# Amalgamations and Equitable Block-Colorings

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**Abstract.** An $H$-decomposition of $G$ is a partition $P$ of $E(G)$ into blocks, each element of which induces a copy of $H$. Amalgamations of graphs have proved to be a valuable tool in the construction of $H$-decompositions. The method can force decompositions to satisfy fairness notions. Here the use of the method is further applied to $(s,p)$-equitable block-colorings of $H$-decompostions: a coloring of the blocks using exactly $s$ colors such that each vertex $v$ is incident with blocks colored with exactly $p$ colors, the blocks containing $v$ being shared out as evenly as possible among the $p$ color classes. Recently interest has turned to the color vector $V(E) = (c_1(E), c_2(E), \ldots, c_s(E))$ of such colorings. Amalgamations are used to construct $(s,p)$-equitable block-colorings of $C_4$-decompositions of $K_n - F$ and $K_2$-decompositions of $K_n$, focusing on one unsolved case with each where $c_1$ is as small as possible and $c_2$ is as large as possible.

## 1 Introduction

An $H$-decomposition of a graph $G$ is an ordered pair $(V,B)$ where $V$ is the vertex set of $G$ and $B$ is a partition of the edges of $G$ into sets, each of which induces a copy of $H$. The elements of $B$ are known as the blocks of the decomposition. An $H$-decomposition $(V,B)$ is said to have an $(s,p)$-equitable block-coloring $E : B \mapsto C = \{1,2,\ldots,s\}$ if:

(i) the blocks in $B$ are colored with exactly $s$ colors,
(ii) for each vertex $u \in V(G)$ the blocks containing $u$ are colored using exactly $p$ colors, and
(iii) for each vertex $u \in V(G)$ and for each $\{i,j\} \subset C(E,u)$, $|b(E,u,i) - b(E,u,j)| \leq 1$,

where $C(E,u) = \{i \mid \text{some block incident with } u \text{ is colored } i \text{ by } E\}$ and $b(E,u,i)$ is the number of blocks in $B$ containing $u$ that are colored $i$ by $E$. Such colorings have been considered by several authors, including L. Gionfriddo, M. Gionfriddo, Hork, Li, Matson, Milazzo, Ragusa and Rodger (see [5–7,13,14]), the work focusing on cases where $H \in \{C_3, C_4\}$ and $G \in \{K_n, K_n - F\}$, where $F$ is a 1-factor of $K_n$. The main focus in these papers was to find the smallest and largest possible values of $s$ for each fixed value of $p$.


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More recently, the research has turned to the structure of such colorings in the form of the color vector $V(E) = (c_1(E), c_2(E), \ldots, c_s(E))$ of an $(s, p)$-equitable block-coloring $E$ of $G$, where $c_i(E)$ is the number of vertices in $G$ incident with a block of color $i$ arranged in non-decreasing order. Of most interest are the extreme values of $c_i(E)$, thus motivating the following definitions.

**Definition 1.** *For any graphs $G$ and $H$ and for $1 \leq i \leq s$, define*

*(i)* $\phi(H, G; s, p, i) = \{c_i(E) \mid E$ *is an $(s, p)$-equitable block-coloring of an $H$-decomposition of $G$*$\}$.
*(ii)* $\psi^{'}(H, G; s, p, i) = \min \phi(H, G; s, p, i)$,
*(iii) and* $\overline{\psi^{'}}(H, G; s, p, i) = \max \phi(H, G; s, p, i)$.

Considering the tightest cases where $s$ is as small as possible for a given value of $p$ is a naturally challenging problem. Often this means that the $s = p$ case is being addressed, and so it is natural to construct such colorings by using path interchange techniques that abound in graph theory. But in rarer cases it turns out that $s$ is always greater than $p$, requiring new methods to make progress to construct the colorings, hence the motivation for proving Theorems 1 and 2 below. (Interchanging colors along paths can introduce new colors at the end blocks, potentially contravening the requirement that exactly $p$ colors appear on blocks at each vertex.) Before stating Theorem 1, some notation needs to be introduced.

Throughout this paper the focus is on the case where $(H, G) = (C_4, K_{v'} - F)$, and the related case where $(H, G) = (K_2, K_v)$ described below. In this context it has been shown that the only situation where $s$ is always greater than $p$ is when $v' \equiv 4t + 2 \pmod{8t}$ (for example, see [14,15]), in which case if $s$ is as small as possible then $(s, p) = (2t + 1, 2t)$ for some integer $t$. So for the rest of the paper we now assume that $(s, p) = (2t + 1, 2t)$ and that $v' = 8tx + 4t + 2$ for some integer $x$; so clearly $v' > 1$ and $t \geq 1$. It is also convenient to define $\psi^{'}(H, G; 2t + 1, 2t, i) = \psi^{'}_i(H, G)$ and $\overline{\psi^{'}}(H, G; 2t + 1, 2t, i) = \overline{\psi^{'}}_i(H, G)$. Since each vertex $u$ in $K_{v'} - F$ obviously has degree $8tx + 4x$, which is divisible by $2p = 4t$, $u$ is contained in exactly $b'(v') = \frac{v'-2}{4t} = 2x + 1$ blocks in each of the $p = 2t$ color classes appearing at $u$ (each block, being a 4-cycle, contains 2 edges incident with $u$). We are now ready to state the following theorem.

**Theorem 1** [15]**.** *Let $v' \equiv 4t + 2 \pmod{8t}$. Let $4t \leq 2b'(v') + 2$. Then*

*(1)* $\psi'_1(C_4, K_{v'} - F) = 2b'(v') + 2$ *and*
*(2) for* $3 \leq i \leq 2t + 1$, $\overline{\psi'_i}(C_4, K_{v'} - F) = v' - 2$.

Notice that there is an unsolved case left in Theorem 1, namely finding $\overline{\psi'_2}(C_4, K_{v'} - F)$; this is the one value of $i$ where $\overline{\psi'_i}(C_4, K_{v'} - F)$ is not always either the obvious lower or upper bound on the size of a color class, so it is particularly challenging to find. In this paper, $\overline{\psi'_2}(C_4, K_{v'} - F)$ is found (see Corollary 1) by solving a related edge-coloring problem in Theorem 4 which is proved using the method of amalgamations of graphs (graph homomorphisms). This construction is then modified in Sect. 4 to provide a new proof of Theorem 1.

Amalgamations provide a versatile proof technique that has been used in the study of factorizations of graphs and Steiner triple systems, but its use in block-colorings is relatively new.

Pursuing this approach in more detail, it is shown in [15] that Theorem 1 is a direct consequence of the existence of a $(2t+1, 2t)$-equitable edge-coloring of $K_v$, where $v = v'/2$ (or, more precisely, a $(2t+1, 2t)$-equitable block-coloring of the obvious $K_2$-decomposition of $K_v$), so $v = 4tx + 2t + 1$ for some integer $x$; clearly if $v > 1$, then $t \geq 1$. Each vertex $u$ has degree $4tx + 2t$, which is clearly divisible by $p = 2t$, so $u$ is contained in exactly $b(v) = \frac{v-1}{2t} = 2x + 1 = b'(v')$ blocks (edges) in each of the $2t$ color classes appearing at $u$. Note that $b(v)$ is odd. In Sect. 4, a new proof of the following result is presented (and by the discussion above, a new proof of Theorem 1 as well).

**Theorem 2** [15]**.** *Let* $v \equiv 2t + 1 \pmod{4t}$ *with* $v > 1$*. Let* $2t \leq b(v) + 1$*. Then,*

*(1)* $\psi'_1(K_2, K_v) = b(v) + 1$ *and*
*(2) for* $3 \leq i \leq 2t + 1$*,* $\overline{\psi'_i}(K_2, K_v) = v - 1$*.*

It is worth noting that a more generalized result in [15] complements Theorems 1 and 2, addressing the cases where $4t \geq 2b'(v') + 2$ and $2t \geq b(v) + 1$, showing that then $\overline{\psi'_2}(C_4, K_{v'} - F) = v' - 1$ and $\overline{\psi'_2}(K_2, K_v) = v - 1$ respectively.

The following notation will be useful throughout the paper. Let $K[R]$ denote the complete graph defined on the vertex set $R$. Color $i$ is said to appear at a vertex $u$ if at least one block incident with $u$ is colored $i$.

## 2 Some Preliminary Results

In order to find $\overline{\psi'_2}(C_4, K_{v'} - F)$ and $\overline{\psi'_2}(K_2, K_v)$, we begin by finding bounds on the value of $c_2$ in the following Lemmas, utilizing some results proved in [14,15]. For ease of notation define $\lfloor x \rfloor_e$ to be the largest even integer less than or equal to $x$.

**Lemma 1.** *For* $v \equiv 2t + 1 \pmod{4t}$ *and* $v' = 2v$*,*

$$\overline{\psi}'_i(K_2, K_v) \leq \left\lfloor \frac{2tv - \sum_{j=1}^{i-1} \psi'_j(K_2, K_v)}{2t + 2 - i} \right\rfloor_e \text{ and}$$

$$\overline{\psi}'_i(C_4, K_{v'} - F) \leq \left\lfloor \frac{2tv' - \sum_{j=1}^{i-1} \psi'_j(C_4, K_{v'} - F)}{2t + 2 - i} \right\rfloor_e .$$

*Proof.* Note the elements of the color vector are listed in non-decreasing order; and since in Lemma 2.5 of [14] it is shown that for any $(2t+1, 2t)$-equitable edge-coloring $E$ of $K_v$ and for any $(2t + 1, 2t)$-equitable $C_4$-coloring $E'$ of $K_{v'} - F$, both $\sum_{i=1}^{2t+1} c_i(E) = 2tv$ and $\sum_{i=1}^{2t+1} c_i(E') = 2tv'$, the above holds. □

**Lemma 2.** *Let $v = v'/2 = 4tx + 2t + 1$ for some integer $x$ and $b(v) + 1 = b'(v') + 1 \geq 2t$. Then,*

$$\overline{\psi}_2'(K_2, K_v) \leq \left\lfloor v - \frac{x+1}{t} \right\rfloor_e \text{ and } \overline{\psi}_2'(C_4, K_{v'} - F) \leq \left\lfloor v' - \frac{2x+2}{t} \right\rfloor_e.$$

*Proof.* Let $b(v)+1 \geq 2t$. By Theorem 3.5 of [15], $\psi_1'(K_2, K_v) = b(v)+1$. Therefore by Lemma 1:

$$\begin{aligned}\overline{\psi}_2'(K_2, K_v) &\leq \lfloor \frac{2tv - (b(v)+1)}{2t+2-2} \rfloor_e \\ &= \lfloor v - \frac{(b(v)+1)}{2t} \rfloor_e \\ &= \lfloor v - \frac{2x+2}{2t} \rfloor_e \\ &= \lfloor v - \frac{x+1}{t} \rfloor_e.\end{aligned}$$

By Corollary 3.6 of [15], $\psi_1'(C_4, K_{v'} - F) = 2b'(v') + 2$. Therefore by Lemma 1:

$$\begin{aligned}\overline{\psi}_2'(C_4, K_{v'} - F) &\leq \left\lfloor \frac{2tv' - (2b'(v')+2)}{2t+2-2} \right\rfloor_e \\ &= \left\lfloor v' - \frac{(2b'(v')+2)}{2t} \right\rfloor_e \\ &= \left\lfloor v' - \frac{2x+4}{2t} \right\rfloor_e \\ &= \left\lfloor v' - \frac{2x+2}{t} \right\rfloor_e.\end{aligned}$$

□

## 3 Settling the Unsolved Cases in Theorems 1 and 2

Apart from completing the open case left in Theorems 1 and 2, in this paper the use of amalgamations in block-decompositions is further demonstrated. Hilton and Rodger [8,9] used this technique to find embeddings of edge-colorings into hamiltonian decompositions. Buchanan [2] used amalgamations to find hamiltonian decompositions of $K_n - E(U)$ for any 2-regular spanning subgraph $U$, and this was extended to various multipartite graphs by Leach and Rodger [10,12]. Leach and Rodger [11] went on to find hamilton decompositions of complete multipartite graphs where each hamilton cycle spreads its edges out as evenly as possible among the pairs of parts of the graph. This notion has recently been extended by Erzurumluoğlu and Rodger [3,4] to factorizations and holey factorizations of complete multipartite graphs and then to $C_4$-decompositions of $K_v - F$ and edge-decompositions of $K_v$ by Matson and Rodger in [15].

Formally, a graph $H$ is said to be an *amalgamation* of a graph $G$ if there exists a function $\psi$ from $V(G)$ onto $V(H)$ and a bijection $\psi' : E(G) \to E(H)$ such that $e = \{u, v\} \in E(G)$ if and only if $\psi'(e) = \{\psi(u), \psi(v)\} \in E(H)$. The function $\psi$ is called an amalgamation function. We say that $G$ is a *detachment* of $H$, where each vertex $u$ of $H$ splits into the vertices of $\psi^{-1}(\{u\})$. An $\eta$-*detachment* of $H$ is a detachment in which each vertex $u$ of $H$ splits into $\eta(u)$ vertices.

To describe the amalgamation result used here more precisely, some notation will be needed. Let $x \approx y$ represent the fact that $\lfloor y \rfloor \leq x \leq \lceil y \rceil$. Furthermore, let $\ell(u)$ denote the number of loops incident with vertex $u$, where each loop contributes twice to the degree of $u$, let $G(j)$ denote the subgraph of $G$ induced by the edges colored $j$, and let $m(u, v)$ denote the number of edges between the pair of vertices $u$ and $v$ in $G$.

The following is a special case of Theorem 3.1 in [1] (omitting the condition that ensures color classes are connected and a balanced property on the color classes for multigraphs since in our case $G$ is simple).

**Theorem 3** (Bahmanian and Rodger [1, Theorem 3.1])**.** *Let $H$ be a $k$-edge-colored graph and let $\eta$ be a function from $V(H)$ into $\mathbb{N}$ such that for each $v \in V(H)$, $\eta(v) = 1$ implies $\ell_H(v) = 0$. Then there exists a loopless $\eta$-detachment $G$ of $H$ in which each $v \in V(H)$ is detached into $v_1, \ldots, v_{\eta(v)}$, such that $G$ satisfies the following conditions:*

1. *$d_G(u_i) \approx d_H(u)/\eta(u)$ for each $u \in V(H)$ and $1 \leq i \leq \eta(u)$;*
2. *$d_{G(j)}(u_i) \approx d_{H(j)}(u)/\eta(u)$ for each $u \in V(H)$, $1 \leq i \leq \eta(u)$, and $1 \leq j \leq k$;*
3. *$m_G(u_i, u_{i'}) \approx \ell_H(u)/\binom{\eta(u)}{2}$ for each $u \in V(H)$ with $\eta(u) \geq 2$ and $1 \leq i < i' \leq \eta(u)$; and*
4. *$m_G(u_i, v_{i'}) \approx m_H(u, v)/(\eta(u)\eta(v))$ for every pair of distinct vertices $u, v \in V(H)$, $1 \leq i \leq \eta(u)$, and $1 \leq i' \leq \eta(v)$.*

We now complete the open case left in Theorem 2 as stated here as Theorem 4. As explained in the introduction, as a result of Theorem 4, we also complete the open case left in Theorem 1, stated here as Corollary 1, using the method of amalgamations in both.

**Theorem 4.** *Let $v \equiv 2t + 1 \pmod{4t}$ with $v > 1$. Let $2t \leq b(v) + 1$. Then*

$$\overline{\psi_2'}(K_2, K_v) = \left\lfloor v - \frac{x+1}{t} \right\rfloor_e .$$

*Proof.* Let $v = 4tx + 2t + 1$ for some integer $x$. Form a complete graph $\mathcal{G}_0$ on the set of vertices $V_0 = \{u_1, \ldots, u_{2x+2}\}$ and color all the edges of $\mathcal{G}_0$ with color $2t + 1$. So each vertex in $\mathcal{G}_0$ is incident with $2x + 1 = b(v)$ edges colored $2t + 1$ as desired. Notice that in the final edge-coloring of $K_v$, each vertex is missing (i.e., is not incident with any edges of) exactly one color. We will arrange for $1 \leq i \leq 2t$, color $m(i) = i$ to be missing from vertex $u_i$, for $2t + 1 \leq i \leq 2x + 2$ color $m(i) = \lceil \frac{i-2t}{2} \rceil \pmod{2t} \in \{1, \ldots, 2t\}$ to be missing from $u_i$, and color $m(\alpha_i) = 2t + 1$ to be missing from the remaining $v - 2x - 2$ vertices (which will be named $\alpha_1, \ldots, \alpha_{\eta(\alpha)}$ below). For $1 \leq i \leq 2t$ let $M(i) = \{u_j \in V_0 \mid m(j) = i\}$. Note for $1 \leq i < j \leq 2t$, $|\,|M(i)| - |M(j)|\,| \in \{0, 2\}$ and $|M(i)|$ is odd for all $i$.

Next form a new edge-colored graph $\mathcal{G}_0^+$ from $\mathcal{G}_0$ as follows. Add a single vertex, $\alpha$. The aim now is to complete the proof by using Theorem 3 with $\eta(u_i) = 1$ for $1 \leq i \leq 2x+2$ and $\eta(\alpha) = v - 2x - 2$. For $1 \leq i \leq 2x+2$ join $u_i$ to $\alpha$ with $b(v)$ edges of each color in $\{1, 2, \ldots, 2t\} \setminus \{m(i)\}$. Thus for $1 \leq i \leq 2x+2$ the number of edges joining $u_i$ to $\alpha$ is $(2t-1)(2x+1) = 4tx+2t+1-(2x+1)-1 = v - 1 - (2x+1) = \eta(\alpha)$, and $d_{\mathcal{G}_0^+}(u_i) = v - 1$.

Let $a(i)$ be the number of vertices in $G_0^+$ where color $i$ appears and let $\epsilon_i = 2$ for $1 \leq i \leq x + 1 - t \pmod{2t}$ and $\epsilon_i = 0$ otherwise. Therefore $a(2t+1) = 2x+2$ and for $1 \leq i \leq 2t$,

$$\begin{aligned} a(i) &= 2x + 3 - |M(i)| \\ &= 2x + 2 - 2\left\lfloor \frac{2x+2-2t}{4t} \right\rfloor - \epsilon_i. \end{aligned}$$

Note since $x \geq 0$ and $t \geq 1$ for $1 \leq i \leq 2t$,

$$\begin{aligned} \eta(\alpha) - (a(i) - 1) &= v - 2x - 2 - \left(2x + 2 - 2\left\lfloor \frac{2x+2-2t}{4t} \right\rfloor - \epsilon_i - 1\right) \\ &= v - 4x - 3 - 2\left\lfloor \frac{2x+2-2t}{4t} \right\rfloor + \epsilon_i \\ &\geq 4tx + 2t - 2 - 4x - \left(\frac{2x+2-2t}{2t}\right) \\ &= 4x(t-1) + 2t - 1 - \frac{x+1}{t} \\ &= (4x+1)(t-1) + t - \frac{x+1}{t} \geq 0. \end{aligned}$$

Thus for $1 \leq i \leq 2t$ add $(b(v)\eta(\alpha) - b(v)(a(i) - 1))/2$ loops colored $i$ to $\alpha$, thus resulting in $d_{\mathcal{G}_0^+(i)}(\alpha) = b(v)\eta(\alpha)$. By the above calculations we know we will be adding a non-negative number of loops for all colors $1, \ldots, 2t$.

Let $l(\alpha)$ be the number of loops incident with $\alpha$ and $E(V(G_0), \alpha)$ be the set of edges from a vertex in $G_0$ to $\alpha$. Therefore,

$$\begin{aligned} l(\alpha) &= \left(d_{G_0^+}(\alpha) - |E(V(G_0), \alpha)|\right)/2 \\ &= \left(\eta(\alpha)b(v)2t - (2x+2)[b(v)(2t-1)]\right)/2 \\ &= \left(\eta(\alpha)b(v)2t - (2x+2)\eta(\alpha)\right)/2 \\ &= \eta(\alpha)\left(b(v)2t - 2x - 2\right)/2 \\ &= \eta(\alpha)\left(4tx + 2t + 1 - 2x - 3\right)/2 \\ &= \eta(\alpha)(v - 2x - 2 - 1)/2 \\ &= \eta(\alpha)(\eta(\alpha) - 1)/2. \end{aligned}$$

Now apply Theorem 3 to form the detachment $\mathcal{G}$ of $\mathcal{G}_0^+$ in which $\alpha$ is detached into the vertices $\alpha_1, \ldots, \alpha_{\eta(\alpha)}$. For $1 \le i \le 2x+2$, since $u_i$ is joined to $\alpha$ with $b(v)$ edges in $\mathcal{G}_0^+$, by condition (3) $u_i$ is joined to each vertex $\alpha_j$ for $1 \le j \le \eta(\alpha)$ by exactly one edge in $\mathcal{G}$. Also, since $\alpha$ is incident with $\eta(\alpha)(\eta(\alpha)-1)/2$ loops in $\mathcal{G}_0^+$, by condition (4) $\alpha_i$ is joined to $\alpha_j$ by exactly one edge for $1 \le i < j \le \eta(\alpha)$ in $\mathcal{G}$. It follows that $\mathcal{G}$ is isomorphic to $K_{2x+2+\eta(\alpha)} = K_v$. By condition (2), for each vertex $u$ in $\mathcal{G}$, each color which appears at $u$ does so on $b(v)$ edges. Therefore the edge-coloring $E$ of $\mathcal{G}$ is $(2t+1, 2t)$-equitable. Furthermore, in $\mathcal{G}$, color $2t+1$ appears at $b(v)+1 \ge 2t$ vertices and for $1 \le i \le 2t$, the number of vertices where color $i$ appears is

$$\begin{aligned} a(i) - 1 + \eta(\alpha) &= (2x+2) - 2\left\lfloor \frac{x+1-t}{2t} \right\rfloor - \epsilon_i - 1 + v - (2x+2) \\ &= v - 1 - 2\left\lfloor \frac{x+1-t}{2t} \right\rfloor - \epsilon_i. \end{aligned}$$

Therefore, since $a(i)$ and $\eta(\alpha)$ are both odd integers, if $2t$ divides $(x+1-t)$, then $\epsilon_1 = 0$ and

$$\begin{aligned} a(1) - 1 + \eta(\alpha) &= v - 1 - \frac{x+1-t}{t} \\ &= v - \frac{x+1}{t} \\ &= \left\lfloor v - \frac{x+1}{t} \right\rfloor_e, \end{aligned}$$

and if $2t$ does not divide $(x+1-t)$ then $\epsilon_1 = 2$ and

$$\begin{aligned} a(1) - 1 + \eta(\alpha) &= v - 1 - \left(2\left\lfloor \frac{x+1-t}{2t} \right\rfloor + 2\right) \\ &= v - 1 - 2\left\lceil \frac{x+1-t}{2t} \right\rceil \\ &= v - 1 + 2\left\lfloor \frac{-(x+1-t)}{2t} \right\rfloor \\ &= 2\left\lfloor \frac{v-1}{2} + \frac{1}{2} - \frac{x+1}{2t} \right\rfloor \\ &= 2\left\lfloor \frac{v}{2} - \frac{x+1}{2t} \right\rfloor \\ &= \left\lfloor v - \frac{x+1}{t} \right\rfloor_e. \end{aligned}$$

Therefore by Lemma 2, $\overline{\psi_2'}(K_2, K_v) = \lfloor v - \frac{x+1}{t} \rfloor_e$ and the proof is complete (after renaming color $2t+1$ with 1 and renaming colors $1, 2, \ldots, 2t$ with $2, 3, \ldots, 2t+1$ respectively). $\square$

**Corollary 1.** *Let* $v' \equiv 4t + 2 \pmod{8t}$*. Let* $2t \leq b'(v') + 1$*. Then*

$$\overline{\psi_2'}(C_4, K_{v'} - F) = \lfloor v - \frac{2x+2}{t} \rfloor_e.$$

*Proof.* By Theorem 4 fo $v = v'/2$ there exists a $(2t+1, 2t)$-equitable edge-coloring $E$ of $K_v$ such that $c_2(E) = \lfloor v - \frac{x+1}{t} \rfloor_e$. So as explained in [15] there exists a $(2t+1, 2t)$-equitable $C_4$-coloring $E'$ of $K_{v'} - F$ such that $c_2(E') = 2c_2(E) = 2\lfloor v - \frac{x+1}{t} \rfloor_e = \lfloor 2\left(v - \frac{x+1}{t}\right) \rfloor_e = \lfloor 2v - \frac{2x+2}{t} \rfloor_e = \lfloor v' - \frac{2x+2}{t} \rfloor_e$. Therefore by Lemma 2, $\overline{\psi_2'}(C_4, K_{v'} - F) = \lfloor v - \frac{2x+2}{t} \rfloor_e$. □

## 4 A New Proof of Theorems 1 and 2

By modifying the proof of Theorem 4 we obtain a new proof of Theorem 2, and as explained in the introduction, a new proof of Theorem 1 as well, using amalgamations.

*Proof.* Let $v = 4tx + 2t + 1$ for some integer $x$. Form $\mathcal{G}_0$ in the same way as in Theorem 4. Here color $m(i) = i$ will be missing from vertex $u_i$ for $1 \leq i \leq 2t-1$, color $m(i) = 2t$ will be missing from vertex $u_i$ for $2t \leq i \leq 2x + 2$, and color $m(\alpha_i) = 2t + 1$ will be missing from the remaining $v - 2x - 2$ vertices (which will be named $\alpha_1, \ldots, \alpha_{\eta(\alpha)}$ below).

Next form a new edge-colored graph $\mathcal{G}_0^+$ as in Theorem 4 and again the aim now is to complete the proof using Theorem 3 with $\eta(u_i) = 1$ for $1 \leq i \leq 2x+2$ and $\eta(\alpha) = v - 2x - 2$. For $1 \leq i \leq 2x+2$ join $u_i$ to $\alpha$ with $b(v)$ edges of each color $\{1, 2, \ldots, 2t\} \setminus \{m(i)\}$ as in Theorem 4; again the number of edges joining $u_i$ to $\alpha$ is $\eta(\alpha)$, and $d_{\mathcal{G}_0^+}(u_i) = v-1$. For $1 \leq i \leq 2t-1$ add $b(v)(\eta(v) - (2x+1))/2$ loops of color $i$ to $\alpha$; so $\alpha$ has degree $b(v)\eta(v)$ in color class $i$ (where loops contribute 2 to the degree of the incident vertex). Also add $b(v)(\eta(v) - (2t-1))/2$ loops of color $2t$ to $\alpha$; so $\alpha$ has degree $b(v)\eta(v)$ in color class $2t$ as well. Notice that the number of loops incident with $\alpha$ is

$$\begin{aligned}
l(\alpha) &= (2t-1)b(v)(\eta(\alpha) - (2x+1)/2) + b(v)(\eta(\alpha) - (2t-1))/2 \\
&= (2t(2x+1)\eta(\alpha) - (2x+1)(2t-1)(2x+2))/2 \\
&= (2x+1)(2t\eta(\alpha) - (4xt - 2x - 4t - 2))/2 \\
&= (2x+1)(2t\eta(\alpha) - (\eta(\alpha) + 2t - 1))/2 \\
&= (2x+1)(\eta(\alpha) - 1)(2t-1))/2 \\
&= \eta(\alpha)(\eta(\alpha) - 1)/2.
\end{aligned}$$

As in the proof of Theorem 4, Theorem 3 allows us to form $\mathcal{G}$ isomorphic to $K_v$ from $\mathcal{G}_0^+$ so that the edge-coloring $E$ of $\mathcal{G}$ is $(2t+1, 2t)$-equitable. Furthermore, in $\mathcal{G}$, color $2t+1$ appears at $b(v)+1$ vertices, color $2t$ appears at $v - 2t - 1$ vertices, and each other color appears at $v-1$ vertices. Since in [15] it is shown in this case that $\psi_i'(K_2, K_v) \geq b(v) + 1$ and that $\overline{\psi_i'}(K_2, K_v) \leq v - 1$ for $1 \leq i \leq 2t+1$, the proof is complete (after renaming the colors $1, 2, \ldots, 2t+1$ with $2t+1, 2t, \ldots, 1$ respectively). □

## References


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