Generalized Statistical Convergence for Sequences of Function in Random 2-Normed Spaces

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Abstract. In this paper, we introduce a new type of convergence for a sequence of function, namely, λ-statistically convergent sequences of functions in random 2-normed space, which is a natural generalization of convergence in random 2-normed space. In particular, following the line of recent work of Karakaya et al. [12], we introduce the concepts of uniform λ-statistical convergence and pointwise λ-statistical convergence in the topology induced by random 2-normed spaces. We define the λ-statistical analog of the Cauchy convergence criterion for pointwise and uniform λ-statistical convergence in a random 2-normed space and give some basic properties of these concepts. In addition, the preservation of continuity by pointwise and uniform λ-statistical convergence is proven.

Keywords: λ-statistical convergence · Random 2-normed space
The sequences of functions

1 Introduction and Preliminaries

Our aim is to propose some new variants of statistical convergence (and more general λ-statistical convergence) for sequences of functions in random 2-normed spaces. We put special attention on functions in random 2-normed spaces, in a sense extending original ideas of Balcerzak et al. [3] and Karakaya et al. [12].

The theory of probabilistic normed (PN) spaces is important area of research in functional analysis. Lots of work have been done by this theory and it has many important applications in real life situations. PN spaces are the vector spaces in which the norms of the vectors are uncertain due to randomness. A PN space is a generalization of an ordinary normed linear space. In a PN space, the norms of the vectors are represented by probability distribution functions instead of non-negative real numbers. If x is an element of a PN space, then its norm is denoted by F_x, and the value $F_x(t)$ is interpreted as the probability that the norm of x is smaller than t. The probabilistic metric space was introduced by Menger [13] which is an interesting and an important generalization of the notion

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of a metric space. The theory of probabilistic normed (or metric) space was initiated and developed in [1,19–21]; further it was extended to random/probabilistic 2-normed spaces by Golet [9] using the concept of 2-norm which is defined by Gähler (see [7]); and Gürdal and Pehlivan [10] studied statistical convergence in 2-Banach spaces.

In order to extend the notion of convergence of sequences, statistical convergence of sequences was introduced by Fast [5]. A lot of developments have been made in this areas after the work of Fridy [6]. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, summability theory and number theory. Recently, Mursaleen [14] studied λ-statistical convergence as a generalization of the statistical convergence, and in [15] he considered the concept of statistical convergence of sequences in random 2-normed spaces. Quite recently, Savaş and Mohiuddine [18] defined λ-statistical convergence for double sequences in probabilistic normed spaces, and also Savaş [16] studied generalized statistical convergence in random 2-normed space (also see [17]).

In another direction the idea of statistical convergence of sequences of real functions was studied in [3], and some important results and references on statistical convergence and function sequences can be found in [4,8]. Recently, in [12], Karakaya et al. studied the statistical convergence of sequences of functions with respect to the intuitionistic fuzzy normed spaces. Also in [11], Karakaya et al. introduced the concept of λ-statistical convergence of sequences of functions in the intuitionistic fuzzy normed spaces.

The notion of λ-statistical convergence of sequences of functions has not been studied previously in the setting of random 2-normed spaces. Motivated by this fact, in this paper, as a variant of statistical convergence, the notion of λ-statistical convergence of sequences of functions is introduced in a random 2-normed space. In Sect. 2, we prove some results concerning to convergence in pointwise λ-statistical convergence and uniform λ-statistical convergence of sequences of functions in a random 2-normed spaces. We demonstrate the λ-statistical analog of the Cauchy convergence criterion for pointwise and uniform λ-statistical convergence in a random 2-normed space and give some basic properties of these concepts. Finally, we prove that pointwise and uniform λ-statistical convergence preserves continuity.

First we recall some of the basic concepts, that will be used in this paper.

The notion of statistical convergence depends on the density of subsets of $\mathbb{N}$, the set of natural numbers. Let K be a subset of $\mathbb{N}$. Then the asymptotic density of the set K denoted by $\delta(K)$ is defined as

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : k \in K\}\right|,$$

where the vertical bars denote the cardinality of the enclosed set. A number sequence $x = (x_k)_{k\in\mathbb{N}}$ is said to be statistically convergent to a point L if for every $\varepsilon > 0$, the set $K(\varepsilon) = \{k \le n : |x_k - L| \ge \varepsilon\}$ has asymptotic density zero, i.e.,

$$\lim_{n\to\infty}\frac{1}{n}\left|\{k\leq n:|x_k-L|\geq\varepsilon\}\right|=0.$$

In this case we write S-$\lim x=L$ or $x_k\to L\,(S)$ (see [5,6]).

The following definitions are due to Mursaleen [14].

Definition 1. *Let K be a subset of $\mathbb{N}$ and $\lambda=(\lambda_n)$ be a non-decreasing sequences of positive real numbers tending to ∞ and such that*

$$\lambda_{n+1}\leq\lambda_n+1,\ \ \lambda_1=0.$$

Let K be a subset of $\mathbb{N}$, the set of natural numbers. The number

$$\delta_\lambda(K)=\lim_n\frac{1}{\lambda_n}\left|\{k\in K:n-\lambda_n+1\leq k\leq n\}\right|,$$

is said to be the λ-density of K.

Definition 2. *A sequence $x=(x_k)$ in X is said to be λ-statistically convergent to $L\in X$ and is denoted by S_λ-$\lim x=L$, if, for every $\varepsilon>0$, the set $K(\varepsilon)$ has λ-density zero, i.e.,*

$$\lim_n\frac{1}{\lambda_n}\left|K_n(\varepsilon)\right|=0,$$

where $K_n(\varepsilon)=\{k\in I_n:|x_k-L|\geq\varepsilon\}$ and $I_n=[n-\lambda_n+1,n]$.

Definition 3 ([7]). *Let X be a real vector space of dimension d, where $2\leq d<\infty$. A 2-norm on X is a function $\|\cdot,\cdot\|:X\times X\to\mathbb{R}$ which satisfies* (**i**) $\|x,y\|=0$ *if and only if x and y are linearly dependent;* (**ii**) $\|x,y\|=\|y,x\|$; (**iii**) $\|\alpha x,y\|=|\alpha|\,\|x,y\|$, $\alpha\in\mathbb{R}$; (**iv**) $\|x,y+z\|\leq\|x,y\|+\|x,z\|$. *The pair $(X,\|\cdot,\cdot\|)$ is then called a 2-normed space.*

As an example of a 2-normed space we may take $X=\mathbb{R}^2$ being equipped with the 2-norm $\|x,y\|:=$ the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x,y\|=|x_1y_2-x_2y_1|,\quad x=(x_1,x_2),\ \ y=(y_1,y_2).$$

All the concepts listed below are studied in depth in the fundamental book by Schweizer and Sklar [19].

Definition 4. *Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+=\{x\in\mathbb{R}:x\geq0\}$ and $S=[0,1]$ the closed unit interval. A mapping $f:\mathbb{R}\to S$ is called a distribution function if, it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}}f(t)=0$ and $\sup_{t\in\mathbb{R}}f(t)=1$.*

We denote the set of all distribution functions by D^+ such that $f(0)=0$. If $a\in\mathbb{R}_+$, then $H_a\in D^+$, where

$$H_a(t)=\begin{cases}1 \text{ if } t>a,\\ 0 \text{ if } t\leq a.\end{cases}$$

It is obvious that $H_0\geq f$ for all $f\in D^+$.

Definition 5. *A triangular norm (t-norm) is a continuous mapping* $* : S \times S \to S$ *be such that* $(S, *)$ *is an abelian monoid with unit one and* $c * d \leq a * b$ *if* $c \leq a$ *and* $d \leq b$ *for all* $a, b, c, d \in S$. *A triangle function* τ *is a binary operation on* D^+ *which is commutative, associative and* $\tau(f, H_0) = f$ *for every* $f \in D^+$.

Definition 6. *Let* X *be a linear space of dimension greater than one,* τ *be a triangle function, and* $F : X \times X \to D^+$. *Then* F *is called a probabilistic* 2*-norm and* (X, F, τ) *a probabilistic* 2*-normed space if the following conditions are satisfied:*

(i) $F(x, y; t) = H_0(t)$ *if* x *and* y *are linearly dependent, where* $F(x, y; t)$ *denotes the value of* $F(x, y)$ *at* $t \in \mathbb{R}$,
(ii) $F(x, y; t) \neq H_0(t)$ *if* x *and* y *are linearly independent,*
(iii) $F(x, y; t) = F(y, x; t)$ *for all* $x, y \in X$,
(iv) $F(\alpha x, y; t) = F(x, y; \frac{t}{|\alpha|})$ *for every* $t > 0$, $\alpha \neq 0$ *and* $x, y \in X$,
(v) $F(x + y, z; t) \geq \tau(F(x, z; t), F(y, z; t))$ *whenever* $x, y, z \in X$, *and* $t > 0$. *If* (v) *is replaced by*
(vi) $F(x+y, z; t_1+t_2) \geq F(x, z; t_1) * F(y, z; t_2)$ *for all* $x, y, z \in X$ *and* $t_1, t_2 \in \mathbb{R}_+$; *then* $(X, F, *)$ *is called a random* 2*-normed space (for short, RTNS).*

We provide the following example.

Example 1. Let $(X, \|.,.\|)$ be a 2-normed space, and let $a * b = ab$ for all $a, b \in S$. For all $x, y \in X$ and every $t > 0$, consider

$$F(x, y; t) = \frac{t}{t + \|x, y\|}.$$

Clearly $(X, F, *)$ is a random 2-normed space.

Let $(X, F, *)$ be a RTN space. Since $*$ is a continuous t-norm, the system of (ε, η)-neighbourhoods of θ (the null vector in X)

$$\left\{\mathcal{N}_{(\theta,z)}(\varepsilon, \eta) : \varepsilon > 0,\ \eta \in (0, 1),\ z \in X\right\},$$

where

$$\mathcal{N}_{(\theta,z)}(\varepsilon, \eta) = \left\{(x, z) \in X \times X : F_{(x,z)}(\varepsilon) > 1 - \eta\right\}.$$

determines a first countable Hausdorff topology on $X \times X$, called the F-topology. Thus, the F-topology can be completely specified by means of F-convergence of sequences. It is clear that $x - y \in \mathcal{N}_{(\theta,z)}$ means $y \in \mathcal{N}_{(x,z)}$ and vice-versa.

A sequence $x = (x_k)$ in X is said to be F-convergent to $L \in X$ if for every $\varepsilon > 0$, $\eta \in (0, 1)$ and for each non-zero $z \in X$ there exists a positive integer N such that;

$$(x_k, z - L) \in \mathcal{N}_{(\theta,z)}(\varepsilon, \eta) \text{ for each } k \geq N$$

or equivalently,

$$(x_k, z) \in \mathcal{N}_{(L,z)}(\varepsilon, \eta) \text{ for each } k \geq N.$$

In this case we write $F\text{-}\lim (x_k, z) = L$.

We also recall that the concept of convergence and Cauchy sequence in a random 2-normed space is studied in [2].

Definition 7. *Let* $(X, F, *)$ *be a RN space. Then, a sequence* $x = \{x_k\}$ *is said to be convergent to* $L \in X$ *with respect to the random norm* F *if, for every* $\varepsilon > 0$ *and* $\eta \in (0,1)$, *there exists* $k_0 \in \mathbb{N}$ *such that* $F_{(x_k - L, z)}(\varepsilon) > 1 - \eta$ *whenever* $k \geq k_0$. *It is denoted by* $F\text{-}\lim x = L$ *or* $x_k \rightarrow_F L$ *as* $k \rightarrow \infty$.

Definition 8. *Let* $(X, F, *)$ *be a RN space. Then, a sequence* $x = \{x_k\}$ *is called a Cauchy sequence with respect to the random norm* F *if, for every* $\varepsilon > 0$ *and* $\eta \in (0,1)$, *there exists* $k_0 \in \mathbb{N}$ *such that* $F_{(x_k - x_m, z)}(\varepsilon) > 1 - \eta$ *for all* $k, m \geq k_0$.

Definition 9. *Let* $(X, F, *)$ *be a RN space. Then, a sequence* $x = \{x_k\}$ *is said to be* λ*-statistically convergent to* $L \in X$ *with respect to the* F*-topology if for every* $\varepsilon > 0$, $\eta \in (0,1)$ *and each non-zero* $z \in X$ *such that;*

$$\delta_\lambda \left(\left\{ k \in \mathbb{N} : F_{(x_k - L, z)}(\varepsilon) \leq 1 - \eta \right\} \right) = 0$$

or equivalently

$$\delta_\lambda \left(\left\{ k \in \mathbb{N} : F_{(x_k - L, z)}(\varepsilon) > 1 - \eta \right\} \right) = 1.$$

In this case we write $S_\lambda^{R2N}\text{-}\lim x = L$ *or* $x_k \rightarrow L \left(S_\lambda^{R2N} \right)$.

If $\lambda_n = n$ for every n then every λ-statistically convergent sequences in random 2-normed space $(X, F, *)$ reduce to statistically convergent sequences in random 2-normed space $(X, F, *)$.

Definition 10. *Let* $(X, F, *)$ *be a RN space. Then, a sequence* $x = \{x_k\}$ *is said to be* λ*-statistical Cauchy to* $L \in X$ *with respect to the* F*-topology if, for every* $\varepsilon > 0$, $\eta \in (0,1)$ *and each non-zero* $z \in X$ *there exists a positive integer* $N = N(\varepsilon)$ *such that*

$$\delta_\lambda \left(\left\{ k \in \mathbb{N} : F_{(x_k - x_N, z)}(\varepsilon) \leq 1 - \eta \right\} \right) = 0$$

or equivalently

$$\delta_\lambda \left(\left\{ k \in \mathbb{N} : F_{(x_k - x_N, z)}(\varepsilon) > 1 - \eta \right\} \right) = 1.$$

In this case we write $S_\lambda^{R2N}\text{-}\lim x = L$ *or* $x_k \rightarrow L \left(S_\lambda^{R2N} \right)$.

2 Kinds of λ-Statistical Convergence for Functions in RTNS

In this section we are concerned with convergence in pointwise λ-statistical convergence and uniform λ-statistical convergence of sequences of functions in a random 2-normed spaces. Particularly, we introduce the λ-statistical analog of the Cauchy convergence criterion for pointwise and uniform λ-statistical convergence in a random 2-normed space. Finally, we prove that pointwise and uniform λ-statistical convergence preserves continuity.

2.1 Pointwise λ-Statistical Convergence in RTNS

Fix a random 2-normed space $(Y, F', *)$. Assume that $(X, F, *)$ is a RTN space and that $\mathcal{N}'_{(\theta,z)}(\varepsilon,\eta) = \left\{x, z \in X \times X : F'_{(x,z)}(\varepsilon) > 1-\eta\right\}$, called the F'-topology, is given.

Let $f_k : (X, F, *) \to (Y, F', *)$, $k \in \mathbb{N}$, be a sequence of functions. A sequences of functions $(f_k)_{k\in\mathbb{N}}$ (on X) is said to be F-convergence to f (on X) if for every $\varepsilon > 0$, $\eta \in (0,1)$ and for each non-zero $z \in X$, there exists a positive integer $N = N(\varepsilon, \eta, x)$ such that

$$(f_k(x) - f(x), z) \in \mathcal{N}'_{\theta,z}(\varepsilon,\eta) = \left\{(x,z) \in X \times X : F'_{((f_k(x)-f(x)),z)}(\varepsilon) > 1-\eta\right\}$$

for each $k \geq N$ and for each $x \in X$ or equivalently,

$$(f_k(x), z) \in \mathcal{N}'_{(f(x),z)}(\varepsilon,\eta) \text{ for each } k \geq N \text{ and for each } x \in X.$$

In this case we write $f_k \to_{F^2} f$.

First let us define pointwise λ-statistical convergence in a random 2-normed space.

Definition 11. *Let $f_k : (X, F, *) \to (Y, F', *)$, $k \in \mathbb{N}$, be a sequence of functions. $(f_k)_{k\in\mathbb{N}}$ is said to be pointwise λ-statistical convergence to a function f (on X) with respect to F-topology if, for every $x \in X$, $\varepsilon > 0$, $\eta \in (0,1)$ and each non-zero $z \in X$ the set*

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : (f_k(x), z) \notin \mathcal{N}'_{(f(x),z)}(\varepsilon,\eta)\right\}\right) = 0,$$

or equivalently

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : (f_k(x), z) \in \mathcal{N}'_{(f(x),z)}(\varepsilon,\eta)\right\}\right) = 1.$$

In this case we write $f_k \to f\left(S_\lambda^{RTN}\right)$.

Theorem 1. *Let $(X, F, *)$, $(Y, F', *)$ be RTN spaces. Assume that $(f_k)_{k\in\mathbb{N}}$ is pointwise convergent (on X) with respect to F-topology where $f_k : (X, F, *) \to (Y, F', *)$, $k \in \mathbb{N}$. Then $f_k \to f\left(S_\lambda^{RTN}\right)$ (on X). However the converse of this is not true.*

Proof. Let $\varepsilon > 0$ and $\eta \in (0,1)$. Suppose $(f_k)_{k\in\mathbb{N}}$ is F-convergent on X. In this case the sequence $(f_k(x))$ is convergent with respect to F'-topology for each $x \in X$. Then, there exists a number $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that $(f_k(x), z) \in \mathcal{N}'_{(f(x),z)}(\varepsilon,\eta)$ for every $k \geq k_0$, every non-zero $z \in X$ and for each $x \in X$. This implies that the set

$$A(\varepsilon,\eta) = \left\{k \in \mathbb{N} : (f_k(x), z) \notin \mathcal{N}'_{f(x),z}(\varepsilon,\eta)\right\} \subseteq \{1, 2, 3, ..., k_0 - 1\}.$$

Since finite subset of $\mathbb{N}$ has λ-density 0, we have $\delta_\lambda(A(\varepsilon,\eta)) = 0$. That is, $f_k \to f\left(S_\lambda^{RTN}\right)$ (on X).

Example 2. Considering X as in Example 1, we have $(X, F, *)$ as a RTN space induced by the random 2-norm $F_{(x,y)}(\varepsilon) = \frac{\varepsilon}{\varepsilon + \|x,y\|}$. Define a sequence of functions $f_k : [0,1] \to \mathbb{R}$ via

$$f_k(x) = \begin{cases} x^k + 1 & \text{if } n - \sqrt{\lambda_n} + 1 \le k \le n \text{ and } x \in [0, \frac{1}{2}) \\ 0 & \text{if otherwise and } x \in [0, \frac{1}{2}) \\ x^k + \frac{1}{2} & \text{if } n - \sqrt{\lambda_n} + 1 \le k \le n \text{ and } x \in [\frac{1}{2}, 1) \\ 1 & \text{if otherwise and } x \in [\frac{1}{2}, 1) \\ 2 & \text{if } x = 1. \end{cases}$$

Then, for every $\varepsilon > 0$, $\eta \in (0,1)$, $x \in [0, \frac{1}{2})$ and each non-zero $z \in X$, let $A_n(\varepsilon, \eta) = \left\{ k \in \mathbb{N} : (f_k(x), z) \notin \mathcal{N}'_{(f(x),z)}(\varepsilon, \lambda) \right\}$. We observe that;

$$\begin{aligned} A_n(\varepsilon, \eta) &= \left\{ k \in I_n : \frac{\varepsilon}{\varepsilon + \|f_k(x), z\|} \le 1 - \eta) \right\} \\ &= \left\{ k \in I_n : \|f_k(x), z\| \ge \frac{\varepsilon\eta}{1-\varepsilon} \right\} \\ &= \left\{ k \in I_n : f_k(x) = x^k + 1 \right\}. \end{aligned}$$

and $|A_n(\varepsilon, \lambda)| \le \sqrt{\lambda_n}$. Thus, for each $x \in [0, \frac{1}{2})$, since

$$\delta_\lambda(A_n(\varepsilon, \eta)) = \lim_{n\to\infty} \frac{|A_n(\varepsilon, \eta)|}{\lambda_n} = \lim_{n\to\infty} \frac{\sqrt{\lambda_n}}{\lambda_n} = 0$$

$(f_k)_{k\in\mathbb{N}}$ is λ-statistically convergent to 0 with respect to F-topology. Similarly, if we take $x \in [\frac{1}{2}, 1)$ and $x = 1$, it can be easily seen that $(f_k)_{k\in\mathbb{N}}$ is λ-statistical convergence to $\frac{1}{2}$ and 2 with respect to F-topology, respectively. Hence $(f_k)_{k\in\mathbb{N}}$ is pointwise λ-statistical convergent with respect to F-topology (on X).

Theorem 2. *Let $(X, F, *)$, $(Y, F', *)$ be RTN spaces and let $f_k : (X, F, *) \to (Y, F', *)$, $k \in \mathbb{N}$, be a sequence of functions. Then the following statements are equivalent:*

(i) $f_k \to f\left(S_\lambda^{RTN}\right)$.
(ii) $\delta_\lambda\left(\left\{k \in \mathbb{N} : (f_k(x), z) \notin \mathcal{N}'_{(f(x),z)}(\varepsilon, \eta)\right\}\right) = 0$ *for every* $\varepsilon > 0$, $\eta \in (0,1)$, *for each* $x \in X$ *and each non-zero* $z \in X$.
(iii) $\delta_\lambda\left(\left\{k \in \mathbb{N} : (f_k(x), z) \in \mathcal{N}'_{(f(x),z)}(\varepsilon, \eta)\right\}\right) = 1$ *for every* $\varepsilon > 0$, $\eta \in (0,1)$, *for each* $x \in X$ *and each non-zero* $z \in X$.
(iv) $S_\lambda\text{-}\lim F'_{(f_k(x)-f(x),z)}(\varepsilon) = 1$ *for every* $x \in X$ *and each non-zero* $z \in X$.

Proof is standard.

Theorem 3. *Let $(f_k)_{k\in\mathbb{N}}$ and $(g_k)_{k\in\mathbb{N}}$ be two sequences of functions from $(X, F, *)$ to $(Y, F', *)$ with $a * a > a$ for every $a \in (0,1)$. If $f_k \to f\left(S_\lambda^{RTN}\right)$ and $g_k \to g\left(S_\lambda^{RTN}\right)$, then $(\alpha f_k + \beta g_k) \to (\alpha f + \beta g)\left(S_\lambda^{RTN}\right)$ where $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$).*

Proof. Let $\varepsilon > 0$ and $\eta \in (0,1)$. Since $f_k \to f\left(S_\lambda^{RTN}\right)$ and $g_k \to g\left(S_\lambda^{RTN}\right)$ for each $x \in X$, if we define

$$A_1 = \left\{k \in \mathbb{N} : (f_k(x), z) \notin \mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{2}, \eta)\right\} \text{ and } A_2 = \left\{k \in \mathbb{N} : (g_k(x), z) \notin \mathcal{N}'_{(g(x),z)}(\frac{\varepsilon}{2}, \eta)\right\}$$

then $\delta_\lambda(A_1) = 0$ and $\delta_\lambda(A_2) = 0$. Since $\delta_\lambda(A_1) = 0$ and $\delta_\lambda(A_2) = 0$, if we represent A by $(A_1 \cup A_2)$ then $\delta_\lambda(A) = 0$. Hence $A_1 \cup A_2 \neq \mathbb{N}$ and there exists $\exists k_0 \in \mathbb{N}$ such that;

$$(f_{k_0}(x), z) \in \mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{2}, \eta) \text{ and } (g_{k_0}(x), z) \in \mathcal{N}'_{(g(x),z)}(\frac{\varepsilon}{2}, \eta)$$

Let

$$B = \left\{k \in \mathbb{N} : ((\alpha f_k(x) + \beta g_k(x)), z) \notin \mathcal{N}'_{((\alpha f(x)+\beta g(x)),z)}(\varepsilon, \eta)\right\}.$$

We shall show that $A^c \subset B$ for each $x \in X$. Let $k_0 \in A^c$. In this case,

$$(f_{k_0}(x), z) \in \mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{2}, \eta) \text{ and } (g_{k_0}(x), z) \in \mathcal{N}'_{(g(x),z)}(\frac{\varepsilon}{2}, \eta).$$

From the above expressions, we have

$$\begin{aligned} F'_{((\alpha f_k(x)+\beta g_k(x)-\alpha f(x)+\beta g(x)),z)}(\varepsilon) &\geq F'_{((\alpha f_k(x)-\alpha f(x)),z)}\left(\frac{\varepsilon}{2}\right) * F'_{((\beta g_k(x)-\beta g(x)),z)}\left(\frac{\varepsilon}{2}\right) \\ &= F'_{((f_k(x)-f(x)),z)}\left(\frac{\varepsilon}{2\alpha}\right) * F'_{((g_k(x)-g(x)),z)}\left(\frac{\varepsilon}{2\beta}\right) \\ &> (1-\eta) * (1-\eta) \\ &> 1-\eta. \end{aligned}$$

This implies $A^c \subset B$. Since $B^c \subset A$ and $\delta_\lambda(A) = 0$, hence $\delta_\lambda(B^c) = 0$. That is

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : ((\alpha f_k(x) + \beta g_k(x)), z) \notin \mathcal{N}'_{((\alpha f(x)+\beta g(x)),z)}(\varepsilon, \eta)\right\}\right) = 0.$$

Definition 12. *Let $(X, F, *)$, $(Y, F', *)$ be RTN spaces and let $f_k : (X, F, *) \to (Y, F', *)$, $k \in \mathbb{N}$, be a sequence of functions. A sequence $(f_k)_{k\in\mathbb{N}}$ is called pointwise λ-statistical Cauchy sequence in RTN space if, for every $\varepsilon > 0$, $\eta \in (0,1)$ and each non-zero $z \in X$ there exists $M = M(\varepsilon, \eta, x) \in \mathbb{N}$ such that;*

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : (f_k(x) - f_M(x), z) \notin \mathcal{N}'_{(\theta,z)}(\varepsilon, \eta)\right\}\right) = 0.$$

Theorem 4. *Let $(X, F, *)$, $(Y, F', *)$ be RTN spaces such that $a * a > a$ for every $a \in (0,1)$ and let $f_k : (X, F, *) \to (Y, F', *)$, $k \in \mathbb{N}$, be a sequence of functions. If $(f_k)_{k\in\mathbb{N}}$ is a pointwise λ-statistical convergent sequence with respect to F-topology, then $(f_k)_{k\in\mathbb{N}}$ is a pointwise λ-statistical Cauchy sequence with respect to F-topology. However the converse of this is not true.*

Proof. Suppose that $(f_k)_{k\in\mathbb{N}}$ is a pointwise λ-statistical convergent to f with respect to F-topology. Let $\varepsilon > 0$ and $\eta \in (0,1)$ be given. If we state A and A^c by

$$A = \left\{k \in \mathbb{N} : (f_k(x), z) \notin \mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{2}, \eta)\right\} \text{ and } A^c = \left\{k \in \mathbb{N} : (f_k(x), z) \in \mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{2}, \eta)\right\},$$

then $\delta_\lambda(A) = 0$ and $\delta_\lambda(A^c) = 1$. Now, for every $k, m \in A^c$,

$$\begin{aligned} F'_{(f_k(x)-f_m(x),z)}(\varepsilon) &\geq F'_{(f_k(x)-f(x),z)}\left(\frac{\varepsilon}{2}\right) * F'_{(f_m(x)-f(x),z)}\left(\frac{\varepsilon}{2}\right) \\ &> (1-\eta) * (1-\eta) \\ &> 1-\eta. \end{aligned}$$

So, $\delta_\lambda\left(\left\{k \in \mathbb{N} : (f_k(x) - f_m(x), z) \in \mathcal{N}'_{(\theta,z)}(\varepsilon, \eta)\right\}\right) = 1$. Therefore

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : (f_k(x) - f_m(x), z) \notin \mathcal{N}'_{(\theta,z)}(\varepsilon, \eta)\right\}\right) = 0,$$

i.e., $(f_k)_{k\in\mathbb{N}}$ is a pointwise λ-statistical Cauchy sequence with respect to F-topology.

The next result is a modification of a well-known result.

Theorem 5. *Let $(X, F, *)$, $(Y, F', *)$ be a RTN spaces such that $a * a > a$ for every $a \in (0,1)$. Assume that $f_k \to f\left(S_\lambda^{RTN}\right)$ (on X) where functions $f_k : (X, F, *) \to (Y, F', *)$, $k \in \mathbb{N}$, are equi-continuous (on X) and $f : (X, F, *) \to (Y, F', *)$. Then f is continuous (on X) with respect to F-topology.*

Proof. We prove that f is continuous with respect to F-topology. Let $x_0 \in X$ and $(x - x_0, z) \in \mathcal{N}_{\theta,z}(\varepsilon, \eta)$ be fixed. By the equi-continuity of f_k's, for every $\varepsilon > 0$ and each non-zero $z \in X$, there exists a $\gamma \in (0,1)$ with $\gamma < \eta$ such that $(f_k(x) - f_k(x_0), z) \in \mathcal{N}'_{(\theta,z)}(\frac{\varepsilon}{3}, \gamma)$ for every $k \in \mathbb{N}$. Since $f_k \to f\left(S_\lambda^{RTN}\right)$, if we state respectively A and B by the sets $A = \left\{k \in \mathbb{N} : (f_k(x_0), z) \notin \mathcal{N}'_{(f(x_0),z)}(\frac{\varepsilon}{3}, \gamma)\right\}$ and $B = \left\{k \in \mathbb{N} : (f_k(x), z) \notin \mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{3}, \gamma)\right\}$, then $\delta_\lambda(A) = 0$ and $\delta_\lambda(B) = 0$. Therefore, $\delta_\lambda(A \cup B) = 0$ and $A \cup B$ is different from $\mathbb{N}$. So, there exists $k \in \mathbb{N}$ such that $(f_k(x_0), z) \in \mathcal{N}'_{(f(x_0),z)}(\frac{\varepsilon}{3}, \gamma)$ and $(f_k(x), z) \in \mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{3}, \gamma)$. We have

$$\begin{aligned} F'_{(f(x_0)-f(x),z)}(\varepsilon) &\geq F'_{(f(x_0)-f_k(x_0),z)}\left(\frac{\varepsilon}{3}\right) * \left[F'_{(f_k(x_0)-f_k(x),z)}\left(\frac{\varepsilon}{3}\right) * F'_{(f_k(x)-f(x),z)}\left(\frac{\varepsilon}{3}\right)\right] \\ &> (1-\gamma) * [(1-\gamma) * (1-\gamma)] \\ &> (1-\gamma) * (1-\gamma) \\ &> 1-\gamma \\ &> 1-\eta \end{aligned}$$

and the continuity of f with respect to F-topology is proved.

2.2 Uniformly λ-Statistical Convergence in RTNS

Let us define uniform λ-statistical convergence in a random 2-normed space.

Definition 13. *Let $(X,F,*)$, $(Y,F',*)$ be RTN spaces. We say that a sequence of functions $f_k:(X,F,*)\to(Y,F',*)$, $k\in\mathbb{N}$, is uniform λ-statistically convergent to a function f (on X) with respect to F-topology if and only if $\forall\varepsilon>0$, $\exists M\subset\mathbb{N}$, $\delta_\lambda(M)=1$, $\exists k_0=k_0(\varepsilon,\eta,x)\in M\ni\forall k>k_0$, $k\in M$, $\forall z\in X$ and $\forall x\in X,\eta\in(0,1)$ $(f_k(x),z)\in\mathcal{N}'_{(f(x),z)}(\varepsilon,\eta)$.*

In this case we write $f_k\rightrightarrows f\left(S_\lambda^{RTN}\right)$.

We state the following result without proof, which can be established using standard technique.

Theorem 6. *Let $(X,F,*)$, $(Y,F',*)$ be RTN spaces and let $f_k:(X,F,*)\to(Y,F',*)$, $k\in\mathbb{N}$, be a sequence of functions. Then for every $\varepsilon>0$ and $\eta\in(0,1)$, the following statements are equivalent:*

(i) $f_k\rightrightarrows f\left(S_\lambda^{RTN}\right)$.

(ii) $\delta_\lambda\left(\left\{k\in\mathbb{N}:(f_k(x),z)\notin\mathcal{N}'_{(f(x),z)}(\varepsilon,\eta)\right\}\right)=0$ *for every* $x\in X$ *and each non-zero* $z\in X$.

(iii) $\delta_\lambda\left(\left\{k\in\mathbb{N}:(f_k(x),z)\in\mathcal{N}'_{(f(x),z)}(\varepsilon,\eta)\right\}\right)=1$ *for every* $x\in X$ *and each non-zero* $z\in X$.

(iv) S_λ-$\lim F'_{(f_k(x)-f(x),z)}(\varepsilon)=1$ *for every* $x\in X$ *and each non-zero* $z\in X$.

Definition 14. *Let $(X,F,*)$ be a RTN space. A subset Y of X is said to be bounded on RTN spaces if for every $\eta\in(0,1)$ there exists $\varepsilon>0$ such that $(x,z)\in\mathcal{N}_{(\theta,z)}(\varepsilon,\eta)$ for all $x\in Y$ and every non-zero $z\in X$.*

Definition 15. *Let $(X,F,*)$, $(Y,F',*)$ be RTN spaces and let $f_k:(X,F,*)\to(Y,F',*)$, $k\in\mathbb{N}$, and $f:(X,F,*)\to(Y,F',*)$ be bounded functions. Then $f_k\rightrightarrows f\left(S_\lambda^{RTN}\right)$ if and only if S_λ-$\lim\left(\inf_{x\in X}F'_{(f_k(x)-f(x),z)}(\varepsilon)\right)=1$.*

Example 3. Let $(X,F,*)$ be as considered in Example 1. Define a sequence of functions $f_k:[0,1)\to\mathbb{R}$ via

$$f_k(x)=\begin{cases}x^k+1 \text{ if } n-\sqrt{\lambda_n}+1\le k\le n\\ 2 \quad\text{otherwise.}\end{cases}$$

Then, for every $\varepsilon>0$, $\eta\in(0,1)$ and each non-zero $z\in X$, let $A_n(\varepsilon,\eta)=\left\{k\in I_n:(f_k(x),z)\notin\mathcal{N}'_{(1,z)}(\varepsilon,\lambda)\right\}$. For all $x\in X$, we have $\delta_\lambda(A_n(\varepsilon,\lambda))=0$. Since $f_k\to1\left(S_\lambda^{RTN}\right)$ for all $x\in X$, $f_k\rightrightarrows1\left(S_\lambda^{RTN}\right)$ (on $[0,1)$).

Remark 1. If $f_k\rightrightarrows f\left(S_\lambda^{RTN}\right)$, then $f_k\to f\left(S_\lambda^{RTN}\right)$. But not necessarily conversely.

We establish the above remark providing the following example.

Example 4. Define the sequence of functions

$$f_k(x)=\begin{cases}0 & \text{if } n-\sqrt{\lambda_n}+1\le k\le n\\ \frac{k^2x}{1+k^3x^2} & \text{otherwise}\end{cases}$$

on $[0,1]$. Since $f_k\left(\frac{1}{k}\right) \to 1\left(S_{\lambda}^{RTN}\right)$ and $f_k(0) \to 0\left(S_{\lambda}^{RTN}\right)$, this sequence of functions is pointwise λ-statistically convergence to 0 with respect to F-topology. But by Definition 11, it is not uniform λ-statistical convergence with respect to F-topology.

Theorem 7. *Let $(X,F,*)$, $(Y,F',*)$ be RTN spaces. Assume that $(f_k)_{k\in\mathbb{N}}$ is uniformly convergent (on X) with respect to F-topology where $f_k:(X,F,*)\to(Y,F',*)$, $k\in\mathbb{N}$. Then $f_k \rightrightarrows f\left(S_{\lambda}^{RTN}\right)$ (on X). However the converse of this is not true.*

Proof. Assume that $(f_k)_{k\in\mathbb{N}}$ is uniformly convergent to f on X with respect to F-topology. In this case, for every $\varepsilon>0$, $\eta\in(0,1)$ and every non-zero $z\in X$, there exists a positive integer $k_0=k_0(\varepsilon,\eta)$ such that $\forall x\in X$ and $\forall k>k_0$, $(f_k(x),z)\in\mathcal{N}'_{(f(x),z)}(\varepsilon,\eta)$. That is, for $k\le k_0$

$$A(\varepsilon,\eta)=\left\{k\in\mathbb{N}:(f_k(x),z)\notin\mathcal{N}'_{(f(x),z)}(\varepsilon,\eta)\right\}\subseteq\{1,2,3,...,k_0\}.$$

Since finite subset of $\mathbb{N}$ has λ-density 0, we have $\delta_{\lambda}(A(\varepsilon,\eta))=0$. That is, $f_k \rightrightarrows f\left(S_{\lambda}^{RTN}\right)$ (on X).

Definition 16. *Let $(X,F,*)$, $(Y,F',*)$ be RTN spaces and let $f_k:(X,F,*)\to(Y,F',*)$, $k\in\mathbb{N}$, be a sequence of functions. Then a sequence $(f_k)_{k\in\mathbb{N}}$ is called uniform λ-statistical Cauchy sequence in RTN space if for every $\varepsilon>0$, $\eta\in(0,1)$ and each non-zero $z\in X$ there exists $N=N(\varepsilon,\eta)\in\mathbb{N}$ such that*

$$\delta_{\lambda}\left(\left\{k\in\mathbb{N}:(f_k(x)-f_N(x),z)\notin\mathcal{N}'_{(\theta,z)}(\varepsilon,\eta)\right\}\right)=0.$$

Theorem 8. *Let $(X,F,*)$, $(Y,F',*)$ be RTN spaces such that $a*a>a$ for every $a\in(0,1)$ and let $f_k:(X,F,*)\to(Y,F',*)$, $k\in\mathbb{N}$, be a sequence of functions. If $(f_k)_{k\in\mathbb{N}}$ is a uniform λ-statistical convergence sequence with respect to F-topology, then $(f_k)_{k\in\mathbb{N}}$ is a uniform λ-statistical Cauchy sequence with respect to F-topology. However the converse of this is not true.*

Proof. Suppose that $f_k \rightrightarrows f\left(S_{\lambda}^{RTN}\right)$. Let $A=\Big\{k\in\mathbb{N}:(f_k(x),z)\in\mathcal{N}'_{f(x),z}$ $(\varepsilon,\eta)\Big\}$. By Definition 9, for every $\varepsilon>0$, $\eta\in(0,1)$ and each non-zero $z\in X$, there exists $A\subset\mathbb{N}$, $\delta_{\lambda}(A)=0$ and $\exists k_0=k_0(\varepsilon,\eta)\in A$ such that $\forall k>k_0$, $k\in A$ and $\forall x\in X$, $(f_k(x),z)\in\mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{2},\eta)$. Choose $N=N(\varepsilon,\eta)\in A$, $N>k_0$. So, $(f_N(x),z)\in\mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{2},\eta)$. For every $k\in A$, we have

$$\begin{aligned}F'_{(f_k(x)-f_N(x),z)}(\varepsilon)&\ge F'_{(f_k(x)-f(x),z)}\left(\frac{\varepsilon}{2}\right)*F'_{(f(x)-f_N(x),z)}\left(\frac{\varepsilon}{2}\right)\\&>(1-\eta)*(1-\eta)\\&>1-\eta.\end{aligned}$$

Hence, $\delta_\lambda\left(\left\{k\in\mathbb{N}:(f_k(x)-f_N(x),z)\in\mathcal{N}'_{(\theta,z)}(\varepsilon,\eta)\right\}\right)=1$. Therefore

$$\delta_\lambda\left(\left\{k\in\mathbb{N}:(f_k(x)-f_N(x),z)\notin\mathcal{N}'_{(\theta,z)}(\varepsilon,\eta)\right\}\right)=0,$$

i.e., (f_k) is an uniformly λ-statistical Cauchy sequence in RTN space.

The next result is a modification of a well-known result.

Theorem 9. *Let $(X,F,*)$, $(Y,F',*)$ be RTN spaces such that $a*a>a$ for every $a\in(0,1)$ and the map $f_k:(X,F,*)\to(Y,F',*)$, $k\in\mathbb{N}$, be continuous (on X) with respect to F-topology. If $f_k\rightrightarrows f\left(S_\lambda^{RTN}\right)$ (on X) then $f:(X,F,*)\to(Y,F',*)$ is continuous (on X) with respect to F-topology. However the converse of this is not true.*

Proof. Let $x_0\in X$ and $(x_0-x,z)\in\mathcal{N}_{(\theta,z)}(\varepsilon,\eta)$ be fixed. By F-continuity of f_k's, for every $\varepsilon>0$ and each non-zero $z\in X$, there exists a $\gamma\in(0,1)$ with $\gamma<\eta$ such that $(f_k(x_0)-f_k(x),z)\in\mathcal{N}'_{(\theta,z)}(\frac{\varepsilon}{3},\gamma)$ for every $k\in\mathbb{N}$. Since $f_k\rightrightarrows f\left(S_\lambda^{RTN}\right)$, for all $x\in X$, if we state respectively $A(\varepsilon,\eta)$ and $B(\varepsilon,\eta)$ by the sets $A=\left\{k\in\mathbb{N}:(f_k(x_0),z)\notin\mathcal{N}'_{(f(x_0),z)}(\frac{\varepsilon}{3},\gamma)\right\}$ and $B=\left\{k\in\mathbb{N}:(f_k(x),z)\notin\mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{3},\gamma)\right\}$, then $\delta_\lambda(A)=0$ and $\delta_\lambda(B)=0$. Therefore, $\delta_\lambda(A\cup B)=0$ and $A\cup B$ is different from $\mathbb{N}$. So, there exists $k\in\mathbb{N}$ such that $(f_k(x_0),z)\in\mathcal{N}'_{(f(x_0),z)}(\frac{\varepsilon}{3},\gamma)$ and $(f_k(x),z)\in\mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{3},\gamma)$. It follows that

$$\begin{aligned}F'_{(f(x)-f(x_0),z)}(\varepsilon)&\geq F'_{(f(x)-f_m(x),z)}\left(\frac{\varepsilon}{3}\right)*\left[F'_{(f_m(x_0)-f_m(x_0),z)}\left(\frac{\varepsilon}{3}\right)*F'_{(f_m(x_0)-f(x_0),z)}\left(\frac{\varepsilon}{3}\right)\right]\\&>(1-\gamma)*[(1-\gamma)*(1-\gamma)]\\&>(1-\gamma)*(1-\gamma)\\&>1-\gamma\\&>1-\eta.\end{aligned}$$

This implies that f is continuous (on X) with respect to F-topology.

References

1. Alsina, C., Schweizer, B., Sklar, A.: On the definition of a probabilistic normed space. Aequationes Math. **46**, 91–98 (1993)
2. Asadollah, A., Nourouz, K.: Convex sets in probabilistic normed spaces. Chaos, Solitons Fractals **36**, 322–328 (2008)
3. Balcerzak, M., Dems, K., Komisarski, A.: Statistical convergence and ideal convergence for sequences of functions. J. Math. Anal. Appl. **328**, 715–729 (2007)
4. Caserta, A., Giuseppe, D., Kočinac, L.: Statistical convergence in function spaces. Abstr. Appl. Anal. **2011**, 1–11 (2011)
5. Fast, H.: Sur la convergence statistique. Colloquium Math. **2**, 241–244 (1951)
6. Fridy, J.: On statistical convergence. Analysis **5**, 301–314 (1985)
7. Gähler, S.: Lineare 2-normietre räume. Math. Nachr. **28**, 1–43 (1964)

8. Giuseppe, D., Kočinac, L.: Statistical convergence in topology. Topol. Appl. **156**, 28–45 (2008)
9. Golet, I.: On probabilistic 2-normed spaces. Novi Sad J. Math. **35**, 95–102 (2006)
10. Gürdal, M., Pehlivan, S.: The statistical convergence in 2-banach spaces. Thai J. Math. **2**, 107–113 (2004)
11. Karakaya, V., Şimşek, N., Ertürk, M., Gürsoy, F.: λ-statistical convergence of sequences of functions with respect to the intuitionistic fuzzy normed spaces. J. Funct. Spaces Appl. **2012**, 1–14 (2012)
12. Karakaya, V., Şimşek, N., Ertürk, M., Gürsoy, F.: Statistical convergence of sequences of functions with respect to the intuitionistic fuzzy normed spaces. Abstr. Appl. Anal. **2012**, 1–19 (2012)
13. Menger, K.: Statistical metrics. Proc. Nat. Acad. Sci. **28**, 535–537 (1942)
14. Mursaleen, M.: λ-statistical convergence. Math. Slovaca **50**, 111–115 (2000)
15. Mursaleen, M.: On statistical convergence in random 2-normed spaces. Acta Sci. Math. (Szeged) **76**, 101–109 (2010)
16. Savaş, E.: On generalized statistical convergence in random 2-normed space. Iran. J. Sci. Technol. **A4**, 417–423 (2012)
17. Savaş, E., Gürdal, M.: Certain summability methods in intuitionistic fuzzy normed spaces. J. Intell. Fuzzy Syst. **27**, 1621–1629 (2014)
18. Savaş, E., Mohiuddine, S.: λ-statistically convergent double sequences in probabilistic normed spaces. Math. Slovaca **62**, 99–108 (2012)
19. Schweizer, B., Sklar, A.: Probabilistic Metric Spaces. Elsevier Science Publishing Co., New York (1983)
20. Sempi, C.: A short and partial history of probabilistic normed spaces. Mediterr. J. Math. **3**, 283–300 (2006)
21. Serstnev, A.: On the notion of a random normed space. Dokl. Akad. Nauk SSSR **149**, 280–283 (1963)