frac{d\mathbf{n}_a}{dt}\right) + \int_0^T \left\langle u^\infty(t), \frac{d\mathbf{n}_s}{d\theta}(t)\right\rangle d\theta(t)).
\end{aligned}$$

Coming back to (11), we get the equality

$$\int_0^T r_{\varphi_\infty^*}\left(\frac{d\mathbf{n}_s}{d\theta}(t)\right) d\theta(t) = \int_0^T \left\langle u^\infty(t), \frac{d\mathbf{n}_s}{d\theta}(t)\right\rangle d\theta(t)).$$

The proof is complete.

Comments Some comments are in order. In Proposition 3.1, using the existence and uniqueness of $W^{2,\infty}_H(]0,T])$ of the approximating second-order equation

$$\begin{cases} 0 = \ddot{u}^n(t) + A(t)\dot{u}^n(t) + \nabla\varphi_n(u^n(t)), t \in [0,T] \\ u^n(0) = u^n_0,\ \dot{u}^n(0) = \dot{u}^n_0, \end{cases}$$

we state the existence of a generalized solution u^∞ to the second-order evolution inclusion

$$\begin{cases} 0 \in \ddot{u}(t) + A(t)\dot{u}(t) + \partial\varphi_\infty(u(t)), t \in [0,T] \\ u(0) = u_0 \in \text{dom } \varphi_\infty,\ \dot{u}(0) = \dot{u}_0 \end{cases}$$

via an epiconvergence approach involving the structure of bounded sequences in $L^1_H([0,T]$ space [22] and describe various properties of such a generalized solution. In particular, we show that such a generalized solution u^∞ is $W^{1,1}_{BV}([0,T])$ and satisfies the energy conservation and there exists a Young measure ν_t with barycenter $\text{bar}(\nu_t) \in L^1_H([0,T])$ such that $-A(t)\dot{u}^\infty(t) - \text{bar}(\nu_t) \in \partial\varphi_\infty(u\infty(t))$ a.e. In this vein, compare with Attouch et al. [4, 27], Paoli [43], and Schatzman [48].

Now we deal at first with $W^{1,1}_{BV}([0,T],H)$ solution for a second-order evolution problem.

Theorem 3.3 *Let for every $t \in [0,T]$, $A(t) : D(A(t)) \subset H \to 2^H$ be a maximal monotone operator with $D(A(t))$ ball compact for every $t \in [0,T]$ satisfying*

($H1$) *there exists a function $r : [0,T] \to [0,+\infty[$ which is continuous on $[0,T[$ and nondecreasing with $r(T) < +\infty$ such that*

$$\text{dis}(A(t), A(s)) \leq dr(]s,t]) = r(t) - r(s) \ \textit{for} \ 0 \leq s \leq t \leq T$$

($H2$) *there exists a nonnegative real number c such that*

$$\|A^0(t,x)\| \leq c(1 + \|x\|) \ \textit{for} \ t \in [0,T],\ x \in D(A(t))$$

Let $f : [0,T] \times H \times H \to H$ be such that for every $x, y \in H \times H$ the mapping $f(.,x,y)$ is Borel-measurable on $[0,T]$ and for every $t \in [0,T]$, $f(t,.,.)$ is continuous on $H \times H$ and satisfying

(i) $||f(t,x,y)|| \leq M(1 + ||x||),\ \forall t, x, y \in [0,T] \times H \times H.$
(ii) $||f(t,x,z) - f(t,y,z)|| \leq M||x - y||,\ \forall t, x, y, z \in [0,T] \times H \times H \times H.$

Then for $u_0 \in D(A(0))$and$y_0 \in H$, there are a BVC mapping $u : [0,T] \to H$ and a $W^{1,1}_{BV}([0,T],H)$ mapping $y : [0,T] \to H$ satisfying

$$y(t) = y_0 + \int_0^t u(s)ds, \quad t \in [0,T],$$

$$-\frac{du}{dr}(t) \in A(t)u(t) + f(t, u(t), y(t))\ dr\text{-}a.e.\ t \in [0, T],$$

$$u(0) = u_0$$

with the property: $|u(t) - u(\tau)| \leq K|r(t) - r(\tau)|$ *for all* $t, \tau \in [0, T]$ *for some constant* $K \in]0, \infty[$.

Proof By [8, Theorem 3.1] and the assumptions on f, for any continuous mapping $h : [0, T] \to H$, there is a unique BVC solution v_h to the inclusion

$$\begin{cases} v_h(0) = u_0 \in D(A(0)) \\ -\dfrac{dv_h}{dr}(t) \in A(t)v_h(t) + f(t, v_h(t), h(t))\ dr\text{-a.e.} \end{cases}$$

with $||v_h(t)|| \leq K,\ t \in [0, T]$ and $||v_h(t) - v_h(\tau)|| \leq K(r(t) - r(\tau)),\ t, \tau \in [0, T]$ so that

$$dv_h = \frac{dv_h}{dr} dr$$

with $\frac{dv_h}{dr} \in K\overline{B}_H$, consequently $\frac{dv_h}{dr} \in L^\infty_H([0, T], dr)$. Let consider the closed convex subset $\mathscr{X}$ in the Banach space $\mathscr{C}_H([0, T])$ defined by

$$\mathscr{X} := \{u : [0, T] \to H : u(t) = u_0 + \int_0^t \dot{u}(s)ds,\ \dot{u} \in S^1_{K\overline{B}_H},\ t \in [0, T]\}$$

where $S^1_{K\overline{B}_H}$ denotes the set of all integrable selections of the convex weakly compact valued constant multifunction $K\overline{B}_H$. Now for each $h \in \mathscr{X}$, let us consider the mapping

$$\Phi(h)(t) := u_0 + \int_0^t v_h(s)ds,\ t \in [0, T].$$

Then it is clear that $\Phi(h) \in \mathscr{X}$. Our aim is to prove the existence theorem by applying some ideas developed in Castaing et al. [24] via a generalized fixed point theorem [36, 44]. Nevertheless this needs a careful look using the estimation of the BVC solution given above. For this purpose, we first claim that $\Phi : \mathscr{X} \to \mathscr{X}$ is continuous and for any $h \in \mathscr{X}$ and for any $t \in [0, T]$ the inclusion holds

$$\Phi(h)(t) \in u_0 + \int_0^t \overline{co}[D(A(s)) \cap K\overline{B}_H]ds.$$

Since $s \mapsto \overline{co}[D(A(s)) \cap K\overline{B}_H]$ is a convex compact valued and integrably bounded multifunction using the ball-compactness assumption, the second member is convex compact valued [14] so that $\Phi(\mathscr{X})$ is equicontinuous and relatively compact in the Banach space $\mathscr{C}_H([0,T])$. Now we check that Φ is continuous. It is sufficient to show that, if (h_n) converges uniformly to h in $\mathscr{X}$, then BVC solution v_{h_n} associated with h_n

$$\begin{cases} v_{h_n}(0) = u_0 \in D(A(0)) \\ -\dfrac{dv_{h_n}}{dr}(t) \in A(t)v_{h_n}(t) + f(t, v_{h_n}(t), h_n(t))\ dr\text{-a.e.} \end{cases}$$

pointwisely converges to the BVC solution v_h associated with h

$$\begin{cases} v_h(0) = u_0 \in D(A(0)) \\ -\dfrac{dv_h}{dr}(t) \in A(t)v_h(t) + f(t, v_h(t), h(t))\ dr\text{-a.e.} \end{cases}$$

As $D(A(t))$ is ball compact, (v_{h_n}) is uniformly bounded, and bounded in variation since $||v_{h_n}(t) - v_{h_n}(\tau)|| \le K(r(t) - r(\tau)),\ t, \tau \in [0, T]$, we may assume that (v_{h_n}) pointwisely converges to a BVC mapping v. As $v_{h_n} = v_0 + \int_{]0,t]} \frac{dv_{h_n}}{dr} dr,\ t \in [0, T]$ and $\frac{dv_{h_n}}{dr}(s) \in K\overline{B}_H,\ s \in [0, T]$, we may assume that $(\frac{dv_{h_n}}{dr})$ converges weakly in $L^1_H([0, T], dr)$ to $w \in L^1_H([0, T], dr)$ with $w(t) \in K\overline{B}_H,\ t \in [0, T]$ so that

$$\text{weak-}\lim_n v_{h_n} = u_0 + \int_{]0,t]} w dr := z(t),\ t \in [0, T].$$

By identifying the limits, we get

$$v(t) = z(t) = u_0 + \int_{]0,t]} w dr$$

with $\frac{dv}{dr} = w$ so that $\lim_n f(t, v_{h_n}(t), h_n(t)) = f(t, v(t), h(t)),\ t \in [0, T]$. Consequently we may assume that $(\frac{dv_{h_n}}{dr} + f(., v_{h_n}(.), h_n(.)))$ Komlos converges to $\frac{dv}{dr} - f(., v(.), h(.))$. For simplicity, set $g_n(t) = f(t, v_{h_n}(t), h_n(t))$ and $g(t) = f(t, v(t), h(t))$. There is a dr-negligible set N such that for $t \in I \setminus N$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left(\frac{dv_{h_j}}{dr}(t) + g_j(t) \right) = \frac{dv}{dr}(t) + g(t).$$

Let $\eta \in D(A(t))$. From

$$\begin{aligned} &\left\langle \frac{dv_{h_n}}{dr}(t) + g_n(t), v(t) - \eta \right\rangle \\ &\quad = \left\langle \frac{dv_{h_n}}{dr}(t) + g_n(t), v_{h_n}(t) - \eta \right\rangle + \left\langle \frac{dv_{h_n}}{dr}(t) + g_n(t), v(t) - v_{h_n}(t) \right\rangle, \end{aligned}$$

let us write

$$\frac{1}{n}\sum_{j=1}^{n}\left\langle\frac{dv_{h_j}}{dr}(t)+g_j(t), v(t)-\eta\right\rangle$$
$$=\frac{1}{n}\sum_{j=1}^{n}\left\langle\frac{dv_{h_j}}{dr}(t)+g_j(t), v_{h_j}(t)-\eta\right\rangle+\frac{1}{n}\sum_{j=1}^{n}\left\langle\frac{dv_{h_j}}{dr}(t)+g_j(t), v(t)-v_{h_j}(t)\right\rangle,$$

so that

$$\frac{1}{n}\sum_{j=1}^{n}\left\langle\frac{dv_{h_j}}{dr}(t)+g_j(t), v(t)-\eta\right\rangle$$
$$\leq \frac{1}{n}\sum_{j=1}^{n}\left\langle A^0(t,\eta), \eta-v_{h_j}(t)\right\rangle+\big(\text{Constant}\big)\frac{1}{n}\sum_{j=1}^{n}\|v(t)-v_{h_j}(t))\|.$$

Passing to the limit when $n\to\infty$, this last inequality gives immediately

$$\left\langle\frac{dv}{dr}(t)+g(t), v(t)-\eta\right\rangle\leq\left\langle A^0(t,\eta), \eta-v(t)\right\rangle \text{ a.e.}$$

As a consequence, by Lemma 2.2, $-\frac{dv}{dr}(t)\in A(t)v(t)+g(t)=A(t)v(t)+f(t,v(t),h(t))$ a.e. with $v(0)=u_0\in D(A(0))$ so that by uniqueness $v=v_h$. Now let us check that $\Phi:\mathscr{X}\to\mathscr{X}$ is continuous. Let $h_n\to h$. We have

$$\Phi(h_n)(t)-\Phi(h)(t)=\int_0^t v_{h_n}(s)ds-\int_0^t v_h(s)ds=\int_0^t[v_{h_n}(s)-v_h(s)]ds$$

As $||v_{h_n}(.)-v_h(.)||\to 0$ pointwisely and is uniformly bounded : $||v_{h_n}(.)-v_h(.)||\leq 2K$, by we conclude that

$$\sup_{t\in[0,T]}||\Phi(h_n)(t)-\Phi(h)(t)||\leq\sup_{t\in[0,T]}\int_0^t||v_{h_n}(.)-v_h(.)||ds\to 0$$

so that $\Phi(h_n)-\Phi(h)\to 0$ in $\mathscr{C}_H([0,T])$. Here one may invoke a general fact that on bounded subsets of L^∞, the topology of convergence in measure coincides with the topology of uniform convergence on uniformly integrable sets, i.e., on relatively weakly compact subsets, alias the Mackey topology. This is a lemma due to Grothendieck [33, Ch.5 §4 no 1 Prop. 1 and exercice] (see also [15] for a more general result concerning the Mackey topology for bounded sequences in $L^\infty_{E^*}$). Since $\Phi:\mathscr{X}\to\mathscr{X}$ is continuous and $\Phi(\mathscr{X})$ is relatively compact in $\mathscr{C}_H([0,T])$, by [36, 44] Φ has a fixed point, say $h=\Phi(h)\in\mathscr{X}$, that means

$$h(t) = \Phi(h)(t) = u_0 + \int_0^t v_h(s)ds, \ \ t \in [0, T],$$

$$\begin{cases} v_h(0) = u_0 \in D(A(0)) \\ -\dfrac{dv_h}{dr}(t) \in A(t)v_h(t) + f(t, v_h(t), h(t)) \ dr\text{-a.e.} \end{cases}$$

The proof is complete.

The following results are sharp variants of Theorem 3.3.

Theorem 3.4 *Let for every $t \in [0, T]$, $A(t) : D(A(t)) \subset H \to 2^H$ be a maximal monotone operator with $D(A(t)$ ball compact for every $t \in [0, T]$ satisfying $(H2)$ and*

$(H1)'$ *there exists a function $\beta \in W^{1,1}([0, T], \mathbb{R}; dt)$ which is nonnegative on $[0, T]$ and non-decreasing with $\beta(T) < \infty$ such that*

$$\text{dis}(A(t), A(s)) \leq |\beta(t) - \beta(s)|, \ \forall s, t \in [0, T].$$

$(H1)^*$ *For any $t \in [0, T]$ and for any $x \in D(A(t))$, $A(t)x$ is cone-valued.*

Let $f : [0, T] \times H \times H \to H$ be such that for every $x, y \in H \times H$ the mapping $f(., x, y)$ is Lebesgue-measurable on $[0, T]$ and for every $t \in [0, T]$, $f(t, ., .)$ is continuous on $H \times H$ and satisfying

(i) $||f(t, x, y)|| \leq M(1 + ||x||), \ \forall t, x, y \in [0, T] \times H \times H.$
(ii) $||f(t, x, z) - f(t, y, z)|| \leq M||x - y||, \ \forall t, x, y, z \in [0, T] \times H \times H \times H.$

Then, for all $u_0 \in D(A(0))$, $y_0 \in H$, there are an absolutely continuous mapping $u : [0, T] \to H$ and an absolutely continuous mapping $y : [0, T] \to H$ satisfying

$$y(t) = y_0 + \int_0^t u(s)ds, \quad t \in [0, T],$$

$$-\frac{du}{dt}(t) \in A(t)u(t) + f(t, u(t), y(t)) \ dt - \text{a.e. } t \in [0, T], u(0) = u_0,$$

with

$$\|\dot{u}(t)\| \leq \big(K + M(1 + K)\big)(\dot{\beta}(t) + 1) + M(1 + K)$$

for a.e. $t \in [0, T]$, for some positive constant K.

Proof By [7, Theorem 3.4] and the assumptions on f, for any continuous mapping $h : [0, T] \to H$, there is a unique AC solution v_h to the inclusion

$$\begin{cases} v_h(0) = u_0 \in D(A(0)) \\ -\dot{v}_h(t) \in A(t)v_h(t) + f(t, v_h(t), h(t)) \ dt\text{-a.e.} \end{cases}$$

with $||\dot{v}_h(t)|| \leq \gamma(t) := (K + M(1+K))(\dot{\beta}(t)+1) + M(1+K)$ a.e. $t \in [0, T]$ so that $\gamma \in L^1_{\mathbb{R}}([0, T])$ and $||v_h(t)|| \leq L =$ Constant, $t \in [0, T]$. Let us consider the closed convex subset $\mathscr{X}$ in the Banach space $\mathscr{C}_H([0, T])$ defined by

$$\mathscr{X} := \{u : [0, T] \to H : u(t) = u_0 + \int_0^t \dot{u}(s)ds,\ \dot{u} \in S^1_{L\overline{B}_H},\ t \in [0, T]\}$$

where $S^1_{L\overline{B}_H}$ denotes the set of all integrable selections of the convex weakly compact valued constant multifunction $L\overline{B}_H$. Now for each $h \in \mathscr{X}$, let us consider the mapping

$$\Phi(h)(t) := u_0 + \int_0^t v_h(s)ds,\ t \in [0, T].$$

Then it is clear that $\Phi(h) \in \mathscr{X}$. Our aim is to prove the existence theorem by applying some ideas developed in Castaing et al. [24] via a generalized fixed point theorem [36, 44]. Nevertheless this needs a careful look using the estimation of the AC solution given above. For this purpose, we first claim that $\Phi : \mathscr{X} \to \mathscr{X}$ is continuous for any $h \in \mathscr{X}$ and for any $t \in [0, T]$, the inclusion holds

$$\Phi(h)(t) \in u_0 + \int_0^t \overline{co}[D(A(s)) \cap L\overline{B}_H]ds.$$

Since $s \mapsto \overline{co}[D(A(s)) \cap L\overline{B}_H]$ is a convex compact valued and integrably bounded multifunction, the second member is convex compact valued [14] so that $\Phi(\mathscr{X})$ is equicontinuous and relatively compact in the Banach space $\mathscr{C}_H([0, T])$. Now we check that Φ is continuous. It is sufficient to show that, if h_n converges uniformly to h in $\mathscr{X}$, then the AC solution v_{h_n} associated with h_n

$$\begin{cases} v_{h_n}(0) = u_0 \in D(A(0)) \\ -\dot{v}_{h_n}(t) \in A(t)v_{h_n}(t) + f(t, v_{h_n}(t), h_n(t))\ dt\text{-a.e.} \end{cases}$$

converges uniformly to the AC solution v_h associated with h

$$\begin{cases} v_h(0) = u_0 \in D(A(0)) \\ -\dot{v}_h(t) \in A(t)v_h(t) + f(t, v_h(t), h(t))\ dt\text{-a.e.} \end{cases}$$

We have

$$-\dot{v}_{h_n}(t) \in A(t)v_{h_n}(t) + f(t, v_{h_n}(t), h_n(t)),\ \text{a.e. } t \in [0, T],$$

with the estimation $||\dot{v}_{h_n}(t)|| \leq \gamma(t)$ and $\gamma \in L^1_{\mathbb{R}}([0, T])$ for all $n \in \mathbb{N}$. As $D(A(t))$ is ball compact and $(\dot{v}_{h_n})$ is relatively weakly compact in

$L^1_H([0, T])$, we may assume that (v_{h_n}) converges uniformly to an absolutely continuous mapping v such that $v(t) = u_0 + \int_0^t \dot{v}(s)ds$, $t \in [0, T]$, $||\dot{v}(t)|| \leq \gamma(t)$, $t \in [0, T]$, and $(\dot{v}_{h_n})\,\sigma(L^1_H, L^\infty_H)$ converges to $\dot{v}$ so that $\lim_n f(t, v_{h_n}(t), h_n(t)) = f(t, v(t), h(t))$, $t \in [0, T]$. Consequently we may assume that $(\dot{v}_{h_n} + f(., v_{h_n}(.), h_n(.)))$ Komlos converges to $\dot{v} - f(., v(.), h(.))$. Let us set $g_n(t) = f(t, v_{h_n}(t), h_n(t))$ and $g(t) = f(t, v(t), h(t))$. There is a negligible set N such that for $t \in [0, T] \setminus N$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left(\dot{v}_{h_j}(t) + g_j(t)\right) = \dot{v}(t) + g(t).$$

Let $\eta \in D(A(t))$. From

$$\left\langle \dot{v}_{h_n}(t) + g_n(t), v(t) - \eta \right\rangle = \left\langle \dot{v}_{h_n}(t) + g_n(t), v_{h_n}(t) - \eta \right\rangle + \left\langle \dot{v}_{h_n}(t) + g_n(t), v(t) - v_{h_n}(t) \right\rangle$$

let us write

$$\frac{1}{n} \sum_{j=1}^{n} \left\langle \dot{v}_{h_j}(t) + g_j(t), v(t) - \eta \right\rangle = \frac{1}{n} \sum_{j=1}^{n} \left\langle \dot{v}_{h_j}(t) + g_j(t), v_{h_j}(t) - \eta \right\rangle + \frac{1}{n} \sum_{j=1}^{n} \left\langle \dot{v}_{h_j}(t) + g_j(t), v(t) - v_{h_j}(t) \right\rangle,$$

so that

$$\frac{1}{n} \sum_{j=1}^{n} \left\langle \dot{v}_{h_j}(t) + g_j(t), v(t) - \eta \right\rangle \leq \frac{1}{n} \sum_{j=1}^{n} \left\langle A^0(t, \eta), \eta - v_{h_j}(t) \right\rangle + \left(\gamma(t) + \text{Constant}\right) \frac{1}{n} \sum_{j=1}^{n} \|v(t) - v_{h_j}(t))\|.$$

Passing to the limit when $n \to \infty$, this last inequality gives immediately

$$\left\langle \dot{v}(t) + g(t), v(t) - \eta \right\rangle \leq \left\langle A^0(t, \eta), \eta - v(t) \right\rangle \text{ a.e.}$$

As a consequence, $-\dot{v}(t) \in A(t)v(t) + g(t) = A(t)v(t) + f(t, v(t), h(t))$ a.e. with $v(0) = u_0 \in D(A(0))$ so that by uniqueness $v = v_h$. Since $\Phi : \mathscr{X} \to \mathscr{X}$ is continuous and $\Phi(\mathscr{X})$ is relatively compact in $\mathscr{C}_H([0, T])$, by [36, 44] Φ has a fixed point, say $h = \Phi(h) \in \mathscr{X}$, that means

$$h(t) = \Phi(h)(t) = u_0 + \int_0^t v_h(s)ds, \ \ t \in [0, T],$$

$$\begin{cases} v_h(0) = u_0 \in D(A(0)) \\ -\dot{v}_h(t) \in A(t)v_h(t) + f(t, v_h(t), h(t)) \ dt\text{-a.e.} \end{cases}$$

The proof is complete.

Comments The use of a generalized fixed point theorem is initiated in [24] dealing with some second-order sweeping process associated with a closed moving set $C(t, u)$. Actually it is possible to obtain a variant of Theorem 3.4 by assuming that $A(t) : D(A(t)) \subset H \to 2^H$ is a maximal monotone operator with $D(A(t)$ ball compact for every $t \in [0, T]$ satisfying $(H2)$ and
$(H1)'$ there exists a function $\beta \in W^{1,2}([0, T], \mathbb{R}; dt)$ which is nonnegative on I and non-decreasing with $\beta(T) < \infty$ such that

$$\text{dis}(A(t), A(s)) \leq |\beta(t) - \beta(s)|, \ \forall s, t \in [0, T].$$

Here using fixed point theorem provides a short proof with new approach involving the continuous dependance of the trajectory v_h associated with the control $h \in \mathscr{X}$ and also the compactness of the integral of convex compact integrably bounded multifunctions [14].

4 Evolution Problems with Lipschitz Variation Maximal Monotone Operator and Application to Viscosity and Control

Now, based on the existence and uniqueness of $W_H^{2,\infty}([0, T])$ solution to evolution inclusion

$$(\mathscr{S}_1) \begin{cases} 0 \in \ddot{u}(t) + A(t)\dot{u}(t) + f(t, u(t)), \ t \in [0, T] \\ u(0) = u_0, \dot{u}(0) = \dot{u}_0 \in D(A(0)) \end{cases}$$

we will present some problems in optimal control in a second-order evolution inclusion driven by a Lipschitz variation maximal monotone operator $A(t)$ in the same vein as in Castaing-Marques-Raynaud de Fitte [25] dealing with the sweeping process. Before going further, we note that $(\mathscr{S}_1)$ contains the evolution problem associated with the sweeping process by a closed convex Lipschitzian mapping $C : [0, T] \to \text{cc}(H)$

$$\begin{cases} 0 \in \ddot{u}(t) + N_{C(t)}(\dot{u}(t)) + f(t, u(t)), \ t \in [0, T] \\ u(0) = u_0, \dot{u}(0) = \dot{u}_0 \in C(0) \end{cases}$$

by taking $A(t) = \partial\Psi_{C(t)}$ in $(\mathscr{S}_1)$.

We need some notations and background on Young measures in this special context. For the sake of completeness, we summarize some useful facts concerning Young measures. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Let X be a Polish space, and let $\mathcal{C}^b(X)$ be the space of all bounded continuous functions defined on X. Let $\mathcal{M}^1_+(X)$ be the set of all Borel probability measures on X equipped with the narrow topology. A Young measure $\lambda : \Omega \to \mathcal{M}^1_+(X)$ is, by definition, a *scalarly measurable* mapping from Ω into $\mathcal{M}^1_+(X)$, that is, for every $f \in \mathcal{C}^b(X)$, the mapping $\omega \mapsto \langle f, \lambda_\omega \rangle := \int_X f(x)\, d\lambda_\omega(x)$ is $\mathcal{F}$-measurable. A sequence (λ^n) in the space of Young measures $\mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}^1_+(X))$ *stably converges* to a Young measure $\lambda \in \mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}^1_+(X))$ if the following holds:

$$\lim_{n\to\infty} \int_A \left[\int_X f(x)\, d\lambda^n_\omega(x)\right] dP(\omega) = \int_A \left[\int_X f(x)\, d\lambda_\omega(x)\right] dP(\omega)$$

for every $A \in \mathcal{F}$ and for every $f \in \mathcal{C}^b(X)$. We recall and summarize some results for Young measures.

Theorem 4.5 ([22, Theorem 3.3.1]) *Assume that S and T are Polish spaces. Let (μ^n) be a sequence in $\mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}^1_+(S))$, and let (ν^n) be a sequence in $\mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}^1_+(T))$. Assume that*

(i) *(μ^n) converges in probability to $\mu^\infty \in \mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}^1_+(S))$,*
(ii) *(ν^n) stably converges to $\nu^\infty \in \mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}^1_+(T))$.*

Then $(\mu^n \otimes \nu^n)$ stably converges to $\mu^\infty \otimes \nu^\infty$ in $\mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}^1_+(S \times T))$.

Theorem 4.6 ([22, Theorem 6.3.5]) *Assume that X and Z are Polish spaces. Let (u^n) be sequence of $\mathcal{F}$-measurable mappings from Ω into X such that (u^n) converges in probability to a $\mathcal{F}$-measurable mapping u^∞ from Ω into X, and let (v^n) be a sequence of $\mathcal{F}$-measurable mappings from Ω into Z such that (v^n) stably converges to $\nu^\infty \in \mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}^1_+(Z))$. Let $h : \Omega \times X \times Z \to \mathbb{R}$ be a Carathéodory integrand such that the sequence $(h(., u_n(.), v_n(.))$ is uniformly integrable. Then the following holds:*

$$\lim_{n\to\infty} \int_\Omega h(\omega, u^n(\omega), v^n(\omega))\, dP(\omega) = \int_\Omega \left[\int_Z h(\omega, u^\infty(\omega), z)\, d\nu^\infty_\omega(z)\right] dP(\omega).$$

In the remainder, Z is a compact metric space, and $\mathcal{M}^1_+(Z)$ is the space of all probability Radon measures on Z. We will endow $\mathcal{M}^1_+(Z)$ with the narrow topology so that $\mathcal{M}^1_+(Z)$ is a compact metrizable space. Let us denote by $\mathcal{Y}([0,T]; \mathcal{M}^1_+(Z))$ the space of all Young measures (alias *relaxed controls*) defined on $[0,T]$ endowed with the stable topology so that $\mathcal{Y}([0,T]; \mathcal{M}^1_+(Z))$ is a compact metrizable space with respect to this topology. By the Portmanteau Theorem for Young measures [22, Theorem 2.1.3], a sequence (ν^n) in $\mathcal{Y}([0,T]; \mathcal{M}^1_+(Z))$ stably converges to $\nu \in \mathcal{Y}([0,T]; \mathcal{M}^1_+(Z))$ if

$$\lim_{n\to\infty} \int_0^T \left[\int_Z h_t(z) d\nu^n_t(z)\right] dt = \int_0^T \left[\int_Z h_t(z) d\nu_t(z)\right] dt$$

for all $h \in L^1_{\mathscr{C}(Z)}([0, T])$, where $\mathscr{C}(Z)$ denotes the space of all continuous real-valued functions defined on Z endowed with the norm of uniform convergence. Finally let us denote by $\mathscr{Z}$ the set of all Lebesgue-measurable mappings (alias *original controls*) $z : [0, T] \to Z$ and $\mathscr{R} := \mathscr{Y}([0, T]; \mathscr{M}^1_+(Z))$ the set of all relaxed controls (alias Young measures) associated with Z. In the remainder, we assume that $H = \mathbb{R}^d$ and Z is a compact subset in H.

For simplicity, let us consider a mapping $f : [0, T] \times H \to H$ satisfying

(i) for every $x \in H \times Z$, $f(., x)$ is Lebesgue-measurable on $[0, T]$,
(ii) there is $M > 0$ such that

$$||f(t, x)|| \leq M(1 + ||x||)$$

for all (t, x) in $[0, T] \times H$, and

$$||f(t, x) - f(t, y)|| \leq M||x - y||$$

for all $(t, x, y) \in [0, T] \times H \times H$.

We consider the $W^{2,\infty}_H([0, T])$ solution set of the two following control problems

$$(\mathscr{S}_{\mathscr{O}}) \begin{cases} 0 \in \ddot{u}_{x,y,\zeta}(t) + A(t)\dot{u}_{x,y,\zeta}(t)) + f(t, u_{x,y,\zeta}(t)) + \zeta(t), \ t \in [0, T] \\ u_{x,y,\zeta}(0) = x \in H, \dot{u}_{x,y,\zeta}(0) = y \in D(A(0)) \end{cases}$$

and

$$(\mathscr{S}_{\mathscr{R}}) \begin{cases} 0 \in \ddot{u}_{x,y,\lambda}(t) + A(t)\dot{u}_{x,y,\lambda}(t)) + f(t, u_{x,y,\lambda}(t)) + \mathrm{bar}(\lambda_t), \ t \in [0, T] \\ u_{x,y,\lambda}(0) = x \in H, \dot{u}_{x,y,\lambda}(0) = y \in D(A(0)) \end{cases}$$

where ζ belongs to the set $\mathscr{Z}$ of all Lebesgue-measurable mappings (alias original controls) $\zeta : [0, T] \to Z$ original and λ belongs to the set $\mathscr{R}$ of all relaxed controls. Taking $(\mathscr{S}_1)$ into account, for each $(x, y, \zeta) \in H \times D(A(0)) \times \mathscr{Z}$ (resp. $(x, y, \lambda) \in H \times D(A(0)) \times \mathscr{R}$, there exists a unique $W^{2,\infty}_H(]0, T])$ solutions, solution $u_{x,y,\zeta}$ (resp. $u_{x,y,\lambda}$), to $(\mathscr{S}_{\mathscr{O}})$ (resp. $(\mathscr{S}_{\mathscr{R}})$). We aim to present some problems in the framework of optimal control theory for the above inclusions. In particular, we state a viscosity property of the value function associated with these evolution inclusions. Similar problems driven by evolution inclusion with perturbation containing Young measures are initiated by [22, 23]. However, the present study deals with a new setting in the sense that it concerns a second-order evolution inclusion involving time-dependent maximal monotone operator.

Now we present a lemma which is useful for our purpose.

Lemma 4.6 *Let for all $t \in [0, T]$, $A(t) : D(A(t)) \subset H \to 2^H$ be a maximal monotone operator satisfying $(H1)$ and $(H2)$. Let $f : [0, T] \times H \to H$ be a mapping satisfying*

(i) *for every* $x \in H \times Z$, $f(., x)$ *is Lebesgue-measurable on* $[0, T]$,
(ii) *there is* $M > 0$ *such that*

$$||f(t, x)|| \leq M(1 + ||x||)$$

for all (t, x) *in* $[0, T] \times H$, *and*

$$||f(t, x) - f(t, y)|| \leq M||x - y||$$

for all $(t, x, y) \in [0, T] \times H \times H$.

Let $h_n, h \in L^\infty_H([0, T], dt)$ *with* $||h_n(t)|| \leq 1$ *for all* $t \in [0, T]$, *for all* $n \in \mathbb{N}$ *and* $||h(t)|| \leq 1$ *for all* $t \in [0, T]$. *Let us consider the two following second-order evolution inclusions:*

$$\mathscr{S}(A, f, h_n, x, y) \begin{cases} 0 \in \ddot{u}_{x,y,h_n}(t) + A(t)\dot{u}_{x,y,h_n}(t) + f(t, u_{x,y,h_n}(t)) + h_n(t), \ t \in [0, T] \\ u_{x,y,h_n}(0) = x, \dot{u}_{x,y,h_n}(0) = y \in D(A(0)) \end{cases}$$

$$\mathscr{S}(A, f, h, x, y) \begin{cases} 0 \in \ddot{u}_{x,y,h}(t) + A(t)\dot{u}_{x,y,h}(t) + f(t, u_{x,y,h}(t)) + h(t), \ t \in [0, T] \\ u_{x,y,h}(0) = x, \dot{u}_{x,y,h}(0) = y \in D(A(0)) \end{cases}$$

where u_{x,y,h_n} *(resp.* $u_{x,y,h}$*) is the unique* $W^{2,\infty}_H([0, T])$ *solution to* $(\mathscr{S}(A, f, h_n, x, y))$ *(resp.* $(\mathscr{S}(A, f, h_n, x, y))$*). Assume that* (h_n) $\sigma(L^1, L^\infty)$ *converges to* h. *Then* (u_{x,y,h_n}) *converges pointwisely to* $u_{x,y,h}$.

Proof We note that $\ddot{u}_{x,y,h_n}$ is *uniformly bounded*, so there is $u \in W^{2,\infty}_H([0, T])$ such that

$u_{x,y,h_n} \to u$ pointwisely with $u(0) = x$,
$\dot{u}_{x,y,h_n} \to \dot{u}$ pointwisely with $\dot{u}(0) = y$,
$\ddot{u}_{x,y,h_n} \to \ddot{u}$ with respect to $\sigma(L^1, L^\infty)$.

Using Lemma 2.3, it is not difficult to see that $\dot{u}(t) \in D(A(t))$ for every $t \in [0, T]$. As $f(t, u_{x,y,h_n}(t)) \to f(t, u(t))$ pointwisely so that $f(., u_{x,y,h_n}(.)) \to f(.., u(.))$ with respect to $\sigma(L^1, L^\infty)$. Since (h_n) $\sigma(L^1, L^\infty)$ converges to h, so that $f(., u_{x,y,h_n}(.)) + h_n \to f(t., u(.)) + h$ with respect to $\sigma(L^1, L^\infty)$. And so $\ddot{u}_{x,y,h_n}(.) + f(., u_{x,y,h_n}(.)) + h_n(.)$ $\sigma(L^1, L^\infty)$ converges to $\dot{u} + f(.., u(.)) + h$. As a consequence, we may also assume that $\ddot{u}_{x,y,h_n}(.) + f(., u_{x,y,h_n}(.)) + h_n(.)$ Komlos converges to $\dot{u} + f(.., u(.)) + h$. Coming back to the inclusion $-\ddot{u}_{x,y,h_n}(t) - f(t, u_{x,y,h_n}(t)) - h_n(t) \in A(t)\dot{u}_{x,y,h_n}(t)$, we have by the monotonicity of $A(t)$

$$\langle \ddot{u}_{x,y,h_n}(t) + f(t, u_{x,y,h_n}(t)) + h_n(t), \dot{u}_{x,y,h_n}(t) - \eta \rangle \leq \langle A^0(t, \eta), \eta - \dot{u}_{x,y,h_n}(t) \rangle$$

for any $\eta \in D(A(t))$. For notational convenience, set

$$v_n(t) = \ddot{u}_{x,y,h_n}(t) + f(t, u_{x,y,h_n}(t)) + h_n(t), \forall t \in [0, T],$$
$$v(t) = \ddot{u}(t) + f(t, u(t)) + h(t), \forall t \in [0, T].$$

There is a negligible set N such that

$$\lim_n \frac{1}{n}\sum_{i=1}^{n} v_i(t) = v(t)$$

for $t \notin N$. Let us write

$$\langle v_n(t), \dot{u}(t)-\eta\rangle = \langle v_n(t), \dot{u}_{x,y,h_n}(t)-\eta\rangle + \langle v_n(t), \dot{u}(t)-\dot{u}_{x,y,h_n}(t)\rangle$$

so that

$$\frac{1}{n}\sum_{i=1}^{n}\langle v_i(t), \dot{u}(t)-\eta\rangle = \frac{1}{n}\sum_{i=1}^{n}\langle v_i(t), \dot{u}_{x,y,h_i}(t)-\eta\rangle + \frac{1}{n}\sum_{i=1}^{n}\langle v_i(t), \dot{u}(t)-\dot{u}_{x,y,h_i}(t)\rangle$$

$$\leq \frac{1}{n}\sum_{i=1}^{n}\langle A^0(t,\eta), \eta-\dot{u}_{x,y,h_i}(t)\rangle + L\frac{1}{n}\sum_{i=1}^{n}||\dot{u}(t)-\dot{u}_{x,y,h_i}(t)||,$$

where L is a positive generic constant. Passing to the limit when n goes to ∞ in this inequality gives immediately

$$\langle v(t), \dot{u}(t)-\eta\rangle \leq \langle A^0(t,\eta), \eta-\dot{u}(t)\rangle$$

so that by Lemma 2.2 we get

$$-\ddot{u}(t) - f(t, u_{x,y,h}(t)) - h(t) \in A(t)\dot{u}(t) \text{ a.e.}$$

with $u(0) = x$ and $\dot{u}(0) = y$. Due to the uniqueness of solution, we get $u(t) = u_{x,y,h}(t)$ for all $t \in [0, T]$. The proof is complete.

The following shows the continuous dependence of the solution with respect to the control.

Theorem 4.7 *Let for all $t \in [0, T]$, $A(t) : D(A(t)) \subset H \to 2^H$ be a maximal monotone operator satisfying $(H1)$ and $(H2)$. Let $f : [0, T] \times H \to H$ be a mapping satisfying*

(i) *for every $x \in H \times Z$, $f(.,x)$ is Lebesgue-measurable on $[0, T]$,*
(ii) *there is $M > 0$ such that*

$$||f(t,x)|| \leq M(1+||x||)$$

for all (t, x) in $[0, T] \times H$, and

$$||f(t,x_1) - f(t,x_2)|| \leq M||x_1 - x_2||$$

for all $(t, x_1, (t, x_2,) \in [0, T] \times H \times H$.

Let Z be a compact subset of H. Let us consider the control problem

$$\begin{cases} 0 \in \ddot{u}_{x,y,\nu}(t) + A(t)\dot{u}_{x,y,\nu}(t) + f(t, u_{x,y,\nu}(t)) + \mathrm{bar}(\nu_t), \ t \in [0, T] \\ u_{x,y,\nu}(0) = x, \dot{u}_{x,y,\nu}(0) = y \in D(A(0)) \end{cases}$$

where $\mathrm{bar}(\nu_t)$ *denotes the barycenter of the measure* $\nu_t \in \mathscr{M}^1_+(Z)$ *and* $u_{x,y,\nu}$ *is the unique* $W^{2,\infty}_H([0, T])$ *solution associated with to* $\mathrm{bar}(\nu_t)$. *Then, for each* $t \in [0, T]$, *the mapping* $\nu \mapsto u_{x,y,\nu}$ *is continuous from* $\mathscr{R}$ *to* $C_H([0, T]$, *where* $\mathscr{R}$ *is endowed with the stable topology and* $C_H([0, T]$ *is endowed with the topology of pointwise convergence.*

Proof (a) Let $\nu \in \mathscr{R}$ and let $\mathrm{bar}(\nu) : t \mapsto \mathrm{bar}(\nu_t), \ t \in [0, T]$. It is easy to check that $\nu \mapsto \mathrm{bar}(\nu)$ from $\mathscr{R}$ to $L^\infty_H([0, T])$ is continuous with respect to the stable topology and the $\sigma(L^1_H, L^\infty_H)$, respectively. Note that $\mathscr{R}$ is compact metrizable for the stable topology. Now let (ν^n) be a sequence in $\mathscr{R}$ which stably converges to $\nu \in \mathscr{R}$. Then $\mathrm{bar}(\nu^n)$ $\sigma(L^1_H, L^\infty_H)$ converges to $\mathrm{bar}(\nu)$. By Lemma 4.6, we see that u_{x,y,ν^n} pointwisely converges to $u_{x,y,\nu}$. The proof is complete.

We are now able to relate the Bolza type problems associated with the maximal monotone operator $A(t)$ as follows:

Theorem 4.8 *With the hypotheses and notations of Theorem 4.7, assume that* $J : [0, T] \times H \times Z \to \mathbb{R}$ *is a Carathéodory integrand, that is,* $J(t, ., .)$ *is continuous on* $H \times Z$ *for every* $t \in [0, T]$ *and* $J(., x, z)$ *is Lebesgue-measurable on* $[0, T]$ *for every* $(x, z) \in H \times Z$*, which satisfies the condition* $(\mathscr{C})$*: for every sequence* (ζ_n) *in* $\mathscr{Z}$*, the sequence* $(J(., u_{x,y,\zeta^n}(.), \zeta^n(.))$ *is uniformly integrable in* $L^1_{\mathbb{R}}([0, T], dt)$*, where* u_{x,y,ζ^n} *denotes the unique* $W^{2,\infty}_H([0, T])$ *solution associated with* ζ^n *to the evolution inclusion*

$$\begin{cases} 0 \in \ddot{u}_{x,y,\zeta^n}(t) + A(t)\dot{u}_{x,y,\zeta^n}(t) + f(t, u_{x,y,\zeta^n}(t)) + \zeta^n(t), \ t \in [0, T] \\ u_{x,y,\zeta^n}(0) = x, \dot{u}_{x,y,\zeta^n}(0) = y \in D(A(0)) \end{cases}$$

Let us consider the control problems

$$\inf(P_{\mathscr{Z}}) := \inf_{\zeta \in \mathscr{Z}} \int_0^T J(t, u_{x,y,\zeta}(t), \zeta(t))\, dt$$

and

$$\inf(P_{\mathscr{R}}) := \inf_{\lambda \in \mathscr{R}} \int_0^T \left[\int_Z J(t, u_{x,y,\lambda}(t), z)\, \lambda_t(dz) \right] dt$$

where $u_{x,y,\zeta}$ *(resp.* $u_{x,y,\lambda}$*) is the unique* $W^{2,\infty}_H([0, T])$ *solution associated with* ζ *(resp.* λ*) to*

$$\begin{cases} 0 \in \ddot{u}_{x,y,\zeta}(t) + A(t)\dot{u}_{x,y,\zeta}(t) + f(t, u_{x,y,\zeta}(t)) + \zeta(t), \ t \in [0, T] \\ u_{x,y,\zeta}(0) = x, \dot{u}_{x,y,\zeta}(0) = y \in D(A(0)) \end{cases}$$

and

$$\begin{cases} 0 \in \ddot{u}_{x,y,\lambda}(t) + A(t)\dot{u}_{x,y,\nu}(t) + f(t, u_{x,y,\nu}(t)) + \text{bar}(\lambda_t), \ t \in [0, T] \\ u_{x,y,\lambda}(0) = x, \dot{u}_{x,y,\lambda}(0) = y \in D(A(0)) \end{cases}$$

respectively. Then one has

$$\inf(P_{\mathscr{Z}}) = \inf(P_{\mathscr{R}}).$$

Proof Take a control $\lambda \in \mathscr{R}$. By virtue of the denseness with respect to the stable topology of $\mathscr{Z}$ in $\mathscr{R}$, there is a sequence $(\zeta^n)_{n\in\mathbb{N}}$ in $\mathscr{Z}$ such that the sequence $(\delta_{\zeta^n})_{n\in\mathbb{N}}$ of Young measures associated with $(\zeta^n)_{n\in\mathbb{N}}$ stably converges to λ. By Theorem 4.7, the sequence (u_{x,y,ζ^n}) of $W_H^{2,\infty}([0,T])$ solutions associated with ζ^n pointwisely converges to the unique $W_H^{2,\infty}([0,T])$ solution $u_{x,y,\lambda}$. As $(J(t, u_{x,y,\zeta^n}(t), \zeta^n(t)))$ is uniformly integrable by assumption ($\mathscr{C}$), using Theorem 4.6 (or [22, Theorem 6.3.5]), we get

$$\lim_{n\to\infty} \int_0^T J(t, u_{x,y,\zeta^n}(t), \zeta^n(t))\, dt = \int_0^T \left[\int_Z J(t, u_{x,y,\lambda}, z) d\lambda_t(z) \right] dt.$$

As

$$\int_0^T J(t, u_{x,y,\zeta^n}(t), \zeta^n(t))\, dt \geq \inf(P_{\mathscr{Z}})$$

for all $n \in \mathbb{N}$, so is

$$\int_0^T \left[\int_Z J(t, u_{x,y,\lambda}, z) d\lambda_t(z) \right] dt \geq \inf(P_{\mathscr{Z}});$$

by taking the infimum on $\mathscr{R}$ in this inequality, we get

$$\inf(P_{\mathscr{R}}) \geq \inf(P_{\mathscr{O}})$$

As $\inf(P_{\mathscr{O}}) \geq \inf(P_{\mathscr{R}})$, the proof is complete.

In the framework of optimal control, the above considerations lead to the study of the value function associated with the evolution inclusion

$$\begin{cases} 0 \in \ddot{u}_{\tau,x,y,\nu}(t) + A(t)\dot{u}_{\tau,x,y,\nu}(t) + f(t, u_{\tau,x,y,\nu}(t)) + \text{bar}(\nu_t), \\ u_{\tau,x,y,\nu}(\tau) = x, \dot{u}_{\tau,x,y,\nu}(\tau) = y \in D(A(\tau)). \end{cases}$$

The following shows that the value function satisfies the dynamic programming principle (DPP).

Theorem 4.9 *(of dynamic programming principle). Assume the hypothesis and notations of Theorem 4.7, and let $x \in E$, $\tau < T$ and $\sigma > 0$ such that $\tau + \sigma < T$. Assume that $J : [0, T] \times H \times Z \to \mathbb{R}$ is bounded and continuous. Let us consider the value function*

$$V_J(\tau, x, y) = \sup_{\nu \in \mathscr{R}} \int_\tau^T \left[\int_Z J(t, u_{\tau,x,y,\nu}(t), z)\nu_t(dz)\right] dt,$$
$$(\tau, x, y) \in [0, T] \times H \times D(A(\tau))$$

where $u_{\tau,x,y,\nu}$ is the $W_H^{2,\infty}([0, T])$ solution to the evolution inclusion defined on $[\tau, T]$ associated with the control $\nu \in \mathscr{R}$ starting from x, y at time τ

$$\begin{cases} 0 \in \ddot{u}_{\tau,x,y,\nu}(t) + A(t)\dot{u}_{\tau,x,y,\nu}(t) + f(t, u_{\tau,x,y,\nu}(t)) + \mathrm{bar}(\nu_t), \\ u_{\tau,x,y,\nu}(\tau) = x, \dot{u}_{\tau,x,y,\nu}(\tau) = y \in D(A(\tau)) \end{cases}$$

Then the following holds:

$$V_J(\tau, x, y) = \sup_{\nu \in \mathscr{R}} \left\{ \int_\tau^{\tau+\sigma} \left[\int_Z J(t, u_{\tau,x,y,\nu}(t), z)\nu_t(dz)\right] dt \right.$$
$$\left. + V_J(\tau + \sigma, u_{\tau,x,y,\nu}(\tau + \sigma), \dot{u}_{\tau,x,y,\nu}(\tau + \sigma)\right\}$$

with

$$V_J(\tau + \sigma, u_{\tau,x,\nu}(\tau + \sigma), \dot{u}_{\tau,x,\nu}(\tau + \sigma))$$
$$= \sup_{\mu \in \mathscr{R}} \int_{\tau+\sigma}^T \left[\int_Z J(t, v_{\tau+\sigma, u_{\tau,x,y,\nu}(\tau+\sigma), \dot{u}_{\tau,x,y,\nu}(\tau+\sigma), \mu}(t), z)\mu_t(dz)\right] dt$$

where $v_{\tau+\sigma, u_{\tau,x,y,\nu}(\tau+\sigma), \dot{u}_{\tau,x,y,\nu}(\tau+\sigma), \mu}$[1] is the $W_H^{2,\infty}(\tau + \sigma, T)$ solution defined on $[\tau + \sigma, T]$ associated with the control $\mu \in \mathscr{R}$ starting from $u_{\tau,x,\nu}(\tau + \sigma)$, $\dot{u}_{\tau,x,\nu}(\tau + \sigma)$ at time $\tau + \sigma$

$$\begin{cases} 0 \in v_{\tau+\sigma, u_{\tau,x,y,\nu}(\tau+\sigma), \dot{u}_{\tau,x,y,\nu}(\tau+\sigma), \mu}(t) + A(t) v_{\tau+\sigma, u_{\tau,x,y\nu}(\tau+\sigma), \dot{u}_{\tau,x,y,\nu}(\tau+\sigma), \mu}(t), \\ \quad + f(t, v_{\tau+\sigma, u_{\tau,x,y,\nu}(\tau+\sigma), \dot{u}_{\tau,x,y,\nu}(\tau+\sigma), \mu}(t)) + \mathrm{bar}(\mu_t), \\ v_{\tau+\sigma, u_{\tau,x,y,\nu}(\tau+\sigma), \dot{u}_{\tau,x,y,\nu}(\tau+\sigma), \mu}(\tau + \sigma) = u_{\tau,x,y,\nu}(\tau + \sigma), \\ \dot{v}_{\tau+\sigma, u_{\tau,x,y,\nu}(\tau+\sigma), \dot{u}_{\tau,x,y,\nu}(\tau+\sigma), \mu}(\tau + \sigma) = \dot{u}_{\tau,x,y,\nu}(\tau + \sigma) \in D(A(\tau + \sigma)). \end{cases} \quad (13)$$

[1]It is necessary to write completely the expression of the trajectory $v_{\tau+\sigma, u_{\tau,x,y,\nu}(\tau+\sigma), \dot{u}_{\tau,x,y,\nu}(\tau+\sigma), \mu}$ that depends on $(\nu, \mu) \in \mathscr{R} \times \mathscr{R}$ in order to get the continuous dependence with respect to $\nu \in \mathscr{R}$ of $V_J(\tau + \sigma, u_{\tau,x,y,\nu}(\tau + \sigma))$.

Proof Let

$$W_J(\tau, x, y) := \sup_{\nu\in\mathscr{R}} \left\{ \int_\tau^{\tau+\sigma} \left[\int_Z J(t, u_{\tau,x,y,\nu}(t), z)\nu_t(dz) \right] dt + V_J(\tau+\sigma, u_{\tau,x,y,\nu}(\tau+\sigma)) \right\}.$$

For any $\nu \in \mathscr{R}$, we have

$$\int_\tau^T \left[\int_Z J(t, u_{\tau,x,y,\nu}(t), z)\nu_t(dz) \right] dt = \int_\tau^{\tau+\sigma} \left[\int_Z J(t, u_{\tau,x,y,\nu}(t), z)\nu_t(dz) \right] dt + \int_{\tau+\sigma}^T \left[\int_Z J(t, u_{\tau,x,y,\nu}(t), z)\nu_t(dz) \right] dt.$$

By the definition of $V_J(\tau+\sigma, u_{\tau,x,y,\nu}(\tau+\sigma), \dot{u}_{\tau,x,y,\nu}(\tau+\sigma)$, we have

$$V_J(\tau+\sigma, u_{\tau,x,y,\nu}(\tau+\sigma), \dot{u}_{\tau,x,y,\nu}(\tau+\sigma) \geq \int_{\tau+\sigma}^T \left[\int_Z J(t, u_{\tau,x,y,\nu}(t), z)\nu_t(dz) \right] dt.$$

It follows that

$$\int_\tau^T \left[\int_Z J(t, u_{\tau,x,y,\nu}(t), z)\nu_t(dz) \right] dt \leq \int_\tau^{\tau+\sigma} \left[\int_Z J(t, u_{\tau,x,y,\nu}(t), z)\nu_t(dz) \right] dt + V_J(\tau+\sigma, u_{\tau,x,y,\nu}(\tau+\sigma), \dot{u}_{\tau,x,y,\nu}(\tau+\sigma)).$$

By taking the supremum on $\nu \in \mathscr{R}$ in this inequality, we get

$$\begin{aligned} V_J(\tau, x, y) \leq& \sup_{\nu\in\mathscr{R}} \left\{ \int_\tau^{\tau+\sigma} \left[\int_Z J(t, u_{\tau,x,y,\nu}(t), z)\nu_t(dz) \right] dt \right. \\ &\left. + V_J(\tau+\sigma, u_{\tau,x,y,\nu}(\tau+\sigma), \dot{u}_{\tau,x,y,\nu}(\tau+\sigma)) \right\} \\ =& W_J(\tau, x, y). \end{aligned}$$

Let us prove the converse inequality.
Main fact: $\nu \mapsto V_J(\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma), \dot{u}_{\tau,x,\nu}(\tau+\sigma))$ is continuous on $\mathscr{R}$.
Let us focus on the expression of $V_J(\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma), \dot{u}_{\tau,x,\nu}(\tau+\sigma))$:

$$V_J(\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma), \dot{u}_{\tau,x,\nu}(\tau+\sigma)) = \sup_{\mu\in\mathscr{R}} \int_{\tau+\sigma}^T \left[\int_Z J(t, v_{\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma), \dot{u}_{\tau,x,\nu}(\tau+\sigma),\mu}(t), z)\mu_t(dz) \right] dt$$

where $v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\dot{u}_{\tau,x,\nu}(\tau+\sigma),\mu}$ denotes the trajectory solution on $[\tau+\sigma,T]$ associated with the control $\mu\in\mathscr{R}$ starting from $u_{\tau,x,\nu}(\tau+\sigma),\dot{u}_{\tau,x,\nu}(\tau+\sigma)$, at time $\tau+\sigma$ in (13). Using the continuous dependence of the solution with respect to the state and the control, it is readily seen that the mapping $(\nu,\mu)\mapsto v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\dot{u}_{\tau,x,\nu}(\tau+\sigma),\mu}(t)$ is continuous on $\mathscr{R}\times\mathscr{R}$ for each $t\in[\tau,T]$, namely, if ν^n stably converges to $\nu\in\mathscr{R}$ and μ^n stably converges to $\mu\in\mathscr{R}$, then $v_{\tau+\sigma,u_{\tau,x,\nu^n}(\tau+\sigma),\dot{u}_{\tau,x,\nu}(\tau+\sigma),\mu^n}$ pointwisely converges to $v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\dot{u}_{\tau,x,\nu}(\tau+\sigma),\mu}$. By using the fiber product of Young measure (see Theorem 4.5 or [22, Theorem 3.3.1]), we deduce that

$$(\nu,\mu)\mapsto\int_{\tau+\sigma}^{T}\left[\int_Z J(t,v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\dot{u}_{\tau,x,\nu}(\tau+\sigma)\mu}(t),z)\mu_t(dz)\right]dt$$

is continuous on $\mathscr{R}\times\mathscr{R}$. Consequently $\nu\mapsto V_J(\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\dot{u}_{\tau,x,\nu}(\tau+\sigma))$ is continuous on $\mathscr{R}$. Hence the mapping $\nu\mapsto\int_\tau^{\tau+\sigma}[\int_Z J(t,u_{\tau,x,\nu}(t),z)\nu_t(dz)]dt+V_J(\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\dot{u}_{\tau,x,\nu}(\tau+\sigma))$ is continuous on $\mathscr{R}$. By compactness of $\mathscr{R}$, there is a maximum point $\nu^1\in\mathscr{R}$ such that

$$W_J(\tau,x,y)=\int_\tau^{\tau+\sigma}\left[\int_Z J(t,u_{\tau,x,y,\nu^1}(t),z)\nu_t^1(dz)\right]dt+V_J(\tau+\sigma,u_{\tau,x,y,\nu^1}(\tau+\sigma)).$$

Similarly there is $\mu^2\in\mathscr{R}$ such that

$$\begin{aligned}&V_J(\tau+\sigma,u_{\tau,x,\nu^1}(\tau+\sigma),\dot{u}_{\tau,x,\nu^1}(\tau+\sigma))\\&\qquad=\int_{\tau+\sigma}^{T}\left[\int_Z J(t,v_{\tau+\sigma,u_{\tau,x,\nu^1}(\tau+\sigma),\dot{u}_{\tau,x,\nu^1}(\tau+\sigma),\mu^2}(t),z)\mu_t^2(dz)\right]dt\end{aligned}$$

where

$$v_{\tau+\sigma,u_{\tau,x,\nu^1}(\tau+\sigma),\dot{u}_{\tau,x,\nu^1}(\tau+\sigma),\mu^2}(t)$$

denotes the trajectory solution associated with the control $\mu^2\in\mathscr{R}$ starting from $u_{\tau,x,\nu^1}(\tau+\sigma),\dot{u}_{\tau,x,\nu^1}(\tau+\sigma)$ at time $\tau+\sigma$ defined on $[\tau+\sigma,T]$

$$\begin{cases}0\in v_{\tau+\sigma,u_{\tau,x,y,\nu^1}(\tau+\sigma),\dot{u}_{\tau,x,y,\nu^1}(\tau+\sigma),\mu^2}(t)+A(t)v_{\tau+\sigma,u_{\tau,x,y\nu^1}(\tau+\sigma),\dot{u}_{\tau,x,y,\nu^1}(\tau+\sigma),\mu^2}(t),\\ \qquad+f(t,v_{\tau+\sigma,u_{\tau,x,y,\nu^1}(\tau+\sigma),\dot{u}_{\tau,x,y,\nu^1}(\tau+\sigma),\mu^2}(t))+\mathrm{bar}(\mu_t^2),\\ v_{\tau+\sigma,u_{\tau,x,y,\nu^1}(\tau+\sigma),\dot{u}_{\tau,x,y,\nu^1}(\tau+\sigma),\mu^2}(\tau+\sigma)=u_{\tau,x,y,\nu^1}(\tau+\sigma),\\ \dot{v}_{\tau+\sigma,u_{\tau,x,y,\nu^2}(\tau+\sigma),\dot{u}_{\tau,x,y,\nu^1}(\tau+\sigma),\mu^2}(\tau+\sigma)=\dot{u}_{\tau,x,y,\nu^1}(\tau+\sigma)\in D(A(\tau+\sigma)).\end{cases}\tag{14}$$

Let us set

$$\overline{\nu} := 1_{[\tau,\tau+\sigma]}\nu^1 + 1_{[\tau+\sigma,T]}\mu^2.$$

Then $\overline{\nu} \in \mathscr{R}$. Let $w_{\tau,x,y,\overline{\nu}}$ be the trajectory solution on $[\tau, T]$ associated with $\overline{\nu} \in \mathscr{R}$, that is,

$$\begin{cases} 0 \in \ddot{w}_{\tau,x,y,\overline{\nu}}(t)+A(t)\dot{w}_{\tau,x,y,\overline{\nu}}(t)+f(t, w_{\tau,x,y,\overline{\nu}}(t))+\operatorname{bar}(\overline{\nu}_t) \quad w_{\tau,x,y,\overline{\nu}}(\tau)=x \\ \dot{w}_{\tau,x,y,\overline{\nu}}(\tau) = y \in D(A(\tau)) \end{cases}$$

By uniqueness of the solution, we have

$$w_{\tau,x,y,\overline{\nu}}(t) = u_{\tau,x,y,\nu^1}(t),\ \forall t \in [\tau, \tau+\sigma],$$
$$w_{\tau,x,y,\overline{\nu}}(t) = v_{\tau+\sigma,u_{\tau,x,y,\nu^1}(\tau+\sigma),\dot{u}_{\tau,x,y,\nu^1}(\tau+\sigma),\mu^2}(t),\ \forall t \in [\tau+\sigma, T].$$

Coming back to the expression of V_J and W_J, we have

$$\begin{aligned} W_J(\tau, x, y) &= \int_\tau^{\tau+\sigma} \left[\int_Z J(t, u_{\tau,x,y,\nu^1}(t), z)\nu_t^1(dz)\right] dt \\ &\quad + \int_{\tau+\sigma}^T \left[\int_Z J(t, v_{\tau+\sigma,u_{\tau,x,\nu^1}(\tau+\sigma),\dot{u}_{\tau,x,\nu^1}(\tau+\sigma),\mu^2}(t), z)\mu_t^2(dz)\right] dt \\ &= \int_\tau^T \left[\int_Z J(t, w_{\tau,x,y,\overline{\nu}}(t), z)\overline{\nu}_t(dz)\right] dt \\ &\le \sup_{\nu\in\mathscr{R}} \left\{\int_\tau^T \left[\int_Z J(t, u_{\tau,x,y,\nu}(t), z)\nu_t(dz)\right] dt\right\} = V_J(\tau, x, y). \end{aligned}$$

The proof is complete.

In the above evolution problem, we deal with second-order inclusion of the form

$$\begin{cases} 0 \in \ddot{u}_{x,y,\lambda}(t) + A(t)\dot{u}_{x,y,\lambda}(t) + f(t, u_{x,y,\lambda}(t)) + \operatorname{bar}(\lambda_t),\ t \in [0, T] \\ u_{x,y,\lambda}(0) = x, \dot{u}_{x,\lambda}(0) = y \in D(A(0)) \end{cases}$$

with perturbed term f and $\operatorname{bar}(\lambda_t)$. Now we focus to the evolution inclusion of the form

$$\begin{cases} 0 \in \dot{u}_{x,\lambda}(t) + A(t)u_{x,\lambda}(t) + f(t, u_{x,\lambda}(t)) + \operatorname{bar}(\lambda_t),\ t \in [0, T] \\ u_{x,\lambda}(0) = x \in D(A(0)) \end{cases}$$

By Theorem 3.1, there is a unique Lipschitz solution $u_{x,\lambda}$ to this inclusion. Using the above techniques and Theorem 3.1, we have a result of dynamic principle that is similar to Theorem 4.9.

Theorem 4.10 (of dynamic programming principle) *Assume the hypothesis and notations of Theorem 3.1, and let $x \in E$, $\tau < T$ and $\sigma > 0$ such that $\tau + \sigma < T$. Assume that $J : [0, T] \times H \times Z \to \mathbb{R}$ is bounded and continuous. Let us consider the value function*

$$V_J(\tau, x, y) = \sup_{\nu \in \mathscr{R}} \int_\tau^T \left[\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz) \right] dt, \quad (\tau, x) \in [0, T] \times D(A(\tau))$$

where $u_{\tau,\nu}$ is the Lipschitz solution to the evolution inclusion defined on $[\tau, T]$ associated with the control $\nu \in \mathscr{R}$ starting from x, at time τ

$$\begin{cases} 0 \in \dot{u}_{\tau,x,\nu}(t) + A(t)u_{\tau,x,\nu}(t) + f(t, u_{\tau,x,\nu}(t)) + \mathrm{bar}(\nu_t), \\ u_{\tau,x,\nu}(\tau) = x \in D(A(\tau)). \end{cases}$$

Then the following holds:

$$V_J(\tau, x) = \sup_{\nu \in \mathscr{R}} \left\{ \int_\tau^{\tau+\sigma} \left[\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz) \right] dt + V_J(\tau+\sigma, u_{\tau,x,\nu}(\tau+\sigma)) \right\}$$

with

$$V_J(\tau + \sigma, u_{\tau,x,\nu}(\tau + \sigma)) = \sup_{\mu \in \mathscr{R}} \int_{\tau+\sigma}^T \left[\int_Z J(t, v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\mu}(t), z)\mu_t(dz) \right] dt$$

where $v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\mu}$[2] *is the Lipschitz solution defined on $[\tau + \sigma, T]$ associated with the control $\mu \in \mathscr{R}$ starting from $u_{\tau,x,\nu}(\tau + \sigma)$ at time $\tau + \sigma$*

$$\begin{cases} 0 \in \dot{v}_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\mu}(t) + A(t)v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma)),\mu}(t) \\ \qquad\qquad + f(t, v_{\tau+\sigma,u_{\tau,x,y,\nu}(\tau+\sigma),\mu}(t)) + \mathrm{bar}(\mu_t), \\ v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\mu}(\tau + \sigma) = u_{\tau,x,\nu}(\tau + \sigma) \in D(A(\tau + \sigma)). \end{cases}$$

Let us mention a useful lemma. See also [16, 22, 23] for related results.

Lemma 4.7 *Assume the hypothesis and notations of Theorem 3.1. Let Z be a compact subset in H, and $\mathscr{M}_+^1(Z)$ is endowed with the narrow topology and $\mathscr{R}$ the space of relaxed controls associated with Z. Let $\Lambda : [0, T] \times H \times \mathscr{M}_+^1(Z) \to \mathbb{R}$ be an upper semicontinuous function such that the restriction of Λ to $[0, T] \times B \times \mathscr{M}_+^1(Z)$ is bounded on any bounded subset B of H. Let $(t_0, x_0) \in [0, T] \times E$. If*

[2] It is necessary to write completely the expression of the trajectory $v_{\tau+\sigma,u_{\tau,x,\nu}(\tau+\sigma),\mu}$ that depends on $(\nu, \mu) \in \mathscr{R} \times \mathscr{R}$ in order to get the continuous dependence with respect to $\nu \in \mathscr{R}$ of $V_J(\tau + \sigma, u_{\tau,x,\nu}(\tau + \sigma))$.

$\max_{\mu\in\mathscr{M}^1_+(Z)} \Lambda(t_0, x_0, \mu) < -\eta < 0$ *for some* $\eta > 0$*, then there exist* $\sigma > 0$ *such that*

$$\sup_{\nu\in\mathscr{R}} \int_{t_0}^{t_0+\sigma} \Lambda(t, u_{t_0,x_0,\nu}(t), \nu_t)\, dt < -\frac{\sigma\eta}{2}$$

where $u_{t_0,x_0,\nu}$ *is the trajectory solution associated with the control* $\nu \in \mathscr{R}$ *and starting from* x_0 *at time* t_0

$$\begin{cases} 0 \in \dot{u}_{t_0,x_0,\nu}(t) + A(t)u_{t_0,x_0,\nu}(t) + f(t, u_{t_0,x_0,\nu}(t) + \mathrm{bar}(\nu_t),\ t \in [t_0, T], \\ u_{t_0,x_0,\nu}(t_0) = x_0 \in D(A(t_0)). \end{cases}$$

Proof By our assumption $\max_{\mu\in\mathscr{M}^1_+(Z)} \Lambda(t_0, x_0, \mu) < -\eta < 0$ for some $\eta > 0$. As the function $(t, x, \mu) \mapsto \Lambda(t, x, \mu)$ is upper semicontinuous, so is the function

$$(t, x) \mapsto \max_{\mu\in\mathscr{M}^1_+(Z)} \Lambda(t, x, \mu).$$

Hence there exists $\zeta > 0$ such that

$$\max_{\mu\in\mathscr{M}^1_+(Z)} \Lambda(t, x, \mu) < -\frac{\eta}{2}$$

for $0 < t - t_0 \le \zeta$ and $||x - x_0|| \le \zeta$. Thus, for small values of σ, we have

$$||u_{t_0,x_0,\nu}(t) - u_{t_0,x_0,\nu}(t_0)|| \le \zeta$$

for all $t \in [t_0, t_0 + \sigma]$ and for all $\nu \in \mathscr{R}$ because $||\dot{u}_{t_0,x_0,\nu}(t)|| \le K =$ Constant for all $\nu \in \mathscr{R}$ and for all $t \in [0, T]$ so that $||u_{t_0,x_0,\nu}(t)|| \le L =$ Constant for all $\nu \in \mathscr{R}$ and for all $t \in [0, T]$ Hence $t \mapsto \Lambda(t, u_{t_0,x_0,\nu}(t), \nu_t)$ is bounded and Lebesgue-measurable on $[t_0, t_0 + \sigma]$. Then by integrating

$$\int_{t_0}^{t_0+\sigma} \Lambda(t, u_{t_0,x_0,\nu}(t), \nu_t)\, dt \le \int_{t_0}^{t_0+\sigma} \left[\max_{\mu\in\mathscr{M}^1_+(Z)} \Lambda(t, u_{t_0,x_0,\nu}(t), \mu)\right] dt < -\frac{\sigma\eta}{2}.$$

The proof is complete.

Now to finish the paper, we provide a direct application to the viscosity solution to the evolution inclusion of the form

$$\begin{cases} 0 \in \dot{u}_{x,\lambda}(t) + A(t)u_{x,\lambda}(t) + f(t, u_{x,\lambda}(t)) + \mathrm{bar}(\lambda_t),\ t \in [0, T] \\ u_{x,\lambda}(0) = x \in D(A(0)) \end{cases}$$

where $A(t)$ is a convex weakly compact valued $H \to cwk(H)$ maximal monotone operator.

Theorem 4.11 *Let for every $t \in [0, T]$, $A(t) : H \to cwk(H)$ be a convex weakly compact valued maximal monotone operator satisfying*

$(H1)$ *there exists a real constant $\alpha \geq 0$ such that*

$$\mathrm{dis}(A(t), A(s)) \leq \alpha(t-s) \ \ for \ \ 0 \leq s \leq t \leq T.$$

$(H2)$ *there exists a nonnegative real number c such that*

$$\|A^0(t,x)\| \leq c(1+\|x\|), t \in [0,T], x \in H$$

$(H3)$ $(t, x) \mapsto A(t)x$ *is scalarly upper semicontinuous on* $[0, T] \times H$.

Let Z be a compact subset in H, and let $\mathscr{R}$ be the space of relaxed controls associated with Z. Let $f : [0, T] \times H \to H$ be a continuous mapping satisfying

(i) *there is $M > 0$ such that $||f(t, x)|| \leq M(1 + ||x||)$ for all (t, x) in $[0, T] \times H$,*
(ii) *$||f(t, x) - f(t, y)|| \leq M||x - y||$ for all $(t, x, y) \in [0, T] \times H \times H$.*

Assume that $J : [0, T] \times H \times Z \to \mathbb{R}$ is bounded and continuous. Let us consider the value function

$$V_J(\tau, x) = \sup_{\nu \in \mathscr{R}} \int_\tau^T \left[\int_Z J(t, u_{\tau,x,\nu}(t), z)\nu_t(dz) \right] dt, \ (\tau, x) \in [0, T] \times H$$

where $u_{\tau,x,\nu}$ is the trajectory solution on $[\tau, T]$ of the evolution inclusion associated with $A(t)$ and the control $\nu \in \mathscr{R}$ and starting from $x \in H$ at time τ

$$\begin{cases} 0 \in \dot{u}_{\tau,x,\nu}(t) + A(t, u_{\tau,x,\nu}(t)) + f(t, u_{\tau,x,\nu}(t)) + \mathrm{bar}(\nu_t), \ t \in [\tau, T] \\ u_{\tau,x,\nu}(\tau) = x \in H \end{cases}$$

and the Hamiltonian

$$H(t, x, \rho)$$
$$= \sup_{\mu \in \mathscr{M}^1_+(Z)} \left\{ -\langle \rho, \mathrm{bar}(\mu) \rangle + \int_Z J(t, x, z)\mu(dz) \right\} + \delta^*(\rho, -f(t, x) - A(t, x))$$

where $(t, x, \rho) \in [0, T] \times H \times H$. Then, V_J is a viscosity subsolution of the HJB equation

$$\frac{\partial U}{\partial t}(t, x) + H(t, x, \nabla U(t, x)) = 0,$$ [3]

[3]Where ∇U is the gradient of U with respect to the second variable.

that is, for any $\varphi \in C^1([0,T]) \times H)$ *for which* $V_J - \varphi$ *reaches a local maximum at* $(t_0, x_0) \in [0,T] \times H$*, we have*

$$H(t_0, x_0, \nabla\varphi(t_0, x_0)) + \frac{\partial\varphi}{\partial t}(t_0, x_0) \geq 0.$$

Proof Assume by contradiction that there exists a $\varphi \in C^1([0,T] \times H)$ and a point $(t_0, x_0) \in [0,T] \times H$ for which

$$\frac{\partial\varphi}{\partial t}(t_0, x_0) + H(t_0, x_0, \nabla\varphi(t_0, x_0)) \leq -\eta < 0 \quad \text{for} \quad \eta > 0.$$

Applying Lemma 3.5 by taking

$$\Lambda(t, x, \mu) = -\langle \nabla\varphi(t, x), \mathrm{bar}(\mu)\rangle + \int_Z J(t, x, z)\mu(dz) + \delta^*(\nabla\varphi(t, x), -f(t, x) - A(t, x)) + \frac{\partial\varphi}{\partial t}(t, x)$$

yields some $\sigma > 0$ such that

$$\begin{aligned}\sup_{\nu\in\mathscr{R}} \Bigg\{ &\int_{t_0}^{t_0+\sigma} \left[\int_Z J(t, u_{t_0,x_0,\nu}(t), z)\nu_t(dz)\right] dt - \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,\nu}(t), \mathrm{bar}(\nu_t)\rangle\, dt \\ &+ \int_{t_0}^{t_0+\sigma} \delta^*(\nabla\varphi(t, u_{t_0,x_0,\nu}(t)), -f(t, u_{t_0,x_0,\nu^n}(t)) - A(t, u_{t_0,x_0,\nu}(t)))\, dt \\ &+ \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,\nu}(t))\, dt\Bigg\} \\ \leq &-\frac{\sigma\eta}{2}\end{aligned} \tag{15}$$

where $u_{t_0,x_0,\nu}$ is the trajectory solution associated with the control $\nu \in \mathscr{R}$ starting from x_0 at time t_0

$$\begin{cases} 0 \in \dot{u}_{t_0,x_0,\nu}(t) + A(t, u_{t_0,x_0,\nu}(t)) + f(t, u_{t_0,x_0,\nu}(t)) + \mathrm{bar}(\nu_t), \;\; t \in [t_0, T] \\ u_{t_0,x_0,\nu}(t_0) = x_0. \end{cases}$$

Applying the dynamic programming principle (Theorem 4.10) gives

$$V_J(t_0, x_0) = \sup_{\nu\in\mathscr{R}} \left\{ \int_{t_0}^{t_0+\sigma} \left[\int_Z J(t, u_{t_0,x_0,\nu}(t), z)\nu_t(dz)\right] dt + V_J(t_0 + \sigma, u_{t_0,x_0,\nu}(t_0+\sigma))\right\}. \tag{16}$$

Since $V_J - \varphi$ has a local maximum at (t_0, x_0), for small enough σ

$$V_J(t_0, x_0) - \varphi(t_0, x_0) \geq V_J(t_0 + \sigma, u_{t_0,x_0,\nu}(t_0 + \sigma)) - \varphi(t_0 + \sigma, u_{t_0,x_0,\nu}(t_0 + \sigma)) \tag{17}$$

for all $\nu \in \mathscr{R}$. By (16), for each $n \in \mathbb{N}$, there exists $\nu^n \in \mathscr{R}$ such that

$$V_J(t_0, x_0) \leq \int_{t_0}^{t_0+\sigma} \left[\int_Z J(t, u_{t_0,x_0,\nu^n}(t)), z)\nu_t^n(dz) \right] dt + V_J(t_0 + \sigma, u_{t_0,x_0,\nu^n}(t_0 + \sigma)) + \frac{1}{n}. \tag{18}$$

From (17) and (18), we deduce that

$$\begin{aligned} V_J(t_0 + \sigma, u_{t_0,x_0,\nu^n}(t_0 + \sigma)) - \varphi(t_0 + \sigma, u_{t_0,x_0,\nu^n}(t_0 + \sigma)) \\ \leq \int_{t_0}^{t_0+\sigma} \left[\int_Z J(t, u_{t_0,x_0,\nu^n}(t)), z)\nu_t^n(dz) \right] dt + \frac{1}{n} \\ - \varphi(t_0, x_0) + V_J(t_0 + \sigma, u_{t_0,x_0,\nu^n}(t_0 + \sigma)). \end{aligned}$$

Therefore we have

$$0 \leq \int_{t_0}^{t_0+\sigma} \left[\int_Z J(t, u_{t_0,x_0,\nu^n}(t)), z)\nu_t^n(dz) \right] dt + \varphi(t_0 + \sigma, u_{t_0,x_0,\nu^n}(t_0 + \sigma)) - \varphi(t_0, x_0) + \frac{1}{n}. \tag{19}$$

As $\varphi \in C^1([0, T] \times H)$, we have

$$\begin{aligned} &\varphi(t_0 + \sigma, u_{t_0,x_0,\nu^n}(t_0 + \sigma)) - \varphi(t_0, x_0) \\ &= \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,\nu^n}(t)), \dot{u}_{t_0,x_0,\nu^n}(t) \rangle \, dt + \int_{t_0}^{t_0+\sigma} \frac{\partial \varphi}{\partial t}(t, u_{t_0,x_0,\nu^n}(t)) \, dt. \end{aligned} \tag{20}$$

Since u_{t_0,x_0,ν^n} is the trajectory solution starting from x_0 at time t_0

$$\begin{cases} 0 \in \dot{u}_{t_0,x_0,\nu^n}(t) + A(t, u_{t_0,x_0,\nu^n}(t)) + f(t, u_{t_0,x_0,\nu^n}(t)) + \text{bar}(\nu_t^n), \; t \in [t_0, T] \\ u_{t_0,x_0,\nu^n}(t_0) = x_0 \end{cases}$$

so that (20) yields the estimate

$$
\begin{aligned}
&\varphi(t_0+\sigma, u_{t_0,x_0,\nu^n}(t_0+\sigma)) - \varphi(t_0, x_0)\\
&= \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,\nu^n}(t)), \dot{u}_{t_0,x_0,\nu^n}(t)\rangle\, dt + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,\nu^n}(t))\, dt\\
&\le -\int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,\nu^n}(t)), \mathrm{bar}(\nu_t^n) + f(t, u_{t_0,x_0,\nu^n}(t))\rangle\, dt\\
&\qquad + \int_{t_0}^{t_0+\sigma} \delta^*(\nabla\varphi(t, u_{t_0,x_0,\nu^n}(t)), -A(t, u_{t_0,x_0,\nu^n}(t)))\, dt\\
&\qquad\qquad + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,\nu^n}(t))\, dt.
\end{aligned}
\tag{21}
$$

Inserting the estimate (21) into (19), we get

$$
\begin{aligned}
0 \le &\int_{t_0}^{t_0+\sigma} \left[\int_Z J(t, u_{t_0,x_0,\nu^n}(t)), z)\nu_t^n(dz)\right] dt\\
&- \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,\nu^n}(t)), \mathrm{bar}(\nu_t^n) + f(t, u_{t_0,x_0,\nu^n}(t))\rangle\, dt\\
&+ \int_{t_0}^{t_0+\sigma} \delta^*(\nabla\varphi(t, u_{t_0,x_0,\nu^n}(t)), -A(t, u_{t_0,x_0,\nu^n}(t))\, dt\\
&+ \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,\nu^n}(t))\, dt + \frac{1}{n}.
\end{aligned}
\tag{22}
$$

Then (15) and (22) yield $0 \le -\frac{\sigma\eta}{2} + \frac{1}{n}$ for all $n \in \mathbb{N}$. By passing to the limit when n goes to ∞ in this inequality, we get a contradiction: $0 \le -\frac{\sigma\eta}{2}$. The proof is therefore complete.

Existence results for evolution inclusion driven by time-dependent maximal monotone operators $A(t)$ with single-valued perturbation f or convex weakly compact valued perturbation F of the form

$$-\dot{u}(t) \in A(t)u(t) + f(t, u(t))$$

or

$$-\dot{u}(t) \in A(t)u(t) + F(t, u(t))$$

are developed in [7, 8], while existence results for convex or nonconvex sweeping process in the form

$$-\dot{u}(t) \in N_{C(t)}(u(t)) + f(t, u(t))$$

or

$$-\dot{u}(t) \in N_{C(t)}(u(t)) + F(t, u(t))$$

where $C(t)$ is a closed convex (or nonconvex) moving set and $N_{C(t)}(u(t))$ is the normal cone of $C(t)$ at the point $u(t)$ is much studied so that our tools developed above allow to treat some further variants on the viscosity solution dealing with some specific maximal monotone operators $A(t)$ or convex or nonconvex sweeping process such as

$$\begin{cases} 0 \in \dot{u}_{t_0,x_0,\nu}(t) + N_{C(t)}(u_{t_0,x_0,\nu}(t)) + f(t, u_{\tau,x,\nu}(t)) + \text{bar}(\nu_t),\ t \in [t_0, T] \\ u_{t_0,x_0,\nu}(t_0) = x_0 \end{cases}$$

using the subdifferential of the distance function $d_{C(t)}x$.

We end the paper with some variational limit results which can be applied to further convergence problems in state-dependent convex sweeping process or second-order state-dependent convex sweeping process. See [1, 3, 34] and the references therein.

Theorem 4.12 *Let $C_n : [0, T] \to H$ and $C : [0, T] \to H$ be a sequence of convex weakly compact valued scalarly measurable bounded mappings satisfying*

(i) $\sup_n \sup_{t\in[0,T]} \mathscr{H}\left(C_n(t), C(t)\right) \le M < \infty$,
(ii) $\lim_n \mathscr{H}\left(C_n(t), C(t)\right) = 0$, *for each* $t \in [0, T]$.

Let (v_n) be a uniformly integrable sequence in $L^1_H([0, T])$ such that v_n converges for $\sigma(L^1_H([0, T]), L^\infty_H([0, T])$ to $v \in L^1_H([0, T])$, and let (u_n) be a uniformly bounded sequence $L^\infty_H([0, T])$ which pointwisely converges to $u \in L^\infty_H([0, T])$. Assume that $-v_n(t) \in N_{C_n(t)}(u_n(t))$ a.e., then

$$u(t) \in C(t) \text{ a.e. and } -v(t) \in N_{C(t)}(u(t)) \text{ a.e.}$$

Proof For simplicity, let $\rho_n(t) = \mathscr{H}\left(C_n(t), C(t)\right)$ for each $t \in [0, T]$. Firstly it is clear that the mappings ρ_n, $t \mapsto \delta^*(-v_n(t), C_n(t))$, $t \mapsto \delta^*(-v_n(t), C(t))$, and $t \mapsto \delta^*(-v(t), C(t))$ are measurable on $[0, T]$ and integrable by boundedness. By the Hormander formula for convex weakly compact set (see [19]), we have

$$|\delta^*(-v_n(t), C_n(t)) - \delta^*(-v_n(t), C(t))| \le ||v_n(t)||\rho_n(t)$$

so that

$$\delta^*(-v_n(t), C_n(t)) - \delta^*(-v_n(t), C(t)) \ge -||v_n(t)||\rho_n(t).$$

By $-v_n(t) \in N_{C_n(t)}(u_n(t))$, we have $\delta^*(-v_n(t), C_n(t)) + \langle v_n(t), u_n(t)\rangle \le 0$ so we get the estimation

$$-||v_n(t)||\rho_n(t) + \delta^*(-v_n(t), C(t)) + \langle v_n(t, u_n(t)\rangle \le 0$$

or

$$\delta^*(-v_n(t), C(t)) + \langle v_n(t), u_n(t)\rangle \le ||v_n(t)||\rho_n(t).$$

Note that the mappings $t \mapsto \delta^*(-v_n(t), C(t)) + \langle v_n(t), u_n(t)\rangle$, and $t \mapsto ||v_n(t)||\rho_n(t)$ are integrable on $[0, T]$. Let B a measurable set in $[0, T]$ and then by integrating

$$\int_B \delta^*(-v_n(t), C(t))dt + \int_B \langle v_n(t), u_n(t)\rangle dt \le \int_B ||v_n(t)||\rho_n(t)dt.$$

We note that the convex integrand $H(t, e) = \delta^*(e, C(t))$ defined on $[0, T] \times H$ is normal because $t \mapsto H(t, e)$ is continuous on $[0, T]$ and $e \mapsto H(t, e)$ is convex continuous on H, with $H(t, e) \ge \langle e, u(t)\rangle$ for all $(t, e) \in [0, T] \times H$. Consequently $H(t, -v_n(t)) = \delta^*(-v_n(t), C(t)) \ge \langle -v_n(t), u(t)\rangle$. But $(\langle -v_n, u\rangle)$ is uniformly integrable in $L^1_{\mathbb{R}}([0, T], dt)$, so that by virtue of the lower semicontinuity of the integral convex functional [22, Theorem 8.1.16], we have

$$\liminf_{n\to\infty} \int_B \delta^*(-v_n(t), C(t))dt \ge \int_B \delta^*(-v(t), C(t))dt. \tag{23}$$

Note that the sequence $\big(u_n(.) - u(.)\big)$ is uniformly bounded and pointwisely converges to 0, so that it converges to 0 uniformly on any uniformly integrable subset of $L^1_H([0, T], dt)$; in other terms, it converges to 0 with respect to the Mackey topology $\tau(L^\infty_H([0, T], dt), L^1_H([0, T], dt))$ (see [15]),[4] so that

$$\lim_{n\to\infty} \int_B \langle v_n(t), u_n(t) - u(t)\rangle dt = 0$$

because (v_n) is uniformly integrable. Consequently

$$\begin{aligned}\lim_{n\to\infty} \int_B \langle v_n(t), u_n(t)\rangle dt &= \lim_{n\to\infty} \int_B \langle v_n(t), u_n(t) - u(t)\rangle dt + \lim_{n\to\infty} \int_B \langle v_n(t), u(t)\rangle dt \\ &= \lim_{n\to\infty} \int_B \langle v_n(t), u(t)\rangle dt = \int_B \langle \dot{v}(t), u(t)\rangle dt. \end{aligned} \tag{24}$$

By our assumptions, $\rho_n(t)$ is bounded measurable and pointwisely converges to 0 and $||v_n(t)||$ is uniformly integrable; then similarly we have

$$\lim_n \int_B ||v_n(t)||\rho_n(t)dt = 0. \tag{25}$$

[4] If $H = \mathbb{R}^d$, one may invoke a classical fact that on bounded subsets of L^∞_H the topology of convergence in measure coincides with the topology of uniform convergence on uniformly integrable sets, i.e. on relatively weakly compact subsets, alias the Mackey topology. This is a lemma due to Grothendieck [33, Ch.5 §4 no 1 Prop. 1 and exercice].

Finally by passing to the limit when n goes to ∞ in

$$\int_B \delta^*(-v_n(t), C(t))dt + \int_B \langle v_n(t), u_n(t)\rangle dt \leq \int_B ||v_n(t)||\rho_n(t)dt$$

and taking into account the above convergence limits (23), (24), and (25), we get

$$\int_B \delta^*(-v(t), C(t))dt + \int_B \langle v(t), u(t)\rangle dt \leq 0.$$

As the function $t \mapsto \delta^*(-v(t), C(t)) + \langle v(t), u(t)$ is integrable on $[0, T]$ and this holds for every B measurable set in $[0, T]$, we get

$$\delta^*(-v(t), C(t))) + \langle v(t), u(t)\rangle \leq 0 \text{ a.e.}$$

Furthermore, it is not difficult to check that $u(t) \in C(t)$ a.e. using (ii) and the fact that $u_n(t) \in C_n(t)$ for all $n \in \mathbb{N}$ and a.e. $t \in [0, T]$; therefore, we conclude that $-v(t) \in N_{C(t)}(u(t))$ a.e. The proof is complete.

Our tools allow to treat the variational limits for further evolution variational inequalities such as

Proposition 4.2 *Let $C_n : [0, T] \to H$ and $C : [0, T] \rightrightarrows H$ be a sequence of convex weakly valued scalarly measurable bounded mappings satisfying*

(i) $\sup_n \sup_{t\in[0,T]} \mathscr{H}\left(C_n(t), C(t)\right) \leq M < \infty$,
(ii) $\lim_n \mathscr{H}\left(C_n(t), C(t)\right) = 0$, *for each* $t \in [0, T]$.

Let $B : H \to H$ be a linear continuous operator such that $\langle Bx, x\rangle > 0$ for all $x \in H \setminus \{0\}$. Let (v_n) be a uniformly bounded sequence in $L^\infty_H([0, T])$ such that v_n $\sigma(L^\infty_H([0, T]), L^1_H([0, T])$ converges to $v \in L^\infty_H([0, T])$, and let (u_n) be a uniformly bounded sequence $L^\infty_H([0, T])$ which pointwisely converges to $u \in L^\infty_H([0, T])$. Assume that $-v_n(t) \in N_{C_n(t)}(u_n(t) + Bv_n(t))$ for all $n \in \mathbb{N}$ and for a.e. $t \in [0, T]$. Then

$$u(t) + Bv(t) \in C(t) \textit{ a.e. and } -v(t) \in N_{C(t)}(u(t) + Bv(t)) \textit{ a.e.}$$

Proof Apply the notations of the proof of Theorem 4.12. Let $\rho_n(t) = \mathscr{H}\left(C_n(t), C(t)\right)$ for each $t \in [0, T]$. It is clear that the mappings ρ_n, $t \mapsto \delta^*(-v_n(t), C_n(t))$, $t \mapsto \delta^*(-v_n(t), C(t))$, and $t \mapsto \delta^*(-v(t), C(t))$ are measurable and integrable on $[0, T]$. By the Hormander formula for convex weakly compact sets (see [19]), we have

$$|\delta^*(-v_n(t), C_n(t)) - \delta^*(-v_n(t), C(t))| \leq ||v_n(t)||\rho_n(t)$$

so that

$$\delta^*(-v_n(t), C_n(t)) - \delta^*(-v_n(t), C(t)) \geq -||v_n(t)||\rho_n(t).$$

By $-v_n(t) \in N_{C_n(t)}(u_n(t) + Bv_n(t))$, we have

$$\delta^*(-v_n(t), C_n(t)) + \langle v_n(t), u_n(t) + Bv_n(t)\rangle \leq 0.$$

Whence

$$\delta^*(-v_n(t), C(t)) + \langle v_n(t), u_n(t) + Bv_n(t)\rangle \leq ||v_n(t)||\rho_n(t)$$

Note that the mappings $t \mapsto \delta^*(-v_n(t), C(t)) + \langle v_n(t), u_n(t) + Bv_n(t)\rangle$, and $t \mapsto ||v_n(t)||\rho_n(t)$ are integrable on $[0, T]$ so that by integrating on any measurable set $L \subset [0, T]$

$$\int_L \delta^*(-v_n(t), C(t))dt + \int_L \langle v_n(t), u_n(t)\rangle dt + \int_L \langle v_n(t), Bv_n(t)\rangle dt \leq \int_L ||v_n(t)||\rho_n(t)dt.$$

Since (v_n) $\sigma(L^\infty_H([0, T]), L^1_H([0, T])$ converges to $v \in L^\infty_H([0, T])$, it is not difficult to check that (Bv_n) converges for $\sigma(L^\infty_H([0, T]), L^1_H([0, T])$ to $Bv \in L^1_H([0, T])$, arguing as in [11, Theorem 4.1]. As a consequence, the sequence $(u_n + Bv_n)$ converges for $\sigma(L^\infty_H([0, T]), L^1_H([0, T])$ to $u + Bv \in L^\infty_H([0, T])$. From $u_n(t) + Bv_n(t) \in C_n(t)$, we deduce

$$\int_L \langle e, u_n(t) + Bv_n(t)\rangle dt \leq \int_L \delta^*(e, C_n(t))dt$$

for every $e \in H$ and for every measurable set $L \subset [0, T]$. By passing to the limit in this inequality, we get

$$\int_L \langle e, u(t) + Bv(t)\rangle dt \leq \limsup_n \int_L \delta^*(e, C_n(t))dt \leq \int_L \delta^*(e, C(t))dt.$$

It follows that

$$\langle e, u(t) + Bv(t)\rangle \leq \delta^*(e, C(t)) \text{ a.e.}$$

By [19, Proposition III.35], we deduce that $u(t) + Bv(t) \in C(t)$ a.e. As in Theorem 3.1, we have already stated that for every measurable set $L \subset [0, T]$,

$$\lim_n \int_L \langle u_n(t), v_n(t)\rangle dt = \int_L \langle u(t), v(t)\rangle dt, \tag{26}$$

$$\lim_n \int_L |v_n(t)||\rho_n(t)dt = 0, \tag{27}$$

$$\liminf_n \int_B \delta^*(-v(t), C_n(t))dt \geq \int_B \delta^*(-v(t), C_n(t))dt. \tag{28}$$

Now set $\varphi(x) = \langle x, Bx\rangle$ for all $x \in H$. Then $\varphi(x)$ is a nonnegative lower semicontinuous and convex function defined on H. So we have

$$\int_L \langle v_n(t), Bv_n(t)\rangle dt = \int_L \varphi(v_n(t))dt.$$

By lower semicontinuity of convex integral functional [19, 22, 23], we get

$$\liminf_n \int_L \langle v_n(t), Bv_n(t)\rangle dt$$
$$= \liminf_n \int_L \varphi(v_n(t))dt \geq \int_L \varphi(v(t))dt = \int_L \langle v(t), Bv(t)\rangle dt.$$

Taking into consideration the above stated limits (26), (27), (28) and passing to the limit when n goes to ∞ in the inequality

$$\int_L \delta^*(-v_n(t), C(t))dt + \int_L \langle v_n(t), u_n(t)\rangle dt + \int_L \langle v_n(t), Bv_n(t)\rangle dt$$
$$\leq \int_L ||v_n(t)||\rho_n(t)dt,$$

we get

$$\int_L \delta^*(-v(t), C(t))dt + \int_L \langle v(t), u(t) + Bv(t)\rangle dt \leq 0$$

for every measurable set $L \subset [0, T]$. Since the mapping $t \mapsto \delta^*(-v(t), C(t)) + \langle v(t), u(t) + Bv(t)\rangle$ is integrable on $[0, T]$, we have

$$\delta^*(-v(t), C(t)) + \langle v(t), u(t) + Bv(t)\rangle \leq 0 \text{ a.e.}$$

As $u(t) + Bv(t) \in C(t)$ a.e., this yields $-v(t) \in N_{C(t)}(u(t) + Bv(t))$ a.e. The proof is complete.

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